

The General Solution for a linear Second Order Homogenous Differential Equations with Variable Coefficients

Rehab A .Shaaban

Dep .of Mathematics .College of Education for girls. University of Kufa . Iraq

Abstract : The main goal in this work to find the general solution for some kind of linear second order homogenous differential equations with variable coefficients which have the general form $y'' + P(x)y' + Q(x)y = 0$, by using the substitution

$y = e^{\int Z(x)dx}$,which transform form the above equation to Riccati equation .

Keyword :- Differential equations , Riccati equation , Bernoulli equation .

1. INTRODUCTION

Many researchers in this field of differential equations , may face a difficult in solving the linear second order differential equations by using known methods . Therefore ,they are trying to solve these equations by using the power series or the Frobenius method [1] .

Kathem [2] gave a method for solving the above equation ,this method depends to find a function $Z(x)$ such that

$y = e^{\int Z(x)dx}$. Kathem [2] only gave examples which enable to find the general solution by using this substitution .

2- Bernoulli Equation [3]

The general form that of **Bernoulli equation** has is written as

$$y' + p(x)y = q(x)y^n \quad n \neq 1$$

where p and q are functions of, x (or constants)

3- Riccati Equation [4]

The general form of Riccati equation is written as

$$y' = f(x) + g(x)y + h(x)y^2 \quad \dots(1)$$

where f , g and h are given functions of x (or constants) . We can solve it , if one or more particular solutions of (1) can

be found by inspection or otherwise . The general solution of (1) is easy to be obtained by the following conditions

i- If y_1 is a known particular solution , then the general solution

can be obtained by the assumption :-

$$U = y - y_1 ,$$

then (1) transformed into Bernoulli equation

$$U'(x) + (g + 2hy_1)U = hU^2 ,$$

so , the general solution of (1) is given by

$$(y - y_1) \left[c - \int h(x) \chi(x) dx \right] = \chi(x) ; \chi(x) = e^{\int (g + 2hy_1) dx}$$

ii- If y_1 and y_2 are two known particular solutions, then the general solution of (1) can be found by the assumption

$$U(y - y_2) = y - y_1,$$

then the general solution is given by :-

$$y - y_1 = C(y - y_2) e^{\int h(x) (y_1 - y_2) dx},$$

where C is an arbitrary constant.

iii- If y_1, y_2 and y_3 , are three known particular solutions, say then the general solution of equation (1) is given as :

$$\frac{(y - y_1)(y_3 - y_2)}{(y_3 - y_1)(y - y_1)} = C,$$

where C is an arbitrary constant

4- How Find The General Solution for the Linear Second Order Differential Equations

We can solve the equation

$$y'' + P(x)y' + Q(x)y = 0 \quad \dots(2)$$

by the following cases:

i- If $P(x)$ and $Q(x)$ are constants say $P(x) = a$ and $Q(x) = b$ then the equation (2) becomes

$$Z'(x) + Z^2(x) + aZ(x) + b = 0 \quad \dots(3),$$

and the solution of (3) is given by :-

a)
$$y = e^{-\frac{a}{2}x} \left(C_1 \cos \sqrt{b - \frac{a^2}{4}}x + C_2 \sin \sqrt{b - \frac{a^2}{4}}x \right),$$

if $b \neq \frac{a^2}{4}$, where α and β are arbitrary constants

b)
$$y = A e^{-\frac{a}{2}x} (x + C),$$

if $b = \frac{a^2}{4}$, where A and C are arbitrary constants.

proof :-

a) Since $Z'(x) + Z^2(x) + aZ(x) + b = 0$, so

$$\frac{dZ}{\left(Z + \frac{a}{2}\right)^2 + b - \frac{a^2}{4}} + dx = 0 \Rightarrow \int \frac{dZ}{\left(Z + \frac{a}{2}\right)^2 + d^2} = -x + C; d^2 = b - \frac{a^2}{4}$$

$$\Rightarrow \frac{1}{d} \tan^{-1}\left(\frac{Z + a/2}{d}\right) = -x + C \Rightarrow Z = d \tan(f - dx) - \frac{a}{2}; f = dc$$

$$y = e^{\int \left(d \tan(f - dx) - \frac{a}{2}\right) dx} = e^{\ln \cos(f - dx) - \frac{a}{2}x + g}$$

$$y = A e^{-\frac{a}{2}x} \cos(f - dx); A = e^g$$

$$y = e^{-\frac{a}{2}x} \left(C_1 \cos \sqrt{b - \frac{a^2}{4}} x + C_2 \sin \sqrt{b - \frac{a^2}{4}} x \right); \text{ where } C_1 = A \cos f \text{ and } C_2 = A \sin f$$

4.1.Example:- For solving the differential equation

$$y'' + 2y' - 3y = 0, a = 2 \text{ and } b = -3,$$

$$\text{since } b \neq \frac{a^2}{4}$$

Then, by using the above formula, we get the general solution which has the form

$$y = e^{-x} (C_1 \cos \sqrt{-3-1} x + C_2 \sin \sqrt{-3-1} x) \Rightarrow y = e^{-x} (C_1 \cos 2ix + C_2 \sin 2ix)$$

$$y = e^{-x} \left(C_1 \frac{e^{-2x} + e^{2x}}{2} + C_2 \frac{e^{-2x} - e^{2x}}{2i} \right) = A e^{-3x} + B e^x$$

$$\text{, where } A = \left(\frac{C_1}{2} + \frac{C_2}{2i} \right), B = \left(\frac{C_1}{2} - \frac{C_2}{2i} \right)$$

b) If

$$b = \frac{a^2}{4} \Rightarrow \frac{dz}{\left(Z + \frac{a}{2}\right)^2} + dx = 0 \Rightarrow -\frac{1}{Z + \frac{a}{2}} = C_1 - x \Rightarrow Z + \frac{a}{2} = \frac{1}{x + C}; C = -C_1$$

Since

$$y = e^{\int Z(x)dx} \Rightarrow y = e^{\int \left(\frac{1}{x+C} - \frac{a}{2} \right) dx} \Rightarrow y = e^{\ln(x+C) - \frac{a}{2}x + C_2}$$

$$y = A e^{-\frac{a}{2}x} (x+C) ; A = e^{C_2}$$

4.2.Example:-For solving the differential equation

$$y'' + 4y' + 4y = 0 ; a = 4, b = 4$$

we will use the general form in the above formula and we get

$$y = e^{-\frac{a}{2}x} (Ax + B) , B = AC \Rightarrow y = e^{-2x} (Ax + B)$$

ii-If $Q(x) = 0$, then the general solution is given by :-

$$y = A \int e^{-\int p dx} dx + B$$

proof:- Since

$$Z'(x) + Z^2(x) + PZ(x) + Q = 0 \Rightarrow Z' + Z^2 + PZ = 0 \Rightarrow Z' + PZ = -Z^2$$

this is like Bernoulli equation , to solve it , let $Z^{-1} = t \Rightarrow t' - Pt = 1$

this equation is linear , and its integrating factor is given by:-

$$I.F = e^{-\int p(x)dx} \Rightarrow e^{-\int p(x)dx} dt - t p(x) e^{-\int p(x)dx} dx = e^{-\int p(x)dx} dx$$

$$\Rightarrow e^{-\int p(x)dx} t = \int e^{-\int p(x)dx} dx \Rightarrow Z = \frac{e^{-\int p(x)dx}}{\int e^{-\int p(x)dx} dx}$$

$$y = e^{\int \frac{e^{-\int p(x)dx}}{\int e^{-\int p(x)dx} dx} dx}$$

$$y = e^{\ln \int e^{-\int p(x)dx} dx + C_1} = A \int e^{-\int p(x)dx} dx + B , A = e^{C_1}$$

4.3.Example :- For solving the differential equation

$$y'' - \frac{2}{x} y' = 0 ,$$

we use the general form in the above formula and we get

$$y = A \int e^{-\int p(x)dx} dx + B \quad \Rightarrow \quad y = A \int e^{\int \frac{2}{x} dx} dx + B$$

$$y = A \int e^{2 \ln x} dx + B \quad \Rightarrow \quad y = A_1 x^3 + B \quad ; A_1 = \frac{A}{3}$$

iii-If $P(x) = 2\sqrt{Q(x)}$, then the equation (3) can be solved by the assumption $u = Z(x) + \sqrt{Q(x)}$, since

$$Z' + Z^2 + PZ + Q = 0 \Rightarrow Z' + Z^2 + 2\sqrt{Q} + Q = 0 \Rightarrow Z' + (Z + \sqrt{Q})^2 = 0$$

to solve this equation, let

$$u = Z + \sqrt{Q} \Rightarrow Z' = u' - \frac{Q'}{2\sqrt{Q}} = u' - \frac{Q'}{P} \Rightarrow u' - \frac{Q'}{P} + u^2 = 0 \Rightarrow u' + u^2 = \frac{Q'}{P}$$

this is Riccati equation, with $f(x) = 1, g(x) = 0$ and $k(x) = \frac{Q'}{P}$

Now, there are many cases

1-If u_1 is a known solution to the last equation, then the general solution is given by

$$y = A e^{\int (u_1 - \sqrt{Q(x)}) dx} e^{-2 \int u_1 dx}$$

proof:-

The assumption $d = u - u_1$ transforms the equation to Bernoulli equation which has the form:-

$$d' + d^2 + 2u_1 d = 0 \Rightarrow d' + 2u_1 d = -d^2$$

to solve it, we set

$d^{-1} = t \Rightarrow -d^{-2} d' = t' \Rightarrow d^{-2} d' = -t' \Rightarrow t' - 2u_1 t = 1$, this is linear equation, and its integrating factor is given by :-

$I.F = e^{-\int 2u_1 dx}$, so the general solution of the last equation is given by :-

$$e^{-\int 2u_1 dx} \frac{d}{dx} \left(e^{\int 2u_1 dx} u \right) = e^{-\int 2u_1 dx} \frac{d}{dx} \left(e^{\int 2u_1 dx} u_1 \right)$$

$$u - u_1 = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} \Rightarrow u = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1$$

$$Z + \sqrt{Q(x)} = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1 \Rightarrow Z = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1 - \sqrt{Q(x)}$$

$$y = e^{\int \left(\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx} dx} + u_1 - \sqrt{Q(x)} \right) dx}$$

$$y = e^{\ln \int e^{-\int 2u_1 dx} dx + \int (u_1 - \sqrt{Q(x)}) dx + C}$$

$$y = A e^{\int (u_1 - \sqrt{Q(x)}) dx} \int e^{-\int 2u_1 dx} dx ; A = e^C$$

2-If u_1 and u_2 are two known solutions , then the general solution of the last equation is given by :-

$$y = e^{\int \left(\frac{u_1 - Cu_2 e^{\int (u_1 - u_2) dx}}{1 - C e^{\int (u_1 - u_2) dx}} - \sqrt{Q} \right) dx} ; C = \text{constant}$$

proof:- From Riccati equation we get

$$u - u_1 = C(u - u_2) e^{\int (u_1 - u_2) dx} ; C \text{ is any arbitrary constant}$$

so

$$u = \frac{u_1 - Cu_2 e^{\int (u_1 - u_2) dx}}{1 - C e^{\int (u_1 - u_2) dx}} \Rightarrow y = e^{\int \left(\frac{u_1 - Cu_2 e^{\int (u_1 - u_2) dx}}{1 - C e^{\int (u_1 - u_2) dx}} - \sqrt{Q} \right) dx}$$

3-if u_1, u_2 and u_3 are three known solutions , then the general solution of the last equation is given by :-

$$y = e^{\int \left(\frac{u_1 - CJ(x)u_2}{1 - CJ(x)} - \sqrt{Q(x)} \right) dx} ; J(x) = \left(\frac{u_3 - u_1}{u_3 - u_2} \right) ; C = \text{constant}$$

proof:- From Riccati equation we get :

$$\begin{aligned} \frac{u - u_1}{u - u_2} &= C \left(\frac{u_3 - u_1}{u_3 - u_2} \right) ; C \text{ is any arbitrary constant} \\ \Rightarrow \frac{u - u_1}{u - u_2} &= C J(x) ; J(x) = \left(\frac{u_3 - u_1}{u_3 - u_2} \right) \\ \Rightarrow u - u_1 &= C J(x)u - C J(x)u_2 \Rightarrow u = \frac{u_1 - CJ(x)u_2}{1 - CJ(x)} \\ \Rightarrow Z &= \frac{u_1 - CJ(x)u_2}{1 - CJ(x)} - \sqrt{Q(x)} \\ y &= e^{\int \left(\frac{u_1 - CJ(x)u_2}{1 - CJ(x)} - \sqrt{Q(x)} \right) dx} ; J(x) = \left(\frac{u_3 - u_1}{u_3 - u_2} \right) \end{aligned}$$

Note:- Some of these equations can be transformed into variable separable equations and don't need the above formula to find the general solution

4.5. Example :- For solving the differential equation

$$y'' + 2xy' + x^2y = 0 ; P(x) = 2x, Q(x) = x^2,$$

by using the equation (3) we get

$$Z' + Z^2 + 2xZ + x^2 = 0 \Rightarrow Z' + (Z + x)^2 = 0,$$

let

$$\begin{aligned} Z + x = t &\Rightarrow Z' = t' - 1 \\ \Rightarrow t' - 1 + t^2 = 0 &\Rightarrow \frac{dt}{1 - t^2} - dx = 0 \Rightarrow \tanh^{-1} t = x + C \Rightarrow t = \tanh(x + C) \\ \Rightarrow Z &= \tanh(x + C) - x \end{aligned}$$

since

$$y = e^{\int Z(x)dx} \Rightarrow y = e^{\int (\tanh(x+C) - x)dx}$$

$$y = e^{\ln \cosh(x+C) - \frac{1}{2}x^2 + a} \Rightarrow y = e^{-\frac{1}{2}x^2 + a} \cosh(x+C)$$

$$y = e^{-\frac{1}{2}x^2} (A \cosh x + B \sinh x) \quad ; A = e^a \cosh C, B = e^a \sinh C$$

iv) If $P(x)$ and $Q(x)$ are not any one of the above cases, then the equation $Z' + Z^2 + P(x)Z + Q(x) = 0$ is like Riccati equation. As a result then there are three cases:

1- If Z_1 is a known solution to it, then the general solution of (1) is given by :-

$$y = A e^{\int Z_1 dx} \int e^{-\int (P+2Z_1) dx} dx \quad ; A = e^a$$

Proof :-

The assumption $Z = Z_1 + u$ transforms the equation to Bernoulli equation which has the form:-

$$u' + (P + 2Z_1)u + u^2 = 0 \quad ,$$

to solve it, we assume $u^{-1} = t$

$\Rightarrow t' - (P + 2Z_1)t = 1$ this is a linear equation, and its integrating factor (I.F) is given by :-

$$I.F = e^{-\int (P+2Z_1) dx} \Rightarrow t \cdot e^{-\int (P+2Z_1) dx} = \int e^{-\int (P+2Z_1) dx} dx$$

$$\Rightarrow Z = \frac{e^{-\int (P+2Z_1) dx}}{\int e^{-\int (P+2Z_1) dx} dx} + Z_1 \Rightarrow y = e^{\int \left(\frac{e^{-\int (P+2Z_1) dx}}{\int e^{-\int (P+2Z_1) dx} dx} + Z_1 \right) dx}$$

$$\Rightarrow y = e^{\ln \int e^{-\int (P+2Z_1) dx} dx} e^{\int Z_1 dx + a}$$

$$\Rightarrow y = A e^{\int Z_1 dx} \int e^{-\int (P+2Z_1) dx} dx \quad ; A = e^a$$

2- If Z_1 and Z_2 are two known solutions of it, then the general solution of this equation is given by :-

$$y = e^{\int \left(\frac{Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx}}{1 - C e^{\int (Z_1 - Z_2) dx}} \right) dx} ; C = \text{constant}$$

proof:- From Riccati equation , we can write

$$Z \left(1 - C e^{\int (Z_1 - Z_2) dx} \right) = Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx} \Rightarrow Z = \frac{Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx}}{1 - C e^{\int (Z_1 - Z_2) dx}}$$

$$\Rightarrow y = e^{\int \left(\frac{Z_1 - C Z_2 e^{\int (Z_1 - Z_2) dx}}{1 - C e^{\int (Z_1 - Z_2) dx}} \right) dx}$$

3-If Z_1, Z_2 and Z_3 are three known solutions of it , then the general solution of this equation is given by :-

$$y = e^{\int \left(\frac{Z_1 - C J(x) Z_2}{1 - C J(x)} \right) dx} ; C = \text{constant and } J(x) = \frac{Z_3 - Z_1}{Z_3 - Z_2}$$

proof:- From Riccati equation , we can write

$$\frac{Z - Z_1}{Z - Z_2} = C \left(\frac{Z_3 - Z_1}{Z_3 - Z_2} \right) ; C \text{ be any arbitrary constant}$$

$$\Rightarrow \frac{Z - Z_1}{Z - Z_2} = C J(x) ; J(x) = \frac{Z_3 - Z_1}{Z_3 - Z_2} \Rightarrow Z - Z_1 = C J(x) Z - C J(x) Z_2$$

$$Z = \frac{Z_1 - C J(x) Z_2}{1 - C J(x)} \Rightarrow y = e^{\int \left(\frac{Z_1 - C J(x) Z_2}{1 - C J(x)} \right) dx}$$

4.6..Example :- For solving the differential equation

$$y'' + \frac{2}{x} y' - \frac{2}{x^2} y = 0 ,$$

we use the general form in the above formula , which is

$$y = A e^{\int Z_1 dx} \int e^{-\int (P+2Z_1) dx} dx ; A = e^a ,$$

now, let $Z_1 = \frac{1}{x}$ (which is a particular solution of Riccati equation)

$$y = A e^{\int \frac{1}{x} dx} \int e^{-\int \frac{4}{x} dx} dx$$

$$y = A e^{\ln x} \int e^{-4 \ln x} dx = Ax \left(\frac{-x^{-3}}{3} + C_1 \right) = -\frac{A}{3} x^{-2} + Bx \quad ; B = AC_1$$

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