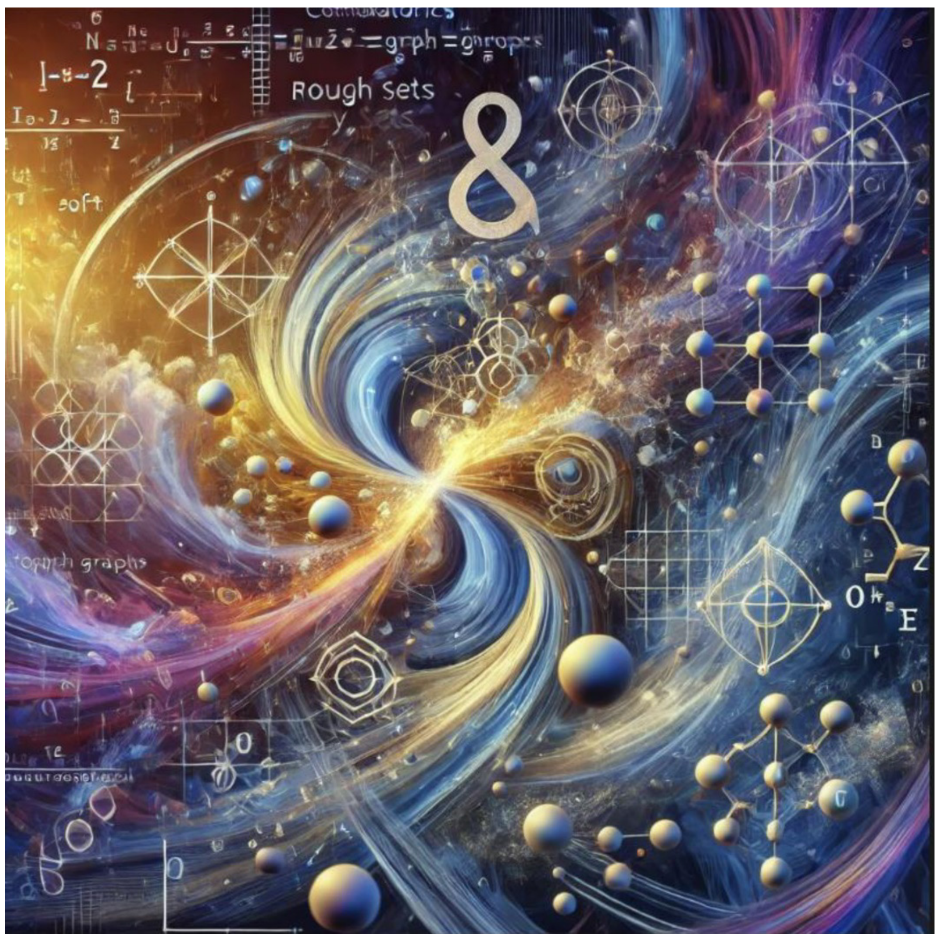
Takaaki Fujita
Florentin Smarandache

Advancing Uncertain Combinatorics

through Graphization, Hyperization, and Uncertainization:
Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Third Volume



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Foreword

The third volume of “Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond” presents an in-depth exploration of the cutting-edge developments in uncertain combinatorics and set theory. This comprehensive collection highlights innovative methodologies such as graphization, hyperization, and uncertainization, which enhance combinatorics by incorporating foundational concepts from fuzzy, neutrosophic, soft, and rough set theories. These advancements open new mathematical horizons, offering novel approaches to managing uncertainty within complex systems.

Combinatorics, a discipline focused on counting, arrangement, and structure, often faces challenges when uncertainty is present. Set theory, which underpins combinatorial problems, has evolved to tackle these challenges. The introduction of fuzzy and neutrosophic sets has expanded the toolkit for modeling uncertainty by incorporating elements of truth, indeterminacy, and falsehood into decision-making processes. These innovations seamlessly intersect with graph theory, providing new ways to represent uncertain structures through "graphized" forms such as hypergraphs and superhypergraphs.

This volume also introduces advanced concepts like Neutrosophic Oversets, Undersets, and Offsets, which push the boundaries of classical graph theory and offer deeper insights into the mathematical and practical challenges posed by real-world systems. By blending combinatorics, set theory, and graph theory, the authors have created a robust framework for addressing uncertainty in both mathematical systems and their real-world applications. This foundation sets the stage for future breakthroughs in combinatorics, set theory, and related fields.

Each chapter in this volume contributes both theoretical foundations and practical applications, demonstrating the power of integrating graph theory, set theory, and uncertainty models. The new ideas, algorithms, and mathematical tools presented here will drive the future of combinatorial research and its applications in uncertain environments.

In the first chapter, “Introduction to Upside-Down Logic: Its Deep Relation to Neutrosophic Logic and Applications”, the authors present Upside-Down Logic, a novel logical framework that systematically transforms truths into falsehoods and vice versa, based on contextual shifts. Introduced by F. Smarandache, this paper provides a mathematical definition of Upside-Down Logic, including applications related to the Japanese language. The chapter also introduces Contextual Upside-Down Logic, an extension that adjusts logical connectives alongside flipped truth values, as well as Indeterm-Upside-Down Logic and Certain Upside-Down Logic to address indeterminacy. A simple algorithm is also proposed to demonstrate the computational aspects of this logic.

In the second chapter, “Local-Neutrosophic Logic and Local-Neutrosophic Sets: Incorporating Locality with Applications”, the authors introduce Local-Neutrosophic Logic and Local-Neutrosophic Sets, which integrate the concept of locality into Neutrosophic Logic. By defining locality as the influence of immediate surroundings on an object or system, this chapter explores how it affects indeterminacy in real-world problems. The paper also examines potential applications and provides mathematical definitions for these new concepts.

The third chapter, “A Review of Fuzzy and Neutrosophic Offsets: Connections to Some Set Concepts and Normalization Function”, extends the concept of offsets in uncertain set-theoretic frameworks, such as Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets. This chapter introduces several advanced types of offsets, including Nonstationary Fuzzy Offset, Multi-valued Plithogenic Offset, and Subset-valued Neutrosophic Offset, offering deeper insights into handling uncertainty in mathematical models.

In the fourth chapter, “Review of Plithogenic Directed, Mixed, Bidirected, and Pangene OffGraph”, the authors build upon Plithogenic Graphs to propose extensions such as Plithogenic Directed OffGraph, Plithogenic BiDirected OffGraph, and Plithogenic Mixed OffGraph. These new concepts, including the Plithogenic Pangene OffGraph, are explored in detail, with a focus on their mathematical properties and potential applications in uncertain graph theory.

The fifth chapter, “Short Note on Neutrosophic Closure Matroids”, explores the extension of matroid concepts into Neutrosophic and Turiyam Neutrosophic set theories, introducing Neutrosophic closure matroids. This concept integrates uncertainty, indeterminacy, and liberal states into matroid theory, enhancing its applicability in optimization and combinatorial problems.

In the sixth chapter, “Some Graph Parameters for Superhypertree-width and Neutrosophic Tree-width”, the authors discuss graph parameters such as Superhypertree-width and Neutrosophic tree-width. These parameters play a crucial role in the study of graph characteristics, particularly in algorithms and real-world applications. The chapter explores the generalization of hypergraphs to SuperHyperGraphs and examines how these concepts extend tree-width parameters within the context of Neutrosophic logic.

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Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Third Volume

This series explores the advancement of uncertain combinatorics through innovative methods such as graphization, hyperization, and uncertainization, incorporating concepts from fuzzy, neutrosophic, soft, and rough set theory, among others. Combinatorics and set theory are fundamental mathematical disciplines that focus on counting, arrangement, and the study of collections under specified rules. While combinatorics excels at solving problems involving uncertainty, set theory has expanded to include advanced concepts like fuzzy and neutrosophic sets, which are capable of modeling complex real-world uncertainties by accounting for truth, indeterminacy, and falsehood. These developments intersect with graph theory, leading to novel forms of uncertain sets in "graphized" structures, such as hypergraphs and superhypergraphs. Innovations like Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing the boundaries of theoretical and practical advancements. This synthesis of combinatorics, set theory, and graph theory provides a strong foundation for addressing the complexities and uncertainties present in mathematical and real-world systems, paving the way for future research and application.

Introduction to Upside-Down Logic: Its Deep Relation to Neutrosophic Logic and Applications

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Abstract

In the study of uncertainty, concepts such as fuzzy sets [113], fuzzy graphs [79], and neutrosophic sets [88] have been extensively investigated. This paper focuses on a novel logical framework known as Upside-Down Logic, which systematically transforms truths into falsehoods and vice versa by altering contexts, meanings, or perspectives. The concept was first introduced by F. Smarandache in [99].

To contribute to the growing interest in this area, this paper presents a mathematical definition of Upside-Down Logic, supported by illustrative examples, including applications related to the Japanese language. Additionally, it introduces and explores Contextual Upside-Down Logic, an advanced extension that incorporates a contextual transformation function, enabling the adjustment of logical connectives in conjunction with flipping truth values based on contextual shifts. Furthermore, the paper introduces Indeterm-Upside-Down Logic and Certain Upside-Down Logic, both of which expand Upside-Down Logic to better accommodate indeterminate values. Finally, a simple algorithm leveraging Upside-Down Logic is proposed and analyzed, providing insights into its computational characteristics and potential applications.

Keywords: Upside-Down Logic, Neutrosophic Logic, Logic, Fuzzy Logic, Japanese

1 Short Introduction

1.1 Uncertain Logic and Upside-Down Logic

Uncertainty in real-world events is modeled using mathematical concepts. In the field of logic (cf. [21, 103]), various frameworks have been developed to address uncertainty, including Fuzzy Logic [113–115], Neutrosophic Logic [88, 90, 94], and Plithogenic Logic [93, 102]. For example, Neutrosophic Logic extends classical logic by introducing three degrees—truth, indeterminacy, and falsity—accommodating uncertainty and contradictions simultaneously. These uncertain logics have also been extended to concepts such as sets [89, 100] and graphs [34, 37–39], resulting in numerous studies parallel to advancements in the logic domain.

This paper focuses on a logical framework called Upside-Down Logic, which systematically transforms truths into falsehoods and vice versa by altering contexts, meanings, or perspectives. This logical concept was introduced by F. Smarandache in [99]. A central focus is the phenomenon of reversals caused by ambiguity. In decision-making, individuals strive to discern what is correct or incorrect. However, ambiguity can lead to situations where something initially perceived as correct ultimately proves to be incorrect, causing misunderstandings or unfortunate outcomes. Simply put, Upside-Down Logic formalizes this phenomenon into a structured logical framework.

1.2 Contributions of This Paper

This subsection explains the contributions of this paper. As discussed above, research on Upside-Down Logic is significant but still in its early stages. This paper aims to advance this emerging field by presenting a mathematical definition of Upside-Down Logic, accompanied by several illustrative examples, including applications related to the Japanese language.

Additionally, as a related concept, this paper introduces and explores Contextual Upside-Down Logic. Contextual Upside-Down Logic extends Upside-Down Logic by incorporating a contextual transformation function that not only flips truth values but also adjusts logical connectives based on the given context.

Moreover, this paper introduces Indeterm-Upside-Down Logic and Certain Upside-Down Logic, both of which extend Upside-Down Logic to better handle indeterminate values. Additionally, a simple algorithm utilizing Upside-Down Logic is examined.

These contributions can be applied to various logical frameworks, such as Neutrosophic Logic, and have potential applications in decision-making and other related fields.

1.3 The Structure of the Paper

The structure of this paper is as follows. Section 2 introduces the concept of Upside-Down Logic. Section 3 delves into Contextual Upside-Down Logic. Section 4 discusses Indeterm-Upside-Down Logic. Section 5 focuses on Certain Upside-Down Logic. Finally, Section 6 outlines future directions for this research.

2 Upside-Down Logic

In this section, we explore the mathematical definition of Upside-Down Logic, real-life examples, basic theorems, its application to Neutrosophic Logic, and its associated algorithms.

2.1 Basic Definition of Formal Language

To explore Upside-Down Logic, several key concepts are introduced below. For further details, readers are encouraged to consult the respective lecture notes and surveys on these topics (ex. [33, 42, 43, 50, 62]).

Definition 2.1 (Set)**.** [50] A *set* is a collection of distinct objects, known as elements, that are clearly defined, allowing any object to be identified as either belonging to or not belonging to the set. If A is a set and x is an element of A, this membership is denoted by $x \in A$. Sets are typically represented using curly brackets.

Definition 2.2 (Formal Language)**.** [42, 78] A *formal language* $\mathcal{L}$ is defined as a set of strings (or sequences) formed from a finite alphabet Σ, subject to specific syntactic rules. Formally:

$$\mathcal{L} \subseteq \Sigma^*,$$

where Σ^* is the set of all finite strings over the alphabet Σ. The strings in $\mathcal{L}$ are called *well-formed formulas (WFFs)*.

A formal language $\mathcal{L}$ is typically accompanied by:

- A set of *symbols* (or *alphabet*) Σ, which may include logical connectives (e.g., $\wedge$, $\vee$, $\neg$), quantifiers (e.g., $\forall$, $\exists$), variables, and parentheses.
- A set of *formation rules* that determine which strings in Σ^* are well-formed.

Example 2.3 (Japanese as a Formal Language)**.** Consider the Japanese language as a formal language $\mathcal{L}$. Note that Readers interested in further details on Japanese linguistics are encouraged to consult references such as [66, 67, 105].

Let:

- Σ be the alphabet, including:
  - Hiragana symbols, e.g., {あ, い, う, え, お, . . . },
  - Katakana symbols, e.g., {ア, イ, ウ, エ, オ, . . . },
  - Kanji characters, e.g., {日, 本, 語, . . . },
  - Romanized characters and punctuation, e.g., $\{a, b, c, \ldots, ., !, ?\}$.
- Σ^* be the set of all possible finite sequences of these symbols.
- Formation rules define grammatically correct sentences in Japanese. Examples include:
  - Subject-Object-Verb order, such as:

    私 (I) + りんご (apple) + を (object marker) + 食べます (eat).

  - Use of particles (e.g., は, が, を) to indicate grammatical roles:

    犬 (dog) + が (subject marker) + 走る (run).

  - Proper use of politeness levels, e.g., です or ます forms.

Thus, $\mathcal{L}$ represents all well-formed Japanese sentences that adhere to these grammatical rules.

Example 2.4 (Mathematical Identity as a Formal Language)**.** Consider the mathematical identity A: "2+2 = 4" as a proposition in a formal language $\mathcal{L}$, where:

- The alphabet Σ consists of symbols for numbers, operators, and equality, e.g., $\Sigma = \{0, 1, 2, 3, 4, +, =, \ldots\}$.
- The formation rules are derived from the axioms and rules of standard arithmetic over the real numbers.

Definition 2.5 (Logical System)**.** (cf. [58]) A *logical system* $\mathcal{M}$ is a mathematical structure that formalizes reasoning. It consists of:

$$\mathcal{M} = (\mathcal{P}, \mathcal{V}, v),$$

where:

- $\mathcal{P}$ is the set of propositions (or statements) in the formal language $\mathcal{L}$.
- $\mathcal{V}$ is the set of truth values, such as $\{\text{True}, \text{False}\}$ for classical logic.
- $v : \mathcal{P} \to \mathcal{V}$ is a *valuation function* (or interpretation function) that assigns a truth value to each proposition in $\mathcal{P}$.

In addition, a logical system may include:

- A set of *axioms* $\mathcal{A} \subseteq \mathcal{P}$ that are assumed to be true within the system.
- A set of *inference rules* $\mathcal{I}$ that define valid transformations of propositions to derive new truths.

2.2 Formal Definition of Upside-Down Logic

In this subsection, we examine the mathematical definition of Upside-Down Logic. The related definitions and concepts are outlined below.

Notation 2.6. *Let $\mathcal{P}$ be a set of propositions, and let C be a set of contexts. Let*

$$T : \mathcal{P} \times C \to \{\mathit{True}, \mathit{False}, \mathit{Indeterminate}\}$$

be a truth valuation function that assigns a truth value to each proposition-context pair.

Notation 2.7. *Let $\mathcal{L}$ be a formal language, and let $\mathcal{M}$ be a logical system with a set of propositions $\mathcal{P}$, a set of truth values $\mathcal{V}$, and a valuation function $v : \mathcal{P} \to \mathcal{V}$.*

Definition 2.8 (Upside-Down Logic)**.** [99] An *Upside-Down Logic* is a logical system $\mathcal{M}'$ derived from $\mathcal{M}$ by introducing a transformation U on propositions and/or contexts such that:

1. For any proposition $A \in \mathcal{P}$ with truth value $v(A)$ in context C, there exists a transformed proposition $U(A)$ and/or transformed context $U(C)$ where:
   - *Falsification of the Truth*: If $v(A) = \text{True}$ in C, then $v(U(A)) = \text{False}$ in $U(C)$.
   - *Truthification of the False*: If $v(A) = \text{False}$ in C, then $v(U(A)) = \text{True}$ in $U(C)$.
2. The transformation U is well-defined and consistent within the logical system $\mathcal{M}'$.

Definition 2.9 (Context)**.** [99] A *context* C is a set of parameters or conditions under which propositions are evaluated. This may include spatial, temporal, semantic, or interpretative settings.

Definition 2.10. The Upside-Down Transformation U may involve one or more of the following operations:

- *Changing the Domain*: Modifying the domain or universe of discourse.
- *Altering Attributes*: Changing properties or characteristics of elements within propositions.
- *Reversing Logical Operations*: Applying dual operators or complement functions.
- *Contextual Shifts*: Altering the context C to $U(C)$ such that the truth value of A changes.
- *Semantic Reinterpretation*: Reinterpreting the meanings of terms or predicates.

2.3 Some Example of Upside-Down Logic

In this subsection, we explore some examples of Upside-Down Logic, such as Mathematical Identity and Physical Laws.

Example 2.11 (Mathematical Identity)**.** Let A: "$2 + 2 = 4$" in the context C of standard arithmetic over the real numbers. In context C:

$$2 + 2 = 4 \implies v(A, C) = \text{True}.$$

Falsification of the Truth: Define a new context $U(C)$ where arithmetic is modulo 3.

In context $U(C)$:

$$2 + 2 = 1 \pmod 3 \implies v(A, U(C)) = \text{False}.$$

Alternatively, consider a proposition A: "$2 + 2 = 0$" in context C where arithmetic is modulo 4.

In context C:

$$2 + 2 = 0 \pmod 4 \implies v(A, C) = \text{True}.$$

Truthification of the False: In standard arithmetic context $U(C)$, where arithmetic is over the integers, $2+2 = 4$, so $v(A, U(C)) = \text{False}$.

Example 2.12 (Physical Laws)**.** Let A: "Water boils at 100°C." In context C: "At standard atmospheric pressure (1 atm)."

$$\text{Boiling point of water at 1 atm is } 100^\circ\text{C} \implies v(A, C) = \text{True}.$$

Falsification of the Truth: Change the context to $U(C)$: "At an altitude of 3,000 meters where atmospheric pressure is approximately 0.7 atm."

At this pressure, water boils at approximately 90°C.

$$v(A, U(C)) = \text{False}.$$

Truthification of the False: Let B: "Water boils at 90°C." In context C: "At sea level (1 atm)."

$$\text{Boiling point is } 100^\circ\text{C} \implies v(B, C) = \text{False}.$$

In transformed context $U(C)$: "At an altitude of 3,000 meters (0.7 atm)."

$$\text{Boiling point is } 90^\circ\text{C} \implies v(B, U(C)) = \text{True}.$$

Example 2.13 (Linguistic Ambiguity)**.** Let A: "The bank is open today." In context C: "Referring to a financial institution on a weekday."

$$\text{If today is a weekday, } v(A, C) = \text{True}.$$

Falsification of the Truth: Change the context to $U(C)$: "Referring to the river bank in a natural reserve area that is closed to the public."

$$v(A, U(C)) = \text{False}.$$

Semantic Reinterpretation: The word "bank" shifts meaning from "financial institution" to "river bank," changing the truth value.

Example 2.14 (Cultural Context)**.** Let A: "Eating beef is acceptable." In context C: "In a Western country where eating beef is a common practice."

$$v(A, C) = \text{True}.$$

Falsification of the Truth: Change the context to $U(C)$: "In India where cows are considered sacred by Hindus."

$$v(A, U(C)) = \text{False}.$$

2.4 Example of Upside-Down Logic on Japanese Culture

In this subsection, we examine examples of Upside-Down Logic within the context of the Japanese language. Known for its high degree of ambiguity, Japanese linguistics have been the subject of extensive research on resolving linguistic ambiguities (cf. [24,70,71,107]). Several illustrative examples are provided below. Readers interested in further details on Japanese linguistics are encouraged to consult references such as [66, 67, 105].

Example 2.15 (Ambiguity in Japanese Expressions)**.** In Japanese, the phrase "*sore wa chotto...*" literally means "that is a little...," trailing off without completing the sentence [65].

Let A: "The speaker is refusing the request."

In context C: The literal interpretation suggests hesitation without explicit refusal:

$$v(A, C) = \text{False}.$$

Context Transformation: Understanding cultural nuances in context $U(C)$, the phrase is recognized as a polite way to decline:

$$v(A, U(C)) = \text{True}.$$

Upside-Down Logic Transformation: Using Upside-Down Logic, the truth value of A flips when shifting from C to $U(C)$.

What appears as hesitation is actually a polite refusal in the transformed context.

Example 2.16 (Proverbs and Contradictions in Japanese)**.** Japanese proverb: *Makeru ga kachi* —"Defeat is victory." [1]

Let A: "Defeat leads to victory."

In context C: Literally interpreted, the proposition appears contradictory:

$$v(A, C) = \text{False}.$$

Context Transformation: In the cultural context $U(C)$, the proverb is understood to mean that yielding can lead to ultimate success:

$$v(A, U(C)) = \text{True}.$$

Upside-Down Logic Transformation: The truth value of A flips when moving from the literal context to the cultural context.

The conventional understanding of winning and losing is inverted through the proverb's cultural meaning.

[1] Japanese proverbs are traditional sayings reflecting cultural values, wisdom, and life lessons, often using metaphorical or poetic expressions [16].

Example 2.17 (Honne and Tatemae)**.** In Japanese culture, *Honne* refers to a person's true feelings, while *Tatemae* refers to the public façade (cf. [104, 111]).

Let A: "He agrees with the proposal."

In context C: Based on his public behavior (*Tatemae*), the proposition appears to be true:

$$v(A, C) = \text{True}.$$

Context Transformation: When shifting to the context of *Honne* (private feelings) $U(C)$, he may actually disagree:

$$v(A, U(C)) = \text{False}.$$

Upside-Down Logic Transformation: Using Upside-Down Logic, we can define the transformed proposition:

$$U(A) : \text{"He disagrees with the proposal internally."}$$

In the context $U(C)$:

$$v(U(A), U(C)) = \text{True}.$$

Agreement in public (*Tatemae*) is inverted to disagreement in private feelings (*Honne*) under the transformed context.

Example 2.18 (Omote and Ura (Front and Back))**.** In Japanese aesthetics, *Omote* means the front or public face, while *Ura* refers to the back or hidden side (cf. [49, 51]).

Let A: "The painting is simple."

In context C: Observing the *Omote* (front) of the painting, the proposition holds true:

$$v(A, C) = \text{True}.$$

Context Transformation: Exploring the *Ura* (hidden meanings) in context $U(C)$, new complexities are revealed:

$$v(A, U(C)) = \text{False}.$$

Upside-Down Logic Transformation: We can redefine the proposition to reflect the depth:

$$U(A) : \text{"The painting is complex with deep symbolism."}$$

In the transformed context $U(C)$:

$$v(U(A), U(C)) = \text{True}.$$

The simplicity observed in the *Omote* is inverted into complexity when considering the *Ura*.

Example 2.19 (The Concept of *Ma*)**.** *Ma* refers to the space or pause between objects or moments, an essential concept in Japanese aesthetics.

Let A: "Silence indicates agreement."

In context C: In some cultures, silence may be interpreted as agreement:

$$v(A, C) = \text{True}.$$

Context Transformation: In the Japanese context $U(C)$, silence can signify various meanings, including disagreement or contemplation:

$$v(A, U(C)) = \text{False}.$$

Upside-Down Logic Transformation: Reformulating the proposition to reflect this ambiguity:

$$U(A) : \text{"Silence does not necessarily indicate agreement."}$$

In the context $U(C)$:

$$v(U(A), U(C)) = \text{True}.$$

The interpretation of silence as agreement is inverted in the Japanese context, highlighting the cultural nuances of *Ma*.

Example 2.20 (The Art of Haiku)**.** Haiku is a form of Japanese poetry that captures a moment with brevity and depth (cf. [80, 109]).

Let A: "Short poems lack depth."

In context C: In the general context of poetry, depth is often associated with longer and more complex forms. Thus:

$$v(A, C) = \text{True}.$$

Context Transformation: Consider the context of Japanese Haiku, $U(C)$, where brevity is seen as a means to achieve profound depth. In this new context, the original proposition A no longer holds true:

$$v(A, U(C)) = \text{False}.$$

Upside-Down Logic Transformation: Using Upside-Down Logic, we transform the proposition to:

$$U(A) : \text{"Short poems convey profound depth."}$$

Under the transformed context $U(C)$, this new proposition aligns with the values of Haiku and becomes true:

$$v(U(A), U(C)) = \text{True}.$$

2.5 Some Basic Theorem of Upside-Down Logic

In this subsection, we consider some basic theorems of Upside-Down Logic. The following theorems hold.

Theorem 2.21 (Invariance under Fixed Context)**.** *Within a fixed context C, the Upside-Down Transformation U must alter the proposition A to change its truth value.*

Proof. Assume that the context C is fixed. To change the truth value of A, the transformation U must alter A to $U(A)$ such that $v(U(A), C) \neq v(A, C)$.

If U does not alter A, then $v(A, C) = v(U(A), C)$, and the truth value remains the same.

Therefore, to achieve Falsification or Truthification within a fixed context, the proposition itself must be altered. □

Theorem 2.22 (Composition of Upside-Down Transformations)**.** *The composition of two Upside-Down Transformations may not necessarily return to the original truth value.*

Proof. Let U_1 and U_2 be two different Upside-Down Transformations.

Applying U_1 and then U_2:

$$v(U_2(U_1(A)), U_2(U_1(C))) = v^{(2)}(A).$$

Since each transformation may alter the context and/or the proposition differently, the final truth value may not be equal to the original $v(A, C)$.

Therefore, in general:

$$v(U_2(U_1(A)), U_2(U_1(C))) \neq v(A, C).$$

□

Theorem 2.23 (Non-Idempotence of Upside-Down Transformations)**.** *An Upside-Down Transformation U is not necessarily idempotent, i.e., applying U twice does not guarantee that the original proposition and context are restored.*

Proof. Consider U that alters the context from C to $U(C)$. Applying U again:

$$U(U(C)) = U^2(C).$$

Unless U is specifically defined such that $U^2(C) = C$ and $U^2(A) = A$, the original proposition and context are not restored.

Therefore, in general, U is not idempotent. □

Corollary 2.24. *If U is involutive (i.e., $U^2 = id$), then applying U twice returns to the original proposition and context.*

Proof. If U is such that $U(U(A)) = A$ and $U(U(C)) = C$, then:

$$v(U(U(A)), U(U(C))) = v(A, C).$$

Thus, the original truth value is restored. □

2.6 Relationship between Neutrosophic Logic and Upside-Down Logic

In this subsection, we explore the relationship between Neutrosophic Logic and Upside-Down Logic. First, we present the definition of Neutrosophic Logic below [36, 88]. Note that Neutrosophic Logic is known to generalize Fuzzy Logic (cf. [88]).

Definition 2.25 (Neutrosophic Logic)**.** [88] Neutrosophic Logic extends classical logic by assigning to each proposition a truth value comprising three components:

$$v(A) = (T, I, F),$$

where $T, I, F \in [0, 1]$ represent the degrees of truth, indeterminacy, and falsity, respectively.

Neutrosophic Logic and related concepts (e.g., Neutrosophic Set) are known for their ability to model a wide range of phenomena [6, 25, 27, 54, 64, 74, 106, 110, 112, 116, 117]. As a reference, several examples of applying Neutrosophic Logic in real-life contexts are provided below.

Example 2.26 (Medical Diagnosis)**.** In medical diagnosis, Neutrosophic Logic is used to model uncertain and incomplete data. For example, a proposition like "The patient has Disease X" can be evaluated as $v(A) = (0.6, 0.2, 0.2)$, where:

- $T = 0.6$: There is a 60% likelihood the patient has the disease.
- $I = 0.2$: 20% of the data is inconclusive.
- $F = 0.2$: There is a 20% likelihood the patient does not have the disease.

This helps in combining test results, symptoms, and expert opinions to make better-informed decisions(cf. [13, 19, 26]).

Example 2.27 (Decision-Making in Business)**.** In business decision-making, Neutrosophic Logic is used to evaluate competing strategies under uncertain conditions. For instance, when deciding whether to invest in a project, a proposition like "The project will yield profit" might have a value $v(A) = (0.7, 0.1, 0.2)$, indicating:

- $T = 0.7$: A 70% chance the project will be profitable.
- $I = 0.1$: A 10% level of uncertainty due to incomplete market data.
- $F = 0.2$: A 20% chance the project will not be profitable.

This allows decision-makers to weigh risks and rewards more effectively (cf. [2, 84]).

Example 2.28 (Artificial Intelligence and Robotics)**.** In AI and robotics, Neutrosophic Logic is used to model complex reasoning in uncertain environments. For instance, a robot navigating a dynamic environment can evaluate a proposition like "The path ahead is clear" as $v(A) = (0.8, 0.1, 0.1)$:

- $T = 0.8$: 80% confidence the path is clear.
- $I = 0.1$: 10% uncertainty due to sensor noise.
- $F = 0.1$: 10% likelihood the path is obstructed.

This helps in planning and adapting to changes in real-time(cf. [7]).

Example 2.29 (Social Network Analysis)**.** In social network analysis, Neutrosophic Logic is applied to measure trustworthiness in online interactions. For a proposition like "User X is trustworthy," a truth value $v(A) = (0.5, 0.3, 0.2)$ might represent:

- $T = 0.5$: A 50% degree of trust based on previous interactions.
- $I = 0.3$: 30% uncertainty due to a lack of sufficient data.
- $F = 0.2$: A 20% indication of untrustworthiness from contradictory feedback.

This helps in filtering content and detecting fraudulent activities (cf. [82, 83]).

The relationship between Neutrosophic Logic and Upside-Down Logic is outlined below. Readers should note that the application examples of Upside-Down Logic presented here are merely illustrative. It is hoped that further concrete investigations into their connection will be undertaken in the future.

Theorem 2.30 (Upside-Down Transformation in Neutrosophic Logic)**.** *In Neutrosophic Logic, the Upside-Down Transformation U corresponds to interchanging the truth and falsity components while possibly adjusting indeterminacy.*

Proof. Define U such that for any proposition A:

$$U(v(A)) = (F, I', T),$$

where I' is determined based on the specific transformation rules (e.g., $I' = I$ or $I' = 1-I$). This transformation flips the degrees of truth and falsity, mirroring the Upside-Down Logic concept. □

Example 2.31 (Upside-Down Transformation in Neutrosophic Logic)**.** Let $v(A) = (0.8, 0.1, 0.1)$. Applying U:

$$U(v(A)) = (0.1, I', 0.8).$$

If we let $I' = I = 0.1$, then:

$$U(v(A)) = (0.1, 0.1, 0.8).$$

Thus, the proposition that was mostly true becomes mostly false under U.

Theorem 2.32 (Preservation of Indeterminacy)**.** *If the Upside-Down Transformation preserves the indeterminacy component, then U is an involutive transformation in Neutrosophic Logic.*

Proof. Assuming $I' = I$ and defining U as:

$$U(v(A)) = (F, I, T).$$

Applying U twice:

$$U(U(v(A))) = U(F, I, T) = (T, I, F) = v(A).$$

Therefore, U is involutive. □

Upside-Down Transformation can similarly be applied to concepts analogous to Neutrosophic Logic. As an example, we consider the application of Upside-Down Transformation to Double-Valued Neutrosophic Logic, as outlined below (cf. [52, 59]). Readers should note that the application examples of Upside-Down Logic presented here are merely illustrative.

Definition 2.33 (Double-Valued Neutrosophic Logic)**.** Let X be a space of points (or objects) where each $x \in X$ represents an element. A *Double-Valued Neutrosophic Set (DVNS) A* is characterized by:

$$A = \{(x, T_A(x), I_T(x), I_F(x), F_A(x)) : x \in X\},$$

where:

- $T_A(x) \in [0, 1]$ is the *truth membership value*,
- $I_T(x) \in [0, 1]$ is the *indeterminacy leaning towards truth*,
- $I_F(x) \in [0, 1]$ is the *indeterminacy leaning towards falsity*,
- $F_A(x) \in [0, 1]$ is the *falsity membership value*.

These values satisfy the condition:

$$0 \leq T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4.$$

As a reference, several examples of utilizing Double-Valued Neutrosophic Logic in real-life contexts are provided below.

Example 2.34 (Medical Treatment Effectiveness Evaluation)**.** Consider a medical study evaluating the effectiveness of a new drug. For each patient x, the Double-Valued Neutrosophic Set A represents the assessment of the statement: ”The drug is effective for the patient.”

$$A = \{(x, T_A(x), I_T(x), I_F(x), F_A(x)) : x \in X\},$$

where:

- $T_A(x) = 0.8$: 80% evidence supports the drug's effectiveness.
- $I_T(x) = 0.1$: 10% uncertainty leans towards effectiveness due to inconclusive lab results.
- $I_F(x) = 0.05$: 5% uncertainty leans towards ineffectiveness due to side effects.
- $F_A(x) = 0.05$: 5% evidence indicates the drug is not effective.

This representation allows researchers to account for both supporting and opposing evidence, as well as uncertainty, in evaluating the drug's performance.

Example 2.35 (Customer Sentiment Analysis)**.** In e-commerce, Double-Valued Neutrosophic Logic can be used to analyze customer sentiments about a product. For each customer x, the Double-Valued Neutrosophic Set A evaluates the statement: ”The customer is satisfied with the product.”

$$A = \{(x, T_A(x), I_T(x), I_F(x), F_A(x)) : x \in X\},$$

where:

- $T_A(x) = 0.7$: 70% of the customer's feedback indicates satisfaction.
- $I_T(x) = 0.15$: 15% uncertainty leans towards satisfaction due to mixed comments.
- $I_F(x) = 0.1$: 10% uncertainty leans towards dissatisfaction due to delivery delays.
- $F_A(x) = 0.05$: 5% feedback explicitly indicates dissatisfaction.

This enables companies to better understand customer opinions by considering both the certainty and directionality of indeterminate feedback.

Proposition 2.36. *A Double-Valued Neutrosophic Logic (DVNL) can be transformed into a standard Neutrosophic Logic (NL) by redefining the indeterminacy membership values. Specifically, let A be a Double-Valued Neutrosophic Set (DVNS) characterized by:*

$$A = \{(x, T_A(x), I_T(x), I_F(x), F_A(x)) : x \in X\}.$$

The corresponding Neutrosophic Set A' is obtained as:

$$A' = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\},$$

where the total indeterminacy $I_A(x)$ is given by:

$$I_A(x) = I_T(x) + I_F(x).$$

Proof. Let A be a DVNS defined as:

$$A = \{(x, T_A(x), I_T(x), I_F(x), F_A(x)) : x \in X\}.$$

By definition, the total indeterminacy $I_A(x)$ in a Neutrosophic Logic framework combines the contributions from $I_T(x)$ and $I_F(x)$. Thus, we define:

$$I_A(x) = I_T(x) + I_F(x).$$

The transformed set A' becomes:

$$A' = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\}.$$

The condition for a DVNS ensures:

$$0 \leq T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4.$$

Since $I_A(x) = I_T(x) + I_F(x)$, the condition for A' simplifies to:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3,$$

which satisfies the standard Neutrosophic Logic framework. Therefore, A' is a valid Neutrosophic Set.

The transformation from DVNL to NL is achieved by aggregating $I_T(x)$ and $I_F(x)$ into a single indeterminacy membership value $I_A(x)$, preserving the consistency of the logical framework. □

Definition 2.37 (Upside-Down Transformation for DVNS)**.** For any $x \in X$, the Upside-Down Transformation U modifies the values as follows:

$$U(T_A(x)) = F_A(x),$$
$$U(F_A(x)) = T_A(x),$$
$$U(I_T(x)) = I_F(x),$$
$$U(I_F(x)) = I_T(x).$$

Example 2.38 (Service Quality Evaluation)**.** Service Quality Evaluation is the process of assessing service performance based on customer expectations, perceptions, and satisfaction across various dimensions (cf. [86]).

Let $X = \{x_1, x_2, x_3\}$, where: x_1: Capability, x_2: Trustworthiness, and x_3: Price.

Define a DVNS A over X:

$$A = \{(x_1, 0.3, 0.4, 0.2, 0.1), (x_2, 0.5, 0.3, 0.1, 0.1), (x_3, 0.7, 0.2, 0.1, 0.0)\}.$$

Before UDL Transformation:

- For x_1: $T_A(x_1) = 0.3, I_T(x_1) = 0.4, I_F(x_1) = 0.2, F_A(x_1) = 0.1$.
- For x_2: $T_A(x_2) = 0.5, I_T(x_2) = 0.3, I_F(x_2) = 0.1, F_A(x_2) = 0.1$.
- For x_3: $T_A(x_3) = 0.7, I_T(x_3) = 0.2, I_F(x_3) = 0.1, F_A(x_3) = 0.0$.

After UDL Transformation:

- For x_1:
$$U(T_A(x_1)) = F_A(x_1) = 0.1, \quad U(F_A(x_1)) = T_A(x_1) = 0.3,$$
$$U(I_T(x_1)) = I_F(x_1) = 0.2, \quad U(I_F(x_1)) = I_T(x_1) = 0.4.$$
Result: $(x_1, 0.1, 0.2, 0.4, 0.3)$.
- For x_2:
$$U(T_A(x_2)) = F_A(x_2) = 0.1, \quad U(F_A(x_2)) = T_A(x_2) = 0.5,$$
$$U(I_T(x_2)) = I_F(x_2) = 0.1, \quad U(I_F(x_2)) = I_T(x_2) = 0.3.$$
Result: $(x_2, 0.1, 0.1, 0.3, 0.5)$.
- For x_3:
$$U(T_A(x_3)) = F_A(x_3) = 0.0, \quad U(F_A(x_3)) = T_A(x_3) = 0.7,$$
$$U(I_T(x_3)) = I_F(x_3) = 0.1, \quad U(I_F(x_3)) = I_T(x_3) = 0.2.$$
Result: $(x_3, 0.0, 0.1, 0.2, 0.7)$.

Observations:

- Truth (T_A) and falsity (F_A) are inverted, reflecting the essence of Upside-Down Logic.
- Indeterminacy leaning towards truth (I_T) and falsity (I_F) are swapped, altering the nuanced interpretation of uncertainty.
- This transformation highlights the flexibility and adaptability of DVN Logic under the influence of UDL, allowing for a richer representation of real-world ambiguity.

Theorem 2.39 (Stability of Membership Sum Constraint). *The sum of the membership values remains within the allowable range* $[0, 4]$ *after the Upside-Down Transformation.*

Proof. Let $x \in X$. For the original Double-Valued Neutrosophic Set (DVNS) A, we have:

$$T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4.$$

After applying the Upside-Down Transformation:

$$U(T_A(x)) = F_A(x), \quad U(F_A(x)) = T_A(x), \quad U(I_T(x)) = I_F(x), \quad U(I_F(x)) = I_T(x).$$

The transformed sum becomes:

$$U(T_A(x)) + U(I_T(x)) + U(I_F(x)) + U(F_A(x)) = F_A(x) + I_F(x) + I_T(x) + T_A(x).$$

Since the sum is a permutation of the original values:

$$F_A(x) + I_F(x) + I_T(x) + T_A(x) = T_A(x) + I_T(x) + I_F(x) + F_A(x).$$

Thus, the sum remains unchanged, and the constraint $0 \leq \text{Sum} \leq 4$ is preserved. □

Theorem 2.40 (Invertibility of the Upside-Down Transformation). *The Upside-Down Transformation U is invertible.*

Proof. Define the transformation U as:

$$U(T_A(x)) = F_A(x), \quad U(F_A(x)) = T_A(x), \quad U(I_T(x)) = I_F(x), \quad U(I_F(x)) = I_T(x).$$

To invert U, apply the transformation again:

$$U(U(T_A(x))) = U(F_A(x)) = T_A(x),$$

$$U(U(F_A(x))) = U(T_A(x)) = F_A(x),$$
$$U(U(I_T(x))) = U(I_F(x)) = I_T(x),$$
$$U(U(I_F(x))) = U(I_T(x)) = I_F(x).$$

Thus, applying U twice returns the original values. Therefore, U is its own inverse, i.e., $U^2 = \text{Identity}$. □

Theorem 2.41 (Truth Preservation in Special Cases). *If a DVNS has zero indeterminacy ($I_T(x) = I_F(x) = 0$), the transformation flips truth and falsity values but preserves their relationship.*

Proof. If $I_T(x) = I_F(x) = 0$, the DVNS simplifies to:

$$A = \{(x, T_A(x), 0, 0, F_A(x)) : x \in X\}.$$

Applying U:

$$U(A) = \{(x, F_A(x), 0, 0, T_A(x)) : x \in X\}.$$

Thus:

$$T_A(x) + F_A(x) \text{ (original sum)} = F_A(x) + T_A(x) \text{ (transformed sum)}.$$

Since the transformation merely swaps $T_A(x)$ and $F_A(x)$, the truth and falsity values are inverted while maintaining consistency. □

Upside-Down Logic is not limited to applications in Neutrosophic Logic or Double-Valued Neutrosophic Logic; it can be extended to various logical and set-based frameworks that handle uncertainty.

Although this paper does not delve into details, related concepts such as Neutrosophic OffLogic, Neutrosophic OverLogic, and Neutrosophic UnderLogic are also well-known. Neutrosophic OffLogic, for instance, operates not within the standard $[0, 1]$ interval but within an extended or alternative range of values, allowing greater flexibility in representing degrees of truth, indeterminacy, and falsity [18, 87, 91, 92, 101]. Similar to standard Neutrosophic Logic, these frameworks support the upside-down transformation between truth and falsity, enabling dynamic and adaptive reasoning under uncertain or contradictory conditions.

Question 2.42. Is it possible to mathematically define concepts of Upside-Down Logic in the context of set theory [50] or graph theory [28,29]?

Question 2.43. Can Upside-Down Logic be applied beyond logic and set concepts? For example:

- In game theory [72], could it be used to model new equilibria or strategies by considering reversals of perspectives or situations?
- In belief revision [41], could it provide a framework for dynamically altering the truth or falsity of beliefs based on specific evidence or contexts?
- In quantum logic [31], could it be utilized to reverse states or transform the interpretation of measurement results?

2.7 Algorithm for Upside-Down Logic

An algorithm is a finite, step-by-step procedure designed to perform a specific task or solve a problem systematically and efficiently (cf. [22, 85]). Below, we present an algorithm for Upside-Down Logic.

Algorithm 1 Upside-Down Logic Algorithm

Require: A set of propositions $\mathcal{P}$, a context C, and a truth valuation function $T : \mathcal{P} \times C \to \{\text{True}, \text{False}, \text{Indeterminate}\}$

Ensure: Transformed propositions $U(A)$, transformed contexts $U(C)$, and updated truth values

Parse the input $\mathcal{P}$, C, and T. Define the transformation function U

for each $A \in \mathcal{P}$ **do**

  Compute $U(A)$ and $U(C)$

  Update truth values:

$$T(U(A), U(C)) = \begin{cases} \text{False}, & \text{if } T(A, C) = \text{True}, \\ \text{True}, & \text{if } T(A, C) = \text{False}, \\ \text{Indeterminate}, & \text{otherwise.} \end{cases}$$

end for

Apply U to logical connectives $(\wedge, \vee, \neg)$ and compute transformed truth tables

return $U(\mathcal{P})$, $U(C)$, and updated truth values

Theorem 2.44 (Correctness of the Upside-Down Logic Algorithm)**.** *The Upside-Down Logic algorithm transforms propositions and contexts according to the defined transformation function U, correctly updates truth values based on the rules of Upside-Down Logic, and preserves logical consistency.*

Proof. The algorithm processes the input as follows:

1. Initialization: The input sets $\mathcal{P}$, C, and the truth valuation function T are parsed correctly. The transformation U is well-defined and invertible, ensuring valid inputs and outputs.

2. Transformation of Propositions and Contexts: For each $A \in \mathcal{P}$, the algorithm computes $U(A)$ and $U(C)$. These transformations are consistent with the Upside-Down Logic framework, ensuring valid outputs $U(\mathcal{P})$ and $U(C)$.

3. Truth Value Update: The algorithm updates truth values as:

$$T(U(A), U(C)) = \begin{cases} \text{False}, & \text{if } T(A, C) = \text{True}, \\ \text{True}, & \text{if } T(A, C) = \text{False}, \\ \text{Indeterminate}, & \text{otherwise.} \end{cases}$$

This ensures that True becomes False, False becomes True, and Indeterminate remains unchanged, respecting the rules of Upside-Down Logic.

4. Adjustment of Logical Connectives: Logical connectives $(\wedge, \vee, \neg)$ are transformed consistently under U, maintaining logical relationships within $U(C)$.

5. Logical Consistency: Since U is invertible and operates independently on propositions and contexts, logical consistency is preserved throughout the transformation process.

The algorithm correctly implements the rules of Upside-Down Logic, updates all truth values systematically, and ensures logical consistency in the transformed system. Therefore, the algorithm is correct. □

Definition 2.45. (cf. [75, 85]) The *Total Time Complexity* of an algorithm is defined as the sum of the time required to execute each step of the algorithm, expressed as a function of the input size. If an algorithm involves multiple steps or operations, the total time complexity is determined by the maximum time required for the most time-consuming operation.

Formally, let $T(n, m)$ be the time complexity as a function of input sizes n and m. The total time complexity is:

$$T(n, m) = \max(T_{\text{operation1}}(n, m), T_{\text{operation2}}(n, m), \ldots, T_{\text{operationk}}(n, m)),$$

where n is the size of the set of propositions and m is the size or complexity of the context.

Definition 2.46. (cf. [75, 85]) The *Space Complexity* of an algorithm is the total amount of memory required to execute the algorithm, expressed as a function of the input size. This includes:

- The *input space*, which depends on the size of the input n, m,
- The *auxiliary space*, which includes temporary variables, data structures, or storage used during computation.

Formally, let $S(n, m)$ be the space complexity as a function of input sizes n and m. The total space complexity is:

$$S(n, m) = S_{\text{input}}(n, m) + S_{\text{auxiliary}}(n, m).$$

Definition 2.47. (cf. [75, 85]) *Big-O notation* is a mathematical concept used to describe the upper bound of the time or space complexity of an algorithm. Let $f(n)$ and $g(n)$ be functions that map non-negative integers to non-negative real numbers. We say:

$$f(n) \in O(g(n))$$

if there exist constants $c > 0$ and $n_0 \geq 0$ such that:

$$f(n) \leq c \cdot g(n) \quad \text{for all } n \geq n_0.$$

Theorem 2.48 (Time and Space Complexity of Upside-Down Logic Algorithm)**.** *In the Upside-Down Logic algorithm, the following complexities hold: Total Time Complexity: $O(n + m)$.*
Space Complexity: $O(n + m)$.

Proof. Step 1: Transformation of Propositions
For a set of n propositions $\mathcal{P}$, each proposition $A \in \mathcal{P}$ is transformed using the Upside-Down transformation U. Since the transformation involves a single operation per proposition, the complexity is $O(n)$.

Step 2: Transformation of Context
The transformation of the context C involves modifying m parameters (e.g., spatial, temporal, semantic attributes). Each parameter requires a constant amount of work, resulting in a complexity of $O(m)$.

Step 3: Truth Value Updates
Each proposition $A \in \mathcal{P}$ is evaluated within the transformed context $U(C)$. The evaluation of truth values requires m operations per context, resulting in a complexity of $O(m)$ for each proposition. Since there are n propositions, the total complexity for truth value updates is $O(n \cdot m)$.

Step 4: Logical Connective Adjustment
Logical connectives ($\wedge, \vee, \neg$) associated with each proposition are adjusted once per proposition. This results in a complexity of $O(n)$.

Total Time Complexity:
Summing the complexities of all steps:

$$\text{Total Time Complexity} = O(n) + O(m) + O(n \cdot m) + O(n).$$

The term $O(n \cdot m)$ dominates for large n and m. However, if m is constant or small relative to n, the total complexity simplifies to $O(n + m)$.

Space Complexity:
The algorithm requires memory to store:

- The transformed propositions $U(\mathcal{P})$ ($O(n)$),
- The transformed context $U(C)$ ($O(m)$),
- The updated truth values ($O(n \cdot m)$).

Combining these, the total space complexity is $O(n + m)$.

Thus, the stated complexities are proven. □

3 Contextual Upside-Down Logic

In this section, We introduce a derivative logic called *Contextual Upside-Down Logic*, which extends Upside-Down Logic by incorporating a contextual transformation function that not only flips truth values but also adjusts the logical connectives based on context.

3.1 Formal Definition of Contextual Upside-Down Logic

The definition of Contextual Upside-Down Logic is provided below. In a highly intuitive sense, it is about defining a system that controls "differences in context" using different logical operations. To put it boldly, it can be seen as a definition aimed at reducing the probability of Upside-Down occurrences caused by various factors. There is no significant operational difference from Upside-Down Logic; the only distinction is that the contextual transformation function is explicitly defined.

Definition 3.1 (Contextual Upside-Down Logic)**.** A *Contextual Upside-Down Logic* is a logical system $\mathcal{M}''$ derived from a base logical system $\mathcal{M}$ by introducing a context-dependent transformation U_C such that:

1. For any proposition $A \in \mathcal{P}$ with truth value $v(A, C)$ in context C, there exists a transformed proposition $U_C(A)$ and transformed context $U_C(C)$ where:
   - *Contextual Falsification*: If $v(A, C) = \text{True}$, then $v(U_C(A), U_C(C)) = \text{False}$.
   - *Contextual Truthification*: If $v(A, C) = \text{False}$, then $v(U_C(A), U_C(C)) = \text{True}$.

2. The logical connectives are adjusted according to the context transformation, allowing for different logical operations in $U_C(C)$.
3. The transformation U_C is well-defined, invertible, and consistent within the logical system $\mathcal{M}''$.

Remark 3.2. The comparison with General Upside-Down Logic and notable points are described below.

- *Context Dependence*: General Upside-Down Logic applies transformations U globally, altering truth values uniformly across all contexts. In contrast, Contextual Upside-Down Logic introduces a transformation U_C that is explicitly dependent on the context C, allowing for adjustments that are sensitive to specific cultural, semantic, or situational factors.
- *Adjustment of Logical Connectives*: In Contextual Upside-Down Logic, not only are truth values flipped based on context, but logical connectives are also adjusted. This means that operations like "and," "or," "implies," etc., may have different interpretations in different contexts, affecting the outcome of logical evaluations.
- *Transformation Scope*: While Upside-Down Logic applies transformations uniformly, Contextual Upside-Down Logic allows transformations to vary between contexts, leading to a more nuanced and accurate modeling of propositions in context-dependent scenarios.

3.2 Some Example of Contextual Upside-Down Logic

In this subsection, we explore some examples of Contextual Upside-Down Logic.

Example 3.3 (Cultural Interpretation of Silence)**.** Let A: "Silence implies agreement."

In context C_{Western}:

In some Western contexts, silence during a discussion can be interpreted as agreement or consent:

$$v(A, C_{\text{Western}}) = \text{True}.$$

Contextual Transformation:

In Japanese cultural context $U_C(C_{\text{Japanese}})$, silence often carries different meanings, such as disagreement, contemplation, or polite refusal.

Applying the context-dependent transformation U_C, we adjust both the proposition and the logical connective "implies":

$$U_C(A) : \text{"Silence does not imply agreement."}$$

The logical connective "implies" is reinterpreted based on the cultural context.

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Japanese}})) = \text{True}.$$

This example demonstrates how Contextual Upside-Down Logic adjusts both the proposition and the logical connective based on cultural context, differing from general Upside-Down Logic by its context-sensitive approach.

Example 3.4 (Double Negatives in Language)**.** Double Negative refers to the use of two negative terms in a sentence, which can create emphasis or reverse the intended meaning(cf. [9]).

Let A: "He didn't do nothing."

In context $C_{\text{Standard English}}$:

In Standard English, double negatives are typically interpreted as a positive statement due to the logical rule that two negatives make a positive:

$$v(A, C_{\text{Standard English}}) = \text{True}.$$

Contextual Transformation:

In certain dialects or colloquial contexts $U_C(C_{\text{Dialectal English}})$, double negatives are used for emphasis and still convey a negative meaning.

Applying the context-dependent transformation U_C, the logical connective (negation) is adjusted:

$$U_C(A) : \text{"He did not do anything."}$$

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Dialectal English}})) = \text{False}.$$

This example shows how Contextual Upside-Down Logic adjusts the interpretation of logical connectives based on linguistic context, leading to different truth values.

Example 3.5 (Logical Connectives in Cultural Context)**.** Let A: "He is smart and hardworking."

In context $C_{\text{Culture A}}$:

In Culture A, the conjunction "and" is used to combine two positive attributes straightforwardly:

$$v(A, C_{\text{Culture A}}) = \text{True}.$$

Contextual Transformation:

In Culture B, where modesty is highly valued, praising someone with multiple positive attributes may be seen as excessive or may imply that one attribute detracts from the other.

Applying the context-dependent transformation U_C, the logical connective "and" is adjusted to reflect this cultural nuance:

$$U_C(A) : \text{"He is smart but not necessarily hardworking."}$$

Or the "and" might be interpreted as an exclusive "or":

$$U_C(A) : \text{"He is smart or hardworking (but not both)."}$$

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Culture B}})) = \text{Uncertain or False}.$$

This example illustrates how the interpretation of logical connectives like "and" can change based on cultural context, a feature captured by Contextual Upside-Down Logic.

Example 3.6 (Indirect Communication in Japanese Culture)**.** Indirect Communication in Japanese Culture emphasizes subtlety, non-verbal cues, and ambiguity, often prioritizing social harmony and avoiding direct confrontation or disagreement (cf. [30, 68]).

Let A: "He is refusing the request."

In context $C_{\text{Direct Communication}}$:

In a culture that values direct communication, saying "I will think about it" suggests consideration, implying that the person may accept the request:

$$v(A, C_{\text{Direct Communication}}) = \text{False}.$$

Contextual Transformation:

In Japanese culture $U_C(C_{\text{Japanese}})$, the phrase "*Kangaete okimasu*" ("I will think about it") is often a polite way to decline a request.

Applying the context-dependent transformation U_C, we adjust the proposition and interpret the logical implications differently:

$$U_C(A) : \text{"He is politely declining the request."}$$

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Japanese}})) = \text{True}.$$

The transformation demonstrates how Contextual Upside-Down Logic captures the nuances of indirect communication by adjusting the proposition and its interpretation based on cultural context.

Example 3.7 (Interpretation of "Yes" in Different Contexts)**.** Let A: "She agrees with the proposal."

In context C_{General}:

In many cultures, saying "yes" directly signifies agreement:

$$v(A, C_{\text{General}}) = \text{True}.$$

Contextual Transformation:

In Japanese culture $U_C(C_{\text{Japanese}})$, saying "*Hai*" can sometimes mean "I hear you" or "I acknowledge what you are saying," without implying agreement.

Applying the context-dependent transformation U_C, we adjust the proposition:

$$U_C(A) : \text{"She acknowledges the proposal but may not agree with it."}$$

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Japanese}})) = \text{Uncertain or False}.$$

This example highlights how Contextual Upside-Down Logic accounts for cultural differences in interpreting affirmations, adjusting both the proposition and its truth value accordingly.

Example 3.8 (Concept of "Honne" and "Tatemae" in Japanese Culture)**.** Let A: "He is expressing his true opinions."

In context $C_{\text{Individualistic}}$:

In an individualistic culture where personal expression is encouraged, the proposition is likely considered true:

$$v(A, C_{\text{Individualistic}}) = \text{True}.$$

Contextual Transformation:

In Japanese culture $U_C(C_{\text{Japanese}})$, the concepts of *Honne* (true feelings) and *Tatemae* (public facade) play a significant role. Individuals often express *Tatemae* to maintain social harmony(cf. [17, 60]).

Applying the context-dependent transformation U_C:

$$U_C(A) : \text{"He is expressing what is expected socially, not his true opinions."}$$

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Japanese}})) = \text{False}.$$

This demonstrates how Contextual Upside-Down Logic captures the impact of cultural concepts on truth values and the interpretation of propositions.

Example 3.9 (Adjusting Logical Connectives Based on Context)**.** Let A: "Taking risks leads to success."

In context $C_{\text{Entrepreneurial}}$:

In an entrepreneurial context where risk-taking is valued, the proposition is considered true:

$$v(A, C_{\text{Entrepreneurial}}) = \text{True}.$$

Contextual Transformation:

In a context where stability is prioritized, such as in certain traditional cultures $U_C(C_{\text{Traditional}})$, the logical connective ”leads to” may be interpreted differently, perhaps even reversed.

Applying the context-dependent transformation U_C:

$$U_C(A) : \text{"Taking risks leads to failure."}$$

Updated Truth Value:

In the transformed context:

$$v(U_C(A), U_C(C_{\text{Traditional}})) = \text{True}.$$

This example shows how Contextual Upside-Down Logic adjusts both the proposition and the logical connective based on context, altering the truth value accordingly.

Remark 3.10. In these examples, Contextual Upside-Down Logic differs from general Upside-Down Logic by:

- **Context-Specific Transformations**: Adjustments are made based on specific contexts rather than applying a uniform transformation across all contexts.
- **Adjustment of Logical Connectives**: Logical connectives are reinterpreted according to the context, affecting how propositions are connected and evaluated.
- **Nuanced Truth Values**: Truth values may become uncertain or require more nuanced interpretation due to context-dependent meanings.

3.3 Some Basic Theorem of Contextual Upside-Down Logic

In this subsection, we consider some basic theorems of Contextual Upside-Down Logic. The following theorems hold.

Definition 3.11 (Bijective Function)**.** A function $f : X \to Y$ is called *bijective* if it is both injective (one-to-one) and surjective (onto). This means:

- f is **injective**: For all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- f is **surjective**: For every $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$.

Thus, every element in Y is mapped to by exactly one element in X.

Definition 3.12 (Invertible Function)**.** A function $f : X \to Y$ is called *invertible* if there exists a function $g : Y \to X$ such that:

$$g(f(x)) = x \quad \text{for all } x \in X, \quad \text{and} \quad f(g(y)) = y \quad \text{for all } y \in Y.$$

The function g is called the *inverse* of f, and is denoted by f^{-1}.

Theorem 3.13 (Contextual Invertibility)**.** *In Contextual Upside-Down Logic, the transformation U_C is invertible if and only if the context mapping $U_C : C \to U_C(C)$ is bijective.*

Proof. If U_C is invertible, there exists a U_C^{-1} such that:

$$U_C^{-1}(U_C(A)) = A, \quad U_C^{-1}(U_C(C)) = C.$$

This requires that U_C is a bijection on both propositions and contexts.

Conversely, if U_C is bijective, then it has an inverse function, and the transformation is invertible. □

Theorem 3.14 (Consistency Preservation)**.** *If the base logic $\mathcal{M}$ is consistent, and the transformation U_C is consistent, then the derived Contextual Upside-Down Logic $\mathcal{M}''$ is consistent.*

Proof. Since U_C transforms propositions and contexts without introducing contradictions, and $\mathcal{M}$ is consistent, any deductions made in $\mathcal{M}''$ will also be consistent. □

3.4 Algorithm for Contextual Upside-Down Transformation

We present an algorithm that, given a proposition A and a context C, computes the transformed proposition $U_C(A)$ and the transformed context $U_C(C)$.

Algorithm 2 Contextual Upside-Down Transformation Algorithm

Require: Proposition A, Context C, Valuation Function v
Ensure: Transformed Proposition $U_C(A)$, Transformed Context $U_C(C)$, Updated Truth Value $v(U_C(A), U_C(C))$

Compute the truth value $v(A, C)$
if $v(A, C)$ = True **then**
    Apply a context transformation function $T_C : C \to U_C(C)$
    Define $U_C(C) = T_C(C)$ such that $v(A, U_C(C))$ = False
else if $v(A, C)$ = False **then**
    Apply a context transformation function $T_C : C \to U_C(C)$
    Define $U_C(C) = T_C(C)$ such that $v(A, U_C(C))$ = True
end if
Define $U_C(A) = A$
Adjust the logical connectives within A to reflect changes in $U_C(C)$, if necessary
return $U_C(A), U_C(C), v(U_C(A), U_C(C))$

Theorem 3.15 (Correctness of the Algorithm)**.** *The Contextual Upside-Down Transformation Algorithm correctly computes the transformed proposition and context such that the truth value of A is inverted in the transformed context, satisfying the definitions of Contextual Upside-Down Logic.*

Proof. Consider the two cases:

1. If $v(A, C)$ = True, the algorithm applies a context transformation T_C such that $U_C(C)$ makes A false. By definition of Contextual Falsification, $v(U_C(A), U_C(C))$ = False.
2. If $v(A, C)$ = False, the algorithm applies a context transformation T_C such that $U_C(C)$ makes A true. By definition of Contextual Truthification, $v(U_C(A), U_C(C))$ = True.

Thus, the algorithm adheres to the principles of Contextual Upside-Down Logic. □

Theorem 3.16 (Time and Space Complexity of the Contextual Upside-Down Transformation Algorithm)**.** *Let n represent the size of the proposition A (e.g., the number of symbols or logical operators), and m represent the complexity of the context C. The following hold for the Contextual Upside-Down Transformation Algorithm:*

1. *The total time complexity is $O(n + m)$.*

2. *The total space complexity is $O(n + m)$.*

Proof. To prove the claims, we analyze each step of the algorithm.

1. *Truth Value Computation*: This step evaluates $v(A, C)$, which involves traversing A and assessing its logical structure within the context C.
   - Traversing A contributes $O(n)$.
   - Accessing contextual information for C contributes $O(m)$.

   Therefore, this step has a complexity of $O(n + m)$.
2. *Context Transformation*: The context C is transformed into $U_C(C)$ using a predefined transformation function T_C.
   - Since this operation is independent of A, the complexity depends solely on C and is $O(m)$.
3. *Logical Connective Adjustment*: Adjusting logical connectives within A requires traversing the structure of A.
   - This contributes $O(n)$.
4. *Total Time Complexity*: Summing up the complexities of the above steps, the total time complexity is:
$$O(n + m) + O(m) + O(n) = O(n + m).$$
5. *Space Complexity*: The algorithm requires space to store the transformed proposition $U_C(A)$ and the transformed context $U_C(C)$.
   - The storage for A contributes $O(n)$.
   - The storage for C contributes $O(m)$.

   Thus, the total space complexity is:
$$O(n + m).$$

This completes the proof of the theorem. □

Example 3.17 (Ambiguity in Language Translation using the Algorithm)**.** We consider following example.

- *Proposition*: A: ”He is cool.”
- *Context*: C: American English, where ”cool” means ”good” or ”impressive.”
- *Valuation*: $v(A, C)$ = True.

Transformation: Transform the context to $U_C(C)$: British English, where ”cool” might mean ”cold” or emotionally distant. $v(U_C(A), U_C(C))$ = False.

Thus, the algorithm successfully inverts the truth value via context transformation.

4 Framework of Indeterm-Upside-Down Logic

We propose a new framework called Indeterm-Upside-Down Logic. This logic is designed to represent real-world phenomena where Indeterminate either decreases or increases.

4.1 Definition and Mathematical Formulation

We examine the definition and mathematical formulation of Indeterm-Upside-Down Logic.

Notation 4.1. *Let $\mathcal{P}$ be a set of propositions, and let C be a set of contexts. Let $T : \mathcal{P} \times C \to \{True, False, Indeterminate\}$ be a truth valuation function that assigns a truth value to each proposition-context pair.*

The following four transformations are defined within the framework of Indeterm-Upside-Down Logic:

- Indeterminate-to-True Transformation: Transforms indeterminate propositions into true ones when evidence strongly supports their truthfulness.
- Indeterminate-to-False Transformation: Converts indeterminate propositions to false when new evidence contradicts their truthfulness.
- True-to-Indeterminate Transformation: Shifts propositions from true to indeterminate when supporting evidence becomes unreliable or insufficient.
- False-to-Indeterminate Transformation: Transforms propositions from false to indeterminate when evidence previously supporting falsity becomes inconclusive.

The framework of Indeterm-Upside-Down Logic, including its definitions and brief examples, is presented below.

Definition 4.2 (Indeterminate-to-True Transformation)**.** Let $A \in \mathcal{P}$, and suppose $T(A, C) = \text{Indeterminate}$. The transformation U_{IT} (Indeterminate-to-True) changes the truth value as follows:

$$T(U_{\text{IT}}(A), U_{\text{IT}}(C)) = \text{True}.$$

This transformation is applied if additional evidence or contextual information **supports the truthfulness** of the proposition.

Example 4.3 (Diagnosing a Medical Condition)**.** Let A: ”The patient has a specific medical condition.”

Initial Context C: The initial test results are inconclusive, making the truth value of A indeterminate:

$$T(A, C) = \text{Indeterminate}.$$

Context Transformation $U_{\text{IT}}(C)$: New diagnostic evidence, such as advanced imaging or biomarkers, strongly supports the presence of the condition:

$$T(U_{\text{IT}}(A), U_{\text{IT}}(C)) = \text{True}.$$

Interpretation: As new evidence emerges, the truth value of the proposition A shifts from indeterminate to true, allowing for a definitive diagnosis.

Example 4.4 (Japanese Linguistic Ambiguity)**.** Let E: ”*Sore wa chotto...*” means ”The speaker is refusing the request.”

Initial Context C: In a literal interpretation, the phrase appears hesitant and non-committal:

$$T(E, C) = \text{Indeterminate}.$$

Context Transformation $U_{\text{IT}}(C)$: Understanding cultural subtleties in context $U_{\text{IT}}(C)$, the phrase is recognized as a polite refusal:

$$T(U_{\text{IT}}(E), U_{\text{IT}}(C)) = \text{True}.$$

Interpretation: Cultural context shifts the interpretation of the ambiguous phrase to convey a clear refusal.

Definition 4.5 (Indeterminate-to-False Transformation)**.** Let $A \in \mathcal{P}$, and suppose $T(A, C)$ = Indeterminate. The transformation U_{IF} (Indeterminate-to-False) changes the truth value as follows:

$$T(U_{\text{IF}}(A), U_{\text{IF}}(C)) = \text{False}.$$

This transformation is applied if additional evidence or contextual information **contradicts the truthfulness** of the proposition.

Example 4.6 (Weather Prediction)**.** Let B: "It will rain tomorrow."

Initial Context C: Early weather models are ambiguous, leading to an indeterminate truth value for B:

$$T(B, C) = \text{Indeterminate}.$$

Context Transformation $U_{\text{IF}}(C)$: Updated weather data shows clear skies, strongly contradicting the likelihood of rain:

$$T(U_{\text{IF}}(B), U_{\text{IF}}(C)) = \text{False}.$$

Interpretation: As more reliable weather information becomes available, the proposition B transitions from indeterminate to false.

Definition 4.7 (True-to-Indeterminate Transformation)**.** Let $B \in \mathcal{P}$, and suppose $T(B, C)$ = True. The transformation U_{TI} (True-to-Indeterminate) changes the truth value as follows:

$$T(U_{\text{TI}}(B), U_{\text{TI}}(C)) = \text{Indeterminate}.$$

This transformation is applied if evidence or contextual information that previously supported the truth of B becomes unreliable or insufficient.

Example 4.8 (Legal Testimony)**.** Let C: "The witness's statement is truthful."

Initial Context C: Based on initial corroboration, the statement is considered true:

$$T(C, C) = \text{True}.$$

Context Transformation $U_{\text{TI}}(C)$: New contradictory evidence or inconsistencies emerge, casting doubt on the testimony:

$$T(U_{\text{TI}}(C), U_{\text{TI}}(C)) = \text{Indeterminate}.$$

Interpretation: The emergence of conflicting information shifts the truth value of C from true to indeterminate, reflecting the uncertainty surrounding the testimony.

Example 4.9 (Scientific Consensus)**.** Let G: "A particular medicine is effective for treating a disease."

Initial Context C: Clinical trials initially support the efficacy of the medicine:

$$T(G, C) = \text{True}.$$

Context Transformation $U_{\text{TI}}(C)$: New studies produce mixed results, creating uncertainty:

$$T(U_{\text{TI}}(G), U_{\text{TI}}(C)) = \text{Indeterminate}.$$

Interpretation: Shifting evidence impacts the certainty of scientific claims, transitioning from true to indeterminate.

Definition 4.10 (False-to-Indeterminate Transformation)**.** Let $C \in \mathcal{P}$, and suppose $T(C, C) = \text{False}$. The transformation U_{FI} (False-to-Indeterminate) changes the truth value as follows:

$$T(U_{\text{FI}}(C), U_{\text{FI}}(C)) = \text{Indeterminate}.$$

This transformation is applied if evidence or contextual information that previously supported the falsity of C becomes unreliable or insufficient.

Example 4.11 (Product Safety Assessment)**.** Let D: "The product is unsafe for use."

Initial Context C: Initial tests indicate that the product is unsafe:

$$T(D, C) = \text{False}.$$

Context Transformation $U_{\text{FI}}(C)$: Additional testing and revised safety guidelines introduce ambiguity regarding its safety:

$$T(U_{\text{FI}}(D), U_{\text{FI}}(C)) = \text{Indeterminate}.$$

Interpretation: The shift from false to indeterminate reflects new uncertainty about the safety of the product under revised conditions.

4.2 Neutrosophic Logic Representation

In Neutrosophic Logic, each proposition $A \in \mathcal{P}$ is assigned a truth value $v(A) = (T, I, F)$, where $T, I, F \in [0, 1]$ represent the degrees of truth, indeterminacy, and falsity, respectively. The transformations can be described mathematically as follows:

Definition 4.12. Given $v(A) = (T, I, F)$ with $I > 0$ and evidence supporting truth:

$$v(U_{\text{IT}}(A)) = (T + I, 0, F).$$

Definition 4.13. Given $v(A) = (T, I, F)$ with $I > 0$ and evidence contradicting truth:

$$v(U_{\text{IF}}(A)) = (T, 0, F + I).$$

Definition 4.14. Given $v(B) = (T, I, F)$ with $T > 0$ and emerging ambiguity:

$$v(U_{\text{TI}}(B)) = (0, T + I, F).$$

Definition 4.15. Given $v(C) = (T, I, F)$ with $F > 0$ and emerging ambiguity:

$$v(U_{\text{FI}}(C)) = (T, I + F, 0).$$

Remark 4.16.

1. **Preservation of Total Degree:** All transformations preserve the sum of the components:

$$T + I + F = T' + I' + F'.$$

2. **Consistency:** The transformations satisfy the constraints of Neutrosophic Logic, where $T, I, F \in [0, 1]$ and $T + I + F \leq 1$.

4.3 Mathematical Basic Theorems in Indeterm-Upside-Down Logic

This subsection presents formal theorems within the framework of Indeterm-Upside-Down Logic.

Theorem 4.17 (Preservation of Total Truth Value)**.** *For any proposition $A \in \mathcal{P}$ with a truth value $T(A, C) \in \{True, False, Indeterminate\}$, every transformation $U_{IT}, U_{IF}, U_{TI}, U_{FI}$ preserves the total degree of truth, falsity, and indeterminacy. That is:*

$$T + I + F = T' + I' + F',$$

where (T, I, F) and (T', I', F') represent the truth value components before and after the transformation.

Proof. Consider the four transformations:

1. **Indeterminate-to-True ($U_{\mathbf{IT}}$):**

$$v(A) = (T, I, F) \quad \rightarrow \quad v(U_{\text{IT}}(A)) = (T + I, 0, F).$$

   Clearly, $T + I + F = (T + I) + 0 + F$.

2. **Indeterminate-to-False ($U_{\mathbf{IF}}$):**

$$v(A) = (T, I, F) \quad \rightarrow \quad v(U_{\text{IF}}(A)) = (T, 0, F + I).$$

   Similarly, $T + I + F = T + 0 + (F + I)$.

3. **True-to-Indeterminate ($U_{\mathbf{TI}}$):**

$$v(A) = (T, I, F) \quad \rightarrow \quad v(U_{\text{TI}}(A)) = (0, T + I, F).$$

   Here, $T + I + F = 0 + (T + I) + F$.

4. **False-to-Indeterminate ($U_{\mathbf{FI}}$):**

$$v(A) = (T, I, F) \quad \rightarrow \quad v(U_{\text{FI}}(A)) = (T, I + F, 0).$$

   Finally, $T + I + F = T + (I + F) + 0$.

In each case, the total sum $T + I + F$ remains invariant. Thus, the theorem holds. □

Theorem 4.18 (Involutivity of U_{TI} and U_{FI})**.** *The transformations U_{TI} (True-to-Indeterminate) and U_{FI} (False-to-Indeterminate) are involutive. That is:*

$$U_{TI}(U_{TI}(A)) = A, \quad U_{FI}(U_{FI}(A)) = A.$$

Proof. Let $v(A) = (T, I, F)$.

1. **True-to-Indeterminate ($U_{\mathbf{TI}}$):**

$$v(A) = (T, I, F) \quad \rightarrow \quad v(U_{\text{TI}}(A)) = (0, T + I, F).$$

   Applying U_{TI} again:

$$v(U_{\text{TI}}(U_{\text{TI}}(A))) = (T, I, F),$$

   which is identical to the original truth value.

2. **False-to-Indeterminate ($U_{\mathbf{FI}}$):**

$$v(A) = (T, I, F) \quad \rightarrow \quad v(U_{\text{FI}}(A)) = (T, I + F, 0).$$

   Applying U_{FI} again:

$$v(U_{\text{FI}}(U_{\text{FI}}(A))) = (T, I, F),$$

   again identical to the original truth value.

Thus, U_{TI} and U_{FI} are involutive. □

Theorem 4.19 (Correctness of U_{IT} and U_{IF})**.** *For any $A \in \mathcal{P}$ with $T(A, C)$ = Indeterminate, the transformations U_{IT} (Indeterminate-to-True) and U_{IF} (Indeterminate-to-False) are consistent with the supporting or contradicting evidence E. That is:*

U_{IT} is applied if and only if E supports A,

U_{IF} is applied if and only if E contradicts A.

Proof. By definition:

1. **Indeterminate-to-True ($U_{\mathbf{IT}}$):** This transformation is applied only if evidence E supports A. Let E be a set of observations or data such that:
$$P(A \mid E) > 0.5,$$
where $P(A \mid E)$ is the probability of A being true given E. In this case:
$$T(U_{\text{IT}}(A), U_{\text{IT}}(C)) = \text{True}.$$

2. **Indeterminate-to-False ($U_{\mathbf{IF}}$):** This transformation is applied only if evidence E contradicts A. Let $P(A \mid E) < 0.5$. In this case:
$$T(U_{\text{IF}}(A), U_{\text{IF}}(C)) = \text{False}.$$

The correctness of the transformations follows from their alignment with the evidence E. □

4.4 Real-Life Examples: Combining Neutrosophic Logic with Indeterm-Upside-Down Logic

When Neutrosophic Logic is combined with Indeterm-Upside-Down Logic, we can model scenarios where ambiguity dynamically shifts, either increasing or decreasing, based on contextual evidence or interpretation. Below are some concrete real-life examples that illustrate these transformations.

Example 4.20 (Medical Diagnosis)**.** Consider a proposition A: "The patient has Disease X."

Initial Neutrosophic Value: $v(A) = (0.4, 0.5, 0.1)$, where:

- $T = 0.4$: Moderate evidence supports the diagnosis.
- $I = 0.5$: There is significant ambiguity due to inconclusive test results.
- $F = 0.1$: Minimal evidence contradicts the diagnosis.

Transformation $U_{\mathbf{IT}}$ (Indeterminate to Truth): New test results strongly support the diagnosis, reducing ambiguity. The updated value is:

$$v(U_{\text{IT}}(A)) = (0.4 + 0.5, 0, 0.1) = (0.9, 0, 0.1).$$

Interpretation: The ambiguity is resolved in favor of the diagnosis, leaving a highly probable truth.

Transformation $U_{\mathbf{IF}}$ (Indeterminate to Falsity): Alternatively, if the new evidence contradicts the diagnosis:

$$v(U_{\text{IF}}(A)) = (0.4, 0, 0.1 + 0.5) = (0.4, 0, 0.6).$$

Interpretation: Ambiguity is resolved against the diagnosis, indicating a stronger falsity.

Example 4.21 (Weather Prediction)**.** Consider a proposition B: "It will rain tomorrow."

Initial Neutrosophic Value: $v(B) = (0.6, 0.3, 0.1)$, where:

- $T = 0.6$: Moderate evidence (e.g., weather models) suggests rain.
- $I = 0.3$: Uncertainty due to fluctuating weather conditions.
- $F = 0.1$: Weak evidence opposes rain.

Transformation U_{TI} (Truth to Indeterminate): Unforeseen atmospheric changes increase uncertainty, reducing confidence in the prediction:

$$v(U_{\mathrm{TI}}(B)) = (0, 0.6 + 0.3, 0.1) = (0, 0.9, 0.1).$$

Interpretation: The truth value diminishes, and ambiguity dominates.

Transformation U_{FI} (Falsity to Indeterminate): If evidence opposing rain becomes unclear, ambiguity increases:

$$v(U_{\mathrm{FI}}(B)) = (0.6, 0.3 + 0.1, 0) = (0.6, 0.4, 0).$$

Interpretation: Falsity diminishes, and ambiguity increases while truth remains steady.

Example 4.22 (Consumer Product Reviews)**.** Consider a proposition C: "The product is high-quality."

Initial Neutrosophic Value: $v(C) = (0.7, 0.2, 0.1)$, where:

- $T = 0.7$: Most reviews are positive.
- $I = 0.2$: Some reviews lack clarity.
- $F = 0.1$: A minority of reviews are negative.

Transformation U_{IT} (Indeterminate to Truth): Additional positive reviews clarify doubts, increasing the truth value:

$$v(U_{\mathrm{IT}}(C)) = (0.7 + 0.2, 0, 0.1) = (0.9, 0, 0.1).$$

Transformation U_{TI} (Truth to Indeterminate): If conflicting reviews emerge, ambiguity grows:

$$v(U_{\mathrm{TI}}(C)) = (0, 0.7 + 0.2, 0.1) = (0, 0.9, 0.1).$$

Remark 4.23. These examples demonstrate how combining Neutrosophic Logic with Indeterm-Upside-Down Logic provides a nuanced framework for modeling dynamic changes in truth, indeterminacy, and falsity. This approach is particularly suited for real-world scenarios where evidence evolves over time.

4.5 Algorithm for Indeterm-Upside-Down Logic

We propose an algorithmic framework for Indeterm-Upside-Down Logic, incorporating the four transformations: Indeterminate-to-True, Indeterminate-to-False, True-to-Indeterminate, and False-to-Indeterminate. The algorithm determines the transformation to apply based on the current truth value, context, and available evidence.

Theorem 4.24 (Correctness of Algorithm)**.** *Algorithm 3 correctly applies the appropriate transformation based on the given inputs.*

Proof. The algorithm handles all cases as defined in the framework of Indeterm-Upside-Down Logic:

- If $T(A, C) =$ Indeterminate, it applies U_{IT} if evidence supports A, or U_{IF} if evidence contradicts A.
- If $T(A, C) =$ True, it applies U_{TI} if evidence becomes unreliable, transitioning the truth value to Indeterminate.
- If $T(A, C) =$ False, it applies U_{FI} if evidence becomes unreliable, transitioning the truth value to Indeterminate.

These conditions align with the definitions of Indeterm-Upside-Down Logic transformations. The algorithm evaluates all necessary conditions to determine the appropriate transformation, ensuring correctness. □

Algorithm 3 Indeterm-Upside-Down Logic Transformation

Require: Proposition $A \in \mathcal{P}$, context C, current truth value $T(A, C) \in \{\text{True}, \text{False}, \text{Indeterminate}\}$, and evidence E.

Ensure: Transformed truth value $T'(A, C')$.

```
if T(A, C) = Indeterminate then
    if Evidence E supports A then
        Apply U_IT: T'(A, C') ← True.
    else
        Apply U_IF: T'(A, C') ← False.
    end if
else if T(A, C) = True then
    if Evidence E is unreliable then
        Apply U_TI: T'(A, C') ← Indeterminate.
    else
        T'(A, C') ← T(A, C).
    end if
else if T(A, C) = False then
    if Evidence E is unreliable then
        Apply U_FI: T'(A, C') ← Indeterminate.
    else
        T'(A, C') ← T(A, C).
    end if
end if
return T'(A, C').
```

Theorem 4.25 (Time Complexity)**.** *The time complexity of Algorithm 3 is* $O(1)$.

Proof. The algorithm evaluates the current truth value $T(A, C)$ and evidence E, performing at most two nested conditional checks. As there are no loops or recursive calls, the operations occur in constant time. Hence, the time complexity is $O(1)$. □

Theorem 4.26 (Space Complexity)**.** *The space complexity of Algorithm 3 is* $O(1)$.

Proof. The algorithm requires constant space to store the inputs ($A, C, T(A, C)$, and E) and the output $T'(A, C')$. No additional data structures are instantiated, so the space complexity is $O(1)$. □

Example 4.27 (Medical Diagnosis)**. Input:**

- A: "The patient has disease X."
- C: Initial diagnostic tests are inconclusive.
- $T(A, C)$ = Indeterminate.
- Evidence E: Advanced imaging strongly indicates disease X.

Algorithm Execution: Since $T(A, C)$ = Indeterminate and E supports A, the algorithm applies U_{IT}:

$$T'(A, C') = \text{True}.$$

Output: The patient's diagnosis transitions to definitive: $T'(A, C') = \text{True}$.

5 Certain Upside-Down Logic

In this section, we explore Certain Upside-Down Logic. This logical framework dynamically adjusts the truth, indeterminacy, and falsity values of propositions based on evidence, redistributing and swapping these components as required.

5.1 Definition of Certain Upside-Down Logic

The definition of Certain Upside-Down Logic is presented below. As a note, in this section, Certain Upside-Down Logic is considered within the framework of Neutrosophic Logic.

Notation 5.1. *In this section, let $\mathcal{P}$ be a set of propositions, and let C be a set of contexts. Let $v : \mathcal{P} \times C \to [0, 1]^3$ be a truth valuation function assigning to each proposition-context pair (A, C) a Neutrosophic truth value $v(A, C) = (T, I, F)$, where T, I, and F represent the degrees of* ***truth***, ***indeterminacy***, *and* ***falsity***, *respectively, satisfying:*

$$T, I, F \in [0, 1], \quad T + I + F = 1.$$

Definition 5.2 (Certain Upside-Down Logic)**.** Certain Upside-Down Logic is an extension of Upside-Down Logic that includes transformations of the truth values based on evidence E. The logic allows for the reallocation of the indeterminacy value I to either the truth value T or the falsity value F, depending on the evidence. Specifically, the transformations are defined as follows:

1. **Weak Evidence Shift (Indeterminacy to Falsity and Swap)**:

   $$v'(A) = (F + I, 0, T),$$

   applied when evidence weakly supports the falsity of A.

2. **Weak Evidence Shift (Indeterminacy to Truth and Swap)**:

   $$v'(A) = (F, 0, T + I),$$

   applied when evidence weakly supports the truth of A.

3. **Strong Evidence Shift (All to Truth)**:

   $$v'(A) = (T + I + F, 0, 0),$$

   applied when evidence strongly supports the truth of A.

4. **Strong Evidence Shift (All to Falsity)**:

   $$v'(A) = (0, 0, T + I + F),$$

   applied when evidence strongly supports the falsity of A.

Remark 5.3 (Transformation Rules of the Certain Upside-Down Logic)**.** The Certain Upside-Down Logic provides a mechanism to update the truth values of propositions based on new evidence, extending the Upside-Down Logic by incorporating the indeterminacy component.

1. **Weak Evidence Shift (Indeterminacy to Falsity and Swap)**:
   - The indeterminacy degree I is shifted to the truth value along with swapping T and F.
2. **Weak Evidence Shift (Indeterminacy to Truth and Swap)**:
   - The indeterminacy degree I is shifted to the falsity value along with swapping T and F.
3. **Strong Evidence Shift (All to Truth)**:
   - All degrees are shifted to the truth value, making $T' = 1$.
4. **Strong Evidence Shift (All to Falsity)**:
   - All degrees are shifted to the falsity value, making $F' = 1$.

5.2 Examples of Real-World Applications of Certain Upside-Down Logic

Certain Upside-Down Logic has broad applicability in scenarios involving dynamic adjustments to truth, indeterminacy, and falsity values based on new evidence. Below are two illustrative examples:

Example 5.4 (Medical Diagnosis)**.** Let A: "The patient has Disease X."

Initial Truth Value: $v(A) = (T, I, F) = (0.3, 0.5, 0.2)$.

Case 1: Weak Evidence Shift (Indeterminacy to Falsity and Swap)

Symptoms weakly suggest the absence of Disease X. Applying the transformation:

$$\begin{aligned} T' &= F + I = 0.2 + 0.5 = 0.7 \\ I' &= 0 \\ F' &= T = 0.3 \end{aligned}$$

Updated Truth Value: $v'(A) = (0.7, 0, 0.3)$.

Case 2: Strong Evidence Shift (All to Falsity)

Tests strongly confirm the absence of Disease X. Applying the transformation:

$$\begin{aligned} T' &= 0 \\ I' &= 0 \\ F' &= T + I + F = 0.3 + 0.5 + 0.2 = 1.0 \end{aligned}$$

Updated Truth Value: $v'(A) = (0, 0, 1.0)$.

Example 5.5 (Weather Prediction)**.** Let B: "It will rain tomorrow."

Initial Truth Value: $v(B) = (T, I, F) = (0.2, 0.4, 0.4)$.

Case 1: Weak Evidence Shift (Indeterminacy to Truth and Swap)

Forecast models weakly support rain. Applying the transformation:

$$\begin{aligned} T' &= F = 0.4 \\ I' &= 0 \\ F' &= T + I = 0.2 + 0.4 = 0.6 \end{aligned}$$

Updated Truth Value: $v'(B) = (0.4, 0, 0.6)$.

Case 2: Strong Evidence Shift (All to Truth)

Satellite data strongly supports rain. Applying the transformation:

$$\begin{aligned} T' &= T + I + F = 0.2 + 0.4 + 0.4 = 1.0 \\ I' &= 0 \\ F' &= 0 \end{aligned}$$

Updated Truth Value: $v'(B) = (1.0, 0, 0)$.

5.3 Basic Theorem of Certain Upside-Down Logic

The Basic Theorem of Certain Upside-Down Logic is described as follows. As a note, in this section, Certain Upside-Down Logic is considered within the framework of Neutrosophic Logic.

Theorem 5.6 (Conservation of Total Degree)**.** *The transformation U_C preserves the total degree of truth, indeterminacy, and falsity:*

$$T + I + F = T' + I' + F' = 1.$$

Proof. As shown in the Algorithm Correctness proof, for each case, the sum $T'+I'+F'$ equals $T+I+F = 1$. □

Theorem 5.7 (Bounds Preservation)**.** *For all transformations, the updated truth values satisfy:*

$$0 \leq T', I', F' \leq 1.$$

Proof. Since $T, I, F \in [0, 1]$ and their sum is 1, and because the transformations are combinations of these values without exceeding their original total, the updated values T', I', and F' remain within $[0, 1]$. □

5.4 Algorithm for Certain Upside-Down Logic

The algorithm updates the Neutrosophic truth value (T, I, F) of a proposition A based on evidence E. The transformation adjusts the degrees of truth, indeterminacy, and falsity according to specific rules, as described below. As a note, in this section, Certain Upside-Down Logic is considered within the framework of Neutrosophic Logic.

Algorithm 4 Update Truth Value in Certain Upside-Down Logic

Require: Proposition A, current truth value $v(A) = (T, I, F)$, evidence E
Ensure: Updated truth value $v'(A) = (T', I', F')$

if E weakly supports the falsity of A **then**
  $T' \leftarrow F + I$
  $I' \leftarrow 0$
  $F' \leftarrow T$
else if E weakly supports the truth of A **then**
  $T' \leftarrow F$
  $I' \leftarrow 0$
  $F' \leftarrow T + I$
else if E strongly supports the truth of A **then**
  $T' \leftarrow T + I + F$
  $I' \leftarrow 0$
  $F' \leftarrow 0$
else if E strongly supports the falsity of A **then**
  $T' \leftarrow 0$
  $I' \leftarrow 0$
  $F' \leftarrow T + I + F$
else
  $T' \leftarrow T$
  $I' \leftarrow I$
  $F' \leftarrow F$
end if
return $v'(A) = (T', I', F')$

Theorem 5.8 (Correctness of Transformation)**.** *For any initial truth value $v(A) = (T, I, F)$ with $T, I, F \geq 0$ and $T + I + F = 1$, the updated truth value $v'(A) = (T', I', F')$ produced by Algorithm 4 satisfies $T', I', F' \geq 0$ and $T' + I' + F' = 1$.*

Proof. We consider each case:

1. **Weak Evidence Shift (Indeterminacy to Falsity and Swap)**:

$$T' = F + I$$
$$I' = 0$$
$$F' = T$$

Since $T + I + F = 1$, we have:

$$T' + I' + F' = (F + I) + 0 + T = T + I + F = 1.$$

All components are non-negative because $T, I, F \geq 0$.

2. **Weak Evidence Shift (Indeterminacy to Truth and Swap)**:

$$T' = F$$
$$I' = 0$$
$$F' = T + I$$

Similarly,

$$T' + I' + F' = F + 0 + (T + I) = T + I + F = 1.$$

3. **Strong Evidence Shift (All to Truth)**:

$$T' = T + I + F = 1$$
$$I' = 0$$
$$F' = 0$$

Thus,

$$T' + I' + F' = 1 + 0 + 0 = 1.$$

4. **Strong Evidence Shift (All to Falsity)**:

$$T' = 0$$
$$I' = 0$$
$$F' = T + I + F = 1$$

Therefore,

$$T' + I' + F' = 0 + 0 + 1 = 1.$$

5. **No Evidence Shift**:

$$T' = T$$
$$I' = I$$
$$F' = F$$

So,

$$T' + I' + F' = T + I + F = 1.$$

In all cases, the updated truth values satisfy $T', I', F' \geq 0$ and $T' + I' + F' = 1$. □

Theorem 5.9 (Time Complexity)**.** *Algorithm 4 runs in constant time, i.e., it has a time complexity of* $O(1)$.

Proof. The algorithm performs a fixed number of operations: conditional checks and assignments. These operations do not depend on the size of the input but are constant. Hence, the time complexity is $O(1)$. □

Theorem 5.10 (Space Complexity)**.** *Algorithm 4 uses constant extra space, i.e., it has a space complexity of* $O(1)$.

Proof. The algorithm requires additional space for variables T', I', and F'. Since the number of variables does not depend on the size of the input, the space complexity is $O(1)$. □

6 Conclusion and Future Work of this Paper

This section presents the conclusion and future directions of this paper.

6.1 Conclusion of this Paper

In this paper, we introduced the following logical frameworks:

- **Upside-Down Logic**: A logical framework that systematically flips truth and falsity by dynamically altering contexts, meanings, or perspectives.
- **Contextual Upside-Down Logic**: An extension of Upside-Down Logic that integrates contextual transformations, enabling the adjustment of logical connectives and truth values based on contextual changes.
- **Indeterm-Upside-Down Logic**: A framework designed to represent real-world phenomena where indeterminacy either increases or decreases, capturing dynamic uncertainty effectively.
- **Certain Upside-Down Logic**: This logic provides a mechanism to update truth values of propositions based on new evidence, extending Upside-Down Logic by incorporating the indeterminacy component for enhanced flexibility.

6.2 Future tasks

Finally, we briefly discuss the future prospects of this research.

6.2.1 Applying Upside-Down Logic to Uncertain Sets and Graphs

As mentioned in the introduction, the study of Upside-Down Logic is still in its early stages. We anticipate further exploration into potential applications of Upside-Down Logic, including considerations such as how to apply Upside-Down Logic to Uncertain Logic. Additionally, we expect advancements in research regarding its applicability to concepts like neutrosophic graphs [**?**, 4, 12, 14, 15, 23, 35, 47, 48, 61, 81], adripartitioned neutrosophic set [76, 77], Neutrosophic Topologies [1, 98], Neutrosophic algebra [53, 95], Heptapartitioned neutrosophic sets [11, 69], Neutrosophic Automata [40, 55–57], Neutrosophic oversets [92, 96, 97], refined neutrosophic logic [90], and Bipolar Neutrosophic Sets [5, 25, 27, 106].

6.2.2 Applying Upside-Down Logic to decision-making

There is also significant potential for applying Upside-Down Logic to decision-making (cf. [3, 46]) and similar fields, which could pave the way for a broader understanding and practical use of this novel logical framework.

6.2.3 Relation to Research on Addressing Indeterminacy

When making decisions or applying theoretical frameworks to real-world scenarios, the ability to effectively handle indeterminacy is often a critical factor. Research aimed at understanding and addressing indeterminacy has been conducted across various fields [20]. Below, we list examples of major research areas:

- **Risk Assessment and Decision-Making:** Approaches to classify and manage uncertainty by analyzing the structural environment of decision-makers. Relevant studies focus on societal risks ([8, 108]).
- **Climate Change Prediction:** Research aims to handle uncertainty in climate change predictions by evaluating variations among climate models and excluding unreliable ones, thereby enhancing projection reliability (cf. [32, 45, 73]).

- **Investment Decision-Making:** The use of real options to evaluate factors such as product demand, production capacity, and investment costs under uncertainty [10, 44, 63], enabling the derivation of optimal investment strategies.

The author is particularly interested in exploring whether combining these research areas with Indeterm-Upside-Down Logic or Neutrosophic Logic could yield intriguing mathematical insights or innovative practical applications.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

References

[1] Ravi P Agarwal, Soheyb Milles, Brahim Ziane, Abdelaziz Mennouni, and Lemnaouar Zedam. Ideals and filters on neutrosophic topologies generated by neutrosophic relations. *Axioms*, 13(5):292, 2024.

[2] Ather Abdulrahman Ageeli. A neutrosophic decision-making methods of the key aspects for supply chain management in international business administrations. *International Journal of Neutrosophic Science*, 23(1):155–167, 2023.

[3] Noraini Ahmad, Zahari Md Rodzi, Faisal Al-Sharqi, Ashraf Al-Quran, and Abdalwali Lutfi. Innovative theoretical approach: Bipolar pythagorean neutrosophic sets (bpnss) in decision-making. *International Journal of Neutrosophic Science*, 2023.

[4] Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.

[5] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *J. Intell. Fuzzy Syst.*, 33:4077–4087, 2017.

[6] Ali Alqazzaz and Karam M Sallam. Evaluation of sustainable waste valorization using treesoft set with neutrosophic sets. *Neutrosophic Sets and Systems*, 65(1):9, 2024.

[7] Mohammed Alshikho, Maissam Jdid, and Said Broumi. Artificial intelligence and neutrosophic machine learning in the diagnosis and detection of covid 19. *Journal Prospects for Applied Mathematics and Data Analysis*, 1(2), 2023.

[8] Terje Aven, Piero Baraldi, Roger Flage, and Enrico Zio. *Uncertainty in risk assessment: the representation and treatment of uncertainties by probabilistic and non-probabilistic methods*. John Wiley & Sons, 2014.

[9] C Lee Baker. Double negatives. *Linguistic inquiry*, 1(2):169–186, 1970.

[10] Nick Bloom, Stephen Bond, and John Van Reenen. Uncertainty and investment dynamics. *The review of economic studies*, 74(2):391–415, 2007.

[11] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[12] Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. An isolated interval valued neutrosophic graphs. *Critical Review*, 13:67–80, 2016.

[13] Said Broumi, Irfan Deli, and Florentin Smarandache. N-valued interval neutrosophic sets and their application in medical diagnosis. *Critical Review, Center for Mathematics of Uncertainty, Creighton University, Omaha, NE, USA*, 10:45–69, 2015.

[14] Said Broumi, Florentin Smarandache, Mohamed Talea, and Assia Bakali. An introduction to bipolar single valued neutrosophic graph theory. *Applied Mechanics and Materials*, 841:184–191, 2016.

[15] Said Broumi, R Sundareswaran, M Shanmugapriya, Assia Bakali, and Mohamed Talea. Theory and applications of fermatean neutrosophic graphs. *Neutrosophic sets and systems*, 50:248–286, 2022.

[16] Daniel Crump Buchanan. *Japanese proverbs and sayings*. University of Oklahoma Press, 1965.

[17] Peter Buzzi and Claudia Megele. Honne and tatemae: a world dominated by a ‘game of masks,’. *Communication across cultures*, 2011.

[18] Erick González Caballero, Florentin Smarandache, and Maikel Leyva Vázquez. On neutrosophic offuninorms. *Symmetry*, 11(9):1136, 2019.

[19] Jia Syuen Chai, Ganeshsree Selvachandran, Florentin Smarandache, Vassilis C Gerogiannis, Le Hoang Son, Quang-Thinh Bui, and Bay Vo. New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. *Complex & Intelligent Systems*, 7:703–723, 2021.

[20] Antonio J Conejo, Miguel Carrión, Juan M Morales, et al. *Decision making under uncertainty in electricity markets*, volume 1. Springer, 2010.

[21] Irving M Copi, Carl Cohen, and Kenneth McMahon. *Introduction to logic*. Routledge, 2016.

[22] Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, and Clifford Stein. *Introduction to algorithms*. MIT press, 2022.

[23] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 50(1):225–238, 2022.

[24] Roger J Davies and Osamu Ikeno. *Japanese mind: Understanding contemporary Japanese culture*. Tuttle Publishing, 2011.

[25] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[26] Irfan Deli, Said Broumi, and Florentin Smarandache. On neutrosophic refined sets and their applications in medical diagnosis. *Journal of new theory*, (6):88–98, 2015.

[27] Irfan Deli, Yusuf Subas, Florentin Smarandache, and Mumtaz Ali. Interval valued bipolar neutrosophic sets and their application in pattern recognition. *viXra*, 2016.

[28] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[29] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[30] Irina-Ana Drobot. Analysing the japanese indirect communication culture. *Crossing Boundaries in Culture and Communication*, 12(2):9–20, 2021.

[31] Kurt Engesser, Dov M Gabbay, and Daniel Lehmann. *Handbook of quantum logic and quantum structures: Quantum structures*. Elsevier, 2011.

[32] AM Fischer, AP Weigel, Christoph M Buser, Reto Knutti, Hans Rudolf Künsch, MA Liniger, Christoph Schär, and C Appenzeller. Climate change projections for switzerland based on a bayesian multi-model approach. *International Journal of Climatology*, 32(15):2348, 2012.

[33] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[34] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[35] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.

[36] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[37] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[38] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[39] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[40] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[41] Peter Gärdenfors. *Belief revision*. Number 29. Cambridge University Press, 2003.

[42] Michael A Harrison. *Introduction to formal language theory*. Addison-Wesley Longman Publishing Co., Inc., 1978.

[43] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[44] Claude Henry. Investment decisions under uncertainty: the" irreversibility effect". *The American Economic Review*, 64(6):1006–1012, 1974.

[45] Mike Hulme. Reducing the future to climate: a story of climate determinism and reductionism. *Osiris*, 26(1):245–266, 2011.

[46] Azmat Hussain, Muhammad Irfan Ali, and Tahir Mahmood. Pythagorean fuzzy soft rough sets and their applications in decision-making. *Journal of Taibah University for Science*, 14:101 – 113, 2019.

[47] S Satham Hussain, R Hussain, and Florentin Smarandache. Domination number in neutrosophic soft graphs. *Neutrosophic Sets and Systems*, 28:228–244, 2019.

[48] S Satham Hussain, Hossein Rashmonlou, R Jahir Hussain, Sankar Sahoo, Said Broumi, et al. Quadripartitioned neutrosophic graph structures. *Neutrosophic Sets and Systems*, 51(1):17, 2022.

[49] Takeshi Ishida. Conflict and its accommodation: Omote-ura and uchi-soto relations. *Conflict in Japan*, pages 16–38, 1984.

[50] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[51] Chalmers Johnson. Omote (explicit) and ura (implicit): Translating japanese political terms. *Journal of Japanese Studies*, 6(1):89–115, 1980.

[52] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[53] WB Vasantha Kandasamy and Florentin Smarandache. *Some neutrosophic algebraic structures and neutrosophic N-algebraic structures*. Infinite Study, 2006.

[54] Faruk Karaaslan and Fatih Hunu. Type-2 single-valued neutrosophic sets and their applications in multi-criteria group decision making based on topsis method. *Journal of Ambient Intelligence and Humanized Computing*, 11:4113 – 4132, 2020.

[55] V. Karthikeyan, , and R. Karuppaiya. Characterizations of submachine of interval neutrosophic automata. *Advances in Mathematics: Scientific Journal*, 2020.

[56] V. Karthikeyan and R. Karuppaiya. Strong subsystems of interval neutrosophic automata. 2020.

[57] V. Karthikeyan and R. Karuppaiya. Reverse subsystems of interval neutrosophic automata. 2021.

[58] Richard Kaye. *Models of Peano Arithmetic*. Clarendon Press, Oxford, 1991.

[59] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[60] Emiko Konishi, Michiko Yahiro, Naoko Nakajima, and Miki Ono. The japanese value of harmony and nursing ethics. *Nursing ethics*, 16(5):625–636, 2009.

[61] S krishna Prabha, Said Broumi, Souhail Dhouib, and Mohamed Talea. Implementation of circle-breaking algorithm on fermatean neutrosophic graph to discover shortest path. *Neutrosophic Sets and Systems*, 72:256–271, 2024.

[62] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[63] Haim Levy. *Stochastic dominance: Investment decision making under uncertainty*. Springer, 2015.

[64] Xiaochun Luo, Zilong Wang, Liguo Yang, Lin Lu, and Song Hu. Sustainable supplier selection based on vikor with single-valued neutrosophic sets. *PLOS ONE*, 18, 2023.

[65] Daniel Maciej Maciejewski. 50 ways to say “no” in japanese. Master’s thesis, 2020.

[66] Samuel E Martin. *A reference grammar of Japanese*. University of Hawaii Press, 2003.

[67] William McClure. *Using Japanese: A guide to contemporary usage*. Cambridge University Press, 2000.

[68] L Mičkova. The japanese indirectness phenomenon. *Asian and African studies*, 12(2):135–147, 2003.

[69] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[70] Sumiyo Nishiguchi. Extended qualia-based lexical knowledge for disambiguation of japanese postposition no. *OF THE EUROPEAN SUMMER SCHOOL FOR LOGIC, LANGUAGE, AND INFORMATION*, page 137, 2009.

[71] Sumiyo Nishiguchi. Possessive disambiguation. *PhD Dissertation, Osaka University*, 2009.

[72] Guillermo Owen. *Game theory*. Emerald Group Publishing, 2013.

[73] Tim N Palmer. Predicting uncertainty in forecasts of weather and climate. *Reports on progress in Physics*, 63(2):71, 2000.

[74] Paolo Paolo. A study using treesoft set and neutrosophic sets on possible soil organic transformations in urban agriculture systems. *International Journal of Neutrosophic Science*.

[75] Christos H Papadimitriou. Computational complexity. In *Encyclopedia of computer science*, pages 260–265. 2003.

[76] Surapati Pramanik. Interval quadripartitioned neutrosophic sets. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 146, 2022.

[77] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[78] Stefano Crespi Reghizzi, Luca Breveglieri, and Angelo Morzenti. *Formal languages and compilation*. Springer, 2013.

[79] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[80] Bruce Ross. The essence of haiku. *Modern Haiku*, 38(3):51–62, 2007.

[81] Muhammad Saeed, Muhammad Khubab Siddique, Muhammad Ahsan, Muhammad Rayees Ahmad, and Atiqe Ur Rahman. A novel approach to the rudiments of hypersoft graphs. *Theory and Application of Hypersoft Set, Pons Publication House, Brussel*, pages 203–214, 2021.

[82] AA Salama, HA El-Ghareeb, Mohamed Esia, and M Lotfy. Social network analysis e-learning systems via neutrosophic techniques.

[83] AA Salama, A Haitham, A Manie, and M Lotfy. Utilizing neutrosophic set in social network analysis e-learning systems. *International Journal of Information Science and Intelligent System*, 3(2):61–72, 2014.

[84] Myriam Paulina Barreno Sánchez, Miriam Pantoja Burbano, and Sary Alvarez Hernández. Neutrosophic insights into strategic decision-making: Navigating complexity in ecuadorian business management. *Neutrosophic Optimization and Intelligent Systems*, 1:46–56, 2024.

[85] Robert Sedgewick and Kevin Wayne. *Algorithms*. Addison-wesley professional, 2011.

[86] Nitin Seth, Sanjeev Gopalrao Deshmukh, and Prem Vrat. Service quality models: a review. *International journal of quality & reliability management*, 22(9):913–949, 2005.

[87] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[88] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[89] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[90] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[91] Florentin Smarandache. Degrees of membership> 1 and< 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[92] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[93] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[94] Florentin Smarandache. Extended nonstandard neutrosophic logic, set, and probability based on extended nonstandard analysis. *Symmetry*, 11(4):515, 2019.

[95] Florentin Smarandache. History of superhyperalgebra and neutrosophic superhyperalgebra (revisited again). *Neutrosophic Algebraic Structures and Their Applications*, page 10, 2022.

[96] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[97] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[98] Florentin Smarandache. New types of topologies and neutrosophic topologies (improved version). *Neutrosophic Sets and Systems*, 57(1):14, 2023.

[99] Florentin Smarandache. *Upside-Down Logics: Falsification of the Truth and Truthification of the False*. Infinite Study, 2024.

[100] Florentin Smarandache. Short introduction to standard and nonstandard neutrosophic set and logic (review paper). *Neutrosophic Sets and Systems*, 77, 2025. Review Paper.

[101] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[102] Florentin Smarandache and Maissam Jdid. An overview of neutrosophic and plithogenic theories and applications. 2023.

[103] Patrick Suppes. *Introduction to logic*. Courier Corporation, 2012.

[104] Genelyn Jane D Trinidad. *Honne and tatemae: exploring the two sides of Japanese society*. PhD thesis, 2014.

[105] Natsuko Tsujimura. *An introduction to Japanese linguistics*. John Wiley & Sons, 2013.

[106] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[107] Tracel Lynne Schnorr von Carolsfeld. Lexical and structural ambiguity in japanese and some comparisons in english. *Working Papers of the Linguistics Circle*, 10(1):63–79, 1991.

[108] Yuanpu Xia, Ziming Xiong, Xin Dong, and Hao Lu. Risk assessment and decision-making under uncertainty in tunnel and underground engineering. *Entropy*, 19(10):549, 2017.

[109] Kenneth Yasuda. *Japanese haiku: Its essential nature and history*. Tuttle Publishing, 2011.

[110] Jun Ye. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.*, 26:2459–2466, 2014.

[111] Naoto Yoshida and Tomoko Yonezawa. “honne and tatemae”: Expression of the agent’s hidden desire through physiological phenomena. *The Transactions of Human Interface Society*, 26(2):249–258, 2024.

[112] Hong yu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615–627, 2016.

[113] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[114] Lotfi A. Zadeh. Fuzzy logic = computing with words. *IEEE Trans. Fuzzy Syst.*, 4:103–111, 1996.

[115] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[116] Aiwu Zhang, Jianguo Du, and Hongjun Guan. Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator. *J. Intell. Fuzzy Syst.*, 29:2697–2706, 2015.

[117] Hongyu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615 – 627, 2015.

Local-Neutrosophic Logic and Local-Neutrosophic Sets: Incorporating Locality with Applications

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Abstract

The study of uncertainty has been a significant area of research, with concepts such as fuzzy sets [87], fuzzy graphs [51], and neutrosophic sets [58] receiving extensive attention. In Neutrosophic Logic, indeterminacy often arises from real-world complexities.

This paper explores the concept of locality as a key factor in determining indeterminacy, building upon the framework introduced by F. Smarandache in [73]. Locality refers to processes constrained within a specific region, where an object or system is directly influenced by its immediate surroundings. In contrast, nonlocality involves effects that transcend spatial or temporal boundaries, where changes in one location have direct implications for another.

This paper introduces the concepts of Local-Neutrosophic Logic and Local-Neutrosophic Set by integrating the notion of locality into Neutrosophic Logic. It provides their mathematical definitions and examines potential applications.

Keywords: Neutrosophic Logic, Neutrosophic Set, Fuzzy Logic, Locality

1 Short Introduction

1.1 Uncertain Logic

Uncertainty is an inherent characteristic of real-world events and is often modeled using mathematical frameworks. In the realm of logic (cf. [12, 81]), several approaches have been developed to address uncertainty, including Fuzzy Logic [87,89,90], Neutrosophic Logic [58,62,68], and Plithogenic Logic [67,77]. For instance, Neutrosophic Logic expands upon classical logic by incorporating three dimensions: truth, indeterminacy, and falsity. This framework allows for the simultaneous handling of uncertainty and contradictions, making it a versatile tool for modeling complex systems.

These uncertain logics have been further generalized to other mathematical concepts, such as sets [59, 75] and graphs [20, 22, 23, 25, 26]. This has led to a proliferation of studies that parallel the development of logical systems, showcasing their broad applicability across various domains.

1.2 Locality in Neutrosophic Logic

In Neutrosophic Logic, indeterminacy often emerges from real-world factors. This paper investigates locality as a key determinant of indeterminacy, building on the framework proposed by F. Smarandache in [73].

Locality describes processes confined to a specific region, where an object is influenced by its immediate surroundings. It can be Total Locality (100 percent, all interactions are local) or Partial Locality (greater than 0 and less than 100 percent). Conversely, Nonlocality involves effects spanning space or time, with changes in one location influencing another. Like locality, nonlocality may be Total or Partial.

Indeterminacy arises when a system is neither fully local nor nonlocal, often due to hidden variables or environmental uncertainty. It too can range from Total to Partial, depending on the extent of ambiguity or mixed characteristics.

1.3 Contributions of This Paper

This paper makes several key contributions:

1. It introduces the novel concepts of Local-Neutrosophic Logic and Local-Neutrosophic Set, incorporating the notion of locality into the framework of Neutrosophic Logic.
2. It provides precise mathematical definitions for these concepts, laying a robust theoretical foundation.
3. It explores potential applications, demonstrating the practicality and relevance of these ideas in addressing uncertainty and contextual dependencies.

1.4 The Structure of the Paper

The structure of this paper is as follows.

2 Preliminaries

2.1 Basic Definition of Formal Language

To explore Upside-Down Logic, several key concepts are introduced below. For further details, readers are encouraged to consult the respective lecture notes and surveys on these topics (ex. [16, 29, 30, 33, 40]).

Definition 2.1 (Set). [33] A *set* is a collection of distinct and clearly defined objects, known as elements, such that any object can be identified as either a member of the set or not. If A is a set and x is an element of A, this membership is denoted by $x \in A$. Sets are commonly represented using curly brackets, for example, $A = \{x_1, x_2, \dots, x_n\}$.

Definition 2.2 (Formal Language). [29, 49] A *formal language* $\mathcal{L}$ is defined as a set of strings (or sequences) formed from a finite alphabet Σ, adhering to specific syntactic rules. Formally:

$$\mathcal{L} \subseteq \Sigma^*,$$

where Σ^* represents the set of all finite strings over the alphabet Σ. The strings in $\mathcal{L}$ are referred to as *well-formed formulas (WFFs).*

A formal language $\mathcal{L}$ is typically characterized by:

- A set of *symbols* (or *alphabet*) Σ, which may include logical connectives (e.g., $\wedge$, $\vee$, $\neg$), quantifiers (e.g., $\forall$, $\exists$), variables, and parentheses.
- A set of *formation rules* specifying which strings in Σ^* qualify as well-formed.

Definition 2.3 (Logical System)**.** (cf. [37]) A *logical system* $\mathcal{M}$ is a mathematical structure used to formalize reasoning. It is defined as:

$$\mathcal{M} = (\mathcal{P}, \mathcal{V}, v),$$

where:

- $\mathcal{P}$ is the set of propositions (or statements) expressed in the formal language $\mathcal{L}$.
- $\mathcal{V}$ is the set of truth values, such as $\{\text{True}, \text{False}\}$ in classical logic.
- $v : \mathcal{P} \to \mathcal{V}$ is a *valuation function* (or interpretation function) that assigns a truth value to each proposition in $\mathcal{P}$.

Additionally, a logical system may include:

- A set of *axioms* $\mathcal{A} \subseteq \mathcal{P}$, propositions assumed to be true within the system.
- A set of *inference rules* $\mathcal{I}$, defining valid methods of deriving new truths from existing propositions.

2.2 Neutrosophic Logic

In this subsection, we explore the relationship between Neutrosophic Logic and Upside-Down Logic. First, we present the definition of Neutrosophic Logic below [21, 58]. Note that Neutrosophic Logic is known to generalize Fuzzy Logic (cf. [58]).

Definition 2.4 (Neutrosophic Logic)**.** [58] Neutrosophic Logic extends classical logic by assigning to each proposition a truth value comprising three components:

$$v(A) = (T, I, F),$$

where $T, I, F \in [0, 1]$ represent the degrees of truth, indeterminacy, and falsity, respectively.

Example 2.5 (Student Performance Evaluation)**.** Student performance evaluation assesses academic progress using metrics like grades, participation, and skills, identifying strengths and areas for improvement (cf. [7,36]).

In education, Neutrosophic Logic can assess student performance when data is uncertain or incomplete. For example, consider the proposition "The student will perform well in the final exam," represented as:

$$v(A) = (0.8, 0.1, 0.1),$$

where:

- $T = 0.8$: An 80% chance of good performance based on past grades and class participation.
- $I = 0.1$: A 10% level of indeterminacy due to unmeasured factors like stress or unforeseen circumstances.
- $F = 0.1$: A 10% chance of poor performance due to lack of preparation or external distractions.

This approach enables educators to provide personalized feedback and prepare targeted interventions to improve student outcomes(cf. [3, 47, 56]).

Example 2.6 (Medical Diagnosis)**.** In the field of medical diagnosis(cf. [5, 43]), Neutrosophic Logic is employed to handle uncertain and incomplete information. For instance, consider the proposition "The patient has Disease X," which can be represented as:

$$v(A) = (0.6, 0.2, 0.2),$$

where:

- $T = 0.6$: A 60% probability that the patient has the disease.
- $I = 0.2$: A 20% of the data is inconclusive due to uncertainty in test results or conflicting evidence.
- $F = 0.2$: A 20% probability that the patient does not have the disease.

By integrating test results, symptoms, and expert opinions, this approach enables healthcare professionals to make more informed diagnostic decisions (cf. [9, 11, 13, 55]).

Example 2.7 (Project Management)**.** Project management involves planning, organizing, and executing tasks to achieve specific goals within constraints like time, budget, and resources(cf. [15, 17, 18, 80]).

In project management, Neutrosophic Logic can aid in handling uncertainties and risks associated with project timelines and outcomes. Consider the proposition "The project will be completed on time," represented as:

$$v(A) = (0.6, 0.25, 0.15),$$

where:

- $T = 0.6$: A 60% chance that the project will be completed on schedule, based on current progress and resource availability.
- $I = 0.25$: A 25% level of indeterminacy due to uncertainties like unexpected delays, resource shortages, or scope changes.
- $F = 0.15$: A 15% chance that the project will not meet the deadline, based on known risks or past trends in similar projects.

By quantifying these components, project managers can better assess risks and devise strategies such as resource reallocation or timeline adjustments to mitigate potential delays. This enhances decision-making under uncertainty and improves project success rates(cf. [2, 6, 34, 45, 50]).

Example 2.8 (Decision-Making in Business)**.** Neutrosophic Logic also plays a critical role in business decision-making, particularly under conditions of uncertainty. For example, when evaluating whether to invest in a project, the proposition "The project will yield profit" can be represented as:

$$v(A) = (0.7, 0.1, 0.2),$$

where:

- $T = 0.7$: A 70% chance that the project will be profitable.
- $I = 0.1$: A 10% level of uncertainty due to incomplete or ambiguous market data.
- $F = 0.2$: A 20% chance that the project will not be profitable.

This representation allows decision-makers to assess risks and rewards quantitatively, facilitating more effective strategy formulation (cf. [4, 52]).

2.3 Neutrosophic Set

In this subsection, we explain the concept of the Neutrosophic Set [58]. Intuitively, a Neutrosophic Set can be understood as the set-theoretic extension of Neutrosophic Logic. It is known as a generalization of several classical and modern set concepts, including the Crisp Set (Classic Set), Fuzzy Set [87,88,91–93], Intuitionistic Fuzzy Set [61, 76], Vague Set [10, 96], and Paraconsistent Set [83, 84]. The related definitions are provided below.

Definition 2.9 (Crisp Set)**.** [48] Let X be a universe set, and let $P(X)$ denote the power set of X, which represents all subsets of X. A *crisp set* $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{0, 1\}$, where:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This function χ_A assigns a value of 1 to elements within the set A and 0 to those outside it, creating a clear boundary. Crisp sets are thus bivalent and follow the principle of binary classification, where each element is either a member of the set or not.

Definition 2.10. [58, 60, 78] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

3 Mathematical Framework of Locality, Indeterminacy, and Nonlocality in Neutrosophic Logic

This section discusses the Mathematical Framework of Locality, Indeterminacy, and Nonlocality in Neutrosophic Logic. It redefines these concepts within the context of Neutrosophic Logic, providing basic considerations and illustrative examples.

3.1 Notations and Definitions

Below, we present the Notations and Definitions of Locality, Indeterminacy, and Nonlocality in Neutrosophic Logic.

Notation 3.1. *Let $\mathcal{P}$ denote the set of propositions, C the set of contexts, and $T : \mathcal{P} \times C \to \{True, False, Indeterminate\}$ a truth valuation function. Each proposition $A \in \mathcal{P}$ is associated with a neutrosophic truth value $v(A) = (T, I, F)$, where:*

$$T, I, F \in [0, 1], \quad T + I + F \le 1,$$

represent the degrees of truth (T), indeterminacy (I), and falsity (F), respectively.

Remark 3.2. The sum $T + I + F$ does not necessarily equal 1, allowing for partial states. This flexibility is a core feature of neutrosophic logic.

Definition 3.3 (Context)**.** [74] A *context* C is a set of parameters or conditions under which propositions are evaluated. This may include spatial, temporal, semantic, or interpretative settings.

Definition 3.4 (Locality)**.** (cf. [73]) A proposition A exhibits *locality* if its truth value depends solely on a single context $C \in C$. Formally:

$$v(A) = T(A, C),$$

where C represents the immediate spatial, temporal, or conceptual domain affecting A.

Definition 3.5 (Indeterminacy)**.** (cf. [73]) A proposition A exhibits *indeterminacy* if its truth value includes a non-zero degree of I due to hidden variables, insufficient information, or ambiguity within the context C. Formally:

$$v(A) = (T, I, F), \quad \text{with } I > 0.$$

Definition 3.6 (Nonlocality)**.** (cf. [73]) A proposition A exhibits *nonlocality* if its truth value depends on multiple, spatially or conceptually separate contexts $C_1, C_2 \in C$. Formally:

$$v(A) = T(A, C_1, C_2), \quad \text{with } C_1 \cap C_2 = \emptyset.$$

Definition 3.7 (Multilocality)**.** (cf. [73]) A proposition A exhibits *multilocality* if its truth value depends on a set of interacting local contexts $\{C_i\}_{i=1}^n$. Formally:

$$v(A) = T(A, \{C_i\}_{i=1}^n),$$

where each C_i is confined to a specific local domain.

Definition 3.8 (Multiindeterminacy)**.** (cf. [73]) A proposition A exhibits *multiindeterminacy* if its truth value includes cumulative indeterminacy across multiple contexts:

$$v(A) = (T, I, F), \quad \text{where } I = \sum_{i=1}^{n} I_i > 0,$$

and I_i represents the degree of indeterminacy in each context C_i.

Definition 3.9 (Multinonlocality)**.** (cf. [73]) A proposition A exhibits *multinonlocality* if its truth value depends on interactions across multiple nonlocal contexts $\{C_i, C_j\}_{i \neq j}$:

$$v(A) = T(A, \{C_i, C_j\}_{i \neq j}),$$

where $C_i \cap C_j = \emptyset$.

3.2 Some Real-Life Examples of Locality, Indeterminacy, and Nonlocality

This subsection presents some real-life examples of locality, indeterminacy, and nonlocality.

Example 3.10 (Traffic Flow (Locality))**.** Let A: "The traffic density on road segment R is high."

Context C: The immediate local parameters affecting R, such as vehicle count, average speed, and weather conditions, determine $v(A)$:

$$v(A) = (0.8, 0.1, 0.1),$$

where $T = 0.8$ indicates high traffic density based on local observations. Locality is evident as A depends solely on C.

Example 3.11 (Quantum Entanglement (Nonlocality))**.** Quantum entanglement is a phenomenon where particles share linked states, such that changing one instantly affects the other, regardless of distance (cf. [31, 85]).

Let A: "The spin of particle P_1 is up."

Contexts C_1, C_2**:** Measurement of P_1 in C_1 instantly determines the spin of P_2 in C_2:

$$v(A) = (1, 0, 0), \quad \text{if } P_2 \text{ spin is down in } C_2.$$

Nonlocality is evident as $v(A)$ spans C_1 and C_2.

Example 3.12 (Stock Market Volatility (Indeterminacy))**.** Stock market volatility measures rapid price fluctuations in financial markets, influenced by economic events, investor behavior, and uncertainty (cf. [14, 41]).

Let B: "The stock price of company X will increase tomorrow."

Context C: Factors such as market trends, global events, and investor sentiment introduce indeterminacy. The truth value is:

$$v(B) = (0.5, 0.4, 0.1),$$

where $I = 0.4$ reflects uncertainty due to incomplete or ambiguous data. Indeterminacy arises from unpredictable market influences.

Example 3.13 (Climate Models (Multiindeterminacy)). Let B: "Global temperature will rise by 1°C in 50 years."

Contexts $\{C_i\}_{i=1}^3$**:** Varying predictions from three models yield:

$$v(B) = (0.6, 0.3, 0.1), \quad \text{where } I = 0.3 = \sum_{i=1}^{3} I_i.$$

Multiindeterminacy arises from differing model assumptions.

Example 3.14 (Global Communications (Multinonlocality)). Let E: "Information is transmitted successfully across the network."

Contexts $\{C_i, C_j\}_{i \neq j}$: The reliability of nodes in separate regions C_i and C_j determines:

$$v(E) = (0.9, 0.05, 0.05),$$

where multinonlocality reflects the interaction between contexts C_i and C_j across the global network.

3.3 Some Basic Theorem of Locality, Indeterminacy, and Nonlocality

In this subsection, we present Some Basic Theorems of Locality, Indeterminacy, and Nonlocality.

Theorem 3.15 (Consistency of Locality). *If a proposition A is local, then the truth value $v(A)$ is unaffected by nonlocal contexts. Formally:*

$$T(A, C_1) = T(A, C_2) \quad \textit{for all } C_1, C_2 \neq C.$$

Proof. Locality assumes A is influenced only by C. For $C_1, C_2 \neq C$, the absence of influence implies:

$$v(A) = T(A, C),$$

and thus $T(A, C_1) = T(A, C_2)$ follows trivially. □

Theorem 3.16 (Additivity of Multiindeterminacy). *For a proposition A exhibiting multiindeterminacy across n contexts $\{C_i\}_{i=1}^n$, the total indeterminacy satisfies:*

$$I = \sum_{i=1}^{n} I_i, \quad \textit{where } I_i > 0 \textit{ for each } C_i.$$

Proof. By definition, multiindeterminacy aggregates indeterminacy from individual contexts:

$$v(A) = (T, I, F), \quad I = \sum_{i=1}^{n} I_i.$$

The constraint $I_i > 0$ ensures that each context contributes to the total indeterminacy. □

4 Mathematical Framework of Partial Locality, Partial Non-Locality, and Partial Indeterminacy

In this section, we examine the Mathematical Framework of Partial Locality, Partial Non-Locality, and Partial Indeterminacy.

An overview of Partial Locality, Partial Non-Locality, and Partial Indeterminacy is provided below:

- *Partial Locality* refers to a situation where an object or system is partially influenced by its immediate surroundings, with limited external interactions or dependencies.
- *Partial Non-Locality* describes a condition where an object or system is partially influenced by distant factors or entities without direct physical contact.
- *Partial Indeterminacy* represents a system exhibiting unclear or mixed characteristics, influenced neither entirely locally nor entirely non-locally.

4.1 Definitions of Partial Locality, Partial Non-Locality, and Partial Indeterminacy

In this subsection, we consider about Definitions of Partial Locality, Partial Non-Locality, and Partial Indeterminacy.

Definition 4.1 (Partial Locality)**.** (cf. [73]) A proposition $A \in \mathcal{P}$ exhibits *partial locality* if its truth value is influenced by its immediate surroundings with a fractional dependency denoted by $\alpha \in [0, 1]$. Formally:

$$v(A) = \alpha \cdot T(A, C_{\text{local}}) + (1 - \alpha) \cdot \text{Residual Effects},$$

where C_{local} is the immediate local context and α represents the degree of locality.

Definition 4.2 (Partial Non-Locality)**.** (cf. [73]) A proposition $A \in \mathcal{P}$ exhibits *partial non-locality* if its truth value is influenced by distant or separate contexts with a fractional dependency $\beta \in [0, 1]$. Formally:

$$v(A) = \beta \cdot T(A, C_{\text{nonlocal}}) + (1 - \beta) \cdot \text{Residual Effects},$$

where C_{nonlocal} refers to nonlocal contexts influencing A, and β represents the degree of non-locality.

Definition 4.3 (Partial Indeterminacy)**.** (cf. [73]) A proposition $A \in \mathcal{P}$ exhibits *partial indeterminacy* if its truth value includes an indeterminate component $\gamma \in [0, 1]$ due to hidden variables or ambiguous influences. Formally:

$$v(A) = (T, \gamma \cdot I, F),$$

where I is the total indeterminacy, and γ represents the degree of partial indeterminacy.

Remark 4.4. Partial locality, non-locality, and indeterminacy may coexist for a single proposition, forming a composite influence model:

$$v(A) = \alpha \cdot T(A, C_{\text{local}}) + \beta \cdot T(A, C_{\text{nonlocal}}) + \gamma \cdot I,$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\alpha + \beta + \gamma \leq 1$.

4.2 Theorems and Proofs of Partial Locality, Partial Non-Locality, and Partial Indeterminacy

In this subsection, we present Some Basic Theorems of artial Locality, Partial Non-Locality, and Partial Indeterminacy.

Theorem 4.5 (Consistency of Partial Locality)**.** *If a proposition A exhibits partial locality with degree α, then the influence from nonlocal contexts diminishes proportionally. Formally:*

$$v(A) = \alpha \cdot T(A, C_{local}) + (1 - \alpha) \cdot T(A, C_{nonlocal}),$$

where $\alpha \to 1$ implies pure locality.

Proof. By definition, α scales the influence of C_{local}, and $1 - \alpha$ scales the complementary effect from C_{nonlocal}. As $\alpha \to 1$, the term $(1 - \alpha) \cdot T(A, C_{\text{nonlocal}})$ vanishes, yielding pure locality. □

Theorem 4.6 (Superposition of Influences)**.** *The total influence on a proposition A can be represented as a superposition of partial locality, partial non-locality, and partial indeterminacy:*

$$v(A) = \alpha \cdot T(A, C_{local}) + \beta \cdot T(A, C_{nonlocal}) + \gamma \cdot I.$$

Proof. The influence components are orthogonal by construction: C_{local} affects A directly, C_{nonlocal} introduces distant dependencies, and I incorporates ambiguity. Thus, the superposition holds. □

4.3 Examples of Partial Locality, Partial Non-Locality, and Partial Indeterminacy

In this subsection, we present Examples of Partial Locality, Partial Non-Locality, and Partial Indeterminacy.

Example 4.7 (Quantum Physics: Aharonov-Bohm Effect)**.** Quantum physics studies the behavior of particles at atomic and subatomic scales, governed by principles like wave-particle duality and superposition (cf. [8,27]).

Let A: "A charged particle is influenced by an electromagnetic potential."

Local Context C_{local}: The particle exists in a region with no magnetic field intensity.

Nonlocal Context C_{nonlocal}: The electromagnetic potential resides outside the particle's local region.

$$v(A) = \alpha \cdot T(A, C_{\text{local}}) + \beta \cdot T(A, C_{\text{nonlocal}}),$$

where $\beta > 0$ captures the nonlocal influence of the potential.

Example 4.8 (Ecology: Migratory Birds)**.** Let B: "Birds impact the nutrient cycle."

Local Context C_{local}: Birds forage and nest locally.

Nonlocal Context C_{nonlocal}: Migratory behavior spreads nutrients across ecosystems.

$$v(B) = \alpha \cdot T(B, C_{\text{local}}) + \beta \cdot T(B, C_{\text{nonlocal}}),$$

where $\beta > 0$ quantifies the nonlocal nutrient transfer.

5 New Definition of Local-Neutrosophic Logic and Set

In this section, we introduce a new concept called Local-Neutrosophic Logic and Set. This concept extends Neutrosophic Logic by incorporating the notion of locality. The definition is provided below.

Definition 5.1 (Local-Neutrosophic Logic)**.** Local-Neutrosophic Logic assigns to each proposition $A \in \mathcal{P}$ a truth value of the form:

$$v(A) = (T, I, L, F),$$

where:

- $T \in [0, 1]$: Degree of truth.
- $I \in [0, 1]$: Degree of indeterminacy.
- $L \in [0, 1]$: Degree of locality, representing the influence of immediate contextual or spatial factors.
- $F \in [0, 1]$: Degree of falsity.

These components satisfy the constraint:

$$T + I + L + F \leq 1.$$

Remark 5.2 (Transformation Rules)**.** Transformation Rules of Local-Neutrosophic Logic are following.

- **Locality-to-Truth Transformation:** When locality L provides strong supporting evidence for truth:

$$v(U_{LT}(A)) = (T + L, I, 0, F).$$

- **Locality-to-Falsity Transformation:** When locality L provides strong evidence against truth:

$$v(U_{LF}(A)) = (T, I, 0, F + L).$$

- **Locality-to-Indeterminacy Transformation:** When locality L introduces ambiguity or uncertainty:

$$v(U_{LI}(A)) = (T, I + L, 0, F).$$

- **Indeterminacy-to-Locality Transformation:** When indeterminacy I is clarified by locality L:

$$v(U_{IL}(A)) = (T, 0, I + L, F).$$

The definition of the Local-Neutrosophic Set, which extends Local-Neutrosophic Logic to sets, is as follows. It is anticipated that future research will explore the specific mathematical structures and applications of this concept.

Definition 5.3 (Local-Neutrosophic Set)**.** Let X be a given universe of discourse. A *Local-Neutrosophic Set* A on X is characterized by four membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad L_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where, for each $x \in X$:

- $T_A(x)$: The degree of truth of x in A.
- $I_A(x)$: The degree of indeterminacy of x in A.
- $L_A(x)$: The degree of locality of x in A, representing the influence of immediate spatial, contextual, or environmental factors.
- $F_A(x)$: The degree of falsity of x in A.

These membership values satisfy the following constraint:

$$0 \leq T_A(x) + I_A(x) + L_A(x) + F_A(x) \leq 4.$$

Remark 5.4 (Local-Neutrosophic Set)**.** Compared to other sets:

- A *Fuzzy Set* has a single membership function, $\mu_A(x)$, representing truth.
- An *Intuitionistic Fuzzy Set* adds a falsity component, $\nu_A(x)$, to truth.
- A *Neutrosophic Set* further includes indeterminacy, $I_A(x)$.
- The *Local-Neutrosophic Set* expands these by adding locality ($L_A(x)$) to model systems influenced by contextual factors.

5.1 Basic Theorem of Local-Neutrosophic Logic

We outline several basic theorems of Local-Neutrosophic Logic below.

Theorem 5.5 (Preservation of Total Degree)**.** *The total degree of the truth valuation remains invariant under transformations:*

$$T + I + L + F = T' + I' + L' + F'.$$

Proof. Each transformation redistributes the components among T, I, L, and F, preserving their total sum. □

Theorem 5.6 (Superposition of Influences)**.** *The truth value of a proposition A in Local-Neutrosophic Logic can be expressed as:*

$$v(A) = \alpha \cdot T + \beta \cdot I + \gamma \cdot L + \delta \cdot F,$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ *represent the relative weights of each component and* $\alpha + \beta + \gamma + \delta = 1$.

Proof. The superposition follows directly from the normalized representation of truth valuation components. □

Theorem 5.7. *Local-Neutrosophic Logic extends Neutrosophic Logic by introducing an additional degree of locality L, which represents the influence of immediate contextual or spatial factors. Specifically, every proposition in Neutrosophic Logic can be represented as a special case of Local-Neutrosophic Logic where* $L = 0$.

Proof. The truth value in Neutrosophic Logic is defined as:

$$v_{\text{NL}}(A) = (T, I, F),$$

where $T, I, F \in [0, 1]$ and satisfy the constraint:

$$T + I + F \leq 1.$$

In Local-Neutrosophic Logic, the truth value is extended to:

$$v_{\text{LNL}}(A) = (T, I, L, F),$$

where $T, I, L, F \in [0, 1]$ and satisfy the constraint:

$$T + I + L + F \leq 1.$$

If we set $L = 0$ in Local-Neutrosophic Logic, the truth value simplifies to:

$$v_{\text{LNL}}(A) = (T, I, 0, F) = (T, I, F),$$

which is identical to the truth value in Neutrosophic Logic. Therefore, every truth value in Neutrosophic Logic is a valid truth value in Local-Neutrosophic Logic.

The addition of L in Local-Neutrosophic Logic allows for the representation of an additional degree of influence from immediate contextual or spatial factors, which is not captured in Neutrosophic Logic. This makes Local-Neutrosophic Logic a generalized framework.

The constraint $T + I + F \leq 1$ in Neutrosophic Logic is preserved in Local-Neutrosophic Logic because setting $L = 0$ satisfies:

$$T + I + L + F \leq 1.$$

Local-Neutrosophic Logic reduces to Neutrosophic Logic when $L = 0$, but it also allows for additional flexibility when $L > 0$. Hence, Local-Neutrosophic Logic is a strict extension of Neutrosophic Logic. □

Theorem 5.8. *A Local-Neutrosophic Set is an extension of a Neutrosophic Set. Specifically, every Neutrosophic Set can be represented as a Local-Neutrosophic Set where the degree of locality* $L_A(x) = 0$ *for all* $x \in X$.

Proof. The membership functions of a Neutrosophic Set A on X are:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

with the constraint:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3, \quad \forall x \in X.$$

For a Local-Neutrosophic Set A on X, the membership functions are extended to include L_A:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad L_A : X \to [0,1], \quad F_A : X \to [0,1],$$

with the constraint:

$$0 \leq T_A(x) + I_A(x) + L_A(x) + F_A(x) \leq 4, \quad \forall x \in X.$$

If $L_A(x) = 0$ for all $x \in X$, the membership functions of a Local-Neutrosophic Set reduce to:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

with the constraint:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3, \quad \forall x \in X,$$

which is identical to the structure of a Neutrosophic Set.

The additional component $L_A(x)$ in a Local-Neutrosophic Set allows for the representation of an additional degree of locality, representing the influence of spatial, contextual, or environmental factors. This flexibility generalizes the concept of Neutrosophic Sets.

For any $x \in X$, the constraint $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ in a Neutrosophic Set is preserved in a Local-Neutrosophic Set when $L_A(x) = 0$, satisfying:

$$0 \leq T_A(x) + I_A(x) + L_A(x) + F_A(x) \leq 4.$$

A Local-Neutrosophic Set reduces to a Neutrosophic Set when $L_A(x) = 0$ for all $x \in X$, but it also allows for additional flexibility when $L_A(x) > 0$. Hence, the Local-Neutrosophic Set is a strict extension of the Neutrosophic Set. □

Theorem 5.9. *Local-Neutrosophic Logic can represent locality, non-locality, and indeterminacy through the values T, I, L, F assigned to propositions in the form $v(A) = (T, I, L, F)$, satisfying $T + I + L + F \leq 1$. Specifically:*

- *Locality is captured by L, representing the influence of immediate surroundings.*
- *Non-locality is represented by the truth or falsity components T and F, influenced by distant contexts.*
- *Indeterminacy is represented by I, capturing the uncertainty or ambiguity.*

Proof. Let $A \in \mathcal{P}$ be a proposition evaluated in Local-Neutrosophic Logic with $v(A) = (T, I, L, F)$.

From the definition of locality:

$$v(A) = (T, I, L, F), \quad L > 0.$$

Here, L explicitly represents the degree to which A is influenced by its immediate surroundings (local context). If $L = 0$, A has no local influence. Thus, locality is embedded within the L component.

Non-locality is captured when L is small ($L \to 0$) and the remaining truth (T) or falsity (F) values depend on distant contexts. Specifically:

$$v(A) = (\beta \cdot T(A, C_{\text{nonlocal}}), I, L, \beta \cdot F(A, C_{\text{nonlocal}})),$$

where C_{nonlocal} represents nonlocal contexts and β $(0 < \beta \leq 1)$ denotes the degree of non-locality. Non-locality arises when T or F depends on contexts spatially or conceptually separated from the local context.

Indeterminacy is directly captured by the I component in $v(A)$. If $I > 0$, there exists a degree of uncertainty or ambiguity in the truth value of A. Indeterminacy arises from hidden variables, conflicting evidence, or incomplete information. Formally:

$$v(A) = (T, \gamma \cdot I, L, F), \quad \text{where } \gamma \in [0,1].$$

Here, γ controls the contribution of indeterminacy to the overall evaluation.

By definition, Local-Neutrosophic Logic satisfies:

$$T + I + L + F \leq 1.$$

This constraint ensures that locality, non-locality, and indeterminacy are mathematically consistent and their contributions to $v(A)$ are bounded.

Hence, Local-Neutrosophic Logic effectively represents locality, non-locality, and indeterminacy through the values T, I, L, F, satisfying the stated constraint. □

5.2 Examples of Local-Neutrosophic Logic in real-life scenarios

In this subsection, we explain examples of Local-Neutrosophic Logic in real-life scenarios.

Example 5.10 (Quantum Physics: Measurement Locality)**.** Quantum physics studies the behavior of particles at atomic and subatomic scales, governed by principles like wave-particle duality and superposition (cf. [8,27]).

Let A: "The spin of a particle is up."

Truth Components:

$$v(A) = (T, I, L, F) = (0.6, 0.2, 0.1, 0.1),$$

where:

- $T = 0.6$: Evidence strongly supports the particle's spin being up.
- $I = 0.2$: Uncertainty due to measurement limitations.
- $L = 0.1$: Local experimental context influences spin alignment.
- $F = 0.1$: Weak evidence against the proposition.

Example 5.11 (Ecology: Pollination Dynamics)**.** Pollination dynamics refers to the interactions between plants and pollinators, such as bees or birds, facilitating plant reproduction and ecosystem stability (cf. [28]).

Let B: "Pollinators improve crop yield in a region."

Truth Components:

$$v(B) = (T, I, L, F) = (0.7, 0.1, 0.15, 0.05),$$

where:

- $T = 0.7$: Direct observations confirm significant pollination effects.
- $I = 0.1$: Uncertainty due to unmeasured ecological variables.
- $L = 0.15$: Local interactions between pollinators and plants are notable.
- $F = 0.05$: Minimal evidence contradicts the proposition.

Example 5.12 (Medical Diagnosis(Locality-to-Truth Transformation))**.** Let A: "The patient has a specific viral infection."

Initial Truth Value:

$$v(A) = (T, I, L, F) = (0.5, 0.3, 0.2, 0.0),$$

where:

- $T = 0.5$: Initial test results partially support the diagnosis.

- $I = 0.3$: Indeterminate due to conflicting symptoms.
- $L = 0.2$: Locality reflects observations by a specialist.
- $F = 0.0$: No evidence against the diagnosis.

Transformation: Given that L provides strong supporting evidence (e.g., specialist confirmation), the transformation U_{LT} is applied:

$$v(U_{LT}(A)) = (T + L, I, 0, F) = (0.7, 0.3, 0.0, 0.0).$$

Interpretation: The patient's diagnosis is now more likely to be true, as locality strongly supports the proposition.

Example 5.13 (Weather Prediction(Locality-to-Falsity Transformation))**.** Let B: "It will rain tomorrow."

Initial Truth Value:

$$v(B) = (T, I, L, F) = (0.4, 0.3, 0.3, 0.0),$$

where:

- $T = 0.4$: Weak prediction models suggest rain.
- $I = 0.3$: Indeterminacy due to uncertainty in weather models.
- $L = 0.3$: Local observations (e.g., clear skies).
- $F = 0.0$: No significant evidence against rain.

Transformation: Since L strongly opposes T (e.g., clear skies observed), the transformation U_{LF} is applied:

$$v(U_{LF}(B)) = (T, I, 0, F + L) = (0.4, 0.3, 0.0, 0.3).$$

Interpretation: The proposition becomes less likely, as local observations indicate no rain.

Example 5.14 (Ecological Impact(Locality-to-Indeterminacy Transformation))**.** Let C: "Reintroducing wolves to a forest will improve biodiversity."

Initial Truth Value:

$$v(C) = (T, I, L, F) = (0.6, 0.2, 0.2, 0.0),$$

where:

- $T = 0.6$: Prior studies support this outcome.
- $I = 0.2$: Some uncertainty due to unknown ecological factors.
- $L = 0.2$: Local reports suggest potential unintended consequences.
- $F = 0.0$: No evidence contradicting the proposition.

Transformation: Local observations introduce ambiguity, applying U_{LI}:

$$v(U_{LI}(C)) = (T, I + L, 0, F) = (0.6, 0.4, 0.0, 0.0).$$

Interpretation: The uncertainty in the proposition increases due to conflicting local data.

Example 5.15 (Supply Chain Disruption (Indeterminacy-to-Locality Transformation))**.** Let D: "A factory shutdown will disrupt global supply chains."

Initial Truth Value:

$$v(D) = (T, I, L, F) = (0.5, 0.4, 0.1, 0.0),$$

where:

- $T = 0.5$: Preliminary analysis suggests a significant impact.
- $I = 0.4$: Indeterminacy due to lack of specific data.
- $L = 0.1$: Localized reports provide clarity on immediate effects.
- $F = 0.0$: No evidence against the proposition.

Transformation: Local evidence reduces indeterminacy, applying U_{IL}:

$$v(U_{IL}(D)) = (T, 0, I + L, F) = (0.5, 0.0, 0.5, 0.0).$$

Interpretation: The proposition's reliance on locality increases as specific data becomes available.

6 Future Tasks of this research

In this section, we consider future tasks of this research.

6.1 Some Extension of Local-Neutrosophic Logic (Open Question)

There is interest in exploring the possibility of extending the above logic using the following set concepts. Further research in this direction is anticipated.

Question 6.1. Can the logic be extended using the following sets? Additionally, what are the mathematical characteristics of these extensions, their relationships with other uncertain concepts, and their potential applications?

- Double-Valued Neutrosophic Sets [35, 38]
- Interval-Valued Neutrosophic Sets [86, 94, 95]
- Plithogenic Sets [25, 66, 67, 79]
- Soft Sets [44, 46]
- Hypersoft Sets [1, 19, 24, 53, 65]
- Neutrosophic Offset [57, 63, 64, 69, 70, 72]

6.2 Neutrosophic Dynamic Systems

A *Neutrosophic Dynamic System* (NDS) is a generalized framework for modeling systems characterized by uncertainty, incompleteness, or contradictions. The definition is provided below [71]. There is particular interest in exploring how the current Local-Neutrosophic Logic can be extended to Neutrosophic Dynamic Systems.

Definition 6.2 (Neutrosophic Dynamic Systems)**.** [71] Let $\mathcal{U}$ be the universe of discourse. A *Neutrosophic Dynamic System* is defined as:

$$\mathcal{D}_N = (\Omega, \mathcal{E}, \mathcal{R}),$$

where:

- $\Omega \subseteq \mathcal{U}$: The neutrosophic space (or state space), representing the elements of the system. It is defined as:
$$\Omega = \{x_i(T_i, I_i, F_i) \mid x_i \in \Omega,\ T_i, I_i, F_i \in [0,1],\ i \in \{1, 2, \ldots, n\}\},$$
where:
  - T_i: The degree of membership (truth) of x_i in Ω.
  - I_i: The degree of indeterminacy (uncertainty) of x_i in Ω.
  - F_i: The degree of non-membership (falsity) of x_i in Ω.
- $\mathcal{E}$: The set of elements within Ω. Each element x_i is associated with time-varying neutrosophic degrees (T_i, I_i, F_i), which evolve over time.
- $\mathcal{R}$: The set of neutrosophic hyperrelationships representing interactions within the system. A neutrosophic hyperrelationship is defined as:
$$\mathcal{R}_{\mathrm{HR}} : \Omega^k \times C(\Omega)^l \to \mathcal{P}([0,1]),$$
where:
  - $\mathcal{R}_{\mathrm{HR}}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, y_{j_1}, y_{j_2}, \ldots, y_{j_l}) = (T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}})$,
  - $T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}} \in [0,1]$: Degrees of truth, indeterminacy, and falsity for the hyperrelationship.
  - k: Number of interacting elements within Ω.
  - l: Number of interacting elements between Ω and $C(\Omega)$, the complement of Ω in $\mathcal{U}$.

Definition 6.3 (Open and Closed Neutrosophic Systems)**.** [71]

- A system is *closed* if $l = 0$, meaning all relationships are confined to Ω.
- A system is *open* if $l \geq 1$, allowing interactions between Ω and $C(\Omega)$.

Definition 6.4 (Time-Dependent Neutrosophic dynamic system)**.** [71] The neutrosophic dynamic system evolves over time, with changes occurring in its space, elements, and relationships:
$$\mathcal{D}_N(t) = (\Omega(t), \mathcal{E}(t), \mathcal{R}(t)).$$

- **Element Dynamics**: The degrees of membership, indeterminacy, and non-membership for each element vary over time:
$$T_i(t), \quad I_i(t), \quad F_i(t),$$
subject to the constraint:
$$T_i(t) + I_i(t) + F_i(t) \leq 1, \quad \forall i,\ \forall t.$$
- **Relationship Dynamics**: The hyperrelationships evolve over time, represented as:
$$\mathcal{R}_{\mathrm{HR}}(t) = (T_{\mathcal{R}}(t), I_{\mathcal{R}}(t), F_{\mathcal{R}}(t)).$$
- **Space Dynamics**: The neutrosophic space Ω may change due to:
  - Addition of new elements to Ω.
  - Removal of existing elements from Ω.
  - Changes in the neutrosophic degrees (T_i, I_i, F_i) of elements within Ω.

Example 6.5 (Ecosystem Dynamics)**.** A biological ecosystem is a community of living organisms interacting with each other and their environment, including air, water, and soil (cf. [54, 82]).

Consider a biological ecosystem modeled as a neutrosophic dynamic system:

- $\Omega = \{x_1, x_2, \ldots, x_n\}$: The set of species in the ecosystem, where each species x_i is characterized by its neutrosophic attributes (T_i, I_i, F_i).

- $T_i(t)$: Degree to which species x_i is adapted to the environment at time t. For example, $T_i(t)$ may increase if x_i develops traits that improve survival under current environmental conditions.
- $I_i(t)$: Degree of uncertainty in species x_i's role or impact in the ecosystem. This reflects incomplete knowledge about how x_i interacts with other species or adapts to environmental changes.
- $F_i(t)$: Degree to which species x_i is maladapted or detrimental to the ecosystem. For instance, $F_i(t)$ may increase if x_i contributes to ecosystem imbalance.

The hyperrelationships $\mathcal{R}_{\mathrm{HR}}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = (T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}})$ capture the dynamic interactions among species:

- $T_{\mathcal{R}}$: Degree to which the interaction benefits the ecosystem (e.g., symbiosis).
- $I_{\mathcal{R}}$: Degree of indeterminacy in the interaction (e.g., uncertain impact of competition).
- $F_{\mathcal{R}}$: Degree to which the interaction harms the ecosystem (e.g., predation imbalance).

As the ecosystem evolves, the following dynamics may occur:

- Species x_i may join or leave Ω due to migration, extinction, or introduction.
- The neutrosophic degrees $T_i(t), I_i(t), F_i(t)$ and hyperrelationship values $T_{\mathcal{R}}(t), I_{\mathcal{R}}(t), F_{\mathcal{R}}(t)$ change over time based on environmental conditions, resource availability, and species interactions.
- External influences, such as human intervention or climate change, may alter the system by introducing new relationships $\mathcal{R}_{\mathrm{HR}}$ or modifying Ω.

This framework provides a dynamic, nuanced representation of ecosystem behavior, accommodating uncertainty and variability in species interactions.

Example 6.6 (Social Networks)**.** Social networks are structures of individuals or groups connected through relationships like communication, collaboration, or shared interests, often facilitated by technology(cf. [32,39, 42]).

Consider a social network modeled as a neutrosophic dynamic system:

- $\Omega = \{x_1, x_2, \ldots, x_n\}$: The set of individuals in the network, where each individual x_i is characterized by neutrosophic attributes (T_i, I_i, F_i).
- $T_i(t)$: Degree to which individual x_i positively contributes to the network at time t. For instance, $T_i(t)$ may increase if x_i actively collaborates or shares valuable information.
- $I_i(t)$: Degree of neutrality or indifference of x_i in the network. This could reflect an individual's limited or ambiguous involvement in network activities.
- $F_i(t)$: Degree to which x_i detracts from the network, such as spreading misinformation or creating conflicts.

The hyperrelationships $\mathcal{R}_{\mathrm{HR}}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = (T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}})$ represent interactions within the network:

- $T_{\mathcal{R}}$: Degree to which the interaction enhances network cohesion (e.g., collaborative projects).
- $I_{\mathcal{R}}$: Degree of indeterminacy in the interaction (e.g., ambiguous communication).
- $F_{\mathcal{R}}$: Degree to which the interaction harms the network (e.g., disputes or competitive behavior).

The dynamic behavior of the network includes:

- Addition or removal of individuals (x_i) to/from Ω, reflecting network growth or attrition.
- Changes in the neutrosophic degrees $T_i(t), I_i(t), F_i(t)$ of individuals based on their evolving roles and contributions.
- Evolution of hyperrelationships $\mathcal{R}_{\mathrm{HR}}(t)$ as collaboration patterns, social dynamics, or external influences (e.g., new policies or technological changes) reshape the network.

This model captures the complexity and variability of social interactions, accommodating the uncertainty and contradictions inherent in human networks.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

References

[1] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[2] Mohamed Abdel-Basset, Asmaa Atef, and Florentin Smarandache. A hybrid neutrosophic multiple criteria group decision making approach for project selection. *Cognitive Systems Research*, 57:216–227, 2019.

[3] Sunday Adesina Adebisi and Said Broumi. Assessing students performance using neutrosophic tool. *Neutrosophic Optimization and Intelligent Systems*, 2:7–18, 2024.

[4] Ather Abdulrahman Ageeli. A neutrosophic decision-making methods of the key aspects for supply chain management in international business administrations. *International Journal of Neutrosophic Science*, 23(1):155–167, 2023.

[5] Qeethara Al-Shayea. Artificial neural networks in medical diagnosis. *International Journal of Research Publication and Reviews*, 2024.

[6] Salah Hasan Saleh Al-Subhi, Iliana Pérez Pupo, Roberto García Vacacela, Pedro Y Piñero Pérez, and Maikel Y Leyva Vázquez. *A new neutrosophic cognitive map with neutrosophic sets on connections, application in project management*. Infinite Study, 2018.

[7] Kathryn M Andolsek. Improving the medical student performance evaluation to facilitate resident selection. *Academic Medicine*, 91:1475–1479, 2016.

[8] Karen Barad. Meeting the universe halfway: Quantum physics and the entanglement of matter and meaning. 2007.

[9] Said Broumi, Irfan Deli, and Florentin Smarandache. N-valued interval neutrosophic sets and their application in medical diagnosis. *Critical Review, Center for Mathematics of Uncertainty, Creighton University, Omaha, NE, USA*, 10:45–69, 2015.

[10] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[11] Jia Syuen Chai, Ganeshsree Selvachandran, Florentin Smarandache, Vassilis C Gerogiannis, Le Hoang Son, Quang-Thinh Bui, and Bay Vo. New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. *Complex & Intelligent Systems*, 7:703–723, 2021.

[12] Irving M Copi, Carl Cohen, and Kenneth McMahon. *Introduction to logic*. Routledge, 2016.

[13] Irfan Deli, Said Broumi, and Florentin Smarandache. On neutrosophic refined sets and their applications in medical diagnosis. *Journal of new theory*, (6):88–98, 2015.

[14] Barkha Dhingra, Shallu Batra, Vaibhav Aggarwal, Mahender Yadav, and Pankaj Kumar. Stock market volatility: a systematic review. *Journal of Modelling in Management*, 2023.

[15] William R. Duncan. A guide to the project management body of knowledge. 1996.

[16] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[17] Takaaki Fujita. Breaking down barriers: Proposals for overcoming challenges in student project management. *European Journal of Management and Marketing Studies*, 2023.

[18] Takaaki Fujita. A consideration of project-based learning: Exploring its potential, effects, challenges, and best practices. *European Journal of Education Studies*, 10(3), 2023.

[19] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[20] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[21] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[22] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[23] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[24] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[25] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[26] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[27] Thierry Giamarchi. Quantum physics in one dimension. 2004.

[28] Adam S. Hadley and Matthew G. Betts. The effects of landscape fragmentation on pollination dynamics: absence of evidence not evidence of absence. *Biological Reviews*, 87, 2012.

[29] Michael A Harrison. *Introduction to formal language theory*. Addison-Wesley Longman Publishing Co., Inc., 1978.

[30] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[31] Ryszard Horodecki, Pawe Horodecki, Micha Horodecki, and Karol Horodecki. Quantum entanglement. 2007.

[32] Naeem Jan, Tahir Mahmood, Lemnaouar Zedam, Kifayat Ullah, José Carlos Rodríguez Alcantud, and Bijan Davvaz. Analysis of social networks, communication networks and shortest path problems in the environment of interval-valued q-rung ortho pair fuzzy graphs. *International Journal of Fuzzy Systems*, 21:1687–1708, 2019.

[33] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[34] Roberto Carlos Jiménez Martínez, César Elías Paucar Paucar, José Ignacio Cruz Arboleda, Miguel Ángel Guambo Llerena, and Erick González Caballero. Neutrosophic matrix games to solve project management conflicts. *Neutrosophic Sets and Systems*, 44(1):2, 2021.

[35] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[36] V. Ganesh Karthikeyan, P. Thangaraj, and S. Karthik. Towards developing hybrid educational data mining model (hedm) for efficient and accurate student performance evaluation. *Soft Computing*, 24:18477 – 18487, 2020.

[37] Richard Kaye. *Models of Peano Arithmetic*. Clarendon Press, Oxford, 1991.

[38] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[39] László T. Kóczy, Naeem Jan, Tahir Mahmood, and Kifayat Ullah. Analysis of social networks and wi-fi networks by using the concept of picture fuzzy graphs. *Soft Computing*, 24:16551 – 16563, 2020.

[40] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[41] Fang Liu, Muhammad Umair, and Junjun Gao. Assessing oil price volatility co-movement with stock market volatility through quantile regression approach. *Resources Policy*, 2023.

[42] Rupkumar Mahapatra, Sovan Samanta, Madhumangal Pal, Tofigh Allahviranloo, and Antonios Kalampakas. A study on linguistic z-graph and its application in social networks. *Mathematics*, 12(18):2898, 2024.

[43] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[44] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[45] Mai Mohamed, Mohamed Abdel-Basset, Abdel-Nasser Hussien, and Florentin Smarandache. Using neutrosophic sets to obtain pert three-times estimates in project management. *Infinite Study*, 2017.

[46] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[47] S Narasimman, M Shanmugapriya, R Sundareswaran, Laxmi Rathour, Lakshmi Narayan Mishra, Vinita Dewangan, and Vishnu Narayan Mishra. Identification of influential factors affecting student performance in semester examinations in the educational institution using score topological indices in single valued neutrosophic graphs. *Neutrosophic Sets and Systems*, 75:224–240, 2025.

[48] Wendy Olsen and Hisako Nomura. Poverty reduction: fuzzy sets vs. crisp sets compared. *Sociological Theory and Methods*, 24(2):219–246, 2009.

[49] Stefano Crespi Reghizzi, Luca Breveglieri, and Angelo Morzenti. *Formal languages and compilation*. Springer, 2013.

[50] Ariel Romero Fernández, Lourdes Viviana Moreira Rosales, Olga Germania Arciniegas Paspuel, Walter Bolívar Jarrín López, and Anthony Rafael Sotolongo León. Neutrosophic statistics for project management. application to a computer system project. *Neutrosophic Sets and Systems*, 44(1):34, 2021.

[51] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[52] Myriam Paulina Barreno Sánchez, Miriam Pantoja Burbano, and Sary Alvarez Hernández. Neutrosophic insights into strategic decision-making: Navigating complexity in ecuadorian business management. *Neutrosophic Optimization and Intelligent Systems*, 1:46–56, 2024.

[53] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.

[54] Corinna Schrum, Irina Alekseeva, and Mike St John. Development of a coupled physical–biological ecosystem model ecosmo: part i: model description and validation for the north sea. *Journal of Marine Systems*, 61(1-2):79–99, 2006.

[55] Gulfam Shahzadi, Muhammad Akram, Arsham Borumand Saeid, et al. An application of single-valued neutrosophic sets in medical diagnosis. *Neutrosophic sets and systems*, 18:80–88, 2017.

[56] Ahmed Mohamed Shitaya, Mohamed El Syed Wahed, Amr Ismail, Mahmoud Y Shams, AA Salama, et al. Predicting student behavior using a neutrosophic deep learning model. *Neutrosophic Sets and Systems*, 76:288–310, 2025.

[57] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[58] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[59] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[60] Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.

[61] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[62] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[63] Florentin Smarandache. Degrees of membership> 1 and< 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[64] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[65] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[66] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[67] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[68] Florentin Smarandache. Extended nonstandard neutrosophic logic, set, and probability based on extended nonstandard analysis. *Symmetry*, 11(4):515, 2019.

[69] Florentin Smarandache. *Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components*. Infinite Study, 2021.

[70] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[71] Florentin Smarandache. Neutrosophic systems and neutrosophic dynamic systems. *Collected Papers. Volume VIII: On Neutrosophic Theory and Applications*, page 50, 2022.

[72] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[73] Florentin Smarandache. The principles of (partial locality, partial indeterminacy, partial nonlocality) and (multi locality, multi indeterminacy, multi nonlocality). *Neutrosophic Sets and Systems*, 72(1):12, 2024.

[74] Florentin Smarandache. *Upside-Down Logics: Falsification of the Truth and Truthification of the False*. Infinite Study, 2024.

[75] Florentin Smarandache. Short introduction to standard and nonstandard neutrosophic set and logic (review paper). *Neutrosophic Sets and Systems*, 77, 2025. Review Paper.

[76] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[77] Florentin Smarandache and Maissam Jdid. An overview of neutrosophic and plithogenic theories and applications. 2023.

[78] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[79] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[80] Ieee Std. Ieee guide adoption of pmi standard a guide to the project management body of knowledge. 2004.

[81] Patrick Suppes. *Introduction to logic*. Courier Corporation, 2012.

[82] Masaharu Tsujimoto, Yuya Kajikawa, Junichi Tomita, and Yoichi Matsumoto. A review of the ecosystem concept-towards coherent ecosystem design. *Technological forecasting and social change*, 136:49–58, 2018.

[83] Zach Weber. Transfinite numbers in paraconsistent set theory. *The Review of Symbolic Logic*, 3:71 – 92, 2010.

[84] Zach Weber. Transfinite cardinals in paraconsistent set theory. *The Review of Symbolic Logic*, 5:269 – 293, 2012.

[85] Shengjun Wu and Jeeva S. Anandan. What is quantum entanglement. 2003.

[86] Jun Ye and Shigui Du. Some distances, similarity and entropy measures for interval-valued neutrosophic sets and their relationship. *International Journal of Machine Learning and Cybernetics*, 10:347 – 355, 2017.

[87] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[88] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[89] Lotfi A. Zadeh. Fuzzy logic = computing with words. *IEEE Trans. Fuzzy Syst.*, 4:103–111, 1996.

[90] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[91] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[92] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[93] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

[94] Hongyu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615 – 627, 2015.

[95] Hongyu Zhang, Jianqiang Wang, and Xiaohong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615–627, 2016.

[96] Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

A Review of Fuzzy and Neutrosophic Offsets: Connections to Some Set Concepts and Normalization Function

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Abstract

To effectively address the uncertainties inherent in real-world scenarios, various extensions of set theory have been developed, including Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets. Among these, the concepts of Fuzzy Offset, Overset, Underset, and their Neutrosophic counterparts have been defined and extensively studied within extended frameworks.

In this paper, we aim to extend the notion of offset to a broader range of uncertain set-theoretic concepts. Specifically, we define and explore the following: Nonstationary Fuzzy Offset, Multi-valued Plithogenic Offset, Subset-valued Neutrosophic Offset, Hesitant fuzzy offset, Spherical Fuzzy OffSet, Fuzzy Off Matroid, Plithogenic Off Matroid, Fuzzy Off Ultrafilter, Neutrosophic Off Ultrafilter, and Plithogenic Off Ultrafilter. Through this investigation, we aim to advance the development of set-theoretic extensions that address uncertainty and complexity in mathematical modeling, thereby enriching the theoretical foundations of these frameworks.

Keywords: Neutrosophic Set, plithogenic set, fuzzy set, Neutrosophic Offset, plithogenic Offset

1 Short Introduction

1.1 Neutrosophic Set and Related Set Theory

Set theory, a foundational branch of mathematics, focuses on the study of "sets," which are defined as collections of objects [42, 96, 213, 214]. Over the years, to better address real-world uncertainties, various extensions of classical sets have been proposed, including Fuzzy Sets [40, 206, 227, 229–232, 237], Vague Sets [4, 28, 35, 87, 235], Soft Sets [11, 13, 63, 126, 134, 224], Hypersoft set [192, 193], Rough set [148–154], Hyperfuzzy set [62, 70, 98, 202], and Neutrosophic Sets [6, 27, 50, 139, 180, 181, 198, 216].

Each of these frameworks is designed to address specific types of ambiguity and uncertainty. For example, a *Fuzzy Set* assigns each element a membership degree within $[0, 1]$, representing partial belonging rather than binary inclusion, thereby enabling flexible modeling [227]. *Neutrosophic Sets*, on the other hand, simultaneously account for truth, indeterminacy, and falsehood, making them highly versatile for handling uncertainty in complex systems [180, 181]. A *Hesitant Fuzzy Set* assigns a set of possible membership degrees within $[0, 1]$ to each element, effectively capturing hesitation or uncertainty [210, 211]. Similarly, a *Spherical Fuzzy Set* defines membership, abstinence, and non-membership degrees that satisfy $s^2 + i^2 + d^2 \leq 1$, where s, i, and d represent the degrees of membership, abstinence, and non-membership, respectively, providing a multidimensional approach to modeling opinions [5, 77]. Beyond these, countless other set concepts and logics have been proposed to address uncertainty. Among them, *Plithogenic Sets*, which generalize these uncertain set frameworks, have recently gained attention as a powerful and versatile extension [1, 73, 167, 179, 186, 187, 195, 199, 204].

Uncertain sets, including the examples mentioned above, have been the focus of extensive research, as highlighted in various studies [67, 99, 180, 181, 187]. Moreover, the study of uncertain graphs, such as Fuzzy Graphs and Neutrosophic Graphs, has experienced significant advancements. These developments have been applied to a wide range of problems, as demonstrated in recent research [58–60, 62–66, 168]. Additionally, these concepts have been actively explored in practical applications, such as Neural Networks [12, 83, 112, 120, 122, 165, 207, 208] and decision-making processes [3, 5, 34, 76, 97, 160, 162], further showcasing their versatility and importance.

One emerging concept in uncertain sets is the notion of "offset" [29, 62, 184, 185, 189, 191]. The offset concept allows for flexible adjustment of membership function values within uncertain sets, making it a compelling subject of recent studies. This flexibility provides a fresh perspective on interpreting and applying uncertain set memberships, thereby driving rapid advancements in research. Additionally, special cases of offsets, known as "overset" and "underset," have also been identified and are similarly under active investigation [17, 130, 190, 215].

1.2 Background on Filters and Matroids

In classical set theory, as well as in Fuzzy and Neutrosophic Sets, mathematical structures such as filters, ultrafilters [16, 24, 57, 86, 86, 173], ideals [92, 201, 201], matroids [52, 56, 144, 166, 212, 221], and antimatroids [128] have been extensively explored.

A filter is a subset of a poset closed under finite intersection and supersets. An ultrafilter is a maximal filter where every subset or its complement belongs to the filter(cf. [61, 84, 86, 135, 173]). An ideal is a subset of a poset closed under finite union and contained in larger sets. A matroid is a combinatorial structure generalizing independence in vector spaces and graph theory [144]. These structures are fundamental to mathematics and exhibit intriguing properties, enhancing the understanding of mathematical systems.

1.3 Our Contribution in This Paper

As discussed, the concepts of Fuzzy Offset, Overset, Underset, and their Neutrosophic counterparts are well-established. These concepts expand traditional frameworks by allowing greater flexibility in membership value adjustments, generalizing the inherent adaptability of Fuzzy and Neutrosophic Sets.

In this paper, we aim to extend the notion of offset to a broader range of set-theoretic concepts. Specifically, we define and explore the following: Nonstationary Fuzzy Offset, Multi-valued Plithogenic Offset, Subset-valued Neutrosophic Offset, Hesitant fuzzy offset, Spherical Fuzzy OffSet, Fuzzy Off Matroid, Plithogenic Off Matroid, Fuzzy Off Ultrafilter, Neutrosophic Off Ultrafilter, and Plithogenic Off Ultrafilter. Additionally, we examine their relationships with other related mathematical structures. Through this work, we aim to contribute to the development of set-theoretic extensions that address uncertainty and complexity in mathematical modeling.

Note that the author believes that when considering real-world concepts within the framework of Neutrosophic Sets or Fuzzy Sets, they may not always naturally conform to the neat interval $[0, 1]$. In such cases, normalization may be necessary. Therefore, this study explores the process of normalizing "offsets" [185, 189, 191] into sets. It is hoped that this investigation will contribute to making Fuzzy Sets and Neutrosophic Sets more accessible and applicable in machine learning and various practical applications.

1.4 The Structure of the Paper

The structure of this paper is as follows.

2 Preliminaries and Definitions

Some foundational concepts from set theory are applied in parts of this work. For further details on these foundational concepts, please consult the relevant references as needed [55, 82, 88, 96, 115]. Additionally, for operations and related topics concerning each concept, please refer to the respective references as necessary.

2.1 Basic Set Theory

Below are some fundamental concepts in set theory. For more comprehensive details, please refer to the relevant references as needed [96].

Definition 2.1 (Set)**.** [96] A *set* is a collection of distinct objects, known as elements, that are clearly defined, allowing any object to be identified as either belonging to or not belonging to the set. If A is a set and x is an element of A, this membership is denoted by $x \in A$. Sets are typically represented using curly brackets.

Definition 2.2 (Subset)**.** [96] A set A is called a *subset* of another set B, denoted as $A \subseteq B$, if every element of A is also an element of B. Formally:

$$A \subseteq B \iff \forall x\, (x \in A \implies x \in B).$$

If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset* of B, denoted $A \subset B$.

Definition 2.3 (Intersection)**.** [96] The *intersection* of two sets A and B, denoted $A \cap B$, is the set of elements that are common to both A and B. Formally:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

If $A \cap B = \emptyset$, then A and B are said to be *disjoint*.

Definition 2.4 (Union)**.** [96] The *union* of two sets A and B, denoted $A \cup B$, is the set of all elements that are in A, B, or both. Formally:

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Definition 2.5 (Empty Set)**.** [96] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. Formally, the empty set is defined as:

$$\emptyset = \{x \mid x \neq x\},$$

indicating that there are no elements x for which the condition $x = x$ fails, thereby resulting in an empty collection. The empty set is a subset of every set and has a cardinality of zero.

Definition 2.6 (Non-Empty Set)**.** A *non-empty set* is a set that contains at least one element. Formally, a set S is non-empty if:

$$\exists x \in S.$$

In contrast to the empty set $\emptyset$, a non-empty set has a cardinality $|S| > 0$.

2.2 Crisp Sets and Neutrosophic Sets

When dealing with Fuzzy Sets or Neutrosophic Sets, they are often discussed alongside their foundational Crisp Sets. The definition of a Crisp Set is provided below.

Definition 2.7 (Universe Set)**.** (cf. [140]) A *universe set*, often denoted by U, is a set that contains all the elements under consideration for a particular discussion or problem domain. Formally, U is defined as a set that encompasses every element within the scope of a given context or framework, so that any subset of interest can be regarded as a subset of U.

In set theory, the universe set U is typically assumed to contain all elements relevant to the discourse, meaning that for any set A, if $A \subseteq U$, then all elements of A are elements of U. Related concepts include underlying sets and whole sets.

Definition 2.8 (Crisp Set)**.** [142] Let X be a universe set, and let $P(X)$ denote the power set of X, which represents all subsets of X. A *crisp set* $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{0, 1\}$, where:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This function χ_A assigns a value of 1 to elements within the set A and 0 to those outside it, creating a clear boundary. Crisp sets are thus bivalent and follow the principle of binary classification, where each element is either a member of the set or not.

The Fuzzy Set is a well-known concept used to handle uncertainty in set theory. The definition is provided below [227].

Definition 2.9. [227–232, 232, 233] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 2.10. [180, 182, 183, 197, 198] Let X be a given set. A (single-valued) Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 2.11 (Medical Diagnosis)**.** In diagnosing a disease D, a patient x may exhibit symptoms that are partially indicative of D. The truth, indeterminacy, and falsity membership functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ could represent:

- $T_A(x) = 0.8$: High likelihood the patient has D.
- $I_A(x) = 0.1$: Indeterminate due to inconclusive test results.
- $F_A(x) = 0.1$: Small chance the patient does not have D.

Medical diagnosis is a concept that has been extensively studied using Neutrosophic Sets [10, 26, 32, 41, 124, 174, 236].

Example 2.12 (Consumer Sentiment Analysis)**.** For a product review x, the degree of positivity, neutrality, and negativity can be represented as:

- $T_A(x) = 0.6$: 60% of users express a positive sentiment.
- $I_A(x) = 0.3$: 30% of users are neutral or indecisive.
- $F_A(x) = 0.1$: 10% of users express negative sentiment.

Sentiment Analysis is a concept that has been extensively studied using Neutrosophic Sets [19,85,100,116,159].

Example 2.13 (Environmental Risk Assessment)**.** When evaluating the risk of pollution in a river x:

- $T_A(x) = 0.7$: 70% chance the river is polluted based on chemical levels.
- $I_A(x) = 0.2$: 20% uncertainty due to fluctuating seasonal factors.
- $F_A(x) = 0.1$: 10% chance the river is not polluted, as some parameters are within safe limits.

Risk Assessment is a concept that has been extensively studied using Neutrosophic Sets [7, 22, 104, 117, 146].

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [186, 187]. The definition of the Plithogenic Set is provided below.

Definition 2.14. [186, 187] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

Example 2.15. (cf. [59]) The following examples of Plithogenic sets are provided.

- When $s = t = 1$, PS is called a *Plithogenic Fuzzy Set*.
- When $s = 2, t = 1$, PS is called a *Plithogenic Intuitionistic Fuzzy Set*.
- When $s = 3, t = 1$, PS is called a *Plithogenic Neutrosophic Set*.
- When $s = 4, t = 1$, PS is called a *Plithogenic quadripartitioned Neutrosophic Set* (cf. [91, 164, 177]).
- When $s = 5, t = 1$, PS is called a *Plithogenic pentapartitioned Neutrosophic Set* (cf. [23, 39, 127]).
- When $s = 6, t = 1$, PS is called a *Plithogenic hexapartitioned Neutrosophic Set* (cf. [147]).
- When $s = 7, t = 1$, PS is called a *Plithogenic heptapartitioned Neutrosophic Set* (cf. [25, 138]).
- When $s = 8, t = 1$, PS is called a *Plithogenic octapartitioned Neutrosophic Set*.
- When $s = 9, t = 1$, PS is called a *Plithogenic nonapartitioned Neutrosophic Set*.

2.3 Plithogenic Offset/Overset/Underset

This section provides an explanation of the *Plithogenic Offset*, *Overset*, and *Underset*. The *Plithogenic Offset* is an extended concept derived from the Plithogenic Set. Restricted versions of the Offset, namely the *Overset* (where only Ω is unrestricted) and the *Underset* (where only Ψ is unrestricted), are also recognized and studied.

While this paper primarily focuses on the definition of the Offset, any concept defined using the Offset can also be defined using the Overset or Underset. For example, a Fuzzy Overset or Fuzzy Underset can define a Fuzzy Offset.

Definition 2.16 (Crisp Offset)**.** (cf. [62]) Let X be a universe of discourse, and let Ψ and Ω represent 0 and 1, respectively. A *Crisp Offset* $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{\Psi, \Omega\}$, where:

$$\chi_A(x) = \begin{cases} \Omega & \text{if } x \in A, \\ \Psi & \text{if } x \notin A. \end{cases}$$

In this context, the function χ_A assigns a value of Ω (1) to elements that are within the set A and Ψ (0) to elements that are outside A. This structure adheres to the principle of binary classification, as each element is either fully included in the set A or completely excluded from it.

The concept of a Crisp Offset, unlike fuzzy or neutrosophic sets, does not allow for intermediate degrees of membership. Instead, membership is strictly limited to the values Ψ and Ω, reflecting the clear-cut, deterministic nature of this classification approach. This discrete boundary is a distinguishing feature of Crisp Offsets, contrasting with the gradual membership levels typical of fuzzy sets.

Definition 2.17 (Fuzzy Offset)**.** (cf. [185]) Let X be a universe of discourse. A *Fuzzy Offset* $\tilde{A}$ in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [\Psi, \Omega]\},$$

where $\Omega > 1$ and $\Psi < 0$. There exist elements $x, y \in X$ such that $\mu_{\tilde{A}}(x) > 1$ and $\mu_{\tilde{A}}(y) < 0$.

Definition 2.18 (Single-Valued Neutrosophic OffSet)**.** (cf. [184, 185, 188–190]) A *Single-Valued Neutrosophic OffSet*, denoted $A_{\text{off}} \subseteq U_{\text{off}}$, is a set within a universe of discourse U_{off} in which certain elements may possess neutrosophic degrees—truth, indeterminacy, or falsity—that extend beyond the standard limits, either above 1 or below 0. It is formally defined as:

$$A_{\text{off}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{off}}, \exists\ (T(x) > 1 \text{ or } F(x) < 0)\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of each $x \in U_{\text{off}}$.
- $T(x), I(x), F(x) \in [\Psi, \Omega]$, where $\Omega > 1$ (termed the *OverLimit*) and $\Psi < 0$ (termed the *UnderLimit*), allow the possibility for $T(x)$, $I(x)$, or $F(x)$ to take values beyond the conventional bounds of $[0, 1]$.

Definition 2.19 (Plithogenic Offset)**.** (cf. [62]) Let S be a universal set, and $P \subseteq S$. A *Plithogenic Offset* PS_{off} is defined as:

$$PS_{\text{off}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.

- $pdf : P \times Pv \to [\Psi_v, \Omega_v]^s$ is the *Degree of Appurtenance Function (DAF)*, where $\Psi_v < 0$ and $\Omega_v > 1$.
- $pCF : Pv \times Pv \to [\Psi_v, \Omega_v]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, the DAF and DCF allow the membership degrees $pdf(x, a)$ to range from below 0 to above 1, between the underlimit Ψ_v and the overlimit Ω_v.

Example 2.20. (cf. [59]) The following examples of Plithogenic offsets are provided.

- When $s = t = 1$, PS_{off} is called a *Plithogenic Fuzzy OffSet*.
- When $s = 2, t = 1$, PS_{off} is called a *Plithogenic Intuitionistic Fuzzy OffSet*.
- When $s = 3, t = 1$, PS_{off} is called a *Plithogenic Neutrosophic OffSet*.
- When $s = 4, t = 1$, PS_{off} is called a *Plithogenic quadripartitioned Neutrosophic OffSet*.
- When $s = 5, t = 1$, PS_{off} is called a *Plithogenic pentapartitioned Neutrosophic OffSet*.
- When $s = 6, t = 1$, PS_{off} is called a *Plithogenic hexapartitioned Neutrosophic OffSet*.
- When $s = 7, t = 1$, PS_{off} is called a *Plithogenic heptapartitioned Neutrosophic OffSet*.
- When $s = 8, t = 1$, PS_{off} is called a *Plithogenic octapartitioned Neutrosophic OffSet*.
- When $s = 9, t = 1$, PS_{off} is called a *Plithogenic nonapartitioned Neutrosophic OffSet*.

Proposition 2.21. *A Plithogenic Set is a subset of a Plithogenic Offset.*

Proof. A Plithogenic Set $PS = (P, v, Pv, pdf, pCF)$ is defined by:

- $pdf : P \times Pv \to [0, 1]^s$,
- $pCF : Pv \times Pv \to [0, 1]^t$.

A Plithogenic Offset $PS_{\text{off}} = (P, v, Pv, pdf_{\text{off}}, pCF_{\text{off}})$ generalizes this definition by allowing:

$$pdf_{\text{off}} : P \times Pv \to [\Psi_v, \Omega_v]^s, \quad \text{where } \Psi_v < 0,\ \Omega_v > 1.$$

When $\Psi_v = 0$ and $\Omega_v = 1$, the Plithogenic Offset reduces to a Plithogenic Set, as pdf_{off} and pCF_{off} restrict to the interval $[0, 1]$. Hence, a Plithogenic Set is a subset of a Plithogenic Offset. □

Please refer to the respective introductory notes for details on the operations of fuzzy sets, neutrosophic sets, and plithogenic sets [48, 68, 186, 196, 238].

3 Result: Some Sets and Some Concepts

In this section, we apply the concept of Offset to several existing sets and examine the relationships between them. It should be noted that Overset and Underset are constrained versions of Offset. Since they can be defined and proven using similar methods as those presented here, they are omitted from this publication.

3.1 Nonstationary Fuzzy offset

We define the concept of a Nonstationary Fuzzy Offset and explore its relationships with existing concepts [15,69,89,90].

Definition 3.1 (Nonstationary Fuzzy Set)**.** [69] Let A be a fuzzy set in the universe of discourse X, characterized by a membership function $\mu_A : X \to [0,1]$. Define a set of time points $T = \{t_1, t_2, \ldots, t_n\}$, which may be finite or infinite, and let $f : T \to \mathbb{R}$ be a perturbation function that introduces time-dependent changes to the parameters of μ_A.

A *Nonstationary Fuzzy Set* A^* is defined as:

$$A^* = \{(\mu_{A^*}(t,x), x, t) \mid x \in X, t \in T\},$$

where:

- $\mu_{A^*} : T \times X \to [0,1]$ is a nonstationary membership function that assigns a time-dependent membership degree to each element $x \in X$ at each time $t \in T$.
- $\mu_{A^*}(t,x)$ represents the membership degree of x at time t.

Definition 3.2 (Nonstationary Membership Function)**.** The *nonstationary membership function* is formally defined as:

$$\mu_{A^*}(t,x) = \mu_A(x, p_1(t), p_2(t), \ldots, p_m(t)),$$

where $\{p_i(t)\}$ are the parameters of the standard membership function $\mu_A(x)$, modified by time-dependent perturbation functions. Specifically, for each parameter $p_i(t)$, we have:

$$p_i(t) = p_i + k_i f_i(t), \quad i = 1, 2, \ldots, m,$$

where:

- p_i are the initial parameters of $\mu_A(x)$,
- k_i are scaling constants that control the magnitude of perturbation,
- $f_i(t)$ are perturbation functions that induce time-based variations in the parameters.

Definition 3.3 (Nonstationary Fuzzy Offset)**.** Let X be a universe of discourse, and let A be a fuzzy set in X characterized by a membership function $\mu_A : X \to [\Psi, \Omega]$. Define a set of time points $T = \{t_1, t_2, \ldots, t_n\}$, which may be finite or infinite, and let $f : T \to \mathbb{R}$ be a perturbation function introducing time-dependent changes to the parameters of μ_A.

A *Nonstationary Fuzzy Offset* $\tilde{A}^*$ is defined as:

$$\tilde{A}^* = \left\{ \left(\mu_{\tilde{A}^*}(t,x), x, t\right) \mid x \in X,\ t \in T \right\},$$

where:

- $\mu_{\tilde{A}^*} : T \times X \to [\Psi, \Omega]$ is a nonstationary membership function, assigning a time-dependent membership degree to each $x \in X$ at each $t \in T$, where $\Omega > 1$ and $\Psi < 0$.
- $\mu_{\tilde{A}^*}(t,x)$ represents the membership degree of x at time t.

The nonstationary membership function $\mu_{\tilde{A}^*}(t,x)$ is formally defined as:

$$\mu_{\tilde{A}^*}(t,x) = \mu_A(x, p_1(t), p_2(t), \ldots, p_m(t)),$$

where $\{p_i(t)\}$ are parameters of the standard membership function $\mu_A(x)$, modified by time-dependent perturbation functions. For each parameter $p_i(t)$, we have:

$$p_i(t) = p_i + k_i f_i(t), \quad i = 1, 2, \ldots, m,$$

with:

- p_i being the initial parameters of $\mu_A(x)$,
- k_i being scaling constants controlling the magnitude of perturbation,
- $f_i(t)$ being perturbation functions inducing time-dependent variations.

Additionally, there exist elements $x, y \in X$ and time points $t_x, t_y \in T$ such that:

$$\mu_{\tilde{A}^*}(t_x, x) > 1 \quad \text{and} \quad \mu_{\tilde{A}^*}(t_y, y) < 0.$$

Theorem 3.4. *A Nonstationary Fuzzy Offset generalizes both the Nonstationary Fuzzy Set and the Fuzzy Offset.*

Proof. Let A be a fuzzy set in the universe of discourse X, characterized by the membership function $\mu_A : X \to [0, 1]$. We examine the relationships between the Nonstationary Fuzzy Offset $\tilde{A}^*$, the Nonstationary Fuzzy Set A^*, and the Fuzzy Offset $\tilde{A}$. *: The Nonstationary Fuzzy Set A^* is defined as:

$$A^* = \{(\mu_{A^*}(t, x), x, t) \mid x \in X, t \in T\},$$

where $\mu_{A^*}(t, x)$ satisfies:

$$\mu_{A^*}(t, x) = \mu_A(x, p_1(t), p_2(t), \ldots, p_m(t)),$$

with $\mu_A(x)$ taking values in the range $[0, 1]$.

The Nonstationary Fuzzy Offset $\tilde{A}^*$ extends A^* by allowing the membership function $\mu_{\tilde{A}^*}(t, x)$ to take values in the broader range $[\Psi, \Omega]$, where $\Psi < 0$ and $\Omega > 1$. By restricting $\Psi = 0$ and $\Omega = 1$, $\tilde{A}^*$ reduces to A^*. Thus, $\tilde{A}^*$ generalizes A^*.

The Fuzzy Offset $\tilde{A}$ is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [\Psi, \Omega]\}.$$

In this case, the membership function $\mu_{\tilde{A}}(x)$ does not depend on time. By setting $T = \{t_0\}$, where t_0 is a single fixed time point, and $p_i(t) = p_i$ (constant parameters), $\tilde{A}^*$ reduces to $\tilde{A}$. Thus, $\tilde{A}^*$ generalizes $\tilde{A}$.

The Nonstationary Fuzzy Offset $\tilde{A}^*$ incorporates both the time-dependent variability of A^* and the extended membership range of $\tilde{A}$, combining their properties into a unified framework. This demonstrates that $\tilde{A}^*$ is a proper generalization of both A^* and $\tilde{A}$.

Therefore, the Nonstationary Fuzzy Offset generalizes both the Nonstationary Fuzzy Set and the Fuzzy Offset. □

3.2 Multi-Valued Plithogenic Offset

A *Multi-Valued Neutrosophic Set (MVNS)* is a mathematical framework extending Neutrosophic Sets to represent truth, indeterminacy, and falsity degrees with multiple possible values [118, 119, 123, 155, 156, 223]. Formally:

Definition 3.5. (cf. [155]) Let X be a universe of discourse, where $x \in X$ is an element. A *Multi-Valued Neutrosophic Set (MVNS) A* in X is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\},$$

where:

- $T_A(x) \subseteq [0, 1]$: The set of possible truth-membership degrees of x.
- $I_A(x) \subseteq [0, 1]$: The set of possible indeterminacy-membership degrees of x.
- $F_A(x) \subseteq [0, 1]$: The set of possible falsity-membership degrees of x.

These values satisfy the following conditions:

1. $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$,
2. For any $t \in T_A(x), i \in I_A(x), f \in F_A(x)$,

$$0 \leq t + i + f \leq 3.$$

Each element $\langle T_A(x), I_A(x), F_A(x)\rangle$ is called a *Multi-Valued Neutrosophic Number (MVNN).*

We extend the concept of a Multi-Valued Neutrosophic Number (MVNN) to a Multi-Valued Plithogenic Set, and further generalize it to an Offset. The definitions and theorems are provided below.

Definition 3.6 (Multi-Valued Plithogenic Set)**.** Let S be a universal set, and $P \subseteq S$.

A *Multi-Valued Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to 2^{[0,1]^s}$ is the *Degree of Appurtenance Function (DAF)*, mapping each pair (x, a) to a set of s-tuples in $[0, 1]$.
- $pCF : Pv \times Pv \to 2^{[0,1]^t}$ is the *Degree of Contradiction Function (DCF).*

In this definition, the DAF allows for multiple membership degrees for each element x and attribute value a, capturing the uncertainty or hesitation in the membership assessment. The DCF allows the contradiction degrees to extend beyond the standard interval $[0, 1]$, accommodating over- and under-contradiction degrees.

Definition 3.7 (Multi-Valued Plithogenic OffSet)**.** Let S be a universal set, and $P \subseteq S$.

A *Multi-Valued Plithogenic OffSet* PS_{off} is defined as:

$$PS_{\text{off}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to 2^{[\Psi_v,\Omega_v]^s}$ is the *Degree of Appurtenance Function (DAF)*, mapping each pair (x, a) to a set of s-tuples in $[\Psi_v, \Omega_v]$, where $\Psi_v < 0$ and $\Omega_v > 1$.
- $pCF : Pv \times Pv \to 2^{[\Psi_v,\Omega_v]^t}$ is the *Degree of Contradiction Function (DCF)*, mapping each pair (a, b) to a set of t-tuples in $[\Psi_v, \Omega_v]$.

In this definition, both the DAF and the DCF allow the degrees to extend beyond the standard interval $[0, 1]$, accommodating over- and under-membership and contradiction degrees.

Example 3.8. (cf. [59]) The following examples of Multi-Valued Plithogenic OffSet are provided.

- When $s = t = 1$, PG is called a *Multi-Valued Plithogenic Fuzzy OffSet*.
- When $s = 2, t = 1$, PG is called a *Multi-Valued Plithogenic Intuitionistic Fuzzy OffSet*.
- When $s = 3, t = 1$, PG is called a *Multi-Valued Plithogenic Neutrosophic OffSet*.
- When $s = 4, t = 1$, PG is called a *Multi-Valued Plithogenic quadripartitioned Neutrosophic OffSet*.
- When $s = 5, t = 1$, PG is called a *Multi-Valued Plithogenic pentapartitioned Neutrosophic OffSet*.
- When $s = 6, t = 1$, PG is called a *Multi-Valued Plithogenic hexapartitioned Neutrosophic OffSet*.
- When $s = 7, t = 1$, PG is called a *Multi-Valued Plithogenic heptapartitioned Neutrosophic OffSet*.
- When $s = 8, t = 1$, PG is called a *Multi-Valued Plithogenic octapartitioned Neutrosophic OffSet*.
- When $s = 9, t = 1$, PG is called a *Multi-Valued Plithogenic nonapartitioned Neutrosophic OffSet*.

Theorem 3.9. *A Multi-Valued Plithogenic Set can be transformed into a Multi-Valued Neutrosophic Set and a Plithogenic Set.*

Proof. Let $PS = (P, v, Pv, pdf, pCF)$ be a Multi-Valued Plithogenic Set.

To transform PS into a Multi-Valued Neutrosophic Set, proceed as follows:

- Choose the attribute v to represent neutrosophic components, such as *Truth*, *Indeterminacy*, and *Falsity*.
- Set $Pv = \{\text{Truth}, \text{Indeterminacy}, \text{Falsity}\}$. $(s = 3, t = 1)$
- Define the DAF pdf such that for each $x \in P$ and $a \in Pv$, $pdf(x, a) \subseteq [0, 1]$ captures the multi-valued membership degrees corresponding to the neutrosophic components.
- Adjust the DCF pCF accordingly, ensuring it maps into $2^{[0,1]^t}$ and reflects standard contradiction degrees.

By this construction, PS encapsulates the structure of a Multi-Valued Neutrosophic Set.

To transform PS into a Plithogenic Set, constrain the DAF and DCF to single-valued functions within $[0, 1]^s$ and $[0, 1]^t$, respectively:

- For each $x \in P$ and $a \in Pv$, restrict $pdf(x, a)$ to a single s-tuple in $[0, 1]^s$.
- Similarly, restrict $pCF(a, b)$ to a single t-tuple in $[0, 1]^t$.

Therefore, the Multi-Valued Plithogenic Set generalizes both the Multi-Valued Neutrosophic Set and the Plithogenic Set. □

Theorem 3.10. *A Multi-Valued Plithogenic OffSet reduces to a Multi-Valued Plithogenic Set when the DAF pdf and DCF pCF are restricted to map into $2^{[0,1]^s}$ and $2^{[0,1]^t}$, respectively.*

Proof. Consider the Multi-Valued Plithogenic OffSet $PS_{\text{off}} = (P, v, Pv, pdf, pCF)$ with:

$$pdf : P \times Pv \to 2^{[\Psi_v, \Omega_v]^s},$$
$$pCF : Pv \times Pv \to 2^{[\Psi_v, \Omega_v]^t}.$$

By restricting the ranges of pdf and pCF to $2^{[0,1]^s}$ and $2^{[0,1]^t}$, respectively (i.e., setting $\Psi_v = 0$ and $\Omega_v = 1$), we obtain mappings that align with the definition of a Multi-Valued Plithogenic Set.

Therefore, PS_{off} becomes a Multi-Valued Plithogenic Set PS under these restrictions. □

Theorem 3.11. *A Multi-Valued Plithogenic OffSet reduces to a Plithogenic OffSet when the DAF pdf and DCF pCF map each pair to single values instead of sets.*

Proof. Starting with the Multi-Valued Plithogenic OffSet $PS_{\text{off}} = (P, v, Pv, pdf, pCF)$, enforce the following conditions:

- The DAF pdf maps each (x, a) to a single s-tuple in $[\Psi_v, \Omega_v]^s$.
- The DCF pCF maps each (a, b) to a single t-tuple in $[\Psi_v, \Omega_v]^t$.

Under these conditions, PS_{off} satisfies the definition of a Plithogenic OffSet as given in [62].

Thus, the Multi-Valued Plithogenic OffSet reduces to a Plithogenic OffSet. □

3.3 Subset-Valued Neutrosophic OffSet

We consider the concept of the Subset-Valued Neutrosophic OffSet as an extension of the Subset-Valued Neutrosophic Set using the framework of Neutrosophic OffSets. The definitions and related theorems are provided below.

Definition 3.12 (Subset-Valued Neutrosophic Set (SVNS))**.** [194] Let $\mathcal{U}$ be a universe of discourse, and let $\mathcal{S} \subseteq \mathcal{U}$. A *Subset-Valued Neutrosophic Set (SVNS)* $\mathcal{S}$ is defined as:

$$\mathcal{S} = \{(x, T(x), I(x), F(x)) \mid x \in \mathcal{U}\},$$

where:

- $T(x), I(x), F(x)$ are subsets of $[0, 1]$, representing the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* degrees of x, respectively.
- The following conditions hold:

  $$0 \leq \inf(T(x)) + \inf(I(x)) + \inf(F(x)) \leq \sup(T(x)) + \sup(I(x)) + \sup(F(x)) \leq 3,$$

  where inf and sup denote the infimum and supremum, respectively, of the subsets $T(x)$, $I(x)$, and $F(x)$.

Example 3.13 (Financial Investment Risk Assessment)**.** Let U_{off} represent a set of financial investments. Each investment $x \in U_{\text{off}}$ is associated with truth (T), indeterminacy (I), and falsity (F) degrees extended to $[\Psi, \Omega]$, where $\Psi < 0$ and $\Omega > 1$. For instance:

$$A_{\text{off}} = \{(\text{Investment A}, \langle T(\text{A}) = 1.2, I(\text{A}) = 0.5, F(\text{A}) = -0.3\rangle), (\text{Investment B}, \langle T(\text{B}) = 0.8, I(\text{B}) = 0.2, F(\text{B}) = 0.1\rangle)\},$$

where:

- $T(\text{A}) = 1.2$: Investment A is expected to yield returns exceeding typical expectations.
- $I(\text{A}) = 0.5$: There is moderate uncertainty regarding Investment A's performance.
- $F(\text{A}) = -0.3$: Negative falsity degree indicates negligible risk for Investment A.

Example 3.14 (Medical Diagnosis)**.** Let U_{off} denote a set of suspected medical conditions. Each condition $x \in U_{\text{off}}$ is assigned neutrosophic degrees $T(x)$, $I(x)$, and $F(x)$ extended to $[\Psi, \Omega]$. For example:

$$A_{\text{off}} = \{(\text{Disease X}, \langle T(\text{X}) = 1.1, I(\text{X}) = 0.4, F(\text{X}) = -0.2\rangle), (\text{Disease Y}, \langle T(\text{Y}) = 0.7, I(\text{Y}) = 0.6, F(\text{Y}) = 0.1\rangle)\},$$

where:

- $T(\text{X}) = 1.1$: High probability of Disease X due to advanced diagnostic tools.

- $I(\mathrm{X}) = 0.4$: Moderate uncertainty caused by overlapping symptoms with other conditions.
- $F(\mathrm{X}) = -0.2$: Negative falsity implies atypical symptoms decrease the likelihood of misdiagnosis.

Example 3.15 (Customer Sentiment Analysis in E-Commerce)**.** Let U_{off} be a set of products in an e-commerce platform. Each product $x \in U_{\text{off}}$ is evaluated using extended neutrosophic degrees. For instance:

$$A_{\text{off}} = \{(\text{Product A}, \langle T(\mathrm{A}) = 1.3, I(\mathrm{A}) = 0.3, F(\mathrm{A}) = -0.1\rangle), (\text{Product B}, \langle T(\mathrm{B}) = 0.9, I(\mathrm{B}) = 0.2, F(\mathrm{B}) = 0.5\rangle)\},$$

where:

- $T(\mathrm{A}) = 1.3$: Overwhelmingly positive sentiment for Product A due to excellent reviews.
- $I(\mathrm{A}) = 0.3$: Minor uncertainty from a small number of inconsistent reviews.
- $F(\mathrm{A}) = -0.1$: Negligible dissatisfaction for Product A.

Definition 3.16 (Subset-Valued Neutrosophic OffSet (SVNOS))**.** Let $\mathcal{U}_{\text{off}}$ be a universe of discourse, and let $\mathcal{S}_{\text{off}} \subseteq \mathcal{U}_{\text{off}}$. A *Subset-Valued Neutrosophic OffSet (SVNOS)* $\mathcal{S}_{\text{off}}$ is defined as:

$$\mathcal{S}_{\text{off}} = \{(x, T(x), I(x), F(x)) \mid x \in \mathcal{U}_{\text{off}}, \exists\ (\sup(T(x)) > 1 \text{ or } \inf(F(x)) < 0)\},$$

where:

- $T(x), I(x), F(x)$ are subsets of $[\Psi, \Omega]$, representing the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* degrees of x, respectively.
- $\Psi < 0$ and $\Omega > 1$ represent the *underlimit* and *overlimit*, allowing $T(x)$, $I(x)$, and $F(x)$ to take values beyond the conventional bounds of $[0, 1]$.
- The following conditions hold:

$$\Psi \leq \inf(T(x)) + \inf(I(x)) + \inf(F(x)) \leq \sup(T(x)) + \sup(I(x)) + \sup(F(x)) \leq \Omega.$$

Theorem 3.17. *A Subset-Valued Neutrosophic OffSet (SVNOS) can be transformed into a Subset-Valued Neutrosophic Set (SVNS) by restricting the membership degrees to the interval* $[0, 1]$.

Proof. By definition, the membership degrees $T(x), I(x), F(x)$ of a Subset-Valued Neutrosophic OffSet (SVNOS) lie in the interval $[\Psi, \Omega]$, where $\Psi < 0$ and $\Omega > 1$.

Restricting these degrees to $[0, 1]$, we ensure that:

$$0 \leq \inf(T(x)) + \inf(I(x)) + \inf(F(x)) \leq \sup(T(x)) + \sup(I(x)) + \sup(F(x)) \leq 3,$$

which satisfies the conditions for a Subset-Valued Neutrosophic Set (SVNS).

Thus, any SVNOS can be converted into an SVNS by simply truncating the values outside $[0, 1]$. □

Theorem 3.18. *A Subset-Valued Neutrosophic OffSet (SVNOS) generalizes both Neutrosophic OffSet and Plithogenic Neutrosophic OffSet.*

Proof. A Neutrosophic OffSet is defined as a set with truth, indeterminacy, and falsity membership degrees

$$T(x), I(x), F(x) \in [\Psi, \Omega]$$

, where $\Psi < 0$ and $\Omega > 1$. These degrees are single values. A Subset-Valued Neutrosophic OffSet extends this definition by allowing $T(x), I(x), F(x)$ to be subsets of $[\Psi, \Omega]$ instead of single values, thereby generalizing the Neutrosophic OffSet. A Plithogenic Neutrosophic OffSet includes additional attributes like the Degree of Contradiction Function (DCF), which evaluates the contradiction between membership degrees. The Subset-Valued Neutrosophic OffSet accommodates these attributes by allowing subsets of membership degrees and can be extended to include DCF as a secondary function. Therefore, the Subset-Valued Neutrosophic OffSet subsumes both the Neutrosophic OffSet and Plithogenic Neutrosophic OffSet as special cases. □

3.4 Hesitant Fuzzy Offset and Spherical Fuzzy Offset

In the field of Fuzzy Set theory, Hesitant Fuzzy Sets [2, 81, 125, 145, 210, 211] and Spherical Fuzzy Sets [18, 77, 132] are well-known concepts. This study explores the extension of these sets to their Offset counterparts. The definitions, along with related theorems and properties, are presented below.

Definition 3.19. [211] Let X be a reference set. A *Hesitant Fuzzy Set (HFS)* on X is defined as a function $h : X \to 2^{[0,1]}$, where $h(x) \subseteq [0, 1]$ represents a set of possible membership values of x in the fuzzy set. Formally:

$$H = \{\langle x, h(x)\rangle : x \in X\},$$

where:

- $h(x) \subseteq [0, 1]$ is the *hesitant membership set*, containing all possible degrees of membership for x.
- If $h(x)$ is a closed interval $[a, b] \subseteq [0, 1]$, then H reduces to an *Intuitionistic Fuzzy Set (IFS)*.

Definition 3.20. Let X be a reference set, and let $\Psi < 0$ and $\Omega > 1$ be underlimit and overlimit values, respectively. A *Hesitant Fuzzy Offset (HFO)* on X is defined as a function $h_{\text{off}} : X \to 2^{[\Psi,\Omega]}$, where $h_{\text{off}}(x) \subseteq [\Psi, \Omega]$ represents a set of possible membership values of x in the fuzzy offset set. Formally:

$$H_{\text{off}} = \{\langle x, h_{\text{off}}(x)\rangle : x \in X\},$$

where:

- $h_{\text{off}}(x) \subseteq [\Psi, \Omega]$ is the *hesitant offset membership set*, allowing values to exceed the bounds of $[0, 1]$.
- If $\Psi = 0$ and $\Omega = 1$, then H_{off} reduces to a Hesitant Fuzzy Set.

Theorem 3.21. *Restricting a Hesitant Fuzzy Offset (HFO) to the interval* $[0, 1]$ *results in a Hesitant Fuzzy Set (HFS).*

Proof. Let $H_{\text{off}} = \{\langle x, h_{\text{off}}(x)\rangle : x \in X\}$ be a Hesitant Fuzzy Offset, where $h_{\text{off}}(x) \subseteq [\Psi, \Omega]$ with $\Psi < 0$ and $\Omega > 1$.

If we restrict the range of $h_{\text{off}}(x)$ to $[0, 1]$, then $h_{\text{off}}(x) \cap [0, 1] = h(x) \subseteq [0, 1]$. The resulting set $H = \{\langle x, h(x)\rangle : x \in X\}$ satisfies the definition of a Hesitant Fuzzy Set, where $h(x) \subseteq [0, 1]$ represents the set of possible membership values for each x.

Thus, the restricted Hesitant Fuzzy Offset becomes a Hesitant Fuzzy Set. □

Definition 3.22. [77] A *Spherical Fuzzy Set (SFS)* is defined on a universal set X as:

$$S = \{(x, s(x), i(x), d(x)) : x \in X\},$$

where:

- $s(x), i(x), d(x) : X \to [0, 1]$ represent the *degree of membership*, *degree of abstinence*, and *degree of non-membership*, respectively,
- These values satisfy the constraint:

$$s(x)^2 + i(x)^2 + d(x)^2 \leq 1.$$

- The *degree of refusal* $r(x)$ is defined as:

$$r(x) = \sqrt{1 - s(x)^2 - i(x)^2 - d(x)^2}.$$

The triplet $(s(x), i(x), d(x))$ is called a *Spherical Fuzzy Number (SFN)*.

Definition 3.23. A *Spherical Fuzzy Offset (SFO)* is an extension of the SFS, where the membership values can exceed the standard range $[0, 1]$. It is defined on a universal set X as:

$$S_{\text{off}} = \{(x, s_{\text{off}}(x), i_{\text{off}}(x), d_{\text{off}}(x)) : x \in X\},$$

where:

- $s_{\text{off}}(x), i_{\text{off}}(x), d_{\text{off}}(x) : X \to [\Psi, \Omega]$, with $\Psi < 0$ and $\Omega > 1$,
- These values satisfy the generalized constraint:

$$s_{\text{off}}(x)^2 + i_{\text{off}}(x)^2 + d_{\text{off}}(x)^2 \leq \Omega^2,$$

- The *degree of extended refusal* $r_{\text{off}}(x)$ is:

$$r_{\text{off}}(x) = \sqrt{\Omega^2 - s_{\text{off}}(x)^2 - i_{\text{off}}(x)^2 - d_{\text{off}}(x)^2}.$$

Theorem 3.24. *A Spherical Fuzzy Offset (SFO) reduces to a Spherical Fuzzy Set (SFS) when the range of* $s_{\text{off}}(x), i_{\text{off}}(x), d_{\text{off}}(x)$ *is restricted to* $[0, 1]$.

Proof. By definition, the membership values $s_{\text{off}}(x), i_{\text{off}}(x), d_{\text{off}}(x)$ in an SFO satisfy:

$$s_{\text{off}}(x)^2 + i_{\text{off}}(x)^2 + d_{\text{off}}(x)^2 \leq \Omega^2.$$

Restricting $\Psi = 0$ and $\Omega = 1$, the range becomes $[0, 1]$, and the constraint simplifies to:

$$s(x)^2 + i(x)^2 + d(x)^2 \leq 1,$$

which is the condition for an SFS.

Therefore, the SFO reduces to an SFS under this restriction. □

Theorem 3.25. *Both Hesitant Fuzzy Offsets (HFO) and Spherical Fuzzy Offsets (SFO) generalize the Fuzzy Offset.*

Proof. Let $H_{\text{off}} = \{\langle x, h_{\text{off}}(x)\rangle : x \in X\}$ be a Hesitant Fuzzy Offset. For each $x \in X$, $h_{\text{off}}(x) \subseteq [\Psi, \Omega]$, representing a set of possible membership values. If $h_{\text{off}}(x) = \{\mu_{\tilde{A}}(x)\}$, where $\mu_{\tilde{A}}(x) \in [\Psi, \Omega]$, then H_{off} reduces to a Fuzzy Offset $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$.

Therefore, the Hesitant Fuzzy Offset generalizes the Fuzzy Offset by allowing multiple possible membership values for each x.

And let $S_{\text{off}} = \{\langle x, (s(x), i(x), d(x))\rangle : x \in X\}$ be a Spherical Fuzzy Offset, where $s(x), i(x), d(x) \in [\Psi, \Omega]$ satisfy the constraint $s^2(x) + i^2(x) + d^2(x) \leq 1$.

If we set $i(x) = 0$ and $d(x) = 0$ for all $x \in X$, then $s(x) \in [\Psi, \Omega]$, and S_{off} reduces to a Fuzzy Offset $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}}(x) = s(x)$.

Thus, the Spherical Fuzzy Offset generalizes the Fuzzy Offset by incorporating additional components for abstinence ($i(x)$) and non-membership ($d(x)$).

Hence, both Hesitant Fuzzy Offsets and Spherical Fuzzy Offsets extend the concept of Fuzzy Offsets. □

Theorem 3.26. *A Plithogenic Offset with* $s = 3$ *and* $t = 1$ *generalizes both the Hesitant Offset and the Spherical Fuzzy Offset.*

Proof. In a Hesitant Offset, the membership degrees $h_{\text{off}}(x)$ are subsets of $[\Psi, \Omega]$. In a Plithogenic Offset, the Degree of Appurtenance Function (DAF) maps:

$$pdf(x, a) : P \times Pv \to [\Psi, \Omega]^s.$$

By setting $s = 1$, the Plithogenic Offset reduces to a Hesitant Offset, where each $pdf(x, a)$ represents the hesitant membership set $h_{\text{off}}(x)$.

Thus, the Hesitant Offset is a special case of the Plithogenic Offset.

In a Spherical Fuzzy Offset, the triplet $(s_{\text{off}}(x), i_{\text{off}}(x), d_{\text{off}}(x))$ satisfies:

$$s_{\text{off}}(x)^2 + i_{\text{off}}(x)^2 + d_{\text{off}}(x)^2 \leq \Omega^2.$$

In a Plithogenic Offset, the DAF maps x to a set of s-tuples in $[\Psi, \Omega]^s$. By setting $s = 3$, the DAF can encode the triplet $(s_{\text{off}}(x), i_{\text{off}}(x), d_{\text{off}}(x))$.

By imposing the spherical constraint on the DAF, the Plithogenic Offset reduces to a Spherical Fuzzy Offset.

A Plithogenic Offset with $s = 3$ allows:

- Triplet encoding for spherical membership (Spherical Fuzzy Offset),
- Hesitant membership for each component of the triplet (Hesitant Offset).

This dual flexibility unifies both structures. □

3.5 Fuzzy Offmatroid

The definition of the Fuzzy Offmatroid is provided below. It is a concept derived by incorporating the notion of offset into fuzzy matroids.

Definition 3.27. [71,72,175,176] Let $F(Y)$ denote the power set of fuzzy subsets on Y. A pair $M = (X, I)$ is called a *fuzzy matroid* if $I \subseteq F(Y)$ satisfies the following conditions:

1. If $\tau_1 \in I$ and $\tau_2 \subset \tau_1$, then $\tau_2 \in I$, where $\tau_2 \subset \tau_1$ means that $\tau_2(y) < \tau_1(y)$ for every $y \in X$. 2. If $\tau_1, \tau_2 \in I$ and $|\operatorname{supp}(\tau_1)| < |\operatorname{supp}(\tau_2)|$, then there exists $\tau_3 \in I$ such that:

- $\tau_1 \subset \tau_3 \subseteq \tau_1 \cup \tau_2$, where for any $y \in X$, $\tau_1 \cup \tau_2(y) = \max\{\tau_1(y), \tau_2(y)\}$,
- $m(\tau_3) \geq \min\{m(\tau_1), m(\tau_2)\}$, where $m(v) = \min\{v(y) : y \in \operatorname{supp}(v)\}$.

Here, I is called the *family of independent fuzzy sets* of the fuzzy matroid $M = (X, I)$.

Definition 3.28 (Fuzzy Offmatroid)**.** Let X be a finite ground set, and $F(X)$ denote the power set of fuzzy subsets on X. A pair $M_{\text{off}} = (X, I_{\text{off}})$ is called a *Fuzzy Offmatroid* if $I_{\text{off}} \subseteq F(X)$ satisfies the following conditions:

1. (*Hereditary Property*): If $\tau_1 \in I_{\text{off}}$ and $\tau_2 \subset \tau_1$, then $\tau_2 \in I_{\text{off}}$, where $\tau_2 \subset \tau_1$ means $\tau_2(x) \leq \tau_1(x)$ for all $x \in X$.
2. (*Exchange Property*): If $\tau_1, \tau_2 \in I_{\text{off}}$ and $|\operatorname{supp}(\tau_1)| < |\operatorname{supp}(\tau_2)|$, then there exists $\tau_3 \in I_{\text{off}}$ such that:
   (a) $\tau_1 \subset \tau_3 \subseteq \tau_1 \cup \tau_2$, where $(\tau_1 \cup \tau_2)(x) = \max\{\tau_1(x), \tau_2(x)\}$ for all $x \in X$,
   (b) $m(\tau_3) \geq \min\{m(\tau_1), m(\tau_2)\}$, where $m(\tau) = \min\{\tau(x) : x \in \operatorname{supp}(\tau)\}$,
   (c) $\tau_3(x) \in [\Psi, \Omega]$ for all $x \in X$, where $\Psi < 0$ and $\Omega > 1$.

Here, I_{off} is called the *family of independent fuzzy offsets* of the Fuzzy Offmatroid $M_{\text{off}} = (X, I_{\text{off}})$.

Theorem 3.29. *A Fuzzy Offmatroid is a Fuzzy Offset.*

Proof. Let $M_{\text{off}} = (X, I_{\text{off}})$ be a Fuzzy Offmatroid. By definition of a Fuzzy Offset:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [\Psi, \Omega]\}.$$

Each fuzzy set $\tau \in I_{\text{off}}$ is defined by its membership function $\tau(x) \in [\Psi, \Omega]$. The hereditary property ensures that I_{off} is closed under restriction to subsets, and the exchange property ensures that I_{off} satisfies the structural conditions of independence within the extended membership range $[\Psi, \Omega]$.

Thus, I_{off} defines a family of fuzzy subsets within the range $[\Psi, \Omega]$, making M_{off} a special case of a Fuzzy Offset. □

Theorem 3.30. *If a Fuzzy Offmatroid $M_{off} = (X, I_{off})$ is restricted such that $\tau(x) \in [0, 1]$ for all $\tau \in I_{off}$ and $x \in X$, then M_{off} becomes a Fuzzy Matroid $M = (X, I)$.*

Proof. Let $M_{\text{off}} = (X, I_{\text{off}})$ be a Fuzzy Offmatroid. By definition, I_{off} satisfies the hereditary and exchange properties, with the membership degrees $\tau(x)$ belonging to the range $[\Psi, \Omega]$, where $\Psi < 0$ and $\Omega > 1$.

Now, impose the restriction $\tau(x) \in [0, 1]$ for all $\tau \in I_{\text{off}}$ and $x \in X$:

- Under this restriction, the hereditary property of I_{off} is preserved because the condition $\tau_2(x) \leq \tau_1(x)$ for all $x \in X$ remains valid within $[0, 1]$.
- Similarly, the exchange property is preserved because $\tau_3(x) \in [0, 1]$ for all $x \in X$, and the minimum and maximum operations used in the definition of τ_3 do not exceed the range $[0, 1]$.

Thus, under the restriction $\tau(x) \in [0, 1]$, the family I_{off} satisfies the axioms of a Fuzzy Matroid:

- (*Hereditary Property*): If $\tau_1 \in I$ and $\tau_2 \subset \tau_1$, then $\tau_2 \in I$, where $I = I_{\text{off}} \cap F(X)$ with $F(X)$ restricted to fuzzy sets with membership degrees in $[0, 1]$.
- (*Exchange Property*): If $\tau_1, \tau_2 \in I$ and $|\operatorname{supp}(\tau_1)| < |\operatorname{supp}(\tau_2)|$, then there exists $\tau_3 \in I$ satisfying:

$$\tau_1 \subset \tau_3 \subseteq \tau_1 \cup \tau_2 \quad \text{and} \quad m(\tau_3) \geq \min\{m(\tau_1), m(\tau_2)\}.$$

Therefore, the restricted Fuzzy Offmatroid $M_{\text{off}} = (X, I_{\text{off}})$ becomes a Fuzzy Matroid $M = (X, I)$ when $\tau(x) \in [0, 1]$. □

3.6 Plithogenic Off Matroid

Define the Plithogenic Matroid and Plithogenic Off Matroid as follows. These are concepts extended using the Plithogenic Set and Plithogenic Offset.

Definition 3.31 (Plithogenic Matroid)**.** Let X be a finite ground set, and s, t be the degrees of appurtenance and contradiction, respectively. A *Plithogenic Matroid* $M = (X, I)$ is defined by a family of *plithogenic sets* $I \subseteq F(X)$, where $F(X)$ is the set of all plithogenic functions $\mu : X \to [0, 1]^s$, satisfying:

1. (*Hereditary Property*) If $\mu_1 \in I$ and $\mu_2 \leq \mu_1$ (component-wise), then $\mu_2 \in I$.
2. (*Exchange Property*) If $\mu_1, \mu_2 \in I$ and $|\operatorname{supp}(\mu_1)| < |\operatorname{supp}(\mu_2)|$, then there exists $\mu_3 \in I$ such that:

$$\mu_1 \leq \mu_3 \leq \mu_1 \cup \mu_2,$$

where $(\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\}$ for all $x \in X$.

Theorem 3.32. *A Plithogenic Matroid is a Plithogenic Set.*

Proof. By definition, a Plithogenic Matroid $M = (X, I)$ consists of a finite ground set X and a family I of plithogenic sets.

The structure I is defined using the Degree of Appurtenance Function (DAF), $pdf : X \to [0, 1]^s$, and Degree of Contradiction Function (DCF), $pCF : [0, 1]^s \to [0, 1]^t$, which are central to the definition of Plithogenic Sets. Thus, the Plithogenic Matroid is a specific instance of a Plithogenic Set, where the family I satisfies additional hereditary and exchange properties. Therefore, M is a Plithogenic Set. □

Theorem 3.33. *A Plithogenic Matroid with $s = 1$ and $t = 1$ reduces to a Fuzzy Matroid.*

Proof. Let $M = (X, I)$ be a Plithogenic Matroid. When $s = 1$ and $t = 1$, each plithogenic function $\mu \in I$ is reduced to:

$$\mu : X \to [0, 1],$$

where $\mu(x)$ is the fuzzy membership value. The hereditary and exchange properties of the Plithogenic Matroid align with those of the Fuzzy Matroid, making M equivalent to a Fuzzy Matroid. □

Definition 3.34 (Plithogenic Off Matroid)**.** Let X be a finite ground set, and s, t be the degrees of appurtenance and contradiction, respectively. A *Plithogenic Off Matroid* $M_{\text{off}} = (X, I_{\text{off}})$ is defined by a family of *plithogenic offset sets* $I_{\text{off}} \subseteq F_{\text{off}}(X)$, where $F_{\text{off}}(X)$ is the set of all plithogenic functions $\mu : X \to [\Psi, \Omega]^s$, satisfying:

1. (*Hereditary Property*) If $\mu_1 \in I_{\text{off}}$ and $\mu_2 \leq \mu_1$ (component-wise), then $\mu_2 \in I_{\text{off}}$.
2. (*Exchange Property*) If $\mu_1, \mu_2 \in I_{\text{off}}$ and $|\operatorname{supp}(\mu_1)| < |\operatorname{supp}(\mu_2)|$, then there exists $\mu_3 \in I_{\text{off}}$ such that:

   $$\mu_1 \leq \mu_3 \leq \mu_1 \cup \mu_2,$$

   where $(\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\}$ for all $x \in X$.

Example 3.35. (cf. [59]) The following examples of Plithogenic Off Matroids are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy off Matroids*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy off Matroids*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic off Matroids*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic off Matroids*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic off Matroids*.
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic off Matroids*.
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic off Matroids*.
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic off Matroids*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic off Matroids*.

Corollary 3.36. *A Plithogenic OffMatroid is a Plithogenic OffSet.*

Proof. The proof can be established using the same method as before. □

Theorem 3.37. *A Plithogenic Off Matroid with $s = 1$ and $t = 1$ reduces to a Fuzzy Off Matroid.*

Proof. Let $M_{\text{off}} = (X, I_{\text{off}})$ be a Plithogenic Off Matroid. When $s = 1$ and $t = 1$, each plithogenic offset function $\mu \in I_{\text{off}}$ is reduced to:

$$\mu : X \to [\Psi, \Omega],$$

where $\Psi < 0$ and $\Omega > 1$ define the extended range of membership. The hereditary and exchange properties of M_{off} align with those of the Fuzzy Off Matroid, making M_{off} equivalent to a Fuzzy Off Matroid. □

Theorem 3.38. *If a Plithogenic Off Matroid $M_{off} = (X, I_{off})$ is restricted such that $\mu(x) \in [0, 1]$ for all $\mu \in I_{off}$ and $x \in X$, then M_{off} becomes a Plithogenic Matroid $M = (X, I)$.*

Proof. Let $M_{\text{off}} = (X, I_{\text{off}})$ be a Plithogenic Off Matroid. By definition, the membership values $\mu(x) \in [\Psi, \Omega]$ satisfy the hereditary and exchange properties.

Restricting $\mu(x) \in [0, 1]$:

- The hereditary property $\mu_2 \leq \mu_1$ is preserved because the restriction does not alter the ordering of membership values.
- The exchange property $\mu_1 \leq \mu_3 \leq \mu_1 \cup \mu_2$ is preserved because the maximum operation remains valid within $[0, 1]$.

Thus, the restricted family I_{off} satisfies the axioms of a Plithogenic Matroid $M = (X, I)$. □

In the future, we hope to see further research and applications related to antimatroids [37, 128] quasi-matroids [102, 103], ultra matroids [56], feeble matroid [9], k-balanced matroids [8], and greedoids [108], in connection with the aforementioned matroids.

3.7 Fuzzy Off Ultrafilter

The definition of the Fuzzy Off Ultrafilter is provided below. The Fuzzy Off Ultrafilter is a concept that extends the Fuzzy Ultrafilter by incorporating the idea of a Fuzzy Offset.

Definition 3.39 (Fuzzy Ultrafilter)**.** [33, 158, 203, 219] Let X be a non-empty set, and let $I = [0, 1]$ denote the closed unit interval. A *fuzzy set* on X is a function $\mu : X \to I$. A *fuzzy ultrafilter* $\mathcal{U} \subseteq \mathcal{P}(I^X)$ is defined as a maximal fuzzy filter satisfying the following conditions:

1. $(U1)$ $\mathcal{U}$ is closed under finite intersections:
$$\mu, \nu \in \mathcal{U} \implies \mu \wedge \nu \in \mathcal{U},$$
where $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$.
2. $(U2)$ The constant zero function $0 \notin \mathcal{U}$.
3. $(U3)$ For every fuzzy set $\mu \in I^X$, either $\mu \in \mathcal{U}$ or $1 - \mu \in \mathcal{U}$, where $(1 - \mu)(x) = 1 - \mu(x)$ for all $x \in X$.
4. $(U4)$ If $\mu, \nu \in I^X$ and $\mu \vee \nu \in \mathcal{U}$, then either $\mu \in \mathcal{U}$ or $\nu \in \mathcal{U}$, where $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in X$.

Definition 3.40 (Fuzzy Off Ultrafilter)**.** Let X be a non-empty set, and let $[\Psi, \Omega]$ be an extended interval where $\Psi < 0$ and $\Omega > 1$. A *fuzzy off ultrafilter* $\mathcal{U}_{\text{off}} \subseteq \mathcal{P}([\Psi, \Omega]^X)$ is defined as a maximal fuzzy filter on $[\Psi, \Omega]^X$, satisfying:

1. $(O1)$ $\mathcal{U}_{\text{off}}$ is closed under finite intersections:
$$\mu, \nu \in \mathcal{U}_{\text{off}} \implies \mu \wedge \nu \in \mathcal{U}_{\text{off}},$$
where $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$.
2. $(O2)$ The constant zero function $0 \notin \mathcal{U}_{\text{off}}$.
3. $(O3)$ For every fuzzy set $\mu \in [\Psi, \Omega]^X$, either $\mu \in \mathcal{U}_{\text{off}}$ or $\Omega - \mu \in \mathcal{U}_{\text{off}}$, where $(\Omega - \mu)(x) = \Omega - \mu(x)$ for all $x \in X$.
4. $(O4)$ If $\mu, \nu \in [\Psi, \Omega]^X$ and $\mu \vee \nu \in \mathcal{U}_{\text{off}}$, then either $\mu \in \mathcal{U}_{\text{off}}$ or $\nu \in \mathcal{U}_{\text{off}}$, where $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in X$.

Theorem 3.41. *A Fuzzy Off Ultrafilter is a Fuzzy Offset.*

Proof. By definition, a Fuzzy Off Ultrafilter $\mathcal{U}_{\text{off}}$ is a maximal fuzzy filter on $[\Psi, \Omega]^X$, where $\Psi < 0$ and $\Omega > 1$. To prove that $\mathcal{U}_{\text{off}}$ is a Fuzzy Offset, we verify that it satisfies the properties of a Fuzzy Offset as defined below:

1. *Membership values in the extended interval*: For any $\mu \in \mathcal{U}_{\text{off}}$, $\mu(x) \in [\Psi, \Omega]$ for all $x \in X$, by the definition of $[\Psi, \Omega]^X$. This matches the condition for a Fuzzy Offset.
2. *Existence of values exceeding standard bounds*: By the properties of the extended interval, there exist $\mu \in \mathcal{U}_{\text{off}}$ such that $\mu(x) > 1$ or $\mu(x) < 0$ for some $x \in X$, as required by the definition of a Fuzzy Offset.
3. *Maximality and closure under operations*: The closure properties ($O1$-$O4$) ensure that $\mathcal{U}_{\text{off}}$ adheres to the operations consistent with fuzzy logic, including closure under minimum ($\wedge$) and maximum ($\vee$) operations and the inclusion of complements. This aligns with the operations defined for Fuzzy Offsets.

Thus, $\mathcal{U}_{\text{off}}$ satisfies all the conditions of a Fuzzy Offset, proving the theorem. □

Theorem 3.42. *A fuzzy off ultrafilter $\mathcal{U}_{off}$ restricted to $\mu(x) \in [0, 1]$ for all $\mu \in \mathcal{U}_{off}$ and $x \in X$ reduces to a fuzzy ultrafilter $\mathcal{U}$.*

Proof. Let $\mathcal{U}_{\text{off}} \subseteq [\Psi, \Omega]^X$ be a fuzzy off ultrafilter. Under the restriction $\mu(x) \in [0, 1]$ for all $x \in X$, the membership degrees of all fuzzy sets in $\mathcal{U}_{\text{off}}$ fall within the interval $[0, 1]$. This restriction ensures:

1. ($U1$) Closure under finite intersections is preserved, as $\mu \wedge \nu \in \mathcal{U}_{\text{off}}$ for all $\mu, \nu \in \mathcal{U}_{\text{off}}$.
2. ($U2$) The zero function $0 \notin \mathcal{U}_{\text{off}}$.
3. ($U3$) For every $\mu \in I^X$, either $\mu \in \mathcal{U}_{\text{off}}$ or $1 - \mu \in \mathcal{U}_{\text{off}}$ (as $\Omega - \mu$ reduces to $1 - \mu$ under the restriction $\mu(x) \in [0, 1]$).
4. ($U4$) Maximality ensures that if $\mu \vee \nu \in \mathcal{U}_{\text{off}}$, then either $\mu \in \mathcal{U}_{\text{off}}$ or $\nu \in \mathcal{U}_{\text{off}}$.

Thus, $\mathcal{U}_{\text{off}}$ restricted to $[0, 1]^X$ satisfies all the axioms of a fuzzy ultrafilter $\mathcal{U}$, completing the proof. □

3.8 Neutrosophic Off Ultrafilter and Plithogenic Off Ultrafilter

The Neutrosophic Off Ultrafilter is a concept that extends the Neutrosophic Ultrafilter by incorporating the idea of an offset. We first introduce the definition of a Neutrosophic Ultrafilter [93, 141, 171, 172, 226], followed by a discussion of the Neutrosophic Off Ultrafilter. The relevant definitions are provided below.

Definition 3.43 (Neutrosophic Filter)**.** [171] Let X be a non-empty set, and let A be a neutrosophic set on X, represented as $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$, where:

- $T_A(x)$ is the truth-membership degree,
- $I_A(x)$ is the indeterminacy-membership degree,
- $F_A(x)$ is the falsity-membership degree.

A *neutrosophic filter* $\mathcal{N} \subseteq \mathcal{P}(X)$ satisfies the following conditions:

1. ($N1$) Every neutrosophic set $A \subseteq X$ containing a member of $\mathcal{N}$ belongs to $\mathcal{N}$.
2. ($N2$) Every finite intersection of members of $\mathcal{N}$ belongs to $\mathcal{N}$.
3. ($N3$) The neutrosophic zero set $\mathbf{0}_N = \{(x, 0, 1, 1) \mid x \in X\} \notin \mathcal{N}$.

Definition 3.44 (Neutrosophic Ultrafilter)**.** [171] A *neutrosophic ultrafilter* $\mathcal{N}_U$ on a set X is a neutrosophic filter on X that is maximal, meaning there is no neutrosophic filter on X strictly finer than $\mathcal{N}_U$.

Formally, $\mathcal{N}_U$ satisfies:

1. For every neutrosophic set $A \subseteq X$, either $A \in \mathcal{N}_U$ or $\neg A \in \mathcal{N}_U$, where $\neg A = \{(x, F_A(x), I_A(x), T_A(x)) \mid x \in X\}$.
2. If $A, B \in \mathcal{N}_U$, then $A \cap B \in \mathcal{N}_U$.
3. If $A \cup B \in \mathcal{N}_U$, then $A \in \mathcal{N}_U$ or $B \in \mathcal{N}_U$.

Corollary 3.45. *A Neutrosophic Ultrafilter is a Neutrosophic Set.*

Proof. The proof can be established using the same method as before. □

Definition 3.46 (Neutrosophic Off Ultrafilter)**.** Let X be a non-empty set, and let $[\Psi, \Omega]$ define the extended interval with $\Psi < 0$ and $\Omega > 1$. A neutrosophic set A on X is represented as:

$$A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\},$$

where:

- $T_A(x) \in [\Psi, \Omega]$ is the truth-membership degree,
- $I_A(x) \in [\Psi, \Omega]$ is the indeterminacy-membership degree,
- $F_A(x) \in [\Psi, \Omega]$ is the falsity-membership degree.

A *neutrosophic off ultrafilter* $\mathcal{N}_{\text{off}}$ on X is defined as a maximal neutrosophic filter $\mathcal{N} \subseteq \mathcal{P}(X)$ satisfying:

1. $(N1_{\text{off}})$ For every neutrosophic set $A \subseteq X$, either $A \in \mathcal{N}_{\text{off}}$ or $\neg A \in \mathcal{N}_{\text{off}}$, where $\neg A = \{(x, F_A(x), I_A(x), T_A(x)) \mid x \in X\}$.
2. $(N2_{\text{off}})$ For every $A, B \in \mathcal{N}_{\text{off}}$, $A \cap B \in \mathcal{N}_{\text{off}}$.
3. $(N3_{\text{off}})$ The neutrosophic zero set $\mathbf{0}_{\text{off}} = \{(x, \Psi, \Omega, \Omega) \mid x \in X\} \notin \mathcal{N}_{\text{off}}$.

Corollary 3.47. *A Neutrosophic Off Ultrafilter is a Neutrosophic OffSet.*

Proof. The proof can be established using the same method as before. □

Theorem 3.48. *A Neutrosophic Off Ultrafilter generalizes a Fuzzy Off Ultrafilter.*

Proof. Let $\mathcal{F}_{\text{off}}$ be a Fuzzy Off Ultrafilter defined over a set X with membership degrees $\mu(x) \in [\Psi, \Omega]$. Consider a Neutrosophic Off Ultrafilter $\mathcal{N}_{\text{off}}$ such that:

$$\mathcal{N}_{\text{off}} = \{A = (x, T_A(x), I_A(x), F_A(x)) \mid T_A(x), I_A(x), F_A(x) \in [\Psi, \Omega]\}.$$

Each fuzzy set $\mu \in \mathcal{F}_{\text{off}}$ can be represented as a neutrosophic set with $T_A(x) = \mu(x), I_A(x) = 0, F_A(x) = 1 - \mu(x)$. Thus, $\mathcal{F}_{\text{off}}$ is a special case of $\mathcal{N}_{\text{off}}$, making $\mathcal{N}_{\text{off}}$ a generalization of $\mathcal{F}_{\text{off}}$. □

Theorem 3.49. *If a Neutrosophic Off Ultrafilter $\mathcal{N}_{off}$ is restricted such that $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for all $x \in X$, it becomes a Neutrosophic Ultrafilter $\mathcal{N}_U$.*

Proof. Let $\mathcal{N}_{\text{off}}$ be a Neutrosophic Off Ultrafilter on X. Restrict the neutrosophic components $T_A(x), I_A(x), F_A(x)$ to the standard interval $[0, 1]$. Under this restriction:

1. $T_A(x), I_A(x), F_A(x) \in [0, 1]$ ensures compatibility with the neutrosophic ultrafilter definition.
2. $(N1)$ For every neutrosophic set $A \subseteq X$, either $A \in \mathcal{N}_U$ or $\neg A \in \mathcal{N}_U$, where $\neg A = \{(x, F_A(x), I_A(x), T_A(x)) \mid x \in X\}$.
3. $(N2)$ For every $A, B \in \mathcal{N}_U$, $A \cap B \in \mathcal{N}_U$.
4. $(N3)$ The neutrosophic zero set $\mathbf{0}_N = \{(x, 0, 1, 1) \mid x \in X\} \notin \mathcal{N}_U$.

Thus, under the restriction $[\Psi, \Omega] \to [0, 1]$, $\mathcal{N}_{\text{off}}$ satisfies all the conditions for being a Neutrosophic Ultrafilter $\mathcal{N}_U$. □

Definition 3.50 (Plithogenic Ultrafilter)**.** Let X be a universal set, and let s and t denote the number of degrees of appurtenance and contradiction, respectively. A *Plithogenic Ultrafilter* $\mathcal{P} \subseteq [0, 1]^X$ is defined as a maximal collection of plithogenic sets satisfying:

1. (*Hereditary Property*) For every $\mu, \nu \in \mathcal{P}$, $\mu \wedge \nu \in \mathcal{P}$.
2. (*Maximality*) For any plithogenic set $\mu \in [0, 1]^X$, either $\mu \in \mathcal{P}$ or $1 - \mu \in \mathcal{P}$, where $(1 - \mu)(x) = 1 - \mu(x)$ for all $x \in X$.
3. (*Non-emptiness*) The zero function $\mathbf{0} \notin \mathcal{P}$.

Here, $\mu \in [0, 1]^X$ is a plithogenic function characterized by s-valued appurtenance degrees and t-valued contradiction functions.

Corollary 3.51. *A Plithogenic Ultrafilter is a Plithogenic Set.*

Proof. The proof can be established using the same method as before. □

Definition 3.52 (Plithogenic Off Ultrafilter)**.** Let X be a universal set, and let s and t denote the number of degrees of appurtenance and contradiction, respectively. A *Plithogenic Off Ultrafilter* $\mathcal{P}_{\text{off}} \subseteq [\Psi, \Omega]^X$ is defined as a maximal collection of plithogenic offset sets satisfying:

1. (*Hereditary Property*) For every $\mu, \nu \in \mathcal{P}_{\text{off}}$, $\mu \wedge \nu \in \mathcal{P}_{\text{off}}$.
2. (*Maximality*) For any plithogenic offset set $\mu \in [\Psi, \Omega]^X$, either $\mu \in \mathcal{P}_{\text{off}}$ or $\Omega - \mu \in \mathcal{P}_{\text{off}}$, where $(\Omega - \mu)(x) = \Omega - \mu(x)$ for all $x \in X$.
3. (*Non-emptiness*) The zero function $\mathbf{0}_{\text{off}} \notin \mathcal{P}_{\text{off}}$.

Example 3.53. (cf. [59]) The following examples of Plithogenic Off Ultrafilter are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy off Ultrafilter*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy off Ultrafilter*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic off Ultrafilter*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic off Ultrafilter*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic off Ultrafilter*.
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic off Ultrafilter*.
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic off Ultrafilter*.
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic off Ultrafilter*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic off Ultrafilter*.

Corollary 3.54. *A Plithogenic Off Ultrafilter is a Plithogenic OffSet.*

Proof. The proof can be established using the same method as before. □

Theorem 3.55. *A Plithogenic Ultrafilter with* $s = 3$ *and* $t = 1$ *reduces to a Neutrosophic Ultrafilter, and a Plithogenic Ultrafilter with* $s = 1$ *and* $t = 1$ *reduces to a Fuzzy Ultrafilter.*

Proof. Let $\mathcal{P}$ be a Plithogenic Ultrafilter. When $s = 3$ and $t = 1$, each plithogenic set in $\mathcal{P}$ can be represented as:

$$\mu(x) = (T(x), I(x), F(x)) \in [0, 1]^3,$$

where $T(x)$, $I(x)$, and $F(x)$ represent truth, indeterminacy, and falsity, respectively. This corresponds to the definition of a Neutrosophic Ultrafilter.

When $s = 1$ and $t = 1$, each plithogenic set in $\mathcal{P}$ reduces to:

$$\mu(x) \in [0, 1],$$

which matches the definition of a Fuzzy Ultrafilter. Therefore, Plithogenic Ultrafilters generalize Neutrosophic and Fuzzy Ultrafilters. □

Theorem 3.56. *A Plithogenic Off Ultrafilter with* $s = 3$ *and* $t = 1$ *reduces to a Neutrosophic Off Ultrafilter, and a Plithogenic Off Ultrafilter with* $s = 1$ *and* $t = 1$ *reduces to a Fuzzy Off Ultrafilter.*

Proof. Let $\mathcal{P}_{\text{off}}$ be a Plithogenic Off Ultrafilter. When $s = 3$ and $t = 1$, each plithogenic offset set in $\mathcal{P}_{\text{off}}$ can be represented as:

$$\mu(x) = (T(x), I(x), F(x)) \in [\Psi, \Omega]^3,$$

which corresponds to the definition of a Neutrosophic Off Ultrafilter. Similarly, when $s = 1$ and $t = 1$, the plithogenic offset set reduces to:

$$\mu(x) \in [\Psi, \Omega],$$

matching the definition of a Fuzzy Off Ultrafilter. Thus, Plithogenic Off Ultrafilters generalize Neutrosophic and Fuzzy Off Ultrafilters. □

Theorem 3.57. *If a Plithogenic Off Ultrafilter* $\mathcal{P}_{\text{off}}$ *is restricted such that* $\mu(x) \in [0, 1]$ *for all* $\mu \in \mathcal{P}_{\text{off}}$ *and* $x \in X$*, it becomes a Plithogenic Ultrafilter* $\mathcal{P}$*.*

Proof. Let $\mathcal{P}_{\text{off}} \subseteq [\Psi, \Omega]^X$ be a Plithogenic Off Ultrafilter. Under the restriction $\mu(x) \in [0, 1]$ for all $\mu \in \mathcal{P}_{\text{off}}$, the hereditary property and maximality are preserved because:

- For any $\mu, \nu \in \mathcal{P}_{\text{off}}$, $\mu \wedge \nu \in \mathcal{P}_{\text{off}}$ remains valid within $[0, 1]$.
- For any $\mu \in [0, 1]^X$, either $\mu \in \mathcal{P}$ or $1 - \mu \in \mathcal{P}$.

Thus, under this restriction, $\mathcal{P}_{\text{off}}$ satisfies all conditions for being a Plithogenic Ultrafilter. □

In the future, we hope to see further research on extending weak-ultrafilters [109–111] and quasi-ultrafilters [31] to uncertain offsets, as well as exploring their potential applications.

4 Discussion

4.1 Offset Framework (Extension from Uncertain Sets to Offsets)

We are considering an Offset Framework that transforms general Uncertain sets into their Offset counterparts. Please note that this framework is not intended to provide exhaustive definitions or a purely mathematical treatment.

There may be additional perspectives or considerations beyond what is presented here, and we hope that capable readers will further refine and enhance the framework as needed.

Note 1 (Offset Framework)**.** *Let X be a universal set, and let U denote an uncertain set defined on X with a membership function $\mu_U : X \to [0, 1]$. The Offset Framework extends the standard uncertain set U into an Offset Uncertain Set, denoted U_{off}, by modifying the range of the membership function.*

$$U_{off} = \{(x, \mu_{off}(x)) \mid x \in X,\ \mu_{off}(x) \in [\Psi, \Omega]\}$$

where:

- $\mu_{off} : X \to [\Psi, \Omega]$ *is the offset membership function, such that* $\Psi < 0$ *and* $\Omega > 1$ *define the extended limits of uncertainty.*
- Ψ *(UnderLimit): Allows membership values to extend below 0, representing negative degrees of membership.*
- Ω *(OverLimit): Allows membership values to extend above 1, representing degrees of membership greater than certainty.*

The transformation from U to U_{off} is defined by the following mapping:

$$\mu_{off}(x) = f(\mu_U(x)) \quad \forall x \in X,$$

where $f : [0, 1] \to [\Psi, \Omega]$ is a continuous scaling or shifting function satisfying:

1. $f(0) = \Psi$ *(maps minimum membership to the UnderLimit).*
2. $f(1) = \Omega$ *(maps maximum membership to the OverLimit).*
3. f *is monotonic and preserves the order of membership values.*

Special Cases

1. *When $\Psi = 0$ and $\Omega = 1$, U_{off} reduces to the original uncertain set U.*
2. *When $\Psi = -\infty$ or $\Omega = \infty$, U_{off} represents an unbounded uncertain set.*
3. *When $\Psi = 0$, U_{off} becomes an OverSet, allowing membership values to extend above 1 while retaining the lower limit of 0.*
4. *When $\Omega = 1$, U_{off} becomes an UnderSet, allowing membership values to extend below 0 while retaining the upper limit of 1.*
5. *When $\Psi = c$ and $\Omega = d$, where $c, d \in \mathbb{R}$, U_{off} defines a Shifted Set with fixed bounds determined by c and d.*

4.2 Other Off Concepts

The concept of Offset may potentially be applicable to frameworks beyond Fuzzy Sets and Neutrosophic Sets. An example is provided below. While related concepts and applications may already exist, this serves as an illustrative example of the approach.

4.2.1 Probability Off Distribution

A Probability Distribution is a mathematical function that describes the likelihood of different outcomes in a random experiment, assigning probabilities to each possible event in a defined sample space.

Definition 4.1 (Probability Distribution)**.** (cf. [36, 169, 170]) A *Probability Distribution* is a function P defined on a sample space Ω that assigns a probability $P(A)$ to each event $A \subseteq \Omega$, satisfying the following axioms:

1. $P(A) \geq 0 \quad \forall A \subseteq \Omega$ (non-negativity),
2. $P(\Omega) = 1$ (normalization),
3. $P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$ for any disjoint events A_i (additivity).

For discrete random variables, the probability distribution is represented by a *Probability Mass Function (PMF)* $p(x)$, where $p(x) = P(X = x)$. For continuous random variables, it is described by a *Probability Density Function (PDF)* $f(x)$, where the probability of an interval is given by:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

Definition 4.2 (Probability Off Distribution)**.** A *Probability Distribution with [Ψ, Ω] and c* extends the classical probability distribution by introducing a constant c such that the total probability satisfies:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = c,$$

where $c \in [\Psi, \Omega]$, with $\Psi < 0$ and $\Omega > 1$.

For a random variable X, the probability density function $f_X(x) : \mathbb{R} \to [\Psi, \Omega]$ must satisfy:

$$\Psi \leq f_X(x) \leq \Omega, \quad \forall x \in \mathbb{R}.$$

This definition extends the classical probability distribution by allowing the integral to sum to a value $c \neq 1$.

4.2.2 Sugeno Off Measure

Sugeno Measure is a non-additive function that evaluates the overlap among sets and is widely applied in fuzzy logic and decision-making processes [75, 114, 205, 217]. Sugeno Measure has also been extensively studied in the contexts of Fuzzy Sets and Neutrosophic Sets [74, 78, 220]. While still in the conceptual stage, its definition is outlined below.

Definition 4.3 (Sugeno Measure)**.** [205] The *Sugeno Measure* is a non-additive set function $g : 2^X \to [0, 1]$ defined on the power set of a finite set $X = \{x_1, x_2, \ldots, x_n\}$. It satisfies the following axioms:

1. **Boundary Conditions:**
$$g(\emptyset) = 0, \quad g(X) = 1.$$
2. **Monotonicity:** For all $A, B \subseteq X$, if $A \subseteq B$, then:
$$g(A) \leq g(B).$$

3. **Non-Additivity:** The measure is not necessarily additive, meaning $g(A \cup B) \neq g(A) + g(B)$ in general, but it satisfies the following pseudo-additivity condition for $A, B \subseteq X$:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B),$$

where $\lambda \geq -1$ is a parameter controlling the interaction between A and B.

The Sugeno Measure is widely used in decision-making, fuzzy integrals, and multi-criteria analysis when independence assumptions are not valid.

Definition 4.4 (Sugeno Off Measure with $[\Psi, \Omega)$**.**] A *Sugeno Measure with $[\Psi, \Omega]$* generalizes the classical measure by allowing the capacity function g to take values in $[\Psi, \Omega]$.

Let X be a finite universal set, and let $g : 2^X \to [\Psi, \Omega]$ be a monotone set function satisfying:

$$g(\emptyset) = \Psi, \quad g(X) = \Omega.$$

The measure satisfies:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B), \quad A \cap B = \emptyset,$$

where $\lambda \in [-1, \infty)$.

4.2.3 Z-Off Number

A Z-Number represents uncertain information by combining a fuzzy value (e.g., "approximately 50") with its associated reliability (e.g., "high confidence") [14,136,161,209,218,234]. Z-Numbers are frequently discussed in the context of fuzzy and neutrosophic concepts. Furthermore, extended notions such as the Neutrosophic Z-Number [47, 101, 225] and Z-Hyper Number [62] have also been introduced.

Definition 4.5. [234] A *Z-Number* is a pair $Z = (A, B)$, where:

- A is a fuzzy subset of the real numbers $\mathbb{R}$, representing an **uncertain value**.
- B is a fuzzy subset of the interval $[0, 1]$, representing the **reliability or certainty** of A.

Explanation

- The first component A captures a linguistic or fuzzy representation of a value, such as "approximately 50."
- The second component B quantifies the degree of confidence or reliability in A, such as "high confidence."

Definition 4.6 (Z-Off Number)**.** A *Z-number with $[\Psi, \Omega]$ and c* generalizes the Z-number concept by introducing a reliability constant $c \in [\Psi, \Omega]$, representing the overall reliability level.

A Z-number $Z = (A, B)$ consists of:

1. A fuzzy number A, representing the constraint.
2. A reliability function $B(x) : X \to [\Psi, \Omega]$, satisfying:

$$\int_X B(x)\, dx = c.$$

Here, $B(x)$ may take values less than 0 or greater than 1, and the total reliability aligns with the constant c.

4.2.4 Jaccard Off Index

The Jaccard Index measures similarity between two sets by dividing the number of shared elements by the total unique elements across both sets [38, 49, 54, 113].

Definition 4.7. (cf. [38, 49, 54, 113]) Let A and B be two finite sets. The Jaccard Index, denoted as $J(A, B)$, is defined as:

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

where:

- $|A \cap B|$ is the cardinality of the intersection of A and B,
- $|A \cup B|$ is the cardinality of the union of A and B.

Properties

- $0 \leq J(A, B) \leq 1$:
  - $J(A, B) = 1$ if A and B are identical.
  - $J(A, B) = 0$ if A and B have no elements in common.
- The Jaccard Index is symmetric: $J(A, B) = J(B, A)$.

The Jaccard Off Index is defined using the concept of Fuzzy Offset. The definition is provided below. It is anticipated that future research will explore the mathematical structures related to these concepts.

Definition 4.8 (Jaccard Off Index)**.** The *Jaccard Off Index with* $[\Psi, \Omega]$ *and* c generalizes the classical Jaccard Index by introducing a scaling constant c, where $c \in [\Psi, \Omega]$. Let A and B be two fuzzy sets defined on a universe X with membership functions $\mu_A, \mu_B : X \to [0, 1]$. The Jaccard Off Index is defined as:

$$J_{\text{off}}(A, B) = c \cdot \frac{\sum_{x \in X} \min\left(\mu_A(x), \mu_B(x)\right)}{\sum_{x \in X} \max\left(\mu_A(x), \mu_B(x)\right)},$$

where:

- $\min(\mu_A(x), \mu_B(x))$ represents the intersection of the fuzzy offsets A and B,
- $\max(\mu_A(x), \mu_B(x))$ represents the union of the fuzzy offsets A and B,
- $c \in [\Psi, \Omega]$ is a scaling constant that adjusts the result of the traditional Jaccard Index.

The classical Jaccard Index is recovered as a special case when $c = 1$, $\Psi = 0$, and $\Omega = 1$.

Theorem 4.9. *The classical Jaccard Index is a special case of the Jaccard Off Index when* $c = 1$*,* $\Psi = 0$*, and* $\Omega = 1$*.*

Proof. Let A and B be two fuzzy sets defined on a universe X with membership functions $\mu_A, \mu_B : X \to [0, 1]$. By definition, the Jaccard Off Index is given as:

$$J_{\text{off}}(A, B) = c \cdot \frac{\sum_{x \in X} \min\left(\mu_A(x), \mu_B(x)\right)}{\sum_{x \in X} \max\left(\mu_A(x), \mu_B(x)\right)}.$$

For the classical Jaccard Index, the scaling constant $c = 1$, and the membership functions are restricted to $[0, 1]$, corresponding to $\Psi = 0$ and $\Omega = 1$. Substituting these values into the Jaccard Off Index:

$$J_{\text{off}}(A, B) = 1 \cdot \frac{\sum_{x \in X} \min\left(\mu_A(x), \mu_B(x)\right)}{\sum_{x \in X} \max\left(\mu_A(x), \mu_B(x)\right)}.$$

This is exactly the definition of the classical Jaccard Index:

$$J(A, B) = \frac{\sum_{x \in X} \min\left(\mu_A(x), \mu_B(x)\right)}{\sum_{x \in X} \max\left(\mu_A(x), \mu_B(x)\right)}.$$

Thus, when $c = 1$, $\Psi = 0$, and $\Omega = 1$, $J_{\text{off}}(A, B)$ reduces to $J(A, B)$. Therefore, the classical Jaccard Index is a special case of the Jaccard Off Index. □

4.3 Off-Range Normalization: Transforming Offset to Set

Normalization is a mathematical process that adjusts data, functions, or values to fit within a specific range or meet defined criteria, facilitating fair comparisons and enhancing algorithm efficiency(cf. [222]). It is a fundamental concept widely applied in fields such as probability, data science, and machine learning [20, 21, 30, 51, 94, 105, 106, 133]. Below, its formal definition and common types are presented.

Definition 4.10. Normalization is a transformation that maps elements from an original domain X to a target domain Y, ensuring that the transformed values adhere to a specific scale or constraint.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of data points or values, and $f : X \to Y$ be a normalization function. The normalization process is defined as:

$$y_i = f(x_i), \quad \forall x_i \in X,$$

where $Y \subseteq \mathbb{R}$.

Normalization often involves one or more of the following types:

1. Range Normalization (cf. [200]) Maps x_i to a value y_i in a target range $[a, b]$:

$$y_i = a + \frac{(x_i - x_{\min})(b - a)}{x_{\max} - x_{\min}},$$

where:

- $x_{\min} = \min(X)$ and $x_{\max} = \max(X)$,
- $a, b \in \mathbb{R}$ define the desired range.

2. Unit Vector Normalization (cf. [107]) Scales a vector $x = (x_1, x_2, \ldots, x_n)$ such that its norm equals 1:

$$y = \frac{x}{\|x\|_p},$$

where:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

is the p-norm of x.

3. Probability Normalization (cf. [143]) Normalizes $x_i \in X$ to represent probabilities (e.g., in discrete distributions):

$$y_i = \frac{x_i}{\sum_{j=1}^n x_j}, \quad x_i \geq 0.$$

4. Standard Score Normalization (Z-Score) (cf. [53, 95]) Normalizes data x_i to have zero mean and unit variance:

$$y_i = \frac{x_i - \mu}{\sigma},$$

where:

- $\mu = \frac{1}{n} \sum_{i=1}^n x_i$ is the mean of X,
- $\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2}$ is the standard deviation of X.

5. Softmax Normalization (cf. [121, 163]) Transforms x_i into a probability-like value using the exponential function:

$$y_i = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad x_i \in \mathbb{R}.$$

Properties

Normalization transformations often preserve the relative order of elements in X and ensure that the transformed set Y satisfies specific criteria, such as:

- Values in Y lie within a bounded range.
- Y is scale-invariant.
- Y adheres to a probability distribution or vector norm.

Definition 4.11. (cf. [131, 137]) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite dataset. The *Standard Min-Max Normalization* maps each $x_i \in X$ to a new value $y_i \in [0, 1]$, using the formula:

$$y_i = \frac{x_i - x_{\min}}{x_{\max} - x_{\min}},$$

where:

- $x_{\min} = \min(X)$: The minimum value in the dataset.
- $x_{\max} = \max(X)$: The maximum value in the dataset.

Theorem 4.12 (Transformation of Offset Sets)**.** *Let U_{off} be a Fuzzy Offset set or Neutrosophic Offset set with membership functions $\mu_{off} : X \to [\Psi, \Omega]$, where $\Psi < 0$ and $\Omega > 1$. Using range normalization, U_{off} can be transformed into a standard Fuzzy Set U or a Neutrosophic Set A with membership functions $\mu : X \to [0, 1]$.*

Proof. The range normalization function $f : [\Psi, \Omega] \to [0, 1]$ is defined as:

$$f(x) = \frac{x - \Psi}{\Omega - \Psi}.$$

For any $x \in X$, the membership function $\mu_{\text{off}}(x) \in [\Psi, \Omega]$ is transformed into:

$$\mu(x) = f(\mu_{\text{off}}(x)) = \frac{\mu_{\text{off}}(x) - \Psi}{\Omega - \Psi}.$$

This ensures $\mu(x) \in [0, 1]$.

Fuzzy Offset to Fuzzy Set The Fuzzy Offset set U_{off} is defined as:

$$U_{\text{off}} = \{(x, \mu_{\text{off}}(x)) \mid x \in X,\ \mu_{\text{off}}(x) \in [\Psi, \Omega]\}.$$

Applying f yields:

$$U = \{(x, \mu(x)) \mid x \in X,\ \mu(x) \in [0, 1]\},$$

which is a Fuzzy Set.

Neutrosophic Offset to Neutrosophic Set The Neutrosophic Offset set A_{off} is defined as:

$$A_{\text{off}} = \{(x, \langle T_{\text{off}}(x), I_{\text{off}}(x), F_{\text{off}}(x)\rangle) \mid x \in X,\ T_{\text{off}}, I_{\text{off}}, F_{\text{off}} \in [\Psi, \Omega]\}.$$

Applying f to each component:

$$T(x) = f(T_{\text{off}}(x)), \quad I(x) = f(I_{\text{off}}(x)), \quad F(x) = f(F_{\text{off}}(x)),$$

yields:

$$A = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in X,\ T(x), I(x), F(x) \in [0, 1]\}.$$

This is a Neutrosophic Set. □

Example 4.13. Consider a survey assessing the satisfaction (T), uncertainty (I), and dissatisfaction (F) levels of customers towards a product. The satisfaction levels are collected as a Single-Valued Neutrosophic Offset:

$$A_{\text{off}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{off}}, T(x) \in [\Psi_T, \Omega_T], I(x) \in [\Psi_I, \Omega_I], F(x) \in [\Psi_F, \Omega_F]\},$$

where:

- $T(x) \in [-0.2, 1.2]$,
- $I(x) \in [-0.1, 1.1]$,
- $F(x) \in [-0.3, 1.5]$.

To normalize:

$$T'(x) = \frac{T(x) - \Psi_T}{\Omega_T - \Psi_T} = \frac{T(x) + 0.2}{1.2 + 0.2},$$
$$I'(x) = \frac{I(x) - \Psi_I}{\Omega_I - \Psi_I} = \frac{I(x) + 0.1}{1.1 + 0.1},$$
$$F'(x) = \frac{F(x) - \Psi_F}{\Omega_F - \Psi_F} = \frac{F(x) + 0.3}{1.5 + 0.3}.$$

After normalization:

$$A = \{(x, \langle T'(x), I'(x), F'(x)\rangle) \mid x \in U, T'(x), I'(x), F'(x) \in [0, 1]\}.$$

This transformed set A can now be analyzed using standard Neutrosophic Set techniques.

Theorem 4.14. *A Plithogenic Offset PS_{off} can be normalized into a standard Plithogenic Set PS through range normalization. The normalization maps the extended degree of appurtenance and contradiction from* $[\Psi_v, \Omega_v]$ *to* $[0, 1]$.

Proof. Let $PS_{\text{off}} = (P, v, Pv, pdf_{\text{off}}, pCF_{\text{off}})$ be a Plithogenic Offset defined as:

$$pdf_{\text{off}} : P \times Pv \to [\Psi_v, \Omega_v]^s, \quad pCF_{\text{off}} : Pv \times Pv \to [\Psi_v, \Omega_v]^t,$$

where $\Psi_v < 0$ and $\Omega_v > 1$.

We normalize pdf_{off} and pCF_{off} to map their ranges to $[0, 1]$ using a linear transformation:

$$pdf(x, a) = \frac{pdf_{\text{off}}(x, a) - \Psi_v}{\Omega_v - \Psi_v}, \quad pCF(a, b) = \frac{pCF_{\text{off}}(a, b) - \Psi_v}{\Omega_v - \Psi_v}.$$

Verification

1. **Range Preservation**:
$$pdf(x,a) \in [0,1], \quad pCF(a,b) \in [0,1].$$
For $pdf_{\text{off}}(x,a) = \Psi_v$, $pdf(x,a) = 0$. For $pdf_{\text{off}}(x,a) = \Omega_v$, $pdf(x,a) = 1$. Similar logic applies to $pCF(a,b)$.

2. **Structure Preservation**: The normalization does not alter the relative ordering of the original values in pdf_{off} or pCF_{off}, ensuring consistency.

3. **Resulting Set**: After normalization, PS_{off} is transformed into a standard Plithogenic Set:
$$PS = (P, v, Pv, pdf, pCF),$$
where pdf and pCF now map to $[0,1]$.

Thus, PS_{off} is successfully normalized into PS. □

Based on the previous discussions, the following Off Range Normalization Function can be defined. Since most real-world concepts are not inherently normalized, it is likely that normalization will be necessary when analyzing practical concepts or data using frameworks like Fuzzy Sets or Neutrosophic Sets. Therefore, the Off Range Normalization Function defined below could prove useful in such scenarios. It is hoped that further research will explore applications and use cases for this function in various fields.

Definition 4.15. The **Off Range Normalization Function** transforms an offset membership function, defined over an extended range $[\Psi, \Omega]$, into a standard membership function over $[0,1]$. This process enables the conversion of an Offset Set into a standard Set.

Formal Definition Let X be a universal set and $\mu_{\text{off}} : X \to [\Psi, \Omega]$ be the offset membership function, where:
$$\Psi < 0 \quad \text{(UnderLimit)}, \quad \Omega > 1 \quad \text{(OverLimit)}.$$
The **Off Range Normalization Function** $f : [\Psi, \Omega] \to [0,1]$ is defined as:
$$f(x) = \frac{x - \Psi}{\Omega - \Psi}.$$

Transformation: For any $x \in X$, the membership value $\mu_{\text{off}}(x) \in [\Psi, \Omega]$ is normalized using $f(x)$ to produce the standard membership value $\mu(x)$:
$$\mu(x) = f(\mu_{\text{off}}(x)) = \frac{\mu_{\text{off}}(x) - \Psi}{\Omega - \Psi}.$$

Properties:

1. **Range Preservation**:
$$\mu_{\text{off}}(x) = \Psi \implies \mu(x) = 0, \quad \mu_{\text{off}}(x) = \Omega \implies \mu(x) = 1.$$

2. **Monotonicity**: If $\mu_{\text{off}}(x_1) \le \mu_{\text{off}}(x_2)$, then $\mu(x_1) \le \mu(x_2)$.

3. **Set Transformation**: For an Offset Set $A_{\text{off}} = \{(x, \mu_{\text{off}}(x)) \mid x \in X\}$, the normalized set A is defined as:
$$A = \{(x, \mu(x)) \mid x \in X, \mu(x) = f(\mu_{\text{off}}(x))\}.$$

5 Future tasks

Regarding future prospects for related research, we plan to extend certain sets from fuzzy sets to neutrosophic sets, examine their relationships, and further expand these concepts to include Offset, Overset, and Underset. Through this approach, we aim to explore their mathematical structures and potential applications.

5.1 B-Neutrosophic sets

The B-Neutrosophic set (Boolean Neutrosophic set) is defined as follows. It is an extension of the Boolean Fuzzy set, and further research into the mathematical structure of these sets is anticipated in the future (cf. [43–46, 129]).

Definition 5.1 (Boolean Algebra)**.** (cf. [79, 80, 178]) A *Boolean algebra* B is a set equipped with two binary operations $\wedge$ (meet) and $\vee$ (join), a unary operation $\neg$ (complement), and two distinguished elements 0_B (zero) and 1_B (unit), satisfying the following axioms:

1. *Associativity*: For all $a, b, c \in B$,
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad a \vee (b \vee c) = (a \vee b) \vee c.$$
2. *Commutativity*: For all $a, b \in B$,
$$a \wedge b = b \wedge a, \quad a \vee b = b \vee a.$$
3. *Distributivity*: For all $a, b, c \in B$,
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$
4. *Identity*: For all $a \in B$,
$$a \wedge 1_B = a, \quad a \vee 0_B = a.$$
5. *Complementarity*: For all $a \in B$,
$$a \wedge \neg a = 0_B, \quad a \vee \neg a = 1_B.$$

Definition 5.2 (Completeness)**.** (cf. [157]) A Boolean algebra B is *complete* if, for every subset $S \subseteq B$, the following exist:

- The *supremum* $\bigvee S$ (least upper bound of S),
- The *infimum* $\bigwedge S$ (greatest lower bound of S).

Remark 5.3 (Properties of Complete Boolean Algebra)**.** (cf. [157])

- Completeness ensures that infinite operations (joins and meets over arbitrary sets) are well-defined.
- The unit element 1_B is the supremum of B, and the zero element 0_B is the infimum of B.

Definition 5.4 (B-fuzzy set(Boolean fuzzy set))**.** [43] Let B be a complete Boolean algebra, and let A be a non-empty set. A *B-fuzzy set* is defined as a mapping $f : A \to B$ such that for all $x, y \in A$, the following conditions hold:
$$x \neq y \implies f(x) \wedge f(y) = 0_B,$$
where 0_B is the zero element of the Boolean algebra B. The set of all such mappings is denoted by $A[B]$ and is referred to as the *Boolean power* of A with respect to B.

Furthermore, a *partition of unity* in B is a family of elements $\{t_i \in B \mid i \in I\}$ such that:
$$t_i \wedge t_j = 0_B \quad \text{for } i \neq j, \quad \text{and} \quad \bigvee_{i \in I} t_i = 1_B.$$

Using a partition of unity, any $f \in A[B]$ can be expressed in the form:

$$f = \bigvee_{i \in I} a_i \cdot t_i,$$

where $a_i \in A$ and $t_i \in B$ for $i \in I$.

In this representation:

- Each t_i acts as a "weight" from the Boolean algebra B, indicating the degree of membership of the element $a_i \in A$ in the B-fuzzy set.
- The mapping f essentially represents a mixing of crisp elements of A with the weights t_i derived from B.

Definition 5.5 (B-Neutrosophic Set(Boolean Neutrosophic Set))**.** Let X be a given set and B a complete Boolean algebra. A *B-Neutrosophic Set A* on X is defined as a triplet of mappings:

$$T_A : X \to B, \quad I_A : X \to B, \quad F_A : X \to B,$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity in the Boolean algebra B. These mappings satisfy the following conditions:

- For all $x \in X$, the values $T_A(x), I_A(x), F_A(x) \in B$.
- The disjointness condition:

  $$T_A(x) \wedge I_A(x) = 0_B, \quad T_A(x) \wedge F_A(x) = 0_B, \quad I_A(x) \wedge F_A(x) = 0_B,$$

  where 0_B is the zero element of B.
- The boundedness condition:

  $$T_A(x) \vee I_A(x) \vee F_A(x) \leq 1_B,$$

  where 1_B is the unity element of B.

For a given $x \in X$, the triplet $(T_A(x), I_A(x), F_A(x))$ describes the Boolean-valued truth, indeterminacy, and falsity levels of x in the B-Neutrosophic Set A.

Remark 5.6. If $T_A(x), I_A(x), F_A(x)$ are defined using a partition of unity $\{t_i \in B : i \in I\}$, where:

$$t_i \wedge t_j = 0_B \quad \text{for } i \neq j, \quad \text{and} \quad \bigvee_{i \in I} t_i = 1_B,$$

then $T_A(x)$, $I_A(x)$, and $F_A(x)$ can be expressed as:

$$T_A(x) = \bigvee_{i \in I} t_i, \quad I_A(x) = \bigvee_{j \in J} t_j, \quad F_A(x) = \bigvee_{k \in K} t_k,$$

where $I, J, K \subseteq I$ are disjoint subsets.

Remark 5.7. When $B = [0, 1]$, the B-Neutrosophic Set reduces to a classical Neutrosophic Set, where:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

and for all $x \in X$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Theorem 5.8. *The B-Neutrosophic Set generalizes the B-fuzzy Set.*

Proof. Let X be a non-empty set and B a complete Boolean algebra. A B-fuzzy Set f is defined as a mapping $f : X \to B$ such that for all $x \in X$, $f(x) \in B$. The membership degree $f(x)$ represents the degree of membership of x in the set, and the disjointness condition holds:

$$f(x) \wedge f(y) = 0_B \quad \text{for } x \neq y.$$

A B-Neutrosophic Set A is defined as a triplet of mappings:

$$T_A : X \to B, \quad I_A : X \to B, \quad F_A : X \to B,$$

where:

- $T_A(x)$ represents the degree of truth of x,
- $I_A(x)$ represents the degree of indeterminacy of x,
- $F_A(x)$ represents the degree of falsity of x,

and the following conditions hold for all $x \in X$:

$$\begin{aligned} T_A(x) \wedge I_A(x) &= 0_B, \\ T_A(x) \wedge F_A(x) &= 0_B, \\ I_A(x) \wedge F_A(x) &= 0_B, \\ T_A(x) \vee I_A(x) \vee F_A(x) &\leq 1_B. \end{aligned}$$

If $I_A(x) = 0_B$ and $F_A(x) = 0_B$ for all $x \in X$, then the B-Neutrosophic Set reduces to a single mapping:

$$T_A(x) = f(x),$$

where $f : X \to B$ satisfies the conditions of a B-fuzzy Set. In this case, the truth-membership function $T_A(x)$ directly represents the membership degree of x in the B-fuzzy Set.

The B-Neutrosophic Set extends the B-fuzzy Set by introducing additional degrees $I_A(x)$ (indeterminacy) and $F_A(x)$ (falsity) for each element $x \in X$. These additional mappings provide a richer framework for modeling uncertainty and ambiguity that cannot be captured by the single membership function of a B-fuzzy Set.

Both B-fuzzy Sets and B-Neutrosophic Sets can be expressed using a partition of unity $\{t_i \in B : i \in I\}$. In the case of a B-fuzzy Set, the membership function $f(x)$ can be written as:

$$f(x) = \bigvee_{i \in I} a_i \cdot t_i,$$

where $a_i \in X$ and $t_i \in B$. For a B-Neutrosophic Set, each of the truth, indeterminacy, and falsity mappings $T_A(x), I_A(x), F_A(x)$ can similarly be expressed in terms of disjoint subsets of the partition:

$$T_A(x) = \bigvee_{i \in I_T} t_i, \quad I_A(x) = \bigvee_{j \in I_I} t_j, \quad F_A(x) = \bigvee_{k \in I_F} t_k,$$

where $I_T, I_I, I_F \subseteq I$ are disjoint.

Since the B-Neutrosophic Set encompasses the B-fuzzy Set as a special case when $I_A(x) = 0_B$ and $F_A(x) = 0_B$, and it extends the framework to include additional degrees of indeterminacy and falsity, it follows that the B-Neutrosophic Set generalizes the B-fuzzy Set. □

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

References

[1] Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.

[2] Zeeshan Ahmad, Tahir Mahmood, Kifayat Ullah, and Naeem Jan. Similarity measures based on the novel interval-valued picture hesitant fuzzy sets and their applications in pattern recognition. *Punjab University Journal of Mathematics*, 2022.

[3] Zhao Aiwu and Guan Hongjun. Fuzzy-valued linguistic soft set theory and multi-attribute decision-making application. *Chaos Solitons & Fractals*, 89:2–7, 2016.

[4] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[5] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[6] Muhammad Akram and Musavarah Sarwar. Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *viXra*, 2017.

[7] Mohammed Abdullah Al-Hagery, Abdalla I Abdalla Musa, et al. Automated credit card risk assessment using fuzzy parameterized neutrosophic hypersoft expert set. *International Journal of Neutrosophic Science*, (1):93–3, 2025.

[8] Talal Al-Hawary. On functions of k-balanced matroids. *Open Journal of Discrete Mathematics*, 7(3):103–107, 2017.

[9] Talal A Al-Hawary. Feeble-matroids. *Italian journal of pure and applied mathematics*, 14:87–94, 2003.

[10] Faisal Al-Sharqi, Ashraf Al-Quran, et al. Similarity measures on interval-complex neutrosophic soft sets with applications to decision making and medical diagnosis under uncertainty. *Neutrosophic Sets and Systems*, 51:495–515, 2022.

[11] José Carlos R Alcantud, Azadeh Zahedi Khameneh, Gustavo Santos-García, and Muhammad Akram. A systematic literature review of soft set theory. *Neural Computing and Applications*, 36(16):8951–8975, 2024.

[12] Ebrahem Ateatullah Algehyne, Muhammad Lawan Jibril, Naseh A Algehainy, Osama Abdulaziz Alamri, and Abdullah Khaled J Alzahrani. Fuzzy neural network expert system with an improved gini index random forest-based feature importance measure algorithm for early diagnosis of breast cancer in saudi arabia. *Big Data Cogn. Comput.*, 6:13, 2022.

[13] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[14] RA Aliev, OH Huseynov, and RR Aliyev. A sum of a large number of z-numbers. *Procedia computer science*, 120:16–22, 2017.

[15] Marcos Antônio Alves, Petrônio Cândido de Lima e Silva, Carlos Alberto Severiano Junior, Gustavo Linhares Vieira, Frederico Gadelha Guimarães, and Hossein Javedani Sadaei. An extension of nonstationary fuzzy sets to heteroskedastic fuzzy time series. In *ESANN*, 2018.

[16] J Angoa, YF Ortiz-Castillo, and A Tamariz-Mascarúa. Ultrafilters and properties related to compactness. In *Topology Proc*, volume 43, pages 183–200, 2014.

[17] Nabeel Ezzulddin Arif et al. Domination (set and number) in neutrosophic soft over graphs. *Wasit Journal for Pure sciences*, 1(3):26–43, 2022.

[18] Shahzaib Ashraf, Saleem Abdullah, Tahir Mahmood, Fazal Ghani, and Tariq Mahmood. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst.*, 36:2829–2844, 2019.

[19] Ibrahim Awajan, Mumtazimah Mohamad, and Ashraf Al-Quran. Sentiment analysis technique and neutrosophic set theory for mining and ranking big data from online reviews. *IEEE Access*, 9:47338–47353, 2021.

[20] Jimmy Lei Ba. Layer normalization. *arXiv preprint arXiv:1607.06450*, 2016.

[21] Peter L Bartlett, Dylan J Foster, and Matus J Telgarsky. Spectrally-normalized margin bounds for neural networks. *Advances in neural information processing systems*, 30, 2017.

[22] Veysi Başhan, Hakan Demirel, and Muhammet Gul. An fmea-based topsis approach under single valued neutrosophic sets for maritime risk evaluation: the case of ship navigation safety. *Soft Computing*, 24(24):18749–18764, 2020.

[23] Pranab Biswas, Surapati Pramanik, and Bibhas Chandra Giri. Single valued bipolar pentapartitioned neutrosophic set and its application in madm strategy. 2022.

[24] David Booth. Ultrafilters on a countable set. *Annals of Mathematical Logic*, 2(1):1–24, 1970.

[25] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[26] Said Broumi, Irfan Deli, and Florentin Smarandache. N-valued interval neutrosophic sets and their application in medical diagnosis. *Critical Review, Center for Mathematics of Uncertainty, Creighton University, Omaha, NE, USA*, 10:45–69, 2015.

[27] Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.

[28] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[29] Erick González Caballero, Florentin Smarandache, and Maikel Leyva Vázquez. On neutrosophic offuninorms. *Symmetry*, 11(9):1136, 2019.

[30] Matteo Carandini and David J Heeger. Normalization as a canonical neural computation. *Nature reviews neuroscience*, 13(1):51–62, 2012.

[31] Susumu Cato. Quasi-decisiveness, quasi-ultrafilter, and social quasi-orderings. *Social Choice and Welfare*, 41(1):169–202, 2013.

[32] Jia Syuen Chai, Ganeshsree Selvachandran, Florentin Smarandache, Vassilis C Gerogiannis, Le Hoang Son, Quang-Thinh Bui, and Bay Vo. New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. *Complex & Intelligent Systems*, 7:703–723, 2021.

[33] KC Chattopadhyay, UK Mukherjee, and SK Samanta. A note on fuzzy ultrafilters. *INDIAN JOURNAL OF PURE AND APPLIED MATHEMATICS*, 29:695–704, 1998.

[34] Liuxin Chen, Yutai Wang, and Dongmei Yang. Picture fuzzy z-linguistic set and its application in multiple attribute group decision-making. *J. Intell. Fuzzy Syst.*, 43:5997–6011, 2022.

[35] Shyi-Ming Chen. Measures of similarity between vague sets. *Fuzzy sets and Systems*, 74(2):217–223, 1995.

[36] Kai Lai Chung. *A course in probability theory*. Elsevier, 2000.

[37] Vašek Chvátal. Antimatroids, betweenness, convexity. *Research Trends in Combinatorial Optimization: Bonn 2008*, pages 57–64, 2009.

[38] Luciano da F Costa. Further generalizations of the jaccard index. *arXiv preprint arXiv:2110.09619*, 2021.

[39] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[40] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[41] Irfan Deli, Said Broumi, and Florentin Smarandache. On neutrosophic refined sets and their applications in medical diagnosis. *Journal of new theory*, (6):88–98, 2015.

[42] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[43] CA Drossos and G Markakis. Boolean fuzzy sets. *Fuzzy Sets and Systems*, 46(1):81–95, 1992.

[44] CA Drossos, G Markakis, and PL Theodoropoulos. B-fuzzy stochastics. *Asymptotics in Statistics and Probability: Papers in Honor of George Gregory Roussas*, pages 155–170, 2000.

[45] Costas Drossos and George Markakis. Boolean representation of fuzzy sets. *Kybernetes*, 22(3):35–40, 1993.

[46] Costas A Drossos and PL Theodoropoulos. B-fuzzy probabilities. *Fuzzy sets and systems*, 78(3):355–369, 1996.

[47] Shigui Du, Jun Ye, Rui Yong, and Fangwei Zhang. Some aggregation operators of neutrosophic z-numbers and their multicriteria decision making method. *Complex & Intelligent Systems*, 7:429 – 438, 2020.

[48] Didier Dubois and Henri Prade. *Fundamentals of fuzzy sets*, volume 7. Springer Science & Business Media, 2012.

[49] Tom Eelbode, Jeroen Bertels, Maxim Berman, Dirk Vandermeulen, Frederik Maes, Raf Bisschops, and Matthew B Blaschko. Optimization for medical image segmentation: theory and practice when evaluating with dice score or jaccard index. *IEEE transactions on medical imaging*, 39(11):3679–3690, 2020.

[50] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[51] Pablo A Estévez, Michel Tesmer, Claudio A Perez, and Jacek M Zurada. Normalized mutual information feature selection. *IEEE Transactions on neural networks*, 20(2):189–201, 2009.

[52] Ulrich Faigle. Matroids in combinatorial optimization, 1987.

[53] Nanyi Fei, Yizhao Gao, Zhiwu Lu, and Tao Xiang. Z-score normalization, hubness, and few-shot learning. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pages 142–151, 2021.

[54] Sam Fletcher, Md Zahidul Islam, et al. Comparing sets of patterns with the jaccard index. *Australasian Journal of Information Systems*, 22, 2018.

[55] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[56] Takaaki Fujita. Relation between ultra matroid and linear decomposition. *Italian Journal of Pure and Applied Mathematics. Accepted.*

[57] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[58] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[59] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.

[60] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). *preprint*, 2024.

[61] Takaaki Fujita. Various properties of various ultrafilters, various graph width parameters, and various connectivity systems. *arXiv preprint arXiv:2408.02299*, 2024.

[62] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[63] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[64] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[65] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[66] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[67] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[68] M Ganesh. *Introduction to fuzzy sets and fuzzy logic*. PHI Learning Pvt. Ltd., 2006.

[69] Jonathan M Garibaldi, Marcin Jaroszewski, and Salang Musikasuwan. Nonstationary fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 16(4):1072–1086, 2008.

[70] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[71] R. Goetschel Jr. and W. Voxman. Fuzzy matroids. *Fuzzy Sets and Systems*, 27:291–301, 1988.

[72] Roy Goetschel Jr and William Voxman. Spanning properties for fuzzy matroids. *Fuzzy sets and systems*, 51(3):313–321, 1992.

[73] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[74] Anastasiya B Goncharova. Preliminary medical diagnostics based on the fuzzy sets theory using the sugeno measure. 2019.

[75] Michel Grabisch, Michio Sugeno, and Toshiaki Murofushi. Fuzzy measures and integrals: Theory and applications. 2000.

[76] Hongjun Guan, Shuang Guan, and Aiwu Zhao. Intuitionistic fuzzy linguistic soft sets and their application in multi-attribute decision-making. *J. Intell. Fuzzy Syst.*, 31:2869–2879, 2016.

[77] Fatma Kutlu Gündogdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *J. Intell. Fuzzy Syst.*, 36:337–352, 2019.

[78] Minghu Ha, Chao Wang, and Witold Pedrycz. The key theorem of learning theory based on sugeno measure and fuzzy random samples. In *International Conference on Intelligent Computing for Sustainable Energy and Environment*, pages 241–249. Springer, 2010.

[79] Paul Halmos and Steven Givant. *Introduction to Boolean algebras*. Springer, 2009.

[80] Paul R Halmos. *Lectures on Boolean algebras*. Courier Dover Publications, 2018.

[81] Zhinan Hao, Zeshui Xu, Hua Zhao, and Zhan Su. Probabilistic dual hesitant fuzzy set and its application in risk evaluation. *Knowl. Based Syst.*, 127:16–28, 2017.

[82] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[83] Wei He and Yiting Dong. Adaptive fuzzy neural network control for a constrained robot using impedance learning. *IEEE Transactions on Neural Networks and Learning Systems*, 29:1174–1186, 2018.

[84] Horst Herrlich, Paul E. Howard, and Kyriakos Keremedis. On extensions of countable filterbases to ultrafilters and ultrafilter compactness. *Quaestiones Mathematicae*, 41:213 – 225, 2017.

[85] Anwer Mustafa Hilal, Jaber S Alzahrani, Hadeel Alsolai, Noha Negm, Faisal Mohammed Nafie, Abdelwahed Motwakel, Ishfaq Yaseen, and Manar Ahmed Hamza. Sentiment analysis technique for textual reviews using neutrosophic set theory in the multi-criteria decision-making system.

[86] Neil Hindman. Ultrafilters and combinatorial number theory. In *Number Theory Carbondale 1979: Proceedings of the Southern Illinois Number Theory Conference Carbondale, March 30 and 31, 1979*, pages 119–184. Springer, 1979.

[87] Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.

[88] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[89] Han-Chen Huang and Xiao-Jun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, 24:213–228, 2016.

[90] Han-Chen Huang and Xiaojun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(02):213–227, 2016.

[91] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[92] I Ibedou, SE Abbas, and S Jafari. Ideals and grills associated with a rough set. In *Advances in Topology and Their Interdisciplinary Applications*, pages 167–181. Springer, 2023.

[93] A Ibrahim, S Karunya Helen Gunaseeli, and Said Broumi. Some types of neutrosophic filters in basic logic algebras. *Neutrosophic Systems with Applications*, 11:31–38, 2023.

[94] Sergey Ioffe. Batch normalization: Accelerating deep network training by reducing internal covariate shift. *arXiv preprint arXiv:1502.03167*, 2015.

[95] Anil Jain, Karthik Nandakumar, and Arun Ross. Score normalization in multimodal biometric systems. *Pattern recognition*, 38(12):2270–2285, 2005.

[96] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[97] Juan juan Peng, Xin Ge Chen, Xiao Kang Wang, Jian Qiang Wang, Qingqing Long, and Lv Jiang Yin. Picture fuzzy decision-making theories and methodologies: a systematic review. *International Journal of Systems Science*, 54:2663 – 2675, 2023.

[98] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[99] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 6(6):87, 2018.

[100] Ilanthenral Kandasamy, WB Vasantha, Jagan M Obbineni, and Florentin Smarandache. Sentiment analysis of tweets using refined neutrosophic sets. *Computers in Industry*, 115:103180, 2020.

[101] Mesut Karabacak. Correlation coefficient for neutrosophic z-numbers and its applications in decision making. *J. Intell. Fuzzy Syst.*, 45:215–228, 2023.

[102] Yukihito Kawahara. Combinatorics coming from hyperplane arrangements and the orlik-solomon algebra (algebraic combinatorics). *RIMS Kokyuroku (Research Institute for Mathematical Sciences)*, 1394:97–108, 2004.

[103] Yukihito Kawahara. On matroids and orlik-solomon algebras. *Annals of Combinatorics*, 8:63–80, 2004.

[104] Shahbaz Khan, Abid Haleem, and Mohd Imran Khan. A risk assessment framework using neutrosophic theory for the halal supply chain under an uncertain environment. *Arab Gulf Journal of Scientific Research*, 42(3):852–870, 2024.

[105] Günter Klambauer, Thomas Unterthiner, Andreas Mayr, and Sepp Hochreiter. Self-normalizing neural networks. *Advances in neural information processing systems*, 30, 2017.

[106] Ivan Kobyzev, Simon JD Prince, and Marcus A Brubaker. Normalizing flows: An introduction and review of current methods. *IEEE transactions on pattern analysis and machine intelligence*, 43(11):3964–3979, 2020.

[107] Stefan Kolašinac, Ilinka Pećinar, Dario Danojević, and Zora Dajić Stevanović. Raman spectroscopy coupled with chemometric modeling approaches for authentication of different paprika varieties at physiological maturity. *LWT*, 162:113402, 2022.

[108] Bernhard Korte, László Lovász, and Rainer Schrader. *Greedoids*, volume 4. Springer Science & Business Media, 2012.

[109] Costas D Koutras, Christos Moyzes, Christos Nomikos, Konstantinos Tsaprounis, and Yorgos Zikos. On weak filters and ultrafilters: Set theory from (and for) knowledge representation. *Logic Journal of the IGPL*, 31(1):68–95, 2023.

[110] Costas D Koutras, Christos Moyzes, Christos Nomikos, and Yorgos Zikos. On the 'in many cases' modality: tableaux, decidability, complexity, variants. In *Artificial Intelligence: Methods and Applications: 8th Hellenic Conference on AI, SETN 2014, Ioannina, Greece, May 15-17, 2014. Proceedings 8*, pages 207–220. Springer, 2014.

[111] Costas D Koutras, Christos Moyzes, and Yorgos Zikos. A modal logic of knowledge, belief and estimation. *Journal of Logic and Computation*, 27(8):2303–2339, 2017.

[112] Hon Keung Kwan and Yaling Cai. A fuzzy neural network and its application to pattern recognition. *IEEE transactions on Fuzzy Systems*, 2(3):185–193, 1994.

[113] Soojung Lee. Improving jaccard index for measuring similarity in collaborative filtering. In *Information Science and Applications 2017: ICISA 2017 8*, pages 799–806. Springer, 2017.

[114] Konrad W. Leszczynski, Pawel A. Penczek, and W. Daniel Grochulski. Sugeno's fuzzy measure and fuzzy clustering. *Fuzzy Sets and Systems*, 15:147–158, 1985.

[115] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[116] Xihua Li, Yun Luo, Hui Wang, Jiong Lin, and Bin Deng. Doctor selection based on aspect-based sentiment analysis and neutrosophic topsis method. *Engineering Applications of Artificial Intelligence*, 124:106599, 2023.

[117] Sheng-Wei Lin and Huai-Wei Lo. An fmea model for risk assessment of university sustainability: using a combined itara with topsis-al approach based neutrosophic sets. *Annals of Operations Research*, pages 1–27, 2023.

[118] Peide Liu and Shufeng Cheng. An extension of aras methodology for multi-criteria group decision-making problems within probability multi-valued neutrosophic sets. *International Journal of Fuzzy Systems*, 21:2472–2489, 2019.

[119] Peide Liu, Shufeng Cheng, and Yuming Zhang. An extended multi-criteria group decision-making promethee method based on probability multi-valued neutrosophic sets. *International journal of fuzzy systems*, 21:388–406, 2019.

[120] Yu-Ting Liu, Yang-Yin Lin, Shang-Lin Wu, Chun-Hsiang Chuang, and Chin-Teng Lin. Brain dynamics in predicting driving fatigue using a recurrent self-evolving fuzzy neural network. *IEEE Transactions on Neural Networks and Learning Systems*, 27:347–360, 2016.

[121] Jiachen Lu, Jinghan Yao, Junge Zhang, Xiatian Zhu, Hang Xu, Weiguo Gao, Chunjing Xu, Tao Xiang, and Li Zhang. Soft: Softmax-free transformer with linear complexity. *Advances in Neural Information Processing Systems*, 34:21297–21309, 2021.

[122] Xuejiao Ma, Yu Jin, and Qingli Dong. A generalized dynamic fuzzy neural network based on singular spectrum analysis optimized by brain storm optimization for short-term wind speed forecasting. *Appl. Soft Comput.*, 54:296–312, 2017.

[123] Asma Mahmood, Mujahid Abbas, and Ghulam Murtaza. Multi-valued multi-polar neutrosophic sets with an application in multi-criteria decision-making. *Neutrosophic Sets and Systems*, 53(1):32, 2023.

[124] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[125] Tahir Mahmood, Ubaid ur Rehman, Zeeshan Ali, and Tariq Mahmood. Hybrid vector similarity measures based on complex hesitant fuzzy sets and their applications to pattern recognition and medical diagnosis. *J. Intell. Fuzzy Syst.*, 40:625–646, 2021.

[126] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[127] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[128] Hua Mao and Sanyang Liu. On antimatroids of infinite character. *Mathematica Pannonica*, 23(2):257–266, 2012.

[129] George Markakis. Boolean fuzzy sets and possibility measures. *Fuzzy sets and systems*, 110(2):279–285, 2000.

[130] Nivetha Martin, Priya Priya.R, and Florentin Smarandache. Decision making on teachers' adaptation to cybergogy in saturated interval- valued refined neutrosophic overset /underset /offset environment. *International Journal of Neutrosophic Science*, 2020.

[131] Soroush Masoudi, Mohammad Sima, and MAJID Tolouei-Rad. Comparative study of ann and anfis models for predicting temperature in machining. *Journal of Engineering Science and Technology*, 13(1):211–225, 2018.

[132] Manoj Mathew, Ripon Kumar Chakrabortty, and Michael J. Ryan. A novel approach integrating ahp and topsis under spherical fuzzy sets for advanced manufacturing system selection. *Eng. Appl. Artif. Intell.*, 96:103988, 2020.

[133] Takeru Miyato, Toshiki Kataoka, Masanori Koyama, and Yuichi Yoshida. Spectral normalization for generative adversarial networks. *ArXiv*, abs/1802.05957, 2018.

[134] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[135] Bernard Monjardet. On the use of ultrafilters in social choice theory. *Social choice and welfare*, 5:73–78, 1983.

[136] Dilnoz Muhamediyeva and Baxtiyor Tagbayev. Method of converting z-number to classic fuzzy number. *Scientific Collection «InterConf+»*, (21 (109)):348–352, 2022.

[137] Lkhagvadorj Munkhdalai, Tsendsuren Munkhdalai, Kwang Ho Park, Heon Gyu Lee, Meijing Li, and Keun Ho Ryu. Mixture of activation functions with extended min-max normalization for forex market prediction. *IEEE Access*, 7:183680–183691, 2019.

[138] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[139] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.

[140] Xuan Thao Nguyen, Doan Dong Nguyen, et al. Rough fuzzy relation on two universal sets. *International Journal of Intelligent Systems and Applications*, 6(4):49, 2014.

[141] Giorgio Nordo, Arif Mehmood, and Said Broumi. *Single valued neutrosophic filters*. Infinite Study, 2020.

[142] Wendy Olsen and Hisako Nomura. Poverty reduction: fuzzy sets vs. crisp sets compared. *Sociological Theory and Methods*, 24(2):219–246, 2009.

[143] CongJie Ou and JinCan Chen. Generalized entropies under different probability normalization conditions. *Chinese Science Bulletin*, 56:3649–3653, 2011.

[144] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[145] Serif Özlü. Multi-criteria decision making based on vector similarity measures of picture type-2 hesitant fuzzy sets. *Granular Computing*, 8:1505–1531, 2023.

[146] Sunay P Pai and Rajesh S Prabhu Gaonkar. Modelling uncertainty using neutrosophic sets for precise risk assessment of marine systems. *International Journal of System Assurance Engineering and Management*, pages 1–8, 2023.

[147] Vasile Patrascu. Penta and hexa valued representation of neutrosophic information. *arXiv preprint arXiv:1603.03729*, 2016.

[148] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[149] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[150] Zdzisław Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[151] Zdzisław Pawlak. *Rough sets: Theoretical aspects of reasoning about data*, volume 9. Springer Science & Business Media, 2012.

[152] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[153] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[154] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[155] Hong-gang Peng, Hong-yu Zhang, and Jian-qiang Wang. Probability multi-valued neutrosophic sets and its application in multi-criteria group decision-making problems. *Neural Computing and Applications*, 30:563–583, 2018.

[156] Juan-juan Peng and Jian-qiang Wang. Multi-valued neutrosophic sets and its application in multi-criteria decision-making problems. *Neutrosophic Sets and Systems*, 10(1):6, 2015.

[157] Richard S Pierce. A note on complete boolean algebras. *Proceedings of the American Mathematical Society*, 9(6):892–896, 1958.

[158] Gerhard Preuss. (weak) compactness and local (weak) compactness in fuzzy preuniform convergence spaces. *Scientiae Mathematicae Japonicae*, 71(3):285–300, 2010.

[159] C Manju Priya. Neutrosophic theory and sentiment analysis technique for mining and ranking big data from online evaluation. 2022.

[160] Jian qiang Wang, Jia ting Wu, Jing Wang, Hong yu Zhang, and Xiao hong Chen. Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. *Inf. Sci.*, 288:55–72, 2014.

[161] Jian qiang Wang, Yong xi Cao, and Hong yu Zhang. Multi-criteria decision-making method based on distance measure and choquet integral for linguistic z-numbers. *Cognitive Computation*, 9:827 – 842, 2017.

[162] Jian qiang Wang, Yu Yang, and Lin Li. Multi-criteria decision-making method based on single-valued neutrosophic linguistic maclaurin symmetric mean operators. *Neural Computing and Applications*, 30:1529 – 1547, 2016.

[163] Zhen Qin, Weixuan Sun, Hui Deng, Dongxu Li, Yunshen Wei, Baohong Lv, Junjie Yan, Lingpeng Kong, and Yiran Zhong. cosformer: Rethinking softmax in attention. *arXiv preprint arXiv:2202.08791*, 2022.

[164] M Ramya, Sandesh Murali, and R.Radha. Bipolar quadripartitioned neutrosophic soft set. 2022.

[165] Jinjun Rao, Bo Li, Zhen Zhang, Dongdong Chen, and Wojciech Giernacki. Position control of quadrotor uav based on cascade fuzzy neural network. *Energies*, 2022.

[166] András Recski. *Matroid theory and its applications in electric network theory and in statics*, volume 6. Springer Science & Business Media, 2013.

[167] Akbar Rezaei, Tahsin Oner, Tugce Katican, Florentin Smarandache, and N Gandotra. *A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets*. Infinite Study, 2022.

[168] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[169] Sheldon M Ross. *Introduction to probability models*. Academic press, 2014.

[170] Sheldon M Ross, Sheldon M Ross, Sheldon M Ross, and Sheldon M Ross. *A first course in probability*, volume 2. Macmillan New York, 1976.

[171] AA Salama and H Elagamy. Neutrosophic filters. *International Journal of Computer Science Engineering and Information Technology Reseearch (IJCSEITR)*, 3(1):307–312, 2013.

[172] AA Salama and Florentin Smarandache. *Filters via neutrosophic crisp sets*. Infinite Study, 2013.

[173] Pierre Samuel. Ultrafilters and compactification of uniform spaces. *Transactions of the American Mathematical Society*, 64(1):100–132, 1948.

[174] Gulfam Shahzadi, Muhammad Akram, Arsham Borumand Saeid, et al. An application of single-valued neutrosophic sets in medical diagnosis. *Neutrosophic sets and systems*, 18:80–88, 2017.

[175] Fu-Gui Shi. (l, m)-fuzzy matroids. *Fuzzy sets and systems*, 160(16):2387–2400, 2009.

[176] Fu-Gui Shi. A new approach to the fuzzification of matroids. *Fuzzy Sets and Systems*, 160(5):696–705, 2009.

[177] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[178] Roman Sikorski et al. *Boolean algebras*, volume 2. Springer, 1969.

[179] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Sciences*, 18(1):57–72, 2022.

[180] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[181] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[182] Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.

[183] Florentin Smarandache. Neutrosophic physics: More problems, more solutions. 2010.

[184] Florentin Smarandache. Degrees of membership> 1 and< 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[185] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[186] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[187] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[188] Florentin Smarandache. *Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components*. Infinite Study, 2021.

[189] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[190] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[191] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[192] Florentin Smarandache. Foundation of the superhypersoft set and the fuzzy extension superhypersoft set: A new vision. *Neutrosophic Systems with Applications*, 11:48–51, 2023.

[193] Florentin Smarandache. *New types of soft sets "hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": an improved version*. Infinite Study, 2023.

[194] Florentin Smarandache. Short introduction to standard and nonstandard neutrosophic set and logic. *Neutrosophic Sets and Systems*, 77:395–404, 2025.

[195] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[196] Florentin Smarandache, Said Broumi, Prem Kumar Singh, Chun-fang Liu, V Venkateswara Rao, Hai-Long Yang, Ion Patrascu, and Azeddine Elhassouny. Introduction to neutrosophy and neutrosophic environment. In *Neutrosophic Set in Medical Image Analysis*, pages 3–29. Elsevier, 2019.

[197] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[198] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[199] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[200] Alireza Soltani, Benedetto De Martino, and Colin Camerer. A range-normalization model of context-dependent choice: A new model and evidence. *PLoS Computational Biology*, 8, 2012.

[201] Seok Zun Song, Hee Sik Kim, and Young Bae Jun. Ideal theory in semigroups based on intersectional soft sets. *The Scientific World Journal*, 2014(1):136424, 2014.

[202] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[203] P Srivastava and RL Gupta. Fuzzy proximity structures and fuzzy ultrafilters. *Journal of Mathematical Analysis and Applications*, 94(2):297–311, 1983.

[204] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[205] M. Sugeno. Theory of fuzzy integrals and its applications. 1975.

[206] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[207] Jinjun Tang, Fang Liu, Wenhui Zhang, Ruimin Ke, and Yajie Zou. Lane-changes prediction based on adaptive fuzzy neural network. *Expert Syst. Appl.*, 91:452–463, 2018.

[208] Jinjun Tang, Fang Liu, Yajie Zou, Weibin Zhang, and Yinhai Wang. An improved fuzzy neural network for traffic speed prediction considering periodic characteristic. *IEEE Transactions on Intelligent Transportation Systems*, 18:2340–2350, 2017.

[209] Ye Tian, Lili Liu, Xiangjun Mi, and Bingyi Kang. Zslf: A new soft likelihood function based on z-numbers and its application in expert decision system. *IEEE Transactions on Fuzzy Systems*, 29:2283–2295, 2021.

[210] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[211] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[212] William Thomas Tutte. Matroids and graphs. *Transactions of the American Mathematical Society*, 90(3):527–552, 1959.

[213] Robert L Vaught. *Set theory: an introduction*. Springer Science & Business Media, 2001.

[214] Nikolai Konstantinovich Vereshchagin and Alexander Shen. *Basic set theory*. Number 17. American Mathematical Soc., 2002.

[215] Dhatchinamoorthy Vinoth and Devarasan Ezhilmaran. An analysis of global and adaptive thresholding for biometric images based on neutrosophic overset and underset approaches. *Symmetry*, 15:1102, 2023.

[216] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[217] Jia Wang and Zhenyuan Wang. Using neural networks to determine sugeno measures by statistics. *Neural Networks*, 10:183–195, 1997.

[218] Tianxing Wang, Huaxiong Li, Xianzhong Zhou, Dun Liu, and Bing Huang. Three-way decision based on third-generation prospect theory with z-numbers. *Inf. Sci.*, 569:13–38, 2021.

[219] Wei Wang and Xiao-long Xin. On fuzzy filters of pseudo bl-algebras. *Fuzzy Sets and Systems*, 162(1):27–38, 2011.

[220] Zhenyuan Wang and George J Klir. *Fuzzy measure theory*. Springer Science & Business Media, 2013.

[221] Dominic JA Welsh. *Matroid theory*. Courier Corporation, 2010.

[222] Jianxin Wu. Introduction to convolutional neural networks. *National Key Lab for Novel Software Technology. Nanjing University. China*, 5(23):495, 2017.

[223] Dongsheng Xu and Lijuan Peng. An improved method based on todim and topsis for multi-attribute decision-making with multi-valued neutrosophic sets. *Cmes-Computer Modeling in Engineering & Sciences*, 129(2), 2021.

[224] Xibei Yang, Dongjun Yu, Jingyu Yang, and Chen Wu. Generalization of soft set theory: from crisp to fuzzy case. In *Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE)*, pages 345–354. Springer, 2007.

[225] Jun Ye. Similarity measures based on the generalized distance of neutrosophic z-number sets and their multi-attribute decision making method. *Soft Computing*, 25:13975 – 13985, 2021.

[226] Fu Yuhua. Neutrosophic examples in physics. *Neutrosophic Sets and Systems*, 1:26–33, 2013.

[227] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[228] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[229] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[230] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[231] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[232] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[233] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[234] Lotfi A Zadeh. A note on z-numbers. *Information sciences*, 181(14):2923–2932, 2011.

[235] Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

[236] Xiaoyan Zhou, Mingwei Lin, and Weiwei Wang. Statistical correlation coefficients for single-valued neutrosophic sets and their applications in medical diagnosis. *AIMS Mathematics*, 2023.

[237] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.

[238] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

Review of Plithogenic Directed, Mixed, Bidirected, and Pangene OffGraph

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Abstract

Set theory, a fundamental branch of mathematics, focuses on the study of "sets," or collections of objects. To address real-world uncertainties more effectively, various extensions of classical set theory have been developed, including Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets. A particularly noteworthy concept in this field is the "offset," which introduces flexibility in defining membership functions within uncertain sets. Related concepts such as Overset and Underset are also well-known. These concepts have been extended to graph theory, with the Offset concept being further developed into OffGraph and studied extensively.

This paper builds upon Plithogenic Graphs to propose new extensions: the Plithogenic Directed OffGraph, the Plithogenic BiDirected OffGraph, and the Plithogenic Mixed OffGraph. Furthermore, we introduce the Plithogenic Pangene OffGraph, a generalization of the Plithogenic BiDirected OffGraph, and explore its mathematical properties.

Keywords: Neutrosophic OffSet, plithogenic Offset, fuzzy Offset, Neutrosophic OffGraph

1 Short Introduction

1.1 Neutrosophic Sets and Related Set Theories

Set theory, a fundamental branch of mathematics, focuses on the study of "sets," or collections of objects [32, 126, 127]. To address real-world uncertainties more effectively, numerous extensions of classical set theory have been developed, including Fuzzy Sets [29, 34, 35, 76, 124, 131, 135, 136], Vague Sets [8, 24, 26, 61, 134], Soft Sets [12, 13, 17, 82, 85, 130], Rough Sets [90, 91], and Hypersoft Expert Sets [2, 63–70].

Among these, Neutrosophic Sets [11, 22, 37, 87, 105, 106, 121, 128] stand out for their ability to simultaneously model truth, indeterminacy, and falsehood, making them especially versatile for dealing with complex systems and uncertainties. Plithogenic Sets, a more recent generalization, aim to further extend these models by incorporating multidimensional uncertainty [1, 54, 98, 104, 108, 109, 118, 122, 123].

A particularly noteworthy concept in this field is the "offset," which introduces flexibility in defining membership functions within uncertain sets [84, 107, 107, 112, 115]. Offsets allow membership values to extend beyond the conventional range, facilitating new ways of interpreting and applying uncertain set theory. Moreover, it is known that offsets generalize both Oversets and Undersets, further expanding their utility.

1.2 Graph Classes and Uncertain Graph Classes

Graph theory, the study of networks comprising nodes (vertices) and their connections (edges), is a well-established field in mathematics [33]. Due to its wide range of applications, including modeling real-world systems, graph theory has been extensively researched [33]. Over the years, numerous graph classes have been introduced based on specific properties and characteristics of graphs [19].

Among these, classical graph classes include undirected graphs, where edges lack orientation; directed graphs, where edges have specific directions [18, 23, 51, 73, 75, 95]; and mixed graphs, which incorporate both directed and undirected edges [39, 100, 101].

Uncertain graph models have emerged as a significant extension of classical graph theory to address uncertainties in various applications. These include Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Plithogenic Graphs, collectively referred to as "uncertain graphs" [4–10, 40, 41, 44, 45, 47–50, 97, 99, 107, 111, 119, 120]. These models introduce degrees of uncertainty into graph properties, allowing for more flexible representations of real-world networks.

Similarly, the concept of "OffGraphs" extends the flexibility of uncertain graphs by generalizing Oversets and Undersets(cf. [15, 30, 31, 46, 81]). OffGraphs enable researchers to reinterpret graph membership functions, further advancing the study of uncertain graph theory.

1.3 Our Contributions in This Paper

This paper builds upon Plithogenic Graphs to propose new extensions: the Plithogenic Directed OffGraph, the Plithogenic BiDirected OffGraph, and the Plithogenic Mixed OffGraph. Additionally, we introduce the Plithogenic Pangene OffGraph, a generalization of the Plithogenic BiDirected OffGraph, and explore its mathematical properties.

1.4 The Structure of the Paper

The structure of this paper is as follows.

2 Preliminaries and Definitions

Some foundational concepts from set theory are applied in parts of this work. For further details on these foundational concepts, please consult the relevant references as needed [38, 58, 62, 71, 78]. Additionally, for operations and related topics concerning each concept, please refer to the respective references as necessary.

2.1 Basic Set Theory

Below are some fundamental concepts in set theory. For more comprehensive details, please refer to the relevant references as needed [27, 38, 58–60, 62, 71, 74, 78, 102, 103].

Definition 2.1 (Set). [71] A *set* is a collection of distinct objects, known as elements, that are clearly defined, allowing any object to be identified as either belonging to or not belonging to the set. If A is a set and x is an element of A, this membership is denoted by $x \in A$. Sets are typically represented using curly brackets. For example, $A = \{1, 2, 3\}$ indicates that the set A contains the elements 1, 2, and 3.

Definition 2.2 (Union). [71] The *union* of two sets A and B is the set of all elements that are in either A, B, or both. The union is denoted by $A \cup B$ and is formally defined as:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

In other words, an element x is in $A \cup B$ if and only if x is in A, in B, or in both.

Definition 2.3 (Intersection). [71] The *intersection* of two sets A and B is the set of all elements that A and B have in common. The intersection is denoted by $A \cap B$ and is formally defined as:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

In other words, an element x is in $A \cap B$ if and only if x is in both A and B.

Definition 2.4 (Empty Set)**.** [71] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. Formally, the empty set is defined as:

$$\emptyset = \{x \mid x \neq x\},$$

indicating that there are no elements x for which the condition $x = x$ fails, thereby resulting in an empty collection. The empty set is a subset of every set and has a cardinality of zero.

Definition 2.5 (Non-Empty Set)**.** A *non-empty set* is a set that contains at least one element. Formally, a set S is non-empty if:

$$\exists x \in S.$$

In contrast to the empty set $\emptyset$, a non-empty set has a cardinality $|S| > 0$.

2.2 Crisp Sets and Neutrosophic Sets

When dealing with Fuzzy Sets or Neutrosophic Sets, they are often discussed alongside their foundational Crisp Sets. The definition of a Crisp Set is provided below.

Definition 2.6 (Universe Set)**.** (cf. [88]) A *universe set*, often denoted by U, is a set that contains all the elements under consideration for a particular discussion or problem domain. Formally, U is defined as a set that encompasses every element within the scope of a given context or framework, so that any subset of interest can be regarded as a subset of U.

In set theory, the universe set U is typically assumed to contain all elements relevant to the discourse, meaning that for any set A, if $A \subseteq U$, then all elements of A are elements of U. Related concepts include underlying sets and whole sets.

Definition 2.7 (Non-empty Universe Set)**.** A *non-empty universe set*, denoted by U, is a set that contains all elements under consideration in a specific context and satisfies $U \neq \emptyset$. Formally:

$$U = \{x \mid x \text{ is relevant to the problem domain}\}, \quad \text{with } U \neq \emptyset.$$

Every subset of interest is considered a subset of U, ensuring that $A \subseteq U$ for any A.

Definition 2.8 (Crisp Set)**.** [89] Let X be a universe set, and let $P(X)$ denote the power set of X, which represents all subsets of X. A *crisp set* $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{0, 1\}$, where:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This function χ_A assigns a value of 1 to elements within the set A and 0 to those outside it, creating a clear boundary. Crisp sets are thus bivalent and follow the principle of binary classification, where each element is either a member of the set or not.

Definition 2.9 (Crisp Set Representation of the Empty Set)**.** The empty set $\emptyset$ in a universe X is represented as a Crisp Set $A \subseteq X$ by the characteristic function $\chi_\emptyset : X \to \{0, 1\}$, where:

$$\chi_\emptyset(x) = 0 \quad \text{for all } x \in X.$$

The Fuzzy Set is a well-known concept used to handle uncertainty in set theory. The definition is provided below [131].

Definition 2.10. [131, 132] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 2.11. [105] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [108, 109]. The definition of the Plithogenic Set is provided below.

Definition 2.12. [108, 109] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

Example 2.13. The following examples are provided.

- When $s = t = 1$, PG is called a **Plithogenic Fuzzy Set**.
- When $s = 2, t = 1$, PG is called a **Plithogenic Intuitionistic Fuzzy Set**.
- When $s = 3, t = 1$, PG is called a **Plithogenic Neutrosophic Set**.
- When $s = 4, t = 1$, PG is called a **Plithogenic Quadripartitioned Neutrosophic Set** (cf. [92, 96]).
- When $s = 5, t = 1$, PG is called a **Plithogenic Pentapartitioned Neutrosophic Set** (cf. [21, 25, 28, 83]).
- When $s = 7, t = 1$, PG is called a **Plithogenic Heptapartitioned Neutrosophic Set** (cf. [20, 86]).

2.3 Plithogenic Offset/Overset/Underset

Related concepts include the Plithogenic Offset. As restricted versions of the Offset, the Overset (where only Ω is unrestricted) and the Underset (where only ψ is unrestricted) are known [46].

While this paper primarily focuses on the definition of the Offset, any concept defined using the Offset can also be defined using the Overset or Underset. For example, a Fuzzy Overset or Fuzzy Underset can define a Fuzzy Offset.

Definition 2.14 (Crisp Offset)**.** (cf. [46]) Let X be a universe of discourse, and let Ψ and Ω represent 0 and 1, respectively. A *Crisp Offset* $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{\Psi, \Omega\}$, where:

$$\chi_A(x) = \begin{cases} \Omega & \text{if } x \in A, \\ \Psi & \text{if } x \notin A. \end{cases}$$

In this context, the function χ_A assigns a value of Ω (1) to elements that are within the set A and Ψ (0) to elements that are outside A. This structure adheres to the principle of binary classification, as each element is either fully included in the set A or completely excluded from it.

The concept of a Crisp Offset, unlike fuzzy or neutrosophic sets, does not allow for intermediate degrees of membership. Instead, membership is strictly limited to the values Ψ and Ω, reflecting the clear-cut, deterministic nature of this classification approach. This discrete boundary is a distinguishing feature of Crisp Offsets, contrasting with the gradual membership levels typical of fuzzy sets.

Definition 2.15 (Fuzzy Offset)**.** (cf. [107]) Let X be a universe of discourse. A *Fuzzy Offset* $\tilde{A}$ in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [\Psi, \Omega]\},$$

where $\Omega > 1$ and $\Psi < 0$. There exist elements $x, y \in X$ such that $\mu_{\tilde{A}}(x) > 1$ and $\mu_{\tilde{A}}(y) < 0$.

Definition 2.16 (Single-Valued Neutrosophic OffSet)**.** [114] A *Single-Valued Neutrosophic OffSet*, denoted $A_{\text{off}} \subseteq U_{\text{off}}$, is a set within a universe of discourse U_{off} in which certain elements may possess neutrosophic degrees—truth, indeterminacy, or falsity—that extend beyond the standard limits, either above 1 or below 0. It is formally defined as:

$$A_{\text{off}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{off}}, \exists\ (T(x) > 1 \text{ or } F(x) < 0)\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of each $x \in U_{\text{off}}$.
- $T(x), I(x), F(x) \in [\Psi, \Omega]$, where $\Omega > 1$ (termed the *OverLimit*) and $\Psi < 0$ (termed the *UnderLimit*), allow the possibility for $T(x)$, $I(x)$, or $F(x)$ to take values beyond the conventional bounds of $[0, 1]$.

Definition 2.17 (Plithogenic Offset)**.** (cf. [46]) Let S be a universal set, and $P \subseteq S$. A *Plithogenic Offset* PS_{off} is defined as:

$$PS_{\text{off}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to [\Psi_v, \Omega_v]^s$ is the *Degree of Appurtenance Function (DAF)*, where $\Psi_v < 0$ and $\Omega_v > 1$.
- $pCF : Pv \times Pv \to [\Psi_v, \Omega_v]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, the DAF allows the membership degrees $pdf(x, a)$ to range from below 0 to above 1, between the underlimit Ψ_v and the overlimit Ω_v.

Example 2.18. The following examples are provided.

- When $s = t = 1$, PG is called a **Plithogenic Fuzzy OffSet**.
- When $s = 2, t = 1$, PG is called a **Plithogenic Intuitionistic Fuzzy OffSet**.
- When $s = 3, t = 1$, PG is called a **Plithogenic Neutrosophic OffSet**.
- When $s = 4, t = 1$, PG is called a **Plithogenic Quadripartitioned Neutrosophic OffSet**.
- When $s = 5, t = 1$, PG is called a **Plithogenic Pentapartitioned Neutrosophic OffSet**.
- When $s = 7, t = 1$, PG is called a **Plithogenic Heptapartitioned Neutrosophic OffSet**.

2.4 Plithogenic Offgraph/Overgraph/Undergraph

Related concepts include Plithogenic Offgraph/Overgraph/Undergraph. As restricted versions of the OffGraph, the OverGraph (where only Ω is unrestricted) and the UnderGraph (where only ψ is unrestricted) are known [46].

Definition 2.19 (Graph)**.** [33] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.20 (Subgraph)**.** [33] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 2.21 (Fuzzy Offgraph)**.** (cf. [46]) A *Fuzzy Offgraph* $G = (V, E, \mu_V, \mu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- A vertex membership function $\mu_V : V \to [\Psi, \Omega]$, with $\Omega > 1$, $\Psi < 0$, and $\exists v \in V$ such that $\mu_V(v) > 1$ or $\mu_V(v) < 0$.
- An edge membership function $\mu_E : E \to [\Psi, \Omega]$, where $\exists e \in E$ such that $\mu_E(e) > 1$ or $\mu_E(e) < 0$.

Definition 2.22. (cf. [31,46]) A *Single-Valued Neutrosophic OffGraph* is a graph $G = (V, E)$ defined over a universe U_{off}, where:

- Each vertex $v \in V$ is assigned degrees $T(v)$, $I(v)$, and $F(v)$, with $T(v) \in [0, \Omega]$ and $F(v) \in [\Psi, \Omega]$, where $\Omega > 1$ and $\Psi < 0$, allowing $T(v) > 1$ and $F(v) < 0$.
- Each edge $e = (u, v) \in E$ is assigned degrees $T(e) \in [\Psi, \Omega]$, $I(e) \in [\Psi, \Omega]$, and $F(e) \in [\Psi, \Omega]$.
- For all $v \in V$, $T(v) + I(v) + F(v) \leq 3\Omega$.

Definition 2.23 (Plithogenic OffGraph)**.** (cf. [46]) A *Plithogenic OffGraph* is a Plithogenic Graph where the membership degrees μ_{A_i} can both exceed 1 and be less than 0. That is, $\mu_{A_i}(x) \in [\Psi_i, \Omega_i]$ with $\Psi_i < 0$ and $\Omega_i > 1$ for all attributes A_i and elements $x \in V \cup E$.

Example 2.24. The following examples are provided.

- When $s = t = 1$, PG is called a **Plithogenic Fuzzy OffGraph**.
- When $s = 2, t = 1$, PG is called a **Plithogenic Intuitionistic Fuzzy OffGraph**.
- When $s = 3, t = 1$, PG is called a **Plithogenic Neutrosophic OffGraph**.
- When $s = 4, t = 1$, PG is called a **Plithogenic Quadripartitioned Neutrosophic OffGraph**.
- When $s = 5, t = 1$, PG is called a **Plithogenic Pentapartitioned Neutrosophic OffGraph**.
- When $s = 7, t = 1$, PG is called a **Plithogenic Heptapartitioned Neutrosophic OffGraph**.

3 Result: Some Concepts

In this paper, we propose several types of graphs by applying the concept of offsets to existing graph and set theories, exploring their mathematical structures. Since Overset and Underset are concepts derived by limiting one side of the range of an offset, any graph defined as an OffGraph can also be defined as an OverGraph or an UnderGraph. We hope that future research will further develop the applications and mathematical structures of these concepts.

3.1 Plithogenic Directed OffGraph

We define the Plithogenic Directed OffGraph. This is a definition that extends the Plithogenic Directed Graph by incorporating the concept of offsets.

Definition 3.1 (Directed Graph)**.** [3, 133] A *Directed Graph*, denoted as $G = (V, E)$, is a mathematical structure consisting of:

- V: A finite set of vertices (also called nodes).
- E: A set of ordered pairs of vertices, called directed edges or arcs. Each directed edge is represented as (u, v), where $u, v \in V$ and $u \neq v$.

Definition 3.2 (Plithogenic Directed Graph)**.** [43] A *Plithogenic Directed Graph* $G_P = (P_M, P_N)$ extends the concept of Plithogenic Graphs to directed graphs. Let $G = (V, E)$ be a crisp directed graph where:

- V is a finite set of vertices.
- $E \subseteq V \times V$ is a set of directed edges.

The Plithogenic Directed Graph $G_P = (P_M, P_N)$ is defined as follows:

1. **Plithogenic Vertex Set** $P_M = (M, l, M_l, \text{adf}, \text{acf})$:
   - $M \subseteq V$: The set of vertices.
   - l: An attribute associated with the vertices.
   - M_l: The range of possible attribute values for l.
   - $\text{adf} : M \times M_l \to [0, 1]^s$: The Degree of Appurtenance Function (DAF) for vertices.
   - $\text{acf} : M_l \times M_l \to [0, 1]^t$: The Degree of Contradiction Function (DCF) for vertices.
2. **Plithogenic Edge Set** $P_N = (N, m, N_m, \text{bdf}, \text{bcf})$:
   - $N \subseteq E$: The set of directed edges.
   - m: An attribute associated with the edges.
   - N_m: The range of possible attribute values for m.
   - $\text{bdf} : N \times N_m \to [0, 1]^s$: The Degree of Appurtenance Function (DAF) for edges.
   - $\text{bcf} : N_m \times N_m \to [0, 1]^t$: The Degree of Contradiction Function (DCF) for edges.

Conditions for Plithogenic Directed Graphs The Plithogenic Directed Graph G_P must satisfy the following conditions:

1. **Edge Appurtenance Constraint**: For all $(x, a), (y, b) \in M \times M_l$:

$$\text{bdf}((xy), (a, b)) \leq \min\{\text{adf}(x, a), \text{adf}(y, b)\},$$

where $xy \in N$ is a directed edge from x to y, and $(a, b) \in N_m \times N_m$ are the corresponding attribute values.

2. **Contradiction Function Constraint**: For all $(a, b), (c, d) \in N_m \times N_m$:

$$\text{bcf}((a, b), (c, d)) \leq \min\{\text{acf}(a, c), \text{acf}(b, d)\}.$$

3. **Reflexivity and Symmetry of Contradiction Functions**:

$$\text{acf}(a, a) = 0, \quad \text{acf}(a, b) = \text{acf}(b, a), \quad \forall a, b \in M_l,$$

$$\text{bcf}(a, a) = 0, \quad \text{bcf}(a, b) = \text{bcf}(b, a), \quad \forall a, b \in N_m.$$

Example 3.3. The Plithogenic Directed Graph G_P can be classified into specific subclasses depending on the parameters s and t. The following are examples of special cases of Plithogenic Directed Graphs:

- When $s = t = 1$, G_P is called a **Plithogenic Fuzzy Directed Graph**. In this case, each vertex and edge has a single membership degree.
- When $s = 2, t = 1$, G_P is called a **Plithogenic Intuitionistic Fuzzy Directed Graph**. Each vertex and edge is associated with two membership degrees: truth and falsity.
- When $s = 3, t = 1$, G_P is called a **Plithogenic Neutrosophic Directed Graph**. Each vertex and edge is characterized by triplets of truth, indeterminacy, and falsity membership degrees.
- When $s = 4, t = 1$, G_P is called a **Plithogenic Quadripartitioned Neutrosophic Directed Graph**. Each vertex and edge is represented by quadruples of membership degrees, including truth, indeterminacy, falsity, and liberation.
- When $s = 5, t = 1$, G_P is called a **Plithogenic Pentapartitioned Neutrosophic Directed Graph**. Each vertex and edge has quintuple membership degrees corresponding to truth, contradiction, ignorance, unknown, and falsity.
- When $s = 7, t = 1$, G_P is called a **Plithogenic Heptapartitioned Neutrosophic Directed Graph**. Each vertex and edge has seven membership degrees representing various nuanced states of uncertainty.

Definition 3.4 (Plithogenic Directed OffGraph)**.** A *Plithogenic Directed OffGraph* $G_{\text{off}} = (P_M, P_N)$ is an extension of the Plithogenic Directed Graph, incorporating the concept of offsets into its structure. Let $G = (V, E)$ be a crisp directed graph where:

- V is a finite set of vertices.
- $E \subseteq V \times V$ is a set of directed edges.

The Plithogenic Directed OffGraph $G_{\text{off}} = (P_M, P_N)$ is defined as follows:

1. **Plithogenic Offset Vertex Set** $P_M = (M, l, M_l, \text{adf}, \text{acf})$:
   - $M \subseteq V$: The set of vertices.
   - l: An attribute associated with the vertices.
   - M_l: The range of possible attribute values for l.
   - $\text{adf} : M \times M_l \to [\Psi_v, \Omega_v]^s$: The Degree of Appurtenance Function (DAF) for vertices, where $\Psi_v < 0$ and $\Omega_v > 1$.
   - $\text{acf} : M_l \times M_l \to [\Psi_v, \Omega_v]^t$: The Degree of Contradiction Function (DCF) for vertices.
2. **Plithogenic Offset Edge Set** $P_N = (N, m, N_m, \text{bdf}, \text{bcf})$:
   - $N \subseteq E$: The set of directed edges.
   - m: An attribute associated with the edges.
   - N_m: The range of possible attribute values for m.
   - $\text{bdf} : N \times N_m \to [\Psi_e, \Omega_e]^s$: The Degree of Appurtenance Function (DAF) for edges, where $\Psi_e < 0$ and $\Omega_e > 1$.
   - $\text{bcf} : N_m \times N_m \to [\Psi_e, \Omega_e]^t$: The Degree of Contradiction Function (DCF) for edges.

The Plithogenic Directed OffGraph satisfies the same constraints as the Plithogenic Directed Graph but allows values of DAF and DCF to extend beyond the $[0, 1]$ range, within $[\Psi_v, \Omega_v]$ and $[\Psi_e, \Omega_e]$.

Theorem 3.5. *A Plithogenic Directed OffGraph $G_{\mathit{off}} = (P_M, P_N)$ can be transformed into a Plithogenic Directed Graph $G_P = (P'_M, P'_N)$ by restricting the Degree of Appurtenance Function (DAF) and Degree of Contradiction Function (DCF) to* $[0, 1]$.

Proof. Let $G_{\text{off}} = (P_M, P_N)$ be a Plithogenic Directed OffGraph, where:

$$\text{adf} : M \times M_l \to [\Psi_v, \Omega_v]^s, \quad \text{acf} : M_l \times M_l \to [\Psi_v, \Omega_v]^t.$$

To transform G_{off} into G_P: 1. Restrict adf to $[0,1]^s$ by defining:

$$\text{adf}'(x,a) = \begin{cases} \text{adf}(x,a), & \text{if adf}(x,a) \in [0,1]^s, \\ \max(0, \min(\text{adf}(x,a), 1)), & \text{otherwise.} \end{cases}$$

2. Similarly, restrict acf to $[0,1]^t$:

$$\text{acf}'(a,b) = \begin{cases} \text{acf}(a,b), & \text{if acf}(a,b) \in [0,1]^t, \\ \max(0, \min(\text{acf}(a,b), 1)), & \text{otherwise.} \end{cases}$$

Perform analogous transformations for the edge-related functions bdf and bcf. The resulting structure $G_P = (P'_M, P'_N)$ is a Plithogenic Directed Graph, completing the proof. □

Theorem 3.6. *A Plithogenic Directed OffGraph $G_{off} = (P_M, P_N)$ can be transformed into a Plithogenic OffGraph $G'_{off} = (P_M, P'_N)$ by treating all directed edges as undirected edges.*

Proof. Let $G_{\text{off}} = (P_M, P_N)$ be a Plithogenic Directed OffGraph. To transform G_{off} into a Plithogenic OffGraph: 1. Replace the directed edge set $N \subseteq V \times V$ with an undirected edge set $N' \subseteq \{\{u,v\} \mid (u,v) \in N\}$. 2. Define $\text{bdf}'(e,a) = \text{bdf}(e,a)$ for undirected edges $e = \{u,v\}$. 3. Similarly, $\text{bcf}'(a,b) = \text{bcf}(a,b)$ for $a, b \in N_m$.

Since the contradiction and appurtenance functions remain well-defined in the undirected context, the resulting graph G'_{off} satisfies the properties of a Plithogenic OffGraph. This transformation completes the proof. □

3.2 Plithogenic Mixed OffGraph

We define the Plithogenic Mixed OffGraph. This is a definition that extends the Plithogenic Mixed Graph by incorporating the concept of offsets.

Definition 3.7 (Mixed Graph)**.** (cf. [100]) A *Mixed Graph* $G = (V, E, A)$ is a mathematical structure that combines the properties of undirected and directed graphs. It consists of the following components:

- V: A non-empty finite set of vertices.
- $E \subseteq \{\{u,v\} \mid u, v \in V, u \neq v\}$: A set of undirected edges, where each edge $e = \{u,v\}$ represents a connection between vertices u and v with no specific direction.
- $A \subseteq \{(u,v) \mid u, v \in V, u \neq v\}$: A set of directed arcs, where each arc $a = (u,v)$ represents a directed connection from vertex u to vertex v.

The graph G satisfies the following properties:

1. Each pair of vertices $u, v \in V$ can have at most one undirected edge $\{u,v\} \in E$.
2. Each pair of vertices $u, v \in V$ can have at most two directed arcs: $(u,v) \in A$ and $(v,u) \in A$.
3. The undirected edges in E do not overlap with the directed arcs in A, meaning $E \cap A = \emptyset$.

Definition 3.8 (Plithogenic Mixed Graph)**.** A *Plithogenic Mixed Graph* $G = (V, E, A, \text{adf}_V, \text{adf}_E, \text{adf}_A, \text{acf}_V, \text{acf}_E, \text{acf}_A)$ is a graph that generalizes both directed and undirected edges within the Plithogenic framework. It is defined as follows:

- V: The set of vertices.
- E: The set of undirected edges, where $E \subseteq \{\{u,v\} \mid u,v \in V, u \neq v\}$.
- A: The set of directed edges (arcs), where $A \subseteq \{(u,v) \mid u,v \in V, u \neq v\}$.
- $\mathrm{adf}_V : V \to [0,1]^s$: The attribute degree function for vertices, assigning an s-tuple $\mathrm{adf}_V(v) = (a_1(v), a_2(v), \ldots, a_s(v))$ to each vertex $v \in V$.
- $\mathrm{adf}_E : E \to [0,1]^s$: The attribute degree function for undirected edges.
- $\mathrm{adf}_A : A \to [0,1]^s$: The attribute degree function for directed edges.
- $\mathrm{acf}_V : V \times V \to [0,1]^t$: The contradiction degree function for vertices.
- $\mathrm{acf}_E : E \times E \to [0,1]^t$: The contradiction degree function for undirected edges.
- $\mathrm{acf}_A : A \times A \to [0,1]^t$: The contradiction degree function for directed edges.

These functions satisfy the following conditions: 1. **Edge Attribute Degree Constraint**:

- For each undirected edge $e = \{u,v\} \in E$:

$$\mathrm{adf}_E(e) \leq \min\{\mathrm{adf}_V(u), \mathrm{adf}_V(v)\},$$

  where the minimum operation is taken component-wise.

- For each directed edge $a = (u,v) \in A$:

$$\mathrm{adf}_A(a) \leq \min\{\mathrm{adf}_V(u), \mathrm{adf}_V(v)\}.$$

2. **Contradiction Function Constraints**:

- The contradiction functions satisfy:

$$\mathrm{acf}_V(u,u) = 0, \quad \mathrm{acf}_V(u,v) = \mathrm{acf}_V(v,u), \quad \forall u,v \in V.$$

- Similar properties hold for acf_E and acf_A:

$$\mathrm{acf}_E(e_1,e_2) = \mathrm{acf}_E(e_2,e_1), \quad \mathrm{acf}_A(a_1,a_2) = \mathrm{acf}_A(a_2,a_1).$$

Example 3.9. The Plithogenic Mixed Graph G_P can be classified into specific subclasses depending on the parameters s and t. The following are examples of special cases of Plithogenic Mixed Graphs:

- When $s = t = 1$, G_P is called a **Plithogenic Fuzzy Mixed Graph**. In this case, each vertex and edge has a single membership degree.
- When $s = 2, t = 1$, G_P is called a **Plithogenic Intuitionistic Fuzzy Mixed Graph**. Each vertex and edge is associated with two membership degrees: truth and falsity.
- When $s = 3, t = 1$, G_P is called a **Plithogenic Neutrosophic Mixed Graph**. Each vertex and edge is characterized by triplets of truth, indeterminacy, and falsity membership degrees.
- When $s = 4, t = 1$, G_P is called a **Plithogenic Quadripartitioned Neutrosophic Mixed Graph**. Each vertex and edge is represented by quadruples of membership degrees, including truth, indeterminacy, falsity, and liberation.
- When $s = 5, t = 1$, G_P is called a **Plithogenic Pentapartitioned Neutrosophic Mixed Graph**. Each vertex and edge has quintuple membership degrees corresponding to truth, contradiction, ignorance, unknown, and falsity.

- When $s = 7, t = 1$, G_P is called a **Plithogenic Heptapartitioned Neutrosophic Mixed Graph**. Each vertex and edge has seven membership degrees representing various nuanced states of uncertainty.

Theorem 3.10. *A Plithogenic Mixed Graph $G = (V, E, A, adf_V, adf_E, adf_A, acf_V, acf_E, acf_A)$ can be transformed into a Plithogenic Directed Graph $G' = (V, A, adf_V, adf_A, acf_V, acf_A)$ by removing all undirected edges E.*

Proof. Remove all undirected edges E from the graph structure. Retain all vertices V, directed edges A, and the associated functions $\text{adf}_V, \text{adf}_A, \text{acf}_V, \text{acf}_A$. Since the removal of E does not affect the properties of A or V, the resulting graph G' satisfies all the conditions of a Plithogenic Directed Graph. This completes the proof. □

Theorem 3.11. *A Plithogenic Mixed Graph $G = (V, E, A, adf_V, adf_E, adf_A, acf_V, acf_E, acf_A)$ can be transformed into a Plithogenic Graph $G' = (V, E', adf_V, adf'_E, acf_V, acf'_E)$ by treating all directed edges A as undirected edges.*

Proof. To transform G into G', Define a new undirected edge set $E' = E \cup \{\{u, v\} \mid (u, v) \in A\}$, effectively treating all directed edges A as undirected. Retain all vertices V and redefine adf'_E and acf'_E to include the attributes and contradictions of both original E and transformed A:

$$\text{adf}'_E(e) = \begin{cases} \text{adf}_E(e), & \text{if } e \in E, \\ \text{adf}_A(a), & \text{if } e = \{u, v\} \text{ and } a = (u, v) \in A. \end{cases}$$

$$\text{acf}'_E(e_1, e_2) = \begin{cases} \text{acf}_E(e_1, e_2), & \text{if } e_1, e_2 \in E, \\ \text{acf}_A(a_1, a_2), & \text{if } e_1, e_2 \text{ are transformed from } A. \end{cases}$$

Since A is transformed into undirected edges in E', the resulting graph G' satisfies the conditions of a Plithogenic Graph. This completes the proof. □

Definition 3.12 (Plithogenic Mixed OffGraph)**.** A *Plithogenic Mixed OffGraph*

$$G_{\text{off}} = (V, E, A, \text{adf}_V, \text{adf}_E, \text{adf}_A, \text{acf}_V, \text{acf}_E, \text{acf}_A)$$

is an extension of the Plithogenic Mixed Graph that incorporates the concept of offsets. It is defined as follows:

- V: The set of vertices.
- E: The set of undirected edges, where $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$.
- A: The set of directed edges (arcs), where $A \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$.
- $\text{adf}_V : V \to [\Psi_v, \Omega_v]^s$: The attribute degree function for vertices, assigning an s-tuple $\text{adf}_V(v) = (a_1(v), a_2(v), \ldots, a_s(v))$ to each vertex $v \in V$, where $\Psi_v < 0$ and $\Omega_v > 1$.
- $\text{adf}_E : E \to [\Psi_e, \Omega_e]^s$: The attribute degree function for undirected edges, where $\Psi_e < 0$ and $\Omega_e > 1$.
- $\text{adf}_A : A \to [\Psi_a, \Omega_a]^s$: The attribute degree function for directed edges, where $\Psi_a < 0$ and $\Omega_a > 1$.
- $\text{acf}_V : V \times V \to [\Psi_v, \Omega_v]^t$: The contradiction degree function for vertices.
- $\text{acf}_E : E \times E \to [\Psi_e, \Omega_e]^t$: The contradiction degree function for undirected edges.
- $\text{acf}_A : A \times A \to [\Psi_a, \Omega_a]^t$: The contradiction degree function for directed edges.

These functions satisfy the following conditions:

1. **Edge Attribute Degree Constraint**:

- For each undirected edge $e = \{u, v\} \in E$:

$$\text{adf}_E(e) \leq \min\{\text{adf}_V(u), \text{adf}_V(v)\},$$

where the minimum operation is taken component-wise.

- For each directed edge $a = (u, v) \in A$:

$$\text{adf}_A(a) \leq \min\{\text{adf}_V(u), \text{adf}_V(v)\}.$$

2. **Contradiction Function Constraints**:

- The contradiction functions satisfy:

$$\text{acf}_V(u, u) = 0, \quad \text{acf}_V(u, v) = \text{acf}_V(v, u), \quad \forall u, v \in V.$$

- Similar properties hold for acf_E and acf_A:

$$\text{acf}_E(e_1, e_2) = \text{acf}_E(e_2, e_1), \quad \text{acf}_A(a_1, a_2) = \text{acf}_A(a_2, a_1).$$

Theorem 3.13. *A Plithogenic Mixed OffGraph G_{off} can be transformed into a Plithogenic Directed OffGraph G'_{off} by removing all undirected edges E.*

Proof. Remove all undirected edges E from the graph structure. Retain all vertices V, directed edges A, and the associated functions $\text{adf}_V, \text{adf}_A, \text{acf}_V, \text{acf}_A$. Since the removal of E does not affect the properties of A or V, the resulting graph G'_{off} satisfies all the conditions of a Plithogenic Directed OffGraph. This completes the proof. □

Theorem 3.14. *A Plithogenic Mixed OffGraph G_{off} can be transformed into a Plithogenic OffGraph G'_{off} by treating all directed edges A as undirected edges.*

Proof. Define a new undirected edge set $E' = E \cup \{\{u, v\} \mid (u, v) \in A\}$, effectively treating all directed edges A as undirected. Retain all vertices V and redefine adf'_E and acf'_E to include the attributes and contradictions of both original E and transformed A:

$$\text{adf}'_E(e) = \begin{cases} \text{adf}_E(e), & \text{if } e \in E, \\ \text{adf}_A(a), & \text{if } e = \{u, v\} \text{ and } a = (u, v) \in A. \end{cases}$$

$$\text{acf}'_E(e_1, e_2) = \begin{cases} \text{acf}_E(e_1, e_2), & \text{if } e_1, e_2 \in E, \\ \text{acf}_A(a_1, a_2), & \text{if } e_1, e_2 \text{ are transformed from } A. \end{cases}$$

The resulting graph G'_{off} satisfies the conditions of a Plithogenic OffGraph. This completes the proof. □

Theorem 3.15. *A Plithogenic Mixed OffGraph $G_{off} = (V, E, A, adf_V, adf_E, adf_A, acf_V, acf_E, acf_A)$ is a Mixed Graph under its structural definition.*

Proof. To prove this, we verify that the structural components of G_{off} align with the definition of a Mixed Graph.

- The set of vertices V in G_{off} is well-defined and finite, satisfying the vertex condition of a Mixed Graph.
- The set $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ represents undirected edges. Each edge $e = \{u, v\}$ connects vertices u and v without a specific direction, meeting the Mixed Graph definition.
- The set $A \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$ represents directed arcs. Each arc $a = (u, v)$ establishes a directed relationship from vertex u to vertex v, fulfilling the Mixed Graph requirement.

- While $\text{adf}_V, \text{adf}_E, \text{adf}_A, \text{acf}_V, \text{acf}_E, \text{acf}_A$ provide additional attributes for vertices and edges, they do not alter the fundamental structure of the graph. These functions extend the standard Mixed Graph by adding appurtenance and contradiction degrees, which are extraneous to the basic connectivity.
- The undirected edges E and directed arcs A satisfy the condition $E \cap A = \emptyset$, ensuring no overlap between the two types of connections.

Thus, the structural components (V, E, A) of G_{off} form a Mixed Graph, with the offset functions acting as additional features that do not interfere with the underlying graph structure. □

3.3 Plithogenic BiDirected OffGraph

We define the Plithogenic BiDirected OffGraph. This is a definition that extends the Plithogenic BiDirected Graph by incorporating the concept of offsets.

Definition 3.16 (Bidirected Graph)**.** [14, 36, 52] A *bidirected graph* (also known as a *bigraph*) is a pair $B = (G, \tau)$, where:

- $G = (V, E)$ is a simple undirected graph, where V is a non-empty set of vertices and E is a set of edges (without parallel edges or loops).
- $\tau : V \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, which assigns a *local orientation* to each vertex-edge pair (v, e) as follows:
  - $\tau(v, e) = 1$: Edge e is directed *towards* vertex v.
  - $\tau(v, e) = -1$: Edge e is directed *away from* vertex v.
  - $\tau(v, e) = 0$: Vertex v is not incident to edge e.

The graph G is referred to as the *underlying graph* of B, and the function τ provides the bidirected structure on G by assigning a direction at each endpoint of every edge in E.

Definition 3.17 (Bidirected Plithogenic Graph)**.** A *Bidirected Plithogenic Graph* is a tuple $BPG = (G, \tau, PM, PN)$, where:

- $G = (V, E)$ is a simple undirected graph.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function as defined above.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic Vertex Set:
  - $M \subseteq V$ is the set of vertices.
  - l is the vertex attribute.
  - Ml is the range of possible vertex attribute values.
  - $adf : M \times Ml \to [0, 1]^s$ is the Degree of Appurtenance Function.
  - $aCf : Ml \times Ml \to [0, 1]^t$ is the Degree of Contradiction Function for vertices.
- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic Edge Set:
  - $N \subseteq E$ is the set of edges.
  - m is the edge attribute.
  - Nm is the range of possible edge attribute values.
  - $bdf : N \times Nm \to [0, 1]^s$ is the Degree of Appurtenance Function.
  - $bCf : Nm \times Nm \to [0, 1]^t$ is the Degree of Contradiction Function for edges.

The functions adf and bdf must satisfy the Appurtenance Constraint, and aCf and bCf must satisfy the Contradiction Function Constraints as defined in Plithogenic Graphs.

Example 3.18. The Plithogenic BiDirected Graph G_P can be classified into specific subclasses depending on the parameters s and t. The following are examples of special cases of Plithogenic BiDirected Graphs:

- When $s = t = 1$, G_P is called a **Plithogenic Fuzzy BiDirected Graph**. In this case, each vertex and edge has a single membership degree.
- When $s = 2, t = 1$, G_P is called a **Plithogenic Intuitionistic Fuzzy BiDirected Graph**. Each vertex and edge is associated with two membership degrees: truth and falsity.
- When $s = 3, t = 1$, G_P is called a **Plithogenic Neutrosophic BiDirected Graph**. Each vertex and edge is characterized by triplets of truth, indeterminacy, and falsity membership degrees.
- When $s = 4, t = 1$, G_P is called a **Plithogenic Quadripartitioned Neutrosophic BiDirected Graph**. Each vertex and edge is represented by quadruples of membership degrees, including truth, indeterminacy, falsity, and liberation.
- When $s = 5, t = 1$, G_P is called a **Plithogenic Pentapartitioned Neutrosophic BiDirected Graph**. Each vertex and edge has quintuple membership degrees corresponding to truth, contradiction, ignorance, unknown, and falsity.
- When $s = 7, t = 1$, G_P is called a **Plithogenic Heptapartitioned Neutrosophic BiDirected Graph**. Each vertex and edge has seven membership degrees representing various nuanced states of uncertainty.

Definition 3.19 (Plithogenic BiDirected OffGraph)**.** A *Plithogenic BiDirected OffGraph* is a tuple $PBGO = (G, \tau, PM, PN)$, where:

- $G = (V, E)$: An undirected graph with vertex set V and edge set E.
- $\tau : V \times E \to \{-1, 0, 1\}$: The bidirection function, defining the orientation of edges relative to vertices.
- $PM = (M, l, M_l, \text{adf}, \text{acf})$: The Plithogenic Offset Vertex Set, where:
  - $M \subseteq V$: The set of vertices.
  - l: An attribute associated with the vertices.
  - M_l: The range of possible attribute values for l.
  - $\text{adf} : M \times M_l \to [\Psi_v, \Omega_v]^s$: The Degree of Appurtenance Function (DAF) for vertices, where $\Psi_v < 0$ and $\Omega_v > 1$.
  - $\text{acf} : M_l \times M_l \to [\Psi_v, \Omega_v]^t$: The Degree of Contradiction Function (DCF) for vertices.
- $PN = (N, m, N_m, \text{bdf}, \text{bcf})$: The Plithogenic Offset Edge Set, where:
  - $N \subseteq E$: The set of edges.
  - m: An attribute associated with the edges.
  - N_m: The range of possible attribute values for m.
  - $\text{bdf} : N \times N_m \to [\Psi_e, \Omega_e]^s$: The Degree of Appurtenance Function (DAF) for edges, where $\Psi_e < 0$ and $\Omega_e > 1$.
  - $\text{bcf} : N_m \times N_m \to [\Psi_e, \Omega_e]^t$: The Degree of Contradiction Function (DCF) for edges.

The Degree of Appurtenance Functions (adf, bdf) and Contradiction Functions (acf, bcf) satisfy the following constraints:

1. **Appurtenance Constraint**:
$$\text{bdf}(e, a) \le \min\{\text{adf}(u, l), \text{adf}(v, l)\},$$
where $e = \{u, v\}$, $u, v \in M$, and $a \in N_m$.

2. **Contradiction Constraint**:

$$\mathrm{bcf}(a,b) \le \min\{\mathrm{acf}(x,y)\},$$

where $a, b \in N_m$, and $x, y \in M_l$.

Theorem 3.20. *A Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM, PN)$ retains the structure of a BiDirected Graph $B = (G, \tau)$.*

Proof. To demonstrate that $PBGO$ retains the structure of a BiDirected Graph, we verify that it satisfies the properties of $B = (G, \tau)$.

The graph $G = (V, E)$ in $PBGO$ is undirected, with V as the vertex set and E as the edge set. This satisfies the definition of the underlying graph of a BiDirected Graph.

The bidirection function $\tau : V \times E \to \{-1, 0, 1\}$ in $PBGO$ assigns local orientations to edges relative to vertices. The values of τ follow the same rules as in a BiDirected Graph:

$$\tau(v,e) = \begin{cases} 1 & \text{if } e \text{ is directed towards } v, \\ -1 & \text{if } e \text{ is directed away from } v, \\ 0 & \text{if } v \text{ is not incident to } e. \end{cases}$$

This ensures that the local orientation of edges is consistent with the structure of a BiDirected Graph.

The sets PM and PN in $PBGO$ introduce Plithogenic parameters (e.g., Degree of Appurtenance and Degree of Contradiction Functions) that enrich the representation of vertices and edges without altering the underlying graph structure or the bidirection function τ.

Since $PBGO$ retains the undirected nature of G and adheres to the bidirection rules of τ, it preserves the structure of a BiDirected Graph $B = (G, \tau)$. This completes the proof. □

Theorem 3.21. *A Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM, PN)$ generalizes the structure of a BiDirected Graph by incorporating Plithogenic offsets.*

Proof. The Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM, PN)$ extends the standard BiDirected Graph $B = (G, \tau)$ by introducing:

- A Degree of Appurtenance Function (DAF) and a Degree of Contradiction Function (DCF) for vertices (PM).
- A DAF and DCF for edges (PN).

These functions allow membership and contradiction values to exceed the interval $[0, 1]$, extending the expressive capacity of the graph. However, these extensions do not alter the fundamental properties of the graph, such as the bidirection function τ or the structure of G. Thus, $PBGO$ generalizes B while preserving its core characteristics. □

Theorem 3.22. *A Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM, PN)$ can be transformed into a Plithogenic BiDirected Graph $PBG = (G, \tau, PM', PN')$ by restricting all functions to* $[0, 1]$.

Proof. Let $PBGO = (G, \tau, PM, PN)$, where:

$$\mathrm{adf} : M \times M_l \to [\Psi_v, \Omega_v]^s, \quad \mathrm{acf} : M_l \times M_l \to [\Psi_v, \Omega_v]^t,$$

$$\mathrm{bdf} : N \times N_m \to [\Psi_e, \Omega_e]^s, \quad \mathrm{bcf} : N_m \times N_m \to [\Psi_e, \Omega_e]^t.$$

To restrict these functions to $[0, 1]$, define:

$$\mathrm{adf}'(x,a) = \max(0, \min(\mathrm{adf}(x,a), 1)),$$

$$\begin{aligned}
\text{acf}'(a, b) &= \max(0, \min(\text{acf}(a, b), 1)),\\
\text{bdf}'(e, c) &= \max(0, \min(\text{bdf}(e, c), 1)),\\
\text{bcf}'(c, d) &= \max(0, \min(\text{bcf}(c, d), 1)).
\end{aligned}$$

The resulting graph $PBG = (G, \tau, PM', PN')$ satisfies all properties of a Plithogenic BiDirected Graph because all functions are now restricted to $[0, 1]$. This completes the proof. □

Theorem 3.23. *A Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM, PN)$ can be transformed into a Plithogenic Directed OffGraph $PDO = (PM, PN')$ by assigning orientations to bidirected edges.*

Proof. Let $PBGO = (G, \tau, PM, PN)$. Assign orientations to each bidirected edge $e \in E$ based on the bidirection function τ:

$$\tau(u, e) = 1, \tau(v, e) = -1 \implies (u, v) \in E'.$$

Retain all vertex and edge functions (adf, acf, bdf, bcf) as defined in $PBGO$. The resulting graph $PDO = (PM, PN')$ is a Plithogenic Directed OffGraph. □

Theorem 3.24. *A Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM, PN)$ can be transformed into a Plithogenic OffGraph $POG = (G', PM, PN')$ by treating all bidirected edges as undirected edges.*

Proof. Let $PBGO = (G, \tau, PM, PN)$. Define a new undirected graph G', where:

$$G' = (V, E'), \quad E' = \{\{u, v\} \mid e = \{u, v\} \in E\}.$$

Retain all vertex and edge functions (adf, acf, bdf, bcf) from $PBGO$. The resulting graph $POG = (G', PM, PN')$ satisfies the properties of a Plithogenic OffGraph. This completes the proof. □

3.4 Plithogenic Pangene OffGraph

A graph known as the Pangene Graph [79] has been recently defined. This graph is recognized as a generalization of both bidirected and directed graphs. In this work, we extend this concept to define the Plithogenic Pangene OffGraph.

Definition 3.25 (Pangene Graph)**.** [79] A *Pangene Graph $G_P = (X, E)$* is a mathematical structure that represents gene-level variations across multiple genomes. It is defined as follows:

- Let V be the set of genes.
- Let $X = V \times \{>, <\}$ be the set of oriented genes, where:
  - $(v, >)$ (denoted $> v$) represents a gene v with a forward orientation.
  - $(v, <)$ (denoted $< v$) represents a gene v with a reverse orientation.
  - For any $x \in X$, $v(x) = v$ gives the underlying gene v.
- Let $E \subseteq X \times X$ be the set of edges, where an edge $(x, y) \in E$ implies y immediately follows x on an input contig or, due to DNA strand symmetry, x immediately follows y.

The graph satisfies the following properties:

1. **Skew Symmetry**: If $(x, y) \in E$, then $(y, x) \in E$.
2. **Inversion Representation**: If x can reach both y and $\bar{y}$ (the reverse orientation of y), $v(x)$ is said to exhibit an inversion.

Theorem 3.26. *A Pangene Graph $G_P = (X, E)$ can be represented as a directed graph $G_D = (X, E_D)$, where $E_D = E$.*

Proof. In a Pangene Graph $G_P = (X, E)$, edges (x, y) satisfy skew symmetry, i.e., $(x, y) \in E \implies (y, x) \in E$. By treating X as the vertex set and E as the directed edge set, we obtain a directed graph $G_D = (X, E)$. The property that $(y, x) \in E$ ensures that the graph represents bidirectional relationships between genes, which aligns with the requirements of a directed graph. Thus, G_P can be represented as G_D. □

Theorem 3.27. *A Pangene Graph $G_P = (X, E)$ can also be represented as a bidirected graph $G_B = (V, E_B)$, where:*

- *V is the set of genes.*
- *$E_B \subseteq V \times V$ is the set of bidirected edges, with each edge $(v, w) \in E_B$ associated with two orientations $(> v, < w)$ or $(< v, > w)$.*

Proof. In $G_P = (X, E)$, the edges represent oriented relationships between genes. Define $G_B = (V, E_B)$, where $V = \{v(x) \mid x \in X\}$ and $E_B = \{(v(x), v(y)) \mid (x, y) \in E\}$. Each bidirected edge $(v, w) \in E_B$ is associated with two orientations, $(> v, < w)$ and $(< v, > w)$, representing the possible connections between v and w in G_P. Since G_B retains the bidirectional nature of relationships and adheres to the bidirected graph properties, G_P can be represented as G_B. □

Definition 3.28 (Plithogenic Pangene OffGraph)**.** A *Plithogenic Pangene OffGraph* $G_{P,\text{off}} = (X, E, PM, PN)$ is an extension of the Pangene Graph that incorporates the concept of offsets within the Plithogenic framework. It is defined as follows:

- V: The set of genes.
- $X = V \times \{>, <\}$: The set of oriented genes, where:
  - $(v, >)$ (denoted $> v$) represents a gene v with a forward orientation.
  - $(v, <)$ (denoted $< v$) represents a gene v with a reverse orientation.
  - For any $x \in X$, $v(x) = v$ gives the underlying gene v.
- $E \subseteq X \times X$: The set of edges, where an edge $(x, y) \in E$ implies y immediately follows x on an input contig, or x immediately follows y due to DNA strand symmetry.
- $PM = (M, l, M_l, \text{adf}, \text{acf})$: The Plithogenic Offset Vertex Set, where:
  - $M \subseteq V$: The set of vertices.
  - l: An attribute associated with the vertices.
  - M_l: The range of possible attribute values for l.
  - $\text{adf} : M \times M_l \to [\Psi_v, \Omega_v]^s$: The Degree of Appurtenance Function (DAF) for vertices, where $\Psi_v < 0$ and $\Omega_v > 1$.
  - $\text{acf} : M_l \times M_l \to [\Psi_v, \Omega_v]^t$: The Degree of Contradiction Function (DCF) for vertices.
- $PN = (N, m, N_m, \text{bdf}, \text{bcf})$: The Plithogenic Offset Edge Set, where:
  - $N \subseteq E$: The set of edges.
  - m: An attribute associated with the edges.
  - N_m: The range of possible attribute values for m.
  - $\text{bdf} : N \times N_m \to [\Psi_e, \Omega_e]^s$: The Degree of Appurtenance Function (DAF) for edges, where $\Psi_e < 0$ and $\Omega_e > 1$.
  - $\text{bcf} : N_m \times N_m \to [\Psi_e, \Omega_e]^t$: The Degree of Contradiction Function (DCF) for edges.

The graph satisfies the following properties:

1. **Skew Symmetry**: If $(x, y) \in E$, then $(y, x) \in E$.

2. **Inversion Representation**: If x can reach both y and $\bar{y}$ (the reverse orientation of y), $\nu(x)$ is said to exhibit an inversion.

3. The Degree of Appurtenance and Contradiction Functions (adf, acf, bdf, bcf) satisfy the Appurtenance and Contradiction Constraints as defined in Plithogenic Graphs.

Example 3.29. The Plithogenic Pangene OffGraph G_P can be classified into specific subclasses depending on the parameters s and t. The following are examples of special cases of Plithogenic Pangene OffGraphs:

- When $s = t = 1$, G_P is called a **Plithogenic Fuzzy Pangene OffGraph**.
- When $s = 2, t = 1$, G_P is called a **Plithogenic Intuitionistic Fuzzy Pangene OffGraph**. Each vertex and edge is associated with two membership degrees: truth and falsity.
- When $s = 3, t = 1$, G_P is called a **Plithogenic Neutrosophic Pangene OffGraph**. Each vertex and edge is characterized by triplets of truth, indeterminacy, and falsity membership degrees.
- When $s = 4, t = 1$, G_P is called a **Plithogenic Quadripartitioned Neutrosophic Pangene OffGraph**. Each vertex and edge is represented by quadruples of membership degrees, including truth, indeterminacy, falsity, and liberation.
- When $s = 5, t = 1$, G_P is called a **Plithogenic Pentapartitioned Neutrosophic Pangene OffGraph**. Each vertex and edge has quintuple membership degrees corresponding to truth, contradiction, ignorance, unknown, and falsity.
- When $s = 7, t = 1$, G_P is called a **Plithogenic Heptapartitioned Neutrosophic Pangene OffGraph**. Each vertex and edge has seven membership degrees representing various nuanced states of uncertainty.

Theorem 3.30. *A Plithogenic Pangene OffGraph $G_{P,\mathit{off}} = (X, E, PM, PN)$ can be transformed into a Plithogenic BiDirected OffGraph $PBGO = (G, \tau, PM', PN')$.*

Proof. To transform $G_{P,\text{off}}$ into $PBGO$:

- Define the underlying graph $G = (V, E')$, where $E' = \{\{u, v\} \mid (x, y) \in E, \nu(x) = u, \nu(y) = v\}$.
- Define the bidirection function τ such that:

$$\tau(u, e) = \begin{cases} 1 & \text{if } (x, y) \in E \text{ and } \nu(x) = u, \\ -1 & \text{if } (y, x) \in E \text{ and } \nu(x) = u, \\ 0 & \text{otherwise.} \end{cases}$$

- Retain all Plithogenic functions (adf, acf, bdf, bcf).

The resulting graph $PBGO$ satisfies all properties of a Plithogenic BiDirected OffGraph. This completes the proof. □

Theorem 3.31. *A Plithogenic Pangene OffGraph $G_{P,\mathit{off}} = (X, E, PM, PN)$ can also be transformed into a Plithogenic Directed OffGraph $PDO = (PM, PN')$.*

Proof. Assign directions to all edges in E based on the orientation of the vertices in X. Retain all Plithogenic functions (adf, acf, bdf, bcf) from $G_{P,\text{off}}$. The resulting graph PDO satisfies all properties of a Plithogenic Directed OffGraph. This completes the proof. □

Theorem 3.32. *A Plithogenic Pangene OffGraph $G_{P,\mathit{off}} = (X, E, PM, PN)$ can be transformed into a Plithogenic OffGraph $POG = (G', PM, PN')$ by treating all bidirected edges as undirected edges.*

Proof. To transform $G_{P,\text{off}}$ into POG:

- Define a new undirected graph $G' = (V, E')$, where $E' = \{\{u, v\} \mid (x, y) \in E, \nu(x) = u, \nu(y) = v\}$.
- Retain all Plithogenic functions (adf, acf, bdf, bcf) from $G_{P,\text{off}}$.

The resulting graph POG satisfies the properties of a Plithogenic OffGraph. This completes the proof. □

Theorem 3.33. *A Plithogenic Pangene OffGraph $G_{P,off} = (X, E, PM, PN)$ retains the structure of a Pangene Graph.*

Proof. To establish that a Plithogenic Pangene OffGraph $G_{P,\text{off}}$ retains the structure of a Pangene Graph G_P, we demonstrate that it satisfies the properties of Pangene Graphs:

1. **Skew Symmetry**: In $G_{P,\text{off}}$, the edges $E \subseteq X \times X$ satisfy the skew symmetry property: $(x, y) \in E \implies (y, x) \in E$. Since this condition is inherited from the definition of a Pangene Graph G_P, $G_{P,\text{off}}$ also satisfies skew symmetry.

2. **Inversion Representation**: Each element $x \in X$ represents an oriented gene. If x can reach both y and $\bar{y}$ (the reverse orientation of y) through edges in E, then the underlying gene $\nu(x)$ exhibits an inversion. The inclusion of additional Plithogenic parameters PM and PN does not alter the representation of inversions, as these parameters extend the membership information without modifying the core graph structure.

3. **Plithogenic Parameters**: The sets $PM = (M, l, M_l, \text{adf}, \text{acf})$ and $PN = (N, m, N_m, \text{bdf}, \text{bcf})$ introduce appurtenance and contradiction functions that extend the attributes of vertices and edges. These extensions enrich the representation of the graph but do not conflict with the core structure defined by G_P.

Since $G_{P,\text{off}}$ retains the properties of skew symmetry and inversion representation, it maintains the structure of a Pangene Graph G_P. This completes the proof. □

Theorem 3.34. *A Plithogenic Pangene OffGraph $G_{P,off}$ generalizes the structure of a Pangene Graph by incorporating Plithogenic offsets.*

Proof. The Plithogenic Pangene OffGraph $G_{P,\text{off}} = (X, E, PM, PN)$ includes additional parameters PM and PN, which extend the representation of vertices and edges through Plithogenic offsets.

The vertex set PM introduces a Degree of Appurtenance Function (DAF) and a Degree of Contradiction Function (DCF) defined over the range $[\Psi_v, \Omega_v]$, allowing for extended membership values beyond the standard interval $[0, 1]$. The edge set PN similarly incorporates DAF and DCF parameters for edges, enabling nuanced representations of relationships between genes. These Plithogenic functions allow $G_{P,\text{off}}$ to model uncertainties, contradictions, and complex relationships more effectively than a standard Pangene Graph G_P.

By enhancing the representation without violating the structural properties of G_P, $G_{P,\text{off}}$ generalizes the concept of a Pangene Graph. This completes the proof. □

4 Future tasks

The future directions of this research are outlined as follows. We intend to delve into the mathematical structures and applications of Directed Hypergraphs [16,80,93,94,129] and Mixed Hypergraphs [72,77,125], particularly within the framework of uncertain directed graphs such as Fuzzy and Neutrosophic graphs. Furthermore, we aim to extend this study by defining analogous graph concepts in the context of Directed Superhypergraphs (cf. [42, 53, 55–57, 110, 111, 113, 116, 117]). Finally, we seek to explore the potential applications of these extended models, paving the way for their integration into diverse fields.

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Data Availability

The datasets generated or analyzed during this study are not publicly accessible due to privacy concerns.

Ethical Approval

This article does not involve any studies with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest related to the publication of this paper.

Disclaimer

Please note that preprints and early-stage research may not have undergone peer review. Additionally, as I am an independent researcher with time constraints, please understand. Sorry.

As research in this field continues to evolve, the findings and interpretations presented in this paper may be subject to change. Readers are encouraged to consult future publications for the latest developments.

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References

[1] Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.

[2] Admin Admin and Abdalla I. Abdalla Musa. Automated credit card risk assessment using fuzzy parameterized neutrosophic hypersoft expert set. *International Journal of Neutrosophic Science*.

[3] Alfred V. Aho, Michael R Garey, and Jeffrey D. Ullman. The transitive reduction of a directed graph. *SIAM Journal on Computing*, 1(2):131–137, 1972.

[4] M. Akram and K. H. Dar. On n-graphs. *Southeast Asian Bulletin of Mathematics*, 38:35–49, 2014.

[5] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[6] Muhammad Akram. Interval-valued fuzzy line graphs. *Neural Computing and Applications*, 21:145–150, 2012.

[7] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[8] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[9] Muhammad Akram, MG Karunambigai, K Palanivel, and S Sivasankar. Balanced bipolar fuzzy graphs. *Journal of advanced research in pure mathematics*, 6(4):58–71, 2014.

[10] Muhammad Akram, Sheng-Gang Li, and KP Shum. Antipodal bipolar fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, 31(56):425–438, 2013.

[11] Muhammad Akram and Musavarah Sarwar. Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *viXra*, 2017.

[12] José Carlos R Alcantud, Azadeh Zahedi Khameneh, Gustavo Santos-García, and Muhammad Akram. A systematic literature review of soft set theory. *Neural Computing and Applications*, 36(16):8951–8975, 2024.

[13] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[14] Kazutoshi Ando, Satoru Fujishige, and Toshio Nemoto. Decomposition of a bidirected graph into strongly connected components and its signed poset structure. *Discrete Applied Mathematics*, 68(3):237–248, 1996.

[15] Nabeel Ezzulddin Arif et al. Domination (set and number) in neutrosophic soft over graphs. *Wasit Journal for Pure sciences*, 1(3):26–43, 2022.

[16] Giorgio Ausiello and Luigi Laura. Directed hypergraphs: Introduction and fundamental algorithms-a survey. *Theoretical Computer Science*, 658:293–306, 2017.

[17] Alireza Bagheri Salec. On soft ultrafilters. *International Journal of Nonlinear Analysis and Applications*, 10(1):131–138, 2019.

[18] Dietmar Berwanger, Anuj Dawar, Paul Hunter, Stephan Kreutzer, and Jan Obdržálek. The dag-width of directed graphs. *Journal of Combinatorial Theory, Series B*, 102(4):900–923, 2012.

[19] Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey*. SIAM, 1999.

[20] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[21] Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.

[22] Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.

[23] Adam L. Buchsbaum, Emden R. Gansner, and Suresh Venkatasubramanian. Directed graphs and rectangular layouts. *2007 6th International Asia-Pacific Symposium on Visualization*, pages 61–64, 2007.

[24] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[25] Tanmoy Chatterjee and Surapati Pramanik. Triangular fuzzy pentapartitioned neutrosophic set and its properties.

[26] Shyi-Ming Chen. Measures of similarity between vague sets. *Fuzzy sets and Systems*, 74(2):217–223, 1995.

[27] Jonathan Pila Contents. Set theory. *Mathematical Statistics with Applications in R*, 2018.

[28] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued bipolar pentapartitioned neutrosophic set and its application in madm strategy. *Neutrosophic Sets and Systems*, 49:145–163, 2022.

[29] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[30] R. Devi. Minimal domination via neutrosophic over graphs. *1ST INTERNATIONAL CONFERENCE ON MATHEMATICAL TECHNIQUES AND APPLICATIONS: ICMTA2020*, 2020.

[31] R Narmada Devi and R Dhavaseelan. *New Type of Neutrosophic Off Graphs*. Infinite Study.

[32] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[33] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[34] Didier Dubois and Henri Prade. A review of fuzzy set aggregation connectives. *Information sciences*, 36(1-2):85–121, 1985.

[35] Didier Dubois and Henri Prade. Fuzzy sets and systems: theory and applications. In *Mathematics in Science and Engineering*, 2011.

[36] Jack Edmonds and Ellis L Johnson. Matching: A well-solved class of integer linear programs. In *Combinatorial Optimization-Eureka, You Shrink! Papers Dedicated to Jack Edmonds 5th International Workshop Aussois, France, March 5–9, 2001 Revised Papers*, pages 27–30. Springer, 2003.

[37] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[38] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[39] Takaaki Fujita. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. June 2024.

[40] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[41] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.

[42] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[43] Takaaki Fujita. Some graph classes for turiyam neutrosophic directed graphs and pentapartitioned neutrosophic directed graphs. 2024.

[44] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *ResearchGate*, July 2024.

[45] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[46] Takaaki Fujita. Advancing uncertain combinatorics through graphization, hyperization, and uncertainization: Fuzzy, neutrosophic, soft, rough, and beyond. 2025.

[47] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[48] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[49] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[50] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[51] Emden R Gansner, Eleftherios Koutsofios, Stephen C North, and K-P Vo. A technique for drawing directed graphs. *IEEE Transactions on Software Engineering*, 19(3):214–230, 1993.

[52] Laura Gellert and Raman Sanyal. On degree sequences of undirected, directed, and bidirected graphs. *European Journal of Combinatorics*, 64:113–124, 2017.

[53] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[54] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[55] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[56] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[57] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[58] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[59] Wilfrid Hodges. Basic set theory. *Bulletin of The London Mathematical Society*, 12:158–159, 1980.

[60] Doeko Homan. Very basic set theory. 2023.

[61] Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.

[62] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[63] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Fuzzy hypersoft expert set with application in decision making for the best selection of product. 2021.

[64] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Hypersoft expert set with application in decision making for recruitment process. 2021.

[65] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Single valued neutrosophic hypersoft expert set with application in decision making. 2021.

[66] Muhammad Ihsan, Muhammad Haris Saeed, Agaeb Mahal Alanzi, and Hamiden Abd El-Wahed Khalifa. An algorithmic multiple attribute decision-making method for heart problem analysis under neutrosophic hypersoft expert set with fuzzy parameterized degree-based setting. *PeerJ Computer Science*, 9, 2023.

[67] Muhammad Ihsan, Muhammad Haris Saeed, Alhanouf Alburaikan, and Hamiden A. Wahed Khalifa. Product evaluation through multi-criteria decision making based on fuzzy parameterized pythagorean fuzzy hypersoft expert set. *AIMS Mathematics*, 2022.

[68] Muhammad Ihsan, Muhammad Haris Saeed, and Atiqe Ur Rahman. An intuitionistic fuzzy hypersoft expert set-based robust decision-support framework for human resource management integrated with modified topsis and correlation coefficient. *Neural Computing and Applications*, pages 1–25, 2023.

[69] Muhammad Ihsan, Muhammad Haris Saeed, Atiqe Ur Rahman, Hüseyin Kamac, Nehad Ali Shah, and Wajaree Weera. An madm-based fuzzy parameterized framework for solar panels evaluation in a fuzzy hypersoft expert set environment. *AIMS Mathematics*, 2022.

[70] Muhammad Ihsan, Muhammad Haris Saeed, Atiqe Ur Rahman, Hamiden A. Wahed Khalifa, and Salwa El-Morsy. An intelligent fuzzy parameterized multi-criteria decision-support system based on intuitionistic fuzzy hypersoft expert set for automobile evaluation. *Advances in Mechanical Engineering*, 14, 2022.

[71] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[72] Tao Jiang, Dhruv Mubayi, Zsolt Tuza, Vitaly Voloshin, and Douglas B West. The chromatic spectrum of mixed hypergraphs. *Graphs and Combinatorics*, 18(2):309–318, 2002.

[73] Donald B Johnson. Finding all the elementary circuits of a directed graph. *SIAM Journal on Computing*, 4(1):77–84, 1975.

[74] Calvin Jongsma. Basic set theory and combinatorics. *Undergraduate Texts in Mathematics*, 2019.

[75] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft directed graphs, some of their operations, and properties. *New Mathematics and Natural Computation*, 20(01):129–155, 2024.

[76] Cengiz Kahraman. *Fuzzy multi-criteria decision making: theory and applications with recent developments*, volume 16. Springer Science & Business Media, 2008.

[77] Daniel Král'. Mixed hypergraphs and other coloring problems. *Discrete mathematics*, 307(7-8):923–938, 2007.

[78] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[79] Heng Li, Maximillian Marin, and Maha R Farhat. Exploring gene content with pangene graphs. *Bioinformatics*, 40(7):btae456, 2024.

[80] Xiaoyi Luo, Jiaheng Peng, and Jun Liang. Directed hypergraph attention network for traffic forecasting. *IET Intelligent Transport Systems*, 16(1):85–98, 2022.

[81] Amir Sabir majeed and Nabeel Ezzulddin Arif. Domination (set and number) in neutrosophic soft over graphs. *Wasit Journal of Pure sciences*, 2022.

[82] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[83] Rama Mallick and Surapati Pramanik. *Pentapartitioned neutrosophic set and its properties*, volume 36. Infinite Study, 2020.

[84] Nivetha Martin, Priya Priya.R, and Florentin Smarandache. Decision making on teachers' adaptation to cybergogy in saturated interval- valued refined neutrosophic overset /underset /offset environment. *International Journal of Neutrosophic Science*, 2020.

[85] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[86] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[87] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.

[88] Xuan Thao Nguyen, Doan Dong Nguyen, et al. Rough fuzzy relation on two universal sets. *International Journal of Intelligent Systems and Applications*, 6(4):49, 2014.

[89] Wendy Olsen and Hisako Nomura. Poverty reduction: fuzzy sets vs. crisp sets compared. *Sociological Theory and Methods*, 24(2):219–246, 2009.

[90] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[91] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[92] Surapati Pramanik. Interval quadripartitioned neutrosophic sets. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 146, 2022.

[93] Daniele Pretolani. A directed hypergraph model for random time dependent shortest paths. *European Journal of Operational Research*, 123(2):315–324, 2000.

[94] Daniele Pretolani. Finding hypernetworks in directed hypergraphs. *European Journal of Operational Research*, 230(2):226–230, 2013.

[95] Roman Rabinovich and Lehr-und Forschungsgebiet. Complexity measures of directed graphs. *Diss., Rheinisch-Westfälische Technische Hochschule Aachen*, page 123, 2008.

[96] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[97] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[98] Akbar Rezaei, Tahsin Oner, Tugce Katican, Florentin Smarandache, and N Gandotra. *A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets*. Infinite Study, 2022.

[99] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[100] Kayvan Sadeghi. Stable mixed graphs. 2013.

[101] Kayvan Sadeghi and Steffen Lauritzen. Markov properties for mixed graphs. 2014.

[102] Bhavanari Satyanarayana, Tumurukota Venkata Pradeep Kumar, and Shaik Mohiddin Shaw. Set theory. *Functional Interpretations*, 2019.

[103] Alexander Shen and Nikolai K. Vereshchagin. Basic set theory. In *The Student Mathematical Library*, 2002.

[104] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Sciences*, 18(1):57–72, 2022.

[105] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[106] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[107] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[108] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[109] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[110] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[111] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[112] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[113] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[114] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[115] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[116] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[117] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[118] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[119] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[120] Florentin Smarandache, Said Broumi, Mohamed Talea, Assia Bakali, and Kishore Kumar. Shortest path problem on single valued neutrosophic graphs. In *International Symposium on Networks, Computers and Communications (ISNCC-2017), Marrakech, Morocco, May 16-18, 2017, www. isncc-conf. org*. ISNCC, 2017.

[121] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[122] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[123] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[124] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[125] Zsolt Tuza and Vitaly Voloshin. Uncolorable mixed hypergraphs. *Discrete Applied Mathematics*, 99(1-3):209–227, 2000.

[126] Robert L Vaught. *Set theory: an introduction*. Springer Science & Business Media, 2001.

[127] Nikolai Konstantinovich Vereshchagin and Alexander Shen. *Basic set theory*. Number 17. American Mathematical Soc., 2002.

[128] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[129] Jinshan Xie and Liqun Qi. Spectral directed hypergraph theory via tensors. *Linear and Multilinear Algebra*, 64(4):780–794, 2016.

[130] Xibei Yang, Dongjun Yu, Jingyu Yang, and Chen Wu. Generalization of soft set theory: from crisp to fuzzy case. In *Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE)*, pages 345–354. Springer, 2007.

[131] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[132] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[133] Ping Zhang and Gary Chartrand. Introduction to graph theory. *Tata McGraw-Hill*, 2:2–1, 2006.

[134] Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

[135] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.

[136] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

Short Note of Neutrosophic Closure matroids

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Abstract: Matroids play a crucial role in discrete optimization, owing to their robust mathematical foundations and applicability in graph theory and computer science. This paper delves into the extension of matroid concepts to Neutrosophic and Turiyam Neutrosophic set theories, introducing Neutrosophic closure matroids and Turiyam Neutrosophic closure matroids, which integrate uncertainty, indeterminacy, and liberal states.

Keywords: Neutrosophic set, Fuzzy set, Matroid, Antimatroid, Greedoid

1. Introduction

A matroid is a combinatorial structure that extends the concept of linear independence from vector spaces to general sets [57]. Matroids are essential in fields like discrete optimization due to their strong mathematical foundations and applicability in areas such as graph theory and computer science [24, 35, 41, 55, 57, 60, 81]. A closure matroid is defined as a matroid where the closure of the union of any two subsets is equal to the union of their individual closures [2, 7].

Various concepts for handling uncertainty are actively studied to address unpredictability [26–34, 70, 72, 73, 75, 76]. This paper focuses on fuzzy, neutrosophic, and Turiyam Neutrosophic sets. A fuzzy set assigns partial membership, a neutrosophic set assigns truth, indeterminacy, and falsity, while a Turiyam Neutrosophic set adds a liberal state[1, 16, 19, 21, 52, 70, 78, 79, 82, 84, 85]. These sets are crucial in modeling uncertainty across fields.

In this paper, we present the mathematical definitions of Neutrosophic closure Matroids and Turiyam Neutrosophic closure Matroids. Neutrosophic Matroids (including Antimatroids and Greedoids) and Turiyam Neutrosophic Matroids (including Antimatroids and Greedoids) have been recently defined in the literature [25]. The purpose of this study is to explore the applications of these concepts. These structures extend classical combinatorial concepts by incorporating Neutrosophic and Turiyam Neutrosophic set theories, enabling the modeling of uncertainty, indeterminacy, and other complex states within mathematical frameworks.

2. The Structure of the Paper

The format of this paper is described below.

3. Basic Matroid Concepts

Here are the key concepts of matroid theory. This paper utilizes fundamental ideas from matroid theory and set theory. For an introduction to these basic concepts, please refer to lecture notes or surveys on set theory [39, 43, 47], graph theory[17, 18], and matroid theory [57].

Definition 1 (Matroid)**.** [57] Let Y be a non-empty universe, and let $I \subseteq \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is the power set of Y. The pair $M = (Y, I)$ is called a *matroid* if the following conditions hold:

1. If $D_1 \in I$ and $D_2 \subseteq D_1$, then $D_2 \in I$.
2. If $D_1, D_2 \in I$ and $|D_1| < |D_2|$, then there exists $D_3 \in I$ such that $D_1 \subseteq D_3 \subseteq D_1 \cup D_2$.

The set I is called the *family of independent sets* of the matroid M.

Example 2. Consider the set $E = \{1, 2, 3, 4\}$. The collection of independent sets $\mathcal{I}$ is defined as follows:

$$\mathcal{I} = \{A \subseteq E \mid |A| \leq 2\}$$

This matroid $M = (E, \mathcal{I})$ is called a **rank-2 uniform matroid**, denoted as $U_{2,4}$.
We consider Verification of Matroid Axioms.

1. **Non-emptiness:** The empty set $\emptyset$ satisfies $|\emptyset| = 0 \leq 2$, so it is included in the collection of independent sets.

2. **Hereditary Property (Axiom 1):** If $A \in \mathcal{I}$ and $B \subseteq A$, then $|B| \leq |A| \leq 2$, implying that $B \in \mathcal{I}$.

3. **Exchange Property (Axiom 2):** If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $|A| < |B| \leq 2$, so $|A| \leq 1$. We can choose an element $e \in B \setminus A$ such that $|A \cup \{e\}| \leq 2$, ensuring $A \cup \{e\} \in \mathcal{I}$.

Matroids have been extensively studied from various perspectives, including graph theory and computer science [4, 6, 24]. In matroid theory, a well-known concept is the rank function [14, 42, 65]. A rank function assigns a non-negative integer to each subset, representing the maximum size of independent elements within that subset. Related concepts, such as rank-width and linear-rank-width, are also widely studied [22, 53, 54].

Definition 3 (Rank)**.** [14,42,65] Let $M = (Y, I)$ be a matroid. The *rank function* $R : \mathcal{P}(Y) \to \{0, 1, 2, \ldots, |Y|\}$ is defined by:

$$R(D) = \max\{|F| : F \subseteq D, F \in I\},$$

for all $D \in \mathcal{P}(Y)$. The value $R(D)$ is called the *rank* of D.

A mapping $\mu : \mathcal{P}(Y) \to [0, \infty)$ is called *submodular* if for all $D, F \in \mathcal{P}(Y)$, the following inequality holds:

$$\mu(D) + \mu(F) \geq \mu(D \cup F) + \mu(D \cap F).$$

Example 4. The rank function $r : 2^E \to \mathbb{N}$ is defined as:

$$r(A) = \min\{|A|, 2\}$$

Examples:

- $r(\{1\}) = 1$
- $r(\{1, 2\}) = 2$
- $r(\{1, 2, 3\}) = 2$
- $r(E) = r(\{1, 2, 3, 4\}) = 2$

In the field of matroid theory, several mathematical structures such as bases[11, 44, 45], circuits[38, 51, 62], cyclic flats[15, 20, 23, 58, 59], and hyperplanes[9, 40, 46] are well-known. Among these, the structure of a "flat" is also considered in this paper. The definition is provided below [12, 56, 57].

Definition 5 (Flat)**.** [57] Let $M = (E, \mathcal{I})$ be a matroid, where E is the ground set and $\mathcal{I}$ is the collection of independent sets of M. A subset $F \subseteq E$ is called a **flat** of the matroid M if it satisfies the following condition:

$$\mathrm{cl}(F) = F,$$

where $\mathrm{cl}(F)$ denotes the *closure* of F in the matroid. This closure $\mathrm{cl}(F)$ is defined as:

$$\mathrm{cl}(F) = \{e \in E \mid r(F \cup \{e\}) = r(F)\},$$

where $r(F)$ is the *rank* of the subset F, which represents the maximum size of an independent subset contained in F.

In other words, a subset F is a flat if adding any element $e \in E \setminus F$ to F increases the rank of F. Equivalently, a flat is a maximal set with respect to having a given rank.

Closure is one of the operations in matroids, and extensive research has been conducted not only in the field of matroid theory but also in other disciplines [48, 56, 80]. The definitions are provided below.

Definition 6 (Closure)**.** [48,56,80] In matroid theory, the closure operator on a set provides the smallest superset that contains all the elements determined by a set of independent elements. Formally, it is defined as:

$$\mathrm{cl}(A) = \{e \in E : r(A \cup \{e\}) = r(A)\}$$

where:

- $A \subseteq E$ is a subset of the ground set E,
- $r(A)$ is the rank of the subset A, which is the maximum size of an independent subset of A.

This means that the closure of a set A in a matroid is the set of elements $e \in E$ that do not increase the rank of A, i.e., they are dependent on the set A.

Example 7. The closure of a subset $A \subseteq E$ is defined as:

$$\mathrm{cl}(A) = \{e \in E \mid r(A \cup \{e\}) = r(A)\}$$

Let us compute the closure of $A = \{1, 2\}$:

- $r(A) = 2$
- For $e \in E \setminus A = \{3, 4\}$:
  - $r(A \cup \{e\}) = r(\{1, 2, e\}) = 2$ (The number of elements is 3, but the rank remains 2)
  - Hence, $e \in \mathrm{cl}(A)$ (since $r(A \cup \{e\}) = r(A)$)

Thus, $\mathrm{cl}(\{1, 2\}) = E$.
Similarly, for $A = \{1\}$:

- $r(\{1\}) = 1$
- For $e \in E \setminus \{1\} = \{2, 3, 4\}$:
  - $r(\{1, e\}) = \min\{2, 2\} = 2$
  - Since $r(\{1, e\}) > r(\{1\})$, $e \notin \mathrm{cl}(\{1\})$

Therefore, $\mathrm{cl}(\{1\}) = \{1\}$.

A closure matroid is a matroid where the closure of the union of any two subsets equals the union of their closures [2,7]. The definitions are provided below.

Definition 8 (Closure matroid)**.** [2,7] A matroid $M = (E, I)$, where E is the ground set and I is the collection of independent sets, is called a *closure matroid* if for any subsets $A, B \subseteq E$:

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$$

This means that the closure of the union of two sets is equal to the union of their closures.

As a related concept, the definition of a Free Matroid is provided below.

Definition 9 (Free Matroid)**.** (cf.[57]) A **Free Matroid** is a matroid $M = (E, \mathcal{I})$ defined on a finite set E, where $\mathcal{I}$ is the collection of all subsets of E. In other words, every subset of E is an independent set. Formally:

$$\mathcal{I} = \{A \subseteq E \mid A \text{ is independent}\}.$$

Thus, the rank function r of the Free Matroid satisfies:

$$r(A) = |A|, \quad \text{for all } A \subseteq E.$$

The Free Matroid is the simplest matroid structure, as it allows complete independence of all subsets of E.

4. Fuzzy Closure Matroid

A Fuzzy Matroid is an extension of the concept of a Matroid using Fuzzy set theory. Like Matroids, it has been the subject of extensive research [8,10,48–50,61,63,64]. First, the definition of a Fuzzy set and Fuzzy Matroid is provided below.

Definition 10. [82,83] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y,z \in Y.$$

Next, the definition of a fuzzy matroid, based on the concept of a fuzzy set, is provided below [36,37, 63,64].

Definition 11. [36,37,63,64] Let $F(Y)$ denote the power set of fuzzy subsets on Y. A pair $M = (X, I)$ is called a *fuzzy matroid* if $I \subseteq F(Y)$ satisfies the following conditions:

1. If $\tau_1 \in I$ and $\tau_2 \subset \tau_1$, then $\tau_2 \in I$, where $\tau_2 \subset \tau_1$ means that $\tau_2(y) < \tau_1(y)$ for every $y \in X$. 2. If $\tau_1, \tau_2 \in I$ and $|\operatorname{supp}(\tau_1)| < |\operatorname{supp}(\tau_2)|$, then there exists $\tau_3 \in I$ such that:

- $\tau_1 \subset \tau_3 \subseteq \tau_1 \cup \tau_2$, where for any $y \in X$, $\tau_1 \cup \tau_2(y) = \max\{\tau_1(y), \tau_2(y)\}$,
- $m(\tau_3) \geq \min\{m(\tau_1), m(\tau_2)\}$, where $m(\nu) = \min\{\nu(y) : y \in \operatorname{supp}(\nu)\}$.

Here, I is called the *family of independent fuzzy sets* of the fuzzy matroid $M = (X, I)$.

Definition 12. [3] A fuzzy set $\lambda \in F(E)$ is called a *fuzzy flat* of the fuzzy matroid $M = (E, \mathcal{I})$ if:

- $\lambda \in \mathcal{F}$, where $\mathcal{F}$ is the family of fuzzy sets satisfying:
  1. $1_E \in \mathcal{F}$, where $1_E(e) = 1$ for all $e \in E$.
  2. If $\mu_1, \mu_2 \in \mathcal{F}$, then $\mu_1 \wedge \mu_2 \in \mathcal{F}$, where $(\mu_1 \wedge \mu_2)(e) = \min\{\mu_1(e), \mu_2(e)\}$.
  3. If $\mu \in \mathcal{F}$ and $\mu_1, \mu_2, \ldots, \mu_n$ are all minimal members of $\mathcal{F}$ such that $\mu \leq \mu_i$ for all $i = 1, 2, \ldots, n$, then:

     $$\bigvee_{i=1}^{n} \mu_i = 1_E,$$

     where $(\bigvee_{i=1}^{n} \mu_i)(e) = \max_{1 \leq i \leq n} \mu_i(e)$.

Definition 13. [5] Given a fuzzy matroid $M = (E, \mathcal{I})$ and a fuzzy set $\mu \in F(E)$, the *fuzzy closure* of μ is defined as:

$$\bar{\mu} = \bigwedge_{\lambda \in \mathcal{F}, \mu \leq \lambda} \lambda,$$

where the meet (infimum) is taken over all fuzzy flats $\lambda \in \mathcal{F}$ such that $\mu \leq \lambda$. The meet operation $\bigwedge$ is defined pointwise:

$$\left(\bigwedge_{\lambda} \lambda\right)(e) = \min_{\lambda} \lambda(e), \quad \forall e \in E.$$

Definition 14. [5] A *fuzzy closure matroid* $M = (E, \mathcal{F})$ is a fuzzy matroid where the following condition holds:

$$\overline{\mu \vee \eta} = \bar{\mu} \vee \bar{\eta}, \quad \forall \mu, \eta \in F(E),$$

where $\bar{\mu}$ and $\bar{\eta}$ are the fuzzy closures of μ and η, respectively, and $\vee$ denotes the fuzzy union (element-wise maximum):

$$(\mu \vee \eta)(e) = \max\{\mu(e), \eta(e)\}, \quad \forall e \in E.$$

This property ensures that the closure of the union of any two fuzzy subsets is equal to the union of their closures.

5. Neutrosophic Matroid

A Neutrosophic Matroid is an extension of the concept of a Matroid using Neutrosophic set theory. The definitions are provided below.

Definition 15. [1,13,21,52,70,71,77,79] Let E be a finite non-empty set (called the ground set). A *single-valued neutrosophic set* (SVNS) μ on E is defined as:

$$\mu = \left\{\langle e, T_\mu(e), I_\mu(e), F_\mu(e)\rangle : e \in E\right\},$$

where $T_\mu(e), I_\mu(e), F_\mu(e) \in [0,1]$ represent the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* degrees of element e, respectively.

Note: There is no restriction on the sum $T_\mu(e) + I_\mu(e) + F_\mu(e)$, i.e., $0 \leq T_\mu(e) + I_\mu(e) + F_\mu(e) \leq 3$.

Definition 16. For neutrosophic sets μ and ν on E:

- *Equality:* $\mu = \nu$ if $T_\mu(e) = T_\nu(e)$, $I_\mu(e) = I_\nu(e)$, and $F_\mu(e) = F_\nu(e)$ for all $e \in E$.
- *Order Relation:* $\mu \leq \nu$ if $T_\mu(e) \leq T_\nu(e)$, $I_\mu(e) \geq I_\nu(e)$, and $F_\mu(e) \geq F_\nu(e)$ for all $e \in E$.
- *Maximum (Join):* $(\mu \vee \nu)(e) = (\max\{T_\mu(e), T_\nu(e)\}, \min\{I_\mu(e), I_\nu(e)\}, \min\{F_\mu(e), F_\nu(e)\})$.
- *Minimum (Meet):* $(\mu \wedge \nu)(e) = (\min\{T_\mu(e), T_\nu(e)\}, \max\{I_\mu(e), I_\nu(e)\}, \max\{F_\mu(e), F_\nu(e)\})$.
- *Support of μ:* $\text{supp}(\mu) = \{e \in E : T_\mu(e) > 0\}$.
- *Minimal Truth-membership Value:* $m_T(\mu) = \min\{T_\mu(e) : e \in \text{supp}(\mu)\}$.

Definition 17. [25] A *neutrosophic matroid* is a pair (E, Ψ), where E is a finite ground set, and Ψ is a non-empty collection of neutrosophic sets on E, called *independent neutrosophic sets*, satisfying the following axioms:

- *Neutrosophic Hereditary Property (NM1):*
  If $\mu \in \Psi$ and $\nu \leq \mu$ (i.e., $T_\nu(e) \leq T_\mu(e)$, $I_\nu(e) \geq I_\mu(e)$, $F_\nu(e) \geq F_\mu(e)$ for all $e \in E$), then $\nu \in \Psi$.
- *Neutrosophic Exchange Property (NM2):*
  If $\mu, \nu \in \Psi$ and $|\text{supp}(\mu)| < |\text{supp}(\nu)|$, then there exists $\omega \in \Psi$ such that:

  $$\mu < \omega \leq \mu \vee \nu,$$

  and

  $$m_T(\omega) \geq \min\{m_T(\mu), m_T(\nu)\}.$$

Definition 18. [25] Neutrosophic Circuits: The minimal dependent neutrosophic subsets in Ψ are called *neutrosophic circuits*. A neutrosophic circuit is a set $\mu \in N(E)$ such that it is minimal with respect to inclusion, i.e., if $\nu \subset \mu$ then $\nu \notin \Psi$.

Neutrosophic Rank Function: The neutrosophic rank function $r_\mu : N(E) \to [0, \infty]$ is defined for any neutrosophic set $\mu \in N(E)$ as:

$$r_\mu(\nu) = \sup\{|\omega| : \omega \subseteq \nu \text{ and } \omega \in \Psi\},$$

where $|\omega|$ is the cardinality of the support of ω, i.e., the number of elements in the support with non-zero truth-membership, indeterminacy-membership, or falsity-membership.

Definition 19. Define the family of *neutrosophic flats* $\mathcal{F} \subseteq N(E)$ as follows:

1. $1_E \in \mathcal{F}$, where $1_E(e) = (1, 0, 0)$ for all $e \in E$.
2. If $\mu_1, \mu_2 \in \mathcal{F}$, then $\mu_1 \wedge \mu_2 \in \mathcal{F}$.
3. If $\mu \in \mathcal{F}$ and $\mu_1, \mu_2, \ldots, \mu_n$ are all minimal members of $\mathcal{F}$ such that $\mu \leq \mu_i$ for all $i = 1, 2, \ldots, n$, then:

$$\bigvee_{i=1}^{n} \mu_i = 1_E.$$

Definition 20. Given a neutrosophic matroid $M = (E, \Psi)$ and a neutrosophic set $\mu \in N(E)$, the *neutrosophic closure* of μ is defined as:

$$\bar{\mu} = \bigwedge_{\lambda \in \mathcal{F},\ \mu \leq \lambda} \lambda,$$

where the meet (infimum) is taken over all neutrosophic flats $\lambda \in \mathcal{F}$ such that $\mu \leq \lambda$. The meet operation $\wedge$ is defined element-wise as:

$$\bar{\mu}(e) = \left(\min_{\lambda} T_{\lambda}(e),\ \max_{\lambda} I_{\lambda}(e),\ \max_{\lambda} F_{\lambda}(e) \right), \quad \forall e \in E.$$

Definition 21. A neutrosophic matroid $M = (E, \Psi)$ is called a *neutrosophic closure matroid* if the closure operator $\bar{\cdot}$ satisfies the following property:

$$\overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}, \quad \forall \mu, \nu \in N(E).$$

This means that the closure of the join (maximum) of any two neutrosophic sets is equal to the join of their closures.

Theorem 22. *Every neutrosophic closure matroid can be transformed into a fuzzy closure matroid via a mapping that preserves the matroid structure.*

Proof. Define a mapping $\varphi : N(E) \to F(E)$ from the set of neutrosophic sets to the set of fuzzy sets by:

$$\varphi(\mu)(e) = T_{\mu}(e), \quad \forall e \in E.$$

This mapping extracts the truth-membership degree of each element e from the neutrosophic set μ to form the fuzzy set $\varphi(\mu)$.

We consider Transformation of Independent Sets. Let $M_N = (E, \Psi_N)$ be a neutrosophic closure matroid with independent neutrosophic sets Ψ_N. We define the corresponding fuzzy matroid $M_F = (E, \Psi_F)$, where:

$$\Psi_F = \{\varphi(\mu) \mid \mu \in \Psi_N\}.$$

We need to show that Ψ_F satisfies the axioms of a fuzzy matroid and that the closure operations correspond under the mapping φ.

We consider Preservation of Matroid Axioms. **Axiom 1 (Hereditary Property)**: Let $\phi \in \Psi_F$ and $\theta \leq \phi$, where $\theta(e) \leq \phi(e)$ for all $e \in E$. Since $\phi = \varphi(\mu)$ for some $\mu \in \Psi_N$, and $\theta \leq \varphi(\mu)$, there exists ν such that $\varphi(\nu) = \theta$ and $\nu \leq \mu$ in $N(E)$ (i.e., $T_{\nu}(e) \leq T_{\mu}(e)$ and $I_{\nu}(e) \geq I_{\mu}(e)$, $F_{\nu}(e) \geq F_{\mu}(e)$). By the neutrosophic hereditary property (NM1), $\nu \in \Psi_N$, so $\theta = \varphi(\nu) \in \Psi_F$.

Axiom 2 (Exchange Property): Let $\phi_1, \phi_2 \in \Psi_F$ with $|\mathrm{supp}(\phi_1)| < |\mathrm{supp}(\phi_2)|$. There exist $\mu_1, \mu_2 \in \Psi_N$ such that $\varphi(\mu_i) = \phi_i$ for $i = 1, 2$. By NM2, there exists $\omega \in \Psi_N$ such that $\mu_1 < \omega \leq \mu_1 \vee \mu_2$ and $m_T(\omega) \geq \min\{m_T(\mu_1), m_T(\mu_2)\}$. Let $\phi_3 = \varphi(\omega)$. Then $\phi_1 \leq \phi_3 \leq \phi_1 \vee \phi_2$, and $\phi_3 \in \Psi_F$.

We consider Preservation of Closure Operator. **Neutrosophic Closure**:

$$\overline{\mu} = \bigwedge_{\lambda \in \mathcal{F}_N, \mu \leq \lambda} \lambda,$$

where $\mathcal{F}_N$ is the family of neutrosophic flats.

Fuzzy Closure:

$$\overline{\phi} = \bigwedge_{\lambda \in \mathcal{F}_F, \phi \leq \lambda} \lambda,$$

where $\mathcal{F}_F$ is the family of fuzzy flats.

Since φ preserves the order ($\mu \leq \nu \implies \varphi(\mu) \leq \varphi(\nu)$), the closures correspond:

$$\varphi(\overline{\mu}) = \overline{\varphi(\mu)}.$$

Therefore, under the mapping φ, the neutrosophic closure matroid M_N transforms into the fuzzy closure matroid M_F, preserving the matroid structure and closure properties. □

6. Turiyam Neutrosophic Matroid

Next, we define the Turiyam Neutrosophic Matroid as follows. Similar to Fuzzy sets, Turiyam Neutrosophic sets have multiple associated concepts that are defined within this framework [66–68]. Note that the Turiyam Neutrosophic Set is, in fact, a specific case of the Quadripartitioned Neutrosophic Set, achieved by replacing "Contradiction" with "Liberal." (cf.[69, 74]) The definitions are provided below.

Definition 23. [66] Let E be a finite non-empty set (the ground set). A *Turiyam Neutrosophic set* μ on E is defined as:

$$\mu = \left\{\langle e, T_\mu(e), I_\mu(e), F_\mu(e), L_\mu(e)\rangle : e \in E\right\},$$

where $T_\mu(e), I_\mu(e), F_\mu(e), L_\mu(e) \in [0, 1]$ represent the *truth-membership*, *indeterminacy-membership*, *falsity-membership*, and *Turiyam Neutrosophic (liberal) state* degrees of element e, respectively.

Note: The sum $T_\mu(e) + I_\mu(e) + F_\mu(e) + L_\mu(e)$ satisfies:

$$0 \leq T_\mu(e) + I_\mu(e) + F_\mu(e) + L_\mu(e) \leq 4, \quad \forall e \in E.$$

Definition 24. For Turiyam Neutrosophic sets μ and ν on E:

- *Order Relation:* $\mu \leq \nu$ if for all $e \in E$:

$$T_\mu(e) \leq T_\nu(e), \quad I_\mu(e) \geq I_\nu(e), \quad F_\mu(e) \geq F_\nu(e), \quad L_\mu(e) \geq L_\nu(e).$$

- *Maximum (Join):*

$$(\mu \vee \nu)(e) = \left(\max\{T_\mu(e), T_\nu(e)\},\ \min\{I_\mu(e), I_\nu(e)\},\ \min\{F_\mu(e), F_\nu(e)\},\ \min\{L_\mu(e), L_\nu(e)\}\right).$$

- *Minimum (Meet):*

$$(\mu \wedge \nu)(e) = \left(\min\{T_\mu(e), T_\nu(e)\},\ \max\{I_\mu(e), I_\nu(e)\},\ \max\{F_\mu(e), F_\nu(e)\},\ \max\{L_\mu(e), L_\nu(e)\}\right).$$

- *Support of μ:*

$$\mathrm{supp}(\mu) = \{e \in E : T_\mu(e) > 0\}.$$

- *Minimal Truth-Membership Value:*

$$m_T(\mu) = \min\{T_\mu(e) : e \in \mathrm{supp}(\mu)\}.$$

The definition of a Turiyam Neutrosophic Matroid using the aforementioned sets is provided below.

Definition 25. [25] A *Turiyam Neutrosophic Matroid* is a pair (E, Ψ), where E is a finite ground set, and Ψ is a non-empty collection of Turiyam Neutrosophic sets on E, called *independent Turiyam Neutrosophic sets*, satisfying the following axioms:

1. *Turiyam Neutrosophic Hereditary Property (TM1):*
   If $\mu \in \Psi$ and $\nu \leq \mu$ (i.e., for all $e \in E$:

$$T_\nu(e) \leq T_\mu(e), \quad I_\nu(e) \geq I_\mu(e), \quad F_\nu(e) \geq F_\mu(e), \quad L_\nu(e) \geq L_\mu(e)),$$

   then $\nu \in \Psi$.

2. *Turiyam Neutrosophic Exchange Property (TM2):*
   If $\mu, \nu \in \Psi$ and $|\mathrm{supp}(\mu)| < |\mathrm{supp}(\nu)|$, then there exists $\omega \in \Psi$ such that:

   - $\mu < \omega \leq \mu \vee \nu$, where $\mu < \omega$ means $\mu \leq \omega$ and $\mu \neq \omega$.
   - $m_T(\omega) \geq \min\{m_T(\mu), m_T(\nu)\}$.

Definition 26. Define the rank function $r : 2^E \to \mathbb{N}$ by:

$$r(A) = \max\{|\operatorname{supp}(\mu)| : \mu \in \Psi, \operatorname{supp}(\mu) \subseteq A\}.$$

This function r satisfies the properties of a matroid rank function:

1. *Non-negativity and Boundedness:*
   $0 \leq r(A) \leq |A|$ for all $A \subseteq E$.
2. *Monotonicity:*
   If $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$.
3. *Submodularity:*
   For $A, B \subseteq E$:

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B).$$

Definition 27. Define the family of *Turiyam Neutrosophic flats* $\mathcal{F} \subseteq T(E)$ as follows:

1. $1_E \in \mathcal{F}$, where $1_E(e) = (1, 0, 0, 0)$ for all $e \in E$.
2. If $\mu_1, \mu_2 \in \mathcal{F}$, then $\mu_1 \wedge \mu_2 \in \mathcal{F}$.
3. If $\mu \in \mathcal{F}$ and $\mu_1, \mu_2, \ldots, \mu_n$ are all minimal members of $\mathcal{F}$ such that $\mu \leq \mu_i$ for all $i = 1, 2, \ldots, n$, then:

$$\bigvee_{i=1}^{n} \mu_i = 1_E.$$

Definition 28. Given a Turiyam Neutrosophic matroid $M = (E, \Psi)$ and a Turiyam Neutrosophic set $\mu \in T(E)$, the *Turiyam Neutrosophic closure* of μ is defined as:

$$\bar{\mu} = \bigwedge_{\lambda \in \mathcal{F},\ \mu \leq \lambda} \lambda,$$

where the meet (infimum) is taken over all Turiyam Neutrosophic flats $\lambda \in \mathcal{F}$ such that $\mu \leq \lambda$. The meet operation $\wedge$ is defined element-wise as:

$$\bar{\mu}(e) = \left(\min_{\lambda} T_\lambda(e),\ \max_{\lambda} I_\lambda(e),\ \max_{\lambda} F_\lambda(e),\ \max_{\lambda} L_\lambda(e)\right), \quad \forall e \in E.$$

Definition 29. A Turiyam Neutrosophic matroid $M = (E, \Psi)$ is called a *Turiyam Neutrosophic closure matroid* if the closure operator $\bar{\cdot}$ satisfies the following property:

$$\overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}, \quad \forall \mu, \nu \in T(E).$$

This means that the closure of the join (maximum) of any two Turiyam Neutrosophic sets is equal to the join of their closures.

Theorem 30. *Every Turiyam Neutrosophic closure matroid can be transformed into a neutrosophic closure matroid and consequently into a fuzzy closure matroid via mappings that preserve the matroid structure.*

Proof. Define a mapping $\psi : T(E) \to N(E)$ from Turiyam Neutrosophic sets to neutrosophic sets by:

$$\psi(\mu)(e) = \left(T_\mu(e),\ I_\mu(e) + L_\mu(e),\ F_\mu(e)\right), \quad \forall e \in E.$$

The indeterminacy in the Turiyam Neutrosophic set is increased by the liberal state $L_\mu(e)$, combining both into the indeterminacy-membership of the neutrosophic set.

Let $M_T = (E, \Psi_T)$ be a Turiyam Neutrosophic closure matroid with independent Turiyam Neutrosophic sets Ψ_T. Define the corresponding neutrosophic matroid $M_N = (E, \Psi_N)$, where:

$$\Psi_N = \{\psi(\mu) \mid \mu \in \Psi_T\}.$$

We consider Preservation of Matroid Axioms.

Axiom 1 (Hereditary Property): If $\mu \in \Psi_T$ and $\nu \leq \mu$ in $T(E)$, then $\nu \in \Psi_T$. Under ψ, $\psi(\nu) \leq \psi(\mu)$ in $N(E)$ and and $\psi(\nu) \in \Psi_N$. Thus, the hereditary property of the Turiyam Neutrosophic matroid is preserved in the neutrosophic matroid.

Axiom 2 (Exchange Property): Let $\mu_1, \mu_2 \in \Psi_T$ with $|\operatorname{supp}(\mu_1)| < |\operatorname{supp}(\mu_2)|$. Then there exists $\omega \in \Psi_T$ such that:

$$\mu_1 < \omega \leq \mu_1 \vee \mu_2 \quad \text{and} \quad m_T(\omega) \geq \min\{m_T(\mu_1), m_T(\mu_2)\}.$$

Applying ψ to this, we obtain:

$$\psi(\mu_1) < \psi(\omega) \leq \psi(\mu_1) \vee \psi(\mu_2) \quad \text{and} \quad m_T(\psi(\omega)) \geq \min\{m_T(\psi(\mu_1)), m_T(\psi(\mu_2))\}.$$

Thus, the exchange property is preserved, and ψ maps the Turiyam Neutrosophic closure matroid into a neutrosophic closure matroid.

Next, we define a mapping $\varphi : N(E) \to F(E)$ that transforms neutrosophic sets to fuzzy sets by extracting the truth-membership values:

$$\varphi(\mu)(e) = T_\mu(e), \quad \forall e \in E.$$

This mapping directly transforms the neutrosophic matroid M_N into the fuzzy closure matroid M_F.

We condsier about Preservation of Closure Operator.

Turiyam Neutrosophic Closure:

$$\overline{\mu} = \bigwedge_{\lambda \in \mathcal{F}_T, \mu \leq \lambda} \lambda,$$

where $\mathcal{F}_T$ is the family of Turiyam Neutrosophic flats.

Neutrosophic Closure:

$$\overline{\psi(\mu)} = \bigwedge_{\lambda \in \mathcal{F}_N, \psi(\mu) \leq \lambda} \lambda,$$

where $\mathcal{F}_N$ is the family of neutrosophic flats.

Fuzzy Closure:

$$\overline{\varphi(\psi(\mu))} = \bigwedge_{\lambda \in \mathcal{F}_F, \varphi(\psi(\mu)) \leq \lambda} \lambda,$$

where $\mathcal{F}_F$ is the family of fuzzy flats.

Since ψ and φ preserve the order relations, the closure operations are preserved across all transformations:

$$\varphi(\overline{\psi(\mu)}) = \overline{\varphi(\psi(\mu))}.$$

Thus, the Turiyam Neutrosophic closure matroid M_T can be transformed into a neutrosophic closure matroid M_N and subsequently into a fuzzy closure matroid M_F, preserving the matroid structure and closure properties. □

7. Some property of Neutrosophic closure matroid

In this section, we consider Some property of Neutrosophic closure matroid.

Theorem 31. *The closure of the intersection of two neutrosophic flats is equal to the intersection of their closures.*

Proof. Let μ and η be two neutrosophic flats in a neutrosophic closure matroid $M = (E, \Psi)$. By definition of the neutrosophic closure, we have:

$$\bar{\mu} \wedge \bar{\eta} = \overline{\mu \wedge \eta}.$$

Since μ and η are neutrosophic flats, $\bar{\mu} = \mu$ and $\bar{\eta} = \eta$. Therefore:

$$\mu \wedge \eta = \overline{\mu \wedge \eta}.$$

This proves that the closure of the intersection of two neutrosophic flats is equal to the intersection of their closures. □

Definition 32. An operation f on a set X is called *idempotent* if applying the operation multiple times has the same effect as applying it once. Formally, f is idempotent if:

$$f(f(x)) = f(x), \quad \forall x \in X.$$

In other words, applying f twice to any element $x \in X$ yields the same result as applying f once.

Example 33. Consider a set S and the closure operation cl in topology, which maps a subset of S to its closure. The closure operation is idempotent because for any subset $A \subseteq S$:

$$\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A).$$

Theorem 34. *The neutrosophic closure operator is idempotent.*

Proof. Let $\mu \in N(E)$ be a neutrosophic set, and let $\bar{\mu}$ be its closure. By the definition of the closure operator, we have:

$$\bar{\bar{\mu}} = \bar{\mu}.$$

This follows directly from the fact that applying the closure operator twice does not change the result. Hence, the neutrosophic closure operator is idempotent. □

Theorem 35. *Let $M_1 = (E_1, \Psi_1)$ and $M_2 = (E_2, \Psi_2)$ be loopless Neutrosophic matroids on disjoint ground sets E_1 and E_2, respectively. Then, $M_1 \oplus M_2$ is a Neutrosophic closure matroid if and only if both M_1 and M_2 are Neutrosophic closure matroids.*

Proof. We will prove both directions of the theorem.

Assume that $M_1 \oplus M_2$ is a Neutrosophic closure matroid. We need to show that both M_1 and M_2 are Neutrosophic closure matroids.

- Since $M_1 \oplus M_2$ is a Neutrosophic closure matroid, the closure operator $\bar{\cdot}$ satisfies the property for any $A_1 \subseteq E_1$ and $A_2 \subseteq E_2$:

$$\overline{A_1 \cup A_2} = \bar{A_1} \cup \bar{A_2}.$$

- Let us now focus on $A_1 \subseteq E_1$. Consider the closure operation within M_1, which corresponds to the restriction of $M_1 \oplus M_2$ to the ground set E_1. Since $A_2 = \emptyset$ in this case, we have:

$$\overline{A_1 \cup \emptyset} = \bar{A_1} \cup \bar{\emptyset}.$$

  But, by the properties of the closure operator, $\bar{\emptyset} = \emptyset$, so:

$$\overline{A_1} = \bar{A_1}.$$

  Hence, M_1 satisfies the closure property, and thus M_1 is a Neutrosophic closure matroid.

- A similar argument holds for M_2. Let $A_2 \subseteq E_2$ and apply the closure operator to $A_1 = \emptyset$, giving:

$$\overline{\emptyset \cup A_2} = \bar{\emptyset} \cup \bar{A_2} = \bar{A_2}.$$

  Thus, M_2 also satisfies the closure property, and therefore M_2 is a Neutrosophic closure matroid.

Next, assume that both M_1 and M_2 are Neutrosophic closure matroids. We need to show that $M_1 \oplus M_2$ is also a Neutrosophic closure matroid.

- Let $A_1 \subseteq E_1$ and $A_2 \subseteq E_2$. Since M_1 and M_2 are Neutrosophic closure matroids, we know that for $A_1 \subseteq E_1$ and $B_1 \subseteq E_1$:

$$\overline{A_1 \cup B_1} = \bar{A_1} \cup \bar{B_1}.$$

  Similarly, for $A_2 \subseteq E_2$ and $B_2 \subseteq E_2$:

$$\overline{A_2 \cup B_2} = \bar{A_2} \cup \bar{B_2}.$$

- Consider the closure operator applied to the union of sets $A_1 \subseteq E_1$ and $A_2 \subseteq E_2$. Since the ground sets are disjoint, the closure operator for $M_1 \oplus M_2$ acts independently on E_1 and E_2:

$$\overline{(A_1 \cup A_2)} = \overline{A_1} \cup \overline{A_2}.$$

- But, by the closure properties of M_1 and M_2, we know that:

$$\overline{A_1} = \bar{A_1} \quad \text{and} \quad \overline{A_2} = \bar{A_2}.$$

Thus:

$$\overline{A_1 \cup A_2} = \bar{A_1} \cup \bar{A_2}.$$

This shows that $M_1 \oplus M_2$ satisfies the Neutrosophic closure property, and hence $M_1 \oplus M_2$ is a Neutrosophic closure matroid.

We have proven both directions, so the theorem is true: $M_1 \oplus M_2$ is a Neutrosophic closure matroid if and only if both M_1 and M_2 are Neutrosophic closure matroids. □

Theorem 36. *The rank function of a neutrosophic closure matroid is submodular.*

Proof. Let A and B be subsets of E, and let r be the rank function of a neutrosophic closure matroid. We need to show that:

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B).$$

Since the neutrosophic closure operator satisfies the submodular inequality, the rank function r, which measures the size of the largest independent set, also satisfies submodularity. Thus, the rank function of a neutrosophic closure matroid is submodular. □

Theorem 37. *If $\mu, \nu \in \Psi$ are independent Neutrosophic sets in a Neutrosophic closure matroid M, and $|\operatorname{supp}(\mu)| < |\operatorname{supp}(\nu)|$, then there exists $\omega \in \Psi$ such that:*

$$\mu < \omega \leq \mu \vee \nu,$$

and

$$m_T(\omega) \geq \min\{m_T(\mu), m_T(\nu)\}.$$

Proof. This follows directly from the Neutrosophic closure properties and the exchange property inherited from Neutrosophic matroids. The closure of the join of two sets $\mu \vee \nu$ maintains the exchange property, where ω is the intermediate set that satisfies the conditions for exchange. □

Theorem 38. *If $\mu \in \Psi$ is an independent Neutrosophic set in a Neutrosophic closure matroid M, and $\nu \leq \mu$, then $\nu \in \Psi$ (i.e., Ψ satisfies the hereditary property).*

Proof. This is a direct consequence of the Neutrosophic closure matroid definition. If $\mu \in \Psi$ is closed, any smaller subset $\nu \leq \mu$, satisfying $T_\nu(e) \leq T_\mu(e), I_\nu(e) \geq I_\mu(e), F_\nu(e) \geq F_\mu(e)$, must also be closed under the Neutrosophic closure operator. Hence, $\nu \in \Psi$. □

Theorem 39. *A Neutrosophic matroid $M = (E, \Psi)$ is a Neutrosophic closure matroid if and only if the unions of Neutrosophic flats of M are again Neutrosophic flats of M.*

Proof. We will prove both directions of the theorem:

If M is a Neutrosophic closure matroid, then the union of Neutrosophic flats is a Neutrosophic flat. By definition, a Neutrosophic flat is a maximal set of elements in the ground set E that has been closed under the Neutrosophic closure operator $\bar{\cdot}$. That is, for any Neutrosophic flat A, we have $\bar{A} = A$, and for any other flat B, we have $\bar{B} = B$.

Let us consider the union $A \cup B$. Applying the Neutrosophic closure operator to this union gives:

$$(A \bar{\cup} B).$$

Since M is assumed to be a Neutrosophic closure matroid, we have the property:

$$(A \bar{\cup} B) = \bar{A} \cup \bar{B} = A \cup B.$$

Thus, the closure of the union of two Neutrosophic flats A and B is equal to their union, implying that $A \cup B$ is a Neutrosophic flat. Hence, the forward implication holds.

If the union of Neutrosophic flats is a Neutrosophic flat, then M is a Neutrosophic closure matroid. Assume that for any two Neutrosophic flats A and B, their union is a Neutrosophic flat. By assumption, $A \cup B$ is a Neutrosophic flat, so:

$$(A \bar{\cup} B) = A \cup B.$$

Moreover, since A and B are Neutrosophic flats, we also have:

$$\bar{A} = A \quad \text{and} \quad \bar{B} = B.$$

Thus, we can rewrite the closure of the union $A \cup B$ as:

$$(A \bar{\cup} B) = \bar{A} \cup \bar{B} = A \cup B.$$

This confirms that M satisfies the defining property of a Neutrosophic closure matroid, namely:

$$\overline{A \cup B} = \bar{A} \cup \bar{B}.$$

Hence, the reverse implication holds as well.

Since both directions have been proven, the theorem is true. □

Definition 40. A *neutrosophic matroid* $M = (E, \Psi)$ is called *neutrosophic modular* if and only if every neutrosophic flat μ in M is neutrosophic modular. Specifically, for every pair of neutrosophic flats μ and η, the following condition holds:

$$r(\mu) + r(\eta) = r(\mu \vee \eta) + r(\mu \wedge \eta),$$

where:

- $r(\mu)$ is the rank of the neutrosophic flat μ,
- $\mu \vee \eta$ is the join (maximum) of μ and η, defined as:

$$(\mu \vee \eta)(e) = \left(\max\{T_\mu(e), T_\eta(e)\},\ \min\{I_\mu(e), I_\eta(e)\},\ \min\{F_\mu(e), F_\eta(e)\}\right),$$

  for all $e \in E$,
- $\mu \wedge \eta$ is the meet (minimum) of μ and η, defined as:

$$(\mu \wedge \eta)(e) = \left(\min\{T_\mu(e), T_\eta(e)\},\ \max\{I_\mu(e), I_\eta(e)\},\ \max\{F_\mu(e), F_\eta(e)\}\right),$$

  for all $e \in E$.

This condition ensures that the sum of the ranks of any two neutrosophic flats equals the rank of their join plus the rank of their meet, reflecting the modularity property in the context of neutrosophic sets.

Theorem 41. *A neutrosophic closure matroid is a neutrosophic modular matroid.*

Proof. Let $M = (E, \Psi)$ be a neutrosophic closure matroid, where E is the ground set and Ψ is the family of independent neutrosophic sets. We aim to prove that every neutrosophic flat in M is neutrosophic modular, i.e., for any two neutrosophic flats μ and η, the following equation holds:

$$r(\mu) + r(\eta) = r(\mu \vee \eta) + r(\mu \wedge \eta),$$

where r is the rank function, $\mu \vee \eta$ is the join (maximum) of μ and η, and $\mu \wedge \eta$ is the meet (minimum) of μ and η.

We start by assuming that M satisfies the neutrosophic closure property. By definition, for any two neutrosophic sets μ and η, we have:

$$\overline{\mu \vee \eta} = \bar{\mu} \vee \bar{\eta},$$

where $\bar{\mu}$ and $\bar{\eta}$ denote the neutrosophic closures of μ and η, respectively. This closure property ensures that the closure of the union of any two neutrosophic sets is equal to the union of their closures.

Let μ and η be neutrosophic flats in M. By definition, neutrosophic flats are closed sets, meaning $\bar{\mu} = \mu$ and $\bar{\eta} = \eta$. Thus, the closure property simplifies to:

$$\mu \vee \eta = \bar{\mu} \vee \bar{\eta}.$$

The rank function r of a neutrosophic closure matroid satisfies the submodular inequality:

$$r(\mu) + r(\eta) \geq r(\mu \vee \eta) + r(\mu \wedge \eta).$$

To show equality, we consider the following: - The rank of the union $r(\mu \vee \eta)$ represents the size of the largest independent set contained in the union of μ and η. - The rank of the intersection $r(\mu \wedge \eta)$ is the size of the largest independent set contained in both μ and η.

Consider maximal chains of flats in μ and η. Let $\mu_1, \mu_2, \ldots, \mu_k$ be a chain of neutrosophic flats in μ, and let $\eta_1, \eta_2, \ldots, \eta_\ell$ be a chain of neutrosophic flats in η. These chains represent the independent sets contained in μ and η, respectively.

By the neutrosophic closure property, the union of these chains forms a chain of neutrosophic flats in $\mu \vee \eta$. Hence, the rank of $\mu \vee \eta$ must be at least the sum of the ranks of μ and η.

Since the submodular inequality holds, and we have constructed a chain showing that $r(\mu \vee \eta) \geq r(\mu) + r(\eta)$, it follows that:

$$r(\mu) + r(\eta) = r(\mu \vee \eta) + r(\mu \wedge \eta),$$

proving that M is neutrosophic modular. □

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Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

Please note that preprints and early-stage research may not have undergone peer review. Additionally, as I am an independent researcher, please understand. Sorry.

As research in this field continues to evolve, the findings and interpretations presented in this paper may be subject to change. Readers are encouraged to consult future publications for the latest developments.

The authors have made every effort to accurately cite and reference all sources used in this paper. However, any discrepancies or omissions are unintentional, and the authors welcome any corrections.

References

[1] Muhammad Akram and Musavarah Sarwar. Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *viXra*, 2017.

[2] T Al-Hawary. Closure matroid properties. *Mu'tah Lil-Buhuth Wad-Dirasat*, 19(3):35–43, 2004.

[3] T Al-Hawary. Fuzzy flats. *Indian J. Mathematics*, 55(2):223–236, 2013.

[4] T Al-Hawary. On modular flats and pushouts of matroids. *Ital. J. Pure Appl. Math*, 43:237–241, 2020.

[5] Talal Al-Hawary. Fuzzy closure matroids. *Matematika*, pages 69–74, 2016.

[6] Talal Al-Hawary. On functions of k-balanced matroids. *Open Journal of Discrete Mathematics*, 7(3):103–107, 2017.

[7] Talal Al-Hawary and Jenny McNulty. Closure matroids. *Congressus Numerantium*, pages 93–96, 2001.

[8] Talal Ali Al-Hawary. On fuzzy matroids. *Mathematical Combinatorics*, 1:13–21, 2012.

[9] Federico Ardila. *Enumerative and algebraic aspects of matroids and hyperplane arrangements*. PhD thesis, Massachusetts Institute of Technology, 2003.

[10] Muhammad Asif, Muhammad Akram, and Ghous Ali. Pythagorean fuzzy matroids with application. *Symmetry*, 12(3):423, 2020.

[11] Yossi Azar, Andrei Z. Broder, and Alan M. Frieze. On the problem of approximating the number of bases of a matroid. *Inf. Process. Lett.*, 50(1):9–11, 1994.

[12] Joseph E Bonin and Anna de Mier. The lattice of cyclic flats of a matroid. *Annals of combinatorics*, 12(2):155–170, 2008.

[13] Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.

[14] Gruia Calinescu, Chandra Chekuri, Martin Pal, and Jan Vondrák. Maximizing a monotone submodular function subject to a matroid constraint. *SIAM Journal on Computing*, 40(6):1740–1766, 2011.

[15] Laszlo Csirmaz. Cyclic flats of a polymatroid. *Annals of Combinatorics*, 24(4):637–648, 2020.

[16] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[17] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[18] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[19] Didier Dubois and Henri Prade. A review of fuzzy set aggregation connectives. *Information sciences*, 36(1-2):85–121, 1985.

[20] Jens Niklas Eberhardt. Computing the tutte polynomial of a matroid from its lattice of cyclic flats. *arXiv preprint arXiv:1407.6666*, 2014.

[21] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[22] Fedor V Fomin, Sang-il Oum, and Dimitrios M Thilikos. Rank-width and tree-width of h-minor-free graphs. *European Journal of Combinatorics*, 31(7):1617–1628, 2010.

[23] Ragnar Freij-Hollanti, Matthias Grezet, Camilla Hollanti, and Thomas Westerbäck. Cyclic flats of binary matroids. *Advances in Applied Mathematics*, 127:102165, 2021.

[24] Takaaki Fujita. Matroid, ideal, ultrafilter, tangle, and so on: Reconsideration of obstruction to linear decomposition. *International Journal of Mathematics Trends and Technology (IJMTT)*, 70:18–29, 2024.

[25] Takaaki Fujita. Neutrosophic matroid, antimatroid, and greedoid. June 2024.

[26] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[27] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.

[28] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *ResearchGate*, July 2024.

[29] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[30] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[31] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[32] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[33] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[34] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[35] Jim Geelen, Bert Gerards, and Geoff Whittle. Obstructions to branch-decomposition of matroids. *Journal of Combinatorial Theory, Series B*, 96(4):560–570, 2006.

[36] R. Goetschel Jr. and W. Voxman. Fuzzy matroids. *Fuzzy Sets and Systems*, 27:291–301, 1988.

[37] Roy Goetschel Jr and William Voxman. Spanning properties for fuzzy matroids. *Fuzzy sets and systems*, 51(3):313–321, 1992.

[38] Rohit Gurjar and Nisheeth K Vishnoi. On the number of circuits in regular matroids (with connections to lattices and codes). In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 861–880. SIAM, 2019.

[39] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[40] AP Heron. A property of the hyperplanes of a matroid and an extension of dilworth's theorem. *Journal of Mathematical Analysis and Applications*, 42(1):119–131, 1973.

[41] Petr Hliněný and Geoff Whittle. Matroid tree-width. *European Journal of Combinatorics*, 27(7):1117–1128, 2006.

[42] Bill Jackson and Tibor Jordán. On the rank function of the 3-dimensional rigidity matroid. *International Journal of Computational Geometry & Applications*, 16(05n06):415–429, 2006.

[43] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[44] Michel Las Vergnas. Bases in oriented matroids. *Journal of Combinatorial Theory, Series B*, 25(3):283–289, 1978.

[45] Michel Las Vergnas. Active orders for matroid bases. *European Journal of Combinatorics*, 22(5):709–721, 2001.

[46] Manoel Lemos and TRB Melo. Connected hyperplanes in binary matroids. *Linear algebra and its applications*, 432(1):259–274, 2010.

[47] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[48] Sheng-Gang Li, Xiu Xin, and Yao-Long Li. Closure axioms for a class of fuzzy matroids and co-towers of matroids. *Fuzzy sets and systems*, 158(11):1246–1257, 2007.

[49] Yao-Long Li, Guo-Jun Zhang, and Ling-Xia Lu. Axioms for bases of closed regular fuzzy matroids. *Fuzzy sets and Systems*, 161(12):1711–1725, 2010.

[50] J Mahalakshmi and M Sudha. On fuzzy g$\delta\beta$ continuity in fuzzy matroids. *Annals of Fuzzy Mathematics and Informatics*, 11(5):737–753, 2016.

[51] Patrick Mullins. *The Circuit and Cocircuit Lattices of a Regular Matroid*. The University of Vermont and State Agricultural College, 2020.

[52] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.

[53] Sang-il Oum. Rank-width is less than or equal to branch-width. *Journal of Graph Theory*, 57(3):239–244, 2008.

[54] Sang-il Oum. Rank-width: Algorithmic and structural results. *Discrete Applied Mathematics*, 231:15–24, 2017.

[55] Sang-il Oum and Paul Seymour. Testing branch-width. *Journal of Combinatorial Theory, Series B*, 97(3):385–393, 2007.

[56] James Oxley. Matroid theory. In *Handbook of the Tutte polynomial and related topics*, pages 44–85. Chapman and Hall/CRC, 2022.

[57] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[58] Kadin Prideaux. *Matroids, cyclic flats, and polyhedra*. PhD thesis, Open Access Te Herenga Waka-Victoria University of Wellington, 2016.

[59] A Alberto Ravagnani, B Benjamin Jany, and RA Rudi Pendavingh. Codes, matroids and cyclic flats. 2024.

[60] András Recski. *Matroid theory and its applications in electric network theory and in statics*, volume 6. Springer Science & Business Media, 2013.

[61] Musavarah Sarwar and Muhammad Akram. New applications of m-polar fuzzy matroids. *Symmetry*, 9(12):319, 2017.

[62] Paul D Seymour. Triples in matroid circuits. *European Journal of Combinatorics*, 7(2):177–185, 1986.

[63] Fu-Gui Shi. (l, m)-fuzzy matroids. *Fuzzy sets and systems*, 160(16):2387–2400, 2009.

[64] Fu-Gui Shi. A new approach to the fuzzification of matroids. *Fuzzy Sets and Systems*, 160(5):696–705, 2009.

[65] Akiyoshi Shioura. Matroid rank functions and discrete concavity. *Japan journal of industrial and applied mathematics*, 29(3):535–546, 2012.

[66] Prem Kumar Singh. Turiyam set a fourth dimension data representation. *Journal of Applied Mathematics and Physics*, 9(7):1821–1828, 2021.

[67] Prem Kumar Singh. Turiyam set and its mathematical distinction from other sets. *Galoitica: Journal of Mathematical Structures and Applications*, 8(1):08–19, 2023.

[68] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[69] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[70] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[71] Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.

[72] Florentin Smarandache. Neutrosophic physics: More problems, more solutions. 2010.

[73] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[74] Florentin Smarandache. *Nidus Idearum. Scilogs, XI: in-turns and out-turns*. Infinite Study, 2023.

[75] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[76] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[77] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[78] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[79] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[80] DJA Welsh et al. Matroids: fundamental concepts. *Handbook of combinatorics*, 1:481–526, 1995.

[81] Dominic JA Welsh. *Matroid theory*. Courier Corporation, 2010.

[82] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[83] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[84] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.

[85] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

Some Graph Parameters for Superhypertree-width and Neutrosophic tree-width

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Abstract. Graph characteristics are often studied through various parameters, with ongoing research dedicated to exploring these aspects. Among these, graph width parameters—such as tree-width—are particularly important due to their practical applications in algorithms and real-world problems. A hypergraph generalizes traditional graph theory by abstracting and extending its concepts [77]. More recently, the concept of a SuperHyperGraph has been introduced as a further generalization of the hypergraph. Neutrosophic logic [133], a mathematical framework, extends classical and fuzzy logic by allowing the simultaneous consideration of truth, indeterminacy, and falsity within an interval. In this paper, we explore Superhypertree-width, Neutrosophic tree-width, and t-Neutrosophic tree-width.

Keywords: Neutrosophic graph; Hypertree width; Superhypertree width; Tree-width; Neutrosophictree-width

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1. Introduction

1.1. *Graph Width Parameters*

Graphs have been extensively researched in recent years [52], with particular emphasis on characterizing their properties. Graph characteristics are frequently analyzed using various parameters, and ongoing studies aim to deepen our understanding of these aspects. Analyzing graph parameters is essential for understanding the structural properties of graphs, which is crucial for solving computational problems in fields like network optimization, database theory, and machine learning applications (cf. [73]).

Among these parameters, graph width parameters such as tree-width [25,27,28,91,126–128], cut-width [85,98], clique-width [45], modular-width [3], tree-cut-width [68,99], boolean-width [4,155], band-width [30,44,124], rank-width [93,116,118], and path-width [48,94,152] are particularly significant due to their practical applications in algorithms and real-world problems. This has fueled active research, focusing on how these parameters influence computational efficiency and problem-solving methods.

For instance, tree-width measures how closely a graph resembles a tree, while path-width shows how similar a graph is to a path. Graph width parameters often correspond to efficient, manageable graph structures, which is why they have been the focus of extensive research for many years (e.g., [79, 85, 124, 152–154]). Given this background, research on graph width parameters is of great importance.

1.2. *Hypergraph and SuperHyperGraph*

A hypergraph is a generalization of a conventional graph, extending concepts from graph theory [22, 77]. A hypergraph is a generalization of a graph where edges, called hyperedges, can connect any number of vertices, not just two. Formally, a hypergraph is defined as $H = (V, E)$, where V is the vertex set and E is the set of hyperedges. Hypergraphs are widely used in fields such as machine learning and network analysis [41, 69, 87, 100].

In many real-world applications, evaluating how closely a graph approximates a tree structure is essential. This has led to significant research on Hypertree-width [6, 72, 74, 103, 161] and Hyperpath-width [5, 105, 115], which quantify how much a hypergraph resembles a tree or a path. Hypertree-width, in particular, plays a key role in database systems and practical applications [73, 74].

Recently, the concept of a SuperHyperGraph has emerged as a further generalization of the hypergraph, attracting significant research interest similar to that seen with hypergraphs [66, 82–84, 137–139, 143, 149]. It generalizes hypergraphs, allowing vertices to be individual elements or subsets (super-vertices), and edges to connect groups of vertices or super-vertices. This structure models complex relationships. Similar to hypergraphs, various applications are also being explored [61, 84].

1.3. *Neutrosophic Graph and Neutrosophic HyperGraph*

In any scientific field, a classical theorem defined within a specific space is a statement that holds true for 100 percent of the elements in that space. However, when applied to real-world scenarios, various constraints often arise, leading to the need for considering uncertain concepts, such as fuzzy sets [162–166], Vague set [29, 86, 167], plithogenic sets [1, 67, 136, 149], Rough sets [119–122], soft sets [106], Hypersoft set [141, 142], and neutrosophic sets [133–135, 146, 147] To address uncertainty and the relationships between concepts, several graph classes have been introduced, including Fuzzy Graphs [109, 125], Vague Graphs [131], and Rough Graphs [49]. Among these, this paper focuses on Neutrosophic Graphs [89].

A Neutrosophic Set generalizes the concept of fuzzy sets by introducing three membership functions: truth, indeterminacy, and falsity, allowing for more nuanced representation of uncertainty [18, 150]. In recent years, Neutrosophic Graphs [11, 34, 76, 88, 130, 133, 140, 147, 148] and Neutrosophic Hypergraphs [8, 12, 102, 112] have been actively studied within Neutrosophic Set Theory. Neutrosophic refers to a mathematical framework that generalizes classical and fuzzy logic, simultaneously handling degrees of truth, indeterminacy, and falsity within an interval. These graphs, as generalizations of Fuzzy Graphs [110, 129], have garnered attention for their potential applications similar to those of Fuzzy Graphs. The concept of a Neutrosophic SuperhyperGraph, which further generalizes Neutrosophic Hypergraphs, has also been the subject of active research in recent years [112, 144]. Due to the significance of Neutrosophic graphs, many other graph classes have also been proposed, such as those in [32, 32, 47, 58, 60, 65].

1.4. *Our Contribution*

Research on tree and path structures in Neutrosophic Graphs and SuperHyperGraphs is still in its early stages, and while several graph parameters have been proposed, there remains significant room for further exploration. In this context, [59] introduced the concept of SuperHyperTree-width.

We also explore Neutrosophic Graphs and Neutrosophic Hypergraphs. By applying tree-width and path-width to these types of graphs, we aim to accelerate research and applications related to graph width parameters, as well as to Neutrosophic Graphs and SuperHyperGraphs.

This paper is structured as follows: Section 2 provides definitions and examples of general graphs, fuzzy graphs, and hypergraphs. Section 3 reviews Neutrosophic Graphs and SuperHyperGraphs, and defines and examines properties such as Neutrosophic tree-width, Neutrosophic hypertree-width, and n-Neutrosophic tree-width. Section 4 discusses future tasks.

2. **Preliminaries and definitions**

In this section, we briefly explain the definitions and notations used in this paper.

2.1. *Basic Graph Concepts*

A graph G is a mathematical structure consisting of nodes (vertices) connected by edges, representing relationships or connections. In a graph G, $V(G)$ denotes the set of vertices, and $E(G)$ denotes the set of edges. The notation $G = (V, E)$ indicates that the graph G is defined by the pair of sets V (vertices) and E (edges).

In this paper, we provide several essential graph theory definitions and simple examples necessary for the discussions that follow.

Definition 2.1. A subgraph is formed by selecting specific vertices and edges from a graph.

Definition 2.2. A graph $G = (V, E)$ is called *connected* if, for every pair of vertices $u, v \in V$, there exists a path in G connecting u and v. In other words, a graph is connected if there is a sequence of edges that allows traversal between any two vertices in the graph. If no such path exists between certain vertices, the graph is called *disconnected*.

Definition 2.3. (cf. [42]) A path is a walk with no repeated vertices, a cycle is a closed path, and a tree is a connected acyclic graph.

Definition 2.4 (Tree)**.** (cf. [158]) A tree is a connected, acyclic undirected graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist. Additionally, a tree with n vertices has $n-1$ edges.

Properties of Trees:

- There is exactly one path between any two vertices in a tree.
- A tree contains no cycles, and removing any edge from a tree will disconnect it.
- Adding an edge to a tree will create exactly one cycle.

For more basic graph notation and concepts, please refer to [19, 42, 52, 75].

2.2. *Hypergraph Concepts*

A hypergraph is a generalization of a graph where edges, called hyperedges, can connect any number of vertices, not just two. This structure is useful for modeling complex relationships in various fields like computer science and biology [56, 71, 123]. The definition is provided below.

Definition 2.5. [31] A hypergraph is a pair $H = (V(H), E(H))$, consisting of a nonempty set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$, called the hyperedges of H. In this paper, we consider only finite hypergraphs.

Example 2.6. Let H be a hypergraph with vertex set $V(H) = \{A, B, C, D, E\}$ and hyperedge set $E(H) = \{e_1, e_2, e_3\}$, where:

$$e_1 = \{A, D\}, \quad e_2 = \{D, E\}, \quad e_3 = \{A, B, C\}.$$

Thus, H is represented by the pair $H(V, E) = (\{A, B, C, D, E\}, \{\{A, D\}, \{D, E\}, \{A, B, C\}\})$.

Definition 2.7. [31] For a hypergraph H and a subset $X \subseteq V(H)$, the subhypergraph induced by X is defined as $H[X] = (X, \{e \cap X \mid e \in E(H)\})$. We denote the hypergraph obtained by removing X from H as $H \setminus X := H[V(H) \setminus X]$.

Definition 2.8. A *hyperpath* in a hypergraph $H = (V(H), E(H))$ is a sequence of alternating vertices and hyperedges:

$$(v_0, e_1, v_1, e_2, v_2, \ldots, e_t, v_t)$$

such that:

- For each consecutive pair v_{i-1} and v_i, there is a hyperedge $e_i \in E(H)$ where both vertices v_{i-1} and v_i are contained in e_i, i.e., $v_{i-1}, v_i \in e_i$,
- $v_{i-1} \neq v_i$, meaning no vertex is repeated consecutively in the path.

A hyperpath is called a *simple hyperpath* if all vertices v_i and all hyperedges e_i are distinct.

Definition 2.9. (cf. [6, 104]) A *hypertree* T is a connected hypergraph in which the removal of any hyperedge from T results in a disconnected hypergraph. Specifically:

- For any hyperedge $e \in E(T)$, if e is removed, the resulting subhypergraph is no longer connected.

- A hypertree can contain cycles, as long as removing any hyperedge disconnects the hypergraph.

Example 2.10. Let T be a hypergraph with vertex set $V(T) = \{A, B, C, D, E\}$ and hyperedge set $E(T) = \{e_1, e_2, e_3\}$, where:

$$e_1 = \{A, B\}, \quad e_2 = \{B, C, D\}, \quad e_3 = \{D, E\}.$$

In this case, T is a hypertree because removing any hyperedge e_1, e_2, or e_3 would disconnect the hypergraph.

For more basic hypergraph notation and concepts, please refer to [31, 46].

2.3. *Tree-width and Hypertree-width*

We now define Tree-width and Hypertree-width. Tree-width measures how closely a graph resembles a tree by representing the graph in a tree-like structure with minimal width [25–27, 126, 128]. Hypertree-width generalizes tree-width to hypergraphs, quantifying how well a hypergraph can be decomposed into a tree-like structure [6, 72, 74, 103, 161]. The formal definitions of Tree-width and Hypertree-width are provided below. For more details on Tree-width, please refer to [25, 26].

Definition 2.11. [128] A tree-decomposition of an undirected graph G is a pair (T, W), where T is a tree, and $W = (W_t \mid t \in V(T))$ is a family of subsets that associates with every node t of T a subset W_t of vertices of G such that:

(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(T2) For each edge $(u, v) \in E(G)$, there exists some node t of T such that $\{u, v\} \subseteq W_t$, and

(T3) For all nodes r, s, t in T, if s is on the unique path from r to t then $W_r \cap W_t \subseteq W_s$.

The width of a tree-decomposition (T, W) is the maximum of $|W_t| - 1$ over all nodes t of T. The tree-width of G is the minimum width over all tree-decompositions of G.

Example 2.12. Consider the following graph G:

$$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E(G) = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_5)\}$$

We want to find a tree-decomposition for this graph G. Let T be a tree with three nodes t_1, t_2, t_3. We define the bags W_t as follows:

$$W_{t_1} = \{v_1, v_2, v_3\}, \quad W_{t_2} = \{v_2, v_3, v_4\}, \quad W_{t_3} = \{v_4, v_5\}$$

- *(T1):* The union of all bags covers all vertices of G:

$$W_{t_1} \cup W_{t_2} \cup W_{t_3} = \{v_1, v_2, v_3, v_4, v_5\} = V(G)$$

- *(T2):* For each edge $(u, v) \in E(G)$, there exists a bag that contains both u and v:
  - (v_1, v_2) is in W_{t_1},
  - (v_1, v_3) is in W_{t_1},
  - (v_2, v_4) is in W_{t_2},
  - (v_3, v_4) is in W_{t_2},
  - (v_4, v_5) is in W_{t_3}.
- *(T3):* For all nodes r, s, t in T, if s lies on the unique path from r to t, then:

$$W_r \cap W_t \subseteq W_s$$

  This is satisfied, as $W_{t_1} \cap W_{t_3} = \{v_4\} \subseteq W_{t_2}$.

The size of each bag is:

$$|W_{t_1}| = 3, \quad |W_{t_2}| = 3, \quad |W_{t_3}| = 2$$

The width of this tree-decomposition is the maximum bag size minus 1:

$$\text{width} = \max(3-1, 3-1, 2-1) = 2$$

Definition 2.13. [6] A generalized hypertree decomposition of H is a triple (T, B, C), where (T, B) is a tree-decomposition of H and $C = (C_t)_{t \in V(T)}$ is a family of subsets of $E(H)$ such that for every $t \in V(T)$ we have $B_t \subseteq \bigcup C_t$. Here $\bigcup C_t$ denotes the union of the sets (hyperedges) in C_t, that is, the set $\{v \in V(H) \mid \exists e \in C_t : v \in e\}$. The sets C_t are called the guards of the decomposition. The width of the decomposition (T, B, C) is $\max\{|C_t| \mid t \in V(T)\}$. The generalized hypertree width of H, denoted by $\text{ghw}(H)$, is the minimum of the widths of the generalized hypertree decompositions of H.

Definition 2.14. [6] A hypertree decomposition of H is a generalized hypertree decomposition (T, B, C) that satisfies the following special condition: $(\bigcup C_t) \cap \bigcup_{u \in V(T_t)} B_u \subseteq B_t$ for all $t \in V(T)$. Recall that T_t denotes the subtree of the T with root t. The hypertree width of H, denoted by $\text{hw}(H)$, is the minimum of the widths of all hypertree decompositions of H.

2.4. *Fuzzy Graph*

A Fuzzy Graph represents relationships involving uncertainty by assigning membership degrees to both vertices and edges, allowing for more flexible and nuanced analysis. Due to its significance, Fuzzy Graphs have been the subject of extensive research [9, 10, 14–17, 23, 24, 92, 101, 107, 111, 151, 157]. The formal definition of a Fuzzy Graph is as follows [108, 129].

Definition 2.15. [129] A fuzzy graph $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.

- $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ, such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ denotes the minimum operation.

The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where:

- $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$
- $\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}$

Definition 2.16. A fuzzy subgraph $H = (\sigma', \mu')$ of G is defined as follows:

- There exists $X \subseteq V$ such that $\sigma' : X \to [0,1]$ is a fuzzy subset.
- $\mu' : X \times X \to [0,1]$ is a fuzzy relation on σ', satisfying $\mu'(x,y) \leq \sigma'(x) \wedge \sigma'(y)$ for all $x, y \in X$.

Example 2.17. (cf. [35]) Consider a fuzzy graph $G = (\sigma, \mu)$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as depicted in the diagram.

The membership degrees of the vertices are as follows:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

The fuzzy relation on the edges is defined by the values of μ, where $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. The fuzzy membership degrees of the edges are as follows:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

In this case, the fuzzy graph G has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying membership degrees.
- The fuzzy relations ensure that $\mu(x,y)$ for any edge (x,y) does not exceed the minimum membership of the corresponding vertices.

In [62], Fuzzy Tree-width was defined as an extension of Tree-width for Fuzzy Graphs, serving as a graph width parameter. The definition is provided below.

Definition 2.18. [62] Let $G = (\sigma, \mu)$ be a fuzzy graph, where $\sigma : V \to [0,1]$ is a fuzzy subset representing vertex membership, and $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ. A fuzzy tree-decomposition of G is a pair $(T, \{B_t\}_{t \in T})$, where:

- $T = (I, F)$ is a tree with nodes I and edges F.
- $\{B_t\}_{t \in T}$ is a collection of fuzzy subsets of V (called bags) associated with the nodes of T such that:
  (1) For each vertex $v \in V$, the set $\{t \in I : v \in B_t\}$ is connected in the tree T.
  (2) For each edge $(u,v) \in V \times V$ with membership degree $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$, there exists some node $t \in I$ such that both u and v are in B_t, and the membership degree of u and v in B_t is at least $\mu(u,v)$.

The width of a fuzzy tree-decomposition $(T, \{B_t\}_{t\in T})$ is defined as

$$\max_{t\in I}\left(\sup_{v\in B_t}\mu(v, B_t) - 1\right),$$

where $\mu(v, B_t)$ represents the maximum membership degree of vertex v in the fuzzy set B_t. The Fuzzy-Tree-Width of the fuzzy graph G is the minimum width among all possible fuzzy tree-decompositions of G.

Example 2.19. Definition of Fuzzy Graph: Consider the following Fuzzy Graph $G = (\sigma, \mu)$, which has vertices and edges defined as follows:

$$V = \{v_1, v_2, v_3, v_4\}$$

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$
$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

Given this Fuzzy Graph G, we aim to construct a Fuzzy Tree-Decomposition $(T, \{B_t\}_{t\in T})$. We will set the following bags $\{B_t\}_{t\in T}$ and the tree T:

Bags:

$$B_1 = \{v_1, v_2\}, \quad B_2 = \{v_2, v_3, v_4\}, \quad B_3 = \{v_4, v_1\}$$

- B_1 and B_2 share the vertex v_2 and are connected.
- B_2 and B_3 share the vertex v_4 and are connected.

Verification of the Connectivity Condition:

- Vertex v_1: Appears in B_1 and B_3, and these bags are connected in T.
- Vertex v_2: Appears in B_1 and B_2, and these bags are connected.
- Vertex v_3: Appears only in B_2.
- Vertex v_4: Appears in B_2 and B_3, and these bags are connected.

Thus, the connectivity condition is satisfied for all vertices.

Next, we verify the edge condition for each edge:

- For (v_1, v_2), v_1 and v_2 appear in B_1, and $\mu(v_1, v_2) = 0.1 \leq \sigma(v_1) \wedge \sigma(v_2) = 0.1$.
- For (v_2, v_3), v_2 and v_3 appear in B_2, and $\mu(v_2, v_3) = 0.1 \leq \sigma(v_2) \wedge \sigma(v_3) = 0.1$.
- For (v_3, v_4), v_3 and v_4 appear in B_2, and $\mu(v_3, v_4) = 0.1 \leq \sigma(v_3) \wedge \sigma(v_4) = 0.2$.
- For (v_4, v_1), v_4 and v_1 appear in B_3, and $\mu(v_4, v_1) = 0.1 \leq \sigma(v_4) \wedge \sigma(v_1) = 0.1$.
- For (v_2, v_4), v_2 and v_4 appear in B_2, and $\mu(v_2, v_4) = 0.3 \leq \sigma(v_2) \wedge \sigma(v_4) = 0.3$.

Thus, the edge condition is satisfied for all edges.

The width of the Fuzzy Tree-Decomposition $(T, \{B_t\})$ is defined as:

$$\max_{t\in T}\left(\sup_{v\in B_t}\mu(v, B_t) - 1\right)$$

- For Bag $B_1 = \{v_1, v_2\}$:

$$\mu(v_1, v_2) = 0.1, \quad \sup_{v \in B_1} \mu(v, B_1) = 0.1$$

Width $= 0.1 - 1 = -0.9$

- For Bag $B_2 = \{v_2, v_3, v_4\}$:

$$\max(\mu(v_2, v_3), \mu(v_3, v_4), \mu(v_2, v_4)) = \max(0.1, 0.1, 0.3) = 0.3$$

Width $= 0.3 - 1 = -0.7$

- For Bag $B_3 = \{v_4, v_1\}$:

$$\mu(v_4, v_1) = 0.1, \quad \sup_{v \in B_3} \mu(v, B_3) = 0.1$$

Width $= 0.1 - 1 = -0.9$

$$\max(-0.9, -0.7, -0.9) = -0.7$$

The width of this Fuzzy Tree-Decomposition is -0.7.

2.5. *Graph Parameter Hierarchy*

The Graph Parameter Hierarchy is frequently studied in research on graph parameters and algorithms. It describes the relationships between graph parameters based on their ability to upper or lower bound one another [63, 78, 132, 156].

Definition 2.20. A graph parameter is a function $f : G \to \mathbb{R}$, where G represents the set of all undirected finite graphs, and the function returns a real number. We say that *Parameter A upper bounds Parameter B* if there exists a non-decreasing function $f_{A,B}$ such that

$$f_{A,B}(A(G)) \geq B(G)$$

for all graphs G. Conversely, if *Parameter A* does not upper bound *Parameter B*, we say that *Parameter B* is *unbounded* with respect to *Parameter A*.

Lemma 2.21. *[132] If a upper bounds b, and b upper bounds c, then a also upper bounds c.*

Proof. Refer to [132] for details as necessary. □

3. **Result in this paper**

In this section, we present the results of this paper. We focus on the study of Tree-width in the context of SuperHypergraphs and Neutrosophic graphs.

3.1. *SuperHyperTree-width*

A SuperHyperGraph is an advanced structure that extends hypergraphs by allowing both vertices and edges to be sets [64, 137, 138]. First, we will explain the definition of a SuperHyperGraph. The definition is provided below.

Definition 3.1 (SuperHyperGraph (SHG)). [137, 138] A *SuperHyperGraph (SHG)* is an ordered pair $\text{SHG} = (G, E)$, where:

(1) $G \subseteq P(V)$ is the set of vertices, and $E \subseteq P(V)$ is the set of edges, with $V = \{V_1, V_2, \ldots, V_m\}$, where $m \geq 0$.
(2) $P(V)$ denotes the power set of V, i.e., all subsets of V. Each element in G is referred to as an SHG-vertex, which can take the following forms:
   - A single vertex (as in classical graphs).
   - A super-vertex, representing a subset of vertices (e.g., a group or organization).
   - An indeterminate-vertex, representing an unclear or unknown entity.
   - A null-vertex, represented by $\emptyset$, signifying a vertex without elements.
(3) $E = \{E_1, E_2, \ldots, E_k\}$, with $k \geq 1$, represents the family of SHG-edges. Each $E_j \in P(V)$ is a subset of V, and SHG-edges can include:
   - A single edge (classical edge), connecting two single vertices.
   - A super-edge, connecting super-vertices.
   - A hyper-super-edge, connecting three or more groups or organizations.
   - A multi-edge, connecting multiple vertices or super-vertices simultaneously.
   - An indeterminate-edge, representing unclear or unknown relationships.
   - A null-edge $\emptyset$, representing no connection between the given vertices.

Definition 3.2 (Elements of a SuperHyperGraph). [137, 138] A *SuperHyperGraph (SHG)* consists of the following elements:

- **Single Vertices** V_i: Individual vertices as in classical graphs, e.g., V_1, V_2.
- **SuperVertices (SubsetVertices)** $\text{SV}_{i,j}$: Subsets of vertices, such as $\text{SV}_{1,3} = \{V_1, V_3\}$. SuperVertices can represent groups or organizations. For example:
  - $\text{SV}_{1,23} = \{V_1, V_{23}\}$, combining single vertices V_1 and V_{23}.
  - $\text{SV}_{1,2,3} = \{V_1, V_2, V_3\}$, combining three single vertices.
- **NullVertex** $\emptyset_V$: A vertex with no elements.
- **Single Edges** $E_{i,j}$: Edges connecting two single vertices, e.g., $E_{1,5} = \{V_1, V_5\}$, $E_{2,3} = \{V_2, V_3\}$.
- **HyperEdges** $\text{HE}_{i,j,k}$: Edges connecting three or more single vertices, e.g., $\text{HE}_{1,4,6} = \{V_1, V_4, V_6\}$.
- **SuperEdges (SubsetEdges)** $\text{SE}_{(i,j),(k,l)}$: Edges connecting super-vertices, e.g., $\text{SE}_{(1,3),(4,5)} = \{\text{SV}_{1,3}, \text{SV}_{4,5}\}$.

- **HyperSuperEdges (or HyperSubsetEdges)** $\mathrm{HSE}_{i,j,k}$: Edges connecting three or more vertices, with at least one being a super-vertex. For example:

$$\mathrm{HSE}_{1,45,23} = \{V_1, \mathrm{SV}_{45}, \mathrm{SV}_{23}\}.$$

- **IndeterminateEdges** $IE_{x,y}$: Edges with unknown or unclear connections between vertices.
- **NullEdges** $\emptyset_E$: Edges representing no connection between vertices.

We consider the relationship between a Hypergraph and a SuperHypergraph. The following holds.

Theorem 3.3. *Any SuperHypergraph can be reduced to a Hypergraph.*

Proof. To map any SuperHypergraph to a Hypergraph, we need to handle both *SuperVertices* and *SuperEdges* from the SHG.

Each super-vertex in the SuperHypergraph corresponds to a set of vertices in a classical graph. Specifically, a super-vertex $\mathrm{SV}_{i,j} \in G$ (e.g., $\{V_i, V_j\}$) can be represented by multiple individual vertices in the corresponding hypergraph. The vertex mapping is therefore straightforward: each element of the super-vertex (which is a set) will be treated as an individual vertex in the hypergraph. Formally:

- For each super-vertex $\mathrm{SV}_{i,j} = \{V_i, V_j\} \in V_{\mathrm{SHG}}$, create individual vertices $V_i, V_j \in V_{\mathrm{H}}$.

SuperEdges or HyperSuperEdges in SHG, which may connect multiple super-vertices or sets, need to be flattened into simple hyperedges in the hypergraph. For each super-edge or hyper-super-edge $\mathrm{SE}_{(i,j),(k,l)}$ connecting super-vertices $\mathrm{SV}_{i,j}$ and $\mathrm{SV}_{k,l}$, create a hyperedge in the hypergraph that connects all the corresponding individual vertices. Formally:

- For each super-edge $\mathrm{SE}_{(i,j),(k,l)} \in E_{\mathrm{SHG}}$, create a hyperedge $e_{\mathrm{H}} \in E_{\mathrm{H}}$, where $e_{\mathrm{H}} = \{V_i, V_j, V_k, V_l\}$.

The general rule is that any *SuperEdge* connecting multiple super-vertices in the SHG corresponds to a single hyperedge connecting the individual vertices (elements) of those super-vertices in the hypergraph. □

Example 3.4. Consider the following SuperHypergraph SHG:

- Vertices: $V_{\mathrm{SHG}} = \{\{V_1, V_2\}, \{V_3\}\}$
- Edges: $E_{\mathrm{SHG}} = \{\{V_1, V_2\}, \{V_1, V_2, V_3\}\}$

To convert this into a Hypergraph:

(1) The super-vertices $\{V_1, V_2\}$ and $\{V_3\}$ are flattened into individual vertices V_1, V_2, V_3.
(2) The super-edge $\{V_1, V_2\}$ becomes a hyperedge $\{V_1, V_2\}$ in the hypergraph.
(3) The super-edge $\{V_1, V_2, V_3\}$ becomes a hyperedge $\{V_1, V_2, V_3\}$ in the hypergraph.

Thus, the resulting Hypergraph H is:

- Vertices: $V(H) = \{V_1, V_2, V_3\}$

- Hyperedges: $E(H) = \{\{V_1, V_2\}, \{V_1, V_2, V_3\}\}$

Next, we consider the SuperHyperTree. The following will be written as definitions.

Definition 3.5 (SuperHyperTree (SHT))**.** A *SuperHyperTree (SHT)* is a specific type of SuperHyperGraph SHT $= (V, E)$ that satisfies the following conditions:

(1) **Host Graph Condition:** There exists a host graph $T = (V, E_T)$, which is a tree, such that:
   - T shares the same vertex set V as SHT.
   - The edges in T represent the connections between vertices in V.

(2) **SuperHyperTree Condition:** Every hyperedge $E_i \in E$ of the SuperHyperGraph corresponds to a connected subtree of the host tree T. Specifically:
   - If E_i is a single edge, it connects two vertices directly within T.
   - If E_i is a super-edge (connecting subsets of vertices), the vertices in each subset must form a connected subtree in T.
   - If E_i is a hyper-edge (connecting more than two vertices), all vertices in E_i must form a connected subtree in T.
   - If E_i is an indeterminate edge, any realization of E_i must satisfy the condition that the vertices involved form a connected subtree in T.

(3) **Acyclic Condition:** The host graph T must be acyclic, which is a fundamental property of trees. Consequently, SHT inherits this acyclic nature through its hyperedges.

Key Properties of a SuperHyperTree:

- **Connectedness:** A SuperHyperTree is connected, meaning there exists a path between any two vertices via a sequence of hyperedges.
- **No Cycles:** Since the host graph T is a tree, SHT does not contain any cycles, including those involving super-vertices or super-edges.
- **Generalization of Trees:** A SuperHyperTree extends the concept of a tree by allowing super-vertices and super-edges while maintaining the acyclic and connected properties of a tree.

Theorem 3.6. *A SuperHyperTree (SHT) is a generalization of a hypertree. Specifically, every hypertree can be represented as a SuperHyperTree, but not every SuperHyperTree is a hypertree.*

Proof. Let SHT $= (V, E)$ be a SuperHyperTree and $HT = (V, E')$ be a hypertree.
1. SuperHyperTree as a Generalization of Hypertree. A hypertree is a connected hypergraph where removing any hyperedge disconnects the hypergraph. For a hypertree HT:

- Each hyperedge $e \in E'(HT)$ connects a set of vertices that maintains the connectedness of the hypergraph.
- Removing any hyperedge e results in a disconnected hypergraph.

In a SuperHyperTree SHT, the following conditions are satisfied:

- *Host Graph Condition:* SHT is embedded within a host tree $T = (V, E_T)$, ensuring an acyclic structure.
- *SuperHyperTree Condition:* Each hyperedge in E corresponds to a connected subtree of T.
- *Acyclicity:* SHT inherits the acyclic property from the host tree T.

Since hypertrees are connected and acyclic, any hypertree HT can be embedded as a SuperHyperTree SHT by constructing a host tree T such that each hyperedge $e \in E'(HT)$ corresponds to a connected subtree of T.
2. Differences Between SuperHyperTrees and Hypertrees. While hypertrees can contain cycles within hyperedges, SuperHyperTrees explicitly disallow cycles due to the acyclic nature of their host tree T. Furthermore, SuperHyperTrees allow for more complex structures such as super-edges (which connect subsets of vertices) and indeterminate edges, which are not defined in hypertrees.
3. Examples.

- *Hypertree as a Special Case of SuperHyperTree:* Consider a hypertree $HT = (V, E')$. By embedding HT within a host tree T, where each hyperedge $e \in E'(HT)$ corresponds to a connected subtree of T, HT satisfies all the conditions of a SuperHyperTree SHT.
- *SuperHyperTree That Is Not a Hypertree:* Consider a SuperHyperTree with a super-edge that connects subsets of vertices. This structure does not satisfy the conditions of a hypertree because hypertrees do not allow for super-edges or indeterminate edges.

Thus, SuperHyperTrees generalize the concept of hypertrees by introducing super-edges, indeterminate edges, and the requirement of a host tree, while hypertrees are a specific subclass of SuperHyperTrees. □

Corollary 3.7. *The hypertree-width of a hypertree HT is equal to its tree-width, whereas the SuperHyperTree-width of a SuperHyperTree SHT may exceed its tree-width due to the inclusion of super-edges and indeterminate edges.*

Proof. For Hypertrees: Since hypertrees can be decomposed into a tree structure where each hyperedge corresponds to a single bag in the tree decomposition, their hypertree-width coincides with their tree-width.

For SuperHyperTrees: In a SuperHyperTree SHT, the inclusion of super-edges and indeterminate edges increases the complexity of its decomposition. The SuperHyperTree-width accounts for these additional structures, leading to a possible increase in width compared to the tree-width. □

Next, we define superhypertree-width by extending the concept of hypertree-width and tree-width(cf. [61]). Following that, we will briefly explore the Graph Parameter Hierarchy.

Definition 3.8 (SuperHyperTree Decomposition)**.** (cf. [61]) Let SHT $= (V, E)$ be a SuperHyperGraph, where V is the set of vertices and E is the set of SuperEdges. A *SuperHyperTree Decomposition* of SHT is defined as a tuple $(T, \mathcal{B}, \mathcal{C})$, where:

- $T = (V_T, E_T)$ is a tree.
- $\mathcal{B} = \{B_t \mid t \in V_T\}$ is a family of subsets of V (called *bags*) associated with the nodes of T, satisfying:
  (1) **Coverage Condition for SuperEdges:** For each SuperEdge $e \in E$, there exists a node $t \in V_T$ such that the entire SuperEdge e is contained in the corresponding bag B_t, i.e., $e \subseteq B_t$.
  (2) **Vertex Connectivity Condition:** For each vertex $v \in V$, the set of nodes $\{t \in V_T \mid v \in B_t\}$ forms a connected subtree of T.
- $\mathcal{C} = \{C_t \mid t \in V_T\}$ is a family of subsets of E (called *guards*) associated with the nodes of T, satisfying:
  (1) **Guard Condition for SuperEdges:** For each $t \in V_T$, $B_t \subseteq \bigcup C_t$, where $\bigcup C_t$ denotes the union of all vertices in the SuperEdges of C_t, i.e., $\bigcup C_t = \{v \in V \mid \exists e \in C_t : v \in e\}$.
  (2) **SuperHyperTree Condition:** For each $t \in V_T$,
  $$(\bigcup C_t) \cap \bigcup_{u \in V(T_t)} B_u \subseteq B_t,$$
  where T_t is the subtree of T rooted at t.

Width of a SuperHyperTree Decomposition: The width of a SuperHyperTree Decomposition $(T, \mathcal{B}, \mathcal{C})$ is defined as:

$$\text{width}(T, \mathcal{B}, \mathcal{C}) = \max_{t \in V_T} |C_t|,$$

where $|C_t|$ is the cardinality of the guard C_t.

SuperHyperTree-width: The SuperHyperTree-width of the SuperHyperGraph SHT, denoted by SHT-width(SHT), is the minimum width over all possible SuperHyperTree Decompositions:

$$\text{SHT-width(SHT)} = \min_{(T, \mathcal{B}, \mathcal{C})} \text{width}(T, \mathcal{B}, \mathcal{C}).$$

- The SuperHyperTree-width measures how closely a SuperHyperGraph resembles a SuperHyperTree.
- For classical graphs, the SuperHyperTree-width coincides with the traditional treewidth.

SuperHyperPath Decomposition: A SuperHyperPath Decomposition is a path-based variant of the SuperHyperTree Decomposition, where the host structure T is a path instead of a tree.

Lemma 3.9. *Let SH be a SuperHypergraph. The following inequalities hold:*

$$\text{shw}(SH) \leq \text{hw}(SH) \leq \text{tw}(G_p(SH)) + 1$$

where:

- $\mathrm{shw}(SH)$ *denotes the SuperHyperTree-width of SH.*
- $\mathrm{hw}(SH)$ *denotes the hypertree-width of SH.*
- $\mathrm{tw}(G_p(SH))$ *denotes the treewidth of the primal graph* $G_p(SH)$ *of SH.*

Proof. Since every SuperHyperTree decomposition is a special case of a hypertree decomposition:

$$\mathrm{shw}(\mathrm{SH}) \leq \mathrm{hw}(\mathrm{SH})$$

Proof of $\mathrm{hw}(\mathbf{SH}) \leq \mathrm{tw}(\mathbf{G}_p(\mathbf{SH})) + 1$:

From [74]:

$$\mathrm{hw}(H) \leq \mathrm{tw}(\mathrm{G}_p(H)) + 1$$

Applying this to SH:

$$\mathrm{hw}(\mathrm{SH}) \leq \mathrm{tw}(\mathrm{G}_p(\mathrm{SH})) + 1$$

Conclusion:

$$\mathrm{shw}(\mathrm{SH}) \leq \mathrm{hw}(\mathrm{SH}) \leq \mathrm{tw}(\mathrm{G}_p(\mathrm{SH})) + 1$$

□

3.2. *NeutrosophicTree-width*

In this subsection, we consider the concept of NeutrosophicTree-width.

First, we introduce the concept of a neutrosophic graph [11, 34, 76, 88, 130, 145]. A neutrosophic graph generalizes traditional graph theory by incorporating degrees of truth, indeterminacy, and falsity in its edges and vertices, enabling more complex and nuanced relationships. It extends the concept of a fuzzy graph [92]. Neutrosophic graph theory is particularly useful in uncertain environments, contributing to fields like human networks and decision-making [7,43]. Fuzzy graph theories have numerous applications in modern science and technology, especially in operations research, neural networks [2, 101], artificial intelligence [2, 101], and decision-making [92]. The definition is as follows.

Definition 3.10. (cf. [148]) A neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as a graph where $\sigma_i : V \to [0, 1]$, $\mu_i : E \to [0, 1]$, and for every $v_i v_j \in E$, the following condition holds: $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$.

(1) σ is called the neutrosophic vertex set.
(2) μ is called the neutrosophic edge set.
(3) $|V|$ is called the order of NTG, and it is denoted by $O(NTG)$.
(4) $\sum_{v \in V} \sigma(v)$ is called the neutrosophic order of NTG, and it is denoted by $On(NTG)$.
(5) $|E|$ is called the size of NTG, and it is denoted by $S(NTG)$.
(6) $\sum_{e \in E} \mu(e)$ is called the neutrosophic size of NTG, and it is denoted by $Sn(NTG)$.

Definition 3.11. (i) A sequence of vertices $P : x_0, x_1, \cdots, x_n$ is called a path where $x_i x_{i+1} \in E$, $i = 0, 1, \cdots, n-1$.

(ii) The strength of the path $P : x_0, x_1, \cdots, x_n$ is $\bigwedge_{i=0,\cdots,n-1} \mu(x_i x_{i+1})$.

(iii) The connectedness between vertices x_0 and x_n is defined as:

$$\mu_\infty(x, y) = \bigwedge_{P:x_0,x_1,\cdots,x_n} \bigwedge_{i=0,\cdots,n-1} \mu(x_i x_{i+1}).$$

The Examples of neutrosophic graph is following.

Example 3.12. (cf. [35]) Consider a neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as shown in the diagram.

The neutrosophic membership degrees of the vertices are as follows:

$$\sigma(v_1) = (0.5, 0.1, 0.4), \quad \sigma(v_2) = (0.6, 0.3, 0.2),$$
$$\sigma(v_3) = (0.2, 0.3, 0.4), \quad \sigma(v_4) = (0.4, 0.2, 0.5)$$

The neutrosophic membership degrees of the edges are as follows:

$$\mu(v_1 v_2) = (0.2, 0.3, 0.4), \quad \mu(v_2 v_3) = (0.3, 0.3, 0.4),$$
$$\mu(v_3 v_4) = (0.2, 0.3, 0.4), \quad \mu(v_4 v_1) = (0.1, 0.2, 0.5)$$

In this case, the neutrosophic graph NTG has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying neutrosophic membership degrees.
- The neutrosophic relations ensure that for every edge $v_i v_j \in E$, $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, where $\wedge$ denotes the minimum operation.

Theorem 3.13. *A Neutrosophic Graph can be transformed into a Fuzzy Graph by mapping the neutrosophic truth-membership values of vertices and edges directly to the fuzzy membership values, effectively disregarding the indeterminacy and falsity components.*

Proof. Let $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ be a neutrosophic graph. We aim to transform NTG into a fuzzy graph $G = (V, \sigma', \mu')$.

In a neutrosophic graph, the vertex membership $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$ includes truth, indeterminacy, and falsity components. To transform this into a fuzzy graph, we only retain the truth-membership component $\sigma_1(v)$. The transformed fuzzy vertex membership is thus:

$$\sigma'(v) = \sigma_1(v)$$

where $\sigma_1(v) \in [0, 1]$ represents the fuzzy membership degree of vertex v.

Similarly, for the edge set, the neutrosophic edge membership $\mu(e) = (\mu_1(e), \mu_2(e), \mu_3(e))$ includes truth, indeterminacy, and falsity components. We retain only the truth-membership component $\mu_1(e)$. The transformed fuzzy edge membership is thus:

$$\mu'(e) = \mu_1(e)$$

where $\mu_1(e) \in [0, 1]$ represents the fuzzy membership degree of edge e.

To ensure that the transformed graph satisfies the properties of a fuzzy graph, we check the following conditions:

(1) *Vertex Membership Condition:* In the fuzzy graph G, the fuzzy vertex membership function $\sigma'(v)$ must satisfy $\sigma'(v) \in [0, 1]$. Since $\sigma_1(v) \in [0, 1]$ in the neutrosophic graph, this condition holds automatically.

(2) *Edge Membership Condition:* In the fuzzy graph G, the fuzzy edge membership function $\mu'(e)$ must satisfy $\mu'(e) \leq \sigma'(v_i) \wedge \sigma'(v_j)$ for all edges $e = (v_i, v_j)$. In the neutrosophic graph, the truth-membership component of an edge $\mu_1(e)$ satisfies the condition $\mu_1(e) \leq \sigma_1(v_i) \wedge \sigma_1(v_j)$, which is equivalent to the fuzzy graph condition.

This proof is completed. □

Next, we consider a neutrosophic tree. The definition is similar to that of a general tree and a fuzzy tree. A neutrosophic tree is defined as follows:

Definition 3.14. (cf. [13, 70, 80]) A **Neutrosophic Tree** $NT = (V, E, \sigma, \mu)$ is a connected acyclic neutrosophic graph satisfying:

(1) **Connectedness**: For every pair of distinct vertices $u, v \in V$, there exists a unique path P from u to v such that for each edge e in P, $\mu(e) > 0$.

(2) **Acyclicity**: The neutrosophic tree contains no cycles.

(3) **Neutrosophic Degree Condition**: For each vertex $v \in V$, the neutrosophic degree $\sigma(v)$ satisfies:

$$\sigma(v) = \sum_{u \in N(v)} \mu(vu),$$

where $N(v)$ is the set of neighbors of v.

Based on the above definitions, we define the Neutrosophic Tree-width.

Definition 3.15. A **Neutrosophic Tree-Decomposition** of a neutrosophic graph $NG = (V, E, \sigma, \mu)$ is a pair $(T, \mathcal{B})$ where:

- $T = (V_T, E_T)$ is a tree.
- $\mathcal{B} = \{B_t \mid t \in V_T\}$ is a family of subsets of V (called *bags*), each associated with a node t of T.

This decomposition satisfies:

(1) **Coverage Condition**: For every edge $e = (v_i, v_j) \in E$, there exists a node $t \in V_T$ such that $\{v_i, v_j\} \subseteq B_t$.

(2) **Connectivity Condition**: For each vertex $v \in V$, the set $\{t \in V_T \mid v \in B_t\}$ forms a connected subtree of T.

The **width** of a Neutrosophic Tree-Decomposition $(T, \mathcal{B})$ is defined as:

$$\text{width}(T, \mathcal{B}) = \max_{t \in V_T} \left(\left\lceil \sum_{v \in B_t} \sigma(v) \right\rceil - 1 \right),$$

where $\lceil x \rceil$ denotes the ceiling function.

The **Neutrosophic Tree-Width** of the neutrosophic graph NG, denoted $\text{NTW}(NG)$, is the minimum width over all possible Neutrosophic Tree-Decompositions of NG:

$$\text{NTW}(NG) = \min_{(T, \mathcal{B})} \text{width}(T, \mathcal{B}).$$

Example 3.16. Consider a neutrosophic graph $NTG = (V, E, \sigma, \mu)$ where the vertex set $V = \{v_1, v_2, v_3, v_4\}$ and the edge set $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$. The neutrosophic vertex membership functions $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and the neutrosophic edge membership functions $\mu = (\mu_1, \mu_2, \mu_3)$ are defined as follows:

- $\sigma(v_1) = (0.9, 0.1, 0.0)$, $\sigma(v_2) = (0.8, 0.2, 0.0)$
- $\sigma(v_3) = (0.7, 0.3, 0.0)$, $\sigma(v_4) = (0.6, 0.4, 0.0)$
- $\mu(v_1v_2) = (0.8, 0.1, 0.1)$, $\mu(v_2v_3) = (0.7, 0.2, 0.1)$, $\mu(v_3v_4) = (0.6, 0.3, 0.1)$

We create a tree $T = (V_T, E_T)$ where $V_T = \{t_1, t_2, t_3\}$ and associate the following bags $\mathcal{B} = \{B_{t_1}, B_{t_2}, B_{t_3}\}$:

- $B_{t_1} = \{v_1, v_2\}$
- $B_{t_2} = \{v_2, v_3\}$
- $B_{t_3} = \{v_3, v_4\}$
- *Coverage Condition:* For each edge in E, there exists a corresponding bag:
  - $(v_1, v_2) \in B_{t_1}$
  - $(v_2, v_3) \in B_{t_2}$
  - $(v_3, v_4) \in B_{t_3}$

  For all edges (v_i, v_j), the condition $\mu(v_iv_j) \leq \min(\sigma(v_i), \sigma(v_j))$ is satisfied.
- *Vertex Connectivity Condition:* Each vertex forms a connected subtree of T:
  - v_1 appears only in B_{t_1}.
  - v_2 appears in both B_{t_1} and B_{t_2}, and these bags are connected in T.
  - v_3 appears in both B_{t_2} and B_{t_3}, and these bags are connected in T.
  - v_4 appears only in B_{t_3}.

For each bag B_t, we calculate the neutrosophic width:

- $\text{width}(B_{t_1}) = \sum_{v \in B_{t_1}} \sigma(v) - 1 = (0.9 + 0.8) - 1 = 1.7 - 1 = 0.7$
- $\text{width}(B_{t_2}) = \sum_{v \in B_{t_2}} \sigma(v) - 1 = (0.8 + 0.7) - 1 = 1.5 - 1 = 0.5$
- $\text{width}(B_{t_3}) = \sum_{v \in B_{t_3}} \sigma(v) - 1 = (0.7 + 0.6) - 1 = 1.3 - 1 = 0.3$

The maximum width is 0.7. Therefore, the Neutrosophic Tree-width of the graph is 0.7.

Theorem 3.17. *The Neutrosophic Tree-Width $NTW(NG)$ of a neutrosophic graph NG satisfies:*

- *If NG is a single vertex with $\sigma(v) = 1$, then $NTW(NG) = 0$.*
- *If NG consists of isolated vertices with $\sigma(v) = 1$ for all $v \in V$, then $NTW(NG) = 0$.*

Proof. For a single vertex v with $\sigma(v) = 1$, construct a tree T with a single node t and set $B_t = \{v\}$. The width is:

$$\text{width}(T, \mathcal{B}) = \lceil \sigma(v) \rceil - 1 = 1 - 1 = 0.$$

For multiple isolated vertices $\{v_1, v_2, \ldots, v_n\}$ with $\sigma(v_i) = 1$, create singleton bags $B_{t_i} = \{v_i\}$ in T. For each bag:

$$\text{width}(T, \mathcal{B}) = \lceil \sigma(v_i) \rceil - 1 = 1 - 1 = 0.$$

Thus, $\text{NTW}(NG) = 0$. □

Theorem 3.18. *The Neutrosophic Tree-width (NT-width) of an empty graph is -1.*

Proof. In an empty graph (no vertices), there are no bags. By convention, we define:

$$\text{NTW}(NG) = -1.$$

□

Theorem 3.19. *For any neutrosophic graph $NG = (V, E, \sigma, \mu)$, the Neutrosophic Tree-Width $NTW(NG)$ satisfies:*

$$NTW(NG) \leq tw(G),$$

where $tw(G)$ is the tree-width of the underlying simple graph $G = (V, E)$.

Proof. Since $\sigma(v) \in [0, 1]$, we have:

$$\sum_{v \in B_t} \sigma(v) \leq |B_t|.$$

Therefore,

$$\left\lceil \sum_{v \in B_t} \sigma(v) \right\rceil - 1 \leq |B_t| - 1.$$

Since $\text{tw}(G)$ is the minimum over all tree-decompositions, we have:

$$\text{NTW}(NG) \leq \text{tw}(G).$$

□

Theorem 3.20. *If all vertices in NG have $\sigma(v) = 1$ and all edges have $\mu(e) = 1$, then:*

$$NTW(NG) = tw(G).$$

Proof. With $\sigma(v) = 1$ and $\mu(e) = 1$, NG behaves like G. For each bag B_t:

$$\sum_{v \in B_t} \sigma(v) = |B_t|.$$

Thus,

$$\text{width}(T, \mathcal{B}) = |B_t| - 1.$$

Therefore, $\text{NTW}(NG) = \text{tw}(G)$. □

Additionally, in the field of Neutrosophic Graphs, several classes such as Single Valued Neutrosophic Graphs [11, 53, 81, 113], Fermatean neutrosophic graphs [33], Single valued pentapartitioned neutrosophic graphs [47], Interval Valued Neutrosophic Graphs [34,36,37,95,160] have been proposed. We plan to explore and characterize these classes in more detail in the future.

3.3. *Neutrosophic HyperTree-width*

Next, we introduce the concept of a Neutrosophic Hypergraph. Similar to Neutrosophic Graphs and Hypergraphs, Neutrosophic Hypergraphs have been the subject of extensive research [8, 51, 102]. The definition is provided below.

Definition 3.21. (cf. [8, 51, 102]) A **Neutrosophic Hypergraph** $NHG = (V, E, \sigma, \mu)$ consists of:

- A finite set V of vertices.
- A set $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ of hyperedges.
- A neutrosophic vertex membership function $\sigma : V \to [0, 1]^3$, where for each $v \in V$:

  $$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)),$$

  representing the truth-membership, indeterminacy-membership, and falsity-membership degrees of v.
- A neutrosophic hyperedge membership function $\mu : E \to [0, 1]^3$, where for each $e \in E$:

  $$\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)),$$

  representing the truth-membership, indeterminacy-membership, and falsity-membership degrees of e.

These functions satisfy the following condition for every hyperedge $e \in E$:

$$\mu_T(e) \le \min_{v \in e} \sigma_T(v), \quad \mu_I(e) \ge \max_{v \in e} \sigma_I(v), \quad \mu_F(e) \ge \max_{v \in e} \sigma_F(v).$$

Next, a Neutrosophic Hypertree is defined as follows.

Definition 3.22. A **Neutrosophic Hypertree** $NHT = (V, E, \sigma, \mu)$ is a connected, acyclic neutrosophic hypergraph satisfying:

(1) **Acyclicity**: The hypergraph contains no cycles. The incidence graph associated with NHT is acyclic.

(2) **Connectedness**: For every pair of distinct vertices $u, v \in V$, there exists a sequence of hyperedges $e_1, e_2, \dots, e_k \in E$ such that:
   - $u \in e_1$ and $v \in e_k$.
   - $e_i \cap e_{i+1} \neq \emptyset$ for $i = 1, \dots, k-1$.
   - $\mu_T(e_i) > 0$ for all i.

(3) **Neutrosophic Degree Condition**: For each $v \in V$:

$$\sum_{e \in E_v} \mu_T(e) = \sigma_T(v),$$

where $E_v = \{e \in E \mid v \in e\}$.

Based on the above, we define Neutrosophic HyperTree-decomposition as follows.

Definition 3.23. A **Neutrosophic HyperTree-Decomposition** of a neutrosophic hypergraph $NHG = (V, E, \sigma, \mu)$ is a triple $(T, \mathcal{B}, \mathcal{C})$ where:

- $T = (V_T, E_T)$ is a tree.
- $\mathcal{B} = \{B_t \subseteq V \mid t \in V_T\}$ is a family of bags.
- $\mathcal{C} = \{C_t \subseteq E \mid t \in V_T\}$ is a family of guards.

This decomposition satisfies:

(1) **Coverage Condition**: For every hyperedge $e \in E$, there exists $t \in V_T$ such that $e \subseteq B_t$ and $\mu_T(e) \leq \min_{v \in e} \sigma_T(v)$.

(2) **Vertex Connectivity Condition**: For each $v \in V$, the set $\{t \in V_T \mid v \in B_t\}$ forms a connected subtree of T.

(3) **Guard Condition**: For each $t \in V_T$, $B_t \subseteq \bigcup_{e \in C_t} e$.

The **width** of a Neutrosophic HyperTree-Decomposition $(T, \mathcal{B}, \mathcal{C})$ is defined as:

$$\text{width}(T, \mathcal{B}, \mathcal{C}) = \max_{t \in V_T} \left(\sum_{e \in C_t} \mu_T(e) \right).$$

The **Neutrosophic HyperTree-Width** of NHG, denoted $\text{NHTW}(NHG)$, is:

$$\text{NHTW}(NHG) = \min_{(T, \mathcal{B}, \mathcal{C})} \text{width}(T, \mathcal{B}, \mathcal{C}).$$

Theorem 3.24. *For any neutrosophic hypergraph NHG:*

$$NHTW(NHG) \leq HTW(NHG),$$

where $HTW(NHG)$ is the standard hypertree-width.

Proof. Since $\mu_T(e) \leq 1$, we have:

$$\sum_{e \in C_t} \mu_T(e) \leq \sum_{e \in C_t} 1 = |C_t|.$$

Thus,

$$\text{width}_{\text{NHT}}(T, \mathcal{B}, \mathcal{C}) \leq \text{width}_{\text{HT}}(T, \mathcal{B}, \mathcal{C}).$$

Taking the minimum over all decompositions:

$$\text{NHTW}(NHG) \leq \text{HTW}(NHG).$$

□

Theorem 3.25. *If $\sigma_T(v) = 1$ for all $v \in V$ and $\mu_T(e) = 1$ for all $e \in E$, then:*

$$NHTW(NHG) = HTW(NHG).$$

Proof. Under these conditions, the width becomes:

$$\text{width}(T, \mathcal{B}, \mathcal{C}) = \max_{t \in V_T} |C_t|.$$

Therefore:

$$\text{NHTW}(NHG) = \text{HTW}(NHG).$$

□

Theorem 3.26. *For any neutrosophic hypergraph NHG:*

$$NHTW(NHG) \leq NTW(G),$$

where $NTW(G)$ is the Neutrosophic Tree-Width of the incidence graph G of NHG.

Proof. By constructing a corresponding Neutrosophic Tree-Decomposition of G, we establish:

$$\text{NHTW}(NHG) \leq \text{NTW}(G).$$

□

Theorem 3.27. *For any neutrosophic hypergraph NHG:*

$$NHTW(NHG) \leq tw(G) + 1,$$

where $tw(G)$ is the tree-width of the primal graph G of NHG.

Proof. Using the inequality $\text{HTW}(NHG) \leq \text{tw}(G) + 1$ and Theorem 1:

$$\text{NHTW}(NHG) \leq \text{HTW}(NHG) \leq \text{tw}(G) + 1.$$

□

We intend to further examine the validity of the above definitions.

3.4. *Definition of t-NeutrosophicTree-width*

The t-neutrosophic approach links different values of the parameter ”t” to various layers of the graph, allowing for multi-level analysis. This method enables a detailed exploration of the relationships within the graph, incorporating varying degrees of confidence. As a result, it provides a more nuanced understanding of the underlying structure [96, 97].

Definition 3.28. [96, 97] Let G be a Neutrosophic Set (NS) over a universal set U with $t \in [0,1]$. The t-Neutrosophic Graph NSG_t of U, also known as a t-Neutrosophic Set (t-NS), is defined for each $u_1 \in U$ as follows:

$$T_{G_t}(u_1) = \min\{T_G(u_1), t\}, \quad I_{G_t}(u_1) = \max\{I_G(u_1), 1-t\}, \quad F_{G_t}(u_1) = \max\{F_G(u_1), 1-t\},$$

where T_G, I_G, and F_G represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. The t-Neutrosophic Set can then be represented as:

$$G_t = \{u_1, T_G(u_1), I_G(u_1), F_G(u_1) \mid u_1 \in U\}.$$

Furthermore, the membership functions satisfy the condition:

$$0 \leq T_G(u_1) + I_G(u_1) + F_G(u_1) \leq 1.$$

Definition 3.29. [96, 97] Let $G = (V, E)$ be a simple graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A t-Neutrosophic Graph G_t is represented as:

$$G_t = (A_t, B_t),$$

where A_t is a t-Neutrosophic Set on the vertices V and B_t is a t-Neutrosophic Set on the edges E.

- $A_t = \{(u_i, T_G(u_i), I_G(u_i), F_G(u_i)) \mid u_i \in V\}$ represents the t-Neutrosophic Set on the vertex set V.
- $B_t = \{((u_i, u_j), T_G(u_i, u_j), I_G(u_i, u_j), F_G(u_i, u_j)) \mid (u_i, u_j) \in E\}$ represents the t-Neutrosophic Set on the edge set $E \subseteq V \times V$.

For each edge $(u_i, u_j) \in E$, the following conditions hold:

$$T_{B_t}(u_i, u_j) \leq \min\{T_{A_t}(u_i), T_{A_t}(u_j)\}, \quad I_{B_t}(u_i, u_j) \leq \max\{I_{A_t}(u_i), I_{A_t}(u_j)\},$$

$$F_{B_t}(u_i, u_j) \leq \max\{F_{A_t}(u_i), F_{A_t}(u_j)\}.$$

The t-Neutrosophic Graph also satisfies the following conditions for both vertices and edges:

$$0 \leq T_{A_t}(u_i) + I_{A_t}(u_i) + F_{A_t}(u_i) \leq 1, \quad \text{for all } u_i \in V,$$

$$0 \leq T_{B_t}(u_i, u_j) + I_{B_t}(u_i, u_j) + F_{B_t}(u_i, u_j) \leq 1, \quad \text{for all } (u_i, u_j) \in E.$$

The t-Neutrosophic Tree and t-Neutrosophic Tree-width extend classical tree and tree-width concepts to account for uncertainty, truth, indeterminacy, and falseness. The threshold parameter t allows for varying levels of confidence in the analysis of graph structures. The definition of a *t-Neutrosophic Tree* is as follows.

Definition 3.30. A *t-Neutrosophic Tree* (t-NTT) is a connected, acyclic t-Neutrosophic graph $G_t = (V, E)$ that satisfies:

- *Acyclicity:* The graph contains no cycles, meaning for every pair of distinct vertices $u, v \in V$, there is exactly one path connecting them with no repeated vertices.
- *Connectedness:* For every pair of distinct vertices $u, v \in V$, there exists a path $P \subseteq G_t$ connecting u and v.

Additionally, the neutrosophic membership conditions hold for all vertices and edges as defined in the t-Neutrosophic Graph framework.

The t-Neutrosophic Tree-width measures how closely the graph resembles a tree in the neutrosophic framework.

Definition 3.31. A *t-Neutrosophic Tree-decomposition* of a t-Neutrosophic graph $G_t = (V, E)$ is a pair $(T, \mathcal{B})$, where:

- $T = (V_T, E_T)$ is a tree.
- $\mathcal{B} = \{B_t \mid t \in V_T\}$ is a collection of subsets (called bags) of vertices from G_t, satisfying:
  (1) For every edge $(u, v) \in E$, there exists a bag $B_t \in \mathcal{B}$ such that $\{u, v\} \subseteq B_t$.
  (2) For each vertex $u \in V$, the set $\{t \in V_T \mid u \in B_t\}$ forms a connected subtree of T.

Definition 3.32. The *width* of a t-Neutrosophic Tree-decomposition is defined as:

$$\text{width}(T, \mathcal{B}) = \max_{t \in V_T} \left(\sum_{v \in B_t} \sigma(v) - 1 \right),$$

where $\sigma(v)$ is the neutrosophic degree of vertex v. The *t-Neutrosophic Tree-width* of a t-Neutrosophic graph G_t, denoted by tNT-width(G_t), is the minimum width over all possible t-Neutrosophic Tree-decompositions of G_t:

$$\text{t-NTT-width}(G_t) = \min_{(T,\mathcal{B})} \text{width}(T, \mathcal{B}).$$

The following theorem holds.

Theorem 3.33. *Let t-NTT $= (V_t, E_t)$ be a t-Neutrosophic Tree, where $t \in [0, 1]$. The t-Neutrosophic Tree-width of t-NTT is less than or equal to the Neutrosophic Tree-width of t-NTT, that is,*

$$\textit{t-NTT-width}(\textit{t-NTT}) \leq \textit{NTT-width}(\textit{t-NTT}).$$

Proof. We prove this theorem by considering the role of the parameter t.

Case 1: $t = 1$

When $t = 1$, the t-Neutrosophic Tree is identical to a standard Neutrosophic Tree, and their widths are equal:

$$\text{t-NTT-width(t-NTT)} = \text{NTT-width(t-NTT)}.$$

Case 2: $t < 1$

When $t < 1$, the t-Neutrosophic Tree incorporates the threshold t, imposing constraints on the relationships between vertices and edges. This results in fewer edges being considered in the decomposition.

Let B_t be the bag in a t-Neutrosophic Tree decomposition, and B_t^* the corresponding bag in a Neutrosophic Tree decomposition (without the threshold t). The difference between B_t and B_t^* arises because t restricts certain vertices and edges based on neutrosophic membership values.

We have:

$$|B_t| \leq |B_t^*| \quad \forall t \in V_T,$$

where B_t^* includes all vertices and edges without the threshold restriction. Therefore, the t-Neutrosophic Tree decomposition bags are equal to or smaller than those in the Neutrosophic Tree decomposition.

Thus, the width of the t-Neutrosophic Tree decomposition is less than or equal to the Neutrosophic Tree decomposition, leading to:

$$\text{t-NTT-width(t-NTT)} \leq \text{NTT-width(t-NTT)}.$$

This proof is completed. □

4. Conclusion and Future Research Goals

We intend to explore the relationship between SuperHyperTree decomposition, Neutrosophic HyperTree decomposition, and Neutrosophic Tree decomposition with other graph width parameters.

In classical graph theory, various width parameters such as boolean-width [4, 21, 38–40], modular-width [3], clique-width [55,90], and rank-width [57,99,116–118] have been extensively explored. Our research aims to investigate whether any new characteristics arise when these concepts are extended to Neutrosophic graphs and SuperHypergraphs.

Additionally, we are interested in extending fundamental concepts like tree-depth [50, 114, 159], tree-length [20, 54], and tree-breadth to fuzzy hypergraphs and fuzzy directed graphs, with the goal of uncovering any unique properties or emerging behaviors.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

References

1. Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.
2. Shigeo Abe. *Neural networks and fuzzy systems: theory and applications*. Springer Science & Business Media, 2012.
3. Faisal N Abu-Khzam, Shouwei Li, Christine Markarian, Friedhelm Meyer auf der Heide, and Pavel Podlipyan. Modular-width: An auxiliary parameter for parameterized parallel complexity. In *Frontiers in Algorithmics: 11th International Workshop, FAW 2017, Chengdu, China, June 23-25, 2017, Proceedings 11*, pages 139–150. Springer, 2017.
4. Isolde Adler, Binh-Minh Bui-Xuan, Yuri Rabinovich, Gabriel Renault, Jan Arne Telle, and Martin Vatshelle. On the boolean-width of a graph: Structure and applications. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 159–170. Springer, 2010.
5. Isolde Adler, Tomáš Gavenčiak, and Tereza Klimošová. Hypertree-depth and minors in hypergraphs. *Theoretical Computer Science*, 463:84–95, 2012.
6. Isolde Adler, Georg Gottlob, and Martin Grohe. Hypertree width and related hypergraph invariants. *European Journal of Combinatorics*, 28(8):2167–2181, 2007.
7. D. Ajay, S. John Borg, and P. Chellamani. Domination in pythagorean neutrosophic graphs with an application in fuzzy intelligent decision making. In *International Conference on Intelligent and Fuzzy Systems*, pages 667–675, Cham, July 2022. Springer International Publishing.
8. Muhammad Akram and Anam Luqman. Intuitionistic single-valued neutrosophic hypergraphs. *Opsearch*, 54:799–815, 2017.

9. Muhammad Akram and Anam Luqman. *Fuzzy hypergraphs and related extensions*. Springer, 2020.
10. Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.
11. Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.
12. Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.
13. Muhammad Akram and Muzzamal Sitara. Novel applications of single-valued neutrosophic graph structures in decision-making. *Journal of Applied Mathematics and Computing*, 56:501–532, 2018.
14. T Al-Hawary, S Al-Shalaldeh, and MUHAMMAD Akram. Certain matrices and energies of fuzzy graphs. *TWMS Journal of Applied and Engineering Mathematics*, 2:2146–1147, 2021.
15. Talal Al-Hawary and Laith AlMomani. -balanced fuzzy graphs. *arXiv preprint arXiv:1804.08677*, 2018.
16. Talal Ali Al-Hawary. Certain classes of fuzzy graphs. *European Journal of Pure and Applied Mathematics*, 10(3):552–560, 2017.
17. TALAL ALI AL-HAWARY and MAREF YM ALZOUBI. β-product of product fuzzy graphs. *Journal of applied mathematics & informatics*, 42(2):283–290, 2024.
18. Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.
19. Rangaswami Balakrishnan and Kanna Ranganathan. *A textbook of graph theory*. Springer Science & Business Media, 2012.
20. Rémy Belmonte, Fedor V Fomin, Petr A Golovach, and MS Ramanujan. Metric dimension of bounded tree-length graphs. *SIAM Journal on Discrete Mathematics*, 31(2):1217–1243, 2017.
21. Rémy Belmonte and Martin Vatshelle. On graph classes with logarithmic boolean-width. *arXiv preprint arXiv:1009.0216*, 2010.
22. Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.
23. Leonid S Bershtein and Alexander V Bozhenyuk. Fuzzy graphs and fuzzy hypergraphs. In *Encyclopedia of Artificial Intelligence*, pages 704–709. IGI Global, 2009.
24. Prabir Bhattacharya. Some remarks on fuzzy graphs. *Pattern recognition letters*, 6(5):297–302, 1987.
25. Hans L Bodlaender. A tourist guide through treewidth. *Acta cybernetica*, 11(1-2):1–21, 1993.
26. Hans L Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical computer science*, 209(1-2):1–45, 1998.
27. Hans L Bodlaender, Michael R Fellows, and Michael T Hallett. Beyond np-completeness for problems of bounded width (extended abstract) hardness for the w hierarchy. In *Proceedings of the twenty-sixth annual ACM symposium on Theory of computing*, pages 449–458, 1994.
28. Hans L Bodlaender and Arie MCA Koster. Treewidth computations i. upper bounds. *Information and Computation*, 208(3):259–275, 2010.
29. Zbigniew Bonikowski and Urszula Wybraniec-Skardowska. Rough sets and vague sets. In *Rough Sets and Intelligent Systems Paradigms: International Conference, RSEISP 2007, Warsaw, Poland, June 28-30, 2007. Proceedings 1*, pages 122–132. Springer, 2007.
30. Julia Böttcher, Klaas P Pruessmann, Anusch Taraz, and Andreas Würfl. Bandwidth, treewidth, separators, expansion, and universality. *Electronic Notes in Discrete Mathematics*, 31:91–96, 2008.
31. Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.
32. Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.
33. Said Broumi, R Sundareswaran, M Shanmugapriya, Assia Bakali, and Mohamed Talea. Theory and applications of fermatean neutrosophic graphs. *Neutrosophic sets and systems*, 50:248–286, 2022.

34. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.
35. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
36. Said Broumi, Mohamed Talea, Assia Bakali, Florentin Smarandache, Quek Shio Gai, and Ganeshsree Selvachandran. Introduction of some new results on interval-valued neutrosophic graphs. *Journal of Information and Optimization Sciences*, 40(7):1475–1498, 2019.
37. Said Broumi, Mohamed Talea, Assia Bakali, Florentin Smarandache, and PK Kishore Kumar. Shortest path problem on single valued neutrosophic graphs. In *2017 international symposium on networks, computers and communications (ISNCC)*, pages 1–6. IEEE, 2017.
38. Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Boolean-width of graphs. *Theoretical Computer Science*, 412(39):5187–5204, 2011.
39. Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Boolean-width of graphs. *Theoretical Computer Science*, 412(39):5187–5204, 2011.
40. BM Bui-Xuan, JA Telle, and M Vatshelle. Fast algorithms for vertex subset and vertex partitioning problems on graphs of low boolean-width. *submitted to IWPEC*, 2009.
41. Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, Shenda Hong, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *IJCAI*, pages 1923–1929, 2022.
42. Gary Chartrand. *Introductory graph theory*. Courier Corporation, 2012.
43. P Chellamani, D Ajay, Mohammed M Al-Shamiri, and Rashad Ismail. *Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making*. Infinite Study, 2023.
44. Phyllis Z Chinn, Jarmila Chvátalová, Alexander K Dewdney, and Norman E Gibbs. The bandwidth problem for graphs and matrices—a survey. *Journal of Graph Theory*, 6(3):223–254, 1982.
45. Bruno Courcelle and Stephan Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101(1-3):77–114, 2000.
46. Qionghai Dai and Yue Gao. Mathematical foundations of hypergraph. In *Hypergraph Computation*, pages 19–40. Springer, 2023.
47. Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 50(1):225–238, 2022.
48. Dariusz Dereniowski. From pathwidth to connected pathwidth. *SIAM Journal on Discrete Mathematics*, 26(4):1709–1732, 2012.
49. R Aruna Devi and K Anitha. Construction of rough graph to handle uncertain pattern from an information system. *arXiv preprint arXiv:2205.10127*, 2022.
50. Matt DeVos, O-joung Kwon, and Sang-il Oum. Branch-depth: Generalizing tree-depth of graphs. *European Journal of Combinatorics*, 90:103186, 2020.
51. PM Dhanya and PB Ramkumar. Text analysis using morphological operations on a neutrosophic text hypergraph. *Neutrosophic Sets and Systems*, 61:337–364, 2023.
52. Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.
53. Juanjuan Ding, Wenhui Bai, and Chao Zhang. A new multi-attribute decision making method with single-valued neutrosophic graphs. *International Journal of Neutrosophic Science*, 11(2):76–86, 2020.
54. Yon Dourisboure, Feodor F Dragan, Cyril Gavoille, and Chenyu Yan. Spanners for bounded tree-length graphs. *Theoretical Computer Science*, 383(1):34–44, 2007.
55. Michael R. Fellows, Frances A. Rosamond, Ulrike Rotics, and Stefan Szeider. Clique-width is np-complete. *SIAM Journal on Discrete Mathematics*, 23(2):909–939, 2009.

56. Song Feng, Emily Heath, Brett Jefferson, Cliff Joslyn, Henry Kvinge, Hugh D Mitchell, Brenda Praggastis, Amie J Eisfeld, Amy C Sims, Larissa B Thackray, et al. Hypergraph models of biological networks to identify genes critical to pathogenic viral response. *BMC bioinformatics*, 22(1):287, 2021.
57. Fedor V Fomin, Sang-il Oum, and Dimitrios M Thilikos. Rank-width and tree-width of h-minor-free graphs. *European Journal of Combinatorics*, 31(7):1617–1628, 2010.
58. Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.
59. Takaaki Fujita. Obstruction for hypertree width and superhypertree width. *preprint(researchgate)*, 2024.
60. Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.
61. Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.
62. Takaaki Fujita. Tree-decomposition on fuzzy graph. *preprint(researchgate)*, 2024.
63. Takaaki Fujita. Various properties of various ultrafilters, various graph width parameters, and various connectivity systems. *arXiv preprint arXiv:2408.02299*, 2024.
64. Takaaki Fujita. Fundamental computational problems and algorithms for superhypergraphs. 2025.
65. Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.
66. Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.
67. Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.
68. Robert Ganian, Eun Jung Kim, and Stefan Szeider. Algorithmic applications of tree-cut width. In *International Symposium on Mathematical Foundations of Computer Science*, pages 348–360. Springer, 2015.
69. Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(3):3181–3199, 2022.
70. Masoud Ghods and Zahra Rostami. *Connectivity index in neutrosophic trees and the algorithm to find its maximum spanning tree*, volume 36. Infinite Study, 2020.
71. Sathyanarayanan Gopalakrishnan, Supriya Sridharan, Soumya Ranjan Nayak, Janmenjoy Nayak, and Swaminathan Venkataraman. Central hubs prediction for bio networks by directed hypergraph-ga with validation to covid-19 ppi. *Pattern Recognition Letters*, 153:246–253, 2022.
72. Georg Gottlob, Gianluigi Greco, Francesco Scarcello, et al. Treewidth and hypertree width. *Tractability: Practical Approaches to Hard Problems*, 1:20, 2014.
73. Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.
74. Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.
75. Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications.* Chapman and Hall/CRC, 2018.
76. Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):203, 2018.
77. Tom Gur, Noam Lifshitz, and Siqi Liu. Hypercontractivity on high dimensional expanders. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 176–184, 2022.
78. Frank Gurski and Carolin Rehs. Comparing linear width parameters for directed graphs. *Theory of Computing Systems*, 63:1358–1387, 2019.

79. Jens Gustedt, Ole A Mæhle, and Jan Arne Telle. The treewidth of java programs. In *Workshop on Algorithm Engineering and Experimentation*, pages 86–97. Springer, 2002.
80. Mohammad Hamidi and Arsham Borumand Saeid. Achievable single-valued neutrosophic graphs in wireless sensor networks. *New Mathematics and Natural Computation*, 14(02):157–185, 2018.
81. Mohammad Hamidi and Florentin Smarandache. Derivable single valued neutrosophic graphs based on km-fuzzy metric. *IEEE Access*, 8:131076–131087, 2020.
82. Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.
83. Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.
84. Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.
85. Pinar Heggernes, Daniel Lokshtanov, Rodica Mihai, and Charis Papadopoulos. Cutwidth of split graphs and threshold graphs. *SIAM Journal on Discrete Mathematics*, 25(3):1418–1437, 2011.
86. Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.
87. Jing Huang and Jie Yang. Unignn: a unified framework for graph and hypergraph neural networks. *arXiv preprint arXiv:2105.00956*, 2021.
88. Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.
89. S Satham Hussain, R Jahir Hussain, and Florentin Smarandache. *On neutrosophic vague graphs*. Infinite Study, 2019.
90. Sang il Oum and Paul Seymour. Approximating clique-width and branch-width. *Journal of Combinatorial Theory, Series B*, 96(4):514–528, 2006.
91. Remie Janssen, Mark Jones, Steven Kelk, Georgios Stamoulis, and Taoyang Wu. Treewidth of display graphs: bounds, brambles and applications. *arXiv preprint arXiv:1809.00907*, 2018.
92. Cengiz Kahraman. *Fuzzy multi-criteria decision making: theory and applications with recent developments*, volume 16. Springer Science & Business Media, 2008.
93. Mamadou Moustapha Kanté and Michaël Rao. -rank-width of (edge-colored) graphs. In *International Conference on Algebraic Informatics*, pages 158–173. Springer, 2011.
94. Haim Kaplan and Ron Shamir. Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM Journal on Computing*, 25(3):540–561, 1996.
95. Faruk Karaaslan and Bijan Davvaz. Properties of single-valued neutrosophic graphs. *Journal of Intelligent & Fuzzy Systems*, 34(1):57–79, 2018.
96. M Kaviyarasu, Hossein Rashmanlou, Saeed Kosari, Said Broumi, R Venitha, M Rajeshwa, and Farshid Mofidnakhaei. Circular economy strategies to promote sustainable development using t-neutrosophic fuzzy graph. *Neutrosophic Sets and Systems*, 72:186–210, 2024.
97. Murugan Kaviyarasu, Luminița-Ioana Cotîrlă, Daniel Breaz, Murugesan Rajeshwari, and Eleonora Rapeanu. A study on complex t-neutrosophic graph with intention to preserve biodiversity. *Symmetry*, 16(8):1033, 2024.
98. Ephraim Korach and Nir Solel. Tree-width, path-width, and cutwidth. *Discrete Applied Mathematics*, 43(1):97–101, 1993.
99. Choongbum Lee, Joonkyung Lee, and Sang-il Oum. Rank-width of random graphs. *Journal of Graph Theory*, 70(3):339–347, 2012.
100. Xiaowei Liao, Yong Xu, and Haibin Ling. Hypergraph neural networks for hypergraph matching. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pages 1266–1275, 2021.

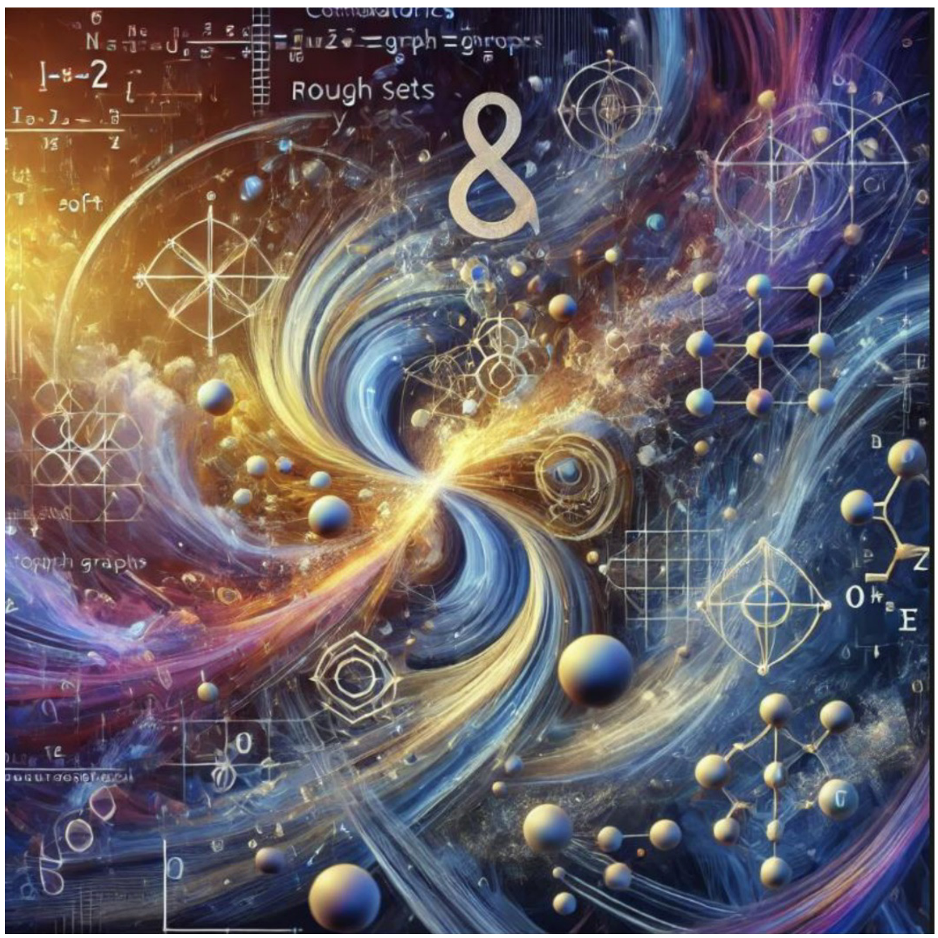
101. Puyin Liu and Hong-Xing Li. *Fuzzy neural network theory and application*, volume 59. World Scientific, 2004.
102. Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: new social network models. *Algorithms*, 12(11):234, 2019.
103. Dániel Marx. Approximating fractional hypertree width. *ACM Transactions on Algorithms (TALG)*, 6(2):1–17, 2010.
104. Rogers Mathew, Ilan Newman, Yuri Rabinovich, and Deepak Rajendraprasad. Boundaries of hypertrees, and hamiltonian cycles in simplicial complexes. *arXiv preprint arXiv:1507.04471*, 2015.
105. Zoltán Miklós. *Understanding Tractable Decompositions for Constraint Satisfaction.* PhD thesis, University of Oxford, 2008.
106. Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.
107. John N Mordeson and Peng Chang-Shyh. Operations on fuzzy graphs. *Information sciences*, 79(3-4):159–170, 1994.
108. John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.
109. John N Mordeson and Premchand S Nair. Applications of fuzzy graphs. In *Fuzzy graphs and fuzzy hypergraphs*, pages 83–133. Springer, 2000.
110. John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.
111. Susana Munoz, M Teresa Ortuno, Javier Ramírez, and Javier Yanez. Coloring fuzzy graphs. *Omega*, 33(3):211–221, 2005.
112. J Muthuerulappan and S Chelliah. A study of the neutrosophic dominating path-coloring number and multivalued star chromatic number in neutrosophic graphs. *Machine Intelligence Research*, 18(1):585–606, 2024.
113. Sumera Naz and Muhammad Aslam Malik. Single-valued neutrosophic line graphs. *TWMS Journal of Applied and Engineering Mathematics*, 8(2):483–494, 2018.
114. Jaroslav Nešetřil and Patrice Ossona De Mendez. Tree-depth, subgraph coloring and homomorphism bounds. *European Journal of Combinatorics*, 27(6):1022–1041, 2006.
115. Dan Olteanu and Jakub Závodnỳ. Factorised representations of query results: size bounds and readability. In *Proceedings of the 15th International Conference on Database Theory*, pages 285–298, 2012.
116. Sang-il Oum. Approximating rank-width and clique-width quickly. *ACM Transactions on Algorithms (TALG)*, 5(1):1–20, 2008.
117. Sang-il Oum. Rank-width is less than or equal to branch-width. *Journal of Graph Theory*, 57(3):239–244, 2008.
118. Sang-il Oum. Rank-width: Algorithmic and structural results. *Discrete Applied Mathematics*, 231:15–24, 2017.
119. Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.
120. Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.
121. Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.
122. Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.
123. Nicole Pearcy, Jonathan J Crofts, and Nadia Chuzhanova. Hypergraph models of metabolism. *International Journal of Biological, Veterinary, Agricultural and Food Engineering*, 8(8):752–756, 2014.
124. Andrzej Proskurowski and Jan Arne Telle. From bandwidth to pathwidth k. *_THEORETICAL-E*, page 90, 1996.
125. Hossein Rashmanlou and Madhumangal Pal. Balanced interval-valued fuzzy graphs. 2013.

126. Neil Robertson and Paul D. Seymour. Graph minors. i. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983.
127. Neil Robertson and Paul D. Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984.
128. Neil Robertson and Paul D. Seymour. Graph minors. x. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.
129. Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.
130. Rıdvan Şahin. An approach to neutrosophic graph theory with applications. *Soft Computing*, 23(2):569–581, 2019.
131. Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.
132. Robert Sasak. Comparing 17 graph parameters. Master's thesis, The University of Bergen, 2010.
133. Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.
134. Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.
135. Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.
136. Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.
137. Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.
138. Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
139. Florentin Smarandache. *Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra*. Infinite Study, 2022.
140. Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.
141. Florentin Smarandache. Foundation of the superhypersoft set and the fuzzy extension superhypersoft set: A new vision. *Neutrosophic Systems with Applications*, 11:48–51, 2023.
142. Florentin Smarandache. *New types of soft sets "hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": an improved version*. Infinite Study, 2023.
143. Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.
144. Florentin Smarandache. *Nidus Idearum. Scilogs, XIV: SuperHyperAlgebra*. Infinite Study, 2024.
145. Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.
146. Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.
147. Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.
148. Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Neutrosophic graphs: A new dimension to graph theory. 2015.
149. Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

150. Metawee Songsaeng and Aiyared Iampan. *Neutrosophic set theory applied to UP-algebras*. Infinite Study, 2019.
151. P Srivastava and RL Gupta. Fuzzy proximity structures and fuzzy ultrafilters. *Journal of Mathematical Analysis and Applications*, 94(2):297–311, 1983.
152. Atsushi Takahashi, Shuichi Ueno, and Yoji Kajitani. Minimal acyclic forbidden minors for the family of graphs with bounded path-width. *Discrete Mathematics*, 127(1-3):293–304, 1994.
153. Atsushi Takahashi, Shuichi Ueno, and Yoji Kajitani. Mixed searching and proper-path-width. *Theoretical Computer Science*, 137(2):253–268, 1995.
154. Atsushi TAKAHASHI, Shuichi UENO, and Yoji KAJITANI. On the proper-path-decomposition of trees. *IEICE transactions on fundamentals of electronics, communications and computer sciences*, 78(1):131–136, 1995.
155. CB ten Brinke. Variations on boolean-width. Master's thesis, 2015.
156. Duc Long Tran. *Expanding the Graph Parameter Hierarchy*. PhD thesis, Institute of Software, 2022.
157. Danyang Wang and Ping Zhu. Fuzzy rough digraph based on strength of connectedness with application. *Neural Computing and Applications*, 35(16):11847–11866, 2023.
158. Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.
159. Marcin Wrochna. Reconfiguration in bounded bandwidth and tree-depth. *Journal of Computer and System Sciences*, 93:1–10, 2018.
160. Lehua Yang, Dongmei Li, and Ruipu Tan. Research on the shortest path solution method of interval valued neutrosophic graphs based on the ant colony algorithm. *IEEE Access*, 8:88717–88728, 2020.
161. Nikola Yolov. Minor-matching hypertree width. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 219–233. SIAM, 2018.
162. Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
163. Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.
164. Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.
165. Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.
166. Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.
167. Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

The third volume of “Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond” presents an in-depth exploration of the cutting-edge developments in uncertain combinatorics and set theory. This comprehensive collection highlights innovative methodologies such as graphization, hyperization, and uncertainization, which enhance combinatorics by incorporating foundational concepts from fuzzy, neutrosophic, soft, and rough set theories. These advancements open new mathematical horizons, offering novel approaches to managing uncertainty within complex systems.

This volume also introduces advanced concepts like Neutrosophic Oversets, Undersets, and Offsets, which push the boundaries of classical graph theory and offer deeper insights into the mathematical and practical challenges posed by real-world systems. By blending combinatorics, set theory, and graph theory, the authors have created a robust framework for addressing uncertainty in both mathematical systems and their real-world applications. This foundation sets the stage for future breakthroughs in combinatorics, set theory, and related fields.



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