Quantum Theory from Probability Conservation

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In this work, we derive the standard formalism of quantum theory by analyzing the behavior of single-variable systems under measurements. These systems, with minimal information capacity, exhibit indeterministic behavior in independent measurements while yielding probabilistically predictable outcomes in dependent measurements. Enforcing probability conservation in the probability transformations leads to the derivation of the Born rule, which subsequently gives rise to the Hilbert space structure and the Schrödinger equation. Additionally, we show that preparing physical systems in coherent states —crucial for observing quantum phenomena— effectively reduces the number of independent variables to one. This first-principles derivation of quantum theory from probability conservation in single-variable systems offers new insights into the physical meaning of quantum theory and clarifies its domain of applicability.

I. INTRODUCTION.

Quantum mechanics is a cornerstone of modern physics, providing fundamental insights into the behavior of particles at atomic and subatomic scales. It has been extraordinarily successful in explaining phenomena such as atomic structure and electron behavior in solids. Despite its overwhelming empirical success, the foundational principles of quantum theory –particularly the derivation of its core equations from fundamental physical laws– remain an open question. Traditional approaches to quantum theory often rely on abstract, axiomatic postulates [1-9], leaving a gap in explanations grounded in first physical principles.

In this paper, we aim to bridge this gap by deriving quantum theory through the study of physical systems with a single independent variable. Due to their extremely limited information capacity, these systems exhibit indeterministic behavior upon measurements, best described probabilistically. By enforcing the principle of probability conservation in the transformations across different measurement scenarios, we derive the core components of quantum theory. Furthermore, we demonstrate that the coherence requirement in conducting quantum experiments effectively reduces the number of independent variables in the physical systems, transforming them into single-variable systems.

Our approach establishes a framework where the standard formalism of quantum theory naturally emerges from the principle of probability conservation in single-variable systems, plays a fundamental role in their behavior. This derivation, grounded in intuitive physical principles, provides new insights into the origins of quantum phenomena. Beyond its theoretical significance, this framework has the potential to drive future advancements in both quantum technologies and fundamental physics.

II. METHODS

A. Informational description of physical systems

Translating concepts from physics into the language of information theory is relatively straightforward. In physics, measurements are performed to obtain outcomes. A physical measurement M^K with N possible outcomes m_1^K to m_N^K and their corresponding probabilities p_1^K to p_N^K (in which $\sum_i^N p_i^K = 1$), can be translated into an informational query Q^{K}, asking, "What is the outcome of measurement M^K?", with N possible answers q_1^K to q_N^K, corresponding to the same probabilities. The relationships between dependent and independent measurements mirror those between dependent and independent questions. These parallels allow the full apparatus of information theory –such as Shannon entropy, information content, information gain– to be applied in the analysis of physical systems.

Physical systems inherently contain and convey pieces of information. Each independent variable within a physical system can be considered a carrier of a single piece of information, or a message. Accordingly, the number of independent messages a physical system can convey is equal to the number of its independent variables. The length of each message is determined by the number of possible configurations of the variable. For instance, a variable with eight possible states can encode a three-bit message.

B. Single-variable physical systems

Microscopic systems often involve only a small number of independent variables. At the most fundamental level, certain systems can be entirely described by a single variable. These systems are inherently limited to carrying only one piece of information at a time, a property that uniquely distinguishes them from classical systems, which can simultaneously convey multiple independent messages.

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A notable consequence arises when attempting to extract two independent pieces of information from a single-variable system that can retain only one. From an information-theoretic perspective, the second piece of information cannot provide any additional value, and zero information equates to randomness. This suggests that performing independent measurements on such a system will inevitably yield random outcomes. The observed randomness is a direct result of the system's inability to store more than one piece of information, rather than being due to any underlying hidden variables.

While independent measurements on single-variable systems yield random outcomes, dependent measurements allow for probabilistic analysis of their behavior. In dependent measurements, the probability of obtaining a specific outcome in one measurement correlates with the outcome in the other. These probabilistic correlations are quantified by the conditional probabilities between the outcomes, with $p(m_i^L|m_j^K)$ representing the probability of obtaining outcome m_i^L in measurement L given outcome m_j^K in measurement K. The law of total probability ensures that for a given outcome in the first measurement, the sum of conditional probabilities equals one, i.e., $\sum_i p(m_i^L|m_j^K) = 1$, for fixed j.

C. Probability calculations

Given the probabilities of the system for measurement K, $\boldsymbol{P}^K = [p_1^K, \dots, p_N^K]^T$, the probabilities for the outcomes of measurement L can be calculated using their conditional probabilities as $p_j^L = \sum_i p(m_j^L|m_i^K)\, p_i^K$. The mapping $P_{ij}^{LK} = p(m_i^L|m_j^K)$ preserves total probability: $\sum_i p_i^L = \sum_i \sum_j P_{ij}^{LK}\, p_j^K = \sum_j p_j^K \sum_i P_{ij}^{LK} = \sum_j p_j^K \sum_i p(m_i^L|m_j^K) = \sum_j p_j^K = 1$.

Conversely, mapping the probabilities from measurement L to measurement K involves $P_{ij}^{KL} = p(m_i^K|m_j^L)$, and the probabilities can be calculated as $p_j^K = \sum_i P_{ji}^{KL}\, p_i^L$.

While these mappings conserve total probability, they are not bidirectionally reversible. Reversibility would require $p_j^K = \sum_i P_{ji}^{KL}\, p_i^L = \sum_i P_{ji}^{KL} \sum_n P_{in}^{LK}\, p_n^K$, for all j, n, or equivalently,

$$\sum_i P_{ji}^{KL} P_{in}^{LK} = \delta_{jn}. \tag{1}$$

This condition, however, is generally not satisfied, since the multiplication of positive values cannot sum to zero. The positivity constraint on the values of the mapping elements is what prevents bidirectional reversibility in the direct mapping of the probabilities.

To establish bidirectional mappings, the mappings $\sigma_j^K = \sum_i \rho_{ij}^{KL}\, \sigma_i^L$ (and similarly for K and L switched) must allow non-positive elements, $0 \le \left|\rho_{ij}^{KL}\right| \le 1$. Consequently, a probability measure capable of handling non-positive values is required. This measure must preserve probabilities at 0 and 1 (i.e., $P(0) = 0$, and $P(1) = 1$), be a monotonic single-parameter function with $P(x) \in [0,1]$, and be scale-invariant (i.e., $P(\lambda x) = \lambda^\alpha P(x)$).

These criteria lead to the following power-function probability measures:

$$P(x) = |x|^\alpha, \alpha \in \mathbb{R}^+. \tag{2}$$

This is the most general form of probability measures satisfying these conditions and allowing non-positive values. The revised mappings are thus given by $\sigma_j^K = \sum_i \rho_{ij}^{KL}\, \sigma_i^L$, in which $\left|\rho_{ij}^{KL}\right|^\alpha = p\left(m_i^K \middle| m_j^L\right)$ and $|\sigma_i^L|^\alpha = p_i^L$.

III. RESULTS

A. The square probability measure

We solve for the exponent α that enables bidirectional reversible mappings, satisfying the condition:

$$\sum_n \rho_{an}^{LK} \rho_{nb}^{KL} = \sum_i \rho_{ai}^{KL} \rho_{ib}^{LK} = \delta_{ab}, \tag{3}$$

while conserving total probability. The law of total probability implies $\sum_n P\left(m_n^{\mathrm{K}} \middle| m_i^{\mathrm{L}}\right) = 1$, or equivalently,

$$\sum_n |\rho_{ni}^{KL}|^\alpha = 1. \tag{4}$$

Equating this with the (i,i) element of $\boldsymbol{\rho}^{LK}\boldsymbol{\rho}^{KL}$ from (3) yields $\sum_n \rho_{in}^{LK} \rho_{ni}^{KL} = \sum_n |\rho_{ni}^{KL}|^\alpha$. This must hold term by term, implying:

$$\rho_{in}^{LK} = |\rho_{ni}^{KL}|^{(\alpha-1)}. \tag{5}$$

Combining this with the law of total probability for $\boldsymbol{\rho}^{LK}$, i.e., $\sum_i |\rho_{in}^{LK}|^\alpha = \sum_i P\left(m_i^{\mathrm{L}} \middle| m_n^{\mathrm{K}}\right) = 1$, leads to:

$$1 = \sum_i |\rho_{in}^{LK}|^\alpha = \sum_i \left||\rho_{ni}^{KL}|^{(\alpha-1)}\right|^\alpha = \sum_i |\rho_{ni}^{KL}|^{(\alpha^2-\alpha)}. \tag{6}$$

Employing (5) along with the (n,n) element of $\boldsymbol{\rho}^{KL}\boldsymbol{\rho}^{LK}$ leads to:

$$1 = \sum_i \rho_{ni}^{KL} \rho_{in}^{LK} = \sum_i \rho_{ni}^{KL} |\rho_{ni}^{KL}|^{(\alpha-1)} = \sum_i |\rho_{ni}^{KL}|^\alpha. \tag{7}$$

Finally, comparing (6) and (7) yields $\alpha^2 - \alpha = \alpha$, which uniquely determines $\alpha = 2$, resulting in the square probability measure:

$$P(x) = |x|^2, \tag{8}$$

based on the square of the probability amplitudes.

Unitarity of the mappings

Identifying the square probability measure as the one that conserves total probability in bidirectional mappings implies that it is the probability amplitude state of the system, $\boldsymbol{\sigma}^K = [\sigma_1^K, \dots, \sigma_N^K]^T$, rather than the probability itself, that can

consistently transform in these mappings. The probability amplitudes are transformed between the measurements according to:

$$\boldsymbol{\sigma}^L = \boldsymbol{\rho}^{LK}\boldsymbol{\sigma}^K, \tag{9}$$

in which

$$\left|\rho_{ij}^{LK}\right|^2 = p\left(m_i^L \middle| m_j^K\right), \tag{10}$$

and

$$|\sigma_i^K|^2 = p_i^K. \tag{11}$$

Moreover, substituting $\alpha = 2$ into (4) and (7) shows that the sum of the squares of the mapping elements in each row and each column equals one, indicating the unitary nature of the mappings.

Mathematically, our goal was to identify a diffeomorphic transformation between probability spaces associated with different measurements. We found that unitary mappings, along with the square probability measure, uniquely preserve total probability within bidirectional mappings. Specifically, the unitary transformations conserve total probability by preserving the length of the probability amplitude vectors in the mappings: $\sum_i^N |\sigma_i^L|^2 = \sum_i^N |\sigma_i^K|^2 = \sum_i^N p_i^K = 1$.

B. The Hilbert space formalism

The Hilbert space formalism seamlessly emerges in our framework from the derived square probability measure and unitary mappings. Specifically, unitary transformations of the system's probability amplitude state correspond to rotations of the system's state vector. Using conventional bra-ket notation, the probability amplitude state of the system can be written as a vector within an N-dimensional vector space spanned by the independent outcomes of a specific measurement m_1^K to m_N^K: $|\sigma^K\rangle = \sum_i \sigma_i^K\, |m_i^K\rangle$, which is of unit length, $\langle\sigma^K|\sigma^K\rangle = \sum_i^N |\sigma_i^K|^2 = 1$. These state vectors are mapped between different measurement's probability spaces through unitary transformations according to $|\sigma^K\rangle = \boldsymbol{\rho}^{\mathrm{LK}}|\sigma^K\rangle$.

C. The Born probability rule

The Born rule for calculating probabilities is inherently embedded in our framework through the derived square probability measure. The probability of a specific measurement outcome is given by the square of the inner product between the system's current state and the outcome state:

$$P(m_a^L|m_b^K) = |\rho_{ab}^{LK}|^2 = |\langle\sigma_b^K|\sigma_a^L\rangle|^2. \tag{12}$$

Our derivation shows that the Born rule is a conservation law, which is a direct consequence of probability conservation in bidirectional transformations between different measurements.

D. The Schrödinger equation

Time evolution can be incorporated into our framework by associating the transformations with a time variable:

$$\boldsymbol{\sigma}^L = \boldsymbol{\rho}^{LK}(t)\boldsymbol{\sigma}^K. \tag{13}$$

Given that the correlations between the measurements are time-independent, the time evolution of $\boldsymbol{\rho}^{LK}(t)$ is governed by a unitary transformation: $\boldsymbol{\rho}^{LK}(t_2) = \mathbf{U}^{-1}(t_2 - t_1)\boldsymbol{\rho}^{LK}(t_1)\mathbf{U}(t_2 - t_1)$. Under standard assumptions about time evolution (continuity and $\mathbf{U}(0) = \mathbf{1}$), this transformation can be expressed as $\mathbf{U}(t_2 - t_1) = e^{-i\boldsymbol{H}(t_2 - t_1)}$, in which $\boldsymbol{H}$ is a self-adjoint operator (the Hamiltonian) governing the system's dynamics. Shifting the focus to the time evolution of the state itself, rather than the transformations (i.e., switching from the *Heisenberg picture* to the *Schrödinger picture*), leads to the following equation for the time evolution of the state:

$$\frac{d}{dt}\boldsymbol{\sigma}^{\mathrm{K}}(t) = -i\boldsymbol{H}\boldsymbol{\sigma}^{\mathrm{K}}(t). \tag{14}$$

In the above, we have developed a mathematical framework to describe the behavior of single-variable physical systems with a finite number of states. This framework can be readily extended to the infinite-dimensional case without complications. Remarkably, our framework directly leads to the core formalism of the standard quantum theory: the Hilbert Space operator algebra, the Born probability rule, and the Schrödinger equation. By identifying the minimal physical principles required to establish the mathematical foundation of quantum theory, we have revealed the principle of probability conservation in single-variable physical systems as the cornerstone of the entire framework.

IV. DISCUSSION

Quantum mechanics (QM) is traditionally seen as the framework for describing microscopic systems and elementary particles. However, this perspective has been contested, as the QM formalism does not inherently impose a scale limit, and there appears to be no clear boundary between classical and quantum domains, often referred to as the *Heisenberg cut*. While Schrödinger's cat thought experiment [10] was originally conceived to critique the application of QM to macroscopic objects, numerous experiments have since expanded the scale at which quantum effect can be observed. This includes interference experiments with increasingly larger macromolecules [11-15]. However, whether QM can serve as a truly universal theory remains an open question.

In this paper, we derived quantum theory as the governing framework for physical systems with a single independent

variable. Our findings not only establish quantum theory from first principles but also offer a novel perspective on the nature of quantum systems. Specifically, we propose that quantum systems can consistently be described as physical systems with no more than a single independent variable. In what follows, we demonstrate how this interpretation aligns with the physical realization of quantum experiments and clarifies the scope of quantum theory's applicability.

A. Coherence

The success of quantum experiments relies on preparing and maintaining systems in coherent states, typically achieved under stringent laboratory conditions, such as extremely low temperatures or high fields. In these conditions, many of the system's variables are effectively constrained, reducing the number of independent variables. We assert that strong coherence, essential for observing quantum phenomena, reduces the number of independent variables of the systems to a single variable, which becomes the primary focus of the measurement.

To illustrate this, consider the structure of electron guns, which, along with lasers, are commonly used in quantum experiments as sources of coherent electrons and photons, respectively. An electron gun (Fig.1) extracts electrons from a heated filament via thermionic emission. The electrons are then accelerated by an anode in a collimator, directing them into a uniform beam. This design ensures that the electrons possess identical energy and identical momentum, leaving spin as their only independent variable. Similarly, a coherent beam of light consists of photons with equal energy and equal direction, leaving polarization as the only variable. While the availability of laser pointers has made the demonstration of interference phenomena very accessible, it is important to note that ordinary incoherent light sources do not produce interference patterns.

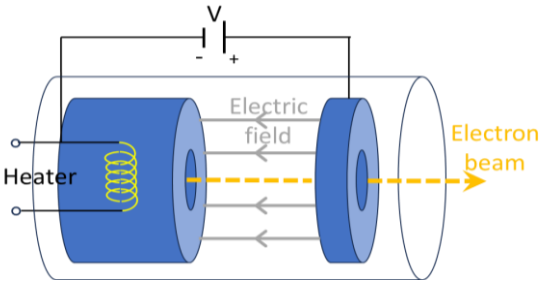

FIG 1. **Schematic of an electron gun structure, a device that produces single-variable systems (not to scale):** Electrons are produced through thermionic emission from a heated filament at the cathode. Attracted to the anode, they travel in parallel lines through a collimator. The applied voltage accelerates the electrons to high velocities. The resulting electron beam consists of electrons with identical energy and identical momentum, leaving spin as the only independent variable. This device effectively generates a stream of single-variable particles.

Achieving coherence involves fixing the values of a system's variables, which becomes increasingly difficult as the size of the system and, concurrently, the number of variables increase. Figure 2 illustrates the patterns observed in interference experiments with particle beams exhibiting varying degrees of coherence. The appearance of interference fringes is directly linked to the presence of coherence within the population. While coherence can be easily achieved in microscopic systems, macroscopic systems, such as golf balls, cannot be prepared in coherent states, and their interference patterns lack undulation.

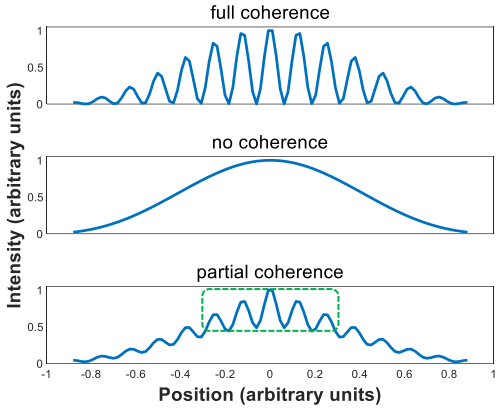

FIG 2. **Effect of coherence on the intensity of interference fringes:** full coherence (top), no coherence (middle), and partial coherence (bottom). In the case of partial coherence, the fringe minima do not reach zero, as observed in interference experiments with macromolecules. However, framing the fringes can make them resemble those in fully coherent systems (see dashed line inset).

At intermediate scales, full coherence remains elusive, although partial coherence is achievable. This results in hybrid interference patterns (Figure 2, bottom), in which interference fringes are superimposed on classical pattern, indicating the presence of quantum effects beside the classical expectations. Unlike in fully coherent cases, interference fringe minima in partially coherent cases do not reach zero, as observed in experiments with macromolecules. The results, however, can be framed or rescaled (see, for example, Refs. [15-17]) to resemble full coherence, creating the impression of full quantum coherence even in macroscopic systems.

B. The Heisenberg cut

Our results offer new insights into the Heisenberg cut, suggesting that the boundary where quantum theory ceases to apply is not determined by the size of a system, but by the number of its independent variables. Systems with many independent variables, such as cats or measuring devices, do not qualify as single-variable systems and thus fall outside of the formalism of quantum theory. On the other hand, quantum theory can be applied on large scales in degenerate stars [18, 19], which exist in coherent states under extreme gravitational

fields. Systems with partial coherence exhibit a blend of quantum and classical behavior, with quantum effects manifesting according to the degree of coherence.

C. The state function

Our derivation also sheds light on the nature of the quantum state function, which we identify as the probability amplitudes of single-variable systems corresponding to different measurement outcomes. Essentially, the state function represents the system's propensity for specific measurement outcomes. This propensity depends on both the system's current state and the type of measurement being performed. While this relational property between the system and the measurement is objective, it does not correspond to any direct physical attribute. Borrowing from philosophical terminology [20], quantum state functions are best understood as epistemic realities, rather than ontological objective realities.

That being said, one must still address where in the physical world these propensities reside. The present derivation suggests that the propensities are embedded in the transformations that map the state of the system to the planned measurement, $\boldsymbol{\rho}^{LK}$, which, in turn, reflect the correlations between two measurements –one that defines the state of the system, and the other, the planned measurement. As our derivation demonstrates, the transformations that map the state function between different measurements involve the conditional probability amplitudes of the measurement outcomes. Therefore, the propensities are contained within the correlations between the measurements, not in the single-variable system itself.

D. Quantum entanglement

An intriguing implication of our findings relates to quantum entanglement. Single-variable physical systems can, in principle, be multipartite and physically extended, with parts separated by large distances. As we shall see, this is the case for entangled quantum systems. Consider, for example, the generalized Greenberger-Horne-Zeilinger (GHZ) state [21], which is a superposition of n subsystems, all in the state $|\uparrow\rangle$ with all in the state $|\downarrow\rangle$, represented as $|\mathrm{GHZ}>= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle^{\otimes n} + |\downarrow\rangle^{\otimes n}\right)$. Although a collection of n single-variable subsystems might suggest n independent variables in the system, the $n-1$ correlations between the subsystems in the GHZ state (i.e., $|S_1\rangle = |S_2\rangle$, $|S_2\rangle = |S_3\rangle$, …, and $|S_{n-1}\rangle = |S_n\rangle$, where $|S_i\rangle$ represents the states of a subsystem) reduce the number of independent variables of the system to just one. Thus, the entire n-component entangled system comprises a single variable.

The unified description of quantum systems –single-variable physical systems that can convey only one piece of information– highlights the fundamental role of shared information in quantum entanglement. In entangled quantum systems, the entire system conveys only one piece of information, enforcing perfect correlations between subsystems, regardless of their physical separation.

V. CONCLUSIONS

This work presents a novel perspective on the foundations of quantum theory, grounded in single-variable physical systems that exhibit inherently indeterministic behavior. By deriving quantum theory from first principles and specifying its domain of applicability, we provide new insights into key concepts such as coherence, the quantum-classical boundary, and entanglement. This new lens can also be applied to reinterpret other well-established quantum concepts.

The implications of our derivation extend beyond theoretical insights. By providing clear criteria for observing quantum effects, our findings deepen understanding of the quantum realm, its limitations, and potential extensions. These insights also offer opportunities for advancing quantum technologies. Ultimately, the consistent framework for understanding quantum physics presented here has the potential to reshape and enrich our comprehension of the fundamental physical principles that govern the universe.

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