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ORIGINAL RESEARCH

Epistemic logic with partial grasp

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Abstract
We have to gain from recognizing a relation between epistemic agents and the parts of subject matters that play a role in their cognitive lives. I call this relation "grasping". Namely, I zone in on one notion of having a partial grasp of a subject matter—that of agents grasping part of the subject matter that they are attending to—and characterize it. I propose that giving up the idealization that we fully grasp the subject matters we attend to allows one to more realistically characterize the epistemic life of agents. To show this, I propose an epistemic logic with partial grasp that has in mind considerations from first-order aboutness theory with the aim of avoiding certain forms of logical omniscience, and which provides an alternative to immanent closure (Yablo Aboutness, Princeton University Press, 2014).

Keywords Subject matter · Aboutness · First order · Immanent closure · Epistemic logic

1 Introduction

There are various ways in which one can intuitively be said to *only* properly partially grasp a subject matter[1] that one is attending to.[2] One first way is when an agent is unable to provide informative answers to questions included in the topic at hand.[3] This

[1] In what follows I assume that what primarily agents grasp or fail to grasp are *subject matters* and not the sentences or propositions that express those same subject matters. This is so as, I take it, if two propositions or sentences φ and ψ share the same subject matter, then an agent grasps φ to the same extent that one grasps ψ. So it seems that the propositional grasping facts are grounded in the subject matter grasping facts, which are then more fundamental (i.e. more basic in our explanation).

[2] That is, "only" or "merely" partially grasping a subject matter as opposed to also fully grasping it (in which case one also partially grasps it).

[3] In what follows I assume that there is a correspondence between topics or subject matters (I use the two notions interchangeably) and questions, so that for each topic it is possible to identify a corresponding question. So, for instance when considering the topic **the number of stars** one is thereby considering the

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is for instance the case of a student who is taking a test on **Newtonian mechanics**[4] and is unable to answer the question *What is Newton's Second Law of motion?*[5]

A second way is when the agent fails to have any propositional attitude towards a topic that is properly included in the one they are said to have a proper grasp of, as in the case of one who has a good grasp of various topics falling under the general topic of **mathematics**, but who is unfamiliar with **topology**. The central idea behind this second way of thinking of partial grasp can be captured nicely into a slogan: partial grasp is grasp of a part.

What sets the second kind of case apart from the first is that agents might be unable to provide informative answers to given questions on a topic simply because they lack knowledge of how things stand in relation to the question at hand, whereas they can also fail to have propositional attitudes towards parts of a topic for other reasons, such as lacking the required conceptual resources.[6]

It is on the second of these ways one might say that someone has a partial grasp of a subject matter that I will focus on in this paper, as I am not directly concerned with whether one's knowledge limits one's grasp of a subject matter in the first sense, but with whether one's grasp of a subject matter (in the second sense) can limit one's knowledge.

In what follows, I further focus on simple questions that admit only of a single correct answer. This is so as to focus primarily on the presentation of the theory of partial grasp and ignore for now some of the details that are orthogonal to the core of the paper.

Having clarified at an informal level what having a partial grasp of a subject matter amounts to, in what follows I present a first formal approximation to the notion (Sect. 2). Afterwards, I briefly present and motivate a view on the topics of first-order formulae and designators that clashes with the principle of immanent closure (Sect. 3). I argue that this conflict arises from the fact that the principle of immanent closure does not make reference to what part of a subject matter a given agent grasps.[7] I then propose an alternative that does and that helps to account for some desiderata for epistemic closure involving general sentences (Sect. 3). Having done so, I present more formally the language and models for an epistemic logic with non-normal worlds

question *How many stars are there?*. In this I follow Lewis (1988a, b) and Yablo (2014). I assume that this correspondence holds at least for *How many* and *What* questions. The case of *How* and *Why* questions where these demand an explanation are perhaps less clear, as these might involve some necessary but asymmetric relation between *explanandum* and *explanans*, which I hope to address in the future. Still, having this connection in mind, by question inclusion, in what follows I will mean the corresponding subject matter inclusion.

4 Henceforth I use boldfaced notation for topics whenever I don't describe them as the topics of specific expressions.

5 A subject matter like that of **Newtonian mechanics** presumably includes various subject matters as parts, as can be seen by considering that one could also ask in such a test "What arc does a projectile trace?". When one asks this question, one is asking about the trajectory of a projectile, but since that is included in the topic of **Newtonian mechanics**, one can be said to also be asking a question *about* **Newtonian mechanics**.

6 This corresponds to the familiar distinction from awareness logics (Schipper, 2014) between agents simply lacking information (case in which they might not be able to give informative answers but still might fully grasp the subject matter) and lacking conception/being unaware of a certain topic.

7 Thanks to Thomas Randriamahazaka for calling my attention to this lack of connection in subject matter theory.

(Sect. 4) as well as adding to them partial grasp, motivating them philosophically and proving a number of desiderata tailor-made to avoid certain kinds of logical omniscience (Sect. 5). Finally, I explore the limits and possible developments of the view at an informal level (Sect. 6).

2 Partial grasp and coarser-grained partitions

Following Lewis (1988a, b), we can as a first approximation take a subject matter to be a partition of the set of possible worlds induced by an equivalence relation, where two worlds are in the same cell of the partition that is the subject matter **m** if and only if they are **m**-wise indistinguishable (adopting some terminology from Yablo, 2014).[8] For instance, if **m** is **the number of stars**, then worlds will be divided into cells according to whether they have 0 stars, 1 star, and so on, regardless of whatever else goes on in them.

Partitions are partially ordered by a relation of refinement, where P_2 refines P_1 if and only if each cell of P_2 is a subset of a cell of P_1 (Lewis, 1988a, b; Yablo, 2014, 2017). Then we say that P_1 is a coarser-grained partition than P_2 and that P_2 is more fine-grained than P_1. When the subject matter **A** refines the subject matter **B**, then we say that **B** is part of **A** ($\mathbf{B} \sqsubseteq \mathbf{A}$). On an intuitive level, **A** includes all the distinctions between ways reality can be that **B** includes, plus some others (i.e when **B** is a proper part of **A**, and so $\mathbf{A} \neq \mathbf{B}$).

Going back to the intuitive definition of having a partial grasp of a subject matter, it was said that to have *only* a partial grasp of a subject matter is to fail to fully grasp some part of the subject matter. On this view, that amounts to the same as not grasping all the possible distinctions between ways for reality to be in relation to the topic at hand. More precisely, for a given subject matter **m** the agent should recognize *some* but not all the ways for the world to be **m**-wise—for if they recognize no way at all, then they do not grasp at all the subject matter **m**.[9] Just like in various works in mereology one recognizes a relation of proper parthood as well as a relation of parthood encompassing the case where the part is the same as the whole, here too I will define partial grasp more generally in terms of a notion that allows for an agent who fully grasps a subject matter to thereby also partially grasp it. I will, however, be focusing on cases of proper partial grasp throughout. The notion of having a partial grasp of a subject matter can then be defined in terms of full grasp:

[8] Here I take this to be just an approximation as there are well-known problems (see for instance Yablo, 2014 and Berto, 2022 for discussion) affecting Lewis's account of subject matters. These are, for the most part, immaterial for the discussion in the first sections of the paper, so I work within this framework as it is both intuitive and simple. See Silva (2024a, b) for my considered view on what subject matters are, as well as how the differences between my preferred view and Lewis's interacts with the issue, also addressed here, of agents conflating between distinct questions. When presenting the formal epistemic logic I will, however, be making use of a modal space that includes non-normal worlds (following Berto and Jago 2019), for reasons that will be clearer later.

[9] I talk here simply of recognizing ways for things to be **m**-wise, but really this should be recognizing them as *distinct* ways for things to be **m**-wise, that is, in such a way that when an agent is able to recognize a way for things to be **m**-wise, they must be able to distinguish that way for things to be from the other ways for things to be **m**-wise. I will, therefore, also talk in what follows of recognizing the distinctions between those same ways.

PARTIAL GRASP: An agent, A, partially grasps a subject matter **m**, if and only if there is a subject matter **n** such that **n** is part of **m** and **n** $\neq$ **o** (where **o** is the trivial subject matter[10]) and A fully grasps **n**.

Full grasp of a subject matter can itself be defined as grasp of every part of a subject matter.[11] Having a grasp of a given subject matter in turn can be understood in terms of whether the subject matter plays a role in an agent's cognitive life. Whether a subject matter plays such a role can be seen for instance by whether the agent is capable of supposing that the world goes one way or the other with respect to the subject matter; by whether they're able to revise or update their beliefs with propositions that are answers to the question that corresponds to the subject matter; whether they could have a specific want that distinguishes between one of the possible answers to the question and the others; and so on, for other propositional attitudes that involve propositions for which the topic at hand is the exact topic of the proposition (Yablo, 2014). This, then, corresponds to the second way of grasping a subject matter considered above. The rough motivating idea behind calling such a relation "grasping" is the assumption that one can only have a propositional attitude towards φ if one is able to entertain (or "grasp") φ's topic.[12]

3 Immanent closure

According to my preferred account of the subject matter of first-order sentences, individual terms, predicates and open formulae (developed in Silva 2024a), the subject matter of $\exists x Fx$ is the same as the subject matter of $\forall x Fx$, and it includes the subject matter of all the sentences Fa where a is a designator that F meaningfully applies to (i.e. such that Fa is truth-evaluable).

Here I am not able to reproduce all the reasons that led me to this view, so I'll have to content myself with providing some highlights, in broad strokes. It is widely assumed in the literature on subject matters (Berto, 2022; Ferguson 2023a, b, c; Fine, 2017, 2020; Hawke, 2018) that the extensional Boolean connectives are subject matter transparent. Take, for instance, the case of negation. Then this requirement, using $\sigma(\cdot)$ for a function outputting a subject matter for a given expression of the language, is just that $\sigma(\neg A) = \sigma(A)$. Transposing to the case of first-order logic, it is also natural to take the quantifiers $\forall x$ and $\exists x$ to be part of the extensional logical vocabulary, and therefore to say for instance that $\sigma(\forall x Fx) = \sigma(Fx)$, and further that $\sigma(\exists x Fx) = \sigma(\forall x Fx)$.

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10 That is, the partition of the set of possible worlds making no distinctions between worlds (what Yablo (2014, p. 39) calls "whatever"): $\{W\}$, where W is the set of all possible worlds.

11 Notice, then, that it is the notion of grasp and not of full grasp that is a primitive in the theory.

12 This is not entirely correct, but serves to paint a general picture. As it will be seen later, I believe this only holds for particular propositions, whereas one might have propositional attitudes towards general (universal or existential) propositions even though one is only on top of (to borrow terminology from Berto, 2022) part of their topic.

This goes very well in hand with the Fregean thought that quantifiers are really second-order predicates.[13]

Taking this thought seriously leads us, then, to consider what the subject matters of predicates/open formulae are. I think that a natural view is to take the topic of a predicate to be the fusion of all the topics of sentences in which that predicate is featured. Consider for instance the predicate "is a square". I believe that two worlds can diverge when it comes to what is true in them concerning "is square" in at least one of two ways: (i) there being different things which in those worlds satisfy the predicate "is a square"; and (ii) there being different things which in those worlds fail to satisfy the predicate "is a square". Two worlds w_1 and w_2 having the same objects that are squares, may yet diverge for instance by one having a circle and the other not, and thereby it being true of one circle in one world that it is not a square. This is something true we can say *about* the predicate "is a square"—namely how it is differently instantiated across modal space.

It is roughly for this reason that I take the subject matter of general sentences to include the topics of all of their instances as parts. A serious worry for this conception of subject matters is that it flies in the face of certain constraints we would like to hold for doxastic and epistemic logic, if we accept the plausible principle of immanent closure (Yablo, 2014) for knowledge and belief (I'll focus on the case for knowledge).

Immanent Closure If an agent S knows that φ, then S knows that ψ if and only if:

(i) $\varphi \vDash \psi$
(ii) $\sigma(\psi) \sqsubseteq \sigma(\varphi)$

That is, knowledge is closed under logical consequence that does not add subject matter. If we accept this principle, we would have the following wrong validity and invalidity for the logic of knowledge (using K as the knowledge operator):

x $K\forall x Fx \vDash KFa$
x $KFa \nvDash K\exists x Fx$.

In the first case, the subject matter of Fa is part of the subject matter of $\forall x Fx$, and further there is a relation of entailment from the former to the latter. In the second case, the subject matter of $\exists x Fx$ will not in general be contained in the subject matter of Fa, so the second condition fails, even though there is entailment from the former to the latter.

My opponent would say that this raises a problem for my account of subject matters. I believe instead that this raises a worry for the principle of immanent closure itself, for the principle purports to tell us something about agents' relation to subject matters

[13] This is not to say that all logical expressions must be subject matter transparent, at least in the way that subject matter transparency is usually conceived of (topics of expressions containing a given expression being just the fusion of the topics of other expressions, not containing the relevant expression). Logan and Ferguson (forthcoming), as well as Ferguson (2023a, b, c) present a strong case for why this is the case. I myself am convinced by their arguments. However, I hope to tackle this topic very soon as I think that the counterexamples to the general thesis might be topic transparent in other ways. Still, the basic claim stands for *extensional* logical vocabulary. Thanks to an anonymous reviewer for pushing me to clarify my position on this.

without taking into account what part of each subject matter the agents *grasp*.[14] Let us refer to the biggest part of a subject matter that an agent grasps as σ_g, which will be unique as it will be the fusion of all parts of the subject matter that the agent grasps. Again, it corresponds to the part of the subject matter that plays a role in the agent's epistemic life.

In the first instance, what I would say is that we have a case where an agent knows a given proposition even though they don't fully grasp its topic, but only part of it, and that's why knowledge shouldn't transmit from premises to conclusion. This is so as the agent might not grasp the part of the topic corresponding to Fa. Plausibly instead, we should capture the condition imposed by Yablo's principle of immanent closure in terms of what's included in the part of the topic that the agent grasps: $\sigma(\psi) \sqsubseteq \sigma_g(\varphi)$. This would block the problematic inference from $K\forall x Fx$ to KFa, and, it seems, precisely in the right cases: when the agent does not have full grasp of the topic of the former.

One might worry at this point that this move is arbitrary. The whole point of introducing the notion of a grasp of a subject matter is to point to the parts of subject matters we attend to that play a role in our cognitive lives. Why should it be, then, that we can have propositional attitudes towards, say, a universal proposition when we don't have a full grasp of its topic? It seems that something has to give.

Even though this strategy might seem arbitrary, I believe there are good reasons to opt for it, due to differences between particular and universal propositions. The first one is that one's propositional attitudes towards universal propositions are often arrived at by induction based on propositional attitudes towards particular instances. When reasoning by induction from a limited number of instances, and if we think that the topics of the instances are included in the topic of the universal proposition (as argued above), then it is natural to think that we may have propositional attitudes towards the universal proposition even though we don't have propositional attitudes towards all the propositions whose topics the topic of the universal proposition includes.

This first reason depends on my view that the topic of a universal proposition contains the topics of its instances as parts. Suppose, then, that was not the case. Then from the knowledge of $\forall x Fx$ it wouldn't follow that one would know that Fa, as

[14] One might resist that there is indeed a problem for immanent closure by denying that it is the business of topical constraints to deal with all phenomena that give rise to failures of logical omniscience. So, only once it's been established that this pattern of validities and invalidities should be dealt with by the machinery of topics instead of by some other method (like fragmentation, or some other possible story one could tell) can it be established that there is a problem for immanent closure. The rest of the paper aims to show that there is a very simple story one can tell without leaving the remits of 'topicology' by simply introducing a partial function from topics to parts of topics, intuitively corresponding to the parts of topics that the agent grasps. The study of subject matters in how they interact with first-order logic is still in its infancy, but so far I can see no alternative way of rescuing immanent closure by other means that are as natural. For instance, it doesn't seem like the failure in this case of an agent who knows that a universal sentence is the case but not all of its instances need at least always be motivated by the fact that they hold beliefs in regard to general facts and to particular facts in different fragments of their belief system. Such an explanation fails to account for the plausible cases in which the agent simply isn't acquainted with all the instances of a predicate to begin with. Such an alternative plausible explanation that doesn't abandon immanent closure forthcoming, I can keep my claim that we should look for an alternative to immanent closure for how to close knowledge in its relation to topics. Thanks to Franz Berto for raising this concern and pushing me to clarify why I reject immanent closure.

wanted. So far, so good. Consider, however, the following inference $K\forall x(Fx \rightarrow Gx), KFa \vDash KGa$. If $\sigma(Fa) \not\sqsubseteq \sigma(\forall x(Fx \rightarrow Gx))$ then the subject matter of the conclusion won't be included in the fusion of the subject matter of the premises,[15] and we will get a violation of immanent closure.[16] But this seems implausible: if an agent knows that all F's are G's and that a is an F, then they seem to thereby know that a is a G as well.

This, I take it, is a second strong reason to accept that the topic of instances is included as part of the topic of universal sentences. But if this is so, then we must accept a revision of immanent closure, for otherwise we will validate the inference from $K\forall x Fx$ to KFa.[17]

Finally, I believe a third reason to take this move relies on a syntactic difference between general sentences and their particular "translations" (i.e. for a universal sentence, the conjunction of all instances, for the existential, the disjunction of all instances). Say that a and b are the only objects in the domain, that they're both F, and that an agent A is able to fix reference to a and not to b (or otherwise that they're able to form beliefs and other propositional attitudes involving a but not b). The intuition I'm trying to gesture at is that the agent is then able nonetheless to have propositional attitudes towards $\forall x Fx$, for they have a propositional attitude involving F, as let us suppose they believe Fa, but they're not able to have a propositional attitude towards $Fa \wedge Fb$ because they don't have any propositional attitudes towards Fb. It is, of course, a hotly debated topic when is it that one's knowledge of particular instances of F when these are not all the F's justify one in believing $\forall x Fx$. But this doesn't stop it from being the case that agents *are* able to have propositional attitudes (whether justified or not) in such propositions. And for that, I take it, one only needs to grasp the topic of an instance of the predicate.

Moving to the case of existential introduction, it might be, on the other hand, that agents' knowledge is extended from φ to ψ even though $\sigma(\psi) \not\sqsubseteq \sigma(\varphi)$. As defended earlier, agents only need to partially grasp the topic of what they know to know a general statement. Still, this must be constrained somehow by what they already

15 A helpful anonymous reviewer asks why is it that if we reject that $\sigma(Fa) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx))$ then we must also reject that $\sigma(Ga) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx)$, which is what is needed for my argument to run. Namely, if we think of $\forall x(Fx \rightarrow Gx)$ as $\forall x(\neg Fx \vee Gx)$, then it seems that more plausibly $\sigma(Ga)$ is included in $\sigma(\forall x(Fx \rightarrow Gx)$ than $\sigma(Fa)$ is. Here I have been assuming that negation is subject-matter transparent and so, even reading the conditional as a disjunction, we have that if $\sigma(\neg Fa \vee Ga) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx))$, then also, by the conditions for how the subject matter of disjunctions depend on the subject matters of disjuncts, $(\sigma(\neg Fa) \sqcup \sigma(Ga)) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx))$, and therefore that $(\sigma(Fa) \sqcup \sigma(Ga)) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx))$, and therefore that both $\sigma(Fa) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx)$ and $\sigma(Ga) \sqsubseteq \sigma(\forall x(Fx \rightarrow Gx)$. So if you want to deny that $\sigma(Fa)$ is part of $\sigma(\forall x(Fx \rightarrow Gx))$, then by parity of reason you should deny that $\sigma(Ga)$ is part of $\sigma(\forall x(Fx \rightarrow Gx))$, for the reasons we have to say one is part of $\sigma(\forall x(Fx \rightarrow Gx))$ is the same as the reasons we have for saying the other is.

16 Here I am presupposing that one could naturally extend immanent closure to a multi-premise principle stating that if $K\varphi$ for all $\varphi \in \Gamma$ and $\sigma(\psi) \sqsubseteq \bigsqcup\{\sigma(\varphi) : \varphi \in \Gamma\}$ and $\Gamma \vDash \psi$, then $K\psi$, where given a set S with members $s_1, \ldots, s_n$, $\bigsqcup S$ is the same as the fusion $s_1 \sqcup \ldots \sqcup s_n$. Later, in Sect. 5, I provide a precise definition of fusion and prove that $\sqsubseteq$ is a complete join semilattice. Thanks to an anonymous referee for helping me to clarify these points.

17 This second argument is closely tied to a form of K-epistemic closure, which some might want to reject. I merely provide it as a second avenue for someone to reject immanent closure, even if they don't agree with me on what the topic inclusion facts are between the topic of a universal sentence and the topics of its instances.

know, otherwise we will end up with the infamous case of disjunction introduction. An initial natural constraint is that $\sigma_g(\varphi) \sqsubseteq \sigma_g(\psi)$, that is, the part of $\sigma(\varphi)$ that the agent grasps is at least part of the part of $\sigma(\psi)$ that the agent grasps. Notice, however, that this condition is also satisfied by disjunction introduction: $K\varphi \vDash K(\varphi \vee \psi)$, as we have $\varphi \vDash \varphi \vee \psi$ and since $\sigma(\varphi) \sqsubseteq \sigma(\varphi \vee \psi)$, we have that $\sigma_g(\varphi) \sqsubseteq \sigma_g(\varphi \vee \psi)$. So, taking for granted that we don't want knowledge to be closed under disjunction introduction (one of the primary motivations for immanent closure), we want to add some constraint.

The thought is to impose that one should grasp the topic of each subformula of the formula one knows. This is not a general requirement on the grasp of a subject matter, but rather a requirement for *knowledge*. More formally, this amounts to the following restriction:

$$K\varphi \rightarrow (\exists x(\sigma_g(\varphi) = x) \text{ and for all } \psi, \ulcorner\psi\urcorner \leq \ulcorner\varphi\urcorner \rightarrow \exists y(\sigma_g(\psi) = y)))$$ [18]

One's grasp of the subject matter of sentences then goes from those of the atomic sentences, to the most complex ones, by a simple relation of fusion, and one's knowledge only demands partial grasp of the subject matters one is attending to.[19]

The motivation for this syntactic restriction is to be found in the general conception of first-order aboutness found in Silva (2023) and briefly gestured at above. There, I argue, we start by assigning topics to atomic sentences and then proceed to assign subject matters to all other expressions of the language (including subsentential expressions) on their basis.[20] This makes the epistemic constraint that agents grasp part of each substence's subject matters of what they know natural, for the former are determined from the latter in a compositional way.

We can then show how to validate the inference $KFa \vDash K\exists xFx$ in a way that doesn't validate disjunction introduction. The premise without the K-operator entails the conclusion, so there isn't any problem there. But we can now also say that a subject matter requirement is met since if the agent grasps $\sigma(Fa)$, then $\sigma_g(Fa) \sqsubseteq \sigma_g(\exists xFx)$. In the case of disjunction introduction, there might be a subsentence $\ulcorner\psi\urcorner$ of $\ulcorner\varphi \vee \psi\urcorner$ such that $\neg\exists x(x = \sigma_g(\psi))$, and therefore for which $\sigma_g(\psi) \sqsubseteq \sigma_g(\exists xFx)$ is undefined. In the case of an existential sentence, however, there is no such subsentence. So the only formula whose subject matter one needs to have a partial grasp of to have a partial grasp of the existential sentence is its corresponding open formula (which is identical to that of the existential sentence). And to have a partial grasp of the subject matter of the sentence, it is enough to grasp the subject matter of one of the instances. Again,

[18] For x a variable ranging over topics (i.e. partitions of a given set of worlds), $\leq$ a relation between formulae when one is a subformula of the other and $\ulcorner\ \urcorner$ quasi-quotation marks that allow us to mention instead of using the related sentences.

[19] When a complex sentence is formed of only non-quantified extensional sentences "all the way down" then this demand is equal to a demand for full grasp of the subject matter of what one comes to know. This goes in line with what was said before in relation to "σ_g" being the cognitively significant part of a subject matter for a given subject. It just so happens that in the case of $\exists x(\varphi(x))$, $\sigma_g(\exists x\varphi(x))$ need not be equal to $\sigma(\exists x\varphi(x))$ for it being possible that an agent knows that $\exists x(\varphi(x))$. These results are formally proven later in the paper.

[20] This goes hand-in-hand with the Fregean thesis that Thoughts, or what we call propositions, are the primary bearers of meaning, and that we arrive at the meaning of subsentential expressions from them by abstraction.

we don't impose this as a general requirement on having a partial grasp of a subject matter of a sentence, but rather for coming to *know* a proposition from propositions one partially grasps.

We are then in a position to give an alternative to immanent closure.

If an agent knows that φ, then they know that ψ iff:

(i) $\varphi \vDash \psi$
(ii) $\sigma(\psi) \sqsubseteq \sigma_g(\varphi)$ or $\sigma_g(\varphi) \sqsubseteq \sigma_g(\psi)$ and the agent partially grasps the subject matter of every subsentence of $\ulcorner \psi \urcorner$.

I believe this goes some way to better characterize the relation between content inclusion and the role it plays in agents' cognitive lives. Still, it makes it so that our epistemic closure principle is disjunctive, which might raise some eyebrows. The reason for this is that this formulation is still too tied to the relation between the subject matters of what one comes to know and what one knew already. I include it here primarily to show my progress from the usual principle of immanent closure to what is my considered and simplified view, presented below. Furthermore, this presentation also aims to show how one might characterize the account in terms of the connection between the subject matters of premise and conclusion. In order to better show how I would further tweak this principle, and to make the wider uses of partial grasp of a subject matter in epistemic logic clearer, I now turn to formally present an epistemic logic with partial grasp, with an eye to invalidate certain forms of logical omniscience.

4 The desiderata and simple epistemic logic

Having presented the motivations so far at a mostly informal level, I want now to present an epistemic logic with subject matters. Before presenting the language and models themselves, I want to say a bit more about the motivation for doing so. Like Berto (2022), Berto and Jago (2019), Hawke (2016), Hawke et al. (2020), Jago (2014), Yablo (2014) and many others, I take it that we should look for an epistemic logic that falls way short of validating principles of logical omniscience. That is, an epistemic logic that doesn't have as a consequence, among other unwanted results: that agents know everything that follows from what they know; and that they know all logical equivalents of what they know. These are just two of the epistemic closure principles known in the literature under the moniker of logical omniscience. We might call the former the closure principle of LOGICAL CONSEQUENCE for knowledge and the latter the closure principle of LOGICAL EQUIVALENCE.

It is due to issues arising from how these seemingly do not fit limited agents such as the average human being that it is particularly important to investigate other principles of closure, like immanent closure. Logical consequence alone (even in both ways, i.e. logical equivalence) is not sufficient to guide us on how knowledge should be closed. So these serve as our two basic negative desiderata, where $\varphi \Rightarrow \psi$ is defined as $\Box(\neg\varphi \vee \psi)$ and $\Leftrightarrow$ is defined as $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

Desideratum 1 Avoid LOGICAL CONSEQUENCE. That is, prove the following invalidity: $K\varphi, \varphi \Rightarrow \psi \nvDash K\psi$.

Desideratum 2 Avoid LOGICAL EQUIVALENCE. That is, prove the following invalidity: $K\varphi, \varphi \Leftrightarrow \psi \nvDash K\psi$.[21]

A third way in which one might be omniscient is with respect to necessary truths, namely, by coming out as knowing every logically necessary truth. This is also a result we would wish to avoid. One can find diverse motivations to avoid this consequence in the literature on minimal rationality (Hoek, 2022; Yalcin, 2018). I am particularly attracted to views that claim that agents might be mistaken about what are the possible worlds, and therefore might consider some impossible worlds possible. Another reason why this principle ought to fail is that agents just might fail to have any beliefs whatsoever about the topic of a given necessary truth. For these reasons, avoiding knowledge of all logically necessary propositions is our third desideratum.

Desideratum 3 Avoid LOGICAL NECESSITY. That is, prove the following invalidity: $\Box\varphi \nvDash K\varphi$.

For very similar reasons, just like one shouldn't come out as knowing all logically necessary truths, one shouldn't come out as knowing that all logically necessary truths are necessary. And perhaps this is even more clearly the case, for aside from being mistaken about its truth-value, one might be mistaken about the modal status of a given claim.

Desideratum 4 Avoid NECESSITY OF LOGICAL NECESSITY. That is, prove the following invalidity: $\Box\varphi \nvDash K\Box\varphi$.

Two other infamous closure principles in epistemic logic for their idealizing nature are the positive introspection, also known as KK-principle, and negative introspection principles. It seems that there might be cases where agents know a given proposition, and yet fail to know that they know it (against positive introspection),[22] and perhaps even more clearly, agents seem to fail to know that they fail to know some propositions. So I add avoiding these closure principles as two further negative desiderata.

Desideratum 5 Avoid POSITIVE INTROSPECTION. That is, prove the following invalidity: $K\varphi \nvDash KK\varphi$.

Desideratum 6 Avoid NEGATIVE INTROSPECTION. That is, prove the following invalidity: $\neg K\varphi \nvDash K\neg K\varphi$.

But there are other principles we should plausibly avoid, two of which we have seen already, and which are both motivated by topical constraints. Suppose that everything is material and that Democritus knew that. Did thereby Democritus know that Neptune's moons were material? Intuitively, no: he had no beliefs *about* Neptune's moons, and insofar as knowledge implies belief, he had no knowledge of that fact. Similarly, even though he knew, presumably, that his left foot was material, it's implausible that he had any state of knowledge like "Either my foot is material, or the moons of Neptune are", for again plausibly he didn't have any beliefs about the latter.

[21] As an anonymous referee rightfully points out, knowledge is therefore non-congruential.

[22] Williamson (2002) has influentially argued against the KK-principle for human-like agents based on our capacity for discrimination and a safety condition on knowledge.

Desideratum 7 Avoid UNIVERSAL INSTANTIATION. That is, prove the following invalidity: $K\forall x(\varphi(x)) \nvDash K\varphi(x/a)$.

Desideratum 8 Avoid DISJUNCTION INTRODUCTION. That is, prove the following invalidity: $K\varphi \nvDash K(\varphi \vee \psi)$.

Still, we don't want our epistemic logic to be trivial: we want it to make predictions about what agents will know based on what they know. What's the point of having an epistemic logic to begin with if it turns out to be trivial? So I will also be presenting some validities that the system should be able to give us. Perhaps the most obvious one is the following.

The first of these principles is CONJUNCTION ELIMINATION. Williamson (2002) and Berto (2022) make a very strong case for the difference between conjunction elimination and disjunction introduction. There is a very strong intuitive sense in which for one to come to know a conjunction, one has to already have come to know both conjuncts, so that there really is no effort (even deductive!) to come to know a conjunct from a conjunction. The content "is already there".

Desideratum 9 CONJUNCTION ELIMINATION. That is, prove $K(\varphi \wedge \psi) \vDash K\varphi$.

Another one we have seen already is EXISTENTIAL INTRODUCTION. The reasoning is very similar to that of conjunction elimination. The thought is that by coming to know an instance of a predicate, one has thereby come to know everything one needs to know, to know that something or other satisfies the predicate. In fact, one knows more than what would be necessary.

Desideratum 10 EXISTENTIAL INTRODUCTION. That is, prove $K\varphi(a) \vDash K\exists x(\varphi(x/a))$.

At this point, an attentive reader might have noticed a curious fact about the listed desiderata. It is often said in the literature that really a universal sentence is just a big conjunction of all of its instances, and similarly that an existential sentence is just a big disjunction of all of its instances.[23] And yet, we want to reject UNIVERSAL INSTANTIATION while accepting CONJUNCTION ELIMINATION; and we want to reject DISJUNCTION INTRODUCTION while accepting EXISTENTIAL INTRODUCTION. How come? If the universal sentence is just a big conjunction, how come we can't know a particular instance from it? From where does the epistemic difference between general sentences and their translation into particular sentences come from?

Here I believe we should trust our intuitive judgements: there *is* an epistemic distinction. Still, as can be surmised from what I've said earlier, I don't believe that it amounts to a distinction in their *content*. Rather, it comes from whether full grasp of a topic is necessary for propositional attitudes. I believe it is so for particular propositions, but not so for general propositions.

Finally, based on previous discussion, we should present the following as a further validity.

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[23] It is often added that one needs an extra (perhaps metasemantic) statement to the fact that such and such conjuncts/disjuncts are all the possible instances. Notable authors making this claim are Wittgenstein (2010), Armstrong (2004) and several others in the metaphysical truthmaker literature, like Rosen (2010). Others, like Yablo (2014) think that the fact that such-and-such facts are all the relevant facts are "truthmaker-makers" and not truthmakers for the initial sentence. Thanks to Franz Berto and to an anonymous referee for encouraging me to establish this link to the truthmaking and grounding literature.

Desideratum 11 UNIVERSAL MODUS PONENS That is, prove $K(\forall x(\varphi(x) \rightarrow \psi(x))), K\varphi(a/x) \vDash K\psi(a/x)$.

Which in the unary atomic case tells us that if an agent knows that all F's are G's, and further knows that a is an F, then they thereby know that a is a G. Intuitively, the knowledge of a particular instance of F guarantees acquaintance with it, sufficient for the knowledge that it is G if one also knows that everything that is F is G.[24]

4.1 The language

Our epistemic language $\mathscr{EL}$ has lowercase letters from the beginning of the Latin alphabet with subscripts if necessary ($a, b, \ldots a_1, b_1$, etc.) that serve as constants for individuals, lowercase letters from the end of the latin alphabet that serve as variables for individuals ($x, y, z, \ldots$). It has lowercase letters from the middle of the alphabet with subscripts for variables for worlds ($w, u, v, \ldots$). It has capital letters from the middle of the alphabet for predicates with numerical superscripts indicating arity (F^1, G^1, R^2, etc.).[25] I will also use the following letters from the greek alphabet in the following way. φ, ψ and ξ as metavariables for formulae. When a formula contains one or more instance of a variable that is not bound by a quantifier, I call it an open formula, otherwise I call it a sentence. If an open formula φ contains one more instances of x free, I may write it as $\varphi(x)$. Whenever a formula, whether open or closed, contains one or more instances of a designator, a, then it may be written as $\varphi(a)$. Whenever we want to take note that we have replaced all occurrences of a in $\varphi(a)$ for the variable x, we write $\varphi(x/a)$. If, on the other hand, we replace all the instances of x in $\varphi(x)$ with the designator a, we write $\varphi(a/x)$. The Greek letter π with subscripts, stands for functions assigning objects in the domain to variables. I will use π_x as a metavariable over x-variant variable assignments, where two variable assignments π_1 and π_2 are x-variant if they assign the same objects to all variables except possibly x. ρ for a valuation relation between a triple of world, variable assignment and formula and a truth-value 1 or 0. σ for a function outputting subject matters from expressions of the language, and with a subscript σ_g to denote the greatest part of a subject matter that the agent grasps. An atomic formula of the language will be a formula of the form $X^n(t_1, \ldots, t_n)$, for $t_1 \ldots t_n$ standing for terms, i.e. either constants or variables of the language.

We can then define the other well-formed formulae using BNF form as follows:

$$\varphi := X^n(a_1, \ldots a_n) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \forall x\varphi(x) \mid \exists x\varphi(x) \mid \Box\varphi \mid K\varphi$$

Even though some desiderata above use the symbol $\rightarrow$, and I make use of it below, $\varphi \rightarrow \psi$ will always be short for $\neg\varphi \vee \psi$, so in interest of space I don't include it in the language. Similarly for $\varphi \leftrightarrow \psi$, which stands for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. The formulae $\varphi \Rightarrow \psi$ and $\varphi \Leftrightarrow \psi$ stand for, respectively, $\Box(\varphi \rightarrow \psi)$ and $\Box(\varphi \leftrightarrow \psi)$.

[24] Of course one might possess both pieces of knowledge in different fragments or frames of mind and fail to put them together, but the model to be presented does not attempt to deal with fragmentation, though it could easily incorporate it.

[25] I'll be loose with notation an omit the arity where it is obvious.

4.2 The models

A simple $\mathscr{E}\mathscr{L}$-model (epistemic logic model), $\mathscr{M}$, is a tuple $\langle W, N, D, R, I, I^+, I^-, \Pi, \rho\rangle$, where: W is a non-empty set of worlds; $N \subseteq W$ is a non-empty set of normal worlds, also called possible worlds; D is a non-empty set of domains of individuals, D_w, one for each $w \in W$; R is a reflexive epistemic accessibility relation on W[26]; I is an interpretation function attributing to individual constants in the language objects in the domain ($I(a, w) \in D_w$); I^+ is a function attributing to predicates n-tuples of objects in the domain ($I^+(P^n, w) \in D^n_w$), intuitively their extensions; and likewise I^- attributes to predicates n-tuples of objects in the domain($I^-(P^n, w) \in D^n_w$), intuitively their anti-extensions; Π is a set of variable assignments π attributing individuals in D_w to variables ($\pi(x, w) \in D_w$); ρ is a *relation* between formulae of the language at worlds and under a variable assignment to the truth-values 1 (True) and 0 (False)—I will write these as, for instance $\rho_{w,\pi}(\varphi, 1)$, to signify that φ is true at world w under variable assignment π.

I now give further conditions for how ρ behaves, for any worlds $w \in W$:

If $\varphi \in$ **Atom** and is of the form $X^n(t_1, \ldots t_n)$, then, defining $I + \pi$ as a function which takes a term t and outputs $I(t, w)$ if t is a constant and outputs $\pi(t, w)$ if t is a variable, we have that:

$$\rho_{w,\pi}(\varphi, 1) \text{ iff } \langle I + \pi(t_1, w), \ldots, I + \pi(t_n, w)\rangle \in I^+(X^n, w).$$
$$\rho_{w,\pi}(\varphi, 0) \text{ iff} \langle I + \pi(t_1, w), \ldots, I + \pi(t_n, w)\rangle \in I^-(X^n, w).$$

And for arbitrary φ and ψ:

- $\rho_{w,\pi}(\neg\varphi, 1)$ iff $\rho_{w,\pi}(\varphi, 0)$.
- $\rho_{w,\pi}(\neg\varphi, 0)$ iff $\rho_{w,\pi}(\varphi, 1)$.
- $\rho_{w,\pi}(\varphi \wedge \psi, 1)$ iff $\rho_{w,\pi}(\varphi, 1)$ and $\rho_{w,\pi}(\psi, 1)$.
- $\rho_{w,\pi}(\varphi \wedge \psi, 0)$ iff $\rho_{w,\pi}(\varphi, 0)$ or $\rho_{w,\pi}(\psi, 0)$.
- $\rho_{w,\pi}(\varphi \vee \psi, 1)$ iff $\rho_{w,\pi}(\varphi, 1)$ or $\rho_{w,\pi}(\psi, 1)$.
- $\rho_{w,\pi}(\varphi \vee \psi, 0)$ iff $\rho_{w,\pi}(\varphi, 0)$ and $\rho_{w,\pi}(\psi, 0)$.
- $\rho_{w,\pi}(\forall x\varphi(x), 1)$ iff for all variable assignments π_x disagreeing with π at most on the value of x, $\rho_{w,\pi_x}(\varphi(x), 1)$.
- $\rho_{w,\pi}(\forall x\varphi(x), 0)$ iff for some variable assignment π_x disagreeing with π at most on the value of x, $\rho_{w,\pi_x}(\varphi(x), 0)$.
- $\rho_{w,\pi}(\exists x\varphi(x), 1)$ iff for some variable assignment π_x disagreeing with π at most on the value of x, $\rho_{w,\pi_x}(\varphi(x), 1)$.
- $\rho_{w,\pi}(\exists x\varphi(x), 0)$ iff for all variable assignments π_x disagreeing with π at most on the value of x, $\rho_{w,\pi_x}(\varphi(x), 0)$.
- $\rho_{w,\pi}(\Box\varphi, 1)$ iff $\rho_{w,\pi}(\varphi, 1)$ or $\rho_{w,\pi}(\varphi, 0)$ and for all worlds $v \in N$ $\rho_{v,\pi}(\varphi, 1)$.[27]

[26] Usually, this would be a set of such relations, each of them agent-indexed, and we would then consider multi-agent epistemic scenarios. In what follows, however, I limit myself to considering single-agent scenarios, so I only consider a single accessibility relation. I'm not imposing any restrictions on R besides reflexivity as I'm interested in exploring a variety of restrictions on knowledge. Minimally, however, I presuppose that knowledge is factive, and so that R is reflexive.

[27] NB that the first condition is trivially satisfied if $w \in N$ given the second condition, but not so if $w \in W - N$, for w might not represent φ as being true or as false.

- $\rho_{w,\pi}(\Box\varphi, 0)$ iff $\rho_{w,\pi}(\varphi, 1)$ or $\rho_{w,\pi}(\varphi, 0)$ and for some world $v \in N$ $\rho_{v,\pi}(\varphi, 0)$.[28]
- $\rho_{w,\pi}(K\varphi, 1)$ iff for all v such that wRv, $\rho_{v,\pi}(\varphi, 1)$.
- $\rho_{w,\pi}(K\varphi, 0)$ iff it's not the case that for all worlds v such that wRv, $\rho_{v,\pi}(\varphi, 1)$.

Providing the semantics in terms of a relation to truth-values, as opposed to a valuation-function is inspired by the work of Priest (1998, 2008) on the liar's paradox (for discussion, see Berto, 2007). For worlds $w \in N$, formulae are related to at most one truth-value and all formulae are related to one of the two truth-values, so that providing both positive and negative clauses is superfluous in such a case, however, to get this result, we have to impose conditions on the extensions and anti-extensions of predicates, namely for any predicate X^n and world $w \in N$, then $I^+(X^n, w) \cap I^-(X^n, w) = \emptyset$ (i.e. the extensions and anti-extensions of predicates are exclusive), and $I^+(X^n_n, w) \cup I^-(X^n_n, w) = D^w$ (i.e. the extensions and anti-extensions of predicates are exhaustive).[29] Finally, as usual, we will say that $\rho_w(\varphi, 1)$ whenever $\rho_{w,\pi}(\varphi, 1)$, for all $\pi \in \Pi$. I will further use $||\varphi||^+$ to refer to the set of worlds (possible or impossible) in which a sentence φ is true and $||\varphi||^-$ for the set of worlds in which it is false, I will use $||\varphi(x)||^{+/-}_\pi$ for the set of worlds where $\varphi(x)$ is true/false under the variable assignment π.

I now present the usual definitions of logical consequence and logical truth. From a given set of formulae Γ, one can derive ($\vDash$) a formula φ if and only if in all *normal* worlds w and under any assignment of individuals to variables π, in any model $\mathscr{M}$, we have that if $\rho_{w,\pi}(\psi, 1)$ for all $\psi \in \Gamma$, then $\rho_{w,\pi}(\varphi, 1)$. Logical truth is then as usual truth in every normal world of every model under any variable assignment. Whenever $\rho_{w,\pi}(\varphi, 1)$ for any variable assignment π, I simply write $\rho_w(\varphi, 1)$ (same for 0). Similarly, if for any two worlds w and w' in a model $\pi(x, w) = \pi(x, w')$, then I'll simply use $\pi(x)$.[30]

4.3 Meeting some of the desiderata

With these simple $\mathscr{E}\mathscr{L}$-models, we can already meet a number of desiderata, following a suggestion from an anonymous referee, I present the results in the form of lemmas.

Lemma 1 LOGICAL CONSEQUENCE *is invalid in $\mathscr{E}\mathscr{L}$-models.*

Proof We prove the failure of closure under LOGICAL CONSEQUENCE by providing a counterexample. So let's suppose we have the following model, $\mathscr{M}$, where $W = \{w_1, w_2\}$, $N = \{w_1\}$, $R = \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_2\rangle\}$ and further that $||\varphi||^+ = W$ and $||\psi||^+ = N$. We have that $\varphi \Rightarrow \psi$, for in all worlds in N in which φ is the case, so is ψ. We also have $\rho_{w_1}(K\varphi, 1)$, for in all worlds accessible from w_1, φ is the case. But it's not the case that $\rho_{w_1}(K\psi, 1)$, for w_1 accesses w_2 and ψ isn't true in w_2. So there is at least one normal world in one model where $K\varphi$ and $\varphi \Rightarrow \psi$, but not $K\psi$. So LOGICAL CONSEQUENCE fails. □

Lemma 2 LOGICAL EQUIVALENCE *is invalid in $\mathscr{E}\mathscr{L}$-models.*

28 Again, the first constraint is trivial if $w \in N$.

29 Thank you to an anonymous reviewer for helping me to clarify this point.

30 This assumption will always hold true in what follows.

Proof We again prove an invalidity by providing a counterexample. The countermodel presented in the proof for Lemma 1 serves as a countermodel to LOGICAL EQUIVALENCE as well. So suppose again that we have the following model, $\mathscr{M}$, where $W = \{w_1, w_2\}$, $N = \{w_1\}$, $R = \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_2\rangle\}$ and further that $||\varphi||^+ = W$ and $||\psi||^+ = N$. We have therefore that $\rho_{w_1}(\varphi \Leftrightarrow \psi, 1)$. Yet, we have that $\rho_{w_1}(K\varphi, 1)$ and $\rho_{w_1}(K\psi, 0)$, for $w_1 R w_2$ and ψ is not true in w_2. So there is at least one normal world in one model such that $\rho_{w_1}(K\varphi, 1)$, $\rho_{w_1}(\varphi \Leftrightarrow \psi, 1)$ and yet $\rho_{w_1}(K\psi, 0)$. This invalidates LOGICAL CONSEQUENCE. □

Lemma 3 LOGICAL NECESSITY *is invalid in $\mathscr{EL}$-models.*

Proof The previous model can also serve as a countermodel to LOGICAL NECESSITY, i.e. to the claim that we know all logically necessary truths (i.e. truth in all normal worlds). As per the model above $\rho_{w_1}(\psi, 1)$, and therefore it is ψ is true in all normal worlds of the model, for $N = \{w_1\}$. Thus, we have that $\rho_{w_1}(\Box\psi, 1)$.[31] But we've seen that $\rho_{w_1}(K\psi, 0)$. So there is one normal world in one model in which $\Box\psi$ is true, but not $K\psi$, providing a counterexample to LOGICAL NECESSITY. □

Lemma 4 NECESSITY OF LOGICAL NECESSITY *is invalid in $\mathscr{EL}$-models.*

Proof To show a failure of NECESSITY OF LOGICAL NECESSITY we only need a model $\mathscr{M}$ with two worlds, w_1 and w_2 such that $w_1 \in N$, $w_2 \in W - N$, $R = \{\langle w_1, w_1\rangle\langle w_1 w_2\rangle, \langle w_2, w_2\rangle\}$, that $||\varphi||^+ = N$ and that $||\varphi||^- = \emptyset$. Then $\rho_{w_1}(\varphi, 1)$ and since $w \in N$ if and only if $w = w_1$, then $\rho_{w_1}(\Box\varphi, 1)$, so $\Box\varphi$ is true in all normal worlds of the model $\mathscr{M}$. Yet it is not the case that $\rho_{w_2}(\Box\varphi, 1)$, for φ is neither true nor false in w_2, and $w_1 R w_2$, so $\rho_{w_1}(K\Box\varphi, 0)$. So there is a normal world in a model $\mathscr{M}$ where $\Box\varphi$ is true but $K\Box\varphi$ is false, which serves as a countermodel to NECESSITY OF LOGICAL NECESSITY. □

Lemma 5 POSITIVE INTROSPECTION *is invalid in $\mathscr{EL}$-models.*

Proof To show a failure of POSITIVE INTROSPECTION we need a model with three worlds (leading to a failure of transitivity). So let's have now $W = \{w_1, w_2, w_3\}$ and $R = \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_2\rangle, \langle w_2, w_3\rangle, \langle w_3, w_3\rangle\}$. We can let N be again $\{w_1\}$. Notably, let's have $||\varphi||^+$ be again $\{w_1, w_2\}$. Again, $\rho_{w_1}(K\varphi, 1)$ because φ is true in w_1 and w_2. But it's not the case that $\rho_{w_2}(K\varphi, 1)$ since φ is not the case in w_3 and $w_2 R w_3$. So there is an accessible world from w_1 where $K\varphi$ is not the case, so $KK\varphi$ is not the case in w_1. There is, then, a normal world in a model where $K\varphi$ but not $KK\varphi$, which is a counterexample to POSITIVE INTROSPECTION. □

Lemma 6 NEGATIVE INTROSPECTION *is invalid in $\mathscr{EL}$-models.*

Proof To prove a failure of negative introspection, only a two-world model is needed, showing that the accessibility relation is not euclidian. Let $W = N = \{w_1, w_2\}$. Further, let $R = \{\langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_2\rangle\}$. Suppose further that $||\varphi||^+ = w_2$, and therefore, given that all worlds are in N, that $||\varphi||^- = w_1$. Therefore, $\rho_{w_1}(\neg K\varphi, 1)$

[31] Formulae where $\Box$ is the main operator will in fact be true in every world, since we're not considering a non-epistemic accessibility relation. Here I'm ignoring issues related to that.

given that $\rho_{w_1}(K\varphi, 0)$, as $\rho_{w_1}(\varphi, 0)$ and R is reflexive. At the same time, however, $\rho_{w_2}(\varphi, 1)$, and since w_2 only accesses itself, $\rho_{w_2}(K\varphi, 1)$, and therefore $\rho_{w_2}(\neg K\varphi, 0)$. But $w_1 R w_2$, so there is a world accessible from w_1 where $\neg K\varphi$ is not the case, so $\rho_{w_1}(K\neg K\varphi, 0)$. So there is a normal world in a model where $\neg K\varphi$ but not $K\neg K\varphi$, and we have our counterexample to NEGATIVE INTROSPECTION. □

So far, these results are very standard. For the remaining desiderata, we need to introduce considerations having to do with subject matters, so we have to enrich our models.

4.4 Impossible worlds and subject matters: why not both?

Before moving on to implement subject matters, I would like to add a quick word on why I use both subject matters and non-normal worlds or states. After all, one might think that using both resources overcomplicates our models and makes them less attractive. At least as much as simplicity and elegance are theoretical virtues we want to preserve.

While I agree that simplicity and elegance are important when providing models, and often a simpler model that is easier to use will trump an overcomplicated model that does the same explanatory work, I take it that models with both impossible worlds and subject matters *do* present explanatory benefits, in that they can account for phenomena that neither can in isolation.

The main motivation for using both tools comes from the following pair of cases: one for why subject matter theories need impossible worlds; one for why impossible worlds theories need subject matters.

First, the case for impossible worlds. Suppose logicians all over the world are trying to prove a complex theorem T, which they don't know yet if it's true or false. Suppose that, as a matter of fact T is true. Being a logical theorem, it's also logically necessary: it could never have been anything but true. Suppose that T is also expressed in a clear way that does not admit of vagueness or indetermination. This can be easily achieved by it being expressed in a formal as opposed to a natural language, using only logical vocabulary. It's then also plausible that logicians know that either T is the case or it isn't the case: $T \vee \neg T$. But by the logical transparency of the extensional propositional connectives, T and $T \vee \neg T$ have the same subject matter, and further they're true in the same logically possible worlds—all of them. So a theory of subject matters without impossible worlds will not do justice to our intuitions in such a case.[32] I see no way around it: we need (impossible) valuation points that make T false, and $T \vee \neg T$ true. It is such an epistemic alternative that is not being ruled out by an agent who says that T might not be true, until they have been shown a proof of T.

[32] This is really an adaptation of a case that is usually presented making use of the particular example of *Goldbach's Conjecture* (for discussion from the possible worlds perspective, see Hawke et al., 2020 and Berto, 2022). But then the discussion might derail into whether mathematical truths are logical necessities and the like, which is beside the point. Our own discipline is ripe with examples of theorems that take great cognitive effort to be proven, why not refer to those and not muddy the waters? I haven't given any examples, but perhaps a good historical case would be Gödel's incompleteness theorems. Prior to their discovery in 1931, were Hilbert to be confronted with the disjunction of each of them with their respective negation, it's plausible he would know the disjunction to be true. But he would probably be convinced of the wrong disjunct.

The case for subject matters is a dilemma. Either impossible worlds' theorists impose no restriction on what their impossible worlds' represent, or restrictions imposed without using subject matters will be inadequate. Let's start with the first horn of the dilemma. Then we end up with Priest's (2005) infamous open worlds. They afford great liberty in forming models, but they end up giving rise to what seem to be syntactic models for attitudes like knowledge and belief (Jago, 2007, 2009). For instance, in open worlds, all formulae are treated as atomic at worlds, and therefore, if no conditions are imposed, we would have an epistemic logic where an agent could know a conjunction, but neither of the conjuncts. I find this very implausible, and to amount to giving up on providing a predictive epistemic logic. On the other hand, if we opt to use more constrained states or worlds, like the FDE-worlds of Berto and Jago (2019) (or like the states of inexact truthmaker semantics (Fine, 2017), if we agree with Berto and Jago (2023) and Silva (forthcoming) that aside from an emphasis on exact truthmaking, the two approaches are notational variants), then we will get implausible closure principles for knowledge. Namely, we have that if an agent knows that φ, they know that $\varphi \vee \psi$, for any ψ whatsoever. The reason why this is an implausible closure principle seems to be due to aboutness constraints: the disjunction includes a topic that φ didn't include, and it might be that the agent has no grasp of such topics. Simple impossible worlds theories also won't do justice to our intuitions. So it seems that we need to add topical constraints to impossible worlds theories. I do this by combining a space of FDE-worlds, which have some structure, with subject matters understood as partitions of subsets of the space of worlds (I have also previously done this in Silva, 2024a).[33]

5 Epistemic logic with partial grasp

Having explained why I work with both non-normal worlds and subject matters, let us now expand the simple $\mathscr{EL}$-models to models with partial grasp, which we might call $\mathscr{ELPG}$-models. These will be tuples $\langle W, N, D, R, I, I^+, I^-, \Pi, \rho, \sigma, \sigma_g \rangle$ extending the previously given models.

W, N, D, R, I, I^+ and I^- and Π are as above. ρ is as above, except for modifications that will be introduced for formulae with the K operator, which will be given once the apparatus on subject matters has been formally detailed.

[33] As a helpful anonymous revi ewer points out, I should qualify and make clear that my claim is not that the failure of disjunction introduction is the single motivator for adding topics to a model with impossible worlds. After all, the advantage of the models below that they avoid disjunction introduction can be recaptured in logics like nine-valued AC (Correia, 2004; Ferguson, 2016), as well as Priest and Daniels' five-valued logic. The reasons for why I prefer to work with models that include both impossible worlds and topics instead of just impossible worlds and more truth-values are three-fold: (1) I find both components of the models (impossible worlds and subject matters) more intuitive than the interpretation of the added truth-values; (2) I take, as stated in the body of the text, that failures of disjunction introduction to be connected to topical in nature, i.e. I think that if $\psi \sqsubseteq \varphi$, then $\varphi \vDash \varphi \vee \psi$ should be valid; and (3) there are other advantages, in particular in the first-order case that are obtained in a very intuitive way by combining topics and impossible worlds and defining a partial function σ_g from topics to topics, intuitively corresponding to the part of the topic that the agent grasps.

The added condition states that there is a part of the subject matter that is grasped for every subsentence of what is known, or conversely, that one might fail to know a proposition, simply because one fails to partially grasp one of its subsentences.

σ now is a function from the union of the set of terms and the set of formulae (open or close) to $\mathscr{P}(\mathscr{P}(W))$ attributing to each expression of the language a subject matter, which is a partition of a subset of W.

We now present conditions on σ. We start from atomic sentences (given the motivations above) and then define the other topics from them.

If $\varphi \in$ **Atom**, then we have two cases, depending on whether it is an atomic sentence, or an open formula. If it is a sentence, then:

$$\sigma(\varphi) = \{||\varphi||^{+}, ||\varphi||^{-}\}$$

If it is an open formula, then we introduce a relativization of σ to a variable assignment and do essentially the same:

$$\sigma_{\pi}(\varphi(x)) = \{||\varphi(x)||^{+}_{\pi}, ||\varphi(x)||^{-}_{\pi}\}$$

Then for arbitrary φ and ψ:

- $\sigma(\neg\varphi) = \sigma(\varphi)$.[34]
- $\sigma(\varphi \vee \psi) = \sigma(\varphi \wedge \psi) = \{x \cap y | x \in \sigma(\varphi) \wedge y \in \sigma(\psi)\}$.[35]
- $\sigma(\forall x\varphi(x)) = \sigma(\exists x\varphi(x)) = \sigma(\varphi(x)) = \bigsqcup\{\sigma_{\pi_x}(\varphi(x)) : \pi_x \in \Pi\}$.[36]
- $\sigma(a) = \bigsqcup\{\sigma(\varphi(a)) : \varphi(a) \in \textbf{Atm}\}$ where $\varphi(a)$ marks that a given formula contains a as a designator.[37]
- $\sigma(\Box\varphi) = \{x \cap N | x \in \sigma(\varphi)\}$.[38]
- $\sigma(K\varphi) = \{||K\varphi||^{+}, ||K\varphi||^{-}\}$.[39]

[34] We can see this intuitively in the case where φ is an atom. By the conditions for $\rho_{w,\pi}$, $||\neg\varphi||^{+} = ||\varphi||^{-}$, and $||\neg\varphi||^{-} = ||\varphi||^{+}$, so $\sigma(\neg\varphi) = \{||\varphi||^{-}, ||\varphi||^{+}\}$.

[35] In the atomic case we have that: $\sigma(\varphi \wedge \psi) = \{||\varphi||^{+} \cap ||\psi||^{+}, ||\varphi||^{+} \cap ||\psi||^{-}, ||\varphi||^{-} \cap ||\psi||^{+}, ||\varphi||^{-} \cap ||\psi||^{-}\}$. For philosophical motivation, see Silva (2024a).

[36] Here note that this makes it so that while for any two variables x and y $\sigma(\varphi(x/y)) = \sigma(x)$, we should not expect the same to be true for $\sigma_{\pi}(\varphi(x/y))$ and $\sigma_{\pi}(\varphi(x))$, for x and y might be assigned different objects of the domain by π. This goes well in line with the thought variables are just notational marks. Thanks to Franz Berto for pressuring me on this.

[37] For philosophical motivation, see Silva (2023). The intuitive idea is that the subject matter of a designator, a, will be the fusion of all the subject matters of sentences where a shows up.

[38] Here I don't provide robust justification for this identification, but a few words are forthcoming on why I pick the subject matter of $\Box\varphi$ to be what it is. The motivation is that alethic modal formulae seem to have to do with the way certain formulae are true or false across *possible worlds* And so an immediate thought is to say that the subject matter of $\Box\varphi$ is in a sense the restriction to N of the subject matter of φ. If one works only with possible worlds, then the two come out identical. This plays nicely with the idea that logical vocabulary (if we're willing to include $\Box$ among the logical vocabulary) should be subject matter transparent.

[39] This identification is admittedly arbitrary. Again, I leave for future work addressing the concern of why or why not we should think that this is the subject matter of a particular knowledge claim. For now, I just want to note that this preserves the intuition that sentences concerning an agent's knowledge are not directly about the content of their knowledge, but rather about their state of knowledge itself. In future work, I hope to explore the significance of this fact for the status of K as a logical constant. Namely, to ascertain whether

It will be useful in what follows to have a notion of parthood ($\sqsubseteq$) between subject matters, which we minimally assume to be a partial order (i.e. a reflexive, transitive and anti-symmetric relation) and a complete join semilattice on the sets of sets of worlds that are subject matters. We don't add this to our models as this is really a defined notion in the model and not a primitive. Namely we define the notion of parthood via the notion of intersection, $\bigcap$:

$$\sigma(\varphi) \sqsubseteq \sigma(\psi) := \forall x(x \in \sigma(\psi) \rightarrow x = \bigcap\{y|y \in \sigma(\varphi)\}$$

Informally, the subject matter of φ is part of the subject matter of ψ if and only if all the members of $\sigma(\psi)$ are intersections of all the members of $\sigma(\varphi)$. This allows us to say immediately that $\sigma(\varphi) \sqsubseteq \sigma(\varphi \wedge \psi)$, per the definition above. Fusion is defined in terms of the subject matter of the conjunction:

$$\sigma(\varphi \wedge \psi) = \sigma(\varphi) \sqcup \sigma(\psi).$$

Let us now show that $\sqsubseteq$ is a complete join semilattice. We do this by showing that any set of subject matters has a least upper bound with respect to $\sqsubseteq$, i.e. for any (possibly empty) set S of subject matters such that $\{\sigma(\varphi_i)|i \in I\} \subseteq S$ (for any index I) then $\bigsqcup\{\sigma(\varphi_i)|i \in I\}$ exists and is a member of S. We can show that this is the case by induction. First we show that this is the case for the empty set of subject matters, then we show that if this is the case for a given arbitrary set, then it is the case for a set with an added element, and we'll be done. Case for the the empty set: Let O be an empty set of subject matters. Recall that subject matters are sets of sets of worlds, and that a subject matter is part of another whenever all members of the latter are intersections of members of the latter. Let $\sigma(O) = \{W\}$, then the condition that $\sigma(O)$ is part of any subject matter is automatically satisfied, for any set of worlds of W is going to intersect W. Further, since O is an empty set of subject matters, it is trivially satisfied that all members of O are part of $\sigma(O)$. Furthermore, W is a subset of W, so $\{W\}$ is a partition of a subset of W. We have found our subject matter. For the induction step, suppose that the set of subject matters, S described above, with $\sigma(\varphi_i)$ for each $i \in I$ as members, has a least upper bound. Suppose we add to it a new member, $\sigma(\varphi_j)$, to form the set S', which is formed with a new index set J such that $I \subseteq J$. We want to prove that there is the subject matter $\sigma(S')$ meeting the constraints listed above. By the conditions for $\sigma(\cdot)$, we can let the subject matter of S' be equal to $\bigsqcup\{\sigma(\varphi_j)|j \in J\} = \bigsqcup\{\sigma(\varphi_i)|i \in I\} \sqcup \sigma(\varphi_j) = \bigcap\{\{x_i|x_i \in \sigma(\varphi_i)\}|i \in I\} \cap \{x_j|x_j \in \varphi_j\}$. We now want to show: (i) that for each $\sigma(\varphi_i)$ that $\sigma(\varphi_i) \sqsubseteq \bigsqcup\{\sigma(\varphi_i)|i \in I\}$; and (ii) that for any other $\sigma(\xi)$, if each $\sigma(\varphi_i) \sqsubseteq \sigma(\xi)$, then $\bigsqcup\{\sigma(\varphi_i)|i \in I\} \sqsubseteq \sigma(\xi)$. Condition (i) follows immediately, because the members of $\{\sigma(\varphi_i)|i \in I\}$ are just intersections of members of each individual $\sigma(\varphi_i)$ taken all together. As for condition (ii), suppose that

Footnote 39 continued
it differs in that regard from the alethic modalities of logical and metaphysical necessities. Recent work by Ferguson (2023a, b, c) explores non-extensional conditionals, as well as alethic and epistemic modalities, and the reader can find there a very engaging and fruitful proposal on the subject matter of these pieces of logical vocabulary. Thanks to an anonymous reviewer for pointing me to these papers, whose ideas unfortunately I could not incorporate in a more fruitful way in this paper.

each $\sigma(\varphi_i) \sqsubseteq \sigma(\xi)$, but that $\bigsqcup\{\sigma(\varphi_i)|i \in I\} \not\sqsubseteq \sigma(\xi)$. Then $\bigsqcup\{\sigma(\varphi_i)|i \in I\}$ contains the intersection of each $\sigma(\varphi_i)$, so $\sigma(\xi)$ necessarily either contains this intersection or contains an even 'bigger' one.

Given that for any arbitrary set of subject matters, there is a join, then it automatically follows that $\sqsubseteq$ is a partial order, from the definition of $\sqsubseteq$ through the special case of a two-membered set $\bigsqcup\{\sigma, \xi\} = \xi$.[40]

Finally, we will need to introduce a partial function from subject matters to subject matters, σ_g, which corresponds to what is above glossed as what the agent grasps of a given subject matter. So, $\sigma_g(\varphi)$, which is guaranteed to exist given that $\sqsubseteq$ is a complete join semilattice, is always part of $\sigma(\varphi)$. That is: $\sigma_g(\varphi)$ is a (potentially) coarser set of set of worlds on a (potentially) larger subset of W. We don't concern ourselves here with how this grasp might change over time or with the presentation and processing of new information. The model, therefore, is static.

We will now impose some conditions on σ_g, aside from it taking a subject matter to a part of itself. The first is that if $\sigma_g(\varphi)$ exists, then there is a ψ (potentially identical to φ) such that $\sigma(\psi) \sqsubseteq \sigma(\varphi)$ and $\sigma_g(\psi) = \sigma(\psi)$. This is the condition glossed above that for there to be partial grasp of a topic, there must be "full grasp" of a topic included in it.

The second condition has to do with how grasp percolates up from smaller topics to more comprehensive topics. The idea is to capture the constraint that $\sigma_g(\varphi)$ is the biggest part of $\sigma(\varphi)$ that the agent grasps. The idea will be to equate $\sigma_g(\varphi)$ to the fusion of $\sigma_g(\psi)$ for all ψ such that $\sigma(\psi) \sqsubseteq \sigma(\varphi)$ and $\sigma_g(\psi) = \sigma(\psi)$. This guarantees, given that $\sqsubseteq$ is a complete join semilattice, that $\sigma_g(\varphi)$ is unique for any φ. Formally:

$$\sigma_g(\varphi) = \bigsqcup\{\sigma_g(\psi) : \sigma(\psi) \sqsubseteq \sigma(\varphi) \wedge \sigma_g(\psi) = \sigma(\psi)\}$$

It follows from this that if an agent partially grasps the topic of a formula, then there is always a formula whose topic is what they fully grasp of the topic they partially grasp. In the finite case, this is either the disjunction or conjunction of sentences whose topics the agent partially grasp and whose topics are part of the starting topic.

If $\sigma(\varphi)$ has finite parts, then:
$\sigma_g(\varphi) = \sigma_g(\psi_1 \wedge \ldots \wedge \psi_n)$, where each ψ_i is such that (i) $\sigma(\psi_i) \sqsubseteq \sigma(\varphi)$ and (ii) $\sigma_g(\psi_i) = \sigma(\psi_i)$.

In the infinite case, this can be the following open formula:

$$\xi(x) := x \in \{\sigma_g(\psi) : \sigma(\psi) \sqsubseteq \sigma(\varphi) \wedge \sigma_g(\psi) = \sigma(\psi)\}$$

From these two conditions it follows that if $\sigma(\varphi) \sqsubseteq \sigma(\psi)$ and $\sigma_g(\varphi)$ exists, then $\sigma_g(\psi)$ must exist too and $\sigma_g(\varphi) \sqsubseteq \sigma_g(\psi)$. Suppose that as per above, $\sigma(\xi)$ is what the agent fully grasps of $\sigma(\varphi)$. Immediately, it follows that $\sigma_g(\xi) = \sigma(\xi)$, and $\sigma(\xi) \sqsubseteq \sigma(\varphi)$. Indeed, we know that $\sigma_g(\varphi) = \sigma(\xi)$. But then $\sigma(\xi) \sqsubseteq \sigma(\psi)$ by the transitivity of $\sqsubseteq$. Furthermore, we know that $\sigma_g(\psi)$ is going to be a fusion of all the parts of $\sigma(\psi)$ that the agent fully grasps, and that includes $\sigma(\xi)$, i.e. $\sigma_g(\varphi)$.

[40] Thanks to an anonymous referee for calling my attention to this relation.

An interesting result is that for particular extensional propositions, partial grasp of every subsentence of a sentence means that one has full grasp of that sentence, i.e. that $\sigma_g(\varphi) = \sigma(\varphi)$. I prove this by induction. Atomic sentences have atomic subject matters. So we automatically have the required result. Our Inductive Hypothesis (IH) will be that: if the agent has partial grasp of every subsentence of φ, then they have full grasp of φ, i.e. $\sigma_g(\varphi) = \sigma(\varphi)$. Case for $\neg\varphi$ is trivial since $\sigma(\neg\varphi) = \sigma(\varphi)$, and thereby we have our result directly from (IH). The cases for conjunction and disjunction are identical, I only present the one for conjunction. Suppose that $\sigma_g(\varphi) = \sigma(\varphi)$ by (IH) and again $\sigma_g(\psi) = \sigma(\psi)$ by (IH). But $\sigma(\varphi \wedge \psi) = \sigma(\psi) \sqcup \sigma(\varphi)$,[41] and therefore $\sigma(\varphi \wedge \psi) = \sigma_g(\varphi) \sqcup \sigma_g(\psi)$. And of course since $\sigma_g(\varphi \wedge \psi) \sqsubseteq \sigma(\varphi \wedge \psi)$ then $\sigma_g(\varphi \wedge \psi) \sqsubseteq (\sigma_g(\varphi) \sqcup \sigma_g(\psi))$. In fact, it must be that $\sigma_g(\varphi \wedge \psi) = (\sigma_g(\varphi) \sqcup \sigma_g(\psi))$. For suppose not. Then by (IH) either $\sigma_g(\varphi \wedge \psi) = \sigma(\varphi)$ or $\sigma_g(\varphi \wedge \psi) = \sigma(\psi)$. The two cases lead to contradiction in the same way, I present the first one. So suppose $\sigma_g(\varphi \wedge \psi) = \sigma(\varphi)$. This directly contradicts the fact that $\sigma_g(\psi)$, which is such that $\sigma_g(\psi) \sqsubseteq \sigma_g(\varphi \wedge \psi)$, exists. So $\sigma_g(\varphi \wedge \psi)$ must be equal to $\sigma_g(\varphi) \sqcup \sigma_g(\psi)$, i.e. to $\sigma(\varphi \wedge \psi)$. No other formula is a particular extensional proposition, so we're done.[42]

Finally, the last condition on σ_g is a principle of coordination for the grasp of quantified formulae. An example illustrates clearly what I have in mind. I wish to say that one might grasp part of the topic of $\forall x Fx$ in $\forall x Fx \wedge \forall x Gx$ without grasping part of the topic of $\forall x Gx$, whereas one cannot do so in the case of $\forall x Fx \wedge Gx$. The reason for this is that in the first case instances of the generalisation are separate for $\forall x Fx$ and $\forall x Gx$, whereas they're not for $\forall x Fx \wedge Gx$. If quantifiers are understood as second-order predicates, then in the first case $\forall x$ applies to Fx and Gx whereas in the second case it applies to $Fx \wedge Gx$. The coordination principle is therefore that if the nucleus (greatest lower bound) of the subject matter of Fx with a given subject matter of a designator is part of $\sigma_g(Fx \wedge Gx)$, then likewise the nucleus of the subject matter of the subject matter of Gx with the subject matter of a given designator is going to be part of $\sigma_g(Fx \wedge Gx)$. Using $\times$ for the g.l.b., the following is the formal translation in the general case:

$\sigma(\xi(x)) \times \sigma(a) \sqsubseteq \sigma_g(\varphi) \rightarrow (\sigma(\psi(x)) \times \sigma(a) \sqsubseteq \sigma_g(\varphi))$, for $\xi(x)$ and $\psi(x)$ any open subformulae of φ.

Having provided now some conditions and results about how σ_g works, we can finally provide the truth-conditions for knowledge:

- $\rho_{w,\pi}(K\varphi, 1)$ iff for all v such that wRv, $\rho_{v,\pi}(\varphi, 1)$ and for all subformulae ψ of φ, $\exists x(\sigma_g(\psi) = x)$.
- $\rho_{w,\pi}(K\varphi, 0)$ iff it's not the case that for all wRv, $\rho_{v,\pi}(\varphi, 1)$ or for some subformula ψ of φ, $\neg\exists x(\sigma_g(\psi) = x)$.

41 Where $\sqcup$ is the usual operator of fusion understood as the least upper bound of the relation of parthood.

42 Here note that even if for particular extensional propositions $\sigma_g(\varphi) = \sigma(\varphi)$, $\sigma_g(\varphi)$ is not guaranteed to exist even if $\sigma_g(\psi)$ exists for all subsentences of φ distinct from φ. The intuitive thought is that the agent might fail to "put together" the grasp they have of the various topics of the subsentences.

5.1 Meeting the remaining desiderata

Let us now consider the extent to which the remaining desiderata are met, as well as some other interesting results we get from the model theory. Let's consider the remaining desiderata in the order by which they appear, in a theorem-like environment.

Lemma 7 UNIVERSAL INSTANTIATION *is invalid in $\mathscr{ELPG}$-models.*

Lemma 7 introduces considerations from first-order logic, and can only be proven by the role of partial grasp in $\mathscr{ELPG}$ models. We want to prove that an agent might know a universal sentence while failing to know one of its instances, which we want to explain by it being possible that $\sigma_g(\varphi(a))$ is included in $\sigma_g(\forall x\varphi(x))$.

Proof Let a model $\mathscr{M}$ be such that $W = N = \{w_1\}$, $R = \{\langle w_1, w_1\rangle\}$. Suppose that $\rho_{w_1}(K\forall x\varphi(x), 1)$, then $\rho_{w_1}(\forall x\varphi(x), 1)$ and $\sigma_g(\forall x\varphi(x)) = \sigma_g(\varphi(x))$ exists. Even supposing that for some assignments of objects to variables π, $\pi(x) = I(a)$, we can suppose that $\sigma_g(\varphi(a))$ does not exist. For instance, it might be that $\sigma_g(\varphi(x)) = \sigma_g(\varphi(b))$, for some other designator b such that for some variable assignment π, $\pi(x) = b$. Since $\sigma_g(\varphi(a))$ doesn't exist, it isn't the case that $\rho_{w_1}(K\varphi(a), 1)$. So there is a normal world in a model where the premise is true and the conclusion isn't, and this is a counterexample to UNIVERSAL INSTANTIATION. □

Lemma 8 DISJUNCTION INTRODUCTION *is invalid in $\mathscr{ELPG}$-models.*

Proof Let $\mathscr{M}$ be a model where $W = N = \{w_1\}$ and that $||\varphi||^+ = W$. Finally, let $R = \{\langle w_1, w_1\rangle\}$ and let it be the case that $\sigma_g(\varphi)$ exists, but not $\sigma_g(\psi)$. We have that $\rho_{w_1}(\varphi, 1)$ and therefore $\rho_{w_1}(K\varphi, 1)$ but at the same time we don't have $\rho_{w_1}(K(\varphi \vee \psi), 1)$, for there is a subformula of $\ulcorner\varphi \vee \psi\urcorner$, namely ψ, such that $\sigma_g(\psi)$ doesn't exist. So there is a normal world in a model where $K\varphi$ but not $K(\varphi \vee \psi)$, which is our counterexample to DISJUNCTION INTRODUCTION.

Lemma 9 CONJUNCTION ELIMINATION *is valid in $\mathscr{ELPG}$-models.*

Proof In order to prove that CONJUNCTION ELIMINATION is valid, i.e. that conjunction elimination under the scope of K is valid, we first suppose that $\rho_w(K(\varphi \wedge \psi), 1)$ for $w \in N$ in an arbitrary model $\mathscr{M}$. We want to show that therefore $\rho_w(K\varphi, 1)$ and $\rho_w(K\psi, 1)$. We know that for all v such that wRv, $\rho_v(\varphi \wedge \psi, 1)$. And we know from the truth-conditions for conjunction, that $\rho_v(\varphi, 1)$ and $\rho_v(\psi, 1)$.[43] Further, we know that $\sigma_g(\varphi \wedge \psi)$ exists, and likewise for all subsentences of $\ulcorner\varphi \wedge \psi\urcorner$, and so $\sigma_g(\varphi)$ and $\sigma_g(\psi)$ both exist. Therefore $\rho_w(K\varphi, 1)$ and $\rho_w(K\psi, 1)$. Since w was an arbitrary normal world in an arbitrary model, we have proven this result for all normal worlds of any model, showing that if $K(\varphi \wedge \psi)$, then $K\varphi$ and $K\psi$. □

Lemma 10 EXISTENTIAL INTRODUCTION *is valid in $\mathscr{ELPG}$-models.*

To prove EXISTENTIAL INTRODUCTION, i.e. that we can apply existential generalization under the scope of the knowledge operator, we suppose that $\rho_w(K\varphi(a), 1)$ for

[43] This first part of the proof also shows that conjunction elimination is valid in $\mathscr{EL}$ models.

w a normal world, and therefore that $\rho_v(\varphi(a), 1)$ for all v such that wRv, further, we suppose that $\sigma_g(\varphi(a))$ exists. We therefore also know that since $\varphi(a)$ is true in all worlds v, $\rho_v(\exists x\varphi(x/a), 1)$. Furthermore, we know that $\sigma(\varphi(a) \sqsubseteq \sigma(\exists x\varphi(x/a))$, and that $\sigma_g(\varphi(a))$ exists. So we know that $\sigma_g(\exists x\varphi(x/a))$ exists and that $\sigma_g(\varphi(a)) \sqsubseteq \sigma_g(\exists x\varphi(x/a))$. So we have that $\rho_w(K\exists x\varphi(x/a), 1)$. and therefore in any normal world in which $K\varphi(a)$, we also have $K\exists x\varphi(x/a)$. □

Lemma 11 UNIVERSAL MODUS PONENS *is valid in* $\mathscr{ELPG}$*-models.*

Proof Suppose $\rho_w(K\forall x(\varphi(x) \to \psi(x)), 1)$ and $\rho_w(K\varphi(a), 1)$ for w a normal world in an arbitrary model $\mathscr{M}$. Given that $\forall x(\varphi(x) \to \psi(x))$ is known, we have that for all v such that wRv, then $\rho_v(\forall x(\varphi(x) \to \psi(x)), 1)$. Furthermore, we know that $\sigma_g(\forall x(\varphi(x) \to \psi(x)))$ exists, as well as $\sigma_g(\varphi(x) \to \psi(x))$, $\sigma_g(\varphi(x))$ and $\sigma_g(\psi(x))$, for these are the subformulae of $\forall x(\varphi(x) \to \psi(x))$. We also know that $\rho_v(\varphi(a), 1)$ for all v such that wRv and further that $\sigma_g(\varphi(a))$ exists. Since $\sigma_g(\varphi(a))$ exists, so does $\sigma_g(\varphi(x)) \times \sigma_g(a)$, for this is just $\sigma_g(\varphi(a))$. Furthermore, since $\sigma_g(\varphi(x)) \sqsubseteq \sigma_g(\varphi(x) \to \psi(x))$,[44] then by transitivity $\sigma_g(\varphi(a) \sqsubseteq \sigma_g(\varphi(x) \to \psi(x))$. By the last condition imposed on σ_g, this entails that $\sigma_g(\psi(a)) \sqsubseteq \sigma_g(\varphi(x) \to \psi(x))$. Finally, by the usual application of propositional *modus ponens*, we get that $\rho_v(\psi(a), 1)$ for any world v accessible from w (for $\rho_v(\varphi(a) \to \psi(a), 1)$). So $\rho_w(K\psi(a), 1)$. Since w was an arbitrary normal world in a model, this holds true for any normal world in any model, yielding the desired result. □

6 The limits of the proposal

The account provided can, then, meet a number of desiderata. In trying to model some phenomena, however, it idealizes in some other respects. Namely, it contains three closure principles that are perhaps not desirable for human-like cognitive agents. The first is an unrestricted form of conjunction introduction under the scope of the knowledge operator, which we may call closure under ADJUNCTION, stating that if an agent knows φ and they know ψ, they thereby know $\varphi \wedge \psi$.

Proposition 1 ADJUNCTION *is valid in* $\mathscr{ELPG}$ *models.*

Proof Let w be a normal world in an arbitrary model $\mathscr{M}$. Suppose that $\rho_w(K\varphi, 1)$ and $\rho_w(K\psi, 1)$. This implies that $\rho_v(\varphi)$ and $\rho_v(\psi, 1)$ for any world v such that wRv. This immediately implies that $\rho_v(\varphi \wedge \psi, 1)$ for any such world v. Furthermore, both $\sigma_g(\varphi)$ and $\sigma_g(\psi)$ exist. Which implies that $\sigma_g(\varphi \wedge \psi)$ exists and is the fusion of $\sigma_g(\varphi)$ and $\sigma_g(\psi)$. This shows that $\rho_w(K(\varphi \wedge \psi), 1)$. Since w was an arbitrary normal world in an arbitrary model, we can generalize, proving the (un)desired result. □

The second result one would like to avoid is monotonicity,[45] it seems that by gaining new evidence, agents might come to lose knowledge that they previously held on to,

[44] This relies on a more general result, proof of which is omitted here for reasons of space, that if φ is a subformula of ψ, then $\sigma(\varphi) \sqsubseteq \sigma(\psi)$.

[45] As an anonymous referee helpfully points out, "monotonicity" in the modal logic literature often refers to the property that if $\varphi \vDash \psi$ then (in the case of knowledge) $K\varphi \vDash K\psi$. This is not what I intend here. Instead, I mean "monotonicity" as it is used in the substructural literature.

for they stopped believing in the relevant propositions. Still, the models here presented both validate various forms of MONOTONICITY. I present only one, framed in terms of adding any proposition to one's knowledge base:

Proposition 2 MONOTONICITY *(i.e.* $K\varphi, K\psi \vDash K\varphi$*) is valid in* $\mathscr{ELPG}$ *models.*

Proof Immediate from the conditions on $K\varphi$. □

The non-monotonicity of knowledge, such that new evidence might lead to the loss of knowledge, is widely accepted in the epistemology literature in the wake of the Kripke–Harman paradox (where it is framed in terms of evidence). For references of similar models in formal epistemology in which monotonicity fails employing subject matters, see Berto (2022), Hawke et al. (2020) and Silva (2024a).

I believe that the epistemic logic presented above could be further complicated to avoid these closure principles. Here I won't provide a full characterization of what I take would be a good way of doing it, but I want to give the highlights of an informal characterization and list some requirements.

Keeping up with an association between questions and subject matters, a natural addition to a model would be the phenomenon of agents entertaining or considering given questions. Formally, this could be an update. This update, when successful, would restrict the worlds in an agents' epistemic space to the ones "speaking to a question": i.e. the worlds $w \in x$ for all $x \in \sigma(\varphi)$, for some φ. One could then impose that one's knowledge is only attained relative to a question or other that one is entertaining, with the added explanation that this need not be a conscious effort that is explicit to agents.

With such a background, one could then module failures of adjunction as failures to update one's epistemic space in the required way: perhaps one only knows a conjunction if one is relevantly fine-tuned with its subject matter. In the same way, failures of monotonicity might result from changes in what one knows when considering distinct questions. Similarly, one could add a number of interesting validities to the logic. Even though DISJUNCTION INTRODUCTION and ADJUNCTION fail, perhaps question-relative versions of the two would fare better? Like so: if the agent knows φ and successfully considers/entertains the subject matter $\sigma(\varphi \vee \psi)$, they know that $\varphi \vee \psi$. This reads to me as a much more plausible version of disjunction introduction, for in this case the agent must possess the relevant concepts to understand ψ, and ψ must be cognitively salient to the agent. The same applies for the case of conjunction.

Here, however, I don't have the space to elaborate on how such an operator would work in detail. I believe that both Hawke (2022, forthcoming), Hawke et al. (2020) and Berto (2022) have very interesting approaches when it comes to the issue of fragmentation and failures of adjunction (like classically presented in Lewis (1982)) and monotonicity, and I take it that such an operator of "considering a question" would have to mimic some features from both frameworks.

It would be further interesting to see what the combination of such approaches would have to say about our intuition that knowledge should be closed under UNIVERSAL MODUS PONENS. Are there cases where we can combine knowledge items without worrying about which fragments they're coming from, because they ought to come from the same fragment? Is a universal claim and a particular instance concerning a predicate F such a pair of sentences? After all, if one is considering whether a is F and hence G, one might be *eo ipso* be considering how being F and being G relate, and vice-versa, if one keeps one's knowledge that a is F present in one's mind. These are interesting questions, that unfortunately I won't be able to explore here.

The last result that might be unwelcome and of which I am a lot less sure that it is not desirable is what is known in the literature simply as K CLOSURE PRINCIPLE. That is, the principle that knowledge is closed under known implication. What we have called UNIVERSAL MODUS PONENS is a generalized, first-order version of it, but it is worth mentioning the propositional variety as well (the proof is omitted as it is very similar).

Proposition 3 K CLOSURE PRINCIPLE *(i.e.* $K\varphi, K(\varphi \rightarrow \psi) \vDash K\psi$*) is valid in* $\mathscr{ELPG}$ *models.*

Even though the model validates both the K CLOSURE PRINCIPLE and UNIVERSAL MODUS PONENS, I don't want to take a stance on the validity of either principle more generally. As stated, here failures of omniscience due to agents grasping or failing to grasp given subject matters are accounted for, but other factors are not taken into consideration. A more 'true-to-life' model would perhaps incorporate other tools and would thereby also invalidate this closure principle, but doing so lies outside the remit of this paper. Unlike monotonicity and adjunction, however, I am more skeptical that our best models should invalidate this principle.

Finally, before wrapping up, I just want to mention a further consequence of the view. Agents will know that $\neg\neg A$ whenever they know that A, and vice-versa, as well as for $\neg\neg\neg\neg A$, and so on for all levels of Double Negation Introduction.[46] Is this plausible? Doesn't closure for knowledge fail even in this kind of case?

While I don't take it to be an adequate account of what goes on in all cases of putative hyperintensionality, here I agree with Stalnaker (1984) metalinguistic approach and would say that there isn't a distinction at the level of *content* between A, $\neg\neg A$ and so on. Rather, I'd say that there might be second-order questions concerning what propositions are being expressed by what sentences, which lead to different patterns of acceptance and rejection on the part of agents. Simply put: double-negation adds syntactic complexity but no content to what is being said.

7 Conclusion

What should we say about how knowledge transmits from universal claims to their instances, and from instances to existential sentences? Further, does knowledge transmit from an instance of the antecedent of a universal generalization to an instance of its consequent through knowledge of the universal sentence?

I believe we can make some headway in approaching questions like these by introducing into our toolbox, as formal epistemologists who accept the usefulness of subject

[46] As an anonymous reviewer pointed out to me, relevant logics like BM are such that the introduction of a double negation might lead to a change in topic. For the reasons below, I prefer to accept a metalinguistic explanation for the intuitive distinction between contents in this case. Still, if the reader is not convinced that the metalinguistic approach succeeds, BM might be a system worth considering.

matters, a notion of "grasp" of a subject matter. My aims in this paper have been mainly to support the usefulness of the notion and to provide philosophical support for it *qua* a new formal component in our models.

Although the epistemic logic provided here was not able to meet all desiderata one could possibly hope for in a logic for knowledge (which was not its purpose), it is my hope that it nonetheless manages to show, perhaps more clearly than more complicated models embedding the ones presented here, the crucial role that a notion like that of partial grasp of a subject matter might play in revising principles of closure for knowledge.

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