Florentin Smarandache
(author and editor)

Collected Papers

(on Neutrosophic Theory and Applications)

Volume VI

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(author and editor)

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Introductory Note

This sixth volume of *Collected Papers* includes 74 papers comprising 974 pages on (theoretic and applied) neutrosophics, written between 2015-2021 by the author alone or in collaboration with the following 121 co-authors from 19 countries: Mohamed Abdel-Basset, Abdel Nasser H. Zaied, Abduallah Gamal, Amir Abdullah, Firoz Ahmad, Nadeem Ahmad, Ahmad Yusuf Adhami, Ahmed Aboelfetouh, Ahmed Mostafa Khalil, Shariful Alam, W. Alharbi, Ali Hassan, Mumtaz Ali, Amira S. Ashour, Asmaa Atef, Assia Bakali, Ayoub Bahnasse, A. A. Azzam, Willem K.M. Brauers, Bui Cong Cuong, Fausto Cavallaro, Ahmet Çevik, Robby I. Chandra, Kalaivani Chandran, Victor Chang, Chang Su Kim, Jyotir Moy Chatterjee, Victor Christianto, Chunxin Bo, Mihaela Colhon, Shyamal Dalapati, Arindam Dey, Dunqian Cao, Fahad Alsharari, Faruk Karaaslan, Aleksandra Fedajev, Daniela Gîfu, Hina Gulzar, Haitham A. El-Ghareeb, Masooma Raza Hashmi, Hewayda El-Ghawalby, Hoang Viet Long, Le Hoang Son, F. Nirmala Irudayam, Branislav Ivanov, S. Jafari, Jeong Gon Lee, Milena Jevtić, Sudan Jha, Junhui Kim, Ilanthenral Kandasamy, W.B. Vasantha Kandasamy, Darjan Karabašević, Songül Karabatak, Abdullah Kargın, M. Karthika, Ieva Meidute-Kavaliauskiene, Madad Khan, Majid Khan, Manju Khari, Kifayat Ullah, K. Kishore, Kul Hur, Santanu Kumar Patro, Prem Kumar Singh, Raghvendra Kumar, Tapan Kumar Roy, Malayalan Lathamaheswari, Luu Quoc Dat, T. Madhumathi, Tahir Mahmood, Mladjan Maksimovic, Gunasekaran Manogaran, Nivetha Martin, M. Kasi Mayan, Mai Mohamed, Mohamed Talea, Muhammad Akram, Muhammad Gulistan, Raja Muhammad Hashim, Muhammad Riaz, Muhammad Saeed, Rana Muhammad Zulqarnain, Nada A. Nabeeh, Deivanayagampillai Nagarajan, Xenia Negrea, Nguyen Xuan Thao, Jagan M. Obbineni, Angelo de Oliveira, M. Parimala, Gabrijela Popovic, Ishaani Priyadarshini, Yaser Saber, Mehmet Șahin, Said Broumi, A. A. Salama, M. Saleh, Ganeshsree Selvachandran, Dönüș Șengür, Shio Gai Quek, Songtao Shao, Dragiša Stanujkić, Surapati Pramanik, Swathi Sundari Sundaramoorthy, Mirela Teodorescu, Selçuk Topal, Muhammed Turhan, Alptekin Ulutaș, Luige Vlădăreanu, Victor Vlădăreanu, Ştefan Vlăduţescu, Dan Valeriu Voinea, Volkan Duran, Navneet Yadav, Yanhui Guo, Naveed Yaqoob, Yongquan Zhou, Young Bae Jun, Xiaohong Zhang, Xiao Long Xin, Edmundas Kazimieras Zavadskas.

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Neutrosophic Axiomatic System

Florentin Smarandache

Florentin Smarandache (2015). Neutrosophic Axiomatic System. *Critical Review*, X, 5-28.

Abstract

In this paper, we introduce for the first time the notions of Neutrosophic Axiom, Neutrosophic Axiomatic System, Neutrosophic Deducibility and Neutrosophic Inference, Neutrosophic Proof, Neutrosophic Tautologies, Neutrosophic Quantifiers, Neutrosophic Propositional Logic, Neutrosophic Axiomatic Space, Degree of Contradiction (Dissimilarity) of Two Neutrosophic Axioms, and Neutrosophic Model. A class of neutrosophic implications is also introduced. A comparison between these innovatory neutrosophic notions and their corresponding classical notions is made. Then, three concrete examples of neutrosophic axiomatic systems, describing the same neutrosophic geometrical model, are presented at the end of the paper.

Keywords

Neutrosophic logic, Neutrosophic Axiom, Neutrosophic Deducibility, Neutrosophic Inference, Neutrosophic Proof, Neutrosophic Tautologies, Neutrosophic Quantifiers, Neutrosophic Propositional Logic, Neutrosophic Axiomatic Space.

1 Neutrosophic Axiom

A *neutrosophic axiom* or *neutrosophic postulate* (α) is a partial premise, which is t% true (degree of truth), i% indeterminate (degree of indeterminacy), and f% false (degree of falsehood), where <t, i, f> are standard or nonstandard subsets included in the non-standard unit interval]-0, 1+[.

The non-standard subsets and non-standard unit interval are mostly used in philosophy in cases where one needs to make distinction between “absolute truth” (which is a truth in all possible worlds) and “relative truth” (which is a truth in at least one world, but not in all possible worlds), and similarly for distinction between "absolute indeterminacy" and "relative indeterminacy", and respectively distinction between "absolute falsehood" and "relative falsehood".

But for other scientific and technical applications one uses standard subsets, and the standard classical unit interval [0, 1].

As a particular case of neutrosophic axiom is the classical axiom. In the classical mathematics an axiom is supposed 100% true, 0% indeterminate, and 0% false. But this thing occurs in idealistic systems, in perfectly closed systems, not in many of the real world situations.

Unlike the classical axiom which is a total premise of reasoning and without any controversy, the neutrosophic axiom is a partial premise of reasoning with a partial controversy.

The neutrosophic axioms serve in approximate reasoning.

The partial truth of a neutrosophic axiom is similarly taken for granting.

The neutrosophic axioms, and in general the neutrosophic propositions, deal with approximate ideas or with probable ideas, and in general with ideas we are not able to measure exactly. That's why one cannot get 100% true statements (propositions).

In our life we deal with approximations. An axiom is approximately true, and the inference is approximately true either.

A neutrosophic axiom is a self-evident assumption in some degrees of truth, indeterminacy, and falsehood respectively.

2 Neutrosophic Deducing and Neutrosophic Inference

The neutrosophic axioms are employed in *neutrosophic deducing* and *neutrosophic inference* rules, which are sort of neutrosophic implications, and similarly they have degrees of truth, indeterminacy, and respectively falsehood.

3 Neutrosophic Proof

Consequently, a *neutrosophic proof* has also a degree of validity, degree of indeterminacy, and degree of invalidity. And this is when we work with not-well determinate elements in the space or not not-well determinate inference rules.

The neutrosophic axioms are at the foundation of various *neutrosophic sciences*.

The approximate, indeterminate, incomplete, partially unknown, ambiguous, vagueness, imprecision, contradictory, etc. knowledge can be neutrosophically axiomized.

4 Neutrosophic Axiomatic System

A set of neutrosophic axioms Ω is called *neutrosophic axiomatic system*, where the neutrosophic deducing and the neutrosophic inference (neutrosophic implication) are used.

The neutrosophic axioms are defined on a given space S. The space can be classical (space without indeterminacy), or neutrosophic space (space which has some indeterminacy with respect to its elements).

A neutrosophic space may be, for example, a space that has at least one element which only partially belongs to the space. Let us say the element x <0.5, 0.2, 0.3> that belongs only 50% to the space, while 20% its appurtenance is indeterminate, and 30% it does not belong to the space.

Therefore, we have three types of neutrosophic axiomatic systems:

[1] Neutrosophic axioms defined on classical space;

[2] Classical axioms defined on neutrosophic space;

[3] Neutrosophic axioms defined on neutrosophic space.

Remark:

The neutrosophic axiomatic system is not unique, in the sense that several different axiomatic systems may describe the same neutrosophic model. This happens because one deals with approximations, and because the neutrosophic axioms represent partial (not total) truths.

5 Classification of the Neutrosophic Axioms

[1] *Neutrosophic Logical Axioms*, which are neutrosophic statements whose truth-value is <t, i, f> within the system of neutrosophic logic. For example: (α or β) neutrosophically implies β.

[2] *Neutrosophic Non-Logical Axioms*, which are neutrosophic properties of the elements of the space. For example: the neutrosophic associativity $a(bc) = (ab)c$, which occurs for some elements, it is unknown (indeterminate) for others, and does not occur for others.

In general, a neutrosophic non-logical axiom is a classical non-logical axiom that works for certain space elements, is indeterminate for others, and does not work for others.

6 Neutrosophic Tautologies

A classical tautology is a statement that is universally true [regarded in a larger way, or lato sensu], i.e. true in all possible worlds (according to Leibniz's definition of "world"). For example, "M = M" in all possible worlds.

A *neutrosophic tautology* is a statement that is true in a narrow way [i.e. regarded in stricto sensu], or it is <1, 0, 0> true for a class of certain parameters and conditions, and <t, i, f> true for another class of certain parameters and conditions, where <t, i, f> $\neq$ <1, 0, 0>. I.e. a neutrosophic tautology is true in some worlds, and partially true in other worlds. For example, the previous assertion: "M = M".

If "M" is a number [i.e. the parameter = number], then a number is always equal to itself in any numeration base.

But if "M" is a person [i.e. the parameter = person], call him Martin, then Martin at time t1 is the same as Martin at time t1 [i.e. it has been considered another parameter = time], but Martin at time t1 is different from Martin at time t2 (meaning for example 20 years ago: hence Martin younger is different from Martin older). Therefore, from the point of view of parameters 'person' and 'time', "M = M" is not a classical tautology.

Similarly, we may have a proposition P which is true locally, but it is untrue non-locally.

A neutrosophic logical system is an approximate minimal set of partially true/indeterminate/false propositions.

While the classical axioms cannot be deduced from other axioms, there are neutrosophic axioms that can be partially deduced from other neutrosophic axioms.

7 Notations regarding the Classical Logic and Set, Fuzzy Logic and Set, Intuitionistic Fuzzy Logic and Set, and Neutrosophic Logic and Set

In order to make distinction between classical (Boolean) logic/set, fuzzy logic/set, intuitionistic fuzzy logic/set, and neutrosophic logic/set, we denote their corresponding operators (negation/complement, conjunction/ intersection, disjunction/union, implication/inclusion, and equivalence/equality), as it follows:

[1] For classical (Boolean) logic and set:

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \tag{1}$$

[2] For fuzzy logic and set:

$$\underset{F}{\neg} \quad \underset{F}{\wedge} \quad \underset{F}{\vee} \quad \underset{F}{\rightarrow} \quad \underset{F}{\leftrightarrow} \tag{2}$$

[3] For intuitionistic fuzzy logic and set:

$$\underset{IF}{\neg} \quad \underset{IF}{\wedge} \quad \underset{IF}{\vee} \quad \underset{IF}{\rightarrow} \quad \underset{IF}{\leftrightarrow} \tag{3}$$

[4] For neutrosophic logic and set:

$$\underset{N}{\neg} \quad \underset{N}{\wedge} \quad \underset{N}{\vee} \quad \underset{N}{\rightarrow} \quad \underset{N}{\leftrightarrow} \tag{4}$$

8 The Classical Quantifiers

The classical *Existential Quantifier* is the following way:

$$\exists x \in A, P(x). \tag{5}$$

In a neutrosophic way we can write it as:

There exist *x<1, 0, 0>* in *A* such that *P(x)<1, 0, 0>*, or:

$$\exists x < 1,0,0 > \in A, P(x) < 1,0,0 >. \tag{6}$$

The classical *Universal Quantifier* is the following way:

$$\forall x \in A, P(x). \tag{7}$$

In a neutrosophic way we can write it as:

For any *x<1, 0, 0>* in *A* one has *P(x)<1, 0, 0>*, or:

$$\forall x < 1,0,0 > \in A, P(x) < 1,0,0 >. \tag{8}$$

9 The Neutrosophic Quantifiers

The *Neutrosophic Existential Quantifier* is in the following way:

There exist $x<t_x, i_x, f_x>$ in A such that $P(x)<t_P, i_P, f_P>$, or:

$$\exists x < t_x, i_x, f_x > \in A, P(x) < t_P, i_P, f_P >, \quad (9)$$

which means that: there exists an element x which belongs to A in a neutrosophic degree $<t_x, i_x, f_x>$, such that the proposition P has the neutrosophic degree of truth $<t_P, i_P, f_P>$.

The *Neutrosophic Universal Quantifier* is the following way:

For any $x<t_x, i_x, f_x>$ in A one has $P(x)<t_P, i_P, f_P>$, or:

$$\forall x < t_x, i_x, f_x > \in A, P(x) < t_P, i_P, f_P >, \quad (10)$$

which means that: for any element x that belongs to A in a neutrosophic degree $<t_x, i_x, f_x>$, one has the proposition P with the neutrosophic degree of truth $<t_P, i_P, f_P>$.

10 Neutrosophic Axiom Schema

A *neutrosophic axiom schema* is a neutrosophic rule for generating infinitely many neutrosophic axioms.

Examples of neutrosophic axiom schema:

[1] *Neutrosophic Axiom Scheme for Universal Instantiation.*

Let Φ(x) be a formula, depending on variable x defined on a domain D, in the first-order language L, and let's substitute x for $a \in$ D. Then the new formula:

$$\forall x \Phi(x) \rightarrow_N \Phi(a) \quad (11)$$

is $< t_{\rightarrow_N}, i_{\rightarrow_N}\ f_{\rightarrow_N}$ -neutrosophically [universally] valid.

This means the following: if one knows that a formula Φ(x) holds $<t_x, i_x, f_x>$-neutrosophically for every x in the domain D, and for $x = a$ the formula $\Phi(a)$ holds $<t_a, i_a, f_a>$-neutrosophically, then the whole new formula (a) holds $< t_{\rightarrow_N}, i_{\rightarrow_N}\ f_{\rightarrow_N}$ -neutrosophically, where $t_{\rightarrow_N}$ means the truth degree, $i_{\rightarrow_N}$ the indeterminacy degree, and $f_{\rightarrow_N}$ the falsehood degree –- all resulted from the neutrosophic implication $\rightarrow_N$.

[2] *Neutrosophic Axiom Scheme for Existential Generalization.*

Let $\Phi(x)$ be a formula, depending on variable x defined on a domain D, in the first-order language L, and let's substitute x for $a \in$ D. Then the new formula:

$$\Phi(a) \rightarrow_N \exists x \Phi(x) \quad (12)$$

is $< t_{\rightarrow_N}, i_{\rightarrow_N}\ f_{\rightarrow_N}$ -neutrosophically [universally] valid.

This means the following: if one knows that a formula $\Phi(a)$ holds $<t_a, i_a, f_a>$-neutrosophically for a given x = a in the domain D, and for every x in the domain formula Φ(x) holds $<t_x, i_x, f_x>$-neutrosophically, then the whole new formula (b) holds $< t_{\rightarrow_N}, i_{\rightarrow_N}\ f_{\rightarrow_N}$ -neutrosophically, where $t_{\rightarrow_N}$ means the truth degree, $i_{\rightarrow_N}$ the indeterminacy degree, and $f_{\rightarrow_N}$ the falsehood degree –- all resulted from the neutrosophic implication $\rightarrow_N$.

These are *neutrosophic metatheorems* of the mathematical neutrosophic theory where they are employed.

11 Neutrosophic Propositional Logic

We have many neutrosophic formulas that one takes as neutrosophic axioms. For example, as extension from the classical logic, one has the following.

Let $P<t_P, i_P, f_P>$, $Q<t_Q, i_Q, f_Q>$, $R<t_R, i_R, f_R>$, $S<t_S, i_S, f_S>$ be neutrosophic propositions, where $<t_P, i_P, f_P>$ is the neutrosophic-truth value of P, and similarly for Q, R, and S. Then:

a) Neutrosophic modus ponens (neutrosophic implication elimination):

$$P \rightarrow_N\ Q \rightarrow_N P) \quad (13)$$

b) Neutrosophic modus tollens (neutrosophic law of contrapositive):

$$((P \rightarrow_N Q) \wedge_N \neg_N Q) \rightarrow_N \neg_N P \quad (14)$$

c) Neutrosophic *disjunctive syllogism* (neutrosophic disjunction elimination):

$$((P \vee_N Q) \wedge_N \neg_N P) \rightarrow_N Q \quad (15)$$

d) Neutrosophic *hypothetical syllogism* (neutrosophic chain argument):

$$((P \rightarrow_N Q) \wedge_N (Q \rightarrow_N R)) \rightarrow_N (P \rightarrow_N R) \quad (16)$$

e) Neutrosophic constructive dilemma (neutrosophic *disjunctive* version of *modus ponens*):

$$(((P \rightarrow_N Q) \wedge_N (R \rightarrow_N S)) \wedge_N (P \vee_N R)) \rightarrow_N (Q \vee_N S) \quad (17)$$

f) Neutrosophic distructive dilemma (neutrosophic *disjunctive* version of *modus tollens*):

$$\begin{aligned}&(((P \rightarrow_N Q) \wedge_N (R \rightarrow_N S)) \wedge_N \\ &(\neg_N Q \vee_N \neg_N S)) \rightarrow_N (\neg_N P \vee_N \neg_N R)\end{aligned} \quad (18)$$

All these neutrosophic formulae also run as *neutrosophic rules of inference*.

These neutrosophic formulas or neutrosophic derivation rules only *partially* preserve the truth, and depending on the neutrosophic implication operator that is employed the indeterminacy may increase or decrease.

This happens for one working with approximations.

While the above classical formulas in classical proportional logic are classical tautologies (i.e. from a neutrosophical point of view they are *100%* true, *0%* indeterminate, and *0%* false), their corresponding neutrosophic formulas are neither classical tautologies nor neutrosophical tautologies, but ordinary neutrosophic propositions whose $< t, i, f >$ – neutrosophic truth-value is resulted from the $\underset{N}{\rightarrow}$ neutrosophic implication

$$A < t_A, i_A, f_A > \underset{N}{\rightarrow} B < (t_B, i_B, f_B) >. \quad (19)$$

12 Classes of Neutrosophic Negation Operators

There are defined in neutrosophic literature classes of neutrosophic negation operators as follows: if $A(t_A, i_A, f_A)$, then its negation is:

$$\underset{N}{\neg} A(f_A, i_A, t_A), \quad (20)$$

$$\text{or } \underset{N}{\neg} A(f_A, 1 - i_A, t_A), \quad (21)$$

$$\text{or } \underset{N}{\neg} A(1 - t_A, 1 - i_A, 1 - f_A), \quad (22)$$

$$\text{or } \underset{N}{\neg} A(1 - t_A, i_A, 1 - f_A), \text{ etc.} \quad (23)$$

13 Classes of Neutrosophic Conjunctive Operators.

Similarly: if $A(t_A, i_A, f_A)$ and $B(t_B, i_B, f_B)$, then

$$A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \quad (24)$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \quad (25)$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle \quad (26)$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, \frac{i_A + i_B}{2}, f_A \overset{\vee}{_F} f_B \rangle, \quad (27)$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, 1 - \frac{i_A + i_B}{2}, f_A \overset{\vee}{_F} f_B \rangle, \quad (28)$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, |i_A - i_B|, f_A \overset{\vee}{_F} f_B \rangle, \text{ etc.} \quad (29)$$

14 Classes of Neutrosophic Disjunctive Operators

And analogously, there were defined:

$$A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle, \quad (30)$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle, \quad (31)$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \quad (32)$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, \frac{i_A + i_B}{2}, f_A \overset{\wedge}{_F} f_B \rangle, \quad (33)$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, 1 - \frac{i_A + i_B}{2}, f_A \overset{\wedge}{_F} f_B \rangle, \quad (34)$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, |i_A - i_B|, f_A \overset{\vee}{_F} f_B \rangle, \text{ etc.} \quad (35)$$

15 Fuzzy Operators

Let $\alpha, \beta \in [0, 1]$.

15.1. The *Fuzzy Negation* has been defined as $\neg_F \alpha = 1 - \alpha$. (36)

15.2. While the class of *Fuzzy Conjunctions* (or *t-norm*) may be:

$$\alpha \overset{\wedge}{_F} \beta = \min\{\alpha, \beta\}, \quad (37)$$

$$\text{or } \alpha \overset{\wedge}{_F} \beta = \alpha \cdot \beta, \quad (38)$$

$$\text{or } \alpha \overset{\wedge}{_F} \beta = \max\{0, \alpha + \beta - 1\}, \text{ etc.} \quad (39)$$

15.3. And the class of *Fuzzy Disjunctions* (or *t-conorm*) may be:

$$\alpha \overset{\vee}{_F} \beta = \max\{\alpha, \beta\}, \tag{40}$$

$$\text{or } \alpha \overset{\vee}{_F} \beta = \alpha + \beta - \alpha\beta, \tag{41}$$

$$\text{or } \alpha \overset{\vee}{_F} \beta = \min\{1, \alpha + \beta\}, \text{ etc.} \tag{42}$$

15.4. Examples of *Fuzzy Implications* $x \underset{F}{\rightarrow} y$, for $x, y \in [0,1]$, defined below:

- Fodor (1993): $I_{FD}(x,y) = \begin{cases} 1, \text{ if } x \le y \\ \max(1-x, y), \text{ if } x > y \end{cases}$ (43)
- Weber (1983): $I_{WB}(x,y) = \begin{cases} 1, \text{if } x < y \\ y, \text{if } x = 1 \end{cases}$ (44)
- Yager (1980): $I_{YG}(x,y) = \begin{cases} 1, \text{if } x = 0 \text{ and } y = 0 \\ y^x, \text{if } x > 0 \text{ or } y > 0 \end{cases}$ (45)
- Goguen (1969): $I_{GG}(x,y) = \begin{cases} 1, \text{if } x \le y \\ \frac{y}{x}, \text{if } x > y \end{cases}$ (46)
- Rescher (1969): $I_{RS}(x,y) = \begin{cases} 1, \text{if } x \le y \\ 0, \text{if } x > y \end{cases}$ (47)
- Kleene-Dienes (1938): $I_{KD}(x,y) = \max(1-x, y)$ (48)
- Reichenbach (1935): $I_{RC}(x,y) = 1 - x + xy$ (49)
- Gödel (1932): $I_{GD}(x,y) = \begin{cases} 1, \text{if } x \le y \\ y, \text{if } x > y \end{cases}$ (50)
- Lukasiewicz (1923): $I_{LK}(x,y) = \min(1, 1-x+y)$, (51)

according to the list made by Michal Baczyński and Balasubramaniam Jayaram (2008).

16 Example of Intuitionistic Fuzzy Implication

Example of *Intuitionistic Fuzzy Implication* $A(t_A, f_A) \underset{IF}{\rightarrow} B(t_B, f_B)$ is:

$$I_{IF} = \left(\left[(1-t_A) \overset{\vee}{_F} t_B \right] \overset{\wedge}{_F} \left[(1-f_B) \overset{\vee}{_F} f_A \right], f_B \overset{\wedge}{_F} (1-t_A) \right), \tag{52}$$

according to Yunhua Xiao, Tianyu Xue, Zhan'ao Xue, and Huiru Cheng (2011).

17 Classes of Neutrosophic Implication Operators

We now propose for the first time *eight new classes of neutrosophic implications* and extend a ninth one defined previously:

$$A(t_A, i_A, f_A) \underset{N}{\rightarrow} B(t_B, i_B, f_B),$$

in the following ways:

17.1-17.2. $I_{N1}\left(t_A \underset{F/IF}{\longrightarrow} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B\right)$, (53)

where $t_A \underset{F/IF}{\longrightarrow} t_B$ is any fuzzy implication (from above or others) or any intuitionistic fuzzy implication (from above or others), while $\overset{\wedge}{_F}$ is any fuzzy conjunction (from above or others);

17.3-17.4. $I_{N2}\left(t_A \underset{F/IF}{\longrightarrow} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\wedge}{_F} f_B\right)$, (54)

where $\overset{\vee}{_F}$ is any fuzzy disjunction (from above or others);

17.5-17.6. $I_{N3}\left(t_A \underset{F/IF}{\longrightarrow} t_B, \frac{i_A+i_B}{2}, f_A \overset{\wedge}{_F} f_B\right)$; (55)

17.7-17.8. $I_{N4}\left(t_A \underset{F/IF}{\longrightarrow} t_B, \frac{i_A+i_B}{2}, \frac{f_A+f_B}{2}\right)$. (56)

17.9. Now we extend another neutrosophic implication that has been defined by S. Broumi & F. Smarandache (2014) and it was based on the classical logical equivalence:

$$(A \rightarrow B) \leftrightarrow (\neg A \vee B). \quad (57)$$

Whence, since the corresponding neutrosophic logic equivalence:

$$\left(A \underset{N}{\rightarrow} B\right) \underset{N}{\leftrightarrow} (\underset{N}{\neg} A \overset{\vee}{_N} B) \quad (58)$$

holds, one obtains another *Class of Neutrosophic Implication Operators* as:

$$(\underset{N}{\neg} A \overset{\vee}{_N} B) \quad (59)$$

where one may use any neutrosophic negation $\underset{N}{\neg}$ (from above or others), and any neutrosophic disjunction $\overset{\vee}{_N}$ (from above or others).

18 Example of Neutrosophic Implication

Let's see an *Example of Neutrosophic Implication*.

Let's have two neutrosophic propositions $A\langle 0.3, 0.4, 0.2\rangle$ and $B\langle 0.7, 0.1, 0.4\rangle$. Then $A \underset{N}{\rightarrow} B$ has the neutrosophic truth value of $\underset{N}{\neg} A \underset{N}{\vee} B$, i.e.:

$\langle 0.2, 0.4, 0.3\rangle \underset{N}{\vee} \langle 0.7, 0.1, 0.4\rangle$,

or $\langle \max\{0.2, 0.7\}, \min\{0.4, 0.1\}, \min\{0.3, 0.4\}\rangle$,

or $\langle 0.7, 0.1, 0.3\rangle$,

where we used the neutrosophic operators defined above: $\underset{N}{\neg}\langle t, i, f\rangle = \langle f, i, t\rangle$ for neutrosophic negation, and $\langle t_1, i_1, f_1\rangle \underset{N}{\vee} \langle t_2, i_2, f_2\rangle = \langle \max\{t_1, t_2\}, \min\{i_1, i_2\}, \min\{f_1, f_2\}\rangle$ for the neutrosophic disjunction.

Using different versions of the neutrosophic negation operators and/or different versions of the neutrosophic disjunction operators, one obtains, in general, different results. Similarly as in fuzzy logic.

18.1. Another Example of Neutrosophic Implication.

Let A have the neutrosophic truth-value (t_A, i_A, f_A) , and B have the neutrosophic truth-value (t_B, i_B, f_B), then:

$$\left[A \underset{N}{\rightarrow} B\right] \underset{N}{\leftrightarrow} [(\underset{N}{\neg} A) \underset{N}{\vee} B], \tag{60}$$

where $\underset{N}{\neg}$ is any of the above neutrosophic negations, while $\underset{N}{\vee}$ is any of the above neutrosophic disjunctions.

19 General Definition of Neutrosophic Operators

We consider that the most general definition of neutrosophic operators shall be the followings:

$$A(t_A, i_A, f_A) \underset{N}{\oplus} B(t_B, i_B, f_B) = A \underset{N}{\oplus} B\langle u(t_A, i_A, f_A, t_B, i_B, f_B), v(t_A, i_A, f_A, t_B, i_B, f_B),\ w(t_A, i_A, f_A, t_B, i_B, f_B)\rangle \tag{61}$$

where $\underset{N}{\oplus}$ is any binary neutrosophic operator, and

$u(x_1, x_2, x_3, x_4, x_5, x_6), v(x_1, x_2, x_3, x_4, x_5, x_6),$

$w(x_1, x_2, x_3, x_4, x_5, x_6)$: $[0,1]^6 \rightarrow [0,1]$.

Even more, the neutrosophic component functions u, v, w may depend, on the top of these six variables, on hidden parameters as well, such as: $h_1, h_2, \dots, h_n$.

For a unary neutrosophic operator (for example, the neutrosophic negation), similarly:

$$\underset{N}{\neg}A(t_A, i_A, f_A) = \langle u'(t_A, i_A, f_A), v'(t_A, i_A, f_A), w'(t_A, i_A, f_A)\rangle, \qquad (62)$$

where $u'(t_A, i_A, f_A), v'(t_A, i_A, f_A), w'(t_A, i_A, f_A)$: $[0, 1]^3 \rightarrow [0,1]$,

and even more u', v', w' may depend, on the top of these three variables, of hidden parameters as well, such as: $h_1, h_2, \dots, h_n$.

{Similarly there should be for a *general definition of fuzzy operators* and *general definition of intuitionistic fuzzy operators*.}

As an example, we have defined [6]: (63)

$$A(t_A, i_A, f_A) \underset{N}{\wedge} B(t_B, i_B, f_B) \\ = \langle t_A t_B, i_A i_B + t_A i_B + t_B i_A, t_A f_B + t_B f_A + i_A f_B + i_B f_A\rangle$$

these result from multiplying

$$(t_A + i_A + f_A) \cdot (t_B + i_B + f_B) \qquad (64)$$

and ordering upon the below pessimistic order:

truth ≺ indeterminacy ≺ falsity,

meaning that to the *truth* only the terms of t's goes, i.e. $t_A t_B$,

to *indeterminacy* only the terms of t's and i's go, i.e. $i_A i_B + t_A i_B + t_B i_A$,

and to *falsity* the other terms left, i.e. $t_A f_B + t_B f_A + i_A f_B + i_B f_A + f_A f_B$.

20 Neutrosophic Deductive System

A *Neutrosophic Deductive System* consists of a set $\mathcal{L}_1$ of neutrosophic logical axioms, and a set $\mathcal{L}_2$ of neutrosophic non-logical axioms, and a set $\mathcal{R}$ of neutrosophic rules of inference – all defined on a neutrosophic space $\mathcal{S}$ that is composed of many elements.

A neutrosophic deductive system is said to be neutrosophically complete, if for any neutrosophic formula φ that is a neutrosophic logical consequence of $\mathcal{L}_1$, i.e. $\mathcal{L}_1 \underset{N}{\vDash} \varphi$, there exists a neutrosophic deduction of φ from $\mathcal{L}_1$, i.e. $\mathcal{L}_1 \underset{N}{\vdash} \varphi$, where $\underset{N}{\vDash}$ denotes neutrosophic logical consequence, and $\underset{N}{\vdash}$ denotes neutrosophic deduction.

Actually, everything that is neutrosophically (partially) true [i.e. made neutrosophically (partially) true by the set $\mathcal{L}_1$ of neutrosophic axioms] is neutrosophically (partially) provable.

The neutrosophic completeness of set $\mathcal{L}_2$ of neutrosophic non-logical axioms is not the same as the neutrosophic completeness of set $\mathcal{L}_1$ of neutrosophic logical axioms.

21 Neutrosophic Axiomatic Space

The space $\mathcal{S}$ is called *neutrosophic space* if it has some indeterminacy with respect to one or more of the following:

a. Its *elements*;

1. At least one element x partially belongs to the set $\mathcal{S}$, or $x(t_x, i_x, f_x)$ with $(t_X, i_X, f_X) \neq (1, 0, 0)$;

2. There is at least an element y in $\mathcal{S}$ whose appurtenance to $\mathcal{S}$ is unknown.

b. Its *logical axioms*;

1. At least a logical axiom $\mathcal{A}$ is partially true, or $\mathcal{A}(t_A, i_A, f_A)$, where similary $(t_A, i_A, f_A) \neq (1, 0, 0)$;

2. There is at least an axiom $\mathcal{B}$ whose truth-value is unknown.

c. Its *non-logical axioms*;

1. At least a non-logical axiom $\mathcal{C}$ is true for some elements, and indeterminate or false or other elements;

2. There is at least a non-logical axiom whose truth-value is unknown for some elements in the space.

d. There exist at least two neutrosophic logical axioms that have some degree of contradiction (strictly greater than zero).

e. There exist at least two neutrosophic non-logical axioms that have some degree of contradiction (strictly greater than zero).

22 Degree of Contradiction (Dissimilarity) of Two Neutrosophic Axioms

Two neutrosophic logical axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ are contradictory (dissimilar) if their semantics (meanings) are contradictory in some degree d_1, while their neutrosophic truth values $<t_1, i_1, f_1>$ and $<t_2, i_2, f_2>$ are contradictory in a different degree d_2 [in other words $d_1 \neq d_2$].

As a particular case, if two neutrosophic logical axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same semantic (meaning) [in other words $d_1 = 0$], but their neutrosophic truth-values are different [in other words $d_2 > 0$], they are contradictory.

Another particular case, if two neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have different semantics (meanings) [in other words $d_1 > 0$], but their neutrosophic truth values are the same $<t_1, i_1, f_1> = <t_2, i_2, f_2>$ [in other words $d_2 = 0$], they are contradictory.

If two neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have the semantic degree of contradiction d_1, and the neutrosophic truth value degree of contradiction d_2, then the total degree of contradiction of the two neutrosophic axioms is $d = |d_1 - d_2|$, where $|\ |$ mean the absolute value.

We did not manage to design a formula in order to compute the semantic degree of contradiction d_1 of two neutrosophic axioms. The reader is invited to explore such metric.

But we can compute the neutrosophic truth value degree of contradiction d_2. If $\langle t_1, i_1, f_1\rangle$ is the neutrosophic truth-value of $\mathcal{A}_1$ and $\langle t_2, i_2, f_2\rangle$ the neutrosophic truth-value of $\mathcal{A}_2$, where $t_1, i_1, f_1, t_2, i_2, f_2$ are single values in $[0, 1]$, then the neutrosophic truth value degree of contradiction d_2 of the neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ is:

$$d_2 = \frac{1}{3}(|t_1 - t_2| + |i_1 - i_2| + |f_1 - f_2|), \tag{65}$$

whence $d_2 \in [0, 1]$.

We get $d_2 = 0$, when $\mathcal{A}_1$ is identical with $\mathcal{A}_2$ from the point of view of neutrosophical truth values, i.e. when $t_1 = t_2, i_1 = i_2, f_1 = f_2$. And we get $d_2 = 1$, when $\langle t_1, i_1, f_1\rangle$ and $\langle t_2, i_2, f_2\rangle$ are respectively equal to:

$\langle 1, 0, 0\rangle$, $\langle 0, 1, 1\rangle$;

or $\langle 0, 1, 0\rangle$, $\langle 1, 0, 1\rangle$;

or $\langle 0, 0, 1\rangle$, $\langle 1, 1, 0\rangle$;

or $\langle 0, 0, 0\rangle$, $\langle 1, 1, 1\rangle$.

23 Neutrosophic Axiomatic System

The neutrosophic axioms are used, in neutrosophic conjunction, in order to derive neutrosophic theorems.

A *neutrosophic mathematical theory* may consist of a neutrosophic space where a neutrosophic axiomatic system acts and produces all neutrosophic theorems within the theory.

Yet, in a *neutrosophic formal system*, in general, the more recurrences are done the more is increased the indeterminacy and decreased the accuracy.

24 Properties of the Neutrosophic Axiomatic System

[1] While in classical mathematics an axiomatic system is consistent, in a neutrosophic axiomatic system it happens to have *partially inconsistent (contradictory) axioms*.

[2] Similarly, while in classical mathematics the axioms are independent, in a neutrosophic axiomatic system they may be *dependent in certain degree*.

[3] In classical mathematics if an axiom is dependent from other axioms, it can be removed, without affecting the axiomatic system.

[4] However, if a neutrosophic axiom is partially dependent from other neutrosophic axioms, by removing it the neutrosophic axiomatic system is affected.

[5] While, again, in classical mathematics an axiomatic system has to be complete (meaning that each statement or its negation is derivable), a neutrosophic axiomatic system is *partially complete and partially incomplete*. It is partially incomplete because one can add extra partially independent neutrosophic axioms.

[6] The *neutrosophic relative consistency* of an axiomatic system is referred to the neutrosophically (partially) undefined terms of a first neutrosophic axiomatic system that are assigned neutrosophic definitions from another neutrosophic axiomatic system in a way that, with respect to both neutrosophic axiomatic systems, is neutrosophically consistent.

25 Neutrosophic Model

A *Neutrosophic Model* is a model that assigns neutrosophic meaning to the neutrosophically (un)defined terms of a neutrosophic axiomatic system.

Similarly to the classical model, we have the following classification:

[1] *Neutrosophic Abstract Model*, which is a neutrosophic model based on another neutrosophic axiomatic system.

[2] *Neutrosophic Concrete Model*, which is a neutrosophic model based on real world, i.e. using real objects and real relations between the objects.

In general, a neutrosophic model is a *<t, i, f>-approximation*, i.e. *T%* of accuracy, *I%* indeterminacy, and *F%* inaccuracy, of a neutrosophic axiomatic system.

26 Neutrosophically Isomorphic Models

Further, *two neutrosophic models are neutrosophically isomorphic* if there is a neutrosophic one-to-one correspondence between their neutrosophic elements such that their neutrosophic relationships hold.

A neutrosophic axiomatic system is called *neutrosophically categorial* (or *categorical*) is any two of its neutrosophic models are neutrosophically isomorphic.

27 Neutrosophic Infinite Regressions

There may be situations of neutrosophic axiomatic systems that generate neutrosophic infinite regressions, unlike the classical axiomatic systems.

28 Neutrosophic Axiomatization

A *Neutrosophic Axiomatization* is referred to an approximate formulation of a set of neutrosophic statements, about a number of neutrosophic primitive terms, such that by the neutrosophic deduction one obtains various neutrosophic propositions (theorems).

29 Example of Neutrosophic Axiomatic System

Let's consider two neighboring countries M and N that have a disputed frontier zone Z:

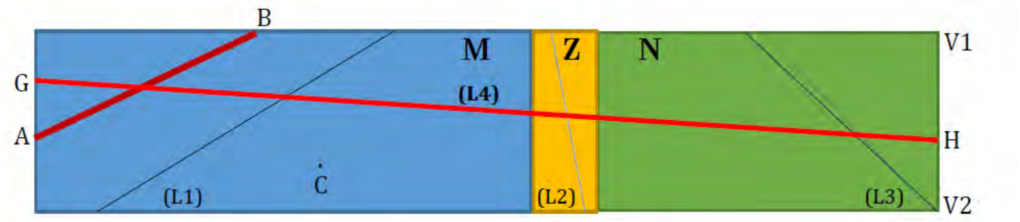

Figure 1: A Neutrosophic Model.

Let's consider the universe of discourse $U = M \cup Z \cup N$; this is a *neutrosophic space* since it has an indeterminate part (the disputed frontier).

The *neutrosophic primitive notions* in this example are: neutrosophic point, neutrosophic line, and neutrosophic plane (space).

And the *neutrosophic primitive relations* are: neutrosophic incidence, and neutrosophic parallel.

The four boundary edges of rectangle Z belong to Z (or Z is a closed set). While only three boundary edges of M (except the fourth one which is common with Z) belong to M, and similarly only three boundaries of N (except the fourth one which is common with Z) belong to N. Therefore M and N are neither closed nor open sets.

Taking a classical point P in U, one has three possibilities:

[1] $P \in M$ (membership with respect to country M);
[2] $P \in Z$ (indeterminate membership with respect to both countries);
[3] or $P \in N$ (nonmembership with respect to country M).

Such points, that can be indeterminate as well, are called *neutrosophic points*.

A *neutrosophic line* is a classical segment of line that unites two neutrosophic points lying on opposite edges of the universe of discourse U. We may have:

[1] determinate line (with respect to country M), that is completely into the determinate part M {for example ($L1$)};
[2] indeterminate line, that is completely into the frontier zone {for example ($L2$)};
[3] determinate line (with respect to country N), that is completely into the determinate part N {for example ($L3$)};

[4] or mixed, i.e. either two or three of the following: partially determinate with respect to *M*, partially indeterminate with respect to both countries, and partially determinate with respect to *N* {for example the red line *(L4)*}.

Through two neutrosophic points there may be passing:

[1] only one neutrosophic line {for example, through *G* and *H* passes only one neutrosophic line (*L4*)};
[2] no neutrosophic line {for example, through *A* and *B* passes no neutrosophic line, since the classical segment of line *AB* does not unite points of opposite edges of the universe of discourse *U*}.

Two *neutrosophic lines* are *parallel* is they have no common neutrosophic points.

Through a neutrosophic point outside of a neutrosophic line, one can draw:

[1] infinitely many neutrosophic parallels {for example, through the neutrosophic point *C* one can draw infinitely many neutrosophic parallels to the neutrosophic line *(L1)*};
[2] only one neutrosophic parallel {for example, through the neutrosophic point *H* that belongs to the edge *(V1V2)* one can draw only one neutrosophic parallel (i.e. V1V2) to the neutrosophic line *(L1)*};
[3] no neutrosophic parallel {for example, through the neutrosophic point *H* there is no neutrosophic parallel to the neutrosophic line *(L3)*}.

For example, the neutrosophic lines (*L1), (L2)* and (*L3)* are parallel. But the neutrosophic line *(L4)* is not parallel with *(L1),* nor with *(L2)* or *(L3)*.

A *neutrosophic polygon* is a classical polygon which has one or more of the following indeterminacies:

[1] indeterminate vertex;
[2] partially or totally indeterminate edge;
[3] partially or totally indeterminate region in the interior of the polygon.

We may construct several neutrosophic axiomatic systems, for this example, referring to incidence and parallel.

a) First neutrosophic axiomatic system

α1) Through two distinct neutrosophic points there is passing a single neutrosophic line.

{According to several experts, the neutrosophic truth-value of this axiom is *<0.6, 0.1, 0.2>,* meaning that having two given neutrosophic points, the chance that only one line (that do not intersect the indeterminate zone *Z*) passes through them is *0.6,* the chance that line that passes through them intersects the indeterminate zone *Z*) is *0.1*, and the chance that no line (that does not intersect the indeterminate zone *Z*) passes through them is *0.2*.}

α2) Through a neutrosophic point exterior to a neutrosophic line there is passing either one neutrosophic parallel or infinitely many neutrosophic parallels.

{According to several experts, the neutrosophic truth-value of this axiom is *<0.7, 0.2, 0.3>,* meaning that having a given neutrosophic line and a given exterior neutrosophic point, the chance that infinitely many parallels pass through this exterior point is *0.7*, the chance that the parallels passing through this exterior point intersect the indeterminate zone *Z* is *0.2*, and the chance that no parallel passes through this point is *0.3.*}

Now, let's apply a first neutrosophic deducibility.

Suppose one has three non-collinear neutrosophic (distinct) points *P, Q,* and *R* (meaning points not on the same line, alike in classical geometry). According to the neutrosophic axiom *(α1),* through *P, Q* passes only one neutrosophic line {let's call it *(PQ)*}, with a neutrosophic truth value *(0.6, 0.1, 0.2).* Now, according to axiom *(α2)*, through the neutrosophic point *R,* which does not lie on *(PQ),* there is passing either only one neutrosophic parallel or infinitely many neutrosophic parallels to the neutrosophic line (PQ), with a neutrosophic truth value *(0.7, 0.2, 0.3).*

Therefore,

$$(\alpha 1) \underset{N}{\wedge} (\alpha 2) = <0.6,\ 0.1,\ 0.2> \underset{N}{\wedge} <0.7,\ 0.2,\ 0.3> = <\min\{0.6,\ 0.7\}, \max\{0.1,\ 0.2\}, \max\{0.2,\ 0.3\}> = <0.6,\ 0.2,\ 0.3>, \quad (66)$$

which means the following: the chance that through the two distinct given neutrosophic points *P* and *Q* passes only one neutrosophic line, and through the exterior neutrosophic point *R* passese either only one neutrosophic parallel or infinitely many parallels to *(PQ)* is *(0.6, 0.2, 0.3),* i.e. *60%* true, *20%* indeterminate, and *30%* false.

Herein we have used the simplest neutrosophic conjunction operator $\underset{N}{\wedge}$ of the form <*min, max, max*>, but other neutrosophic conjunction operator can be used as well.

A second neutrosophic deducibility:

Again, suppose one has three non-collinear neutrosophic (distinct) points *P, Q,* and *R* (meaning points not on the same line, as in classical geometry).

Now, let's compute the neutrosophic truth value that through *P* and *Q* is passing one neutrosophic line, but through *Q* there is no neutrosophic parallel to *(PQ)*.

$$\alpha 1 \underset{N}{\wedge} (\underset{N}{\neg} \alpha 2) = <0.6, 0.1, 0.2> \underset{N}{\wedge} (\underset{N}{\neg} <0.7, 0.2, 0.3>) = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.3, 0.2, 0.7> = <0.3, 0.2, 0.7>. \quad (67)$$

b) Second neutrosophic axiomatic system

β1) Through two distinct neutrosophic points there is passing either a single neutrosophic line or no neutrosophic line. {With the neutrosophic truth-value *<0.8, 0.1, 0.0>*}.

β2) Through a neutrosophic point exterior to a neutrosophic line there is passing either one neutrosophic parallel, or infinitely many neutrosophic parallels, or no neutrosophic parallel. {With the neutrosophic truth-value *<1.0, 0.2, 0.0>*}.

In this neutrosophic axiomatic system the above propositions W1 and W2:

W1: Through two given neutrosophic points there is passing only one neutrosophic line, and through a neutrosophic point exterior to this neutrosophic line there is passing either one neutrosophic parallel or infinitely many neutrosophic parallels to the given neutrosophic line; and W2: Through two given neutrosophic points there is passing only one neutrosophic line, and through a neutrosophic point exterior to this neutrosophic line there is passing no neutrosophic parallel to the line; are not deducible.

c) Third neutrosophic axiomatic system

γ1) Through two distinct neutrosophic points there is passing a single neutrosophic line.

{With the neutrosophic truth-value *<0.6, 0.1, 0.2>*}.

γ2) Through two distinct neutrosophic points there is passing no neutrosophic line.

{With the neutrosophic truth-value *<0.2, 0.1, 0.6>*}.

δ1) Through a neutrosophic point exterior to a neutrosophic line there is passing only one neutrosophic parallel.

{With the neutrosophic truth-value *<0.1, 0.2, 0.9>*}.

δ2) Through a neutrosophic point exterior to a neutrosophic line there are passing infinitely many neutrosophic parallels.

{With the neutrosophic truth-value *<0.6, 0.2, 0.4>*}.

δ3) Through a neutrosophic point exterior to a neutrosophic line there is passing no neutrosophic parallel.

{With the neutrosophic truth-value *<0.3, 0.2, 0.7>*}.

In this neutrosophic axiomatic system we have contradictory axioms:

- *(γ1)* is in *100%* degree of contradiction with *(γ2);*
- and similarly *(δ3)* is in *100%* degree of contradiction with *[(δ1)* together with *(δ2)].*

Totally or partially contradictory axioms are allowed in a neutrosophic axiomatic systems, since they are part of our imperfect world and since they approximately describe models that are - in general - partially true.

Regarding the previous two neutrosophic deducibilities one has: (68)

$$\gamma 1 \underset{N}{\wedge} (\delta 1 \underset{N}{\vee} \delta 2) = <0.6, 0.1, 0.2> \underset{N}{\wedge} \left(<0.1, 0.2, 0.9> \underset{N}{\vee} <0.6, 0.2, 0.4>\right) = <0.6, 0.1, 0.2> \underset{N}{\wedge} <\max\{0.1, 0.6\}, \min\{0.2, 0.2\}, \min\{0.9, 0.4\}> = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.6, 0.2, 0.4> = <0.6, 0.2, 0.4>,$$

which is slightly different from the result we got using the first neutrosophic axiomatic system *<0.6, 0.2, 0.3>*, and respectively:

$$\gamma 1 \underset{N}{\wedge} \delta 3 = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.3, 0.2, 0.7> = <0.3, 0.2, 0.7>, \quad (69)$$

which is the same as the result we got using the first neutrosophic axiomatic system.

The third neutrosophic axiomatic system is a refinement of the first and second neutrosophic axiomatic systems. From a deducibility point of view it is better and easier to work with a refined system than with a rough system.

30 Conclusion

This paper proposes a new framework to model interdependencies in project portfolio. NCM representation model is used for modeling relation among risks.

In many real world situations, the spaces and laws are not exact, not perfect. They are inter-dependent. This means that in most cases they are not 100% true, i.e. not universal. For example, many physical laws are valid in ideal and perfectly closed systems. However, perfectly closed systems do not exist in our heterogeneous world where we mostly deal with approximations. Also, since in the real world there is not a single homogenous space, we have to use the multispace for any attempt to unify various theories.

We do not have perfect spaces and perfect systems in reality. Therefore, many physical laws function approximatively (see [5]). The physical constants are not universal too; variations of their values depend from a space to another, from a system to another. A physical constant is t% true, i% indeterminate, and f% false in a given space with a certain composition, and it has a different neutrosophical truth value <t’, i’, f’> in another space with another composition.

A neutrosophic axiomatic system may be dynamic: new axioms can be added and others excluded.

The neutrosophic axiomatic systems are formed by axioms than can be partially dependent (redundant), partially contradictory (inconsistent), partially incomplete, and reflecting a partial truth (and consequently a partial indeterminacy and a partial falsehood) - since they deal with approximations of reality.

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Neutrosophic Actions, Prevalence Order, Refinement of Neutrosophic Entities, and Neutrosophic Literal Logical Operators

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Abstract

In this paper we define for the first time three neutrosophic actions and their properties. We then introduce the prevalence order on $\{T, I, F\}$ with respect to a given neutrosophic operator "o", which may be subjective - as defined by the neutrosophic experts. And the refinement of neutrosophic entities <A>, <neutA>, and <antiA>. Then we extend the classical logical operators to neutrosophic literal logical operators and to refined literal logical operators, and we define the refinement neutrosophic literal space.

Keywords

neutrosophy, neutrosophics, neutrosophic actions, prevalence order, neutrosophic operator, refinement of neutrosophic entities, neutrosophic literal logical operators, refined literal logical operators, refinement neutrosophic literal space.

1 Introduction

In Boolean Logic, a proposition $\mathcal{P}$ is either true (T), or false (F). In Neutrosophic Logic, a proposition $\mathcal{P}$ is either true (T), false (F), or indeterminate (I).

For example, in Boolean Logic the proposition $\mathcal{P}_1$:

"1+1=2 (in base 10)"

is true, while the proposition $\mathcal{P}_2$:

"1+1=3 (in base 10)"

is false.

In neutrosophic logic, besides propositions $\mathcal{P}_1$ (which is true) and $\mathcal{P}_2$ (which is false), we may also have proposition $\mathcal{P}_3$:

"1+1= ?(in base 10)",

which is an incomplete/indeterminate proposition (neither true, nor false).

1.1 Remark

All conjectures in science are indeterminate at the beginning (researchers not knowing if they are true or false), and later they are proved as being either true, or false, or indeterminate in the case they were unclearly formulated.

2 Notations

In order to avoid confusions regarding the operators, we note them as:

Boolean (classical) logic:

$$\neg, \quad \wedge, \quad \vee, \quad \underline{\vee}, \quad \rightarrow, \quad \leftrightarrow$$

Fuzzy logic:

$$\underset{F}{\neg}, \quad \underset{F}{\wedge}, \quad \underset{F}{\vee}, \quad \underset{F}{\underline{\vee}}, \quad \underset{F}{\rightarrow}, \quad \underset{F}{\leftrightarrow}$$

Neutrosophic logic:

$$\underset{N}{\neg}, \quad \underset{N}{\wedge}, \quad \underset{N}{\vee}, \quad \underset{N}{\underline{\vee}}, \quad \underset{N}{\rightarrow}, \quad \underset{N}{\leftrightarrow}$$

3 Three Neutrosophic Actions

In the frame of neutrosophy, we have considered [1995] for each entity ⟨A⟩, its opposite ⟨antiA⟩, and their neutrality ⟨neutA⟩ {i.e. neither ⟨A⟩, nor ⟨antiA⟩}. Also, by ⟨nonA⟩ we mean what is not ⟨A⟩, i.e. its opposite ⟨antiA⟩, together with its neutral(ity) ⟨neutA⟩; therefore:

$$\langle \text{non}A\rangle = \langle \text{neut}A\rangle \vee \langle \text{anti}A\rangle.$$

Based on these, we may straightforwardly introduce for the first time the following neutrosophic actions with respect to an entity <A>:

1. To neutralize (or to neuter, or simply to neut-ize) the entity <A>. [As a noun: neutralization, or neuter-ization, or simply neut-ization.] We denote it by <neutA> or neut(A).

2. To antithetic-ize (or to anti-ize) the entity <A>. [As a noun: antithetic-ization, or anti-ization.] We denote it by <antiA> ot anti(A).

This action is 100% opposition to entity <A> (strong opposition, or strong negation).

3. To non-ize the entity <A>. [As a noun: non-ization]. We denote it by <nonA> or non(A).
It is an opposition in a percentage between (0, 100]% to entity <A> (weak opposition).

Of course, not all entities <A> can be neutralized, or antithetic-ized, or non-ized.

3.1 Example

Let

⟨A⟩="Phoenix Cardinals beats Texas Cowboys".

Then,

⟨neutA⟩="\"Phoenix Cardinals has a tie game with Texas Cowboys\"";

⟨antiA⟩="\"Phoenix Cardinals is beaten by Texas Cowboys\"";

⟨nonA⟩="\"Phoenix Cardinals has a tie game with Texas Cowboys,"

"or Phoenix Cardinals is beaten by Texas Cowboys\"."

3.2 Properties of the Three Neutrosophic Actions

$$\text{neut}(\langle \text{anti}A \rangle) = \text{neut}(\langle \text{neutA} \rangle) = \text{neut}(A);$$

$$\text{anti}(\langle \text{anti}A \rangle) = A;\ \text{anti}(\langle \text{neut}A \rangle) = \langle A \rangle \text{ or } \langle \text{anti}A \rangle;$$

$$\text{non}(\langle \text{anti}A \rangle) = \langle A \rangle \text{ or } \langle \text{neut}A \rangle;\ \text{non}(\langle \text{neut}A \rangle) = \langle A \rangle \text{ or } \langle \text{anti}A \rangle.$$

4 Neutrosophic Actions' Truth-Value Tables

Let's have a logical proposition P, which may be true (T), Indeterminate (I), or false (F) as in previous example. One applies the neutrosophic actions below.

4.1 Neutralization (or Indetermination) of P

neut(P)	T	I	F
	I	*I*	*I*

4.2 Antitheticization (Neutrosophic Strong Opposition to P)

anti(P)	T	I	F
	F	$T \vee F$	T

4.3 Non-ization (Neutrosophic Weak Opposition to P):

non(P)	T	I	F
	$I \vee F$	$T \vee F$	$T \vee I$

5 Refinement of Entities in Neutrosophy

In neutrosophy, an entity ⟨A⟩ has an opposite ⟨antiA⟩ and a neutral ⟨neutA⟩. But these three categories can be refined in sub-entities $\langle A\rangle_1, \langle A\rangle_2, \dots, \langle A\rangle_m$, and respectively $\langle \text{neut}A\rangle_1, \langle \text{neut}A\rangle_2, \dots, \langle \text{neut}A\rangle_n$, and also $\langle \text{anti}A\rangle_1$, $\langle \text{anti}A\rangle_2, \dots, \langle \text{anti}A\rangle_p$, where m, n, p are integers ≥1, but $m + n + p \geq 4$ (meaning that at least one of ⟨A⟩, ⟨antiA⟩ or ⟨neutA⟩ is refined in two or more sub-entities).

For example, if ⟨A⟩=white color, then

⟨antiA⟩=black color,

while ⟨neutA⟩=colors different from white and black.

If we refine them, we get various nuances of white color: $\langle A\rangle_1, \langle A\rangle_2, \dots$, and various nuances of black color: $\langle \text{anti}A\rangle_1, \langle \text{anti}A\rangle_2, \dots$, and the colors in between them (red, green, yellow, blue, etc.): $\langle \text{neut}A\rangle_1, \langle \text{neut}A\rangle_2, \dots$.

Similarly as above, we want to point out that not all entities <A> and/or their corresponding (if any) <neutA> and <antiA> can be refined.

6 The Prevalence Order

Let's consider the classical literal (symbolic) truth (T) and falsehood (F).

In a similar way, for neutrosophic operators we may consider the literal (symbolic) truth (T), the literal (symbolic) indeterminacy (I), and the literal (symbolic) falsehood (F).

We also introduce the prevalence order on $\{T, I, F\}$ with respect to a given binary and commutative neutrosophic operator "o".

The neutrosophic operators are: neutrosophic negation, neutrosophic conjunction, neutrosophic disjunction, neutrosophic exclusive disjunction, neutrosophic Sheffer's stroke, neutrosophic implication, neutrosophic equivalence, etc.

The prevalence order is partially objective (following the classical logic for the relationship between T and F), and partially subjective (when the indeterminacy I interferes with itself or with T or F).

For its subjective part, the prevalence order is determined by the neutrosophic logic expert in terms of the application/problem to solve, and also depending on the specific conditions of the application/problem.

For $X \neq Y$, we write $XⓅY$, or $X \succ_o Y$, and we read X prevails to Y with respect to the neutrosophic binary commutative operator "o", which means that $XoY = X$.

Let's see the below examples. We mean by "o": conjunction, disjunction, exclusive disjunction, Sheffer's stroke, and equivalence.

7 Neutrosophic Literal Operators & Neutrosophic Numerical Operators

7.1 If we mean by neutrosophic literal proposition, a proposition whose truth value is a letter: either T or I or F. The operators that deal with such logical propositions are called neutrosophic literal operators.

7.2 And by neutrosophic numerical proposition, a proposition whose truth value is a triple of numbers (or in general of numerical subsets of the interval [0, 1]), for examples A(0.6, 0.1, 0.4) or B([0, 0.2], {0.3, 0.4, 0.6}, (0.7, 0.8)). The operators that deal with such logical propositions are called neutrosophic numerical operators.

8 Truth-Value Tables of Neutrosophic Literal Operators

In Boolean Logic, one has the following truth-value table for negation:

8.1 Classical Negation

¬	T	F
	F	T

In Neutrosophic Logic, one has the following neutrosophic truth-value table for the neutrosophic negation:

8.2 Neutrosophic Negation

$\neg_N$	T	I	F
	F	I	T

So, we have to consider that the negation of I is I, while the negations of T and F are similar as in classical logic.

In classical logic, one has:

8.3 Classical Conjunction

∧	T	F
T	T	F
F	F	F

In neutrosophic logic, one has:

8.4 Neutrosophic Conjunction (AND_N), version 1

$\wedge_N$	T	I	F
T	T	I	F
I	I	I	I
F	F	I	F

The objective part (circled literal components in the above table) remains as in classical logic, but when indeterminacy I interferes, the neutrosophic expert may choose the most fit prevalence order.

There are also cases when the expert may choose, for various reasons, to entangle the classical logic in the objective part. In this case, the prevalence order will be totally subjective.

The prevalence order works for classical logic too. As an example, for classical conjunction, one has $F \succ_c T$, which means that $F \wedge T = F$. While the prevalence order for the neutrosophic conjunction in the above tables was:

$$I \succ_c F \succ_c T,$$

which means that $I \wedge_N F = I$, and $I \wedge_N T = I$.

Other prevalence orders can be used herein, such as:

$$F \succ_c I \succ_c T,$$

and its corresponding table would be:

8.5 Neutrosophic Conjunction (AND_N), version 2

$\wedge_N$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

which means that $F_{\wedge_N} I = F$ and $I_{\wedge_N} I = I$; or another prevalence order:

$$F \succ_c T \succ_c I,$$

and its corresponging table would be:

8.6 Neutrosophic Conjunction (AND_N), version 3

$\wedge_N$	T	I	F
T	T	T	F
I	T	I	F
F	F	F	F

which means that $F_{\wedge_N} I = F$ and $T_{\wedge_N} I = T$.

If one compares the three versions of the neutrosophic literal conjunction, one observes that the objective part remains the same, but the subjective part changes.

The subjective of the prevalence order can be established in an optimistic way, or pessimistic way, or according to the weights assigned to the neutrosophic literal components *T, I, F* by the experts.

In a similar way, we do for disjunction. In classical logic, one has:

8.7 Classical Disjunction

∨	T	F
T	T	T
F	T	F

In neutrosophic logic, one has:

8.8 Classical Disjunction (OR_N)

$\vee_N$	T	I	F
T	(T)	T	(T)
I	T	I	F
F	(T)	F	(F)

where we used the following prevalence order:

$$T \succ_d F \succ_d I,$$

but the reader is invited (as an exercise) to use another prevalence order, such as:

$$T \succ_d I \succ_d F,$$

Or

$$I \succ_d T \succ_d F, \text{ etc.,}$$

for all neutrosophic logical operators presented above and below in this paper.

In classical logic, one has:

8.9 Classical Exclusive Disjunction

$\underline{\vee}$	T	F
T	*F*	*T*
F	*T*	*F*

In neutrosophic logic, one has:

8.10 Neutrosophic Exclusive Disjunction

$\underline{\vee}_N$	T	I	F
T	*F*	*T*	*T*
I	*T*	*I*	*F*
F	*T*	*F*	*F*

using the prevalence order

$$T \succ_d F \succ_d I.$$

In classical logic, one has:

8.11 Classical Sheffer's Stroke

|	T	F
T	*F*	*T*
F	*T*	*T*

In neutrosophic logic, one has:

8.12 Neutrosophic Sheffer's Stroke

$\|_N$	T	I	F
T	*F*	*T*	*T*
I	*T*	*I*	*I*
F	*T*	*I*	*T*

using the prevalence order

$$T \succ_d I \succ_d F.$$

In classical logic, one has:

8.13 Classical Implication

$\rightarrow$	T	F
T	T	F
F	T	T

In neutrosophic logic, one has:

8.14 Neutrosophic Implication

$\rightarrow_N$	T	I	F
T	T (circled)	I	F (circled)
I	T	T	F
F	T (circled)	T	T (circled)

using the subjective preference that $I \rightarrow_N T$ is true (because in the classical implication T is implied by anything), and $I \rightarrow_N F$ is false, while $I \rightarrow_N I$ is true because is similar to the classical implications $T \rightarrow T$ and $F \rightarrow F$, which are true.

The reader is free to check different subjective preferences.

In classical logic, one has:

8.15 Classical Equivalence

$\leftrightarrow$	T	F
T	T	F
F	F	T

In neutrosophic logic, one has:

8.15 Neutrosophic Equivalence

$\leftrightarrow_N$	T	I	F
T	T	I	F
I	I	T	I
F	F	I	T

using the subjective preference that $I \leftrightarrow_N I$ is true, because it is similar to the classical equivalences that $T \to T$ and $F \to F$ are true, and also using the prevalence:

$$I \succ_e F \succ_e T.$$

9 Refined Neutrosophic Literal Logic

Each particular case has to be treated individually.

In this paper, we present a simple example. Let's consider the following neutrosophic logical propositions:

T = Tomorrow it will rain or snow.
T is split into
→ Tomorrow it will rain.
→ Tomorrow it will snow.
F = Tomorrow it will neither rain nor snow.
F is split into
→ Tomorrow it will not rain.
→ Tomorrow it will not snow.
I = Do not know if tomorrow it will be raining, nor if it will be snowing.
I is split into
→ Do not know if tomorrow it will be raining or not.
→ Do not know if tomorrow it will be snowing or not.

Then:

$\neg_N$	T_1	T_2	I_1	I_2	F_1	F_2
	F_1	F_2	$T_1 \vee F_1$	$T_2 \vee F_2$	T_1	T_2

It is clear that the negation of T_1 (Tomorrow it will raining) is F_1 (Tomorrow it will not be raining). Similarly for the negation of T_2, which is F_2.

But, the negation of I_1 (Do not know if tomorrow it will be raining or not) is "Do know if tomorrow it will be raining or not", which is equivalent to "We know that tomorrow it will be raining" (T_1), or "We know that tomorrow it will not be raining" (F_1).

Whence, the negation of I_1 is $T_1 \vee F_1$, and similarly, the negation of I_2 is $T_2 \vee F_2$.

9.1 Refined Neutrosophic Literal Conjunction Operator

$\wedge_N$	T_1	T_2	I_1	I_2	F_1	F_2
T_1	T_1	$T_{1\,2}$	I_1	I_2	F_1	F_2
T_2	$T_{1\,2}$	T_2	I_1	I_2	F_1	F_2
I_1	I_1	I_1	I_1	I	F_1	F_2
I_2	I_2	I_2	I	I_2	F_1	F_2
F_1	F_1	F_1	F_1	F_1	F_1	F
F_2	F_2	F_2	F_2	F_2	F	F_2

where $T_{1\,2} = T_1 \wedge T_2 =$ "Tomorrow it will rain and it will snow".

Of course, other prevalence orders can be studied for this particular example.

With respect to the neutrosophic conjunction, F_l prevail in front of I_k, which prevail in front of T_j, or $F_l \succ I_k \succ T_j$, for all $l, k, j \in \{1, 2\}$.

9.2 Refined Neutrosophic Literal Disjunction Operator

$\vee_N$	T_1	T_2	I_1	I_2	F_1	F_2
T_1	T_1	T	T_1	T_1	T_1	T_1
T_2	T	T_2	T_2	T_2	T_2	T_2
I_1	T_1	T_2	I_1	I	F_1	F_2
I_2	T_1	T_2	I	I_2	F_1	F_2
F_1	T_1	T_2	F_1	F_1	F_1	$F_1 \vee F_2$
F_2	T_1	T_2	F_2	F_2	$F_1 \vee F_2$	F_2

With respect to the neutrosophic disjunction, T_j prevail in front of F_l, which prevail in front of I_k, or $T_j \succ F_l \succ I_k$, for all $j, l, k \in \{1, 2\}$.

For example, $T_1 \vee T_2 = T$, but $F_1 \vee F_2 \notin \{T, I\ F\} \cup \{T_1, \mathrm{T}_2, I_1, \mathrm{I}_2, F_1, \mathrm{F}_2\}$.

9.3 Refined Neutrosophic Literal Space

The Refinement Neutrosophic Literal Space $\{T_1, T_2, I_1, I_2, F_1, F_2\}$ is not closed under neutrosophic negation, neutrosophic conjunction, and neutrosophic disjunction. The reader can check the closeness under other neutrosophic literal operations.

A neutrosophic refined literal space

$$S_N = \{T_1, T_2, \dots, T_p;\ I_1, I_2, \dots, I_r;\ F_1, F_2, \dots, F_s\},$$

where p, r, s are integers ≥ 1, is said to be closed under a given neutrosophic operator "θ_N", if for any elements $X, Y \in S_N$ one has $X_{\theta_N} Y \in S_N$.

Let's denote the extension of S_N with respect to a single θ_N by:

$$S_{N_1}^C = (S_N, \theta_N).$$

If S_N is not closed with respect to the given neutrosophic operator θ_N, then $S_{N_1}^C \neq S_N$, and we extend S_N by adding in the new elements resulted from the operation $X\theta_N Y$, let's denote them by $A_1, A_2, \dots A_m$.

Therefore,

$$S_{N_1}^C \neq S_N \cup \{A_1, A_2, \dots A_m\}.$$

$S_{N_1}^C$ encloses S_N.

Similarly, we can define the closeness of the neutrosophic refined literal space S_N with respect to the two or more neutrosophic operators $\theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}$, for $w \geq 2$.

S_N is closed under $\theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}$ if for any $X, Y \in S_N$ and for any $i \in \{1, 2, \dots, w\}$ one has $X_{\theta_{i_N}} Y \in S_N$.

If S_N is not closed under these neutrosophic operators, one can extend it as previously.

Let's consider: $S_{N_w}^C = (S_N, \theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N})$, which is S_N closed with respect to all neutrosophic operators $\theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}$, then $S_{N_w}^C$ encloses S_N.

10 Conclusion

We have defined for the first time three neutrosophic actions and their properties. We have introduced the prevalence order on $\{T, I, F\}$ with respect to a given neutrosophic operator "o", the refinement of neutrosophic entities <A>, <neutA>, and <antiA>, and the neutrosophic literal logical operators and refined literal logical operators, and the refinement neutrosophic literal space.

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An Introduction to Bipolar Single Valued Neutrosophic Graph Theory

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Keywords: Single valued neutrosophic set, bipolar neutrosophic set, single valued neutrosophic graph, bipolar single valued neutrosophic graphs.

Abstract. In this paper, we first define the concept of bipolar single neutrosophic graphs as the generalization of bipolar fuzzy graphs, N-graphs, intuitionistic fuzzy graph, single valued neutrosophic graphs and bipolar intuitionistic fuzzy graphs.

1. Introduction

Zadeh [9] coined the term 'degree of membership' and defined the concept of fuzzy set in order to deal with uncertainty. Atanassov [8] incorporated the degree of non-membership in the concept of fuzzy set as an independent component and defined the concept of intuitionistic fuzzy set. Smarandache [2] grounded the term 'degree of indeterminacy' as an independent component and defined the concept of neutrosophic set from the philosophical point of view to deal with incomplete, indeterminate and inconsistent information in real world. The concept of neutrosophic set is a generalization of the theory of fuzzy set, intuitionistic fuzzy set. Each element of a neutrosophic set has three membership degrees including a truth membership degree, an indeterminacy membership degree, and a falsity membership degree which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. Therefore, if their range is restrained within the real standard unit interval [0, 1], the neutrosophic set is easily applied to engineering problems. For this purpose, Wang et al. [6] introduced the concept of the single-valued neutrosophic set (SVNS) as a subclass of the neutrosophic set. Recently, Deli et al. [7] defined the concept of bipolar neutrosophic, as a generalization of single valued neutrosophic set, and bipolar fuzzy graph, also studying some of their related properties. The neutrosophic set theory of and their extensions have been applied in various domains [22] (refer to the site http://fs.gallup.unm.edu/NSS/).

When the relations between nodes (or vertices) in problems are indeterminate, the concept of fuzzy graphs [15] and its extensions, such as intuitionistic fuzzy graphs [11, 16], N-graphs [13], bipolar fuzzy graphs [11, 12, 14], bipolar intuitionistic fuzzy graphs [1] are not suitable. For this purpose, Smarandache [3] defined four main categories of neutrosophic graphs, two based on literal indeterminacy (I), calling them I-edge neutrosophic graph and I-vertex neutrosophic graph; these concepts are deeply studied and gained popularity among some researchers [4, 5, 19, 20, 21] due to their applications in the real world problems. The two others graphs are based on (t, i, f) components, and are called: (t, i, f)-edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph; but these new concepts are not developed at all yet. Later on, Broumi et al. [18] introduced a third neutrosophic graph model. The single valued neutrosophic graph is the generalization of fuzzy graph and intuitionstic fuzzy graph. Also, the same authors [17] introduced neighborhood degree of a vertex and closed neighborhood degree of a vertex in the single valued neutrosophic graph, as a generalization of neighborhood degree of a vertex and closed neighborhood degree of vertex in fuzzy graph and intuitionistic fuzzy graph.

In this paper, motivated by the works of Deli et al. [7] and Broumi et al. [18], we introduced the concept of bipolar single valued neutrosophic graph and proved some propositions.

2. Preliminaries

In this section, we mainly recall some notions, which we are also going to use in the rest of the paper. The readers are referred to [6, 7, 10, 11, 13, 15, 18] for further details and background.

Definition 2.1 [6]

Let U be an universe of a discourse; then, the neutrosophic set A is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ U}, where the functions T, I, F: U→]$^-0,1^+$[define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ U to the set A with the condition: $^-0 \le T_A(x)+ I_A(x)+ F_A(x) \le 3^+$.

Definition 2.2 [7]

A bipolar neutrosophic set A in X is defined as an object of the form A={<x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)>: x ∈ X}, where T^P, I^P, F^P:X→ [1, 0] and T^N, I^N, F^N: X→ [-1, 0]. The positive membership degree T^P(x), I^P(x), F^P(x) denotes the truth membership, indeterminate membership and false membership of an element ∈ X corresponding to a bipolar neutrosophic set A, and the negative membership degree T^N(x), I^N(x), F^N(x) denotes the truth membership, indeterminate membership and false membership of an element ∈ X to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Example 2.1

Let X = $\{x_1, x_2, x_3\}$;

A = $\left\{\begin{matrix} <x_1, 0.5, 0.3, 0.1, -0.6, -0.4, -0.05> \\ <x_2, 0.3, 0.2, 0.7, -0.02, -0.3, -0.02> \\ <x_3, 0.8, 0.05, 0.4, -0.6, -0.6, -0.03> \end{matrix}\right\}$ is a bipolar neutrosophic subset of X.

Definition 2.3 [7]

Let A_1 = {<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2 = {<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets. Then, $A_1 \subseteq A_2$ if and only if $T_1^P(x) \le T_2^P(x)$, $I_1^P(x) \le I_2^P(x)$, $F_1^P(x) \ge F_2^P(x)$ and $T_1^N(x) \ge T_2^N(x)$, $I_1^N(x) \ge I_2^N(x)$, $F_1^N(x) \le F_2^N(x)$ for all x ∈ X.

Definition 2.4 [15]

A fuzzy graph with V as the underlying set is a pair G = (σ, μ), where σ: V → [0, 1] is a fuzzy subset and μ: V × V → [0, 1] is a fuzzy relation on σ such that μ(x, y) ≤ σ(x) ∧ σ(y) for all x, y ∈ V where ∧ stands for minimum.

Definition 2.5 [13]

By a *N*-graph G of a graph G^*, we mean a pair G= (μ_1, μ_2) where μ_1 is an *N*-function in V and μ_2 is an *N*-relation on E such that μ_2(u,v) ≥ max (μ_1(u), μ_1(v)) all u, v ∈ V.

Definition 2.6 [10]

An intuitionistic fuzzy graph is of the form G = (V, E), where

i. V = {v_1, v_2,…., v_n} such that μ_1: V→ [0,1] and γ_1: V → [0,1] denoting the degree of membership and non-membership of the element v_i ∈ V, respectively, and 0≤ $\mu_1(v_i)$ + $\gamma_1(v_i)$) ≤ 1 for every v_i ∈ V, (i = 1, 2, ……. n), (1)

ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that $\mu_2(v_i, v_j) \le$ min [$\mu_1(v_i)$, $\mu_1(v_j)$] and $\gamma_2(v_i, v_j) \ge$ max [$\gamma_1(v_i)$, $\gamma_1(v_j)$] and $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \le 1$ for every (v_i, v_j) ∈ E, (i, j = 1,2, ……. n). (2)

Definition 2.7 [11]

Let X be a non-empty set. A bipolar fuzzy set A in X is an object having the form A = {(x, μ_A^P(x), μ_A^N(x)) | x ∈ X}, where μ_A^P(x): X → [0, 1] and μ_A^N(x): X → [−1, 0] are mappings.

Definition 2.8 [11]

A bipolar fuzzy graph of a graph G^*= (V, E) is a pair G = (A,B), where A = (μ_A^P, μ_A^N) is a bipolar fuzzy set in V and B = (μ_B^P, μ_B^N) is a bipolar fuzzy set on E $\subseteq$ V x V such that μ_B^P(xy) $\leq$ min{μ_A^P(x), μ_A^P(y)} for all xy $\in E$, μ_B^N(xy) $\geq$ min{μ_A^N(x), μ_A^N(y)} for all xy $\in E$ and μ_B^P(xy) = μ_B^N(xy) = 0 for all xy $\in \tilde{V}^2$ –E. Here A is called bipolar fuzzy vertex set of V, and B - the bipolar fuzzy edge set of E.

Definition 2.9 [18]

A single valued neutrosophic graph (SVNG) of a graph G^*= (V, E) is a pair G = (A, B), where:

i. V = {v_1, v_2,…, v_n} such that T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and 0≤ $T_A(v_i)$ + $I_A(v_i)$ +$F_A(v_i)$ ≤3 for every $v_i \in$ V (i=1, 2, …, n). (3)

ii. E $\subseteq$ V x V, where T_B:V x V →[0, 1], I_B:V x V →[0, 1] and F_B:V x V →[0, 1] are such that $T_B(v_i, v_j) \leq$ min [$T_A(v_i)$, $T_A(v_j)$], $I_B(v_i, v_j) \geq$ max [$I_A(v_i)$, $I_A(v_j)$] and $F_B(v_i, v_j) \geq$ max [$F_A(v_i)$, $F_A(v_j)$] and 0≤ $T_B(v_i, v_j)$ + $I_B(v_i, v_j)$+ $F_B(v_i, v_j)$ ≤3, for every $(v_i, v_j) \in$ E (i, j = 1, 2,…, n). (4)

3. Bipolar Single Valued Neutrosophic Graphs

In this section, we firstly define the concept of a bipolar single valued neutrosophic relation.

Definition 3.1

Let X be a non-empty set. Then we call a mapping A = (x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)):X × X → [−1, 0] × [0, 1] a bipolar single valued neutrosophic relation on X such that T_A^P(x, y) ∈ [0, 1], I_A^P(x, y) ∈ [0, 1], F_A^P(x, y) ∈ [0, 1], and T_A^N(x, y) ∈ [−1, 0], I_A^N(x, y) ∈ [−1, 0], F_A^N(x, y) ∈ [−1, 0].

Definition 3.2

Let A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) and B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) be a bipolar single valued neutrosophic graph on a set X. If B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) is a bipolar single valued neutrosophic relation on A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) then:

$T_B^P(x, y) \leq \min(T_A^P(x), T_A^P(y))$, $T_B^N(x, y) \geq \max(T_A^N(x), T_A^N(y))$ (5)

$I_B^P(x, y) \geq \max(I_A^P(x), I_A^P(y))$, $I_B^N(x, y) \leq \min(I_A^N(x), I_A^N(y))$ (6)

$F_B^P(x, y) \geq \max(F_A^P(x), F_A^P(y))$, $F_B^N(x, y) \leq \min(F_A^N(x), F_A^N(y))$, for all $x, y \in X$. (7)

A bipolar single valued neutrosophic relation B on X is called symmetric if $T_B^P(x, y) = T_B^P(y, x)$, $I_B^P(x, y) = I_B^P(y, x)$, $F_B^P(x, y) = F_B^P(y, x)$ and $T_B^N(x, y) = T_B^N(y, x)$, $I_B^N(x, y) = I_B^N(y, x)$, $F_B^N(x, y) = F_B^N(y, x)$, for all $x, y \in X$.

Definition 3.3

A bipolar single valued neutrosophic graph of a graph G^* = (V, E) is a pair G = (A, B), where A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) is a bipolar single valued neutrosophic set in V, and B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) is a bipolar single valued neutrosophic set in $\tilde{V}^2$, such that

$T_B^P(x, y) \leq \min(T_A^P(v_i), T_A^P(v_j))$, $T_B^N(x, y) \geq \max(T_A^N(v_i), T_A^N(v_j))$ (8)

$I_B^P(x, y) \geq \max(I_A^P(v_i), I_A^P(v_j))$, $I_B^N(x, y) \leq \min(I_A^N(v_i), I_A^N(v_j))$, and (9)

$F_B^P(x, y) \geq \max(F_A^P(v_i), F_A^P(v_j))$, $F_B^N(x, y) \leq \min(F_A^N(v_i), F_A^N(v_j))$, for all xy $\in \tilde{V}^2$. (10)

Notation

An edge of BSVNG is denoted by e_{ij} ∈ E or v_iv_j ∈ E.

Here, the sextuple (v_i, T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) denotes the positive degree of truth-membership, the positive degree of indeterminacy-membership, the positive degree of falsity-membership, the negative degree of truth-membership, the negative degree of indeterminacy-membership, the negative degree of falsity- membership of the vertex v_i.

The sextuple (e_{ij} , T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) denotes the positive degree of truth-membership, the positive degree of indeterminacy-membership, the positive degree of falsity-membership, the negative degree of truth-membership, the negative degree of indeterminacy-membership, the negative degree of falsity- membership of the edge relation $e_{ij} = (v_i, v_j)$ on V× V.

Notes

i. When $T_A^P = I_A^P = F_A^P = 0$ and $T_A^N = I_A^N = F_A^N = 0$ for some i and j, then there is no edge between v_i and v_j . Otherwise there exists an edge between v_i and v_j .

ii. If one of the inequalities is not satisfied, then G is not a BSVNG.

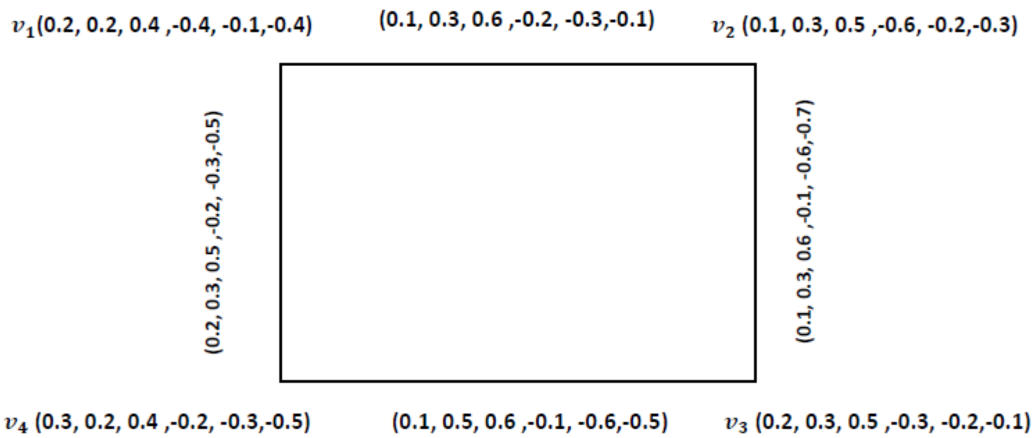

Fig.1: Bipolar single valued neutrosophic graph.

Proposition 3.1

A bipolar single valued neutrosophic graph is the generalization of the fuzzy graph.

Proof

Suppose G = (A, B) is a bipolar single valued neutrosophic graph. Then, by setting the positive indeterminacy-membership, positive falsity-membership and negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero, it reduces the bipolar single valued neutrosophic graph to a fuzzy graph.

Example 3.1

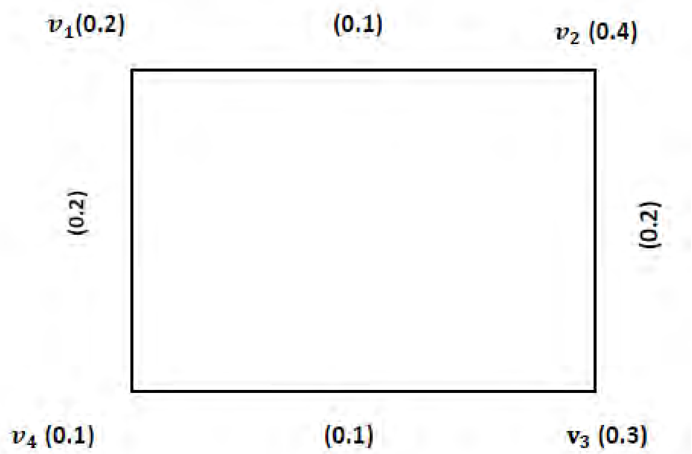

Fig. 2: Fuzzy graph

Proposition 3.2

A bipolar single valued neutrosophic graph is the generalization of the bipolar intuitionstic fuzzy graph.

Proof

Suppose G = (A, B) is a bipolar single valued neutrosophic graph. Then, by setting the positive indeterminacy-membership, negative indeterminacy-membership values of vertex set and edge set equals to zero, it reduces the bipolar single valued neutrosophic graph to a bipolar intuitionistic fuzzy graph.

Example 3.2

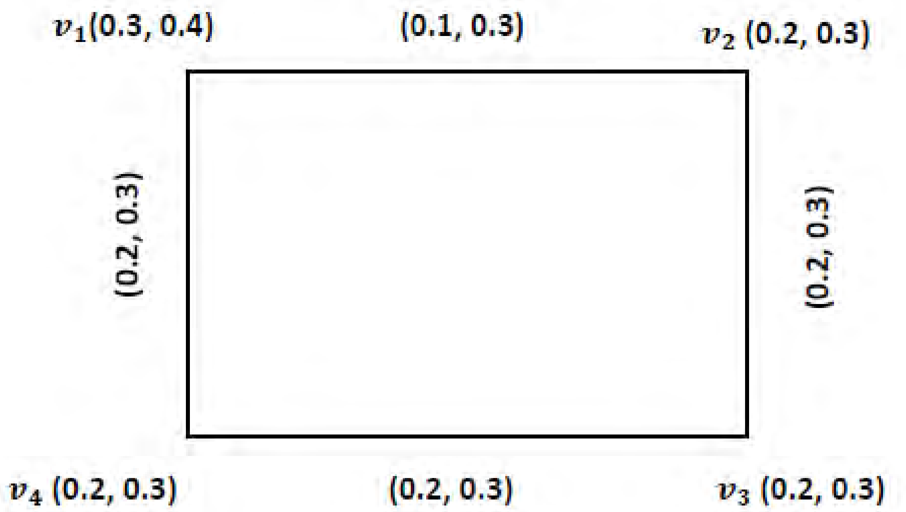

Fig.3: Intuitionistic fuzzy graph.

Proposition 3.3

A bipolar single valued neutrosophic graph is the generalization of the single valued neutrosophic graph.

Proof

Suppose G = (A, B) is a bipolar single valued neutrosophic graph. Then, by setting the negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero, it reduces the bipolar single valued neutrosophic graph to a single valued neutrosophic graph.

Example 3.3

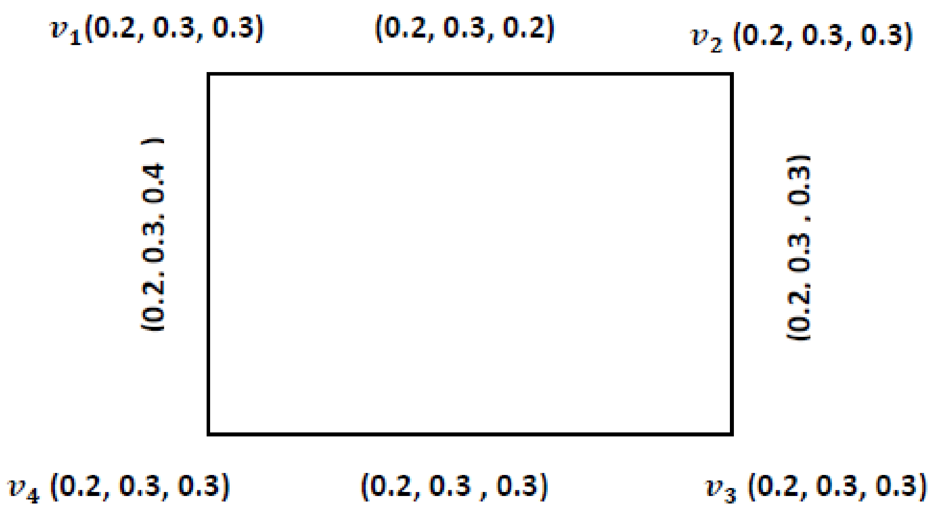

Fig. 4: Single valued neutrosophic graph.

Proposition 3.4

A bipolar single valued neutrosophic graph is the generalization of the bipolar intuitionstic fuzzy graph.

Proof

Suppose G = (A, B) is a bipolar single valued neutrosophic graph. Then, by setting the positive indeterminacy-membership, negative indeterminacy-membership values of vertex set and edge set equals to zero, it reduces the bipolar single valued neutrosophic graph to a bipolar intuitionistic fuzzy graph.

Example 3.4

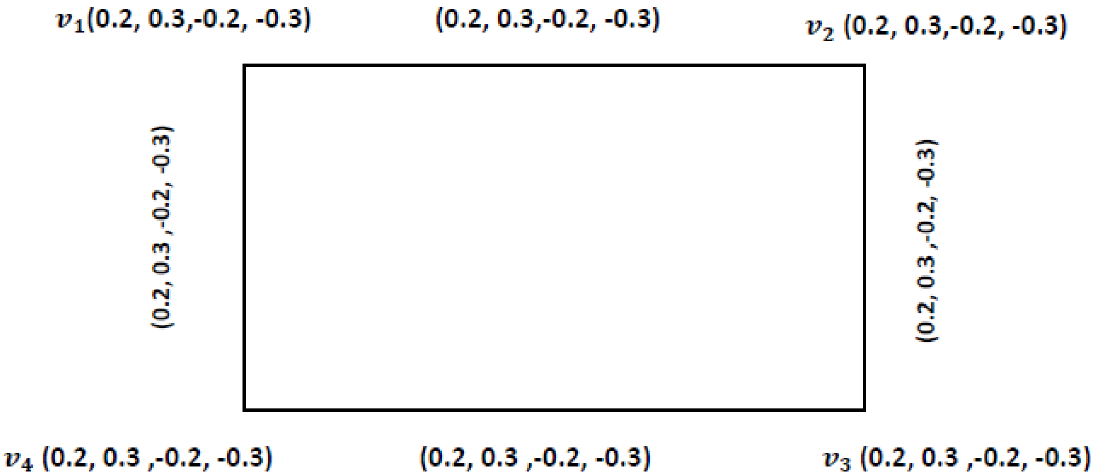

Fig.5: Bipolar intuitionistic fuzzy graph.

Proposition 3.5

A bipolar single valued neutrosophic graph is the generalization of the *N*-graph.

Proof

Suppose G = (A, B) is a bipolar single valued neutrosophic graph. Then, by setting the positive degree membership such truth-membership, indeterminacy-membership, falsity-membership and negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero, it reduces the single valued neutrosophic graph to a *N*-graph.

Example 3.5

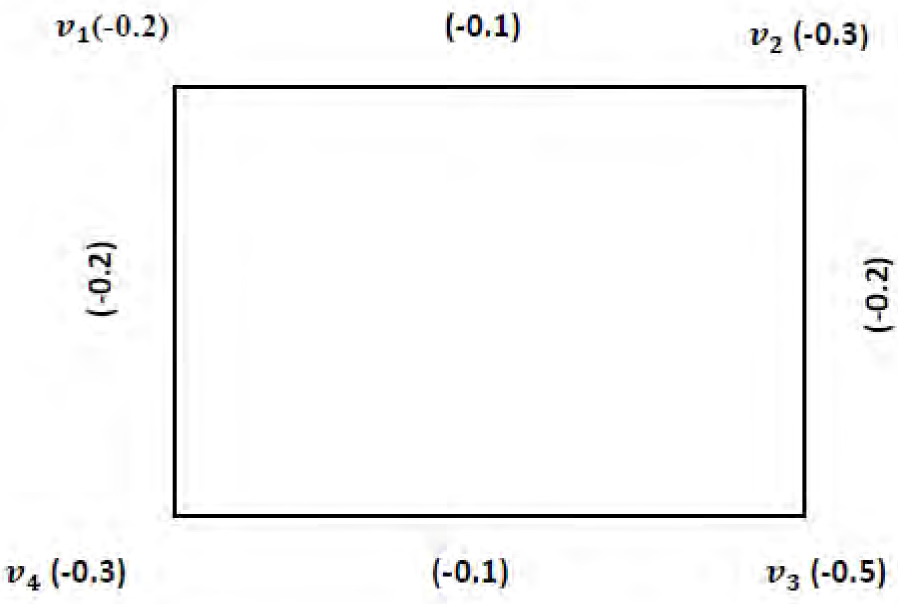

Fig. 6: *N*-graph.

4. Conclusion

In this paper, we have introduced the concept of bipolar single valued neutrosophic graphs and also proved that the most widely used extensions of fuzzy graphs are particular cases of bipolar single valued neutrosophic graphs. So our future work will focus on: (1) The study of certains types of bipolar single valued neutrosophic graphs such as, complete bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs. (2) The concept of energy of bipolar single valued neutrosophic graphs. (3) The study about applications, especially in traffic light problem.

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Neutrosophic Crisp Probability Theory & Decision Making Process

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Abstract

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. For that purpose, it is natural to adopt the value from the selected set with highest degree of truth-membership, indeterminacy membership and least degree of falsity-membership on the decision set. These factors indicate that a decision making process takes place in neutrosophic environment. In this paper, we introduce and study the probability of neutrosophic crisp sets. After giving the fundamental definitions and operations, we obtain several properties and discuss the relationship between them. These notions can help researchers and make great use in the future in making algorithms to solve problems and manage between these notions to produce a new application or new algorithm of solving decision support problems. Possible applications to mathematical computer sciences are touched upon.

Keyword

Neutrosophic set, Neutrosophic probability, Neutrosophic crisp set, Intuitionistic neutrosophic set.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts [1, 2, 3, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 42] such as the neutrosophic set theory. The fundamental concepts of neutrosophic set, introduced by Smarandache in [37, 38, 39, 40], and Salama et al. in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], provides a natural foundation for treating mathematically the neutrosophic phenomena which pervasively exist in our real world and for building new branches of neutrosophic mathematics.

In this paper, we introduce and study the probability of neutrosophic crisp sets. After giving the fundamental definitions and operations, we obtain several properties, and discuss the relationship between neutrosophic crisp sets and others.

2 Terminology

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [37, 38, 39, 40], and Salama et al. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Smarandache introduced the neutrosophic components *T, I, F* — which represent the membership, indeterminacy and non-membership values respectively, which are included into the nonstandard unit interval.

2.1 Example 2.1 [37, 39]

Let us consider a neutrosophic set, a collection of possible locations (positions) of particle *x* and let A and B two neutrosophic sets.

One can say, by language abuse, that any particle *x* neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between $^{-}0$ and 1^{+}.

For example: *x* (0.5, 0.2, 0.3) belongs to A (which means a probability of 50% that the particle *x* is in A, a probability of 30% that *x* is not in A, and the rest is undecidable); or *y* (0, 0, 1) belongs to A (which normally means *y* is not for sure in A); or *z* (0, 1, 0) belongs to A (which means one does know absolutely nothing about *z* affiliation with A).

More general, *x*((0.2-0.3), (0.4—0.45) $\cup$ [0.50-0.51,{0.2,0.24,0.28}) belongs to the set, which means: with a probability in between 20-30%, the particle *x* is in a position of A (one cannot find an exact approximation because of various sources used); with a probability of 20% or 24% or 28%, *x* is not in A; the indeterminacy related to the appurtenance of *x* to A is in between 40-45% or between 50-51% (limits included).

The subsets representing the appurtenance, indeterminacy, and falsity may overlap, and, in this case, *n-sup* = 30% + 51% + 28% > 100.

Definition 2.1 [14, 15, 21]

A neutrosophic crisp set (NCS for short) $A = \langle A_1, A_2, A_3 \rangle$ can be identified to an ordered triple $\langle A_1, A_2, A_3 \rangle$ which are subsets on X, and every crisp set in X is obviously a NCS having the form $\langle A_1, A_2, A_3 \rangle$.

Definition 2.2 [21]

The object having the form $A = \langle A_1, A_2, A_3 \rangle$ is called

1) *Neutrosophic Crisp Set with Type I* if it satisfies $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$ (NCS-Type I for short).
2) *Neutrosophic Crisp Set with Type II* if it satisfies $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = X$ (NCS-Type II for short).
3) *Neutrosophic Crisp Set with Type III* if it satisfies $A_1 \cap A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = X$ (NCS-Type III for short).

Definition 2.3

1. *Neutrosophic Set* [7]: Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$, where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$$0^- \leq \mu_A(x), \sigma_A(x\ , \nu_A(x) \leq 1^+$$

and

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ .$$

2. *Neutrosophic Intuitionistic Set of Type 1* [8]: Let X be a non-empty fixed set. A neutrosophic intuitionistic set of type 1 (NIS1 for short) set A is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$, where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$$0^- \leq \mu_A(x), \sigma_A(x\ , \nu_A(x) \leq 1^+$$

and the functions satisfy the condition

$$\mu_A(x) \wedge \sigma_A(x) \wedge \nu_A(x) \leq 0.5$$

and

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ .$$

3. *Neutrosophic Intuitionistic Set of Type 2* [41]: Let X be a non-empty fixed set. A neutrosophic intuitionistic set of type 2 A (NIS2 for short) is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of

indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$$0.5 \le \mu_A(x), \sigma_A(x), \nu_A(x)$$

and the functions satisfy the condition

$$\mu_A(x) \wedge \sigma_A(x) \le 0.5,\ \mu_A(x) \wedge \nu_A(x) \le 0.5,\ \sigma_A(x) \wedge \nu_A(x) \le 0.5,$$

and

$$^-0 \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 2^+.$$

A neutrosophic crisp with three types the object $A = \langle A_1, A_2, A_3 \rangle$ can be identified to an ordered triple $\langle A_1, A_2, A_3 \rangle$ which are subsets on X, and every crisp set in X is obviously a NCS having the form $\langle A_1, A_2, A_3 \rangle$. Every neutrosophic set $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ on X is obviously a NS having the form $\langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$.

Salama et al in [14, 15, 21] constructed the tools for developed neutrosophic crisp set and introduced the NCS ϕ_N, X_N in X.

Remark 2.1

The neutrosophic intuitionistic set is a neutrosophic set, but the neutrosophic set is not a neutrosophic intuitionistic set in general. Neutrosophic crisp sets with three types are neutrosophic crisp set.

3 The Probability of Neutrosophic Crisp Sets

If an experiment produces indeterminacy, that is called a neutrosophic experiment. Collecting all results, including the indeterminacy, we get the neutrosophic sample space (or the neutrosophic probability space) of the experiment. The neutrosophic power set of the neutrosophic sample space is formed by all different collections (that may or may not include the indeterminacy) of possible results. These collections are called neutrosophic events.

In classical experimental, the probability is

$$\left(\frac{\text{number of times event A occurs}}{\text{total number of trials}} \right).$$

Similarly, Smarandache in [16, 17, 18] introduced the Neutrosophic Experimental Probability, which is:

$$\left(\frac{\text{number of times event A occurs}}{\text{total number of trials}}, \frac{\text{number of times indeterminacy occurs}}{\text{total number of trials}}, \frac{\text{number of times event A does not occur}}{\text{total number of trials}}\right)$$

Probability of NCS is a generalization of the classical probability in which the chance that an event $A = \langle A_1, A_2, A_3 \rangle$ to occur is:

$$P(A_1)\ true,\ P(A_2)\ \text{indeterminate},\ P(A_3)\ false,$$

on a sample space X, or $NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle$.

A subspace of the universal set, endowed with a neutrosophic probability defined for each of its subsets, forms a probability neutrosophic crisp space.

Definition 3.1

Let X be a non- empty set and A be any type of neutrosophic crisp set on a space X, then the neutrosophic probability is a mapping $NP : X \to [0,1]^3$, $NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle$, that is the probability of a neutrosophic crisp set that has the property that —

$$NP(A) = \begin{cases} (p_1, p_2, p_3) & \text{where } p_{1,2,3} \in [0,1] \\ 0 & \text{if } p_1, p_2, p_3 < o \end{cases}.$$

Remark 3.1

1. In case if $A = \langle A_1, A_2, A_3 \rangle$ is NCS, then

$$^-0 \le P(A_1) + P(A_2) + P(A_3) \le 3^+.$$

2. In case if $A = \langle A_1, A_2, A_3 \rangle$ is NCS-Type I, then $0 \le P(A_1) + P(A_2) + P(A_3) \le 2$.

3. The Probability of NCS-Type II is a neutrosophic crisp set where

$$^-0 \le P(A_1) + P(A_2) + P(A_3) \le 2^+.$$

4. The Probability of NCS-Type III is a neutrosophic crisp set where

$$^-0 \le P(A_1) + P(A_2) + P(A_3) \le 3^+.$$

Probability Axioms of NCS Axioms

1. The Probability of neutrosophic crisp set and NCS-Type III A on X $NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle$ where $P(A_1) \ge 0, P(A_2) \ge 0, P(A_3) \ge 0$ or

$$NP(A) = \begin{cases} (p_1, p_2, p_3) & \text{where } p_{1,2,3} \in [0,1] \\ 0 & \text{if } p_1, p_2, p_3 < o \end{cases}.$$

2. The probability of neutrosophic crisp set and NCS-Type IIIs A on X $NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle$ where $^-0 \le p(A_1) + p(A_2) + p(A_3) \le 3^+$.

3. Bounding the probability of neutrosophic crisp set and NCS-Type III
$NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle$ where $1 \geq P(A_1) \geq 0, P(A_2) \geq 0, P(A_3) \geq 0.$

4. Addition law for any two neutrosophic crisp sets or NCS-Type III

$$NP(A \cup B) =< (P(A_1) + P(B_1) - P(A_1 \cap B_1),$$
$$(P(A_2) + P(B_2) - P(A_2 \cap B_2),\ (P(A_3) + P(B_3) - P(A_3 \cap B_3) >$$

if

$A \cap B = \phi_N$, then $NP(A \cap B) = NP(\phi_N)$.

$$NP(A \cup B) =< NP(A_1) + NP(B_1) - NP(\phi_{N_1}), NP(A_2) + NP(B_2) - NP(\phi_{N_2}),$$
$$NP(A_3) + NP(B_3) - NP(\phi_{N_3}).$$

Since our main purpose is to construct the tools for developing probability of neutrosophic crisp sets, we must introduce the following —

1. Probability of neutrosophic crisp empty set with three types ($NP(\phi_N)$ for short) may be defined as four types:

Type 1: $NP(\phi_N) = \langle P(\phi), P(\phi), P(X) \rangle =< 0,0,1 >$;

Type 2: $NP(\phi_N) = \langle P(\phi), P(X), P(X) \rangle =< 0,1,1 >$;

Type 3: $NP(\phi_N) = \langle P(\phi), P(\phi), P(\phi) \rangle =< 0,0,0 >$;

Type 4: $NP(\phi_N) = \langle P(\phi), P(X), P(\phi) \rangle =< 0,1,0 >$.

2. Probability of neutrosophic crisp universal and NCS-Type III universal sets ($NP(X_N)$ for short) may be defined as four types —

Type 1: $NP(X_N) = \langle P(X), P(\phi), P(\phi) \rangle =< 1,0,0 >$;

Type 2: $NP(X_N) = \langle P(X), P(X), P(\phi) \rangle =< 1,1,0 >$;

Type 3: $NP(X_N) = \langle P(X), P(X), P(X) \rangle =< 1,1,1 >$;

Type 4: $NP(X_N) = \langle P(X), P(\phi), P(X) \rangle =< 1,0,1 >$.

Remark 3.2

$NP(X_N) = 1_N$, $NP(\phi_N) = O_N$, where $1_N, O_N$ are in Definition 2.1 [6], or equals any type for 1_N .

Definition 3.2 (Monotonicity)

Let X be a non-empty set, and NCSS A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$ with

$$NP(A) = \langle P(A_1), P(A_2), P(A_3) \rangle, NP(B) = \langle P(B_1), P(B_2), P(B_3) \rangle,$$

then we may consider two possible definitions for subsets ($A \subseteq B$) —

Type1:

$$NP(A) \le NP(B) \Leftrightarrow P(A_1) \le P(B_1), P(A_2) \le P(B_2) \text{ and } P(A_3) \ge P(B_3),$$

or Type2:

$$NP(A) \le NP(B) \Leftrightarrow P(A_1) \le P(B_1), P(A_2) \ge P(B_2) \text{ and } P(A_3) \ge P(B_3).$$

Definition 3.3

Let X be a non-empty set, and NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$ be NCSs.

Then —

1. $NP(A \cap B)$ may be defined two types as —

Type1:

$$NP(A \cap B) = \langle P(A_1 \cap B_1), P(A_2 \cap B_2), P(A_3 \cup B_3) \rangle, \text{ or}$$

Type2:

$$NP(A \cap B) = \langle P(A_1 \cap B_1), P(A_2 \cup B_2), P(A_3 \cup B_3) \rangle.$$

2. $NP(A \cup B)$ may be defined two types as:

Type1:

$$NP(A \cup B) = \langle P(A_1 \cup B_1), P(A_2 \cap B_2), P(A_3 \cap B_3) \rangle,$$

or Type 2:

$$NP(A \cup B) = \langle P(A_1 \cup B_1), P(A_2 \cup B_2), P(A_3 \cap B_3) \rangle.$$

3. $NP(A^c)$ may be defined by three types:

Type1:

$$NP(A^c) = \left\langle P(A_1{}^c), P(A_2^c), P(A_3^c) \right\rangle = <(1 - A_1), (1 - A_2), (1 - A_3)>$$

or Type2:

$$NP(A^c) = \left\langle P(A_3), P(A_2^c), P(A_1) \right\rangle$$

or Type3:

$$NP(A^c) = \langle P(A_3), P(A_2), P(A_1) \rangle.$$

Proposition 3.1

Let A and B in the form $A=\langle A_1,A_2,A_3\rangle$, $B=\langle B_1,B_2,B_3\rangle$ be NCSs on a non-empty set X.

Then –

$$NP(A)^c + NP(A) =< (1,1,1> \text{ or } NP(X_N)=1_N, \text{ or = any type of } 1_N.$$

$$NP(A-B) =< (P(A_1)-P(A_1\cap B_1)),(P(A_2)-P(A_2\cap B_2)),$$
$$(P(A_3)-P(A_3\cap B_3)>$$

$$NP(A/B) =< \frac{NP(A_1)}{NP(A_1\cap B_1)},\frac{NP(A_2)}{NP(A_2\cap B_2)},\frac{NP(A_3)}{NP(A_3\cap B_3)}>.$$

Proposition 3.1

Let A and B in the form $A=\langle A_1,A_2,A_3\rangle$, $B=\langle B_1,B_2,B_3\rangle$ are NCSs on a non-empty set X and p , p_N are NCSs.
Then

$$NP(p)=\left\langle \frac{1}{n(X)},\frac{1}{n(X)},\frac{1}{n(X)}\right\rangle;$$

$$NP(p_N)=\left\langle 0,\frac{1}{n(X)},1-\frac{1}{n(X)}\right\rangle.$$

Example 3.1

1. Let $X=\{a,b,c,d\}$ and A, B are two neutrosophic crisp events on X defined by $A=\langle\{a\},\{b,c\},\{c,d\}\rangle$, $B=\langle\{a,b\},\{a,c\},\{c\}\rangle$, $p=\langle\{a\},\{c\},\{d\}\rangle$ then see that $NP(A)=\langle 0.25,0.5,0.5\rangle$, $NP(B)=\langle 0.5,0.5,0.25\rangle$, $NP(p)=\langle 0.25,0.25,0.25\rangle$, one can compute all probabilities from definitions.

2. If $A=\langle\{\phi\},\{b,c\},\{\phi\}\rangle$ and $B=\langle\{\phi\},\{d\},\{\phi\}\rangle$ are neutrosophic crisp sets on X. Then –

$$A\cap B=\langle\{\phi\},\{\phi\},\{\phi\}\rangle \text{ and } NP(A\cap B)=\langle 0,0,0\rangle=0_N,$$

$$A\cap B=\langle\{\phi\},\{b,c,d\},\{\phi\}\rangle \text{ and } NP(A\cap B)=\langle 0,0.75,0\rangle\neq 0_N.$$

Example 3.2

Let $X=\{a,b,c,d,e,f\}$,

$A=\langle\{a,b,c,d\},\{e\},\{f\}\rangle$, $D=\langle\{a,b\},\{e,c\},\{f,d\}\rangle$ be a NCS-Type 2,

$B = \langle \{a,b,c\},\{d\},\{e\} \rangle$ be a NCT-Type I but not NCS-Type II, III,

$C = \langle \{a,b\},\{c,d\},\{e,f,a\} \rangle$ be a NCS-Type III, but not NCS-Type I, II,

$E = \langle \{a,b,c,d,e\},\{c,d\},\{e,f,a\} \rangle$,

$F = \langle \{a,b,c,d,e\},\phi,\{e,f,a,d,c,b\} \rangle$.

We can compute the probabilities for NCSs by the following:

$$NP(A) = \left\langle \frac{4}{6}, \frac{1}{6}, \frac{1}{6} \right\rangle,$$

$$NP(D) = \left\langle \frac{2}{6}, \frac{2}{6}, \frac{2}{6} \right\rangle,$$

$$NP(B) = \left\langle \frac{3}{6}, \frac{1}{6}, \frac{1}{6} \right\rangle,$$

$$NP(C) = \left\langle \frac{2}{6}, \frac{2}{6}, \frac{3}{6} \right\rangle,$$

$$NP(E) = \left\langle \frac{4}{6}, \frac{2}{6}, \frac{3}{6} \right\rangle,$$

$$NP(F) = \left\langle \frac{5}{6}, 0, - \right\rangle.$$

Remark 3.3

The probabilities of a neutrosophic crisp set are neutrosophic sets.

Example 3.3

Let $X = \{a,b,c,d\}$, $A = \langle \{a,b\},\{c\},\{d\} \rangle$, $B = \langle \{a\},\{c\},\{d,b\} \rangle$ are NCS-Type I on X and $U_1 = \langle \{a,b\},\{c,d\},\{a,d\} \rangle$, $U_2 = \langle \{a,b,c\},\{c\},\{d\} \rangle$ are NCS-Type III on X; then we can find the following operations —

1. *Union, intersection, complement, difference and its probabilities.*

a) Type1: $A \cap B = \langle \{a\},\{c\},\{d,b\} \rangle$, $NP(A \cap B) = \langle 0.25, 0.25, 0.5\} \rangle$ and
Type 2,3: $A \cap B = \langle \{a\},\{c\},\{d,b\} \rangle$, $NP(A \cap B) = \langle 0.25, 0.25, 0.5\} \rangle$.

2. $NP(A-B)$ *may be equals.*

Type1: $NP(A-B) =< 0.25,0,0 >$, Type 2: $NP(A-B) =< 0.25,0,0 >$,
Type 3: $NP(A-B) =< 0.25,0,0 >$,
b) Type 2: $A \cup B = \langle \{a,b\},\{c\},\{d\} \rangle$, $NP(A \cup B) = \langle 0.5, 0.25, 0.25\} \rangle$ and
Type 2: $A \cup B = \langle \{a.b\},\{c\},\{d\} \rangle$ $NP(A \cup B) = \langle 0.5, 0.25, 0.25\} \rangle$.

c) Type1: $A^c = \langle \{c,d\} \;, \{a,b,d\}, \{a,b,c\} \rangle$ NCS-Type III set on X,

$NP(A^c) = \langle 0.5, 0.75, 0.75 \rangle$.

Type2: $A^c = \langle \{d\} \;, \{a,b,d\}, \{a,b\} \rangle$ NCS-Type III on X,

$NP(A^c) = \langle 0.25, 0.75, 0.5 \rangle$.

Type3: $A^c = \langle \{d\} \;, \{c\}, \{a,b\} \rangle$ NCS-Type III on X,

$NP(A^c) = \langle 0.75, 0.75, 0.5 \rangle$.

d) Type 1: $B^c = \langle \{b,c,d\}, \{a,b,d\}, \{a,c\} \rangle$ be NCS-Type III on X ,

$NP(B^c) = \langle 0.75, 0.75, 0.5 \rangle$

Type 2: $B^c = \langle \{b,d\}, \{c\}, \{a\} \rangle$ NCS-Type I on X, and $NP(B^c) =$ $\langle 0.5, 0.25, 0.25 \rangle$.

Type 3: $B^c = \langle \{b,d\}, \{a,b,d\}, \{a\} \rangle$ NCS-Type III on X and $NP(B^c) =$ $\langle 0.5, 0.75, 0.25 \rangle$.

e) Type 1: $U_1 \cup U_2 = \langle \{a,b,c\}, \{c,d\}, \{a,d\} \rangle$, NCS-Type III,

$NP(U_1 \cup U_2) = \langle \{0.75, 0.5, 0.5 \rangle$,

Type 2: $U_1 \cup U_2 = \langle \{a,b,c\}, \{c\}, \{a,d\} \rangle$, $NP(U_1 \cup U \;) = \langle \{0.75, 0.25, 0.5 \rangle$.

f) Type1: $U_1 \cap U_2 = \langle \{a,b\}, \{c,d\}, \{a,d\} \rangle$, NCS-Type III,

$NP(U_1 \cap U_2) = \langle 0.5, 0.5, 0.5 \rangle$,

Type2: $U_1 \cap U_2 = \langle \{a,b\}, \{c\}, \{a,d\} \rangle$, NCS-Type III, and

$NP(U_1 \cap U_2) = \langle 0.5, 0.25, 0.5 \rangle$,

g) Type 1: ${U_1}^c = \langle \{c,d\}, \{a,b\}, \{c,b\} \rangle$, NCS-Type III and

$NP({U_1}^c) = \langle 0.5, 0.5, 0.5 \rangle$

Type 2: ${U_1}^c = \langle \{a,d\}, \{c,d\}, \{a,b\} \rangle$, NCS-Type III and

$NP({U_1}^c) = \langle 0.5, 0.5, 0.5 \rangle$

Type3: ${U_1}^c = \langle \{a,d\}, \{a,b\}, \{a,b\} \rangle$, NCS-Type III and

$NP({U_1}^c) = \langle 0.5, 0.5, 0.5 \rangle$.

h) Type1: ${U_2}^c = \langle \{d\}, \{a,b,d\}, \{a,b,c\} \rangle$ NCS-Type III and

$NP({U_2}^c) = \langle 0.25, 0.75, 0.75 \rangle$, Type2: $U^c{}_2 = \langle \{d\}, \{c\}, \{a,b,c\} \rangle$ NCS-Type III and $NP({U_2}^c) = \langle 0.25, 0.25, 0.75 \rangle$, Type3: $U^c{}_2 = \langle \{d\}, \{a,b,d\}, \{a,b,c\} \rangle$ NCS-Type III. $NP({U_2}^c) = \langle 0.25, 0.75, 0.75 \rangle$.

3. *Probabilities for events.*

$NP(A) = \langle 0.5, 0.25, 0.25 \rangle$, $NP(B) = \langle 0.25, 0.25, 0.5 \rangle$, $NP(U_1) = \langle 0.5, 0.5, 0.5 \rangle$,

$NP(U_2) = \langle 0.75, 0.25, 0.25 \rangle$

$NP(U_1{}^c) = \langle 0.5, 0.5, 0.5 \rangle$, $NP(U_2{}^c) = \langle 0.25, 0.75, 0.75 \rangle$.

e) $(A \cap B)^c = \langle \{b, c, d\}, \{a, b, d\}, \{a, c\} \rangle$ be a NCS-Type III.

$NP(A \cap B)^c = \langle 0.75, 0.75, 0.25 \rangle$ be a neutrosophic set.

f) $NP(A)^c \cap NP(B)^c = \langle 0.5, 0.75, 0.75 \rangle$,

$NP(A)^c \cup NP(B)^c = \langle 0.75, 0.75, 0.5 \rangle$

g) $NP(A \cup B) = NP(A) + NP(B) - NP(A \cap B) = \langle 0.5, 0.25, 0.25 \} \rangle$

h) $NP(A) = \langle 0.5, 0.25, 0.25 \rangle$, $NP(A)^c = \langle 0.5, 0.75, 0.75 \rangle$,

$NP(B) = \langle 0.25, 0.25, 0.5 \rangle$, $NP(B^c) = \langle 0.75, 0.75, 0.5 \rangle$

4. *Probabilities for Products.* The product of two events given by —

$A \times B = \langle \{(a, a), (b, a)\}, \{(c, c)\}, \{(d, d), (d, b)\} \rangle$,

and $NP(A \times B) = \langle 2/16, 1/16, 2/16 \rangle$

$B \times A = \langle \{(a, a), (a, b)\}, \{(c, c)\}, \{(d, d), (b, d)\} \rangle$

and $NP(B \times A) = \langle 2/16, 1/16, 2/16 \rangle$

$A \times U_1 = \langle \{(a, a), (b, a), (a, b), (b, b)\}, \{(c, c), (c, d)\}, \{(d, d), (d, a)\} \rangle$,

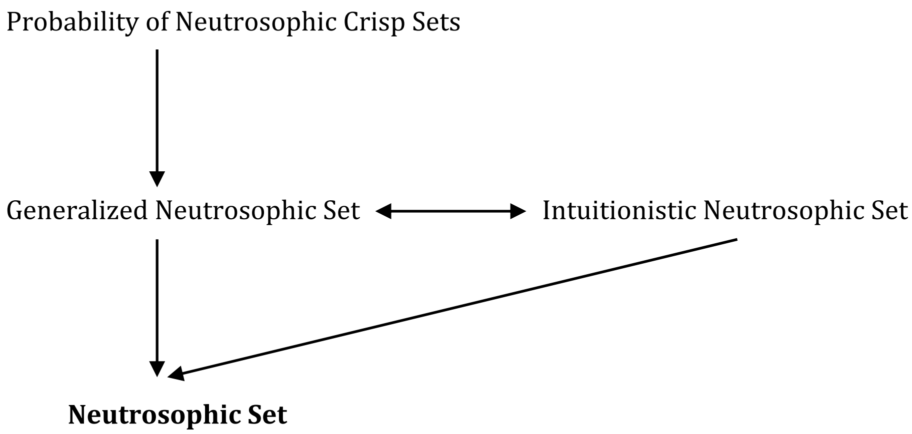
and $NP(A \times U_1) = \langle 4/16, 2/16, 2/16 \rangle$

$U_1 \times U_2 = \langle \{(a, a), (b, a), (a, b), (b, b), (a, c), (b, c)\}, \{(c, c), (d, c)\}, \{(d, d), (a, d)\} \rangle$

and $NP(U_1 \times U_2) = \langle 6/16, 2/16, 2/16 \rangle$.

Remark 3.4

The following diagram represents the relation between neutrosophic crisp concepts and neutrosphic sets:



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Decision-Making Method based on the Interval Valued Neutrosophic Graph

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***Abstract*—In this article, we extend the concept of neutrosophic graph-based multicriteria decision making method (NGMCDM) to the case of interval valued neutrosophic graph theory. The new concept is called interval valued neutrosophic graph-based multicriteria decision making method (IVNGMCDM for short). Finally, an illustrative example is given and a comparison analysis is conducted between the proposed approach and other existing methods, to verify the feasibility and effectiveness of the developed approach.**

Keywords—interval valued neutrosophic set; interval valued neutrosophic graph; influence coefficient; decision making problem

I. INTRODUCTION

The Neutrosophic Set (NS), proposed by Smarandache [1, 2] as a generalization of fuzzy sets theory [3], intuitionistic fuzzy set [4, 5], interval-valued fuzzy set [6] and interval-valued intuitionistic fuzzy set [7], is a powerful mathematical tool for dealing with incomplete, indeterminate and inconsistent information in real world. The neutrosophic sets are characterized by a truth-membership function (t), an indeterminacy-membership function (i) and a falsity-membership function (f) independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. In order to conveniently apply NS in real life applications, Wang et al. [8] introduced the concept of single-valued neutrosophic set (SVNS), a subclass of the neutrosophic sets. The same authors [9] also introduced the concept of interval valued neutrosophic set (IVNS), which is more precise and more flexible than the single valued neutrosophic set. The IVNS is a generalization of the single valued neutrosophic set, in which three membership functions are independent, and their value belong to the unit interval [0, 1].

The theory of single valued neutrosophic set and interval valued neutrosophic set have been applied in a wide diversity of fields [10, 11, 12, 13, 14, 15, 16]. Multi-criteria decision making attempts to handle problems with imprecise goals, referring to a number of individual criteria by a set of alternatives at choice. Many scholars have begun to study the practical application of neutrosophic sets and interval valued neutrosophic sets in multi-attribute decision-making problems. [17]

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. The graph is a widely used tool for solving combinatorial problems in different areas, such as geometry, algebra, number theory, topology, optimization and computer science. When the relations between nodes (or vertices) in problems are indeterminate, the fuzzy graphs and their extensions [18, 19, 20, 21, 22] fail. For this purpose, Smarandache [23, 24, 25] defined four main categories of neutrosophic graphs. Two of them, called I-edge neutrosophic graph and I-vertex neutrosophic graph, are based on literal indeterminacy (I); these concepts are deeply studied and gained popularity among the researchers due to applications via real world problems [26, 27, 28, 29]. The two other categories of graphs, called (t, i, f)-Edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph, are based on (t, i, f) components, but they not at all developed. Later on, Broumi et al. [30, 31] introduced a third neutrosophic graph model, called single valued neutrosophic graph (SVNG), and investigated some of its properties. This model allows the attachment of truth-membership (t), indeterminacy–membership (i) and falsity-membership(f) degrees both to vertices and edges. The single valued neutrosophic graph is a generalization of fuzzy graph and intuitionistic fuzzy graph. Also, the same authors [32] introduced neighborhood degree of a vertex and closed neighborhood degree of a vertex in single valued neutrosophic graph, as a generalization of neighborhood degree of a vertex and closed neighborhood degree of a vertex in fuzzy graph and intuitionistic fuzzy graph. Moreover, Broumi et al. [33] introduced the concept of interval valued neutrosophic graph as a generalization fuzzy graph, intuitionistic fuzzy graph, interval valued fuzzy graph, interval valued intuitionistic fuzzy graph and single valued neutrosophic graph, discussing some properties with proofs and examples. In addition, Broumi et al. [34] proposed some operations - such as Cartesian product, composition, union and join - on interval valued neutrosophic graphs, and investigated some properties. Withal, Broumi et al. [35] discussed a sub class of interval valued neutrosophic graph, called strong interval valued neutrosophic graph, and introduced as well some operations - such as Cartesian product, composition and join of two strong interval valued neutrosophic graph - with proofs. Recently, Broumi et al. [36, 37] propounded the concept of bipolar single valued neutrosophic graph as a generalization of fuzzy graph, intuitionistic fuzzy graph, N-graph [38], bipolar fuzzy graph [39] and single valued neutrosophic graph, and studied some properties. In this paper, we extend the concept of neutrosophic graph-based multicriteria decision making (NGMADM) method, introduced by Shain [40] to solve MCDM problems with interval valued neutrosophic information.

The paper is organized as follows: in the 2nd section, we give all the basic definitions to be employed in later sections, related to single valued neutrosophic graph and interval valued neutrosophic graph; in the 3rd section, we present the neutrosophic graph-based multicriteria decision making (NGMCDM) method; in the 4thsection, we present an application of interval valued neutrosophic graphs in decision making; in the 5th section, an illustrative example is given, and then a comparison analysis is conducted between the proposed approach and other existing methods, in order to verify its feasibility and effectiveness. Finally, the conclusions are drawn in the 7th section.

II. PRELIMINARIES

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, single valued neutrosophic graphs and interval valued neutrosophic graphs, relevant to the present paper. The readers are referred to [1, 8, 9, 14, 30, 31].

Definition 2.1 [1]. Let X be a space of points (objects) with generic elements in X denoted by x; then, the neutrosophic set A (NS A) is an object having the form $A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}$, where the functions T, I, F: $X \to]^-0,1^+[$ define respectively a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A, with the condition:

$$^-0 \le T_A(x)+ I_A(x)+ F_A(x)\le 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0,1^+[$.

Since it is difficult to apply NSs to practical problems, Wang et al. [8] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [8]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X,$T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$.

A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}. \quad (2)$$

Definition 2.3 [9]. Let X be a universe of discourse and Int [0,1] be the set of all closed subsets of [0,1]. Then, an interval neutrosophic set is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}, \quad (3)$$

Where $T_A: X \to \mathrm{Int}[0,1]$, $I_A: X \to \mathrm{Int}[0,1]$ and $F_A: X \to \mathrm{Int}[0,1]$ with $0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$, for all x ∈ X.

The intervals $T_A(x), I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of xto A, respectively.

For convenience, if $T_A(x) = [T_A^L(x), T_A^U(x)]$, $I_A(x) = [I_A^L(x), I_A^U(x)]$and $F(x) = [F_A^L(x), F_A^U(x)]$, then:

$$A = \{\langle x, [T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [I_A^L(x), I_A^U(x)]\rangle : x \in X\}, \quad (4)$$

with the condition, $0 \le \sup T_A^U(x) + \sup I_A^U(x) + \sup F_A^U(x) \le 3$, for all x ∈ X.

Definition 2.4 [14]. Let $\alpha = \langle [t^l, t^u], [i^l, i^u], [f^l, f^u]\rangle$ be an interval neutrosophic number; a score function S of the interval valued neutrosophic number can be defined by

$$S(\alpha) = \frac{2+t^l+t^u-2i^l-2i^u-f^l-f^u}{4} \quad (5)$$

where$S(\alpha) \in [-1,1]$.

Definition 2.5 [30]. Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be single valued neutrosophic sets on a set *X*.

If $A = (T_A, I_A, F_A)$ is a single valued neutrosophic relation on a set *X*, then $A =(T_A, I_A, F_A)$ is called a single valued neutrosophic relation on $B = (T_B, I_B, F_B)$, if:

$T_B(x, y) \le \min(T_A(x),T_A(y))$
$I_B(x, y) \ge \max(I_A(x),I_A(y))$
and
$$F_B(x, y) \ge \max(F_A x),F_A(y)), \text{ for all } x, y \in X. \quad (6)$$

A single valued neutrosophic relation *A* on *X*is called symmetric, if:

$T_A(x, y) = T_A(y, x)$, $I_A(x, y) = I_A(y, x)$, $F_A(x, y) = F_A(y, x)$,
$T_B(x, y) = T_B(y, x)$, $I_B(x, y) = I_B(y, x)$ and $F_B(x, y) = F_B(y, x)$, for all $x, y \in X$.

Definition 2.6 [30]. A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G= (A, B) where:

1) The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and

$0\leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$ for all $v_i \in$ V (i=1, 2, …, n). (7)

2) The functions T_B: E ⊆ V x V →[0, 1], I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by

$T_B(v_i, v_j) \leq \min [T_A(v_i), T_A(v_j)]$, $I_B(v_i, v_j) \geq \max [I_A(v_i), I_A(v_j)]$, and $F_B(v_i, v_j) \geq \max [F_A(v_i), F_A(v_j)]$ (8)

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i,v_j) ∈ E respectively, where:

$0\leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$, for all $(v_i, v_j) \in$ E (i, j = 1, 2,…, n) (9)

they called A the single valued neutrosophic vertex set of V, B the single valued neutrosophic edge set of E, respectively; note that B is a symmetric single valued neutrosophic relation on A.

Example 2.7 [30] Figure 1 is an example for SVNG, G=(A, B) defined on a graph G^* = (V, E) such that V = {v_1, v_2, v_3,v_4}, E = {v_1v_2, v_2v_3, v_3v_4, v_4v_1}, A is single valued neutrosophic set of V

A={ < v_1,(0.5, 0.1 ,0.4)>, <v_2, (0.6, 0.3,0.2) >, <v_3, (0.2, 0.3 ,0.4) >, < v_4 , (0.4, 0.2 ,0.5)>}, and B single valued neutrosophic set of E ⊆ V x V

B={ < v_1v_2,(0.5, 0.4 ,0.5)>, <v_2v_3, (0.2, 0.3,0.4) >, <v_3v_4, (0.2, 0.4 ,0.5) >, <v_4v_1, (0.4, 0.3 ,0.6)>}

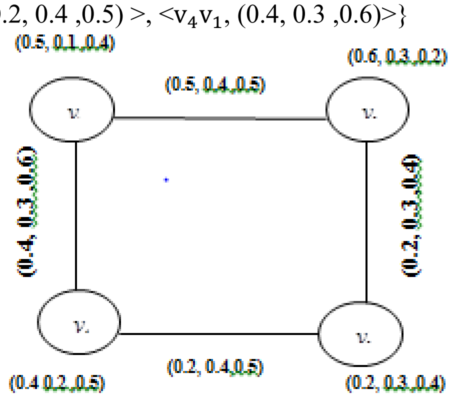

Fig. 1. Single valued neutrosophic graph

Definition 2.8 [30]. A single valued neutrosophic graph G=(A, B) of G^* = (V, E) is called strong single valued neutrosophic graph, if:

$T_B(v_i, v_j) = \min [T_A(v_i), T_A(v_j)]$, $I_B(v_i, v_j) = \max [I_A(v_i), I_A(v_j)]$, $F_B(v_i, v_j) = \max [F_A(v_i), F_A(v_j)]$, for all $(v_i, v_j) \in$ E. (10)

Definition 2.9 [30]. A single valued neutrosophic graph G= (A, B) is called complete, if:

$T_B(v_i, v_j) = \min [T_A(v_i), T_A(v_j)]$, $I_B(v_i, v_j) = \max [I_A(v_i), I_A(v_j)]$ and $F_B(v_i, v_j) = \max [F_A(v_i), F_A(v_j)]$, for all $v_i, v_j \in$ V. (11)

Definition 2.10 [31]. By an interval-valued neutrosophic graph of a graph G* = (V, E) we mean a pair G = (A, B), where A =< [T_{AL}, T_{AU}], [I_{AL}, I_{AU}], [F_{AL}, F_{AU}]> is an interval-valued neutrosophic set on V, and B =<[T_{BL}, T_{BU}], [I_{BL}, I_{BU}], [F_{BL}, F_{BU}]> is an interval-valued neutrosophic relation on E satisfying the following condition:

1) V= {v_1, v_2 ,…,v_n} such that T_{AL}:V→[0, 1], T_{AU}:V→[0, 1], I_{AL}:V→[0, 1], I_{AU}:V→[0, 1], and F_{AL}:V→[0, 1], F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element $y \in$ V, respectively, and

$0\leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$ for all $v_i \in$ V (i=1, 2, …, n). (12)

2) The functions T_{BL}:V x V →[0, 1], T_{BU}:V x V →[0, 1], I_{BL}:V x V →[0, 1], I_{BU}:V x V →[0, 1] and F_{BL}:V x V →[0,1], F_{BU}:V x V →[0, 1] are such that:

$T_{BL}(v_i, v_j) \leq \min [T_{AL}(v_i), T_{AL}(v_j)]$, $T_{BU}(v_i, v_j) \leq \min [T_{AU}(v_i), T_{AU}(v_j)]$,

$I_{BL}(v_i, v_j) \geq \max [I_{BL}(v_i), I_{BL}(v_j)]$, $I_{BU}(v_i, v_j) \geq \max[I_{BU}(v_i), I_{BU}(v_j)]$ and

$F_{BL}(v_i, v_j) \geq \max [F_{BL}(v_i), F_{BL}(v_j)]$, $F_{BU}(v_i, v_j) \geq \max[F_{BU}(v_i), F_{BU}(v_j)]$ (13)

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i,v_j) ∈ E respectively, where:

$0\leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$, for all $(v_i, v_j) \in$ E (i, j = 1, 2,…, n). (14)

They called A the interval valued neutrosophic vertex set of V, and B the interval valued neutrosophic edge set of E, respectively; note that B is a symmetric interval valued neutrosophic relation on A.

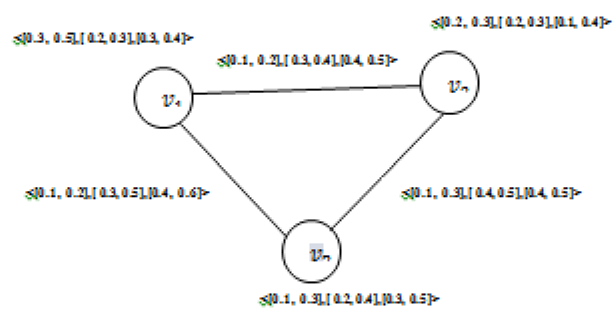

Fig. 2. Interval valued neutrosophic graph

Example 2.11 [33] Figure 2 is an example for IVNG, G=(A, B) defined on a graph G* = (V, E) such that V= {v_1, v_2, v_3}, E= {v_1v_2, v_2v_3, v_3v_4, v_4v_1}, A is an interval valued neutrosophic set of V

A={ < v_1,([0.3, 0.5], [0.2, 0.3],[0.3, 0.4])>, <v_2, ([0.2, 0.3], [0.2, 0.3],[0.1, 0.4]) >, <v_3, ([0.1, 0.3], [0.2, 0.4],[0.3, 0.5]) >}, and B an interval valued neutrosophic set of E ⊆ V x V

B={ < v_1v_2 ,([0.1, 0.2], [0.3, 0.4],[0.4, 0.5])>, <v_2v_3 , ([0.1, 0.3], [0.4, 0.5],[0.4, 0.5]) >, <v_3v_1, ([0.1, 0.2], [0.3, 0.5],[0.4, 0.6]) >}

Remark 2.12: -The underlying set V is vertex set of usual graph that we use it in neutrosophic graph as vertex.

- G*=(V,E) denoted a usual graph where a neutrosophic graphs obtained from it that truth –membership, indeterminacy –membership and non-membership values are 0 to 1.

III. Neutrosophic Graph-based Multicriteria Decision Making (NGMCDM) Method

Shain [40] proposed a procedure for the decision-maker to select the best choice with neutrosophic information. The method implies the following steps:

Step1. Compute the influence coefficient between the criteria α_i and $\alpha_j(i, j = 1,2, \dots , n)$ in decision process by

$$\xi ij=\frac{t_{ij}+(1-i_{ij})(1-f_{ij})}{3}, \quad (15)$$

Where $\varphi_{ij}= (t_{ij}, i_{ij}, f_{ij})$is the neutrosophic neutrosophic edge between the vertexes $\alpha_i$and $\alpha_j(i, j = 1,2, \dots , n)$. We have $\xi ii = 1$ and $\xi ij= \xi ji$ for $i = j$.

The (t, i, f) is a neutrosophic number. Because truth degree prove a positive impact while indeterminacy degree and falsity degree prove a negative impact in the relationship. If this relationship has maximum i.e., (t, i, f)=(1,0,0) then we should have the biggest impact, $\xi_{ij} = 1$. If two criteria are independent, this relationships should be (0,1,1) i.e., $\xi_{ij} =0$

Step 2. Obtain the overall criterion value of the alternative p_k ($k = 1,2, \dots , m$) by

$$\tilde{p}_k=\sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{n} e_{ki}\xi_{ij}\right), \quad (16)$$

Where $e_{ki} = (t_{ki}, i_{ki}, f_{ki})$ is clearly a neutrosophic number.

Step 3: Compute the score value of the alternative p_k ($k = 1,2, \dots , m$) which is defined by:

$$s(\tilde{p}_k) = \frac{1+\tilde{t}-2\tilde{i}-\tilde{f}}{2}. \quad (17)$$

Step 4. Rank all the alternatives p_k ($k = 1,2, \dots, m$) and select the best one(s) in concordance with $s(\tilde{p}_k)$.

Step5. End.

IV. Decision-Making Method based on the Interval Valued Neutrosophic Graph

The interval valued neutrosophic set proposed by Wang et al. [9] is independently characterized by the truth-membership, the indeterminacy-membership and the falsity-membership, which is a powerful tool to deal with incomplete, indeterminate and inconsistent information. Recently, the interval valued neutrosophic set became an interesting topic research and attracted wide attention. The interval valued neutrosophic graph can well describe the uncertainly in real-life world. Therefore, we will extend the NGMCDM method introduced by Shain [40] to solve MCDM problems with interval valued neutrosophic information. The new model to solve the decision-making problems is called interval valued neutrosophic graph-based multicriteria decision making (IVNGMCDM) method.

We firstly describe the decision making problem.

Suppose that $P = \{p_1, p_2, \dots , p_m\}$ is a collection of alternatives, $B = \{\alpha_1, \alpha_2, \dots , \alpha_n\}$ is a collection of criteria, which weight vector is $\omega =(\omega_1, \omega_2, \dots , \omega_n)^T$satisfying $w_j \in [0, 1]$, $\sum_{j=1}^{n}\omega_j = 1$. If the decision maker provide a neutrosophic value for the alternative $p_k(k = 1,2, \dots , m)$ under the criteria α_j ($j = 1,2, \dots , n$), these values can be characterized as an IVNN e_{kj} ={[t_{kj}^l ,t_{kj}^u], [i_{kj}^l ,i_{kj}^u], [f_{kj}^l ,f_{kj}^u]}($j = 1,2, \dots , n$; $k =1,2, \dots , m$). Assume that $E = [e_{kj}]_{m\times n}$is the decision matrix, where e_{kj} is expressed by an interval valued neutrosophic element. If there exists an interval valued neutrosophic relation between two criteria $\alpha_i = \langle[t_i^l,t_i^u], [i_i^l,i_i^u], [f_i^l,f_i^u]\rangle$ and $\alpha_j = \langle[t_j^l, t_j^u], [i_j^l, i_j^u], [f_j^l, f_j^u]\rangle$, we denote the interval valued neutrosophic relation as φ_{ij} = {[t_{ij}^l ,t_{ij}^u], [i_{ij}^l ,i_{ij}^u], [f_{ij}^l ,f_{ij}^u]},with the properties:

$t_{ij}^l \le \min(t_i^l , t_j^l)$, $t_{ij}^u \le \min(t_i^u , t_j^u)$,
$i_{ij}^l \ge \max(i_i^l , i_j^l)$, $i_{ij}^u \ge \max (i_i^u , i_j^u)$,
$f_{ij}^l \ge \max(f_i^l , f_j^l)$, $f_{ij}^u \ge \max(f_i^u , f_j^u)$,

for all ($i, j = 1,2, \dots , m$); otherwise, $\varphi_{ij} = \langle[0, 0], [1, 1], [1, 1]\rangle$.

On the basis of the developed graph structure, we can propose a procedure for the decision-maker to select the best choice with interval valued neutrosophic information.

The method is described by the following steps:

Step 1. Compute the influence coefficient between the criteria α_i and α_j ($i, j = 1,2, \dots, n$) in decision process by

$$\xi_{ij} =\frac{(t_{ij}^l+t_{ij}^u)+(2-(i_{ij}^l+i_{ij}^u))(2-(f_{ij}^l+f_{ij}^u))}{6}, \quad (18)$$

Where φ_{ij} = {[t_{ij}^l ,t_{ij}^u], [i_{ij}^l ,i_{ij}^u], [f_{ij}^l ,f_{ij}^u]} is the interval valued neutrosophic edge between the vertexes α_i and $\alpha_j(i, j = 1,2, \dots , n)$. We have $\xi ii = 1$and $\xi ij = \xi ji$for $i = j$.

Step 2. Obtain the overall criterion value of the alternative p_k ($k = 1,2, \dots , m$) by

$$\tilde{p}_k =\sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{n} e_{ki}\xi_{ij}\right), \quad (19)$$

Where e_{ki} = <[t_{ki}^l ,t_{ki}^u], [i_{ki}^l ,i_{ki}^u], [f_{ki}^l ,f_{ki}^u]>is clearly an interval valued neutrosophic number.

Step 3. Compute the score value of the alternative p_k ($k = 1,2, \dots , m$), which is defined by:

$$s(\tilde{p}_k) = \frac{2+\tilde{t}^l+\tilde{t}^u-2\tilde{i}^l-2\tilde{i}^u-\tilde{f}^l-\tilde{f}^u}{4} \quad (20)$$

Step 4. Rank all the alternatives p_k ($k = 1,2, \dots , m$)and select the best one(s) in concordance with $s(\tilde{p}_k)$

Step 5. End.

V. An Illustrative Example

In this section, an example for an IVNGMCDM problem with interval-valued neutrosophic information is used to prove the application and effectiveness of the proposed decision-making method.

Let us consider the decision-making problem adapted from Zhao et al. [41].

Example 5.1. An investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money: (1) p_1 is a car company, (2) p_2 is a food company, (3) p_3 is a computer company, and (4) p_4 is an armament company. The investment company must take a decision according to three criteria: (1) α_1 is the risk analysis; (2) α_2 is the growth analysis, and (3) α_3 is the environmental impact analysis. Then, the weight vector of the criteria is given by ω = $(0.2, 0.25, 0.55)^{\mathrm{T}}$. The four possible alternatives are to be evaluated under these three criteria and presented in the form of interval valued neutrosophic information by decision maker, consistent to criteria α_j (j = 1, 2, 3) and the information evaluation on the alternative p_k(k = 1, 2, 3, 4) under the factors α_j (j = 1, 2, 3); it results the interval valued valued neutrosophic decision matrix D:

$$D = \begin{bmatrix} \langle[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle & \langle[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]\rangle & \langle[0.4, 0.5], [0.2, 0.3], [0.7, 0.9]\rangle \\ \langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\rangle & \langle[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\rangle & \langle[0.8, 0.9], [0.3, 0.5], [0.3, 0.6]\rangle \\ \langle[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle & \langle[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle & \langle[0.7, 0.9], [0.2, 0.4], [0.4, 0.5]\rangle \\ \langle[0.7, 0.8], [0.0, 0.1], [0.1, 0.2]\rangle & \langle[0.6, 0.7], [0.1, 0.2], [0.1, 0.3]\rangle & \langle[0.8, 0.9], [0.3, 0.4], [0.6, 0.7]\rangle \end{bmatrix}$$

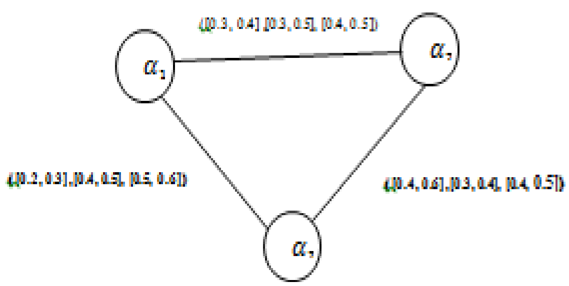

Fig. 3. The graph relationship among the criteria

Moreover, we assume that the relationships among the factors α_j (j = 1,2,3) can be described by a complete graph G = (A, E), where $A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $E = \{\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_2\alpha_3\}$ (see Fig. 3). Employing Eq. (18), we can obtain all influence coefficients to quantify the relationships among the criteria.

Suppose that the neutrosophic edges denoting the connection among the criteria are described as follows:

e_{12}= $\langle[t_{12}^l, t_{12}^u], [i_{12}^l, i_{12}^u], [f_{12}^l, f_{12}^u]\rangle$ = ⟨[0.3, 0.4] ,[0.3, 0.5], [0.4, 0.5]⟩,
e_{13}= $\langle[t_{13}^l, t_{13}^u], [i_{13}^l, i_{13}^u], [f_{13}^l, f_{13}^u]\rangle$= ⟨[0.2, 0.3] ,[0.4, 0.5], [0.5, 0.6]⟩,
e_{23}= $\langle[t_{23}^l, t_{23}^u], [i_{23}^l, i_{23}^u], [f_{23}^l, f_{23}^u]\rangle$ = ⟨[0.4, 0.6] ,[0.3, 0.4], [0.4, 0.5]⟩.

Note that G = (A, E) describes an interval valued neutrosophic graph according to the relationship among criteria for each alternative.

To get the best alternative(s), the following steps are involved:

Step 1. We apply all computations only in the alternative p_1. Others can be similarly proved.

The influence coefficients between criteria was computed as follows:

$$\xi_{12}^1 = \frac{(t_{12}^l + t_{12}^u) + (2 - (i_{12}^l + i_{12}^u))(2 - (f_{12}^l + f_{12}^u))}{6} = \frac{(0.3+0.4) + (2-(0.3+0.5)(2-(0.4+0.5))}{6} = 0.337,$$

$$\xi_{13}^1 = \frac{(t_{13}^l + t_{13}^u) + (2 - (i_{13}^l + i_{13}^u))(2 - (f_{13}^l + f_{13}^u))}{6} = \frac{(0.2+0.3) + (2-(0.4+0.5)(2-(0.5+0.6))}{6} = 0.248,$$

$$\xi_{23}^1 = \frac{(t_{23}^l + t_{23}^u) + (2 - (i_{23}^l + i_{23}^u))(2 - (f_{23}^l + f_{23}^u))}{6} = \frac{(0.4+0.6) + (2-(0.3+0.4)(2-(0.4+05))}{6} = 0.405.$$

Step 2. By applying Eq.(19) we can obtain the overall criterion value of the alternative p_1 as follows:

$\tilde{p}_1 = w_1 \times (e_{11}\xi_{11} + e_{12}\xi_{21} + e_{13}\xi_{31}) + w_2 \times (e_{11}\xi_{12} + e_{12}\xi_{22} + e_{13}\xi_{32}) + w_3 \times (e_{11}\xi_{13} + e_{12}\xi_{23} + e_{13}\xi_{33})$= 0.2 × (⟨[0.4, 0.5] , [0.2, 0.3], [0.3, 0.4]⟩ + 0.337 × ⟨[0.4, 0.6] , [0.1, 0.3], [0.2, 0.4]⟩+ 0.248 × ⟨[0.4, 0.5] , [0.2, 0.3], [0.7, 0.9]⟩)+0.25 × (0.337 × ⟨[0.4, 0.5] , [0.2, 0.3], [0.3, 0.4]⟩+ ⟨[0.4, 0.6] , [0.1, 0.3], [0.2, 0.4]⟩ + 0.405 × ⟨[0.4, 0.5] , [0.2, 0.3], [0.7, 0.9]⟩)+0.55 × (0.248 × (⟨[0.4, 0.5] , [0.2, 0.3], [0.3, 0.4]⟩ + 0.405 × ⟨[0.4, 0.6] , [0.1, 0.3], [0.2, 0.4]⟩+ ⟨[0.4, 0.5] , [0.2, 0.3], [0.7, 0.9]⟩))

$\tilde{p}_1$=<[0.6718, 0.8937], [0.2819, 0.5038], [0.7373, 1.0311]>.

Similarly,

$\tilde{p}_2$<[1.1514, 1.3193], [0.3117, 0.5515], [0.4077, 0.7194]>,

$\tilde{p}_3$ = <[0.8993,1.2232], [0.3359, 0.5757], [0.5757, 0.7436]>

and

$\tilde{p}_4$ = <[1.1934,1.3614], [0.2696, 0.4375], [0.5272, 0.7792]>.

Step 3. By applying Eq.(20) we can obtain $s(\tilde{p}_i)$ (i=1, 2, 3, 4) as follows:

$s(\tilde{p}_1)$=0.0564, $s(\tilde{p}_2)$ = 0.4043, $s(\tilde{p}_3)$ = 0.2450 and $s(\tilde{p}_4)$ =0.4660.

Step 4.Since $s(\tilde{p}_4) > s(\tilde{p}_2) > s(\tilde{p}_3) > s(\tilde{p}_1)$, the ranking order of four alternatives is $p_4 > p_2 > p_3 > p_1$. Therefore, we can see that the alternative p_4 is the best choice among all the alternatives

VI. Comparison Analysis

In order to verify the feasibility and effectiveness of the proposed decision-making approach, a comparison analysis with interval valued neutrosophic decision method, used by Zhao et al. [41], is given, based on the same illustrative example.

Clearly, the ranking order results are consistent with the result obtained in [41]; however, the best alternative is the same as A_4. because the ranking principle is different, these two methods produced the same best and worst alternatives.

VII. CONCLUSION

The interval neutrosophic set, as a concept combining single valued neutrosophic set and interval fuzzy set, provides additional capability to deal with uncertainty, inconsistent, incomplete and imprecise information by including a truth-membership interval, an indeterminacy-membership interval and a falsity membership interval. Therefore, it plays a significant role in the uncertainty system. An Interval valued neutrosophic models provide more precision, flexibility and compatibility to the system in comparison to classic, fuzzy models and neutrosophic model. In this study, we consider the importance of relationships among criteria in decision process, we developed a new model, called interval valued neutrosophic graph-based multicriteria decision making (IVNGMADM) method, to solve complex problems within the interval valued neutrosophic information. That is, the relationships among criteria for each alternative are included by this method in the decision process. In this case, we can select the alternative(s) according to the overall criteria values resulting from the model. Finally, an illustrative example was given to prove the application of proposed method. The developed method is more suitable to handle indeterminate information and inconsistent information in complex decision making problems with interval valued neutrosophic information.

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Interval Valued Neutrosophic Graphs

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Abstract

The notion of interval valued neutrosophic sets is a generalization of fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets, interval valued intuitionstic fuzzy sets and single valued neutrosophic sets. We apply for the first time to graph theory the concept of interval valued neutrosophic sets, an instance of neutrosophic sets. We introduce certain types of interval valued neutrosophc graphs (IVNG) and investigate some of their properties with proofs and examples.

Keyword

Interval valued neutrosophic set, Interval valued neutrosophic graph, Strong interval valued neutrosophic graph, Constant interval valued neutrosophic graph, Complete interval valued neutrosophic graph, Degree of interval valued neutrosophic graph.

1 Introduction

Neutrosophic sets (NSs) proposed by Smarandache [13, 14] are powerful mathematical tools for dealing with incomplete, indeterminate and inconsistent information in real world. They are a generalization of fuzzy sets [31], intuitionistic fuzzy sets [28, 30], interval valued fuzzy set [23] and interval-valued intuitionistic fuzzy sets theories [29].

The neutrosophic sets are characterized by a truth-membership function (t), an indeterminacy-membership function (i) and a falsity-membership function (f) independently, which are within the real standard or nonstandard unit interval $]^-0, 1^+[$. In order to conveniently practice NS in real life applications, Smarandache [53] and Wang et al. [17] introduced the concept of single-valued neutrosophic set (SVNS), a subclass of the neutrosophic sets.

The same authors [16, 18] introduced as well the concept of interval valued neutrosophic set (IVNS), which is more precise and flexible than the single valued neutrosophic set. The IVNS is a generalization of the single valued neutrosophic set, in which three membership functions are independent, and their values included into the unit interval [0, 1].

More on single valued neutrosophic sets, interval valued neutrosophic sets and their applications may be found in [3, 4, 5,6, 19, 20, 21, 22, 24, 25, 26, 27, 39, 41, 42, 43, 44, 45, 49].

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving a combinatorial problem in different areas, such as geometry, algebra, number theory, topology, optimization or computer science. Most important thing which is to be noted is that, when we have uncertainty regarding either the set of vertices or edges, or both, the model becomes a fuzzy graph.

The extension of fuzzy graph [7, 9, 38] theory have been developed by several researchers, including intuitionistic fuzzy graphs [8, 32, 40], considering the vertex sets and edge sets as intuitionistic fuzzy sets. In interval valued fuzzy graphs [33, 34], the vertex sets and edge sets are considered as interval valued fuzzy sets. In interval valued intuitionstic fuzzy graphs [2, 48], the vertex sets and edge sets are regarded as interval valued intuitionstic fuzzy sets. In bipolar fuzzy graphs [35, 36], the vertex sets and edge sets are considered as bipolar fuzzy sets. In m-polar fuzzy graphs [37], the vertex sets and edge sets are regarded as m-polar fuzzy sets.

But, when the relations between nodes (or vertices) in problems are indeterminate, the fuzzy graphs and their extensions fail. In order to overcome the failure, Smarandache [10, 11, 12, 51] defined four main categories of neutrosophic graphs: I-edge neutrosophic graph, I-vertex neutrosophic graph [1, 15, 50, 52], (*t, i, f*)-edge neutrosophic graph and (*t, i, f*)-vertex neutrosophic graph. Later on, Broumi et al. [47] introduced another neutrosophic graph model. This model allows the attachment of truth-membership (*t*), indeterminacy –membership (*i*) and falsity-membership (*f*) degrees both to vertices and edges. A neutrosophic graph model that generalizes the fuzzy graph and intuitionstic fuzzy graph is called single valued neutrosophic graph (SVNG). Broumi [46] introduced as well the neighborhood degree of a vertex and closed neighborhood degree of a vertex in single valued neutrosophic graph, as generalizations of neighborhood degree of a vertex and closed neighborhood degree of a vertex in fuzzy graph and intuitionistic fuzzy graph.

In this paper, we focus on the study of interval valued neutrosophic graphs.

2 Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, fuzzy graph, intuitionistic fuzzy graph, single valued neutrosophic graphs, relevant to the present work. See especially [2, 7, 8, 13, 18, 47] for further details and background.

Definition 2.1 [13]

Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X→$]^-0,1^+[$ define respectively the a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x)\leq 3^+.$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0,1^+[$.

Since it is difficult to apply NSs to practical problems, Wang et al. [16] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [17]

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}$$

Definition 2.3 [7]

A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1], such that μ(uv) ≤ σ(u) ⋀ σ(v) for all u, v ∈ V where *uv* denotes the edge between *u* and *v* and σ(u) ⋀ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

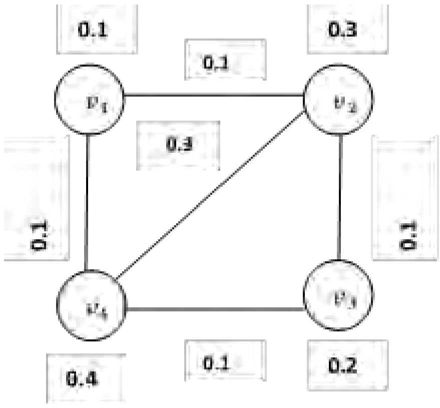

Figure 1: Fuzzy Graph

Definition 2.4 [7]

The fuzzy subgraph H = (τ, ρ) is called a fuzzy subgraph of G = (σ, μ), if τ(u) ≤ σ(u) for all u ∈ V and ρ(u, v) ≤ μ(u, v) for all u, v ∈ V.

Definition 2.5 [8]

An Intuitionistic fuzzy graph is of the form G =(V, E), where

i. V={v_1, v_2,…., v_n} such that μ_1: V→ [0,1] and γ_1: V → [0,1] denote the degree of membership and nonmembership of the element v_i ∈ V, respectively, and 0 ≤ $\mu_1(v_i) + \gamma_1(v_i)$) ≤ 1 , for every v_i ∈ V, (i = 1, 2, ……. n);

ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that $\mu_2(v_i, v_j) \leq \min [\mu_1(v_i), \mu_1(v_j)]$ and $\gamma_2(v_i, v_j) \geq \max [\gamma_1(v_i), \gamma_1(v_j)]$ and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every (v_i, v_j) ∈ E, (i, j = 1,2, ……. n)

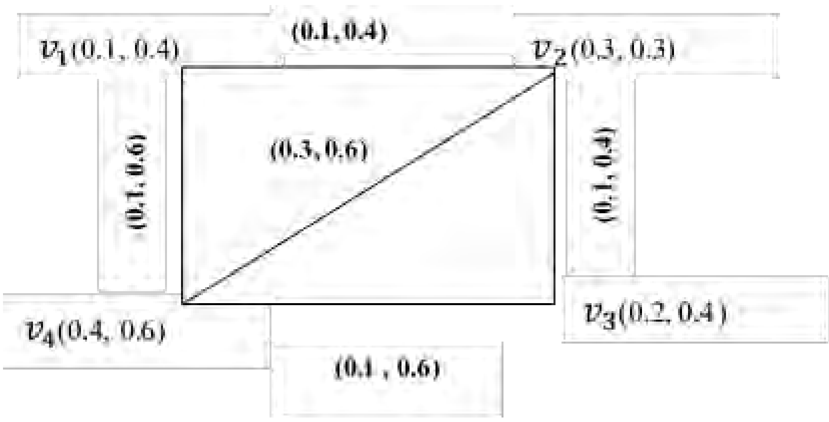

Figure 2: Intuitionistic Fuzzy Graph

Definition 2.6 [2]

An interval valued intuitionistic fuzzy graph with underlying set V is defined to be a pair G= (A, B), where

1) The functions M_A : V→ D [0, 1] and N_A : V→ D [0, 1] denote the degree of membership and non-membership of the element x ∈ V, respectively, such that 0≤M_A(x)+ N_A(x) ≤ 1 for all x ∈ V.

2) The functions $M_B : E \subseteq V \times V \rightarrow$ D [0, 1] and $N_B : : E \subseteq V \times V \rightarrow$ D [0, 1] are defined by:

$$M_{BL}(x, y))\leq \min (M_{AL}(x), M_{AL}(y)),$$

$$N_{BL}(x, y)) \geq \max (N_{AL}(x), N_{AL}(y)),$$

$$M_{BU}(x, y))\leq \min (M_{AU}(x), M_{AU}(y)),$$

$$N_{BU}(x, y)) \geq \max (N_{AU}(x), N_{AU}(y)),$$

such that

$$0\leq M_{BU}(x, y))+ N_{BU}(x, y)) \leq 1, \text{ for all } (x, y) \in E.$$

Definition 2.7 [47]

Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be single valued neutrosophic sets on a set X. If $A = (T_A, I_A, F_A)$ is a single valued neutrosophic relation on a set X, then $A =(T_A, I_A, F_A)$ is called a single valued neutrosophic relation on $B = (T_B, I_B, F_B)$, if

$$T_B(x, y) \leq \min(T_A(x), T_A(y)),$$

$$I_B(x, y) \geq \max(I_A(x), I_A(y)),$$

$$F_B(x, y) \geq \max(F_A x), F_A(y)),$$

for all x, y ∈ X.
A single valued neutrosophic relation A on X *is* called symmetric if

$$T_A(x, y) = T_A(y, x), I_A(x, y) = I_A(y, x), F_A(x, y) = F_A(y, x)$$

$$T_B(x, y) = T_B(y, x), I_B(x, y) = I_B(y, x)$$

$$F_B(x, y) = F_B(y, x),$$

for all $x, y \in X$.

Definition 2.8 [47]

A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G = (A, B), where

1) The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and

$$0\leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3,$$

for all $v_i \in$ V (i=1, 2, …, n).

2) The functions T_B: E ⊆ V x V →[0, 1], I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by

$$T_B(\{v_i, v_j\}) \le \min [T_A(v_i), T_A(v_j)],$$

$$I_B(\{v_i, v_j\}) \ge \max [I_A(v_i), I_A(v_j)],$$

$$F_B(\{v_i, v_j\}) \ge \max [F_A(v_i), F_A(v_j)],$$

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i, v_j) ∈ E respectively, where

$$0 \le T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \le 3,$$

for all $\{v_i, v_j\}$ ∈ E (i, j = 1, 2, ..., n).

We call A the single valued neutrosophic vertex set of V, and B the single valued neutrosophic edge set of E, respectively. Note that B is a symmetric single valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is a single valued neutrosophic graph of G*= (V, E) if

$$T_B(v_i, v_j) \le \min [T_A(v_i), T_A(v_j)],$$

$$I_B(v_i, v_j) \ge \max [I_A(v_i), I_A(v_j)],$$

$$F_B(v_i, v_j) \ge \max [F_A(v_i), F_A(v_j)],$$

for all (v_i, v_j) ∈ E.

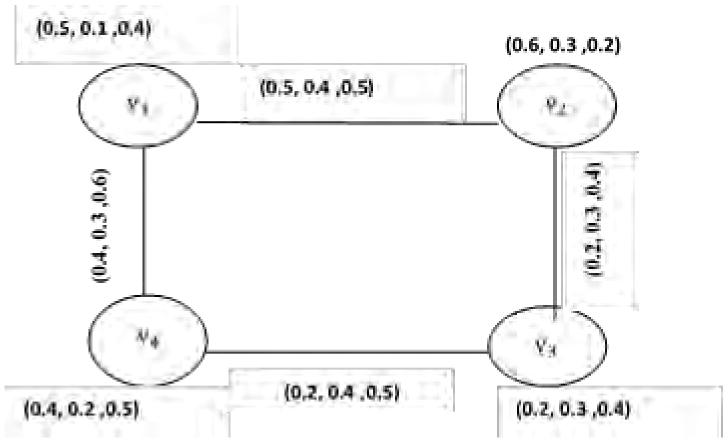

Figure 3: Single valued neutrosophic graph

Definition 2.9 [47]

A partial SVN-subgraph of SVN-graph G= (A, B) is a SVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that

(i) $\boldsymbol{V' \subseteq V}$, where $\boldsymbol{T'_A(v_i) \le T_A(v_i)}$, $\boldsymbol{I'_A(v_i) \ge I_A(v_i)}$, $\boldsymbol{F'_A(v_i) \ge F_A(v_i)}$, for all $\boldsymbol{v_i \in V}$.

(ii) $\boldsymbol{E' \subseteq E}$, where $\boldsymbol{T'_B(v_i, v_j) \le T_B(v_i, v_j)}$, $\boldsymbol{I'_{Bij} \ge I_B(v_i, v_j)}$, $\boldsymbol{F'_B(v_i, v_j) \ge F_B(v_i, v_j)}$, for all $\boldsymbol{(v_i\ v_j) \in E}$.

Definition 2.10 [47]

A SVN-subgraph of SVN-graph G= (V, E) is a SVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that

(i) $\boldsymbol{V'} = \boldsymbol{V}$, where $\boldsymbol{T'_A(v_i)} = \boldsymbol{T_A(v_i)}$, $\boldsymbol{I'_A(v_i)} = \boldsymbol{I_A(v_i)}$, $\boldsymbol{F'_A(v_i)} = \boldsymbol{F_A(v_i)}$ for all $\boldsymbol{v_i}$ in the vertex set of $\boldsymbol{V'}$.

(ii) $\boldsymbol{E'} = \boldsymbol{E}$, where $\boldsymbol{T'_B(v_i, v_j)} = \boldsymbol{T_B(v_i, v_j)}$, $\boldsymbol{I'_B(v_i, v_j)} = \boldsymbol{I_B(v_i, v_j)}$, $\boldsymbol{F'_B(v_i, v_j)} = \boldsymbol{F_B(v_i, v_j)}$ for every $(\boldsymbol{v_i}\ \boldsymbol{v_j}) \in \boldsymbol{E}$ in the edge set of $\boldsymbol{E'}$.

Definition 2.11 [47]

Let G= (A, B) be a single valued neutrosophic graph. Then the degree of any vertex v is the sum of degree of truth-membership, sum of degree of indeterminacy-membership and sum of degree of falsity-membership of all those edges which are incident on vertex v denoted by d(v) = ($d_T(v)$, $d_I(v)$, $d_F(v)$), where

$d_T(v)=\sum_{u \neq v} T_B(u, v)$ denotes degree of truth-membership vertex,

$d_I(v)=\sum_{u \neq v} I_B(u, v)$ denotes degree of indeterminacy-membership vertex,

$d_F(v)=\sum_{u \neq v} F_B(u, v)$ denotes degree of falsity-membership vertex.

Definition 2.12 [47]

A single valued neutrosophic graph G=(A, B) of G^*= (V, E) is called strong single valued neutrosophic graph, if

$$T_B(v_i, v_j) = \min\,[T_A(v_i),\ T_A(v_j)],$$

$$I_B(v_i, v_j) = \max\,[I_A(v_i),\ I_A(v_j)],$$

$$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)],$$

for all $(v_i, v_j) \in$ E.

Definition 2.13 [47]

A single valued neutrosophic graph G = (A, B) is called complete if

$$T_B(v_i, v_j) = \min\,[T_A(v_i),\ T_A(v_j)],$$

$$I_B(v_i, v_j) = \max\,[I_A(v_i),\ I_A(v_j)],$$

$$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)],$$

for all $v_i, v_j \in$ V.

Definition 2.14 [47]

The complement of a single valued neutrosophic graph G (A, B) on G^* is a single valued neutrosophic graph $\bar{G}$ on G^*, where:

1. $\bar{A}$ =A

2. $\overline{T_A}(v_i)$= $T_A(v_i)$, $\overline{I_A}(v_i)$= $I_A(v_i)$, $\overline{F_A}(v_i)$ = $F_A(v_i)$, for all $v_j \in$ V.

3. $\overline{T_B}(v_i, v_j)$= min $[T_A(v_i), T_A(v_j)] - T_B(v_i, v_j)$

$\overline{I_B}(v_i, v_j)$= max $[I_A(v_i), I_A(v_j)] - I_B(v_i, v_j)$, and

$\overline{F_B}(v_i, v_j)$= max $[F_A(v_i), F_A(v_j)] - F_B(v_i, v_j)$,

for all $(v_i, v_j) \in$ E.

Definition 2.15 [18]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS A) A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x)$= $[T_{AL}(x), T_{AU}(x)]$, $I_A(x)$ = $[I_{AL}(x), I_{AU}(x)]$, $F_A(x)$ = $[F_{AL}(x), F_{AU}(x)] \subseteq$ [0, 1] and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.16 [18]

An IVNS A is contained in the IVNS B, A ⊆ B, if and only if $T_{AL}(x) \leq T_{BL}(x)$, $T_{AU}(x) \leq T_{BU}(x)$, $I_{AL}(x) \geq I_{BL}(x)$, $I_{AU}(x) \geq I_{BU}(x)$, $F_{AL}(x) \geq F_{BL}(x)$ and $F_{AU}(x) \geq F_{BU}(x)$ for any x in X.

Definition 2.17 [18]

The union of two interval valued neutrosophic sets A and B is an interval neutrosophic set C, written as C = A ∪ B, whose truth-membership, indeterminacy-membership, and false membership are related to A and B by

$T_{CL}(x) = \max(T_{AL}(x), T_{BL}(x))$

$T_{CU}(x) = \max(T_{AU}(x), T_{BU}(x))$

$I_{CL}(x) = \min(I_{AL}(x), I_{BL}(x))$

$I_{CU}(x) = \min(I_{AU}(x), I_{BU}(x))$

$F_{CL}(x) = \min(F_{AL}(x), F_{BL}(x))$

$F_{CU}(x) = \min(F_{AU}(x), F_{BU}(x))$

for all x in X.

Definition 2.18 [18]

Let X and Y be two non-empty crisp sets. An interval valued neutrosophic relation R(X, Y) is a subset of product space X × Y, and is characterized by the truth membership function $T_R(x, y)$, the indeterminacy membership function $I_R(x, y)$, and the falsity membership function $F_R(x, y)$, where x ∈ X and y ∈ Y and $T_R(x, y)$, $I_R(x, y)$, $F_R(x, y) \subseteq [0, 1]$.

3 Interval Valued Neutrosophic Graphs

Throughout this paper, we denote G^* = (V, E) a crisp graph, and G = (A, B) an interval valued neutrosophic graph.

Definition 3.1

By an interval-valued neutrosophic graph of a graph G^* = (V, E) we mean a pair G = (A, B), where A =< $[T_{AL}, T_{AU}]$, $[I_{AL}, I_{AU}]$, $[F_{AL}, F_{AU}]$> is an interval-valued neutrosophic set on V; and B =< $[T_{BL}, T_{BU}]$, $[I_{BL}, I_{BU}]$, $[F_{BL}, F_{BU}]$> is an interval-valued neutrosophic relation on E satisfying the following condition:

1) V = $\{v_1, v_2, \ldots, v_n\}$, such that T_{AL}:V→[0, 1], T_{AU}:V→[0, 1], I_{AL}:V→[0, 1], I_{AU}:V→[0, 1] and F_{AL}:V→[0, 1], F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element $y \in$ V, respectively, and

$$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3,$$

for all $v_i \in$ V (i=1, 2, …,n)

2) The functions T_{BL}:V x V →[0, 1], T_{BU}:V x V →[0, 1], I_{BL}:V x V →[0, 1], I_{BU}:V x V →[0, 1] and F_{BL}:V x V →[0,1], F_{BU}:V x V →[0, 1] are such that

$$T_{BL}(\{v_i, v_j\}) \leq \min\,[T_{AL}(v_i), T_{AL}(v_j)],$$

$$T_{BU}(\{v_i, v_j\}) \leq \min\,[T_{AU}(v_i), T_{AU}(v_j)],$$

$$I_{BL}(\{v_i, v_j\}) \geq \max[I_{BL}(v_i), I_{BL}(v_j)],$$

$$I_{BU}(\{v_i, v_j\}) \geq \max[I_{BU}(v_i), I_{BU}(v_j)],$$

$$F_{BL}(\{v_i, v_j\}) \geq \max[F_{BL}(v_i), F_{BL}(v_j)],$$

$$F_{BU}(\{v_i, v_j\}) \geq \max[F_{BU}(v_i), F_{BU}(v_j)],$$

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where

$$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3$$

for all $\{v_i, v_j\} \in$ E (i, j = 1, 2,…, n).

We call A the interval valued neutrosophic vertex set of V, and B the interval valued neutrosophic edge set of E, respectively. Note that B is a symmetric interval valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is an interval valued neutrosophic graph of G*= (V, E) if

$$T_{BL}(v_i, v_j) \leq \min[T_{AL}(v_i), T_{AL}(v_j)],$$

$$T_{BU}(v_i, v_j) \leq \min[T_{AU}(v_i), T_{AU}(v_j)],$$

$$I_{BL}(v_i, v_j) \geq \max[I_{BL}(v_i), I_{BL}(v_j)],$$

$$I_{BU}(v_i, v_j) \geq \max[I_{BU}(v_i), I_{BU}(v_j)],$$

$$F_{BL}(v_i, v_j) \geq \max[F_{BL}(v_i), F_{BL}(v_j)],$$

$$F_{BU}(v_i, v_j) \geq \max[F_{BU}(v_i), F_{BU}(v_j)] \text{ — for all } (v_i, v_j) \in \text{E}.$$

Example 3.2

Consider a graph G^*, such that V = $\{v_1, v_2, v_3, v_4\}$, E = $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be a interval valued neutrosophic subset of V and B a interval valued neutrosophic subset of E, denoted by

	v_1	v_2	v_3
T_{AL}	0.3	0.2	0.1
T_{AU}	0.5	0.3	0.3
I_{AL}	0.2	0.2	0.2
I_{AU}	0.3	0.3	0.4
F_{AL}	0.3	0.1	0.3
F_{AU}	0.4	0.4	0.5

	v_1v_2	v_2v_3	v_3v_1
T_{BL}	0.1	0.1	0.1
T_{BU}	0.2	0.3	0.2
I_{BL}	0.3	0.4	0.3
I_{BU}	0.4	0.5	0.5
F_{BL}	0.4	0.4	0.4
F_{BU}	0.5	0.5	0.6

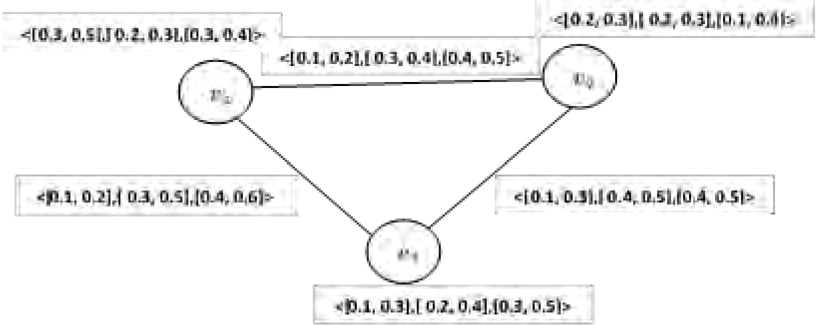

Figure 4: G: Interval valued neutrosophic graph

In *Figure 4*,

(i) (v_1, <[0.3, 0.5],[0.2, 0.3],[0.3, 0.4]>) is an interval valued neutrosophic vertex or IVN-vertex.

(ii) (v_1v_2, <[0.1, 0.2], [0.3, 0.4], [0.4, 0.5]>) is an interval valued neutrosophic edge or IVN-edge.

(iii) (v_1, <[0.3, 0.5], [0.2, 0.3], [0.3, 0.4]>) and (v_2, <[0.2, 0.3],[0.2, 0.3],[0.1, 0.4]>) are interval valued neutrosophic adjacent vertices.

(iv) (v_1v_2, <[0.1, 0.2], [0.3, 0.4], [0.4, 0.5]>) and (v_1v_3, <[0.1, 0.2],[0.3, 0.5], [0.4, 0.6]>) are an interval valued neutrosophic adjacent edge.

Remarks

(i) When $T_{BL}(v_i, v_j) = T_{BU}(v_i, v_j) = I_{BL}(v_i, v_j) = I_{BU}(v_i, v_j) = F_{BL}(v_i, v_j) = F_{BU}(v_i, v_j)$ for some *i* and *j*, then there is no edge between v_i and v_j . Otherwise there exists an edge between v_i and v_j .

(ii) If one of the inequalities is not satisfied in (1) and (2), then G is not an IVNG.

The interval valued neutrosophic graph G depicted in *Figure 3* is represented by the following adjacency matrix $\boldsymbol{M_G}$ —

$$\boldsymbol{M_G} = \begin{bmatrix} <[0.3,0.5],[0.2,0.3],[0.3,0.4]> & <[0.1,0.2],[0.3,0.4],[0.4,0.5]> & <[0.1,0.2],[0.3,0.5],[0.4,0.6]> \\ <[0.1,0.2],[0.3,0.4],[0.4,0.5]> & <[0.2,0.3],[0.2,0.3],[0.1,0.4]> & <[0.1,0.3],[0.4,0.5],[0.4,0.5]> \\ <[0.1,0.2],[0.3,0.5],[0.4,0.6]> & <[0.1,0.3],[0.4,0.5],[0.4,0.5]> & <[0.1,0.3],[0.2,0.4],[0.3,0.5]> \end{bmatrix}$$

Definition 3.3

A partial IVN-subgraph of IVN-graph G= (A, B) is an IVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that —

(i) $\boldsymbol{V' \subseteq V}$, where $\boldsymbol{T'_{AL}(v_i) \le T_{AL}(v_i)}$, $\boldsymbol{T'_{AU}(v_i) \le T_{AU}(v_i)}$, $\boldsymbol{I'_{AL}(v_i) \ge I_{AL}(v_i)}$, $\boldsymbol{I'_{AU}(v_i) \ge I_{AU}(v_i)}$, $\boldsymbol{F'_{AL}(v_i) \ge F_{AL}(v_i)}$, $\boldsymbol{F'_{AU}(v_i) \ge F_{AU}(v_i)}$, for all $\boldsymbol{v_i \in V}$.

(ii) $\boldsymbol{E' \subseteq E}$, where $\boldsymbol{T'_{BL}(v_i, v_j) \le T_{BL}(v_i, v_j)}$, $\boldsymbol{T'_{BU}(v_i, v_j) \le T_{BU}(v_i, v_j)}$, $\boldsymbol{I'_{BL}(v_i, v_j) \ge I_{BL}(v_i, v_j)}$, $\boldsymbol{I'_{BU}(v_i, v_j) \ge I_{BU}(v_i, v_j)}$, $\boldsymbol{F'_{BL}(v_i, v_j) \ge F_{BL}(v_i, v_j)}$, $\boldsymbol{F'_{BU}(v_i, v_j) \ge F_{BU}(v_i, v_j)}$, for all $\boldsymbol{(v_i\, v_j) \in E}$.

Definition 3.4

An IVN-subgraph of IVN-graph G= (V, E) is an IVN-graph H = ($\boldsymbol{V'}$, $\boldsymbol{E'}$) such that

(i) $\boldsymbol{T'_{AL}(v_i) = T_{AL}(v_i)}$, $\boldsymbol{T'_{AU}(v_i) = T_{AU}(v_i)}$, $\boldsymbol{I'_{AL}(v_i) = I_{AL}(v_i)}$, $\boldsymbol{I'_{AU}(v_i) = I_{AU}(v_i)}$, $\boldsymbol{F'_{AL}(v_i) = F_{AL}(v_i)}$, $\boldsymbol{F'_{AU}(v_i) = F_{AU}(v_i)}$, for all $\boldsymbol{v_i}$ in the vertex set of $\boldsymbol{V'}$.

(ii) $\boldsymbol{E' = E}$, where $\boldsymbol{T'_{BL}(v_i, v_j) = T_{BL}(v_i, v_j)}$, $\boldsymbol{T'_{BU}(v_i, v_j) = T_{BU}(v_i, v_j)}$,

$\boldsymbol{I'_{BL}(v_i, v_j) = I_{BL}(v_i, v_j)}$, $\boldsymbol{I'_{BU}(v_i, v_j) = I_{BU}(v_i, v_j)}$, $\boldsymbol{F'_{BL}(v_i, v_j) = F_{BL}(v_i, v_j)}$, $\boldsymbol{F'_{BU}(v_i, v_j) = F_{BU}(v_i, v_j)}$, for every $(\boldsymbol{v_i\ v_j}) \in \boldsymbol{E}$ in the edge set of $\boldsymbol{E'}$.

Example 3.5

$\mathbf{G_1}$ in *Figure 5* is an IVN-graph, $\mathbf{H_1}$ in *Figure 6* is a partial IVN-subgraph and $\mathbf{H_2}$ in *Figure 7* is a IVN-subgraph of $\mathbf{G_1}$.

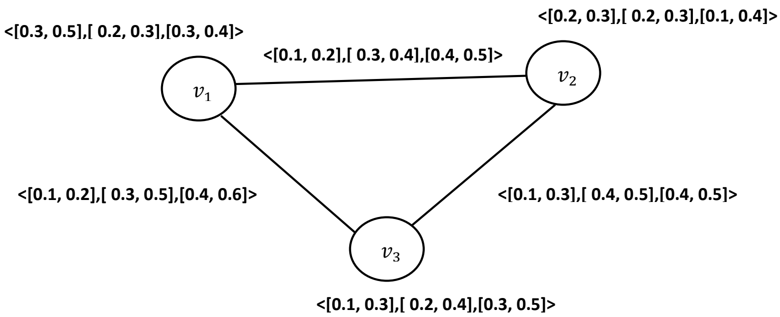

Figure 5: G_1, an interval valued neutrosophic graph

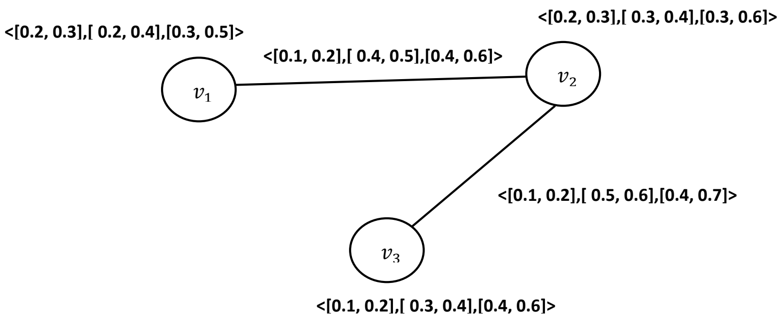

Figure 6: H_1, a partial IVN-subgraph of G_1

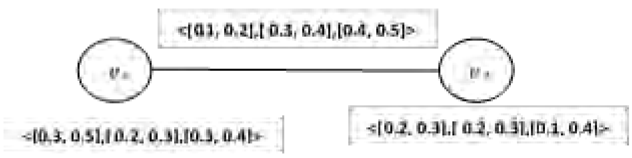

Figure 7: H_2, an IVN-subgraph of G_1

Definition 3.6

The two vertices are said to be adjacent in an interval valued neutrosophic graph G= (A, B) if —

$$T_{BL}(v_i, v_j) = \min[T_{AL}(v_i), T_{AL}(v_j)],$$
$$T_{BU}(v_i, v_j) = \min[T_{AU}(v_i), T_{AU}(v_j)],$$
$$I_{BL}(v_i, v_j) = \max[I_{AL}(v_i), I_{AL}(v_j)]$$
$$I_{BU}(v_i, v_j) = \max[I_{AU}(v_i), I_{AU}(v_j)]$$

$$F_{BL}(v_i, v_j) = \max[F_{AL}(v_i), F_{AL}(v_j)]$$
$$F_{BU}(v_i, v_j) = \max[F_{AU}(v_i), F_{AU}(v_j)]$$

In this case, v_i and v_j are said to be neighbours and (v_i, v_j) is incident at v_i and v_j also.

Definition 3.7

A path P in an interval valued neutrosophic graph G= (A, B) is a sequence of distinct vertices $v_0, v_1, v_3,... v_n$ such that $T_{BL}(v_{i-1}, v_i) > 0$, $T_{BU}(v_{i-1}, v_i) > 0$, $I_{BL}(v_{i-1}, v_i) > 0$, $I_{BU}(v_{i-1}, v_i) > 0$ and $F_{BL}(v_{i-1}, v_i) > 0$, $F_{BU}(v_{i-1}, v_i) > 0$ for $0 \leq i \leq 1$. Here $n \geq 1$ is called the length of the path P. A single node or vertex v_i may also be considered as a path. In this case, the path is of the length ([0, 0], [0, 0], [0, 0]). The consecutive pairs (v_{i-1}, v_i) are called edges of the path. We call P a cycle if $v_0 = v_n$ and $n \geq 3$.

Definition 3.8

An interval valued neutrosophic graph G= (A, B) is said to be connected if every pair of vertices has at least one interval valued neutrosophic path between them, otherwise it is disconnected.

Definition 3.9

A vertex $v_j \in V$ of interval valued neutrosophic graph G= (A, B) is said to be an isolated vertex if there is no effective edge incident at v_j.

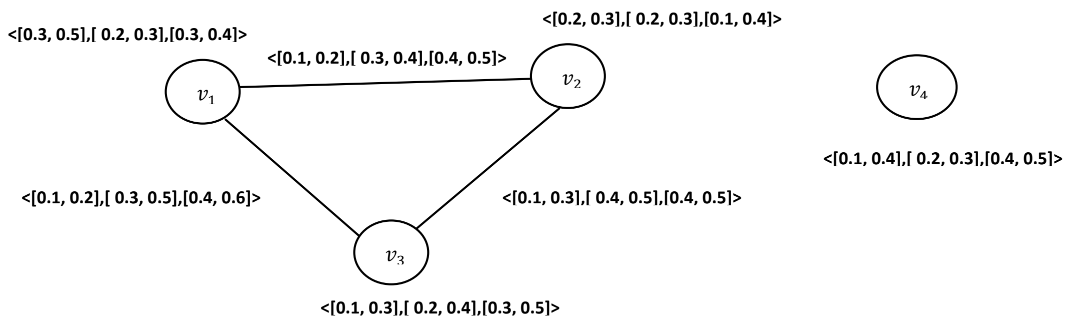

Figure 8. Example of interval valued neutrosophic graph

In *Figure 8*, the interval valued neutrosophic vertex v_4 is an isolated vertex.

Definition 3.10

A vertex in an interval valued neutrosophic G = (A, B) having exactly one neighbor is called *a pendent vertex*. Otherwise, it is called *non-pendent vertex*. An edge in an interval valued neutrosophic graph incident with a pendent vertex is called a *pendent edge*. Otherwise it is called *non-pendent edge*. A vertex in an interval valued neutrosophic graph adjacent to the pendent vertex is called a support of the pendent edge.

Definition 3.11

An interval valued neutrosophic graph G = (A, B) that has neither self-loops nor parallel edge is called simple interval valued neutrosophic graph.

Definition 3.12

When a vertex $\mathbf{v_i}$ is end vertex of some edges $(\mathbf{v_i}, \mathbf{v_j})$ of any IVN-graph G = (A, B). Then $\mathbf{v_i}$ and $(\mathbf{v_i}, \mathbf{v_j})$ are said to be incident to each other.

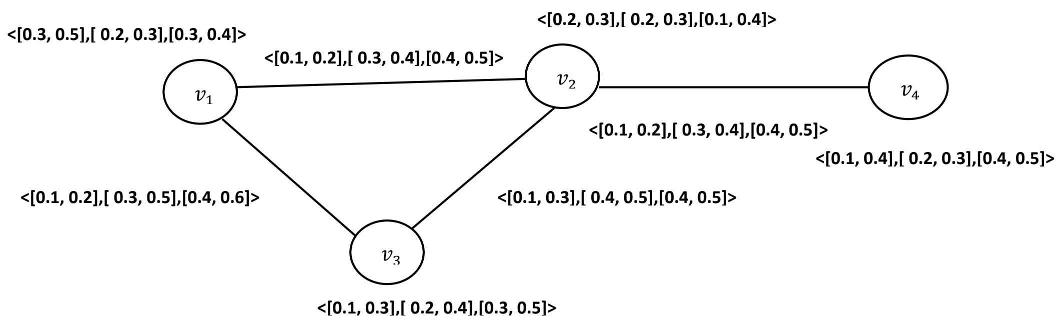

Figure 9. Incident IVN-graph.

In this graph v_2v_1, v_2v_3 and v_2v_4 are incident on v_2.

Definition 3.13

Let G = (A, B) be an interval valued neutrosophic graph. Then the degree of any vertex v is sum of degree of truth-membership, sum of degree of indeterminacy-membership and sum of degree of falsity-membership of all those edges which are incident on vertex v denoted by —

$$d(v) = ([d_{TL}(v), d_{TU}(v)], [d_{IL}(v), d_{IU}(v)], [d_{FL}(v), d_{FU}(v)]),$$

where:

$d_{TL}(v)=\sum_{u \neq v} T_{BL}(u, v)$ denotes the degree of lower truth-membership vertex;

$d_{TU}(v) = \sum_{u \neq v} T_{BU}(u, v)$ denotes the degree of upper truth-membership vertex;

$d_{IL}(v) = \sum_{u \neq v} I_{BL}(u, v)$ denotes the degree of lower indeterminacy-membership vertex;

$d_{IU}(v) = \sum_{u \neq v} I_{BU}(u, v)$ denotes the degree of upper indeterminacy-membership vertex;

$d_{FL}(v) = \sum_{u \neq v} F_{BL}(u, v)$ denotes the degree of lower falsity-membership vertex;

$d_{FU}(v) = \sum_{u \neq v} F_{BU}(u, v)$ denotes the degree of upper falsity-membership vertex.

Example 3.14

Let us consider an interval valued neutrosophic graph G = (A, B) of G^* = (V, E) where V = {v_1, v_2, v_3, v_4} and E = {v_1v_2, v_2v_3, v_3v_4 , v_4v_1}.

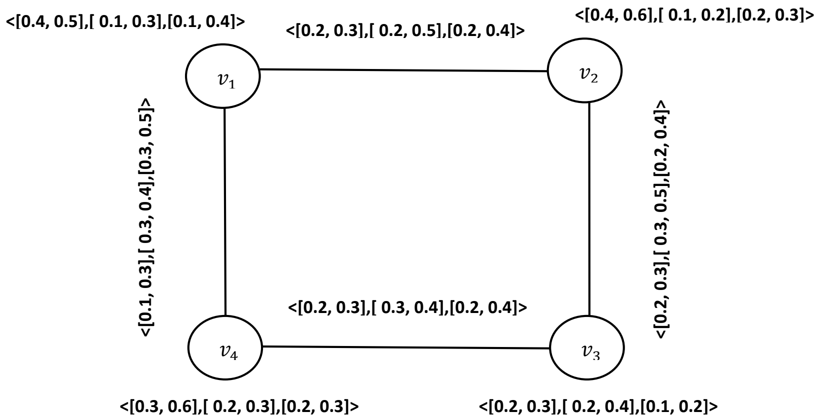

Figure 10: Degree of vertex of interval valued neutrosophic graph

We have the degree of each vertex as follows:

$d(v_1)$= ([0.3, 0.6], [0.5, 0.9], [0.5, 0.9]), $d(v_2)$= ([0.4, 0.6], [0.5, 1.0], [0.4, 0.8]), $d(v_3)$= ([0.4, 0.6], [0.6, 0.9], [0.4, 0.8]), $d(v_4)$= ([0.3, 0.6], [0.6, 0.8], [0.5, 0.9]).

Definition 3.15

An interval valued neutrosophic graph G= (A, B) is called constant if degree of each vertex is k =($[k_{1L}, k_{1U}]$, $[k_{2L}, k_{2U}]$, $[k_{3L}, k_{3U}]$). That is d(v) =($[k_{1L}, k_{1U}]$, $[k_{2L}, k_{2U}]$, $[k_{3L}, k_{3U}]$), for all $v \in$ V.

Example 3.16

Consider an interval valued neutrosophic graph G such that V = {v_1, v_2, v_3, v_4} and E = {v_1v_2, v_2v_3, v_3v_4 , v_4v_1}.

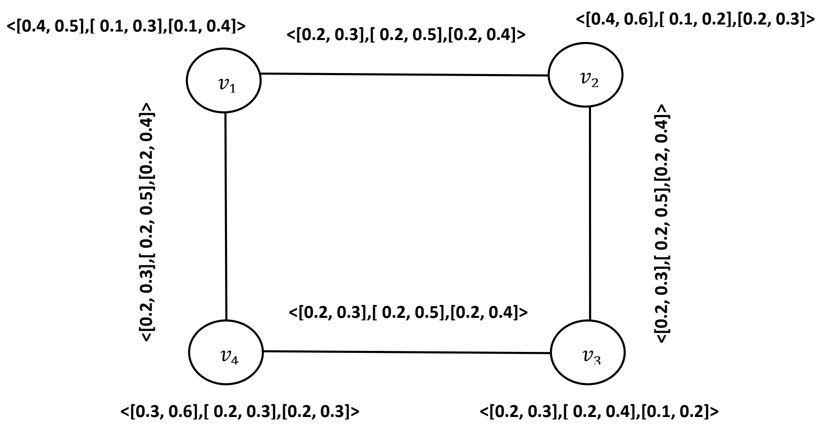

Figure 11. Constant IVN-graph.

Clearly, G is constant IVN-graph since the degree of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$ and $\boldsymbol{v_4}$ is ([0.4, 0.6], [0.4, 1], [0.4, 0.8])

Definition 3.17

An interval valued neutrosophic graph G = (A, B) of G^*= (V, E) is called strong interval valued neutrosophic graph if

$T_{BL}(v_i, v_j) = \min[T_{AL}(v_i),\ T_{AL}(v_j)], T_{BU}(v_i, v_j) = \min[T_{AU}(v_i),\ T_{AU}(v_j)]$

$I_{BL}(v_i, v_j) = \max[I_{AL}(v_i),\ I_{AL}(v_j)],\ I_{BU}(v_i, v_j) = \max[I_{AU}(v_i),\ I_{AU}(v_j)]$

$F_{BL}(v_i, v_j) = \max[F_{AL}(v_i), F_{AL}(v_j)],\ F_{BU}(v_i, v_j) = \max[F_{AU}(v_i),\ F_{AU}(v_j)]$, for all $(v_i, v_j) \in$ E.

Example 3.18

Consider a graph G^* such that V = $\{v_1, v_2, v_3, v_4\}$, E = $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be an interval valued neutrosophic subset of V and let B an interval valued neutrosophic subset of E denoted by:

	v_1	v_2	v_3
T_{AL}	0.3	0.2	0.1
T_{AU}	0.5	0.3	0.3
I_{AL}	0.2	0.2	0.2
I_{AU}	0.3	0.3	0.4
F_{AL}	0.3	0.1	0.3
F_{AU}	0.4	0.4	0.5

	v_1v_2	v_2v_3	v_3v_1
T_{BL}	0.2	0.1	0.1
T_{BU}	0.3	0.3	0.3
I_{BL}	0.2	0.2	0.2
I_{BU}	0.3	0.4	0.4
F_{BL}	0.3	0.3	0.3
F_{BU}	0.4	0.4	0.5

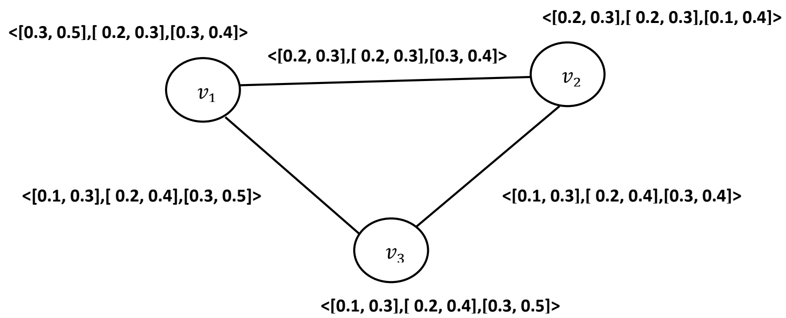

Figure 12. Strong IVN-graph.

By routing computations, it is easy to see that G is a strong interval valued neutrosophic of G^*.

Proposition 3.19

An interval valued neutrosophic graph is the generalization of interval valued fuzzy graph

Proof

Suppose G = (V, E) be an interval valued neutrosophic graph. Then by setting the indeterminacy-membership and falsity-membership values of vertex set and edge set equals to zero reduces the interval valued neutrosophic graph to interval valued fuzzy graph.

Proposition 3.20

An interval valued neutrosophic graph is the generalization of interval valued intuitionistic fuzzy graph

Proof

Suppose G = (V, E) is an interval valued neutrosophic graph. Then by setting the indeterminacy-membership values of vertex set and edge set equals to zero reduces the interval valued neutrosophic graph to interval valued intuitionistic fuzzy graph.

Proposition 3.21

An interval valued neutrosophic graph is the generalization of intuitionistic fuzzy graph.

Proof

Suppose G = (V, E) is an interval valued neutrosophic graph. Then by setting the indeterminacy-membership, upper truth-membership and upper falsity-membership values of vertex set and edge set equals to zero reduces the interval valued neutrosophic graph to intuitionistic fuzzy graph.

Proposition 3.22

An interval valued neutrosophic graph is the generalization of single valued neutrosophic graph.

Proof

Suppose G = (V, E) is an interval valued neutrosophic graph. Then by setting the upper truth-membership equals lower truth-membership, upper indeterminacy-membership equals lower indeterminacy-membership and upper falsity-membership equals lower falsity-membership values of vertex set and edge set reduces the interval valued neutrosophic graph to single valued neutrosophic graph.

Definition 3.23

The complement of an interval valued neutrosophic graph G (A, B) on G^* is an interval valued neutrosophic graph $\bar{G}$ on G^* where:

1. $\bar{A}$ =A

2. $\overline{T_{AL}}(v_i)= T_{AL}(v_i)$, $\overline{T_{AU}}(v_i)= T_{AU}(v_i)$, $\overline{I_{AL}}(v_i)= I_{AL}(v_i)$, $\overline{I_{AU}}(v_i)= I_{AU}(v_i)$, $\overline{F_{AL}}(v_i) = F_{AL}(v_i)$, $\overline{F_{AU}}(v_i) = F_{AU}(v_i)$,

for all $v_j \in$ V.

3. $\overline{T_{BL}}(v_i, v_j)= \min\left[T_{AL}(v_i), T_{AL}(v_j)\right] - T_{BL}(v_i, v_j)$,
$\overline{T_{BU}}(v_i, v_j)= \min\left[T_{AU}(v_i), T_{AU}(v_j)\right] - T_{BU}(v_i, v_j)$,

$\overline{I_{BL}}(v_i, v_j)= \max\left[I_{AL}(v_i), I_{AL}(v_j)\right] - I_{BL}(v_i, v_j)$, $\overline{I_{BU}}(v_i, v_j)= \max\left[I_{AU}(v_i), I_{AU}(v_j)\right] - I_{BU}(v_i, v_j)$,

and

$\overline{F_{BL}}(v_i, v_j)= \max\left[F_{AL}(v_i), F_{AL}(v_j)\right] - F_{BL}(v_i, v_j)$,
$\overline{F_{BU}}(v_i, v_j)= \max\left[F_{AU}(v_i), F_{AU}(v_j)\right] - F_{BU}(v_i, v_j)$,

for all $(v_i, v_j) \in$ E

Remark 3.24

If G = (V, E) is an interval valued neutrosophic graph on G^*. Then from above definition, it follow that $\bar{\bar{G}}$ is given by the interval valued neutrosophic graph $\bar{\bar{G}} = (\bar{\bar{V}}, \bar{\bar{E}})$ on G^* where $\bar{\bar{V}}$ =V and —

$$\overline{\overline{T_{BL}}}(v_i, v_j)= \min\left[T_{AL}(v_i), T_A(v_j)\right] - T_{BL}(v_i, v_j),$$

$$\overline{\overline{T_{BU}}}(v_i, v_j)= \min\left[T_{AU}(v_i), T_A(v_j)\right] - T_{BU}(v_i, v_j),$$

$$\overline{\overline{I_{BL}}}(v_i, v_j)= \max\left[I_{AL}(v_i), I_{AL}(v_j)\right] - I_{BL}(v_i, v_j),$$
$$\overline{\overline{I_{BU}}}(v_i, v_j)= \max\left[I_{AU}(v_i), I_{AU}(v_j)\right] - I_{BU}(v_i, v_j),$$

and

$$\overline{\overline{F_{BL}}}(v_i, v_j) = \max\left[F_{AL}(v_i), F_{AL}(v_j)\right] - F_{BL}(v_i, v_j), \overline{\overline{F_{BU}}}(v_i, v_j) = \max\left[F_{AU}(v_i), F_{AU}(v_j)\right] - F_{BU}(v_i, v_j), \text{ For all } (v_i, v_j) \in E.$$

Thus $\overline{\overline{T_{BL}}} = T_{BL}$, $\overline{\overline{T_{BU}}} = T_{BUL}$, $\overline{\overline{I_{BL}}} = I_{BL}$, $\overline{\overline{I_{BU}}} = I_{BU}$, and $\overline{\overline{F_{BL}}} = F_{BL}$, $\overline{\overline{F_{BU}}} = F_{BU}$ on V, where E =($[T_{BL}, T_{BU}]$, $[I_{BL}, I_{BU}]$, $[F_{BL}, F_{BU}]$) is the interval valued neutrosophic relation on V. For any interval valued neutrosophic graph G, $\bar{G}$ is strong interval valued neutrosophic graph and $G \subseteq \bar{G}$.

Proposition 3.25

G= $\overline{\overline{G}}$ if and only if G is a strong interval valued neutrosophic graph.

Proof

It is obvious.

Definition 3.26

A strong interval valued neutrosophic graph G is called self complementary if G$\cong \bar{G}$, where $\bar{G}$ is the complement of interval valued neutrosophic graph G.

Example 3.27

Consider a graph G^* = (V, E) such that V ={v_1, v_2, v_3, v_4}, E={v_1v_2, v_2v_3, v_3v_4, v_1v_4}. Consider an interval valued neutrosophic graph G.

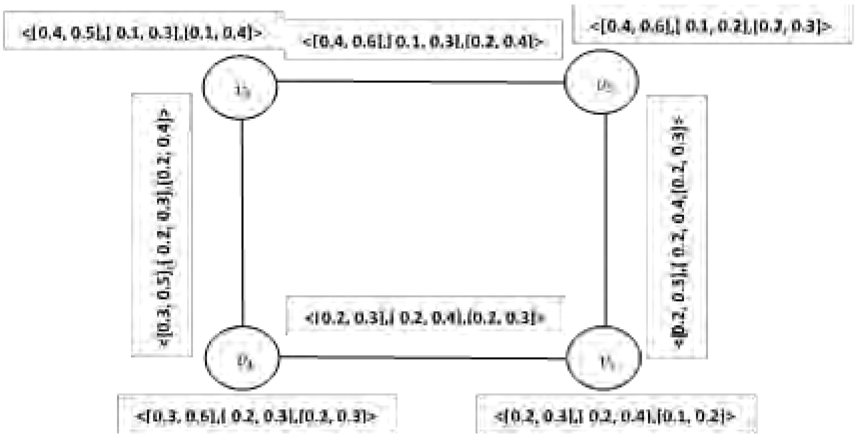

Figure 13. G: Strong IVN- graph

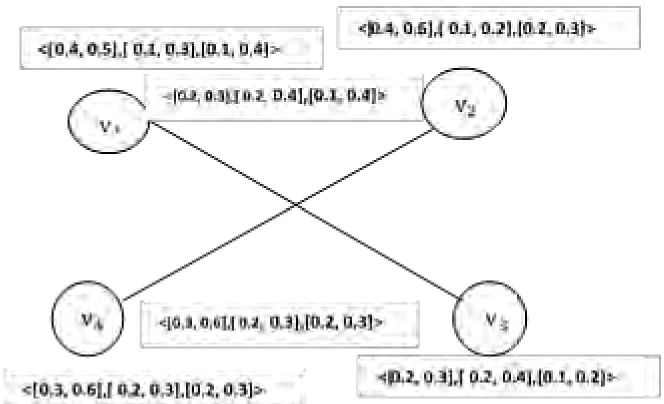

Figure 14. $\bar{G}$ Strong IVN- graph

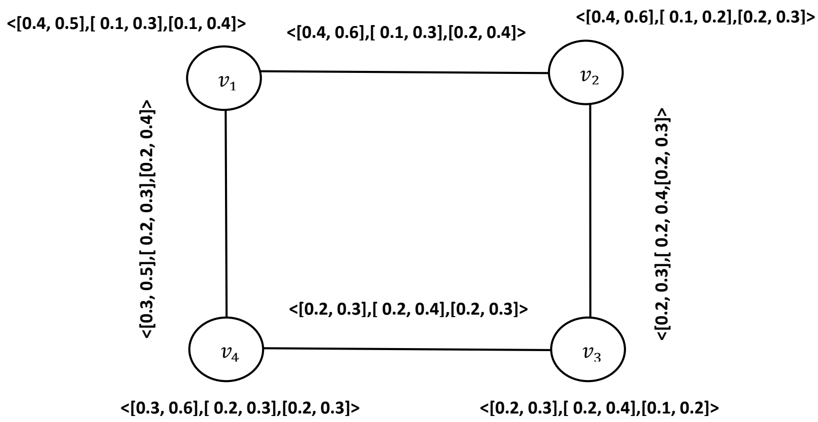

Figure 15. $\overline{\overline{G}}$ Strong IVN- graph

Clearly, G≅ $\overline{\overline{G}}$, hence G is self complementary.

Proposition 3.26

Let G=(A, B) be a *strong* interval valued neutrosophic graph. If –

$$T_{BL}(v_i, v_j) = \min\,[T_{AL}(v_i),\ T_{AL}(v_j)]$$

$$T_{BU}(v_i, v_j) = \min\,[T_{AU}(v_i),\ T_{AU}(v_j)]$$

$$I_{BL}(v_i, v_j) = \max\,[I_{AL}(v_i),\ I_{AL}(v_j)]$$

$$I_{BU}(v_i, v_j) = \max\,[I_{AU}(v_i),\ I_{AU}(v_j)]$$

$$F_{BL}(v_i, v_j) = \max\,[F_{AL}(v_i), F_{AL}(v_j)]$$

$$F_{BU}(v_i, v_j) = \max\,[F_{AU}(v_i), F_{AU}(v_j)]$$

for all $v_i, v_j \in$ V, then G is self complementary.

Proof

Let G = (A, B) be a strong interval valued neutrosophic graph such that –

$$T_{BL}(v_i, v_j) = \min\,[T_{AL}(v_i),\ T_{AL}(v_j)];$$

$$T_{BU}(v_i, v_j) = \min\,[T_{AU}(v_i),\ T_{AU}(v_j)];$$

$$I_{BL}(v_i, v_j) = \max\,[I_{AL}(v_i),\ I_{AL}(v_j)];$$

$$I_{BU}(v_i, v_j) = \max\,[I_{AU}(v_i),\ I_{AU}(v_j)];$$

$$F_{BL}(v_i, v_j) = \max\,[F_{AL}(v_i), F_{AL}(v_j)];$$

$$F_{BU}(v_i, v_j) = \max\,[F_{AU}(v_i), F_{AU}(v_j)],$$

for all $v_i, v_j \in$ V, then G≈ $\overline{\overline{G}}$ under the identity map *I*: V →V, hence G is self complementary.

Proposition 3.27

Let G be a self complementary interval valued neutrosophic graph. Then —

$$\sum_{v_i \neq v_j} T_{BL}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \min\,[T_{AL}(v_i), T_{AL}(v_j)]$$

$$\sum_{v_i \neq v_j} T_{BU}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \min\,[T_{AU}(v_i), T_{AU}(v_j)]$$

$$\sum_{v_i \neq v_j} I_{BL}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max\,[I_{AL}(v_i), I_{AL}(v_j)]$$

$$\sum_{v_i \neq v_j} I_{BU}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max\,[I_{AU}(v_i), I_{AU}(v_j)]$$

$$\sum_{v_i \neq v_j} F_{BL}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max\,[F_{AL}(v_i), F_{AL}(v_j)]$$

$$\sum_{v_i \neq v_j} F_{BU}(v_i, v_j) = \frac{1}{2} \sum_{v_i \neq v_j} \max\,[F_{AU}(v_i), F_{AU}(v_j)].$$

Proof

If G be a self complementary interval valued neutrosophic graph. Then there exist an isomorphism $f: V_1 \to V_1$ satisfying

$$\overline{T_{V_1}}(f(v_i)) = T_{V_1}(f(v_i)) = T_{V_1}(v_i)$$

$$\overline{I_{V_1}}(f(v_i)) = I_{V_1}(f(v_i)) = I_{V_1}(v_i)$$

$$\overline{\overline{F_{V_1}}}(f(v_i)) = F_{V_1}(f(v_i)) = F_{V_1}(v_i)$$

for all $v_i \in V_1$, and —

$$\overline{T_{E_1}}(f(v_i), f(v_j)) = T_{E_1}(f(v_i), f(v_j)) = T_{E_1}(v_i, v_j)$$

$$\overline{I_{E_1}}(f(v_i), f(v_j)) = I_{E_1}(f(v_i), f(v_j)) = I_{E_1}(v_i, v_j)$$

$$\overline{F_{E_1}}(f(v_i), f(v_j)) = F_{E_1}(f(v_i), f(v_j)) = F_{E_1}(v_i, v_j)$$

for all $(v_i, v_j) \in E_1$.

We have

$$\overline{T_{E_1}}(f(v_i), f(v_j)) = \min\,[\overline{T_{V_1}}(f(v_i)), \overline{T_{V_1}}(f(v_j))] - T_{E_1}(f(v_i), f(v_j))$$

i.e, $T_{E_1}(v_i, v_j) = \min\,[T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(f(v_i), f(v_j))$

$$T_{E_1}(v_i, v_j) = \min\,[T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(v_i, v_j).$$

That is —

$$\sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \min\,[T_{V_1}(v_i), T_{V_1}(v_j)]$$

$$\sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max\,[I_{V_1}(v_i), I_{V_1}(v_j)]$$

$$\sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) + \sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max\,[F_{V_1}(v_i), F_{V_1}(v_j)]$$

$$2\sum_{v_i \neq v_j} T_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \min\,[T_{V_1}(v_i), T_{V_1}(v_j)]$$
$$2\sum_{v_i \neq v_j} I_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max\,[I_{V_1}(v_i), I_{V_1}(v_j)]$$
$$2\sum_{v_i \neq v_j} F_{E_1}(v_i, v_j) = \sum_{v_i \neq v_j} \max\,[F_{V_1}(v_i), F_{V_1}(v_j)].$$

From these equations, *Proposition 3.27* holds.

Proposition 3.28

Let G_1 and G_2 be strong interval valued neutrosophic graph, $\overline{G_1} \approx \overline{G_2}$ (isomorphism).

Proof

Assume that G_1 and G_2 are isomorphic, there exists a bijective map $f: V_1 \to V_2$ satisfying

$$T_{V_1}(v_i) = T_{V_2}(f(v_i)),$$
$$I_{V_1}(v_i) = I_{V_2}(f(v_i)),$$
$$F_{V_1}(v_i) = F_{V_2}(f(v_i)),$$

for all $v_i \in V_1$, and

$$T_{E_1}(v_i, v_j) = T_{E_2}(f(v_i), f(v_j)),$$
$$I_{E_1}(v_i, v_j) = I_{E_2}(f(v_i), f(v_j)),$$
$$F_{E_1}(v_i, v_j) = F_{E_2}(f(v_i), f(v_j)),$$

for all $(v_i, v_j) \in E_1$.

By *Definition 3.21*, we have

$$\overline{T_{E_1}}(v_i, v_j) = \min\,[T_{V_1}(v_i), T_{V_1}(v_j)] - T_{E_1}(v_i, v_j)$$
$$= \min\,[T_{V_2}(f(v_i)),\ T_{V_2}(f(v_j))] - T_{E_2}(f(v_i), f(v_j)),$$
$$= \overline{T_{E_2}}(f(v_i), f(v_j)),$$

$$\overline{I_{E_1}}(v_i, v_j) = \max\,[I_{V_1}(v_i),\ I_{V_1}(v_j)] - I_{E_1}(v_i, v_j)$$
$$= \max[I_{V_2}(f(v_i)), I_{V_2}(f(v_j))] - I_{E_2}(f(v_i), f(v_j)),$$
$$= \overline{I_{E_2}}(f(v_i), f(v_j)),$$

$$\overline{F_{E_1}}(v_i, v_j) = \min\,[F_{V_1}(v_i), F_{V_1}(v_j)] - F_{E_1}(v_i, v_j)$$
$$= \min\,[F_{V_2}(f(v_i)), F_{V_2}(f(v_j))] - F_{E_2}(f(v_i), f(v_j)),$$
$$= \overline{F_{E_2}}(f(v_i), f(v_j)),$$

for all $(v_i, v_j) \in E_1$, hence $\overline{G_1} \approx \overline{G_2}$. The converse is straightforward.

4 Complete Interval Valued Neutrosophic Graphs

Definition 4.1

An interval valued neutrosophic graph G= (A, B) is called complete if

$T_{BL}(v_i, v_j) = \min(T_{AL}(v_i), T_{AL}(v_j))$, $T_{BU}(v_i, v_j) = \min(T_{AU}(v_i), T_{AU}(v_j))$,

$I_{BL}(v_i, v_j) = \max(I_A(v_i), I_A(v_j))$, $I_{BU}(v_i, v_j) = \max(I_{AU}(v_i), I_{AU}(v_j))$,

and

$F_{BL}(v_i, v_j) = \max(F_A(v_i), F_A(v_j))$, $F_{BU}(v_i, v_j) = \max(F_{AU}(v_i), F_{AU}(v_j))$,

for all $v_i, v_j \in$ V.

Example 4.2

Consider a graph G^* = (V, E) such that V = {v_1, v_2, v_3, v_4}, E = {v_1v_2, v_1v_3 , v_2v_3, v_1v_4, v_3v_4 , v_2v_4}, then G = (A, B) is a complete interval valued neutrosophic graph of G^*.

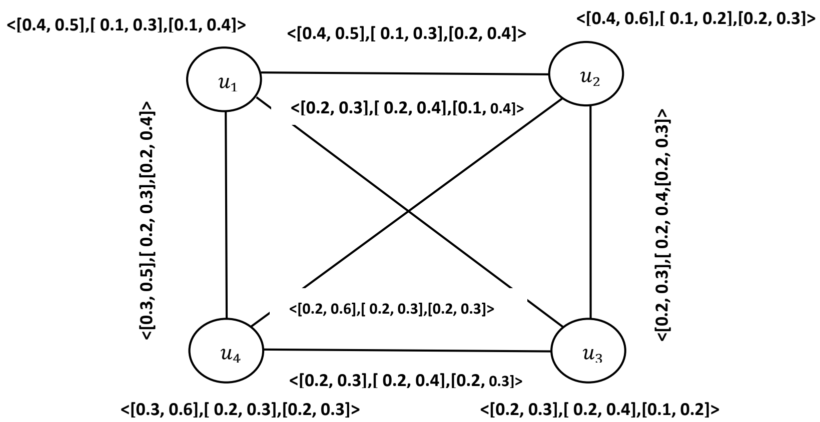

Figure17: Complete interval valued neutrosophic graph

Definition 4.3

The complement of a complete interval valued neutrosophic graph G = (A, B) of G^*= (V, E) is an interval valued neutrosophic complete graph $\bar{G}$= ($\bar{A}$, $\bar{B}$) on G^*= (V, $\bar{E}$), where

1. $\bar{V}$ =V

2. $\overline{T_{AL}}(v_i)$= $T_{AL}(v_i)$, $\overline{T_{AU}}(v_i)$= $T_{AU}(v_i)$, $\overline{I_{AL}}(v_i)$= $I_{AL}(v_i)$, $\overline{I_{AU}}(v_i)$= $I_{AU}(v_i)$, $\overline{F_{AL}}(v_i)$ = $F_{AL}(v_i)$, $\overline{F_{AU}}(v_i)$ = $F_{AU}(v_i)$, for all $v_j \in$ V.

3. $\overline{T_{BL}}(v_i, v_j)$= min $[T_{AL}(v_i), T_{AL}(v_j)]$ $- T_{BL}(v_i, v_j)$,

$\overline{T_{BU}}(v_i, v_j)$= min $[T_{AU}(v_i), T_{AU}(v_j)]$ $- T_{BU}(v_i, v_j)$,

$\overline{I_{BL}}(v_i, v_j)$= max $[I_{AL}(v_i), I_{AL}(v_j)]$ $- I_{BL}(v_i, v_j)$,

$\overline{I_{BU}}(v_i, v_j)$= max $[I_{AU}(v_i), I_{AU}(v_j)]$ $- I_{BU}(v_i, v_j)$,

and

$\overline{F_{BL}}(v_i, v_j)$= max $[F_{AL}(v_i), F_{AL}(v_j)]$ $- F_{BL}(v_i, v_j)$,

$\overline{F_{BU}}(v_i, v_j)$= max $[F_{AU}(v_i), F_{AU}(v_j)]$ $- F_{BU}(v_i, v_j)$,

for all $(v_i, v_j) \in$ E.

Proposition 4.4

The complement of complete IVN-graph is a IVN-graph with no edge. Or if G is a complete, then in $\bar{G}$ the edge is empty.

Proof

Let G = (A, B) be a complete IVN-graph. So

$T_{BL}(v_i, v_j)$ = min($T_{AL}(v_i)$, $T_{AL}(v_j)$), $T_{BU}(v_i, v_j)$ = min($T_{AU}(v_i)$, $T_{AU}(v_j)$), $I_{BL}(v_i, v_j)$ = max ($I_{AL}(v_i)$, $I_{AL}(v_j)$), $I_{BU}(v_i, v_j)$ = max ($I_{AU}(v_i)$, $I_{AU}(v_j)$) and $F_{BL}(v_i, v_j)$ = max ($F_{AL}(v_i)$, $F_{AL}(v_j)$), $F_{BU}(v_i, v_j)$ = max ($F_{AU}(v_i)$, $F_{AU}(v_j)$), for all $v_i, v_j \in$ V

Hence in $\bar{G}$,

$\bar{T}_{BL}(v_i, v_j)$= min($T_{AL}(v_i), T_{AL}(v_j)$) $- T_{BL}(v_i, v_j)$ for all i, j, ..., n

= min($T_{AL}(v_i), T_{AL}(v_j)$) $-$ min($T_{AL}(v_i)$, $T_{AL}(v_j)$) for all i, j, …, n

= 0 for all i, j, ..., n

$\bar{T}_{BU}(v_i, v_j)$= min($T_{AU}(v_i), T_{AU}(v_j)$) $- T_{BU}(v_i, v_j)$ for all i, j, ..., n

= min($T_{AU}(v_i), T_{AU}(v_j)$) $-$ min($T_{AU}(v_i)$, $T_{AU}(v_j)$) for all i, j, ..., n

= 0 for all i, j, .., n.

and

$\bar{I}_{BL}(v_i, v_j)$= max($I_{AL}(v_i), I_{AL}(v_j)$) $- I_{BL}(v_i, v_j)$ for all i, j, ..., n

= max($I_{AL}(v_i), I_{AL}(v_j)$) $-$ max($I_{AL}(v_i)$, $I_{AL}(v_j)$) for all i, j, …, n

= 0 for all i, j, .., n

$\bar{I}_{BU}(v_i, v_j)$= max($I_{AU}(v_i), I_{AU}(v_j)) - I_{BU}(v_i, v_j)$ for all i, j, .., n

= max($I_{AU}(v_i), I_{AU}(v_j)) -$ max($I_{AU}(v_i), I_{AU}(v_j)$) for all i, j, …, n

= 0 for all i, j, ..., n.

Also

$\bar{F}_{BL}(v_i, v_j)$= max($F_{AL}(v_i), F_{AL}(v_j)) - F_{BL}(v_i, v_j)$ for all i, j, ..., n.

= max($F_{AL}(v_i), I_{AL}(v_j)) -$ max($F_{AL}(v_i), F_{AL}(v_j)$) for all i, j, …, n

= 0, for all i, j, ..., n.

$\bar{F}_{BU}(v_i, v_j)$= max($F_{AU}(v_i), F_{AU}(v_j)) - F_{BU}(v_i, v_j)$ for all i, j, ..., n

= max($F_{AU}(v_i), F_{AU}(v_j)) -$ max($F_{AU}(v_i), F_{AU}(v_j)$) for all i, j, ...,n

= 0, for all i, j, ..., n.

Thus

([$\bar{T}_{BL}(v_i, v_j), \bar{T}_{BU}(v_i, v_j)$], [$\bar{I}_{BL}(v_i, v_j), \bar{I}_{BU}(v_i, v_j)$],
[$\bar{F}_{BL}(v_i, v_j), \bar{F}_{BU}(v_i, v_j)$]) = ([0, 0], [0, 0], [0, 0]).

Hence, the edge set of $\bar{G}$ is empty if G is a complete IVN-graph.

5 Conclusion

Interval valued neutrosophic sets is a generalization of the notion of fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets, interval valued intuitionstic fuzzy sets and single valued neutrosophic sets.

Interval valued neutrosophic model gives more precisions, flexibility and compatibility to the system as compared to the classical, fuzzy, intuitionistic fuzzy and single valued neutrosophic models.

In this paper, we have defined for the first time certain types of interval valued neutrosophic graphs, such as strong interval valued neutrosophic graph, constant interval valued neutrosophic graph and complete interval valued neutrosophic graphs.

In future study, we plan to extend our research to regular interval valued neutrosophic graphs and irregular interval valued neutrosophic.

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An Isolated Interval Valued Neutrosophic Graph

Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache

Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache (2016). An Isolated Interval Valued Neutrosophic Graph. *Critical Review*, XIII, 67-80

Abstract

The interval valued neutrosophic graphs are generalizations of the fuzzy graphs, interval fuzzy graphs, interval valued intuitionstic fuzzy graphs, and single valued neutrosophic graphs. Previously, several results have been proved on the isolated graphs and the complete graphs. In this paper, a necessary and sufficient condition for an interval valued neutrosophic graph to be an isolated interval valued neutrosophic graph is proved.

Keyword

interval valued neutrosophic graphs, complete interval valued neutrosophic graphs, isolated interval valued neutrosophic graphs.

1 Introduction

To express indeterminate and inconsistent information which exists in real world, Smarandache [9] originally proposed the concept of the neutrosophic set from a philosophical point of view. The concept of the neutrosophic set (NS) is a generalization of the theories of fuzzy sets [14], intuitionistic fuzzy sets [15], interval valued fuzzy set [12] and interval-valued intuitionistic fuzzy sets [14].

The neutrosophic sets are characterized by a truth-membership function (t), an indeterminacy-membership function (i) and a falsity-membership function (f) independently, which are within the real standard or nonstandard unit interval $]^-0, 1^+[$.

Further on, Wang et al. [10] introduced the concept of a single-valued neutrosophic sets (SVNS), a subclass of the neutrosophic sets. The same authors [11] introduced the interval valued neutrosophic sets (IVNS), as a generalization of the single valued neutrosophic sets, in which three membership functions are independent and their value belong to the unit interval [0, 1]. Some more work on single valued neutrosophic sets, interval valued neutrosophic sets, and their applications, may be found in [1, 5, 7,8, 29, 30, 31, 37, 38].

Graph theory has become a major branch of applied mathematics, and it is generally regarded as a branch of combinatorics. Graph is a widely-used tool for solving combinatorial problems in different areas, such as geometry, algebra, number theory, topology, optimization and computer science. Most important thing which is to be noted is that, when we have uncertainty regarding either the set of vertices, or edges, or both, the model becomes a fuzzy graph.

In the literature, many extensions of fuzzy graphs have been deeply studied by several researchers, such as intuitionistic fuzzy graphs, interval valued fuzzy graphs, interval valued intuitionistic fuzzy graphs [2, 3, 16, 17, 18, 19, 20, 21, 22, 34].

But, when the relations between nodes (or vertices) in problems are indeterminate and inconsistent, the fuzzy graphs and their extensions fail. To overcome this issue Smarandache [5, 6, 7, 37] have defined four main categories of neutrosophic graphs: two are based on literal indeterminacy (I), (the I-edge neutrosophic graph and the I-vertex neutrosophic graph, [6, 36]), and the two others graphs are based on (t, i, f) components (the (t, i, f)-edge neutrosophic graph and the (t, i, f)-vertex neutrosophic graph, not developed yet).

Later, Broumi et al. [23] presented the concept of single valued neutrosophic graphs by combining the single valued neutrosophic set theory and the graph theory, and defined different types of single valued neutrosophic graphs (SVNG) including the strong single valued neutrosophic graph, the constant single valued neutrosophic graph, the complete single valued neutrosophic graph, and investigated some of their properties with proofs and suitable illustrations.

Concepts like size, order, degree, total degree, neighborhood degree and closed neighborhood degree of vertex in a single valued neutrosophic graph are introduced, along with theoretical analysis and examples, by Broumi al. in [24]. In addition, Broumi et al. [25] introduced the concept of isolated single valued neutrosophic graphs. Using the concepts of bipolar neutrosophic sets, Broumi et al. [32] also introduced the concept of bipolar single neutrosophic graph, as the generalization of the bipolar fuzzy graphs, N-graphs, intuitionistic fuzzy graph, single valued neutrosophic graphs and bipolar intuitionistic fuzzy graphs. Same authors [33] proposed different types of bipolar single valued neutrosophic graphs, such as bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs, studying some of their related properties. Moreover, in [26, 27, 28], the authors introduced the concept of interval valued neutrosophic graph as a generalization of fuzzy graph, intuitionistic fuzzy graph and single valued neutrosophic graph, and discussed some of their properties with examples.

The aim of this paper is to prove a necessary and sufficient condition for an interval valued neutrosophic graph to be an isolated interval valued neutrosophic graph.

2 Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, fuzzy graph, intuitionistic fuzzy graph, single valued neutrosophic graphs and interval valued neutrosophic graph, relevant to the present work. See especially [2, 9, 10, 22, 23, 26] for further details and background.

Definition 2.1 [9]

Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X→]−0,1+[define respectively the a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^{-}0 \le T_A(x)+ I_A(x)+ F_A(x)\le 3^{+}. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^{-}0,1^{+}[$.

Since it is difficult to apply NSs to practical problems, Wang et al. [10] introduced the concept of a SVNS, which is an instance of a NS, and can be used in real scientific and engineering applications.

Definition 2.2 [10]

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by the truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\} \quad (2)$$

Definition 2.3 [2]

A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, i.e. σ : V → [0,1] and μ: V x V → [0,1] such that μ(uv) ≤ σ(u) ⋀ σ(v), for all u, v ∈ V, where uv denotes the edge between u and v and σ(u) ⋀ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

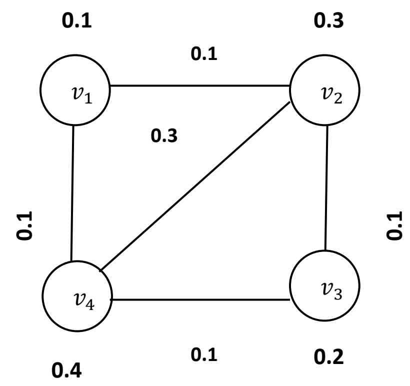

Figure 1. Fuzzy Graph.

Definition 2.4 [2]

The fuzzy subgraph H = (τ, ρ) is called a fuzzy subgraph of G = (σ, μ) if τ(u) ≤ σ(u) for all u ∈ V and ρ(u, v) ≤ μ(u, v) for all u, v ∈ V.

Definition 2.5 [22]

An intuitionistic fuzzy graph is of the form G = (V, E), where:

i. V=$\{v_1, v_2, \ldots, v_n\}$ such that μ_1: V→ [0,1] and γ_1: V → [0,1] denote the degree of membership and nonmembership of the element $v_i \in V$, respectively, and $0 \le \mu_1(v_i) + \gamma_1(v_i)) \le 1$ for every $v_i \in V$, (i = 1, 2, ..., n);

ii. E ⊆ V x V where μ_2: VxV→[0,1] and γ_2: VxV→ [0,1] are such that $\mu_2(v_i, v_j) \le \min [\mu_1(v_i), \mu_1(v_j)]$ and $\gamma_2(v_i, v_j) \ge \max [\gamma_1(v_i), \gamma_1(v_j)]$ and $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \le 1$ for every $(v_i, v_j) \in E$, (i, j = 1, 2, ..., n).

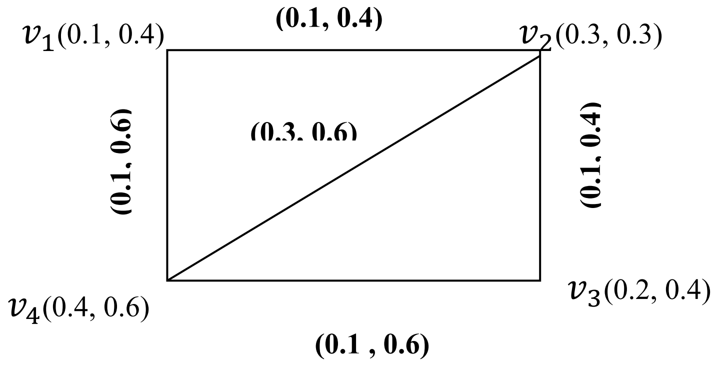

Figure 2. Intuitionistic Fuzzy Graph.

Definition 2.5 [23]

Let $A = (T_A,\ I_A, F_A)$ and $B = (T_B,\ I_B, F_B)$ be two single valued neutrosophic sets on a set X. If $A = (T_A,\ I_A, F_A)$ is a single valued neutrosophic relation on a set X, then $A = (T_A,\ I_A, F_A)$ is called a single valued neutrosophic relation on $B = (T_B,\ I_B, F_B)$ if

$$T_B(x, y) \leq \min(T_A(x), T_A(y)), \quad (3)$$

$$I_B(x, y) \geq \max(I_A(x), I_A(y)), \quad (4)$$

$$F_B(x, y) \geq \max(F_A x), F_A(y)), \quad (5)$$

for all x, y ∈ X.

A single valued neutrosophic relation A on X is called symmetric if $T_A(x, y) = T_A(y, x)$, $I_A(x, y) = I_A(y, x)$, $F_A(x, y) = F_A(y, x)$ and $T_B(x, y) = T_B(y, x)$, $I_B(x, y) = I_B(y, x)$ and $F_B(x, y) = F_B(y, x)$, for all $x, y \in X$.

Definition 2.6 [23]

A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G = (A, B), where:

1. The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and:

$$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3 \quad (6)$$

for all $v_i \in$ V (i = 1, 2, …,n).

2. The functions T_B: E ⊆ V x V →[0, 1], I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by:

$$T_B(\{v_i, v_j\}) \leq \min\,[T_A(v_i), T_A(v_j)], \quad (7)$$

$$I_B(\{v_i, v_j\}) \geq \max\,[I_A(v_i), I_A(v_j)], \tag{8}$$

$$F_B(\{v_i, v_j\}) \geq \max\,[F_A(v_i), F_A(v_j)], \tag{9}$$

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where:

$$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3 \text{ for all } \{v_i, v_j\} \in \text{E (i, j = 1, 2,…, n)} \tag{10}$$

We have A - the single valued neutrosophic vertex set of V, and B - the single valued neutrosophic edge set of E, respectively. Note that B is a symmetric single valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is a single valued neutrosophic graph of G*= (V, E) if:

$$T_B(v_i, v_j) \leq \min\,[T_A(v_i), T_A(v_j)], \tag{11}$$

$$I_B(v_i, v_j) \geq \max\,[I_A(v_i), I_A(v_j)], \tag{12}$$

$$F_B(v_i, v_j) \geq \max\,[F_A(v_i), F_A(v_j)], \tag{13}$$

for all $(v_i, v_j) \in$ E.

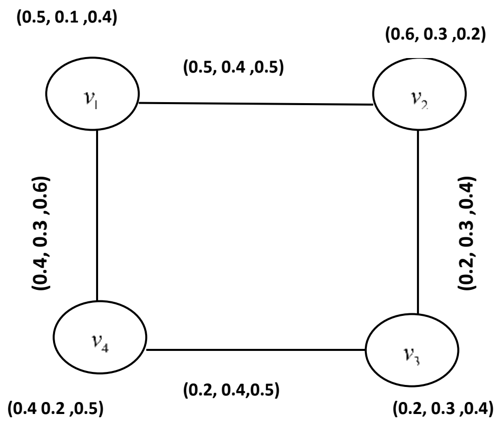

Figure 3. Single valued neutrosophic graph.

Definition 2.7 [23]

A single valued neutrosophic graph G= (A, B) is called complete if:

$$\mathrm{T_B(v_i, v_j) = \min\,[T_A(v_i),\ T_A(v_j)]} \tag{14}$$

$$\mathrm{I_B(v_i, v_j) = \max\,[I_A(v_i),\ I_A(v_j)]} \tag{15}$$

$$\mathrm{F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)]} \tag{16}$$

for all $\mathrm{v_i, v_j} \in$ V.

Definition 2.8 [23]

The complement of a single valued neutrosophic graph G (A, B) on G^* is a single valued neutrosophic graph $\overline{G}$ on G^*, where:

1. $\overline{A}$ =A. (17)

2. $\overline{T_A}(v_i)= T_A(v_i)$, $\overline{I_A}(v_i)= I_A(v_i)$, $\overline{F_A}(v_i) = F_A(v_i)$, (18)

for all $v_j \in V$.

3. $\overline{T_B}(v_i, v_j)= \min\left[T_A(v_i), T_A(v_j)\right] - T_B(v_i, v_j)$, (19)

$\overline{I_B}(v_i, v_j)= \max\left[I_A(v_i), I_A(v_j)\right] - I_B(v_i, v_j)$, (20)

$\overline{F_B}(v_i, v_j)= \max\left[F_A(v_i), F_A(v_j)\right] - F_B(v_i, v_j)$, (21)

for all $(v_i, v_j) \in E$.

Definition 2.9 [26]

By an interval-valued neutrosophic graph of a graph G^* = (V, E) we mean a pair G = (A, B), where A = < $[T_{AL}, T_{AU}]$, $[I_{AL}, I_{AU}]$, $[F_{AL}, F_{AU}]$> is an interval-valued neutrosophic set on V and B =< $[T_{BL}, T_{BU}]$, $[I_{BL}, I_{BU}]$, $[F_{BL}, F_{BU}]$> is an interval valued neutrosophic relation on E, satisfying the following condition:

1. V = $\{v_1, v_2, \ldots, v_n\}$ such that $T_{AL}:V\rightarrow[0, 1]$, $T_{AU}:V\rightarrow[0, 1]$, $I_{AL}:V\rightarrow[0, 1]$, $I_{AU}:V\rightarrow[0, 1]$ and $F_{AL}:V\rightarrow[0, 1]$, $F_{AU}:V\rightarrow[0, 1]$, denoting the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element y ∈ V, respectively, and:

$0\leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$, (22)

for all $v_i \in V$ (i=1, 2, …,n)

2. The functions $T_{BL}:V \times V \rightarrow[0, 1]$, $T_{BU}:V \times V \rightarrow[0, 1]$, $I_{BL}:V \times V \rightarrow[0, 1]$, $I_{BU}:V \times V \rightarrow[0, 1]$ and $F_{BL}:V \times V \rightarrow[0,1]$, $F_{BU}:V \times V \rightarrow[0, 1]$ are such that:

$T_{BL}(\{v_i, v_j\}) \leq \min[T_{AL}(v_i), T_{AL}(v_j)]$, (23)

$T_{BU}(\{v_i, v_j\}) \leq \min[T_{AU}(v_i), T_{AU}(v_j)]$, (24)

$I_{BL}(\{v_i, v_j\}) \geq \max[I_{BL}(v_i), I_{BL}(v_j)]$, (25)

$I_{BU}(\{v_i, v_j\}) \geq \max[I_{BU}(v_i), I_{BU}(v_j)]$, (26)

$F_{BL}(\{v_i, v_j\}) \geq \max[F_{BL}(v_i), F_{BL}(v_j)]$, (27)

$F_{BU}(\{v_i, v_j\}) \geq \max[F_{BU}(v_i), F_{BU}(v_j)]$, (28)

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in E$ respectively, where:

$$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3, \quad (29)$$

for all $\{v_i, v_j\} \in E$ (i, j = 1, 2, …, n).

We have A - the interval valued neutrosophic vertex set of V, and B - the interval valued neutrosophic edge set of E, respectively. Note that B is a symmetric interval valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is an interval valued neutrosophic graph of G^*= (V, E), if:

$$T_{BL}(v_i, v_j) \leq \min [T_{AL}(v_i), T_{AL}(v_j)], \quad (30)$$

$$T_{BU}(v_i, v_j) \leq \min [T_{AU}(v_i), T_{AU}(v_j)], \quad (31)$$

$$I_{BL}(v_i, v_j) \geq \max [I_{BL}(v_i), I_{BL}(v_j)], \quad (32)$$

$$I_{BU}(v_i, v_j) \geq \max [I_{BU}(v_i), I_{BU}(v_j)], \quad (33)$$

$$F_{BL}(v_i, v_j) \geq \max [F_{BL}(v_i), F_{BL}(v_j)], \quad (34)$$

$$F_{BU}(v_i, v_j) \geq \max [F_{BU}(v_i), F_{BU}(v_j)], \quad (35)$$

for all $(v_i, v_j) \in E$.

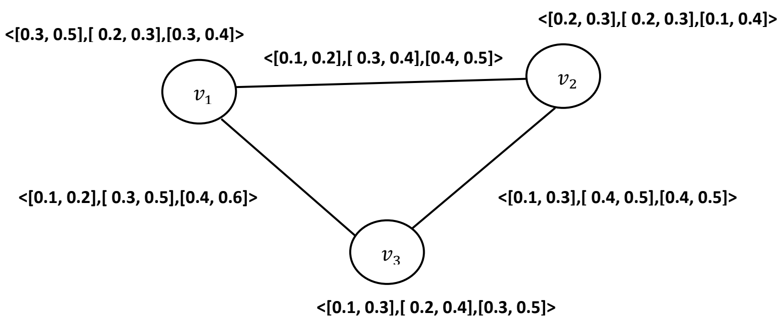

Figure 4. Interval valued neutrosophic graph.

Definition 2.10 [26]

The complement of a complete interval valued neutrosophic graph G = (A, B) of G^*= (V, E) is a complete interval valued neutrosophic graph $\overline{G}$= ($\overline{A}$, $\overline{B}$) = (A, $\overline{B}$) on G^*= (V, $\overline{E}$), where:

$$1.\ \overline{V} = V \quad (36)$$

$$2.\ \overline{T_{AL}}(v_i) = T_{AL}(v_i), \quad (37)$$

$$\overline{T_{AU}}(v_i) = T_{AU}(v_i), \quad (38)$$

$$\overline{I_{AL}}(v_i) = I_{AL}(v_i), \quad (39)$$

$$\overline{I_{AU}}(v_i) = I_{AU}(v_i), \quad (40)$$

$$\overline{F_{AL}}(v_i) = F_{AL}(v_i), \tag{41}$$

$$\overline{F_{AU}}(v_i) = F_{AU}(v_i), \tag{42}$$

for all $v_j \in V$.

3. $$\overline{T_{BL}}(v_i, v_j) = \min\left[T_{AL}(v_i), T_{AL}(v_j)\right] - T_{BL}(v_i, v_j), \tag{43}$$

$$\overline{T_{BU}}(v_i, v_j) = \min\left[T_{AU}(v_i), T_{AU}(v_j)\right] - T_{BU}(v_i, v_j), \tag{44}$$

$$\overline{I_{BL}}(v_i, v_j) = \max\left[I_{AL}(v_i), I_{AL}(v_j)\right] - I_{BL}(v_i, v_j), \tag{45}$$

$$\overline{I_{BU}}(v_i, v_j) = \max\left[I_{AU}(v_i), I_{AU}(v_j)\right] - I_{BU}(v_i, v_j), \tag{46}$$

$$\overline{F_{BL}}(v_i, v_j) = \max\left[F_{AL}(v_i), F_{AL}(v_j)\right] - F_{BL}(v_i, v_j), \tag{47}$$

$$\overline{F_{BU}}(v_i, v_j) = \max\left[F_{AU}(v_i), F_{AU}(v_j)\right] - F_{BU}(v_i, v_j), \tag{48}$$

for all $(v_i, v_j) \in E$.

Definition 2.11 [26]

An interval valued neutrosophic graph G= (A, B) is called complete, if:

$$T_{BL}(v_i, v_j) = \min(T_{AL}(v_i), T_{AL}(v_j)), \tag{49}$$

$$T_{BU}(v_i, v_j) = \min(T_{AU}(v_i), T_{AU}(v_j)), \tag{50}$$

$$I_{BL}(v_i, v_j) = \max(I_A(v_i), I_A(v_j)), \tag{51}$$

$$I_{BU}(v_i, v_j) = \max(I_{AU}(v_i), I_{AU}(v_j)), \tag{52}$$

$$F_{BL}(v_i, v_j) = \max(F_A(v_i), F_A(v_j)), \tag{53}$$

$$F_{BU}(v_i, v_j) = \max(F_{AU}(v_i), F_{AU}(v_j)), \tag{54}$$

for all $v_i, v_j \in V$.

3 Main Result

Theorem 3.1:

An interval valued neutrosophic graph G = (A, B) is an isolated interval valued neutrosophic graph if and only if its complement is a complete interval valued neutrosophic graph.

Proof

Let G= (A, B) be a complete interval valued neutrosophic graph.

Therefore:

$$T_{BL}(v_i, v_j) = \min(T_{AL}(v_i),\ T_{AL}(v_j)), \tag{55}$$

$$T_{BU}(v_i, v_j) = \min(T_{AU}(v_i), T_{AU}(v_j)), \tag{56}$$

$$I_{BL}(v_i, v_j) = \max\ (I_{AL}(v_i), I_{AL}(v_j)), \tag{57}$$

$$I_{BU}(v_i, v_j) = \max\ (I_{AU}(v_i), I_{AU}(v_j)), \tag{58}$$

$$F_{BL}(v_i, v_j) = \max\ (F_{AL}(v_i), F_{AL}(v_j)), \tag{59}$$

$$F_{BU}(v_i, v_j) = \max\ (F_{AU}(v_i), F_{AU}(v_j)), \tag{60}$$

for all $v_i, v_j \in V$.

Hence in $\overline{G}$,

$$\overline{T}_{BL}(v_i, v_j) = \min(T_{AL}(v_i), T_{AL}(v_j)) - T_{BL}(v_i, v_j) \tag{61}$$

for all i, j, ..., n.

$$= \min(T_{AL}(v_i), T_{AL}(v_j)) - \min(T_{AL}(v_i), T_{AL}(v_j)) \tag{62}$$

for all i, j, ..., n.

$$= 0 \tag{63}$$

for all i, j, ..., n.

$$\overline{T}_{BU}(v_i, v_j) = \min(T_{AU}(v_i), T_{AU}(v_j)) - T_{BU}(v_i, v_j) \tag{64}$$

for all i, j, ..., n.

$$= \min(T_{AU}(v_i), T_{AU}(v_j)) - \min(T_{AU}(v_i), T_{AU}(v_j)) \tag{65}$$

for all i, j, ..., n.

$$= 0 \tag{66}$$

for all i, j, ..., n.

And:

$$\overline{I}_{BL}(v_i, v_j) = \max(I_{AL}(v_i), I_{AL}(v_j)) - I_{BL}(v_i, v_j) \tag{67}$$

for all i, j, ..., n.

$$= \max(I_{AL}(v_i), I_{AL}(v_j)) - \max(I_{AL}(v_i), I_{AL}(v_j)) \tag{68}$$

for all i, j, ..., n.

$$= 0 \tag{69}$$

for all i, j, ..., n.

$$\overline{I}_{BU}(v_i, v_j) = \max(I_{AU}(v_i), I_{AU}(v_j)) - I_{BU}(v_i, v_j) \tag{70}$$

for all i, j, ..., n.

$$= \max(I_{AU}(v_i), I_{AU}(v_j)) - \max(I_{AU}(v_i), I_{AU}(v_j)) \tag{71}$$

for all i, j, ..., n.

$$= 0 \quad (72)$$

for all i, j, ..., n.

Also:

$$\bar{F}_{BL}(v_i, v_j) = \max(F_{AL}(v_i), F_{AL}(v_j)) - F_{BL}(v_i, v_j) \quad (73)$$

for all i, j, ..., n.

$$= \max(F_{AL}(v_i), F_{AL}(v_j)) - \max(F_{AL}(v_i), F_{AL}(v_j)) \quad (74)$$

for all i, j, ..., n.

$$= 0 \quad (75)$$

for all i, j, ..., n.

$$\bar{F}_{BU}(v_i, v_j) = \max(F_{AU}(v_i), F_{AU}(v_j)) - F_{BU}(v_i, v_j) \quad (76)$$

for all i, j, ..., n.

$$= \max(F_{AU}(v_i), F_{AU}(v_j)) - \max(F_{AU}(v_i), F_{AU}(v_j)) \quad (77)$$

for all i, j, ..., n.

$$= 0 \quad (78)$$

for all i, j, ..., n.

Thus,

$$([\bar{T}_{BL}(v_i, v_j), \bar{T}_{BU}(v_i, v_j)],\ [\bar{I}_{BL}(v_i, v_j), \bar{I}_{BU}(v_i, v_j)],\ [\bar{F}_{BL}(v_i, v_j), \bar{F}_{BU}(v_i, v_j)]) =$$

$$= ([0, 0], [0, 0], [0, 0]). \quad (79)$$

Hence, G = (A, B) is an isolated interval valued neutrosophic graph.

4 Conclusions

In this paper, we extended the concept of isolated single valued neutrosophic graph to an isolated interval valued neutrosophic graph. In future works, we plan to study the concept of isolated bipolar single valued neutrosophic graph.

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P-Union and P-Intersection of Neutrosophic Cubic Sets

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Abstract

Conditions for the P-intersection and P-intersection of falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic sets to be an falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic set are provided. Conditions for the P-union and the P-intersection of two truth-external (resp. indeterminacy-external and falsity-external) neutrosophic cubic sets to be a truth-internal (resp. indeterminacy-internal and falsity-internal) neutrosophic cubic set are discussed.

Key Words: Truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cu-bic set, truth-external (indeterminacy-external, falsity-external) neutrosophic cubic set, P-union, P-intersection.

1 Introduction

The concept of neutrosophic set (NS) developed by Smarandache ([3, 4]) is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site http://fs.gallup.unm.edu/ neutrosophy.htm). Jun et al. [2] extended the concept of cubic sets to the neutrosophic sets. They introduced the notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) neutrosophic cubic sets, and investigate related properties. Generally, the P-intersection of falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic sets may not be a falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic set (see [2]). As a continuation of the paper [2], we provide a condition for the P-intersection of falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic sets to be a falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic set. We provide examples to show that the P-union of falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic sets may not be a falsity-external (resp. indeterminacy-external and truth-external) neutrosophic cubic set. We consider a condition for the P-union of truth-external (resp. indeterminacy-external and falsity-external) neutrosophic cubic sets to be a truth-external (resp. indeterminacy-external and falsity-external) neutrosophic cubic set. We also give a condition for the P-intersection of two neutrosophic cubic sets to be both a truth-internal (resp. indeterminacy-internal and falsity-internal) neutrosophic cubic set and a truth-external (resp. indeterminacy-external and falsity-external) neutrosophic cubic set. Generally, the P-union of two truth-external (resp. indeterminacy-external and falsity-external) neutrosophic cubic sets may not be a truth-internal (resp. indeterminacy-internal and falsity-internal) neutrosophic cubic set. We provide conditions for the P-union and the P-intersection of two truth-external (resp. indeterminacy-external and falsity-external) neutrosophic cubic sets to be a truth-internal (resp. indeterminacy-internal and falsity-internal) neutrosophic cubic set.

2 Preliminaries

Jun et al. [1] have defined the cubic set as follows:

Let X be a non-empty set. A cubic set in X is a structure of the form:

$$\mathbf{C} = \{(x, A(x), \lambda(x)) \mid x \in X\}$$

where A is an interval-valued fuzzy set in X and λ is a fuzzy set in X.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [3]) is a structure of the form:

$$\Lambda := \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$$

where $\lambda_T : X \to [0,1]$ is a truth membership function, $\lambda_I : X \to [0,1]$ is an indeterminate membership function, and $\lambda_F : X \to [0,1]$ is a false membership function.

Let X be a non-empty set. An interval neutrosophic set (INS) in X (see [5]) is a structure of the form:

$$\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where A_T, A_I and A_F are interval-valued fuzzy sets in X, which are called an interval truth membership function, an interval indeterminacy membership function and an interval falsity membership function, respectively.

Jun et al. [2] considered the notion of neutrosophic cubic sets as an extension of cubic sets.

Let X be a non-empty set. A neutrosophic cubic set (NCS) in X is a pair $\mathscr{A} = (\mathbf{A}, \Lambda)$ where $\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$ is an interval neutrosophic set in X and $\Lambda := \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$ is a neutrosophic set in X.

Definition 2.1 ([2]). Let X be a non-empty set. A neutrosophic cubic set $\mathscr{A} = (\mathbf{A}, \Lambda)$ in X is said to be

- truth-internal (briefly, T-internal) if the following inequality is valid

$$(\forall x \in X)\left(A_T^-(x) \leq \lambda_T(x) \leq A_T^+(x)\right), \tag{2.1}$$

- indeterminacy-internal (briefly, I-internal) if the following inequality is valid

$$(\forall x \in X)\left(A_I^-(x) \leq \lambda_I(x) \leq A_I^+(x)\right), \tag{2.2}$$

- falsity-internal (briefly, F-internal) if the following inequality is valid

$$(\forall x \in X)\left(A_F^-(x) \leq \lambda_F(x) \leq A_F^+(x)\right). \tag{2.3}$$

Definition 2.2 ([2]). Let X be a non-empty set. A neutrosophic cubic set $\mathscr{A} = (\mathbf{A}, \Lambda)$ in X is said to be

- truth-external (briefly, T-external) if the following inequality is valid

$$(\forall x \in X)\left(\lambda_T(x) \notin (A_T^-(x), A_T^+(x))\right), \tag{2.4}$$

- indeterminacy-external (briefly, I-external) if the following inequality is valid

$$(\forall x \in X)\left(\lambda_I(x) \notin (A_I^-(x), A_I^+(x))\right), \tag{2.5}$$

- falsity-external (briefly, F-external) if the following inequality is valid

$$(\forall x \in X)\left(\lambda_F(x) \notin (A_F^-(x), A_F^+(x))\right). \tag{2.6}$$

3 P-union and P-intersection of neutrosophic cubic sets

Note that P-intersection of F-external (resp. I-external and T-external) neutrosophic cubic sets may not be an F-external (resp. I-external and T-external) neutrosophic cubic set (see [2]). We provide a condition for the P-intersection of F-external (resp. I-external and T-external) neutrosophic cubic sets to be an F-external (resp. I-external and T-external) neutrosophic cubic set.

Theorem 3.1. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be T-external neutrosophic cubic sets in X such that*

$$\begin{aligned} &\max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\} < (\lambda_T \wedge \psi_T)(x) \\ &\leq \min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\} \end{aligned} \tag{3.1}$$

for all $x \in X$. Then the P-intersection $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ is a T-external neutrosophic cubic set in X.

Proof. For any $x \in X$, let

$$a_x := \min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\}$$

and

$$b_x := \max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\}.$$

Then $a_x = A_T^-(x)$, $a_x = B_T^-(x)$, $a_x = A_T^+(x)$, or $a_x = B_T^+(x)$. It is possible to consider the cases $a_x = A_T^-(x)$ and $a_x = A_T^+(x)$ only because the remaining cases are similar to these cases. If $a_x = A_T^-(x)$, then

$$B_T^-(x) \leq B_T^+(x) \leq A_T^-(x) \leq A_T^+(x).$$

Thus $b_x = B_T^+(x)$, and so

$$\begin{aligned} B_T^-(x) = (A_T \cap B_T)^-(x) &\leq (A_T \cap B_T)^+(x) \\ &= B_T^+(x) = b_x < (\lambda_T \wedge \psi_T)(x). \end{aligned}$$

Hence $(\lambda_T \wedge \psi_T)(x) \notin ((A_T \cap B_T)^-(x), (A_T \cap B_T)^+(x))$. If $a_x = A_T^+(x)$, then $B_T^-(x) \leq A_T^+(x) \leq B_T^+(x)$ and thus $b_x = \max\{A_T^-(x), B_T^-(x)\}$. Suppose that $b_x = A_T^-(x)$. Then

$$B_T^-(x) \leq A_T^-(x) < (\lambda_T \wedge \psi_T)(x) \leq A_T^+(x) \leq B_T^+(x). \tag{3.2}$$

It follows that

$$B_T^-(x) \leq A_T^-(x) < (\lambda_T \wedge \psi_T)(x) < A_T^+(x) \leq B_T^+(x) \tag{3.3}$$

or

$$B_T^-(x) \leq A_T^-(x) < (\lambda_T \wedge \psi_T)(x) = A_T^+(x) \leq B_T^+(x). \tag{3.4}$$

The case (3.3) induces a contradiction. The case (3.4) implies that

$$(\lambda_T \wedge \psi_T)(x) \notin \left((A_T \cap B_T)^-(x), (A_T \cap B_T)^+(x)\right)$$

since $(\lambda_T \wedge \psi_T)(x) = A_T^+(x) = (A_T \cap B_T)^+(x)$. Now, if $b_x = B_T^-(x)$, then

$$A_T^-(x) \leq B_T^-(x) < (\lambda_T \wedge \psi_T)(x) \leq A_T^+(x) \leq B_T^+(x). \tag{3.5}$$

Hence we have

$$A_T^-(x) \leq B_T^-(x) < (\lambda_T \wedge \psi_T)(x) < A_T^+(x) \leq B_T^+(x) \tag{3.6}$$

or

$$A_T^-(x) \leq B_T^-(x) < (\lambda_T \wedge \psi_T)(x) = A_T^+(x) \leq B_T^+(x). \tag{3.7}$$

The case (3.6) induces a contradiction. The case (3.7) induces

$$(\lambda_T \wedge \psi_T)(x) \notin \left((A_T \cap B_T)^-(x), (A_T \cap B_T)^+(x)\right).$$

Therefore the P-intersection $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ is a T-external neutrosophic cubic set in X. □

Similarly, we have the following theorems.

Theorem 3.2. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be I-external neutrosophic cubic sets in X such that*

$$\begin{aligned}&\max\left\{\min\{A_I^+(x), B_I^-(x)\}, \min\{A_I^-(x), B_I^+(x)\}\right\} < (\lambda_I \wedge \psi_I)(x)\\ &\leq \min\left\{\max\{A_I^+(x), B_I^-(x)\}, \max\{A_I^-(x), B_I^+(x)\}\right\}\end{aligned} \tag{3.8}$$

for all $x \in X$. Then the P-intersection $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ is an I-external neutrosophic cubic set in X.

Theorem 3.3. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be F-external neutrosophic cubic sets in X such that*

$$\begin{aligned}&\max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\} < (\lambda_F \wedge \psi_F)(x)\\ &\leq \min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\}\end{aligned} \tag{3.9}$$

for all $x \in X$. Then the P-intersection $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ is an F-external neutrosophic cubic set in X.

Corollary 3.4. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be external neutrosophic cubic sets in X. Then the P-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an external neutrosophic cubic set in X when the conditions* (3.1)*,* (3.8) *and* (3.9) *are valid.*

The following example shows that the P-union of F-external (resp. I-external and T-external) neutrosophic cubic sets may not be an F-external (resp. I-external and T-external) neutrosophic cubic set.

Example 3.5. (1) Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $X = [0, 1]$ with the tabular representations in Tables 1 and 2, respectively.

Table 1: Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
$0 \leq x < 0.5$	$([0.25, 0.26], [0.2, 0.3], [0.15, 0.25])$	$(0.25, 0.15, 0.5x + 0.5)$
$0.5 \leq x \leq 1$	$([0.5, 0.7], [0.5, 0.6], [0.6, 0.7])$	$(0.55, 0.75, 0.30)$

Table 2: Tabular representation of $\mathscr{B} = (\mathbf{B}, \Psi)$

X	$\mathbf{B}(x)$	$\Psi(x)$
$0 \leq x < 0.5$	$([0.25, 0.26], [0.2, 0.3], [0.8, 0.9])$	$(0.25, 0.15, 0.40)$
$0.5 \leq x \leq 1$	$([0.5, 0.7], [0.5, 0.6], [0.1, 0.2])$	$(0.55, 0.75, x)$

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are F-external neutrosophic cubic sets in $X = [0, 1]$, and the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is given by Table 3.

Table 3: Tabular representation of $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$

X	$(\mathbf{A} \cup \mathbf{B})(x)$	$(\Lambda \vee \Psi)(x)$
$0 \leq x < 0.5$	$([0.25, 0.26], [0.2, 0.3], [0.8, 0.9])$	$(0.25, 0.15, 0.5x + 0.5)$
$0.5 \leq x \leq 1$	$([0.5, 0.7], [0.5, 0.6], [0.6, 0.7])$	$(0.55, 0.75, x)$

Then

$$(\lambda_F \vee \psi_F)(0.67) = 0.67 \in (0.6, 0.7)$$
$$= \left((A_F \cup B_F)^-(0.67), (A_F \cup B_F)^+(0.67)\right),$$

and so the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is not an F-external neutrosophic cubic set in $X = [0, 1]$.

(2) Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $X = [0, 1]$ with the tabular representations in Tables 4 and 5, respectively.

Table 4: Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
$0 \leq x \leq 0.3$	$([0.3, 0.6], [0.3, 0.5], [0.6, 1])$	$(x + 0.6, 0.15, \frac{1}{2}x + \frac{1}{2})$
$0.3 < x \leq 1$	$([0.4, 0.9], [0.5, 0.6], [0.6, 0.7])$	$(-\frac{2}{5}x + 0.4, 0.75, 0.30)$

Table 5: Tabular representation of $\mathscr{B} = (\mathbf{B}, \Psi)$

X	$\mathbf{B}(x)$	$\Psi(x)$
$0 \leq x \leq 0.3$	$([0.4, 0.8], [0.2, 0.3], [0.8, 0.9])$	$(\frac{1}{2}x + 0.8, 0.15, 0.40)$
$0.3 < x \leq 1$	$([0.3, 0.5], [0.5, 0.6], [0.1, 0.2])$	$(\frac{1}{3}x + 0.5, 0.75, x)$

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are T-external neutrosophic cubic sets in $X = [0, 1]$. Note that

$$(A_T \cup B_T)^-(x) = \begin{cases} [0.4,\ 0.8] & \text{if } 0 \leq x \leq 0.3, \\ [0.4, 0.9] & \text{if } 0.3 < x \leq 1, \end{cases}$$
$$(\lambda_T \vee \psi_T)(x) = \begin{cases} \frac{1}{2}x + 0.8 & \text{if } 0 \leq x \leq 0.3, \\ \frac{1}{3}x + 0.5 & \text{if } 0.3 < x \leq 1, \end{cases}$$

and so the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is not a T-external neutrosophic cubic set in $X = [0, 1]$ since

$$(\lambda_T \vee \psi_T)(0.6) = 0.7 \in (0.4, 0.9) = \left((A_T \cup B_T)^-(0.6), (A_T \cup B_T)^+(0.6)\right).$$

(3) Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $X = [0, 1]$ with the tabular representations in Tables 6 and 7, respectively.

Table 6: Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
$0 \leq x \leq 0.5$	$([0.3, 0.6], [0.2, 0.7], [0.6, 1.0])$	$(0.4, \frac{1}{5}x + 0.7, \frac{1}{2}x + \frac{1}{2})$
$0.5 < x \leq 1$	$([0.4, 0.9], [0.3, 1.0], [0.6, 0.7])$	$(0.3, -\frac{1}{10}x + 0.3, 0.30)$

Table 7: Tabular representation of $\mathscr{B} = (\mathbf{B}, \Psi)$

X	$\mathbf{B}(x)$	$\Psi(x)$
$0 \leq x \leq 0.5$	$([0.4, 0.8], [0.3, 0.8], [0.8, 0.9])$	$(0.3, -\frac{1}{5}x + 0.3, 0.40)$
$0.5 < x \leq 1$	$([0.3, 0.5], [0.5, 0.9], [0.1, 0.2])$	$(0.5, -\frac{1}{10}x + 1.0, x)$

It is routine to verify that $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are I-external neutrosophic cubic sets in $X = [0, 1]$, but their P-union is not an I-external neutrosophic cubic sets in $X = [0, 1]$ since

$$(\lambda_I \vee \psi_I)(0.7) = 0.93 \in (0.5, 1.0) = \left((A_I \cup B_I)^-(0.7), (A_I \cup B_I)^+(0.7)\right).$$

We consider a condition for the P-union of T-external (resp. I-external and F-external) neutrosophic cubic sets to be a T-external (resp. I-external and F-external) neutrosophic cubic set.

Theorem 3.6. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be F-external neutrosophic cubic sets in X such that*

$$\begin{aligned} &\max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\} \leq (\lambda_F \vee \psi_F)(x) \\ &< \min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\} \end{aligned} \tag{3.10}$$

for all $x \in X$. Then the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is an F-external neutrosophic cubic set in X.

Proof. For any $x \in X$, let

$$a_x := \min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\}$$

and

$$b_x := \max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\}.$$

Then $a_x = A^-_F(x)$, $a_x = B^-_F(x)$, $a_x = A^+_F(x)$, or $a_x = B^+_F(x)$. It is possible to consider the cases $a_x = A^-_F(x)$ and $a_x = A^+_F(x)$ only because the remaining cases are similar to these cases. If $a_x = A^-_F(x)$, then

$$B^-_F(x) \le B^+_F(x) \le A^-_F(x) \le A^+_F(x),$$

and so $b_x = B^+_F(x)$. Thus

$$(A_F \cup B_F)^-(x) = A^-_F(x) = a_x > (\lambda_F \vee \psi_F)(x),$$

and hence $(\lambda_F \vee \psi_F)(x) \notin ((A_F \cup B_F)^-(x), (A_F \cup B_F)^+(x))$. If $a_x = A^+_F(x)$, then $B^-_F(x) \le A^+_F(x) \le B^+_F(x)$ and thus $b_x = \max\{A^-_F(x), B^-_F(x)\}$. Suppose that $b_x = A^-_F(x)$. Then

$$B^-_F(x) \le A^-_F(x) \le (\lambda_F \vee \psi_F)(x) < A^+_F(x) \le B^+_F(x), \tag{3.11}$$

and so

$$B^-_F(x) \le A^-_F(x) < (\lambda_F \vee \psi_F)(x) < A^+_F(x) \le B^+_F(x) \tag{3.12}$$

or

$$B^-_F(x) \le A^-_F(x) = (\lambda_F \vee \psi_F)(x) < A^+_F(x) \le B^+_F(x). \tag{3.13}$$

The case (3.12) induces a contradiction. The case (3.13) implies that

$$(\lambda_F \vee \psi_F)(x) \notin \big((A_F \cup B_F)^-(x), (A_F \cup B_F)^+(x)\big)$$

since $(\lambda_F \vee \psi_F)(x) = A^-_F(x) = (A_F \cup B_F)^-(x)$. Now, if $b_x = B^-_F(x)$, then

$$A^-_F(x) \le B^-_F(x) \le (\lambda_F \vee \psi_F)(x) \le A^+_F(x) \le B^+_F(x). \tag{3.14}$$

Hence we have

$$A^-_F(x) \le B^-_F(x) < (\lambda_F \vee \psi_F)(x) \le A^+_F(x) \le B^+_F(x) \tag{3.15}$$

or

$$A^-_F(x) \le B^-_F(x) = (\lambda_F \vee \psi_F)(x) \le A^+_F(x) \le B^+_F(x). \tag{3.16}$$

The case (3.15) induces a contradiction. The case (3.16) induces

$$(\lambda_F \vee \psi_F)(x) \notin \big((A_F \cup B_F)^-(x), (A_F \cup B_F)^+(x)\big)\,.$$

Therefore the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is an F-external neutrosophic cubic set in X. □

Similarly, we have the following theorems.

Theorem 3.7. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be T-external neutrosophic cubic sets in X such that*

$$\begin{aligned}&\max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\} \leq (\lambda_T \vee \psi_T)(x)\\&< \min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\}\end{aligned} \tag{3.17}$$

for all $x \in X$. Then the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is a T-external neutrosophic cubic set in X.

Theorem 3.8. *Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be I-external neutrosophic cubic sets in X such that*

$$\begin{aligned}&\max\left\{\min\{A_I^+(x), B_I^-(x)\}, \min\{A_I^-(x), B_I^+(x)\}\right\} \leq (\lambda_I \vee \psi_I)(x)\\&< \min\left\{\max\{A_I^+(x), B_I^-(x)\}, \max\{A_I^-(x), B_I^+(x)\}\right\}\end{aligned} \tag{3.18}$$

for all $x \in X$. Then the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is an I-external neutrosophic cubic set in X.

We give a condition for the P-intersection of two neutrosophic cubic sets to be both a T-internal (resp. I-internal and F-internal) neutrosophic cubic set and a T-external (resp. I-external and F-external) neutrosophic cubic set.

Theorem 3.9. *If neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ X satisfy the following condition*

$$\begin{aligned}&\min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\} = (\lambda_F \wedge \psi_F)(x)\\&= \max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\}\end{aligned} \tag{3.19}$$

for all $x \in X$, then the P-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is both an F-internal neutrosophic cubic set and an F-external neutrosophic cubic set in X.

Proof. For any $x \in X$, take

$$a_x := \min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\}$$

and

$$b_x := \max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\}.$$

Then a_x is one of $A_F^-(x)$, $B_F^-(x)$, $A_F^+(x)$ and $B_F^+(x)$. We consider $a_x = A_F^-(x)$ or $a_x = A_F^+(x)$ only. For remaining cases, it is similar to these cases. If $a_x = A_F^-(x)$, then

$$B_F^-(x) \leq B_F^+(x) \leq A_F^-(x) \leq A_F^+(x)$$

and thus $b_x = B_F^+(x)$. This implies that

$$A_F^-(x) = a_x = (\lambda_F \wedge \psi_F)(x) = b_x = B_F^+(x).$$

Hence $B_F^-(x) \leq B_F^+(x) = (\lambda_F \wedge \psi_F)(x) = A_F^-(x) \leq A_F^+(x)$, which implies that

$$(\lambda_F \wedge \psi_F)(x) = B_F^+(x) = (A_F \cap B_F)^+(x).$$

Hence $(\lambda_F \wedge \psi_F)(x) \notin ((A_F \cap B_F)^-(x), (A_F \cap B_F)^+(x))$ and

$$(A_F \cap B_F)^-(x) \leq (\lambda_F \wedge \psi_F)(x) \leq (A_F \cap B_F)^+(x).$$

If $a_x = A_F^+(x)$, then $B_F^-(x) \leq A_F^+(x) \leq B_F^+(x)$ and thus

$$(\lambda_F \wedge \psi_F)(x) = A_F^+(x) = (A_F \cap B_F)^+(x).$$

Hence $(\lambda_F \wedge \psi_F)(x) \notin ((A_F \cap B_F)^-(x), (A_F \cap B_F)^+(x))$ and

$$(A_F \cap B_F)^-(x) \leq (\lambda_F \wedge \psi_F)(x) \leq (A_F \cap B_F)^+(x).$$

Consequently, the P-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is both an F-internal neutrosophic cubic set and an F-external neutrosophic cubic set in X. □

Similarly, we get the following theorems.

Theorem 3.10. *If neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ X satisfy the following condition*

$$\begin{aligned} &\min\left\{\max\{A_I^+(x), B_I^-(x)\}, \max\{A_I^-(x), B_I^+(x)\}\right\} = (\lambda_I \wedge \psi_I)(x) \\ &= \max\left\{\min\{A_I^+(x), B_I^-(x)\}, \min\{A_I^-(x), B_I^+(x)\}\right\} \end{aligned} \tag{3.20}$$

for all $x \in X$, then the P-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is both an I-internal neutrosophic cubic set and an I-external neutrosophic cubic set in X.

Theorem 3.11. *If neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ X satisfy the following condition*

$$\begin{aligned} &\min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\} = (\lambda_T \wedge \psi_T)(x) \\ &= \max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\} \end{aligned} \tag{3.21}$$

for all $x \in X$, then the P-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is both a T-internal neutrosophic cubic set and a T-external neutrosophic cubic set in X.

Corollary 3.12. *If neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ X satisfy conditions 3.19, 3.20 and 3.21, then the P-intersection of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is both an internal neutrosophic cubic set and an external neutrosophic cubic set in X.*

Given two neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X where

$$\begin{aligned}
\mathbf{A} &:= \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\},\\
\Lambda &:= \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\},\\
\mathbf{B} &:= \{\langle x; B_T(x), B_I(x), B_F(x)\rangle \mid x \in X\},\\
\Psi &:= \{\langle x; \psi_T(x), \psi_I(x), \psi_F(x)\rangle \mid x \in X\},
\end{aligned}$$

we try to exchange Λ and Ψ, and make new neutrosophic cubic sets $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ in X. Under these circumstances, we have questions.

Question. 1. If two neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X are T-external (resp., I-external and F-external), then are the new neutrosophic cubic sets $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ T-internal (resp., I-internal and F-internal) neutrosophic cubic sets in X?

2. If two neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X are T-external (resp., I-external and F-external), then are the new neutrosophic cubic sets $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ T-external (resp., I-external and F-external) neutrosophic cubic sets in X?

The answer to the question above is negative as seen in the following example.

Example 3.13. (1) Let $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $[0, 1]$ where

$$\begin{aligned}
\mathbf{A} &= \{\langle x; [0.6, 0.7], [0.5, 0.7], [0.3, 0.5]\rangle \mid x \in [0, 1]\},\\
\Lambda &= \{\langle x; 0.8, 0.4, 0.8\rangle \mid x \in [0, 1]\},\\
\mathbf{B} &= \{\langle x; [0.3, 0.4], [0.4, 0.7], [0.7, 0.9]\rangle \mid x \in [0, 1]\},\\
\Psi &= \{\langle x; 0.2, 0.3, 0.4\rangle \mid x \in [0, 1]\}.
\end{aligned}$$

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ are both T-external and F-external neutrosophic cubic sets in $[0, 1]$. It is easy to verify that $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are F-internal neutrosophic cubic sets in $[0, 1]$, but not T-internal neutrosophic cubic sets in $[0, 1]$.

(2) For $X = \{a, b\}$, let $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in X with the tabular representations in Tables 8 and 9, respectively.
Then $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ are I-external neutrosophic cubic sets in X, and $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are represented as Tables 10 and 11, respectively,

Table 8: Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
a	$([0.3, 0.6], [0.2, 0.3], [0.2, 0.5])$	$(0.25, 0.15, 0.40)$
b	$([0.5, 0.7], [0.5, 0.6], [0.3, 0.4])$	$(0.55, 0.75, 0.35)$

Table 9: Tabular representation of $\mathscr{B} = (\mathbf{B}, \Psi)$

X	$\mathbf{B}(x)$	$\Psi(x)$
a	$([0.3, 0.7], [0.4, 0.5], [0.1, 0.5])$	$(0.35, 0.95, 0.60)$
b	$([0.5, 0.8], [0.7, 0.9], [0.2, 0.5])$	$(0.45, 0.35, 0.30)$

Table 10: Tabular representation of $\mathscr{A}^* := (\mathbf{A}, \Psi)$

X	$\mathbf{A}(x)$	$\Psi(x)$
a	$([0.3, 0.6], [0.2, 0.3], [0.2, 0.5])$	$(0.35, 0.95, 0.60)$
b	$([0.5, 0.7], [0.5, 0.6], [0.3, 0.4])$	$(0.45, 0.35, 0.30)$

Table 11: Tabular representation of $\mathscr{B}^* := (\mathbf{B}, \Lambda)$

X	$\mathbf{B}(x)$	$\Lambda(x)$
a	$([0.3, 0.7], [0.4, 0.5], [0.1, 0.5])$	$(0.25, 0.15, 0.40)$
b	$([0.5, 0.8], [0.7, 0.9], [0.2, 0.5])$	$(0.55, 0.75, 0.35)$

which are not I-internal neutrosophic cubic sets in X.

(3) For $X = \{a, b, c\}$, let $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in X with the tabular representations in Tables 12 and 13, respectively.

Then $\mathscr{A} = (\mathbf{A}, \Lambda)$, and $\mathscr{B} = (\mathbf{B}, \Psi)$ are F-external neutrosophic cubic sets in X. Note that $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are represented as Tables 14 and 15, respectively,

and they are not F-internal neutrosophic cubic sets in X.

Table 12: Tabular representation of $\mathscr{A} = (\mathbf{A}, \Lambda)$

X	$\mathbf{A}(x)$	$\Lambda(x)$
a	$([0.2, 0.3], [0.3, 0.5], [0.31, 0.51])$	$(0.35, 0.25, 0.75)$
b	$([0.4, 0.7], [0.1, 0.4], [0.22, 0.41])$	$(0.35, 0.50, 0.65)$
c	$([0.6, 0.9], [0.0, 0.2], [0.33, 0.42])$	$(0.50, 0.60, 0.75)$

Table 13: Tabular representation of $\mathscr{B} = (\mathbf{B}, \Psi)$

X	$\mathbf{B}(x)$	$\Psi(x)$
a	$([0.3, 0.7], [0.3, 0.5], [0.61, 0.81])$	$(0.25, 0.25, 0.35)$
b	$([0.5, 0.8], [0.5, 0.6], [0.25, 0.55])$	$(0.45, 0.30, 0.10)$
c	$([0.4, 0.9], [0.4, 0.7], [0.71, 0.85])$	$(0.35, 0.10, 0.40)$

Table 14: Tabular representation of $\mathscr{A}^* := (\mathbf{A}, \Psi)$

X	$\mathbf{A}(x)$	$\Psi(x)$
a	$([0.2, 0.3], [0.3, 0.5], [0.31, 0.51])$	$(0.25, 0.25, 0.35)$
b	$([0.4, 0.7], [0.1, 0.4], [0.22, 0.41])$	$(0.45, 0.30, 0.10)$
c	$([0.6, 0.9], [0.0, 0.2], [0.33, 0.42])$	$(0.35, 0.10, 0.40)$

Table 15: Tabular representation of $\mathscr{B}^* := (\mathbf{B}, \Lambda)$

X	$\mathbf{B}(x)$	$\Lambda(x)$
a	$([0.3, 0.7], [0.3, 0.5], [0.61, 0.81])$	$(0.35, 0.25, 0.75)$
b	$([0.5, 0.8], [0.5, 0.6], [0.25, 0.55])$	$(0.35, 0.50, 0.65)$
c	$([0.4, 0.9], [0.4, 0.7], [0.71, 0.85])$	$(0.50, 0.60, 0.75)$

Generally, the P-union of two T-external (resp. I-external and F-external) neutrosophic cubic sets may not be a T-internal (resp. I-internal and F-internal) neutrosophic cubic set.

Example 3.14. Consider the F-external neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$

and $\mathscr{B} = (\mathbf{B}, \Psi)$ in Example 3.13(3). The P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is represented by Table 16, and it is not an F-internal neutrosophic cubic set in X.

Table 16: Tabular representation of $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$

X	$(\mathbf{A} \cup \mathbf{B})(x)$	$(\Lambda \vee \Psi)(x)$
a	$([0.3, 0.7], [0.3, 0.5], [0.61, 0.81])$	$(0.35, 0.25, 0.75)$
b	$([0.5, 0.8], [0.5, 0.6], [0.25, 0.55])$	$(0.45, 0.50, 0.65)$
c	$([0.6, 0.9], [0.4, 0.7], [0.71, 0.85])$	$(0.50, 0.60, 0.75)$

We provide conditions for the P-union of two T-external (resp. I-external and F-external) neutrosophic cubic sets to be a T-internal (resp. I-internal and F-internal) neutrosophic cubic set.

Theorem 3.15. *For any T-external neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X, if $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are T-internal neutrosophic cubic sets in X, then the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is a T-internal neutrosophic cubic set in X.*

Proof. Assume that $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are T-internal neutrosophic cubic sets in X for any T-external neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X. Then

$$\lambda_T(x) \notin \left(A_T^-(x), A_T^+(x)\right), \ \psi_T(x) \notin \left(B_T^-(x), B_T^+(x)\right),$$
$$B_T^-(x) \leq \lambda_T(x) \leq B_T^+(x), \ A_T^-(x) \leq \psi_T(x) \leq A_T^+(x)$$

for all $x \in X$. We now consider the following cases.

(1) $\lambda_T(x) \leq A_T^-(x) \leq \psi_T(x) \leq A_T^+(x)$ and $\psi_T(x) \leq B_T^-(x) \leq \lambda_T(x) \leq B_T^+(x)$.

(2) $A_T^-(x) \leq \psi_T(x) \leq A_T^+(x) \leq \lambda_T(x)$ and $B_T^-(x) \leq \lambda_T(x) \leq B_T^+(x) \leq \psi_T(x)$.

(3) $\lambda_T(x) \leq A_T^-(x) \leq \psi_T(x) \leq A_T^+(x)$ and $B_T^-(x) \leq \lambda_T(x) \leq B_T^+(x) \leq \psi_T(x)$.

(2) $A_T^-(x) \leq \psi_T(x) \leq A_T^+(x) \leq \lambda_T(x)$ and $\psi_T(x) \leq B_T^-(x) \leq \lambda_T(x) \leq B_T^+(x)$.

First case implies that $\psi_T(x) = A_T^-(x) = B_T^-(x) = \lambda_T(x)$. Since $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are T-internal neutrosophic cubic sets in X, we have $\psi_T(x) \leq A_T^+(x)$ and $\lambda_T(x) \leq B_T^+(x)$. It follows that

$$\begin{aligned}(A_T \cup B_T)^-(x) &= \max\{A_T^-(x), B_T^-(x)\} = (\lambda_T \vee \psi_T)(x) \\ &\leq \max\{A_T^+(x), B_T^+(x)\} = (A_T \cup B_T)^+(x)\end{aligned}$$

for all $x \in X$. Therefore the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ is a T-internal neutrosophic cubic set in X. We can prove the other cases by the similar to the first case. □

Similarly, we have the following theorems.

Theorem 3.16. *For any I-external neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X, if $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are I-internal neutrosophic cubic sets in X, then the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an I-internal neutrosophic cubic set in X.*

Theorem 3.17. *For any F-external neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X, if $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are F-internal neutrosophic cubic sets in X, then the P-union $\mathscr{A} \cup_P \mathscr{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \vee \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is a F-internal neutrosophic cubic set in X.*

We provide conditions for the P-intersection of two T-external (resp. I-external and F-external) neutrosophic cubic sets to be a T-internal (resp. I-internal and F-internal) neutrosophic cubic set.

Theorem 3.18. *For any T-external (resp., I-external and F-external) neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X, if $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are T-internal (resp., I-internal and F-internal) neutrosophic cubic sets in X, then the P-intersection $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is a T-internal (resp., I-internal and F-internal) neutrosophic cubic set in X.*

Proof. It is similar to the proof of Theorem 3.15. □

Corollary 3.19. *For any external neutrosophic cubic sets $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ in X, if $\mathscr{A}^* := (\mathbf{A}, \Psi)$ and $\mathscr{B}^* := (\mathbf{B}, \Lambda)$ are internal neutrosophic cubic sets in X, then the P-intersection $\mathscr{A} \cap_P \mathscr{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \wedge \Psi)$ of $\mathscr{A} = (\mathbf{A}, \Lambda)$ and $\mathscr{B} = (\mathbf{B}, \Psi)$ is an internal neutrosophic cubic set in X.*

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Neutrosophy, a Sentiment Analysis Model

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ABSTRACT
This paper describes the importance of Neutrosophy Theory in order to find a method that could solve the uncertainties arising on discursive analysis. The aim of this pilot study is to find a procedure to diminish the uncertainties from public discourse induced, especially, by humans (politicians, journalists, etc.). We consider that Neutrosophy Theory is a sentiment analysis specific case regarding processing of the three states: positive, negative, and neutral. The study is intended to identify a method to answer to uncertainties solving in order to support politician's staff, NLP specialists, artificial intelligence researchers and generally the electors.

KEYWORDS
neutrosophy, sentiment analysis, communication, true, false, uncertainty.

1. INTRODUCTION

This study is the first step of a research that points out the uncertainties solving in discursive analysis. The research is based on Neutrosophy Theory[1] (Smarandache, 2005), which studies the neutrality as an essentially disputed concept with a generous applicability in sciences, like artificial intelligence (Vlădăreanu et al., 2014). This article explains the role of neutrality starting from the political context and the voters' decision.

In fact, the novelty of neutrosophy[2] consists of approaching the indeterminacy status that we can associate to neutral class of sentiment analysis (SA) (Gîfu and Scutelnicu, 2013), usually ignored. Moreover, some researchers associate neutral class with objective class in SA, but they consider it being less informative, preferring subjective class. SA, known as *opinion mining* (Pang and Lee, 2008), is a very important task of Natural Language Processing (NLP), the most known SA classification of texts is a binary one: subjective and objective (Pang and Lee, 2002), most often more difficult to undertake than polarity classification (Mihalcea et al., 2007). For other researchers the neutrality is determined the first one and sentiment polarity is determined the second one (Wilson et al., 2005).

We believe that Neutrosophy Theory seen as SA model would be useful for NLP specialists, linguistics, journalists, politicians, PR, and other scientists interested to find a method of uncertainties solving.

The paper is structured as follows: after a brief introduction, section 2 describes the background related to neutrosophy applicability; section 3 discusses the annotations regarding Neutrosophy Theory described in transposed algebraic structures and algorithms, section 4 introduces the relation between neutrosophy and sentiment analysis and finally, section 5 depicts some conclusions and directions for the future work.

2. BACKGROUND

According to the Neutrosophy Theory (NT), the neutral (uncertainty) instances can be analyzed and accordingly, reduced. There are some spectacular results of applying neutrosophy in practical application such as artificial intelligence (Gal et al., 2011). Extending these results, neutrosophy theory can be applied for solving uncertainty on other domains; in Robotics there are confirmed results of neutrosophics logics applied to make decisions when appear situations of uncertainty (Okuyama el al., 2013; Smarandache, 2011).

The real-time adaptive networked control of rescue robots is another project that used neutrosophic logic to control the robot movement in a surface with uncertainties (Smarandache, 2014). Starting with this point, we are confident that Neutrosophy Theory can help to analyse, evaluate and make the right decision in discursive analysis taking into account all sources that can generate uncertainty, of not informed voters, lack of information in candidates' politic campaign, not a strong candidate's propaganda, etc.

3. THE FUNDAMENTALS OF NEUTROSOPHY

The specialty literature reveals Zadeh introduced the degree of membership/truth (t), so the rest would be $(1-t)$ equal to f, their sum being 1, and he defined the fuzzy set in 1965.

In 1986, Atanassov introduced the degree of nonmembership/falsehood (f) and defined the intuitionistic fuzzy set.

[1] This theory was revealed by Smarandache in 1995 (published in 1998) it also was defined the neutrosophic set. Smarandache has coined the words "neutrosophy" and "neutrosophic".

[2] The etymology of *Neutrosophy* [in French, neutre and Latin, neuter - neutral, and in Greek, sophia - skill/wisdom] means knowledge of neutral thought.

if

$$0 <= t+f <= 1$$

and

$$0 <= 1-t-f$$

would be interpreted as indeterminacy

$$t+f <= 1$$

Why was it necessary to extend the *fuzzy logic*?

The indeterminacy state, as proposition, cannot be described in fuzzy logic, is missing the uncertainty state; the neutrosophic logic helps to make a distinction between a 'relative truth' and an 'absolute truth', while fuzzy logic does not.

As novelty to previous theory, Smarandache introduced and defined explicitly the degree of indeterminacy/neutrality (i) as independent component, where:

$$0 <= t+i+f <= 3$$

a) if

$$t+i+f < 1$$

we have incomplete information;

b) if

$$t+i+f =1$$

we have complete information (thus we get intuitionistic fuzzy set);

c) if

$$t+i+f > 1$$

we have paraconsistente information (contradictory).

In neutrosophy set, the three components `t, i, f` are independent because it is possible from a source to get (t), from another independent source to get (i) and from the third source to get (f). Smarandache goes further; he refined the range (Smarandache, 1995).

If there are some dependent sources (or respectively some dependent subcomponents), we can treat those dependent subcomponents together.

YSIS

A logic in which each proposition is estimated to have the percentage of truth in a subset `T`, the percentage of indeterminacy in a subset `I`, and the percentage of falsity in a subset `F`, where `T, I, F` are defined above, is called *Neutrosophic Logic*.

Similarly sentiment analysis defines states as positive, negative and neutral.

NT	**SA**
T	positive
I	neutral
F	negative

Statically `T, I, F` are subsets, but dynamically the components `T, I, F` are set-valued vector functions/operators depend- ing on many parameters, such as: time, space, etc. (some of them are hidden parameters, i.e. unknown parameters):

`T(t, s, …), I(t, s, …), F(t, s, …)`

where

`t = time, s = space …`

that is why the neutrosophic logic can be used also in quantum physics. If the Dynamic Neutrosophic Calculus can be used in discursive analysis, neutrosophics tries to reflect the dynamics of things and ideas.

We try to show an example of neutrosophic set from socio-human sciences.

Example: During an election process with 2 candidates C1 and C2, we have the following options:

E = {E1, E2, E3, E4} where we define:

E1 – Poll voting candidate C1;
E2 – Poll voting candidate C2;
E3 – Hesitant Poll who generates uncertainties;
E4 – Absent poll.

The initial neutrosophic space looks like (see Figure 1):

E1- represents the poll voting the candidate C1:
E1 = (t_{11}, i_{11}, f_{11}) => (28.65, 0, 0)

E2- represents the poll voting the candidate C2:
E2 = (t_{21}, i_{21}, f_{21}) => (18.7, 0, 0)

E3- represents the hesitant, neutral, uncertainty poll:
E3 = (t_{31}, i_{31}, f_{31}) => (0, 11.3, 0)

E4- represents the absent poll from election process:
E4 = (t_{41}, i_{41}, f_{41}) => (10.1, 15.4, 15.3) – they can vote both C1 candidate and C2 candidate, but also can be undecided; we exclude this aggregate from discussion.

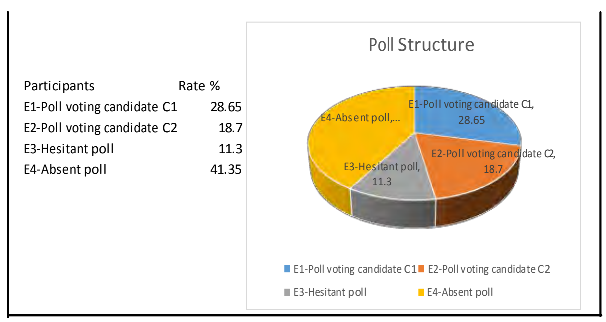

Figure 1. Initial Poll Structure

Analyzing these data, it can be summarized:

NT	C1	C2	SA
T	28.65	18.7	positive
I	7.2	4.1	neutral
F	18.7	28.65	positive

The purpose of neutrosophy is to investigate the uncertainties. In our case, the space generating uncertainties is E3 with its subset (t_{31}, i_{31}, f_{31}). Our purpose is to reduce the rate of "i_{31}" and to increase rate of "f_{31}" and "t_{31}", minimizing the uncertainties, this means a refining method for the process. Taking into account that we analyze a socio-human process belonging to politics communication, the applied techniques are methods of persuasion and conviction belonging to involved actors.

Through elective process we got data from ballot paper after refining uncertainties (see Figure 2):

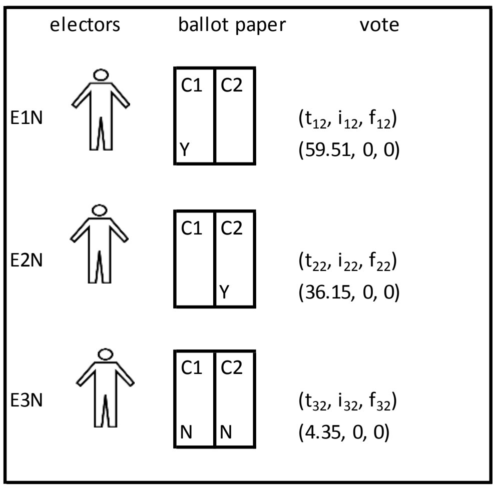

Figure 2. Elective process by ballot paper

Analyzing the process:

E1N(t_{12} =59.51, i_{12} =0, f_{12}=0) means that elector E1N voted only candidate C1 in rate of 59.51%;

E2N(t_{22}=36.15, i_{22} =0, f_{22}=0) means that elector E2N voted only candidate C2 in rate of 36.15%;

E3N(t_{32} =4.35, i_{32}=0.5, f_{32}=0) means that elector E3N gave a blind vote both for candidate C1 and C2 in rate of 4.35%.

In an election process, uncertainties reveal not only the null votes, blind votes, but also not participating voters to election process, because we cannot interpret their decision. There are situations when the percentage of this part of elector is high. The refined process proved that is possible to modify the rate of uncertainties, neutral status. We find the data in T and F as stable status (see Figure 3).

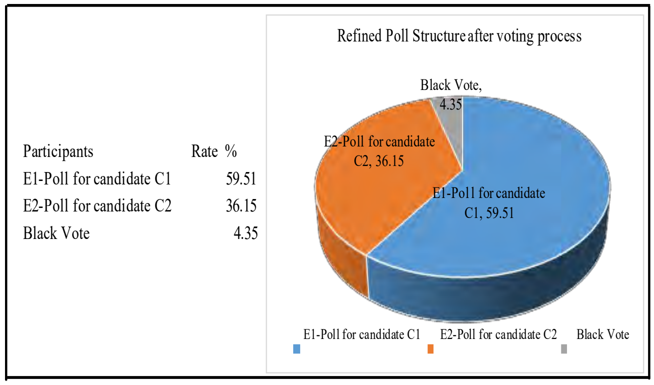

Figure 3. Refined Process

5. CONCLUSIONS AND FUTURE WORK

In this paper, it is presented a way of correcting the uncertainties arising in discursive analysis applying Neutrosophy Theory in relation with sentiment analysis. The Neutrosophy Theory could be considered a sentiment analysis model for solving the uncertainty (neutral), extended in IT applications, logistics, and human resource.

In the future work we will be oriented to find an algorithm to achieve the objectives to improve the percentage of stable statuses, by evaluation and interpret the neutrality/uncertainty state, in order to reduce it.

ACKNOWLEDGMENTS

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A Neutrosophic Extension of the MULTIMOORA Method

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Abstract. The aim of this manuscript is to propose a new extension of the MULTIMOORA method adapted for usage with a neutrosophic set. By using single valued neutrosophic sets, the MULITIMOORA method can be more efficient for solving complex problems whose solving requires assessment and prediction, i.e. those problems associated with inaccurate and unreliable data. The suitability of the proposed approach is presented through an example.

Key words: neutrosophic set, single valued neutrosophic set, MULTIMOORA, MCDM.

1. Introduction

The MULTIMOORA (Multi-Objective Optimization by a Ratio Analysis plus the Full Multiplicative Form) was proposed by Brauers and Zavadskas (2010).

The ordinary MULTIMOORA method has been proposed for usage with crisp numbers. In order to enable its use in solving a larger number of complex decision-making problems, several extensions have been proposed, out of which only the most prominent are mentioned: Brauers *et al.* (2011) proposed a fuzzy extension of the MULTIMOORA method; Balezentis and Zeng (2013) proposed an interval-valued fuzzy extension; Balezentis *et al.* (2014) proposed an intuitionistic fuzzy extension and Zavadskas *et al.* (2015) proposed an interval-valued intuitionistic extension of the MULTIMOORA method.

The MULTIMOORA method has been applied for the purpose of solving a wide range of problems.

As some of the most cited, the studies that consider different problems in economics (Brauers and Zavadskas, 2010, 2011; Brauers, 2010), personnel selection (Balezentis *et al.*, 2012a, 2012b), construction (Kracka *et al.*, 2015), risk management (Liu *et al.*, 2014a) and waste treatment (Liu *et al.*, 2014b) can be mentioned.

As some of the newest studies in which the MULTIMOORA method is used for solving various decision-making problems, the following ones can be mentioned: material selection (Hafezalkotob and Hafezalkotob, 2016; Hafezalkotob *et al.*, 2016) and the CNC machine tool evaluation (Sahu *et al.*, 2016).

A significant approach in solving complex decision-making problems was formed by adapting the multiple criteria decision-making methods for the purpose of using fuzzy numbers, proposed by Zadeh in the fuzzy set theory (Zadeh, 1965).

Based on the fuzzy set theory, some extensions are also proposed, such as: interval-valued fuzzy sets (Turksen, 1986), intuitionistic fuzzy sets (Atanassov, 1986) and interval-valued intuitionistic fuzzy sets (Atanassov and Gargov, 1989).

In addition to the membership function proposed in fuzzy sets, Atanassov (1986) introduced the non-membership function that expresses non-membership to a set, thus having created the basis for solving a much larger number of decision-making problems.

The intuitionistic fuzzy set is composed of membership (the so-called truth-membership) $T_A(x)$ and non-membership (the so-called falsity-membership) $F_A(x)$, which satisfies the conditions $T_A(x), F_A(x) \in [0, 1]$ and $0 \leqslant T_A(x) + F_A(x) \leqslant 1$. Therefore, intuitionistic fuzzy sets are capable of operating with incomplete pieces of information, but do not include intermediate and inconsistent information (Li *et al.*, 2016).

In intuitionistic fuzzy sets, indeterminacy $\pi_A(x)$ is $1 - T_A(x) - F_A(x)$ by default. Smarandache (1998, 1999) further extended intuitionistic fuzzy sets by proposing Neutrosophic, also introducing independent indeterminacy-membership.

Such a proposed neutrosophic set is composed of three independent membership functions named the truth-membership $T_A(x)$, the falsity-membership $F_A(x)$ and the indeterminacy-membership $I_A(x)$ functions.

Smarandache (1999) and Wang *et al.* (2010) further proposed a single valued neutrosophic set, by modifying the conditions $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0, 1]$ and $0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3$, which are more suitable for solving scientific and engineering problems (Li *et al.*, 2016).

Compared with the fuzzy set and its extensions, the single valued neutrosophic set can be identified as more flexible, for which reason an extension of the MULTIMOORA method adapted for the purpose of using the single valued neutrosophic set is proposed in this approach.

Therefore, the rest of this paper is organized as follows: in Section 2, some basic definitions related to the single valued neutrosophic set are given. In Section 3, the ordinary MULTIMOORA method is presented, whereas in Section 4, the Single Valued Neutrosophic Extension of the MULTIMOORA method is proposed. In Section 5, an example is considered with the aim to explain in detail the proposed methodology. The conclusions are presented in the final section.

2. The Single Valued Neutrosophic Set

Definition 1. (See Smarandache, 1999.) Let X be the universe of discourse, with a generic element in X denoted by x. Then, the Neutrosophic Set (NS) A in X is as follows:

$$A = \{x\langle T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}, \tag{1}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively,

$$T_A, I_A, F_A : X \to]^-0, 1^+[\quad \text{and} \quad ^-0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3^+.$$

Definition 2. (See Smarandache, 1999; Wang *et al.*, 2010.) Let X be the universe of discourse. The Single valued neutrosophic set (SVNS) A over X is an object having the following form:

$$A = \{x\langle T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}, \tag{2}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively,

$$T_A, I_A, F_A : X \to [0, 1] \quad \text{and} \quad 0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3.$$

Definition 3. (See Smarandache, 1999.) For an SVNS A in X, the triple $\langle t_A, i_A, f_A\rangle$ is called the single valued neutrosophic number (SVNN).

Definition 4. Let $x_1 = \langle t_1, i_1, f_1\rangle$ and $x_2 = \langle t_2, i_2, f_2\rangle$ be two SVNNs and $\lambda > 0$; then the basic operations are defined as follows:

$$x_1 + x_2 = \langle t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2\rangle, \tag{3}$$

$$x_1 \cdot x_2 = \langle t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2\rangle, \tag{4}$$

$$\lambda x_1 = \langle 1 - (1 - t_1)^\lambda, i_1^\lambda, f_1^\lambda\rangle, \tag{5}$$

$$x_1^\lambda = \langle t_1^\lambda, 1 - (1 - i_1)^\lambda, 1 - (1 - f_1)^\lambda\rangle. \tag{6}$$

Definition 5. (See Sahin, 2014.) Let $x = \langle t_x, i_x, f_x\rangle$ be an SVNN; then the score function s_x of x can be as follows:

$$s_x = (1 + t_x - 2i_x - f_x)/2, \tag{7}$$

where $s_x \in [-1, 1]$.

Definition 6. Let $x_1 = \langle t_1, i_1, f_1\rangle$ and $x_2 = \langle t_2, i_2, f_2\rangle$ be two SVNNs. Then the maximum distance between x_1 and x_2 is as follows:

$$d_{\max}(x_1, x_2) = \begin{cases} |t_1 - t_2|, & x_1, x_2 \in \Omega_{\max}, \\ |f_1 - f_2|, & x_1, x_2 \in \Omega_{\min}. \end{cases} \tag{8}$$

Definition 7. (See Sahin, 2014.) Let $A_j = \langle t_j, i_j, f_j\rangle$ be a collection of SVNSs and $W = (w_1, w_2, \dots, w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Average (SVNWA) operator of A_j is as follows:

$$\begin{aligned} &SVNWA(A_1, A_2, \dots, A_n) \\ &= \sum_{j=1}^{n} w_j A_j = \left(1 - \prod_{j=1}^{n}(1 - t_j)^{w_j}, \prod_{j=1}^{n}(i_j)^{w_j}, \prod_{j=1}^{n}(f_j)^{w_j}\right). \end{aligned} \tag{9}$$

where: w_j is the element j of the weighting vector, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 8. (See Sahin, 2014.) Let $A_j = \langle t_j, i_j, f_j\rangle$ be a collection of SVNSs and $W = (w_1, w_2, \dots, w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Geometric (SVNWG) operator of A_j is as follows:

$$\begin{aligned} &SVNWG(A_1, A_2, \dots, A_n) \\ &= \prod_{j=1}^{n}(A_j)^{w_j} = \left(\prod_{j=1}^{n}(t_j)^{w_j}, 1 - \prod_{j=1}^{n}(1 - i_j)^{w_j}, 1 - \prod_{j=1}^{n}(1 - f_j)^{w_j}\right). \end{aligned} \tag{10}$$

where: w_j is the element j of the weighting vector, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. The MULTIMOORA Method

The MULTIMOORA method consists of three approaches named as follows: the Ratio System (RS) Approach, the Reference Point (RP) Approach and the Full Multiplicative Form (FMF).

The considered alternatives are ranked based on all three approaches and the final ranking order and the final decision is made based on the theory of dominance. In other words, the alternative with the highest number of appearances in the first positions on all ranking lists is the best-ranked alternative.

The ratio system approach. In this approach, the overall importance of the alternative i can be calculated as follows:

$$y_i = y_i^+ - y_i^-, \tag{11}$$

with:

$$y_i^+ = \sum_{j \in \Omega_{\max}} w_j r_{ij}, \quad \text{and} \tag{12}$$

$$y_i^- = \sum_{j \in \Omega_{\min}} w_j r_{ij}, \tag{13}$$

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{n} x_{ij}^2}}, \tag{14}$$

where: y_i denotes the overall importance of the alternative i, obtained on the basis of all the criteria; y_i^+ and y_i^- denote the overall importance of the alternative i, obtained on the basis of the benefit and cost criteria, respectively; r_{ij} denotes the normalized performance of the alternative i with respect to the criterion j; x_{ij} denotes the performance of the alternative i to the criterion j; $\Omega_{\max}$ and $\Omega_{\min}$ denote the sets of the benefit cost criteria, respectively; $i = 1, 2, \ldots, m$; m is the number of the alternatives, $j = 1, 2, \ldots, n$; n is the number of the criteria.

In this approach, the compared alternatives are ranked based on y_i in descending order and the alternative with the highest value of y_i is considered to be the best-ranked.

The reference point approach. The optimization based on this approach can be shown as follows:

$$d_i^{\max} = \max_j \left(w_j \left| r_j^* - r_{ij} \right| \right), \tag{15}$$

where: $d_i^{\max}$ denotes the maximum distance of the alternative i to the reference point and r_j^* denotes the coordinate j of the reference point as follows:

$$r_j^* = \begin{cases} \max\limits_i r_{ij}, & j \in \Omega_{\max}, \\ \min\limits_i r_{ij}, & j \in \Omega_{\min}. \end{cases} \tag{16}$$

In this approach, the compared alternatives are ranked based on $d_i^{\max}$ in ascending order and the alternative with the lowest value of $d_i^{\max}$ is considered the best-ranked.

The full multiplicative form. In the FMF, the overall utility of the alternative i can be determined in the following manner:

$$u_i = \frac{a_i}{b_i}, \tag{17}$$

with:

$$a_i = \prod_{j \in \Omega_{\max}} w_j r_{ij}, \tag{18}$$

$$b_i = \prod_{j \in \Omega_{\min}} w_j r_{ij}, \tag{19}$$

where: u_i denotes the overall utility of the alternative i, a_i denotes the product of the weighted performance ratings of the benefit criteria and b_i denotes the product of the weighted performance ratings of the cost criteria of the alternative i.

As in the RSA, the compared alternatives are ranked based on their u_i in descending order and the alternative with the highest value of u_i is considered the best-ranked.

The final ranking of alternatives based on the MULTIMOORA method. As a result of evaluation by applying the MULTIMOORA method, three ranking lists of the considered alternatives are obtained. Based on Brauers and Zavadskas (2011), the final ranking order of the alternatives is determined based on the theory of dominance.

4. An Extension of the MULTIMOORA Method Based on Single Valued Neutrosophic Numbers

For an MCDM problem involving m alternatives and n criteria, whereby the performances of the alternatives are expressed by using SVNS, the calculation procedure of the extended MULTIMOORA method can be expressed as follows:

Step 1. Determine the ranking order of the alternatives based on the RS approach. The ranking of the alternatives and the selection of the best one based on this approach in the proposed extension of the MULTIMOORA method can be expressed through the following sub steps:

Step 1.1. *Calculate* Y_i^+ and Y_i^- by using the SVNWA operator, as follows:

$$Y_i^+ = \left(1 - \prod_{j \in \Omega_{\max}} (1 - t_j)^{w_j}, \prod_{j \in \Omega_{\max}} (i_j)^{w_j}, \prod_{j \in \Omega_{\max}} (f_j)^{w_j}\right), \tag{20}$$

$$Y_i^- = \left(1 - \prod_{j \in \Omega_{\min}} (1 - t_j)^{w_j}, \prod_{j \in \Omega_{\min}} (i_j)^{w_j}, \prod_{j \in \Omega_{\min}} (f_j)^{w_j}\right), \tag{21}$$

where: Y_i^+ and Y_i^- denote the importance of the alternative i obtained based on the benefit and cost criteria, respectively; Y_i^+ and Y_i^- are SVNNs.

Step 1.2. *Calculate* y_i^+ and y_i^- by using the Score Function, as follows:

$$y_i^+ = s\left(Y_i^+\right), \tag{22}$$

$$y_i^- = s\left(Y_i^-\right). \tag{23}$$

Step 1.3. *Calculate the overall importance* for each alternative, as follows:

$$y_i = y_i^+ - y_i^-. \tag{24}$$

Step 1.4. *Rank the alternatives and select the best one*. The ranking of the alternatives can be performed in the same way as in the RS approach of the ordinary MULTIMOORA method.

Step 2. Determine the ranking order of the alternatives based on the RP approach. The ranking of the alternatives and the selection of the best one, based on the RP approach, can be expressed through the following substeps:

Step 2.1. *Determine the reference point*. In this approach, each coordinate of the reference point $r^* = \{r_1^*, r_2^*, \ldots, r_n^*\}$ is an SVNN, $r_j^* = \langle t_j^*, i_j^*, f_j^* \rangle$, whose values are determined as follows:

$$r_j^* = \begin{cases} \left\langle \max\limits_i t_{ij}, \min\limits_i i_{ij}, \min\limits_i f_{ij} \right\rangle, & j \in \Omega_{\max}, \\ \left\langle \min\limits_i t_{ij}, \min\limits_i i_{ij}, \max\limits_i f_{ij} \right\rangle, & j \in \Omega_{\min}, \end{cases} \tag{25}$$

where: r_j^* denotes the coordinate j of the reference point.
For the sake of simplicity, r_j^* could be determined as follows:

$$r_j^* = \begin{cases} \langle 1, 0, 0 \rangle, & j \in \Omega_{\max}, \\ \langle 0, 0, 1 \rangle, & j \in \Omega_{\min}. \end{cases} \tag{25a}$$

Step 2.2. *Determine the maximum distance from each alternative to all the coordinates of the reference point* as follows:

$$d_{ij}^{\max} = d_{\max}\left(r_{ij}, r_j^*\right) w_j, \tag{26}$$

where $d_{ij}^{\max}$ denotes the maximum distance of the alternative i obtained based on the criterion j determined by Eq. (8).

Step 2.3. *Determine the maximum distance of each alternative*, as follows:

$$d_i^{\max} = \max_j d_{ij}^{\max}. \tag{27}$$

Step 2.4. *Rank the alternatives and select the best one*. At this step, the ranking of the alternatives can be done in the same way as in the RPA of the ordinary MULTIMOORA method.

Step 3. Determine the ranking order of the alternatives and select the best one based on the FMF. The ranking of the alternatives and the selection of the best one can be expressed through the following sub steps:

Step 3.1. *Calculate* A_i and B_i as follows:

$$A_i = \left(\prod_{j \in \Omega_{\max}} (t_j)^{w_j}, 1 - \prod_{j \in \Omega_{\max}} (1 - i_j)^{w_j}, 1 - \prod_{j \in \Omega_{\max}} (1 - f_j)^{w_j} \right), \tag{28}$$

$$B_i = \left(\prod_{j \in \Omega_{\min}} (t_j)^{w_j}, 1 - \prod_{j \in \Omega_{\min}} (1 - i_j)^{w_j}, 1 - \prod_{j \in \Omega_{\min}} (1 - f_j)^{w_j} \right), \tag{29}$$

where: $A_i = \langle t_{Ai}, i_{Ai}, f_{Ai} \rangle$ and $B_i = \langle t_{Bi}, i_{Bi}, f_{Bi} \rangle$ are SVNNs.

Step 3.2. *Calculate* a_i and b_i by using the Score Function as follows:

$$a_i = s(A_i), \tag{30}$$

$$b_i = s(B_i). \tag{31}$$

Step 3.3. *Determine the overall utility for each alternative* as follows:

$$u_i = \frac{a_i}{b_i}. \tag{32}$$

Sep 3.4. *Rank the alternatives and select the best one*. The ranking of the alternatives can be performed in the same way as in the FMF of the ordinary MULTIMOORA method.

Step 4. Determine the final ranking order of the alternatives. The final ranking order of the alternatives can be determined as in the case of the ordinary MULTIMOORA method, i.e. based on the dominance theory.

5. A Numerical Example

In order to demonstrate the applicability and efficiency of the proposed approach, an example has been adopted from Stanujkic *et al.* (2015). In order to briefly demonstrate the advantages of the proposed methodology, this example has been slightly modified.

Suppose that a mining and smelting company has to build a new flotation plant, for which reason an expert has been engaged to evaluate the three Comminution Circuit Designs (CCDs) listed below:

- A_1, the CCDs based on the combined use of rod mills and ball mills;
- A_2, the CCDs based on the use of ball mills; and
- A_3, the CCDs based on the use of semi-autogenous mills.

For the purpose of conducting an evaluation, the following criteria have been chosen:

- C_1, Grinding efficiency;
- C_2, Economic efficiency;
- C_3, Technological reliability;
- C_4, Capital investment costs; and
- C_5, Environmental impact.

The ratings obtained from the expert are shown in Table 1.

The ranking based on the RS approach. The ranking results and the ranking order of the alternatives obtained based on the RS approach, i.e. by applying Eqs. (19) to (23), are accounted for in Table 2.

The ranking based on the RPA. The ranking of the alternatives based on the RP approach begins by determining the reference point, as it is shown in Table 3.

Table 1
The ratings of the three generic CCDs obtained from an expert.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle 0.9, 0.1, 0.2\rangle$	$\langle 0.7, 0.2, 0.3\rangle$	$\langle 0.9, 0.1, 0.2\rangle$	$\langle 0.9, 0.1, 0.2\rangle$	$\langle 0.9, 0.1, 0.2\rangle$
A_2	$\langle 0.8, 0.1, 0.3\rangle$	$\langle 0.8, 0.1, 0.3\rangle$	$\langle 0.8, 0.1, 0.3\rangle$	$\langle 0.9, 0.1, 0.2\rangle$	$\langle 0.8, 0.1, 0.3\rangle$
A_3	$\langle 1.0, 0.1, 0.3\rangle$	$\langle 0.9, 0.1, 0.2\rangle$	$\langle 0.9, 0.1, 0.2\rangle$	$\langle 0.7, 0.2, 0.5\rangle$	$\langle 0.7, 0.2, 0.3\rangle$

Table 2
The ranking orders of the alternatives obtained on the basis of the RS approach.

	Y_i^+	Y_i^-	y_i^+	y_i^-	y_i	Rank
A_1	$\langle 0.73, 0.25, 0.38\rangle$	$\langle 0.55, 0.45, 0.57\rangle$	0.425	0.045	0.380	2
A_2	$\langle 0.65, 0.22, 0.46\rangle$	$\langle 0.51, 0.45, 0.60\rangle$	0.372	0.006	0.366	3
A_3	$\langle 1.0, 0.22, 0.39\rangle$	$\langle 0.34, 0.57, 0.73\rangle$	0.583	−0.263	0.845	1

Table 3
The reference point.

	C_1	C_2	C_3	C_4	C_5
r_j^*	$\langle 1.0, 0.1, 0.3\rangle$	$\langle 0.9, 0.2, 0.3\rangle$	$\langle 0.9, 0.1, 0.3\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.7, 0.1, 0.2\rangle$

Table 4
The ranking order of the alternatives obtained based on the RP approach.

I	II	III	IV	V	VI	VII	VI
	r_1^*	r_2^*	r_3^*	r_4^*	r_5^*	$d_i^{\max}$	Rank
A_1	0.02	0.03	0.00	0.00	0.00	0.034	1
A_2	0.05	0.02	0.02	0.00	0.01	0.048	2
A_3	0.00	0.00	0.00	0.06	0.01	0.063	3

Table 5
The ranking order of the alternatives obtained based on the FMF.

	A_i	B_i	a_i	b_i	u_i	Rank
A_1	$\langle 0.89, 0.25, 0.15\rangle$	$\langle 0.96, 0.45, 0.08\rangle$	0.618	0.498	1.242	3
A_2	$\langle 0.86, 0.22, 0.21\rangle$	$\langle 0.95, 0.45, 0.09\rangle$	0.605	0.481	1.258	2
A_3	$\langle 0.96, 0.22, 0.16\rangle$	$\langle 0.88, 0.57, 0.18\rangle$	0.674	0.283	2.379	1

The maximum distances from each alternative to the coordinate j of the reference point obtained by using Eq. (25) and the maximum distance of each alternative obtained by using Eq. (26) are presented in Table 4. The ranking order of the alternatives is also presented in Table 4.

The ranking based on the FMF. The ranking results and the ranking order of the alternatives obtained on the basis of the FMF approach, i.e. by applying Eqs. (27) to (31), are demonstrated in Table 5.

Table 6
The final ranking order of the alternatives according to the MULTIMOORA method.

	RS	RP	FMF	Rank
A_1	2	1	3	3
A_2	3	2	2	2
A_3	1	3	1	1

The final ranking order of the alternatives which summarizes the three different ranks provided by the respective parts of the MULTIMOORA method is shown in Table 6.

As it can be seen from Table 6, all three approaches, integrated in the MULTIMOORA, have resulted in different ranking orders, for which reason the final ranking order is determined based on the dominance theory.

6. Conclusion

The MULTIMOORA method has been proven in solving different decision-making problems. In order to enable its application in the solving of a larger number of complex decision-making problems, numerous extensions have been proposed for the MULTIMOORA method.

Compared to crisp, fuzzy, interval-valued and intuitionistic fuzzy numbers, the neutrosophic set provides significantly greater flexibility, which can be conducive to solving decision-making problems associated with uncertainty, estimations and predictions.

Therefore, an extension of the MULTIMOORA method enabling the use of single valued neutrosophic numbers is proposed in this paper.

The usability and efficiency of the proposed extension is presented in the example of the comminution circuit design selection.

Finally, it should be noted that the proposed extension of the MULTIMOORA method can be used for solving a much larger number of complex decision-making problems. A number of real-world decision making problems which have to be solved is based on the data acquired from respondents can be identified as one of the areas where the proposed extension of the MULTIMOORA method can reach its advantages.

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Shortest Path Problem under Trapezoidal Neutrosophic Information

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***Abstract*—In this research paper, a new approach is proposed for computing the shortest path length from source node to destination node in a neutrosophic environment. The edges of the network are assigned by trapezoidal fuzzy neutrosophic numbers. A numerical example is provided to show the performance of the proposed approach.**

***Keywords*—neutrosophic sets; trapezoidal neutrosophic sets; shortest path problem; score function**

I. Introduction

Smarandache [1] proposed the concept of neutrosophic sets (in short NSs) as a means of expressing the inconsistencies and indeterminacies that exists in most real-life problems. The proposed concept generalized fuzzy sets and intuitionistic fuzzy set theory [3, 4]. The notion of NSs is described with three functions: truth, an indeterminacy and a falsity, where the functions are totally independent, the three functions are inside the unit interval]−0, 1+[. To practice NSs in real life situations efficiently. A new version of NSs named Single valued Neutrosophic Sets (in short SVNSs) was defined by Smarandache in [1]. Subsequently Wang et al. [5] defined various operations and operators for the SVNS model. Additional literature on single valued neutrosophic sets can be found in [6-14, 16]. Also later on, Smarandache extended the neutrosophic set to neutrosophic overset, underset, and offset [15]. Ye [17] presented the concept of trapezoidal fuzzy neutrosophic set (in short TrFNSs) and studied some interesting results with proofs and examples. In TrFNSs, the truth, the indeterminate and the false membership degrees are expressed with Trapezoidal Fuzzy Numbers (TrFN) instead of real numbers. Smarandache and Kandasamy [25, 28-29] introduced the concept of neutrosophic graph based on the indeterminacy component (I). Later on, in [18-23, 26-27] Broumi et al. introduced different types of neutrosophic graph based on the neutrosophic values (T, I, F) including single valued neutrosophic graphs, interval valued neutrosophic graphs and bipolar neutrosophic graphs. In graph theory, the shortest path problems (in short SPP) is one of the known famous problems studied in the numerous discipline including operation research, computer science, communication network and so on. In the literature, many research papers have been focused seriously on fuzzy shortest path problems and their extensions [30-39]. Till now, few research papers deal with shortest path problems in neutrosophic environment. In [40-44], Broumi et al. presented some algorithms for solving the shortest path problems in neutrosophic environment. All these algorithms are based on the score functions. In this paper, the addition operation and the order relation have been given by Ye [17]. In this research paper, our main objective is to solving the shortest path problems in a network, where the edges weight are represented by trapezoidal fuzzy neutrosophic numbers.

This paper is constructed as follows: In Section 2, some basic definitions of neutrosophic sets, SVN-sets and trapezoidal fuzzy neutrosophic sets are introduced. In section 3, a new proposed algorithm for computing the trapezoidal fuzzy neutrosophic shortest path problem on a network is presented. In Section 4, a numerical example is given for computing the shortest path and shortest distance from the source node to destination node. We conclude the paper in Section 5.

II. Preliminaries

In this section, some definitions related to the concept of neutrosophic sets, single valued neutrosophic and trapezoidal fuzzy neutrosophic sets are taken from [2, 5, 17]

Definition 2.1 [2] Let ζ be a universal set. The neutrosophic set A on the universal set ζ categorized into three membership functions called the true $T_A(x)$, indeterminate $I_A(x)$ and false $F_A(x)$ contained in real standard or non–standard subset of]$^-$0, 1$^+$[respectively and denoted as following

$$A= \{<x: T_A(x), I_A(x), F_A(x)> x \in \zeta \} \qquad (1)$$

Definition 2.2 [5] Let ζ be a universal set. The single valued neutrosophic sets (in short SVNS) A on the universal is denoted as following:

$$A = \{<x: T_A(x), I_A(x), F_A(x)>, x \in \zeta\} \quad (2)$$

The function $T_A(x) \in [0, 1]$, $I_A(x) \in [0, 1]$ and $I_A(x) \in [0, 1]$ are named "degree of truth, indeterminacy and falsity membership of x in A", satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \quad (3)$$

Definition 2.3 [17]. Let ζ be a universal set and ψ [0, 1] be *the sets of all trapezoidal fuzzy numbers on [0, 1]*. The trapezoidal fuzzy neutrosophic sets (in short TrFNSs) $\breve{A}$ on the universal is denoted as following:

$$\breve{A} = \{<x: \breve{T}_A(x), \breve{I}_A(x), \breve{F}_A(x)>, x \in \zeta\} \quad (4)$$

Where $\breve{T}_A(x): \zeta \to \psi[0,1]$, $\breve{I}_A(x): \zeta \to \psi[0,1]$ and $\breve{F}_A(x): \zeta \to \psi[0,1]$. The trapezoidal fuzzy numbers

$$\breve{T}_A(x) = (T_A^1(x), T_A^2(x), T_A^3(x), T_A^4(x)), \quad (5)$$

$$\breve{I}_A(x) = (I_A^1(x), I_A^2(x), I_A^3(x), I_A^4(x)) \text{ and} \quad (6)$$

$\breve{F}_A(x) = (F_A^1(x), F_A^2(x), F_A^3(x), F_A^4(x))$, respectively denotes degree of truth, indeterminacy and falsity membership of x in $\breve{A}$ $\forall x \in \zeta$.

$$0 \le T_A^4(x) + I_A^4(x) + F_A^4(x) \le 3. \quad (7)$$

For notational convenience, the trapezoidal fuzzy neutrosophic value (TrFNV) $\tilde{A}$ is denoted by $\breve{A} = \langle (t_1,t_2,t_3,t_4),(i_1,i_2,i_3,i_4),(f_1,f_2,f_3,f_4)\rangle$ where,

$$(T_A^1(x), T_A^2(x), T_A^3(x), T_A^4(x)) = (t_1,t_2,t_3,t_4), \quad (8)$$

$$(I_A^1(x), I_A^2(x), I_A^3(x), I_A^4(x)) = (i_1,i_2,i_3,i_4), \text{ and} \quad (9)$$

$$(F_A^1(x), F_A^2(x), F_A^3(x), F_A^4(x)) = (f_1,f_2,f_3,f_4) \quad (10)$$

with $t_1 \le t_2 \le t_3 \le t_4$, $i_1 \le i_2 \le i_3 \le i_4$ and $f_1 \le f_2 \le f_3 \le f_4$

where, the truth membership function is given as bellow:

$$\breve{T}_A(x) = \begin{cases} \frac{x-t_1}{t_2-t_1} & t_1 \le x \le t_2 \\ 1 & t_2 \le x \le t_3 \\ \frac{x-t_1}{t_2-t_1} & t_3 \le x \le t_4 \\ 0 & otherwise \end{cases} \quad (11)$$

The indeterminacy membership is given as below:

$$\breve{I}_A(x) = \begin{cases} \frac{x-i_1}{i_2-i_1} & i_1 \le x \le i_2 \\ 1 & i_2 \le x \le i_3 \\ \frac{i_4-x}{i_4-i_3} & i_3 \le x \le i_4 \\ 0 & otherwise \end{cases} \quad (12)$$

And the falsity membership function is given as below:

$$\breve{F}_A(x) = \begin{cases} \frac{x-f_1}{f_2-f_1} & f_1 \le x \le f_2 \\ 1 & f_2 \le x \le f_3 \\ \frac{f_4-x}{f_4-f_3} & f_3 \le x \le f_4 \\ 0 & otherwise \end{cases} \quad (13)$$

Definition 2.4 [17]. The trapezoidal fuzzy neutrosophic number $\breve{A} = \langle (t_1,t_2,t_3,t_4),(i_1,i_2,i_3,i_4),(f_1,f_2,f_3,f_4)\rangle$ is said to be trapezoidal fuzzy neutrosophic zero if and only if

$(t_1,t_2,t_3,t_4) = (0, 0, 0, 0)$, $(i_1,i_2,i_3,i_4) = (1, 1, 1, 1)$ and $(f_1,f_2,f_3,f_4) = (1, 1, 1, 1)$ (14)

Definition 2.5 [17]. Let $\breve{A}_1$ and $\breve{A}_2$ two TrFNVs defined on the set of real numbers, denoted as :

$\breve{A}_1 = \langle (a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle$ and

$\breve{A}_2 = \langle (e_1,e_2,e_3,e_4),(f_1,f_2,f_3,f_4),(g_1,g_2,g_3,g_4)\rangle$ and $\eta > 0$.

Hence , the operations rules are defined as following:

$$\text{(i)} \quad \breve{A}_1 \oplus \breve{A}_2 = \left\langle \begin{array}{l} \left(\begin{array}{l} (a_1+e_1-a_1e_1),(a_2+e_2-a_2e_2), \\ (a_3+e_3-a_3e_3),(a_4+e_4-a_4e_4) \end{array} \right), \\ ((b_1 f_1),(b_2 f_2),(b_3 f_3),(b_4 f_4)), \\ (c_1g_1),(c_2g_2),(c_3g_3),(c_4g_4)) \end{array} \right\rangle \quad (15)$$

$$\text{(ii)} \quad \breve{A}_1 \otimes \breve{A}_2 = \left\langle \begin{matrix} (a_1e_1, a_2e_2, a_3e_3, a_4e_4), \\ \begin{pmatrix} (b_1 + f_1 - b_1 f)_1, (b_2 + f_2 - b_2 f_2), \\ (b_3 + f_3 - b_3 f_3), (b_4 + f_4 - b_4 f_4) \end{pmatrix}, \\ \begin{pmatrix} (c_1 + g_1 - c_1 g_1), (c_2 + g_2 - c_2 g_2), \\ (c_3 + g_3 - c_3 g_3), (c_4 + g_4 - c_4 g_4) \end{pmatrix} \end{matrix} \right\rangle \quad (16)$$

$$\text{(iii)} \quad \eta \breve{A} = \left\langle \begin{matrix} \begin{pmatrix} (1-(1-a_1)^{\eta}), (1-(1-a_2)^{\eta}), \\ (1-(1-a_3)^{\eta}), (1-(1-a_4)^{\eta}) \end{pmatrix} \\ (b_1^{\eta}, b_2^{\eta}, b_3^{\eta}, b_4^{\eta}), (c_1^{\eta}, c_2^{\eta}, c_3^{\eta}, c_4^{\eta}) \end{matrix} \right\rangle \quad (17)$$

$$\text{(iv)} \quad \breve{A}_1^{\eta} = \left\langle \begin{matrix} (a_1^{\eta}, a_2^{\eta}, a_3^{\eta}, a_4^{\eta}), \\ \left((1-(1-b_1)^{\eta}), (1-(1-b_2)^{\eta}), (1-(1-b_3)^{\eta}), 1-(1-b_4)^{\eta})\right), \\ \left((1-(1-c_1)^{\eta}), (1-(1-c_2)^{\eta}), (1-(1-c_3)^{\eta})), (1-(1-c_4)^{\eta})\right) \end{matrix} \right\rangle$$

where $\eta > 0$ (18)

Ye [17] gave the definition of score function $s(\breve{A}_1)$ and accuracy function $H(\breve{A}_1)$ to compare the grades of TrFNS. These functions shows that greater is the value, the greater is the TrFNS and by using these concept paths can be ranked.

Definition 2.6. Let $\breve{A}_1$ be a TrFNV denoted as

$\breve{A}_1 = \langle (t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4) \rangle$ Hence , the score function and the accuracy function of TrFNV are denoted as below:

$$\text{(i)} \quad s(\breve{A}_1) = \frac{1}{12} \begin{bmatrix} 8 + (t_1 + t_2 + t_3 + t_4) - (i_1 + i_2 + i_3 + i_4) \\ -(f_1 + f_2 + f_3 + f_4) \end{bmatrix} \quad (19)$$

$$\text{(ii)} \quad H(\breve{A}_1) = \frac{1}{4}\left[(t_1 + t_2 + t_3 + t_4) - (f_1 + f_2 + f_3 + f_4)\right] \quad (20)$$

In order to make a comparisons between two TrFNV, Ye **[17]**, presented the order relations between two TrFNVs.

Definition 2.7 Let $\breve{A}_1$ and $\breve{A}_2$ be two TrFNV defined on the set of real numbers , denoted as

$\breve{A}_1 = \langle (t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4) \rangle$ and $\breve{A}_2 = \langle (p_1, p_2, p_3, p_4), (q_1, q_2, q_3, q_4), (r_1, r_2, r_3, r_4) \rangle$. Hence , the ranking method is defined as follows:

1) If $s(\breve{A}_1) \succ s(\breve{A}_2)$, then $\breve{A}_1$ is greater than $\breve{A}_2$, that is, $\breve{A}_1$ is superior to $\breve{A}_2$, denoted by $\breve{A}_1 \succ \breve{A}_2$

2) If $s(\breve{A}_1) = s(\breve{A}_2)$, and $H(\breve{A}_1) \succ H(\breve{A}_2)$ then $\breve{A}_1$ is greater than $\breve{A}_2$, that is, $\breve{A}_1$ is superior to $\breve{A}_2$, denoted by $\breve{A}_1 \succ \breve{A}_2$.

III. TRFN- SHORTEST PATH PROBLEM

In this section, the edge length in a network is considered to be trapezoidal fuzzy neutrosophic number. To find the shortest path in a network , where the edges are characterized by trapezoidal fuzzy neutrosophic number. We present the following procedure:

Step 1 Suppose $\tilde{d}_1$ = <(0, 0, 0, 0) (1, 1, 1, 1), (1, 1, 1, 1)> and label the source node1 as [$\tilde{d}_1$ = < (0, 0, 0, 0),(1, 1, 1,1), (1, 1, 1, 1)>, -]

Let n is the destination node.

Step 2: Select $\tilde{d}_j = \min \{ \tilde{d}_i \oplus \tilde{d}_{ij} \}$ for all j= 2,3,…,n.

Step 3: If the minimum provided correspond to one value of i then label node j as [$\tilde{d}_j$, i]. If the minimum provided correspond to several values of i, then it indicate that there exist more than one TrFN-path between the source node and the node j. Hence the TrFN-distance along path is $\tilde{d}_j$, so select any value of i.

Step 4: Set the destination node n be labeled as [$\tilde{d}_n$,*l*], then the TrFN-shortest path distance from source node to destination node is $\tilde{d}_n$.

Step 5: Since the destination node n is labeled [$\tilde{d}_n$, *l*]. In order to find the TrFN-shortest path connecting the source node and the destination node, identify the label of the node *l*. Set it as [$\tilde{d}_l$,*p*], Repeat step 2 and step 3 until the node 1 is obtained.

Step 6: To obtain the TrFN-shortest path, we should joining all the nodes provided by the step 5.

IV. ILLUSTRATIVE EXAMPLE

Consider a small network shown in the following figure 1 in which each edge length is represented by a trapezoidal fuzzy neutrosophic number (see table 1). This network includes 6 nodes and 8 directed edges. This problem is to compute the shortest path between source node and destination node in the given network.

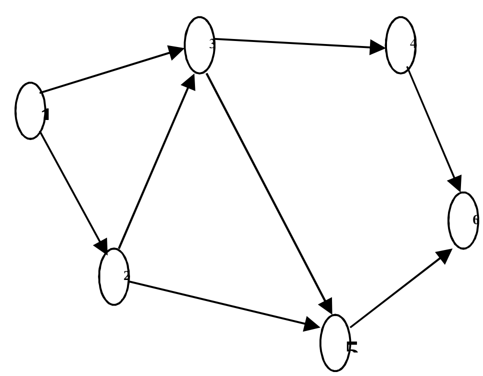

Fig. 1. Trapezoidal fuzzy neutrosophic network

The edges weight of the trapezoidal fuzzy neutrosophic network are represented by trapezoidal fuzzy neutrosophic numbers.

TABLE I. THE EDGES WEIGHT OF THE TRAPEZOIDAL FUZZY NEUTROSOPHIC GRAPHS

Edges	Trapezoidal fuzzy neutrosophic distance
1-2	<(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>
1-3	<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)>
2-3	<(0.3, 0.4, 0.6, 0.7), (0.1, 0.2, 0.3, 0.5), (0.3, 0.5, 0.7, 0.9)>
2-5	<(0.1, 0.3, 0.4, 0.5), (0.3, 0.4, 0.5, 0.7), (0.2, 0.3, 0.6, 0.7)>
3-4	<(0.2, 0.3, 0.5, 0.6), (0.2, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.8)>
3-5	<(0.3, 0.6, 0.7, 0.8), (0.1, 0.2, 0.3, 0.4), (0.1, 0.4, 0.5, 0.6)>
4-6	<(0.4, 0.6, 0.8, 0.9), (0.2, 0.4, 0.5, 0.6), (0.1, 0.3, 0.4, 0.5)>
5-6	<(0.2, 0.3, 0.4, 0.5), (0.3, 0.4, 0.5, 0.6), (0.1, 0,3, 0.5, 0.6)>

Using the algorithm proposed in section 2, we can determine the shortest path between any two nodes. Let node 1 is the source node and node 6 is the destination node.

Suppose $\tilde{d}_1$ = <(0, 0, 0, 0), (1, 1, 1, 1), (1, 1, 1, 1)> and label the source node 1 as [<(0, 0, 0, 0), (1, 1, 1, 1), (1, 1, 1, 1)> ,-], the value $\tilde{d}_2$, $\tilde{d}_3$, $\tilde{d}_4$, $\tilde{d}_5$ and $\tilde{d}_5$ can be computed following the iterations described below:

Iteration1: The node 2 has on predecessor, which is node 2. Following the step 2 in the proposed algorithm, we put i=1and j=2, hence the value of $\tilde{d}_2$ can be computed as follows:

$\tilde{d}_2 = \min\{\tilde{d}_1 \oplus \tilde{d}_{12}\}$ = min{<(0, 0, 0), (1, 1, 1), (1, 1, 1)> ⊕ <(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>= <(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>

So, the minimum provided correspond to the node 1.Hence, the node 2 is labeled as

[<(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>, 1]

$\tilde{d}_2$ = <(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>

Iteration 2: The node 3 has two predecessors, which are node 1 and node 2. Following the step 2 in the proposed algorithm, we put i=1, 2and j=3, hence the value of $\tilde{d}_3$ can be computed as follows:

$\tilde{d}_3 = \min\{\tilde{d}_1 \oplus \tilde{d}_{13}, \tilde{d}_2 \oplus \tilde{d}_{23}\}$= min {<((0, 0, 0, 0), (1, 1, 1, 1), (1, 1, 1, 1)> ⊕ <(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)> , <(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>⊕ <(0.3, 0.4, 0.6, 0.7), (0.1, 0.2, 0.3, 0.5), (0.3, 0.5, 0.7, 0.9)>} = min{<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)> , <(0.37, 0.52, 0.72, 0.85), (0.02, 0.06, 0.15, 0.3), (0.12, 0.25, 0.42, 0.72)>}

S (<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)>) using Eq.19, we have

$$s(\tilde{A}_1) = \frac{1}{12}\begin{bmatrix} 8 + (t_1 + t_2 + t_3 + t_4) - (i_1 + i_2 + i_3 + i_4) \\ -(f_1 + f_2 + f_3 + f_4) \end{bmatrix} = 0.54$$

S (<(0.37, 0.52, 0.72, 0.85), (0.02, 0.06, 0.15,0.3), (0.12, 0.25, 0.42, 0.72)>) = 0.70

Since S (<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)>) < S (<(0.37, 0.52, 0.72, 0.85), (0.02, 0.06, 0.15,0.3), (0.12, 0.25, 0.42, 0.72)>)

So, min{<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)> , <(0.37, 0.52, 0.72, 0.85), (0.02, 0.06, 0.15, 0.3), (0.12, 0.25, 0.42, 0.72)>}

$\tilde{d}_3$ = <(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)>

So, the minimum provided correspond to the node 1.Hence, the node 3 is labeled as [(<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)>) , 1]

$\tilde{d}_3$= <(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)>

Iteration 3: The node 4 has one predecessor, which is node 3. Following the step 2 in the proposed algorithm, we put i=1and j=4, hence the value of $\tilde{d}_4$ can be computed as follows:

$\tilde{d}_4 = \min\{\tilde{d}_3 \oplus \tilde{d}_{34}\}$= min {<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3,0.4)> ⊕ <(0.2, 0.3, 0.5, 0.6), (0.2, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.8)>} = <(0.36, 0.58, 0.75, 0.88), (0.06, 0.25, 0.36, 0.63), (0.04, 0.1, 0.18, 0.32)>

So min{<(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3,0.4)> ⊕ <(0.2, 0.3, 0.5, 0.6), (0.2, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.8)>}= <(0.36, 0.58, 0.75, 0.88), (0.06, 0.25, 0.36, 0.63), (0.04, 0.1, 0.18, 0.32)>

So, the minimum provided correspond to the node 3.Hence, the node 4 is labeled as [<(0.36, 0.58, 0.75, 0.88), (0.06, 0.25, 0.36, 0.63), (0.04, 0.1, 0.18, 0.32)> ,3]

$\tilde{d}_4$ =<(0.36, 0.58, 0.75, 0.88), (0.06, 0.25, 0.36, 0.63), (0.04, 0.1, 0.18, 0.32)>

Iteration 4: The node 5 has two predecessors, which are node 2 and node 3. Following the step 2 in the proposed algorithm, we put i=2, 3 and j=5, hence the value of $\tilde{d}_5$ can be computed as follows:

$\tilde{d}_5 = \min\{\tilde{d}_2 \oplus \tilde{d}_{25}, \tilde{d}_3 \oplus \tilde{d}_{35}\}$ = min{<(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>⊕ <(0.1, 0.3, 0.4, 0.5), (0.3, 0.4, 0.5, 0.7), (0.2, 0.3, 0.6, 0.7)>, <(0.2, 0.4, 0.5, 0.7), (0.3, 0.5, 0.6, 0.9), (0.1, 0.2, 0.3, 0.4)> ⊕ <(0.3, 0.6, 0.7, 0.8), (0.1, 0.2, 0.3, 0.4), (0.1, 0.4, 0.5, 0.6)>} =

min{<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)> , <(0.44, 0.76, 0.85, 0.94), (0.03, 0.1, 0.18, 0.36), (0.01, 0.08, 0.15, 0.42)>}

S (<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)>) = 0.69

S (<(0.44, 0.76, 0.85, 0.94), (0.03, 0.1, 0.18, 0.36), (0.01, 0.08, 0.15, 0.42)>) = 0.81

Since S (<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)>) S (<(0.44, 0.76, 0.85, 0.94), (0.03, 0.1, 0.18, 0.36), (0.01, 0.08, 0.15, 0.42)>)

min{<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)> , <(0.44, 0.76, 0.85, 0.94), (0.03, 0.1, 0.18, 0.36), (0.01, 0.08, 0.15, 0.42)>}

= <(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)>

So, the minimum provided correspond to the node 2.Hence, the node 5 is labeled as [<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)>, 2]

$\tilde{d}_5$ = <(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25,0.42), (0.02, 0.06, 0.18, 0.56)>

Iteration 5: The node 6 has two predecessors, which are node 4 and node 5. Following the step 2 in the proposed algorithm, we put i=4, 5 and j=6, hence the value of $\tilde{d}_6$ can be computed as follows:

$\tilde{d}_6$ = min{ $\tilde{d}_4 \oplus \tilde{d}_{46}, \tilde{d}_5 \oplus \tilde{d}_{56}$ }= min{<(0.36, 0.58, 0.75,0.88), (0.06, 0.25, 0.36, 0.63), (0.04, 0.1, 0.18, 0.32)>⊕ <(0.4, 0.6, 0.8, 0.9), (0.2, 0.4, 0.5, 0.6), (0.1, 0.3, 0.4, 0.5)>, <(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25, 0.42), (0.02, 0.06, 0.18, 0.56)> ⊕ <(0.2, 0.3, 0.4, 0.5), (0.3, 0.4, 0.5, 0.6), (0.1, 0.5,0.3, 0.6)>} = min{<(0.616, 0.832, 0.95, 0.98), (0.012, 0.1, 0.18, 0.37), (0.004, 0.03, 0.072, 0.16)>, <(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)> }

S (<(0.616, 0.832, 0.95, 0.98), (0.012, 0.1, 0.18, 0.37), (0.004, 0.03, 0.072, 0.16)>) = 0.87

S (<(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>) = 0.81

Since S (<(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>) < S (<(0.616, 0.832, 0.95, 0.98), (0.012, 0.1, 0.18, 0.37), (0.004, 0.03, 0.072, 0.16)>)

min{<(0.616, 0.832, 0.95, 0.98), (0.012, 0.1, 0.18, 0.37), (0.004, 0.03, 0.072, 0.16)>, <(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)> }= <(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>

$\tilde{d}_6$ = <(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>

So, the minimum provided correspond to the node 5.Hence, the node 6 is labeled as [<(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>, 5]

Since the destination node of the proposed network is the node 6. Hence, the TrFN- shortest distance between source node 1 and destination node is <(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>

So, the TrFN-shortest path between the source node 1 and the destination node 6 can be determined using the following method:

The node 6 takes the label [<(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>, 5], which indicate that we are moving from node 5. The node 5 takes the label [<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25,0.42), (0.02, 0.06, 0.18, 0.56)>, 2] , which indicate that we are moving from node 2. The node 2 takes the label [<(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>, 1] which indicate that we are moving from node 1.So, joining all the provided nodes, we get the TrFN-shortest path between the source node 1 and the destination node 6. Hence the TrFN-shortest path is given as follows: $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

Following the algorithm described in section 2, the computational results for finding the TrFN-shortest path from source node 1 to destination node 6 are summarized in table 2.

TABLE II. SUMMARIZE OF TRAPEZOIDAL FUZZY NEUTROSOPHIC DISTANCE AND SHORTEST PATH

Node	$\tilde{d}_i$	shortest path between the i-th and 1st node
2	<(0.1, 0.2, 0.3, 0.5), (0.2, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.8)>	$1 \rightarrow 2$
3	<(0.2, 0.4, 0.5), (0.3, 0.5, 0.6), (0.1, 0.2, 0.3)>	$1 \rightarrow 3$
4	<(0.36, 0.58, 0.75, 0.88), (0.06, 0.25, 0.36, 0.63), (0.04, 0.1, 0.18, 0.32)>	$1 \rightarrow 3 \rightarrow 4$
5	<(0.19, 0.44, 0.58, 0.75), (0.06, 0.12, 0.25,0.42), (0.02, 0.06, 0.18, 0.56)>	$1 \rightarrow 2 \rightarrow 5$
6	<(0.352, 0.608, 0.748, 0.88), (0.018, 0.048, 0.125, 0.25), (0.002, 0.03, 0.054, 0.34)>	$1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

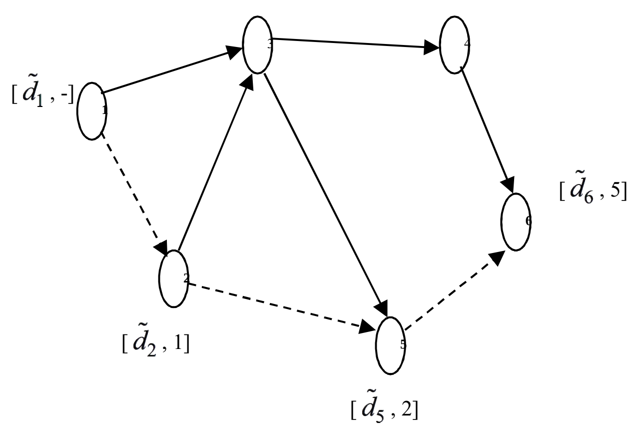

Fig. 2. TrFN-shortest path from source node 1 to destination node 6

V. CONCLUSION

In this research paper, a new algorithm based on trapezoidal fuzzy neutrosophic numbers is presented for finding the shortest path problem in a network where the edges weight are represented by TrFNN. A numerical example is introduced to show the efficacy of the proposed algorithm. So in the next work, we plan to implement this approach practically.

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A Critical Path Problem Using Triangular Neutrosophic Number

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***Abstract*— The Critical Path Method (CPM) is one of several related techniques for planning and managing of complicated projects in real world applications. In many situations, the data obtained for decision makers are only approximate, which gives rise of neutrosophic critical path problem. In this paper, the proposed method has been made to find the critical path in network diagram, whose activity time uncertain. The vague parameters in the network are represented by triangular neutrosophic numbers, instead of crisp numbers. At the end of paper, two illustrative examples are provided to validate the proposed approach.**

***Keywords*— Neutrosophic Sets, Project Management, CPM, Score and Accuracy Functions**.

I. INTRODUCTION

Project management is concerned with selecting, planning, execution and control of projects in order to meet or exceed stakeholders' need or expectation from project. Two techniques of project management, namely Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) where developed in 1950s. [1] The successful implementation of CPM requires clear determined time duration for each activity.

Steps involved in CPM include [2]:

1. Develop Work Breakdown Structure of a project, estimate the resources needed and establish precedence relationship among activities.
2. Translate the activities into network.
3. Carry out network computation and prepare schedule of the activities.

In CPM the main problem is wrongly calculated activity durations, of large projects that have many activities. The planned value of activity duration time may change under certain circumstances and may not be presented in a precise manner due to the error of the measuring technique or instruments etc. It has been obvious that neutrosophic set theory is more appropriate to model uncertainty that is associated with parameters such as activity duration time and resource availability in CPM.

This paper is organized as follows:

In section 2, the basic concepts neutrosophic sets are briefly reviewed. In section 3, the mathematical model of neutrosophic CPM and the proposed algorithm is presented. In section 4, two numerical examples are illustrated. Finally section 5 concludes the paper with future work.

II. PRELIMINARIES

In this section, the basic definitions involving neutrosophic set, single valued neutrosophic sets, triangular neutrosophic numbers and operations on triangular neutrosophic numbers are outlined.

Definition 1. [3,5-7] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$ and $F_A(x): X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2. [3,7] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A=\{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$ and $F_A(x): X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is denoted by $A = (a, b, c)$, where $a, b, c \in [0, 1]$ and $a+b+c \leq 3$.

Definition 3. [4,5] Let $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \epsilon [0,1]$ and $a_1, a_2, a_3 \epsilon \boldsymbol{R}$ such that $a_1 \leq a_2 \leq a_3$. Then a single valued triangular neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set $\boldsymbol{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are given as follows:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\frac{x-a_1}{a_2-a_1}\right) & if\ a_1 \le x \le a_2 \\ \alpha_{\tilde{a}} & if\ x = a_2 \\ \alpha_{\tilde{a}}\left(\frac{a_3-x}{a_3-a_2}\right) & if\ a_2 < x \le a_3 \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{\left(a_2 - x + \theta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1 \le x \le a_2 \\ \theta_{\tilde{a}} & if\ x = a_2 \\ \frac{(x - a_2 + \theta_{\tilde{a}}(a_3-x))}{(a_3-a_2)} & if\ a_2 < x \le a_3 \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \frac{\left(a_2 - x + \beta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1 \le x \le a_2 \\ \beta_{\tilde{a}} & if\ x = a_2 \\ \frac{(x - a_2 + \beta_{\tilde{a}}(a_3-x))}{(a_3-a_2)} & if\ a_2 < x \le a_3 \\ 1 & otherwise \end{cases} \quad (3)$$

Where $\alpha_{\tilde{a}}$,$\theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued triangular neutrosophic number $\tilde{a}$=$\langle(a_1, a_2, a_3); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ may express an ill-defined quantity about a,which is approximately equal to a.

Definition 4. [4] Let $\tilde{a}$=$\langle(a_1, a_2, a_3); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ and $\tilde{b}$ =$\langle(b_1, b_2, b_3); \alpha_{\tilde{b}}, \theta_{\tilde{b}}, \beta_{\tilde{b}}\rangle$ be two single valued triangular neutrosophic numbers and $\gamma \ne 0$ be any real number. Then,

$$\tilde{a}+\tilde{b} = \langle(a_1+b_1, a_2+b_2, a_3+b_3);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle$$

$$\tilde{a}-\tilde{b} = \langle(a_1-b_3, a_2-b_2, a_3-b_1);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle$$

$$\tilde{a}\tilde{b} = \begin{cases} \langle(a_1b_1, a_2b_2, a_3b_3);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle \\ \quad if\ (a_3 > 0, b_3 > 0) \\ \langle(a_1b_3, a_2b_2, a_3b_1);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle \\ \quad if\ (a_3 < 0, b_3 > 0) \\ \langle(a_3b_3, a_2b_2, a_1b_1);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle \\ \quad if\ (a_3 < 0, b_3 < 0) \end{cases}$$

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \langle\left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & if (a_3 > 0, b_3 > 0) \\ \langle\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & if (a_3 < 0, b_3 > 0) \\ \langle\left(\frac{a_3}{b_1}, \frac{a_2}{b_2}, \frac{a_1}{b_3}\right);\ \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & if (a_3 < 0, b_3 < 0) \end{cases}$$

$$\gamma\tilde{a} = \begin{cases} \langle(\gamma a_1, \gamma a_2, \gamma a_3);\ \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle\ if (\gamma > 0) \\ \langle(\gamma a_3, \gamma a_2, \gamma a_1);\ \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle\ if\ (\gamma < 0) \end{cases}$$

$$\tilde{a}^{-1} = \langle\left(\frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle, \quad \text{Where } (\tilde{a} \ne 0)$$

III. CRITICAL PATH METHOD IN NEUTROSOPHIC ENVIRONMENT AND THE PROPOSED ALGORITHM

Project network is a set of activities that must be performed according to precedence constraints determining which activities must start after the completion of specified other activities. Let us define some terms used in drawing network diagram of CPM:

1. Activity: It is any portion of a project that has a definite beginning and ending and may use some resources such as time, labor, material, equipment, etc.
2. Event or Node: Beginning and ending points of activities denoted by circles are called nodes or events.
3. Critical Path: Is the longest path in the network.

The problems of determining critical activities, events and paths are easy ones in a network with deterministic (crisp) duration of activities and for this reason; in this section we convert the neutrosophic CPM to its equivalent crisp model.

The CPM in neutrosophic environment takes the following form:

A network N= $\langle E, A, \tilde{T}\rangle$, being a project model, is given. E is asset of events (nodes) and A$\subset E \times E$ is a set of activities. $\tilde{T}$ is a triangular neutrosophic number and stand for activity duration.

To obtain crisp model of neutrosophic CPM we should use the following equations:

We defined a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle(a_1, b_1, c_1), \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a_1 + b_1 + c_1] \times \left(2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}\right) \quad (4)$$

And

$$A(\tilde{a}) = \frac{1}{16}[a_1 + b_1 + c_1] \times \left(2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}\right) \quad (5)$$

Is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic CPM model can be represented by a crisp model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of $\tilde{a}$, using equations (1), (2), (3) and (4), (5) respectively.

Then the CPM with crisp activity times becomes:

A network N= $\langle E, A, T\rangle$, being a project model, is given. E is asset of events (nodes) and A$\subset E \times E$ is a set of activities. The set E=$\{1,2, \dots, n\}$ is labeled in such a way that the following

condition holds: $(i, j) \in A$ and $i < j$. The activity times in the network are determined by T_{ij}.
Notations of CPM solution:
T_i^e=Earliest occurrence time of predecessor event i,
T_i^l= Latest occurrence time of predecessor event i,
T_j^e=Earliest occurrence time of successor event j,
T_j^l= Latest occurrence time of successor event j,
T_{ij}^e/Start= Earliest start time of an activity *ij*,
T_{ij}^e/Finisht=Earliest finish time of an activity *ij*,
T_{ij}^l/Start=Latest start time of an T_i^lactivity *ij*,
T_{ij}^l/Finisht= Latest finish time of an activity *ij*,
T_{ij} = Duration time of activity *ij*,
Earliest and Latest occurrence time of an event:
T_j^e=maximum $(T_j^e + T_{ij})$, calculate all T_j^e for jth event, select maximum value.
T_i^l=minimum $(T_j^l - T_{ij})$, calculate all T_i^l for ith event, select minimum value.
T_{ij}^e/Start=T_i^e,
T_{ij}^e/Finisht=$T_i^e + T_{ij}$,
T_{ij}^l/Finisht=T_j^l,
T_{ij}^l/Start=$T_j^l - T_{ij}$,
Critical path is the longest path in the network. At critical path, T_i^e=T_i^l, for all *i*.
Slack or Float is cushion available on event/ activity by which it can be delayed without affecting the project completion time.
Slack for ith event = $T_i^l - T_i^e$, for events on critical path, slack is zero.

From the previous steps we can conclude the proposed algorithm as follows:

1. To deal with uncertain, inconsistent and incomplete information about activity time, we considered activity time of CPM technique as triangular neutrosophic number.
2. Calculate membership functions of each triangular neutrosophic number, using equation 1, 2 and 3.
3. Obtain crisp model of neutrosophic CPM using equation (4) and (5) as we illustrated previously.
4. Draw CPM network diagram.
5. Determine floats and critical path, which is the longest path in network.
6. Determine expected project completion time.

IV. ILLUSTRATIVE EXAMPLES

To explain the proposed approach in a better way, we solved two numerical examples and steps of solution are determined clearly.

A. NUMERICAL EXAMPLE 1

An application deals with the realization of a road connection between two famous cities in Egypt namely Cairo and Zagazig. Linguistics terms such as "approximately between" and "around" can be properly represented by approximate reasoning of neutrosophic set theory. Here triangular neutrosophic numbers are used to describe the duration of each task of project. As a real time application of this model, the following example is considered. The project manager wishes to construct a possible route from Cairo (*s*) to Zagazig (*d*). Given a road map of Egypt on which the times taken between each pair of successive intersection are marked, to determine the critical path from source vertex (s) to the destination vertex (*d*). Activities and their neutrosophic durations are presented in table 1.

TABLE 1. INPUT DATA FOR NEUTROSOPHIC CPM.

Activity	**Neutrosophic Activity Time(days)**	**Immediate predecessors**
A	About 2 days (1,2,3;0.8,0.5,0.3)	-
B	About 3 days (2,3,8;0.6,0.3,0.5)	-
C	About 3 days (1,3,10;0.9,0.7,0.6)	A
D	About 2 days (1,2,6;0.5,0.6,0.4)	B
E	About 5 days (2,5,11;0.8,0.6,0.7)	B
F	About 4 days (1,4,8;0.4,0.6,0.8)	C
G	About 5 days (3,5,20;0.8,0.3,0.2)	C
H	About 6 days (4,6,10;0.8,0.5,0.3)	D
I	About 7 days (5,7,15;0.3,0.5,0.4)	F,E
J	About 5 days (3,5,7;0.8,0.5,0.7)	H,G

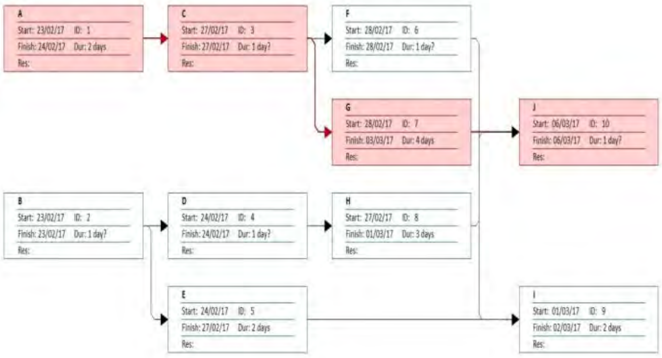
Step 1: Neutrosophic model of project take the following form: N= $\langle E, A, \tilde{T} \rangle$, where E is asset of events (nodes) and A$\subset E \times E$ is a set of activities. $\tilde{T}$ is a triangular neutrosophic number and stand for activity time.
Step 2: Obtaining crisp model of problem by using equations (4) and (5). Activities and their crisp durations are presented in table 2.

TABLE 2. INPUT DATA FOR CRISP CPM.

Activity	**Activity Time(days)**	**Immediate predecessors**
A	2	-
B	1	-
C	1	A
D	1	B
E	2	B
F	1	C
G	4	C
H	3	D
I	2	F,E
J	1	H,G

Step 3: Draw network diagram of CPM.
Network diagram of CPM using Microsoft Project 2010 presented in Fig.1.

Fig. 1. Network of activities with critical path



Step 4: Determine critical path, which is the longest path in the network.
From Fig.1, we find that the critical path is A-C-G-J and is denoted by red line.
Step 5: Calculate project completion time.
The expected project completion time = $t_A + t_C + t_G + t_J$ = 8 days.

B. NUMERICAL EXAMPLE 2

Let us consider neutrosophic CPM and try to obtain crisp model from it. Since you are given the following data for a project.

TABLE 3. INPUT DATA FOR NEUTROSOPHIC CPM.

Activity	Neutrosophic Activity Time(days)	Immediate predecessors
A	$\tilde{2}$	-
B	$\tilde{4}$	A
C	$\tilde{5}$	A
D	$\tilde{8}$	B
E	$\tilde{6}$	C
F	$\widetilde{10}$	D,E

Time in the previous table considered as a triangular neutrosophic numbers.
Let,
$\tilde{2} = \langle(0,2,4); 0.8,0.6,0.4\rangle, \tilde{8} = \langle(4,8,15);\ \ 0.2,0.3,0.5\rangle$,
$\tilde{4} = \langle(1,4,12); 0.2,0.5,0.6\rangle, \tilde{6} = \langle(2,6,18); 0.5,0.4,0.9\rangle$,
$\tilde{5} = \langle(1,5,10); 0.8,0.2,0.4\rangle, \widetilde{10} = \langle(2,10,22); 0.7,0.2,0.5\rangle$.

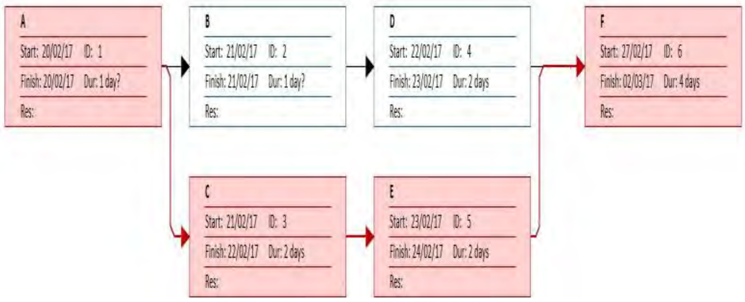
To obtain crisp values of each triangular neutrosophic number, we should calculate score function of each neutrosophic number using equation (4).
The expected time of each activity are presented in table 4.

TABLE4. INPUT DATA FOR CRISP CPM.

Activity	Immediate Predecessors	Activity Time(days)
A	-	1
B	A	1
C	A	2
D	B	2
E	C	2
F	D,E	4

After obtaining crisp values of activity time we can solve the critical path method easily, and determine critical path efficiently.
To draw network of activities with critical path we used Microsoft project program.

Fig. 2. Network of activities with critical path



From Fig.2, we find that the critical path is A-C-E-F and is denoted by red line.
The expected project completion time = $t_A + t_C + t_E + t_F$ = 9 days.

V. CONCLUSION

Neutrosophic set is a generalization of classical set, fuzzy set and intuitionistic fuzzy set because it not only considers the truth-membership and falsity- membership but also an indeterminacy function which is very obvious in real life situations. In this paper, we have considered activity time of CPM as triangular neutrosophic numbers and we used score function to obtain crisp values of activity time. In future, the research will be extended to deal with different project management techniques.

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A Lattice Theoretic Look: A Negated Approach to Adjectival (Intersective, Neutrosophic and Private) Phrases

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***Abstract*—The aim of this paper is to provide a contribution to Natural Logic and Neutrosophic Theory. This paper considers lattice structures built on noun phrases. Firstly, we present some new negations of intersective adjectival phrases and their set-theoretic semantics such as *non-red non-cars* and *red non-cars*. Secondly, a lattice structure is built on positive and negative nouns and their positive and negative intersective adjectival phrases. Thirdly, a richer lattice is obtained from previous one by adding neutrosophic prefixes *neut* and *anti* to intersective adjectival phrases. Finally, the richest lattice is constructed via extending the previous lattice structures by private adjectives (fake, counterfeit). We call these lattice classes *Neutrosophic Linguistic Lattices* (*NLL*).**

Keywords: Logic of natural languages; neutrosophy; pre-orders, orders and lattices; adjectives; noun phrases; negation

I. Introduction

One of the basic subfields of the foundations of mathematics and mathematical logic, lattice theory, is a powerful tool of many areas such as linguistics, chemistry, physics, and information science. Especially, with a set theoretical view, lattice applications of mathematical models in linguistics are a common occurrence.

Fundamentally, Natural Logic [1], [2] is a human reasoning discipline that explores inference patterns and logics in natural language. Those patterns and logics are constructed on relations between syntax and semantics of sentences and phrases. In order to explore and identify the entailment relations among sentences by mathematical structures, it is first necessary to determine the relations between words and clauses themselves. We would like to find new connections between natural logic and neutrosophic by discovering the phrases and neutrosophic clauses. In this sense, we will associate phrases and negated phrases to neutrosophic concepts.

Recently, a theory called Neutrosophy, introduced by Smarandache [4], [6], [5] has widespread mathematics, philosophy and applied sciences coverage. Mathematically, it offers a system which is an extension of intuitionistic fuzzy system. Neutrosophy considers an entity, "A" in relation to its opposite, "$anti - A$" and that which is not A, "$non - A$", and that which is neither "A" nor "$anti - A$", denoted by "$neut - A$".

Up to section 3.3, we will obtain various negated versions of phrases (intersective adjectival) because Neutrosophy considers opposite property of concepts and we would like to associate the phrases and Neutrosophic phrases. We will present the first *NLL* in section 3.3. Notice that all models and interpretations of phrases will be finite throughout the paper.

II. Negating Intersective Adjectival Phrases

Phrases such as "red cars" can be interpreted the intersection of the set of *red things* with the set of *cars* and get the set of "red cars". In the sense of model-theoretic semantics, the interpretation of a phrase such as *red cars* would be the intersection of the interpretation of *cars* with a set of *red individuals* (the region b in Figure 1). Such adjectives are called intersective adjectives or intersecting adjectives. As to negational interpretation, Keenan and Faltz told that "similarly, intersective adjectives, like common nouns, are negatable by non-: non-Albanian (cf. non-student) "in their book [7]. In this sense, *non-red cars* would interpret the intersection of the of *non-red things* and the set of *cars*. Negating intersective adjectives without nouns (red things) would be complements of the set of *red things*, in other words, *non-red things*. We mean by *non-red things* are which the things are which are *not red*. Remark that *non-red things* does not guarantee that *those individuals* have to have a colour property or something else. It is changeable under incorporating situations but we will might say something about it in another paper. On the other hand, negating nouns (cars) would be complements of the set of *cars*, in other words, *non-cars*. We mean by non-cars that the things are which are not cars. Adhering to the spirit of intersective adjectivity, we can explore new meanings and their interpretations from negated intersective adjectival phrases by intersecting negated (or not) adjectives with negated (or not) nouns. As was in the book, *non-red cars* is the intersection the set of things that are not red with cars. In other words,

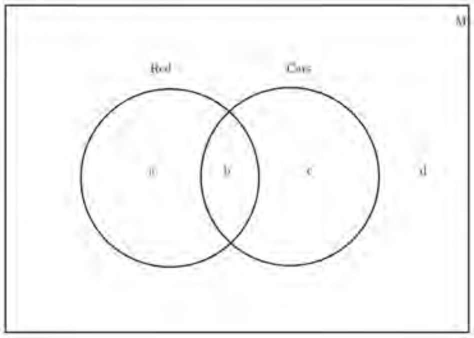

Fig. 1: An example of $cars$ and red in a discourse universe

non-red cars are the cars but not red (the region c in Figure 1). Another candidate for the negated case, *non-red non-cars* refers to intersect the set of non-red things (things that are not red) with non-cars (the region d in Figure 1). The last one, *red non-cars* has meaning that is the set of intersection of the set of red things and the set of non-cars (the region a in Figure 1). $red\bar{x}$ is called *noun level partially semantic complement*. $\overline{red}x$ is called *adjective level partially semantic complement*. $\overline{red}\,\bar{x}$ is called *full phrasal semantic complement*. In summary, we obtain *non-red cars*, *red non-cars* and *non-red non-cars* from *red cars* we already had.

The intersective theory and conjunctives suits well into boolean semantics [7], [8] which proposes very close relationship between *and* and *or* in natural language, as conjunction and disjunction in propositional and predicate logics that have been applied to natural language semantics. In these logics, the relationship between conjunction and disjunction corresponds to the relationship between the set-theoretic notions of intersection and union [9], [10]. On the other hand, correlative conjunctions might help to interpret negated intersective adjectival phrases within boolean semantics because the conjunctions are paired conjunctions (neither/nor, either/or, both/and,) that link words, phrases, and clauses. We might reassessment those negated intersective adjectival phrases in perspective of correlative conjunctions. "*neither A nor B* " and "*both non-A and non-B* can be used interchangeably where *A* is an intersective adjective and *B* is a noun. Therefore, we say "*neither red (things) nor pencils* "and "*both non-red (things) and non-pencils* "equivalent sentences. An evidence for the interchangeability comes from equivalent statements in propositional logic, that is, $\neg(R \vee C)$ is logically equivalent to $\neg R \wedge \neg C$ [11]. Other negated statements would be $\neg R \wedge C$ and $R \wedge \neg C$. Semantically, $\neg R \wedge \neg C$ is *full phrasal semantic complement* of $R \vee C$, and also $\neg R \wedge C$ and $R \wedge \neg C$ are partially semantic complements of $R \vee C$.

We will explore full and partially semantic complements of several adjectival phrases. We will generally negate the phrases and nouns by adding prefix "*non*", "*anti*" and "*neut*". We will use interpretation function [[ ]] from set of phrases (Ph) to power set of universe ($\mathcal{P}(M)$) (set of individuals) to express phrases with understanding of a set-theoretic viewpoint. Hence, $[[p]] \subseteq M$ for every $p \in Ph$. For an adjective a (negated or not) and a plural noun n (negated or not) , $a\ n$ will be interpreted as $[[a]] \cap [[n]]$. If n is a positive plural noun, $non - n$ will be interpreted as $[[non - n]] = [[\bar{n}]] = M \setminus [[n]]$. Similarly, if a is a positive adjective, $non - a$ will be interpreted as $[[non - a]] = [[\bar{a}]] = M \setminus [[a]]$. While we will add *non* to both nouns and adjectives as prefix, "*anti*" and "*neut*" will be added in front of only adjectives. Some adjectives themselves have negational meaning such as *fake*. Semantics of phrases with *anti*, *neut* and *fake* will be mentioned in next sections.

III. Lattice Theoretic Look

We will give some fundamental definitions before we start to construct lattice structures from those adjectival phrases.

A lattice is an algebraic structure that consists of a partially ordered set in which every two elements have a unique supremum (a least upper bound or join) and a unique infimum (a greatest lower bound or meet) [12]. The most classical example is on sets by interpreting set intersection as meet and union as join. For any set A, the power set of A can be ordered via subset inclusion to obtain a lattice bounded by A and the empty set. We will give two new definitions in subsection 3.2 to start constructing lattice structures.

Remark 3.1: We will use the letter *a* and *red* for intersective adjectives, and the letter *x*, *n* and *cars* for common plural nouns in the name of abbreviation and space saving throughout the paper.

A. Individuals

Each element of $[[ax]]$ and $[[\bar{a}x]]$ is a distinct individual and belongs to $[[x]]$. It is already known that $[[a\,x]] \cap [[\bar{a}\,x]] = \emptyset$ and $[[a\,x]] \overset{\star}{\bigcup} [[\bar{a}\,x]] = [[x]]$. It means that no common elements exist in $[[a\,x]]$ and $[[\bar{a}\,x]]$. Hence, every element of those sets can be considered as individual objects such as Larry, John, Meg,.. etc. Uchida and Cassimatis [13] already gave a lattice structure on power set of all of individuals (a domain or a universe).

B. Lattice $\mathcal{L}_{IA}$

Intersective adjectives (red) provide some properties for nouns (cars). Excluding (complementing) a property from an intersective adjective phrase also provide another property for nouns. In this direction, "*red*" is an property for a noun, "$non - red$" is another property for the noun as well. red and $non - red$ have discrete meaning and sets as can be seen in Figure 1. Naturally, every set of restricted objects with a property (red cars) is a subset of those objects without the properties (cars). $[[red\ x]]$ and $[[\overline{red}\ x]]$ are always subsets of $[[x]]$. Neither $[[red\ x]] \leq_\star [[\overline{red}\ x]]$ nor $[[\overline{red}\ x]] \leq_\star [[red\ x]]$ since $[[red\ x]] \cap [[\overline{red}\ x]]$ by assuming $[[\overline{red}\ x]] \neq \emptyset$ and $[[red\ x]] \neq \emptyset$. Without loss of generality, for negative (complement) of the noun x and the intersective adjective red (positive and negative) are $\bar{x}, red\,\bar{x}$

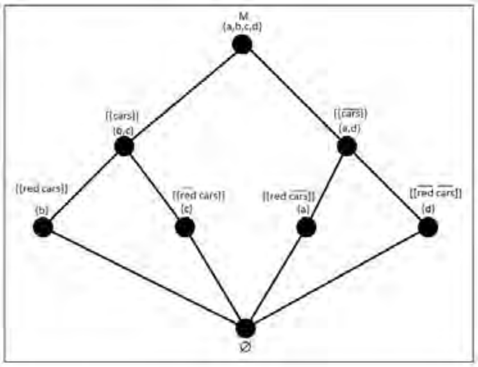

Fig. 2: Lattice on $cars$ and red

and $\overline{red}\,\bar{x}$. $[[red\ \bar{x}]]$ and $[[\overline{red}\ \bar{x}]]$ are always subsets of $[[\bar{x}]]$. Neither $[[red\ \bar{x}]] \leq_\star [[\overline{red}\ \bar{x}]]$ nor $[[\overline{red}\ \bar{x}] \leq_\star [[red\ \bar{x}]]$ since $[[red\ \bar{x}]] \cap [[\overline{red}\ \bar{x}]]$ by assuming $[[\overline{red}\ \bar{x}]] \neq \emptyset$ and $[[red\ \bar{x}]] \neq \emptyset$. On the other hand, $[[x]] \cap [[\bar{x}]] = \emptyset$ and $[[x]]\overset{\star}{\bigcup}[[\bar{x}]] = M$ (M is the universe of discourse) and also $[[red\ x]]$, $[[\overline{red}\ x]]$, $[[red\ \bar{x}]]$ and $[[\overline{red}\ \bar{x}]]$ are by two discrete. We do not allow $[[red\ x]]\overset{\star}{\bigcup}[[\overline{red}\ x]]$ and $[[red\ x]]\overset{\star}{\bigcup}[[\overline{red}\ \bar{x}]]$ and $[[\overline{red}\ x]]\overset{\star}{\bigcup}[[red\ \bar{x}]]$ and $[[\overline{red}\ x]]\overset{\star}{\bigcup}[[\overline{red}\ \bar{x}]]$ to take places in the lattice in Figure 2 because we try to build the lattice from phrases only in our language. To do this, we define a set operation $\overset{\star}{\bigcup}$ and an order relation $\leq_\star$ as the follows:

Definition 3.2: We define a binary set operator $\overset{\star}{\bigcup}$ for our languages as the follow: Let S be a set of sets and $A,\ B \in S$. $A \overset{\star}{\bigcup} B = \ C \ :\Leftrightarrow C$ is the smallest set which includes both A and B, and also $C \in S$.

Definition 3.3: We define a partial order $\leq_\star$ on sets as the follow:

$$A \leq_\star B \text{ if } B = A \overset{\star}{\bigcup} B$$
$$A \leq_\star B \text{ if } A = A \bigcap B$$

Example 3.4: Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{1,2,4\}$, $D = \{1,2,3,4\}$ and $S = \{A, B, C, D\}$.

$A \overset{\star}{\bigcup} A = A$, $A \overset{\star}{\bigcup} C = C$, $A \overset{\star}{\bigcup} B = D$, $B \overset{\star}{\bigcup} C = D$, $C \overset{\star}{\bigcup} D = D$.

$C \leq_\star C,\ A \leq_\star C,\ A \leq_\star D,\ B \leq_\star D,\ C \leq_\star D$

Notice that $\leq_\star$ is a reflexive, transitive relation (pre-order) and $\overset{\star}{\bigcup}$ is a reflexive, symmetric relation.

Figure 3 illustrates a diagram on $cars$ and red. The diagram does not contain sets $\{b,d\}, \{a,b\}, \{a,c\}$ and $\{c,d\}$ because the sets do not represent linguistically any phrases in the language. Because of this reason, $\{a\}\overset{\star}{\bigcup}\{c\}$ and $\{a\}\overset{\star}{\bigcup}\{b\}$ and $\{d\}\overset{\star}{\bigcup}\{c\}$ and $\{d\}\overset{\star}{\bigcup}\{b\}$ are $\{a,b,c,d\} = M$. This structure builds a lattice up by $\overset{\star}{\bigcup}$ and $\bigcap$ that is the classical set intersection operation.

$\mathcal{L_{IA}} = (L,\ \emptyset,\ \bigcap, \overset{\star}{\bigcup})$ is a lattice where $L = \{M,\ x,\ \bar{x},\ red\ x,\ red\ \bar{x},\ \overline{red}\ x,\ \overline{red}\ \bar{x}\}$. Remark that $\mathcal{L_{IA}} = (L,\ \emptyset,\ \bigcap, \overset{\star}{\bigcup}) = (L,\ \emptyset,\ \leq_\star)$. We call this lattice briefly $\mathcal{L_{IA}}$.

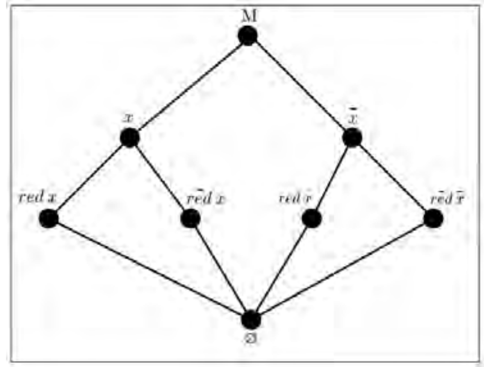

Fig. 3: Hasse Diagram of lattice of $\mathcal{L_{IA}} = (L, \emptyset, \bigcap, \overset{\star}{\bigcup})$

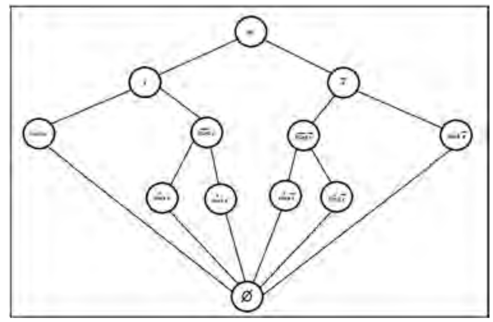

Fig. 4: The Lattice $\mathcal{L}^N_{IA}$

C. Lattice $\mathcal{L}^N_{IA}$

In this section, we present first *NLL*. Let A be the color $white$. Then, $non{-}A = \{black, red, yellow, blue, ...\}$, $anti{-}A$ points at $black$, and $neut - A = \{red, yellow, blue, ...\}$. In our interpretation base, $anti - black\ cars$ ($\overset{a}{black}\ cars$) is a specific set of $cars$ which is a subset of set $non - black\ cars$ ($\overline{black}\ cars$). $neut - black\ cars$ ($\overset{n}{black}\ cars$) is a subset of $\overline{black}\ cars$ which is obtained by excluding sets $black\ cars$ and $\overset{a}{black}\ cars$ from $\overline{black}\ cars$. Similarly, $anti - black\, cars$ ($\overset{a}{black}\ \overline{cars}$) is a specific set of $\overline{cars}$ which is a subset of set $non - black\, non - cars$ ($\overline{black}\ \overline{cars}$). $neut - black\ \overline{cars}$ ($\overset{n}{black}\ \overline{cars}$) is a subset of $\overline{black}\ \overline{cars}$ which is obtained by excluding sets of $black\ \overline{cars}$ and $\overset{a}{black}\ \overline{cars}$ from $\overline{black}\ \overline{cars}$. The new structure represents an extended lattice equipped with $\leq_\star$ as can be seen in Figure 4. We call this lattice $\mathcal{L}^N_{IA}$.

D. Lattice $\mathcal{L}^N_{IA}(F)$

Another *NLL* is an extended version of $\mathcal{L}^N_{IA}$ by private adjectives. Those adjectives have negative effects on nouns such *fake* and *counterfeit*. The adjectives are representative elements of, called *private*, a special class of adjectives [14], [15], [16].

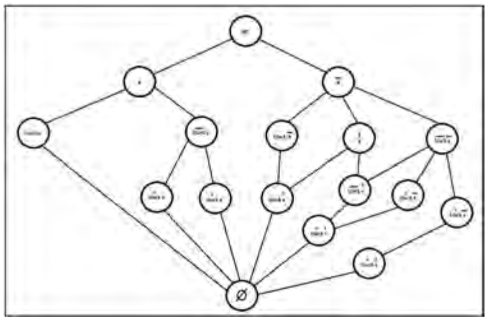

Fig. 5: The lattice $\mathcal{L}_{IA}^{N}(F)$

Chatzikyriakidis and Luo treated transition from the adjectival phrase to noun as $Private\ Adj(N) \Rightarrow \neg N$ in inferential base. Furthermore, they gave an equivalence "$real\_gun(g)$ iff $\neg\ fake\_gun(g)$" where $[|g\ is\ a\ real\ gun|] = real\ \_\ gun(g)$ and $[|f\ is\ not\ a\ real\ gun|] = \neg\ real\_\ gun(f)$ in order to constitute a modern type-theoretical setting. In light of these facts, *fake car is not a car (real)* and plural form: *fake cars are not cars*. Hence, set of fake cars is a subset of set of non-cars in our treatment.

On the one hand, compositions with private adjectives and intersective adjectival phrases do not effect the intersective adjectives negatively but nouns as usual. Then, interpretation of "*fake red cars*" would be intersection of set of *red things* and set of $non - cars$.

Applying "non" to private adjectival phrases, $non - fake\ cars$ are cars (real), $[[non - fake\ cars]] = [[cars]]$ whereas $[[fake\ cars]] \subseteq [[non-cars]]$. $non-fake\ cars$ will be not given a place in the lattice. Remark that phrase "non-fake non-cars" is ambiguous since $fake$ is not a intersective adjective. We will not consider this phrase in our lattice.

$\overset{f}{x}$ is incomparable both $bl\bar{a}ck\ x$ and $bl\bar{a}ck\ \bar{x}$ except $\bar{x}$ as can be seen in Figure 5. So, we can not determine that set of $fake\ cars$ is a subset or superset of a set of any adjectival phrases. But we know that $[[fake\ cars]] \subseteq [[non - cars]]$. Then, we can see easily $[[fake\ black\ cars]]] \subseteq [[blacks\ non-cars]]$ by using $[[fake\ cars]] \bigcap [[black\ things]] \subseteq [[c\bar{a}rs]] \bigcap [[black\ things]]$.

Without loss of generality, set of $fake\ black\ cars$ is a subset of set $black\ non - cars$ and also set of $fake\ non - black\ cars$ is a subset of set $non-black\ non-cars$. Continuing with *neut* and *anti*, set of $fake\ neut\ black\ cars$ is a subset of set of $neut\ black\ non-cars$ and also $fake\ anti\ black\ cars$ is a subset of set of $anti\ black\ non - cars$. Those phrases build the lattice $\mathcal{L}_{IA}^{N}(F)$ in Figure 5.

Notice that when M and empty set are removed from lattices will construct, the structures lose property of lattice. The structures will be hold neither join nor meet semi-lattice property as well. On the other hand, set of $\{\overset{n}{bl\bar{a}ck}\overset{f}{x},\ bl\bar{a}ck\overset{f}{x},\ \overset{n}{bl\bar{a}ck}\bar{x},\ bl\bar{a}ck\bar{x}\}$ equipped with $\leq_{\star}$ is the only one sub-lattice of $\mathcal{L}_{IA}^{N}(F)$ without using M and empty set.

IV. Conclusion and Future Work

In this paper, we have proposed some new negated versions of set and model theoretical semantics of intersective adjectival phrases (plural). After we first have obtained the lattice structure $\mathcal{L}_{IA}$, two lattices $\mathcal{L}_{IA}^{N}$ and $\mathcal{L}_{IA}^{N}(F)$ have been built from the proposed phrases by adding 'neut', 'anti' and 'fake' step by step.

It might be interesting that lattices in this paper can be extended with incorporating coordinates such as *light red cars* and *red cars*. One might work on algebraic properties as filters and ideals of the lattices considering the languages. Some decidable logics might be investigated by extending syllogistic logics with the phrases. Another possible work in future, this idea can be extended to complex neutrosophic set, bipolar neutrosophic set, interval neutrosophic set [17], [18], [19], [20].

We hope that linguists, computer scientists and logicians might be interested in results in this paper and the results will help with other results in several areas.

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Generalized Interval Valued Neutrosophic Graphs of First Type

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***Abstract*— In this paper, motivated by the notion of generalized single valued neutrosophic graphs of first type, we defined a new neutrosophic graphs named generalized interval valued neutrosophic graphs of first type (GIVNG1) and presented a matrix representation for it and studied few properties of this new concept. The concept of GIVNG1 is an extension of generalized fuzzy graphs (GFG1) and generalized single valued neutrosophic of first type (GSVNG1).**

Keywords— Interval valued neutrosophic graph; Generalized Interval valued neutrosophic graphs of first type; Matrix representation.

I. Introduction

Smarandache [7] grounded the concept of neutrosophic set theory (NS) from philosophical point of view by incorporating the degree of indeterminacy or neutrality as independent component to deal with problems involving imprecise, indeterminate and inconsistent information. The concept of neutrosophic set theory is a generalization of the theory of fuzzy set [17], intuitionistic fuzzy sets [14, 15], interval-valued fuzzy sets [13] and interval-valued intuitionistic fuzzy sets [16]. In neutrosophic set every element has three membership degrees including a true membership degree T, an indeterminacy membership degree I and a falsity membership degree F independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. Therefore, if their range is restrained within the real standard unit interval [0, 1], Nevertheless, NSs are hard to be apply in practical problems since the values of the functions of truth, indeterminacy and falsity lie in]−0, 1+[.The single valued neutrosophic set was introduced for the first time by Smarandache in his book [7] . Later on, Wang et al.[10] studied some properties related to single valued neutrosophic sets. In fact sometimes the degree of truth-membership, indeterminacy-membership and falsity- membership about a certain statement cannot be defined exactly in the real situations, but expressed by several possible interval values. So the interval valued neutrosophic set (IVNS) was required. For this purpose, Wang et al.[11] introduced the concept of interval valued neutrosophic set (IVNS for short), which is more precise and more flexible than the single valued neutrosophic set. The interval valued neutrosophic sets (IVNS) is a generalization of the concept of single valued neutrosophic set, in which three membership (T, I, F) functions are independent, and their values belong to the unite interval [0 , 1]. Some more literature about neutrosophic sets, interval valued neutrosophic sets and their applications in various fields can be found in [32, 34, 46].

Graphs are the most powerful and handful tool used in representing information involving relationship between objects and concepts. In a crisp graphs two vertices are either related or not related to each other, mathematically, the degree of relationship is either 0 or 1. While in fuzzy graphs, the degree of relationship takes values from [0, 1].The concept fuzzy graphs, intuitionistic fuzzy graphs and their extensions such interval valued fuzzy graphs [2, 3, 12, 20], interval valued intuitionitic fuzzy graphs [41], and so on, have been studied deeply in over hundred papers. All these types of graphs have a common property that each edge must have a membership value less than or equal to the minimum membership of the nodes it connects.

In 2016, Samanta et al [37] proposed a new concept called the generalized fuzzy graphs (GFG) and studied some major properties such as completeness and regularity with proved results. The authors classified the GFG into two type. The first type is called generalized fuzzy graphs of first type (GFG1). The second is called generalized fuzzy graphs of second type 2 (GFG2). Each type of GFG are represented by matrices similar to fuzzy graphs. The authors have claimed that fuzzy graphs defined by several researches are limited to represent for some systems such as social network.

When description of the object or their relations or both is indeterminate and inconsistent, it cannot be handled by fuzzy, intuitionistic fuzzy, interval valued fuzzy and interval valued intuitionstic fuzzy graphs . So, for this purpose, Smaranadache [9] proposed the concept of neutrosophic graphs based on literal indeterminacy (I) to deal with such situations. Many book on neutrosophic graphs based on literal indeterminacy (I) was completed by Smarandache and Vandasamy [45]. Later on, Smarandache [5, 6] gave another definition for neutrosphic graph theory using the neutrosophic truth-values (T, I, F) without and constructed three structures of neutrosophic graphs: neutrosophic edge graphs, neutrosophic vertex graphs and neutrosophic vertex-edge graphs. Later on Smarandache [8] proposed new version of neutrosophic graphs such as neutrosophic offgraph, neutrosophic bipolar/tripola/ multipolar graph. In a short period of time, few authors have focused deeply on the study of neutrosophic vertex-edge graphs and explored diverse types of different neutrosophic graphs.

In 2016, using the concepts of single valued neutrosophic sets, Broumi et al.[27] introduced the concept of single valued neutrosophic graphs, and introduced certain types of single valued neutrosophic graphs (SVNG) such as strong single valued neutrosophic graph, constant single valued neutrosophic graph, complete single valued neutrosophic graph and investigate some of their properties with proofs and examples. Later on, Broumi et al.[28] also introduced neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of vertex in fuzzy graph and intuitionistic fuzzy graph. In addition, Broumi et al.[29] proved a necessary and sufficient condition for a single valued neutrosophic graph to be an isolated single valued neutrosophic graph. The same authors [35] defined the concept of bipolar single neutrosophic graphs as the generalization of bipolar fuzzy graphs, N-graphs, intuitionistic fuzzy graph, single valued neutrosophic graphs and bipolar intuitionistic fuzzy graphs. In addition, the same authors [36] proposed different types of bipolar single valued neutrosophic graphs such as bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and investigate some of their related properties. In [30, 31, 47], the authors initiated the idea of interval valued neutrosophic graphs and the concept of strong interval valued neutrosophic graph, where different operations such as union, join, intersection and complement have been investigated.

Nasir et al. [22, 23] proposed a new type of graph called neutrosophic soft graphs and have established a link between graphs and neutrosophic soft sets. The authors also, defined some basic operations of neutrosophic soft graphs such as union, intersection and complement.

Akram et al.[18] proposed a new type of single valued neutrosophic graphs different that the concepts proposed in [22,27] and presented some fundamental operations on single-valued neutrosophic graphs. Also, the authors presented some interesting properties of single-valued neutrosophic graphs by level graphs.

In [19] Malik and Hassan introduced the concept of single valued neutrosophic trees and studied some of their properties. Also, Hassan et Malik [1] proposed some classes of bipolar single valued neutrosophic graphs and investigated some of their properties.

Dhavaseelan et al. [26] introduced the concept of strong neutrosophic graph and studied some interesting properties of strong neutrosophic graphs. P. K. Singh [24] has discussed adequate analysis of uncertainty and vagueness in medical data set using the properties of three-way fuzzy concept lattice and neutrosophic graph introduced by Broumi et al. [27].

Fathhi et al.[43] computed the dissimilarity between two neutrosophic graphs based on the concept of Haussdorff distance.

Ashraf et al.[40], proposed some novels concepts of edge regular, partially edge regular and full edge regular single valued neutrosophic graphs and investigated some of their properties. Also the authors, introduced the notion of single valued neutrosophic digraphs (SVNDGs) and presented an application of SVNDG in multi-attribute decision making.

Mehra and Singh [39] introduced the concept of single valued neutrosophic signed graphs and examined the properties of this concept with examples. Ulucay et al.[44] introduced the concept of neutrosophic soft expert graph and have established a link between graphs and neutrosophic soft expert sets [21] and studies some basic operations of neutrosophic soft experts graphs such as union, intersection and complement.

Recently, Naz et al. [42] defined basic operations on SVNGs such as direct product, Cartesian product, semi-strong product, strong product, lexicographic product, union, ring sum and join and provided an application of single valued neutrosophic digraph (SVNDG) in travel time.

Similar to the interval valued fuzzy graphs and interval valued intuitionistic fuzzy graphs, which have a common property that each edge must have a membership value less than or equal to the minimum membership of the nodes it connects. Also, the interval valued neutrosophic graphs presented in the literature [30, 31] have a common property, that edge membership value is less than the minimum of it's end vertex values. Whereas the edge indeterminacy-membership value is less than the maximum of it's end vertex values or is greater than the maximum of its's end vertex values. And the edge non-membership value is less than the minimum of it's end vertex values or is greater than the maximum of its's end vertex values.

Broumi et al [38] have discussed the removal of the edge degree restriction of single valued neutrosophic graphs and presented a new class of single valued neutrosophic graph called generalized single valued neutrosophic graph type1, which is a is an extension of generalized fuzzy graph type1 [37]. with the following

Based on generalized single valued neutrosophic graph of type1(GSVNG1) introduced in [38]. The main objective of this paper is to extend the concept of generalized single valued neutrosophic graph of first type to interval valued neutrosophic graphs first type (GIVNG1) to model systems having an indeterminate information and introduced a matrix representation of GIVNG1. This paper has been arranged as the following:

In Section 2, some fundamental concepts about neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic graph and generalized single valued neutrosophic graphs type 1 are presented which will employed in later sections. In Section 3, the concept of generalized interval valued neutrosophic graphs type 1 is given with an illustrative example. In section 4 a representation matrix of generalized interval valued neutrosophic graphs type 1 is introduced.. Conclusion is also given at the end of section 5.

II. Preliminaries

This section contains some basic definitions from [7, 10, 30, 38] about neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic graphs and generalized single valued neutrosophic graphs type 1, which will helpful for rest of the sections.

Definition 2.1 [7]. Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x ∈ X}, where the functions T, I, F: X→]$^-$0,1$^+$[define respectively the truth-membership function, indeterminacy-membership function, and falsity-membership function of the element x ∈ ∈ X to the set A with the condition:

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-$0,1$^+$[.

Since it is difficult to apply NSs to practical problems, Smarandache [7] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [10]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

$$A=\{< x: T_A(x),\ I_A(x),\ F_A(x) >,\ x \in X\} \quad (2)$$

Definition 2.10[30]. By an interval-valued neutrosophic graph of a graph G^* = (V, E) we mean a pair G = (A, B), where A =< [T_{AL}, T_{AU}], [I_{AL}, I_{AU}], [F_{AL}, F_{AU}]> is an interval-valued neutrosophic set on V, and B =<[T_{BL}, T_{BU}], [I_{BL}, I_{BU}], [F_{BL}, F_{BU}]> is an interval-valued neutrosophic relation on E satisfying the following condition:

1.V= {$v_1, v_2, \ldots, v_n$} such that T_{AL}:V→[0, 1], T_{AU}:V→[0, 1], I_{AL}:V→[0, 1], I_{AU}:V→[0, 1], and F_{AL}:V→[0, 1], F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element y ∈ V, respectively, and

$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$ for all v_i ∈ V (i=1, 2, …, n). (3)

2. The functions T_{BL}:V x V →[0, 1], T_{BU}:V x V →[0, 1], I_{BL}:V x V →[0, 1], I_{BU}:V x V →[0, 1] and F_{BL}:V x V →[0,1], F_{BU}:V x V →[0, 1] are such that:

$T_{BL}(v_i, v_j) \leq \min[T_{AL}(v_i, T_{AL}(v_j)]$, $T_{BU}(v_i, v_j) \leq \min[T_{AU}(v_i), T_{AU}(v_j)]$,

$I_{BL}(v_i, v_j) \geq \max[I_{BL}(v_i), I_{BL}(v_j)]$, $I_{BU}(v_i, v_j) \geq \max[I_{BU}(v_i), I_{BU}(v_j)]$ and

$F_{BL}(v_i, v_j) \geq \max[F_{BL}(v_i), F_{BL}(v_j)]$, $F_{BU}(v_i, v_j) \geq \max[F_{BU}(v_i), F_{BU}(v_j)]$ (4)

denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i, v_j) ∈ E respectively, where:

$0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$, for all (v_i, v_j) ∈ E (i, j = 1, 2,…, n). (5)

They called A the interval valued neutrosophic vertex set of V, and B the interval valued neutrosophic edge set of E, respectively; note that B is a symmetric interval valued neutrosophic relation on A.

Example 2.3 [30] Figure 1 is an example for IVNG, G=(A, B) defined on a graph G^* = (V, E) such that V= {v_1, v_2, v_3}, E= {$v_1v_2, v_2v_3, v_3v_4, v_4v_1$}, A is an interval valued neutrosophic set of V

A={ < v_1,([0.3, 0.5], [0.2, 0.3],[0.3, 0.4])>, <v_2, ([0.2, 0.3], [0.2, 0.3],[0.1, 0.4]) >, <v_3, ([0.1, 0.3], [0.2, 0.4],[0.3, 0.5]) >}, and B an interval valued neutrosophic set of E ⊆ V x V

B={ < v_1v_2,([0.1, 0.2], [0.3, 0.4],[0.4, 0.5])>, <v_2v_3, ([0.1, 0.3], [0.4, 0.5],[0.4, 0.5]) >, <v_3v_1, ([0.1, 0.2], [0.3, 0.5],[0.4, 0.6]) >}

Remark 2.4: -The underlying set V is vertex set of usual graph that we use it in neutrosophic graph as vertex.

- G*=(V,E) denoted a usual graph where a neutrosophic graphs obtained from it that truth –membership, indeterminacy –membership and non-membership values are 0 to 1.

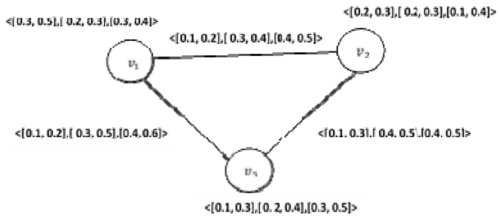

Fig.1. interval valued neutrosophic graph

Definition 2.5 [38]. Let V be a non-void set. Two function are considered as follows:

$\rho=(\rho_T, \rho_I, \rho_F)$:V → $[0,1]^3$and

$\omega=(\omega_T, \omega_I, \omega_F)$:VxV → $[0,1]^3$. We suppose

A= $\{(\rho_T(x), \rho_T(y)) \mid \omega_T(x, y) \geq 0\}$,
B= $\{(\rho_I(x), \rho_I(y)) \mid \omega_I(x, y) \geq 0\}$,
C= $\{(\rho_F(x), \rho_F(y)) \mid \omega_F(x, y) \geq 0\}$,
We have considered ω_T, ω_I and $\omega_F \geq 0$ for all set A,B, C , since its is possible to have edge degree = 0 (for T, or I, or F).
The triad (V, ρ, ω) is defined to be generalized single valued neutrosophic graph of first type (GSVNG1) if there are functions
α:A→ [0, 1] , β:B→ [0, 1] and δ:C→ [0, 1] such that
$\omega_T(x, y) = \alpha((\rho_T(x), \rho_T(y)))$
$\omega_I(x, y) = \beta((\rho_I(x), \rho_I(y)))$
$\omega_F(x, y) = \delta((\rho_F(x), \rho_F(y)))$
Where x, y∈ V.
Here $\rho(x)$=($\rho_T(x)$, $\rho_I(x)$, $\rho_F(x)$), x∈ V are the membership, indeterminacy and non-membership of the vertex x and $\omega(x, y)$=($\omega_T(x, y)$, $\omega_I(x, y)$, $\omega_F(x, y)$), x, y∈ V are the membership, indeterminacy and non-membership values of the edge (x, y).

III. Generalized Interval Valued Neutrosophic Graph of First Type

In this section, based on the generalized single valued neutrosophic graphs of first type proposed by Broumi et al.[38], the definition of generalized interval valued neutrosophic graphs first type is defined as follow:

Definition 3.1. Let V be a non-void set. Two function are considered as follows:
ρ=($[\rho_T^L, \rho_T^U]$, $[\rho_I^L, \rho_I^U]$, $[\rho_F^L, \rho_F^U]$):V → $[0, 1]^3$and
ω=($[\omega_T^L, \omega_T^U]$, $[\omega_I^L, \omega_I^U]$, $[\omega_F^L, \omega_F^U]$):VxV → $[0, 1]^3$. We suppose
A= $\{([\rho_T^L(x), \rho_T^U(x)], [\rho_T^L(y), \rho_T^U(y)]) \mid \omega_T^L(x, y) \geq 0$ and $\omega_T^U(x, y) \geq 0\}$,
B= $\{([\rho_I^L(x), \rho_I^U(x)], [\rho_I^L(y), \rho_I^U(y)]) \mid \omega_I^L(x, y) \geq 0$ and $\omega_I^U(x, y) \geq 0\}$,
C= $\{([\rho_F^L(x), \rho_F^U(x)], [\rho_F^L(y), \rho_F^U(y)]) \mid \omega_F^L(x, y) \geq 0$ and $\omega_F^U(x, y) \geq 0\}$,
We have considered $\omega_T^L, \omega_T^U, \omega_I^L, \omega_I^U, \omega_F^L, \omega_F^U \geq 0$ for all set A, B, C , since its is possible to have edge degree = 0 (for T, or I, or F).
The triad (V, ρ, ω) is defined to be generalized interval valued neutrosophic graph of first type (GIVNG1) if there are functions
α:A→ [0, 1] , β:B→ [0, 1] and δ:C→ [0, 1] such that
$\omega_T^L(x, y) = \alpha((\rho_T^L(x), \rho_T^L(y)))$, $\omega_T^U(x, y) = \alpha((\rho_T^U(x), \rho_T^U(y)))$,
$\omega_I^L(x, y) = \beta((\rho_I^L(x), \rho_I^L(y)))$, $\omega_I^U(x, y) = \beta((\rho_I^U(x), \rho_I^U(y)))$,
$\omega_F^L(x, y) = \delta((\rho_F^L(x), \rho_F^L(y)))$, $\omega_F^L(x, y) = \delta((\rho_F^U(x), \rho_F^U(y)))$
Where x, y∈ V.
Here $\rho(x)$=($\rho_T(x)$, $\rho_I(x)$, $\rho_F(x)$), x∈ V are the interval membership, interval indeterminacy and interval non-membership of the vertex x and $\omega(x, y)$=($\omega_T(x, y)$, $\omega_I(x, y)$, $\omega_F(x, y)$), x, y∈ V are the interval membership, interval indeterminacy membership and interval non-membership values of the edge (x, y).

Example 3.2 : Let the vertex set be V={x, y, z, t} and edge set be E={(x, y),(x, z),(x, t),(y, t)}

	x	y	z	t
$[\rho_T^L, \rho_T^U]$	[0.5, 0.6]	[0.9 , 1]	[0.3, 0.4]	[0.8, 0.9]
$[\rho_I^L, \rho_I^U]$	[0.3, 0.4]	[0.2, 0.3]	[0.1, 0.2]	[0.5, 0.6]
$[\rho_F^L, \rho_F^U]$	[0.1, 0.2]	[0.6, 0.7]	[0.8, 0.9]	[0.4, 0.5]

Table 1: interval membership, interval indeterminacy and interval non-membership of the vertex set.

Let us consider functions $\alpha(m, n)$= m ∨ m= $\beta(m, n)$= $\delta(m, n)$
Here, A={([0.5, 0.6], [0.9, 1]), ([0.5, 0.6], [0.3, 0.4]), ([0.5, 0.6], [0.8, 0.9]), ([0.9, 1.0], [0.8, 0.9])}
B = {([0.3, 0.4], [0.2, 0.3]), ([0.3, 0.4], [0.1, 0.2]), ([0.3, 0.4], [0.5, 0.6]), ([0.2, 0.3], [0.5, 0.6])}
C = {([0.1, 0.2], [0.6, 0.7]), ([0.1, 0.2], [0.8, 0.9]), ([0.1, 0.2], [0.4, 0.5]), ([0.6, 0.7], [0.4, 0.5])}.Then

ω	(x, y)	(x, z)	(x, t)	(y, t)
$[\omega_T^L, \omega_T^U]$	[0.9, 1]	[0.5, 0.6]	[0.8,0.9]	[0.9,1]
$[\omega_I^L, \omega_I^U]$	[0.3,0.4]	[0.3,0.4]	[0.5,0.6]	[0.5, 0.6]
$[\omega_F^L, \omega_F^U]$	[0.6, 0.7]	[0.8,0.9]	[0.4,0.5]	[0.6,0.7]

Table 2: membership, indeterminacy and non-membership of the edge set.

The corresponding generalized single valued neutrosophic graph is shown in Fig.2

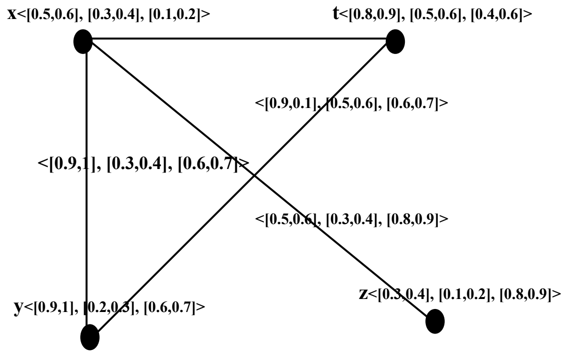

Fig 2.GIVNG of first type.

The easier way to represent any graph is to use the matrix representation. The adjacency matrices, incident matrices are the widely matrices used. In the following section GIVNG1 is represented by adjacency matrix.

IV. Matrix Representation of Generalized Interval Valued Neutrosophic Graph of First Type

Because Interval membership, interval indeterminacy membership and interval non-membership of the vertices are considered independents. In this section, we extended the representation matrix of generalized single valued neutrosophic graphs first type proposed in [38] to the case of generalized interval valued neutrosophic graphs of first type.
The generalized interval valued neutrosophic graph (GIVNG1) has one property that edge membership values (T, I, F) depends on the membership values (T, I, F) of adjacent

vertices . Suppose ξ=(V, ρ, ω) is a GIVNG1 where vertex set V={v_1, v_2,…, v_n}. The functions

α :A→ (0, 1] is taken such that

$\omega_T^L(x, y) = \alpha((\rho_T^L(x), \rho_T^L(y)))$, $\omega_T^U(x, y) = \alpha((\rho_T^U(x), \rho_T^U(y)))$,

Where x, y∈ V and A= {([$\rho_T^L(x)$, $\rho_T^U(x)$], [$\rho_T^L(y)$, $\rho_T^U(y)$]) |ω_T^L(x, y) ≥ 0 and ω_T^U(x, y) ≥ 0 }

,

β :B→ (0, 1] is taken such that

$\omega_I^L(x, y) = \beta((\rho_I^L(x), \rho_I^L(y)))$, $\omega_I^U(x, y) = \beta((\rho_I^U(x), \rho_I^U(y)))$,

Where x, y∈ V and B= {([$\rho_I^L(x)$, $\rho_I^U(x)$], [$\rho_I^L(y)$, $\rho_I^U(y)$]) |ω_I^L(x, y) ≥ 0 and ω_I^U(x, y) ≥ 0 }

and

δ :C→ (0, 1] is taken such that

$\omega_F^L(x, y) = \delta((\rho_F^L(x), \rho_F^L(y)))$, $\omega_F^U(x, y) = \delta((\rho_F^U(x), \rho_F^U(y)))$,

Where x, y∈ V and C= {([$\rho_F^L(x)$, $\rho_F^U(x)$], [$\rho_F^L(y)$, $\rho_F^U(y)$]) |ω_F^L(x, y) ≥ 0 and ω_F^U(x, y) ≥ 0 }. The GIVNG1 can be represented by (n+1) x (n+1) matrix $M_{G_1}^{T,I,F}$=[$a^{T,I,F}$(i, j)] as follows:

The interval membership (T), interval indeterminacy-membership (I) and the interval non-membership (F) values of the vertices are provided in the first row and first column. The (i+1, j+1)- th-entry are the membership (T), indeterminacy-membership (I) and the non-membership (F) values of the edge (x_i, x_j), i, j=1,…,n if i≠j.

The (i, i)-th entry is $\rho(x_i)$=($\rho_T(x_i)$, $\rho_I(x_i)$, $\rho_F(x_i)$), where i=1,2,…,n. The interval membership (T), interval indeterminacy-membership (I) and the interval non-membership (F) values of the edge can be computed easily using the functions α, β and δ which are in (1,1)-position of the matrix. The matrix representation of GIVNG1, denoted by $M_{G_1}^{T,I,F}$, can be written as three matrix representation $M_{G_1}^T$, $M_{G_1}^I$ and $M_{G_1}^F$. For convenience representation $v_i(\rho_T(v_i))$ =[$\rho_T^L(v_i)$, $\rho_T^U(v_i)$], for i=1, …., n

The $M_{G_1}^T$ can be represented as follows

α	$v_1(\rho_T(v_1))$	$v_2(\rho_T(v_2))$	$v_n(\rho_T(v_n))$
$v_1(\rho_T(v_1))$	$[\rho_T^L(v_1), \rho_T^U(v_1)]$	$\alpha(\rho_T(v_1), \rho_T(v_2))$	$\alpha(\rho_T(v_1), \rho_T(v_n))$
$v_2(\rho_T(v_2))$	$\alpha(\rho_T(v_2), \rho_T(v_1))$	$[\rho_T^L(v_2), \rho_T^U(v_2)]$	$\alpha(\rho_T(v_2), \rho_T(v_2))$
…	….	…	…
$v_n(\rho_T(v_n))$	$\alpha(\rho_T(v_n), \rho_T(v_1))$	$\alpha(\rho_T(v_n), \rho_T(v_2))$	$[\rho_T^L(v_n), \rho_T^U(v_n)]$

Table3. Matrix representation of T-GIVNG1

The $M_{G_1}^I$ can be represented as follows

β	$v_1(\rho_I(v_1))$	$v_2(\rho_I(v_2))$	$v_n(\rho_I(v_n))$
$v_1(\rho_I(v_1))$	$[\rho_I^L(v_1), \rho_I^U(v_1)]$	$\beta(\rho_I(v_1), \rho_I(v_2))$	$\beta(\rho_I(v_1), \rho_I(v_n))$
$v_2(\rho_I(v_2))$	$\beta(\rho_I(v_2), \rho_I(v_1))$	$[\rho_I^L(v_2), \rho_I^U(v_2)]$	$\beta(\rho_I(v_2), \rho_I(v_2))$
…	….	…	…
$v_n(\rho_I(v_n))$	$\beta(\rho_I(v_n), \rho_I(v_1))$	$\beta(\rho_T(v_n), \rho_I(v_2))$	$[\rho_I^L(v_n), \rho_I^U(v_n)]$

Table4. Matrix representation of I-GIVNG1

The $M_{G_1}^I$ can be represented as follows

δ	$v_1(\rho_F(v_1))$	$v_2(\rho_F(v_2))$	$v_n(\rho_F(v_n))$
$v_1(\rho_F(v_1))$	$[\rho_F^L(v_1), \rho_F^U(v_1)]$	$\delta(\rho_F(v_1), \rho_F(v_2))$	$\delta(\rho_F(v_1), \rho_F(v_n))$
$v_2(\rho_F(v_2))$	$\delta(\rho_F(v_2), \rho_F(v_1))$	$[\rho_F^L(v_2), \rho_F^U(v_2)]$	$\delta(\rho_F(v_2), \rho_F(v_2))$
…	….	…	…
$v_n(\rho_F(v_n))$	$\delta(\rho_F(v_n), \rho_F(v_1))$	$\delta(\rho_F(v_n), \rho_F(v_2))$	$[\rho_F^L(v_n), \rho_F^U(v_n)]$

Table5. Matrix representation of F-GIVNG1

Remark1 : if $\rho_I^L(x)$= $\rho_I^U(x)$=0 and $\rho_F^L(x)$= $\rho_F^U(x) = 0$ the generalized interval valued neutrosophic graphs type 1 is reduced to generalized fuzzy graphs type 1 (GFG1).

Remark 2: if $\rho_T^L(x)$ = $\rho_T^U(x)$, $\rho_I^L(x)$ = $\rho_I^U(x)$ and $\rho_F^L(x)$= $\rho_F^U(x)$, the generalized interval valued neutrosophic graphs type 1 is reduced to generalized single valued graphs type 1 (GSVNG1).

Here the generalized Interval valued neutrosophic graph of first type (GIVNG1) can be represented by the matrix representation depicted in table 9. The matrix representation can be written as three interval matrices one containing the entries as T, I, F (see table 6, 7 and 8).

α = max(x, y)	x([0.5,0.6])	y([0.9,1])	z(0.3,0.4])	t([0.8,0.9])
x([0.5,0.6])	[0.5,0.6]	[0.9, 1.0]	[0.5, 0.6]	[0.8,0.9]
y([0.9,1])	[0.9, 1.0]	[0.9,1]	[0, 0]	[0.9,1.0]
z([0.3,0.4])	[0.5, 0.6]	[0, 0]	[0.3,0.4]	[0, 0]
t([0.8,0.9])	[0.8, 0.9]	[0.9, 1.0]	[0, 0]	[0.8,0.9]

Table 6:Lower and upper Truth- matrix representation of GIVNG1

β = max(x, y)	x([0.3,0.4])	y([0.2,0.3])	z([0.1,0.2])	t([0.5,0.6])
x([0.3,0.4])	[0.3,0.4]	[0.3,0.4]	[0.3,0.4]	[0.5,0.6]
y([0.2,0.3])	[0.3,0.4]	[0.2,0.3]	[0, 0]	[0.5,0.6]
z([0.1,0.2])	[0.3,0.4]	[0, 0]	[0.1,0.2]	[0, 0]
t([0.5,0.6])	[0.5,0.6]	[0.5,0.6]	[0, 0]	[0.5,0.6]

Table 7: lower and upper Indeterminacy- matrix representation of GIVNG1

δ= max(x, y)	x([0.1,0.2])	y([0.6,0.7])	z([0.8,0.9])	t([0.4,0.6])
x([0.1,0.2])	[0.1,0.2]	[0.6,0.7]	[0.8,0.9]	[0.4,0.6]
y([0.6,0.7])	[0.6,0.7]	[0.6,0.7]	[0, 0]	[0.6,0.7]
z([0.8,0.9])	[0.8,0.9]	[0, 0]	[0.8,0.9]	[0, 0]
t([0.4,0.6])	[0.4,0.6]	[0.6,0.7]	[0, 0]	[0.4,0.6]

Table 8: Lower and upper Falsity- matrix representation of GIVNG1

The matrix representation of GIVNG1 can be represented as follows:

(α,β,δ)	x(0.5,0.3,0.1)	y(0.9,0.2,0.6)	z(0.3,0.1,0.8)	t(0.8,0.5,0.4)
x(**[0.5,0.6], [0.3,0.4], [0.1,0.2]**)	**<[0.5,0.6], [0.3,0.4], [0.1,0.2]>**	**<[0.9,1.0], [0.3,0.4], [0.6,0.7]>**	**<[0.5,0.6], [0.3,0.4], [0.8,0.9]>**	**<[0.8,0.9], [0.5,0.6], [0.4,0.6]>**
y(**[0.9, 1.0], [0.2, 0.3], [0.6, 0.7]**)	**<[0.9,1.0], [0.3,0.4], [0.6,0.7]>**	**<[0.9, 1.0], [0.2, 0.3], [0.6, 0.7]>**	**<[0, 0], [0, 0], [0, 0]>**	**<[0.9, 1.0], [0.5, 0.6], [0.6, 0.7]>**
z(**[0.3,0.4], [0.1,0.2], [0.8,0.9]**)	**<[0.5,0.6], [0.3,0.4], [0.8,0.9]>**	**<[0, 0], [0, 0], [0, 0]>**	**<[0.3,0.4], [0.1,0.2], [0.8,0.9]>**	**<[0, 0], [0, 0], [0, 0]>**
t(**[0.8,0.9], [0.5,0.6], [0.4,0.6]**)	**<[0.8,0.9], [0.5,0.6], [0.4,0.6]>**	**<[0.9,1.0], [0.5,0.6], [0.6,0.7]>**	**<[0, 0], [0, 0], [0, 0]>**	**<[0.8,0.9], [0.5,0.6], [0.4,0.6]>**

Table 9: Matrix representation of GIVNG1.

Theorem 1. Let $M_{G_1}^{T}$ be matrix representation of T-GIVNG1, then the degree of vertex $D_T(x_k)$ =$[\sum_{j=1,j\neq k}^{n} a_T^L(k+1,j+1), \sum_{j=1,j\neq k}^{n} a_T^U(k+1,j+1)]$, $x_k \in$ V or
$D_T(x_p)$ = $[\sum_{i=1,i\neq p}^{n} a_T^L(i+1,p+1)$, $\sum_{i=1,i\neq p}^{n} a_T^U(i+1,p+1)$ $x_p \in$ V.

Proof :is similar as in theorem 1 of [37].

Theorem 2. Let $M_{G_1}^{I}$ be matrix representation of I-GIVNG1, then the degree of vertex $D_I(x_k)$ =$[\sum_{j=1,j\neq k}^{n} a_I^L(k+1,j+1), \sum_{j=1,j\neq k}^{n} a_I^U(k+1,j+1)]$, $x_k \in$ V
or $D_I(x_p)$ =$[\sum_{i=1,i\neq p}^{n} a_I^L(i+1,p+1)$,$\sum_{i=1,i\neq p}^{n} a_I^U(i+1,p+1)$ $x_p \in$ V.
Proof :is similar as in theorem 1 of [37].

Theorem 3. Let $M_{G_1}^{F}$ be matrix representation of F-GIVNG1, then the degree of vertex
$D_F(x_k)$ =[$\sum_{j=1,j\neq k}^{n} a_F^L(k+1,j+1)$, $\sum_{j=1,j\neq k}^{n} a_F^U(k+1,j+1)$], $x_k \in$ V or
$D_F(x_p)$ = $[\sum_{i=1,i\neq p}^{n} a_F^L(i+1,p+1)$, $\sum_{i=1,i\neq p}^{n} a_F^U(i+1,p+1)$ $x_p \in$ V.

Proof :is similar as in theorem 1 of [37].

Theorem4. Let $M_{G_1}^{T,I,F}$ be matrix representation of GIVNG1, then the degree of vertex D(x_k) =($D_T(x_k), D_I(x_k), D_F(x_k)$) where
$D_T(x_k)$ =[$\sum_{j=1,j\neq k}^{n} a_T^L(k+1,j+1)$, $\sum_{j=1,j\neq k}^{n} a_T^U(k+1,j+1)$] $x_k \in$ V.
$D_I(x_k)$ ==[$\sum_{j=1,j\neq k}^{n} a_I^L(k+1,j+1)$, $\sum_{j=1,j\neq k}^{n} a_I^U(k+1,j+1)$] $x_k \in$ V.
$D_F(x_k)$ ==[$\sum_{j=1,j\neq k}^{n} a_F^L(k+1,j+1)$, $\sum_{j=1,j\neq k}^{n} a_F^U(k+1,j+1)$] $x_k \in$ V.
Proof: the proof is obvious.

V. CONCLUSION

In this article, we have extended the concept of generalized single valued neutrosophic graph type 1 (GSVNG1) to generalized interval valued neutrosophic graph type 1 (GIVNG1) and presented a matrix representation of it. In the future works, we plan to study the concept of completeness, the concept of regularity and to define the concept of generalized interval valued neutrosophic graphs type 2.

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Neutrosophic Application for Decision Logic in Robot Intelligent Control Systems

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***Abstract*—The paper presents the research undertaken for extending the functionality of a regular fuzzy multi-input decision support system through the application of basic neutrosophy principles. The application is designed and simulated using tools already available for the fuzzy implementation, adapted to the requirements of the neutrosophic extension. The results show an increase in the decision classification index, while improving the environmental accuracy of the considered model.**

Keywords—robotics, neutrosophic set, intelligent control systems, decision making, computational intelligence

I. Introduction

The paper presents a proposed solution for the implementation of a generalized neutrosophic inference system (NSIS) by extending the already available tools for the implementation of fuzzy logic, amended to the particularities of neutrosophic logic. The end result is a neutrosophic logic controller (NSLC). As will be discussed in the appropriate section, its implementation can function either on the original neutrosophic principles (T-norms and T-conorms), or on a fuzzy-T2 schema, which can also be seen as a subset of neutrosophic logic. That is to say, the uncertainty dimension can be either used in the original formulae, or directly as a determinant of the fuzzy degree of membership.

The history of fuzzy logic starts with the paper published in 1965 by L.A. Zadeh, entitled 'Fuzzy sets', in which the author introduces his new approach to set theory. Essentially, a fuzzy set is an extension of a classical bivalent (crisp) set with 'a membership function which assigns to each object a grade of membership between 0 and 1 [1]. Subsequent detailed investigations made by Mamdani [2] and Takagi and Sugeno [3] have led to Fuzzy Logic becoming an increasingly appealing alternative to classical control for an array of systems [4].

Fuzzy Logic has long been used in academia and in industry and is one of the more palpable staples of artificial intelligence in use in the world today. Fuzzy logic controllers have been proven to be robust, relatively easy to design [5] and, although a unified algorithm for parameter selection and optimization is still sought after [6, 7], they seem to suffer from no one major flaw while providing a number of important benefits (expert knowledge emulation being perhaps chief among them). There are a number of implementations of various algorithms for the optimization of fuzzy inference systems' parameters, such as genetic algorithms and neural networks.

Neutrosophy extends fuzzy logic by adding the dimension of uncertainty to the considered model. This is especially useful in information fusion dealing with multiple sources of sensor data. While still in its beginning, neutrosophy enjoys remarkable interest from world-wide research teams due to a proven record of improving inference system models [8, 9]. With applications in artificial intelligence, business, marketing, planning, control theory and image processing, it is one of the fastest developing new fields of study in the world today. Neutrosophic reasoning components work very well with database expert systems and provide a large boost to the current knowledge in decision support systems [10], making them particularly well suited for robotic applications.

In order to determining the angular error of the actuator drive control loop in the robot joints is known the Vladareanu-Smarandache method for hybrid force-position robot control [11] by applying Neutroscopic logic.

The generalized area where a robot works can be defined in a constraint space with six degrees of freedom (DOF), with position constrains along the normal force of this area and force constrains along the tangents. On the basis of these two constrains there is described the general scheme of hybrid position and force control in figure 1. Variables X_C and F_C represent the Cartesian position and the Cartesian force exerted onto the environment. Considering X_C and F_C expressed in specific frame of coordinates, the selection matrices S_x and S_f can be determined, which are diagonal matrices with 0 and 1 diagonal elements, and which satisfy relation: $S_x + S_f = I_d$, where S_x and S_f are methodically deduced from kinematics constrains imposed by the working environment [16-18].

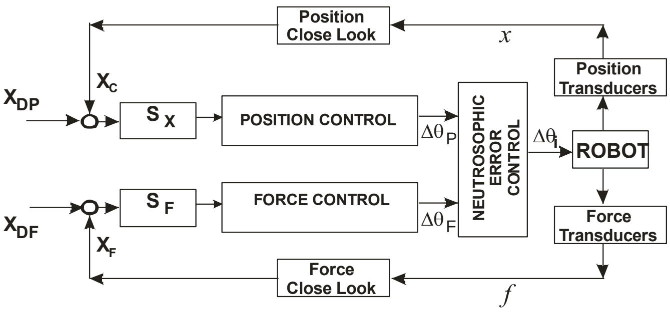

Figure 1. Architecture of the Robot Neutrosophic Control (RNC)

In order to determine the control relations in this situation, ΔX_P – the measured deviation of Cartesian coordinate command system is split in two sets: ΔX^F corresponds to force controlled component and ΔX^P corresponds to position control with axis actuating in accordance with the selected matrixes S_f and S_x. If there is considered only positional control on the directions established by the selection matrix S_x there can be determined the desired end - effector differential motions that correspond to position control in the relation: $\Delta X_P = K_P\,\Delta X^P$, where K_P is the gain matrix, respectively desired motion joint on position controlled axis: $\Delta\theta_P = J^{-1}(\theta)\cdot\Delta X_P$ [19 - 21].

Now taking into consideration the force control on the other directions left, the relation between the desired joint motion of end-effector and the force error ΔX_F is given by the relation: $\Delta\theta_F = J^{-1}(\theta)\cdot\Delta X_F$, where the position error due to force ΔX_F is the motion difference between ΔX^F– current position deviation measured by the control system that generates position deviation for force controlled axis and ΔX_D – position deviation because of desired residual force. Noting the given desired residual force as F_D and the physical rigidity K_W there is obtained the relation: $\Delta X_D = K_W^{-1}\cdot F_D$.

Thus, ΔX_F can be calculated from the relation: $\Delta X_F = K_F(\Delta X^F - \Delta X_D)$, where K_F is the dimensionless ratio of the stiffness matrix. Finally, the motion variation on the robot axis matched to the motion variation of the end-effectors is obtained through the relation: $\Delta\theta = J^{-1}(\theta)\,\Delta X_F + J^{-1}(\theta)\,\Delta X_P$. Starting from this representation the architecture of the hybrid position – force control system was developed with the corresponding coordinate transformations applicable to systems with open architecture and a distributed and decentralized structure.

This paper presents a robot position control application on a single axis, as hybrid force-position control may be developed in future works by generalizing neutrosophic logic to bi-dimensional space. Respectively for hybrid force-position control on ***n*** degrees of freedom (n DOF) by generalizing NSL to 2***n***-dimensional space.

The designed system is tested on a controller application for the position control of a simple direct current motor, representing the position actuator for one joint of a small robot. The inputs to the motor are the voltage demand used to control the system and the value of the load applied to the motor, which will be a negative value in this layout. The outputs are the speed of the motor, which will later be fed into an Integral block in order to obtain position control, and the value of the armature current. This will be sunk into a scope which allows for monitoring the armature current.

The implementation is done using coded functions and classes, but is also shown in a schematic diagram for purposes of visualization. It should be noted that both coded versions take little time to run, while the models will of course take longer, especially for the double fuzzy version. Both versions of the code draw heavily from the existing fuzzy logic implementations in the Matlab / Octave environment. The opportunity for actual practical implementation of both versions is discussed in the conclusions section.

The rest of the paper is divided as follows. Chapter 2 presents the generalized neutrosophic model with a focus on the elements required for the present application. Chapter 3 outlines the controller logic, NSLC implementation and its functionality, while also discussing an experimental application of the controller and details the actual use-case simulation. Chapter 4 discusses the obtained results and attempts to draw conclusion for the present and future research involving neutrosophic models and decision support systems.

II. Neutrosophic Model

The operating algorithm for a FLC consists of three stages: Fuzzification, Inference and Defuzzification. All FLCs used for the simulations have a set input and output range of ±1 which is then adjusted to fit the particulars of the system being controlled. Therefore, the FLC is preceded and succeeded by input and output scaling, respectively. Fuzzification is the process of mapping the inputs to ‘linguistic variables’ (fuzzy sets) and determine the degree of membership for each respective set [12]. This membership value is of course entirely dependent upon the shape and layout of the MFs.

The Fuzzy Inference System (FIS) is the key point of fuzzy logic, which is designed to mimic human reasoning [7]. It connects the fuzzified inputs with the fuzzified outputs through a set of “IF – THEN” rules using the previously defined linguistic variables. A ‘rule base’ is merely a table representation of the rules used within a FIS. A number of rule bases are employed depending on the number of inputs, desired controller performance, controller and process particulars and are the defining characteristic of the respective FLCs they are used in.

The FIS rules use *and* and *or* operators to connect the various inputs (linguistic variables) to a prescribed output fuzzy set (also a linguistic variable).The degree of support for each rule and therefore for its respective output fuzzy set is to be found among the degrees of support for the input linguistic variables that are part of that rule. When the *and* operator is used, the minimum of all degrees of support from the input will be used as the degree of support for the output, while the *or* operator will determine the maximum of those values to be used.

The overlapping of MFs means that more than one rule will be used (‘fired’). This leads to a number of output fuzzy sets, which must then be aggregated into a final output fuzzy set. While there are other methods for aggregation, the ‘max’ method is the general standard and as such it was used in all simulations for this paper. It implies mapping the resulted output sets onto the output range and taking the maximum value in areas where they overlap.

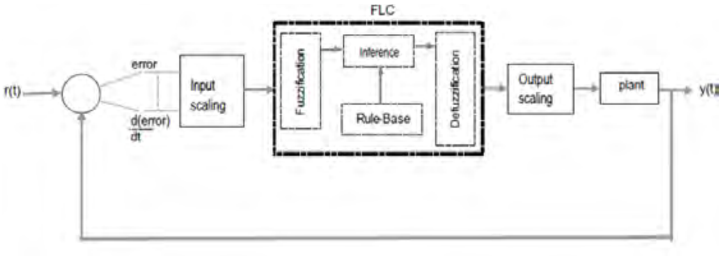

Figure 2. FLC implementation diagram

Fuzzy set theory expands conventional ('crisp') sets with a 'membership value' (between 0 and 1) which expresses the degree to which a certain element belongs to a set. The relation between a value that is part of a set and its membership value is called a Membership Function (MF). The MFs for a given input must cover the entire universe of discourse [12] and they can and do overlap. Fuzzy Logic Controllers (FLCs) use fuzzy logic to mimic the way an experienced operator would go about controlling the process. A diagram of the implementation of a FLC is shown in Figure 2 [12].

By contrast, neutrosophy allows for an increase in the dimensionality of each input parameter, providing more information that can be coherently modelled and input to the inference system [10].

Neutrosophy is meant to be a unifying theory for the design and implementation of decision support systems. As such, it is a generalization of fuzzy logic. A neutrosophic logic statement includes values for truth, falsehood and indeterminacy, where the appropriate memberships are real values. Similarly to fuzzy logic and in contrast to probability theory, there are no restrictions placed on the sum of the resulting components – in probability theory the sum of all possible outcomes must be 1. The truth and falsehood parameters tie neutrosophy to the existing treatment of decision support problems, to which is added support for modelling the indeterminacy, which expresses the percentage of unknown parameters or states [13].

Inside a universe of discourse *U* and with respect to a set *M* included in *U*, and element x is noted as *x(T,I,F)*, with the following properties [14,15]:

- *x* has a value of truth *t* of belonging to the set *M*
- *x* has a value of indeterminacy *i* of belonging to the set *M*
- *x* has a value of falsehood *f* of belonging to the set *M*

The transformation from a standard operating model using fuzzy logic to a neutrosophic model is done using the formulae set forth in the Dezert-Smarandache Theory (DSmT) [14]. Let $\theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be a finite set made up of *n* exhaustive elements (this can be assumed without loss of generality, see [14]). The DSmT hyper-power set D^Θ is then defined as the set of all composite propositions built from those elements using the *reunion* and *intersection* operators. This means that

$$\emptyset, \theta_1, \dots, \theta_n \in D^\theta$$

$$\forall A, B \in D^\theta \rightarrow A \cap B \in D^\theta, A \cup B \in D^\theta$$

By defining an application $m(.): D^\theta \rightarrow [0,1]$ within this set, there results $m(\emptyset) = 0$ and $\sum_{A \in D^\theta} m(A) = 1$, where *m(A)* is the generalized basic belief assignment of *A* [14,15].

In working with multiple sources of information, the DSm rule of combination states:

$$\forall C \in D^\theta, m_{M^T(\theta)}(C) \equiv m(C) = \sum_{\substack{A,B \subset D^\theta \\ A \cap B = C}} m_1(A) \cdot m_2(B)$$

With D^θ closed under the set operators of reunion and intersection, the DSm rule of combination results in *m(.)* being a proper belief mass, with $m(.): D^\theta \rightarrow [0,1]$, being commutative, associative and extendable to an unlimited number of sources [14,15].

Alternatively, the transformation can be done using fuzzy-T2 style operators, whereby the third dimension is actually the fuzzification of the fuzzy degree of membership (i.e. the second dimension). This relates well to applications where the uncertainty specifically models the reliability of a sensor network or input device, which can be statically estimated and traced to each individual output [13].

This can be exemplified in the test application model, where feedback inputs are sampled from a Gaussian distribution with an appropriate variance for each for the inputs. This allows the model to incorporate the uncertainty normally found in such practical applications. It is also completely feasible that such values should be known and traced back individually to their respective field equipment. The exact expression of the third dimension of each tuple need not be specifically the variance, as long as it correctly models the uncertainty in the system. However, this is a possible topic for future research and is outside the scope of the current application.

III. Simulation and Testing / Implementation

The implementation of the proposed generalized model is achieved by using the packages already available for fuzzy implementations in a Matlab / Octave environment, both for visualization and coding.

Each input value maps to an object of a class defined to handle the multiple dimensions as properties, thereby forming and coherently expressing the tuple needed. This representation was also chosen for its ease of expansion, as may be the case with future research – neutrosophic logic systems can be extended to four- or five- dimensional logic sets [13]. The coded functions which create the neutrosophic logic objects are implemented to be a parallel representation of those already existing as part of the fuzzy logic toolbox or package, where possible. The second implementation version, using the double-fuzzy style operators, particularly makes full use of the existing *evalfis, readfis,* etc. functions. As such, the *fuzzy-logic-toolkit* package and *fuzzy logic* toolbox can be considered dependencies for the Octave and, respectively, Matlab environments.

The controller schematic implementation is shown in Figure 4. The chosen test application is the position control of a simple direct current motor representing a robotic joint actuator. The two input feeds are the position error and its derivative, calculated independently. This is possible because the derivative of the position is actually the motor speed and is in fact a more accurate representation of an actual practical application, with available date from both types of transducers. The third dimension of indeterminacy takes into account the standard reliability of position and speed sensors along the feedback path.

With a view to making the simulations as realistic as possible, the effects of disturbance and noise along the circuit, integrator windup and maximum armature current demand are included. The system is subjected to a step input of value *RV* (Reference Value). Also included in the simulation are the Disturbance (*Ds*), Noise (*N*) along the feedback path, Load (*TL*) and the time at which the load occurs (*TLs*) which can be found in the Load Block, and the sample time (*Ts*) in the case of digital systems. These are all declared as global variables and only need to be updated once at the beginning of each session.

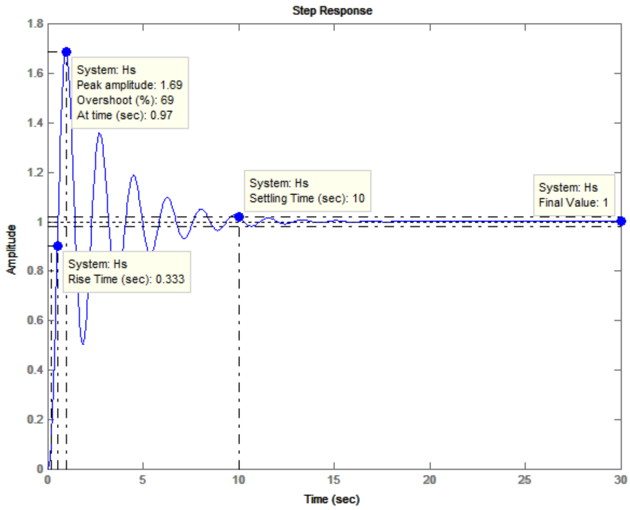

Figure 3. General stable system response

The overall transfer function of a DC Motor is

$$P'(s) = \frac{K}{LJs^2+(RJ+BL)s+RB+K^2}$$

for speed control, which is then divided by s for position control. In the end, the transfer function of the system will be

$$P(s) = \frac{K}{LJs^3+(RJ+BL)s^2+(RB+K^2)s}.$$

All of the parameters are declared within the environment as global variables and are initiated once at the start of the session.

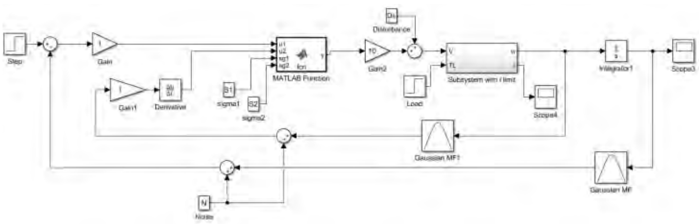

Figure 4. Controller schematics / native norm

After a simulation is run, the transient system response is investigated based on the following performance metrics: overshoot, rise time, settling time and steady state error. Figure 3 shows a random stable system response to a step input for 30 seconds, indicating those characteristics.

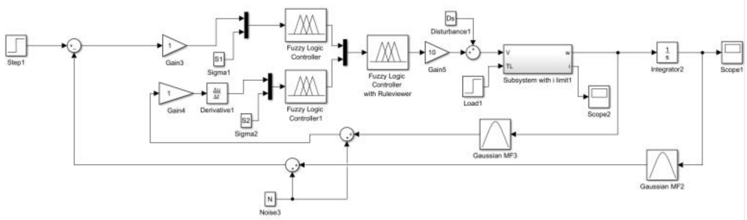

Figure 5. Controller schematics / fuzzy style norm

The actual implementation of the controller schematics is hard coded using the described functions and classes for the neutrosophic logic model. For visualization purposes, Figures 4 and 5 show the controller schematics in Matlab/Simulink for the native norm version and the fuzzy-T2 style version, respectively.

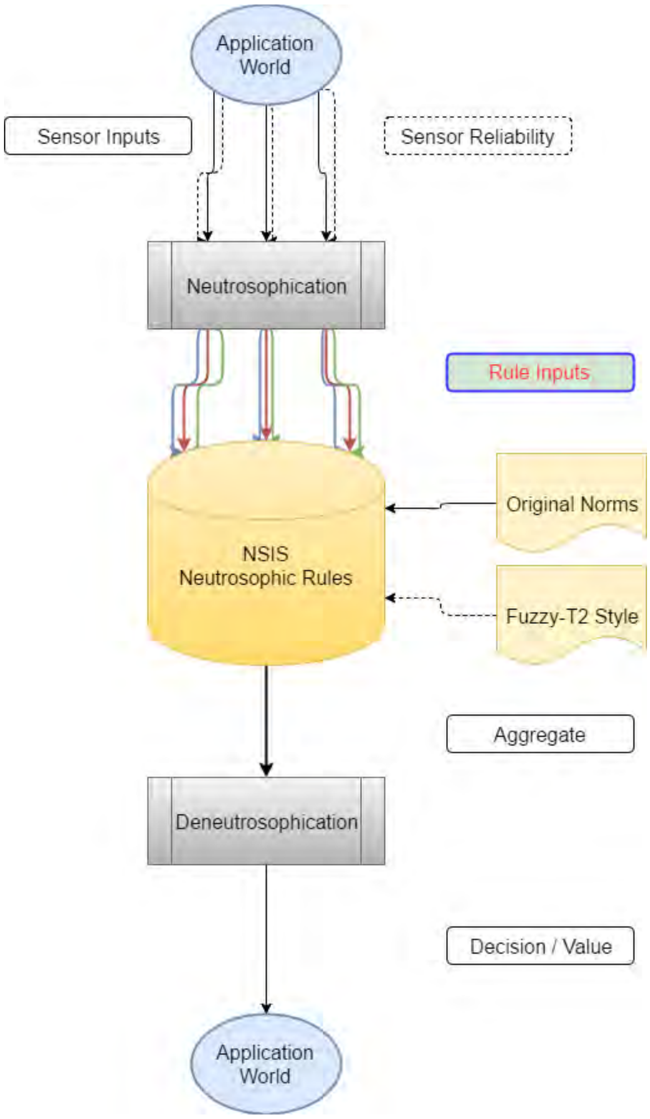

Figure 6. Use-case diagram

It is worth noting the similarities and discrepancies between the two implementation versions. For both, the position and speed feedbacks are resampled using a Gaussian function form with the expected variance of an actual field transducer. In a practical implementation, the actual values for this are parameterized and can be adjusted according to the known specifications provided by the manufacturer. This allows us to introduce uncertainty into the simulation in a way that corresponds to actual practical scenarios.

The difference between the two versions mainly consists in the treatment of the third dimension, uncertainty, as discussed in the theoretical approach. The known variance of sensor inputs is inserted directly into the coded function in the first version (notice the m-function interpreter block that acts as the central controller of the design). In the second, they are a direct expression of the degree of membership of the second dimension values (which models fuzziness). Each input is treated separately together with its own reliability rating and fuzzified, the output of which is then re-fuzzified into the final FLC controller.

The use-case diagram shown in Figure 6 describes the generalized algorithm for the decision making process.

As much as possible, the implementation is parameterized to allow for a wide array of re-application and future development.

It is clear from the onset of neutrosophic theory that, as with fuzzy, various norms and applications can be defined over the considered set and universe of discourse. These can be passed as user-defined functions to the object creator of the neutrosophic class, if needed, and could be the object of future research scenarios for different decision support systems and controllers.

IV. Conclusions

The paper shows an inference system application using neutrosophic logic for the control of a small direct current motor used as a robotic actuator. The theoretical aspects of neutrosophy and the practical considerations of decision support systems and controllers are explored, with the application providing a practical demonstration of the proof of concept. Standard controller concerns and design parameters are explained and discussed briefly, so as to give context to the implementation.

Also of note is the programming framework for neutrosophic applications which is implemented, with a view toward generalization, parameterization and reusability. The code builds upon the existing libraries and toolboxes available for fuzzy logic, of which neutrosophy can be seen as a generalized, unifying theory.

As with the more traditional fuzzy controllers, the neutrosophic logic controller needs to be tuned for the particular application and context it is working in. There is currently no single algorithm guaranteed to find the best configuration among the many different options that would map to search space dimensions in an optimization problem.

A possible solution is to use an evolutionary algorithm to tune the scaling gains of the input membership functions, the fuzzy rule base, or both. While there are a number of authors, such as Byrne who look at GA – tuned fuzzy structures, papers using a PSO algorithm are few and far between. Both these approaches provide intriguing options for the future of both fuzzy and neutrosophic logic control, as well as further theoretical and applicative research.

The uniqueness of neutrosophic theory allows possibilities for further increasing the dimensionality of the neutrosophic object. It could be used, for example, for modelling the general noise and disturbance within the system. This would be particularly useful for industrial applications, in which such factors rise in importance considerably.

Acknowledgment

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Computation of Shortest Path Problem in a Network with SV-Triangular Neutrosophic Numbers

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Abstract— **In this article, we present an algorithm method for finding the shortest path length between a paired nodes on a network where the edge weights are characterized by single valued triangular neutrosophic numbers. The proposed algorithm gives the shortest path length from source node to destination node based on a ranking method. Finally a numerical example is presented to illustrate the efficiency of the proposed approach.**

Keywords— Single valued triangular neutrosophic number; Score function; Network; Shortest path problem.

I. INTRODUCTION

In 1995, the concept of the neutrosophic sets (NS for short) and neutrosophic logic were introduced by Smarandache in [1, 2] in order to efficiently handle the indeterminate and inconsistent information which exist in real world. Unlike fuzzy sets which associate to each member of the set a degree of membership T and intuitionstic fussy sets which associate a degree of membership T and a degree of non-membership F, T, F ∈ [0, 1], Neutrosophic sets characterize each member x of the set with a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$ each of which belongs to the non-standard unit interval]−0, 1+[. Thus, although in some case intuitionstic fuzzy sets consider a particular indeterminacy or hesitation margin, $\pi = 1 - T - F$. Neutrosophic set has the ability of handling uncertainty in a better way since in case of neutrosophic set indeterminacy is taken care of separately. Neutrosophic sets is a generalization of the theory of fuzzy set [3], intuitionistic fuzzy sets [4], interval-valued fuzzy sets [5] and interval-valued intuitionistic fuzzy sets [6]. However, the neutrosophic theory is difficult to be directly applied in real scientific and engineering areas. To easily use it in science and engineering areas, in 2005, Wang et al. [7] proposed the concept of SVNS, which differ from neutrosophic sets only in the fact that in the former's case, the of truth, indeterminacy and falsity membership functions belongs to [0, 1]. Recent research works on neutrosophic set theory and its applications in various fields are progressing rapidly [8]. Very recently Subas et al.[9] presented the concept of triangular and trapezoidal neutrosophic numbers and applied to multiple-attribute decision making problems. Then, Biswas et al. [10] presented a special case of trapezoidal neutrosophic numbers including triangular fuzzy numbers neutrosophic sets and applied to multiple-attribute decision making problems by introducing the cosine similarity measure. Deli and Subas [11] presented the single valued triangular neutrosophic numbers (SVN-numbers) as a generalization of the intuitionistic triangular fuzzy numbers and proposed a methodology for solving multiple-attribute decision making problems with SVN-numbers.

The shortest path problem (SPP) which concentrates on finding a shortest path from a source node to other node, is a fundamental network optimization problem that has been appeared in many domain including, road networks application, transportation, routing in communication channels and scheduling problems and various fields. The main objective of the shortest path problem is to find a path with minimum length between starting node and terminal node which exist in a given network. The edge (arc) length (weight) of the network may ~~represent~~ the real life quantities such as, cost, time, etc. In conventional shortest path, the distances of the edge between different nodes of a network are assumed to be certain. In the literature, many algorithms have been developed with the weights on edges on network being fuzzy numbers, intuitionistic fuzzy numbers, type-2 fuzzy numbers vague numbers [12-17].

In more recent times, Broumi et al. [18-24] presented the concept of neutrosophic graphs, interval valued neutrosophic graphs and bipolar single valued neutrosophic graphs and studied some of their related properties. Also, Smarandache [25-26] proposed another variant of neutrosophic graphs based on literal indeterminacy. Up to date, few papers dealing with shortest path problem in neutrosophic environment have been developed. The paper proposed by Broumi et al. [27] is one of the first on this subject. The authors proposed an algorithm for solving neutrosophic shortest path problem based on score function. The same authors [28] proposed another algorithm for solving shortest path problem in a bipolar neutrosophic environment. Also, in [29] they proposed the shortest path algorithm in a network with its edge lengths as interval valued neutrosophic numbers. However, till now, single valued triangular neutrosophic numbers have not been applied to shortest path problem. The main objective of this paper is to propose an approach for solving shortest path problem in anetwork where the edge weights are represented by single valued triangular neutrosophic numbers.

In order to do, the paper is organized as follows: In Section 2, we firstly review some basic concepts about neutrosophic sets, single valued neutrosophic sets and single valued triangular neutrosophic sets. In Section 3, we propose some modified operations of single valued triangular neutrosophic numbers. In Section 5, we propose an algorithm for finding the shortest path and shortest distance in single valued triangular neutrosophic graph. In section 6, we presented an hypothetical example which is solved by the proposed algorithm. Finally, some concluding remarks are presented in section 7.

II. Preliminaries

In this section, some basic concepts and definitions on neutrosophic sets, single valued neutrosophic sets and single valued triangular neutrosophic sets are reviewed from the literature.

Definition 2.1 [1]. Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x ∈ X}, where the functions T, I, F: X→]$^-0,1^+$[define respectively the truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-0,1^+$[.

Since it is difficult to apply NSs to practical problems, Wang et al. [7] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [7]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X\} \quad (2)$$

Definition 2.3 [11]. A single valued triangular neutrosophic number (SVTrN-number) $\tilde{a} =< (a_1, b_1, c_1); T_a, I_a, F_a >$ is a special neutrosophic set on the real number set *R*, whose truth membership, indeterminacy-membership, and a falsity-membership are given as follows

$$T_a(x) = \begin{cases} (x-a_1)T_a/(b_1-a_1) & (a_1 \le x \le b_1) \\ T_a & (x = b_1) \\ (c_1-x)T_a/(c_1-b_1) & (b_1 \le x \le c_1) \\ 0 & otherwise \end{cases} \quad (3)$$

$$I_a(x) = \begin{cases} (b_1 - x + I_a(x-a_1))/(b_1-a_1) & (a_1 \le x \le b_1) \\ I_a & (x = b_1) \\ (x - b_1 + I_a(c_1-x))/(c_1-b_1) & (b_1 \le x \le c_1) \\ 1 & otherwise \end{cases} \quad (4)$$

$$F_a(x) = \begin{cases} (b_1 - x + F_a(x-a_1))/(b_1-a_1) & (a_1 \le x \le b_1) \\ F_a & (x = b_1) \\ (x - c_1 + F_a(c_1-x))/(c_1-b_1) & (b_1 \le x \le c_1) \\ 1 & otherwise \end{cases} \quad (5)$$

Where 0≤ T_a ≤ 1; 0≤ I_a ≤ 1; 0≤ F_a ≤ 1 and

0≤ $T_a + I_a + F_a$ ≤ 3; $a_1, b_1, c_1 \in R$

Definition 2.4 [11]. Let $\tilde{A}_1 =< (a_1, a_2, a_3); T_1, I_1, F_1 >$ and $\tilde{A}_2 =< (b_1, b_2, b_3); T_2, I_2, F_2 >$ be two single valued triangular neutrosophic numbers. Then, the operations for SVTrN-numbers are defined as below;

(i)

$$\tilde{A}_1 \oplus \tilde{A}_2 =< (a_1+b_1, a_2+b_2, a_3+b_3); \min(T_1,T_2), \max(I_1,I_2), \max(F_1,F_2) > \quad (6)$$

(ii)

$$\tilde{A}_1 \otimes \tilde{A}_2 =< (a_1 b_1, a_2 b_2, a_3 b_3); \min(T_1,T_2), \max(I_1,I_2), \max(F_1,F_2)) \quad (7)$$

(iii)

$$\lambda\tilde{A}_1 =< (\lambda a_1, \lambda a_2, \lambda a_3); \min(T_1,T_2), \max(I_1,I_2), \max(F_1,F_2) > \quad (8)$$

A convenient method for comparing two single valued triangular neutrosophic numbers is by using of score function and accuracy function.

Definition 2.5[11]. Let $\tilde{A}_1 =< (a_1, a_2, a_3); T_1, I_1, F_1 >$ be a single valued triangular neutrosophic number. Then, the score function $s(\tilde{A}_1)$ and accuracy function $a(\tilde{A}_1)$ of a SVTrN-numbers are defined as follows:

(i) $s(\tilde{A}_1) = \left(\frac{1}{12}\right)\left[a_1 + 2a_2 + a_3\right] \times \left[2 + T_1 - I_1 - F_1\right]$ (9)

(ii) $a(\tilde{A}_1) = \left(\frac{1}{12}\right)\left[a_1 + 2a_2 + a_3\right] \times \left[2 + T_1 - I_1 + F_1\right]$ (10)

Definition 2.6 [11]. Let $\tilde{A}_1$ and $\tilde{A}_2$ be two SVTrN-numbers the ranking of $\tilde{A}_1$ and $\tilde{A}_2$ by score function and accuracy function are defined as follows :

(i) If $s(\tilde{A}_1) \prec s(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$

(ii) If $s(\tilde{A}_1) = s(\tilde{A}_2)$ and if

(1) $a(\tilde{A}_1) \prec a(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$

(2) $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1 \succ \tilde{A}_2$

(3) $a(\tilde{A}_1) = a(\tilde{A}_2)$ then $\tilde{A}_1 = \tilde{A}_2$

III. ARITHMETIC OPERATIONS BETWEEN TWO SV-TRIANGULAR NEUTROSOPHIC NUMBERS

In this subsection, a slight modification has been made on some operations between two single valued triangular neutrosophic numbers proposed by Deli and Subas [11], required for the proposed algorithm.

Let $\tilde{A}_1 =< (a_1, a_2, a_3); T_1, I_1, F_1 >$ and $\tilde{A}_2 =< (b_1, b_2, b_3); T_2, I_2, F_2 >$ are two single valued triangular neutrosophic numbers,. Then the operations for SVTrNNs are defined ad below:

(i) $\tilde{A}_1 \oplus \tilde{A}_2 =< (a_1 + b_1, a_2 + b_2, a_3 + b_3); T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2 >$ (11)

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 =< (a_1 b_1, a_2 b_2, a_3 b_3); T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2) >$ (12)

(iii) $\lambda\tilde{A}_1 =< (\lambda a_1, \lambda a_2, \lambda a_3); 1 - (1 - T_1)^{\lambda}), I_1^{\lambda}, F_1^{\lambda}) >$ (13)

IV. NETWORK TERMINOLOGY

Consider a directed network G = (V, E) consisting of a finite set of nodes V={1, 2,…,n} and a set of m directed edges E $\subseteq$ V x V. Each edge is denoted by an ordered pair (i, j) where i, j $\in$ V and $i \neq j$. In this network, we specify two nodes, denoted by s and t, which are the source node and the destination node, respectively. We define a path P_{ij} ={i= i_1 , (i_1, i_2) , i_2 ,…, i_{l-1} , (i_{l-1}, i_l) , i_l =j}as sequence that joins two nodes of edges. The existence of at least one path P_{si} in G (V, E) is assumed for every i $\in$ V-{s}.

d_{ij} denotes a single valued triangular neutrosophic number associated with the edge (i ,j), corresponding to the length necessary to traverse (i, j) from i to j. In real problems, the lengths correspond to the cost, the time, the distance, etc. Then, neutrosophic distance along the path P is denoted as d(P) is defined as

$$d(P) = \sum_{(i,j \in P)} d_{ij} \quad (14)$$

Remark : A node i is said to be predecessor node of node j if

(i) Node i is directly connected to node j.

(ii) The direction of path connecting node i and j from i to j.

V . SINGLE VALUED TRIANGULAR NEUTROSOPHIC PATH PROBLEM

In this section, motivated by the work of Kumar [15], an algorithm is presented for finding the shortest path between the source node (i) and the destination node (j) in a network where the edges weight are characterized by a single valued triangular neutrosophic numbers.

The steps of the algorithm are:

Step1: Assume $\tilde{d}_1 =< (0,0,0); 0,1,1 >$ and label the source node (say node1) as [$\tilde{d}_1 =< (0,0,0); 0,1,1 >$,-]. The label indicating that the node has no predecessor.

Step 2: Find $\tilde{d}_j$ = minimum{ $\tilde{d}_i \oplus \tilde{d}_{ij}$ };j=2,3,…,n.

Step 3: If minimum occurs corresponding to unique value of i i.e., i= r then label node j as [$\tilde{d}_j$,r]. If minimum occurs corresponding to more than one values of i then it represents that there are more than one single valued triangular neutrosophic path between source node and node j but single valued triangular neutrosophic distance along path is $\tilde{d}_j$, so choose any value of i.

Step 4: Let the destination node (node n) be labeled as [$\tilde{d}_n$, *l*], then the single valued triangular neutrosophic shortest distance between source node and destination node is $\tilde{d}_n$.

Step 5: Since destination node is labeled as [$\tilde{d}_n$,*l*], so, to find the single valued triangular neutrosophic shortest path between source node and destination node, check the label of node *l*. Let it be [$\tilde{d}_l$, p], now check the label of node p and so on. Repeat the same procedure until node 1 is obtained.

Step 6: Now the single valued triangular neutrosophic shortest path can be obtained by combining all the nodes obtained by the step 5.

Remark 5.1 Let $\tilde{A}_i$; i =1, 2,…, n be a set of single valued triangular neutrosophic numbers, if S($\tilde{A}_k$) < S($\tilde{A}_i$), for all i, the single valued triangular neutrosophic number is the minimum of $\tilde{A}_k$.

After describing the proposed algorithm, in next section, an hypothetical example is presented and the proposed method is explained completely.

VI. ILLUSTRATIVE EXAMPLE

In this section an hypothetical example is introduced to verify the proposed. Consider the network shown in figure 1; we want to obtain the shortest path from node 1 to node 6 where edges have a single valued triangular neutrosophic numbers. Let us now apply the proposed algorithm to the network given in figure 1.

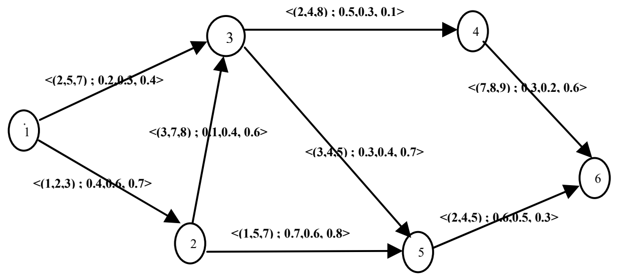

Fig. 1. A network with single valued triangular neutrosophic edges

In this network each edge have been assigned to single valued triangular neutrosophic number as follows:

Edges	Single valued triangular neutrosophic distance
1-2	<(1, 2,3);0.4,0.6,0.7>
1-3	<(2,5,7);0.2,0.3,0.4>
2-3	<(3,7,8);0.1,0.4,0.6>
2-5	<(1,5,7);0.7,0.6,0.8>
3-4	<(2,4,8);0.5,0.3,0.1>
3-5	<(3, 4,5);0.3,0.4,0.7>
4-6	<(7, 8,9);0.3,0.2,0.6>
5-6	<(2,4,5);0.6,0.5,0.3>

Table 1. Weights of the graphs

The calculations for this problem are as follows:
since node 6 is the destination node, so n= 6.
Assume $\tilde{d}_1 =<(0,\ 0,\ 0);\ 0,\ 1,\ 1>$ and label the source node (say node 1) as [$<(0,\ 0,\ 0);\ 0,\ 1,\ 1>$,-], the value of $\tilde{d}_j$; j= 2, 3, 4, 5 ,6 can be obtained as follows:

Iteration1: Since only node 1 is the predecessor node of node 2, so putting i=1 and j= 2 in step of the proposed algorithm, the value of $\tilde{d}_2$ is

$\tilde{d}_2$ = minimum{ $\tilde{d}_1 \oplus \tilde{d}_{12}$ }=minimum{<(0, 0, 0); 0, 1, 1>⊕ <(1, 2, 3); 0.4, 0.6, 0.7>= <(1, 2, 3); 0.4, 0.6, 0.7>

Since minimum occurs corresponding to i=1, so label node 2 as [<(1, 2, 3); 0.4, 0.6, 0.7>, 1]

$\tilde{d}_2$ = <(1, 2, 3); 0.4, 0.6, 0.7>

Iteration 2: The predecessor node of node 3 are node 1 and node 2, so putting i= 1, 2 and j= 3 in step 2 of the proposed algorithm, the value of $\tilde{d}_3$ is

$\tilde{d}_3$=minimum{ $\tilde{d}_1 \oplus \tilde{d}_{13}, \tilde{d}_2 \oplus \tilde{d}_{23}$ }=minimum{<(0, 0, 0); 0, 1, 1>⊕ <(2, 5, 7); 0.2, 0.3, 0.4>, <(1, 2, 3); 0.4, 0.6, 0.7>⊕ <(3, 7, 8); 0.1, 0.4, 0.6>}= minimum{<(2, 5, 7); 0.2, 0.3, 0.4> , <(4, 9, 11); 0.46, 0.24, 0.42>}

Using Eq.9, we have

S ({<(2, 5, 7); 0.2, 0.3, 0.4>)

$=\left(\frac{1}{12}\right)\left[a_1+2a_2+a_3\right]\times\left[2+T_1-I_1-F_1\right]$=2.38

S (<(4, 9, 11); 0.46, 0.24, 0.42>) =4.95

Since S ({<(2, 5, 7); 0.2, 0.3, 0.4>) < S (<(4, 9, 11); 0.46, 0.24, 0.42>)

So minimum{<(2, 5, 7); 0.2, 0.3, 0.4> , <(4, 9, 11); 0.46, 0.24, 0.42>} = <(2, 5, 7); 0.2, 0.3, 0.4>

Since minimum occurs corresponding to i=1, so label node 3 as [<(2, 5, 7); 0.2, 0.3, 0.4>, 1]

$\tilde{d}_3$=<(2, 5, 7); 0.2, 0.3, 0.4>

Iteration 3: The predecessor node of node 4 is node 3, so putting i= 3 and j= 4 in step 2 of the proposed algorithm, the value of $\tilde{d}_4$ is $\tilde{d}_4$ =minimum { $\tilde{d}_3 \oplus \tilde{d}_{34}$ }=minimum{<(2, 5, 7); 0.2, 0.3, 0.4>, ⊕ <(2, 4, 8); 0.5, 0.3, 0.1> }=<(4, 9, 15,); 0.6, 0.09, 0.04>

So minimum {<(2, 5, 7); 0.2, 0.3, 0.4>, ⊕ <(2, 4, 8); 0.5, 0.3, 0.1> }= <(4, 9, 15); 0.6, 0.09, 0.04>

Since minimum occurs corresponding to i=3, so label node 4 as [<(4, 9, 15); 0.6, 0.09, 0.04>,3]

$\tilde{d}_4$ =<(4, 9, 15); 0.6, 0.09, 0.04>

Iteration 4: The predecessor node of node 5 are node 2 and node 3, so putting i= 2, 3and j= 5 in step 2 of the proposed algorithm, the value of $\tilde{d}_5$ is

$\tilde{d}_5$ =minimum{ $\tilde{d}_2 \oplus \tilde{d}_{25}, \tilde{d}_3 \oplus \tilde{d}_{35}$ }=minimum{<(1, 2, 3); 0.4, 0.6, 0.7>⊕ <(1, 5, 7); 0.7, 0.6, 0.8>, <(2, 5, 7); 0.2, 0.3, 0.4>⊕ <(3, 4, 5); 0.3, 0.4, 0.7>}=
minimum{<(2, 7, 10); 0.82, 0.36, 0.56>, <(5, 9, 12); 0.44, 0.12, 0.28>}

Using Eq.9, we have

S (<(2, 7, 10); 0.82, 0.36, 0.56>) =4.12

S (<(5, 9, 12); 0.44, 0.12, 0.28>) =5.13

Since S (<(2, 7, 10); 0.82, 0.36, 0.56>) < S (<(5, 9, 12); 0.44, 0.12, 0.28>)

minimum{<(2, 7, 10); 0.82, 0.36, 0.56>, <(5, 9, 12); 0.44, 0.12, 0.28>}
= <(2, 7, 10); 0.82, 0.36, 0.56>

$\tilde{d}_5$ = <(2, 7, 10); 0.82, 0.36, 0.56>

Since minimum occurs corresponding to i=2, so label node 5 as [<(2, 7, 10); 0.82, 0.36, 0.56>, 2]

Iteration 5: The predecessor node of node 6 are node 4 and node 5, so putting i= 4, 5and j= 6 in step 2 of the proposed algorithm, the value of $\tilde{d}_6$ is

$\tilde{d}_6$ =minimum{ $\tilde{d}_4 \oplus \tilde{d}_{46}, \tilde{d}_5 \oplus \tilde{d}_{56}$ }=minimum{<(4, 9, 15); 0.6, 0.09, 0.04>⊕ <(7, 8, 9); 0.3, 0.2, 0.6>, <(2, 7, 10); 0.82, 0.36, 0.56>⊕ <(2, 4, 5); 0.6, 0.5, 0.3>}= minimum{<(11, 17, 24); 0.72, 0.018, 0.024>, <(4, 11, 15); 0.93, 0.18, 0.17>}

Using Eq.9, we have

S (<(11, 17, 24); 0.72, 0.018, 0.024>) =15.40

S (<(4, 11, 15); 0.93, 0.18, 0.17>) =8.82

Since S (<(4, 11, 15); 0.93, 0.18, 0.17>) < S (<(11, 17, 24); 0.72, 0.018, 0.024>)
So minimum {<(11, 17, 24); 0.72, 0.018, 0.024>, <(4, 11, 15); 0.93, 0.18, 0.17>}
= <(4, 11, 15); 0.93, 0.18, 0.17>
$\tilde{d}_6$ =<(4, 11, 15); 0.93, 0.18, 0.17>
Since minimum occurs corresponding to i= 5, so label node 6 as [<(4, 11, 15); 0.93, 0.18, 0.17>, 5]
Since node 6 is the destination node of the given network, so the single valued triangular neutrosophic shortest distance between node 1 and node 6 is <(4, 11, 15); 0.93, 0.18, 0.17>.
Now the single valued triangular neutrosophic shortest path between node 1 and node 6 can be founded by using the following procedure:
Since node 6 is labeled by [<(4, 11, 15); 0.93, 0.18, 0.17>, 5], which represents that we are coming from node 5. Node 5 is labeled by [<(2, 7, 10); 0.82, 0.36, 0.56>, 2], which represent that we are coming from node 2. Node 2 is labeled by [<(1, 2, 3); 0.4, 0.6, 0.7>, 1], which represents that we are coming from node 1. Now the single valued triangular neutrosophic shortest path between node 1 and node 6 is obtaining by joining all the obtained nodes. Hence the single valued triangular neutrosophic shortest path $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$.
The single valued triangular neutrosophic shortest distance and the single valued triangular neutrosophic shortest path of all nodes from node 1 is depicted in the table 2 and the labeling of each node is shown in figure 2.

Node No.(j)	$\tilde{d}_i$	Single valued triangular Neutrosophic shortest path between jth and 1st node
2	<(1, 2, 3); 0.4, 0.6, 0.7>	$1 \rightarrow 2$
3	<(2, 5, 7); 0.2, 0.3, 0.4>	$1 \rightarrow 3$
4	<(4, 9, 15); 0.6, 0.09, 0.04>	$1 \rightarrow 3 \rightarrow 4$
5	<(2, 7, 10); 0.82, 0.36, 0.56>	$1 \rightarrow 2 \rightarrow 5$
6	<(4, 11, 15); 0.93, 0.18, 0.17>	$1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

Table 2. Tabular representation of different single valued triangular neutrosophic shortest paths

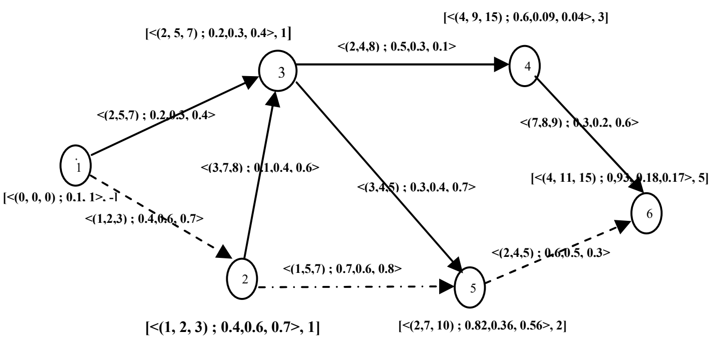

FIG 2. Network with single valued triangular neutrosophic shortest distance of each node from node 1

VI. CONCLUSION

In this article, an algorithm has been developed for solving shortest path problem on a network where the edges weight are characterized by a neutrosophic numbers called single valued triangular neutrosophic numbers. To show the performance of the proposed methodology for the shortest path problem, an hypothetical example was introduced. In future works, we studied the shortest path problem in a single valued trapezoidal neutrosophic environment and we will research the application of this algorithm.

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Complex Neutrosophic Graphs of Type 1

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Abstract—In this paper, we introduced a new neutrosophic graphs called complex neutrosophic graphs of type1 (CNG1) and presented a matrix representation for it and studied some properties of this new concept. The concept of CNG1 is an extension of generalized fuzzy graphs of type 1 (GFG1) and generalized single valued neutrosophic graphs of type 1 (GSVNG1).

Keywords complex neutrosophic set; Complex neutrosophic graph; Matrix representation.

I. Introduction

Smarandache [7] in 1998, introduced a new theory called Neutrosophic, which is basically a branch of philosophy that focus on the origin, nature, and scope of neutralities and their interactions with different ideational spectra. On the basis of neutrosophy, Smarandache defined the concept of neutrosophic set which is characterized by a degree of truth membership T, a degree of indeterminacy membership I and a degree falsehood membership F. The concept of neutrosophic set theory generalizes the concept of classical sets, fuzzy sets [14], intuitionistic fuzzy sets [13], interval-valued fuzzy sets [12]. In fact this mathematical tool is used to handle problems like imprecision, indeterminacy and inconsistency of data. Specially, the indeterminacy presented in the neutrosophic sets is independent on the truth and falsity values. To easily apply the neutrosophic sets to real scientific and engineering areas, Smarandache [7] proposed the single valued neutrosophic sets as subclass of neutrosophic sets. Later on, Wang et al. [11] provided the set-theoretic operators and various properties of single valued neutrosophic sets. The concept of neutrosophic sets and their particular types have been applied successfully in several fields [40]

Graphs are the most powerful and handful tool used in representing information involving relationship between objects and concepts. In a crisp graphs two vertices are either related or not related to each other, mathematically, the degree of relationship is either 0 or 1. While in fuzzy graphs, the degree of relationship takes values from [0, 1]. Later on Atanassov [2] defined intuitionistic fuzzy graphs (IFGs) using five types of Cartesian products. The concept fuzzy graphs and their extensions have a common property that each edge must have a membership value less than or equal to the minimum membership of the nodes it connects.

When description of the object or their relations or both is indeterminate and inconsistent, it cannot be handled by fuzzy intuitionistic fuzzy, bipolar fuzzy, vague and interval valued fuzzy graphs. So, for this reason, Smarandache [10] proposed the concept of neutrosophic graphs based on literal indeterminacy (I) to deal with such situations. Then, Smarandache [4, 5] gave another definition for neutrosophic graph theory using the neutrosophic truth-values (T, I, F) and constructed three structures of neutrosophic graphs: neutrosophic edge graphs, neutrosophic vertex graphs and neutrosophic vertex-edge graphs. Later on Smarandache [9] proposed new version of neutrosophic graphs such as neutrosophic offgraph, neutrosophic bipolar/tripola/ multipolar graph. Presently, works on neutrosophic vertex-edge graphs and neutrosophic edge graphs are progressing rapidly. Broumi et al.[24] combined the concept of single valued neutrosophic sets and graph theory, and introduced certain types of single valued neutrosophic graphs (SVNG) such as strong single valued neutrosophic graph, constant single valued neutrosophic graph, complete single valued neutrosophic graph and investigate some of their properties with proofs and examples. Also, Broumi et al. [25] also introduced neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of vertex in fuzzy graph and intuitionistic fuzzy graph. In addition, Broumi et al.[26] proved a necessary and sufficient condition for a single valued neutrosophic graph to be an isolated single valued neutrosophic graph. After Broumi, the studies on the single valued neutrosophic graph theory have been studied increasingly [1, 16-20, 27-34, 36-38].

Recently, Smarandache [8]-initiated the idea of removal of the edge degree restriction of fuzzy graphs, intuitionistic fuzzy graphs and single valued neutrosophic graphs. Samanta et al [35] proposed a new concept named the generalized fuzzy graphs (GFG) and defined two types of GFG, also the authors studied some major properties such as completeness and regularity with proved results. In this paper, the authors claims that fuzzy graphs and their extension defined by many researches are limited to represent for some systems such as social network. Later on Broumi et al. [34] have discussed the removal of the edge degree restriction of single valued neutrosophic graphs and presented a new class of single valued neutrosophic graph called generalized single valued neutrosophic graph of type1, which is a is an extension of generalized fuzzy graph of type1 [35]. Since complex fuzzy sets was introduced by Ramot [3], few extension of complex fuzzy set have been widely discussed [22, 23]. Ali and Smarandache [15] proposed the concept of complex neutrosophic set which is a generalization of complex fuzzy set and complex intuitionstic fuzzy sets. The concept of complex neutrosophic set is defined by a complex-valued truth membership function, complex-valued indeterminate membership function, and a complex-valued falsehood membership function. Therefore, a complex-valued truth membership function is a combination of traditional truth membership function with the addition of an extra term.

Similar to the fuzzy graphs, which have a common property that each edge must have a membership value less than or equal to the minimum membership of the nodes it connects. Also, complex fuzzy graphs presented in [21] have the same property. Until now, to our best knowledge, there is no research on complex neutrosophic graphs.

The main objective of this paper is to introduce the concept of complex neutrosophic graph of type 1 and introduced a matrix representation of CNG1.

The remainder of this paper is organized as follows. In Section 2, we review some basic concepts about neutrosophic sets, single valued neutrosophic sets, complex neutrosophic sets and generalized single valued neutrosophic graphs of type 1. In Section 3, the concept of complex neutrosophic graphs of type 1 is proposed with an illustrative example. In section 4 a representation matrix of complex neutrosophic graphs of type 1 is introduced. Finally, Section 5outlines the conclusion of this paper and suggests several directions for future research.

II. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, single valued neutrosophic graphs and generalized fuzzy graphs relevant to the present work. See especially [7,11, 15, 34] for further details and background.

Definition 2.1 [7]. Let X be a space of points and let x ∈ X. A neutrosophic set A in X is characterized by a truth membership function T, an indeterminacy membership function I, and a falsity membership function F. T, I, F are real standard or nonstandard subsets of $]^-0,1^+[$, and T, I, F: $X \rightarrow]^-0,1^+[$. The neutrosophic set can be represented as

$$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\} \qquad (1)$$

There is no restriction on the sum of T, I, F, So

$$^-0 \leq T_A(x)+ I_A(x)+F_A(x) \leq 3^+. \qquad (2)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. Thus it is necessary to take the interval [0, 1] instead of $]^-0,1^+[$. For technical applications. It is difficult to apply $]^-0,1^+[$ in the real life applications such as engineering and scientific problems.

Definition 2.2 [11]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$. A SVNS A can be written as

$$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\} \ (3)$$

Definition 2.3 [15]

A complex neutrosophic set A defined on a universe of discourse X, which is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$ that assigns a complex-valued grade of $T_A(x)$, $I_A(x)$, and $F_A(x)$ in A for any x ∈ X. The values $T_A(x)$, $I_A(x)$, and $F_A(x)$and their sum may all within the unit circle in the complex plane and so is of the following form,

$T_A(x)= p_A(x).e^{j\mu_A(x)}$, $I_A(x)= q_A(x).e^{j\nu_A(x)}$ and $F_A(x)= r_A(x).e^{j\omega_A(x)}$

Where, $p_A(x)$, $q_A(x)$, $r_A(x)$ and $\mu_A(x)$, $\nu_A(x)$, $\omega_A(x)$ are respectively, real valued and $p_A(x)$, $q_A(x)$, $r_A(x) \in [0, 1]$ such that

$$0 \leq p_A(x)+ q_A(x) + r_A(x) \leq 3$$

The complex neutrosophic set A can be represented in set form as

$$A=\{(x, T_A(x)=a_T, I_A(x)=a_I, F_A(x)=a_F): x \in X\}$$

where $T_A: X \rightarrow \{a_T: a_T \in C, |a_T| \leq 1\}$,
$I_A: X \rightarrow \{a_I: a_I \in C, |a_I| \leq 1\}$,
$F_A: X \rightarrow \{a_F: a_F \in C, |a_F| \leq 1\}$ and
$|T_A(x) + I_A(x) + F_A(x)| \leq 3$.

Definition 2.4 [15] The union of two complex neutrosophic sets as follows:

Let A and B be two complex neutrosophic sets in X, where

$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\}$ and
$B=\{(x, T_B(x), I_B(x), F_B(x)): x \in X\}$.

Then, the union of A and B is denoted as $A \cup_N B$ and is given as

$$A \cup_N B=\{(x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)): x \in X\}$$

Where the truth membership function $T_{A\cup B}(x)$, the indeterminacy membership function $I_{A\cup B}(x)$ and the falsehood membership function $F_{A\cup B}(x)$ is defined by

$T_{A\cup B}(x) = [(p_A(x) \vee p_B(x))].e^{j.\mu_{T_{A\cup B}}(x)}$,
$I_{A\cup B}(x) = [(q_A(x) \wedge q_B(x))].e^{j.\mu_{I_{A\cup B}}(x)}$,
$F_{A\cup B}(x) = [(r_A(x) \wedge r_B(x))].e^{j.\mu_{F_{A\cup B}}(x)}$

Where $\vee$ and $\wedge$ denotes the max and min operators respectively. The phase term of complex truth membership function, complex indeterminacy membership function and complex falsity membership function belongs to $(0, 2\pi)$ and, they are defined as follows:

a) Sum:

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x),$$
$$\nu_{A\cup B}(x) = \nu_A(x) + \nu_B(x),$$
$$\omega_{A\cup B}(x) = \omega_A(x) + \omega_B(x).$$

b) Max:

$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)),$$
$$\nu_{A\cup B}(x) = \max(\nu_A(x), \nu_B(x)),$$
$$\omega_{A\cup B}(x) = \max(\omega_A(x), \omega_B(x)).$$

c) Min:

$$\mu_{A\cup B}(x) = \min(\mu_A(x), \mu_B(x)),$$
$$\nu_{A\cup B}(x) = \min(\nu_A(x), \nu_B(x)),$$
$$\omega_{A\cup B}(x) = \min(\omega_A(x), \omega_B(x)).$$

d) "The game of winner, neutral, and loser":

$$\mu_{A\cup B}(x) = \begin{cases} \mu_A(x) & if \quad p_A > p_B \\ \mu_B(x) & if \quad p_B > p_A \end{cases},$$

$$\nu_{A\cup B}(x) = \begin{cases} \nu_A(x) & if \quad q_A < q_B \\ \nu_B(x) & if \quad q_B < q_A \end{cases},$$

$$\omega_{A\cup B}(x) = \begin{cases} \omega_A\ x) & if \quad r_A < r_B \\ \omega_B\ x) & if \quad r_B < r_A \end{cases}.$$

The game of winner, neutral, and loser is the generalization of the concept "winner take all" introduced by Ramot et al. in [3] for the union of phase terms.

Definition 2.5 [15] intersection of complex neutrosophic sets

Let A and B be two complex neutrosophic sets in X,
A=$\{(x, T_A(x), I_A(x), F_A(x)): x \in X\}$ and
B=$\{(x, T_B(x), I_B(x), F_B(x)): x \in X\}$.

Then the intersection of A and B is denoted as $A \cap_N B$ and is define as

$$A \cap_N B = \{(x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)): x \in X\}$$

Where the truth membership function $T_{A\cap B}(x)$, the indeterminacy membership function $I_{A\cap B}(x)$ and the falsehood membership function $F_{A\cap B}(x)$ is given as:

$T_{A\cap B}(x) = [(p_A(x) \wedge p_B(x))].e^{j.\mu_{T_{A\cap B}}(x)}$,
$I_{A\cap B}(x) = [(q_A(x) \vee q_B(x))].e^{j.\mu_{I_{A\cap B}}(x)}$,
$F_{A\cap B}(x) = [(r_A(x) \vee r_B(x))].e^{j.\mu_{F_{A\cap B}}(x)}$

Where $\vee$ and $\wedge$ denotes denotes the max and min operators respectively

The phase terms $e^{j.\mu_{T_{A\cap B}}(x)}$, $e^{j.\mu_{I_{A\cap B}}(x)}$ and $e^{j.\mu_{F_{A\cap B}}(x)}$ was calculated on the same lines by winner, neutral, and loser game.

Definition 2.6 [34]. Let V be a non-void set. Two function are considered as follows:
$\rho=(\rho_T, \rho_I, \rho_F)$:V → $[0, 1]^3$and
$\omega= (\omega_T, \omega_I, \omega_F)$:VxV → $[0, 1]^3$. We suppose
A= $\{(\rho_T(x),\rho_T(y)) \mid \omega_T(x, y) \geq 0\}$,
B= $\{(\rho_I(x),\rho_I(y)) \mid \omega_I(x, y) \geq 0\}$,
C= $\{(\rho_F(x),\rho_F(y)) \mid \omega_F(x, y) \geq 0\}$,
We have considered ω_T, ω_I and $\omega_F \geq 0$ for all set A,B, C , since its is possible to have edge degree = 0 (for T, or I, or F).
The triad (V, ρ, ω) is defined to be generalized single valued neutrosophic graph of type 1 (GSVNG1) if there are functions
α:A→ $[0, 1]$, β:B→ $[0, 1]$ and δ:C→ $[0, 1]$ such that
$\omega_T(x, y) = \alpha((\rho_T(x),\rho_T(y)))$
$\omega_I(x, y) = \beta((\rho_I(x),\rho_I(y)))$
$\omega_F(x, y) = \delta((\rho_F(x),\rho_F(y)))$ where x, y∈ V.
Here $\rho(x)$= ($\rho_T(x)$, $\rho_I(x)$, $\rho_F(x)$), x ∈ V are the truth-membership, indeterminate-membership and false-membership of the vertex x and $\omega(x, y)$=($\omega_T(x, y)$, $\omega_I(x, y)$, $\omega_F(x, y)$), x, y∈ V are the truth-membership, indeterminate-membership and false-membership values of the edge (x, y).

III. Complex Neutrosophic Graph of Type 1

By using the concept of complex neutrosophic sets [15] and the concept of generalized single valued neutrosophic graph of type 1 [34], we define the concept of complex neutrosophic graph of type 1 as follows:

Definition 3.1. Let V be a non-void set. Two functions are considered as follows:
ρ=(ρ_T, ρ_I, ρ_F):V → $[0, 1]^3$and
ω=(ω_T, ω_I, ω_F):VxV → $[0, 1]^3$. We suppose
A= $\{(\rho_T(x), \rho_T(y)) \mid \omega_T(x, y) \geq 0\}$,
B= $\{(\rho_I(x), \rho_I(y)) \mid \omega_I(x, y) \geq 0\}$,
C= $\{(\rho_F(x), \rho_F(y)) \mid \omega_F(x, y) \geq 0\}$,
We have considered ω_T, ω_I and $\omega_F \geq 0$ for all set A,B, C , since its is possible to have edge degree = 0 (for T, or I, or F).
The triad (V, ρ, ω) is defined to be complex neutrosophic graph of type 1 (CNG1) if there are functions
α:A→ $[0, 1]$, β:B→ $[0, 1]$ and δ:C→ $[0, 1]$ such that
$\omega_T(x, y) = \alpha((\rho_T(x), \rho_T(y)))$
$\omega_I(x, y) = \beta((\rho_I(x), \rho_I(y)))$
$\omega_F(x, y) = \delta((\rho_F(x), \rho_F(y)))$
Where x, y ∈ V.
Here $\rho(x)$=($\rho_T(x)$, $\rho_I(x)$, $\rho_F(x)$), x∈ V are the complex truth-membership, complex indeterminate-membership and complex false-membership of the vertex x and $\omega(x, y)$=($\omega_T(x, y)$, $\omega_I(x, y)$, $\omega_F(x, y)$), x, y ∈ V are the complex truth-membership, complex indeterminate-membership and complex false-membership values of the edge (x, y).

Example 3.2: Let the vertex set be V={x, y, z, t} and edge set be E={(x, y),(x, z),(x, t),(y, t)}

	x	y	z	t
ρ_T	$0.5\ e^{j.0.8}$	$0.9\ e^{j.0.9}$	$0.3\ e^{j.0.3}$	$0.8\ e^{j.0.1}$
ρ_I	$0.3\ e^{j.\frac{3\pi}{4}}$	$0.2\ e^{j.\frac{\pi}{4}}$	$0.1e^{j.2\pi}$	$0.5\ e^{j.\pi}$
ρ_F	$0.1\ e^{j.0.3}$	$0.6\ e^{j.0.5}$	$0.8\ e^{j.0.5}$	$0.4\ e^{j.0.7}$

Table 1: Complex truth-membership, complex indeterminate-membership and complex false-membership of the vertex set.

Let us consider the functions $\alpha(m,n)=(\ m_T \vee n_T).\ e^{j.\mu_{T_{m\cup n}}}$, $\beta(m,n)=(\ m_I \wedge n_I).\ e^{j.\mu_{I_{m\cup n}}}$ and $\delta(m,n)=(\ m_F \wedge n_F).\ e^{j.\mu_{F_{m\cup n}}}$.

Here, A={(0.5 $e^{j.0.8}$, 0.9 $e^{j.0.9}$), (0.5 $e^{j.0.8}$, 0.3 $e^{j.0.3}$), (0.5 $e^{j.0.8}$, 0.8 $e^{j.0.1}$), (0.9 $e^{j.0.9}$, 0.8 $e^{j.0.1}$)}

B = {(0.3 $e^{j.\frac{3\pi}{4}}$, 0.2 $e^{j.\frac{\pi}{4}}$), (0.3 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.2\pi}$), (0.3 $e^{j.\frac{3\pi}{4}}$, 0.5 $e^{j.\pi}$), (0.2 $e^{j.\frac{\pi}{4}}$, 0.5 $e^{j.\pi}$)}

C = {(0.1 $e^{j.0.3}$, 0.6 $e^{j.0.5}$), (0.1 $e^{j.0.3}$, 0.8 $e^{j.0.5}$), (0.1 $e^{j.0.3}$, 0.4 $e^{j.0.7}$), (0.6 $e^{j.0.5}$, 0.4 $e^{j.0.7}$)}.Then

ω	(x,y)	(x,z)	(x,t)	(y,t)
ω_T	$0.9\ e^{j.0.9}$	$0.5\ e^{j.0.8}$	$0.8\ e^{j.0.8}$	$0.9\ e^{j.0.9}$
ω_I	$0.2\ e^{j.\frac{3\pi}{4}}$	$0.1\ e^{j.2\pi}$	$0.3e^{j.\pi}$	$0.2\ e^{j.\pi}$
ω_F	$0.1\ e^{j.0.5}$	$0.1\ e^{j.0.5}$	$0.1\ e^{j.0.7}$	$0.4\ e^{j.0.7}$

Table 2: Complex truth-membership, complex indeterminate-membership and complex false-membership of the edge set.

The corresponding complex neutrosophic graph is shown in Fig.2

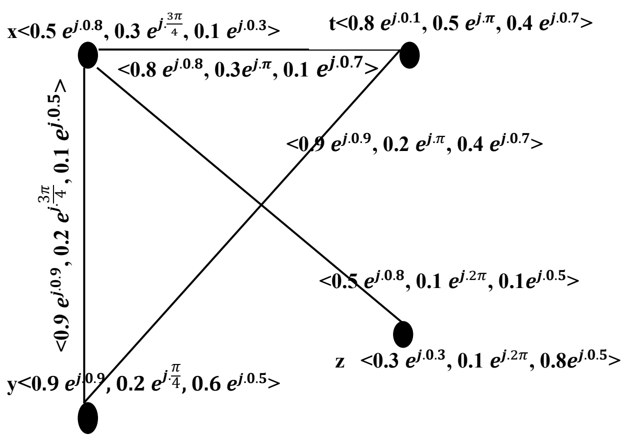

Fig 2.CNG of type 1.

The easier way to represent any graph is to use the matrix representation. The adjacency matrices, incident matrices are the widely matrices used. In the following section CNG1 is represented by adjacency matrix.

IV. Matrix Representation of Complex Neutrosophic Graph of Type 1

In this section, complex truth-membership, complex indeterminate-membership, and complex false-membership are considered independent. So, we adopted the representation matrix of generalized single valued neutrosophic graphs presented ~~in [34].~~

The complex neutrosophic graph (CNG1) has one property that edge membership values (T, I, F) depend on the membership values (T, I, F) of adjacent vertices. Suppose ξ = (V, ρ, ω) is a CNG1 where vertex set V={v_1, v_2,…, v_n}. The functions:

α :A→ (0, 1] is taken such that $\omega_T(x,y)=\alpha((\rho_T(x),\rho_T(y)))$ where x, y∈ V and A= {($\rho_T(x)$, $\rho_T(y)$) | ω_T(x, y) ≥ 0},

β :B→ (0, 1] is taken such that $\omega_I(x,y)=\beta((\rho_I(x),\rho_I(y)))$ where x, y∈ V and B= {($\rho_I(x)$, $\rho_I(y)$) | ω_I(x, y) ≥ 0}, and

δ :C→ (0, 1] is taken such that $\omega_F(x,y)=\delta((\rho_F(x),\rho_F(y)))$ where x, y∈ V and C= {($\rho_F(x)$, $\rho_F(y)$) | ω_F(x, y) ≥ 0}.

The CNG1 can be represented by a (n+1)x(n+1) matrix $M_{G_1}^{T,I,F}$=[$a^{T,I,F}$(i, j)] as follows:

complex truth-membership (T), complex indeterminate-membership (I) and complex false-membership (F) values of the vertices are provided in the first row and first column.

The (i+1, j+1)-th entry are the complex truth-membership (T), complex indeterminate-membership (I), and complex false-membership (F) values of the edge (x_i, x_j), i, j=1,…,n if i≠j.

The (i, i)-th entry is $\rho(x_i)$=($\rho_T(x_i)$, $\rho_I(x_i)$, $\rho_F(x_i)$), where i=1,2,…,n. The Complex truth-membership (T), complex indeterminate-membership (I) and complex false-membership (F) values of the edge can be computed easily using the functions α, β and δ which are in (1,1)-position of the matrix. The matrix representation of CNG1, denoted by $M_{G_1}^{T,I,F}$, can be written as three matrix representation $M_{G_1}^{T}$, $M_{G_1}^{I}$ and $M_{G_1}^{F}$.

The $M_{G_1}^{T}$ can be represented as follows

α	$v_1(\rho_T(v_1))$	$v_2(\rho_T(v_2))$	$v_n(\rho_T(v_n))$
$v_1(\rho_T(v_1))$	$\rho_T(v_1)$	$\alpha(\rho_T(v_1),\rho_T(v_2))$	$\alpha(\rho_T(v_1),\rho_T(v_n))$
$v_2(\rho_T(v_2))$	$\alpha(\rho_T(v_2),\rho_T(v_1))$	$\rho_T(v_2)$	$\alpha(\rho_T(v_2),\rho_T(v_2))$
…	….	…	…
$v_n(\rho_T(v_n))$	$\alpha(\rho_T(v_n),\rho_T(v_1))$	$\alpha(\rho_T(v_n),\rho_T(v_2))$	$\rho_T(v_n)$

Table3. Matrix representation of T-CNG1

The $M_{G_1}^{I}$ can be represented as follows

β	$v_1(\rho_I(v_1))$	$v_2(\rho_I(v_2))$	$v_n(\rho_I(v_n))$
$v_1(\rho_I(v_1))$	$\rho_I(v_1)$	$\beta(\rho_I(v_1),\rho_I(v_2))$	$\beta(\rho_I(v_1),\rho_I(v_n))$
$v_2(\rho_I(v_2))$	$\beta(\rho_I(v_2),\rho_I(v_1))$	$\rho_I(v_2)$	$\beta(\rho_I(v_2),\rho_I(v_2))$
…	….	…	…
$v_n(\rho_I(v_n))$	$\beta(\rho_I(v_n),\rho_I(v_1))$	$\beta(\rho_T(v_n),\rho_I(v_2))$	$\rho_I(v_n)$

Table4. Matrix representation of I-CNG1

The $M_{G_1}^I$ can be represented as follows

δ	$v_1(\rho_F(v_1))$	$v_2(\rho_F(v_2))$	$v_n(\rho_F(v_n))$
$v_1(\rho_F(v_1))$	$\rho_F(v_1)$	$\delta(\rho_F(v_1), \rho_F(v_2))$	$\delta(\rho_F(v_1), \rho_F(v_n))$
$v_2(\rho_F(v_2))$	$\delta(\rho_F(v_2), \rho_F(v_1))$	$\rho_F(v_2)$	$\delta(\rho_F(v_2), \rho_F(v_2))$
…	….	…	…
$v_n(\rho_F(v_n))$	$\delta(\rho_F(v_n), \rho_F(v_1))$	$\delta(\rho_F(v_n), \rho_F(v_2))$	$\rho_F(v_n)$

Table5. Matrix representation of F-CNG1

Remark 1 : If the complex indeterminacy-membership and complex non-membership values of vertices equals zero, and phase term of complex truth membership of vertices equals 0, the complex neutrosophic graphs of type 1 is reduced to generalized fuzzy graphs type 1 (GFG1).
Remark 2: If the phase term of complex truth membership, complex indeterminacy membership and complex falsity membership values of vertices equals 0, the complex neutrosophic graphs of type 1 is reduced to generalized single valued neutrosophic graphs of type 1 (GSVNG1).
Here the complex neutrosophic graph of type 1 (CNG1) can be represented by the matrix representation depicted in table 9. The matrix representation can be written as three matrices one containing the entries as T, I, F (see table 6, 7 and 8).

α	x(0.5 $e^{j.0.8}$)	y(0.9 $e^{j.0.9}$)	z(0.3 $e^{j.0.3}$)	t(0.8 $e^{j.0.1}$)
x(0.5 $e^{j.0.8}$)	0.5 $e^{j.0.8}$	0.9 $e^{j.0.9}$	0.5 $e^{j.0.8}$	0.8 $e^{j.0.8}$
y(0.9 $e^{j.0.9}$)	0.9 $e^{j.0.9}$	0.9 $e^{j.0.9}$	0	0.9 $e^{j.0.9}$
z(0.3 $e^{j.0.3}$)	0.5 $e^{j.0.8}$	0	0.3 $e^{j.0.3}$	0
t(0.8 $e^{j.0.1}$)	0.8 $e^{j.0.8}$	0.9 $e^{j.0.9}$	0	0.8 $e^{j.0.1}$

Table 6: Complex truth-matrix representation of CNG1

β	x(0.3 $e^{j.\frac{3\pi}{4}}$)	y(0.2 $e^{j.\frac{\pi}{4}}$)	z(0.1 $e^{j.2\pi}$)	t(0.5 $e^{j.2\pi}$)
x(0.3 $e^{j.\frac{3\pi}{4}}$)	0.3 $e^{j.\frac{3\pi}{4}}$	0.2 $e^{j.\frac{3\pi}{4}}$	0.1 $e^{j.2\pi}$	0.3 $e^{j.2\pi}$
y(0.2 $e^{j.\frac{\pi}{4}}$)	0.2 $e^{j.\frac{3\pi}{4}}$	0.2 $e^{j.\frac{\pi}{4}}$	0	0.2 $e^{j.2\pi}$
z(0.1 $e^{j.2\pi}$)	0.1 $e^{j.2\pi}$	0	0.1 $e^{j.2\pi}$	0
t(0.5 $e^{j.2\pi}$)	0.3 $e^{j.2\pi}$	0.2 $e^{j.2\pi}$	0	0.5 $e^{j.2\pi}$

Table 7: Complex indeterminate- matrix representation of CNG1.

δ	x(0.1 $e^{j.0.3}$)	y(0.6 $e^{j.0.5}$)	z(0.8 $e^{j.0.5}$)	t(0.8 $e^{j.0.7}$)
x(0.1 $e^{j.0.3}$)	0.1 $e^{j.0.3}$	0.1 $e^{j.0.6}$	0.1 $e^{j.0.3}$	0.1 $e^{j.0.8}$
y(0.6 $e^{j.0.5}$)	0.1 $e^{j.0.5}$	0.6 $e^{j.0.5}$	0	0.6 $e^{j.0.7}$
z(0.8 $e^{j.0.5}$)	0.1 $e^{j.0.5}$	0	0.8 $e^{j.0.5}$	0
t(0.8 $e^{j.0.7}$)	0.1 $e^{j.0.7}$	0.6 $e^{j.0.7}$	0	0.8 $e^{j.0.7}$

Table 8: Complex falsity- matrix representation of CNG1
The matrix representation of CNG1 can be represented as follows:

(α, β, δ)	**x(0.5 $e^{j.0.8}$, 0.3 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.0.3}$)**	**y(0.9 $e^{j.0.9}$, 0.2 $e^{j.\frac{\pi}{4}}$, 0.6 $e^{j.0.5}$)**	**z(0.3 $e^{j.0.3}$, 0.1 $e^{j.2\pi}$, 0.8 $e^{j.0.5}$)**	**t(0.8 $e^{j.0.1}$, 0.5 $e^{j.1\pi}$, 0.4 $e^{j.0.7}$)**
x(0.5 $e^{j.0.8}$, 0.3 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.0.3}$)	**(0.5 $e^{j.0.8}$, 0.3 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.0.3}$)**	**(0.9 $e^{j.0.9}$, 0.2 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.0.5}$)**	**(0.5 $e^{j.0.8}$, 0.1 $e^{j.2\pi}$, 0.1 $e^{j.0.5}$)**	**(0.8 $e^{j.0.8}$, 0.3 $e^{j.1\pi}$, 0.1 $e^{j.0.7}$)**
y(0.9 $e^{j.0.9}$, 0.2 $e^{j.\frac{\pi}{4}}$, 0.6 $e^{j.0.5}$)	**(0.9 $e^{j.0.9}$, 0.2 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.0.5}$)**	**(0.9 $e^{j.0.9}$, 0.2 $e^{j.\frac{\pi}{4}}$, 0.6 $e^{j.0.5}$)**	(0,0,0)	**(0.9 $e^{j.0.9}$, 0.2 $e^{j.1\pi}$, 0.4 $e^{j.0.7}$)**
z(0.3 $e^{j.0.3}$, 0.1 $e^{j.2\pi}$, 0.8 $e^{j.0.5}$)	**(0.5 $e^{j.0.8}$, 0.1 $e^{j.2\pi}$, 0.1 $e^{j.0.5}$)**	(0,0,0)	**(0.3 $e^{j.0.3}$, 0.1 $e^{j.2\pi}$, 0.8 $e^{j.0.5}$)**	(0,0,0)
t(0.8 $e^{j.0.1}$, 0.5 $e^{j.1\pi}$, 0.4 $e^{j.0.7}$)	**(0.8 $e^{j.0.8}$, 0.3 $e^{j.\frac{3\pi}{4}}$, 0.1 $e^{j.0.7}$)**	**(0.9 $e^{j.0.8}$, 0.2 $e^{j.2\pi}$, 0.4 $e^{j.0.5}$)**	(0,0,0)	**(0.8 $e^{j.0.1}$, 0.5 $e^{j.1\pi}$, 0.4 $e^{j.0.7}$)**

Table 9: Matrix representation of CNG1.

Theorem 1. Let $M_{G_1}^T$ be a matrix representation of complexT-CNG1, then the degree of vertex $D_T(x_k) = \sum_{j=1, j\neq k}^{n} a^T(k+1, j+1)$, $x_k \in$ V or $D_T(x_p) = \sum_{i=1, i\neq p}^{n} a^T(i+1, p+1)$, $x_p \in$ V.

Proof: It is similar as in theorem 1 of [34].

Theorem 2. Let $M_{G_1}^I$ be a matrix representation of complex I-CNG1, then the degree of vertex $D_I(x_k) = \sum_{j=1, j\neq k}^{n} a^I(k+1, j+1)$, $x_k \in$ V or $D_I(x_p) = \sum_{i=1, i\neq p}^{n} a^I(i+1, p+1)$, $x_p \in$ V.

Proof: It is similar as in theorem 1 of [34].

Theorem 3. Let $M_{G_1}^F$ be a matrix representation of complex F-CNG1, then the degree of vertex $D_F(x_k) = \sum_{j=1, j\neq k}^{n} a^F(k+1, j+1)$, $x_k \in$ V or $D_F(x_p) = \sum_{i=1, i\neq p}^{n} a^F(i+1, p+1)$, $x_p \in$ V.

Proof: It is similar as in theorem 1 of [34].

Theorem4. Let $M_{G_1}^{T,I,F}$ be matrix representation of CNG1, then the degree of vertex D(x_k) =($D_T(x_k), D_I(x_k), D_F(x_k)$) where
$D_T(x_k) = \sum_{j=1, j\neq k}^{n} a^T(k+1, j+1)$, $x_k \in$ V.
$D_I(x_k) = \sum_{j=1, j\neq k}^{n} a^I(k+1, j+1)$, $x_k \in$ V.
$D_F(x_k) = \sum_{j=1, j\neq k}^{n} a^F(k+1, j+1)$, $x_k \in$ V
Proof: the proof is obvious.

V. Conclusion

In this article, we presented a new concept of neutrosophic graph called complex neutrosophic graphs of type 1 and presented a matrix representation of it. The concept of complex neutrosophic graph of type 1 (CNG1) can be applied to the case of bipolar complex neutrosophic graphs (BCNG1). In the future works, we plan to study the concept of completeness, the concept of regularity and to define the concept of complex neutrosophic graphs of type 2.

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Using Neutrosophic Sets to Obtain PERT Three-Times Estimates in Project Management

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***Abstract*— Neutrosophic sets have been introduced as a generalization of crisp sets, fuzzy sets, and intuitionistic fuzzy sets to represent uncertain, inconsistent and incomplete information about real world problems. Elements of neutrosophic set are characterized by a truth-membership, falsity-membership and indeterminacy membership functions. For the first time, this paper attempts to introduce the mathematical representation of Program Evaluation and Review Technique (PERT) in neutrosophic environment. Here the elements of three-times estimates of PERT are considered as neutrosophic elements. Score and accuracy functions are used to obtain crisp model of problem. The proposed method has been demonstrated by a suitable numerical example.**

Keywords*— Neutrosophic Sets, Project, Project Management, Gantt chart, CPM, PERT, Three-Time Estimate.

I. INTRODUCTION

A project is a one time job that has a definite starting and ending dates, a clearly specified objective, a scope of work to be performed and a predefined budget. Each part of project have an effect on overall project execution time, so project completion on time depends on rightly scheduled plan. The main problem here is wrongly calculated activity durations due to lack of knowledge and experience. Lewis [1] defines project management as "the planning, scheduling and controlling of project activities to achieve project objectives-performance, cost and time for a given scope of work". The most popularly used techniques for project management are Gantt chart, Program Evaluation and Review Technique (PERT) and Critical Path Method (CPM). Gantt chart is an early technique of planning and controlling projects. Gantt charts are simple to construct, easy to understand and change. They can show plan and actual progress. However, it does not show interrelationships of activities. To overcome the limitation of Gantt chart, two project planning techniques-PERT and CPM were developed in 1950s. Both use a network and graphical model of a project, showing the activities, their interrelationships and starting and ending dates. In case of CPM, activity time can be estimated accurately and it does not vary much. In recent years, by depending on the fuzzy set theory for managing projects there were different PERT methods. However, the existing methods of fuzzy PERT have some drawbacks [2]:

- Cannot find a critical path in a fuzzy project network.
- The increasing of the possible critical paths, which is the higher risk path.
- Can't determine indeterminacy, which exist in real life situations.

In case of PERT, time estimates vary significantly [3][4]. Here three time estimates which are optimistic(a), pessimistic(b) and most likely(m) are used. In practice, a question often arises as to how obtain good estimates of a, b, and m. The person who responsible for determining values of a, b, and m often face real problem due to uncertain, inconsistent and incomplete information about real world. It is obvious that neutrosophic set theory is more appropriate than fuzzy set in modeling uncertainty that is associated with parameters such as activity duration time and resource availability in PERT. By using neutrosophic set theory in PERT technique, we can also overcome the drawbacks of fuzzy PERT methods. This paper is organized as follows:

In section 2, the basic concepts neutrosophic sets are briefly reviewed. In section 3, the mathematical model of neutrosophic PERT and the proposed algorithm is presented. In section 4, a suitable numerical example is illustrated. Finally section 5 concludes the paper with future work .

II. PRELIMINARIES

In this section, the basic definitions involving neutrosophic set, single valued neutrosophic sets, trapezoidal neutrosophic numbers and operations on trapezoidal neutrosophic numbers are outlined.

Definition 1. [5] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$ and $F_A(x): X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2. [5] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$ and $F_A(x): X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is denoted by $A = (a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c \leq 3$.

Definition 3. [6] Let $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3, a_4 \in \boldsymbol{R}$ such that $a_1 \leq a_2 \leq a_3 \leq a_4$. Then a single valued trapezoidal neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set $\boldsymbol{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are given as follows[8]:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\dfrac{x-a_1}{a_2-a_1}\right) & if\ a_1 \leq x \leq a_2 \\ \alpha_{\tilde{a}} & if\ a_2 \leq x \leq a_3 \\ \alpha_{\tilde{a}}\left(\dfrac{a_4-x}{a_4-a_3}\right) & if\ a_3 < x \leq a_4 \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \dfrac{(a_2 - x + \theta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & if\ a_1 \leq x \leq a_2 \\ \theta_{\tilde{a}} & if\ a_2 \leq x \leq a_3 \\ \dfrac{(x - a_3 + \theta_{\tilde{a}}(a_4-x))}{(a_4-a_3)} & if\ a_3 < x \leq a_4 \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \dfrac{(a_2 - x + \beta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & if\ a_1 \leq x \leq a_2 \\ \beta_{\tilde{a}} & if\ a_2 \leq x \leq a_3 \\ \dfrac{(x - a_3 + \beta_{\tilde{a}}(a_4-x))}{(a_4-a_3)} & if\ a_3 < x \leq a_4 \\ 1 & otherwise \end{cases} \quad (3)$$

Where $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued trapezoidal neutrosophic number $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ may express an ill-defined quantity about a, which is approximately equal to $[a_2, a_3]$.

Definition 4. [7] Let $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3, b_4); \alpha_{\tilde{b}}, \theta_{\tilde{b}}, \beta_{\tilde{b}} \rangle$ be two single valued trapezoidal neutrosophic numbers and $\gamma \neq 0$ be any real number [9]. Then,

$\tilde{a} + \tilde{b} =$

$$\langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle$$

$\tilde{a} - \tilde{b} =$

$$\langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle$$

$\tilde{a}\tilde{b} =$

$$\begin{cases} \langle (a_1b_1, a_2b_2, a_3b_3, a_4b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & if\ (a_4 > 0, b_4 > 0) \\ \langle (a_1b_4, a_2b_3, a_3b_2, a_4b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & if\ (a_4 < 0, b_4 > 0) \\ \langle (a_4b_4, a_3b_3, a_2b_2, a_1b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & if\ (a_4 < 0, b_4 < 0) \end{cases}$$

$\dfrac{\tilde{a}}{\tilde{b}} =$

$$\begin{cases} \left\langle \left(\dfrac{a_1}{b_4}, \dfrac{a_2}{b_3}, \dfrac{a_3}{b_2}, \dfrac{a_4}{b_1}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \right\rangle & if (a_4 > 0, b_4 > 0) \\ \left\langle \left(\dfrac{a_4}{b_4}, \dfrac{a_3}{b_3}, \dfrac{a_2}{b_2}, \dfrac{a_1}{b_1}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \right\rangle & if (a_4 < 0, b_4 > 0) \\ \left\langle \left(\dfrac{a_4}{b_1}, \dfrac{a_3}{b_2}, \dfrac{a_2}{b_3}, \dfrac{a_1}{b_4}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \right\rangle & if (a_4 < 0, b_4 < 0) \end{cases}$$

$\gamma\tilde{a} =$

$$\begin{cases} \langle (\gamma a_1, \gamma a_2, \gamma a_3, \gamma a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle & if (\gamma > 0) \\ \langle (\gamma a_4, \gamma a_3, \gamma a_2, \gamma a_1); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle & if (\gamma < 0) \end{cases}$$

$$\tilde{a}^{-1} = \left\langle \left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \right\rangle, \quad \text{Where } (\tilde{a} \neq 0)$$

III. PERT IN NEUTROSOPHIC ENVIRONMENT AND THE PROPOSED MODEL

Like CPM, PERT uses network model. However, PERT has been traditionally used in new projects which have large uncertainty in respect of design, technology and construction. To take care of associated uncertainties, we adopts neutrosophic environment for PERT activity duration.

The three time estimates for activity duration are:

1. Optimistic time ($\tilde{a}$): it is the minimum time needed to complete the activity if everything goes well.
2. Pessimistic time($\tilde{b}$) : it is the maximum time needed to complete the activity if one encounters problems at every turn.
3. Most likely time, i.e. Mode ($\tilde{m}$) : it is the time

required to complete the activity in normal circumstances.

Where $\tilde{a}, \tilde{b}, \tilde{m}$ are single valued trapezoidal neutrosophic numbers.

Based on three time estimates $(\tilde{a}, \tilde{b}, \tilde{m})$, expected time and standard deviation of each activity should be calculated , and to do this we should first obtain crisp values of three time estimates.

To obtain crisp values of three time estimates, we should use score functions and accuracy functions as follows:

Let $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ be a single valued trapezoidal neutrosophic number; then

1. Score function

$$S(\tilde{a}) = \left(\frac{1}{16}\right)[a_1 + a_2 + a_3 + a_4] \times [\alpha_{\tilde{a}} + (1 - \theta_{\tilde{a}}) + (1 - \beta_{\tilde{a}})].$$

2. accuracy function

$$A(\tilde{a}) = \left(\frac{1}{16}\right)[a_1 + a_2 + a_3 + a_4] \times [\alpha_{\tilde{a}} + (1 - \theta_{\tilde{a}}) + (1 + \beta_{\tilde{a}})].$$

After obtaining crisp values of each time estimate by using score function, the expected time and standard deviation of each activity calculated as follows;

$$T_{ij} = \frac{a + 4m + b}{6}$$

And

$$\sigma_{ij} = \frac{b - a}{6}$$

Where,

a, m, b are crisp values of optimistic, most likely and pessimistic time respectively,

T_{ij}= Expected time of ij activity and

$\sigma_{ij} =$ Standard deviation of ij activity.

Once the expected time and standard deviation of each activity are calculated, PERT network is treated like CPM network for the purpose of calculation of network parameters like earliest/latest occurrence time of activity, critical path and floats.

Let a network N= $\langle E, A, T \rangle$, being a project model, is given. E is asset of events (nodes) and A$\subset E \times E$ is a set of activities. The set E=$\{1, 2, \ldots, n\}$ is labeled in such a way that the following condition holds: $(i, j) \in A$ and $i < j$. The activity times in the network are determined by T_{ij}.

Notations of network solution and its calculations as follows:

T_i^e=Earliest occurrence time of predecessor event i,

T_i^l= Latest occurrence time of predecessor event i,

T_j^e=Earliest occurrence time of successor event j,

T_j^l= Latest occurrence time of successor event j,

T_{ij}^e/Start= Earliest start time of an activity ij,

T_{ij}^e/Finish=Earliest finish time of an activity ij,

T_{ij}^l/Start=Latest start time of an T_i^l activity ij,

T_{ij}^l/Finish= Latest finish time of an activity ij,

T_{ij} = Duration time of activity ij,

Earliest and Latest occurrence time of an event:

T_j^e=maximum $(T_j^e + T_{ij})$, calculate all T_j^e for jth event, select maximum value.

T_i^l=minimum $(T_j^l - T_{ij})$, calculate all T_i^l for ith event, select minimum value.

T_{ij}^e/Start=T_i^e,

T_{ij}^e/Finish=$T_i^e + T_{ij}$,

T_{ij}^l/Finish=T_j^l,

T_{ij}^l/Start=$T_j^l - T_{ij}$,

Critical path is the longest path in the network. At critical path, T_i^e=T_i^l, for all i.

Slack or Float is cushion available on event/ activity by which it can be delayed without affecting the project completion time.

Slack for ith event = $T_i^l - T_i^e$, for events on critical path, slack is zero.

The expected time of critical path (μ) and its variance (σ^2)calculated as follows;

$\mu = \sum T_{ij,}$ for all ij on critical path.

$\sigma^2 = \sum \sigma_{ij}^2$, for all ij on critical path.

From the previous steps we can conclude the proposed algorithm as follows:

1. To deal with uncertain, inconsistent and incomplete information about activity time, we considered three time estimates of PERT technique as a single valued trapezoidal neutrosophic numbers.
2. Calculate membership functions of each single valued trapezoidal neutrosophic number, using equation 1, 2 and 3.
3. Obtain crisp model of PERT three time estimates using score function equation as we illustrated previously.
4. Use crisp values of three time estimates to calculate expected time and standard deviation of each activity.
5. Draw PERT network diagram.
6. Determine floats and critical path, which is the longest path in network as we illustrated previously with details.
7. Calculate expected time and variance of critical path.
8. Determine expected project completion time.

IV. ILLUSTRATIVE EXAMPLE

Let us consider neutrosophic PERT and try to obtain crisp model from it. Since you are given the following data for a project:

TABLE 1. INPUT dATA FOR NEUTROSOPHIC PERT.

Activity	Immediate Predecessors	Time (days)		
		$\tilde{a}$	$\tilde{m}$	$\tilde{b}$
A	------	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$
B	------	$\tilde{2}$	$\tilde{4}$	$\tilde{6}$
C	A	$\tilde{7}$	$\tilde{9}$	$\widetilde{15}$
D	A	$\tilde{8}$	$\widetilde{10}$	$\widetilde{16}$
E	B	$\tilde{6}$	$\tilde{9}$	$\widetilde{14}$
F	C,D	$\widetilde{11}$	$\widetilde{15}$	$\widetilde{17}$
G	D,E	$\tilde{6}$	$\widetilde{11}$	$\widetilde{19}$
H	F,G	$\tilde{8}$	$\widetilde{12}$	$\widetilde{20}$

In the previous table $\tilde{a}$, $\tilde{m}$ and $\tilde{b}$ are optimistic, most likely and pessimistic time in neutrosophic environment, and considered as a single valued trapezoidal neutrosophic numbers.

Let,
$\tilde{1} = \langle(0,2,4,5); 0.8,0.6,0.4\rangle$, $\tilde{2} = \langle(1,3,5,6); 0.2,0.3,0.5\rangle$,
$\tilde{3} = \langle(1,2,5,6); 0.2,0.5,0.6\rangle$, $\tilde{4} = \langle(1,2,5,7); 0.5,0.4,0.9\rangle$,
$\tilde{5} = \langle(2,4,7,10); 0.8,0.2,0.4\rangle$, $\tilde{6} = \langle(3,7,9,12); 0.7,0.2,0.5\rangle$,
$\tilde{7} = \langle(5,8,9,13); 0.4,0.6,0.8\rangle$, $\tilde{8} = \langle(1,6,10,13); 0.9,0.1,0.3\rangle$
$\tilde{9} = \langle(6,8,10,15); 0.6,0.4,0.7\rangle$,
$\widetilde{10} = \langle(1,6,11,15); 0.7,0.6,0.3\rangle$,
$\widetilde{11} = \langle(5,8,15,20); 0.8,0.2,0.5\rangle$,
$\widetilde{12} = \langle(4,8,17,25); 0.3,0.6,0.4\rangle$,
$\widetilde{14} = \langle(7,10,19,30); 0.8,0.4,0.7\rangle$,
$\widetilde{15} = \langle(8,10,20,35); 0.5,0.2,0.4\rangle$,
$\widetilde{16} = \langle(5,15,25,30); 0.7,0.5,0.6\rangle$,
$\widetilde{17} = \langle(10,15,20,25); 0.2,0.4,0.6\rangle$,
$\widetilde{19} = \langle(15,17,23,25); 0.9,0.7,0.8\rangle$,
$\widetilde{20} = \langle(10,12,27,30); 0.2,0.3,0.5\rangle$.

Step 1: To obtain crisp values of each single valued trapezoidal neutrosophic number, we should calculate score function as follows:

$S(\tilde{1}) = \left(\frac{1}{16}\right)[0 + 2 + 4 + 5] \times [0.8 + (1 - 0.6) + (1 - 0.4)] = 1.24$

$S(\tilde{2}) = \left(\frac{1}{16}\right)[1 + 3 + 5 + 6] \times [0.2 + (1 - 0.3) + (1 - 0.5)] = 1.31$

$S(\tilde{3}) = \left(\frac{1}{16}\right)[1 + 2 + 5 + 6] \times [0.2 + (1 - 0.5) + (1 - 0.6)] = 0.96$

$S(\tilde{4}) = \left(\frac{1}{16}\right)[1 + 2 + 5 + 7] \times [0.5 + (1 - 0.4) + (1 - 0.9)] = 1.12$

$S(\tilde{5}) = \left(\frac{1}{16}\right)[2 + 4 + 7 + 10] \times [0.8 + (1 - 0.2) + (1 - 0.4)] = 3.16$

$S(\tilde{6}) = \left(\frac{1}{16}\right)[3 + 7 + 9 + 12] \times [0.7 + (1 - 0.2) + (1 - 0.5)] = 3.87$

$S(\tilde{7}) = \left(\frac{1}{16}\right)[5 + 8 + 9 + 13] \times [0.4 + (1 - 0.6) + (1 - 0.8)] = 2.19$

$S(\tilde{8}) = \left(\frac{1}{16}\right)[1 + 6 + 10 + 13] \times [0.9 + (1 - 0.1) + (1 - 0.3)] = 4.68$

$S(\tilde{9}) = \left(\frac{1}{16}\right)[6 + 8 + 10 + 15] \times [0.6 + (1 - 0.4) + (1 - 0.7)] = 3.66$

$S(\widetilde{10}) = \left(\frac{1}{16}\right)[1 + 6 + 11 + 15] \times [0.7 + (1 - 0.6) + (1 - 0.3)] = 3.71$

$S(\widetilde{11}) = \left(\frac{1}{16}\right)[5 + 8 + 15 + 20] \times [0.8 + (1 - 0.2) + (1 - 0.5)] = 6.3$

$S(\widetilde{12}) = \left(\frac{1}{16}\right)[4 + 8 + 17 + 25] \times [0.3 + (1 - 0.6) + (1 - 0.4)] = 4.39$

$S(\widetilde{14}) = \left(\frac{1}{16}\right)[7 + 10 + 19 + 30] \times [0.8 + (1 - 0.4) + (1 - 0.7)] = 7.01$

$S(\widetilde{15}) = \left(\frac{1}{16}\right)[8 + 10 + 20 + 35] \times [0.5 + (1 - 0.2) + (1 - 0.4)] = 8.67$

$S(\widetilde{16}) = \left(\frac{1}{16}\right)[5 + 15 + 25 + 30] \times [0.7 + (1 - 0.5) + (1 - 0.6)] = 7.5$

$S(\widetilde{17}) = \left(\frac{1}{16}\right)[10 + 15 + 20 + 25] \times [0.2 + (1 - 0.4) + (1 - 0.6)] = 5.25$

$S(\widetilde{19}) = \left(\frac{1}{16}\right)[15 + 17 + 23 + 25] \times [0.9 + (1 - 0.7) + (1 - 0.8)] = 7$

$S(\widetilde{20}) = \left(\frac{1}{16}\right)[10 + 12 + 27 + 30] \times [0.2 + (1 - 0.3) + (1 - 0.5)] = 6.91$

Step 2: By putting score functions values as crisp values of each time estimate, we can calculate the expected time and variance of each activity as we illustrated with equations in the previous section. The expected time of each activity has been calculated and presented in table 2.

TABLE 2. THE EXPECTED TIME OF EACH ACTIVITY IN THE PROJECT.

Activity	Immediate Predecessors	Expected Time(days)
A	------	1
B	------	2
C	A	3
D	A	4
E	B	4
F	C,D	8
G	D,E	6
H	F,G	5

Step 3: Draw the network diagram by using Microsoft Project 2010.

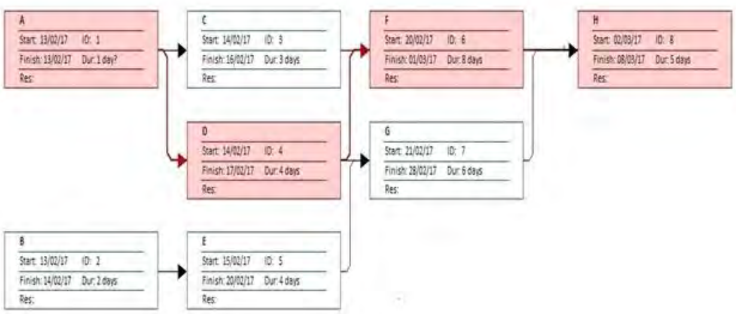

Fig. 1. Network of activities with critical path

From figure 1, we find that the critical path is A-D-F-H and is denoted by red line. The expected project completion time = $t_A + t_D + t_F + t_H$ = 18 days.

V. CONCLUSION

Neutrosophic set is a generalization of classical set, fuzzy set and intuitionistic fuzzy set because it not only considers the truth-membership and falsity- membership but also an indeterminacy membership which is very obvious in real life situations. In this paper, we have considered the three time estimates of PERT as a single valued trapezoidal neutrosophic numbers and we used score function to obtain crisp values of three time estimates. In future, the research will be extended to deal with different project management techniques.

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An Approach for Assessing The Reliability of Data Contained in A Single Valued Neutrosophic Number

Florentin Smarandache, Dragisa Stanujkic, Darjan Karabasevic

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Abstract

Neutrosophic sets, as the generalization of many types of sets, including classical and fuzzy sets, are becoming more and more important for solving a number of complex decision-making problems. On the other hand, the reliability of the information used to solve a problem also has an impact on the selection of the most appropriate solution. Therefore, in this paper, an approach for assessing the reliability of information contained in single valued neutrosophic numbers has been proposed. The usability of the proposed approach is considered in the case of determining customer satisfaction of users of traditional Serbian restaurants in the city of Zajecar.

Keywords: *neutrosophy, SVNN, estimating data reliability*

Introduction

In order to provide methodology for solving complex decision-making problems, Zadeh (1965) introduced fuzzy set theory. Based on the fuzzy set theory, a number of extensions of this theory was proposed.

The membership function to the set, introduced in the fuzzy set theory, in the case of solving some complex decision-making problems has not been sufficient, or its determination was difficult. Therefore, some extensions of the fuzzy set theory are proposed.

For example, Atanassov (1986) proposed intuitionistic fuzzy sets by introducing non-membership function. After that, Atanassov and Gargov, (1989) proposed the moreefficient use of the intuitionistic fuzzy set theory by introducing more flexible approach for determining boundaries of membership function, or more precisely said, they introduced the usage of intervals for determining boundaries of membership function and so they made intuitionistic fuzzy setsmore flexible and practical for solving complex decision-making problems.

The lack of non-membership function, identified in fuzzy set theory, has been successfully solved in anintuitionistic set theory. However, the lack of a measure that would show a gap between membership and non-membership functions remains present in intuitionistic set theory, where it is determined as difference between membership functions.

Finally, Smarandache (1998) introduced the neutrosophic set as generalization the concepts of the classical sets, fuzzy sets and other fuzzy sets based theories, and so provide very flexible approach for dealing with membership, non-membership and indeterminacy functions.Smarandache(1998) and Wang*et al*. (2010) further introduced the single valued neutrosophic sets that are more suitable for solving many real-world decision-making problems.

However, various types of fuzzy numbers, including neutrosophic numbers, are becoming more and more complex compared to crisp numbers. It is certain that the mentioned types of numbers have their advantages. However, the use of such numbers can become rather complex in the case of data collection, especially when data are collected by interviewing respondents who are not pre-prepared for the use of such numbers.

In the past period, many researchers are dedicated to the use of neutrosophic numbers for solving a number of different problems, while problems related to data collection and assessment of their reliability are marginalized.

Therefore, the rest of the manuscript is organized as follows: in Section 2, the basic elements of neutrosophic sets are considered and in Section 3, a procedure for estimating data reliabilityis proposed. Section 4 presents a new innovative procedure for evaluating alternatives whereas in Section 5 its usability is demonstrated in numerical illustration. Finally, the conclusion are given.

Preliminaries

Definition 1. *Neutrosophic set*. Let Xbe the universe of discourse, with a generic element in X denoted by x. Then, the neutrosophic set A in X is as follows(Smarandache, 1999):

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \tag{1}$$

Where$T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to]^-0,1^+[$ and $^-0 \leq T_A(\text{x})+I_A(\text{x})+U_A(\text{x}) \leq 3^+$.

Definition 2.*Single valued neutrosophic*set.Let X be the universe of discourse. The Single Valued Neutrosophic Set(SVNS) A over X is an object having the form(Smarandache, 1998, Wang *et al.* 2010):

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \tag{2}$$

where$T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to [0,1]$ and $0 \leq T_A(\text{x})+I_A(\text{x})+U_A(\text{x}) \leq 3$.

Definition 3.*Single valued neutrosophic number*.For an SVNS A in X, the triple $< t_A, i_A, f_A >$ is called the single valued neutrosophic number (SVNN)(Smarandache, 1999).

Definition 4. *Basic operations onSVNNs*. Let $x_1 = < t_1, i_1, f_1 >$ and $x_2 = < t_2, i_2, f_2 >$ be two SVNNs and $\lambda > 0$; then, the basic operations are defined as follows:

$$x_1 + x_2 =< t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 >. \quad (3)$$
$$x_1 \cdot x_2 =< t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 >. \quad (4)$$

$$\lambda x_1 =< 1-(1-t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda} >. \quad (5)$$

$$x_1^{\lambda} =< t_1^{\lambda}, i_1^{\lambda}, 1-(1-f_1)^{\lambda} >. \quad (6)$$

Definition 6.*Single valued neutrosophic average*.Let $a_i =< t_i, i_1, f_i >$ be a collection of SVNNs and $W = (w_1, w_2, ..., w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Average (SVNWA)operatorof$a_j$is as follows(Smarandache, 2014):

$$SVNWA(a_1, a_2, ..., A_n) = \sum_{j=1}^{n} w_j a_j = \left(1 - \prod_{j=1}^{n} (1-t_j)^{w_j}, \prod_{j=1}^{n} i_j^{w_j}, \prod_{j=1}^{n} f_j^{w_j} \right), \quad (7)$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Procedure for estimating data reliability

In this section, an approach for estimating reliability of SVNN, as well as thecollection of SVNNs, is introduced.

Definition7. *Reliability of information contained in a SVNN*. Let $x =< t, i, f >$ be a SVNN. Then, the reliability of information contained in SVNN x is as follows:

$$r_{(x)} = \frac{t-f}{1+i^{1/n}} \quad (8)$$

where: $r_{(x)} \in [-1,1]$ and $r_{(x)} \to 0$ indicates the lack of the reliability of the information contained in$x$.

Example. Let x=<0.80, 0.10, 0.30> be a SVNN. Then$r_{(x)}$is 0.45 for$n$=1. For higher values of n, such as: 2, 3, 4, 5 and 10, $r_{(x)}$ is as follows: 0.38, 0.34, 0.31 and 0.28.
It is evident that by increasing the value of parameter n the value of r decreases, which could be very successful for analyzing different decision-making scenarios.

Definition 8. *Reliability of information contained in a collection of SVNNs*.Let $x_i =< t_i, i_1, f_i >$ be a collection of SVNNs. Then, the average reliability of collection $x_i$is as follows:

$$ra_{(x_i)} = \frac{1}{L} \sum_{l=1}^{L} r_{(x_i)} \quad (9)$$

where$L$ denotes the number of elements of the collection.

Example. Let $x_i$be a collection of SVNNs. The collection x_i and the values of their $r$and ra functions are accounted for in Table 1,

Table 1: *The reliability and overall reliability of the collection of SVNNs*

		r_i
x_1	<0.80, 0.10, 0.30>	0.45
x_2	<0.70, 0.10, 0.20>	0.45
x_3	<0.70, 0.10, 0.10>	0.55
	ra	0.48

Procedure for assessing the reliability of the information contained in an evaluation matrix can be precisely described by the following steps:

Step 1. Determine the reliability of data contained in each element of the evaluation matrix, using Eq. (8).

Step 2. Determine the reliability of data contained in the evaluation matrix, or its rows or columns, using Eq. (9).

A new innovative procedure for evaluating alternatives

A group multiple criteria decision-making procedure usually begins with a team of experts and / or decision-makers who will perform the evaluation. At the very beginning they define goal, or goals, that should be reached by the evaluation, define a set of evaluation criteria and identify available alternatives. By this time, they also determine the significances, often called weights, of criteria.

The remaining part of the evaluation procedure can be precisely described by the following steps:

- Evaluate alternatives in relation to the select of evaluation criteria. In this step each expert and / or decision-maker forms its individual evaluation matrix, which elements are SVNNs.
- Check the data reliability. In this step, based on the procedure for estimating data reliability, reliability of each expert and/or decision-maker is calculated. If the reliability of any evaluation matrix is under minimally acceptable level, it should be reconsidered or omitted from further calculations.
- Construct a group evaluation matrix. The group evaluation matrix is formed on the basis of evaluation matrix formed by using Eq. (8).
- Calculate the overall rating for each alternative by using Eq (9).
- Determine the ideal point.
- Determine the Hamming distance of each alternative to the ideal point.
- Rank the alternatives according to their distances to the ideal point and select the most appropriate ones. In this approach, the alternative with least distance to the ideal point is the most appropriate one.

Numerical Illustration

In this section, the usability of the proposed approach is demonstrated on the basis of a numerical illustration adopted Stanujkic et al. (2016). In this numerical illustration, three traditional restaurants were evaluated based on the following criteria:

- C_1: the interior of the building and the friendly atmosphere,
- C_2: the helpfulness and friendliness of the staff,
- C_3: the variety of traditional food and drinks,

- C_4: the quality and the taste of the food and drinks, including the manner of serving them, and
- C_5: the appropriate price for the quality of the services provided.

In order to explain the proposed approach, three completed surveys have been selected. The ratings of the evaluated alternativesobtained on the basis of the three surveys are given in Tables 2 to 4.

Table 2: *The ratings obtained from the first of three respondents expressed in the form of SVNN*

	C_1	C_2	C_3	C_4	C_5
A_1	<0.80, 0.10, 0.30>	<0.70, 0.20, 0.20>	<0.80, 0.10, 0.10>	<1.00, 0.01, 0.01>	<0.80, 0.10, 0.10>
A_2	<0.70, 0.10, 0.20>	<1.00, 0.10, 0.10>	<1.00, 0.20, 0.10>	<1.00, 0.01, 0.01>	<0.80, 0.10, 0.10>
A_3	<0.70, 0.10, 0.10>	<1.00, 0.10, 0.10>	<0.70, 0.10, 0.10>	<0.90, 0.20, 0.01>	<0.90, 0.10, 0.10>

Source: *Authors' calculation*

Table 3: *The ratings obtained from the second of three respondents expressed in the form of SVNN*

	C_1	C_2	C_3	C_4	C_5
A_1	<0.80, 0.10, 0.40>	<0.90, 0.15, 0.30>	<0.90, 0.20, 0.20>	<0.85, 0.10, 0.25>	<1.00, 0.10, 0.20>
A_2	<0.90, 0.15, 0.30>	<0.90, 0.15, 0.20>	<1.00, 0.30, 0.20>	<0.70, 0.20, 0.10>	<0.80, 0.20, 0.30>
A_3	<0.60, 0.15, 0.30>	<0.55, 0.20, 0.30>	<0.55, 0.30, 0.30>	<0.60, 0.30, 0.20>	<0.70, 0.20, 0.30>

Source: *Authors' calculation*

Table 4: *The ratings obtained from the third of three respondents expressed in the form of SVNN*

	C_1	C_2	C_3	C_4	C_5
A_1	<1.00, 0.10, 0.10>	<0.90, 0.15, 0.20>	<1.00, 0.20, 0.10>	<0.80, 0.10, 0.10>	<0.90, 0.10, 0.20>
A_2	<0.80, 0.15, 0.30>	<0.90, 0.15, 0.20>	<1.00, 0.20, 0.20>	<0.70, 0.20, 0.10>	<0.80, 0.20, 0.30>
A_3	<0.60, 0.15, 0.30>	<0.55, 0.20, 0.30>	<0.55, 0.30, 0.30>	<0.60, 0.30, 0.20>	<0.70, 0.20, 0.30>

Source: *Authors' calculation*

The reliabilityof the data obtained from the first respondent are accounted for in Table 5.

Table 5: *The reliability data obtained from the first of three respondents*

	C_1	C_2	C_3	C_4	C_5	*Reliability*
A_1	0.45	0.42	0.98	0.64	0.62	0.45
A_2	0.45	0.82	0.98	0.64	0.73	0.45
A_3	0.55	0.82	0.74	0.73	0.68	0.55
Reliability	0.48	0.68	0.90	0.67	0.00	0.48
					Overall reliability	0.68

Source: *Authors' calculation*

The reliability of the data obtained from three respondents is accounted for in Table 6.

Table 6:*The reliability data obtained from three respondents*

	Reliability
E_1	0.68
E_2	0.45
E_3	0.49
Average reliability	0.54

Source: *Authors' calculation*

For the presented evaluation it was decided that the achieved level of data reliability is satisfactory, which is why it was continued with the evaluation. Contrary, in cases when the achieved level of data reliability is not satisfactory surveys with lower values of data reliability must be done again or omitted from the further calculations.

In the next step the group decision matrix is formed by using Eq. (7). The group decision matrix is shown in Table 7.

Table 7: *The group ratings of alternatives*

	C_1	C_2	C_3	C_4	C_5
w_j	0.17	0.17	0.19	0.23	0.24
A_1	<1.00, 0.10, 0.28>	<0.86, 0.17, 0.23>	<1.00, 0.17, 0.13>	<1.00, 0.07, 0.13>	<1.00, 0.10, 0.17>
A_2	<0.82, 0.13, 0.27>	<1.00, 0.13, 0.17>	<1.00, 0.23, 0.17>	<1.00, 0.14, 0.07>	<0.80, 0.17, 0.24>
A_3	<0.64, 0.13, 0.24>	<1.00, 0.17, 0.24>	<0.61, 0.24, 0.24>	<0.75, 0.27, 0.14>	<0.79, 0.17, 0.24>

Source: *Authors' calculation*

Table 6 also shows the weights of the criteria. The overall ratings of the alternatives calculated by using Eq. (7) are shown in Table 8.

Table 8: *The ranking order of alternatives*

	Overall ratings	*Distance*	*Rank*
A_1	<1.00, 0.11, 0.17>	0.00	1
A_2	<1.00, 0.16, 0.16>	0.02	2
A_3	<1.00, 0.19, 0.21>	0.04	3
Ideal point	<1.00, 0.11, 0.16>		

Source: *Authors' calculation*

Table 8 also shows the ideal point, distances of alternatives to the ideal point, as well as the ranking order of alternatives.

As it can be seen from Table 8, the best placed alternative is alternative denoted as A_1.

Conclusion

In this article, an innovative multiple criterion decision making approach for evaluating alternatives based on the use of single valued neutrosophic numbers is presented. The main advantage of this approach is the use of a procedure for estimating the reliability of the collected data, which can be especially useful when the data is collected by the survey.

Using the proposed procedure for estimating data reliability, respondents who inadequately filled out surveys can be identified and further they can be asked to fill out surveys again or their surveys can be omitted from further calculations.

As the second significant characteristic of the proposed approach is the use of Hamming distance to the ideal point for ranking alternatives.

The usability and efficiency of the proposed approach is successfully demonstrated on an example of evaluating customer satisfaction in traditional Serbian restaurants in city of Zajecar.

Finally, developing the similar procedure for estimating data reliability of bipolarneutrosophic number can be identified as a continuation of this research.

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An Approach to FDI Location Choice Based on The Use of Single Valued Neutrosophic Numbers: Case of Non-EU Balkan Countries

Aleksandra Fedajev, Dragisa Stanujkic, Florentin Smarandache

Aleksandra Fedajev, Dragisa Stanujkic, Florentin Smarandache (2018). An Approach to FDI Location Choice Based on The Use of Single Valued Neutrosophic Numbers: Case of Non-EU Balkan Countries. International Scientific & Professional Conference MEFkon, Belgrade, December 6th 2018, 198-207

Abstract

The beginning of transition in former command economies was characterised by deep recession and numerous structural imbalances. Some of transition economies have overcome these problems relatively fast and some of them are still struggling to find their way to growth and development. One of the key drivers of economic expansion in advanced transition economieswereFDI. Foreign investors had different motives for investment. In accordance with them and business environment characteristics in these countries they chose the location of their investment. Having in mind that FDI are still very important generator of economic growth, the growing number of authors is dealing with the development of most efficient decision making method for FDI location choice. This paper presents a single valued neutrosophic numbers approach for selecting the most suitable country for investment. The effectiveness and usability of the proposed approach were demonstrated in the case of non-EU Balkan countries, bearing in mind that these countries are still lagging behind CEE economies in terms of growth and development.

Introduction

The internationalization of businesses is one of the most important global trends in contemporary business conditions and one of the biggest challenges for MNEs Aleksandruk and Forte (2016).Investing money in new projects, as well as selecting a country for new investment, are real problems that deserve great attention, especially in the case of long-term investments, as in a case of FDI.Because of that, special attention is devoted to these problems in scientific and professional literature. As a proof of that, from numerous published articles, some of the most cited articles are listed: Yiu *et al.* (2007), Beim and Levesque (2006), Moen *et al.* (2004), Manigart, *et al.* (2002), Chung and Enderwick (2001), Wells *et al.* (1990).

Motives of foreign investors are different, but most of authors categorized them in these groups: market-seeking, efficiency-seeking, resource-seeking, strategic asset seeking Aleksandruk and Forte (2016), Maza and Villaverde (2015), Estrin and Uvalic (2014), Altomonte and Guagliano (2003), Tampakoudis et al. (2017). Having in mind that mentioned groups of investors have different investment aims, they also have different preferences about characteristics of business environmentin CEE countries. The mostly cited determinants of

FDI in transition countries are: market size and attractiveness, institutional environment, political risk, transaction costs, bilateral exports, transition progress, financial market development, infrastructure, macroeconomic stability, administrative procedures, tax system, labor market and regulations, knowledge resources, natural resources, trade opennessDauti (2015), Obradović et al. (2012), Wilson and Baack (2012), Hengel, E. (2010), Tampakoudis et al. (2017), Wach and Wojciechowski (2016) .Bearing in mind all these criteria, it can be concluded that evaluation of any investment project location involves at least three mutually opposite criteria. So, problem of selecting the most appropriate investment projects can be expressed as follows: How to achieve as much as possible revenue in as is possible shorter period of time with as is possible smaller investments? Of course, the risk of investment should not be ignored here. Therefore, any investment project can be considered as a multiple criteria decision-making problem, and as some evidence for such an approach, the following: Altuntas and Dereli (2015), Kiliç and Kaya (2015), Popović *et al*. (2012), Ginevičius and Zubrecovas (2009), Dimova *et al*. (2006), Tzeng and Teng (1993), and so on.

In order to enable solving of complex problems of decision-making problems, Zadeh (1965) introduced fuzzy set theory. Based on the fuzzy set theory, a number of authors lather proposed some its extensions as follows: intuitionistic (Atanassov, 1986), interval-valued (Turksen 1986) and interval-valued intuitionistic (Atanassov and Gergov, 1989) fuzzy set theory. Further, Smarandache (1998) introduced the neutrosophic set as general framework generalizing the concepts of the classical, and all above mentioned fuzzy theories.In addition to the membership function, or the so-called truth-membership $T_A(x)$, proposed in fuzzy sets, Atanassov (986) introduced the non-membership function, or the so-called falsity-membership $F_A(x)$, which expresses non-membership to a set, thus creating the basis for the solving of a much larger number of decision-making problems. Finally, Smarandache (1999) introduced independent indeterminacy-membership $I_A(x)$, thus making the neutrosopic sets most suitable for solving some complex decision-making problems. In the next step, Smarandache (1998) and Wang*et al*. (2010) further introduced the single valued neutrosophic sets that are more suitable for solving many real-world decision-making problems.

Therefore, the rest of the manuscript is organized as follows: in Section 2, the basic elements of neutrosophic sets are considered and in Section 3, a procedure for evaluating investment projects is proposed. In Section 4, its usability is demonstrated. Finally, the conclusion is given.

Preliminaries

Definition 1. *Neutrosophic set*. Let *X*be the universe of discourse, with a generic element in *X* denoted by *x*. Then, the neutrosophic set *A* in *X* is as follows(Smarandache, 1999):

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \tag{1}$$

where$T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to]^-0, 1^+[$ and $^-0 \leq T_A(x)+I_A(x)+U_A(x) \leq 3^+$.

Definition 2. *Single valued neutrosophic set*.Let *X* be the universe of discourse. The Single Valued Neutrosophic Set(SVNS) *A* over *X* is an object having the form (Smarandache, 1998, Wang *et al*. 2010):

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \tag{2}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \rightarrow [0,1]$ and $0 \leq T_A(x)+I_A(x)+U_A(x) \leq 3$.

Definition 3. *Single valued neutrosophic number*. For an SVNS A in X, the triple $< t_A, i_A, f_A >$ is called the single valued neutrosophic number (SVNN) (Smarandache, 1999).

Definition 4. *SVNNs*. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs and $\lambda > 0$; then, the basic operations are defined as follows:

$$x_1 + x_2 =< t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 >. \quad (3)$$

$$x_1 \cdot x_2 =< t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 >. \quad (4)$$

$$\lambda x_1 =< 1-(1-t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda} >. \quad (5)$$

$$x_1^{\lambda} =< t_1^{\lambda}, i_1^{\lambda}, 1-(1-f_1)^{\lambda} >. \quad (6)$$

Definition 5. *Score function*. Let $x =< t, i, f >$ be a SVNN, then the score function $s_{(x)}$ of x is as follows (Smarandache, 1998):

$$s_{(x)} = (1+t-2i-f)/2, \quad (7)$$

where $s_x \in [-1, 1]$.

Definition 6. *Single valued neutrosophic average*. Let $a_i =< t_i, i_i, f_i >$ be a collection of SVNNs and $W = (w_1, w_2, \ldots, w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Average (SVNWA) operator of a_j is as follows (Smarandache, 2014):

$$SVNWA(a_1, a_2, \ldots, A_n) = \sum_{j=1}^{n} w_j a_j = \left(1 - \prod_{j=1}^{n} (1-t_j)^{w_j}, \prod_{j=1}^{n} i_j^{w_j}, \prod_{j=1}^{n} f_j^{w_j}\right), \quad (8)$$

where: w_j is the element j of the weighting vector, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Framework for evaluating the strategies

Many complex decision-making problems require the participation of more experts and/or decision-makers in selection of the most appropriate alternative. Therefore, in this section, a framework for the evaluation of countries for new investment, based on group decision-making and the SVNNs method, is considered.

The selection process involving m alternatives that are evaluated on the basis of n criteria by K decision maker can be presented in detail using the following steps:

Step 1. Form a team of experts and / or decision-makers who will evaluate potential countries.

Step 2. Define the objectives that need to be achieved by the investment objectives. In this steep, the team of experts and / or decision-makers define the objectives to be achieved.

Step 3. Identify the possible countries. In this step, the team of experts and / or decision-makers identify countries - potential candidates for investment.

Step 4. Form a set of evaluation criteria. In this step, the team of experts and / or decision-makers selects the set of criteria on which basis the evaluation will be carried out.

Step 5. Determine the significance of the criteria. In the literature, many techniques for determining the weights of criteria are proposed, such as pair-wise comparisons (Saaty, 1977), SWARA (Kersulieneet *al*. 2010), Best-worst method (Rezaei, 2015), PIPRECIA (Stanujkic *et al*., 2017).

In this approach, each expert and / or decision-maker evaluates the criteria by applying one of the above-mentioned techniques, after which the group weights are determined as follows:

$$w_j = \frac{1}{K} \sum_{k=1}^{K} w_j^k , \qquad (11)$$

where w_j^k denotes the weight of criterion *j* obtained from expert / decision-maker *k*.

Step 6. Evaluate the strategies in relation to the set of criteria. In this step, each expert forms his / her decision matrix, whose elements are SVNNs.

Step 7. Evaluate alternatives. The selection procedure can be described as follows:

- Form a group decision matrix, based on individual decision-making matrices formed by experts, using Eq. (8).
- Calculate the overall performance of each alternative, based on the group decision matrix, also using Eq. (8)
- Determine the value of the Score function for each alternative using Eq, (7).
- Rank the alternatives in relation to the value of the Score function, where the alternative with the highest value of the Score function is the most appropriate alternative.

Numerical Illustration

In order to present the usability of the SVNNs for solving different decision-making problems in the economics, a numerical illustration is presented below.In this numerical illustration, five Balkan countries, which are not members of the European Union, have been evaluated from the point of view of potential investors with different motives for investment.

At the very beginning of the evaluation, a team of three experts was formed. Based on the FDI determinants shown in Table 1, as well as their experiences and motives for investment, the experts performed out the evaluation the alternatives in relation to the selected set of evaluation criteria.

Table 1: *FDI determinants of the business environment*

		FDI Determinants				
		Market Size (GDP)	Average Salary	Rent	Tax Rate	Property Protection
		€	€		%	
A_1	Albania	4538	390	2	37.3	54.0
A_2	Bosnia	5181	440	1.1	23.7	41.2
A_3	Macedonia	5443	377	1.5	13.0	67.0
A_4	Montenegro	7670	512	0.8	22.1	58.0
A_5	Serbia	5900	459	1.5	39.7	50.3

Source: *Authors' calculation*

In the performed evaluation, the first of the three experts carried out evaluation from the market–seeking investor point of view, while the second and third experts were made evaluations from the point of view of resource-seeking and efficiency-seeking, respectively.

The performances of the alternative in relation to the evaluation criteria, as well as the weight of the criteria, obtained from a team of three experts are shown in Tables 2 to 4, whereby they evaluated the alternatives using a five-pointLikert scale, after which these values, for the purpose of further calculation, are transformed to the corresponding values in the interval [0, 1].

Table 2: *The ratings and weights obtained from the first of three experts*

	C_1			C_2			C_3			C_4			C_5		
w_j	0.25			0.19			0.13			0.21			0.21		
	t	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*
A_1	2.00	0.00	0.00	3.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.10	3.00	0.00	0.10
A_2	3.00	0.00	0.00	4.00	0.00	0.00	1.00	0.00	0.00	2.00	0.00	0.00	2.00	0.00	0.00
A_3	3.00	0.00	1.00	3.00	0.00	0.00	1.00	0.00	0.00	4.00	0.00	0.00	5.00	0.00	0.00
A_4	5.00	0.00	2.00	4.00	0.00	1.00	1.00	0.00	0.00	2.00	0.00	0.10	4.00	0.00	0.00
A_5	4.00	0.00	0.00	4.00	0.00	2.00	1.00	0.00	0.00	1.00	0.00	0.00	3.00	0.00	0.00

Source: *Authors' calculation*

The ratings of alternatives obtained from the first of three experts, expressed using the SVNN, are shown in Table 5, while the overall ratings, the values of score function, as well as the ranking order of alternatives are shown in table 6.

Table 3: *The ratings and weights obtained from the second of three experts*

	C_1			C_2			C_3			C_4			C_5		
w_j	0.08			0.13			0.38			0.26			0.15		
	t	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*
A_1	0.00	0.00	0.00	5.00	0.00	0.00	2.00	0.00	0.00	2.00	0.00	1.00	3.00	0.00	2.00
A_2	0.00	0.00	0.00	4.00	0.00	0.00	4.00	0.00	2.00	3.00	0.00	0.00	1.00	0.00	0.00
A_3	0.00	0.00	0.00	5.00	0.00	0.00	3.00	0.00	0.00	4.00	0.00	0.00	4.00	0.00	0.00
A_4	0.00	0.00	0.00	3.00	0.00	1.00	5.00	0.00	0.00	3.00	0.00	0.00	3.00	0.00	0.00
A_5	0.00	0.00	0.00	4.00	0.00	2.00	3.00	0.00	2.00	2.00	0.00	1.00	2.00	0.00	0.00

Source: *Authors' calculation*

Table 4: *The ratings and weights obtained from the third of three experts*

	C_1			C_2			C_3			C_4			C_5		
w_j	0.09			0.43			0.08			0.29			0.12		
	t	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*	*t*	*i*	*f*
A_1	1.00	0.00	0.00	5.00	0.00	0.00	1.00	0.00	0.00	3.00	0.00	0.00	3.00	0.00	1.00
A_2	2.00	0.00	0.00	4.00	2.00	2.00	3.00	0.00	0.00	4.00	0.00	0.00	1.00	0.00	1.00
A_3	2.00	0.00	0.00	5.00	0.00	0.00	2.00	0.00	0.00	5.00	0.00	0.00	5.00	0.00	0.00
A_4	3.00	0.00	0.00	2.00	0.00	0.00	4.00	0.00	1.00	4.00	0.00	1.00	4.00	0.00	1.00
A_5	2.00	0.00	0.00	3.00	0.00	2.00	2.00	0.00	0.50	2.00	0.00	0.00	2.00	0.00	0.00

Source: *Authors' calculation*

Table 5: *The ratings obtained from the first of three experts expressed in the form of SVNN*

	C_1	C_2	C_3	C_4	C_5
	0.25	0.19	0.13	0.21	0.21
A_1	<0.4,0.0,0.0>	<0.6,0.0,0.0>	<0.2,0.0,0.0>	<0.2,0.0,0.0>	<0.6,0.0,0.0>
A_2	<0.6,0.0,0.0>	<0.8,0.0,0.0>	<0.2,0.0,0.0>	<0.4,0.0,0.0>	<0.4,0.0,0.0>
A_3	<0.6,0.0,0.2>	<0.6,0.0,0.0>	<0.2,0.0,0.0>	<0.8,0.0,0.0>	<1.0,0.0,0.0>
A_4	<1.0,0.0,0.4>	<0.8,0.0,0.2>	<0.2,0.0,0.0>	<0.4,0.0,0.0>	<0.8,0.0,0.0>
A_5	<0.8,0.0,0.0>	<0.8,0.0,0.4>	<0.2,0.0,0.0>	<0.2,0.0,0.0>	<0.6,0.0,0.0>

Source: *Authors' calculation*

Table 6: *The overall ratings, the values of score function, and the ranking order of alternatives obtained on the basis of responses of the first of three experts*

		Overall	S_i	*Rank*
A_1	Albania	<0.4,0.0,0.0>	0.718	5
A_2	Bosnia	<0.5,0.0,0.0>	0.772	4
A_3	Macedonia	<1.0,0.0,0.0>	0.999	1
A_4	Montenegro	<1.0,0.0,0.0>	0.999	1
A_5	Serbia	<0.6,0.0,0.0>	0.813	3

Source: *Authors' calculation*

As it can bee seen from Table 6, the most appropriate alternatives for market-seeking investors are alternatives denotes as A_3 and A_4.

The values of score function, as well as appropriate ranking order of alternatives obtained from three experts, are accounted for in Table 7.

Table 7: *The ranking orders obtained from three experts*

		E_1		E_2		E_3	
		S_i	*Rank*	S_i	*Rank*	S_i	*Rank*
A_1	Albania	0.718	5	0.999	1	0.999	1
A_2	Bosnia	0.772	4	0.816	4	0.862	3
A_3	Macedonia	0.999	1	0.999	1	0.999	1
A_4	Montenegro	0.999	1	0.999	1	0.830	4
A_5	Serbia	0.813	3	0.749	5	0.747	5

Source: *Authors' calculation*

As previously mentioned, alternatives denoted as A_3 and A_4 are the most appropriate for market-seeking investors, while alternatives denoted as A_1, A_3 and A_4 are more suitable for resources-seeking investors. Finally, alternatives denoted as A_1 andA_3 and most suitable for efficiency-seeking investors.

The group ratings of considered alternatives, obtained by using Eq. (10), are encountered for in Table 8, whereby the experts, as well as potential investors, had the following weights:w_1=0.45, w_2=0.25 and w_3=0.30.

Table 8: *The overall ratings and weights obtained from three experts*

	C_1	C_2	C_3	C_4	C_5
	0.14	0.25	0.20	0.22	0.16
A_1	<0.2,0.0,0.0>	<1.0,0.0,0.0>	<0.2,0.0,0.0>	<0.3,0.0,0.0>	<0.6,0.0,0.0>
A_2	<0.4,0.0,0.0>	<0.8,0.0,0.0>	<0.5,0.0,0.0>	<0.6,0.0,0.0>	<0.2,0.0,0.0>
A_3	<0.4,0.0,0.0>	<1.0,0.0,0.0>	<0.3,0.0,0.0>	<1.0,0.0,0.0>	<1.0,0.0,0.0>
A_4	<1.0,0.0,0.0>	<0.6,0.0,0.0>	<1.0,0.0,0.0>	<0.6,0.0,0.0>	<0.7,0.0,0.0>
A_5	<0.5,0.0,0.0>	<0.7,0.0,0.4>	<0.3,0.0,0.0>	<0.3,0.0,0.0>	<0.5,0.0,0.0>

Source: *Authors' calculation*

The final ranking order of considered alternatives is accounting for in Table 9.

Table 9: *The final ranking order of alternatives*

		Overall	S_i	*Rank*
A_1	Albania	<1.0,0.0,0.0>	0.999	1
A_2	Bosnia	<0.5,0.0,0.0>	0.796	4
A_3	Macedonia	<1.0,0.0,0.0>	0.999	1
A_4	Montenegro	<1.0,0.0,0.0>	0.999	1
A_5	Serbia	<0.5,0.0,0.0>	0.766	5

Source: *Authors' calculation*

As it can be from Table 9, the most suitable business environment for investment is the three Balkan countries: Albania, Macedonia and Montenegro, with Bosnia and Herzegovina ranked at fourth position and Serbia ranked in the fifth position.

Conclusion

In this article, an easy-to-use multiple criteria decision-making approach for evaluating potential investment countries is considered. The proposed approach is based on the use of single valued neutrosophic numbers, which should provide easier expression of the preferences, doubt and uncertainty of the information on which basis the evaluation should be carried out.

The considered example of the investment country selection is characterized by a low level of doubt and uncertainty, that is, it is a rather well-structured investment decision-making problem, and it is chosen with the aim of easier presenting usability and efficiency of the proposed approach. Certainly, the mentioned approach can be also used for solving similar problems with greater imprecision and unreliability of the available information, in which case a more complex ranking procedure should be used.

Finally, the ranking results obtained in the presented evaluation indicate that Serbia is in unfavorable position in comparison to other considered countries and that something should be undertaken to improve, or at least mitigate some negative characteristic of the existing business environment in Serbia, first of all to reform tax system and to improve and enforce implementation of regulation in the area of property protection.However, it should be mentioned that some characteristics of business environment are unfavorable for foreign investors, but favorable for wellbeing of citizens and economy as a whole, and that contributed to such position of Serbia in final rankings. Namely, investors prefer to pay low wages in order to lower their labor costs, but lowering wages will lead to lowering of living standard in the country. In addition, high rents on natural resources is also something that is not in favor of foreign investors, but it prevents the exploitation of natural resources for the needs of foreign companies and leave it for future generations in country. At the very end, it should be said that in this case small number of FDI determinants are taken into account and rankings will be certainly somewhat different if more of them are considered, so in the future researches authors will present more characteristics of business environment to foreign investors in order to let them know more about observed countries, as a potential location for their investments.

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Application of Neutrosophic Soft Sets to K-Algebras

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Abstract: Neutrosophic sets and soft sets are two different mathematical tools for representing vagueness and uncertainty. We apply these models in combination to study vagueness and uncertainty in K-algebras. We introduce the notion of single-valued neutrosophic soft (SNS) K-algebras and investigate some of their properties. We establish the notion of $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras and describe some of their related properties. We also illustrate the concepts with numerical examples.

Keywords: K-algebras; single-valued neutrosophic soft K-algebras; $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras

1. Introduction

The notion of a K-algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram [1] in 2003 and published in 2005. A K-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G, which is attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is, in general, non-commutative and non-associative with a right identity e, if $(G, \cdot, e)$ is non-commutative. For a given group G, the K-algebra is proper if G is not an elementary abelian two-group. Thus, a K-algebra is abelian, and being non-abelian purely depends on the base group G. In 2004, Dar and Akram [2] further renamed a K-algebra on a group G as a $K(G)$-algebra due to its structural basis G. The $K(G)$-algebras have been characterized by their left and right mappings in [2] when the group is abelian. The K-algebras have also been characterized by their left and right mappings in [3] when the group is non-abelian. In 2007, Dar and Akram [4] also studied K-homomorphisms of K-algebras.

Logic is an essential tool for giving applications in mathematics and computer science, and it is also a technique for laying a foundation. Non-classical logic takes advantage of the classical logic to handle information with various facts of uncertainty, including the fuzziness and randomness. In particular, non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain in formation. Among all kinds of uncertainties, the incomparability is the most important one that is frequently encountered in our daily lives. Fuzzy set theory, a generalization of classical set theory introduced by Zadeh [5], has drawn the attention of many researchers who have extended the fuzzy sets to intuitionistic fuzzy sets [6], interval-valued intuitionistic fuzzy sets [6], and so on, which are also applied to some decision-making process. On the other hand, Molodtsov [7] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields, including game theory, operations research, Riemann integration and Perron integration. In 1998, Smarandache [8] proposed the idea of neutrosophic sets. He mingled tricomponent logic, non-standard analysis and philosophy. It is a branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. For convenient and advantageous usage of neutrosophic sets in science and engineering, Wang et al. [9] proposed the single-valued neutrosophic sets, whose three independent components have values in the standard unit interval.

Garg and Nancy [10–12] developed a hybrid aggregation operator by using the two instances of the neutrosophic sets, single-valued neutrosophic sets and interval-valued neutrosophic sets. They introduced the concept of some new linguistic prioritized aggregation operators to deal with uncertainty in linguistic terms. To aggregate single-valued neutrosophic information, they developed some new operators to resolve the multi-criteria decision-making problems such as the Muirhead mean, the single-valued neutrosophic prioritized Muirhead mean and the single-valued neutrosophic prioritized Muirhead dual. Maji in [13] initiated the concept of neutrosophic soft sets. Certain notions of fuzzy K-algebras have been studied in [14–18]. Recently, Akram et al. [19,20] introduced single-valued neutrosophic K-algebras and single-valued neutrosophic topological K-algebras. In this paper, we introduce the notion of single-valued neutrosophic soft K-algebras and investigate some of their properties. We establish the notion of $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras and describe some of their related properties. We also illustrate the concepts with numerical examples. The remaining research article is arranged as follows. Section 2 consists of some basic definitions related to K-algebras and single-valued neutrosophic soft sets. In Section 3, the notion of single-valued neutrosophic soft K-algebras is proposed. To have a generalized viewpoint of single-valued neutrosophic soft K-algebras, Section 4 poses the concept of $(\in, \in \vee q)$-single-valued neutrosophic K-algebras with some examples. Finally, some concluding remarks are given in Section 5.

2. Preliminaries

Definition 1 ([1]). *Let $(G, \cdot, e)$ be a group such that each non-identity element is not of order two. Let a binary operation $\odot$ be introduced on the group G and defined by $\odot(s,t) = s \odot t = st^{-1}$ for all $s, t \in G$. If e is the identity of the group G, then:*

(1) e takes the shape of the right $\odot$-identity and not that of the left $\odot$-identity.
(2) Each non-identity element ($s \neq e$) is $\odot$-involutionary because $s \odot s = ss^{-1} = e$.
(3) G is $\odot$-nonassociative because $(s \odot t) \odot u = s \odot (u \odot t^{-1}) \neq s \odot (t \odot u)$ for all $s, y, u \in G$.
(4) G is $\odot$-noncommutative since $s \odot t \neq t \odot s$ for all $s, t \in G$.
(5) If G is an elementary Abelian two-group, then $s \odot t = s \cdot t$.

Definition 2 ([1]). *A K-algebra is a structure $(G, \cdot, \odot, e)$ on a group G, where $\odot : G \times G \to G$ is defined by $\odot(s,t) = s \odot t = s.t^{-1}$, if it satisfies the following axioms:*

(i) $((s \odot t) \odot (s \odot u)) = (s \odot (u^{-1} \odot t^{-1}) \odot s)$,
(ii) $(s \odot (s \odot t)) = ((s \odot t^{-1}) \odot s)$,
(iii) $s \odot s = e$,
(iv) $s \odot e = s$,
(v) $e \odot s = s^{-1}$ for all $s, t, u \in G$.

Definition 3 ([1]). *Let $\mathcal{K}$ be a K-algebra, and let $\mathcal{H}$ be a nonempty subset of $\mathcal{K}$. Then, $\mathcal{H}$ is called a subalgebra of $\mathcal{K}$ if $u \odot v \in \mathcal{H}$ for all $u, v \in \mathcal{H}$.*

Definition 4 ([1]). *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two K-algebras. A mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ is called a homomorphism if it satisfies the following condition:*

- *$f(u \odot v) = f(u) \odot f(v)$ for all $u, v \in \mathcal{K}$.*

Definition 5 ([7])**.** *Let Z be an initial universe, and let $\mathbb{R}$ be a universe of parameters. Then, $(\zeta, \mathbb{M})$ is called a soft set (SS), where $\mathbb{M} \subset \mathbb{R}$, $P(Z)$ is the power set of Z and ζ is a set-valued function, which is defined as $\zeta : \mathbb{M} \to P(Z)$.*

Definition 6 ([7])**.** *Let for two soft sets $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ over the common universe Z; the pair $(\zeta, \mathbb{M})$ is a soft subset of $(\eta, \mathbb{N})$, denoted by $(\zeta, \mathbb{M}) \subseteq (\eta, \mathbb{N})$ if it satisfies the following conditions:*

(a) $\mathbb{M} \subseteq \mathbb{N}$,
(b) $\zeta(\theta) \subseteq \eta(\theta)$ for any $\theta \in \mathbb{M}$.

Definition 7 ([9])**.** *Let Z be a universal set of objects. A single-valued neutrosophic set (SNS) $\mathcal{A}$ in Z is characterized by three membership functions, i.e., the $(\mathcal{T}_{\mathcal{A}})$-truth membership function, $(\mathcal{I}_{\mathcal{A}})$-indeterminacy membership function and $(\mathcal{F}_{\mathcal{A}})$-falsity membership function, where $\mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(s) \in [0,1]$ for all $s \in Z$. There is no restriction on the sum of these three components. Therefore, $0 \leq \mathcal{T}_{\mathcal{A}}(s) + \mathcal{I}_{\mathcal{A}}(s) + \mathcal{F}_{\mathcal{A}}(s) \leq 3$.*

Definition 8. *A single-valued neutrosophic set $\mathcal{A}$ in a non-empty set Z is called a single-valued neutrosophic point if:*

$$\mathcal{T}_{\mathcal{A}}(v) = \begin{cases} \alpha \in (0,1], & \text{if } v = u \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{I}_{\mathcal{A}}(v) = \begin{cases} \beta \in (0,1], & \text{if } v = u \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{F}_{\mathcal{A}}(v) = \begin{cases} \gamma \in [0,1), & \text{if } v = u \\ 0, & \text{otherwise}, \end{cases}$$

with support u and value (α, β, γ), denoted by $u(\alpha, \beta, \gamma)$. This single-valued neutrosophic point is said to "belong to" a single-valued neutrosophic set $\mathcal{A}$, written as $u(\alpha, \beta, \gamma) \in \mathcal{A}$ if $\mathcal{T}_{\mathcal{A}}(u) \geq \alpha, \mathcal{I}_{\mathcal{A}}(u) \geq \beta, \mathcal{F}_{\mathcal{A}}(u) \leq \gamma$ and said to be "quasicoincident with" a single-valued neutrosophic set $\mathcal{A}$, written as $u(\alpha, \beta, \gamma)\ q\ \mathcal{A}$ if $\mathcal{T}_{\mathcal{A}}(u) + \alpha > 1, \mathcal{I}_{\mathcal{A}}(u) + \beta > 1, \mathcal{F}_{\mathcal{A}}(u) + \gamma < 1$.

Definition 9 ([19])**.** *Let $\mathcal{K}$ be a K-algebra, and let $\mathcal{A}$ be a single-valued neutrosophic set in $\mathcal{K}$ such that $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$. Then, $\mathcal{A}$ is called a single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if the following conditions hold:*

(1) $\mathcal{T}_{\mathcal{A}}(e) \geq \mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(e) \geq \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(e) \leq \mathcal{F}_{\mathcal{A}}(s)$ for all $s \neq e \in \mathcal{K}$.
(2) $\mathcal{T}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{T}_{\mathcal{A}}(s), \mathcal{T}_{\mathcal{A}}(t)\}$,
$\mathcal{I}_{\mathcal{A}}(s \odot t) \geq \min\{\mathcal{I}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(t)\}$,
$\mathcal{F}_{\mathcal{A}}(s \odot t) \leq \max\{\mathcal{F}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(t)\}$ for all $s, t \in \mathcal{K}$.

Definition 10 ([19])**.** *A single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ in $\mathcal{K}$ is called an $(\tilde{a}, \tilde{b})$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following condition:*

- $u_{(\alpha_1, \beta_1, \gamma_1)}\ \tilde{a}\ \mathcal{A},\ v_{(\alpha_2, \beta_2, \gamma_2)}\ \tilde{a}\ \mathcal{A} \Rightarrow (u \odot v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))}\ \tilde{b}\ \mathcal{A}$,

  for all $u, v \in G, \alpha_1, \alpha_2 \in (0,1], \beta_1, \beta_2 \in (0,1], \gamma_1, \gamma_2 \in [0,1)$ and $\tilde{a}, \tilde{b} \in \{\in, q, \in \vee q, \in \wedge q\}$.

Definition 11 ([19])**.** *A single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_{\mathcal{A}}, \mathcal{I}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ in a K-algebra $\mathcal{K}$ is called an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if it satisfies the following conditions:*

(a) $e_{(\alpha, \beta, \gamma)} \in \mathcal{A} \Rightarrow (u)_{(\alpha, \beta, \gamma)} \in \vee q\ \mathcal{A}$,
(b) $u_{(\alpha_1, \beta_1, \gamma_1)} \in \mathcal{A}, v_{(\alpha_2, \beta_2, \gamma_2)} \in \mathcal{A} \Rightarrow (u \odot v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2)} \in \vee q\ \mathcal{A}$,

for all $u, v \in G, \alpha, \alpha_1, \alpha_2 \in (0,1], \beta, \beta_1, \beta_2 \in (0,1], \gamma, \gamma_1, \gamma_2 \in [0,1)$.

Example 1. *Consider a K-algebra* $\mathcal{K} = (G, \cdot, \odot, e)$, *where* $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$ *is the cyclic group of order eight and Caley's table for* $\odot$ *is given as:*

$\odot$	e	x	x^2	x^3	x^4	x^5	x^6	x^7
e	e	x^7	x^6	x^5	x^4	x^3	x^2	x
x	x	e	x^7	x^6	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^7	x^6	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^7	x^6	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^7	x^6	x^5
x^5	x^5	x^4	x^3	x^2	x	e	x^7	x^6
x^6	x^6	x^5	x^4	x^3	x^2	x	e	x^7
x^7	x^7	x^6	x^5	x^4	x^3	x^2	x	e

We define a single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ *in* $\mathcal{K}$ *as follows:*

$$\mathcal{T}_\mathcal{A}(s) = \begin{cases} 0.9 & \text{if } s = e \\ 0.7 & \textit{for all } s \neq e \in G, \end{cases}$$
$$\mathcal{I}_\mathcal{A}(s) = \begin{cases} 0.8 & \text{if } s = e \\ 0.6 & \textit{for all } s \neq e \in G, \end{cases}$$
$$\mathcal{F}_\mathcal{A}(s) = \begin{cases} 0 & \text{if } s = e \\ 0.4 & \textit{for all } s \neq e \in G. \end{cases}$$

We take

$\alpha = 0.3, \alpha_1 = 0.6, \alpha_2 = 0.3,$
$\beta = 0.4, \beta_1 = 0.5, \beta_2 = 0.3,$
$\gamma = 0.5, \gamma_1 = 0.5, \gamma_2 = 0.6,$
where $\alpha, \alpha_1, \alpha_2 \in (0,1], \beta, \beta_1, \beta_2 \in (0,1], \gamma, \gamma_1, \gamma_2 \in [0,1)$.

By direct calculations, it is easy to see that $\mathcal{A}$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$.

Theorem 1. *Let* $\mathcal{A} = (\mathcal{T}_\mathcal{A}, \mathcal{I}_\mathcal{A}, \mathcal{F}_\mathcal{A})$ *be a single-valued neutrosophic set in* $\mathcal{K}$. *Then,* $\mathcal{A}$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$ *if and only if:*

$$\text{(i) } \mathcal{T}_\mathcal{A}(u) \geq \min(\mathcal{T}_\mathcal{A}(e), 0.5), \quad \mathcal{I}_\mathcal{A}(u) \geq \min(\mathcal{I}_\mathcal{A}(e), 0.5), \quad \mathcal{F}_\mathcal{A}(u) \leq \max(\mathcal{F}_\mathcal{A}(e), 0.5).$$

$$\text{(ii) } \mathcal{T}_\mathcal{A}(u \odot v) \geq \min(\mathcal{T}_\mathcal{A}(u), \mathcal{T}_\mathcal{A}(v), 0.5), \quad \mathcal{I}_\mathcal{A}(u \odot v) \geq \min(\mathcal{I}_\mathcal{A}(u), \mathcal{I}_\mathcal{A}(v), 0.5), \quad \mathcal{F}_\mathcal{A}(u \odot v) \leq \max(\mathcal{F}_\mathcal{A}(u), \mathcal{F}_\mathcal{A}(v), 0.5) \text{ for all } u, v \in G.$$

Definition 12 ([13]). *Suppose an initial universe* Z *and a universe of parameters* $\mathbb{R}$. *A single-valued neutrosophic soft set (SNSS) is a pair* $(\zeta, \mathbb{M})$, *where* $\mathbb{M} \subset \mathbb{R}$, ζ *is a set-valued function defined as* $\zeta : \mathbb{M} \to P(Z)$. $P(Z)$ *is a set containing all single-valued neutrosophic (SN) subsets of* Z. *Each parameter in* $\mathbb{R}$ *is considered as a neutrosophic word or a sentence containing a neutrosophic word.*

Definition 13 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two single-valued neutrosophic soft sets over a common universe* Z*, then the pair* $(\zeta, \mathbb{M})$ *is a single-valued neutrosophic soft subset of* $(\eta, \mathbb{N})$*, denoted by* $(\zeta, \mathbb{M}) \subseteq (\eta, \mathbb{N})$*, if it satisfies the following conditions:*

(a) Parametric set $\mathbb{M}$ *is a subset of parametric set* $\mathbb{N}$*,*
(b) $\zeta(\theta)$ *is a subset of* $\eta(\theta)$*, for any* $\theta \in \mathbb{M}$*.*

Definition 14 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two SNSSs over* Z*, then the extended intersection is denoted by* $(\zeta, \mathbb{M}) \cap_{ex} (\eta, \mathbb{N}) = (\vartheta, \mathbb{Q})$*, where* $\mathbb{Q} = \mathbb{M} \cup \mathbb{N}$ *and defined as:*

$$\vartheta(\theta) = \begin{cases} \zeta(\theta) & \text{if } \theta \in \mathbb{M} - \mathbb{N}, \\ \eta(\theta) & \text{if } \theta \in \mathbb{N} - \mathbb{M}, \\ \zeta(\theta) \cap \eta(\theta) & \text{if } \theta \in \mathbb{M} \cap \mathbb{N} \quad \text{for all } \theta \in \mathbb{Q}. \end{cases}$$

Definition 15 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two SNSSs over* Z*. We denote their extended union by* $(\zeta, \mathbb{M}) \cup_{ex} (\eta, \mathbb{N}) = (\vartheta, \mathbb{Q})$*, where* $\mathbb{Q} = \mathbb{M} \cup \mathbb{N}$ *and defined as:*

$$\vartheta(\theta) = \begin{cases} \zeta(\theta) & \text{if } \theta \in \mathbb{M} - \mathbb{N}, \\ \eta(\theta) & \text{if } \theta \in \mathbb{N} - \mathbb{M}, \\ \zeta(\theta) \cup \eta(\theta) & \text{if } \theta \in \mathbb{M} \cap \mathbb{N} \quad \text{for all } \theta \in \mathbb{Q}. \end{cases}$$

Definition 16 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two SNSSs over* Z*; the restricted intersection is an SNSS over* Z *and denoted by* $(\vartheta, \mathbb{M} \cap \mathbb{N})$ *with* $\mathbb{M} \cap \mathbb{N} \neq \emptyset$*, where* $(\vartheta, \mathbb{M} \cap \mathbb{N}) = (\zeta, \mathbb{M}) \cap (\eta, \mathbb{N})$ *and* $\vartheta(\theta) = \zeta(\theta) \cap \eta(\theta)$ *for all* $\theta \in \mathbb{M} \cap \mathbb{N}$*.*

Definition 17 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two SNSSs over* Z*, then their restricted union is an SNSS over* Z *and denoted by* $(\vartheta, \mathbb{M} \cap \mathbb{N})$ *with* $\mathbb{M} \cap \mathbb{N} \neq \emptyset$*, where* $(\zeta, \mathbb{M}) \cup (\eta, \mathbb{N}) = (\vartheta, \mathbb{Q})$ *and* $\vartheta(\theta) = \zeta(\theta) \cup \eta(\theta)$ for all $\theta \in \mathbb{M} \cap \mathbb{N}$.

Definition 18 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two SNSSs over* Z*; the "AND" operation is denoted by* $(\zeta, \mathbb{M})$ *AND* $(\eta, \mathbb{N})$= $(\zeta, \mathbb{M}) \wedge (\eta, \mathbb{N}) = (\vartheta, \mathbb{M} \times \mathbb{N})$*, where for all* $(l, m) \in \mathbb{M} \times \mathbb{N}$*,* $\vartheta(l, m) = \zeta(l) \cap \eta(m)$*, and the truth-membership, indeterminacy-membership and falsity-membership of* $\vartheta(l, m)$ *are defined as* $\mathcal{T}_{\vartheta(l,m)} = \min\{\mathcal{T}_{\vartheta(l)}, \mathcal{T}_{\vartheta(m)}\}, \mathcal{I}_{\vartheta(l,m)} = \min\{\mathcal{I}_{\vartheta(l)}, \mathcal{I}_{\vartheta(m)}\}, \mathcal{F}_{\vartheta(l,m)} = \max\{\mathcal{F}_{\vartheta(l)}, \mathcal{F}_{\vartheta(m)}\}$*, for all* $l \in \mathbb{M}$ *and for all* $m \in \mathbb{N}$*.*

Definition 19 ([13])**.** *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two SNSSs over* Z*; the "OR" operation is denoted by* $(\zeta, \mathbb{M})$ *OR* $(\eta, \mathbb{N})$= $(\zeta, \mathbb{M}) \vee (\eta, \mathbb{N}) = (\vartheta, \mathbb{M} \times \mathbb{N})$*, where for all* $(l, m) \in \mathbb{M} \times \mathbb{N}$*,* $\vartheta(l, m) = \zeta(l) \cup \eta(m)$*, and the truth-membership, indeterminacy-membership and falsity-membership of* $\vartheta(l, m)$ *are defined as* $\mathcal{T}_{\vartheta(l,m)} = \max\{\mathcal{T}_{\vartheta(l)}, \mathcal{T}_{\vartheta(m)}\}, \mathcal{I}_{\vartheta(l,m)} = \max\{\mathcal{I}_{\vartheta(l)}, \mathcal{I}_{\vartheta(m)}\}, \mathcal{F}_{\vartheta(l,m)} = \min\{\mathcal{F}_{\vartheta(l)}, \mathcal{F}_{\vartheta(m)}\}$*, for all* $l \in \mathbb{M}$ *and for all* $m \in \mathbb{N}$*.*

3. Single-Valued Neutrosophic Soft K-Algebras

Definition 20. *Let* $(\zeta, \mathbb{M})$ *be a single-valued neutrosophic soft set (SNSS) over* $\mathcal{K}$*. The pair* $(\zeta, \mathbb{M})$ *is called a single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$ *if the following conditions are satisfied:*

$$\begin{aligned} &\text{(i)} \quad \mathcal{T}_{\zeta_\theta}(s \odot t) \geq \min\{\mathcal{T}_{\zeta_\theta}(s), \mathcal{T}_{\zeta_\theta}(t)\}, \\ &\text{(ii)} \quad \mathcal{I}_{\zeta_\theta}(s \odot t) \geq \min\{\mathcal{I}_{\zeta_\theta}(s), \mathcal{I}_{\zeta_\theta}(t)\}, \\ &\text{(iii)} \ \mathcal{F}_{\zeta_\theta}(s \odot t) \leq \max\{\mathcal{F}_{\zeta_\theta}(s), \mathcal{F}_{\zeta_\theta}(t)\} \text{ for all } s, t \in G. \end{aligned}$$

A single-valued neutrosophic soft K-algebra also satisfies the following properties:

$$\mathcal{T}_{\zeta_\theta}(e) \geq \mathcal{T}_{\zeta_\theta}(s),$$
$$\mathcal{I}_{\zeta_\theta}(e) \geq \mathcal{I}_{\zeta_\theta}(s),$$
$$\mathcal{F}_{\zeta_\theta}(e) \leq \mathcal{F}_{\zeta_\theta}(s) \text{ for all } s \neq e \in G.$$

Example 2. *Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where G is the cyclic group of order nine given as $G = \{e, w, w^2, w^3, w^4, w^5, w^6, w^7, w^8\}$. Consider the following Cayley's table:*

$\odot$	e	w	w^2	w^3	w^4	w^5	w^6	w^7	w^8
e	e	w^8	w^7	w^6	w^5	w^4	w^3	w^2	w
w	w	e	w^8	w^7	w^6	w^5	w^4	w^3	w^2
w^2	w^2	w	e	w^8	w^7	w^6	w^5	w^4	w^3
w^3	w^3	w^2	w	e	w^8	w^7	w^6	w^5	w^4
w^4	w^4	w^3	w^2	w	e	w^8	w^7	w^6	w^5
w^5	w^5	w^4	w^3	w^2	w	e	w^8	w^7	w^6
w^6	w^6	w^5	w^4	w^3	w^2	w	e	w^8	w^7
w^7	w^7	w^6	w^5	w^4	w^3	w^2	w	e	w^8
w^8	w^8	w^7	w^6	w^5	w^4	w^3	w^2	w	e

Consider a set of parameters $\mathbb{M} = \{l_1, l_2, l_3,\}$ and a set-valued function $\zeta : \mathbb{M} \to P(G)$, where the membership, indeterminacy-membership and non-membership values of the elements of G at parameters l_1, l_2, l_3 are given as:

$$\begin{aligned}
\text{(i) } \mathcal{T}_{\zeta_{l_1}}(e) = 0.9,\ \mathcal{I}_{\zeta_{l_1}}(e) = 0.3,\ \mathcal{F}_{\zeta_{l_1}}(e) = 0.3,\\
\mathcal{T}_{\zeta_{l_1}}(s) = 0.6,\ \mathcal{I}_{\zeta_{l_1}}(s) = 0.2,\ \mathcal{F}_{\zeta_{l_1}}(s) = 0.4,\\
\text{(ii) } \mathcal{T}_{\zeta_{l_2}}(e) = 0.8,\ \mathcal{I}_{\zeta_{l_2}}(e) = 0.7,\ \mathcal{F}_{\zeta_{l_2}}(e) = 0.4,\\
\mathcal{T}_{\zeta_{l_2}}(s) = 0.7,\ \mathcal{I}_{\zeta_{l_2}}(s) = 0.6,\ \mathcal{F}_{\zeta_{l_2}}(s) = 0.5,\\
\text{(iii) } \mathcal{T}_{\zeta_{l_3}}(e) = 0.9,\ \mathcal{I}_{\zeta_{l_3}}(e) = 0.6,\ \mathcal{F}_{\zeta_{l_3}}(e) = 0.6,\\
\mathcal{T}_{\zeta_{l_3}}(s) = 0.8,\ \mathcal{I}_{\zeta_{l_3}}(s) = 0.5,\ \mathcal{F}_{\zeta_{l_3}}(s) = 0.7
\end{aligned}$$

for all $s \neq e \in G$. The function ζ is defined as:

$$\begin{aligned}
\zeta(l_1) = \{&(e, 0.9, 0.3, 0.3), (w, 0.6, 0.2, 0.4), (w^2, 0.6, 0.2, 0.4), (w^3, 0.6, 0.2, 0.4),\\
&(w^4, 0.6, 0.2, 0.4), (w^5, 0.6, 0.2, 0.4), (w^6, 0.6, 0.2, 0.4),\\
&(w^7, 0.6, 0.2, 0.4), (w^8, 0.6, 0.2, 0.4)\},
\end{aligned}$$

$$\begin{aligned}
\zeta(l_2) = \{&(e, 0.8, 0.7, 0.4), (w, 0.7, 0.6, 0.5), (w^2, 0.7, 0.6, 0.5), (w^3, 0.7, 0.6, 0.5),\\
&(w^4, 0.7, 0.6, 0.5), (w^5, 0.7, 0.6, 0.5), (w^6, 0.7, 0.6, 0.5),\\
&(w^7, 0.7, 0.6, 0.5), (w^8, 0.7, 0.6, 0.5)\},
\end{aligned}$$

$$\begin{aligned}
\zeta(l_3) = \{&(e, 0.9, 0.6, 0.6), (w, 0.8, 0.5, 0.7), (w^2, 0.8, 0.5, 0.7), (w^3, 0.8, 0.5, 0.7),\\
&(w^4, 0.8, 0.5, 0.7), (w^5, 0.8, 0.5, 0.7), (w^6, 0.8, 0.5, 0.7),\\
&(w^7, 0.8, 0.5, 0.7), (w^8, 0.8, 0.5, 0.7)\}.
\end{aligned}$$

Consider a set $\mathbb{N} = \{l_1, l_2\}$ *of parameters and a set-valued function* $\eta : \mathbb{N} \to P(G)$*, where the membership, indeterminacy-membership and non-membership values of the elements of G at parameters* l_1, l_2 *are defined as:*

$$\begin{aligned}
\text{(i) } \mathcal{T}_{\eta_{l_1}}(e) = 0.9,\ \mathcal{I}_{\eta_{l_1}}(e) = 0.8,\ \mathcal{F}_{\eta_{l_1}}(e) = 0.2,\\
\mathcal{T}_{\eta_{l_1}}(s) = 0.5,\ \mathcal{I}_{\eta_{l_1}}(s) = 0.2,\ \mathcal{F}_{\eta_{l_1}}(s) = 0.5,\\
\text{(ii) } \mathcal{T}_{\eta_{l_2}}(e) = 0.3,\ \mathcal{I}_{\eta_{l_2}}(e) = 0.5,\ \mathcal{F}_{\eta_{l_2}}(e) = 0.6,\\
\mathcal{T}_{\eta_{l_2}}(s) = 0.1,\ \mathcal{I}_{\eta_{l_2}}(s) = 0.4,\ \mathcal{F}_{\eta_{l_2}}(s) = 0.8
\end{aligned}$$

for all $s \neq e \in G$*. The function* η *is defined as:*

$$\begin{aligned}
\eta(l_1) = \{&(e, 0.9, 0.8, 0.2), (w, 0.5, 0.2, 0.5), (w^2, 0.5, 0.2, 0.5), (w^3, 0.5, 0.2, 0.5),\\
&(w^4, 0.5, 0.2, 0.5), (w^5, 0.5, 0.2, 0.5), (w^6, 0.5, 0.2, 0.5),\\
&(w^7, 0.5, 0.2, 0.5), (w^8, 0.5, 0.2, 0.5)\},\\
\eta(l_2) = \{&(e, 0.3, 0.5, 0.6), (w, 0.1, 0.4, 0.8), (w^2, 0.1, 0.4, 0.8), (w^3, 0.1, 0.4, 0.8),\\
&(w^4, 0.1, 0.4, 0.8), (w^5, 0.1, 0.4, 0.8), (w^6, 0.1, 0.4, 0.8),\\
&(w^7, 0.1, 0.4, 0.8), (w^8, 0.1, 0.4, 0.8)\}.
\end{aligned}$$

Evidently, the set $(\zeta, \mathbb{M})$ *and the set* $(\eta, \mathbb{N})$ *comprises SNSSs. Since* $\zeta(\theta)$*,* $\eta(\theta)$ *are single-valued neutrosophic K-subalgebras for all* $\theta \in \mathbb{M}$ *and* $\theta \in \mathbb{N}$*. It is concluded that the pairs* $(\zeta, \mathbb{M})$*,* $(\eta, \mathbb{N})$ *are single-valued neutrosophic soft K-subalgebras.*

Example 3. *Consider K-algebra on dihedral group D4 given as* $G = \{e, a, b, c, w, x, y, z\}$*, where* $c = ab, w = a^2, x = a^3, y = a^2b, z = a^3b$*, and Caley's table for* $\odot$ *is given as:*

$\odot$	e	a	b	c	w	x	y	z
e	e	x	b	c	w	a	y	z
a	a	e	c	y	x	w	z	b
b	b	c	e	x	y	z	w	a
c	c	y	a	e	z	b	x	w
w	w	a	y	z	e	x	b	c
x	x	w	z	b	a	e	c	y
y	y	z	w	a	b	c	e	x
z	z	b	x	w	c	y	a	e

Consider a set of parameters $\mathbb{M} = \{l_1, l_2, l_3,\}$ *and a set-valued function* $\zeta : \mathbb{M} \to P(G)$*, where the membership, indeterminacy-membership and non-membership values of the elements of G at parameters* l_1, l_2, l_3 *are given as:*

$$\begin{aligned}
(i)\ \mathcal{T}_{\zeta_{l_1}}(e) = 0.7,\ \mathcal{I}_{\zeta_{l_1}}(e) = 0.7,\ \mathcal{F}_{\zeta_{l_1}}(e) = 0.3,\\
\mathcal{T}_{\zeta_{l_1}}(s) = 0.5,\ \mathcal{I}_{\zeta_{l_1}}(s) = 0.2,\ \mathcal{F}_{\zeta_{l_1}}(s) = 0.7,\\
(ii)\ \mathcal{T}_{\zeta_{l_2}}(e) = 0.9,\ \mathcal{I}_{\zeta_{l_2}}(e) = 0.8,\ \mathcal{F}_{\zeta_{l_2}}(e) = 0.4,\\
\mathcal{T}_{\zeta_{l_2}}(s) = 0.2,\ \mathcal{I}_{\zeta_{l_2}}(s) = 0.2,\ \mathcal{F}_{\zeta_{l_2}}(s) = 0.9,\\
(iii)\ \mathcal{T}_{\zeta_{l_3}}(e) = 0.5,\ \mathcal{I}_{\zeta_{l_3}}(e) = 0.5,\ \mathcal{F}_{\zeta_{l_3}}(e) = 0.3,\\
\mathcal{T}_{\zeta_{l_3}}(s) = 0.1,\ \mathcal{I}_{\zeta_{l_3}}(s) = 0.3,\ \mathcal{F}_{\zeta_{l_3}}(s) = 0.8
\end{aligned}$$

for all $s \neq e \in G$. The function ζ is defined as:

$$\zeta(l_1) = \{(e, 0.7, 0.7, 0.3), (a, 0.5, 0.2, 0.7), (b, 0.5, 0.2, 0.7), (c, 0.5, 0.2, 0.7), (w, 0.5, 0.2, 0.7), (x, 0.5, 0.2, 0.7), (y, 0.5, 0.2, 0.7), (z, 0.5, 0.2, 0.7)\},$$

$$\zeta(l_2) = \{(e, 0.9, 0.8, 0.4), (a, 0.2, 0.2, 0.9), (b, 0.2, 0.2, 0.9), (c, 0.2, 0.2, 0.9), (w, 0.2, 0.2, 0.9), (x, 0.2, 0.2, 0.9), (y, 0.2, 0.2, 0.9), (z, 0.2, 0.2, 0.9)\},$$

$$\zeta(l_3) = \{(e, 0.5, 0.5, 0.3), (a, 0.1, 0.3, 0.8), (b, 0.1, 0.3, 0.8), (c, 0.1, 0.3, 0.8), (w, 0.1, 0.3, 0.8), (x, 0.1, 0.3, 0.8), (y, 0.1, 0.3, 0.8), (z, 0.1, 0.3, 0.8)\}.$$

Consider a set $\mathbb{N} = \{l_1, l_2\}$ of parameters and a set-valued function $\eta : \mathbb{N} \to P(G)$, where the truth, indeterminacy and falsity membership values of the elements of G at parameters l_1, l_2 are defined as:

$$(i)\ \mathcal{T}_{\eta_{l_1}}(e) = 0.8,\ \mathcal{I}_{\eta_{l_1}}(e) = 0.8,\ \mathcal{F}_{\eta_{l_1}}(e) = 0.2,$$
$$\mathcal{T}_{\eta_{l_1}}(s) = 0.6,\ \mathcal{I}_{\eta_{l_1}}(s) = 0.3,\ \mathcal{F}_{\eta_{l_1}}(s) = 0.7,$$
$$(ii)\ \mathcal{T}_{\eta_{l_2}}(e) = 0.6,\ \mathcal{I}_{\eta_{l_2}}(e) = 0.4,\ \mathcal{F}_{\eta_{l_2}}(e) = 0.3,$$
$$\mathcal{T}_{\eta_{l_2}}(s) = 0.5,\ \mathcal{I}_{\eta_{l_2}}(s) = 0.4,\ \mathcal{F}_{\eta_{l_2}}(s) = 0.9$$

for all $s \neq e \in G$. The function η is defined as:

$$\eta(l_1) = \{(e, 0.8, 0.8, 0.2), (a, 0.6, 0.3, 0.7), (b, 0.6, 0.3, 0.7), (c, 0.6, 0.3, 0.7), (w, 0.6, 0.3, 0.7), (x, 0.6, 0.3, 0.7), (y, 0.6, 0.3, 0.7), (z, 0.6, 0.3, 0.7)\},$$

$$\eta(l_2) = \{(e, 0.6, 0.4, 0.3), (a, 0.5, 0.4, 0.9), (b, 0.5, 0.4, 0.9), (c, 0.5, 0.4, 0.9), (w, 0.5, 0.4, 0.9), (x, 0.5, 0.4, 0.9), (y, 0.5, 0.4, 0.9), (z, 0.5, 0.4, 0.9)\}.$$

Obviously, the set $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ comprises SNSSs. Since for $\theta \in \mathbb{M}$ and $\theta \in \mathbb{N}$, the sets $\zeta(\theta)$, $\eta(\theta)$ are single-valued neutrosophic K-subalgebras. This concludes that the pair $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ are single-valued neutrosophic soft K-subalgebras.

Proposition 1. *Let $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ be two single-valued neutrosophic soft K-subalgebras. Then, the extended intersection of $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ is a single-valued neutrosophic soft K-subalgebra.*

Proof. For any $\theta \in \mathbb{Q}$, the following three cases arise.

First case: If $\theta \in \mathbb{M} - \mathbb{N}$, then $\vartheta(\theta) = \zeta(\theta)$ and $\zeta(\theta)$ being single-valued neutrosophic K-subalgebra implies that $\vartheta(\theta)$ is also a single-valued neutrosophic K-subalgebra since $(\zeta, \mathbb{M})$ is an SNS K-subalgebra.

Second case: If $\theta \in \mathbb{N} - \mathbb{M}$, then $\vartheta(\theta) = \eta(\theta)$ and $\eta(\theta)$ being single-valued neutrosophic K-subalgebra implies that $\vartheta(\theta)$ is a single-valued neutrosophic K-subalgebra since $(\eta, \mathbb{N})$ is an SNS K-subalgebra.

Third case: Now, if $\theta \in \mathbb{M} \cap \mathbb{N}$, then $\vartheta(\theta) = \zeta(\theta) \cap \eta(\theta)$, which is again a single-valued neutrosophic K-subalgebra of $\mathcal{K}$. Thus, in any case, $\vartheta(\theta)$ is a single-valued neutrosophic K-subalgebra. Consequently, $(\zeta, \mathbb{M}) \cap_{ex} (\eta, \mathbb{N})$ is a K-subalgebraover $\mathcal{K}$.

□

Proposition 2. *If $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ are two SNS K-subalgebras over $\mathcal{K}$, then $(\zeta, \mathbb{M}) \wedge (\eta, \mathbb{N})$ is an SNS K-subalgebra.*

Proof. Let $(l,m) \in \mathbb{Q}, \zeta(l), \zeta(m)$ be single-valued neutrosophic K-subalgebras of $\mathcal{K}$, where $\mathbb{Q} = \mathbb{M} \times \mathbb{N}$, which implies that $\vartheta(l,m) = \zeta(l) \cap \eta(m)$ is also a single-valued neutrosophic K-subalgebra over $\mathcal{K}$. Hence, $(\zeta,\mathbb{M}) \wedge (\eta,\mathbb{N})$ is an SNS K-subalgebra of $\mathcal{K}$. □

Proposition 3. *If $(\zeta,\mathbb{M})$ and $(\eta,\mathbb{N})$ are two SNS K-subalgebras and $\zeta(l) \subseteq \eta(l)$ for all $l \in \mathbb{M}$, then $(\zeta,\mathbb{M})$ is an SNS K-subalgebra of $(\eta,\mathbb{N})$.*

Proof. Since $(\zeta,\mathbb{M})$ and $(\eta,\mathbb{N})$ are SNS K-subalgebras and $\zeta(l), \eta(l)$ are two single-valued neutrosophic K-subalgebras, also $\zeta(l) \subseteq \eta(l)$. Therefore, $(\zeta,\mathbb{M})$ is an SNS K-subalgebra of $(\eta,\mathbb{N})$. □

Proposition 4. *Let $(\zeta,\mathbb{M})$, $(\eta,\mathbb{N})$ be two SNS K-subalgebras. If $\mathbb{M} \cap \mathbb{N} = \emptyset$, then $(\zeta,\mathbb{M}) \cup_{ex} (\eta,\mathbb{N})$ is an SNS K-subalgebra over $\mathcal{K}$.*

Proof. The proof follows from Definition 15. □

Theorem 2. *If $(\zeta,\mathbb{M})$ is an SNS K-subalgebra, then for a non-empty collection $\{(\vartheta_i,\mathbb{N}_i) \mid i \in \Omega\}$ of SNS K-subalgebras of $(\zeta,\mathbb{M})$, the following results hold:*

(i) $\bigcap_{i\in\Omega} {}_{ex}(\vartheta_i,\mathbb{N}_i)$ *is an SNS K-subalgebra of $(\zeta,\mathbb{M})$.*

(ii) $\bigwedge_{i\in\Omega}$ *is an SNS K-subalgebra of $\bigwedge_{i\in\Omega}(\zeta,\mathbb{M})$.*

(iii) For the disjoint intersection of two parametric sets $\mathbb{N}_i, \mathbb{N}_j$, $\forall i,j \in \Omega$, $\bigvee_{i\in\Omega} {}_{ex}(\vartheta_i,\mathbb{N}_i)$ is an SNS K-subalgebra of $\bigvee_{i\in\Omega}(\zeta,\mathbb{M})$.

Definition 21 ([13]). *Let $(\zeta,\mathbb{M})$ be a single-valued neutrosophic soft set over Z. Then, for each $\alpha,\beta,\gamma \in [0,1]$, the set $(\zeta,\mathbb{M})^{(\alpha,\beta,\gamma)} = (\zeta^{(\alpha,\beta,\gamma)},\mathbb{M})$ is called an (α,β,γ)-level soft set of $(\zeta,\mathbb{M})$ and defined as:*
$\zeta_\theta^{(\alpha,\beta,\gamma)} = \{\mathcal{T}_{\zeta_\theta} \geq \alpha, \mathcal{I}_{\zeta_\theta} \geq \beta, \mathcal{F}_{\zeta_\theta} \leq \gamma\}$, for all $\theta \in \mathbb{M}$.

Theorem 3. *If $(\zeta,\mathbb{M})$ is a single-valued neutrosophic soft set over $\mathcal{K}$, then $(\zeta,\mathbb{M})$ is a single-valued neutrosophic soft K-subalgebra if and only if $(\zeta,\mathbb{M})^{(\alpha,\beta,\gamma)}$ is a soft K-subalgebra for all $\alpha,\beta,\gamma \in [0,1]$.*

Proof. Consider that $(\zeta,\mathbb{M})$ is an SNS K-subalgebra. Then, for all $\alpha,\beta,\gamma \in [0,1]$, $\theta \in \mathbb{M}$ and $u_1,u_2 \in \zeta_\theta{}^{(\alpha,\beta,\gamma)}$, $\mathcal{T}_{\zeta_\theta}(u_1) \geq \alpha$, $\mathcal{T}_{\zeta_\theta}(u_2) \geq \alpha, \mathcal{I}_{\zeta_\theta}(u_1) \geq \beta, \mathcal{I}_{\zeta_\theta}(u_2) \geq \beta, \mathcal{F}_{\zeta_\theta}(u_1) \leq \gamma, \mathcal{F}_{\zeta_\theta}(u_2) \leq \gamma$. It follows that $\mathcal{T}_{\zeta_\theta}(u_1 \odot u_2) \geq \min(\mathcal{T}_{\zeta_\theta}(u_1), \mathcal{T}_{\zeta_\theta}(u_2)) \geq \alpha, \mathcal{I}_{\zeta_\theta}(u_1 \odot (u_2)) \geq \min(\mathcal{I}_{\zeta_\theta}(u_1), \mathcal{I}_{\zeta_\theta}(u_2)) \geq \beta, \mathcal{F}_{\zeta_\theta}(u_1 \odot (u_2)) \leq \max(\mathcal{F}_{\zeta\theta}(u_1), \mathcal{F}_{\zeta_\theta}(u_2)) \leq \gamma$; which implies that $u_1 \odot u_2 \in \zeta_\theta{}^{(\alpha,\beta,\gamma)}$. Hence, $\zeta_\theta{}^{(\alpha,\beta,\gamma)}$ is a soft K-subalgebra for all $\alpha,\beta,\gamma \in [0,1]$. The converse part is obvious. □

Definition 22. *Let φ and ρ be two functions, where $\varphi: S_1 \to S_2$ and $\rho: \mathbb{M} \to \mathbb{N}$ and $\mathbb{M}$ and $\mathbb{N}$ are subsets of the universe of parameters $\mathbb{R}$ from S_1 and S_2, respectively. The pair (φ,ρ) is said to be a single-valued neutrosophic soft function from S_1 to S_2.*

Definition 23. *Let the pair (φ,ρ) be a single-valued neutrosophic soft function from $\mathcal{K}_1$ into $\mathcal{K}_2$, then the pair (φ,ρ) is called a single-valued neutrosophic soft homomorphism if φ is a homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$ and is said to be a single-valued neutrosophic soft bijective homomorphism if φ is an isomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$ and ρ is an injective map from $\mathbb{M}$ to $\mathbb{N}$.*

Definition 24 ([13]). *Let $(\zeta,\mathbb{M})$ and $(\eta,\mathbb{N})$ be two single-valued neutrosophic soft sets over G_1 and G_2, respectively, and let (φ,ρ) be an SNS function from G_1 into G_2. Then, under the single-valued neutrosophic soft function (φ,ρ), the image of $(\zeta,\mathbb{M})$ is a single-valued neutrosophic soft set on $\mathcal{K}_2$, denoted by $(\varphi,\rho)(\zeta,\mathbb{M})$ and defined as: for all $l \in \rho(\mathbb{M})$ and $v \in G_2$, $(\varphi,\rho)(\zeta,\mathbb{M}) = (\varphi(\zeta),\rho(\mathbb{M}))$, where:*

$$\mathcal{T}_{\varphi(\zeta)_l}(v) = \begin{cases} \bigvee_{\varphi(u)=v} \bigvee_{\rho(a)=l} \zeta_a(u) & \text{if } u \in \rho^{-1}(v), \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{\varphi(\zeta)_l}(v) = \begin{cases} \bigvee_{\varphi(u)=v} \bigvee_{\rho(a)=l} \zeta_a(u) & \text{if } u \in \rho^{-1}(v), \\ 1, & \text{otherwise,} \end{cases}$$
$$\mathcal{F}_{\varphi(\zeta)_l}(v) = \begin{cases} \bigwedge_{\varphi(u)=v} \bigwedge_{\rho(a)=l} \zeta_a(u) & \text{if } u \in \rho^{-1}(v), \\ 0, & \text{otherwise.} \end{cases}$$

Under the single-valued neutrosophic soft function (φ,ρ), the preimage of $(\eta,\mathbb{N})$ is denoted as $(\varphi,\rho)^{-1}(\eta,\mathbb{N})$ and defined as for all $a \in \rho^{-1}(\mathbb{N})$ and for all $u \in G_1$, $(\varphi,\rho)^{-1}(\eta,\mathbb{N}) = (\varphi^{-1}(\eta),\rho^{-1}(\mathbb{N}))$, where:

$$\mathcal{T}_{\varphi^{-1}(\eta)_a}(u) = \mathcal{T}_{\eta_{\rho(a)}}(\varphi(u)),$$
$$\mathcal{I}_{\varphi^{-1}(\eta)_a}(u) = \mathcal{I}_{\eta_{\rho(a)}}(\varphi(u)),$$
$$\mathcal{F}_{\varphi^{-1}(\eta)_a}(u) = \mathcal{F}_{\eta_{\rho(a)}}(\varphi(u)).$$

Theorem 4. *Let (φ,ρ) be a single-valued neutrosophic soft homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$ and $(\eta,\mathbb{N})$ be a single-valued neutrosophic soft K-subalgebra on $\mathcal{K}_2$. Then, $(\varphi,\rho)^{-1}(\eta,\mathbb{N})$ is an SNS K-subalgebra on $\mathcal{K}_1$.*

Proof. Assume that $u_1, u_2 \in \mathcal{K}_1$, then we have:

$$\varphi^{-1}(\mathcal{T}_{\eta_\theta})(u_1 \odot u_2) = \mathcal{T}_{\eta_{\rho(\theta)}}(\varphi(u_1 \odot u_2)) = \mathcal{T}_{\eta_{\rho(\theta)}}(\varphi(u_1) \odot \varphi(u_1))$$
$$\varphi^{-1}(\mathcal{T}_{\eta_\theta})(u_1 \odot u_2) \geq \min\{\mathcal{T}_{\eta_{\rho(\theta)}}(\varphi(u_1)), \mathcal{T}_{\eta_{\rho(\theta)}}(\varphi(u_2))\}$$
$$\varphi^{-1}(\mathcal{T}_{\eta_\theta})(u_1 \odot u_2) \geq \min\{\varphi^{-1}(\mathcal{T}_{\eta_\theta})(u_1), \varphi^{-1}(\mathcal{T}_{\eta_\theta})(u_2)\},$$

$$\varphi^{-1}(\mathcal{I}_{\eta_\theta})(u_1 \odot u_2) = \mathcal{I}_{\eta_{\rho(\theta)}}(\varphi(u_1 \odot u_2)) = \mathcal{I}_{\eta_{\rho(\theta)}}(\varphi(u_1) \odot \varphi(u_1))$$
$$\varphi^{-1}(\mathcal{I}_{\eta_\theta})(u_1 \odot u_2) \geq \min\{\mathcal{I}_{\eta_{\rho(\theta)}}(\varphi(u_1)), \mathcal{I}_{\eta_{\rho(\theta)}}(\varphi(u_2))\}$$
$$\varphi^{-1}(\mathcal{I}_{\eta_\theta})(u_1 \odot u_2) \geq \min\{\varphi^{-1}(\mathcal{I}_{\eta_\theta})(u_1), \varphi^{-1}(\mathcal{I}_{\eta_\theta})(u_2)\},$$

$$\varphi^{-1}(\mathcal{F}_{\eta_\theta})(u_1 \odot u_2) = \mathcal{F}_{\eta_{\rho(\theta)}}(\varphi(u_1 \odot u_2)) = \mathcal{F}_{\eta_{\rho(\theta)}}(\varphi(u_1) \odot \varphi(u_1))$$
$$\varphi^{-1}(\mathcal{F}_{\eta_\theta})(u_1 \odot u_2) \leq \max\{\mathcal{F}_{\eta_{\rho(\theta)}}(\varphi(u_1)), \mathcal{F}_{\eta_{\rho(\theta)}}(\varphi(u_2))\}$$
$$\varphi^{-1}(\mathcal{F}_{\eta_\theta})(u_1 \odot u_2) \leq \max\{\varphi^{-1}(\mathcal{F}_{\eta_\theta})(u_1), \varphi^{-1}(\mathcal{F}_{\eta_\theta})(u_2)\}.$$

Therefore, $(\varphi,\rho)^{-1}(\eta,\mathbb{N})$ is an SNS K-subalgebra over $\mathcal{K}_1$. □

Remark 1. *Let $(\zeta,\mathbb{M})$ be a single-valued neutrosophic soft K-subalgebra, and let (φ,ρ) be a single-valued neutrosophic soft homomorphism from $\mathcal{K}_1$ into $\mathcal{K}_2$. Then, $(\varphi,\rho)(\zeta,\mathbb{M})$ may not be a single-valued neutrosophic soft K-subalgebra over $\mathcal{K}_2$.*

4. $(\in, \in \vee q)$-Single-Valued Neutrosophic Soft *K*-Algebras

Definition 25. *Suppose $\mathcal{K}$ is a K-algebra. Let $(\zeta,\mathbb{M})$ be a single-valued neutrosophic soft set. The pair $(\zeta,\mathbb{M})$ is called an $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebra if $\zeta(\theta)$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ for all $\theta \in \mathbb{M}$.*

Example 4. *Consider two cyclic groups $G_1 = \{< u >: u^6 = e\}$ and $G_2 = \{< v >: v^2 = e\}$, where $G = G_1 \times G_2 = \{(e,e),(e,v),(u,e),(u,v),(u^2,e),(u^2,v),(u^3,e),(u^3,v),(u^4,e),(u^4,v),(u^5,e),(u^5,v)\}$ is a group. Consider a K-algebra $\mathcal{K}$ on $G = \{e, \acute{x}_1, \acute{x}_2, \acute{x}_3, \acute{x}_4, \acute{x}_5, \acute{x}_6, \acute{x}_7, \acute{x}_8, \acute{x}_9, \acute{x}_{10}, \acute{x}_{11}\}$, where $e = (e,e)$, $\acute{x}_1 = (e,v)$, $\acute{x}_2 = (u,e)$, $\acute{x}_3 = (u,v)$, $\acute{x}_4 = (u^2,e)$, $\acute{x}_5 = (u^2,v)$, $\acute{x}_6 = (u^3,e)$, $\acute{x}_7 = (u^3,v)$, $\acute{x}_8 = (u^4,e)$, $\acute{x}_9 = (u^4,v)$, $\acute{x}_{10} = (u^5,e)$, $\acute{x}_{11} = (u^5,v)$ and $\odot$ is defined by Caley's table as:*

$\odot$	e	$\acute{x}_1$	$\acute{x}_2$	$\acute{x}_3$	$\acute{x}_4$	$\acute{x}_5$	$\acute{x}_6$	$\acute{x}_7$	$\acute{x}_8$	$\acute{x}_9$	$\acute{x}_{10}$	$\acute{x}_{11}$
e	e	$\acute{x}_1$	$\acute{x}_{10}$	$\acute{x}_{11}$	$\acute{x}_8$	$\acute{x}_9$	$\acute{x}_6$	$\acute{x}_7$	$\acute{x}_4$	$\acute{x}_5$	$\acute{x}_2$	$\acute{x}_3$
$\acute{x}_1$	$\acute{x}_1$	e	$\acute{x}_{11}$	$\acute{x}_{10}$	$\acute{x}_9$	$\acute{x}_8$	$\acute{x}_7$	$\acute{x}_6$	$\acute{x}_5$	$\acute{x}_4$	$\acute{x}_3$	$\acute{x}_2$
$\acute{x}_2$	$\acute{x}_2$	$\acute{x}_3$	e	$\acute{x}_1$	$\acute{x}_{10}$	$\acute{x}_{11}$	$\acute{x}_8$	$\acute{x}_9$	$\acute{x}_6$	$\acute{x}_7$	$\acute{x}_4$	$\acute{x}_5$
$\acute{x}_3$	$\acute{x}_3$	$\acute{x}_2$	$\acute{x}_1$	e	$\acute{x}_{11}$	$\acute{x}_{10}$	$\acute{x}_9$	$\acute{x}_8$	$\acute{x}_7$	$\acute{x}_6$	$\acute{x}_5$	$\acute{x}_4$
$\acute{x}_4$	$\acute{x}_4$	$\acute{x}_5$	$\acute{x}_2$	$\acute{x}_3$	e	$\acute{x}_1$	$\acute{x}_{10}$	$\acute{x}_{11}$	$\acute{x}_8$	$\acute{x}_9$	$\acute{x}_6$	$\acute{x}_7$
$\acute{x}_5$	$\acute{x}_5$	$\acute{x}_4$	$\acute{x}_3$	$\acute{x}_2$	$\acute{x}_1$	e	$\acute{x}_{11}$	$\acute{x}_{10}$	$\acute{x}_9$	$\acute{x}_8$	$\acute{x}_7$	$\acute{x}_6$
$\acute{x}_6$	$\acute{x}_6$	$\acute{x}_7$	$\acute{x}_4$	$\acute{x}_5$	$\acute{x}_2$	$\acute{x}_3$	e	$\acute{x}_1$	$\acute{x}_{10}$	$\acute{x}_{11}$	$\acute{x}_8$	$\acute{x}_9$
$\acute{x}_7$	$\acute{x}_7$	$\acute{x}_6$	$\acute{x}_5$	$\acute{x}_4$	$\acute{x}_3$	$\acute{x}_2$	$\acute{x}_1$	e	$\acute{x}_{11}$	$\acute{x}_{10}$	$\acute{x}_9$	$\acute{x}_8$
$\acute{x}_8$	$\acute{x}_8$	$\acute{x}_9$	$\acute{x}_6$	$\acute{x}_7$	$\acute{x}_4$	$\acute{x}_5$	$\acute{x}_2$	$\acute{x}_3$	e	$\acute{x}_1$	$\acute{x}_{10}$	$\acute{x}_{11}$
$\acute{x}_9$	$\acute{x}_9$	$\acute{x}_8$	$\acute{x}_7$	$\acute{x}_6$	$\acute{x}_5$	$\acute{x}_4$	$\acute{x}_3$	$\acute{x}_2$	$\acute{x}_1$	e	$\acute{x}_{11}$	$\acute{x}_{10}$
$\acute{x}_{10}$	$\acute{x}_{10}$	$\acute{x}_{11}$	$\acute{x}_8$	$\acute{x}_9$	$\acute{x}_6$	$\acute{x}_7$	$\acute{x}_4$	$\acute{x}_5$	$\acute{x}_2$	$\acute{x}_3$	e	$\acute{x}_1$
$\acute{x}_{11}$	$\acute{x}_{11}$	$\acute{x}_{10}$	$\acute{x}_9$	$\acute{x}_8$	$\acute{x}_7$	$\acute{x}_6$	$\acute{x}_5$	$\acute{x}_4$	$\acute{x}_3$	$\acute{x}_2$	$\acute{x}_1$	e

Let $\mathbb{M} = \{l_1, l_2\}$ be a set of parameters and $\zeta : \mathbb{M} \to P(G)$ be a set-valued function defined as follows:

$$\begin{aligned}\zeta(l_1) = \{&(e, 0.9, 0.8, 0.5), (\acute{x}_1, 0.5, 0.8, 0.5), (\acute{x}_2, 0.5, 0.8, 0.5), (\acute{x}_3, 0.5, 0.8, 0.5),\\ &(\acute{x}_4, 0.5, 0.8, 0.5), (\acute{x}_5, 0.5, 0.8, 0.5), (\acute{x}_6, 0.5, 0.8, 0.5), (\acute{x}_7, 0.5, 0.8, 0.5),\\ &(\acute{x}_8, 0.5, 0.8, 0.5), (\acute{x}_9, 0.5, 0.8, 0.5), (\acute{x}_{10}, 0.5, 0.8, 0.5), (\acute{x}_{11}, 0.5, 0.8, 0.5)\},\end{aligned}$$

$$\begin{aligned}\zeta(l_2) = \{&(e, 0.7, 0.8, 0.4), (\acute{x}_1, 0.6, 0.5, 0.5), (\acute{x}_2, 0.6, 0.5, 0.5), (\acute{x}_3, 0.6, 0.5, 0.5),\\ &(\acute{x}_4, 0.6, 0.5, 0.5), (\acute{x}_5, 0.6, 0.5, 0.5), (\acute{x}_6, 0.6, 0.5, 0.5), (\acute{x}_7, 0.6, 0.5, 0.5),\\ &(\acute{x}_8, 0.6, 0.5, 0.5), (\acute{x}_9, 0.6, 0.5, 0.5), (\acute{x}_{10}, 0.6, 0.5, 0.5), (\acute{x}_{11}, 0.6, 0.5, 0.5)\}.\end{aligned}$$

We can see that $(\zeta, \mathbb{M})$ is an SNSS over $\mathcal{K}$. By Theorem 1, it is evident that $\zeta(\theta)$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra for all $\theta \in \mathbb{M}$. Since $\mathcal{T}_{\mathcal{A}}(u) \geq \min(\mathcal{T}_{\mathcal{A}}(e), 0.5)$, $\mathcal{I}_{\mathcal{A}}(u) \geq \min(\mathcal{I}_{\mathcal{A}}(e), 0.5)$, $\mathcal{F}_{\mathcal{A}}(u) \leq \max(\mathcal{F}_{\mathcal{A}}(e), 0.5)$ and $\mathcal{T}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{T}_{\mathcal{A}}(u), \mathcal{T}_{\mathcal{A}}(v), 0.5)$, $\mathcal{I}_{\mathcal{A}}(u \odot v) \geq \min(\mathcal{I}_{\mathcal{A}}(u), \mathcal{I}_{\mathcal{A}}(v), 0.5)$ and $\mathcal{F}_{\mathcal{A}}(u \odot v) \leq \max(\mathcal{F}_{\mathcal{A}}(u), \mathcal{F}_{\mathcal{A}}(v), 0.5)$, for all $u, v \in G$. This implies that $(\zeta, \mathbb{M})$ is an $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebra of $\mathcal{K}$.

Theorem 5. *If the pair $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ are two $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebras, then $(\zeta, \mathbb{M}) \wedge (\eta, \mathbb{N})$ is also an $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebra of $\mathcal{K}$.*

Proof. Consider a K-algebra $\mathcal{K}$. Let for any $(l, m) \in \mathbb{Q}$, $\zeta(l)$ and $\eta(m)$ be two $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebras, where $\mathbb{Q} = \mathbb{M} \times \mathbb{N}$. This implies that $\vartheta(l, m) = \zeta(l) \cap \eta(m)$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$. Hence, $(\zeta, \mathbb{M}) \wedge (\eta, \mathbb{N})$ is an $(\in, \in, \vee q)$-single-valued neutrosophic soft K-subalgebra over $\mathcal{K}$. □

Example 5. *Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where G is the cyclic group of order nine given as $G = \{e, w, w^2, w^3, w^4, w^5, w^6, w^7, w^8\}$, and Cayley's table for $\odot$ is given in Example 2.*
Consider a set of parameters $\mathbb{M} = \{l_1, l_2\}$ and a set-valued function $\zeta : \mathbb{M} \to P(G)$ defined as:

$$\begin{aligned}\zeta(l_1) = \{&(e, 0.9, 0.3, 0.3), (w, 0.6, 0.3, 0.4), (w^2, 0.6, 0.3, 0.4), (w^3, 0.6, 0.3, 0.4),\\ &(w^4, 0.6, 0.3, 0.4), (w^5, 0.6, 0.3, 0.4), (w^6, 0.6, 0.3, 0.4),\\ &(w^7, 0.6, 0.3, 0.4), (w^8, 0.6, 0.3, 0.4)\},\end{aligned}$$

$$\begin{aligned}\zeta(l_2) = \{&(e, 0.9, 0.8, 0.4), (w, 0.8, 0.5, 0.5), (w^2, 0.8, 0.5, 0.5), (w^3, 0.8, 0.5, 0.5),\\ &(w^4, 0.8, 0.5, 0.5), (w^5, 0.8, 0.5, 0.5), (w^6, 0.8, 0.5, 0.5),\\ &(w^7, 0.8, 0.5, 0.5), (w^8, 0.8, 0.5, 0.5)\},\end{aligned}$$

Now, consider a set $\mathbb{N} = \{t_1, t_2\}$ of parameters and a set-valued function $\eta : \mathbb{N} \to P(G)$ defined as:

$$\eta(t_1) = \{(e, 0.9, 0.8, 0.3), (w, 0.6, 0.7, 0.4), (w^2, 0.6, 0.7, 0.4), (w^3, 0.6, 0.7, 0.4), (w^4, 0.6, 0.7, 0.4), (w^5, 0.6, 0.7, 0.4), (w^6, 0.6, 0.7, 0.4), (w^7, 0.6, 0.7, 0.4), (w^8, 0.6, 0.7, 0.4)\},$$
$$\eta(t_2) = \{(e, 0.7, 0.7, 0.5), (w, 0.5, 0.6, 0.3), (w^2, 0.5, 0.6, 0.3), (w^3, 0.5, 0.6, 0.3), (w^4, 0.5, 0.6, 0.3), (w^5, 0.5, 0.6, 0.3), (w^6, 0.5, 0.6, 0.3), (w^7, 0.5, 0.6, 0.3), (w^8, 0.5, 0.6, 0.3)\}.$$

Clearly, the set $(\zeta, \mathbb{M})$ and the set $(\eta, \mathbb{N})$ comprises $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras. By Theorem 1, the sets $\zeta(l)$, $\eta(t)$ are $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebras for all $l \in \mathbb{M}$ and for all $t \in \mathbb{N}$. For all $(l, t) \in \mathbb{M} \times \mathbb{N}$, $(\zeta, \mathbb{M})$ AND $(\eta, \mathbb{N})$= $(\zeta, \mathbb{M})\wedge$ $(\eta, \mathbb{N}) = (\vartheta, \mathbb{M} \times \mathbb{N})$, where a set-valued function $\vartheta : \mathbb{M} \times \mathbb{N} \to P(G)$ is defined as:

$$\vartheta(l_1, t_1) = \{(e, 0.9, 0.3, 0.3), (w, 0.6, 0.3, 0.4), (w^2, 0.6, 0.3, 0.4), (w^3, 0.6, 0.3, 0.4), (w^4, 0.6, 0.3, 0.4), (w^5, 0.6, 0.3, 0.4), (w^6, 0.6, 0.3, 0.4), (w^7, 0.6, 0.3, 0.4), (w^8, 0.6, 0.3, 0.4)\},$$

$$\vartheta(l_2, t_2) = \{(e, 0.7, 0.7, 0.5), (w, 0.5, 0.5, 0.5), (w^2, 0.5, 0.5, 0.5), (w^3, 0.5, 0.5, 0.5), (w^4, 0.5, 0.5, 0.5), (w^5, 0.5, 0.5, 0.5), (w^6, 0.5, 0.5, 0.5), (w^7, 0.5, 0.5, 0.5), (w^8, 0.5, 0.5, 0.5)\},$$

$$\vartheta(l_1, t_2) = \{(e, 0.7, 0.3, 0.5), (w, 0.5, 0.3, 0.4), (w^2, 0.5, 0.3, 0.4), (w^3, 0.5, 0.3, 0.4), (w^4, 0.5, 0.3, 0.4), (w^5, 0.5, 0.3, 0.4), (w^6, 0.5, 0.3, 0.4), (w^7, 0.5, 0.3, 0.4), (w^8, 0.5, 0.3, 0.4)\},$$

$$\vartheta(l_2, t_1) = \{(e, 0.9, 0.8, 0.4), (w, 0.6, 0.5, 0.5), (w^2, 0.6, 0.5, 0.5), (w^3, 0.6, 0.5, 0.5), (w^4, 0.6, 0.5, 0.5), (w^5, 0.6, 0.5, 0.5), (w^6, 0.6, 0.5, 0.5), (w^7, 0.6, 0.5, 0.5), (w^8, 0.6, 0.5, 0.5)\}.$$

Clearly, $(\vartheta, \mathbb{M} \times \mathbb{N}) = \vartheta(l, t) = \zeta(l) \cap \eta(t)$for all $(l, t) \in \mathbb{M} \times \mathbb{N}$ is an $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras.

Theorem 6. *If $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ are two $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebras of $\mathcal{K}$ with $\mathbb{M} \cap \mathbb{N} \neq \emptyset$, then $(\zeta, \mathbb{M})\cap(\zeta, \mathbb{N})$ is an $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebras over $\mathcal{K}$.*

Proof. By Definition 16, for any $l \in \mathbb{Q}$, both $\zeta(l)$ and $\eta(l)$ are $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebras since $(\zeta, \mathbb{M})$ and $(\eta, \mathbb{N})$ are $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebras. Therefore, $\vartheta(l) = \zeta(l) \cap \eta(l)$ is an $(\in, \in \vee q)$-single-valued neutrosophic K-subalgebra. Consequently, $(\zeta, \mathbb{M})\cap(\eta, \mathbb{N})$ is an $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebra of $\mathcal{K}$. □

Example 6. *Consider a K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$, where G is the cyclic group of order nine given as $G = \{e, w, w^2, w^3, w^4, w^5, w^6, w^7, w^8\}$, and Cayley's table for $\odot$ is given in Example 2. Consider a set of parameters $\mathbb{M} = \{l_1, l_2\}$ and set-valued function $\zeta : \mathbb{M} \to P(G)$ and a set of parameters $\mathbb{N} = \{t_1, t_2\}$ with set-valued functions $\eta : \mathbb{N} \to P(G)$, which are defined in Example 5.*

We show that if $\mathbb{M} \cap \mathbb{N} \neq \emptyset$, then $(\zeta, \mathbb{M})\cap(\zeta, \mathbb{N})$ is an $(\in, \in \vee q)$-single-valued neutrosophic soft K-subalgebra over $\mathcal{K}$. Now, if $\mathbb{M} = \{l_1, l_2\}$, $\mathbb{N} = \{t_1, t_2\}$, $\mathbb{M} \cap \mathbb{N} \neq \emptyset$, $(\zeta, \mathbb{M})\cap(\eta, \mathbb{N}) = (\vartheta, \mathbb{Q})$ and $\vartheta : \mathbb{Q} \to P(Z)$ is a set-valued function, where $\mathbb{Q} = \mathbb{M} \cap \mathbb{N}$, then the following cases can be considered.

(i) $\mathbb{Q} = \mathbb{M} \cap \mathbb{N} = \{l_1\}$, *whenever* $t_1 = l_1$ *or* $t_2 = l_1$.
(ii) $\mathbb{Q} = \mathbb{M} \cap \mathbb{N} = \{l_2\}$, *whenever* $t_1 = l_2$ *or* $t_2 = l_2$.
(iii) $\mathbb{Q} = \mathbb{M} \cap \mathbb{N} = \{t_1\}$, *whenever* $l_1 = t_1$ *or* $l_2 = t_1$.
(iv) $\mathbb{Q} = \mathbb{M} \cap \mathbb{N} = \{t_2\}$, *whenever* $l_1 = t_2$ *or* $l_2 = t_2$.

Now, in each case, set-valued function ϑ *is defined as:*

$\vartheta(l_1) = \zeta(l_1) \cap \eta(t_1) = \vartheta(t_1) = \{(e, 0.9, 0.3, 0.3), (w, 0.6, 0.3, 0.4), (w^2, 0.6, 0.3, 0.4), (w^3, 0.6, 0.3, 0.4), (w^4, 0.6, 0.3, 0.4), (w^5, 0.6, 0.3, 0.4), (w^6, 0.6, 0.3, 0.4), (w^7, 0.6, 0.3, 0.4), (w^8, 0.6, 0.3, 0.4)\}$,
where $t_1 = l_1$ *or* $l_1 = t_1$,

$\vartheta(l_2) = \zeta(l_2) \cap \eta(t_1) = \{(e, 0.9, 0.8, 0.4), (w, 0.6, 0.5, 0.5), (w^2, 0.6, 0.5, 0.5), (w^3, 0.6, 0.5, 0.5), (w^4, 0.6, 0.5, 0.5), (w^5, 0.6, 0.5, 0.5), (w^6, 0.6, 0.5, 0.5), (w^7, 0.6, 0.5, 0.5), (w^8, 0.6, 0.5, 0.5)\}$, *where* $t_1 = l_2$,

$\vartheta(t_2) = \zeta(l_1) \cap \eta(t_2) = \{(e, 0.7, 0.3, 0.5), (w, 0.5, 0.3, 0.4), (w^2, 0.5, 0.3, 0.4), (w^3, 0.5, 0.3, 0.4), (w^4, 0.5, 0.3, 0.4), (w^5, 0.5, 0.3, 0.4), (w^6, 0.5, 0.3, 0.4), (w^7, 0.5, 0.3, 0.4), (w^8, 0.5, 0.3, 0.4)\}$, *where* $l_1 = t_2$.

Clearly $\vartheta(\theta)$, *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra for all* $\theta \in \mathbb{Q}$, *which implies that* $(\vartheta, \mathbb{Q})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$, *where* $\mathbb{Q} = \mathbb{M} \cap \mathbb{N}$.

Theorem 7. *If* $(\zeta, \mathbb{M})$ *is an* $(\in, \in \vee q)$ *K-subalgebra of* $\mathcal{K}$, *then for a non-empty collection* $\{(\vartheta_i, \mathbb{N}_i) \mid i \in \Omega\}$ *of* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebras of* $(\zeta, \mathbb{M})$, *the following results hold:*

(i) $\bigcap_{i \in \Omega} {}_{ex} (\vartheta_i, \mathbb{N}_i)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $(\zeta, \mathbb{M})$.
(ii) $\bigwedge_{i \in \Omega} (\vartheta_i, \mathbb{N}_i)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\bigwedge_{i \in \Omega} (\zeta, \mathbb{M})$.
(iii) *For the disjoint intersection of two parametric sets* $\mathbb{N}_i, \mathbb{N}_j$, $\forall i, j \in \Omega$, $\bigvee_{i \in \Omega} {}_{ex} (\vartheta_i, \mathbb{N}_i)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\bigvee_{i \in \Omega} (\zeta, \mathbb{M})$.

Proof. The proof follows from Definitions 14, 18 and 19. □

Theorem 8. *Let* $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *be two* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebras, then* $(\zeta, \mathbb{M}) \cap_{ex} (\eta, \mathbb{N})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$.

Proof. By Definition 14, let for any $l \in \mathbb{Q}$ the following three conditions arise:

(1) If $l \in \mathbb{M} - \mathbb{N}$, then $\vartheta(l) = \zeta(l)$ is an $(\in, \in, \vee q)$-single-valued neutrosophic K-subalgebra since $(\zeta, \mathbb{M})$ is an $(\in, \in, \vee q)$-single-valued neutrosophic soft K-subalgebra of $\mathcal{K}$.
(2) If $l \in \mathbb{N} - \mathbb{M}$, then we have $\vartheta(l) = \eta(l)$, which is an $(\in, \in, \vee q)$-single-valued neutrosophic K-subalgebra since $(\eta, \mathbb{N})$ is an $(\in, \in, \vee q)$-single-valued neutrosophic soft K-subalgebra of $\mathcal{K}$.
(3) Now, if $l \in \mathbb{M} \cap \mathbb{N}$, then $\vartheta(l) = \zeta(l) \cap \eta(l)$, which is also an $(\in, \in, \vee q)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$. Therefore, in each case, $\vartheta(l)$ is an $(\in, \in, \vee q)$-single-valued neutrosophic K-subalgebra. Consequently, $(\zeta, \mathbb{M}) \cap_{ex} (\eta, \mathbb{N})$ is an $(\in, \in, \vee q)$-single-valued neutrosophic soft K-subalgebras of $\mathcal{K}$.

□

Example 7. *Consider a K-algebra* $\mathcal{K} = (G, \cdot, \odot, e)$, *where G is the cyclic group of order nine given as* $G = \{e, w, w^2, w^3, w^4, w^5, w^6, w^7, w^8\}$, *and Cayley's table for* $\odot$ *is given in Example 2. Consider a set of* *parameters* $\mathbb{M} = \{l_1, l_2\}$*, set-valued function* $\zeta : \mathbb{M} \to P(G)$ *and a set of parameters* $\mathbb{N} = \{t_1, t_2\}$ *with set-valued functions* $\eta : \mathbb{N} \to P(G)$*, where* ζ *at parameters* l_1, l_2 *and* η *at parameters* t_1, t_2 *are defined as:*

$$\zeta(l_1) = \{(e, 0.9, 0.3, 0.3), (w, 0.6, 0.3, 0.4), (w^2, 0.6, 0.3, 0.4), (w^3, 0.6, 0.3, 0.4), (w^4, 0.6, 0.3, 0.4), (w^5, 0.6, 0.3, 0.4), (w^6, 0.6, 0.3, 0.4), (w^7, 0.6, 0.3, 0.4), (w^8, 0.6, 0.3, 0.4)\},$$

$$\zeta(l_2) = \{(e, 0.9, 0.8, 0.4), (w, 0.8, 0.5, 0.5), (w^2, 0.8, 0.5, 0.5), (w^3, 0.8, 0.5, 0.5), (w^4, 0.8, 0.5, 0.5), (w^5, 0.8, 0.5, 0.5), (w^6, 0.8, 0.5, 0.5), (w^7, 0.8, 0.5, 0.5), (w^8, 0.8, 0.5, 0.5)\},$$

and

$$\eta(t_1) = \{(e, 0.9, 0.8, 0.3), (w, 0.6, 0.7, 0.4), (w^2, 0.6, 0.7, 0.4), (w^3, 0.6, 0.7, 0.4), (w^4, 0.6, 0.7, 0.4), (w^5, 0.6, 0.7, 0.4), (w^6, 0.6, 0.7, 0.4), (w^7, 0.6, 0.7, 0.4), (w^8, 0.6, 0.7, 0.4)\},$$
$$\eta(t_2) = \{(e, 0.7, 0.7, 0.5), (w, 0.5, 0.6, 0.3), (w^2, 0.5, 0.6, 0.3), (w^3, 0.5, 0.6, 0.3), (w^4, 0.5, 0.6, 0.3), (w^5, 0.5, 0.6, 0.3), (w^6, 0.5, 0.6, 0.3), (w^7, 0.5, 0.6, 0.3), (w^8, 0.5, 0.6, 0.3)\}.$$

Clearly, by Example 5, $(\zeta, \mathbb{M})$ *and* $(\eta, \mathbb{N})$ *are* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebras. Now, to show that* $(\zeta, \mathbb{M}) \cap_{ex} (\eta, \mathbb{N}) = (\vartheta, \mathbb{Q})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$*, where* $\mathbb{Q} = \mathbb{M} \cup \mathbb{N} = \{l_1, l_2, t_1, t_2\}$*, then by Definition 14, the following conditions can be considered:*

(i) *If* $\theta \in \mathbb{M} - \mathbb{N}$*, then* $\theta = \{l_1, l_2\}$ *and set-valued function* ϑ *at parameters* l_1, l_2 *is defined as:*
$\vartheta(l_1) = \zeta(l_1) = \{(e, 0.9, 0.3, 0.3), (w, 0.6, 0.3, 0.4), (w^2, 0.6, 0.3, 0.4), (w^3, 0.6, 0.3, 0.4), (w^4, 0.6, 0.3, 0.4), (w^5, 0.6, 0.3, 0.4), (w^6, 0.6, 0.3, 0.4), (w^7, 0.6, 0.3, 0.4), (w^8, 0.6, 0.3, 0.4)\}$,
$\vartheta(l_2) = \zeta(l_2) = \{(e, 0.9, 0.8, 0.4), (w, 0.8, 0.5, 0.5), (w^2, 0.8, 0.5, 0.5), (w^3, 0.8, 0.5, 0.5), (w^4, 0.8, 0.5, 0.5), (w^5, 0.8, 0.5, 0.5), (w^6, 0.8, 0.5, 0.5), (w^7, 0.8, 0.5, 0.5), (w^8, 0.8, 0.5, 0.5)\}$.
Since $\zeta(\theta)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra, therefore* $\vartheta(\theta)$ *is also an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$*, for all* $\theta \in \mathbb{M} - \mathbb{N}$.

(ii) *If* $\theta \in \mathbb{N} - \mathbb{M}$*, then* $\theta = \{t_1, t_2\}$ *and set-valued function* ϑ *at parameters* t_1, t_2 *is defined as:*
$\vartheta(t_1) = \eta(t_1) = \{(e, 0.9, 0.8, 0.3), (w, 0.6, 0.7, 0.4), (w^2, 0.6, 0.7, 0.4), (w^3, 0.6, 0.7, 0.4), (w^4, 0.6, 0.7, 0.4), (w^5, 0.6, 0.7, 0.4), (w^6, 0.6, 0.7, 0.4), (w^7, 0.6, 0.7, 0.4), (w^8, 0.6, 0.7, 0.4)\}$,
$\vartheta(t_2) = \eta(t_2) = \{(e, 0.7, 0.7, 0.5), (w, 0.5, 0.6, 0.3), (w^2, 0.5, 0.6, 0.3), (w^3, 0.5, 0.6, 0.3), (w^4, 0.5, 0.6, 0.3), (w^5, 0.5, 0.6, 0.3), (w^6, 0.5, 0.6, 0.3), (w^7, 0.5, 0.6, 0.3), (w^8, 0.5, 0.6, 0.3)\}$.
Since $\eta(\theta)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra, therefore* $\vartheta(\theta)$ *is also an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$*, for all* $\theta \in \mathbb{N} - \mathbb{M}$.

(iii) *Now, if* $\theta \in \mathbb{M} \cap \mathbb{N}$*, then* $\vartheta(\theta) = \zeta(\theta) \cap \eta(\theta)$*. By Example 6, it follows that* $\vartheta(\theta)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$*, for all* $\theta \in \mathbb{M} \cap \mathbb{N}$*. Therefore,* $(\zeta, \mathbb{M}) \cap_{ex} (\eta, \mathbb{N}) = (\vartheta, \mathbb{Q})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$.

Theorem 9. *Let* $(\zeta, \mathbb{M})$*,* $(\eta, \mathbb{N})$ *be two* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebras with* $\mathbb{M} \cap \mathbb{N} = \emptyset$*, then* $(\zeta, \mathbb{M}) \cup_{ex} (\eta, \mathbb{N})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$.

Proof. The proof follows from Definition 15. □

We denote the set of all $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras of $\mathcal{K}$ by $\mathbb{N}(G, \mathbb{R})$.

Theorem 10. *Under the ordering relation* $\subset$, $(\mathbb{SN}(G,\mathbb{R}), \cup_{ex}, \cap)$ *is a complete distributive lattice.*

Proof. Suppose that $(\zeta,\mathbb{M}),(\eta,\mathbb{N}) \in \mathbb{SN}(G,\mathbb{R})$, $(\zeta,\mathbb{M}) \cup_{ex} (\eta,\mathbb{N}) \in \mathbb{SN}(G,\mathbb{R})$ and $(\zeta,\mathbb{M}) \cap (\eta,\mathbb{N}) \in \mathbb{SN}(G,\mathbb{R})$. Consider $\{(\zeta,\mathbb{M}),(\eta,\mathbb{N})\}$ are an arbitrary collection of $(\mathbb{SN}(G,\mathbb{R}), \cup_{ex}, \cap)$, since $(\zeta,\mathbb{M}) \cup_{ex} (\eta,\mathbb{N})$ is the supremum of $(\zeta,\mathbb{M})$ and $(\zeta,\mathbb{M}) \cap (\eta,\mathbb{N})$ the infimum of $(\eta,\mathbb{N})$, which shows that $(\mathbb{SN}(G,\mathbb{R}), \cup_{ex}, \cap)$ is a complete lattice.

In order to show that it is a complete distributive lattice, i.e., for all $(\zeta,\mathbb{M}), (\eta,\mathbb{N}), (\vartheta,\mathbb{Q}) \in \mathbb{SN}(G,\mathbb{R})$, $(\zeta,\mathbb{M}) \cap ((\eta,\mathbb{N}) \cup_{ex} (\vartheta,\mathbb{Q})) = ((\zeta,\mathbb{M}) \cap (\eta,\mathbb{N})) \cup_{ex} ((\zeta,\mathbb{M}) \cap (\vartheta,\mathbb{Q}))$; let us suppose that $(\zeta,\mathbb{M}) \cap ((\eta,\mathbb{N}) \cup_{ex} (\vartheta,\mathbb{Q})) = (\mathbb{I}, \mathbb{M} \cap (\mathbb{N} \cup \mathbb{Q}))$, $((\zeta,\mathbb{M}) \cap (\eta,\mathbb{N})) \cup_{ex} ((\zeta,\mathbb{M}) \cap (\vartheta,\mathbb{Q})) = (\mathbb{J}, (\mathbb{M} \cap \mathbb{N}) \cup (\mathbb{M} \cap \mathbb{Q})) = (\mathbb{J}, \mathbb{M} \cap (\mathbb{N} \cup \mathbb{Q}))$. For any $\theta \in \mathbb{M} \cap (\mathbb{N} \cup \mathbb{Q})$, $\theta \in \mathbb{M}$ and $\theta \in \mathbb{N} \cup \mathbb{Q}$, the following cases arise:

(i) $\theta \in \mathbb{M}$, $\theta \notin \mathbb{N}$ and $\theta \in \mathbb{Q}$. Then, $\mathbb{K}(\theta) = \zeta(\theta) \cap \vartheta(\theta) = \mathbb{J}(\theta)$,

(ii) $\theta \in \mathbb{M}$, $\theta \in \mathbb{N}$ and $\theta \notin \mathbb{Q}$. Then, $\mathbb{K}(\theta) = \zeta(\theta) \cap \eta(\theta) = \mathbb{J}(\theta)$,

(iii) $\theta \in \mathbb{M}$, $\theta \in \mathbb{N}$ and $\theta \in \mathbb{Q}$. Then, $\mathbb{K}(\theta) = \zeta(\theta) \cap (\eta(\theta) \cup \vartheta(\theta))$
$= (\zeta(\theta) \cap \eta(\theta)) \cup (\zeta(\theta) \cap \vartheta(\theta)) = \mathbb{J}(\theta)$.

Both $\mathbb{J}$ and $\mathbb{K}$ being the same operators implies that $(\zeta,\mathbb{M}) \cap ((\eta,\mathbb{N}) \cup_{ex} (\vartheta,\mathbb{Q})) = ((\zeta,\mathbb{M}) \cap (\eta,\mathbb{N})) \cup_{ex} ((\zeta,\mathbb{M}) \cap (\vartheta,\mathbb{Q}))$. This completes the proof. □

Definition 26. *The extended product of two single-valued neutrosophic soft sets is denoted by* $(\zeta,\mathbb{M}) \odot (\eta,\mathbb{N}) = (\zeta \circ \eta, \mathbb{Q})$, *where* $\mathbb{Q} = \mathbb{M} \cup \mathbb{N}$ *and* $(\zeta,\mathbb{M})$ *and* $(\eta,\mathbb{N})$ *are two single-valued neutrosophic soft sets over Z, defined as for all* $\theta \in \mathbb{Q}$.

$$(\zeta \circ \eta)(\theta) = \begin{cases} \zeta(\theta) & \text{if } \theta \in \mathbb{M} - \mathbb{N}, \\ \eta(\theta) & \text{if } \theta \in \mathbb{N} - \mathbb{M}, \\ \zeta(\theta) \circ \eta(\theta) & \text{if } \theta \in \mathbb{M} \cap \mathbb{N}. \end{cases}$$

Here, $\zeta(\theta) \circ \eta(\theta)$ *is the product of two single-valued neutrosophic sets.*

Lemma 1. $(\zeta_1,\mathbb{M}),(\zeta_2,\mathbb{M}),(\eta_1,\mathbb{N}),(\eta_2,\mathbb{N})$ *are two SNSSs over* $\mathcal{K}$ *such that* $(\zeta_1,\mathbb{M}) \subset (\zeta_2,\mathbb{M})$ *and* $(\eta_1,\mathbb{N}) \subset (\eta_2,\mathbb{N})$. *Then:*

$$\begin{aligned}
&(a)\ (\zeta_1,\mathbb{M}) \odot (\eta_1,\mathbb{N}) \subset (\zeta_2,\mathbb{M}) \odot (\eta_2,\mathbb{N}),\\
&(b)\ (\zeta_1,\mathbb{M}) \cap (\eta_1,\mathbb{N}) \subset (\zeta_2,\mathbb{M}) \cap (\eta_2,\mathbb{N}),\\
&(c)\ (\zeta_1,\mathbb{M}) \cap_{ex} (\eta_1,\mathbb{N}) \subset (\zeta_2,\mathbb{M}) \cap_{ex} (\eta_2,\mathbb{N}),\\
&(d)\ (\zeta_1,\mathbb{M}) \cup (\eta_1,\mathbb{N}) \subset (\zeta_2,\mathbb{M}) \cup (\eta_2,\mathbb{N}),\\
&(e)\ (\zeta_1,\mathbb{M}) \cup_{ex} (\eta_1,\mathbb{N}) \subset (\zeta_2,\mathbb{M}) \cup_{ex} (\eta_2,\mathbb{N}).
\end{aligned}$$

Lemma 2. *Let* $(\zeta,\mathbb{M}),(\eta,\mathbb{N})$ *and* $(\vartheta,\mathbb{Q})$ *be SNSSs over* $\mathcal{K}$. *Then,* $(\zeta,\mathbb{M}) \odot ((\eta,\mathbb{N}) \odot (\vartheta,\mathbb{Q})) = ((\zeta,\mathbb{M}) \odot (\eta,\mathbb{N})) \odot (\vartheta,\mathbb{Q})$, *where* $\odot$ *is the operation of the product of SNSSs over* $\mathcal{K}$.

Theorem 11. *xxx Let* $\mathcal{K}$ *be a* K*-algebra. If* $(\zeta,\mathbb{M})$ *and* $(\eta,\mathbb{N})$ *are* $(\in, \in, \vee q)$*-single-valued neutrosophic soft* K*-subalgebras, then* $(\zeta,\mathbb{M}) \odot (\eta,\mathbb{N})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft* K*-subalgebras of* $\mathcal{K}$.

Proof. The proof follows from Definition 26. □

Example 8. *Consider a K-algebra* $\mathcal{K} = (G, \cdot, \odot, e)$, *where G is the cyclic group of order nine given as* $G = \{e, w, w^2, w^3, w^4, w^5, w^6, w^7, w^8\}$, *and Cayley's table for* $\odot$ *is given in Example 2. Consider a set of parameters* $\mathbb{M} = \{l_1, l_2\}$ *and a set-valued function* $\zeta : \mathbb{M} \to P(G)$ *defined as:*

$$\zeta(l_1) = \{(e, 0.9, 0.3, 0.3), (s, 0.6, 0.3, 0.4)\}, \text{ for all } s \neq e \in G.$$

$$\zeta(l_2) = \{(e, 0.9, 0.8, 0.4), (s^{'}, 0.8, 0.5, 0.5)\}, \text{ for all } s^{'} \neq e \in G.$$

Now, we consider a set $\mathbb{N} = \{t_1, t_2\}$ *of parameters and a set-valued function* $\eta : \mathbb{N} \to P(G)$*, which is defined as:*

$$\eta(t_1) = \{(e, 0.9, 0.8, 0.3), (w, 0.6, 0.7, 0.4)\}, \text{ for all } w \neq e \in G.$$

$$\eta(t_2) = \{(e, 0.7, 0.7, 0.5), (w^{'}, 0.5, 0.6, 0.3)\}, \text{ for all } w^{'} \neq e \in G.$$

Clearly, the set $(\zeta, \mathbb{M})$ *and the set* $(\eta, \mathbb{N})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra for all* $l_1, l_2 \in \mathbb{M}$ *and* $t_1, t_2 \in \mathbb{N}$*. Now, we show that* $(\zeta, \mathbb{M}) \odot (\eta, \mathbb{N}) = (\zeta \circ \eta, \mathbb{Q})$*, where* $\mathbb{Q} = \mathbb{M} \cup \mathbb{N}$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$*. By Definition 26, the following conditions can be considered:*

(i) *If* $\theta \in \mathbb{M} - \mathbb{N}$*, then* $\theta = \{l_1, l_2\}$ *and* $(\zeta \circ \eta)(\theta) = \zeta(\theta)$*. Since* $\zeta(\theta)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra, therefore* $(\zeta \circ \eta)$ *is also an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$*, for all* $\theta \in \mathbb{M} - \mathbb{N}$*.*

(ii) *If* $\theta \in \mathbb{N} - \mathbb{M}$*, then* $\theta = \{t_1, t_2\}$ *and* $(\zeta \circ \eta)(\theta) = \eta(\theta)$*. Therefore,* $(\zeta \circ \eta)$ *is also an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$ *since* $\eta(\theta)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra for all* $\theta \in \mathbb{N} - \mathbb{M}$*.*

(iii) *If* $\theta \in \mathbb{M} \cap \mathbb{N}$*, then* $(\zeta \circ \eta)(\theta) = \zeta(\theta) \circ \eta(\theta)$*, where* $\zeta(\theta) \circ \eta(\theta)$ *is the product of two single-valued neutrosophic sets at parameter* θ*. Then, by Example 6, four conditions can be considered since* $\theta \in \mathbb{M} \cap \mathbb{N}$ *and corresponding to each condition product can be calculated as:*
$(\zeta \circ \eta)(l_1) = \zeta(l_1) \circ \eta(t_1) = \{\langle (e,e), 0.9, 0.3, 0.3\rangle, \langle (e,w), 0.6, 0.3, 0.4\rangle, \langle (s,e), 0.6, 0.3, 0.4\rangle, \langle (s,w), 0.6, 0.3, 0.4\rangle\}$,
$(\zeta \circ \eta)(l_2) = \zeta(l_2) \circ \eta(t_1) = \{\langle (e,e), 0.9, 0.8, 0.4\rangle, \langle (e,w), 0.6, 0.7, 0.4\rangle, \langle (s^{'},e), 0.8, 0.5, 0.5\rangle, \langle (s^{'},w), 0.6, 0.5, 0.5\rangle\}$,
$(\zeta \circ \eta)(t_2) = \zeta(l_1) \circ \eta(t_2) = \{\langle (e,e), 0.7, 0.3, 0.5\rangle, \langle (e,w^{'}), 0.5, 0.3, 0.3\rangle, \langle (s,e), 0.6, 0.3, 0.5\rangle, \langle (s,w^{'}), 0.5, 0.3, 0.4\rangle\}$.

Clearly, $(\zeta \circ \eta)(\theta)$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic K-subalgebra of* $\mathcal{K}$*, for all* $\theta \in \mathbb{M} \cap \mathbb{N}$*, which shows that* $(\zeta, \mathbb{M}) \odot (\eta, \mathbb{N})$ *is an* $(\in, \in \vee q)$*-single-valued neutrosophic soft K-subalgebra of* $\mathcal{K}$*.*

Theorem 12. *Let* $\mathcal{K}$ *be a K-algebra. Then, under the ordering relation* $\subset$*,* $(\mathbb{SN}(G, \mathbb{R}), \odot, \cap)$ *is a complete lattice.*

Proof. The proof is straightforward. □

5. Conclusions

The world of science and its related fields have accomplished such complicated processes for which consistent and complete information is not always conceivable. For the last few decades, a number of theories and postulates have been introduced by many researchers to handle indeterminate constituents in science and technologies. These theories include the theory of probability, interval mathematics, fuzzy set theory, intuitionistic fuzzy set theory, neutrosophic set theory, etc. Among all these theories, a powerful mathematical tool to deal with indeterminate and inconsistent data is the neutrosophic set theory introduced by Smarandache in 1998. This theory provides a mathematical model to cope up with executions having complex phenomena towards uncertainty. In 1999, Molodtsov introduced the concept of soft set theory to deal with the problems involving indeterminacy without setting the membership function. This theory provides a parameterized consideration to uncertainties. We have applied these mathematical representations in collaboration to scrutinize the factor of uncertainty in K-algebras. A K-algebra is a new kind of non-classic logical algebra. We have introduced the notion of single-valued neutrosophic soft K-algebras and studied related properties. To give a generalized point of view of single-valued neutrosophic soft

K-algebras, we have proposed the concept of $(\in, \in \vee q)$-single-valued neutrosophic soft K-algebras and investigated various conclusive results with some numerical examples. In our opinion, the future study of K-algebras can be connected with: (1) K-modules and single-valued neutrosophic K-modules; (2) rough K-algebras and single-valued neutrosophic rough K-algebras; (3) hyper-K-algebras and single-valued neutrosophic K-algebras.

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Applications of Neutrosophic Bipolar Fuzzy Sets in HOPE Foundation for Planning to Build a Children Hospital with Different Types of Similarity Measures

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Abstract: In this paper we provide an application of neutrosophic bipolar fuzzy sets in daily life's problem related with HOPE foundation that is planning to build a children hospital, which is the main theme of this paper. For it we first develop the theory of neutrosophic bipolar fuzzy sets which is a generalization of bipolar fuzzy sets. After giving the definition we introduce some basic operation of neutrosophic bipolar fuzzy sets and focus on weighted aggregation operators in terms of neutrosophic bipolar fuzzy sets. We define neutrosophic bipolar fuzzy weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$) and neutrosophic bipolar fuzzy ordered weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operators. Next we introduce different kinds of similarity measures of neutrosophic bipolar fuzzy sets. Finally as an application we give an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators with a numerical example related with HOPE foundation.

Keywords: neutrosophic set; bipolar fuzzy set; neutrosophic bipolar fuzzy set; neutrosophic bipolar fuzzy weighted averaging operator; similarity measure; algorithm; multiple attribute decision making problem

1. Introduction

Zadeh [1] started the theory of fuzzy set and since then it has been a significant tool in learning logical subjects. It is applied in many fields, see [2]. There are numbers of over simplifications/generalization of Zadeh's fuzzy set idea to interval-valued fuzzy notion [3], intuitionistic fuzzy set [4], L-fuzzy notion [5], probabilistic fuzzy notion [6] and many others. Zhang [7,8], provided the generality of fuzzy sets as bipolar fuzzy sets. The extensions of fuzzy sets with membership grades from $[-1, 1]$, are the bipolar fuzzy sets. The membership grade $[-1, 0)$ of a section directs in bipolar fuzzy set that the section fairly fulfils the couched stand-property, the membership grade $]0, 1]$ of a section shows that the section fairly fulfils the matter and the membership grade 0 of a section resources that the section is unrelated to the parallel property. While bipolar fuzzy sets and intuitionistic fuzzy sets aspect parallel to one another, they are really distinct sets (see [3]). When we calculate the place of an objective in a universe, positive material conveyed for a collection of thinkable spaces and negative material conveyed for a collection of difficult spaces [9]. Naveed et al. [10–12], discussed theoretical aspects of bipolar fuzzy sets in detail. Smarandache [13], gave the notion of neutrosophic sets as a generalization of intutionistic fuzzy sets. The applications of Neutrosophic set theory are found in many fields (see http://fs.gallup.unm.edu/neutrosophy.htm). Recently Zhang et al. [14], Majumdar et al. [15], Liu et al. [16,17], Peng et al. [18] and Sahin et al. [19] have discussed various uses of neutrosophic set theory in deciding problems. Now a days, neutrosophic sets are very actively used in applications and MCGM problems. Bausys and Juodagalviene [20], Qun et al. [21], Zavadskas et al. [22], Chan and Tan [23], Hong and Choi [24], Zhan et al. [25] studied the applications of neutrosophic cubic sets in multi-criteria decision making in different directions. Anyhow, these approaches use the maximum, minimum operations to workout the aggregation procedure. This leads to subsequent loss of data and, therefore, inaccurate last results. How ever this restriction can be dealt by using famous weighted averaging (WA) operator [26] and the ordered weighted averaging (OWA) operator [27]. Medina and Ojeda-Aciego [28], gave t-notion lattice as a set of triples related to graded tabular information explained in a non-commutative fuzzy logic. Medina et al. [28] introduces a new frame work for the symbolic representation of informations which is called to as signatures and given a very useful technique in fuzzy modelling. In [29], Nowaková et al., studied a novel technique for fuzzy medical image retrieval (FMIR) by vector quantization (VQ) with fuzzy signatures in conjunction with fuzzy S-trees. In [30] Kumar et al., discussed data clustering technique, Fuzzy C-Mean algorithem and moreover Artificial Bee Colony (ABC) algorithm. In [31] Scellato et al.,discuss the rush of vehicles in urban street networks. Recently Gulistan et al. [32], combined neutrosophic cubic sets and graphs and gave the concept of neutrosophic cubic graphs with practical life applications in different areas. For more application of neutrosophic sets, we refer the reader to [33–37]. Since, the models presented in literature have different limitations in different situations. We mainly concern with the following tools:

(1) Neutrosophic sets are the more summed up class by which one can deal with uncertain informations in a more successful way when contrasted with fuzzy sets and all other versions of fuzzy sets. Neutrosophic sets have the greater adaptability, accuracy and similarity to the framework when contrasted with past existing fuzzy models.
(2) And bipolar fuzzy sets are proved to very affective in uncertain problems which can characterized not only the positive characteristics but also the negative characteristics of a certain problem.

We try to blend these two concepts together and try to develop a more powerful tool in the form of neutrosophic bipolar fuzzy sets. In this work we initiate the study of neutrosophic bipolar fuzzy sets which are the generalization of bipolar fuzzy sets and neutrosophic sets. After introducing the definition we give some basic operations, properties and applications of neutrosophic bipolar fuzzy sets. And the rest of the paper is structured as follows; Section 2 provides basic material from the existing literature to understand our proposal. Section 3 consists of the basic notion and properties of neutrosophic bipolar fuzzy set. Section 4 gives the role of weighted aggregation operator in terms of neutrosophic bipolar fuzzy sets. We define neutrosophic bipolar fuzzy weighted averaging operator ($\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$) and neutrosophic bipolar fuzzy ordered weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operators. Section 5 includes different kinds of similarity measures. In Section 6, an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of similarity measures of neutrosophic bipolar fuzzy sets and neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators is proposed. In Section 7, we provide a daily life example related with HOPE foundation, which shows the applicability of the algorithm provided in Section 6. In Section 8, we provide a comparison with the previous existing methods. In Section 9, we discuss conclusion and some future research directions.

2. Preliminaries

Here we provide some basic material from the literature for subsequent use.

Definition 1. *Let $\mathcal{Y}$ be any nonempty set. Then a bipolar fuzzy set [7,8], is an object of the form*

$$B = \langle u, \langle \mu^{+}(u),\ \mu^{-}(u)\rangle : u \in \mathcal{Y}\rangle,$$

and $\mu^{+}(u) : \mathcal{Y} \to [0,1]$ and $\mu^{-}(u) : \mathcal{Y} \to [-1,0]$, $\mu^{+}(u)$ is a positive material and $\mu^{-}(u)$ is a negative material of $u \in \mathcal{Y}$. For simplicity, we donate the bipolar fuzzy set as $B = \langle \mu^{+},\ \mu^{-}\rangle$ in its place of $B = \langle u, \langle \mu^{+}(u), \mu^{-}(u)\rangle : u \in \mathcal{Y}\rangle$.

Definition 2. *Let $B_1 = \langle \mu_1^{+}, \mu_1^{-}\rangle$ and $B_2 = \langle \mu_2^{+}, \mu_2^{-}\rangle$ be two bipolar fuzzy sets [7,8], on $\mathcal{Y}$. Then we define the following operations.*

(1) $B_1^{'} = \{\langle 1 - \mu_1^{+}(u), -1 - \mu_1^{-}(u)\rangle\}$;
(2) $B_1 \cup B_2 = \langle \max(\mu_1^{+}(u), \mu_2^{+}(u)), \min(\mu_1^{-}(u), \mu_2^{-}(u))\rangle$;
(3) $B_1 \cap B_2 = \langle \min(\mu_1^{+}(u), \mu_1^{+}(u)), \max(\mu_1^{-}(u), \mu_2^{-}(u))\rangle$.

Definition 3. *A neutrosophic set [13], is define as:*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x)\rangle : x \in X\},$$

where X is a universe of discoveries and L is characterized by a truth-membership function $\boldsymbol{Tru}_L : X \to]0^{-}, 1^{+}[$, an indtermency-membership function $\boldsymbol{Ind}_L : X \to]0^{-}, 1^{+}[$ and a falsity-membership function $\boldsymbol{Fal}_L : X \to]0^{-}, 1^{+}[$ such that $0 \le \boldsymbol{Tru}_L(x) + \boldsymbol{Ind}_L(x) + \boldsymbol{Fal}_L(x) \le 3$.

Definition 4. *A single valued neutrosophic set [16], is define as:*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x)\rangle : x \in X\},$$

where X is a universe of discoveries and L is characterized by a truth-membership function $\boldsymbol{Tru}_L : X \to [0,1]$, an indtermency-membership function $\boldsymbol{Ind}_L : X \to [0,1]$ and a falsity-membership function $\boldsymbol{Fal}_L : X \to [0,1]$ such that $0 \le \boldsymbol{Tru}_L(x) + \boldsymbol{Ind}_L(x) + \boldsymbol{Fal}_L(x) \le 3$.

Definition 5. *Let [16]*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x)\rangle : x \in X\},$$

and

$$B = \{\langle x, \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_B(x)\rangle : x \in X\},$$

be two single valued neutrosophic sets. Then

(1) *$L \subset B$ if and only if $\boldsymbol{Tru}_L(x) \le \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_L(x) \le \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_L(x) \ge \boldsymbol{Fal}_B(x)$.*
(2) *$L = B$ if and only if $\boldsymbol{Tru}_L(x) = \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_L(x) = \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_L(x) = \boldsymbol{Fal}_B(x)$, for any $x \in X$.*
(3) *The complement of L is denoted by L^c and is defined by*

$$L^c = \{\langle x, \boldsymbol{Fal}_L(x), 1 - \boldsymbol{Ind}_L(x), \boldsymbol{Tru}_L(x)\rangle / x \in X\}.$$

(4) *The intersection*

$$L \cap B = \{\langle x, \min\{\boldsymbol{Tru}_L(x), \boldsymbol{Tru}_B(x)\},\ \max\{\boldsymbol{Ind}_L(x), \boldsymbol{Ind}_B(x)\}, \max\{\boldsymbol{Fal}_L(x), \boldsymbol{Fal}_B(x)\}\rangle : x \in X\}.$$

(5) The Union

$$L \cup B \;= \{\langle x, \max\{\boldsymbol{Tru}_L(x), \boldsymbol{Tru}_B(x)\},\; \min\{\boldsymbol{Ind}_L(x), \boldsymbol{Ind}_B(x)\}, \min\{\boldsymbol{Fal}_L(x), \boldsymbol{Fal}_B(x)\}\rangle : x \in X\}.$$

Definition 6. *Let $\tilde{A}_1 = \langle \boldsymbol{Tru}_1, \boldsymbol{Ind}_1, \boldsymbol{Fal}_1\rangle$ and $\tilde{A}_2 = \langle \boldsymbol{Tru}_2, \boldsymbol{Ind}_2, \boldsymbol{Fal}_2\rangle$ be two single valued neutrosophic number [16]. Then, the operations for NNs are defined as below:*

(1) $\lambda\tilde{A} = \left\langle 1-(1-\boldsymbol{Tru}_1)^{\lambda}, \boldsymbol{Ind}_1^{\lambda}, \boldsymbol{Fal}_1^{\lambda}\right\rangle$;
(2) $\tilde{A}_1^{\lambda} = \left\langle \boldsymbol{Tru}_1^{\lambda}, 1-(1-\boldsymbol{Ind}_1)^{\lambda}, 1-(1-\boldsymbol{Fal}_1)^{\lambda}\right\rangle$;
(3) $\tilde{A}_1 + \tilde{A}_2 = \langle \boldsymbol{Tru}_1 + \boldsymbol{Tru}_2 - \boldsymbol{Tru}_1\boldsymbol{Tru}_2, \boldsymbol{Ind}_1\boldsymbol{Ind}_2, \boldsymbol{Fal}_1\boldsymbol{Fal}_2\rangle$;
(4) $\tilde{A}_1\tilde{A}_2 = \langle \boldsymbol{Tru}_1\boldsymbol{Tru}_2, \boldsymbol{Ind}_1 + \boldsymbol{Ind}_{2-}\boldsymbol{Ind}_1\boldsymbol{Ind}_2, \boldsymbol{Fal}_1 + \boldsymbol{Fal}_2 - \boldsymbol{Fal}_1\boldsymbol{Fal}_2\rangle$ *where* $\lambda > 0$.

Definition 7. *Let $\tilde{A}_1 = \langle \boldsymbol{Tru}_1, \boldsymbol{Ind}_1, \boldsymbol{Fal}_1\rangle$ be a single valued neutrosophic number [16]. Then, the score function $s(\tilde{A}_1)$,accuracy function $L(\tilde{A}_1)$, and certainty function $c(\tilde{A}_1)$, of an NNs are define as under:*

(1) $s(\tilde{A}_1) = \frac{(\boldsymbol{Tru}_1 + 1 - \boldsymbol{Ind}_1 + 1 - \boldsymbol{Fal}_1)}{3}$;
(2) $L(\tilde{A}_1) = \boldsymbol{Tru}_1 - \boldsymbol{Fal}_1$;
(3) $c(\tilde{A}_1) = \boldsymbol{Tru}_1$.

3. Neutrosophic Bipolar Fuzzy Sets and Operations

In this section we apply bipolarity on neutrosophic sets and initiate the notion of neutrosophic bipolar fuzzy set with the help of Section 2, which is the generalization of bipolar fuzzy set. We also study some basic operation on neutrosophic bipolar fuzzy sets.

Definition 8. *A neutrosophic bipolar fuzzy set is an object of the form $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ where*

$$\begin{aligned} \mathcal{N}^{\mathcal{B}+} &= \langle u, \langle \boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}\rangle : u \in \mathcal{Y}\rangle, \\ \mathcal{N}^{\mathcal{B}-} &= \langle u, \langle \boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}\rangle : u \in \mathcal{Y}\rangle, \end{aligned}$$

where $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}} : \mathcal{Y} \to [0,1]$ and $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}} : \mathcal{Y} \to [-1,0]$.

Note: In the Definition 8, we see that a neutrosophic bipolar fuzzy sets $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$, consists of two parts, positive membership functions $\mathcal{N}^{\mathcal{B}+}$ and negative membership functions $\mathcal{N}^{\mathcal{B}-}$. Where positive membership function $\mathcal{N}^{\mathcal{B}+}$ denotes what is desirable and negative membership function $\mathcal{N}^{\mathcal{B}-}$ denotes what is unacceptable. Desirable characteristics are further characterize as: $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in past, $\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in future and $\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in present time. Similarly $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in past, $\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in future and $\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in present time.

Definition 9. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets. Then we define the following operations:*

(1) $\mathcal{N}_1^{\mathcal{B}^c} = \left\{\left\langle 1 - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, 1 - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, -1 - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}} \text{ and } 1 - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, 1 - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, -1 - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right\rangle\right\}$;
(2)

$$\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}} = \left\langle \begin{array}{c} \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}), \\ \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}) \end{array} \right\rangle;$$

(3)

$$\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}} = \left\langle \begin{array}{c} \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}), \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}), \\ \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}), \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}). \end{array} \right\rangle.$$

Definition 10. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets. Then we define the following operations:*

(1)

$$\mathcal{N}_1^{\mathcal{B}+} \oplus \mathcal{N}_2^{\mathcal{B}+} = \left\langle \begin{array}{c} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}} - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}} - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}, \\ -(\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}\right|) \end{array} \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} \oplus \mathcal{N}_2^{\mathcal{B}-} = \left\langle \begin{array}{c} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}} - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}} - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}, \\ -(\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}\right|) \end{array} \right\rangle;$$

(2)

$$\mathcal{N}_1^{\mathcal{B}+} \otimes \mathcal{N}_2^{\mathcal{B}+} = \left\langle \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}} - (\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}\right|) \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} \otimes \mathcal{N}_2^{\mathcal{B}-} = \left\langle \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}} - (\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}\right|) \right\rangle;$$

(3)

$$\mathcal{N}_1^{\mathcal{B}+} - \mathcal{N}_2^{\mathcal{B}+} = \left\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}),\ \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}) \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} - \mathcal{N}_2^{\mathcal{B}-} = \left\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}),\ \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}) \right\rangle.$$

Definition 11. *Let $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ be a neutrosophic bipolar fuzzy set and $\lambda > 0$. Then,*

(1)

$$\begin{aligned} \lambda\mathcal{N}^{\mathcal{B}+} &= \langle 1-(1-\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, 1-(1-\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, -|\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}|^{\lambda}\rangle, \\ \lambda\mathcal{N}^{\mathcal{B}-} &= \langle 1-(1-\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, 1-(1-\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, -|\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}|^{\lambda}\rangle. \end{aligned}$$

(2)

$$\begin{aligned} \mathcal{N}^{\mathcal{B}+^{\lambda}} &= \langle (\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}\ , (\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, -1+|-1+\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}|^{\lambda}\rangle, \\ \mathcal{N}^{\mathcal{B}-\lambda} &= \langle (\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, (\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, -1+|-1+\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}(u)|^{\lambda}\rangle. \end{aligned}$$

Theorem 1. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$, $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ and $\mathcal{N}_3^{\mathcal{B}} = (\mathcal{N}_3^{\mathcal{B}+}, \mathcal{N}_3^{\mathcal{B}-})$ be neutrosophic bipolar fuzzy sets. Then, the following properties hold:*

(1) *Complementary law: $(\mathcal{N}_1^{\mathcal{B}c})^c = \mathcal{N}_1^{\mathcal{B}}$.*

(2) *Idempotent law:*

$$\begin{aligned} (i)\ \mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_1^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}}, \\ (ii)\ \mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_1^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}}. \end{aligned}$$

(3) *Commutative law:*

$$\begin{aligned} (i)\ \mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_1^{\mathcal{B}}, \\ (ii)\ \mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_1^{\mathcal{B}}, \\ (iii)\ \mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \oplus \mathcal{N}_1^{\mathcal{B}}, \\ (iv)\ \mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \otimes \mathcal{N}_1^{\mathcal{B}}. \end{aligned}$$

(4) *Associative law:*

$$\begin{aligned} (i)\ (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}}) \cup \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \cup (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}), \\ (ii)\ (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}) \cap \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \cap \left(\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}\right), \\ (iii)\ (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \oplus \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}), \\ (iv)\ (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \otimes \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}). \end{aligned}$$

(5) *Distributive law:*

$$\begin{aligned} (i)\ \mathcal{N}_1^{\mathcal{B}} \cup (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}), \\ (ii)\ \mathcal{N}_1^{\mathcal{B}} \cap (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}), \\ (iii)\ \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}), \\ (iv)\ \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}), \\ (v)\ \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}), \\ (vi)\ \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}). \end{aligned}$$

(6) *De Morgan$'$s laws:*

$$\begin{aligned} (i)\ (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}})^c &= \mathcal{N}_1^{\mathcal{B}c} \cap \mathcal{N}_2^{\mathcal{B}c}, \\ (ii)\ \left(\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}\right)^c &= \mathcal{N}_1^{\mathcal{B}c} \cup \mathcal{N}_2^{\mathcal{B}c}, \\ (iii)\ (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}})^c &\neq \mathcal{N}_1^{\mathcal{B}c} \otimes \mathcal{N}_2^{\mathcal{B}c}, \\ (iv)\ \left(\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}\right)^c &\neq \mathcal{N}_1^{\mathcal{B}c} \oplus \mathcal{N}_2^{\mathcal{B}c}. \end{aligned}$$

Proof. Straightforward. □

Theorem 2. *Let* $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ *and* $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ *be two neutrosophic bipolar fuzzy sets and let* $\mathcal{N}_3^{\mathcal{B}} = \mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}$ *and* $\mathcal{N}_4^{\mathcal{B}} = \lambda \mathcal{N}_1^{\mathcal{B}}$ $(\lambda > 0)$*. Then both* $\mathcal{N}_3^{\mathcal{B}}$ *and* $\mathcal{N}_4^{\mathcal{B}}$ *are also neutrosophic bipolar fuzzy sets.*

Proof. Straightforward. □

Theorem 3. *Let* $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ *and* $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ *be two neutrosophic bipolar fuzzy sets,* $\lambda, \lambda_1, \lambda_2 > 0$*. Then, we have:*

$$(i)\ \lambda(\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) = \lambda\mathcal{N}_1^{\mathcal{B}} \oplus \lambda\mathcal{N}_2^{\mathcal{B}},$$
$$(ii)\ \lambda_1\mathcal{N}_1^{\mathcal{B}} \oplus \lambda_2\mathcal{N}_2^{\mathcal{B}} = (\lambda_1 \oplus \lambda_2)\mathcal{N}_1^{\mathcal{B}}.$$

Proof. Straightforward. □

4. Neutrosophic Bipolar Fuzzy Weighted/Fuzzy Ordered Weighted Aggregation Operators

After defining neutrosophic bipolar fuzzy sets and some basic operations in Section 3. We in this section as applications point of view we focus on weighted aggregation operator in terms of neutrosophic bipolar fuzzy sets. We define $(\mathcal{N}^{\mathcal{B}}\mathcal{FWA})$ and $(\mathcal{N}^{\mathcal{B}}\mathcal{FOWA})$ operators.

Definition 12. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then we define $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ as a mapping $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k : \Omega^n \to \Omega$ by*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}} \oplus, ..., \oplus k_n\mathcal{N}_n^{\mathcal{B}}.$$

If $k = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ then the $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ operator is reduced to

$$\mathcal{N}^{\mathcal{B}}\mathcal{FA}\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \frac{1}{n}\left(\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}} \oplus, ..., \oplus \mathcal{N}_n^{\mathcal{B}}\right).$$

Theorem 4. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then*

$$\left.\begin{array}{l} \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_j^{\mathcal{B}+}\right) = \left[\begin{array}{c} 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^n\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{array}\right] \\ \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_j^{\mathcal{B}-}\right) = \left[\begin{array}{c} 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^n\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{array}\right] \end{array}\right\}. \quad (1)$$

Proof. Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be a collection of neutrosophic bipolar fuzzy values. We first prove the result for $n = 2$. Since

$$\begin{aligned} k_1\mathcal{N}_L^{\mathcal{B}+} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}\right], \\ k_1\mathcal{N}_L^{\mathcal{B}-} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}\right], \\ k_1\mathcal{N}_b^{\mathcal{B}+} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}\right], \\ k_1\mathcal{N}_b^{\mathcal{B}+} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}\right], \end{aligned}$$

then

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}},\mathcal{N}_b^{\mathcal{B}}\right) &= k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= k_1\mathcal{N}_1^{\mathcal{B}+} \oplus k_2\mathcal{N}_2^{\mathcal{B}+},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= k_1\mathcal{N}_1^{\mathcal{B}-} \oplus k_2\mathcal{N}_2^{\mathcal{B}-},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
2-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}\right),\\
2-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}\right),\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}
\end{array}\right],\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2},1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2},\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}
\end{array}\right],\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= k_1\mathcal{N}_1^{\mathcal{B}-} \oplus k_2\mathcal{N}_2^{\mathcal{B}-},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
2-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}\right),\\
2-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}\right),\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}
\end{array}\right],\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2},1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2},\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}
\end{array}\right].
\end{aligned}$$

So $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}},\mathcal{N}_b^{\mathcal{B}}\right) = k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}}$. If result is true for $n = k$, that is

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+},...,\mathcal{N}_j^{\mathcal{B}+}\right) = \left[\begin{array}{c}
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j},\\
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j},\\
-\Pi_{j=1}^{k}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right|
\end{array}\right],$$

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-},...,\mathcal{N}_j^{\mathcal{B}-}\right) = \left[\begin{array}{c}
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j},\\
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j},\\
-\Pi_{j=1}^{k}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right|
\end{array}\right],$$

then, when $k+1$, we have

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+},...,\mathcal{N}_j^{\mathcal{B}+}\right) = \begin{bmatrix} 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right), \\ 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right), \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix},$$

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-},...,\mathcal{N}_j^{\mathcal{B}-}\right) = \begin{bmatrix} 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right), \\ 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right), \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}.$$

So result holds for $n=k+1$. □

Theorem 5. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+},\mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values and $k = (k_1,k_2,...,k_n)^T$ is the weight vector of $\mathcal{N}_j^{\mathcal{B}}$ $(j=1,2,...,n)$, with $k_j \in [0,1]$ and $\Sigma_{j=1}^{n}k_j = 1$. Then we have the following:*

(1) (Idempotency): If all $\mathcal{N}_j^{\mathcal{B}\sim}$ $(j=1,2,...,n)$ are equal, i.e.,$\mathcal{N}_j^{\mathcal{B}} = \mathcal{N}_j^{\mathcal{B}}$, for all j, then

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) = \mathcal{N}^{\mathcal{B}}.$$

(2) (Boundary):

$$\mathcal{N}^{\mathcal{B}^-} \le \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) \le \mathcal{N}^{\mathcal{B}^+}, \text{ for every } k.$$

(3) (Monotonicity) If $\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \le \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+*}}$, $\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \le \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+*}}$ and $\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \ge \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-*}}$, for all j, then

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) \le \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_{1^*}^{\mathcal{B}},\mathcal{N}_{2^*}^{\mathcal{B}},...,\mathcal{N}_{n^*}^{\mathcal{B}}\right), \text{ for every } k.$$

Definition 13. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ be a collection of neutrosophic bipolar fuzzy values. An neutrosophic bipolar fuzzy OWA($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operator of dimension is a mapping $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA} : \Omega^n \to \Omega$ defined by*

$$\begin{aligned} \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= k_1\mathcal{N}_{\sigma(1)}^{\mathcal{B}+} \oplus k_2\mathcal{N}_{\sigma(2)}^{\mathcal{B}+} \oplus, ..., \oplus k_n\mathcal{N}_{\sigma(n)}^{\mathcal{B}+}, \\ \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_n^{\mathcal{B}-}\right) &= k_1\mathcal{N}_{\sigma(1)}^{\mathcal{B}-} \oplus k_2\mathcal{N}_{\sigma(2)}^{\mathcal{B}-} \oplus, ..., \oplus k_n\mathcal{N}_{\sigma(n)}^{\mathcal{B}-}, \end{aligned}$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$ such that $\mathcal{N}_{\sigma(j-1)}^{\mathcal{B}} \geq \mathcal{N}_{\sigma(j)}^{\mathcal{B}}$ for all j. If $k = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$ then BFOWA operator is reduced to BFA operator having dimension n.

Theorem 6. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then*

$$\left.\begin{aligned} \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= \begin{bmatrix} 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Ind}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^n\left|\left(\boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix}, \\ \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= \begin{bmatrix} 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Ind}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^n\left|\left(\boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}, \end{aligned}\right\}, \tag{2}$$

where

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weight vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ operator with $k_j \in [0, 1]$ and $\Sigma_{j=1}^n k_j = 1$, for all $j = 1, 2, ..., n$, i.e., all $\mathcal{N}_j^{\mathcal{B}\sim}$ $(j = 1, 2, ..., n)$, are reduced to the following form:

$$\begin{aligned} \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\ \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_n^{\mathcal{B}-}\right) &= 1 - \Pi_{j=1}^n\left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}. \end{aligned}$$

Theorem 7. *Let $\mathcal{N}_j^{\mathcal{B}\sim} = \langle\mathcal{N}_{\mathcal{N}_j^{\mathcal{B}\sim}}^{\mathcal{B}+}, \mathcal{N}_{\mathcal{N}_j^{\mathcal{B}\sim}}^{\mathcal{B}-}\rangle$ $(j = 1, 2, ..., n)$ be a collection of neutrosophic bipolar fuzzy values and*

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weighting vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ operator with $k_j \in [0, 1]$ and $\Sigma_{j=1}^n k_j = 1$; then we have the following.

(1) Idempotency: If all $\mathcal{N}_j^{\mathcal{B}}$ $(j = 1, 2, ..., n)$ are equal, i.e., $\mathcal{N}_j^{\mathcal{B}} = \mathcal{N}^{\mathcal{B}}$, for all j, then

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \mathcal{N}^{\mathcal{B}}.$$

(2) Boundary:

$$\mathcal{N}^{\mathcal{B}^-} \leq \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}^+},$$

for where k, where $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be the* $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ $\mathcal{N}_j^{\mathcal{B}+} = \langle \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}} \rangle$ $(j = 1, 2, ..., n)$ *and* $\mathcal{N}_j^{\mathcal{B}-} = \langle \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of neutrosophic bipolar fuzzy values*

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}^-} &= \left[\min_j \left(\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}} \right), \min_j \left(\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}} \right), -\max_j \left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \right) \right], \\
\mathcal{N}^{\mathcal{B}^+} &= \left[\max_j \left(\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \right), \max_j \left(\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \right), -\min_j \left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}} \right) \right].
\end{aligned}$$

(3) *Monotonicity: Let* $\mathcal{N}_j^{\mathcal{B}+^*}$ *and* $\mathcal{N}_j^{\mathcal{B}-^*}$ $(j = 1, 2, ..., n)$ *be a collection of neutrosophic bipolar fuzzy values. If* $\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+^*}}$, $\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+^*}}$ *and* $\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \geq \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-^*}}$, *for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) \leq BFWL_k \left(\mathcal{N}_{1^*}^{\mathcal{B}}, \mathcal{N}_{2^*}^{\mathcal{B}}, ..., \mathcal{N}_{n^*}^{\mathcal{B}} \right), \text{ for every } k.$$

(4) *Commutativity: Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be a collection of neutrosophic bipolar fuzzy values. Then*

$$BFOWL_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) = BFOWL_k \left(\mathcal{N}_1^{\mathcal{B}'}, \mathcal{N}_2^{\mathcal{B}'}, ..., \mathcal{N}_n^{\mathcal{B}'} \right),$$

for every w, where $\left(\mathcal{N}_1^{\mathcal{B}'}, \mathcal{N}_2^{\mathcal{B}'}, ..., \mathcal{N}_n^{\mathcal{B}'} \right)$ *is any permutation of* $\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right)$.

Theorem 8. *Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be a collection of neutrosophic bipolar fuzzy values*

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weighting vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ *operator with*

$$k_j \in [0, 1] \text{ and } \Sigma_{j=1}^n k_j = 1;$$

then we have the following:

(1) *If* $k = (1, 0, ..., 0)^T$, *then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) = \max_j \left(\mathcal{N}_j^{\mathcal{B}} \right).$$

(2) *If* $k = (0, 0, ..., 1)^T$, *then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k \left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}} \right) = \min_j \left(\mathcal{N}_j^{\mathcal{B}} \right).$$

(3) *If* $k_j = 1, k_i = 0$, *and* $i \neq j$, *then*

$$\mathcal{BFOWA}_k \left(\mathcal{N}_1^{\mathcal{B}\sim}, \mathcal{N}_2^{\mathcal{B}\sim}, ..., \mathcal{N}_n^{\mathcal{B}\sim} \right) = \mathcal{N}_{\sigma(j)}^{\mathcal{B}\sim},$$

where $\mathcal{N}_{\sigma(j)}^{\mathcal{B}}$ *is the largest of* $\mathcal{N}_i^{\mathcal{B}}$ $(i = 1, 2, ..., n)$.

5. Similarity Measures of Neutrosophic Bipolar Fuzzy Sets

In Section 4 we define different aggregation operators with the help of operations defined in Section 3. Next in this section we are aiming to define some similarity measures which will be used in the next Section 6. A comparisons of several different fuzzy similarity measures as well as their aggregations have been studied by Beg and Ashraf [38,39]. Theoretical and computational properties of the measures was further investigated with the relationships between them [15,40–42]. A review, or even a listing of all these similarity measures is impossible. Here in this section we define different kinds of similarity measures of neutrosophic bipolar fuzzy sets.

5.1. Neutrosophic Bipolar Fuzzy Distance Measures

Definition 14. *A function $E : \mathcal{N}^{\mathcal{B}}FSs\,(X) \to [0,1]$ is called an entropy for $\mathcal{N}^{\mathcal{B}}FSs\,(X)$,*

(1) $E\left(\mathcal{N}^{\mathcal{B}}\right) = 1 \Leftrightarrow \mathcal{N}^{\mathcal{B}}$ *is a crisp set.*
(2) $E\left(\mathcal{N}^{\mathcal{B}}\right) = 0 \Leftrightarrow$

$$\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x), \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x), \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x)\ \forall\ x \in X.$$

(3) $E\left(\mathcal{N}^{\mathcal{B}}\right) = E\left(\mathcal{N}^{\mathcal{B}c}\right)$ *for each* $\forall \mathcal{N}^{\mathcal{B}} \in BFSs\,(X)$.
(4) $E\left(\mathcal{N}_1^{\mathcal{B}}\right) \le E\left(\mathcal{N}_2^{\mathcal{B}}\right)$ *if* $\mathcal{N}_1^{\mathcal{B}}$ *is less than* $\mathcal{N}_2^{\mathcal{B}}$, *that is,*

$$\begin{aligned}
\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) &\le \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x), \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \le \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\ \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \ge \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\\
\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x) &\le \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x), \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \le \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x),\ \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \ge \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x),
\end{aligned}$$

for $\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x) \le \left|\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x)\right|$

$$\text{or } \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \ge \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\ \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \ge \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x),$$

and

$$\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \le \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x) \le \mathcal{N}_{B_2}^{\mathcal{B}-}(x) \text{ for } \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \ge \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x).$$

Definition 15. *Let $X = \{x_1, x_2, ..., x_n\}$ and $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ be an $\mathcal{N}^{\mathcal{B}}FS$. The entropy of $\mathcal{N}^{\mathcal{B}}FS$ is denoted by $E(\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ and given by*

$$\left.\begin{aligned}
E(\mathcal{N}^{\mathcal{B}+}) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\min((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})\right|)}{\max((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})\right|)}\\
E(\mathcal{N}^{\mathcal{B}-}) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\min((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})\right|)}{\max((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})\right|)}
\end{aligned}\right\}, \tag{3}$$

and for a neutrosophic bipolar fuzzy number $\mathcal{N}^{\mathcal{B}} = \langle \mathcal{N}_L^{\mathcal{B}+}, \mathcal{N}_L^{\mathcal{B}-} \rangle$, the bipolar fuzzy entropy is given by

$$\left.\begin{aligned}
E(\mathcal{N}_L^{\mathcal{B}+}) &= \frac{\min((\boldsymbol{Tru}_{L_1^+}(\mathbf{x}),\min(\boldsymbol{Ind}_{\mathbf{L}_1^+}(\mathbf{x})),\left|\boldsymbol{Fal}_{L_1^+}(\mathbf{x})\right|)}{\max(\boldsymbol{Tru}_{L_1^+}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathbf{L}_1^+}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathbf{L}_1^+}(\mathbf{x})\right|)}\\
E(\mathcal{N}_L^{\mathcal{B}-}) &= \frac{\min((\boldsymbol{Tru}_{L_1^-}(\mathbf{x}),\min(\boldsymbol{Ind}_{\mathbf{L}_1^-}(\mathbf{x})),\left|\boldsymbol{Fal}_{L_1^-}(\mathbf{x})\right|)}{\max(\boldsymbol{Tru}_{L_1^-}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathbf{L}_1^-}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathbf{L}_1^-}(\mathbf{x})\right|)}
\end{aligned}\right\}. \tag{4}$$

Definition 16. *Let $X = \{x_1, x_2, ..., x_n\}$. We define the Hamming distance between $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ belonging to $\mathcal{N}^{\mathcal{B}}FSs(X)$ defined as follows:*

(1) *The Hamming distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{1}{2}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \textit{Hamming distance for positive neutrosophic bipolar sets} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{1}{2}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)||) \\ & \textit{Hamming distance for negative neutrosophic bipolar sets} \end{array}\right\}. \quad (5)$$

(2) *The normalized Hamming distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{1}{2n}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \textit{normalized Hamming distance for positive neutrosophic bipolar sets} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{1}{2n}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \textit{normalized Hamming distance for negative neutrosophic bipolar sets} \end{array}\right\}. \quad (6)$$

(3) *The Euclidean distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \quad (7)$$

(4) *The normalized Euclidean distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \sqrt{\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \sqrt{\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \quad (8)$$

(5) *Based on the geometric distance formula, we have*

$$\left.\begin{aligned} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \left[\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ \quad +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ \quad +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \left[\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ \quad +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ \quad +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{aligned}\right\}. \tag{9}$$

(6) *Normalized geometric distance formula:*

$$\left.\begin{aligned} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \left[\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ \quad +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ \quad +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \left[\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ \quad +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ \quad +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{aligned}\right\}, \tag{10}$$

where $\alpha > 0$.

(i) *If* $\alpha = 1$, *then Equations (9) and (10), reduce to Equations (5) and (6).*
(ii) *If* $\alpha = 2$, *then Equations (9) and (10), reduce to Equations (7) and (8).*
(iii) *We define a weighted distance as follows:*

$$\left.\begin{aligned} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{l} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{l} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \end{aligned}\right\}, \tag{11}$$

where $k = (k_1, k_2, ..., k_n)^{\boldsymbol{T}}$ *is the weight vector of* $x_j (j = 1, 2, ..., n)$, *and* $\alpha > 0$.

(i) *Especially, if* $\alpha = 1$, *then Equation (11) is reduced as*

$$\left.\begin{aligned} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{l} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right| \end{array}\right)\right] \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{l} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right| \end{array}\right)\right] \end{aligned}\right\}. \tag{12}$$

If $k = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then Equation (11) goes to Equation (10), and Equation (12) goes to Equation (6).

(ii) If $\alpha = 2$, then Equation (11) is reduced to the as:

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) & = \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) & = \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \quad (13)$$

If $k = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^{\boldsymbol{T}}$, then Equation (13) is reduced to Equation (8).

5.2. Similarity Measures of Neutrosophic Bipolar Fuzzy Set

Definition 17. *Let $\hat{s}$ be a mapping $\hat{s} : \Omega(X)^2 \to [0,1]$, then the degree of similarity between $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$ is defined as $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}})$, which satisfies the following properties: [43,44].*

(1) $0 \leq \hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) \leq 1$;

(2) $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = 1$ if $\mathcal{N}_1^{\mathcal{B}} = \mathcal{N}_2^{\mathcal{B}}$;

(3) $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = \hat{s}(\mathcal{N}_2^{\mathcal{B}}, \mathcal{N}_1^{\mathcal{B}})$;

(4) If $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = 0$ and $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_3^{\mathcal{B}}) = 0$, $\mathcal{N}_3^{\mathcal{B}} \in \Omega(X)$, then $\hat{s}(\mathcal{N}_2^{\mathcal{B}}, \mathcal{N}_3^{\mathcal{B}}) = 0$. We define a similarity measure of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ as:

$$\left.\begin{array}{ll} \hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) & = 1 - \left[\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) & = 1 - \left[\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{array}\right\}, \quad (14)$$

where $\alpha > 0$, and $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}})$ is the degree of similarity of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$. Now by considering the weight of every element we have,

$$\left.\begin{array}{ll} \hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) & = 1 - \left[\frac{1}{2}\sum_{j=1}^{n} k_j \left(\begin{array}{l} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) & = 1 - \left[\frac{1}{2}\sum_{j=1}^{n} k_j \left(\begin{array}{l} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \end{array}\right\}. \quad (15)$$

If we give equal importance to every member then Equation (15) is reduced to Equation (14). Similarly we may use

$$\left\{\begin{aligned} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{L}\\ +\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)\right)\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{L}\\ \left.+\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)\right)\right|^{L}\right)\end{array}}\right]^{\frac{1}{\alpha}}\\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{L}\\ +\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)\right)\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{L}\\ \left.+\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)\right)\right|^{L}\right)\end{array}}\right]^{\frac{1}{\alpha}}\end{aligned}\right\}. \tag{16}$$

Now by considering the weight of every element we have

$$\left\{\begin{aligned} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}k_j\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{L}\\ +\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{L}\\ \left.+\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\right)\right|^{L}\right)\end{array}}\right]^{\frac{1}{\alpha}}\\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= 1-\left[\frac{\begin{array}{c}\sum_{j=1}^{n}k_j\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{L}\\ +\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{L}\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j\left(\left|\left(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{\alpha}\right.\\ +\left|\left(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{L}\\ \left.+\left|\left(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\right)\right|^{L}\right)\end{array}}\right]^{\frac{1}{\alpha}}\end{aligned}\right\}. \tag{17}$$

If we give equal importance to every member, then Equation (17) is reduced to Equation (16).

5.3. Similarity Measures Based on the Set-Theoretic Approach

Definition 18. *Let $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$. Then, we define a similarity measure $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ from the point of set-theoretic view as:*

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \frac{\begin{array}{c}\sum_{j=1}^{n}\langle\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\\ +\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\\ +\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)\end{array}}{\begin{array}{c}\sum_{j=1}^{n}\langle\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\\ +\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\\ +\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)\rangle\end{array}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \frac{\begin{array}{c}\sum_{j=1}^{n}\langle\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\\ +\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\\ +\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))|)\rangle\end{array}}{\begin{array}{c}\sum_{j=1}^{n}\langle\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\\ +\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\\ +\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)\rangle\end{array}}
\end{aligned}\right\}. \tag{18}$$

Now by considering the weight of every element we have

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \frac{\begin{array}{c}\sum_{j=1}^{n}k_j(\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)+\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{array}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \frac{\begin{array}{c}\sum_{j=1}^{n}k_j(\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+\min(\boldsymbol{Ind}_{N\mathcal{N}_1^{\mathcal{B}-}}(x_j),\\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+\min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n}k_j(\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j),\\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)+\max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|,|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{array}}
\end{aligned}\right\}. \tag{19}$$

If we give equal importance to every member, then Equation (19) is reduced to Equation (18).

5.4. Similarity Measures Based on the Matching Functions

We cover the matching function to agreement through the similarity measure of $\mathcal{N}^{\mathcal{B}}FSs$.

Definition 19. *Let $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$, formerly we explain the degree of similarity of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ based on the matching function as:*

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= \frac{\sum_{j=1}^{n}((\boldsymbol{Tru}_{N\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)}{\begin{array}{c}\max\langle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)),\\ \sum_{j=1}^{n}k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)))\end{array}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= \frac{\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)}{\begin{array}{c}\max\langle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)),\\ \sum_{j=1}^{n}k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)))\end{array}}
\end{aligned}\right\}. \tag{20}$$

Now by considering the weight of every element we have

$$\left.\begin{array}{rcl} \hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \frac{\sum_{j=1}^{n} k_j(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)}{\max\langle\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)), \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j))\rangle} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \frac{\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)}{\max\langle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)), \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j))\rangle} \end{array}\right\}. \quad (21)$$

(1) *If we give equal importance to every member, then Equation (21) is reduced to Equation (20).*
(2) *If the value of $\hat{s}(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}})$ is larger then its mean $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ are more closer to each other.*

6. Application

In this Section 5 after defining some similarity measures we proceed towards the main section namely application of the developed model. In this section we provide an algorithm for solving a multiatribute decision making problem related with the HOPE foundation with the help of neutrosophic bipolar fuzzy aggregation operators, neutrosophic bipolar similarity measures under the neutrosophic bipolar fuzzy sets. For detail see [13,42].

Definition 20. *Let $L = \{L_1, L_2, ..., L_m\}$ consists of alternatives, and let $P = \{P_1, P_2, ..., P_n\}$ containing the attributes and $k = (k_1, k_2, ..., k_n)^T$ be the weight vector that describe the importance of attributes such that $k_j \in [0,1]$ and $\sum_{j=1}^{n} k_j = 1$. Let us use the neutrosophic bipolar fuzzy sets for L_i as under:*

$$\left.\begin{array}{ll} L_i^+ & = \{\langle P_j, (\boldsymbol{Tru})_{Li}^+(P_j), (\boldsymbol{Ind})_{Li}^+(P_j), (\boldsymbol{Fal})_{Li}^+(P_j)\rangle | P_j \in P\},\ i = 1,2,3,...,m \\ L_i^- & = \{\langle P_j, (\boldsymbol{Tru})_{Li}^-(P_j), (\boldsymbol{Ind})_{Li}^- P_j), (\boldsymbol{Fal})_{Li}^-(P_j)\rangle | P_j \in P\},\ i = 1,2,3,...,m \end{array}\right\}. \quad (22)$$

such that

$$\begin{array}{rcl} (\boldsymbol{Tru})_{Li}^+(P_j) & \in & [0,3], (\boldsymbol{Ind})_{Li}^+(P_j) \in [0,3], (\boldsymbol{Fal})_{Li}^+(P_j) \in [0,3], \\ 0 & \leq & (\boldsymbol{Tru})_{Li}^+(P_j), (\boldsymbol{Ind})_{Li}^+ P_j), (\boldsymbol{Fal})_{Li}^+(P_j)\rangle \leq 3. \\ (\boldsymbol{Tru})_{Li}^-(P_j) & \in & [-3,0], (\boldsymbol{Ind})_{Li}^-(P_j) \in [-3,0], (\boldsymbol{Fal})_{Li}^-(P_j) \in [-3,0], \\ -3 & \leq & (\boldsymbol{Tru})_{Li}^-(P_j), (\boldsymbol{Ind})_{Li}^- P_j), (\boldsymbol{Fal})_{Li}^-(P_j)\rangle \leq 0. \end{array}$$

Now we define the positive and negative ideal solutions as under:

$$\left.\begin{array}{ll} L_i^+ & = \{\langle P_j, (\boldsymbol{Tru})_{L+}^+(P_j), (\boldsymbol{Ind})_{L+}^+(P_j), (\boldsymbol{Fal})_{L+}^+(P_j)\rangle | P_j \in P\}\} \\ L_i^- & = \{\langle P_j, (\boldsymbol{Tru})_{L+}^-(P_j), (\boldsymbol{Ind})_{L+}^-(P_j), (\boldsymbol{Fal})_{L+}^-(P_j)\rangle | P_j \in P\}\} \end{array}\right\}, \quad (23)$$

and

$$\left.\begin{array}{ll} L^+ & = \{\langle P_j, (\boldsymbol{Tru})_{L-}^+(P_j), (\boldsymbol{Ind})_{L-}^+(P_j), (\boldsymbol{Fal})_{L-}^+(P_j)\rangle | P_j \in P\}\} \\ L^- & = \{\langle P_j, (\boldsymbol{Tru})_{L-}^-(P_j), (\boldsymbol{Ind})_{L-}^-(P_j), (\boldsymbol{Fal})_{L-}^-(P_j)\rangle | P_j \in P\}\} \end{array}\right\}, \quad (24)$$

where

$$\begin{array}{rcl} (\boldsymbol{Tru})_{L+}^+(P_j) & = & \max_i\{(\boldsymbol{Tru})_{Li}^+(P_j),\ (\boldsymbol{Tru})_{L+}^-(P_j)\} = \min_i\{(\boldsymbol{Tru})_{Li}^+(P_j)\}, (\boldsymbol{Tru})_{Li}^-(P_j) \\ & = & \max_i\{(\boldsymbol{Tru})_{Li}^-(P_j),\ (\boldsymbol{Tru})_{L+}^+(P_j)\} = \min_i\{(\boldsymbol{Tru})_{Li}^-(P_j)\}(\boldsymbol{Ind})_{L+}^+(P_j) \\ & = & \max_i\{(\boldsymbol{Ind})_{Li}^+(P_j),\ (\boldsymbol{Ind})_{L+}^-(P_j)\} = \min_i\{(\boldsymbol{Ind})_{Li}^+(P_j)\}(\boldsymbol{Ind})_{L+}^-(P_j) \\ & = & \max_i\{(\boldsymbol{Ind})_{Li}^-(P_j),\ (\boldsymbol{Ind})_{L+}^+(P_j)\} = \min_i\{(\boldsymbol{Ind})_{Li}^+(P_j)\}. \end{array}$$

$$(\boldsymbol{Fal})^{+}_{Li}(P_j) = \min_i\{(\boldsymbol{Fal})^{+}_{Li}(P_j), (\boldsymbol{Fal})^{-}_{L^+}(P_j)\} = \max_i\{(\boldsymbol{Fal})^{+}_{Li}(P_j)\}.$$
$$(\boldsymbol{Fal})^{-}_{Li}(P_j) = \min_i\{(\boldsymbol{Fal})^{-}_{Li}(P_j), (\boldsymbol{Fal})^{+}_{L^+}(P_j)\} = \max_i\{(\boldsymbol{Fal})^{-}_{Li}(P_j)\}.$$

Now using Equation (15), we find the degree of similarity for L^+, L_i, and L^-, L_i, as under:

$$\left.\begin{aligned} \hat{s}_1(L^+, L_i^+) &= 1 - \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n} k_j(|(\boldsymbol{Tru})^{+}_{L^+}(x_j) - (\boldsymbol{Tru})^{+}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Ind})^{+}_{L^+}(x_j) - (\boldsymbol{Ind})^{+}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Fal})^{+}_{L^+}(x_j) - (\boldsymbol{Fal})^{+}_{Li}(x_j)|^{\alpha}) \end{array}\right]^{\frac{1}{\alpha}} \\ \hat{s}_1(L^+, L_i^-) &= 1 - \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n} k_j(|(\boldsymbol{Tru})^{-}_{L^+}(x_j) - (\boldsymbol{Tru})^{-}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Ind})^{-}_{L^+}(x_j) - (\boldsymbol{Ind})^{-}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Fal})^{-}_{L^+}(x_j) - (\boldsymbol{Fal})^{-}_{Li}(x_j)|^{\alpha}) \end{array}\right]^{\frac{1}{\alpha}} \end{aligned}\right\}, \tag{25}$$

and

$$\left.\begin{aligned} \hat{s}_1(L^-, L_i^+) &= 1 - \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n} k_j(|(\boldsymbol{Tru})^{+}_{L^-}(x_j) - (\boldsymbol{Tru})^{+}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Ind})^{+}_{L^-}(x_j) - (\boldsymbol{Ind})^{+}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Fal})^{+}_{L^-}(x_j) - (\boldsymbol{Fal})^{+}_{Li}(x_j)|^{\alpha}) \end{array}\right]^{\frac{1}{\alpha}} \\ \hat{s}_1(L^-, L_i^-) &= 1 - \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n} k_j(|(\boldsymbol{Tru})^{-}_{L^-}(x_j) - (\boldsymbol{Tru})^{-}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Ind})^{-}_{L^-}(x_j) - (\boldsymbol{Ind})^{-}_{Li}(x_j)|^{\alpha} \\ +|(\boldsymbol{Fal})^{-}_{L^-}(x_j) - (\boldsymbol{Fal})^{-}_{Li}(x_j)|^{\alpha}) \end{array}\right]^{\frac{1}{\alpha}} \end{aligned}\right\}. \tag{26}$$

Using Equations (25) and (26), calculate di of L_i as under:

$$\left.\begin{aligned} d_i^+ &= \frac{s_1(L^+, L_i^+)}{s_1(L^+, L_i^+) + s_1(L^-, L_i^+)},\ i = 1, 2, \ldots, n. \\ d_i^- &= \frac{s_1(L^+, L_i^-)}{s_1(L^+, L_i^-) + s_1(L^-, L_i^-)},\ i = 1, 2, \ldots, n. \end{aligned}\right\}. \tag{27}$$

If the value of d_i is greater, then the alternative L_i is better.

Also using Equations (17), (19) and (21), we find the degree of similarity for L^+, L_i, and L^-, L_i, as under:

(1) Based on Equation (17), we define the following: We define the following:

$$\left.\begin{aligned} \hat{s}_1(L^+, L_i^+) &= 1 - \left[\frac{\begin{array}{c} \sum_{j=1}^{n} k_j(\left|(\boldsymbol{Tru}_{L^+}(x_j) - \boldsymbol{Tru}_{L_i^+}(x_j))\right|^{\alpha} \\ + \left|(\boldsymbol{Ind}_{L^+}(x_j) - \boldsymbol{Ind}_{L_i^+}(x_j))\right|^{L} \\ + \left|(\boldsymbol{Fal}_{L^+}(x_j) - \boldsymbol{Fal}_{L_i^+}(x_j))\right|^{L} \end{array}}{\begin{array}{c} \sum_{j=1}^{n} k_j(\left|(\boldsymbol{Tru}_{L^+}(x_j) - \boldsymbol{Tru}_{L_i^+}(x_j))\right|^{\alpha} \\ + \left|(\boldsymbol{Ind}_{L^+}(x_j) - \boldsymbol{Ind}_{L_i^+}(x_j))\right|^{L} \\ + \left|(\boldsymbol{Fal}_{L^+}(x_j) - \boldsymbol{Fal}_{L_i^+}(x_j))\right|^{L}) \end{array}}\right]^{\frac{1}{\alpha}} \\ \hat{s}_3(L^+, L_i^-) &= 1 - \left[\frac{\begin{array}{c} \sum_{j=1}^{n} k_j(\left|(\boldsymbol{Tru}_{L^-}(x_j) - \boldsymbol{Tru}_{L_i^-}(x_j))\right|^{\alpha} \\ + \left|(\boldsymbol{Ind}_{L^-}(x_j) - \boldsymbol{Ind}_{L_i^-}(x_j))\right|^{L} \\ + \left|(\boldsymbol{Fal}_{L^-}(x_j) - \boldsymbol{Fal}_{L_i^-}(x_j))\right|^{L} \end{array}}{\begin{array}{c} \sum_{j=1}^{n} k_j(\left|(\boldsymbol{Tru}_{L^-}(x_j) - \boldsymbol{Tru}_{L_i^-}(x_j))\right|^{\alpha} \\ + \left|(\boldsymbol{Ind}_{L^-}(x_j) - \boldsymbol{Ind}_{L_i^-}(x_j))\right|^{L} \\ + \left|(\boldsymbol{Fal}_{L^-}(x_j) - \boldsymbol{Fal}_{L_i^-}(x_j))\right|^{L}) \end{array}}\right]^{\frac{1}{\alpha}} \end{aligned}\right\}. \tag{28}$$

(2) Based on Equation (19), we define the following: We define the following:

$$\left.\begin{array}{ll}\hat{s}_2(L^+,L_i^+) & = \frac{\sum_{j=1}^n k_j(\min(\boldsymbol{Tru}_{L^+}(x_j),\boldsymbol{Tru}_{L_i^+}(x_j))+\min(\boldsymbol{Ind}_{L^+}(x_j),\boldsymbol{Ind}_{L_i^+}(x_j))+\min(|\boldsymbol{Fal}_{L^+}(x_j)|,|\boldsymbol{Fal}_{L_i^+}(x_j)|))}{\sum_{j=1}^n k_j(\max(\boldsymbol{Tru}_{L^+}(x_j),\boldsymbol{Tru}_{L_i^+}(x_j))+\max(\boldsymbol{Ind}_{L^+}(x_j),\boldsymbol{Ind}_{L_i^+}(x_j)+\max(|\boldsymbol{Fal}_{L^+}(x_j)|,|\boldsymbol{Fal}_{L_i^+}(x_j)|))} \\ \hat{s}_2(L^-,L_i^-) & = \frac{\sum_{j=1}^n k_j(\min(\boldsymbol{Tru}_{L^-}(x_j),\boldsymbol{Tru}_{L_i^-}(x_j)) + \min(\boldsymbol{Ind}_{L^-}(x_j),\boldsymbol{Ind}_{L_i^-}(x_j)) + \min(|\boldsymbol{Fal}_{L^-}(x_j)|,|\boldsymbol{Fal}_{L_i^-}(x_j)|))}{\sum_{j=1}^n k_j(\max(\boldsymbol{Tru}_{L^-}(x_j),\boldsymbol{Tru}_{L_i^-}(x_j)) + \max(\boldsymbol{Ind}_{L^-}(x_j),\boldsymbol{Ind}_{L_i^-}(x_j) + \max(|\boldsymbol{Fal}_{L^-}(x_j)|,|\boldsymbol{Fal}_{L_i^-}(x_j)|))}\end{array}\right\}. \quad (29)$$

(3) Based on Equation (21), we define the following: We define the following:

$$\left.\begin{array}{ll}\hat{s}_3(L^+,L_i^+) & = \frac{\sum_{j=1}^n k_j(\min((\boldsymbol{Tru})^+_{L^+}(x_j),(\boldsymbol{Tru})^+_{Li}(x_j)) + \min((\boldsymbol{Ind})^+_{L^+}(x_j),(\boldsymbol{Ind})^+_{Li}(x_j)) + \min(|(\boldsymbol{Fal})^+_{L^+}(x_j)|,|(\boldsymbol{Fal})^+_{Li}(x_j)|))}{\sum_{j=1}^n k_j(\max((\boldsymbol{Tru})^+_{L^+}(x_j),(\boldsymbol{Tru})^+_{Li}(x_j)) + (\max((\boldsymbol{Ind})^+_{L^+}(x_j),(\boldsymbol{Ind})^+_{Li}(x_j)) + \max(|(\boldsymbol{Fal})^+_{L^+}(x_j)|,|(\boldsymbol{Fal})^+_{Li}(x_j)|)} \\ \hat{s}_3(L^+,L_i^-) & = \frac{\sum_{j=1}^n k_j(\min((\boldsymbol{Tru})^-_{L^+}(x_j),(\boldsymbol{Tru})^-_{Li}(x_j)) + \min((\boldsymbol{Ind})^-_{L^+}(x_j),(\boldsymbol{Ind})^-_{Li}(x_j)) + \min(|(\boldsymbol{Fal})^-_{L^+}(x_j)|,|(\boldsymbol{Fal})^-_{Li}(x_j)|))}{\sum_{j=1}^n k_j(\max((\boldsymbol{Tru})^-_{L^+}(x_j),(\boldsymbol{Tru})^-_{Li}(x_j)) + (\max((\boldsymbol{Ind})^-_{L^+}(x_j),(\boldsymbol{Ind})^-_{Li}(x_j)) + \max(|(\boldsymbol{Fal})^-_{L^+}(x_j)|,|(\boldsymbol{Fal})^-_{Li}(x_j)|)}\end{array}\right\}. \quad (30)$$

Then use (27).

7. Numerical Example

Now we provide a daily life example which shows the applicability of the algorithm provided in Section 6.

Example 1. *The HOPE foundation is an international organization which provides the financial support to the health sector of children of many families in round about 22 different countries in southwest Missouri. This organization provides the support when other organization does not play their role. Every day a child is diagnosed with a severe illness, sustains a debilitating injury, and a family loses the battle with an illness. With these emergencies come unexpected expenses. Here we discuss a problem related with HOPE foundation as:*

HOPE foundation is planning to build a children hospital and they are planning to fit a suitable air conditioning system in the hospital. Different companies offers them different systems. Companies offer three feasible alternatives $L_i = (i = 1,2,3)$, *by observing the hospital' physical structures. Assume that* P_1 *and* P_2, *are the two attributes which are helpful in the installation of air conditioning system with the weight vector as* $k = (0.4, 0.6)^{\boldsymbol{T}}$ *for the attributes. Now using neutrosophic bipolar fuzzy sets for the alternatives* $L_i = (i = 1,2,3)$ *by examining the different characteristics as under:*

$$\begin{aligned} L_1^+ &= \{\langle P_1, 0.3, 0.4, 0.7\rangle, \langle P_2, 0.8, 0.8, 0.6\rangle\}, \\ L_1^- &= \{\langle P_1, -0.3, -0.2, -0.1\rangle, \langle P_2, -0.4, -0.6, -0.8\rangle\}. \end{aligned}$$

$$\begin{aligned} L_2^+ &= \{\langle P_1, 0.4, 0.6, 0.2\rangle, \langle P_2, 0.3, 0.9, 0.2\rangle\}, \\ L_2^- &= \{\langle P_1, -0.1, -0.3, -0.4\rangle, \langle P_2, -0.8, -0.7, -0.1\rangle\}. \end{aligned}$$

$$L_3^+ = \{\langle P_1, 0.3, 0.5, 0.7\rangle, \langle P_2, 0.2, 0.30.6\rangle\},$$
$$L_3^- = \{\langle P_1, -0.5, -0.1, -0.4\rangle, \langle P_2, -0.3, -0.2, -0.8\rangle\}.$$

where $L_1^+ = \{\langle P_1, 0.3, 0.4, 0.7\rangle, \langle P_2, 0.8, 0.8, 0.6\rangle\}$ *means that the alternative* L_1 *has the positive preferences which is desirable:* 0.3, 0.8 *as a truth function for past,* 0.4, 0.8 *as a indeterminacy function for future and* 0.7, 0.6 *as a falsity function for present time with respect to the attributes* P_1 *and* P_2 *respectively.*

Similarly $L_1^- = \{\langle P_1, -0.3, -0.2, -0.1\rangle, \langle P_2, -0.4, -0.6, -0.8\rangle\}$ means that the alternative L_1 has the negative preferences which is unacceptable: $-0.3, -0.4$ as a truth function for past, $-0.2, -0.6$ as a indeterminacy function for future and $-0.1, -0.8$ as a falsity function for present time with respect to the attributes P_1 and P_2 respectively.

(1) By Equations (23) and (24) we first calculate L^+ and L^- of the alternatives $L_i = (i = 1, 2, 3)$, as

$$L^+ = \{\langle P_1, 0.4, 0.6, 0.7\rangle, \langle P_2, 0.5, 0.9, 0.6\rangle\},$$
$$L^- = \{\langle P_1, 0.3, 0.4, 0.2\rangle, \langle P_2, 0.2, 0.3, 0.2\rangle\},$$

and

$$L^+ = \{\langle P_1, -0.1, -0.1, -0.1\rangle, \langle P_2, -0.3, -0.2, -0.1\rangle\},$$
$$L^- = \{\langle P_1, -0.5, -0.3, -0.4\rangle, \langle P_2, -0.8, -0.7, -0.8\rangle\}.$$

Then by using Equations (25)–(27), (suppose that $\alpha = 2$ and $k = 1$), we have

$$\hat{s}_1(L^+, L_1^+) = 0.8267, \hat{s}_1(L^+, L_2^+) = 0.775, \hat{s}_1(L^+, L_3^+) = 0.5152,$$
$$\hat{s}_1(L^+, L_1^-) = -0.5732, \hat{s}_1(L^+, L_2^-) = -0.8721, \hat{s}_1(L^+, L_3^-) = -0.7776.$$

$$\hat{s}_1(L^-, L_1^+) = 0.3876, \hat{s}_1(L^-, L_2^+) = 0.5, \hat{s}_1(L^-, L_3^+) = 0.5417,$$
$$\hat{s}_1(L^-, L_1^-) = -0.1038, \hat{s}_1(L^-, L_2^-) = -0.2449, \hat{s}_1(L^-, L_3^-) = -0.1119,$$

and

$$\hat{s}_1(L^+, L_1^+) = -0.2609, \hat{s}_1(L^+, L_2^+) = -0.1157, \hat{s}_1(L^+, L_3^+) = -0.2439,$$
$$\hat{s}_1(L^+, L_1^-) = -0.1485, \hat{s}_1(L^+, L_2^-) = -0.075, \hat{s}_1(L^+, L_3^-) = -0.0243.$$

$$\hat{s}_1(L^-, L_1^+) = -0.6229, \hat{s}_1(L^-, L_2^+) = -0.7146, \hat{s}_1(L^-, L_3^+) = -0.7958,$$
$$\hat{s}_1(L^-, L_1^-) = 0.6062, \hat{s}_1(L^-, L_2^-) = 0.3636, \hat{s}_1(L^-, L_3^-) = 0.4803.$$

Now by Equation (27), we have

$$\left.\begin{array}{c} d_1^+ = 0.7207, d_2^+ = 0.1393, d_3^+ = 0.9093, \\ L_1 > L_2 > L_3 \end{array}\right\}, \tag{31}$$

$$\left.\begin{array}{c} d_1^- = -0.3244, d_2^- = -0.2598, d_3^- = -0.0532, \\ L_3 > L_1 > L_2 \end{array}\right\}, \tag{32}$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2813, d_2^+ = 0.4031, d_3^+ = 0.4728, \\ L_3 > L_2 > L_1 \end{array}\right\}, \tag{33}$$

$$\left.\begin{array}{c} d_1^- = 0.06184, d_2^- = 0.1190, d_3^- = 0.1942, \\ L_3 > L_2 > L_1 \end{array}\right\}. \quad (34)$$

(2) Now by Equations (28) and (29) (suppose that $\alpha = 3$), we have

$$\begin{aligned} \hat{s}_2(L^+, L_1^+) &= 0.9051, \hat{s}_2(L^+, L_2^+) = 0.7283, \hat{s}_2(L^+, L_3^+) = 0.6873, \\ \hat{s}_2(L^+, L_1^-) &= -1.9845, \hat{s}_2(L^+, L_2^-) = -2.338, \hat{s}_2(L^+, L_3^-) = -1.3894. \end{aligned}$$

$$\begin{aligned} \hat{s}_2(L^-, L_1^+) &= 0.6940, \hat{s}_2(L^-, L_2^+) = 0.4952, \hat{s}_2(L^-, L_3^+) = 0.577, \\ \hat{s}_2(L^-, L_1^-) &= -1.0988, \hat{s}_2(L^-, L_2^-) = -1.0717, \hat{s}_2(L^-, L_3^-) = -1.004, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_2(L^+, L_1^+) &= -0.6210, \hat{s}_2(L^+, L_2^+) = -0.6086, \hat{s}_2(L^+, L_3^+) = -0.4944, \\ \hat{s}_2(L^+, L_1^-) &= 0.3714, \hat{s}_2(L^+, L_2^-) = 0.5139, \hat{s}_2(L^+, L_3^-) = 0.3358. \end{aligned}$$

$$\begin{aligned} \hat{s}_2(L^-, L_1^+) &= -2.3840, \hat{s}_2(L^-, L_2^+) = -1.968, \hat{s}_2(L^-, L_3^+) = -2.2632, \\ \hat{s}_2(L^-, L_1^-) &= 0.6972, \hat{s}_2(L^-, L_2^-) = 0.5752, \hat{s}_2(L^-, L_3^-) = 0.6691. \end{aligned}$$

Now again using Equation (27), we have

$$\left.\begin{array}{c} d_1^+ = 0.5660, d_2^+ = 0.5952, d_3^+ = 0.5436, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (35)$$

$$\left.\begin{array}{c} d_1^- = 0.6436, d_2^- = 0.6856, d_3^- = 0.5805, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (36)$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2066, d_2^+ = 0.2362, d_3^+ = 0.179, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (37)$$

$$\left.\begin{array}{c} d_1^- = 0.3475, d_2^- = 0.4719, d_3^- = 0.3341, \\ L_2 > L_1 > L_3 \end{array}\right\}. \quad (38)$$

(3) Thus, by Equations (27), (30) and (31), we have

$$\begin{aligned} \hat{s}_3(L^+, L_1^+) &= 0.4285, \hat{s}_3(L^+, L_2^+) = 0.5675, \hat{s}_3(L^+, L_3^+) = 0.7027, \\ \hat{s}_3(L^+, L_1^-) &= -0.6468, \hat{s}_3(L^+, L_2^-) = -0.6486, \hat{s}_3(L^+, L_3^-) = -0.6316, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_3(L^-, L_1^+) &= 0.4848, \hat{s}_3(L^-, L_2^+) = 0.1538, \hat{s}_3(L^-, L_3^+) = 0.6153, \\ \hat{s}_3(L^-, L_1^-) &= -1.375, \hat{s}_3(L^-, L_2^-) = -1.0625, \hat{s}_2(L^-, L_3^-) = -1.4375. \end{aligned}$$

By Equations (30)–(32) we have

$$\begin{aligned} \hat{s}_3(L^+, L_1^+) &= -0.2727, \hat{s}_3(L^+, L_2^+) = -0.3913, \hat{s}_3(L^+, L_3^+) = -0.3461, \\ \hat{s}_3(L^+, L_1^-) &= 2.6666, \hat{s}_3(L^+, L_2^-) = 2.6666, \hat{s}_3(L^+, L_3^-) = 2.5555. \end{aligned}$$

$$\begin{aligned} \hat{s}_3(L^-, L_1^+) &= -1.060, \hat{s}_3(L^-, L_2^+) = -1.3461, \hat{s}_3(L^-, L_3^+) = -1.4000, \\ \hat{s}_3(L^-, L_1^-) &= 1.4585, \hat{s}_3(L^-, L_2^-) = 1.7500, \hat{s}_3(L^-, L_3^-) = 5217. \end{aligned}$$

By Equations (30)–(32), we have

$$\left. \begin{array}{c} d_1^+ = 0.4691, d_2^+ = 0.7868, d_3^+ = 0.5331, \\ L_2 > L_3 > L_1 \end{array} \right\}, \tag{39}$$

$$\left. \begin{array}{c} d_1^- = 0.3199, d_2^- = 0.3790, d_3^- = 0.3018, \\ L_2 > L_1 > L_3 \end{array} \right\}, \tag{40}$$

and

$$\left. \begin{array}{c} d_1^+ = 0.2046, d_2^+ = 0.2252, d_3^+ = 0.1982, \\ L_2 > L_1 > L_3 \end{array} \right\}, \tag{41}$$

$$\left. \begin{array}{c} d_1^- = 0.3475, d_2^- = 0.6037, d_3^- = 0.6267, \\ L_2 > L_3 > L_1 \end{array} \right\}. \tag{42}$$

From the Equations (35)–(42), we have that the alternative L_2 (feasible alternative) is the best one obtained by all the similarity measures. Thus we conclude that air-conditioning system L_2 is better to installed in the hospital after considering its negative and the positive preferences for past, future and present time.

8. Comparison Analysis

There are a lot of different techniques used so for in decision making problems. For example Chen et al. [23] used fuzzy sets, Atanassov [26] used intutionistic fuzzy sets, Dubios et al. [9], used bipolar fuzzy sets, Zavadskas et al. [37] used neutrosophic sets, Zhan et al. [25], used neutrosophic cubic sets, Ali et al. [33] used bipolar neutrosophic soft sets and so many others discuss decision making problems with respect to the different versions of fuzzy sets. Beg et al., and Xu [38,39,41] discussed similarity measures for fuzzy sets, intutionistic fuzzy sets respectively. In this paper by applying bipolarity to neutrosophic sets allow us to distinguish between the negative and the positive preferences with respect to the past, future and present time which is the unique future of our model. Negative preferences denote what is unacceptable while positive preferences are less restrictive and express what is desirable with respect to the past, future and present time. If we consider only one time frame from the set {past, future and present} one can see our model coincide with bipolar fuzzy sets in decision making as Dubios et al. [9] and Xu [41].

9. Conclusions

We define neutrosophic bipolar fuzzy sets, aggregation operators for neutrosophic bipolar fuzzy sets, similarity measures for neutrosophic bipolar fuzzy sets and produce a real life application in decision making problems. This model can easily used in many directions such as,

(1) Try to solve traffic optimization in transport networks based on local routing using neutrosophic bipolar fuzzy sets.
(2) A hybrid clustering method based on improved artificial bee colony and fuzzy C-Means algorithm using neutrosophic bipolar fuzzy sets.
(3) Hybrid multiattribute group decision making based on neutrosophic bipolar fuzzy sets information and GRA method.
(4) Signatures theory by using neutrosophic bipolar fuzzy sets.
(5) Risk analysis using neutrosophic bipolar fuzzy sets.

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Evaluation of Websites of IT Companies from the Perspective of IT Beginners

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Abstract: *The Internet has brought almost limitless possibilities for the promotion of services and products and thereby caused a significant change in the world. It also allowed beginners to get more important information about their future jobs. Based on this idea, the research and writing of this work began. In this paper, we investigate how much IT companies' websites provide information on company products, used technology and relationships with employees. The paper presents a multicriteria model for evaluating IT companies' websites from the point of view of young IT experts.*

Key words: *Website analysis, Single valued neutrosophic set, Multiple criteria evaluation.*

1. INTRODUCTION

The website can help you in gaining information about the company, products and services. However, the existence of a website does not automatically provide a competitive advantage. So, there is one important question: How much website actually meets the requirements of its users and how to measure its quality?

In the literature, numerous studies have been devoted to the evaluation of web site quality. Boyd Collins developed the first formal approach to the evaluation of websites in late 1995. His model, intended for librarians, has been based on six criteria, developed by combining evaluation criteria for printed media, and considering what was relevant for websites (Merwe & Bekker, 2003). The model contains the following criteria: contents, authority, organizations, searchability, graphic design and innovation use.

Studies that are intended for the identification of key evaluation criteria, and / or their significances, are still actual. For example, Dumitrache (2010) gives an overview of criteria used for evaluation of e-Commerce sites in Romania, during the period 2006 and 2009. As very important criteria it defines Response Time, Navigability, Personalization, Tele-presence and Security. Davidaviciene and Tolvaisas (2011) identify the list of criterions for quality evaluation of e-Commerce website. They also provide a comprehensive overview of the criteria that have been recently proposed by different authors. In accordance with (Davidaviciene & Tolvaisas, 2011) criteria: Easy to use, Navigation, Security assurance, Help (real time) and Design have been discussed by numerous authors, such as (Loiacono *et al.*, 2007; Parasuraman *et al.*, 2007; Cao *et al.*, 2005).

Compared to different types of e-commerce, the IT companies have its own peculiarities. Therefore, for the evaluation of IT companies' websites, we must use the appropriate set of criteria, and their significances.

The Internet has also brought significant opportunities for many less known IT companies.

In many countries, the development IT industry is often mentioned as one of the priority directions of development. A similar situation exists in Serbia, which has a number of attractive but also almost unknown IT companies.

After selecting certain IT companies, we probably want to learn more about them. Here, some questions arise: How much website of these IT companies provide the necessary information? To what extent IT companies use the benefits that the Internet provides? What information is provided to interested young IT experts?

The answer to the above questions we get with measuring the quality of websites of some IT companies that are located in Serbia.

Therefore, the rest of the paper is organized as follows: In the second section of the paper, some basic definitions related to the SVNS are given. In the third section of the paper, is proposed the criteria for evaluating websites from the standpoint of the new it professionals. In the fourth section is proposed the procedure for evaluating websites based on the use of adapted SWARA method and SVNS. In the fifth section, the proposed model has been applied to the evaluation of five IT companies that are in Serbia. At the end of the paper, the conclusions are presented.

2. SINGLE VALUED NEUTROSOPHIC SET AND NUMBERS

Definition. *Single valued neutrosophic set (SVNS)*. Let X be the universe of discourse (Wang *et al.*, 2010). The SVNS A over X is an object having the form

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \quad (1)$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \rightarrow [^-0, 1^+]$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Definition. *Single valued neutrosophic number*. For the SVNS A in X the triple $< t_A, i_A, f_A >$ is called the single valued neutrosophic number (SVNN) (Smarandache, 1999).

Definition. *Basic operations on SVNNs*. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs, then additive and multiplication operations are defined as follows (Smarandache, 1998):

$$x_1 + x_2 =< t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 >, \quad (2)$$

$$x_1 \cdot x_2 =< t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 >. \quad (3)$$

Definition. *Scalar multiplication*. Let $x =< t_x, i_x, f_x >$ be a SVNN and $\lambda > 0$, then scalar multiplication is defined as follows (Smarandache, 1998):

$$\lambda x_1 =< 1 - (1 - t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda} >. \quad (4)$$

Definition. *Power*. Let $x =< t_x, i_x, f_x >$ be a SVNN and $\lambda > 0$, then power is defined as follows:

$$x_1^{\lambda} =< t_1^{\lambda}, i_1^{\lambda}, 1 - (1 - f_1)^{\lambda} >. \quad (5)$$

Definition. *Score function*. Let $x =< t_x, i_x, f_x >$ be a SVNN, then the score function s_x of x can be as follows (Smarandache, 1998):

$$s_x = (1 + t_x - 2i_x - f_x)/2, \quad (6)$$

where $s_x \in [-1, 1]$.

Definition. *Single Valued Neutrosophic Weighted Average Operator*. Let $A_j =< t_j, i_j, f_j >$ be a collection of SVNSs and $W = (w_1, w_2, ..., w_n)^T$ is an associated weighting vector. Then, the Single Valued Neutrosophic Weighted Average (SVNWA) operator of A_j is as follows (Sahin, 2014):

$$SVNWA(A_1, A_2, ..., A_n) = \sum_{j=1}^{n} w_j A_j$$
$$= \left(1 - \prod_{j=1}^{n} (1 - t_j)^{w_j}, \prod_{j=1}^{n} (i_j)^{w_j}, \prod_{j=1}^{n} (f_j)^{w_j}, \right), \quad (7)$$

where w_j is the element *j* of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. THE CRITERIA FOR EVALUATING WEBSITES FROM THE STANDPOINT OF THE NEW IT PROFESSIONALS

In the literature, numerous studies have been devoted to the evaluation of website quality. As a result, a number of criteria have been proposed for the evaluation of websites.

However, it should be taken into account that these studies were designed for the evaluation of different types of websites.

Therefore, in this case, a set of criteria that realistically reflects the goals of the IT beginners is selected. Due to the use of SVNN, a set of the evaluation criteria that containing a smaller number of criteria more complex was selected, as follows:

- C_1 - About Us,
- C_2 - Products and Services,
- C_3 - Technologies,
- C_4 - Carrier and benefits,

where**:**

- the criterion C_1 includes general information about a company that can serve to assess its relevance in the IT industry;
- the criterion C_2 more precisely defines the scope of the company's business; and
- criteria C_3 and C_4 indicates IT technologies that used in a company, as well as the possibility of advancement, which can be very important for new IT beginners.

4. THE PROCEDURE FOR EVALUATING WEBSITES BASED ON THE USE OF ADAPTED SWARA METHOD AND SVNS

In each multiple criteria evaluation process, the following three important activities could be identified:

- determining the importance of the evaluation criteria,
- evaluation, where the alternatives are evaluated in relation to the selected set of evaluation criteria, and
- aggregation and ranking alternatives.

In this approach an adaptation of the SWARA method is accepted for determining criteria weights. The SWARA method is proposed by Keršuliene *et al.* (2010), and this method can be considered as efficient and easy to use.

Therefore, this method is used to solve a number of decision-making problems. Unfortunately, the computational procedure of the SWARA method is based on the usage of an ordered list of evaluation criteria, presorted according to their expected significances, which can be a real limitation when it is necessary to collect the real attitudes of in advice unprepared respondents. Therefore, Stanujkic *et al.* (2017) proposed an extension of the SWARA method that does not require the use of a pre-sorted list of evaluation criteria, on which basis the weight of the criteria are determined in this approach.

However, the use of a large number of criteria can lead to forming a more complex MCDM models that could be less practical for the use in the cases when researches are based on gathering real attitudes of in-advice unprepared respondents. Therefore, this approach is based on the use of a smaller number of criteria that are evaluated using SVNN.

Finally, for the aggregation phase, a procedure based on the application of the SVNWA operator and the Score function is selected.

5. A NUMERICAL ILLUSTRATION

In order to explain the proposed approach in detail, below is considered an example of evaluation of the websites of five IT companies'.

In the conducted research, the evaluation of the websites of the following IT companies was carried out:
- **Comtrade**, available at: https://www.comtrade.com/;
- **Levi Nine**, available at: https://www.levi9.com/;
- **NIRI IC**, available at: www.niri-ic.com/;
- **AB Soft**, available at: www.absoft.rs/; and
- **Informatika AD**, available at: www.informatika.com/.

It should be stated here that the aim of this article is not to promote any of the above listed companies, because of which the order of the alternatives in the presented example does not correspond to the order of the above companies.

The responses obtained from the first of three considered respondents, and weights of criteria, obtained by using extended SWARA method, are encountered in Table 1.

The attitudes obtained from the three examinees, as well as the appropriate weights and group criteria weights, are presented in Table 2 as well.

Table 1. The responses and weights of the criteria obtained from the first of three evaluated respondents

Criteria		s_j	k_j	q_j	w_j
C_1	About Us		1	1	0.22
C_2	Products and Services	0.90	1.10	0.91	0.20
C_3	Technologies	1.20	0.80	1.14	0.25
C_4	Carrier and benefits	0.60	1.40	0.81	0.18
			$\sum$	5.27	1.00

Table 2. The attitudes and weights obtained from the three examinees

	E_1		E_2		E_3		
	s_j	w_j	s_j	w_j	s_j	w_j	w_j
C_1		0.19		0.05		0.19	0.14
C_2	1.20	0.24	1.80	0.27	1.20	0.23	0.25
C_3	1.15	0.28	0.80	0.23	1.10	0.26	0.25
C_4	1.05	0.29	1.50	0.45	1.20	0.32	0.36

The following are the responses obtained from the three examinees regarding the evaluation of the websites.

Table 3. The ratings obtained from the first of the three examinees

	C_1	C_2	C_3	C_4
A_1	<0.5,0,0.2>	<0.7,0,0>	<0.7,0,0>	<0.8,0,0>
A_2	<0.7,0,0.5>	<0.8,0,0>	<0.9,0,0>	<0.9,0,0>
A_3	<0.15,0,0.2>	<0.3,0,0.15>	<0.2,0,0.3>	<0.1,0,0.4>
A_4	<0.25,0,0.3>	<0.2,0,0>	<0.15, 0,0.2>	<0.05,0,0.3>
A_5	<0.2,0,0.4>	<0.4,0,0>	<0.1, 0, 0.5>	<0.05,0,0.2>

Table 4. The ratings obtained from the second of the three examinees

	C_1	C_2	C_3	C_4
A_1	<0.8, 0, 0.2>	<0.9, 0, 0>	<0.8, 0, 0>	<0.8, 0, 0>
A_2	<0.5, 0, 0.6>	<0.8, 0, 0>	<0.6, 0, 0>	<0.7, 0, 0>
A_3	<0.1, 0, 0.8>	<0.3, 0, 0>	<0.35, 0,0.8>	<0.2, 0, 0>
A_4	<0.2, 0, 0.6>	<0.2, 0, 0>	<0.3, 0, 0>	<0.1, 0, 0>
A_5	<0.6, 0, 0.3>	<0.1, 0, 0.9>	<0.2, 0, 0>	<0.1, 0, 0>

Table 5. The ratings obtained from the third of the three examinees

	C_1	C_2	C_3	C_4
A_1	<1, 0, 0.1>	<0.5, 0, 0>	<0.9, 0, 0.1>	<0.8,0.1,0.1>
A_2	<0.5,0.1,0.3>	<0.4, 0, 0>	<1, 0, 0>	<0.8,0.1,0.1>
A_3	<0.4, 0, 0>	<0.4, 0, 0.1>	<0.4, 0.1, 1>	<0.4,0.3,0.2>
A_4	<0.2,0.2,0.2>	<0.1, 0, 0>	<0.3,0.3,0.1>	<0.1, 0, 0.4>
A_5	<0.5,0.2,0.1>	<0.1,0.2,0.1>	<0, 0.3, 0.5>	<0.1, 0, 0.4>

The group ratings and overall ratings, shown in Table 6 and Table 7, are obtained by using SVNWA operator, or more precisely by using Eq. (7).

Table 6. The group ratings

	C_1	C_2	C_3	C_4
A_1	<1,0, 0.16>	<0.75, 0, 0>	<0.82, 0, 0>	<0.8, 0, 0>
A_2	<0.58,0,0.45>	<0.71, 0, 0>	<1, 0, 0>	<0.82, 0, 0>
A_3	<0.23,0,0.03>	<0.34,0,0.01>	<0.32,0,0.62>	<0.24,0,0.02>
A_4	<0.22,0,0.33>	<0.17, 0, 0>	<0.25,0,0.01>	<0.08,0,0.02>
A_5	<0.46,0,0.23>	<0.21,0,0.02>	<0.1,0,0.03>	<0.08,0,0.02>

The ranking order of considered alternatives, obtained based on the values of the score function, calculated by using Eq. (6), is also presented in Table 7.

Table 7. The overall ratings

	Overall ratings	S_i	Rank
A_1	<1, 0, 0>	0.9992	2
A_2	<1, 0, 0>	0.9993	1
A_3	<0.29, 0, 0.04>	0.6207	3
A_4	<0.17, 0,0 .01>	0.5807	4
A_5	<0.19, 0, 0.03>	0.5767	5

As it can be concluded on the basis of the data presented in Table 7, the most promising company from the perspective of an IT beginner is the company labelled as A_2, which is somewhat more promising than a company designated as A_1.

6. CONCLUSION

This paper proposes a simple but also effective framework, which can be used for measuring the quality of IT companies' websites from the perspective of IT beginners.

The proposed procedure for evaluating websites based on the use of adapted SWARA method and SVNS has been successfully applied to the evaluation of five IT companies.

Using this procedure, the managers of IT companies, can evaluate their and competing websites and compare them. This would continually influence the improvement of the quality of the websites and thus make it easier for youngsters to get to the desired information about the IT company.

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Multi-Granulation Neutrosophic Rough Sets on a Single Domain and Dual Domains with Applications

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Abstract: It is an interesting direction to study rough sets from a multi-granularity perspective. In rough set theory, the multi-particle structure was represented by a binary relation. This paper considers a new neutrosophic rough set model, multi-granulation neutrosophic rough set (MGNRS). First, the concept of MGNRS on a single domain and dual domains was proposed. Then, their properties and operators were considered. We obtained that MGNRS on dual domains will degenerate into MGNRS on a single domain when the two domains are the same. Finally, a kind of special multi-criteria group decision making (MCGDM) problem was solved based on MGNRS on dual domains, and an example was given to show its feasibility.

Keywords: neutrosophic rough set; MGNRS; dual domains; inclusion relation; decision-making

1. Introduction

As we all know, Pawlak first proposed a rough set in 1982, which was a useful tool of granular computing. The relation is an equivalent in Pawlak's rough set. After that, many researchers proposed other types of rough set theory (see the work by the following authors [1–8]).

In 1965, Zadeh presented a new concept of the fuzzy set. After that, a lot of scholars studied it and made extensions. For example, Atanassov introduced an intuitionistic fuzzy set, which gives two degrees of membership of an element; it is a generalization of the fuzzy set. Smarandache introduced a neutrosophic set in 1998 [9,10], which was an extension of the intuitionistic fuzzy set. It gives three degrees of membership of an element (T.I.F). Smarandache and Wang [11] proposed the definition of a single valued neutrosophic set and studied its operators. Ye [12] proposed the definition of simplified neutrosophic sets and studied their operators. Zhang et al. [13] introduced a new inclusion relation of the neutrosophic set and told us when it was used by an example, and its lattice structure was studied. Garg and Nancy proposed neutrosophic operators and applied them to decision-making problems [14–16]. Now, some researchers have combined the fuzzy set and rough set and have achieved many running results, such as the fuzzy rough set [17] and rough fuzzy set. Broumi and Smarandache [18] proposed the definition of a rough neutrosophic set and studied their operators and properties. In 2016, Yang et al. [19] proposed the definition of single valued neutrosophic rough sets and studied their operators and properties.

Under the perspective of granular computing [20], the concept of a rough set is shown by the upper and lower approximations of granularity. In other words, the concept is represented by the known knowledge, which is defined by a single relationship. In fact, to meet the user's needs or achieve the goal of solving the problem, it is sometimes necessary to use multiple relational representation concepts on the domain, such as illustrated by the authors of [21]. In a grain calculation, an equivalence relation in the domain is a granularity, and a partition is considered as a granularity space [22]. The approximation that is defined by multiple equivalence relationships is a multi-granularity approximation and multiple partitions are considered as multi-granularity spaces; the resulting rough set is named a multi-granularity rough set, which has been proposed by Qian and Liang [23]. Recently, many scholars [24,25] have studied it and made extensions. Huang et al. [26] proposed the notion of intuitionistic fuzzy multi-granulation rough sets and studied their operators. Zhang et al. [27] introduced two new multi-granulation rough set models and investigated their operators. Yao et al. [28] made a summary about the rough set models on the multi-granulation spaces.

Although there have been many studies regarding multi-granulation rough set theory, there have been fewer studies about the multi-granulation rough set model on dual domains. Moreover, a multi-granulation rough set on dual domains is more convenient, for example, medical diagnosis in clinics [22,29]. The symptoms are the uncertainty index sets and the diseases are the decision sets. They are associated with each other, but they belong to two different domains. Therefore, it is necessary to use two different domains when solving the MCGDM problems. Sun et al. [29] discussed the multi-granulation rough set models based on dual domains; their properties were also obtained.

Although neutrosophic sets and multi-granulation rough sets are both useful tools to solve uncertainty problems, there are few research regarding their combination. In this paper, we proposed the definition of MGNRS as a rough set generated by multi-neutrosophic relations. It is useful to solve a kind of special group decision-making problem. We studied their properties and operations and then built a way to solve MCGDM problems based on the MGNRS theory on dual domains.

The structure of the article is as follows. In Section 2, some basic notions and operations are introduced. In Section 3, the notion of MGNRS is proposed and their properties are studied. In Section 4, the model of MGNRS on dual domains is proposed and their properties are obtained. Also, we obtained that MGNRS on dual domains will degenerate into MGNRS on a single domain when the two domains are same. In Section 5, an application of the MGNRS to solve a MCGDM problem was proposed. Finally, Section 6 concludes this paper and provides an outlook.

2. Preliminary

In this section, we review several basic concepts and operations of the neutrosophic set and multi-granulation rough set.

Definition 1 ([11])**.** *A single valued neutrosophic set B is denoted by* $\forall\, y \in Y$*, as follows:*

$$B(y) = (T_B(y), I_B(y), F_B(y))$$

$T_B(y), I_B(y), F_B(y) \in [0,1]$ *and satisfies* $0 \leq T_B(y) + I_B(y) + F_B(y) \leq 3$.

As a matter of convenience, 'single valued neutrosophic set' is abbreviated to 'neutrosophic set' later. In this paper, $NS(Y)$ denotes the set of all single valued neutrosophic sets in Y, and $NR(Y \times Z)$ denotes the set of all of the neutrosophic relations in $Y \times Z$.

Definition 2 ([11])**.** *If A and C are two neutrosophic sets, then the inclusion relation, union, intersection, and complement operations are defined as follows:*

(1) $A \subseteq C$ *iff* $\forall\, y \in Y$, $TA(y) \leq TC(y)$, $IA(y) \geq IC(y)$ *and* $FA(y) \geq FC(y)$
(2) $A^c = \{(y, F_A(y), 1 - I_A(y), T_A(y)) \mid y \in Y\}$
(3) $A \cap C = \{(y, T_A(y) \wedge T_C(y), I_A(y) \vee I_C(y), F_A(y) \vee F_C(y)) \mid y \in Y\}$
(4) $A \cup C = \{(y, T_A(y) \vee T_C(y), I_A(y) \wedge I_C(y), F_A(y) \wedge F_C(y)) \mid y \in Y\}$

Definition 3 ([19]). *If (U, R) is a single valued neutrosophic approximation space. Then ∀ B ϵ SVNS(U), the lower approximation* $\underline{N}(B)$ *and upper approximation* $\overline{N}(B)$ of B are defined as follows:

$$T_{\underline{N}(B)}(y) = \min_{z\in U}[\max(F_R(y,z), T_B(z))],\ I_{\underline{N}(B)}(y) = \max_{z\in U}[\min((1 - I_R(y,z)), I_B(z))],$$
$$F_{\underline{N}(B)}(y) = \max_{z\in U}[\min(T_R(y,z), F_B(z))]$$

$$T_{\overline{N}(B)}(y) = \max_{z\in U}[\min(T_R(y,z), T_B(z))],\ I_{\overline{N}(B)}(y) = \min_{z\in U}[\max(I_R(y,z), I_B(z))],$$
$$F_{\overline{N}(B)}(y) = \min_{z\in U}[\max(F_R(y,z), F_B(z))]$$

The pair $\left(\underline{N}(B), \overline{N}(B)\right)$ is called the single valued neutrosophic rough set of B, with respect to (U, R).

According to the operation of neutrosophic number in [16], the sum of two neutrosophic sets in U is defined as follows.

Definition 4. *If C and D are two neutrosophic sets in U, then the sum of C and D is defined as follows:*

$$C + D = \{\langle y, C(y) \oplus D(y)\rangle \mid y \in U\}.$$

Definition 5 ([30]). *If b = (T_b, I_b, F_b) is a neutrosophic number, n* = (T_{b^*}, I_{b^*}, F_{b^*}) = (1, 0, 0) is an ideal neutrosophic number. Then, the cosine similarity measure is defined as follows:*

$$S(b, b^*) = \frac{T_b \cdot T_{b^*} + I_b \cdot I_{b^*} + F_b \cdot F_{b^*}}{\sqrt{{T_b}^2 + {I_b}^2 + {F_b}^2} \cdot \sqrt{(T_{b^*})^2 + (I_{b^*})^2 + (F_{b^*})^2}}$$

3. Multi-Granulation Neutrosophic Rough Sets

In this part, we propose the concept of MGNRS and study their characterizations. MGNRS is a rough set generated by multi-neutrosophic relations, and when all neutrosophic relations are same, MGNRS will degenerated to neutrosophic rough set.

Definition 6. *Assume U is a non-empty finite domain, and R_i ($1 \le i \le n$) is the binary neutrosophic relation on U. Then, (U, R_i) is called the multi-granulation neutrosophic approximation space (MGNAS).*

Next, we present the multi-granulation rough approximation of a neutrosophic concept in an approximation space.

Definition 7. *Let the tuple ordered set (U, R_i) ($1 \le i \le n$) be a MGNAS. For any B ϵ NS (U), the three memberships of the optimistic lower approximation* $\underline{M}^o(B)$*and optimistic upper approximation*$\overline{M}^o(B)$ *in (U, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z\in U}\left(\max\left(F_{R_i}(y,z), T_B(z)\right)\right)\ I_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z\in U}\left(\min\left((1 - I_{R_i}(y,z)), I_B(z)\right)\right),$$
$$F_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z\in U}\left(\min\left(T_{R_i}(y,z), F_B(z)\right)\right), T_{\overline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z\in U}\left(\min\left(T_{R_i}(y,z), T_B(z)\right)\right),$$
$$I_{\overline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z\in U}\left(\max\left(I_{R_i}(y,z), I_B(z)\right)\right), F_{\overline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z\in U}\left(\max\left(F_{R_i}(y,z), F_B(z)\right)\right).$$

Then, $\underline{M}^o(B)$, $\overline{M}^o(B)$ ϵ NS(U). In addition, B is called a definable neutrosophic set on (U, R_i) when $\underline{M}^o(B) = \overline{M}^o(B)$. Otherwise, the pair $\left(\underline{M}^o(B), \overline{M}^o(B)\right)$ is called an optimistic MGNRS.

Definition 8. *Let the tuple ordered set (U, R_i) ($1 \le i \le n$) be a MGNAS. For any B ϵ NS(U), the three memberships of pessimistic lower approximation $\underline{M}^p(B)$ and pessimistic upper approximation $\overline{M}^p(B)$ in (U, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z\in U}\big(\max\big(F_{R_i}(y,z), T_B(z)\big)\big),\ I_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z\in U}\big(\min\big((1 - I_{R_i}(y,z)), I_B(z)\big)\big),$$
$$F_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z\in U}\big(\min\big(T_{R_i}(y,z), F_B(z)\big)\big),\ T_{\overline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z\in U}\big(\min\big(T_{R_i}(y,z), T_B(z)\big)\big),$$
$$I_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z\in U}\big(\max\big(I_{R_i}(y,z), I_B(z)\big)\big),\ F_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z\in U}\big(\max\big(F_{R_i}(y,z), F_B(z)\big)\big).$$

Similarly, B is called a definable neutrosophic set on (U, R_i) when $\underline{M}^p(B) = \overline{M}^p(B)$. Otherwise, the pair $\left(\underline{M}^p(B), \overline{M}^p(B)\right)$ is called a pessimistic MGNRS.

Example 1. *Define MGNAS (U, R_i), where U = {z_1, z_2, z_3} and R_i ($1 \le i \le 3$) are given in Tables 1–3*

Table 1. Neutrosophic relation R_1.

R_1	z_1	z_2	z_3
z_1	(0.4, 0.5, 0.4)	(0.5, 0.7, 0.1)	(1, 0.8, 0.8)
z_2	(0.5, 0.6, 1)	(0.2, 0.6, 0.4)	(0.9, 0.2, 0.4)
z_3	(1, 0.2, 0)	(0.8, 0.9, 1)	(0.6, 1, 0)

Table 2. Neutrosophic relation R_2.

R_2	z_1	z_2	z_3
z_1	(0.9, 0.2, 0.4)	(0.3, 0.9, 0.1)	(0.1, 0.7, 0)
z_2	(0.4, 0.5, 0.1)	(0, 0.1, 0.7)	(1, 0.8, 0.8)
z_3	(1, 0.5, 0)	(0.4, 0.4, 0.2)	(0.1, 0.5, 0.4)

Table 3. Neutrosophic relation R_3.

R_3	z_1	z_2	z_3
z_1	(0.7, 0.7, 0)	(0.4, 0.8, 0.9)	(1, 0.4, 0.5)
z_2	(0.8, 0.2, 0.1)	(1, 0.1, 0.8)	(0.1, 0.3, 0.5)
z_3	(0, 0.8, 1)	(1, 0, 1)	(1, 1, 0)

Suppose a neutrosophic set on U is as follows: $C(z_1)$ = (0.2, 0.6, 0.4), $C(z_2)$ = (0.5, 0.4, 1), $C(z_3)$ = (0.7, 0.1, 0.5); by Definitions 7 and 8, we can get the following:

$$\underline{M}^o(C)(z_1) = (0.4, 0.3, 0.4),\ \underline{M}^o(C)(z_2) = (0.5, 0.4, 0.5),\ \underline{M}^o(C)(z_3) = (0.7, 0.4, 0.4)$$
$$\overline{M}^o(C)(z_1) = (0.3, 0.6, 0.4),\ \overline{M}^o(C)(z_2) = (0.5, 0.4, 0.5),\ \overline{M}^o(C)(z_3) = (0.4, 0.6, 0.5)$$
$$\underline{M}^p(C)(z_1) = (0.2, 0.6, 0.5),\ \underline{M}^p(C)(z_2) = (0.2, 0.6, 0.1),\ \underline{M}^p(C)(z_3) = (0.2, 0.6, 0.1)$$
$$\overline{M}^p(C)(z_1) = (0.7, 0.4, 0.4),\ \overline{M}^p(C)(z_2) = (0.7, 0.2, 0.4),\ \overline{M}^p(C)(z_3) = (0.7, 0.4, 0.4)$$

Proposition 1. *Assume (U, R_i) is MGNAS, R_i ($1 \le i \le n$) is the neutrosophic relations. ∀ C ϵ NS(U), $\underline{M}^o(C)$ and $\overline{M}^o(C)$ are the optimistic lower and upper approximation of C. Then,*

$$\underline{M}^o(C) = \bigcup_{i=1}^{n} \underline{N}(C) \overline{M}^o(C) = \bigcap_{i=1}^{n} \overline{N}(C)$$

where

$$\underline{N}(C)(y) = \bigcap_{z\in U}(R_i{}^c(y,z) \cup C(z)), \overline{N}(C)(y) = \bigcup_{z\in U}(R_i(y,z) \cap C(z))$$

Proof. They can be proved by Definitions 7.

Proposition 2. *Assume (U, R_i) be MGNAS, R_i ($1 \le i \le n$) be neutrosophic relations. $\forall$ C ϵ NS(U), $\underline{M}^p(C)$ and $\overline{M}^p(C)$ are the pessimistic lower and upper approximation of C. Then*

$$\underline{M}^p(C) = \overset{n}{\underset{i=1}{\cap}} \underline{N}(C)\overline{M}^p(C) = \overset{n}{\underset{i=1}{\cup}} \overline{N}(C)$$

where

$$\underline{N}(C)(y) = \underset{z\in U}{\cap}(R_i{}^c(y,z) \cup C(z)), \overline{N}(C)(y) = \underset{z\in U}{\cup}(R_i(y,z) \cap C(z))$$

Proof. Proposition 2 can be proven by Definition 8.

Proposition 3. *Assume (U, R_i) is MGNAS, R_i ($1 \le i \le n$) is the neutrosophic relations. $\forall$ C, D ϵ NS(U), we have the following:*

(1) $\underline{M}^o(C) = \sim \overline{M}^o(\sim C), \underline{M}^p(C) = \sim \overline{M}^p(\sim C)$;
(2) $\overline{M}^o(C) = \sim \underline{M}^o(\sim C), \overline{M}^p(C) = \sim \underline{M}^p(\sim C)$;
(3) $\underline{M}^o(C \cap D) = \underline{M}^o(C) \cap \underline{M}^o(D), \underline{M}^p(C \cap D) = \underline{M}^p(C) \cap \underline{M}^p(D)$;
(4) $\overline{M}^o(C \cup D) = \overline{M}^o(C) \cup \overline{M}^o(D), \overline{M}^p(C \cup D) = \overline{M}^p(C) \cup \overline{M}^p(D)$;
(5) $C \subseteq D \Rightarrow \underline{M}^o(C) \subseteq \underline{M}^o(D), \underline{M}^p(C) \subseteq \underline{M}^p(D)$;
(6) $C \subseteq D \Rightarrow \overline{M}^o(C) \subseteq \overline{M}^o(D), \overline{M}^p(C) \subseteq \overline{M}^p(D)$;
(7) $\underline{M}^o(C \cup D) \supseteq \underline{M}^o(C) \cup \underline{M}^o(D), \underline{M}^p(C \cup D) \supseteq \underline{M}^p(C) \cup \underline{M}^p(D)$;
(8) $\overline{M}^o(C \cap D) \subseteq \overline{M}^o(C) \cap \overline{M}^o(D), \overline{M}^p(C \cap D) \subseteq \overline{M}^p(C) \cap \overline{M}^p(D)$.

Proof. (1), (2), (5), and (6) can be taken directly from Definitions 7 and 8. We only show (3), (4), (7), and (8).

(3) From Proposition 1, we have the following:

$$\begin{aligned}\underline{M}^o(C \cap D)(y) = &\ \overset{n}{\underset{i=1}{\cup}}\left(\underset{z\in U}{\cap}(R_i{}^c(y,z) \cup (C \cap D)(z))\right)\\ &= \overset{n}{\underset{i=1}{\cup}}\left(\underset{z\in U}{\cap}((R_i{}^c(y,z) \cup C(z)) \cap (R_i{}^c(y,z) \cup D(z)))\right)\\ &= \left(\overset{n}{\underset{i=1}{\cup}}\left(\underset{z\in U}{\cap}(R_i{}^c(y,z) \cup C(z))\right)\right) \cap \left(\overset{n}{\underset{i=1}{\cup}}\left(\underset{z\in U}{\cap}(R_i{}^c(y,z) \cup D(z))\right)\right)\\ &= \underline{M}^o C(y) \cap \underline{M}^o D(y).\end{aligned}$$

Similarly, from Proposition 2, we can get the following:

$$\underline{M}^p(C \cap D)(y) = \underline{M}^p C(y) \cap \underline{M}^p D(y).$$

(4) According to Propositions 1 and 2, in the same way as (3), we can get the proof.

(7) From Definition 7, we have the following:

$$\begin{aligned}T_{\underline{M}^o(C\cup D)}(y) &= \overset{n}{\underset{i=1}{\max}}\ \underset{z\in U}{\min}\{\max[F_{R_i}(y,z), (\max(T_C(z), T_D(z)))]\}\\ &= \overset{n}{\underset{i=1}{\max}}\ \underset{z\in U}{\min}\{\max[(\max(F_{R_i}(y,z), T_C(z))), (\max(F_{R_i}(y,z), T_D(z)))]\}\\ &\ge \max\left\{\left[\overset{n}{\underset{i=1}{\max}}\ \underset{z\in U}{\min}(\max(F_{R_i}(y,z), T_C(z)))\right], \left[\overset{n}{\underset{i=1}{\max}}\ \underset{z\in U}{\min}(\max(F_{R_i}(y,z), T_D(z)))\right]\right\}\\ &= \max\left(T_{\underline{M}^o(C)}(y), T_{\underline{M}^o(D)}(y)\right).\end{aligned}$$

$$\begin{aligned}
I_{\underline{M}^o(C\cup D)}(y) &= \min_{i=1}^{n}\max_{z\in U}\{\min[(1-I_{R_i}(y,z)),(\min(I_C(z),I_D(z)))]\}\\
&= \min_{i=1}^{n}\max_{z\in U}\{\min[(\min((1-I_{R_i}(y,z)),I_C(z))),(\min((1-I_{R_i}(y,z)),I_D(z)))]\}\\
&\le \min\left\{\left[\min_{i=1}^{n}\max_{z\in U}(\min((1-I_{R_i}(y,z)),I_C(z)))\right],\left[\min_{i=1}^{n}\max_{z\in U}(\min((1-I_{R_i}(y,z)),I_D(z)))\right]\right\}\\
&= \min\left(I_{\underline{M}^o(C)}(y),I_{\underline{M}^o(D)}(y)\right).\\
F_{\underline{M}^o(C\cup D)}(y) &= \min_{i=1}^{n}\max_{z\in U}\{\min[T_{R_i}(y,z),(\min(F_C(z),F_D(z)))]\}\\
&= \min_{i=1}^{n}\max_{z\in U}\{\min[\min(T_{R_i}(y,z),F_C(z))],[\min(T_{R_i}(y,z),F_D(z))]\}\\
&\le \min\left\{\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),F_C(z)))\right],\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),F_D(z)))\right]\right\}\\
&= \min\left(F_{\underline{M}^o(C)}(y),F_{\underline{M}^o(D)}(y)\right).
\end{aligned}$$

Hence, $\underline{M}^o(C\cup D)\supseteq \underline{M}^o(C)\cup\underline{M}^o(D)$.

Also, according to Definition 8, we can get $\underline{M}^p(C\cup D)\supseteq \underline{M}^p(C)\cup\underline{M}^p(D)$.

(8) From Definition 7, we have the following:

$$\begin{aligned}
T_{\overline{M}^o(C\cap D)}(y) &= \min_{i=1}^{n}\max_{z\in U}\{\min[T_{R_i}(y,z),(\min(T_C(z),T_D(z)))]\}\\
&= \min_{i=1}^{n}\max_{z\in U}\{\min[(\min(T_{R_i}(y,z),T_C(z))),(\min(T_{R_i}(y,z),T_D(z)))]\}\\
&\le \min\left\{\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),T_C(z)))\right],\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),T_D(z)))\right]\right\}\\
&= \min\left(T_{\overline{M}^o(C)}(y),T_{\overline{M}^o(D)}(y)\right).
\end{aligned}$$

$$\begin{aligned}
I_{\overline{M}^o(C\cap D)}(y) &= \max_{i=1}^{n}\min_{z\in U}\{\max[I_{R_i}(y,z),(\max(I_C(z),I_D(z)))]\}\\
&= \max_{i=1}^{n}\min_{z\in U}\{\max[(\max(I_{R_i}(y,z),I_C(z))),(\max(I_{R_i}(y,z),I_D(z)))]\}\\
&\le \min\left\{\left[\max_{i=1}^{n}\min_{z\in U}(\max(I_{R_i}(y,z),I_C(z)))\right],\left[\max_{i=1}^{n}\min_{z\in U}(\max(I_{R_i}(y,z),I_D(z)))\right]\right\}\\
&= \min\left(I_{\overline{M}^o(C)}(y),I_{\overline{M}^o(D)}(y)\right).\\
F_{\overline{M}^o(C\cap D)}(y) &= \max_{i=1}^{n}\min_{z\in U}[F_{R_i}(y,z)\vee(F_C(z)\vee F_D(z))]\\
&= \max_{i=1}^{n}\min_{z\in U}[(F_{R_i}(y,z)\vee F_C(z))\vee(F_{R_i}(y,z)\vee F_D(z))]\\
&\ge \left[\max_{i=1}^{n}\min_{z\in U}(F_{R_i}(y,z)\vee F_C(z))\right]\vee\left[\max_{i=1}^{n}\min_{z\in U}(F_{R_i}(y,z)\vee F_D(z))\right]\\
&= \max\left(F_{\overline{M}^o(C)}(y),F_{\overline{M}^o(D)}(y)\right).
\end{aligned}$$

Hence, $\overline{M}^o(C\cap D)\subseteq\overline{M}^o(C)\cap\overline{M}^o(D)$.

Similarly, according Definition 8, we can get $\overline{M}^p(C\cap D)\subseteq\overline{M}^p(C)\cap\overline{M}^p(D)$.

Next, we will give an example to show that maybe $\underline{M}^o(C\cup D)\neq\underline{M}^o(C)\cup\underline{M}^o(D)$.

Example 2. *Define MGNAS (U, R_i), where U = {z_1, z_2, z_3} and R_i ($1\le i\le 3$) are given in Example 1.*

Suppose there are two neutrosophic sets on universe U, as follows: C(z_1) = (0.5, 0.1, 0.2), C(z_2) = (0.5, 0.3, 0.2), C(z_3) = (0.6, 0.2, 0.1), D(z_1) = (0.7, 0.2, 0.1), D(z_2) = (0.4, 0.2, 0.1), D(z_3) = (0.2, 0.2, 0.5), we have (C ∪ D)(z_1) = (0.7, 0.1, 0.1), (C ∪ D)(z_2) = (0.5, 0.2, 0.1), (C ∪ D)(z_3) = (0.6, 0.2, 0.1), (C ∩ D)(z_1) = (0.5, 0.1, *0.2), (C ∩ D)(z_2) = (0.4, 0.2, 0.2), (C ∩ D)(z_3) = (0.2, 0.2, 0.5). Then, from Definitions 7 and 8, we can get the following:*

$$\underline{M}^o(C)(z_1) = (0.5, 0, 0.2), \underline{M}^o(C)(z_2) = (0.5, 0.1, 0.2), \underline{M}^o(C)(z_3) = (0.5, 0.1, 0.2);$$
$$\underline{M}^o(D)(z_1) = (0.4, 0, 0.1), \underline{M}^o(D)(z_2) = (0.2, 0.1, 0.2), \underline{M}^o(D)(z_3) = (0.4, 0.1, 0.2);$$
$$\underline{M}^o(C \cup D)(z_1) = (0.5, 0, 0.1), \underline{M}^o(C \cup D)(z_2) = (0.5, 0.1, 0.1), \underline{M}^o(C \cup D)(z_3) = (0.5, 0.1, 0.1)$$
$$(\underline{M}^o(C) \cup \underline{M}^o(D))(z_1) = (0.5, 0, 0.1), (\underline{M}^o(C) \cup \underline{M}^o(D))(z_2) = (0.5, 0.1, 0.2),$$
$$(\underline{M}^o(C) \cup \underline{M}^o(D))(z_3) = (0.5, 0.1, 0.2)$$

So, $\underline{M}^o(C \cup D) \neq \underline{M}^o(C) \cup \underline{M}^o(D)$.
Also, there are examples to show that maybe $\underline{M}^p(C \cup D) \neq \underline{M}^p(C) \cup \underline{M}^p(D)$,
$\overline{M}^o(C \cap D) \neq \overline{M}^o(C) \cap \overline{M}^o(D)$, $\overline{M}^p(C \cap D) \neq \overline{M}^p(C) \cap \overline{M}^p(D)$. *We do not say anymore here.*

4. Multi-Granulation Neutrosophic Rough Sets on Dual Domains

In this section, we propose the concept of MGNRS on dual domains and study their characterizations. Also, we obtain that the MGNRS on dual domains will degenerate into MGNRS, defined in Section 3, when the two domains are same.

Definition 9. *Assume that U and V are two domains, and R_i ϵ NS(U × V) (1 ≤ i ≤ n) is the binary neutrosophic relations. The triple ordered set (U, V, R_i) is called the (two-domain) MGNAS.*

Next, we present the multi-granulation rough approximation of a neutrosophic concept in an approximation space on dual domains.

Definition 10. *Let (U, V, R_i) (1 ≤ i ≤ n) be (two-domain) MGNAS. ∀ B ϵ NS(V) and y ϵ U, the three memberships of the optimistic lower and upper approximation* $\underline{M}^o(B)$, $\overline{M}^o(B)$ *in (U, V, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), T_B(z)\right)\right] I_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in V}\left[\min\left((1 - I_{R_i}(y,z)), I_B(z)\right)\right]$$

$$F_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), F_B(z)\right)\right] T_{\overline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), T_B(z)\right)\right]$$
$$I_{\overline{\sum_{i=1}^{n} R_i}^{o}(B)}(y) = \max_{i=1}^{n} \min_{z \in V}\left[\max\left(I_{R_i}(y,z), I_B(z)\right)\right] F_{\overline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), F_B(z)\right)\right]$$

Then $\underline{M}^o(B)$, $\overline{M}^o(B)$ ϵ *NS(U)*. In addition, *B* is called a definable neutrosophic set on (*U*, *V*, R_i) on dual domains when $\underline{M}^o(B) = \overline{M}^o(B)$. Otherwise, the pair $\left(\underline{M}^o(B), \overline{M}^o(B)\right)$ is called an optimistic MGNRS on dual domains.

Definition 11. *Assume (U, V, R_i) (1 ≤ i ≤ n) is (two-domain) MGNAS. ∀ B ϵ NS(V) and y ϵ U, the three memberships of the pessimistic lower and upper approximation* $\underline{M}^p(B)$, $\overline{M}^p(B)$ *in (U, V, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), T_B(z)\right)\right], I_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in V}\left[\min\left((1 - I_{R_i}(y,z)), I_B(z)\right)\right],$$

$$F_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), F_B(z)\right)\right], T_{\overline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), T_B(z)\right)\right],$$
$$I_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in V}\left[\max\left(I_{R_i}(y,z), I_B(z)\right)\right], F_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), F_B(z)\right)\right].$$

Then, B is called a definable neutrosophic set on (U, V, R_i) when $\underline{M}^p(B) = \overline{M}^p(B)$. Otherwise, the pair $\left(\underline{M}^p(B), \overline{M}^p(B)\right)$ is called a pessimistic MGNRS on dual domains.

Remark 1. *Note that if U = V, then the optimistic and pessimistic MGNRS on the dual domains will be the same with the optimistic and pessimistic MGNRS on a single domain, which is defined in Section 3*

Proposition 4. *Assume (U, V, R_i) ($1 \leq i \leq n$) is (two-domain) MGNAS, R_i ($1 \leq i \leq n$) is the neutrosophic relations. ∀ C, D ϵ NS(U), we have the following:*

(1) $\underline{M}^o(C) = \sim \overline{M}^o(\sim C), \underline{M}^p(C) = \sim \overline{M}^p(\sim C);$
(2) $\overline{M}^o(C) = \sim \underline{M}^o(\sim C), \overline{M}^p(C) = \sim \underline{M}^p(\sim C);$
(3) $\underline{M}^o(C \cap D) = \underline{M}^o(C) \cap \underline{M}^o(D), \underline{M}^p(C \cap D) = \underline{M}^p(C) \cap \underline{M}^p(D);$
(4) $\overline{M}^o(C \cup D) = \overline{M}^o(C) \cup \overline{M}^o(D), \overline{M}^p(C \cup D) = \overline{M}^p(C) \cup \overline{M}^p(D);$
(5) $C \subseteq D \Rightarrow \underline{M}^o(C) \subseteq \underline{M}^o(D), \underline{M}^p(C) \subseteq \underline{M}^p(D);$
(6) $C \subseteq D \Rightarrow \overline{M}^o(C) \subseteq \overline{M}^o(D), \overline{M}^p(C) \subseteq \overline{M}^p(D);$
(7) $\underline{M}^o(C \cup D) \supseteq \underline{M}^o(C) \cup \underline{M}^o(D), \underline{M}^p(C \cup D) \supseteq \underline{M}^p(C) \cup \underline{M}^p(D);$
(8) $\overline{M}^o(C \cap D) \subseteq \overline{M}^o(C) \cap \overline{M}^o(D), \overline{M}^p(C \cap D) \subseteq \overline{M}^p(C) \cap \overline{M}^p(D).$

Proof. These propositions can be directly proven from Definitions 10 and 11.

5. An Application of Multi-Granulation Neutrosophic Rough Set on Dual Domains

Group decision making [31] is a useful way to solve uncertainty problems. It has developed rapidly since it was first proposed. Its essence is that in the decision-making process, multiple decision makers (experts) are required to participate and negotiate in order to settle the corresponding decision-making problems. However, with the complexity of the group decision-making problems, what we need to deal with is the multi-criteria problems, that is, multi-criteria group decision making (MCGDM). The MCGDM problem is to select or rank all of the feasible alternatives in multiple, interactive, and conflicting standards.

In this section, we build a neo-way to solve a kind of special MCGDM problem using the MGNRS theory. We generated the rough set according the multi-neutrosophic relations and then used it to solve the decision-making problems. We show the course and methodology of it.

5.1. Problem Description

Firstly, we describe the considered problem and we show it using a MCGDM example of houses selecting.

Let $U = \{x_1, x_2, \ldots, x_m\}$ be the decision set, where x_1 represents very good, x_2 represents good, x_3 represents less good, ... , and x_m represents not good. Let $V = \{y_1, y_2, \ldots, y_n\}$ be the criteria set to describe the given house, where y_1 represents texture, y_2 represents geographic location, y_3 represents price, ... , and y_n represents solidity. Suppose there are k evaluation experts and all of the experts give their own evaluation for criteria set y_j (y_j ϵ V) (j = 1, 2, ... , n), regarding the decision set elements x_i (x_i ϵ U) (i = 1, 2, ... , m). In this paper, let the evaluation relation $R_1, R_2, \ldots, R_k$ between V and U given by the experts, be the neutrosophic relation, $R_1, R_2, \ldots, R_k$ ϵ SNS ($U \times V$). That is, R_l (x_i, y_j) (l = 1, 2, ... , k) represents the relation of the criteria set y_j and the decision set element x_i, which is given by expert l, based on their own specialized knowledge and experience. For a given customer, the criterion of the customer is shown using a neutrosophic set, C, in V, according to an expert's opinion. Then, the result of this problem is to get the opinion of the given house for the customer.

Then, we show the method to solve the above problem according to the theory of optimistic and pessimistic MGNRS on dual domains.

5.2. New Method

In the first step, we propose the multi-granulation neutrosophic decision information system based on dual domains for the above problem.

According to Section 5.1's description, we can get the evaluation of each expert as a neutrosophic relation. Then, all of the binary neutrosophic relations R_l given by all of the experts construct a relation set $\mathcal{R}$ (i.e., $R_l \in \mathcal{R}$). Then, we get the multi-granulation neutrosophic decision information systems based on dual domains, denoted by $(U, V, \mathcal{R})$.

Secondly, we compute $\underline{M}^o(C), \overline{M}^o(C), \underline{M}^p(C), \overline{M}^p(C)$ for the given customer, regarding $(U, V, \mathcal{R})$.

Thirdly, according to Definition 4, we computed the sum of the optimistic and pessimistic multi-granulation neutrosophic lower and upper approximation.

Next, according Definition 5, we computed the cosine similarity measure. Define the choice $x*$ with the idea characteristics value $\alpha * = (1, 0, 0)$ as the ideal choice. The bigger the value of $S(\alpha_{x_i}, \alpha^*)$ is, the closer the choice x_i with the ideal alternative $x *$, so the better choice x_i is.

Finally, we compared $S(\alpha_{x_i}, \alpha^*)$ and ranked all of the choices that the given customer can choose from and we obtained the optimal choice.

5.3. Algorithm and Pseudo-Code

In this section, we provide the algorithm and pseudo-code given in table Algorithm 1.

Algorithm 1. Multi-granulation neutrosophic decision algorithm.

Input Multi-granulation neutrosophic decision information systems $(U, V, \mathcal{R})$.
Output The optimal choice for the client.
Step 1 Computing $\underline{M}^o(C), \overline{M}^o(C), \underline{M}^p(C), \overline{M}^p(C)$ of neutrosophic set C about $(U, V, \mathcal{R})$;
Step 2 From Definition 4., we get $\underline{M}^o(C) + \overline{M}^o(C)$ and $\underline{M}^p(A) + \overline{M}^p(A)$;
Step 3 From Definition 5., we computer $S^o(\alpha_{x_i}, \alpha^*)$ and $S^p(\alpha_{x_i}, \alpha^*)$ $(i = 1, 2, \ldots, m)$;
Step 4 The optimal decision-making is to choose x_h if
$S(\alpha_{x_h}, \alpha^*) = \max_{i\in\{1,2,\cdots,m\}}(S(\alpha_{x_i}, \alpha^*))$.
pseudo-code
Begin
Input $(U, V, \mathcal{R})$, where U is the decision set, V is the criteria set, and $\mathcal{R}$ denotes the binary neutrosophic relation between criteria set and decision set.
Calculate $\underline{M}^o(C), \overline{M}^o(C), \underline{M}^p(C), \overline{M}^p(C)$. Where $\underline{M}^o(C), \overline{M}^o(C), \underline{M}^p(C), \overline{M}^p(C)$, which represents the optimistic and pessimistic multi-granulation lower and upper approximation of C, which is defined in Section 4.
Calculate $\underline{M}^o(C) + \overline{M}^o(C)$ and $\underline{M}^p(C) + \overline{M}^p(C)$, which is defined in Definition 4.
Calculate $S^o\left(\underline{M}^o(C) + \overline{M}^o(C), \alpha^*\right)$ and $S^p\left(\underline{M}^p(C) + \overline{M}^p(C), \alpha^*\right)$, which is defined in Definition 5.
For $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; l = 1, 2, \ldots, k;$
If $S^o(\alpha_{x_i}, \alpha^*) < S^o\left(\alpha_{x_j}, \alpha^*\right)$, then $S^o\left(\alpha_{x_j}, \alpha^*\right) \to$ Max,
else $S^o(\alpha_{x_i}, \alpha^*) \to$ Max,
If $S^o(\alpha_{x_l}, \alpha^*) > Max$, then $S^o(\alpha_{x_l}, \alpha^*) \to$ Max;
Print Max;
End

5.4. An Example

In this section, we used Section 5.2's way of solving a MCGDM problem, using the example of buying houses.

Let $V = \{y_1, y_2, y_3, y_4\}$ be the criteria set, where y_1 represents the texture, y_2 represents the geographic location, y_3 represents the price, and y_4 represents the solidity. Let $U = \{z_1, z_2, z_3, z_4\}$ be a decision set, where z_1 represents very good, z_2 represents good, z_3 represents less good, and z_4 represents not good.

Assume that there are three experts. They provide their opinions about all of the criteria sets y_j ($y_j \in V$) (j = 1, 2, 3, 4) regarding the decision set elements z_i ($x_i \in U$) (i = 1, 2, 3, 4). Like the discussion in Section 5.1, the experts give three evaluation relations, R_1, R_2, and R_3, which are neutrosophic relations between V and U, that is, $R_1, R_2, R_3 \in NR(U \times V)$. $T_{Rk}(z_i, y_j)$ shows the expert, k, give the truth membership of y_j to z_i; $I_{Rk}(z_i, y_j)$ shows the expert, k, give the indeterminacy membership of y_j to z_i; $F_{Rk}(z_i, y_j)$ shows the expert, k, give the falsity membership of y_j to z_i. For example, the first value (0.2, 0.3, 0.4) in Table 4, of 0.2 shows that the truth membership of the texture for the given house is very good, 0.3 shows that the indeterminacy membership of the texture for the given house is very good, and 0.4 shows that the falsity membership of the texture for the given house is very good.

Table 4. Neutrosophic relation R_1.

R_1	y_1	y_2	y_3	y_4
z_1	(0.2, 0.3, 0.4)	(0.3, 0.5, 0.4)	(0.4, 0.6, 0.2)	(0.1, 0.3, 0.5)
z_2	(0.8, 0.7, 0.1)	(0.2, 0.5, 0.6)	(0.6, 0.6, 0.7)	(0.4, 0.6, 0.3)
z_3	(0.5, 0.7, 0.2)	(0.6, 0.2, 0.1)	(1, 0.9, 0.4)	(0.5, 0.4, 0.3)
z_4	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.4)	(0.3, 0.8, 0.4)	(0.2, 0.9, 0.8)

So, we build the multi-granulation neutrosophic decision information system (U, V, $\mathcal{R}$) for the example.

Assume that the three experts give three evaluation relations, the results are given in Tables 4–6.

Table 5. Neutrosophic relation R_2.

R_2	y_1	y_2	y_3	y_4
z_1	(0.3, 0.4, 0.5)	(0.6, 0.7, 0.2)	(0.1, 0.8, 0.3)	(0.5, 0.3, 0.4)
z_2	(0.5, 0.5, 0.4)	(1, 0, 1)	(0.8, 0.1, 0.8)	(0.7, 0.8, 0.5)
z_3	(0.7, 0.2, 0.1)	(0.3, 0.5, 0.4)	(0.6, 0.1, 0.4)	(1, 0, 0)
z_4	(1, 0.2, 0)	(0.8, 0.1, 0.5)	(0.1, 0.2, 0.7)	(0.2, 0.2, 0.8)

Table 6. Neutrosophic relation R_3.

R_3	y_1	y_2	y_3	y_4
z_1	(0.6, 0.2, 0.2)	(0.3, 0.1, 0.7)	(0, 0.2, 0.9)	(0.8, 0.3, 0.2)
z_2	(0.1, 0.1, 0.7)	(0.2, 0.3, 0.8)	(0.7, 0.1, 0.2)	(0, 0, 1)
z_3	(0.8, 0.4, 0.1)	(0.9, 0.5, 0.3)	(0.2, 0.1, 0.6)	(0.7, 0.2, 0.3)
z_4	(0.6, 0.2, 0.2)	(0.2, 0.2, 0.8)	(1, 1, 0)	(0.5, 0.3, 0.1)

Assume C is the customer's evaluation for each criterion in V, and is given by the following:

$$C(y_1) = (0.6, 0.5, 0.5), C(y_2) = (0.7, 0.3, 0.2), C(y_3) = (0.4, 0.5, 0.9), C(y_4) = (0.3, 0.2, 0.6).$$

From Definitions 10 and 11, we can compute the following:

$$\underline{M}^o(C)(z_1) = (0.4, 0.5, 0.4), \underline{M}^o(C)(z_2) = (0.5, 0.4, 0.6), \underline{M}^o(C)(z_3) = (0.3, 0.3, 0.6), \underline{M}^o(C)(z_4) = (0.6, 0.4, 0.4)$$

$$\overline{M}^o(C)(z_1) = (0.4, 0.3, 0.5), \overline{M}^o(C)(z_2) = (0.4, 0.5, 0.7), \overline{M}^o(C)(z_3) = (0.6, 0.3, 0.4), \overline{M}^o(C)(z_4) = (0.5, 0.5, 0.5)$$
$$\underline{M}^p(C)(z_1) = (0.3, 0.5, 0.6), \underline{M}^p(C)(z_2) = (0.3, 0.5, 0.8), \underline{M}^p(C)(z_3) = (0.3, 0.5, 0.9), \underline{M}^p(C)(z_4) = (0.3, 0.5, 0.9)$$
$$\overline{M}^o(C)(z_1) = (0.6, 0.3, 0.2), \overline{M}^o(C)(z_2) = (0.7, 0.2, 0.5), \overline{M}^o(C)(z_3) = (0.7, 0.2, 0.2), \overline{M}^o(C)(z_4) = (0.7, 0.2, 0.4)$$

According Definition 4, we have the following:

$$\left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_1)=(0.64,0.15,0.2),\ \left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_2)=(0.7,0.2,0.42),$$
$$\left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_3)=(0.72,0.09,0.24),\ \left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_4)=(0.8,0.2,0.2)$$

$$\left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_1)=(0.72,0.15,0.12),\ \left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_2)=(0.79,0.1,0.4),$$
$$\left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_3)=(0.79,0.1,0.18),\ \left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_4)=(0.79,0.1,0.36)$$

Then, according Definition 5, we have the following:

$$S^o(\alpha_{z_1},\alpha^*)=0.9315, S^o(\alpha_{z_2},\alpha^*)=0.8329, S^o(\alpha_{z_3},\alpha^*)=0.8588, S^o(\alpha_{z_4},\alpha^*)=0.9428. \quad (1)$$

$$S^p(\alpha_{z_1},\alpha^*)=0.9662, S^p(\alpha_{z_2},\alpha^*)=0.8865, S^p(\alpha_{z_3},\alpha^*)=9677, S^p(\alpha_{z_4},\alpha^*)=0.9040. \quad (2)$$

Then, we have the following:

$$S^o(\alpha_{z_4},\alpha^*)>S^o(\alpha_{z_1},\alpha^*)>S^o(\alpha_{z_3},\alpha^*)>S^o(\alpha_{z_2},\alpha^*). \quad (3)$$

$$S^p(\alpha_{z_3},\alpha^*)>S^p(\alpha_{z_1},\alpha^*)>S^p(\alpha_{z_4},\alpha^*)=S^p(\alpha_{z_2},\alpha^*). \quad (4)$$

So, the optimistic optimal choice is to choose x_4, that is, this given house is "not good" for the customer; the pessimistic optimal choice is to choose x_3, that is, this given house is "less good" for the customer.

6. Conclusions

In this paper, we propose the concept of MGNRS on a single domain and dual domains, and obtain their properties. I addition, we obtain that MGNRS on dual domains will be the same as the MGNRS on a single domain when the two domains are same. Then, we solve a kind of special group decision-making problem (based on neutrosophic relation) using MGNRS on dual domains, and we show the algorithm and give an example to show its feasibility.

In terms of the future direction, we will study other types of combinations of multi-granulation rough sets and neutrosophic sets and obtain their properties. At the same time, exploring the application of MGNRS in totally dependent-neutrosophic sets (see [32]) and related algebraic systems (see [33–35]), and a new aggregation operator, similarity measure, and distance measure (see [36–39]), are also meaningful research directions for the future.

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Neutrosophic Approach to Grayscale Images Domain

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Abstract. In this paper, we propose a new technique for the enhancing images. It will work on removing the noise contained in the image as well as improving its contrast based on three different enhancing transforms, we commence by embedding the image into a neutrosophic domain; where the image will be mapped in three different levels, a level of trueness, a level of falseness and a level of indeterminacy. Hence, we act separately on each level using the enhancement transforms. Finally, we introduce a new analysis in the field of analysis and processing of images using the neutrosophic crisp set theory via Mat lab program where has been obtained three images, which helps in a new analysis to improve and retrieve images.

Keywords: Image analysis, Image Enhancement, Image processing, Neutrosophic Crisp Set, Gaussian Distribution, Logarithmic Transform, Neutrosophic Crisp Mathematical Morphology

1. Introduction

As a discipline, neutrosophic is an active and growing area of image processing and analysis. Mathematically, a gray scale image is represented by an $m \times n$ array $I_m = [g(i,j)]_{m\times n}$ with entities $g(i,j)$ corresponding to the intensity of the pixel located at (i,j). Presently applications require different kinds of images as sources of information for interpretation and analysis. Whenever an image is converted from one form to another (such as digitizing, scanning, transmitting, storing, etc.) some form of declination occurs at the output. Hence, the output image has to undergo a process called image enhancement which consists of a collection of techniques that seek to improve the visual appearance of an image [12]. Image enhancement is a process which mainly used to improve the quality of images, removing noise from the images.It has important role in many fields like high definition TV (HDTV), X-rayprocessing, motion detection, remote sensing and in studying medical images [8]. The fundamental concepts of neutrosophic set, introduced by Smarandache in [22, 23] and many applications, introduced by Salama et al. in [14-21],[27, 28] provides a natural foundation for treating mathematically the neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics, as an extension of the concept of the fuzzy set theory introduced by Zadeh [25].

2. Preliminaries

we recall some definitions for essential concepts of neutrosophic sets and its operations, which were introduced by Smarandache in [22, 23] and many applications by Salama et al. in [14-21].

2.1. Image Enhancement

Recent applications are in need of different kinds of images as a source of information for interpretation and analysis. Whenever an image is transformed from one structure to another, such as: digitizing, scanning, and transmitting, some kind of distortion might occur to the output image. Hence, a process called image enhancement must be done. The process of an image enhancement contains a collection of techniques with the aim of providing a better visual appearance of the image; it is to improve the image quality so that the resultant image is better than the original image for a specific application. In other words, to convert the image to an appropriate form for analysis by either a human eye or a machine. Currently, the image enhancement research covers wide topics such as: algorithms based on the human visual system [6], histograms with hue-preservation [9], JPEG-based enhancement for the visually impaired [24], and histogram modification techniques [5]. Additive noise, Gaussian noise, Impulse noise and Poisson noise represent several types of noises that corrupt the image, to remove any of such there are various filters available. For instance: Gaussian filter, Median filter, High pass filter and Low pass filter; each of these can be used to remove the image noise and, hence, enhance the image. The applications of image enhancement are in every field where images are needed to be understood and analyzed, as in medical image analysis, and analysis of images from satellites. Generally, the enhancement techniques can be categorized into two main groups, which are the Spatial Domain Methods and the Frequency Domain Methods [26].

2.2. Spatial Domain for Image Enhancement

The spatial domain is the normal image space, which is a direct handling of image pixels [2]. It is the manipulation or the change of image representations. Moreover, spatial domain is used in several applications as smoothing, sharpening and filtering images .Spatial domain techniques such as the logarithmic transforms[7], power law transforms[11], and histogram equalization[13], are basically to perform on the direct manipulation of the image pixels. In practice, spatial techniques are useful for directly changing the gray level intensities of individual pixels and consequently the contrast of the entire image. Usually, the spatial domain techniques enhance the whole image uniformly, which in various cases produces undesirable results and do not make it possible to efficiently enhance edges or other required information.

2.3 Frequency Domain for Image Enhancement

While in the spatial domain an image is treated as it is, and the value of the pixels of the image changes with respect to the scene, in the frequency domain we are dealing with the rate at which the values of the pixel are changing in the spatial domain. In all the methods applied, a Fourier transform of the image is firstly computed so that the image is transferred into the frequency domain. Hence, any operation used for the purpose of image enhancement will be performed on the Fourier transform of the image. Afterward an Inverse Fourier transform is performed to obtain the resultant image. The main objective of all the enhancement operations is to modify the image contrast, brightness or the grey levels distribution. Therefore, the value of the pixels of the output image will be changed according to the transformation applied on the input values. In image processing and image analysis, the image transform is a mathematical tool which is used for detecting the rough or unclear area in the image and fix it. The image transformation allows us to move from frequency domain to time domain to perform the desired task in an easy manner. Various types of image transforms are available such as Fourier Transform [1], Walsh Transform [10], Hadamard Transform, Stant Transform, and Wavelet Transform [4]. The image transformation to neutrosophic
domain in [3]

3. Hesitancy Degrees with Neutrosophic Image Domain

Salama et al. in [27, 28] presented the texture features for images embedded in the neutrosophic domain with Hesitancy degree.

Definition 3.1 [15,27,28]:

Let $A = \{(\mu_A(x), v_A(x), \gamma_A(x)), x\epsilon X\}$ on $X = \{x_1, x_2, x_3, \dots\dots, x_n\}$. Then for a Neutrosophic set $A = \{(\mu_A(x), v_A(x), \gamma_A(x)), x\epsilon X\}$ in X, We call $\pi_A(x) = 3 - \mu_A(x) - v_A(x) - \gamma_a(x)$, the Neutrosophic index of x in A, It is a hesitancy degree of x to A it is obvious that $0 \le \pi_A(x) \le 3$.

In this section we are transforming the image I_m into a neutrosophic domain using four functions: T, I, F and π . A pixel $P(i,j)$ in the image is described by a forth ($T(i,j)$; $I(i,j)$; $F(i,j)$; $\pi(i,j)$). Where $T(i,j)$ is the membership degree of the pixel in the white set, and $F(i,j)$ is its membership degree in the non-white (black) set; while $I(i,j)$ is how much it is neither white nor black; k and $\pi(i,j)$ is hesitancy degree. The values of $T(i,j)$, $I(i,j)$, $F(i,j)$ and $\pi(i,j)$ are defined as follows:

$$T(i,j) = \frac{\bar{g}(i,j) - \bar{g}_{\min}}{\bar{g}_{\max} - \bar{g}_{\min}}, \quad I(i,j) = 1 - \frac{\delta(i,j) - \delta_{\min}}{\delta_{\max} - \delta_{\min}},$$

$$F(i,j) = 1 - T(i,j),$$

$\pi(i,j) = 3-(T(i,j)+I(i,j)+F(i,j)$, where $\bar{g}(i,j)$ is the local mean intensity in some neighborhood w of the pixel, $\bar{g}(i,j) = \frac{1}{w \times w} \sum_{u=i-w/2}^{u=i+w/2} \sum_{v=j-w/2}^{v=j+w/2} g(u,v)$, $\delta(i,j)$ is the homogeneity value computed by the absolute value of difference between the intensity and its local mean value $\delta(i,j) = abs(g(i,j) - \bar{g}(i,j))$.

4. A Neutrosophic Image Enhancement Filter

Consider an Image G in the neutrosophic domain with four functions (T, I, F, π) describing the three levels of trueness, indeterminacy and falseness with hesitancy degree as previously explained in 2. The filter we propose to enhance G is two fold. In one hand it aims to remove the noise from the image, in the other hand it improves the image contrast. To do so, we will work on each level separately.

Firstly, in the indeterminacy level, we will force the stability of this blur area around the mean using the Gaussian distribution. A general form of the Gaussian distribution is $\phi\left(\frac{t-\mu}{\sigma}\right)$, where σ is the standard deviation and μ is the mean value. Secondly, in the falseness level, a logarithmic transform is applied to enhance the details in ; 2 the dark areas while considering the brighter ones. Its general form is, c log (1 + t), where t is assumed to be non-negative; $t \geq 0$, and c is a scaling parameter.

Thirdly, a power-law transform is working over the shattered areas in the trueness level. The power law transformations include the n^{th} power and the n^{th} root transformation, these transformations are also known as gamma transformation and can be given by the general expression, cr^{γ} . Variation in the value of γ varies the enhancement of the images. Finally, we have got the output image, $\bar{G}$ of the enhancement process with the triple $(\bar{T}, \bar{I}, \bar{F}, \bar{\pi})$ where

$$\bar{T}(i,j) = CT^{\gamma}(i,j),$$

$$\bar{I}(i,j) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{(I(i,j)-\mu)^2}{2\sigma^2}\right), (2)$$

$$\bar{F}(i,j) = C\ln(1+F(i,j)),$$

$$\bar{\pi} = 3-(\bar{T}(i,j)+\bar{I}(i,j)+\bar{F}(i,j))$$

5. A Neutrosophic Crisp Operators for Grayscale Image

5.1. Grayscale Image via Neutrosophic Crisp Domain.

In this section, we introduce a new analysis in the field of analysis and processing of images using the neutrosophic crisp set theory due to Salama et al. in [14,17] via Matlab program where has been obtained three images representing, which helps in a new analysis to improve and retrieve images

A grayscale image in a 2D Cartesian domain

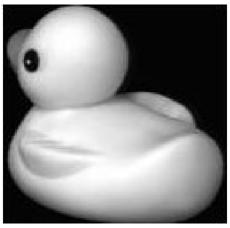

Fig. 1: a) Grayscale image

The following figure shows a grayscale image in a neutrosophic crisp components.

Fig. 1: b) Neutrosophic Crisp Components $\langle A_1, A_2, A_3 \rangle$ respectively

At this point, we have noticed that there exist some crisp sets which having the neutrosophic triple structure and are not classified in either categories of the neutrosophic crisp sets' classification. In this case, the three components of those sets may overlap. In this section, we deduced a new triple structured set; where the three components are disjoint.

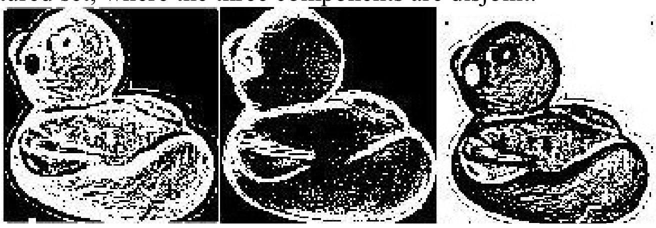

Fig. 2: b) Neutrosophic Crisp Components $\langle A_1, A_2, A_3 \rangle$ respectively

The following figure shows a grayscale image in star neutrosophic crisp components.

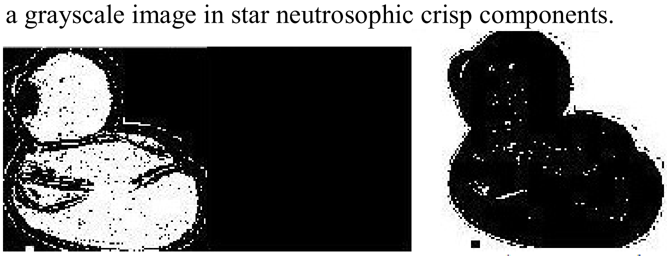

Fig. 3 b) Star Neutrosophic Crisp Components $\langle A_1, A_2, A_3 \rangle$ respectively

Definition 5.1

For any triple structured crisp set A, of the form $A = \langle A_1, A_2, A_3 \rangle$;the retract neutrosophic crisp set A^{r} is the structure $A^{r} = \langle A_1{}^{r}, A_2{}^{r}, A_3{}^{r} \rangle$,where $A_1{}^{r} = A_1 \cap co(A_2 \cup A_3)$, $A_2{}^{r} = A_2 \cap co(A_1 \cup A_3)$ and $A_3{}^{r} = A_3 \cap co(A_1 \cup A_2)$. Furthermore, the three components $A_1{}^{r}$, $A_2{}^{r}$ and $A_3{}^{r}$ are disjoint and $A_i^{r} \subseteq A_i$ $\forall i = 1,2,3$.

The following figure shows a grayscale image in a neutrosophic retract crisp components.

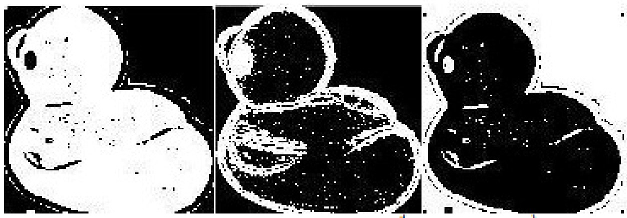

Fig.4: b) Neutrosophic Retract Components $\langle A_1, A_2, A_3 \rangle$ respectively

5.2. A Grayscale Image & Neutrosophic Crisp Operators

Salama et al. [17] extended the definitions of some morphological filters using the neutrosophic crisp sets concept. The idea behind the new introduced operators and filters is to act on the image in the neutrosophic crisp domain instead of the spatial domain.

The following figure shows a grayscale image in a neutrosophic crisp Dilation components.

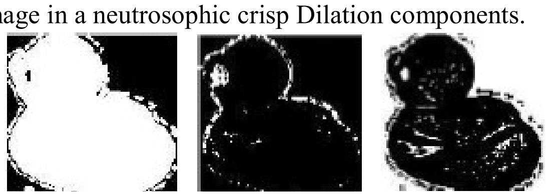

Fig.5: Neutrosophic Crisp Dilation components in type 1 $\langle A_1, A_2, A_3 \rangle$ respectively

Fig.6: Neutrosophic Crisp Dilation components in type 2 $\langle A_1, A_2, A_3 \rangle$ respectively

The following figure shows a grayscale image in a neutrosophic crisp **Erosion** components.

Fig.7: Neutrosophic Crisp Erosion components in type 1 $\langle A_1, A_2, A_3 \rangle$ respectively

Fig.8: Neutrosophic Crisp Erosion components in type2 $\langle A_1, A_2, A_3 \rangle$ respectively

The following figure shows a grayscale image in a neutrosophic crisp **Opening** components.

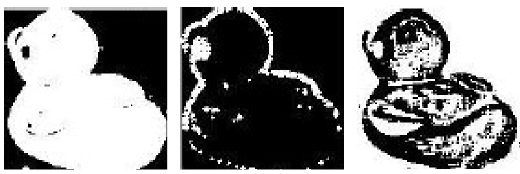

Fig.9: Neutrosophic Crisp opening components in type1 $\langle A_1, A_2, A_3 \rangle$ respectively

Fig.10: Neutrosophic Crisp opening components in type2 $\langle A_1, A_2, A_3 \rangle$ respectively

The following figure shows a grayscale image in a neutrosophic crisp **Closing** components.

Fig.11: Neutrosophic Crisp closing components in type1 $\left\langle A^1, A^2, A^3 \right\rangle$ *respectively*

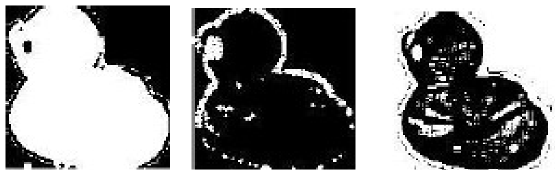

Fig.12: *Neutrosophic Crisp closing components in type2* $\left\langle A^1, A^2, A^3 \right\rangle$ *respectively*

Conclusion

As a discipline, neutrosophic is an active and growing area of image processing and analysis. In this work, we introduce a neutrosophic technique for the image processing, analysis and enhancement. The two fold proposed technique aims to remove the noise from the image, as well as improving the image contrast. To commence, we construct the embedding of the image in the neutrosophic domain; in which the image is mapped into three different levels, describing the levels of trueness, falseness and indeterminacy. Using the Power-law, Logarithmic and Gaussian transforms, the proposed a technique acts on each level of the image separately. Our plan next is to experiment our technique on different types of images, such as medical images.

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Neutrosophic Association Rule Mining Algorithm for Big Data Analysis

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Abstract: Big Data is a large-sized and complex dataset, which cannot be managed using traditional data processing tools. Mining process of big data is the ability to extract valuable information from these large datasets. Association rule mining is a type of data mining process, which is indented to determine interesting associations between items and to establish a set of association rules whose support is greater than a specific threshold. The classical association rules can only be extracted from binary data where an item exists in a transaction, but it fails to deal effectively with quantitative attributes, through decreasing the quality of generated association rules due to sharp boundary problems. In order to overcome the drawbacks of classical association rule mining, we propose in this research a new neutrosophic association rule algorithm. The algorithm uses a new approach for generating association rules by dealing with membership, indeterminacy, and non-membership functions of items, conducting to an efficient decision-making system by considering all vague association rules. To prove the validity of the method, we compare the fuzzy mining and the neutrosophic mining. The results show that the proposed approach increases the number of generated association rules.

Keywords: neutrosophic association rule; data mining; neutrosophic sets; big data

1. Introduction

The term 'Big Data' originated from the massive amount of data produced every day. Each day, Google receives cca. 1 billion queries, Facebook registers more than 800 million updates, and YouTube counts up to 4 billion views, and the produced data grows with 40% every year. Other sources of data are mobile devices and big companies. The produced data may be structured, semi-structured, or unstructured. Most of the big data types are unstructured; only 20% of data consists in structured data. There are four dimensions of big data:

(1) Volume: big data is measured by petabytes and zettabytes.
(2) Velocity: the accelerating speed of data flow.
(3) Variety: the various sources and types of data requiring analysis and management.
(4) Veracity: noise, abnormality, and biases of generated knowledge.

Consequently, Gartner [1] outlines that big data's large volume requires cost-effective, innovative forms for processing information, to enhance insights and decision-making processes.

Prominent domains among applications of big data are [2,3]:

(1) Business domain.
(2) Technology domain.
(3) Health domain.
(4) Smart cities designing.

These various applications help people to obtain better services, experiences, or be healthier, by detecting illness symptoms much earlier than before [2]. Some significant challenges of managing and analyzing big data are [4,5]:

(1) Analytics Architecture: The optimal architecture for dealing with historic and real-time data at the same time is not obvious yet.
(2) Statistical significance: Fulfill statistical results, which should not be random.
(3) Distributed mining: Various data mining methods are not fiddling to paralyze.
(4) Time evolving data: Data should be improved over time according to the field of interest.
(5) Compression: To deal with big data, the amount of space that is needed to store is highly relevant.
(6) Visualization: The main mission of big data analysis is the visualization of results.
(7) Hidden big data: Large amounts of beneficial data are lost since modern data is unstructured data.

Due to the increasing volume of data at a matchless rate and of various forms, we need to manage and analyze uncertainty of various types of data. Big data analytics is a significant function of big data, which discovers unobserved patterns and relationships among various items and people interest on a specific item from the huge data set. Various methods are applied to obtain valid, unknown, and useful models from large data. Association rule mining stands among big data analytics functionalities. The concept of association rule (AR) mining already returns to H'ajek et al. [6]. Each association rule in database is composed from two different sets of items, which are called antecedent and consequent. A simple example of association rule mining is "if the client buys a fruit, he/she is 80% likely to purchase milk also". The previous association rule can help in making a marketing strategy of a grocery store. Then, we can say that association rule-mining finds all of the frequent items in database with the least complexities. From all of the available rules, in order to determine the rules of interest, a set of constraints must be determined. These constraints are support, confidence, lift, and conviction. Support indicates the number of occurrences of an item in all transactions, while the confidence constraint indicates the truth of the existing rule in transactions. The factor "lift" explains the dependency relationship between the antecedent and consequent. On the other hand, the conviction of a rule indicates the frequency ratio of an occurring antecedent without a consequent occurrence. Association rules mining could be limited to the problem of finding large itemsets, where a large itemset is a collection of items existing in a database transactions equal to or greater than the support threshold [7–20]. In [8], the author provides a survey of the itemset methods for discovering association rules. The association rules are positive and negative rules. The positive association rules take the form $X \to Y$, $X \subseteq I$, $Y \subseteq I$ and $X \cap Y = \varphi$, where X, Y are antecedent and consequent and I is a set of items in database. Each positive association rule may lead to three negative association rules, $\to Y$, $X \to Y$, and $X \to Y$. Generating association rules in [9] consists of two problems. The first problem is to find frequent itemsets whose support satisfies a predefined minimum value. Then, the concern is to derive all of the rules exceeding a minimum confidence, based on each frequent itemset. Since the solution of the second problem is straightforward, most of the proposed work goes in for solving the first problem. An a priori algorithm has been proposed in [19], which was the basis for many of the forthcoming algorithms. A two-pass algorithm is presented in [11]. It consumes only two database scan passes, while a priori is a multi-pass algorithm and needs up to c+1 database scans, where c is the number of items (attributes). Association rules mining is applicable in numerous database communities. It has large applications in the retail industry to improve market basket analysis [7]. Streaming-Rules is an algorithm developed by [9] to report an association between pairs of elements in streams for predictive caching and detecting the previously undetectable hit inflation attacks in advertising networks. Running mining algorithms on numerical attributes may result in a large set of candidates. Each candidate has small support and many rules have been generated with useless information, e.g., the age attribute, salary attribute, and students' grades. Many partitioning algorithms have been developed to solve the numerical attributes problem. The proposed algorithms faced two problems. The first problem was the partitioning of attribute domain into meaningful partitions. The second problem was the loss of many useful rules due to the sharp boundary problem. Consequently, some rules may fail to achieve the minimum support threshold because of the separating of its domain into two partitions.

Fuzzy sets have been introduced to solve these two problems. Using fuzzy sets make the resulted association rules more meaningful. Many mining algorithms have been introduced to solve the quantitative attributes problem using fuzzy sets proposed algorithms in [13–27] that can be separated into two types related to the kind of minimum support threshold, fuzzy mining based on single-minimum support threshold, and fuzzy mining based on multi-minimum support threshold [21]. Neutrosophic theory was introduced in [28] to generalize fuzzy theory. In [29–32], the neutrosophic theory has been proposed to solve several applications and it has been used to generate a solution based on neutrosophic sets. Single-valued neutrosophic set was introduced in [33] to transfer the neutrosophic theory from the philosophic field into the mathematical theory, and to become applicable in engineering applications. In [33], a differentiation has been proposed between intuitionistic fuzzy sets and neutrosophic sets based on the independence of membership functions (truth-membership function, falsity-membership function, and indeterminacy-membership function). In neutrosophic sets, indeterminacy is explicitly independent, and truth-membership function and falsity-membership function are independent as well. In this paper, we introduce an approach that is based on neutrosophic sets for mining association rules, instead of fuzzy sets. Also, a comparison resulted association rules in both of the scenarios has been presented. In [34], an attempt to express how neutrosophic sets theory could be used in data mining has been proposed. They define SVNSF (single-valued neutrosophic score function) to aggregate attribute values. In [35], an algorithm has been introduced to mining vague association rules. Items properties have been added to enhance the quality of mining association rules. In addition, almost sold items (items has been selected by the customer, but not checked out) were added to enhance the generated association rules. AH-pair Database consisting of a traditional database and the hesitation information of items was generated. The hesitation information was collected, depending online shopping stores, which make it easier to collect that type of information, which does not exist in traditional stores. In this paper, we are the first to convert numerical attributes (items) into neutrosophic sets. While vague association rules add new items from the hesitating information, our framework adds new items by converting the numerical attributes into linguistic terms. Therefore, the vague association rule mining can be run on the converted database, which contains new linguistic terms.

Research Contribution

Detecting hidden and affinity patterns from various, complex, and big data represents a significant role in various domain areas, such as marketing, business, medical analysis, etc. These patterns are beneficial for strategic decision-making. Association rules mining plays an important role as well in detecting the relationships between patterns for determining frequent itemsets, since classical association rules cannot use all types of data for the mining process. Binary data can only be used to form classical rules, where items either exist in database or not. However, when classical association rules deal with quantitative database, no discovered rules will appear, and this is the reason for innovating quantitative association rules. The quantitative method also leads to the sharp boundary problem, where the item is below or above the estimation values. The fuzzy association rules are introduced to overcome the classical association rules drawbacks. The item in fuzzy association rules has a membership function and a fuzzy set. The fuzzy association rules can deal with vague rules, but not in the best manner, since it cannot consider the indeterminacy of rules. In order to overcome drawbacks of previous association rules, a new neutrosophic association rule algorithm has been introduced in this research. Our proposed algorithm deals effectively and efficiently with vague rules by considering not only the membership function of items, but also the indeterminacy and the falsity functions. Therefore, the proposed algorithm discovers all of the possible association rules and minimizes the losing processes of rules, which leads to building efficient and reliable decision-making system. By comparing our proposed algorithm with fuzzy approaches, we note that the number of association rules is increased, and negative rules are also discovered. The separation of negative association rules from positive ones is not a simple process, and it helps in various fields. As an example, in the medical domain, both positive and negative association rules help not only in the diagnosis of disease, but also in detecting prevention manners.

The rest of this research is organized as follows. The basic concepts and definitions of association rules mining are presented in Section 2. A quick overview of fuzzy association rules is described in Section 3. The neutrosophic association rules and the proposed model are presented in Section 4. A case study of Telecom Egypt Company is presented in Section 5. The experimental results and comparisons between fuzzy and proposed association rules are discussed in Section 6. The conclusions are drawn in Section 7.

2. Association Rules Mining

In this section, we formulate the $|D|$ transactions from the mining association rules for a database D. We used the following notations:

(i) $I = \{i_1, i_2, \ldots i_m\}$ represents all the possible data sets, called items.
(ii) Transaction set T is the set of domain data resulting from transactional processing such as $T \subseteq I$.
(iii) For a given itemset $X \subseteq I$ and a given transaction T, we say that T contains X if and only if $X \subseteq T$.
(iv) σ_X: the support frequency of X, which is defined as the number of transactions out of D that contain X.
(v) s: the support threshold.

X is considered a large itemset, if $\sigma_X \geq |D| \times s$. Further, an association rule is an implication of the form $X \Rightarrow Y$, where $X \subseteq I$, $Y \subseteq I$ and $X \cap Y = \varphi$.

An association rule $X \Rightarrow Y$ is addressed in D with confidence c if at least c transactions out of D contain both X and Y. The rule $X \Rightarrow Y$ is considered as a large itemset having a minimum support s if: $\sigma_{X \cup Y} \geq |D| \times s$.

For a specific confidence and specific support thresholds, the problem of mining association rules is to find out all of the association rules having confidence and support that is larger than the corresponding thresholds. This problem can simply be expressed as finding all of the large itemsets, where a large itemset L is:

$$L = \{X | X \subseteq I \wedge \sigma_X \geq |D| \times s\}.$$

3. Fuzzy Association Rules

Mining of association rules is considered as the main task in data mining. An association rule expresses an interesting relationship between different attributes. Fuzzy association rules can deal with both quantitative and categorical data and are described in linguistic terms, which are understandable terms [26].

Let $T = \{t_1, \ldots, t_n\}$ be a database transactions. Each transaction consists of a number of attributes (items). Let $I = \{i_1, \ldots, i_m\}$ be a set of categorical or quantitative attributes. For each attribute i_k, $(k = 1, \ldots, m)$, we consider $\{n_1, \ldots, n_k\}$ associated fuzzy sets. Typically, a domain expert determines the membership function for each attribute.

The tuple $< X, A >$ is called the fuzzy itemset, where $X \subseteq I$ (set of attributes) and A is a set of fuzzy sets that is associated with attributes from X.

Following is an example of fuzzy association rule:

IF salary is high and age is old THEN insurance is high

Before the mining process starts, we need to deal with numerical attributes and prepare them for the mining process. The main idea is to determine the linguistic terms for the numerical attribute and define the range for every linguistic term. For example, the temperature attribute is determined by the linguistic terms {very cold, cold, cool, warm, hot}. Figure 1 illustrates the membership function of the temperature attribute.

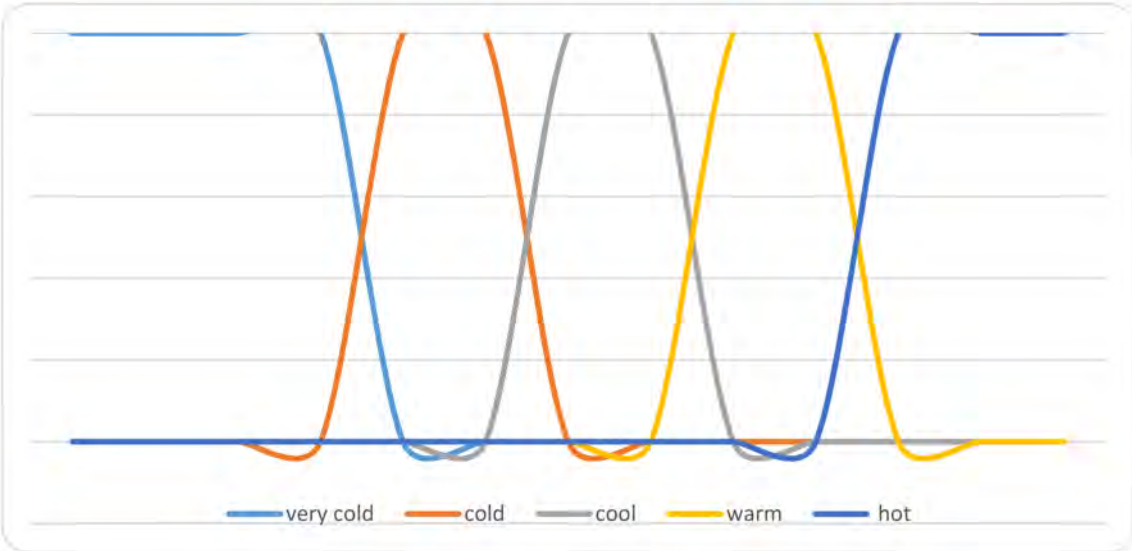

Figure 1. Linguistic terms of the temperature attribute.

The membership function has been calculated for the following database transactions illustrated in Table 1.

Table 1. Membership function for Database Transactions.

Transaction	Temp.	Membership Degree
T1	18	1 cool
T2	13	0.6 cool, 0.4 cold
T3	12	0.4 cool, 0.6 cold
T4	33	0.6 warm, 0.4 hot
T5	21	0.2 warm, 0.8 cool
T6	25	1 warm

We add the linguistic terms {very cold, cold, cool, warm, hot} to the candidate set and calculate the support for those itemsets. After determining the linguistic terms for each numerical attribute, the fuzzy candidate set have been generated.

Table 2 contains the support for each itemset individual one-itemsets. The count for every linguistic term has been calculated by summing its membership degree over the transactions. Table 3 shows the support for two-itemsets. The count for the fuzzy sets is the summation of degrees that resulted from the membership function of that itemset. The count for two-itemset has been calculated by summing the minimum membership degree of the 2 items. For example, {cold, cool} has count 0.8, which resulted from transactions T2 and T3. For transaction T2, membership degree of cool is 0.6 and membership degree for cold is 0.4, so the count for set {cold, cool} in T2 is 0.4. Also, T3 has the same count for {cold, cool}. So, the count of set {cold, cool} over all transactions is 0.8.

Table 2. 1-itemset support.

1-itemset	Count	Support
Very cold	0	0
Cold	1	0.17
Cool	2.8	0.47
Warm	1.6	0.27
Hot	0.6	0.1

Table 3. 2-itemset support.

2-itemset	Count	Support
{Cold, cool}	0.8	0.13
{Warm, hot}	0.4	0.07
{warm, cool}	0.2	0.03

In subsequent discussions, we denote an itemset that contains k items as k-itemset. The set of all k-itemsets in L is referred as L_k.

4. Neutrosophic Association Rules

In this section, we overview some basic concepts of the NSs and SVNSs over the universal set X, and the proposed model of discovering neutrosophic association rules.

4.1. Neutrosophic Set Definitions and Operations

Definition 1 ([33]). *Let X be a space of points and x∈X. A neutrosophic set (NS) A in X is definite by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x)$: $X \to]^-0, 1^+[$, $I_A(x)$: $X \to]^-0, 1^+[$ and $F_A(x)$: $X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.*

Neutrosophic is built on a philosophical concept, which makes it difficult to process during engineering applications or to use it to real applications. To overcome that, Wang et al. [31], defined the SVNS, which is a particular case of NS.

Definition 2. *Let X be a universe of discourse. A single valued neutrosophic set (SVNS) A over X is an object taking the form $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x)$: $X \to [0, 1]$, $I_A(x)$: $X \to [0, 1]$ and $F_A(x)$: $X \to [0, 1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all x∈X. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is represented by A = (a, b, c), where a, b, c∈[0, 1] and $a + b + c \leq 3$.*

Definition 3 (Intersection) ([31]). *For two SVNSs $A = \langle T_A(x), I_A(x), F_A(x)\rangle$ and $B = \langle T_B(x), I_B(x), F_B(x)\rangle$, the intersection of these SVNSs is again an SVNSs which is defined as $C = A \cap B$ whose truth, indeterminacy and falsity membership functions are defined as $T_C(x) = min(T_A(x), T_B(x))$, $I_A(x) = min(I_A(x), I_B(x))$ and $F_C(x) = max(F_A(x), F_B(x))$.*

Definition 4 (Union) ([31]). *For two SVNSs $A = \langle T_A(x), I_A(x), F_A(x)\rangle$ and $B = \langle T_B(x), I_B(x), F_B(x)\rangle$, the union of these SVNSs is again an SVNSs which is defined as $C = A \cup B$ whose truth, indeterminacy and falsity membership functions are defined as $T_C(x) = max(T_A(x), T_B(x))$, $I_A(x) = max(I_A(x), I_B(x))$ and $F_C(x) = min(F_A(x), F_B(x))$.*

Definition 5 (*Containment*) ([31]). A single valued neutrosophic set A contained in the other SVNS B, denoted by $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$ for all x in X.

Next, we propose a method for generating the association rule under the SVNS environment.

4.2. Proposed Model for Association Rule

In this paper, we introduce a model to generate association rules of form:
$X \rightarrow Y$ where $X \cap Y = \varphi$ and X, Y are neutrosophic sets.

Our aim is to find the frequent itemsets and their corresponding support. Generating an association rule from its frequent itemsets, which are dependent on the confidence threshold, are also discussed here. This has been done by adding the neutrosophic set into I, where I is all of the possible data sets, which are referred as items. So $I = N \cup M$ where N is neutrosophic set and M is classical set of items. The general form of an association rule is an implication of the form $X \rightarrow Y$, where $X \subseteq I$, $Y \subseteq I$, $X \cap Y = \varphi$.

Therefore, an association rule $X \rightarrow Y$ is addressed in Database D with confidence 'c' if at least c transactions out of D contains both X and Y. On the other hand, the rule $X \rightarrow Y$ is considered a large item set having a minimum support s if $\sigma_{X \cup Y} \geq |D| \times s$. Furthermore, the process of converting the quantitative values into the neutrosophic sets is proposed, as shown in Figure 2.

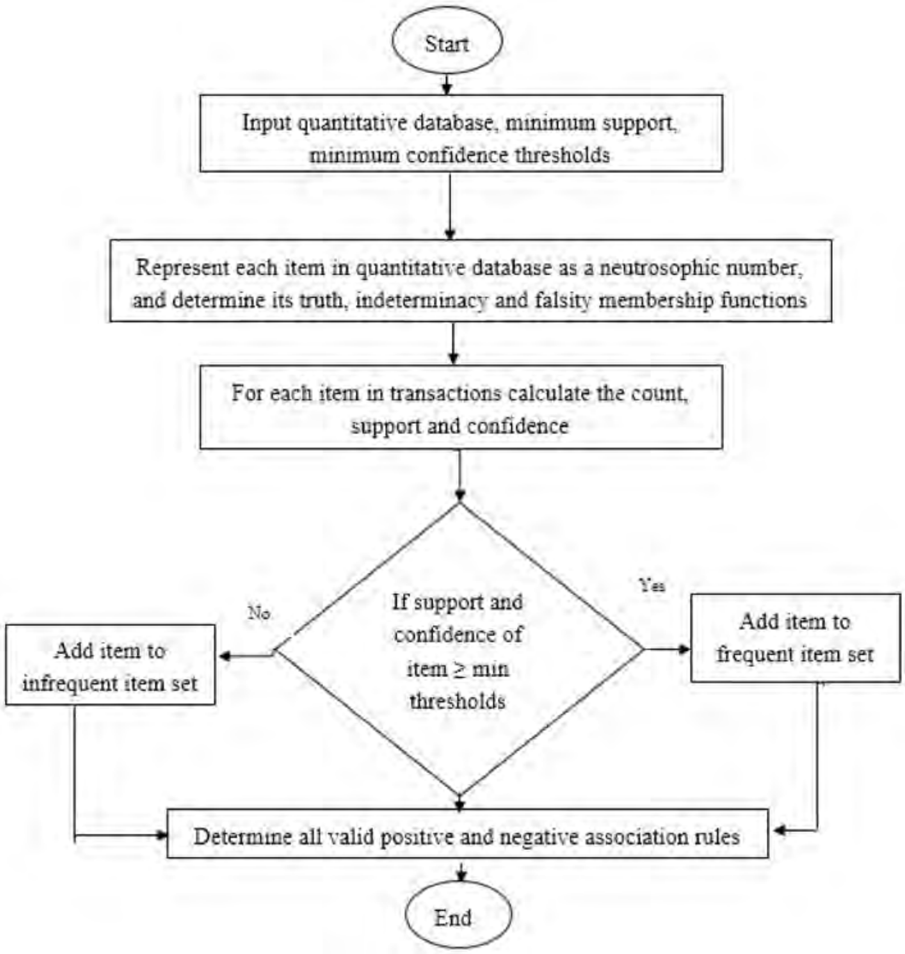

Figure 2. The proposed model.

The proposed model for the construction of the neutrosophic numbers is summarized in the following steps:

Step 1 Set linguistic terms of the variable, which will be used for quantitative attribute.

Step 2 Define the truth, indeterminacy, and the falsity membership functions for each constructed linguistic term.

Step 3 For each transaction t in T, compute the truth-membership, indeterminacy-membership and falsity-membership degrees.

Step 4 Extend each linguistic term l in set of linguistic terms L into T_L, I_L, and F_L to denote truth-membership, indeterminacy-membership, and falsity-membership functions, respectively.

Step 5 For each k-item set where $k = \{1, 2, \ldots, n\}$, and n number of iterations.

- calculate count of each linguistic term by summing degrees of membership for each transaction as $Count(A) = \sum_{i=1}^{i=t} \mu_A(x)$ where μ_A is T_A, I_A or F_A.
- calculate support for each linguistic term $s = \frac{Count(A)}{No.\ of\ trnsactions}$.

Step 6 The above procedure has been repeated for every quantitative attribute in the database.

In order to show the working procedure of the approach, we consider the temperature as an attribute and the terms "very cold", "cold", "cool", "warm", and "hot" as their linguistic terms to represent the temperature of an object. Then, following the steps of the proposed approach, construct their membership function as below:

Step 1 The attribute temperature' has set the linguistic terms "very cold", "cold", "cool", "warm", and "hot", and their ranges are defined in Table 4.

Table 4. Linguistic terms ranges.

Linguistic Term	Core Range	Left Boundary Range	Right Boundary Range
Very Cold	−∞–0	N/A	0–5
Cold	5–10	0–5	10–15
Cool	15–20	10–15	20–25
Warm	25–30	20–25	30–35
Hot	35–∞	30–35	N/A

Step 2 Based on these linguistic term ranges, the truth-membership functions of each linguistic variable are defined, as follows:

$$T_{very-cold}(x) = \begin{cases} 1 & ; for\ x \leq 0 \\ (5-x)/5 & ; for\ 0 < x < 5 \\ 0 & ; for\ x \geq 5 \end{cases}$$

$$T_{cold}(x) = \begin{cases} 1 & ; for\ 5 \leq x \leq 10 \\ (15-x)/5 & ; for\ 10 < x < 15 \\ x/5 & ; for\ 0 < x < 5 \\ 0 & ; for\ x \geq 15\ or\ x \leq 0 \end{cases}$$

$$T_{cool}(x) = \begin{cases} 1 & ; for\ 15 \leq x \leq 20 \\ (25-x)/5 & ; for\ 20 < x < 25 \\ (x-10)/5 & ; for\ 10 < x < 15 \\ 0 & ;\ otherwise \end{cases}$$

$$T_{warm}(x) = \begin{cases} 1 & ; for\ 25 \leq x \leq 30 \\ (35-x)/5 & ; for\ 30 < x < 35 \\ (x-20)/5 & ; for\ 20 < x < 25 \\ 0 & ;\ otherwise \end{cases}$$

$$T_{hot}(x) = \begin{cases} 1 & ;for\ x \geq 35 \\ (x-30)/5 & ;for\ 30 < x < 35 \\ 0 & ;\ otherwise \end{cases}$$

The falsity-membership functions of each linguistic variable are defined as follows:

$$F_{very-cold}(x) = \begin{cases} 0 & ;for\ x \leq 0 \\ x/5 & ;for\ 0 < x < 5 \\ 1 & ;for\ x \geq 5 \end{cases};$$

$$F_{cold}(x) = \begin{cases} 0 & ;for\ 5 \leq x \leq 10 \\ (x-10)/5 & ;for\ 10 < x < 15 \\ (5-x)/5 & ;for\ 0 < x < 5 \\ 1 & ;for\ x \geq 15\ or\ x \leq 0 \end{cases}$$

$$F_{cool}(x) = \begin{cases} 0 & ;for\ 15 \leq x \leq 20 \\ (x-20)/5 & ;for\ 20 < x < 25 \\ (15-x)/5 & ;for\ 10 < x < 15 \\ 1 & ;\ otherwise \end{cases}$$

$$F_{warm}(x) = \begin{cases} 0 & ;for\ 25 \leq x \leq 30 \\ (x-30)/5 & ;for\ 30 < x < 35 \\ (25-x)/5 & ;for\ 20 < x < 25 \\ 1 & ;\ otherwise \end{cases}$$

$$F_{hot}(x) = \begin{cases} 0 & ;\ for\ x \geq 35 \\ (35-x)/5 & ;\ for\ 30 < x < 35 \\ 1 & ;\ otherwise \end{cases}$$

The indeterminacy membership functions of each linguistic variables are defined as follows:

$$I_{very-cold}(x) = \begin{cases} 0 & ;for\ x \leq -2.5 \\ (x+2.5)/5 & ;for -2.5 \leq x \leq 2.5 \\ (7.5-x)/5 & ;for\ 2.5 \leq x \leq 7.5 \\ 0 & ;for\ x \geq 7.5 \end{cases}$$

$$I_{cold}(x) = \begin{cases} (x+2.5)/5 & ;for\ 2.5 \leq x \leq 2.5 \\ (7.5-x)/5 & ;for\ 2.5 \leq x \leq 7.5 \\ (x-7.5)/5 & ;for\ 7.5 \leq x \leq 12.5 \\ (17.5-x)/5 & ;for\ 12.5 \leq x \leq 17.5 \\ 0 & ;otherwise \end{cases}$$

$$I_{cool}(x) = \begin{cases} (x-7.5)/5 & ;for\ 7.5 \leq x \leq 12.5 \\ (17.5-x)/5 & ;for\ 12.5 \leq x \leq 17.5 \\ (x-17.5)/5 & ;for\ 17.5 \leq x \leq 22.5 \\ (27.5-x)/5 & ;for\ 22.5 \leq x \leq 27.5 \\ 0 & ;otherwise \end{cases}$$

$$I_{warm}(x) = \begin{cases} (x-17.5)/5 & ;for\ 17.5 \leq x \leq 22.5 \\ (27.5-x)/5 & ;for\ 22.5 \leq x \leq 27.5 \\ (x-27.5)/5 & ;for\ 27.5 \leq x \leq 32.5 \\ (37.5-x)/5 & ;for\ 32.5 \leq x \leq 37.5 \\ 0 & ;otherwise \end{cases}$$

$$I_{hot}(x) = \begin{cases} (x - 27.5)/5 & ; for\ 27.5 \leq x \leq 32.5 \\ (37.5 - x)/5 & ; for\ 32.5 \leq x \leq 37.5 \\ 0 & ; otherwise \end{cases}$$

The graphical membership degrees of these variables are summarized in Figure 3. The graphical falsity degrees of these variables are summarized in Figure 4. Also, the graphical indeterminacy degrees of these variables are summarized in Figure 5. On the other hand, for a particular linguistic term, 'Cool' in the temperature attribute, their neutrosophic membership functions are represented in Figure 6.

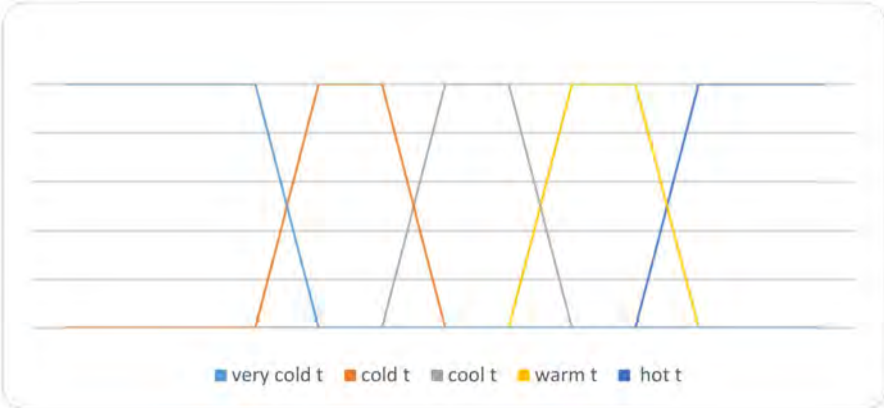

Figure 3. Truth-membership function of temperature attribute.

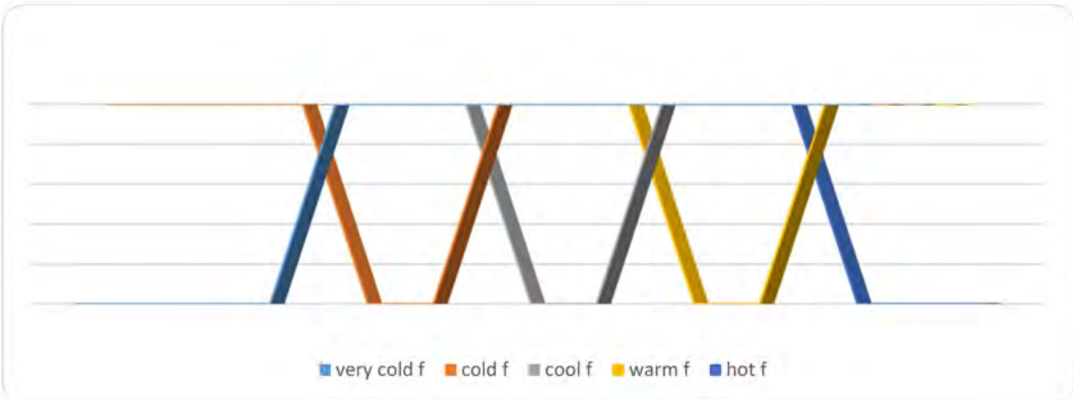

Figure 4. Falsity-membership function of temperature attribute.

Figure 5. Indeterminacy-membership function of temperature attribute.

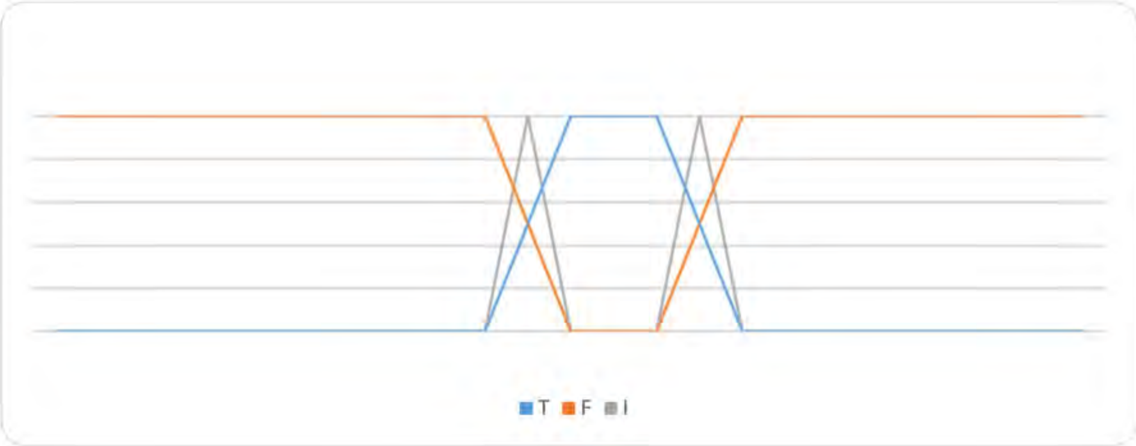

Figure 6. Cool (T, I, F) for temperature attribute.

Step 3 Based on the membership grades, different transaction has been set up by taking different sets of the temperatures. The membership grades in terms of the neutrosophic sets of these transactions are summarized in Table 5.

Table 5. Membership function for database Transactions.

Transaction	Temp.	Membership Degree
T1	18	Very-cold <0,0,1> cold <0,0,1> cool <1,0.1,0> warm <0,0.1,1> hot <0,0,1>
T2	13	Very cold <0,0,1> cold <0.4,0.9,0.6> cool <0.6,0.9,0.4> warm <0,0,1> hot <0,0,1>
T3	12	Very cold <0,0,1> cold <0.6,0.9,0.4> cool <0.4,0.9,0.6> warm <0,0,1> hot <0,0,1>
T4	33	Very cold <0,0,1> cold <0,0,1> cool <0,0,1> warm <0.4,0.9,0.6> hot <0.6,0.9,0.4>
T5	21	Very cold <0,0,1> cold <0,0,1> cool <0.8,0.7,0.2> warm <0.2,0.7,0.8> hot <0,0,1>
T6	25	Very cold <0,0,1> cold <0,0,1> cool <0,0,1> warm <1,0.5,0> hot <0,0,1>

Step 4 Now, we count the set of linguistic terms {very cold, cold, cool, warm, hot} for every element in transactions. Since the truth, falsity, and indeterminacy-memberships are independent functions, the set of linguistic terms can be extended to $\left\{ T_{very-cold}, T_{cold}, T_{cool}, T_{warm}, T_{hot} \; F_{very-cold}, F_{cold}, F_{cool}, F_{warm}, F_{hot} \; I_{very-cold}, I_{cold}, I_{cool}, I_{warm}, I_{hot} \right\}$ where F_{warm} means not worm and I_{warm} means not sure of warmness. This enhances dealing with negative association rules, which is handled as positive rules without extra calculations.

Step 5 By using the membership degrees that are given in Table 5 for candidate 1-itemset, the count and support has been calculated, respectively. The corresponding results are summarized in Table 6.

Table 6. Support for candidate 1-itemset neutrosophic set.

1-itemset	Count	Support
$T_{verycold}$	0	0
T_{Cold}	1	0.17
T_{Cool}	2.8	0.47
T_{Warm}	1.6	0.27
T_{Hot}	0.6	0.1
$I_{verycold}$	0	0
I_{Cold}	1.8	0.3
I_{Cool}	2.6	0.43
I_{Warm}	2.2	0.37
I_{Hot}	0.9	0.15
$F_{verycold}$	6	1
F_{Cold}	5	0.83
F_{Cool}	3.2	0.53
F_{Warm}	4.4	0.73
F_{Hot}	5.4	0.9

Similarly, the two-itemset support is illustrated in Table 7 and the rest of itemset generation (k-itemset for $k = 3, 4 \ldots 8$) are obtained similarly. The count for k-item set in database record is defined by minimum count of each one-itemset exists.

For example: $\{T_{Cold}, T_{Cool}\}$ count is 0.8

Because they exists in both T2 and T3.

In T2: T_{Cold} = 0.4 and T_{Cool} = 0.6 so, count for $\{T_{Cold}, T_{Cool}\}$ in T2 = 0.4

In T3: T_{Cold} = 0.6 and T_{Cool} = 0.4 so, count for $\{T_{Cold}, T_{Cool}\}$ in T2 = 0.4

Thus, count of $\{T_{Cold}, T_{Cool}\}$ in (Database) DB is 0.8.

Table 7. Support for candidate 2-itemset neutrosophic set.

2-itemset	Count	Support	2-itemset	Count	Support
$\{T_{Cold}, T_{Cool}\}$	0.8	0.13	$\{I_{Cold}, I_{Cool}\}$	1.8	0.30
$\{T_{Cold}, I_{Cold}\}$	1	0.17	$\{I_{Cold}, F_{verycold}\}$	1.8	0.30
$\{T_{Cold}, I_{Cool}\}$	1	0.17	$\{I_{Cold}, F_{Cold}\}$	1	0.17
$\{T_{Cold}, F_{verycold}\}$	1	0.17	$\{I_{Cold}, F_{Cool}\}$	1	0.17
$\{T_{Cold}, F_{Cold}\}$	0.8	0.13	$\{I_{Cold}, F_{Warm}\}$	1.8	0.30
$\{T_{Cold}, F_{Cool}\}$	1	0.17	$\{I_{Cold}, F_{Hot}\}$	1.8	0.30
$\{T_{Cold}, F_{Warm}\}$	1	0.17	$\{I_{Cool}, I_{Warm}\}$	0.8	0.13
$\{T_{Cold}, F_{Hot}\}$	1	0.17	$\{I_{Cool}, F_{verycold}\}$	2.6	0.43
$\{T_{Cool}, T_{Warm}\}$	0.2	0.03	$\{I_{Cool}, F_{Cold}\}$	1.8	0.30
$\{T_{Cool}, I_{Cold}\}$	1	0.17	$\{I_{Cool}, F_{Cool}\}$	1.2	0.20
$\{T_{Cool}, F_{Cool}\}$	1.8	0.30	$\{I_{Cool}, F_{Warm}\}$	2.6	0.43
$\{T_{Cool}, I_{Warm}\}$	0.8	0.13	$\{I_{Cool}, F_{Hot}\}$	2.6	0.43
$\{T_{Cool}, F_{verycold}\}$	2.8	0.47	$\{I_{Warm}, I_{Hot}\}$	0.9	0.15

Table 7. *Cont.*

$\{T_{Cool}, F_{Cold}\}$	2.8	0.47	$\{I_{Warm}, F_{verycold}\}$	2.2	0.37
$\{T_{Cool}, F_{Cool}\}$	1	0.17	$\{I_{Warm}, F_{Cold}\}$	2.2	0.37
$\{T_{Cool}, F_{Warm}\}$	2.8	0.47	$\{I_{Warm}, F_{Cool}\}$	1.6	0.27
$\{T_{Cool}, F_{Hot}\}$	2.8	0.47	$\{I_{Warm}, F_{Warm}\}$	1.4	0.23
$\{T_{Warm}, T_{Hot}\}$	0.4	0.07	$\{I_{Warm}, F_{Hot}\}$	1.7	0.28
$\{T_{Warm}, I_{Cool}\}$	0.2	0.03	$\{I_{Hot}, F_{verycold}\}$	0.9	0.15
$\{T_{Warm}, I_{Warm}\}$	1.1	0.18	$\{I_{Hot}, F_{Cold}\}$	0.9	0.15
$\{T_{Warm}, I_{Hot}\}$	0.4	0.07	$\{I_{Hot}, F_{Cool}\}$	0.9	0.15
$\{T_{Warm}, F_{verycold}\}$	1.6	0.27	$\{I_{Hot}, F_{Warm}\}$	0.6	0.10
$\{T_{Warm}, F_{Cold}\}$	1.6	0.27	$\{I_{Hot}, F_{Hot}\}$	0.4	0.07
$\{T_{Warm}, F_{Cool}\}$	1.6	0.27	$\{F_{verycold}, F_{Cold}\}$	5	0.83
$\{T_{Warm}, F_{Warm}\}$	0.6	0.10	$\{F_{verycold}, F_{Cool}\}$	3.2	0.53
$\{T_{Warm}, F_{Hot}\}$	1.6	0.27	$\{F_{verycold}, F_{Warm}\}$	4.4	0.73
$\{T_{Hot}, I_{Warm}\}$	0.6	0.10	$\{F_{verycold}, F_{Hot}\}$	5.4	0.90
$\{T_{Hot}, I_{Hot}\}$	0.6	0.10	$\{F_{Cold}, F_{Cool}\}$	3	0.50
$\{T_{Hot}, F_{verycold}\}$	0.6	0.10	$\{F_{Cold}, F_{Warm}\}$	3.4	0.57
$\{T_{Hot}, F_{Cold}\}$	0.6	0.10	$\{F_{Cold}, F_{Hot}\}$	4.4	0.73
$\{T_{Hot}, F_{Cool}\}$	0.6	0.10	$\{F_{Cool}, F_{Warm}\}$	1.8	0.30
$\{T_{Hot}, F_{Warm}\}$	0.6	0.10	$\{F_{Cool}, F_{Hot}\}$	2.6	0.43
$\{T_{Hot}, F_{Hot}\}$	0.4	0.07	$\{F_{Warm}, F_{Hot}\}$	4.2	0.70

5. Case Study

In this section, the case of Telecom Egypt Company stock records has been studied. Egyptian stock market has many companies. One of the major questions for stock market users is when to buy or to sell a specific stock. Egyptian stock market has three indicators, EGX30, EGX70, and EGX100. Each indicator gives a reflection of the stock market. Also, these indicators have an important impact on the stock market users, affecting their decisions of buying or selling stocks. We focus in our study on the relation between the stock and the three indicators. Also, we consider the month and quarter of the year to be another dimension in our study, while the sell/buy volume of the stock per day is considered to be the third dimension.

In this study, the historical data has been taken from the Egyptian stock market program (Mist) during the program September 2012 until September 2017. For every stock/indicator, Mist keeps a daily track of number of values (opening price, closing price, high price reached, low price reached, and volume). The collected data of Telecom Egypt Stock are summarized in Figure 7.

Telecom Egypt

Symbolid	Ts_date	Open	Close	High	Low	Volume
S48031C016	012-09-13	13.98	14.07	14.29	13.85	490,997
S48031C016	012-09-16	14.08	14.09	14.20	14.02	347,258
S48031C016	012-09-17	14.11	14.60	15.01	14.11	2,050,226
S48031C016	012-09-18	14.52	14.72	15.00	14.52	500,290
S48031C016	012-09-19	14.90	14.68	14.95	14.41	362,893
S48031C016	012-09-20	14.77	14.84	15.00	14.50	913,683
S48031C016	012-09-23	14.56	15.09	15.37	14.56	283,008
S48031C016	012-09-24	14.80	14.73	15.00	14.68	896,914
S48031C016	012-09-25	14.80	14.39	14.80	14.00	2,604,812
S48031C016	012-09-26	14.25	14.53	14.71	14.25	648,102

Figure 7. Telecom Egypt stock records.

In this study, we use the open price and close price values to get price change rate, which are defined as follows:

$$\text{price change rate} = \frac{\text{close price} - \text{open price}}{\text{open price}} \times 100$$

and change the volume to be a percentage of total volume of the stock with the following relation:

$$\text{percentage of volume} = \frac{\text{volume}}{\text{total volume}} \times 100$$

The same was performed for the stock market indicators. Now, we take the attributes as "quarter", "month", "stock change rate", "volume percentage", and "indicators change rate". Table 8 illustrates the segment of resulted data after preparation.

Table 8. Segment of data after preparation.

Ts_Date	Month	Quarter	Change	Volume	Change30	Change70	Change100
13 September 2012	September	3	0.64	0.03	−1.11	0.01	−0.43
16 September 2012	September	3	0.07	0.02	2.82	4.50	3.67
17 September 2012	September	3	3.47	0.12	1.27	0.76	0.81
18 September 2012	September	3	1.38	0.03	−0.08	−0.48	−0.43
19 September 2012	September	3	−1.48	0.02	0.35	−1.10	−0.64
20 September 2012	September	3	0.47	0.05	−1.41	−1.64	−1.55
23 September 2012	September	3	3.64	0.02	−0.21	1.00	0.41
24 September 2012	September	3	−0.47	0.05	0.27	−0.09	0.03
25 September 2012	September	3	−2.77	0.15	2.15	1.79	1.85
26 September 2012	September	3	1.96	0.04	0.22	0.96	0.57
27 September 2012	September	3	0.90	0.05	−1.38	−0.88	−0.92
30 September 2012	September	3	−0.14	0.00	−1.11	−0.79	−0.75
1 October 2012	October	4	−1.60	0.02	−2.95	−4.00	−3.51

Based on these linguistic terms, define the ranges under the SVNSs environment. For this, corresponding to the attribute in "change rate" and "volume", the truth-membership functions by defining their linguistic terms as {"high up", "high low", "no change", "low down", "high down"} corresponding to attribute "change rate", while for the attribute "volume", the linguistic terms are (low, medium, and high) and their ranges are summarized in Figures 8 and 9, respectively. The falsity-membership function and indeterminacy-membership function have been calculated and applied as well for change rate attribute.

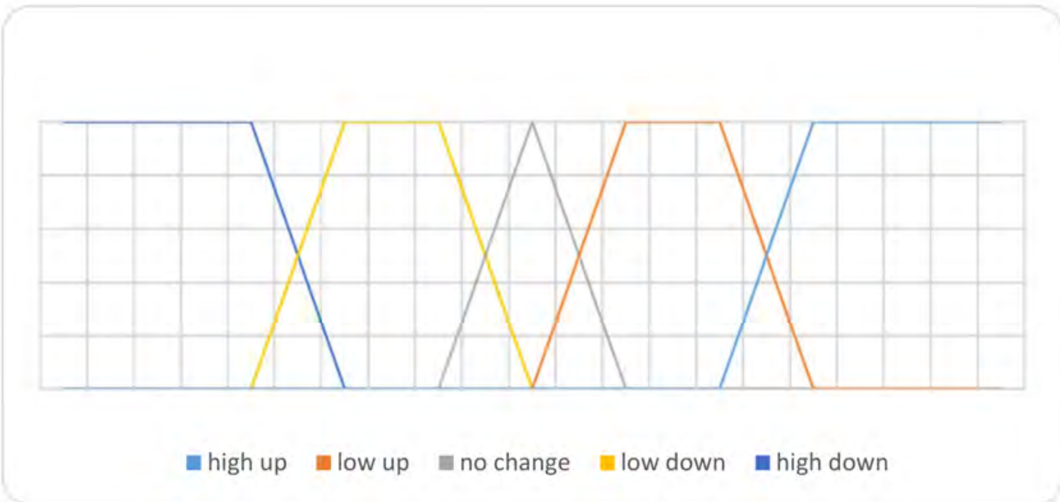

Figure 8. Change rate attribute truth-membership function.

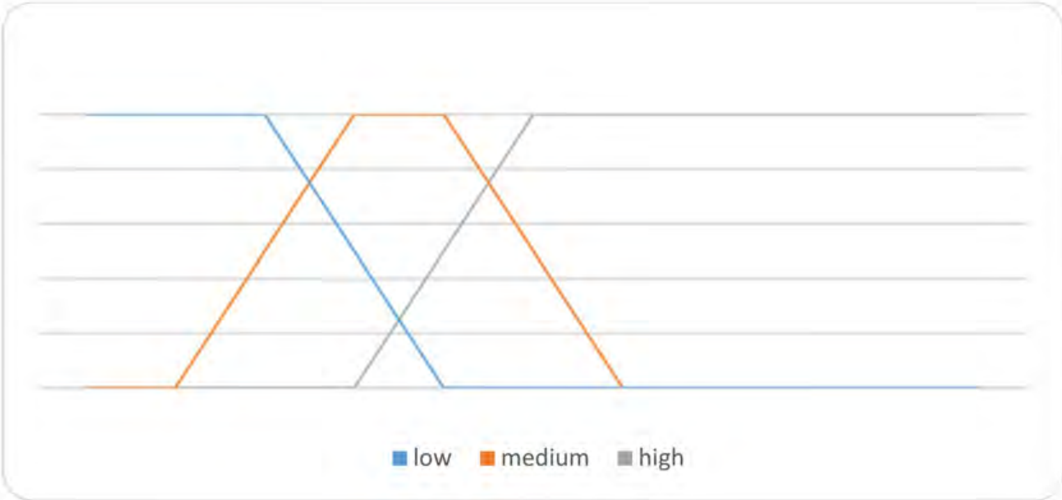

Figure 9. Volume attribute truth-membership function.

6. Experimental Results

We proceeded to a comparison between fuzzy mining and neutrosophic mining algorithms, and we found out that the number of generated association rules increased in neutrosophic mining.

A program has been developed to generate large itemsets for Telecom Egypt historical data. VB.net has been used in creating this program. The obtained data have been stored in an access database. The comparison depends on the number of generated association rules in a different min-support threshold. It should be noted that the performance cannot be part of the comparison because of the number of items (attributes) that are different in fuzzy vs. neutrosophic association rules mining. In fuzzy mining, the number of items was 14, while in neutrosophic mining it is 34. This happens because the number of attributes increased. Spreading each linguistic term into three (True, False, Indeterminacy) terms make the generated rules increase. The falsity-generated association rules can be considered a negative association rules. As pointed out in [36], the conviction of a rule $conv(X \rightarrow Y)$ is defined as the ratio of the expected frequency that X happened without Y falsity-association rules to be used to generate negative association rules if $T(x) + F(x) = 1$. In Table 9, the number of generated fuzzy rules in each $k-$itemset using different min-support threshold are reported, while the total generated fuzzy association rule is presented in Figure 10.

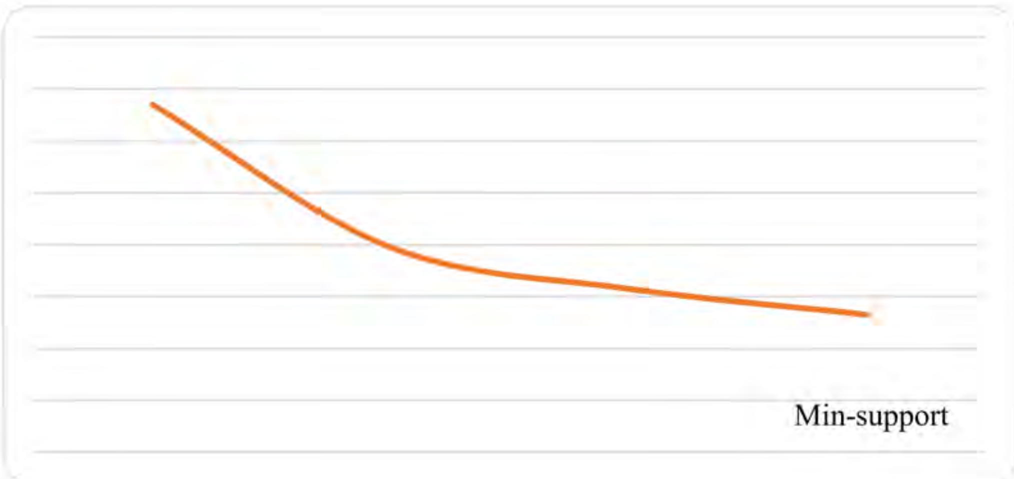

Figure 10. No. of fuzzy association rules with different min-support threshold.

Table 9. No. of resulted fuzzy rules with different min-support.

Min-Support	0.02	0.03	0.04	0.05
1-itemset	10	10	10	10
2-itemset	37	36	36	33
3-itemset	55	29	15	10
4-itemset	32	4	2	0

As compared to the fuzzy approach, by applying the same min-support threshold, we get a huge set of neutrosophic association rules. Table 10 illustrates the booming that happened to generated neutrosophic association rules. We stop generating itemsets at iteration 4 due to the noted expansion in the results shown in Figure 11, which shows the number of neutrosophic association rules.

Table 10. No. of neutrosophic rules with different min-support threshold.

Min-Support	0.02	0.03	0.04	0.05
1-itemset	26	26	26	26
2-itemset	313	311	309	300
3-itemset	2293	2164	2030	1907
4-itemset	11,233	9689	8523	7768

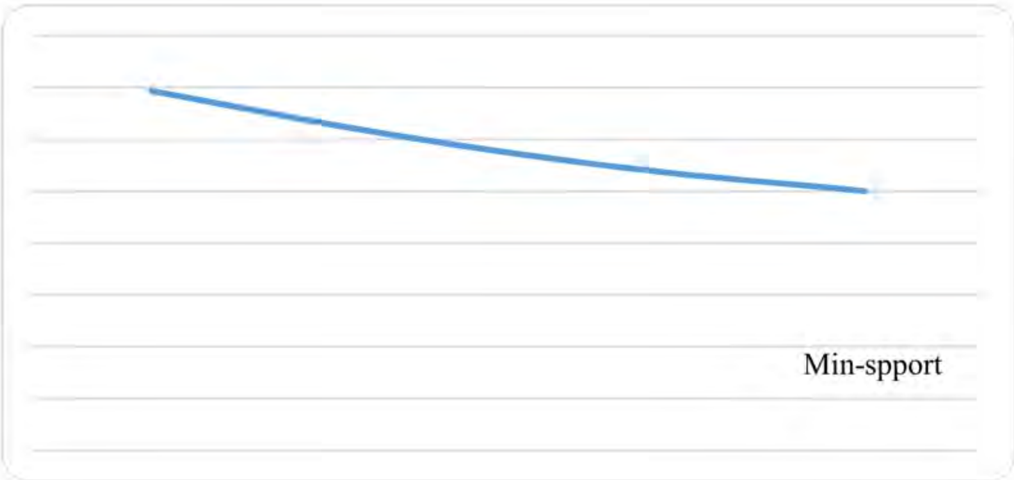

Figure 11. No. of neutrosophic association rules with different min-support threshold.

Experiment has been re-run using different min-support threshold values and the resulted neutrosophic association rules counts has been noted and listed in Table 11. Note the high values that are used for min-support threshold. Figure 12 illustrates the generated neutrosophic association rules for min-support threshold from 0.5 to 0.9.

Table 11. No. of neutrosophic rules with different min-support threshold.

Min-Support	0.5	0.6	0.7	0.8	0.9
1-itemset	11	9	9	6	5
2-itemset	50	33	30	11	10
3-itemset	122	64	50	10	10
4-itemset	175	71	45	5	5
5-itemset	151	45	21	1	1
6-itemset	88	38	8	0	0

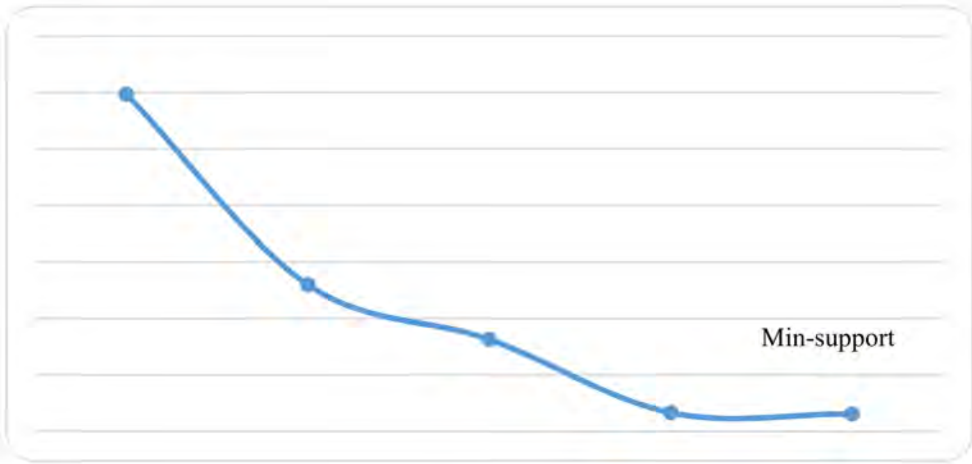

Figure 12. No. of neutrosophic rules for min-support threshold from 0.5 to 0.9.

Using the neutrosophic mining approach makes association rules exist for most of the min-support threshold domain, which may be sometimes misleading. We found that using the neutrosophic approach is useful in generating negative association rules beside positive association rules minings. Huge generated association rules provoke the need to re-mine generated rules (mining of mining association rules). Using suitable high min-support values may help in the neutrosophic mining process.

7. Conclusions and Future Work

Big data analysis will continue to grow in the next years. In order to efficiently and effectively deal with big data, we introduced in this research a new algorithm for mining big data using neutrosophic association rules. Converting quantitative attributes is the main key for generating such rules. Previously, it was performed by employing the fuzzy sets. However, due to fuzzy drawbacks, which we discussed in the introductory section, we preferred to use neutrosophic sets. Experimental results showed that the proposed approach generated an increase in the number of rules. In addition, the indeterminacy-membership function has been used to prevent losing rules from boundaries problems. The proposed model is more effective in processing negative association rules. By comparing it with the fuzzy association rules mining approaches, we conclude that the proposed model generates a larger number of positive and negative association rules, thus ensuring the construction of a real and efficient decision-making system. In the future, we plan to extend the comparison between the neutrosophic association rule mining and other interval fuzzy association rule minings. Furthermore, we seized the falsity-membership function capacity to generate negative association rules. Conjointly, we availed of the indeterminacy-membership function to prevent losing rules from boundaries problems. Many applications can emerge by adaptions of truth-membership function, indeterminacy-membership function, and falsity-membership function. Future work will benefit from the proposed model in generating negative association rules, or in increasing the quality of the generated association rules by using multiple support thresholds and multiple confidence thresholds for each membership function. The proposed model can be developed to mix positive association rules (represented in the truth-membership function) and negative association rules (represented in the falsity-membership function) in order to discover new association rules, and the indeterminacy-membership function can be put forth to help in the automatic adoption of support thresholds and confidence thresholds. Finally, yet importantly, we project to apply the proposed model in the medical field, due to its capability in effective diagnoses through discovering both positive and negative symptoms of a disease. All future big data challenges could be handled by combining neutrosophic sets with various techniques.

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Neutrosophic Computability and Enumeration

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Abstract: We introduce oracle Turing machines with neutrosophic values allowed in the oracle information and then give some results when one is permitted to use neutrosophic sets and logic in relative computation. We also introduce a method to enumerate the elements of a neutrosophic subset of natural numbers.

Keywords: computability; oracle Turing machines; neutrosophic sets; neutrosophic logic; recursive enumerability; oracle computation; criterion functions

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1. Introduction

In classical computability theory, algorithmic computation is modeled by Turing machines, which were introduced by Alan M. Turing [1]. A *Turing machine* is an abstract model of computation defined by a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F, \{L, R\})$, where Q is a finite set of states, Σ is the alphabet, Γ is the tape alphabet, $q_0 \in Q$ is the starting state, $F \subset Q$ is a set of halting states, the set $\{L, R\}$ denotes the possible left (L) and right (R) move of the tape head, and δ is the transition function, defined as:

$$\delta : Q \times \Gamma \to Q \times \Sigma \times \{L, R\}.$$

Each transition is a step of the computation. Let w be a string over the alphabet Σ. We say that a Turing machine on input w *halts* if the computation ends with some state $q \in F$. The output of the machine, in this case, is whatever was written on the tape at the end of the computation. If a Turing machine M on input w halts, then we say that M is defined on w. Since there is a one-to-one correspondence between the set of all finite strings over Σ and the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, without loss of generality we may assume that Turing machines are defined from $\mathbb{N}$ to $\mathbb{N}$.

Standard Turing machines admit partial functions, i.e., functions that may not be defined on every input. The class of functions computable by Turing machines are called *partial recursive (computable)* functions. We shall not delve into the details about what is meant by a function or set that is computable by a Turing machine. We assume that the reader is familiar with the basic terminology. However, for a detailed account, the reader may refer to Reference [2–4]. Using a well known method called *Gödel numbering*, originated from Gödel's celebrated 1931 paper [5], it is possible to have an algorithmic enumeration of all partial recursive functions. We let Ψ_i denote the ith partial recursive function, i.e., the ith Turing machine.

If a partial recursive function is defined on every argument we say that it is *total*. Total recursive functions are simply called *recursive* or *computable*. Since there are countable infinitely many Turing machines, there are countable infinitely many computable functions. Computable sets and functions are widely used in mathematics and computer science. However, nearly all functions are non-computable.

Since there are $2^{\aleph_0}$ functions from $\mathbb{N}$ to $\mathbb{N}$ and only $\aleph_0$ many computable functions, there are uncountably many non-computable functions.

Oracle Turing machines, introduced by Alan Turing [6], are used for relativizing the computation with respect to a given set of natural numbers. An *oracle Turing machine* is a Turing machine with an extra *oracle tape* containing the characteristic function of a given set of natural numbers. The *characteristic function* of a set $S \subset \mathbb{N}$ is defined as:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

We may also think of χ_S as an infinite binary sequence and call it the *characteristic sequence* of S. For a set $S \subset \mathbb{N}$, we let $S(i)$ denote $\chi_S(i)$. So the characteristic sequence of a set S simply gives the membership information about natural numbers regarding S. For a given an oracle Turing machine with the characteristic sequence of a set S provided in the oracle tape, functions are denoted as computable by the machine relative to the *oracle* S. If the oracle Turing machine with an oracle S computes a function f, then we say that f is *computable in* S or we say S *computes* f. We denote the ith oracle Turing machine with an oracle A by $\Psi_i(A)$. Then, it makes sense to write $\Psi_i(A) = B$ if A computes B.

Now we shall look at a non-standard Turing machine model based on neutrosophic sets. Neutrosophic logic, first introduced by Smarandache [7,8], is a generalization of classical, fuzzy and intuitionistic fuzzy logic. The key assumption of neutrosophy is that every idea not only has a certain degree of truth, as is generally taken in many-valued logic contexts, but also has degrees of falsity and indeterminacy, which need to be considered independently from each other. A neutrosophic set relies on the idea that there is a degree of probability that an element is a member of the given set, a degree that the very same element is *not* a member of the set, and a degree that the membership of the element is indeterminate for the set. For our purpose we take subsets of natural numbers. Roughly speaking, if n were a natural number and if A were a neutrosophic set, then there would be a probability distribution $p_\in(n) + p_\notin(n) + p_I(n) = 1$, where $p_\in(n)$ denotes the probability of n being a member of A, $p_\notin(n)$ denotes the probability of n not being a member of A, and $p_I(n)$ denotes the degree of probability that the membership of n is indeterminate in A. Since the probability distribution is expected to be normalized, the summation of all probabilities must be equal to unity. We should note however that the latter requirement can be modified depending on the application.

The above interpretation of a neutrosophic set can be in fact generalized to any multi-dimensional collection of attributes. That is, our attributes did not need to be merely about membership, non-membership, and indeterminacy, but it could range over any finite set of attributes $a_0, a_1, \ldots, a_k$ and $b_0, b_1, \ldots b_k$ so that the value of an element would range over (x, y, I) such that $x \in a_m$ and $y \in b_m$ for $0 \leq m \leq k$. The set of attributes can also be countably infinite or even uncountable. However, we are not concerned with these cases. We shall only consider the membership attribute discussed above.

We are particularly interested in subsets of natural numbers $A \subset \mathbb{N}$, in our study. Any neutrosophic subset A of natural numbers (we shall occasionally denote such a set by A^N) is defined in the form of ordered triplets:

$$\{\langle p_\in(0), p_\notin(0), p_I(0)\rangle, \langle p_\in(1), p_\notin(1), p_I(1)\rangle, \langle p_\in(2), p_\notin(2), p_I(2)\rangle, \ldots\},$$

where, for each $i \in \mathbb{N}$, $p_\in(i)$ denotes the degree of probability of i being an element of A, $p_\notin(i)$ denotes the probability of i being not an element of A, and $p_I(i)$ denotes the probability of i being indetermined. Since we assume a normalized probability distribution, we have that for every $i \in \mathbb{N}$:

$$p_\in(i) + p_\notin(i) + p_I(i) = 1.$$

2. Oracle Turing Machines with Neutrosophic Values

Now we can extend the notion of relativized computation based on neutrosophic sets and neutrosophic logic. For this we introduce oracle Turing machines with neutrosophic oracle tape. The general idea is as follows. Standard oracle tape contains the information of the characteristic sequence of a given set $A \subset \mathbb{N}$. We extend the definition of the characteristic function to neutrosophic sets as follows.

Definition 1. *Let $A \subset \mathbb{N}$ be a set. A neutrosophic oracle tape is a countably infinite sequence $t_0, t_1, \ldots$ where $t_i = \langle a, b, c\rangle$ is an ordered triplet and $a, b, c \in \mathbb{Q}$, so that a is the probability value of i such that $i \in A$, b is the probability value of i such that $i \notin A$, and c is the probability of i being indeterminate for A.*

The overall picture of a neutrosophic oracle tape can be seen in Figure 1. Now we need to modify the notion of the characteristic sequence accordingly.

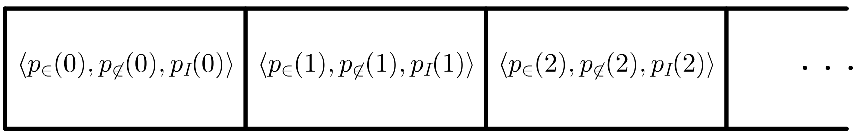

Figure 1. Neutrosophic oracle tape.

Definition 2. *Let $S \subset \mathbb{N}$ be a set and let B denote the blank symbol in the alphabet of the oracle tape. The neutrosophic characteristic function of S is defined by*

$$\chi_S^N(x) = \begin{cases} \langle 1, 0, I\rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) > 0 \\ \langle B, 0, I\rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) > 0 \\ \langle B, B, I\rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) > 0 \\ \langle B, B, B\rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) = 0 \\ \langle 1, B, I\rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) > 0 \\ \langle 1, B, B\rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) = 0 \\ \langle 1, 0, B\rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) = 0 \\ \langle B, 0, B\rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) = 0 \end{cases}$$

The idea behind this definition is to label the distributions which have significant probability value with respect to a pre-determined probability threshold value, in this case we assume this value to be 0 by default. Note that this threshold value could be defined for any $r \in \mathbb{Q}$ so that instead of being greater than 0, we would require the probability for that attribute to be greater than r in order to be labelled. We will talk about the properties of defining an arbitrary threshold value and its relation to neutrosophic computations in the next section.

Definition 3. *A neutrosophic oracle Turing machine is a Turing machine with an additional neutrosophic oracle tape $(Q, \Sigma, \Gamma, \Gamma', \delta, q_0, F, \{L, R\})$, where Q is a finite set of states, Σ is the alphabet, Γ is the tape alphabet, Γ' is the neutrosophic oracle tape alphabet containing the blank symbol B, $q_0 \in Q$ is the starting state, $F \subset Q$ is a set of halting states, the set $\{L, R\}$ denotes the possible left (L) and right (R) move of the tape head, and δ is the transition function defined as:*

$$\delta : Q \times \Gamma \times \Gamma' \to Q \times \Sigma \times \{L, R\}^2.$$

Theorem 1. *Any neutrosophic oracle Turing machine can be simulated by a standard Turing machine.*

Proof. Assuming the Church–Turing thesis in the proof, we only need to argue that standard oracle tapes can in theory represent neutrosophic oracle tapes. In fact any neutrosophic oracle tapes can be represented by three standard oracle tapes each of which contains one and only one attribute. The *i*th cell of the first oracle tape contains the probability value $p_{\in}(i)$. The *i*th cell of the second oracle tape contains the value $p_{\notin}(i)$. Similarly, the *i*th cell of the third oracle tape contains $p_I(i)$.

We also need to argue that a three-tape oracle standard Turing machine can be simulated by a single tape oracle Turing machine. Let Γ be the oracle alphabet. We define an extension Γ' of Γ by introducing a delimeter symbol # to separate each attribute for a given number i. We define another delimiter symbol $\perp$ to separate each $i \in \mathbb{N}$. Let $\Gamma' = \Gamma \cup \{\#, \perp\}$. Then, a neutrosophic oracle tape can be represented by a single oracle tape with the tape alphabet Γ'. The oracle tape will be in the form:

$$p_{\in}(0)\#p_{\notin}(0)\#p_I(0)\perp p_{\in}(1)\#p_{\notin}(1)\#p_I(1)\perp \dots$$

The symbol $\perp$ determines a counter for i, whereas for each i, the symbol # determines a counter for the attribute. □

A neutrosophic set A *computes* another neutrosophic set B if using finitely many pieces of information of the characteristic sequence of A determines the ith entry of the characteristic sequence of B given any index $i \in \mathbb{N}$. Then, based on this definition, a set $B \subset \mathbb{N}$ is *neutrosophically computable* in A if $B = \Psi_e^N(A)$ for some $e \in \mathbb{N}$, where Ψ_e^N denotes the e-th neutrosophic oracle Turing machine. If $B = \Psi_e^N(A)$ for some $e \in \mathbb{N}$, we denote this by $B \leq_N A$. If $B \leq_N A$ and $A \leq_N B$, then we say that A and B are *neutrosophically equivalent* and denote this by $A \equiv_N B$. Intuitively, $A \equiv_N B$ means that A and B are neutrosophic subsets of natural numbers, and they have the same level of neutrosophic information complexity. We leave the discussion on the properties of the equivalence classes induced by $\equiv_N$ for another study as it is beyond the scope of this paper.

3. Neutrosophic Enumeration and Criterion Functions

We now introduce the concept of neutrosophic enumeration of the members of neutrosophic subsets of natural numbers. Since we talk about enumeration, we must only take countable sets into consideration. It is known from classical computability that, given a set $A \subset \mathbb{N}$, A is called *recursively enumerable* if there exists some $e \in \mathbb{N}$ such that A is the domain of Ψ_e. We want to define the neutrosophic counterpart of this notion, but we need to be careful about the indeterminate cases, an intrinsic property in neutrosophic logic.

Definition 4. *A set A is called neutrosophic Turing enumerable if there exists some $e \in \mathbb{N}$ such that A is the domain of Ψ_e^N restricted to elements whose probability degree of membership is greater than a given probability threshold. More precisely, if $r \in \mathbb{Q}$ is a given probability threshold, then A is neutrosophic Turing enumerable if A is the domain of $\Psi_e^N(\emptyset)$ restricted to those elements i such that $p_{\in}^A(i) \geq r$, where $p_{\in}^A(i)$ denotes the degree of probability of membership of i in A.*

If the eth Turing machine is defined on the argument i, we denote this by $\Psi_e(i)\downarrow$. The *halting set* in classical computability theory is defined as:

$$K = \{e : \Psi_e(e)\downarrow\}.$$

It is known that K is recursively enumerable but not recursive. Unlike in classical Turing computability, we show that neutrosophically computable sets allow us to neutrosophically compute the halting set. The way to do this goes as follows. A single neutrosophic subset of natural numbers is not enough to compute the halting set. Instead, we take the union of all neutrosophically computable subsets of natural numbers by taking an infinite join which will code the information of the halting set.

Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable sequence of subsets of $\mathbb{N}$. The *infinite join* is defined by

$$\bigoplus\{A_i\} = \{\langle i, x\rangle : x \in A_i\},$$

where $\langle i, j\rangle$ is mapped to a natural number using a uniform pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Theorem 2. *Let A^N be a neutrosophic subset of $\mathbb{N}$. Then, $\bigoplus\{A_i^N\} \equiv_N K$*

Proof. We first show that $\bigoplus\{A_i^N\} \geq_N K$. The infinite join of all neutrosophically computable sets computes the halting set. Let $\{\Psi_i^N\}_{i\in\mathbb{N}}$ be an effective enumeration of neutrosophic Turing functionals. Let $\{A_i^N\}$ be the corresponding neutrosophic sets, each of which is computable by Ψ_i^N. To compute K, we let $\bigoplus\{A_i^N\}$ be the infinite join of all A_i^N. To know whether $\Psi_i(i)$ is defined or not, we see if $p_\in(i) + p_\notin(i) > 0.5$. If so, then $\Psi_i(i) \downarrow$. Otherwise it must be that $p_I(i) > 0.5$. In this case, $\Psi_i(i)$ is undefined.

Next, we show $\bigoplus\{A_i^N\} \leq_N K$. To prove this, we assume that there exists an oracle for K. If $i \in K$, then there exist indices $x, y \in \mathbb{N}$ such that $\langle x, y\rangle = i$ and it must be that $p_\in(i) + p_\notin(i) > 0.5$ since $\Psi_i(i)$ is defined, but we may not know whether $i \in A_i$ or $i \notin A_i$. If $i \notin K$, then the same argument holds to prove this case as well. □

The use of the probability ratio 0.5 is for convenience. This notion will be generalized later on. Classically speaking, given a subset of natural numbers, we can easily convert it to a neutrosophic set preserving the membership information of the given classical set. Suppose that we are given a set $A \subset \mathbb{N}$ and we want to convert it to a neutrosophic set with the same characteristic sequence. The neutrosophic counterpart A^N is defined, for each $i \in \mathbb{N}$, as:

$$A^N(i) = \begin{cases} \langle i, 1-i, 0\rangle & \text{if } A(i) = 1 \\ \langle 1-i, i, 0\rangle & \text{otherwise.} \end{cases}$$

We now introduce the tree representation of neutrosophic sets and give a method, using trees, to approximate its classical counterpart. Suppose that we are given a neutrosophic subset of natural numbers in the form:

$$A^N = \{\langle p_\in(i), p_\notin(i), p_I(i)\rangle\}_{i\in\mathbb{N}}.$$

We use the probability distribution to decide which element will be included in the classical counterpart. If A^N is a neutrosophic subset of natural numbers, the *classical counterpart of A^N* is defined as:

$$A(i) = \begin{cases} 1 & \text{if } p_\in(i) > p_\notin(i) \\ 0 & \text{if } p_\in(i) < p_\notin(i). \end{cases}$$

Now we introduce a simple conversion using trees. The aim is to approximate to the classical counterpart of a given neutrosophic set A^N in a computable fashion. For this we start with a full ternary tree, as given in Figure 2, coding all possible combinations.

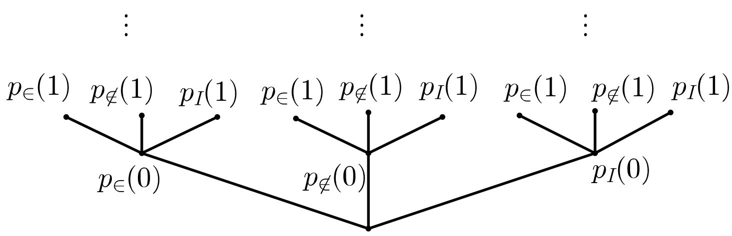

Figure 2. Approximating a neutrosophic set with a classical set through a ternary tree.

The correct interpretation of this tree is as follows. Each branch represent a possible element of the set we want to construct. For instance, if $p_{\in}(0)$ has the largest probability value among $p_{\in}(0)$, $p_{\notin}(0)$, $p_I(0)$, then we choose $p_{\in}(0)$ and define 0 to be an element of the classical set we construct. If either $p_{\notin}(0)$ or $p_I(0)$ is greater than $p_{\in}(0)$, then we know 0 is not an element of the constructed set. Since we are defining a classical set, the only time when some $i \in \mathbb{N}$ is in the constructed set is if $p_{\in}(i) > p_{\notin}(i)$ and $p_{\in}(i) > p_I(i)$. Continuing along this line, if $p_{\notin}(1)$, say, has the greatest probability value among $p_{\in}(1)$, $p_{\notin}(1)$, $p_I(1)$, then 1 will not be an element. So far, 0 is an element and 1 is not an element. So in the tree we choose the leftmost branch and then next we choose the middle branch. Repeating this procedure for every $i \in \mathbb{N}$, we end up defining a computable infinite path on this ternary tree which defines elements of the set being constructed. At each step, we simply take the maximum probability value and select that attribute. The infinite path defines a computable approximation to the classical counterpart of A^N using the tree method.

Earlier we defined Turing machines with a neutrosophic oracle tape. Suppose that the characteristic sequence of a neutrosophic set A can be considered as an oracle. Then, the e-th neutrosophic oracle Turing machine can compute a function of the same characteristic. That is, not only can Turing machines with neutrosophic oracles compute classical sets, but they can also compute neutrosophic sets. It is important to note that we need to modify the definition of standard oracle Turing machines in order to use neutrosophic sets. We add the symbol I to the alphabet of the oracle tape. The transition function δ^N is then defined as:

$$\delta^N : Q \times \Sigma \times \Gamma \to Q \times \Sigma \cup \{I\} \times \{L, R\}^2.$$

We say that the neutrosophic oracle Turing machine, say Ψ_e^N, *computes* a neutrosophic set B if $\Psi_e^N = B$.

We now turn to the problem of enumerating members of a neutrosophic subset A of natural numbers. Normally, general intuition suggests that we pick elements $i \in \mathbb{N}$ such that $p_{\in}(i) > 0.5$. It is important to note that, given $A = \{\langle p_{\in}(i), p_{\notin}(i), p_I(i)\rangle\}_{i\in\mathbb{N}}$, not every i will be enumerated if we use this probability criterion. However, changing the criterion depending on what aspect of the set we want to look at and depending on the application, would also change the enumerated set. Therefore, we would need a kind of criterion function to set a probability threshold regarding which elements of the neutrosophic set are to be enumerated.

In practice, one often encounters a situation where the given information is not directly used but rather analyzed under the criterion determined by a function. We examine how the computation behaves when we impose a function on the neutrosophic oracle tape. That is, suppose that $f : \mathbb{N} \to \{a, b, c\}$ is a function, where $a, b, c \in \mathbb{Q}$, which maps each cell of the neutrosophic oracle tape to a probability value. For example, f could be defined as a constant non-membership function which assigns every triplet in the cells to the non-membership $\notin$ attribute. In this case, the probability of any natural number not being an element of the considered oracle A is just 1. When these kinds of functions are used in the oracle information of A, we may be able to compute some useful information.

The intuition in using criterion functions is to select, under a previously determined probability threshold, a natural number from the probability distribution which is available in a given neutrosophic subset of natural numbers. As an example, let us imagine a neutrosophic subset A of natural numbers. Suppose for simplicity that A is finite and is defined as:

$$A = \{\langle 0.1, 0.4, 0.5\rangle, \langle 0.6, 0.3, 0.1\rangle, \langle 0, 0.9, 0.1\rangle\}.$$

First of all, we should read this as follows: A has neutrosophic information about the first three natural numbers $0, 1, 2$. In this example, $p_{\in}(0) = 0.1$, $p_{\notin}(0) = 0.4$, $p_I(0) = 0.5$. For the natural number 1, we have that $p_{\in}(1) = 0.6$, $p_{\notin}(1) = 0.3$, $p_I(1) = 0.1$. Finally, for the natural number 2, we have $p_{\in}(2) = 0$, $p_{\notin}(2) = 0.9$, $p_I(2) = 0.1$. Now if we want to know which natural numbers are in A, normally we would only pick the number 2 since $p_{\in}(2) > 0.5$. Our criterion of enumeration in this case is 0.5. In general, this probability value may not be always applicable. Moreover, this probability threshold value may not be constant. That is, we may want to have a different probability threshold for every natural number i. If our criterion were to select the ith element whose probability exceeds p_i, we would enumerate those numbers. For example, if the criterion is defined as

$$f(0) = 0, \quad f(1) = 0.8, \quad f(2) = 0.2,$$

then for the first triple, the probability threshold is 0, meaning that we enumerate the natural number 0 if $p_{\in}(0) > 0$. Obviously 0 will be enumerated in this case since $p_{\in}(0) = 0.1 > 0$. The probability threshold for enumerating the number 1 is 0.8, so it will not be enumerated since $p_{\in}(1) = 0.6 < 0.8$. Finally, the probability threshold for enumerating the number 2 is 0.2. In this case, 2 will not be enumerated since $p_{\in}(2) = 0 < 0.2$. So the enumeration set for A under the criterion f will be $\{0\}$.

We are now ready to give the formal definition of a criterion function.

Definition 5. *A criteria function is a mapping $f : \mathbb{N} \to \mathbb{Q}$ which, given a neutrosophic subset A of natural numbers, determines a probability threshold for each triple $\langle p_{\in}(i), p_{\notin}(i), p_I(i)\rangle$ in A.*

We first note a simple observation that if the criterion function is the constant function $f(n) = 0$ for any $n \in \mathbb{N}$, the enumeration set will be equal to $\mathbb{N}$ itself. However, this does not mean that the enumeration set will be empty if $f(n) = 1$. Given a neutrosophic set A, if $p_{\in}(i) = 1$ for all i, then the enumeration set for A will also be equal to $\mathbb{N}$.

We shall next give the following theorem. First we remind the reader that we call a function f *strictly decreasing* if $f(i+1) < f(i)$.

Theorem 3. *Let f be a strictly decreasing criterion function for a neutrosophic set A such that $p_{\in}(i) < p_{\in}(i+1)$ for every $i \in \mathbb{N}$, and let $\mathcal{E}_A$ be the enumeration set for A under the criterion f. Then, there exists some $k \in \mathbb{N}$ such that $|\mathcal{E}_A| < k$.*

Proof. Clearly, given A and that for each i, $p_{\in}(i) < p_{\in}(i+1)$, only those numbers i which satisfy $p_{\in}(i) > f(i)$ will be enumerated. Since the probability distribution of membership degrees of elements of A strictly increases and f is strictly decreasing, there will be some number $j \in \mathbb{N}$ such that $p_{\in}(i) \leq f(j)$. Moreover, for the same reason $p_{\in}(m) \leq f(m)$ for every $m > j$. Therefore, the number of elements enumerated is less than j. That is, $|\mathcal{E}_A| < j$. □

We denote the complement of a neutrosophic subset of natural numbers A by A^c and we define it as follows. Let $p^A_{\in}(i)$ denote the probability of i being an element of A and let $p^A_{\notin}(i)$ denote the probability of i being not an element of A. In addition, $p^A_I(i)$ denotes the probability of the membership of i being indeterminate. Then:

$$A^c(i) = \langle p^A_{\notin}(i), p^A_{\in}(i), p^A_I(i)\rangle.$$

So the complement of a neutrosophic set in consideration is formed by simply interchanging the probabilities of membership and non-membership for all $i \in \mathbb{N}$. Notice that the probability of indeterminacy remains the same. Our next observation is as follows. Suppose that A and A^c are neutrosophic subsets of natural numbers and f is a criterion function. If $\mathcal{E}_A \subset \mathcal{E}_{A^c}$, then clearly $p_{\notin}(i) \geq p_{\in}(i)$ for all $i \in \mathbb{N}$.

The probability distribution of members of a neutrosophic set can be also be given by a function $g(i, j)$ such that $i \in \mathbb{N}$ and $j \in \{1, 2, 3\}$ where j is the index for denoting the *membership* probability by 1, *non-membership* probability by 2, and *indeterminacy* probability by 3, respectively. For instance, for $i \in \mathbb{N}$, $g(i, 2)$ denotes the probability of the non-membership of i generated by the function g. Now g being a computable function means, for any i, j, there is an algorithm to find the value of $g(i, j)$. We give the following theorem.

Theorem 4. *Let A be a neutrosophic set. If g is a computable function, then there exists a computable criterion function f such that $\mathcal{E}_A$ is the enumeration set of A under the criterion function f and, moreover, $\mathcal{E}_A = \mathbb{N}$.*

Proof. Suppose that we are given A. If g is a computable function that generates the probability distribution of members of A, then we can computably find $g(i, 1) = k$. We then simply let $f(i)$ be some $m \leq k$. Since $p_{\in}(i) = k \geq f(i)$, every i will be a member of $\mathcal{E}_A$. Since i is arbitrary, $\mathcal{E}_A = \mathbb{N}$. □

We say that a function f *majorizes* a function g if $f(x) > g(x)$ for all x. Suppose now that g is a quickly growing function in the sense that it majorizes every computable function. That is, assume that $g(i, j) \geq f(i)$ for every $i, j \in \mathbb{N}$ and every computable function f. Now in this case g is necessarily non-computable. Otherwise we would be able to construct a function h where $h(i)$ is chosen to be some $s > t$ such that $g(i, j)$ is defined at step t. So if g is not computable, we cannot apply the previous theorem on g. The only way to enumerate A is by using relative computability rather than giving a plain computable procedure. Suppose that we are given such a function g. Let $\Psi_i(A; i)$ denote the ith Turing machine with oracle A and input i. We define $g' = \{x : \Psi_x(g; x) \downarrow\}$ to be the *jump* of g, where $x = \langle i, j \rangle$ for a uniform pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. The jump of g is basically the halting set relativized to g. If we want to enumerate members of A, we can then use g' as an oracle. Since, by definition, g' computes g, we enumerate members of A computably *in* g'.

We shall also note an observation regarding the relationship between A and A^c. Given a function f, unless $f(i)$ is strictly between $p^A_{\in}(i)$ and $p^A_{\notin}(i)$, we have that $\mathcal{E}_A = \mathcal{E}_{A^c}$. That is, the only case when $\mathcal{E}_A \neq \mathcal{E}_{A^c}$ is if $p^A_{\in}(i) < f(i) < p^A_{\notin}(i)$ or $p^A_{\notin}(i) < f(i) < p^A_{\in}(i)$. Let us examine each case. In the first case, since $f(i) > p^A_{\in}(i)$, i will not be enumerated into $\mathcal{E}_A$, but since $p^{A^c}_{\in}(i) > f(i)$, it will be enumerated into $\mathcal{E}_{A^c}$. The second case is just the opposite. That is, i will be enumerated into $\mathcal{E}_A$ but not into $\mathcal{E}_{A^c}$.

What about the cases where i is enumerated into both enumeration sets? It depends on how we allow our criteria function to operate over probability distributions. If we only want to enumerate those elements i such that $p_{\in}(i) \geq f(i)$, then we may have equal probability distribution among membership and non-membership attributes. We may have that $p_{\in}(i) = p_{\notin}(i) = 0.5$ and $p_I(i) = 0$. In this case, we get to enumerate i both into $\mathcal{E}_A$ and $\mathcal{E}_{A^c}$. However, if we allow the criterion function to operate in a way that i is enumerated if and only if $p_{\in}(i) > f(i)$, then it must be the case that $p_{\notin}(i) < f(i)$ so i will only be enumerated into $\mathcal{E}_A$.

The use of the criterion function may vary depending on the application and which aspect of the given neutrosophic set we want to analyze.

4. Conclusions

We introduced the neutrosophic counterpart of oracle Turing machines with neutrosophic values allowed in the oracle tape. For this we presented a new type of oracle tape where each cell contains a triplet of three probability values, namely for the membership, non-membership, and indeterminacy. The notion of neutrosophic oracle Turing machine is interesting in its own right since oracle information is used in relative computability of sets and enables us to investigate the computability theoretic properties of sets relative to one another. In this paper, we also introduced a method to enumerate the elements of a neutrosophic subset of natural numbers. For this we defined a criterion function to choose elements which satisfy a certain probability degree. This defines a method that can be used in many applications of neutrosophic sets, particularly in decision making problems, solution space searching, and many more. We proved some results about the relationship between the enumeration sets of a given neutrosophic subset of natural numbers and the criterion function. A future work of this study is to investigate the properties of equivalence classes induced by the operator $\equiv_N$. We may call this equivalence class, neutrosophic degree of computability. It would be interesting to study the relationship between neutrosophic degrees of computability and classical Turing degrees. The results also arise further developments in achieving of new generation of computing machines such as fuzzy cellular nonlinear networks paradigm or the memristor-based cellular nonlinear networks [9]. The latter of course has practical benefits.

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Neutrosophic Soft Set Decision Making for Stock Trending Analysis

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Abstract

In this paper, we point out a major issue of stock market regarding trending scenario of trades where data exactness, accuracy of expressing data and uncertainty of values (closing point of the day) are lacked. We use neutrosophic soft sets (NSS) consisting of three factors (True, Uncertainty and False) to deal with exact state of data in several directions. A new approach based on NSS is proposed for stock value prediction based on real data from last 7 years. It calculates the stock price based on the factors like "open", "high", "low" and "adjacent close". The highest score value retrieved from the score function is used to determine which opening price and high price decide the closing price from the above mentioned four factors.

Keywords Neutrosophic soft sets · Soft sets · Stock trending · Stock parameters · Open · Close · High · Low · Adjacent close

1 Introduction

Many fields may not be effectively demonstrated by traditional expression since vulnerability is excessively muddled. They can be demonstrated by various distinctive methodologies including the likelihood theory, fuzzy set (FS) (Zadeh 1965), rough set (Pawlak 1982), neutrosophic set (NS) (Smarandache 2005) and soft set (Molodtsov 1999). NSs can deal with uncertain and conflicting data, which exist, regularly in conviction frameworks (Wang et al. 2005). In this manner, Maji proposed neutrosophic delicate set with operations, which is free of the challenges specified (Maji 2013). He additionally, connected to basic leadership issues (Maji 2012). After Maji, the investigations on the neutrosophic delicate set theory have been considered progressively (Broumi 2013; Broumi and Smarandache 2013). From scholastic perspective, the NS should be determined on the grounds that are connected to genuine applications (Deli 2017). In NS, indeterminacy is evaluated expressly and other membership degrees are free. This presumption is essential in numerous applications, for example, data combination in which the information is consolidated from various sensors. As of late, NSs had for the most part been connected to picture preparing (Cheng and Guo 2008; Guo and Cheng 2009) (Table 1).

Along these lines, Wang et al. (2010) proposed a single-valued NS (SVNS) and set-theoretic operations and properties. Ye (2013a) proposed similarity measures between interim NSs and connected them to multi-criteria decision-making issues. On one hand, a SVNS is an example of a NS, which gives us an extra probability to deal with vulnerability, inadequacy and conflicting data. It would be more appropriate to apply uncertain and conflicting data measures in decision-making. In any case, the connector in the FS is portrayed concerning *T*, i.e. participation just; hereafter the information of indeterminacy and non-enrollment is lost. While in the SVNS, they can be characterized regarding any of them. Thus, the idea of SVNSs is broader and overcomes the previously mentioned issues. Then, SVNSs can be utilized for applications to handle dubious, imprecision and conflicting data. Because of its capacity, SVNS is reasonable for catching loose, indeterminate, and conflicting data in the multi-criteria decision-making.

Broumi and Smarandache (2013) presented correlation coefficients of interval valued NS (INS). Ye (2013b) exhibited the correlation coefficient of single-valued NSs (SVNSs). Ye (2014a, b) presented the idea of streamlined NSs (SNSs), which are a subclass of NSs, and characterized operational laws of SNSs. The authors proposed some accumulation administrators, including a rearranged Neutrosophic weighted number juggling normal administrator and an improved neutrosophic weighted geometric normal administrator. Peng et al. (2014, 2015) showed another outranking approach for multi-criteria decision-making (MCDM) to an improved neutrosophic condition, where reality enrollment degree, indeterminacy-participation degree and misrepresentation participation degree for every component are singleton sub-sets in Zadeh (1965). Ma et al. (2017) proposed cosine similarity measures of SNSs whereas Deli and Şubaş (2017) presented a medicinal treatment determination technique in view of an interval neutrosophic phonetic condition, in which criteria and decision-producers are doled out various levels of need. Ye (2015) introduced a philosophy for tackling multi-trait decision-making issues with SVN-numbers. Pramanik et al. (2017) proposed new vector similitude measures of single valued and interval NSs by hybridizing the ideas of Dice and cosine closeness measures. Mostly often, we see that data lack exactness, lack accuracy of expression. It is indeed necessary to use NS and its extension for dealing with these factors (Thanh et al. 2018a, b; Ali et al. 2017, 2018a, b, c; Nguyen et al. 2018; Broumi et al. 2017a, b; Dey et al. 2018; Thao et al. 2018; Thong et al. 2018; Son et al. 2017; Son and Thong 2017; Son 2015, 2016, 2017; Thong and Son 2015, 2016a, b, c; Angelov and Sotirov 2015).

This paper proposes a model for stock trend prediction based on neutrosophic soft set (NSS). Sections 2 and 3 present preliminary and the proposal. Sections 4 and 5 dive discussion, conclusions and further research respectively.

2 Background

Definition 1 (Molodtsov 1999) A soft set is defined as below,

$$f : S \to \text{power\_of (U)}$$

where, *U* and *S* are the universal and soft sets having parameters like *R* = redundancy contradiction, *Inc* = inconsistency, *In* = incompleteness, U_n = uncertainty, *V* = vagueness, *A* = ambiguity, and *I* = imprecision undefined. Here *f* is a mapping function to *S*.

Definition 2 (Molodtsov 1999) Let *T* be the fuzzy soft set and ך be the Cross so that the Fusion and Cross can be defined as

(Ʈ∪ T)(*x*) = Ʈ(*x*)∨ T(*x*)
(Ʈ∩ T)(*x*) = Ʈ(*x*)∧ T(*x*)
T ⊆Ʈ
if T(*x*) ≤ Ʈ(*x*),*x*∈*U*

where *T* denotes the family of fuzzy soft sets T (upper case), and ך is the Cross to define the Fusion.

Definition 3 (Molodtsov 1999) If (*f*, X) and (*f*′, Y) ∈ U then (*f*, X) is fuzzy soft subset of (*f*′, Y) or (*f*, X) ⊆ (*f*′, Y) if

1. X ⊆ Y
2. f(a) ≤ f(a), a ∈ A.

Table 1 Neutrosophic soft set (*f*, A) representing the stock-trending

ST	High price	Low price	Adj close price
Para1	(35.57571, 35.07286, 0, 35.56)	(37.17857, 36.53286, 37.03143)	(38.10714, 36.6, 38.06714)
Para2	(35.87714, 34.93572, 35.34)	(37.61286, 36.67857, 36.95429)	(36.87714, 35.53286, 36.57)
Para3	(35.41286, 34.53857, 35.29572)	(36.89286, 28.46429, 35.17857)	(35.22429, 32.17286, 33.69429)
Para4	(35.60714, 34.70857, 34.94143)	(36.37857, 35.50428, 36.28429)	(37.12714, 35.78571, 36.64571)

where (f, X) is fuzzy soft set. The difference between (f, A) and (f, X) is that (f, A) represents Universal Neutrosophic Set whereas (f, X) represents the fuzzy soft set.
All neutrosophic sets of X are denoted f_N (X).

Definition 4 (Wang et al. 2010) NSS *X* is contained in NSS *Y* if

1. $X \subseteq Y$
2. $TX(x) \le TY(x)$, $IX(x) \le IY(x)$, $FX(x) \ge FY(x)$ for all x $\in$ X.

Definition 5 (Wang et al. 2010) The complement of NSS is a NSS (fc, ¬ X):

1. f_c: $\neg A \to f N(X)$,
2. $f c(a) = < x, Tf\ c\ (x) = ff\ (x), If\ c\ (x) = 1 - If\ (x), ff\ c\ (x) = Tf\ (x)>$, a $\in$ A and x $\in$ X.

Definition 6 (Wang et al. 2010) Assume $Z = < x, TX(x), IX(x), f''X(x) >$ and $A \le x, T_A(x), I_A(x), f''A(x) >$ is NSS. We have

$X \vee A \le x$, $\max(TX(x), T_A(x))$, $\max(IX(x), I_A(x))$, $\min(f''X(x), f''A(x))>$

$X \wedge A \le x$, $\min(TX(x), T_A(x))$, $\min(IX(x), I_A(x))$, $\max(f''X(x), f''A(x))>$

Definition 7 (Wang et al. 2010) A set Z - ¬(f, X) is said to be non-empty over U if $T_f(a) = 0, I_f(a) : 0, f_f(a) : 1, a \in A$.

Definition 8 (Wang et al. 2010) Fuzzy set (f, A) is a Universal Neutrosophic Set over *U* if $T_f(a) = 1$, $I_f(a) : 1$, $F_f(a) : 0$ for all a $\in$ A.where (f, A) represents a fuzzy set over the universal neutrosophic set.

3 Proposed model

We now derive Fusion, Cross and Structure as below:

(a) Fusion of NSSs (f, A) and (f'', B) is (H, C) = (f, A) $\cup$ (f'', B) over U with $C = A \cup B$ and H(C) = f(C) if c $\in$ A\B.
(b) Cross of NSSs (f, A) and (g, B) is (H, C) = (f, A) $\cap$ (g, B) with $C = A \cap B$, $H(c) = F(c) \wedge G(c)$, c $\in$ C.
(c) Relation is computed by the following steps:

1. Let $L \subseteq A \times B$: $L_e(a, b) = f(a) \wedge f''(b)$, a $\in$ A, b $\in$ B, L_e: K $\to$ FN (U).
2. L_{f1} in relation with (f, A) to (f'', B) and L_{f2} in relation with (f, B) to (f'', C). Then, the composition of relations L_{f1} and L_{f2} is defined by $(L_{f1} \circ L_{f2})(a, c) = L_{f1}(a, b) \wedge L_{f2}(b, c)$, a $\in$ A, b $\in$ B, c $\in$ C.
3. The *fusion* and *cross* of L_{f1} and L_{f2} of (f, A) and (f'', B) over U are:

$$\text{fusion} \to L_{f1} \cup L_{f2}(a, b) = \text{MAX}\{L_{f1}(a, b), L_{f2}(a, b)\},$$
$$\text{cross} \to L_{f1} \cap L_{f2}(a, b) = \text{MIN}\{L_{f1}(a, b), L_{f2}(a, b)\}.$$

4. The MAX→MIN→MAX composition for set is expressed with the relation $L \circ A$.
5. The associative (AL), non-deterministic (NL) and non-associative (NAL) functions can be derived as below:

$$AL \circ A(y) = \cup x[AL(x) \wedge AA(x, y)],$$
$$NL \circ A(y) = \cup x[NL(x) \wedge NA(x, y)],$$
$$NAL \circ A(y) = \wedge x[NAL(x) \vee NALA(x, y)].$$

6. (f, A) conclusively can be defined as: V (F, A) = TA + (1 − UA)—FA where TA → *True* value, UA → *Uncertain* value and FA → denotes the *False* value. The TA, UA and FA are the values with respect to (F, A).
7. Score function → S1 = V (F, A) − V (G, B). The score function for (L,A) → $TA_i - Ua_i * FA_i$.

Now, trade trends are identified using NSSs with variables below: *Date, Open*—opening price of particular date, *High*—highest price at particular date, *Low*—lowest price at particular date, *Close*—closing price at particular date, *Adj Close*—adj. close price at particular date, *Volume*—volume of stock traded.NSS is applied for identification, detection and determination of which stock is getting affected from various parameters. The effect *E* relates to a closing price *C*.

NSS Algorithm for determining the decision for closing of stock trending
Input: variables
Output: Actual and Predicated Values
Algorithm:

1. *Split the data into train and test NSS data*
2. *The Date is the features and the Open price is our target values which need to be predicted*
   X(features) = [Year, Month, Day], y(target) = [Open]
3. *Feature scaling is done on data for faster convergence rate of algorithms and to maintain standardization in data*
4. *Function to plot graph is written which takes dataset as parameters and plots the graph*
5. *Train the model using Regression*
6. *The Train data are fitted into the algorithm*
7. *The model's accuracy is then computed by giving Test data as input and evaluating the predicted result against the known value from Test data*
8. *The graph is plotted between Actual value vs. Predicted value*

Table 2 Obtaining the relation *L*

Factors	Open	High	Low	Close	Adj close
Para1	12.327143	12.368571	11.7	11.971429	10.770167
Para2	12.007143	12.278571	11.97	12.237143	11.00922
Para3	12.252857	12.314285	12.05	12.15	10.930819
Para4	12.28	12.361428	12.18	12.21	10.984798

Table 3 Performing the transformation operation using relation *T*

Factors	Open	High	Low	Close
Para1	12.444285	12.642858	12.29	12.321428
Para2	12.444285	12.481428	12.141429	12.197143
Para3	12.328571	12.378572	12.218572	12.277143
Para4	12.347143	12.355714	12.178572	12.221429

4 Result and discussion

There are sets of 7 years which include the various parameters to predict the next day opening value (https://in.finance.yahoo.com). The parameters achieved here are High (highest price at particular date), Low (lowest price at particular date) and Adj Close (adj. close price at particular date) Let the possible reasons relating to these ups and downs be Gold Price (g_1), Petrol Price (pp_1), Interest Rates (ir_1), the value of the US dollar (USD_1) and Economic/Political shocks (sh_1). The following demonstrates the processing of the algorithm and achieved results.

1. Parameters of stock are sent to association *Q*.
2. Trading relating to open and closing is given in Table 2.
3. Structure *T* of "open and close" and "high and low" is found in Step 5 (Table 3).
4. Supplement of Table 1 is shown in Table 4.
5. Supplement of Table 2 is in Table 5.
6. Estimation of Tables 4 and 5 is in Table 6.
7. Outputs of Tables 3 and 6 are given in Tables 7 and 8 independently.
8. Score associated for the qualities in Tables 7 and 8 is found in Table 9.
9. Score for Table 3 is in Table 10.
10. Find highest score for stock closing affected by different opening and closing.

From all the columns, we extract Date and Open from the table to make a new dataset.

Table 4 The complement of $Q \circ R$

ST	High price	Low price	Adj close price
Para1	(37.02143, 35.79286, 35.84857, 32.25138)	(36.25714, 35.09286, 35.61857, 32.04446)	(35.98571, 34.64143, 34.74286, 31.25662)
Para2	(35.85429, 34.6, 35.78714, 32.19612)	(36.26571, 35.33857, 36.21571, 32.58168)	(37.02143, 36.28714, 36.32571, 32.68065)
Para3	(37.12143, 36.5, 37.09857, 33.37596)	(38.25, 37.23286, 38.17857, 34.34757)	(38.98571, 38.5, 38.83857, 34.94136)
Para4	(39.28571, 38.77428, 39.15286, 35.22411)	(39.85857, 38.39, 38.59572, 34.72287)	(39.42429, 38.78571, 39.12143, 35.19582)

Table 5 The complement of R′

R′	High	Low	Close	Adj close
Para1	(39.23714, 38.27143, 38.71)	(39.02857, 38.3, 38.42857)	(38.61, 37.97286, 38.1)	(12.184286, 12.042857, 12.1, 10.885842)
Para2	(39.23714, 38.27143, 38.71)	(39.02857, 38.3, 38.42857)	(38.61, 37.97286, 38.1)	(12.184286, 12.042857, 12.1, 10.885842)
Para3	(12.234285, 12.081429, 12.185715, 10.962953)	(12.231428, 12.111428, 12.172857, 10.951384)	(12.201428, 12.094286, 12.118571, 10.902546)	(12.308572, 12.022857, 12.271428, 11.040065)
Para4	(12.784286, 12.28, 12.742857, 11.46419)	(12.972857, 12.647142, 12.787143, 11.504031)	(12.905714, 12.692857, 12.724286, 11.447481)	(12.857142, 12.515715, 12.644286, 11.375507)

Table 6 Composition values of Tables 4 and 5

R′	High	Low	Close	Adj close
Para1	(12.44, 11.915714, 11.99, 10.786877)	(12.228572, 11.857142, 12.087143, 10.874274)	(12.615714, 11.964286, 12.437143, 11.189151)	(12.505714, 12.172857, 12.201428, 10.977087)
Para2	(12.664286, 12.251429, 12.331429, 11.094046)	(12.615714, 12.485714, 12.598572, 11.334381)	(12.71, 12.492857, 12.531428, 11.273974)	(12.674286, 12.494286, 12.571428, 11.309962)
Para3	(12.692857, 12.485714, 12.567142, 11.306105)	(12.855714, 12.57, 12.838572, 11.550298)	(12.201428, 12.094286, 12.118571, 10.902546)	(12.308572, 12.022857, 12.271428, 11.040065)
Para4	(12.784286, 12.28, 12.742857, 11.46419)	(12.972857, 12.647142, 12.787143, 11.504031)	(12.905714, 12.692857, 12.724286, 11.447481)	(12.857142, 12.515715, 12.644286, 11.375507)

After decomposition of date			
Year (YYYY) integer	Month (MM) integer	Day (DD) integer	Open (float)

Now, let us take Para = {para1, para2, para3, para4} where para1 = g_1, para2 = pp_1, para3 = ir_1, and para4 = USD_1 as the universal set where g_1, pp_1, ir_1, USD_1 sh_1 are Gold Price (g_1), Petrol Price (pp_1), Interest Rates (ir_1), the value of the US dollar (USD_1) and Economic/Political shocks (sh_1) as explained in the previous statements.

Next, consider the set *K = {k1, k2, k3, k4}* as universal set where *k1, k2, k3, k4* represent consequence/results like open, high, low and adj close respectively and the set $F = \{f_1, f_2, f_3, f_4\}$ where f_1, f_2, f_3 and f_4 represent the faster convergence rate, actual value, trending value and predicted value respectively.

We construct stock-trending relation and trending-close relation as follows:

F (para1) = {k1/(0.7, 0.4, 0.1), k2/(0.8, 0.6, 0.7), k3/(0.4, 0.8, 0.5)},

F (para2) = {k1/(0.6, 0.5, 0.3), k2/(0.6, 0.5, 0.2), k3/(0.7, 0.9, 0.0)},

F (para3) = {k1/(0.8, 0.4, 0.2), k2/(0.5, 0.1, 0.5), k3/(1.0, 0.5, 1.0)},

F (para4) = {k1/(0.4, 0.6, 0.3), k2/(0.5, 0.4, 0.8), k3/(0.5, 0.6, 0.9)}.

(*f*, A) results into a collection of generalized stock-trending in the stock market. It represents stock-trending relation given by,

Next, G(k1) = {f1/(37.59,36.95714,37.44143), f2/(37.85714,36.62857,36.90857), f3/(36.64,35.64286,36.26), f4/(34.92857,33.05,34.61714)}, G(k2) = {f1/(36.59714,35.38714,36.31714), f2/(36.93572,35.75143,36.05143), f3/(34.83571,33.74429,33.96571), f4/(36.13143,34.97857,35.47714)}, G(k4) = {f1/(37.82857,37.19,37.70714,33.92345), f2/(37.93572,37.20153,37.58857,33.81678), f3/(37.41429,36.37571,36.56572,32.89656)}. We realize *{G(f1), G(f2), G(f3)}* of all *S* where $G: S \to FN(D)$. Thus, *(G, S)* is represented by a relation matrix (stock_high-stock_lowmatrix) R given in Table 2.

The trading knowledge relating the parameters with the set of open and closing values under consideration is in Table 2. We perform transformation operation $Q \circ R$ to get the stocks' high and low value relation in Table 3.

Likewise, $Q \circ R$ is calculated to give the stocks' high and low value relation *T*. These set of values are now complimented in Tables 4 and 5. The composition values are calculated in Table 6.

Table 6 uses the composition of relations L_{f1} and L_{f2} which are defined by,

$(L_{f1} \circ L_{f2})(a, c) = L_{f1}(a, b) \wedge L_{f2}(b, c),\ a \in A,\ b \in B,\ c \in C.$

Value functions for Tables 3 and 6 are calculated in Tables 7 and 8.

Table 7 The value function for cross of $L_{f1}(a, b) \wedge L_{f2}(b, c)$

Value	High	Low	Close	Adj close
Para1	12.78, 12.28	12.74	11.46	Para1
Para2	12.97	12.64	12.78	11.50
Para3	12.90	12.69	12.72	11.44
Para4	12.85	12.51	12.64	11.37

Table 8 The value function for fusion of $L_{f1}(a, b) \cup L_{f2}(b, c)$

Value	High	Low	Close	Adj close
Para1	13.68	12.38	12.34	12.46
Para2	12.79	12.46	12.78	12.39
Para3	12.60	12.46	12.27	11.44
Para4	12.42	12.24	12.46	11.37

Table 9 Score function for Tables 7 and 8

Value	High	Low	Close	Adj close
Para1	−0.01	−0.1	−0.04	0.3
Para2	0.4	0.3	0.7	0.6
Para3	0.1	0.3	0.5	0.5
Para4	−0.7	−0.03	−0.04	0

Table 10 Score function for the Table 3

Value	High	Low	Close	Adj close
Para1	0.14	0.3	0.35	0.65
Para2	0.46	0.46	0.55	0.45
Para3	0.55	0.55	0.75	0.38
Para4	0.08	0.2	0.18	0.38

Score function $(L,A) \rightarrow TA_i - Ua_i * FA_i$ for Table 3 is in Table 10. As explained earlier, the score function comprises of values of high, low, close and Adjclose. The closest score function is depicted below.

It is clear from Tables 9 and 10 that stock price at para1 and para4 are absolute alteration due to *k1, k2, k3, k4* represent consequence/results like open, high, low and adj close respectively.

From Table 3, $Q \circ R$ is performed to give the stocks' high and low value relation. These set of values are now complimented in Tables 4 and 5.

5 Conclusion

This paper applied neutrosophic soft sets to predict the stock price. Based upon the factors like open, high, low and adj close, and the score value, we have developed a technique to determine which opening price and high price decide the closing price from what factors. Since there is no competing interest exists in the field of applied NSS, there are various scopes using fuzzy theory to determine the predictability of stock parameterized values at the specific time. In our work, we have focused on the value for the opening and closing points. The work can be extended to trace the decision at any point of time. Yet, it indeed needs huge datasets for testing the model rigorously. We keep this as the reference for the future.

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Neutrosophic triplet group

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Abstract Groups are the most fundamental and rich algebraic structure with respect to some binary operation in the study of algebra. In this paper, for the first time, we introduced the notion of neutrosophic triplet which is a group of three elements that satisfy certain properties with some binary operation. These neutrosophic triplets highly depends on the defined binary operation. Further, in this paper, we utilized these neutrosophic triplets to introduce the innovative notion of neutrosophic triplet group which is completely different from the classical group in the structural properties. A big advantage of neutrosophic triplet is that it gives a new group (neutrosophic triplet group) structure to those algebraic structures which are not group with respect to some binary operation in the classical group theory. In neutrosophic triplet group, we apply the fundamental law of Neutrosophy that for an idea A, we have neutral of A denoted as *neut*(a) and anti of A denoted as *anti*(A) to capture this beautiful picture of neutrosophic triplet group in algebraic structures. We also studied some interesting properties of this newly born structure. We further defined neutro-homomorphisms for neutrosophic triplet groups. A neutron-homomorphism is the generalization of the classical homomorphism with two extra conditions. As a further generalization, we gave rise to a new field or research called Neutrosophic Triplet Structures (such as neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, etc.). In the end, we gave main distinctions and comparison of neutrosophic triplet group with the classical Molaei's generalized group as well as the possible application areas of the neutrosophic triplet groups.

Keywords Groups · Homomorphism · Neutrosophic triplet · Neutrosophic triplet group · Neutro-homomorphism

1 Introduction

Neutrosophy is a new branch of philosophy which studies the nature, origin and scope of neutralities as well as their interaction with ideational spectra. Florentin Smarandache [8] in 1995, first introduced the concept of neutrosophic logic and neutrosophic set where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic especially to intuitionistic fuzzy logic. In fact neutrosophic set is the generalization of classical sets [9], fuzzy set [12], intuitionistic fuzzy set [1, 9] and interval valued fuzzy set [9], etc. This mathematical tool is used to handle problems consisting uncertainty, imprecision, indeterminacy, inconsistency, incompleteness and falsity. By utilizing the idea of neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache studied neutrosophic algebraic structures in [4–6] by inserting an indeterminate element "I" in the algebraic structure and then combine "I" with each element of the structure with respect to corresponding binary operation *. They call it neutrosophic element, and the generated algebraic structure is then termed as neutrosophic algebraic structure. They further study several neutrosophic algebraic structures such as neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids and neutrosophic bigroupoids and so on.

Groups [2, 3, 11] are so much important in algebraic structures as they play the role of back bone in almost all algebraic structures theory. Groups are thought as old algebra due to its rich structure than any other notion. In many algebraic structures, groups provide concrete foundation such as, rings, fields, vector spaces, etc. Groups are also important in many other areas like physics, chemistry, combinatorics, biology, etc., to study the symmetries and other behavior among their elements. The most important aspect of a group is group action. There are many types of groups such as permutation groups, matrix groups, transformation groups, lie groups, etc., which are highly used as a practical perspective in our daily life. Generalized groups [7] are important in this aspect.

In this paper, for the first time, we introduced the idea of neutrosophic triplet. The newly born neutrosophic triplets are highly dependable on the proposed binary operation. These neutrosophic triplets have been discussed by Smarandache and Ali in Physics [10]. Moreover, we utilized these neutrosophic triplets to introduce neutrosophic triplet group which is different from the classical group both in structural and foundational properties from all aspects. Furthermore, we gave some interesting and fundamental properties and notions with illustrative examples. We also introduced a new type of homomorphism called as neutro-homomorphism which is in fact a generalization of the classical homomorphism under some conditions. We also study neutro-homomorphism for neutrosophic triplet groups. The rest of the paper is organized as follows. After the literature review in Sect. 1, we introduced neutrosophic triplets in Sect. 2. Section 3 is dedicated to the introduction of neutrosophic triplet groups with some of its interesting properties. In Sect. 4, we developed neutron-homomorphism, and in Sect. 5, we gave distinction and comparison of neutrosophic triplet group with the classical Molaei's generalized group. We also draw a brief sketch of the possible applications of neutrosophic triplet group in other research areas. Conclusion is given in Sect. 6.

2 Neutrosophic triplet

Definition 2.1 Let N be a set together with a binary operation *. Then, N is called a neutrosophic triplet set if for any $a \in N$, there exist a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a,$$

and

$$a * anti(a) = anti(a) * a = neut(a).$$

The elements a, $neut(a)$ and $anti(a)$ are collectively called as neutrosophic triplet, and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean *neutral* of a and apparently, a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet. For the same element *"a"* in N, there may be more neutrals to it *neut(a)* and more opposites of it *anti(a).*

Definition 2.2 The element b in $(N, *)$ is the second component, denoted as $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).

Definition 2.3 The element c in $(N, *)$ is the third component, denoted as $anti(\cdot)$, of a neutrosophic triplet, if there exist other elements a and b in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).

Example 2.4 Consider Z_6 under multiplication modulo 6, where

$$Z_6 = \{0, 1, 2, 3, 4, 5\}$$

Then, 2 gives rise to a neutrosophic triplet because $neut(2) = 4$, as $2 \times 4 = 8 \equiv 2(\text{mod}6)$. Also $anti(2) = 2$ because $2 \times 2 = 4$. Thus, $(2, 4, 2)$ is a neutrosophic triplet. Similarly 4 gives rise to a neutrosophic triplet because $neut(4) = anti(4) = 4$. So $(4, 4, 4)$ is a neutrosophic triplet. 3 does not give rise to a neutrosophic triplet as $neut(3) = 5$, but $anti(3)$ does not exist in Z_6, and last but not the least 0 gives rise to a trivial neutrosophic triplet as $neut(0) = anti(0) = 0$. The trivial neutrosophic triplet is denoted by $(0, 0, 0)$.

Theorem 2.5 *If $(a, neut(a), anti(a))$ form a neutrosophic triplet, then*

1. *$(anti(a), neut(a), a)$ also form a neutrosophic triplet, and similarly*
2. *$(neut(a), neut(a), neut(a))$ form a neutrosophic triplet.*

Proof We prove both 1 and 2.

1. Of course, $anti(a) * a = neut(a)$.

We need to prove that: $anti(a) * neut(a) = anti(a)$. Multiply by a to the left, and we get:

$a * anti(a) * neut(a) = a * anti(a)$

or

$[a * anti(a)] * neut(a) = neut(a)$

or

$neut(a) * neut(a) = neut(a).$

Again multiply by a to the left and we get:

$a * neut(a) * neut(a) = a * neut(a)$

or

$[a * neut(a)] * neut(a) = a$

or
$a * neut(a) = a.$

2. To show that ($neut(a)$, $neut(a)$, $neut(a)$) is a neutrosophic triplet, it results from the fact that
$neut(a) * neut(a) = neut(a).$

3 Neutrosophic triplet group

Definition 3.1 Let (N, *) be a neutrosophic triplet set. Then, N is called a neutrosophic triplet group, if the following conditions are satisfied.

(1) If (N, *) is well-defined, i.e. for any a, $b \in N$, one has $a * b \in N$.
(2) If (N, *) is associative, i.e. $(a * b) * c = a * (b * c)$ for all a, b, $c \in N$.

The neutrosophic triplet group, in general, is not a group in the classical algebraic way.

We consider, as the neutrosophic neutrals replacing the classical unitary element, and the neutrosophic opposites as replacing the classical inverse elements.

Example 3.2 Consider (Z_{10}, #), where # is defined as $a\#b = 3ab(\text{mod}10)$. Then, (Z_{10}, #) is a neutrosophic triplet group under the binary operation # with the following table (Tables 1, 2).

It is also associative, i.e.

$(a\#b)\#c = a\#(b\#c)$

Now take L. H. S to prove the R. H. S, so

$(a\#b)\#c = (3ab)\#c,$

$= 3(3ab)c = 9abc,$

$= 3a(3bc) = 3a(b\#c),$

$= a\#(b\#c).$

For each $a \in Z_{10}$, we have $neut(a)$ in Z_{10}. That is $neut(0) = 0$, $neut(1) = 7$, $neut(2) = 2$, $neut(3) = 7$, $neut(4) = 2$, and so on.

Table 1 Cayley table of neutrosophic triplet group (Z_{10}, #)

#	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

Table 2 Cayley table of a non-commutative neutrosophic triplet group (Z_{10}, *)

*	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	5	6	7	8	9	0	1	2	3	4
2	0	1	2	3	4	5	6	7	8	9
3	5	6	7	8	9	0	1	2	3	4
4	0	1	2	3	4	5	6	7	8	9
5	5	6	7	8	9	0	1	2	3	4
6	0	1	2	3	4	5	6	7	8	9
7	5	6	7	8	9	0	1	2	3	4
8	0	1	2	3	4	5	6	7	8	9
9	5	6	7	8	9	0	1	2	3	4

Similarly, for each $a \in Z_{10}$, we have $anti(a)$ in Z_{10}. That is $anti(0) = 0$, $anti(1) = 9$, $anti(2) = 2$, $anti(3) = 3$, $anti(4) = 1$, and so on. Thus, (Z_{10}, #) is a neutrosophic triplet group with respect to #.

Definition 3.3 Let (N, *) be a neutrosophic triplet group. Then, N is called a commutative neutrosophic triplet group if for all a, $b \in N$, we have $a * b = b * a$.

Example 3.4 Consider (Z_{10}, *), where * is defined as $a * b = 5a + b(\text{mod}10)$ for all a, $b \in Z_{10}$. Then, (Z_{10}, *) is the neutrosophic triplet group which is given by the following Table 2: Then, $(\mathbb{Z}_{10}, *)$ is a non-commutative neutrosophic triplet group.

Theorem 3.5 *Every idempotent element gives rise to a neutrosophic triplet.*

Proof Let a be an idempotent element. Then, by definition $a^2 = a$. Since $a^2 = a$, which clearly implies that $neut(a) = a$ and $anti(a) = a$. Hence a gives rise to a neutrosophic triplet

$(a, a, a).$

Theorem 3.6 *There are no neutrosophic triplets in Z_n with respect to multiplication if n is a prime.*

Proof It is obvious.

Remark 3.7 Let (*N*, *) be a neutrosophic triplet group under * and let $a \in N$. Then, *neut*(*a*) is not unique in *N,* and also *neut*(*a*) depends on the element *a* and the operation *. To prove the above remark, let's take a look to the following example.

Example 3.8 Let $N = \{0, 4, 8, 9\}$ be a neutrosophic triplet group under multiplicationmodulo 12 in (Z_{12}, $\times$). Then $neut(4) = 4$, $neut(8) = 4$ and $neut(9) = 9$. This shows that*neut*(*a*) is not unique.

Remark 3.9 Let (*N*, *) be a neutrosophic triplet group with respect to * and let $a \in N$. Then, *anti*(*a*) is not unique in *N* and also *anti*(*a*) depends on the element *a* and the operation *. To prove the above remark, let's take a look to the following example.

Example 3.10 Let *N* be a neutrosophic triplet group in above example. Then, $anti(4) = 4$, $anti(8) = 8$ and $anti(9) = 9$.

Proposition 3.11 *Let* (*N*, *) *be a neutrosophic triplet group with respect to * and let*

$a, b, c \in N$.

then

(1) $a * b = a * c$ *if and only if* $neut(a) * b = neut(a) * c$.
(2) $b * a = c * a$ *if and only if* $b * neut(a) = c * neut(a)$.

Proof

1. Suppose that $a * b = a * c$. Since *N* is a neutrosophic triplet group, so $anti(a) \in N$. Multiply *anti*(*a*) to the left side with $a * b = a * c$.

$anti(a) * a * b = anti(a) * a * c$

$[anti(a) * a] * b = [anti(a) * a] * c$

$neut(a) * b = neut(a) * c$

Conversely suppose that $neut(a) * b = neut(a) * c$.Multiply *a* to the left side, we get:

$a * neut(a) * b = a * neut(a) * c$

$[a * neut(a)] * b = [a * neut(a)] * c$

$a * b = a * c$

2. The proof is similar to 1.

Proposition 3.12 *Let* (*N*, *) *be a neutrosophic triplet group with respect to * and let*a, b, $c \in N$.

1. *If* $anti(a) * b = anti(a) * c$, *then* $neut(a) * b = neut(a) * c$.
2. *If* $b * anti(a) = c * anti(a)$, *then* $b * neut(a) = c * neut(a)$.

Proof

1. Suppose that $anti(a) * b = anti(a) * c$. Since *N* is a neutrosophic triplet group with respect to *, so $a \in N$. Multiply *a* to the left side with $anti(a) * - b = anti(a) * c$, we get:

$a * anti(a) * b = a * anti(a) * c$

$[a * anti(a)] * b = [a * anti(a)] * c$

$neut(a) * b = neut(a) * c$.

2. The proof is same as (1).

Theorem 3.13 Let (*N*, *) be a commutative neutrosophic triplet group with respect to * and $a, b \in N$. Then

$neut(a) * neut(b) = neut(a * b)$.

Proof Consider left hand side, *neut*(*a*) * *neut*(*b*). Now multiply to the left with *a* and to the right with *b*, we get:

$a * neut(a) * neut(b) * b = [a * neut(a)] * [neut(b) * b]$

$= a * b$.

Now consider right hand side, we have *neut*(*a* * *b*). Again multiply to the left with *a* and to the right with *b*, we get: $a * neut(a * b) * b = [a * b] * [neut(a * b)]$, as * is associative.

$= a * b$.

This completes the proof.

Theorem 3.14 *Let* (*N*, *) *be a commutative neutrosophic triplet group with respect to * and* $a, b \in N$. *Then*

$anti(a) * anti(b) = anti(a * b)$.

Proof Consider left hand side, *anti*(*a*) * *anti*(*b*). Multiply to the left with *a* and to the right with *b*, we get:

$a * anti(a) * anti(b) * b = [a * anti(a)] * [anti(b) * b]$

$= neut(a) * neut(b)$

$= neut(a * b)$, By above theorem.

Now consider right hand side, which is *anti*(*a* * *b*).

Multiply to the left with *a* and to the right with *b*, we get: $a * anti(a * b) * b = [a * b] * [anti(a * b)]$, since * is associative.

$= neut(a * b)$.

This shows that $anti(a) * anti(b) = anti(a * b)$ is true for all $a, b \in N$.

Theorem 3.15 *Let (N, *) be a commutative neutrosophic triplet group under * and*

$a, b \in N$.

then

1. $neut(a) * neut(b) = neut(b) * neut(a)$.
2. $anti(a) * anti(b) = anti(b) * anti(a)$.

Proof

1. Consider right hand side $neut(b)$ * $neut(a)$. Since by Theorem 3, we have

$neut(b) * neut(a) = neut(b * a)$

= $neut(a * b)$, as N is commutative. = $neut(a)$ * $neut(b)$, again by theorem 3. Hence $neut(a)$ * $neut(b) = neut(b)$ * $neut(a)$.

2. On similar lines, one can easily obtained the proof of (2).

Definition 3.16 Let (N, *) be a neutrosophic triplet group under *, and let H be a subset of N. Then, H is called a neutrosophic triplet subgroup of N if H itself is a neutrosophic triplet group with respect to *.

Example 3.17 Consider (Z_{10}, #) be a neutrosophic triplet group in Example 3.2, and $H = \{0, 2, 4, 6, 8\}$ be a subset of Z_{10}. Then, clearly H is a neutrosophic triplet subgroup of Z_{10}.

Proposition 3.18 *Let (N, *) be a neutrosophic triplet group and H be a subset of N. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions hold.*

1. $a * b \in H$ *for all* $a, b \in H$.
2. $neut(a) \in H$ *for all* $a \in H$.
3. $anti(a) \in H$ *for all* $a \in H$.

Proof The proof is straightforward.

Definition 3.19 Let N be a neutrosophic triplet group and let $a \in N$. A smallest positive integer $n \geq 1$ such that $a^n = neut(a)$ is called neutrosophic triplet order. It is denoted by $nto(a)$.

Example 3.20 Let N be a neutrosophic triplet group under multiplication modulo 10 in (Z_{10}, $\times$), where

$N = \{0, 2, 4, 6, 8\}$.

then

$nto(2) = 4$, $nto(4) = 2$,
$nto(6) = 2, nto(8) = 4$.

Theorem 3.21 *Let (N, *) be a neutrosophic triplet group with respect to * and let* $a \in N$. *Then*

1. $neut(a) * neut(a) = neut(a)$.

In general $(neut(a))^n = neut(a)$, *where n is a nonzero positive integer.*

2. $neut(a) * anti(a) = anti(a) * neut(a) = anti(a)$.

Proof

1. Consider $neut(a) * neut(a) = neut(a)$.

Multiply a to the left side, we get;

$a * neut(a) * neut(a) = a * neut(a)$

$[a * neut(a)] * neut(a) = [a * neut(a)]$

$a * neut(a) = a$

$a = a$.

On the same lines, we can see that $(neut(a))^n = neut(a)$ for a nonzero positive integer n.

2. Consider $neut(a) * anti(a) = anti(a)$.

Multiply to the left with a, we get

$a * neut(a) * anti(a) = a * anti(a)$

$[a * neut(a)] * anti(a) = neut(a)$

$a * anti(a) = neut(a)$

$neut(a) = neut(a)$.

Similarly $anti(a) * neut(a) = anti(a)$.

Definition 3.22 Let N be a neutrosophic triplet group and $a \in N$. Then, N is called neutro-cyclic triplet group if $N = \langle a \rangle$. We say that a is a generator part of the neutrosophic triplet.

Example 3.23 Let $N = \{2, 4, 6, 8\}$ be a neutrosophic triplet group with respect to multiplication modulo 10 in (Z_{10}, $\times$). Then, clearly N is a neutro-cyclic triplet group as $N = \langle 2 \rangle$. Therefore, 2 is the generator part of the neutrosophic triplet (2, 6, 8).

Theorem 3.24 *Let N be a neutro-cyclic triplet group and let a be a generator part of the neutrosophic triplet. Then*

1. $\langle neut(a) \rangle$ *generates neutro-cyclic triplet subgroup of N.*
2. $\langle anti(a) \rangle$ *generates neutro-cyclic triplet subgroup of N.*

Proof Straightforward.

4 Neutro-homomorphism

In this section, we introduced neutron-homomorphism for the neutrosophic triplet groups. We also studied some of their properties. Further, we defined neutro-isomorphisms.

Definition 4.1 Let $(N_1, *_1)$ and $(N_2, *_2)$ be two neutrosophic triplet groups. Let

$f : N_1 \to N_2$

be a mapping. Then, f is called neutro-homomorphism if for all $a, b \in N_1$, we have

1.
$$f(a *_1 b) = f(a) *_2 f(b),$$

2.
$$f(neut(a)) = neut(f(a))$$
and

3.
$$f(anti(a)) = anti(f(a)).$$

Example 4.2 Let N_1 be a neutrosophic triplet group with respect to multiplication modulo 6 in $(Z_6, \times)$, where

$N_1 = \{0, 2, 4\}$.

And let N_2 be another neutrosophic triplet group with respect to multiplication modulo 10 in $(Z_{10}, \times)$, where

$N_2 = \{0, 2, 4, 6, 8\}$.

Let $f{:}N_1 \to N_2$ be a mapping defined as

$f(0) = 0, f(2) = 4, f(4) = 6.$

Then, clearly f is a neutro-homomorphism because conditions (1), (2) and (3) are satisfied easily.

Proposition 4.3 *Every neutro-homomorphism is a classical homomorphism by neglecting the unity element in classical homomorphism.*

Proof First we neglect the unity element that classical homomorphism maps unity element to the corresponding unity element. Now suppose that f is a neutro-homomorphism from a neutrosophic triplet group N_1 to a neutrosophic triplet group N_2. Then, by condition (1), it follows that f is a classical homomorphism.

Definition 4.4 A neutro-homomorphism is called neutro-isomorphism if it is one–one and onto.

5 Distinctions and comparison

The distinctions between Molaei's Generalized Group [7] and Neutrosophic Triplet Group are:

1. in MGG for each element there exists a unique neutral element, which can be the group neutral element, while in NTG each element may have many neutral elements, and also the neutral elements have to be different from the unique group neutral element;
2. in MGG the associativity applies, and in NTG the associativity is not required;
3. in MGG there exists a unique inverse of an element, while in NTG there may be many inverses for the same given element;
4. MGG has a weaker structure than NTG.

So far the applications of neutrosophic triplet sets are in Z, modulon, $n \geq 2$. But new applications can be found, for example in social science: One person $< A >$ that has an enemy $\langle anti(A_{d_1})\rangle$ (enemy in a degree d_1 of enemy city), and a neutral person $\langle neut(A_{d_1})\rangle$ with respect to $\langle anti(A_{d_1})\rangle$. Then, another enemy $\langle anti(A_{d_2})\rangle$ in a different degree of enemy city, and a neutral $\langle anti(A_{d_2})\rangle$, and so on. Hence one has the neutrosophic triplets:

$(A, \langle neut(A_{d_1})\rangle, \langle anti(A_{d_1})\rangle),$

$(A, \langle neut(A_{d_2})\rangle, \langle anti(A_{d_2})\rangle)$, and so on.

Then, we take another person B in the same way…

$(B, \langle neut(B_{d_1})\rangle, \langle anti(B_{d_1})\rangle),$

$(B, \langle neut(B_{d_2})\rangle, \langle anti(B_{d_2})\rangle)$

etc.

More applications can be found, if we deeply think about cases where we have neutrosophic triplets $(A, \langle neut(A)\rangle, \langle anti(A)\rangle)$ in technology and in science.

6 Conclusion

Inspiring from the Neutrosophic philosophy, we defined neutrosophic triplet. Basically A neutrosophic triplet in a set is a group of certain elements which satisfy certain conditions that highly depends upon the proposed binary operation. The main theme of this paper is first to introduced the neutrosophic triplets which are completely new notions and then utilize these neutrosophic triplets to introduce the neutrosophic triplet groups. This neutrosophic triplet group has several extraordinary properties as compared to the classical group. We also studied some interesting properties of this newly born structure. We further defined neutro-homomorphisms for neutrosophic triplet groups. A neutron-homomorphism is the generalization of the classical homomorphism with two extra conditions. As a further generalization, we gave rise to a new field or research called Neutrosophic Triplet Structures (such as neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, etc.). In the end, we gave main distinctions and comparison of neutrosophic triplet group with the classical Molaei's generalized group as well as the possible application areas of the neutrosophic triplet groups.

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Neutrosophic Triplet G-Module

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Abstract: In this study, the neutrosophic triplet G-module is introduced and the properties of neutrosophic triplet G-module are studied. Furthermore, reducible, irreducible, and completely reducible neutrosophic triplet G-modules are defined, and relationships of these structures with each other are examined. Also, it is shown that the neutrosophic triplet G-module is different from the G-module.

Keywords: neutrosophic triplet G-module; neutrosophic triplet group; neutrosophic triplet vector space

1. Introduction

Neutrosophy is a branch of philosophy, firstly introduced by Smarandache in 1980. Neutrosophy [1] is based on neutrosophic logic, probability, and set. Neutrosophic logic is a generalized form of many logics such as fuzzy logic, which was introduced by Zadeh [2], and intuitionistic fuzzy logic, which was introduced by Atanassov [3]. Furthermore, Bucolo et al. [4] studied complex dynamics through fuzzy chains; Chen [5] introduced MAGDM based on intuitionistic 2–Tuple linguistic information, and Chen [6] obtain some q–Rung Ortopair fuzzy aggregation operators and their MAGDM. Fuzzy set has function of membership; intuitionistic fuzzy set has function of membership and function of non-membership. Thus; they do not explain the indeterminancy states. However, the neutrosophic set has a function of membership, a function of indeterminacy, and a function of non-membership. Also, many researchers have studied the concept of neutrosophic theory in [7–12]. Recently, Olgun et al. [13] studied the neutrosophic module; Şahin et al. [14] introduced Neutrosophic soft lattices; Şahin et al. [15] studied the soft normed ring; Şahin et al. [16] introduced the centroid single-valued neutrosophic triangular number and its applications; Şahin et al. [17] introduced the centroid points of transformed single-valued neutrosophic number and its applications; Ji et al. [18] studied multi-valued neutrosophic environments and their applications. Also, Smarandache et al. [19] studied neutrosophic triplet (NT) theory and [20,21] neutrosophic triplet groups. A NT has a form <x, neut(x), anti(x)>, in which neut(x) is neutral of "x" and anti(x) is opposite of "x". Furthermore, neut(x) is different from the classical unitary element. Also, the neutrosophic triplet group is different from the classical group. Recently, Smarandache et al. [22] studied the NT field and [23] the NT ring; Şahin et al. [24] introduced the NT metric space, the NT vector space, and the NT normed space; Şahin et al. [25] introduced the NT inner product.

The concept of G-module [26] was introduced by Curties. G-modules are algebraic structures constructed on groups and vector spaces. The concept of group representation was introduced by Frobenious in the last two decades of the 19th century. The representation theory is an important algebraic structure that makes the elements, which are abstract concepts, more evident. Many important results could be proved only for representations over algebraically closed fields. The module theoretic approach is better suited to deal with deeper results in representation theory. Moreover, the module theoretic approach adds more elegance to the theory. In particular, the G-module structure has been extensively used for the study of representations of finite groups. Also, the representation theory of groups describes all the ways in which group G may be embedded in any linear group GL (V). The G-module also holds an important place in the representation theory of groups. Recently some researchers have been dealing with the G-module. For example, Fernandez [27] studied fuzzy G-modules. Sinho and Dewangan [28] studied isomorphism theory for fuzzy submodules of G-modules. Şahin et al. [29] studied soft G-modules. Sharma and Chopra [30] studied the injectivity of intuitionistic fuzzy G-modules.

In this paper, we study neutrosophic triplet G-Modules in order to obtain a new algebraic constructed on neutrosophic triplet groups and neutrosophic triplet vector spaces. Also we define the reducible neutrosophic triplet G-module, the irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module. In this study, in Section 2, we give some preliminary results for neutrosophic triplet sets, neutrosophic triplet groups, the neutrosophic triplet field, the neutrosophic triplet vector space, and G-modules. In Section 3, we define the neutrosophic triplet G-module, and we introduce some properties of a neutrosophic triplet G-module. We show that the neutrosophic triplet G-module is different from the G-module, and we show that if certain conditions are met, every neutrosophic triplet vector space or neutrosophic triplet group can be a neutrosophic triplet G-module at the same time. Also, we introduce the neutrosophic triplet G-module homomorphism and the direct sum of neutrosophic triplet vector space. In Section 4, we define the reducible neutrosophic triplet G-module, the irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module, and we give some properties and theorems for them. Furthermore, we examine the relationships of these structures with each other, and we give some properties and theorems. In Section 5, we give some conclusions.

2. Preliminaries

Definition 1. *Let N be a set together with a binary operation *. Then, N is called a neutrosophic triplet set if for any a ∈ N there exists a neutral of "a" called neut(a) that is different from the classical algebraic unitary element and an opposite of "a" called anti(a) with neut(a) and anti(a) belonging to N, such that [21]:*

*a*neut(a) = neut(a)* a = a,*
and
*a*anti(a) = anti(a)* a = neut(a).*

Definition 2. *Let (N,*) be a neutrosophic triplet set. Then, N is called a neutrosophic triplet group if the following conditions are satisfied [21].*

(1) If (N,) is well-defined, i.e., for any a, b ∈ N, one has a*b ∈ N.*
(2) If (N,) is associative, i.e., (a*b)*c = a*(b*c) for all a, b, c ∈ N.*

Theorem 1. *Let (N,*) be a commutative neutrosophic triplet group with respect to * and a, b ∈ N, in which a and b are both cancellable [21],*

*(i) neut(a)*neut(b) = neut(a*b).*
*(ii) anti(a)*anti(b) = anti(a*b).*

Definition 3. *Let (NTF,*, #) be a neutrosophic triplet set together with two binary operations * and #. Then (NTF,*, #) is called neutrosophic triplet field if the following conditions hold [22].*

1. *(NTF,*) is a commutative neutrosophic triplet group with respect to *.*
2. *(NTF, #) is a neutrosophic triplet group with respect to #.*
3. *a#(b*c) = (a#b)*(a#c) and (b*c)#a = (b#a)*(c#a) for all a, b, c ∈ NTF.*

Theorem 2. *Let (N,*) be a neutrosophic triplet group with respect to *. For (left or right) cancellable $a \in N$, one has the following [24]:*

(i) neut(neut(a)) = neut(a);
(ii) anti(neut(a)) = neut(a);
(iii) anti(anti(a)) = a;
(iv) neut(anti(a)) = neut(a).

Definition 4. *Let (NTF, $*_1$, $\#_1$) be a neutrosophic triplet field, and let (NTV, $*_2$, $\#_2$) be a neutrosophic triplet set together with binary operations " $*_2$" and "$\#_2$". Then (NTV, $*_2$, $\#_2$) is called a neutrosophic triplet vector space if the following conditions hold. For all $u, v \in NTV$, and for all $k \in NTF$, such that $u*_2v \in NTV$ and $u \#_2k \in NTV$ [24];*

(1) $(u*_2v) *_2t = u*_2 (v*_2t)$; $u, v, t \in NTV$;
(2) $u*_2v = v*_2u$; $u, v \in NTV$;
(3) $(v*_2u) \#_2k = (v\#_2k) *_2(u\#_2k)$; $k \in NTF$ and $u, v \in NTV$;
(4) $(k*_1t) \#_2u = (k\#_2v) *_1(u\#_2v)$; $k, t \in NTF$ and $u \in NTV$;
(5) $(k\#_1t) \#_2u = k\#_1(t\#_2u)$; $k, t \in NTF$ and $u \in NTV$;
(6) There exists any $k \in NTF$ such that $u\#_2neut(k) = neut(k) \#_2u = u$; $u \in NTV$.

Definition 5. *Let G be a finite group. A vector space M over a field K is called a G-module if for every $g \in G$ and $m \epsilon M$ there exists a product (called the action of G on M) $m.g \in M$ satisfying the following axioms [26]:*

(i) $m.1_G = m$, $\forall$ $m \epsilon M$ (1_G being the identity element in G);
(ii) $m.(g.h) = (m.g).h$, $\forall$ $m \epsilon M$; $g, h \epsilon G$;
(iii) $(k_1m_1 + k_2m_2).g = k_1(m_1.g)+ k_2(m_2.g)$; $k_1, k_2 \epsilon K$; $m_1, m_2 \epsilon M$, and $g \epsilon G$.

Definition 6. *Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G [26].*

Definition 7. *Let M and M^* be G-modules. A mapping ϕ [26]: $M \to M^*$ is a G-module homomorphism if*

(i) $\phi(k_1.m_1 + k_2.m_2) = k_1. \phi(m_1) + k_2.\phi(m_2)$;
(ii) $\phi(m.g) = \phi(m).g$; $k_1, k_2 \epsilon K$; $m, m_1, m_2 \epsilon M$; $g \epsilon G$.

Further, if ϕ is 1-1, then ϕ is an isomorphism. The G-modules *M* and *M** are said to be isomorphic if there exists an isomorphism ϕ of *M* onto *M**. Then we write $M \cong M^*$.

Definition 8. *Let M be a nonzero G-module. Then, M is irreducible if the only G-submodules of M are M and {0}. Otherwise, M is reducible [26].*

Definition 9. *Let $M_1, M_2, M_3, \ldots, M_n$ be vector spaces over a field K [31]. Then, the set $\{m_1 + m_2 + \ldots + m_n; m_i \epsilon M_i\}$ becomes a vector space over K under the operations*

$$(m_1+m_2+\ldots+m_n)+(m_1{}'+m_2{}'+\ldots+m_n{}') = (m_1+m_1{}')+(m_2+m_2{}')+\ldots+(m_n+m_n{}') \text{ and}$$

$$\alpha(m_1+m_2+\ldots+m_n) = \alpha m_1+\alpha m_2+\ldots+\alpha m_n;\ \alpha \epsilon K, m_n{}' \epsilon M_i$$

It is the called direct sum of the vector spaces $M_1, M_2, M_3, \ldots, M_n$ and is denoted by ${}^n_{i=1}\oplus M_i$.

Remark 1. *The direct sum $M = {}^n_{i=1}\oplus M_i$ of vector spaces M_i has the following properties [31]:*

(i) Each element $m \epsilon M$ has a unique expression as the sum of elements of M_i.
(ii) The vector subspaces $M_1, M_2, M_3, \ldots, M_n$ of M are independent.
(iii) For each $1 \le i \le n$, $M_j \cap (M_1 + M_2 + \ldots + M_{j-1} + M_{j+1} + \ldots + M_n) = \{0\}$.

Definition 10. *A nonzero G-module M is completely reducible if for every G-submodule N of M there exists a G-submodule N* of M such that M = N ⊕ N* [26].*

Proposition 1. *A G-submodule of a completely reducible G-module is completely reducible [26].*

3. Neutrosophic Triplet G-Module

Definition 11. *Let (G, *) be a neutrosophic triplet group, (NTV,$*_1$, $\#_1$) be a neutrosophic triplet vector space on a neutrosophic triplet field (NTF,$*_2$, $\#_2$), and g*m ϵ NTV for g ϵ G, m ϵ NTV. If the following conditions are satisfied, then (NTV,$*_1$, $\#_1$) is called neutrosophic triplet G-module.*

(a) *There exists g ∈ G such that m*neut(g) = neut(g)*m = m for every m ∈ NTV;*
(b) *$m*_1(g*_1h) = (m*_1g) *_1h$, ∀ m ∈ NTV; g, h ∈ G;*
(c) *$(k_1\#_1m_1 *_1 k_2 \#_1m_2)*g = k_1\#_1 (m_1*g)*_1k_2\#_1 (m_2*g)$, ∀$k_1$, k_2ϵ NTF; m_1, m_2 ϵ NTV; g ϵ G.*

Corollary 1. *Neutrosophic G-modules are generally different from the classical G-modules, since there is a single unit element in classical G-module. However, the neutral element neut(g) in neutrosophic triplet G-module is different from the classical one. Also, neutrosophic triplet G-modules are different from fuzzy G-modules, intuitionistic fuzzy G-modules, and soft G-modules, since neutrosophic triplet set is a generalized form of fuzzy set, intuitionistic fuzzy set, and soft set.*

Example 1. *Let X be a nonempty set and let P(X) be set of all subset of X. From Definition 4, (P(X), ∪, ∩) is a neutrosophic triplet vector space on the (P(X), ∪, ∩) neutrosophic triplet field, in which the neutrosophic triplets with respect to ∪; neut(A) = A and anti(A) = B, such that A, B ∈ P(X); A ⊆ B; and the neutrosophic triplets with respect to ∩; neut(A) = A and anti(A) = B, such that A, B ∈ P(X); B ⊇ A. Furthermore, (P(X), ∪) is a neutrosophic triplet group with respect to ∪, in which neut(A) = A and anti(A) = B such that A, B ⊂ P(X); A ⊆ B. We show that (P(X), ∪, ∩) satisfies condition of neutrosophic triplet G-module. From Definition 11:*

(a) *It is clear that if A = B, there exists any A ∈ P(X) for every B ∈ P(X), such that B ∪ neut(A) = neut(A) ∪ B = A.*
(b) *It is clear that A ∪ (B ∪ C) = (A ∪ B) ∪ C, ∀ A ∈ P(X); B, C ∈ P(X).*
(c) *It is clear that*

$((A_1 \cap B_1) \cup (A_2 \cap B_2)) \cup C = (A_1 \cap B_1) \cup C)) \cup (A_2 \cap B_2) \cup C))$, ∀$A_1$, A_2 ϵ P(X); B_1, B_2 ϵ P(X); C ϵ P(X). Thus, (P(X), ∪, ∩) is a neutrosophic triplet G-module.

Corollary 2. *If G = NTV, * = $*_1$, then each (NTV,$*_1$, $\#_1$) neutrosophic triplet vector space is a neutrosophic triplet G-module at the same time. Thus, if G = NTV and * = $*_1$, then every neutrosophic triplet vector space or neutrosophic triplet group can be a neutrosophic triplet G-module at the same time. It is not provided by classical G-module.*

Proof of Corollary 1. *If G = NTV, * = $*_1$;*

(a) *There exists a g ϵ NTV such that m*neut(g) = neut(g)*m = m, ∀m ϵ NTV;*
(b) *It is clear that m*(g*h) = (m*g)*h, as (NTV,*) is a neutrosophic triplet group; ∀ m, g, h ∈ NTV;*
(c) *It is clear that $(k_1\#_1m_1 *_1 k_2 \#_1m_2)*g = k_1\#_1(m_1*g)*_1k_2\#_1 (m_2*g)$, since (NTV,$*_1$, $\#_1$) is a neutrosophic triplet vector space; ∀ g, k_1, k_2 ϵ NTF; m_1, m_2 ϵ NTV.*

Definition 12. *Let (NTV,$*_1$, $\#_1$) be a neutrosophic triplet G-module. A neutrosophic triplet subvector space (N, $*_1$, $\#_1$) of (NTV,$*_1$, $\#_1$) is a neutrosophic triplet G-submodule if (N,$*_1$, $\#_1$) is also a neutrosophic triplet G-module.*

Example 2. *From Example 1; for N ⊆ X, (P(N), ∪, ∩) is a neutrosophic triplet subvector space of (P(X), ∪, ∩). Also, (P(N), ∪, ∩) is a neutrosophic triplet G-module. Thus, (P(N), ∪, ∩) is a neutrosophic triplet G-submodule of (P(X), ∪, ∩).*

Example 3. *Let (NTV,$*_1$, $\#_1$) be a neutrosophic triplet G-module. N = {neut(x)} ∈ NTV is a neutrosophic triplet subvector space of (NTV,$*_1$, $\#_1$). Also, N = {neut(x) = x} ∈ NTV is a neutrosophic triplet G-submodule of (NTV,$*_1$, $\#_1$).*

Definition 13. *Let (NTV,$*_1$, $\#_1$) and (NTV^*,$*_3$, $\#_3$) be neutrosophic triplet G-modules on neutrosophic triplet field (NTF,$*_2$, $\#_2$) and (G, *) be a neutrosophic triplet group. A mapping φ: NTV→NTV^* is a neutrosophic triplet G-module homomorphism if*

(i) $\phi(neut(m)) = neut(\phi(m))$
(ii) $\phi(anti(m)) = anti(\phi(m))$
(iii) $\phi((k_1\#_1m_1) *_1 (k_2\#_1m_2)) = (k_1\#_3\phi(m_1))*_3(k_2\#_3\phi(m_2))$
(iv) $\phi(m*g) = \phi(m)*g$; $\forall\, k_1, k_2 \in NTF$; $m, m_1, m_2 \in M$; $g \in G$.

Further, if ϕ is 1-1, then ϕ is an isomorphism. The neutrosophic triplet G-modules (NTV,$*_1$, $\#_1$) and (NTV^*,$*_3$, $\#_3$) are said to be isomorphic if there exists an isomorphism ϕ: $NTV \to NTV^*$. Then, we write $NTV \cong NTV^*$.

Example 4. *From Example 1, (P(X), ∪, ∩) is neutrosophic triplet vector space on neutrosophic triplet field (P(X), ∪, ∩). Furthermore, (P(X), ∪, ∩) is a neutrosophic triplet G-module. We give a mapping φ: P(X) → P(X), such that φ(A) = neut(A). Now, we show that φ is a neutrosophic triplet G-module homomorphism.*

(i) $\phi(neut(A)) = neut(neut(A)) = neut(\phi(A))$
(ii) $\phi(anti(A)) = neut(anti(A))$; *from Theorem 2,* $neut(anti(A)) = neut(A)$.

$anti(\phi(A)) = anti(neut(A))$; *from Theorem 2,* $anti(neut(A)) = neut(A)$. *Then* $\phi(anti(A)) = anti(\phi(A))$.
(iii) $\phi((A_1 \cap B_1) \cup (A_2 \cap B_2)) = neut(A_1 \cap B_1) \cup (A_2 \cap B_2))$; *from Theorem 1, as* $neut(a)*neut(b) = neut(a*b)$;

$neut(A_1 \cap B_1) \cup (A_2 \cap B_2)) = neut(A_1 \cap B_1) \cup neut(A_2 \cap B_2) =$

$((neut(A_1) \cap neut(B_1)) \cup ((neut(A_2) \cap neut(B_2))$. *From Example 1, as* $neut(A) = A$,

$((neut(A_1) \cap neut(B_1)) \cup ((neut(A_2) \cap neut(B_2)) = (A_1\cap neut(B_1)) \cup (A_2 \cap neut(B_2)) =$

$(A_1\cap neut(B_1)) \cup (A_2 \cap neut(B_2)) = (A_1\cap\phi(B_1)) \cup (A_2 \cap \phi(B_2))$.
(iv) $\phi(A*B) = neut(A*B)$; *from Theorem 1, as* $neut(a)*neut(b) = neut(a*b)$, $neut(A*B) = neut(A)* neut(B)$. *From Example 1, as* $neut(A) = A$, $neut(A)* neut(B) = A* neut(B) = A* \phi(B)$.

4. Reducible, Irreducible, and Completely Reducible Neutrosophic Triplet G-Modules

Definition 14. *Let (NTV,$*_1$, $\#_1$) be neutrosophic triplet G-modules on neutrosophic triplet field (NTF,$*_2$, $\#_2$). Then, (NTV,$*_1$, $\#_1$) is irreducible neutrosophic triplet G-modules if the only neutrosophic triplet G-submodules of (NTV,$*_1$, $\#_1$) are (NTV,$*_1$, $\#_1$) and {neut(x) = x}, x ∈ NTV. Otherwise, (NTV,$*_1$, $\#_1$) is reducible neutrosophic triplet G-module.*

Example 5. *From Example 2, for N = {1,2} ⊆ {1,2,3} = X, (P(N), ∪, ∩) is a neutrosophic triplet subvector space of (P(X), ∪, ∩). Also, (P(N), ∪, ∩) is a neutrosophic triplet G-module. Thus, (P(N), ∪, ∩) is a neutrosophic triplet G-submodule of (P(X), ∪, ∩). Also, from Definition 14, (P(X), ∪, ∩) is a reducible neutrosophic triplet G-module.*

Example 6. *Let X = G = {1, 2} and P(X) be power set of X. Then, (P(X), *, ∩) is a neutrosophic triplet vector space on the (P(X), *, ∩) neutrosophic triplet field and (G, *) is a neutrosophic triplet group, in which*

$$A^*B = \begin{cases} B\backslash A, s(A) < s(B)\Lambda B \supset A \wedge A' = B \\ A\backslash B, s(B) < s(A)\Lambda A \supset B \wedge B' = A \\ (A\backslash B)', s(A) > s(B) \wedge A \supset B \wedge B\prime \neq A \\ (B\backslash A)', s(B) > s(A) \wedge B \supset A \wedge A\prime \neq B \\ X, s(A) = s(B) \wedge A \neq B \\ \varnothing, A = B \end{cases}$$

Here, s(A) means the cardinal of A, and A′ means the complement of A.
*The neutrosophic triplets with respect to *;*
neut(∅) = ∅, anti(∅) = ∅; neut({1}) = {1, 2}, anti({1}) = {2}; neut({2}) = {1, 2}, anti({2}) = {1}; neut({1, 2}) = ∅, anti({1,2}) = {1, 2};
The neutrosophic triplets with respect to ∩;
neut(A) = A and anti(A) = B, in which B ⊃ ⋂ A.
*Also, (P(X), *, ∩) is a neutrosophic triplet G-module. Here, only neutrosophic triplet G-submodules of (P(X), *, ∩) are (P(X), *, ∩) and {neut(∅) = ∅}. Thus, (P(X),*, ∩) is a irreducible neutrosophic triplet G-module.*

Definition 15. *Let (NTV_1,$*_1$, $\#_1$), (NTV_2,$*_1$, $\#_1$), ... , (NTV_n,$*_1$, $\#_1$) be neutrosophic triplet vector spaces on (NTF,$*_2$, $\#_2$). Then, the set {m_1 + m_2 + ... + m_n; m_i ϵ NTV_i} becomes a neutrosophic triplet vector space on (NTF,$*_2$, $\#_2$), such that*

$$(m_1 *_1 m_2 *_1 \ldots *_1 m_n) \ *_1 \ (m_1{}' *_1 m_2{}' *_1 \ldots *_1 m_n{}') = (m_1 *_1 m_1{}') *_1 (m_2 \ *_1 m_2{}') \ *_1 \ldots *_1 \\ (m_n *_1 m_n{}') \text{ and}$$

$$\alpha\#_1(m_1 *_1 m_2 *_1 \ldots *_1 m_n) = (\alpha\#_1 m_1) *_1 \alpha\#_1 m_2) *_1 \ldots *_1 (\alpha\#_1 m_n);\ \alpha \epsilon\ NTf,\ m_n{}' \epsilon\ NTV_i.$$

It is called the direct sum of the neutrosophic triplet vector spaces NTV_1, NTV_2, NTV_3, ... , NTV_n and is denoted by ${}^{n}_{i=1}\oplus NTV_i$.

Remark 2. *The direct sum NTV = ${}^{n}_{i=1}\oplus NTV_i$ of neutrosophic triplet vector spaces NTV_i has the following properties.*

(i) Each element m ϵ NTV has a unique expression as the sum of elements of NTV_i.
(ii) For each 1 ≤ i ≤ n, NTV_j ∩ (NTV_1 + NTV_2 + ... + NTV_{j-1} + NTV_{j+1} + ... + NTV_n) = {x: neut(x) = x}.

Definition 16. *Let (NTV, $*_1$, $\#_1$) be neutrosophic triplet G-modules on neutrosophic triplet field (NTF, $*_2$, $\#_2$), such that NTV ≠ {neut(x) = x}. Then, (NTV, $*_1$, $\#_1$) is a completely reducible neutrosophic triplet G-module if for every neutrosophic triplet G-submodule (N_1, $*_1$, $\#_1$) of (NTV, $*_1$, $\#_1$) there exists a neutrosophic triplet G-submodule (N_2, $*_1$, $\#_1$) of (NTV, $*_1$, $\#_1$), such that NTV = N_1 ⊕ N_2.*

Example 7. *From Example 5, for N = {1, 2}, (P(N), ∪, ∩) is a neutrosophic triplet vector space on (P(N), ∪, ∩) and a neutrosophic triplet G-module. Also, the neutrosophic triplet G-submodules of (P(N), ∪, ∩) are (P(N), ∪, ∩), (P(M), ∪, ∩), (P(K), ∪, ∩), and (P(L), ∪, ∩). Here, M = {1}, K = {2}, and T = {∅}, in which P(M)⊕P(K) = P(N), P(K)⊕P(M) = P(N), P(N)⊕P(T) = P(N), and P(T)⊕P(N) = P(N). Thus, (P(N), ∪, ∩) is a completely reducible neutrosophic triplet G-module.*

Theorem 3. *A neutrosophic triplet G-submodule of a completely reducible neutrosophic triplet G-module is completely neutrosophic triplet G-module.*

Proof of Theorem 1. *Let (NTV, $*_1$,$\#_1$) is a completely reducible neutrosophic triplet G-module on neutrosophic triplet field (NTF, $*_2$, $\#_2$). Assume that (N, $*_1$, $\#_1$) is a neutrosophic triplet G-submodule of (NTV, $*_1$, $\#_1$) and (M, $*_1$, $\#_1$) is a neutrosophic triplet G-submodule of (N, $*_1$, $\#_1$). Then, (M, $*_1$,$\#_1$) is a neutrosophic triplet*

G-submodule of (NTV, $*_1$, $\#_1$). There exists a neutrosophic triplet G-submodule (T, $*_1$, $\#_1$), such that NTV = M$\oplus$T, since (NTV, $*_1$, $\#_1$) is a completely reducible neutrosophic triplet G-module. Then, we take N′ = T $\cap$ N. From Remark 2,

$$N' \cap M \subset M \cap T = \{x : \text{neut}(x) = x\} \tag{1}$$

Then, we take y $\in$ N. If y $\in$ N, y $\in$ NTV and y = m $*_1$ t, in which m $\in$ M; t $\in$ T. Therefore, we obtain t $\in$ N. Thus,

$$tN' = T \cap N \text{ and } y = m *_1 tN' \oplus M \tag{2}$$

From (i) and (ii), we obtain N = N′$\oplus$M. Thus, (N, $*_1$, $\#_1$) is completely reducible neutrosophic triplet G-module.

Theorem 4. *Let (NTV, $*_1$, $\#_1$) be a completely reducible neutrosophic triplet G-module on neutrosophic triplet field (NTF, $*_2$, $\#_2$). Then, there exists a irreducible neutrosophic triplet G-submodule of (NTV, $*_1$, $\#_1$).*

Proof of Theorem 2. *Let (NTV, $*_1$, $\#_1$) be a completely reducible neutrosophic triplet G-module and (N, $*_1$, $\#_1$) be a neutrosophic triplet G-submodule of (NTV, $*_1$, $\#_1$). We take y $\neq$ neut(y) $\in$ N, and we take collection sets of neutrosophic triplet G-submodules of (N, $*_1$, $\#_1$) such that do not contain element y. This set is not empty, because there is {x: x = neut(x)} neutrosophic triplet G-submodule of (N, $*_1$, $\#_1$). From Zorn's Lemma, the collection has maximal element (M, $*_1$, $\#_1$). From Theorem 3, (N, $*_1$, $\#_1$) is a completely reducible neutrosophic triplet G-module, and there exists a (N_1, $*_1$, $\#_1$) neutrosophic triplet G-submodule, such that N = M$\oplus N_1$. We show that ($N_1 *_1$, $\#_1$) is a irreducible neutrosophic triplet G–submodule. Assume that (N_1, $*_1$, $\#_1$) is a reducible neutrosophic triplet G–submodule. Then, there exists (K_1, $*_1$, $\#_1$) and (K_2, $*_1$, $\#_1$) neutrosophic triplet G-submodules of (N_1, $*_1$, $\#_1$), such that y $\in$ N_1, N_2, and from Theorem 3, N_1 = $K_1 \oplus K_2$, in which, as N = M$\oplus N_1$, N = M$\oplus K_1 \oplus K_2$. From Remark 2, $(M *_1 K_1) \cap K_2$ = {neut(x) = x} or $(M *_1 K_2) \cap K_1$ = neut(x) = x}. Then, y $\notin$ $(M *_1 K_1) \cap K_2$ or y $\notin$ $(M *_1 K_2) \cap K_1$. Hence, y $\notin$ $(M *_1 K_1)$ or y $\notin$ $(M *_1 K_2)$. This is a contraction. Thus, ($N_1 *_1$, $\#_1$) is an irreducible neutrosophic triplet G-submodule.*

Theorem 5. *Let (NTV, $*_1$, $\#_1$) be a completely reducible neutrosophic triplet G-module. Then, (NTV, $*_1$, $\#_1$) is a direct sum of irreducible neutrosophic triplet G-modules of (NTV, $*_1$, $\#_1$).*

Proof of Theorem 3. *From Theorem 3, (N_i,$*_1$, $\#_1$) (i = 1, 2, . . . , n), neutrosophic triplet G-submodules of (NTV, $*_1$, $\#_1$) are completely reducible neutrosophic triplet G-modules, such that NTV = $N_{i-k} \oplus N_k$ (k = 1, 2, . . . , $i-1$). From Theorem 4, there exists (M_i, $*_1$, $\#_1$) irreducible neutrosophic triplet G-submodules of (N_i, $*_1$, $\#_1$). Also, from Theorem 3, (M_i, $*_1$, $\#_1$) are completely reducible neutrosophic triplet G-modules, such that N_i = $N_{i-k} \oplus N_k$ (k = 1, 2, . . . , $i-1$). If these steps are followed, we obtained (NTV, $*_1$, $\#_1$), which is a direct sum of irreducible neutrosophic triplet G-modules of (NTV, $*_1$, $\#_1$).*

5. Conclusions

In this paper; we studied the neutrosophic triplet G-module. Furthermore, we showed that neutrosophic triplet G-module is different from the classical G-module. Also, we introduced the reducible neutrosophic triplet G-module, the irreducible neutrosophic triplet G-module, and the completely reducible neutrosophic triplet G-module. The neutrosophic triplet G-module has new properties compared to the classical G-module. By using the neutrosophic triplet G-module, a theory of representation of neutrosophic triplet groups can be defined. Thus, the usage areas of the neutrosophic triplet structures will be expanded.

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Novel System and Method for Telephone Network Planing based on Neutrosophic Graph

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Abstract- Telephony is gaining momentum in the daily lives of individuals and in the activities of all companies. With the great trend towards telephony networks, whether analogue or digital known as Voice over IP (VoIP), the number of calls an individual can receive becomes considerably high. However, effective management of incoming calls to subscribers becomes a necessity. Recently, much attention has been paid towards applications of single-valued neutrosophic graphs in various research fields. One of the suitable reason is it provides a generalized representation of fuzzy graphs (FGs) for dealing with human nature more effectively when compared to existing models i.e. intuitionistic fuzzy graphs (IFGs), inter-valued fuzzy graphs (IVFGs) and bipolar-valued fuzzy graphs (BPVFGs) etc. In this paper we focused on precise analysis of useful information extracted by calls received, not received due to some reasons using the properties of SVNGs. Hence the proposed method introduced one of the first kind of mathematical model for precise analysis of instantaneous traffic beyond the Erlang unit. To achieve this goal an algorithm is proposed for a neutrosophic mobile network model (NMNM) based on a hypothetical data set. In addition, the drawback and further improvement of proposed method with a mathematical proposition is established for it precise applications.

Keywords: fuzzy graph, intuitionistic fuzzy graph, information extraction, single-valued neutrosophic graph, mobile networks.

I. Introduction

Telephony, appeared in the 1830s, it was based on music notes, for the exchange of messages. It then became a communication system essentially ensuring the transmission and reproduction of speech. Telephony also enables more advanced services such as voicemail, conference calling or voice services. Telephony is based on a telecommunications network, typically, telephony network consists of four main types of equipment: terminals, central systems, ancillary servers, and the access media. we mainly distinguish three types of access media: (i) Land line network, known as Public Switched Telephone Network (PSTN), (ii) wireless network - mobile networks, and (iii) private network, whose companies have their own call centre.

According to the last report published by the National Telecommunications Regulatory Agency (ANRT) of the kingdom of morocco, the rate of possession of individuals (12 to 65 years) by mobile phone is slightly increasing in May 2017 (95% against 94.4% in 2015).the use of smartphones by individuals recorded a notable evolution and increased to 67% instead of 54.7% in 2015 [1]. with the rapid explosion on access to the telephone network, the number of calls received becomes considerable. Nowadays, the terms "priority of incoming call", "priority of numbers", "trust of calling equipment", etc. are used [2]. Guarantee a quality of experience (QoE) for the customer is therefore becoming a necessity and especially a promoter axis. However, the amount of information that the service provider must process to ensure QoE is very high, and the decision to route, hold, or reject the incoming call must be at real-time.

Recent time the theory of graph is utilized for various process to deal with uncertainty and vagueness in data sets. It is a mathematical tool which deals with large number of data or information in efficient manner. Graph theory is one of the richest research area in mathematics as it has applications in enormous fields including management sciences [3], social sciences [4], computer sciences [5], communication networks [6], in description of group structures [7], database theory [8], economics [9] etc.

L. A. Zadeh [10] introduced the theory of fuzzy sets (FSs) in 1965 as a tool to deal with uncertainties. It was Kaufmann [11] who define FG but an illustrated work on FGs was done by Rosenfeld in [12]. The theory of FGs is of great importance and in the recent decades, it has been used extensively in many areas such as cluster analysis [13-16], slicing [16], in the solution of fuzzy intersecting equations [17, 18], data base theory [8], networking [19], group structures

[20, 21], chemical structures [22], navigations [23], traffic controlling [24] etc. The concept of FGs have worth in graph theory as it is the best tool to deal with uncertainties. K. Atanassov [25, 26] proposed the concept of intuitionistic fuzzy sets, an extension of FSs which creates space for IFGs. The concept of IFGs were proposed by R. Parvathi and M. G. Karunambigai [27]. The structure of IFGs is successfully applied in social networks [28], clustering [29], radio coverage network [30] and shortest path problems [31] etc. IFGs effectively deals with uncertainties due to its advance structure. In 1995 F. Smarandache proposed neutrosophic logic which provides a base for neutrosophic set (NS) theory [32, 50]. NS theory is a generalization of IFSs and among one of the best structures of fuzzy logics describing the uncertain situations soundly. To apply NS theory in real life situations a discrete form of NSs is introduced known as single-valued neutrosophic set (SVNS) [33] which give rise to the theory of SVNGs [34, 35]. SVNG is of more advanced structure than IFGs and successfully applied in navigations [36], minimum spanning tree problem [37], shortest path problem [38] so far. Some potential work for SVNGs have been done in [39-50] for partial ignorance in the given information at different granulation [51-52].In this paper, we have focused on analysis of mobile network for extracting some information to describe the offered or carried network for multi-decision analytics.

Although FG theory has been applied to many real-life problems as discussed earlier however literature provide very less attention has been paid about a mobile network model (MNM) and its analysis for information processing. In a mobile network, there are variable factors such as: receiving a call either from known or unknown number, ignoring a call or couldn't attend due to enormous reasons, and rejecting a call for some reasons. In this case, extracting some useful information or pattern to take a particular decision is a major problem for the researchers. To solve this problem the current paper aimed at developing a neutrosophic set based mobilephone network by presenting NMNM in the field off SVNGs. It is proposed that, how SVNGs can be utilized to store the record of incoming or outgoing calls and how neutrosophic logic can be considered a best tool for such type of problems.

This article is organized as follows: Section 2 consists of some basic ideas. The complete description of NMNM is presented in section 3. In section 4, an algorithm is proposed while in section 5 the proposed NMNM is illustrated by a flow chart. At the end a hypothetical example is discussed in section 6. Some special circumstances and significance of neutrosophic mobile network model are presented in section 7. The article ended with some advantages of proposed model and some concluding remark and discussion.

II. Basic Concepts

In this section, some elementary concepts are demonstrated related to graphs including FGs, IFGs and SVNGs. For undefined terms and notions, one may refer to [34-46, 50].

Definition 1[50]. Neutrosophic Set (NS)

Let X be a space of points and let $x \in X$. A neutrosophic set $\overline{S}$ in X is characterized by a truth membership function $T_{\bar{S}}$, an indeterminacy membership function $I_{\bar{S}}$, and a falsehood membership function $F_{\bar{S}}$. $T_{\bar{S}}$, $I_{\bar{S}}$ and $F_{\bar{S}}$ are real standard or non-standard subsets of $]0^-,1^+[$. The neutrosophic set can be represented as

$$\overline{S}=\left\{\left(x,T_{\bar{S}}(x),I_{\bar{S}}(x),F_{\bar{S}}(x)\right):x\in X\right\}$$

The sum of $T_{\bar{S}}(x)$,$I_{\bar{S}}(x)$and $F_{\bar{S}}(x)$ is

$$0^-\le T_{\bar{S}}(x)+I_{\bar{S}}(x)+F_{\bar{S}}(x)\le 3^+.$$

To use neutrosophic set in the real life applications such as engineering and scientific problems, it is necessary to consider the interval [0, 1] instead of $]0^-,1^+[$ for technical applications.

Definition 2: A pair $G=(V,E)$ is known as

1. Fuzzy graph if
   a) $V=\{v_i: i\in I\}$ and $T_1: V\rightarrow[0,1]$ is the association degree of $v_i\in V$.
   b) $E=\{(v_i,v_j):(v_i,v_j)\in V\times V\}$ and $T_2: V\times V\rightarrow[0,1]$ is defined as $T_2(v_i,v_j)\le \min[T_1(v_i),T_1(v_j)]$ for all $(v_i,v_j)\in E$.
2. Intuitionistic fuzzy graph if
   a) $V=\{v_i: i\in I\}$ such as $T_1: V\rightarrow[0,1]$ is the association degree and $F_1: V\rightarrow[0,1]$ is the disassociation degree of $v_i\in V$ subject to condition $0\le T_1+F_1\le 1$.
   b) $E=\{(v_i,v_j):(v_i,v_j)\in V\times V\}$ $T_2: V\times V\rightarrow[0,1]$ is the association degree and $F_2: V\times V\rightarrow[0,1]$ is the disassociation degree of $(v_i,v_j)\in E$ defined as $T_2(v_i,v_j)\le \min[T_1(v_i),T_1(v_j)]$ and $F_2(v_i,v_j)\le \max[F_1(v_i),F_1(v_j)]$ subject to condition $0\le T_2+F_2\le 1$ for all $(v_i,v_j)\in E$.
3. Single-valued neutrosophic graph if
   a) $V=\{v_i: i\in I\}$ such as $T_1: V\rightarrow[0,1]$ is the association degree, $I_1: V\rightarrow[0,1]$ is the indeterminacy degree and $F_1: V\rightarrow[0,1]$ is the disassociation degree of $v_i\in V$ subject to condition $0\le T_1+I_1+F_1\le 3$.

b) $E = \{(v_i, v_j): (v_i, v_j) \in V \times V\}$ $T_2: V \times V \to [0, 1]$ is the association degree, $I_2: V \times V \to [0, 1]$ is the indeterminacy degree and $F_2: V \times V \to [0, 1]$ is the disassociation degree of $(v_i, v_j) \in E$ defined as $T_2(v_i, v_j) \leq \min[T_1(v_i), T_1(v_j)]$, $I_2(v_i, v_j) \geq \max[I_1(v_i), I_1(v_j)]$ and $F_2(v_i, v_j) \geq \max[F_1(v_i), F_1(v_j)]$ subject to condition $0 \leq T_2 + I_2 + F_2 \leq 3$ for all $(v_i, v_j) \in E$.

Example: The following figures $1(a, b, c)$ are the examples of FG, IFG and SVNG respectively.

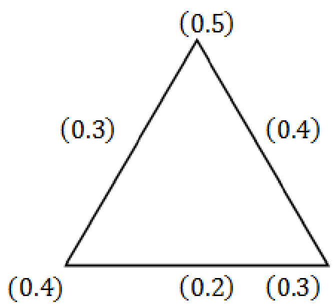

Figure 1 (a): Fuzzy graph.

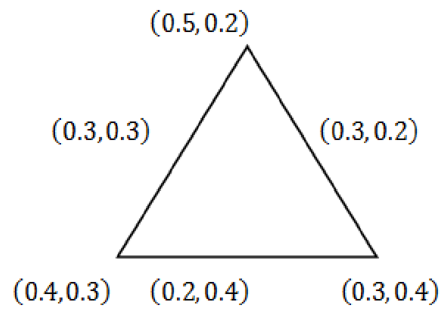

Figure 1 (b): Intuitionistic fuzzy graph.

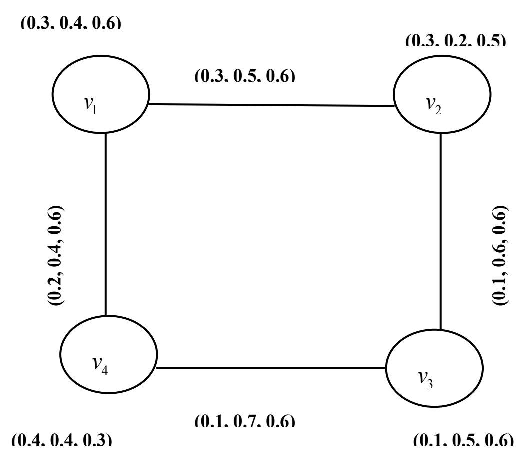

Figure 2 (c): Single valued neutrosophic graph.

III. A Neutrosophic Mobile Netowrk Model

Computing the load of a given Telephone network is one of the major issue for the researchers to extract some useful information for descriptive analysis of carried or offered traffic. It used to measure by "Erlang Unit" which represents the average number of concurrent calls carried by the given telephone network. As for example a radio channel is busy at all time can be considered as load of 1 Erlang. Similarly, an office having two telephone operators and both are busy on each time. It means the office is having two Erlangs. It means the Erland unit represents the offered traffic value followed by average number of concurrent calls which is basically depends on call arrival rate, λ, and the average call-holding time (the average time of a phone call), h, given by: (https://en.wikipedia.org/wiki/Erlang_(unit)).

$$E = \lambda h \quad (1)$$

Where h and λ are represented by the same units of time (seconds and calls per second, or minutes and calls per minute).

The problem arises when the user or expert want to analyze the instantaneous traffic to find the exact number of calls received, not received or uncertain due to some reasons to know the level of traffic, recording devices, or solving other security issues. In this case, characterizing the uncertainty and vagueness in telephone network based on its acceptation, rejection and indeterminacy is major problem. To solve this problem current paper introduces a mathematical representation of telephone network using SVNGs where (T, I, F) can further be divided into some situations as given below:

$\boldsymbol{T}$ can be considered as received calls and is divided into subcases $[\boldsymbol{T_1}, \boldsymbol{T_2}, \dots \boldsymbol{T_n}]$ where $\boldsymbol{T_1}$ represents calls coming from a saved number and $\boldsymbol{T_2}$ represents calls made from some unknown numbers or these can be calls from family member or from friend's circle or from unknown number etc.

$\boldsymbol{I}$ can be considered as calls which couldn't be answered due to many reasons $[\boldsymbol{I_1}, \boldsymbol{I_2}, \boldsymbol{I_3}, \dots \boldsymbol{I_n}]$ represents calls not attended due to driving, busy schedule or meeting or incoming call is from unknown number or any other reason.

$\boldsymbol{F}$ represents those calls which are rejected due to numerous reasons such as $[\boldsymbol{F_1}, \boldsymbol{F_2}, \boldsymbol{F_3}, \dots \boldsymbol{F_n}]$ stand for rejected calls as incoming call is from unknown number or due to hate or behavior of caller etc.

It is clear from the above explanation that in NMNM, all possibilities can be described effectively. Such a model based on SVNGs described uncertain situation better than crisp graphs or fuzzy graphs or intuitionistic fuzzy graphs due to diverse nature of the NS theory. Moreover, it should be noted that in this network the total number of incoming calls is equal to $\boldsymbol{T} + \boldsymbol{I} + \boldsymbol{F}$ denoted by $\boldsymbol{S}$.

The value of **Truth, neutral and falsity** membership grades can be calculated as

$$\left(\frac{\textbf{No. ofcallsattended}}{\textbf{S}}, \frac{\textbf{No. ofcallsleftunattended}}{\textbf{S}}, \frac{\textbf{No. ofcallsrejected}}{\textbf{S}}\right) \quad (2)$$

where S is the total number of incoming calls.

Neutrosophic mobile network model is presented in the following figure 3.

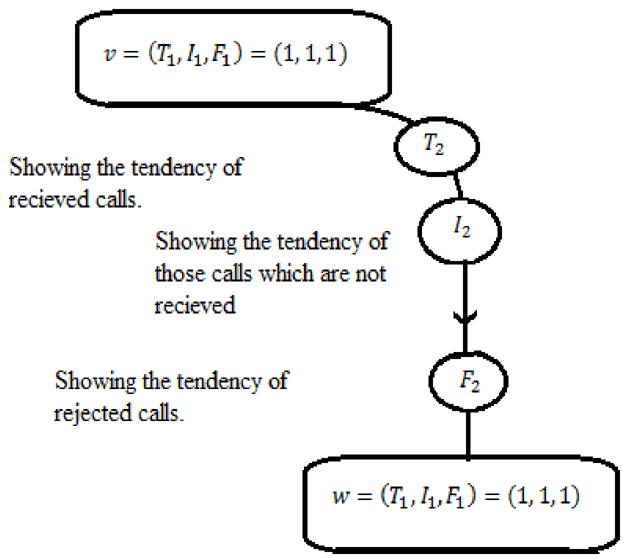

Figure 1: Neutrosophic Mobile Network Model

The figure 3 represents a neutrosophic mobile network model. Using the formula (2), the values of T_2, I_2, F_2 changes in different situations. This value becomes $(0, 1, 0)$ when no calls is received and it becomes $(0, 0, 1)$ when all calls are rejected.

The following example illustrate NMNM in a better way.

Example 1: Let us suppose 100 calls came on a mobile at end of the day and described in form of following information:

1. 60 calls were received truly among them 50 numbers are saved and 10 were unsaved in mobile. In this case these 60 calls will be considered as truth membership i.e. 0.6.
2. 30 calls were not-received by mobile holder. Among them 20 calls which are saved in mobile contacts were not received due to driving, meeting, or phone left in home, car or bag and 10 were not received due to uncertain numbers. In this case all 30 not received numbers by any cause (i.e. driving, meeting or phone left in home) will be considered as Indeterminacy membership i.e. 0.3.
3. 10 calls were those number which was rejected calls intentionally by mobile holder due to behavior of those saved numbers, not useful calls, marketing numbers or other cases for that he/she do not want to pick or may be blocked numbers. In all cases these calls can be considered as false i.e. 0.1 membership value.

The above situation can be represented as:

– neutrosophic set: (0.6, 0.3, 0.1)

– or hesitant neutrosophic set: ({0.5, 0.6}, {0.2, 0.3}, {0.1}) or interval valued neutrosophic set: ([0.5, 0.6], [0.2, 0.3], [0.1, 0.1])

IV. ALGORITHM

In this section, an algorithm is proposed describing the flow of NMNM. Here a network of some neutrosophic mobile phones is assumed and the quantity of received, not attended and rejected calls is expressed in the form of single-valued neutrosophic numbers. The NMNM is not limited to store the data of small networks but it can be applied to large networks as well.

It is assumed that the number of incoming calls received or not received or rejected could be unlimited in this case. In order to calculate the membership grades of T, I and F, formula given in(2) could be of use. The edges in NMNM enables us to get the percentage of calls attended, ignored or rejected at any instant between two mobile numbers. To enable the caller for making or receiving unlimited number of calls, we must assign a neutrosophic number $(1, 1, 1)$ to each vertex.

- Let $v_j = (1, 1, 1)$ and $v_k = (1, 1, 1)$ be two vertices representing two mobile phone numbers.
- $e_{jk} = \left(T_{jk}, I_{jk}, F_{jk}\right)$ be the edge of v_j and v_k.
- Let S denote the number of all calls between two neutrosophic mobile numbers.
- $T_{jk} = \frac{numberofcallsrecevied}{S}$
- $I_{jk} = \frac{numberofcallsleftunattended}{S}$
- $F_{jk} = \frac{numberofcallsrejected}{S}$

This can be written as following propositions:

Let us suppose, total number of all calls between two neutrosophic mobile number =s, m= total number of calls received, n-total number of calls rejected then the number of unattended calls are (s-m-n). This can be written as $(\frac{m}{s}, \frac{s-m-n}{s}, \frac{n}{s})$ neutrosophic number for determining the n^th call.

Initially one call is made and received then truth value is $\frac{1}{1} = 1$, indeterminacy value is 0, falsity value is 0. In case two calls are made and received then too truth value is $\frac{2}{2} = 1$ and so on…

If two calls are made and 1 is received and 1 ignored, then truth is $\frac{1}{2} = 0.5$ and indeterminacy is $\frac{1}{2} = 0.5$ so we may say that 50% calls are received and 50% are ignored. If 3 calls are made and number of received, ignored and rejected calls are 1 so we have $(0.33, 0.33, 0.33)$ which make sense that 33% calls are received, 33% calls are ignored and 33% calls are rejected. Similarly, the algorithm works for n^th calls.

The algorithm proposed here explain every possibility that might be happen in a mobile network proving the worth of SVNGs as the most suitable tool for modeling such type of network.

V. FLOWCHART

A flowchart below described the NMNM step by step. It is assumed here that the total number of call could possibly be received or ignored or rejected is 100 (For the sack of simplicity). Here it is also assumed that initially the number of phone calls made so far is zero. In other words, it may be assumed that initially there is no edge between two nodes $\mathrm{v_j}$ and $\mathrm{v_k}$. The illustrated flowchart is described as follows:

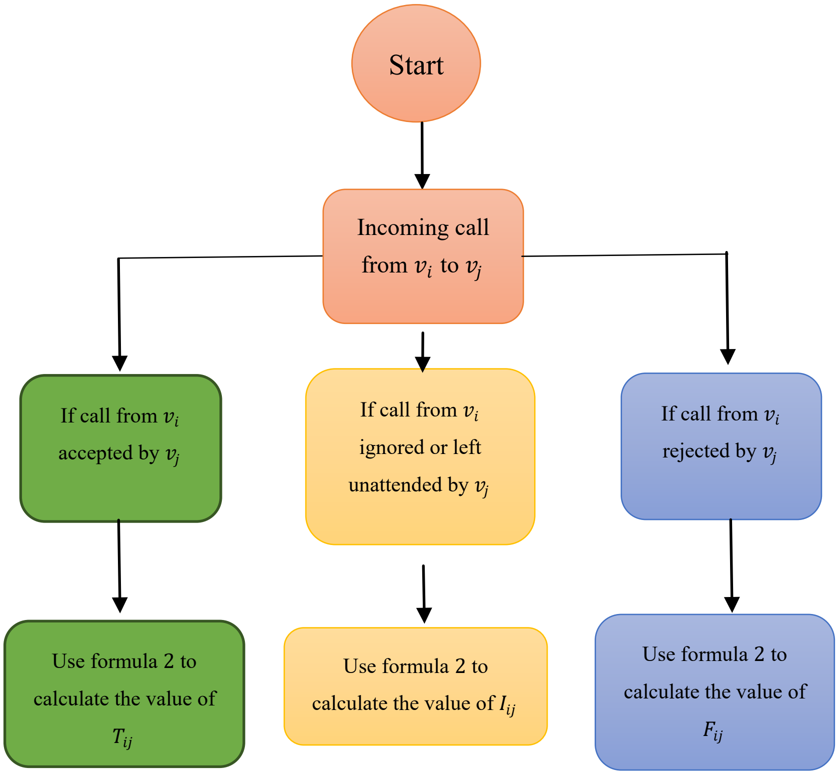

Figure 2: Flow chart describing algorithm of NMNM

In this flow chart, we keep the number of calls limited to 100 but in large networks or in real-life this number of calls cannot be restricted to 100. So, one may set the desired range of calls by their own consent.

VI. Illustrated Example

Consider a network of three people connect to each other via mobile phones which are represented by vertices of SVNGs. The following table 1 describe the calling data (total number of calls, received calls, calls not attended and rejected calls) of these three peoples.

Table 1: Specifying the calling data of a group

Pair	Total calls	Received calls	Not attended calls	Rejected calls	Corresponding Edge
John-Aslam	24	15	5	4	$(0.625, 0.208333, 0.166667)$
Aslam-Chris	15	7	5	3	$(0.466667, 0.3333, 0.2)$
Chris-Aslam	19	15	4	0	$(0.789474, 0.210526, 0)$
Chris-John	5	0	5	0	$(0, 1, 0)$
John-Chris	8	4	3	1	$(0.5, 0.375, 0.125)$

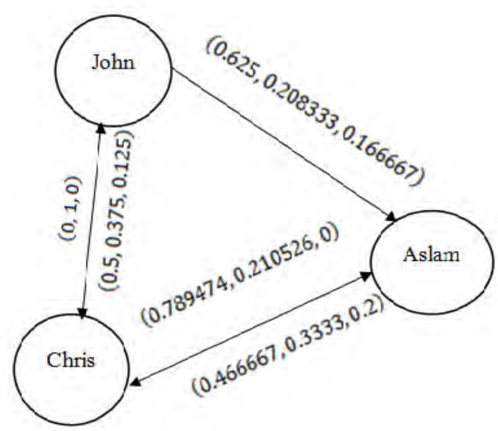

Figure 3: A network of people connected via mobile numbers

In this example, a network of finite number of people is illustrated. The edges in this network is in the form single-valued neutrosophic numbers showing the percentage of number of calls received, left unattended or rejected. The Figure 5 shows that maximum true calls happens among Chris and Aslam due to maximal true membership-values, minimum indeterminacy and minimum falsity membership-values when compared to others. Similarly, other information can be extracted from the proposed method.

VII. Some Special Circumstances and Significance of Neutrosophic Mobile Network Model

In this part of the article, some special cases are listed to extract meaningful information from the proposed method. It is discussed how proposed model is capable of dealing with such kind of situations. This is done in the following way:

Question 1: Is there any difference between saved and unsaved numbers? Did it influence the membership?

Answer: The answer to this question should be of the following form:

When both saved and unsaved numbers are

- *Received:* Then truth valued is increased by an amount.
- *Left Unattended:* Then indeterminacy values in increased by an amount.
- *Rejected:* Then falsity value is increased by an amount.

So saved and unsaved numbers are treated equally in such scenario. But In case the number is saved most probably the holder knows the person and pick the phone or reject it most of time. However, when number is unsaved then many times holders do not want to pick which affects indeterminacy membership-values a lot.

Question 2: How the proposed model deals with marketing numbers as they are important some time while some other time they are meaningless.

Answer: We have introduced a unique scenario to understand the telephone network using single-valued neutrosophic set and its properties as a first basic algorithm when none of the approaches are exists in this regard. Of course, we can control this issue by two cases. The first way is when we do not know that the incoming call is marketing call so it may be rejected or ignored. In second case, when we want to pick the same marketing call in some other time then the number can be saved in the phone as useful number. In this case the first time its membership-values will affect the indeterminacy or falsity value whereas in second case it affects the truth membership-values.

Question 3: When a person is in comma, then all calls on his/her mobile shall be left unattended similarly when a person is kidnapped, then all calls on his/her mobile gets rejection. How the proposed method explains such situation?

Answer: This is an impressive question towards one of the useful applications of our motive to introduce neutrosophic set in telephone network.

We will first try to understand the first case that is Coma means holder is in the operating system. In this case the call may go but holder cannot pick it due to uncertainty. Hence all the incoming call on holder's mobile will be unreceived (not rejected only unreceived) which can be clearly shown by (0, 1, 0). For example, suppose 10 calls came on to his/her mobile and are left unattended…. i.e. $s = 10, m = 0$and $n = 0$. Then $\left(\frac{m}{s}, \frac{s-m-n}{s}, \frac{n}{s}\right) = \left(0, \frac{10-0-0}{10}, 0\right) = (0, 1, 0)$

Now we can understand the case of kidnapping. In this case, the call can be rejected by kidnapper or switch off the phone. It is well known that the kidnapper will not pick the phone or allow to ring the bell several times to understand the location. Hence all calls will be rejected and can be represented as (0, 0, 1) for all time. For example, if 10 calls made and rejected. Then $n = 10, s = 10. \left(\frac{0}{10}, \frac{10-0-10}{10}, \frac{10}{10}\right) = (0, 0, 1)$.

Hence the proposed NMNM can deal with every possibility than one my face. It shows its significance in extracting some meaningful information from mobile network based on their calls received and rejected. The analysis derived from the proposed method will be helpful in making an intelligent system.

In this article, the mobile network is discussed in the environment of SVNGs. It is observed that such a network cannot be established by ordinary FSs i.e. by FGs as FS theory only deals with association degree. Similarly, such a network is difficult to establish in the environment of IFS theory as it describes the association and dissociation degree of elements but in mobile network models we face several types of situations as described earlier. Therefore, the space of SVNG is so far, a best tool for describing such type of situation and for establishing a mobile network model.

VIII. Conclusion and Discussion

In this article, a method for information analysis in mobile network model is described using SVNGs, known as NMNM for precise representation of instantaneous traffic in an alternative way when compared to Erlang Number. The proposed method also describes the structure of FSs and IFSs to make it less resourceful in establishing such type of network for extracting some useful information. A mathematical proposition is also derived for restructuring the SVNGs to represent the received, un-received as well as uncertain calls when compared for depth analysis. The proposed NMNM model is explained using an illustrative example for better understanding. However, the analysis derived from the proposed method is not implemented in any real data sets. To solve this problem in near future the author will focus on comparative study of the proposed method.

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NeutrosophicWeighted Support Vector Machines for the Determination of School Administrators Who Attended an Action Learning Course Based on Their Conflict-Handling Styles

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Abstract: In the recent years, school administrators often come across various problems while teaching, counseling, and promoting and providing other services which engender disagreements and interpersonal conflicts between students, the administrative staff, and others. Action learning is an effective way to train school administrators in order to improve their conflict-handling styles. In this paper, a novel approach is used to determine the effectiveness of training in school administrators who attended an action learning course based on their conflict-handling styles. To this end, a Rahim Organization Conflict Inventory II (ROCI-II) instrument is used that consists of both the demographic information and the conflict-handling styles of the school administrators. The proposed method uses the Neutrosophic Set (NS) and Support Vector Machines (SVMs) to construct an efficient classification scheme neutrosophic support vector machine (NS-SVM). The neutrosophic c-means (NCM) clustering algorithm is used to determine the neutrosophic memberships and then a weighting parameter is calculated from the neutrosophic memberships. The calculated weight value is then used in SVM as handled in the Fuzzy SVM (FSVM) approach. Various experimental works are carried in a computer environment out to validate the proposed idea. All experimental works are simulated in a MATLAB environment with a five-fold cross-validation technique. The classification performance is measured by accuracy criteria. The prediction experiments are conducted based on two scenarios. In the first one, all statements are used to predict if a school administrator is trained or not after attending an action learning program. In the second scenario, five independent dimensions are used individually to predict if a school administrator is trained or not after attending an action learning program. According to the obtained results, the proposed NS-SVM outperforms for all experimental works.

Keywords: action learning; school administrator; SVM; neutrosophic classification

1. Introduction

Support Vector Machine (SVM) is a widely used supervised classifier, which has provided better achievements than traditional classifiers in many pattern recognition applications in the last two decades [1]. SVM is also known as a kernel-based learning algorithm where the input features are transformed into a high-dimensional feature space to increment the class separability of the input features. Then SVM seeks a separating optimal hyperplane that maximizes the margin between two classes in high-dimensional feature space [2]. Maximizing the margin is an optimization problem which can be solved using the Lagrangian multiplier [2]. In addition, some of the input features, which are called support vectors, can also be used to determine the optimal hyperplane [2].

Although SVM outperforms many classification applications, in some applications, some of the input data points may not be truly classified [3]. This misclassification may arise due to noises or other conditions. To handle such a problem, Lin et al. proposed Fuzzy SVMs (FSVMs), in which a fuzzy membership is assigned to each input data point [3]. Thus, a robust SVM architecture is constructed by combining the fuzzy memberships into the learning of the decision surface. Another fuzzy-based improved SVMs approach was proposed by Wang et al. The authors applied it to a credit risk analysis of consumer lending [4]. Ilhan et al. proposed a hybrid method where a genetic algorithm (GA) and SVM were used to predict Single Nucleotide Polymorphisms (SNP) [5]. In other words, GA was used to select the optimum C and γ parameters in order to predict the SNP. The authors also used a particle swarm optimization (PSO) algorithm to optimize C and γ parameters of SVMs. Peng et al. proposed an improved SVM for heterogeneous datasets [6]. To do so, the authors used a mapping procedure to map nominal features to another space via the minimization of the predicted generalization errors. Ju et al. proposed neutrosophic logic to improve the efficiency of the SVMs classifier (N-SVM) [7]. More specifically, the proposed N-SVM approach was applied to image segmentation. The authors used the diverse density support vector machine (DD-SVM) to improve its efficiency with neutrosophic set theory [8]. Almasi et al. proposed a new fuzzy SVM method, which was based on an optimization method [9]. The proposed method simultaneously generated appropriate fuzzy memberships and solved the model selection problem for the SVM family in linear/nonlinear and separable/non-separable classification problems. In Reference [10], Tang et al. proposed a novel fuzzy membership function for linear and nonlinear FSVMs. The structural information of two classes in the input space and in the feature space was used for the calculation of the fuzzy memberships. Wu et al. used an artificial immune system (AIS) in the optimization of SVMs [11]. The authors used the AIS algorithm to optimize the C and γ parameters of SVMs and developed an efficient scheme called AISSVM. Chen et al. optimized the parameters of the SVM by using the artificial bee colony (ABC) approach [12]. Specifically, the authors used an enhanced ABC algorithm where cat chaotic mapping initialization and current optimum were used to improve the ABC approach. Zhao et al. used an ant colony algorithm (ACA) to improve the efficiency of SVMs [13]. The ACA optimization method was used to select the kernel function parameter and soft margin constant C penalty parameter. Guraksin et al. used particle swarm optimization (PSO) to tune SVM parameters to improve its efficiency [14]. The improved SVM approach was applied to a bone age determination system.

In this paper, a new approach is proposed: Neutrosophic SVM (NS-SVM). The neutrosophic set (NS) is defined as the generalization of the fuzzy set [15]. NS is quite effective in dealing with outliers and noises. The noises and outlier samples in a dataset can be treated as a kind of indeterminacy. NS has been successfully applied for indeterminate information processing, and demonstrates advantages to deal with the indeterminacy information of data [16–18]. NS employs three memberships to measure the degree of truth (T), indeterminacy (I), and falsity (F) of each dataset. The neutrosophic c-means (NCM) algorithm is used to produce T, I, and F memberships [16,17]. In recent years, school administrators often come across various problems while teaching, counseling, and promoting and providing other services which engender disagreements and interpersonal conflicts between students, the administrative staff, and others. Action learning is an effective way to train school administrators in order to improve their conflict-handling styles. To this end, the developed NS-SVM approach is applied to determine the effectiveness of training in school administrators who attended an action learning course based on their conflict-handling styles. A Rahim Organization Conflict Inventory II (ROCI-II) instrument is used that consists of both the demographic information and the conflict-handling styles of the school administrators. A five-fold cross-validation test is applied to evaluate the proposed method. The classification accuracy is calculated for performance measure. The proposed method is also compared with SVM and FSVM.

The paper is organized as follows. In the next section, a summarization of the present works on this topic is given. The proposed NS-SVM is introduced in Section 3. Section 4 gives the experimental work and results. We conclude the paper in Section 5.

2. Related Works

As mentioned earlier, there have been a number of presented works about the feature weighting for improving the efficiency of classifiers. To this end, Akbulut et al. proposed an NS-based Extreme Learning Machine (ELM) approach for imbalanced data classification [18]. They initially employed an NS-based clustering algorithm to assign a weight for each input data point and then the obtained weights were linked to the ELM formulation to improve its efficiency. In the experiments, the proposed scheme highly improved the classification accuracy. Ju et al. proposed a similar work and applied it to improve image segmentation performance [7]. The authors opted to construct the NS weights based on the formulations given in Reference [7]. The obtained weights were then used in SVM equations. In other words, the authors used the DD-SVM to improve its efficiency with neutrosophic logic. Guo et al. proposed an unsupervised approach for data clustering [16]. The authors combined NS theory in an unsupervised data clustering which can be seen as a weighting procedure. Thus, the indeterminate data points were also considered in the classification process more efficiently. An NS-based k-NN approach was proposed by Akbulut et al. [19]. The authors used the NS memberships to improve the classification performance of the k-NN classifier. The proposed scheme calculated the NS memberships based on a supervised neutrosophic c-means (NCM) algorithm. A final belonging membership U was calculated from the NS triples. A final voting scheme as given in fuzzy k-NN was considered for class label determination. Budak et al. proposed an NS-based efficient Hough transform [20]. The authors initially transferred the Hough space into the NS space by calculating the NS membership triples. An indeterminacy filtering was constructed where the neighborhood information was used to remove the indeterminacy in the spatial neighborhood of the neutrosophic Hough space. The potential peaks were detected based on thresholding on the neutrosophic Hough space, and these peak locations were then used to detect the lines in the image domain.

3. Proposed Neutrosophic Set Support Vector Machines (NS-SVM)

In this section, we briefly introduce the theories of SVM and NS. The readers may refer to related references for detailed information [1,3]. Then, the proposed neutrosophic set support vector machine is presented in detail below.

3.1. Support Vector Machine (SVM)

SVM is an important and efficient supervised classification algorithm [1,2]. Given a set of N training data points $\{(x_i, y_i)_{n=1}^{N}\}$ where x_i is a multidimensional feature vector and $y_i \in \{-1, 1\}$ is the corresponding label, an SVM models a decision boundary between classes of training data as a separating hyperplane. SVM aims to find an optimal solution by maximizing the margin around the separating hyperplane, which is equivalent to minimizing $||w||$ with the constraint:

$$y_i(w.x_i + b) \geq 1 \tag{1}$$

SVM employs non-linear mapping to transform the input data into a higher dimensional space. Thus, the hyperplane can be found in the higher dimensional space with a maximum margin as:

$$w.\varphi(x) + b = 0 \tag{2}$$

such that for each data sample $(\varphi(x_i), y_i)$:

$$y_i(w.\varphi(x_i) + b) \geq 1, \qquad i = 1, \ldots, N. \tag{3}$$

when the input dataset is not linearly separable, then the soft margin is allowed by defining N non-negative variables, denoted by $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$, such that the constraint for each sample in Equation (3) is rewritten as:

$$y_i(w.\varphi(x_i) + b) \geq 1 - \xi_i, \qquad i = 1, \ldots, N \tag{4}$$

where the optimal hyperplane is determined as;

$$\text{minimum}\left(\frac{1}{2}w^2 + C\sum_{i=1}^{N}\xi_i\right) \tag{5}$$

$$\text{subjected to } y_i(w.\varphi(x_i) + b) \geq 1 - \xi_i, \qquad i = 1, \ldots, N \tag{6}$$

where C is a constant parameter that tunes the balance between the maximum margin and the minimum classification error.

3.2. Neutrosophic c-Means Clustering

In this section, a weighting function is defined by samples using the neutrosophic c-means (NCM) clustering. Let A $= \{A_1,\ A_2,\ ,\ \ldots.,\ A_m\}$ be a set of alternatives in the neutrosophic set. A sample A_i is represented as $\{\mathrm{T}(A_i),\ \mathrm{I}(A_i), \mathrm{F}(A_i)\}/A_i$, where $\mathrm{T}(A_i), \mathrm{I}(A_i)$ and $\mathrm{F}(A_i)$ are the membership values to the true, indeterminate, and false sets. $\mathrm{T}(A_i)$ is used to measure the belonging degree of the sample to the center of the labeled class, $\mathrm{I}(A_i)$ for indiscrimination degree between two classes, and $\mathrm{F}(A_i)$ for the belonging degree to the outliers.

The NCM clustering overcomes the disadvantages of handling indeterminate points in other algorithms [16]. Here we improve the NCM by only computing neutrosophic memberships to the true and indeterminate sets based on the samples' distribution.

Using NCM, the truth and indeterminacy memberships are defined as:

$$K = \left[\frac{1}{\varpi_1}\sum_{j=1}^{C}(x_i - c_j)^{-\frac{2}{m-1}} + \frac{1}{\varpi_2}(x_i - \bar{c}_{imax})^{-\left(\frac{2}{m-1}\right)} + \frac{1}{\varpi_3}\delta^{-\left(\frac{2}{m-1}\right)}\right] \tag{7}$$

$$T_{ij} = \frac{K}{\varpi_1}(x_i - c_j)^{-\left(\frac{2}{m-1}\right)} \tag{8}$$

$$I_i = \frac{K}{\varpi_2}(x_i - \bar{c}_{imax})^{-\left(\frac{2}{m-1}\right)} \tag{9}$$

where T_{ij} and I_i are the true and indeterminacy membership values of point i, and the cluster center is denoted as c_j. $\bar{c}_{\text{imax}}$ is obtained from indexes of the largest and second largest value of T_{ij}. ϖ_1, ϖ_2, and ϖ_3 are constant weights. T_{ij} and I_i are updated at each iteration until $\left|T_{ij}^{(k+1)} - T_{ij}^{(k)}\right| < \varepsilon$, where ε is a termination criterion.

3.3. Proposed Neutrosophic Set Support Vector Machine (NS-SVM)

In the fuzzy support vector machine (FSVM), a membership g_i is assigned for each input data point $\{(x_i, y_i)_{n=1}^{N}\}$, where $0 < g_i < 1$ [3]. As g_i and ξ_i shows the membership and the error of SVM for input data point x_i , respectively, the term $g_i\xi_i$ shows the measure of error with different weighting. Thus, the optimal hyperplane problem can be re-solved as;

$$\text{minimum}\left(\frac{1}{2}w^2 + C\sum_{i=1}^{N}g_i\xi_i\right) \tag{10}$$

$$\text{subjected to } y_i(w.\varphi(x_i) + b) \geq 1 - \xi_i, \qquad i = 1, \ldots, N \tag{11}$$

In the proposed method, a weighting function is defined in the NS based on the memberships to truth and indeterminacy and then used to remove the effect of indeterminacy information for classification.

$$g_{Ni} = \sum_{j=1}^{C} T_{ij} \cdot I_i \tag{12}$$

Then we use the newly defined weight function g_{Ni} to replace the weight function in Equation (4), and an optimization procedure is employed to minimize the cost function as:

$$\text{minimum}\left(\frac{1}{2}w^2 + C\sum_{i=1}^{N} g_{Ni} \cdot \xi_i\right) \tag{13}$$

$$\text{subjected to } y_i(w.\varphi(x_i) + b) \geq 1 - \xi_i, \qquad i = 1, \ldots, N \tag{14}$$

Finally, the support vectors are identified and their weights are obtained for classification. The semantic algorithm of the proposed method is given as:

Input: Labeled training dataset.

Output: Predicted class labels.

Step 1: Calculate the cluster centers according to the labeled dataset and employ NCM algorithm to determine NS memberships T and I for each data point.

Step 2: Calculate g_{Ni} by using T and I components according to Equation (8).

Step 3: Optimize NS-SVM by minimizing the cost function according to Equation (9).

Step 4: Calculate the labels of test data.

4. Experimental Work and Results

In this study, a new approach NS-SVM is proposed and applied to determine if an action learning experience resulted in school administrators being more productive in their conflict-management skills [21]. To this end, an experimental organization was constructed where 38 administrators from various schools in Elazig/Turkey were administered a pre-test and a post-test of the Rahim Organization Conflict Inventory II (ROCI-II) [22]. The pre-test was applied to the administrators before the action learning experience and the post-test was applied after the action learning experience. The ROCI-II contains 28 scale items. These scale items are grouped into five dimensions: integrating, obliging, dominating, avoiding, and compromising. The dataset, which was used in this work, is given in Appendix A. The MATLAB software is used in construction of the NS-SVM approach. In the evaluation of the proposed method, a five-fold cross-validation test is used and the mean accuracy value is recorded. During the experimental work, two different scenarios are considered. In the first one, all 28 scale items are used to determine the trained and non-trained school administrators. In the second scenario, each dimension of ROCI-II is used to determine trained and non-trained administrators in order to determine the relationship between the dimensions and the trained and non-trained school administrators. The NS-SVM parameter C is searched in the range of [10^{-3}, 10^2] at a step size of 10^{-1}. In addition, for NCM the following parameters are chosen: $\varepsilon = 10^{-3}$, $\varpi_1 = 0.75$, $\varpi_2 = 0.125$, $\varpi_3 = 0.125$, which were obtained from trial and error. The δ parameter of NCM method is also searched in the range of $\{2^{-10}, 2^{-8}, \ldots, 2^8, 2^{10}\}$. The dataset is normalized with zero mean and unit variance. Table 1 shows the obtained accuracy scores for the first scenario. The obtained results are further compared with FSVM and other SVM types such as Linear, Quadratic, Cubic, Fine Gaussian, Medium Gaussian, and Coarse Gaussian SVMs.

As seen in Table 1, 81.2% accuracy is obtained with the proposed NS-SVM method, which is the highest among all compared classifier types. The second highest accuracy, 76.9%, is obtained by the FSVM method. An accuracy score of 73.7% is produced by both linear and medium Gaussian SVM methods. In addition, quadratic and cubic SVM techniques produce 68.4% accuracy scores. An accuracy score of 63.2% is obtained by the coarse Gaussian SVM method and finally, the worst accuracy score, 48.7%, is obtained by the fine Gaussian SVM method. Generally speaking, contributing memberships as weighting to SVM highly increases the efficiency. Both FSVM and NS-SVM produce better results than traditional SVM methods. The experimental results that cover the second scenario are given in Tables 2–6. Table 2 shows the obtained accuracy scores when the integrating dimension is used as input. The integrating dimension has six scale items.

Table 1. Prediction accuracies for the first scenario. The bold case shows the highest accuracy. SVM: Support Vector Machines; FSVM: Fuzzy Support Vector Machines; NS-SVM: Neutrosophic Support Vector Machines.

Classifier Type	Accuracy (%)
Linear SVM	73.7
Quadratic SVM	68.4
Cubic SVM	68.4
Fine Gaussian SVM	48.7
Medium Gaussian SVM	73.7
Coarse Gaussian SVM	63.2
FSVM	76.9
NS-SVM	**81.2**

As seen in Table 2, the highest accuracy score, 80.3%, is obtained by the proposed method. This score is 4% better than that achieved by FSVM. The FSVM method produces a 76.3% accuracy score, which is the second highest. Linear and medium Gaussian SVM methods produce 73.7% accuracy scores, which are the third highest. In addition, linear and medium Gaussian SVM methods achieve the best accuracy among the ordinary SVM techniques. It is worth mentioning that cubic SVM has the lowest accuracy score, with an achievement of 53.9%.

Table 2. Prediction accuracies for the second scenario. The integrating dimension is used as input. The bold case shows the highest accuracy.

Classifier Type	Accuracy (%)
Linear SVM	73.7
Quadratic SVM	57.9
Cubic SVM	53.9
Fine Gaussian SVM	60.5
Medium Gaussian SVM	73.7
Coarse Gaussian SVM	67.1
FSVM	76.3
NS-SVM	**80.3**

Table 3 shows the achievements obtained when the obliging dimension is used as input to the classifiers. The obliging dimension covers five scale items and 73.8% accuracy score, which is the highest, obtained by the NS-SVM method. FSVM also produces a 71.3% accuracy score, which is the second-best achievement. The worst accuracy score is obtained by quadratic SVM, for which the accuracy score is 50.0%. One important inference from Table 3 is that ordinary SVM techniques produce almost similar achievements, while weighting with memberships highly improves the accuracy.

The dominating dimension also covers five scale items and the produced results are shown in Table 4. As seen in Table 4, the highest accuracy, 70.0%, is produced by the proposed NS-SVM method. In addition, the second-best accuracy score, 65.0%, is obtained by the FSVM method. The linear SVM obtains 59.2% accuracy, which is the third highest accuracy score. When one considers the ordinary SVM's achievements, an obvious improvement can be seen easily that is achieved by the NS-SVM method.

Table 3. Prediction accuracies for the second scenario. The obliging dimension is used as input. The bold case shows the highest accuracy.

Classifier Type	Accuracy (%)
Linear SVM	61.8
Quadratic SVM	50.0
Cubic SVM	51.3
Fine Gaussian SVM	52.6
Medium Gaussian SVM	61.8
Coarse Gaussian SVM	55.3
FSVM	71.3
NS-SVM	**73.8**

Table 4. Prediction accuracies for the second scenario. The dominating dimension is used as input. The bold case shows the highest accuracy.

Classifier Type	Accuracy (%)
Linear SVM	59.2
Quadratic SVM	57.9
Cubic SVM	52.6
Fine Gaussian SVM	55.3
Medium Gaussian SVM	52.6
Coarse Gaussian SVM	55.3
FSVM	65.0
NS-SVM	**70.0**

The avoiding dimension covers six scale items and the produced results are given in Table 5. As one evaluates the obtained results given in Table 5, it can be observed that the avoiding dimension is not efficient enough in discriminating trained and non-trained participants. In other words, the ordinary SVM techniques do not achieve better accuracy scores. Among them, the highest accuracy, 53.9%, is produced by the cubic SVM method. On the other hand, both FSVM and the proposed NS-SVM methods produce better accuracy scores, with achievements of 63.8% and 66.3%, respectively. Once more, the best accuracy is obtained by the proposed NS-SVM method.

Table 5. Prediction accuracies for the second scenario. The avoiding dimension is used as input. The bold case shows the highest accuracy.

Classifier Type	Accuracy (%)
Linear SVM	50.0
Quadratic SVM	43.4
Cubic SVM	53.9
Fine Gaussian SVM	48.7
Medium Gaussian SVM	44.7
Coarse Gaussian SVM	42.1
FSVM	63.8
NS-SVM	**66.3**

Finally, the compromising dimension covers six scale items and the produced results are given in Table 6. As seen in Table 6, the compromising dimension is quite efficient in the determination of trained and non-trained participants, where better accuracy scores are visible when compared with the avoiding dimension's accuracy scores. A 75.0% accuracy score, the highest among all methods, is obtained by NS-SVM. A 73.8% accuracy score is obtained by the FSVM method. The highest third accuracy score is produced by medium Gaussian SVM.

Table 6. Prediction accuracies for the second scenario. The compromising dimension is used as input. The bold case shows the highest accuracy.

Classifier Type	Accuracy (%)
Linear SVM	67.1
Quadratic SVM	67.1
Cubic SVM	57.9
Fine Gaussian SVM	65.8
Medium Gaussian SVM	71.1
Coarse Gaussian SVM	68.4
FSVM	73.8
NS-SVM	**75.0**

We further analyze the results obtained from the first scenario by considering a statistical measure and the running time. To this end, the f-measure metric was considered. The f-measure calculates the weighted harmonic mean of recall and precision [23]. The results are tabulated in Table 7.

Table 7. Calculated f-measure and running times for the first scenario. The bold cases show the better achievements.

Classifier Type	f-Measure (%)	Time (s)
Linear SVM	73.50	0.314
Quadratic SVM	68.50	0.129
Cubic SVM	68.50	0.122
Fine Gaussian SVM	48.50	0.119
Medium Gaussian SVM	71.00	0.130
Coarse Gaussian SVM	61.00	0.129
FSVM	76.50	0.089
NS-SVM	**80.00**	**0.065**

In Table 7, the best f-measure achievement score, 80.00%, was achieved by the proposed NS-SVM method. The second-best f-measure score, 76.50%, was produced by FSVM. The other SVM techniques also produced reasonable f-measure scores when their accuracy achievements were considered (Table 1). In addition, the running time of the proposed method was less than those of the other SVM methods. The proposed method achieved its process at 0.065 s. In other words, this running time is almost half the running times of the non-weighted SVM methods. Thus, it is evident that the proposed NS-SVM performed more accurate results in a very short time, demonstrating its efficiency.

5. Conclusions

In this paper, neutrosophic set theory and SVM is used to construct an efficient classification approach called NS-SVM. It is then applied to an educational problem. More specifically, the determination of the effectiveness of training in school administrators who attended an action learning course based on their conflict-handling styles is achieved. To this end, a ROCI-II instrument is used that consists of both the demographic information and the conflict-handling styles of the school administrators. Six various SVM approaches and FSVM are used in performance comparison. The experimental works are carried out with a five-fold cross-validation technique and the classification accuracy is measured to evaluate the performance of the proposed NS-SVM approach. The experiments are conducted based on two scenarios. In the first one, all statements are used to predict if a school administrator is trained or not after attending an action learning program. In the second scenario, five independent dimensions are used individually to predict if a school administrator is educated or not after attending an action learning program. According to the obtained results, the first scenario achieves the best performance with the NS-SVM method, resulting in an accuracy score of 81.2%. In addition, for all experiments in the second scenario, the proposed NS-SVM achieves the highest accuracy scores as given in Tables 2–6. Furthermore, FSVM achieved the second highest accuracy scores for all experiments that are handled in scenarios 1 and 2. This situation shows that embedding the membership degrees into the SVM method highly improves its discriminatory ability. To further analyze the efficiency of the proposed method, we used the f-measure test and the running times of the methods. The proposed NS-SVM yielded the highest f-measure score. In addition, the running time of the proposed method was much less than those of the traditional SVM techniques.

This study revealed important results for both educational research and determining the effectiveness of educational practices. First, this research showed that the NS-SVM technique can be used in pre-test and post-test comparisons in experimental educational research. In addition, this study demonstrated that the effectiveness levels of training courses can be determined by examining the NS-SVM discrimination accuracy of individuals who attended training courses compared to those who did not.

Appendix A

The dataset was used in the experimental works is given in Figure A1. The features are in the columns and the last column shows the class labels. Moreover, the rows show the number of samples.

This dataset was originally constructed based on the questionnaire that was based on the ROCI-II instrument [24]. As mentioned earlier, the ROCI-II instrument contains 28 scale items which are grouped into five dimensions; integrating (six scale items, Features 1–6), obliging (five scale items, Features 7–11), dominating (five scale items, Features 12–16), avoiding (six scale items, Features 17–22), and compromising (six scale items, Features 23–28). The school administrators were asked to fill out this questionnaire by assigning a five-point Likert scale (1–5) for each feature before and after a action learning course. Thus, 76 questionnaires were obtained. In scenario 1, the 28 scale items were used in the prediction of trained and non-trained school administrators and in scenario 2, each dimension of the ROCI-II instrument was used to predict trained and non-trained school administrators.

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NS-Cross Entropy-Based MAGDM under Single-Valued Neutrosophic Set Environment

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Abstract: A single-valued neutrosophic set has king power to express uncertainty characterized by indeterminacy, inconsistency and incompleteness. Most of the existing single-valued neutrosophic cross entropy bears an asymmetrical behavior and produces an undefined phenomenon in some situations. In order to deal with these disadvantages, we propose a new cross entropy measure under a single-valued neutrosophic set (SVNS) environment, namely NS-cross entropy, and prove its basic properties. Also we define weighted NS-cross entropy measure and investigate its basic properties. We develop a novel multi-attribute group decision-making (MAGDM) strategy that is free from the drawback of asymmetrical behavior and undefined phenomena. It is capable of dealing with an unknown weight of attributes and an unknown weight of decision-makers. Finally, a numerical example of multi-attribute group decision-making problem of investment potential is solved to show the feasibility, validity and efficiency of the proposed decision-making strategy.

Keywords: neutrosophic set; single-valued neutrosophic set; NS-cross entropy measure; multi-attribute group decision-making

1. Introduction

To tackle the uncertainty and modeling of real and scientific problems, Zadeh [1] first introduced the fuzzy set by defining membership measure in 1965. Bellman and Zadeh [2] contributed important research on fuzzy decision-making using max and min operators. Atanassov [3] established the intuitionistic fuzzy set (IFS) in 1986 by adding non-membership measure as an independent component to the fuzzy set. Theoretical and practical applications of IFSs in multi-criteria decision-making (MCDM) have been reported in the literature [4–12]. Zadeh [13] introduced entropy measure in the fuzzy environment. Burillo and Bustince [14] proposed distance measure between IFSs and offered an axiomatic definition of entropy measure. In the IFS environment, Szmidt and Kacprzyk [15] proposed a new entropy measure based on geometric interpretation of IFS. Wei et al. [16] developed an entropy measure for interval-valued intuitionistic fuzzy set (IVIFS) and presented its applications in pattern recognition and MCDM. Li [17] presented a new multi-attribute decision-making (MADM) strategy combining entropy and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) in an IVIFS environment. Shang and Jiang [18] introduced the cross entropy in the fuzzy environment. Vlachos and Sergiadis [19] presented intuitionistic fuzzy cross entropy by extending fuzzy cross entropy [18]. Ye [20] defined a new cross entropy under an IVIFS environment and presented an optimal decision-making strategy. Xia and Xu [21] put forward a new entropy and a cross entropy and employed them for multi-attribute criteria group decision-making (MAGDM) strategy under an IFS environment. Tong and Yu [22] defined cross entropy under an IVIFS environment and applied it to MADM problems.

The study of uncertainty took a new direction after the publication of the neutrosophic set (NS) [23] and single-valued neutrosophic set (SVNS) [24]. SVNS appeals more to researchers for its applicability in decision-making [25–54], conflict resolution [55], educational problems [56,57], image processing [58–60], cluster analysis [61,62], social problems [63,64], etc. The research on SVNS gained momentum after the inception of the international journal "Neutrosophic Sets and Systems". Combining with the neutrosophic set, a number of hybrid neutrosophic sets such as the neutrosophic soft set [65–72], the neutrosophic soft expert set [73–75], the neutrosophic complex set [76], the rough neutrosophic set [77–86], the rough neutrosophic tri complex set [87], the neutrosophic rough hyper complex set [88], the neutrosophic hesitant fuzzy sets/multi-valued neutrosophic set [89–97], the bipolar neutrosophic set [98–103], the rough bipolar neutrosophic set [104], the neutrosophic cubic set [105–113], and the neutrosophic cubic soft set [114,115] has been reported in the literature. Wang et al. [116] defined the interval neutrosophic set (INS). Different interval neutrosophic hybrid sets and their theoretical development and applications have been reported in the literature, such as the interval-valued neutrosophic soft set [117], the interval neutrosophic complex set [118], the interval neutrosophic rough set [119–121], and the interval neutrosophic hesitant fuzzy set [122]. Other extensions of neutrosophic sets, such as trapezoidal neutrosophic sets [123,124], normal neutrosophic sets [125], single-valued neutrosophic linguistic sets [126], interval neutrosophic linguistic sets [127,128], simplified neutrosophic linguistic sets [129], single-valued neutrosophic trapezoid linguistic sets [130], interval neutrosophic uncertain linguistic sets [131–133], neutrosophic refined sets [134–139], linguistic refined neutrosophic sets [140] bipolar neutrosophic refined sets [141], and dynamic single-valued neutrosophic multi-sets [142] have been proposed to enrich the study of neutrosophics. So the field of neutrosophic study has been steadily developing.

Majumdar and Samanta [143] defined an entropy measure and presented an MCDM strategy under SVNS environment. Ye [144] proposed cross entropy measure under the single-valued neutrosophic set environment, which is not symmetric straight forward and bears undefined phenomena. To overcome the asymmetrical behavior of the cross entropy measure, Ye [144] used a symmetric discrimination information measure for single-valued neutrosophic sets. Ye [145] defined cross entropy measures for SVNSs to overcome the drawback of undefined phenomena of the cross entropy measure [144] and proposed a MCDM strategy.

The aforementioned applications of cross entropy [144,145] can be effective in dealing with neutrosophic MADM problems. However, they also bear some limitations, which are outlined below:

i. The strategies [144,145] are capable of solving neutrosophic MADM problems that require the criterion weights to be completely known. However, it can be difficult and subjective to offer exact criterion weight information due to neutrosophic nature of decision-making situations.
ii. The strategies [144,145] have a single decision-making structure, and not enough attention is paid to improving robustness when processing the assessment information.
iii. The strategies [144,145] cannot deal with the unknown weight of the decision-makers.

Research gap:

MAGDM strategy based on cross entropy measure with unknown weight of attributes and unknown weight of decision-makers.

This study answers the following research questions:

i. Is it possible to define a new cross entropy measure that is free from asymmetrical phenomena and undefined behavior?

ii. Is it possible to define a new weighted cross entropy measure that is free from the asymmetrical phenomena and undefined behavior?
iii. Is it possible to develop a new MAGDM strategy based on the proposed cross entropy measure in single-valued neutrosophic set environment, which is free from the asymmetrical phenomena and undefined behavior?
iv. Is it possible to develop a new MAGDM strategy based on the proposed weighted cross entropy measure in the single-valued neutrosophic set environment that is free from the asymmetrical phenomena and undefined behavior?
v. How do we assign unknown weight of attributes?
vi. How do we assign unknown weight of decision-makers?

Motivation:

The above-mentioned analysis describes the motivation behind proposing a comprehensive NS-cross entropy-based strategy for tackling MAGDM under the neutrosophic environment. This study develops a novel NS-cross entropy-based MAGDM strategy that can deal with multiple decision-makers and unknown weight of attributes and unknown weight of decision-makers and free from the drawbacks that exist in [144,145].

The objectives of the paper are:

1. To define a new cross entropy measure and prove its basic properties, which are free from asymmetrical phenomena and undefined behavior.
2. To define a new weighted cross measure and prove its basic properties, which are free from asymmetrical phenomena and undefined behavior.
3. To develop a new MAGDM strategy based on weighted cross entropy measure under single-valued neutrosophic set environment.
4. To develop a technique to incorporate unknown weight of attributes and unknown weight of decision-makers in the proposed NS-cross entropy-based MAGDM under single-valued neutrosophic environment.

To fill the research gap, we propose NS-cross entropy-based MAGDM, which is capable of dealing with multiple decision-makers with unknown weight of the decision-makers and unknown weight of the attributes.

The main contributions of this paper are summarized below:

1. We define a new NS-cross entropy measure and prove its basic properties. It is straightforward symmetric and it has no undefined behavior.
2. We define a new weighted NS-cross entropy measure in the single-valued neutrosophic set environment and prove its basic properties. It is straightforward symmetric and it has no undefined behavior.
3. In this paper, we develop a new MAGDM strategy based on weighted NS cross entropy to solve MAGDM problems with unknown weight of the attributes and unknown weight of decision-makers.
4. Techniques to determine unknown weight of attributes and unknown weight of decisions makers are proposed in the study.

The rest of the paper is presented as follows: Section 2 describes some concepts of SVNS. In Section 3 we propose a new cross entropy measure between two SVNS and investigate its properties. In Section 4, we develop a novel MAGDM strategy based on the proposed NS-cross entropy with SVNS information. In Section 5 an illustrative example is solved to demonstrate the applicability and efficiency of the developed MAGDM strategy under SVNS environment. In Section 6 we present comparative study and discussion. Section 7 offers conclusions and the future scope of research.

2. Preliminaries

This section presents a short list of mostly known definitions pertaining to this paper.

Definition 1 [23] NS. *Let U be a space of points (objects) with a generic element in U denoted by u, i.e., u ∈ U. A neutrosophic set A in U is characterized by truth-membership measure $T_A(u)$, indeterminacy-membership measure $I_A(u)$ and falsity-membership measure $F_A(u)$, where $T_A(u)$, $I_A(u)$, $F_A(u)$ are the measures from U to $]^-0, 1^+$ [i.e., $T_A(u)$, $I_A(u)$, $F_A(u)$:U→$]^-0, 1^+[$ NS can be expressed as A = {<u; ($T_A(u)$, $I_A(u)$, $F_A(u)$)>: ∀ u ∈U}. Since $T_A(u)$, $I_A(u)$, $F_A(u)$ are the subsets of $]^-0, 1^+$ [there the sum ($T_A(u) + I_A(u) + F_A(u)$) lies between $^-0$ and 3^+.*

Example 1. *Suppose that U = {u_1, u_2, u_3, ...} be the universal set. Let R_1 be any neutrosophic set in U. Then R_1 expressed as R_1 = {<u_1; (0.6, 0.3, 0.4)>: u_1 ∈ U}.*

Definition 2 [24] SVNS. *Assume that U be a space of points (objects) with generic elements u ∈ U. A SVNS H in U is characterized by a truth-membership measure $T_H(u)$, an indeterminacy-membership measure $I_H(u)$, and a falsity-membership measure $F_H(u)$, where $T_H(u)$, $I_H(u)$, $F_H(u)$ ∈ [0, 1] for each point u in U. Therefore, a SVNS A can be expressed as H = {u, ($T_H(u)$, $I_H(u)$, $F_H(u)$) ∣ ∀u ∈ U}, whereas, the sum of $T_H(u)$, $I_H(u)$ and $F_H(u)$ satisfy the condition $0 \le T_H(u) + I_H(u) + F_H(u) \le 3$ and H(u) = <($T_H(u)$, $I_H(u)$, $F_H(u)$> call a single-valued neutrosophic number (SVNN).*

Example 2. *Suppose that U = {u_1, u_2, u_3, ...} be the universal set. A SVNS H in U can be expressed as: H = {u_1, (0.7, 0.3, 0.5) ∣ u_1 ∈ U} and SVNN presented H = <0.7, 0.3, 0.5>.*

Definition 3 [24] Inclusion of SVNSs. *The inclusion of any two SVNS sets H_1 and H_2 in U is denoted by $H_1 \subseteq H_2$ and defined as follows:*

$$H_1 \subseteq H_2,\ T_{H_1}(u)\ \le\ T_{H_2}(u),\ I_{H_1}(u) \ge I_{H_2}(u),\ F_{H_1}(u) \ge F_{H_2}(u)\ iff\ for\ all\ u \in U.$$

Example 3. *Let H_1 and H_2 be any two SVNNs in U presented as follows: H_1 = <(0.7, 0.3, 0.5)> and H_2 = <(0.8, 0.2, 0.4)> for all u ∈ U. Using the property of inclusion of two SVNNs, we conclude that $H_1 \subseteq H_2$.*

Definition 4 [24] Equality of two SVNSs. *The equality of any two SVNS H_1 and H_2 in U denoted by $H_1 = H_2$ and defined as follows:*

$$T_{H_1}(u) = T_{H_2}(u),\ I_{H_1}(u) = I_{H_2}(u)\ and\ F_{H_1}(u) = F_{H_2}(u)\ for\ all\ u \in U.$$

Definition 5 Complement of any SVNSs. *The complement of any SVNS H in U denoted by H^c and defined as follows:*

$$H^c\ =\ \{u,\ 1 - T_H, 1 - I_H,\ 1 - F_H \mid u \in U\}.$$

Example 4. *Let H be any SVNN in U presented as follows: H = < (0.7, 0.3, 0.5) >. Then compliment of H is obtained as H^c = <(0.3, 0.7, 0.5)>.*

Definition 6 [24] Union. *The union of two single-valued neutrosophic sets H_1 and H_2 is a neutrosophic set H_3 (say) written as*

$H_3 = H_1 \cup H_2$.
$T_{H_3}(u)$ = max {$T_{H_1}(u)$, $T_{H_2}(u)$}, $I_{HJ_3}(u)$ = min {$I_{H_1}(u)$, $I_{H_2}(u)$}, $F_{H_3}(u)$ = min {$F_{H_1}(u)$, $F_{H_2}(u)$}, ∀ u ∈ U.

Example 5. *Let H_1 and H_2 be two SVNSs in U presented as follows:*

H_1 = <(0.6, 0.3, 0.4)> *and* H_2 = <(0.7, 0.3, 0.6)>. *Then union of them is presented as:*

$$H_1 \cup H_2 = < (0.7, 0.3, 0.4) > .$$

Definition 7 [24] Intersection. *The intersection of two single-valued neutrosophic sets H_1 and H_2 denoted by H_4 and defined as*

$H_4 = H_1 \cap H_2$
$T_{H_4}(u) = \min\{T_{H_1}(u), T_{H_2}(u)\}$, $I_{H_4}(u) = \max\{I_{H_1}(u), I_{H_2}(u)\}$
$F_{H_4}(u) = \max\{F_{H_1}(u), F_{H_2}(u)\}$, $\forall\, u \in U$.

Example 6. *Let H_1 and H_2 be two SVNSs in U presented as follows:*

H_1 = <(0.6, 0.3, 0.4)> *and* H_2 = <(0.7, 0.3, 0.6)>.

Then intersection of H_1 and H_2 is presented as follows:

$H_1 \cap H_2$ = <(0.6, 0.3, 0.6)>

3. NS-Cross Entropy Measure

In this section, we define a new single-valued neutrosophic cross-entropy measure for measuring the deviation of single-valued neutrosophic variables from an a priori one.

Definition 8 NS-cross entropy measure. *Let H_1 and H_2 be any two SVNSs in $U = \{u_1, u_2, u_3, \ldots, u_n\}$. Then, the single-valued cross-entropy of H_1 and H_2 is denoted by CE_{NS} (H_1, H_2) and defined as follows:*

$$\mathrm{CE_{NS}}\,(\mathrm{H_1},\,\mathrm{H_2}) = \frac{1}{2}\left\{\sum_{i=1}^{n}\left\langle\left[\frac{2\,|T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\left[\frac{2|I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u)|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+\left[\frac{2|F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\rangle\right\} \quad (1)$$

Example 7. *Let H_1 and H_2 be two SVNSs in U, which are given by $H_1 = \{u, (0.7, 0.3, 0.4) \mid u \in U\}$ and $H_2 = \{u, (0.6, 0.4, 0.2) \mid u \in U\}$. Using Equation (1), the cross entropy value of H_1 and H_2 is obtained as $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) = 0.707$.*

Theorem 1. *Single-valued neutrosophic cross entropy $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2})$ for any two SVNSs $\mathrm{H_1}$, $\mathrm{H_2}$, satisfies the following properties:*

i. $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) \geq 0$.
ii. $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) = 0$ if and only if $T_{H_1}(u_i) = T_{H_2}(u_i)$, $I_{H_1}(u_i) = I_{H_2}(u_i)$, $F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall\, u_i \in U$.
iii. $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) = \mathrm{CE_{NS}}\,(\mathrm{H_1^c}, \mathrm{H_2^c})$
iv. $\mathrm{CE_{NS}}\,(\mathrm{H_1}, \mathrm{H_2}) = \mathrm{CE_{NS}}\,(\mathrm{H_2}, \mathrm{H_1})$

Proof. (i) For all values of $u_i \in U$, $|T_{H_1}(u_i)| \geq 0$, $|T_{H_2}(u_i)| \geq 0$, $|T_{H_1}(u_i) - T_{H_2}(u_i)| \geq 0$, $\sqrt{1+|T_{H_1}(u_i)|^2} \geq 0$, $\sqrt{1+|T_{H_2}(u_i)|^2} \geq 0$, $|(1-T_{H_1}(u_i))| \geq 0$, $|(1-T_{H_2}(u_i))| \geq 0$, $|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))| \geq 0$, $\sqrt{1+|(1-T_{H_1}(u_i))|^2} \geq 0$, $\sqrt{1+|(1-T_{H_2}(u_i))|^2} \geq 0$.

Then, $\left[\frac{2|T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}} + \frac{2|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))|}{\sqrt{1+|(1-T_{H_1}(u_i))|^2}+\sqrt{1+|(1-T_{H_2}(u_i))|^2}}\right] \geq 0.$

Similarly, $\left[\frac{2|I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u)|^2}} + \frac{2|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))|}{\sqrt{1+|(1-I_{H_1}(u_i))|^2}+\sqrt{1+|(1-I_{H_2}(u_i))|^2}}\right] \geq 0$,

and $\left[\frac{2|F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}} + \frac{2\,|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))|}{\sqrt{1+|(1-F_{H_1}(u_i))|^2}+\sqrt{1+|(1-F_{H_2}(u_i))|^2}}\right] \geq 0.$

Therefore, $CE_{NS}\ (H_1,\ H_2) \geq 0$.

Hence complete the proof.

(ii) $\left[\frac{2\ |T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}} + \frac{2\ |(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))|}{\sqrt{1+|(1-T_{H_1}(u_i))|^2}+\sqrt{1+|(1-T_{H_2}(u_i))|^2}}\right] = 0, \Leftrightarrow T_{H_1}(u_i) = T_{H_2}(u_i)$,

$\left[\frac{2\ |I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u)\ |^2}} + \frac{2\ |(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))|}{\sqrt{1+|(1-I_{H_1}(u_i))|^2}+\sqrt{1+|(1-I_{H_2}(u_i))|^2}}\right] = 0 \Leftrightarrow I_{H_1}(u_i) = I_{H_2}(u_i)$, and,

$\left[\frac{2\ |F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}} + \frac{2\ |(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))|}{\sqrt{1+|(1-F_{H_1}(u_i))|^2}+\sqrt{1+|(1-F_{H_2}(u_i))|^2}}\right] = 0, \Leftrightarrow F_{H_1}(u_i) = F_{H_2}(u_i)$

Therefore, $CE_{NS}(H_1,\ H_2) = 0$, iff $T_{H_1}(u_i) = T_{H_2}(u_i)$, $I_{H_1}(u_i) = I_{H_2}(u_i)$, $F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall\ u_i \in U$.

Hence complete the proof.

(iii) Using Definition 5, we obtain the following expression

$$\begin{aligned}
CE_{NS}\ (H_1^c,\ H_2^c) = \frac{1}{2}\Bigg\{\sum_{i=1}^{n}\Bigg\langle &\left[\frac{2\ |(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))|}{\sqrt{1+|(1-T_{H_1}(u_i))|^2}+\sqrt{1+|(1-T_{H_2}(u_i))|^2}} + \frac{2\ |T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}}\right] + \\
&\left[\frac{2|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))|}{\sqrt{1+|(1-I_{H_1}(u_i))|^2}+\sqrt{1+|(1-I_{H_2}(u_i))|^2}} + \frac{2|I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u)|^2}}\right] + \\
&\left[\frac{2|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))|}{\sqrt{1+|(1-F_{H_1}(u_i))|^2}+\sqrt{1+|(1-F_{H_2}(u_i))|^2}} + \frac{2|F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}}\right]\Bigg\rangle\Bigg\} \\
= \frac{1}{2}\Bigg\{\sum_{i=1}^{n}\Bigg\langle &\left[\frac{2|T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}} + \frac{2|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))|}{\sqrt{1+|(1-T_{H_1}(u_i))|^2}+\sqrt{1+|(1-T_{H_2}(u_i))|^2}}\right] + \\
&\left[\frac{2|I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u)|^2}} + \frac{2|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))|}{\sqrt{1+|(1-I_{H_1}(u_i))|^2}+\sqrt{1+|(1-I_{H_2}(u_i))|^2}}\right] + \\
&\left[\frac{2|F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}} + \frac{2|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))|}{\sqrt{1+|(1-F_{H_1}(u_i))|^2}+\sqrt{1+|(1-F_{H_2}(u_i))|^2}}\right]\Bigg\rangle\Bigg\} = CE_{SN}(H_1,\ H_2)
\end{aligned}$$

Therefore, $CE_{NS}(H_1,\ H_2) = CE_{NS}(H_1^c,\ H_2^c)$.

Hence complete the proof.

(iv) Since, $|T_{H_1}(u_i) - T_{H_2}(u_i)| = |T_{H_2}(u_i) - T_{H_1}(u_i)|$, $|I_{H_1}(u_i) - I_{H_2}(u_i)| = |I_{H_2}(u_i) - I_{H_1}(u_i)|$, $|F_{H_1}(u_i) - F_{H_2}(u_i)| = |F_{H_2}(u_i) - F_{H_1}(u_i)|$, $|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))| = |(1-T_{H_2}(u_i)) - (1-T_{H_1}(u_i))|$, $|(1-I_{H_1}(u_i)) - (1-I_{H_2}(u_i))| = |(1-I_{H_2}(u_i)) - (1-I_{H_1}(u_i))|$,

$\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right| = \left|(1-F_{H_2}(u_i))-(1-F_{H_1}(u_i))\right|$, then, $\sqrt{1+|T_{H_1}(u_i)|^2} + \sqrt{1+|T_{H_2}(u_i)|^2} = \sqrt{1+|T_{H_2}(u_i)|^2} + \sqrt{1+|T_{H_1}(u_i)|^2}$, $\sqrt{1+|I_{H_1}(u_i)|^2} + \sqrt{1+|I_{H_2}(u_i)|^2} = \sqrt{1+|I_{H_2}(u_i)|^2} + \sqrt{1+|I_{H_1}(u_i)|^2}$, $\sqrt{1+|F_{H_1}(u_i)|^2} + \sqrt{1+|F_{H_2}(u_i)|^2} = \sqrt{1+|F_{H_2}(u_i)|^2} + \sqrt{1+|F_{H_1}(u_i)|^2}$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2} + \sqrt{1+|(1-T_{H_2}(u_i))|^2} = \sqrt{1+|(-T_{H_2}(u_i)\ )|^2} + \sqrt{1+|(1-T_{H_1}(u_i))|^2}$, $\sqrt{1+|(1-I_{H_1}(u_i))|^2} + \sqrt{1+|(1-I_{H_2}(u_i))|^2} = \sqrt{1+|(1-I_{H_2}(u_i))|^2} + \sqrt{1+|(1-I_{H_1}(u_i))|^2}$, $\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2} + \sqrt{1+|(1-F_{H_2}(u_i))|^2} = \sqrt{1+|(1-F_{H_2}(u_i))|^2} + \sqrt{1+|(1-F_{H_1}(u_i))|^2}$, $\forall\ u_i \in U$.

Therefore, $\mathrm{CE_{NS}}(\mathrm{H_1},\ \mathrm{H_2}) = \mathrm{CE_{NS}}\ (\mathrm{H_2},\ \mathrm{H_1})$.

Hence complete the proof. □

Definition 9 Weighted NS-cross entropy measure. *We consider the weight w_i (i = 1, 2, ..., n) for the element u_i (i = 1, 2, .., n) with the conditions* $\mathrm{w_i} \in [0,\ 1]$ *and* $\sum_{i=1}^{n} w_i = 1$.

Then the weighted cross entropy between SVNSs H_1 and H_2 can be defined as follows:

$$\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = \frac{1}{2}\left\langle \sum_{i\,=\,1}^{n} w_i \left\{ \left[\frac{2\ |T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}} + \frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}} \right] + \left[\frac{2\ |I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)\ |^2}+\sqrt{1+|I_{H_2}(u)|^2}} + \frac{2\left|(1-I_{H_1}(u_i))\ -(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1\ -I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}} \right] + \left[\frac{2\ |F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}} + \frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}} \right] \right\} \right\rangle \tag{2}$$

Theorem 2. *Single-valued neutrosophic weighted NS-cross-entropy (defined in Equation (2)) satisfies the following properties:*

i. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) \geq 0$.
ii. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = 0$, if and only if $T_{H_1}(u_i) = T_{H_2}(u_i)\ I_{H_1}(u_i) = I_{H_2}(u_i)$, $F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall\ u_i \in U$.
iii. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = \mathrm{CE_{NS}^{w}}\ (\mathrm{H_1^c},\ \mathrm{H_2^c})$
iv. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = \mathrm{CE_{NS}^{w}}\ (\mathrm{H_2},\ \mathrm{H_1})$

Proof. (i). For all values of $u_i \in U$, $|T_{H_1}(u_i)\ | \geq 0\ |T_{H_2}(u_i)\ | \geq 0$, $|T_{H_1}(u_i)-T_{H_2}(u_i)| \geq 0$, $\sqrt{1+|T_{H_1}(u_i)|^2} \geq 0$, $\sqrt{1+|T_{H_2}(u_i)|^2} \geq 0$, $\left|(1-T_{H_1}(u_i))\right| \geq 0$, $|(1-T_{H_2}(u_i))\ | \geq 0$, $\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-\ T_{H_2}(u_i))\ \right|^2} \geq 0$, then,

$$\left[\frac{2\ |T_{H_1}(u_i)-T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}} + \frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}} \right] \geq 0.$$

Similarly, $\left[\frac{2\ |I_{H_1}(u_i)-I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u)|^2}} + \frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}} \right] \geq 0$,

and $\left[\frac{2\ |F_{H_1}(u_i)-F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}} + \frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}} \right] \geq 0$.

Since $w_i \in [0,\ 1]$ and $\sum_{i=1}^{n} w_i = 1$, therefore, $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) \geq 0$.

Hence complete the proof.

(ii) Since, $\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]=0,\Leftrightarrow\ T_{H_1}(u_i)=T_{H_2}(u_i)$,
$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]=0,\Leftrightarrow I_{H_1}(u_i)=I_{H_2}(u_i)$,
$\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]=0,\Leftrightarrow F_{H_1}(u_i)=F_{H_2}(u_i)$
and $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$, $w_i\geq 0$. Therefore, $CE_{NS}^{w}(H_1,H_2)=0$ iff $T_{H_1}(u_i)=T_{H_2}(u_i)$, $I_{H_1}(u_i)=I_{H_2}(u_i)$, $F_{H_1}(u_i)=F_{H_2}(u_i)$, $\forall\, u_i\in U$.

Hence complete the proof.

(iii) Using Definition 5, we obtain the following expression

$$CE_{NS}^{w}(H_1^c,H_2^c)=\frac{1}{2}\left\{\sum_{i=1}^{n}w_i\left\langle\left[\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}+\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}\right]+\left[\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}+\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}\right]+\left[\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}+\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}\right]\right\rangle\right\}$$

$$=\frac{1}{2}\left\{\sum_{i=1}^{n}w_i\left\langle\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\rangle\right\}=CE_{NS}^{w}(H_1,H_2)$$

Therefore, $CE_{NS}^{w}(H_1,H_2)=CE_{NS}^{w}(H_1^c,H_2^c)$.

Hence complete the proof.

(iv) Since $\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|=\left|T_{H_2}(u_i)-T_{H_1}(u_i)\right|$, $\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|=\left|I_{H_2}(u_i)-I_{H_1}(u_i)\right|$, $\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|=\left|F_{H_2}(u_i)-F_{H_1}(u_i)\right|$, $\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|=\left|(1-T_{H_2}(u_i))-(1-T_{H_1}(u_i))\right|$, $\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|=\left|(1-I_{H_2}(u_i))-(1-I_{H_1}(u_i))\right|$, $\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|=\left|(1-F_{H_2}(u_i))-(1-F_{H_1}(u_i))\right|$, we obtain $\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}=\sqrt{1+\left|T_{H_2}(u_i)\right|^2}+\sqrt{1+\left|T_{H_1}(u_i)\right|^2}$, $\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u_i)\right|^2}=\sqrt{1+\left|I_{H_2}(u_i)\right|^2}+\sqrt{1+\left|I_{H_1}(u_i)\right|^2}$, $\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}=\sqrt{1+\left|F_{H_2}(u_i)\right|^2}+\sqrt{1+\left|F_{H_1}(u_i)\right|^2}$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(-T_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}$, $\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}$, $\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}$, $\forall\, u_i\in U$ and $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$.

Therefore, $CE_{NS}^{w}(H_1,H_2)=CE_{NS}^{w}(H_2,H_1)$.

Hence complete the proof. □

4. MAGDM Strategy Using Proposed Ns-Cross Entropy Measure under SVNS Environment

In this section, we develop a new MAGDM strategy using the proposed NS-cross entropy measure.

Description of the MAGDM Problem

Assume that $A = \{A_1, A_2, A_3, \ldots, A_m\}$ and $G = \{G_1, G_2, G_3, \ldots, G_n\}$ be the discrete set of alternatives and attributes respectively and $W = \{w_1, w_2, w_3, \ldots, w_n\}$ be the weight vector of attributes $G_j(j = 1, 2, 3, \ldots, n)$, where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Assume that $E = \{E_1, E_2, E_3, \ldots, E_\rho\}$ be the set of decision-makers who are employed to evaluate the alternatives. The weight vector of the decision-makers E_k $(k = 1, 2, 3, \ldots, \rho)$ is $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_\rho\}$ (where, $\lambda_k \geq 0$ and $\sum_{k=1}^{\rho} \lambda_k = 1$), which can be determined according to the decision-makers' expertise, judgment quality and domain knowledge.

Now, we describe the steps of the proposed MAGDM strategy (see Figure 1) using NS-cross entropy measure.

MAGDM Strategy Using Ns-Cross Entropy Measure

Step 1. Formulate the decision matrices

For MAGDM with SVNSs information, the rating values of the alternatives A_i $(i = 1, 2, 3, \ldots, m)$ based on the attribute G_j $(j = 1, 2, 3, \ldots, n)$ provided by the k-th decision-maker can be expressed in terms of SVNN as $a_{ij}^k = < T_{ij}^k, I_{ij}^k, F_{ij}^k >$ $(i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n; k = 1, 2, 3, \ldots, \rho)$. We present these rating values of alternatives provided by the decision-makers in matrix form as follows:

$$M^k = \begin{pmatrix} & G_1 & G_2 & \ldots. & G_n \\ A_1 & a_{11}^k & a_{12}^k & \ldots & a_{1n}^k \\ A_2 & a_{21}^k & & a_{2n}^k & a_{22}^k \\ . & . & \ldots & . & \\ A_m & a_{m1}^k & a_{m2}^k & \ldots & a_{mn}^k \end{pmatrix} \quad (3)$$

Step 2. Formulate priori/ideal decision matrix

In the MAGDM, the a priori decision matrix has been used to select the best alternatives among the set of collected feasible alternatives. In the decision-making situation, we use the following decision matrix as a priori decision matrix.

$$P = \begin{pmatrix} & G_1 & G_2 & \ldots. & G_n \\ A_1 & a_{11}^* & a_{12}^* & \ldots & a_{1n}^* \\ A_2 & a_{21}^* & a_{22}^* & & a_{2n}^* \\ . & . & \ldots & . & \\ A_m & a_{m1}^* & a_{m2}^* & \ldots & a_{mn}^* \end{pmatrix} \quad (4)$$

where, $a_{ij}^* = < \max_i (T_{ij}^k), \min_i (I_{ij}^k), \min_i (F_{ij}^k) >$) corresponding to benefit attributes and $a_{ij}^* = < \min_i (T_{ij}^k), \max_i (I_{ij}^k), \max_i (F_{ij}^k) >$ corresponding to cost attributes, and $(i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n; k = 1, 2, 3, \ldots, \rho)$.

Step 3. Determinate the weights of decision-makers

To find the decision-makers' weights we introduce a model based on the NS-cross entropy measure. The collective NS-cross entropy measure between M^k and P (Ideal matrix) is defined as follows:

$$CE_{NS}^c(M^k, P) = \frac{1}{m}\sum_{i=1}^{m} CE_{NS}\Big(M^k(A_i), P(A_i)\Big) \quad (5)$$

where, $CE_{NS}\left(M^k(A_i),\ P(A_i)\right) = \sum_{j=1}^{n} CE_{NS}(M^k(A_i(G_j)),\ P(A_i(G_j)))$.

Thus, we can introduce the following weight model of the decision-makers:

$$\lambda_K = \frac{\left(1 \div CE_{NS}^{c}(M^k,\ P)\right)}{\sum_{k=1}^{\rho}\left(1 \div CE_{NS}^{c}(M^k,\ P)\right)} \tag{6}$$

where, $0 \le \lambda_K \le 1$ and $\sum_{k=1}^{\rho} \lambda_K = 1$ for $k = 1, 2, 3, \ldots, \rho$.

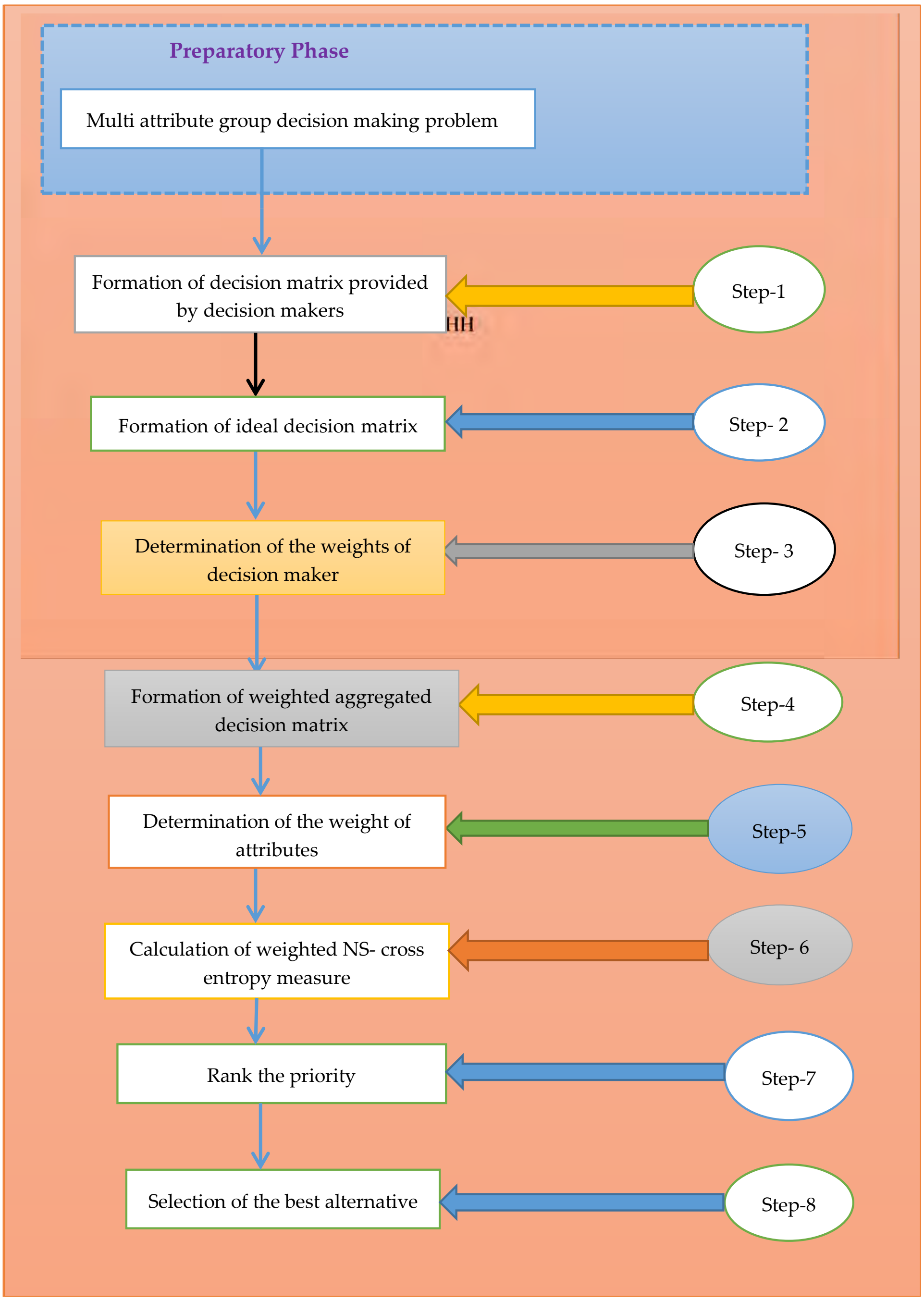


Figure.1 Decision making procedure of the proposed MAGDM strategy
Figure 1. Decision-making procedure of the proposed MAGDM strategy.

Step 4. Formulate the weighted aggregated decision matrix

For obtaining one group decision, we aggregate all the individual decision matrices (M^k) to an aggregated decision matrix (M) using single valued neutrosophic weighted averaging (SVNWA) operator ([51]) as follows:

$$a_{ij} = SVNSWA_{\lambda}(\, a_{ij}^1,\, a_{ij}^2,\, a_{ij}^3,\, \ldots,\, a_{ij}^{\rho}) = (\lambda_1 a_{ij}^1 \oplus \lambda_2 a_{ij}^2 \oplus \lambda_3 a_{ij}^3 \oplus \ldots \oplus \lambda_{\rho} a_{ij}^{\rho}) = \\ <\, 1 - \prod_{k=1}^{\rho} (1 \,- T_{ij}^k)^{\lambda_k},\, \prod_{k=1}^{\rho} (I_{ij}^k)^{\lambda_k},\, \prod_{k=1}^{\rho} (F_{ij}^k)^{\lambda_k} > \tag{7}$$

Therefore, the aggregated decision matrix is defined as follows:

$$M = \begin{pmatrix} & G_1 & G_2 & \ldots. & G_n \\ A_1 & a_{11} & a_{12} & \ldots & a_{1n} \\ A_2 & a_{21} & a_{22} & & a_{2n} \\ . & . & \ldots & . & \\ A_m & a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \tag{8}$$

where, $a_{ij} =< T_{ij},\, I_{ij},\, F_{ij}\ >$, ($i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$; $k = 1, 2, 3, \ldots, \rho$).

Step 5. Determinate the weight of attributes

To find the attributes weight we introduce a model based on the NS-cross entropy measure. The collective NS-cross entropy measure between M (Weighted aggregated decision matrix) and P (Ideal matrix) for each attribute is defined by

$$CE_{NS}^j(M,\, P) = \frac{1}{m}\sum_{i=1}^{m} CE_{NS}\,\big(M(A_i(G_j)),\, P(A_i(G_j))\big) \tag{9}$$

where, $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$.

Thus, we defined a weight model for attributes as follows:

$$w_j = \frac{\left(1 \div CE_{NS}^j(M,\, P)\right)}{\sum_{J=1}^{n}\left(1 \div CE_{NS}^j(M,\, P)\right)} \tag{10}$$

where, $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j = 1$ for $j = 1, 2, 3, \ldots, n$.

Step 6. Calculate the weighted NS-cross entropy measure

Using Equation (2), we calculate weighted cross entropy value between weighted aggregated matrix and priori matrix. The cross entropy values can be presented in matrix form as follows:

$$^{NS}M_{CE}^{w} = \begin{pmatrix} \mathrm{CE_{NS}^{w}}\,(A_1) \\ \mathrm{CE_{NS}^{w}}\,(A_2) \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{CE_{NS}^{w}}\,(A_m) \end{pmatrix} \tag{11}$$

Step 7. Rank the priority

Smaller value of the cross entropy reflects that an alternative is closer to the ideal alternative. Therefore, the preference priority order of all the alternatives can be determined according to the increasing order of the cross entropy values $CE_{NS}^{w}(A_i)$ (i = 1, 2, 3, ... , m). Smallest cross entropy value indicates the best alternative and greatest cross entropy value indicates the worst alternative.

Step 8. Select the best alternative

From the preference rank order (from step 7), we select the best alternative.

5. Illustrative Example

In this section, we solve an illustrative example adapted from [12] of MAGDM problems to reflect the feasibility, applicability and efficiency of the proposed strategy under the SVNS environment.

Now, we use the example [12] for cultivation and analysis. A venture capital firm intends to make evaluation and selection of five enterprises with the investment potential:

(1) Automobile company (A_1)
(2) Military manufacturing enterprise (A_2)
(3) TV media company (A_3)
(4) Food enterprises (A_4)
(5) Computer software company (A_5)

On the basis of four attributes namely:

(1) Social and political factor (G_1)
(2) The environmental factor (G_2)
(3) Investment risk factor (G_3)
(4) The enterprise growth factor (G_4).

The investment firm makes a panel of three decision-makers.

The steps of decision-making strategy (4.1.1.) to rank alternatives are presented as follows:

Step: 1. Formulate the decision matrices

We represent the rating values of alternatives A_i (i = 1, 2, 3, 4, 5) with respects to the attributes G_j (j = 1, 2, 3, 4) provided by the decision-makers E_k (k = 1, 2, 3) in matrix form as follows:

Decision matrix for E_1 decision-maker

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.9, 0.5, 0.4) & (0.7, 0.4, 0.4) & (0.7, 0.3, 0.4) & (0.5, 0.4, 0.9) \\ A_2 & (0.7, 0.2, 0.3) & (0.8, 0.4, 0.3) & (0.9, 0.6, 0.5) & (0.9, 0.1, 0.3) \\ A_3 & (0.8, 0.4, 0.4) & (0.7, 0.4, 0.2) & (0.9, 0.7, 0.6) & (0.7, 0.3, 0.3) \\ A_4 & (0.5, 0.8, 0.7) & (0.6, 0.3, 0.4) & (0.7, 0.2, 0.5) & (0.5, 0.4, 0.7) \\ A_5 & (0.8, 0.4, 0.3) & (0.5, 0.4, 0.5) & (0.6, 0.4, 0.4) & (0.9, 0.7, 0.5) \end{pmatrix} \quad (12)$$

Decision matrix for E_2 decision-maker

$$M^2 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.7, 0.2, 0.3) & (0.5, 0.4, 0.5) & (0.9, 0.4, 0.5) & (0.6, 0.5, 0.3) \\ A_2 & (0.7, 0.4, 0.4) & (0.7, 0.3, 0.4) & (0.7, 0.3, 0.4) & (0.6, 0.4, 0.3) \\ A_3 & (0.6, 0.4, 0.4) & (0.5, 0.3, 0.5) & (0.9, 0.5, 0.4) & (0.6, 0.5, 0.6) \\ A_4 & (0.7, 0.5, 0.3) & (0.6, 0.3, 0.6) & (0.7, 0.4, 0.4) & (0.8, 0.5, 0.4) \\ A_5 & (0.9, 0.4, 0.3) & (0.6, 0 .4, 0.5) & (0.8, 0.5, 0.6) & (0.5, 0.4, 0.5) \end{pmatrix} \quad (13)$$

Decision matrix for E_3 decision-maker

$$M^3 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.7,\ 0.2,\ 0.5) & (0.6,\ 0.4,\ 0.4) & (0.7,\ 0.4,\ 0.5) & (0.9,\ 0.4,\ 0.3) \\ A_2 & (0.6,\ 0.5,\ 0.5) & (0.9,\ 0.3,\ 0.4) & (0.7,\ 0.4,\ 0.3) & (0.8,\ 0.4,\ 0.5) \\ A_3 & (0.8,\ 0.3,\ 0.5) & (0.9,\ 0.3,\ 0.4) & (0.8,\ 0.3,\ 0.4) & (0.7,\ 0.3,\ 0.4) \\ A_4 & (0.9,\ 0.3,\ 0.4) & (0.6,\ 0.3,\ 0.4) & (0.5,\ 0.2,\ 0.4) & (0.7,\ 0.3,\ 0.5) \\ A_5 & (0.8,\ 0.3,\ 0.3) & (0.6,\ 0.4,\ 0.3) & (0.6,\ 0.3,\ 0.4) & (0.7,\ 0.3,\ 0.5) \end{pmatrix} \quad (14)$$

Step: 2. Formulate priori/ideal decision matrix

A priori/ideal decision matrix Please provide a sharper picture

$$P = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.9,\ 0.2,\ 0.3) & (0.7,\ 0.4,\ 0.4) & (0.9,\ 0.3,\ 0.4) & (0.9,\ 0.4,\ 0.3) \\ A_2 & (0.7,\ 0.2,\ 0.3) & (0.9,\ 0.3,\ 0.3) & (0.9,\ 0.3,\ 0.3) & (0.9,\ 0.1,\ 0.3) \\ A_3 & (0.8,\ 0.3,\ 0.4) & (0.9,\ 0.3,\ 0.2) & (0.9,\ 0.3,\ 0.4) & (0.7,\ 0.3,\ 0.3) \\ A_4 & (0.9,\ 0.3,\ 0.3) & (0.6,\ 0.3,\ 0.4) & (0.7,\ 0.2,\ 0.4) & (0.7,\ 0.3,\ 0.4) \\ A_5 & (0.9,\ 0.3,\ 0.3) & (0.6,\ 0.4,\ 0.3) & (0.8,\ 0.3,\ 0.4) & (0.9,\ 0.3,\ 0.5) \end{pmatrix} \quad (15)$$

Step: 3. Determine the weight of decision-makers

By using Equations (5) and (6), we determine the weights of the three decision-makers as follows:

$$\lambda_1 = \frac{(1 \div\ 0.9)}{3.37} \approx 0.33,\ \lambda_2 = \frac{(1 \div\ 1.2)}{3.37} \approx 0.25,\ \lambda_1 = \frac{(1 \div\ .07)}{3.37} \approx 0.42.$$

Step: 4. Formulate the weighted aggregated decision matrix

Using Equation (7) the weighted aggregated decision matrix is presented as follows:
Weighted Aggregated decision matrix

$$M = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.8, 0.3,\ 0.4) & (0.6,\ 0.4,\ 0.4) & (0.8, 0.4,\ 0.4) & (0.7, 0.4,\ 0.5) \\ A_2 & (0.7,\ 0.3, 0\ .4) & (0.8,\ 0.3,\ 0.4) & (0.8,\ 0.4,\ 0.4) & (0.8,\ 0.2,\ 0.3) \\ A_3 & (0.8,\ 0.4,\ 0.4) & (0.8, 0.3,\ 0.3) & (0.9,\ 0.5,\ 0.5) & (0.7,\ 0.3,\ 0.4) \\ A_4 & (0.7, 0.5,\ 0.5) & (0.6,\ 0.3,\ 0.4) & (0.6,\ 0.2, 0.4) & (0.7,\ 0.4,\ 0.5) \\ A_5 & (0.8,\ 0.4,\ 0.4) & (0.6,\ 0.4,\ 0.4) & (0.7,\ 0.4,\ 0.4) & (0.8,\ 0.5,\ 0.5) \end{pmatrix} \quad (16)$$

Step: 5. Determinate the weight of the attributes

By using Equations (9) and (10), we determine the weights of the four attribute as follows:

$$w_1 = \frac{(1 \div 0.26)}{25} \approx 0.16,\ w_2 = \frac{(1 \div 0.11)}{25} \approx 0.37,\ w_3 = \frac{(1 \div 0.20)}{25} \approx 0.20,\ w_4 = \frac{(1 \div 0.15)}{25} \approx 0.27.$$

Step: 6. Calculate the weighted SVNS cross entropy matrix

Using Equation (2) and weights of attributes, we calculate the weighted NS-cross entropy values between ideal matrix and weighted aggregated decision matrix.

$$^{NS}M_{CE}^{w} = \begin{pmatrix} 0.195 \\ 0.198 \\ 0.168 \\ 0.151 \\ 0.184 \end{pmatrix} \quad (17)$$

Step: 7. Rank the priority

The cross entropy values of alternatives are arranged in increasing order as follows:

$$0.151 < 0.168 < 0.184 < 0.195 < 0.198.$$

Alternatives are then preference ranked as follows:

$$A_4 > A_3 > A_5 > A_1 > A_2.$$

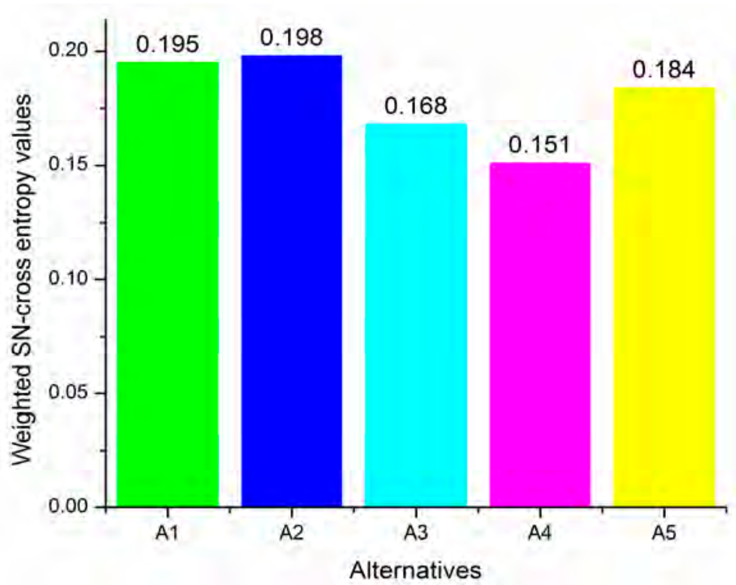
Step: 8. Select the best alternative

From step 7, we identify A_4 is the best alternative. Hence, Food enterprises (A_4) is the best alternative for investment.

In Figure 2, we draw a bar diagram to represent the cross entropy values of alternatives which shows that A_4 is the best alternative according our proposed strategy.

In Figure 3, we represent the relation between cross entropy values and acceptance values of alternatives. The range of acceptance level for five alternatives is taken by five points. The high acceptance level of alternatives indicates the best alternative for acceptance and low acceptance level of alternative indicates the poor acceptance alternative.

We see from Figure 3 that alternative A_4 has the smallest cross entropy value and the highest acceptance level. Therefore A_4 is the best alternative for acceptance. Figure 3 indicates that alternative A_2 has highest cross entropy value and lowest acceptance value that means A_2 is the worst alternative. Finally, we conclude that the relation between cross entropy values and acceptance value of alternatives



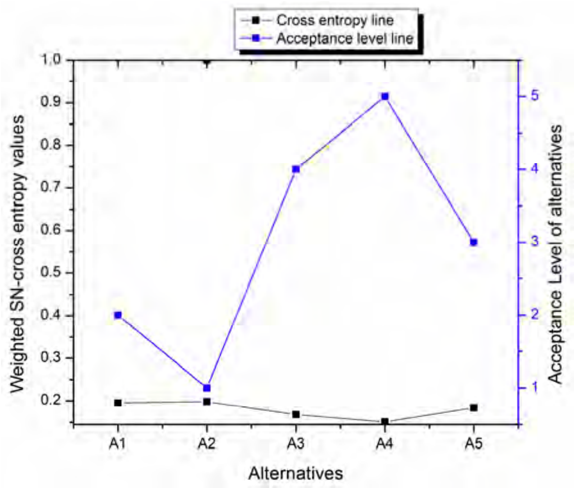


Figure 3. Relation between weighted NS-cross entropy values and acceptance level line of alternatives.

6. Comparative Study and Discussion

In literature only two MADM strategies [144,145] have been proposed. No MADGM strategy is available. So the proposed MAGDM is novel and non-comparable with the existing cross entropy under SVNS for numerical example.

i. The MADM strategies [144,145] are not applicable for MAGDM problems. The proposed MAGDM strategy is free from such drawbacks.
ii. Ye [144] proposed cross entropy that does not satisfy the symmetrical property straightforward and is undefined for some situations but the proposed strategy satisfies symmetric property and is free from undefined phenomenon.
iii. The strategies [144,145] cannot deal with the unknown weight of the attributes whereas the proposed MADGM strategy can deal with the unknown weight of the attributes
iv. The strategies [144,145] are not suitable for dealing with the unknown weight of decision-makers, whereas the essence of the proposed NS-cross entropy-based MAGDM is that it is capable of dealing with the unknown weight of the decision-makers.

7. Conclusions

In this paper, we have defined a novel cross entropy measure in SVNS environment. The proposed cross entropy measure in SVNS environment is free from the drawbacks of asymmetrical behavior and undefined phenomena. It is capable of dealing with the unknown weight of attributes and the unknown weight of decision-makers. We have proved the basic properties of the NS-cross entropy measure. We also defined weighted NS-cross entropy measure and proved its basic properties. Based on the weighted NS-cross entropy measure, we have developed a novel MAGDM strategy to solve neutrosophic multi-attribute group decision-making problems. We have at first proposed a novel MAGDM strategy based on NS-cross entropy measure with technique to determine the unknown weight of attributes and the unknown weight of decision-makers. Other existing cross entropy measures [144,145] can deal only with the MADM problem with single decision-maker and known weight of the attributes. So in general, our proposed NS-cross entropy-based MAGDM strategy is not comparable with the existing cross-entropy-based MADM strategies [144,145] under the single-valued neutrosophic environment. Finally, we solve a MAGDM problem to show the feasibility, applicability and efficiency of the proposed MAGDM strategy. The proposed NS-cross entropy-based MAGDM can be applied in teacher selection, pattern recognition, weaver selection, medical treatment selection options, and other practical problems. In future study, the proposed NS-cross entropy-based MAGDM strategy can be also extended to the interval neutrosophic set environment.

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Rough Standard Neutrosophic Sets: an Application on Standard Neutrosophic Information Systems

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Abstract: A rough fuzzy set is the result of approximation of a fuzzy set with respect to a crisp approximation space. It is mathematical tool for the knowledge discovery in the fuzzy information systems. In this paper, we introduce the concepts of rough standard neutrosophic sets, standard neutrosophic information system and give the knowledge discovery on standard neutrosophic information system based on rough standard neutrosophic sets.

Keywords: rough set, standard neutrosophic set, rough standard neutrosophic set, standard neutrosophic information systems

1. INTRODUCTION

Rough set theory was introduced by Pawlak in 1980s [1]. It becomes a usefully mathematical tool for data mining, especially for redundant and uncertain data. At first, the establishment of the rough set theory is based on equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, is the basis for the construction of upper and lower approximation of the subset of universal set.

Fuzzy set theory was introduced by Zadeh since 1965 [2]. Immediately, it became a useful method to study in the problems of imprecision and uncertainty. Since, a lot of new theories treating imprecision and uncertainty have been introduced. For instance, Intuitionistic fuzzy sets were introduced in1986, by K. Atanassov [3], which is a generalization of the notion of a fuzzy set. When fuzzy set give the degree of membership of an element in a given set, Intuitionistic fuzzy set give a degree of membership and a degree of non-membership of an element in a given set. In 1999 [14], Sarandache gave the concept of neutrosophic set which generalized fuzzy set and intuitionistic fuzzy set. This new concept is difficult to apply in the real appliction. It is a set in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F). Over time, the subclass of neutrosophic sets was proposed. They are also more advantageous in the practical application. Wang et al. [15] proposed interval neutrosophic sets and some operators of them. Wang et al. [16] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [17] defined the concept of simplified neutrosophic sets, It is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and which lies between [0, 1] and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013, Cuong and Kreinovich introduced the concept of picture fuzzy set [4], in which a given set were to be in with three memberships: a degree of positive membership, a degree of negative membership, and a degree of neutral membership of an element in this set. After that, Son given the application of the picture fuzzy set in the clustering problem [5]. We also regard picture fuzzy set as standard neutrosophic set. In addition, combining rough set and fuzzy set has also many interesting results. The approximation of rough (or fuzzy) sets in fuzzy approximation space give us the fuzzy rough set [6, 7, 8]; and the approximation of fuzzy sets in crisp approximation space give us the rough fuzzy set [6, 7]. Wu et al, [8] present a general framework for the study of fuzzy rough sets in both constructive and axiomatic approaches. By the same, Wu and Xu were investigated the fuzzy topological structures on the rough fuzzy sets [9], in which both constructive and axiomatic approaches are used. In 2012, Xu and Wu were also investigated the rough intuitionistic fuzzy set and the intuitionistic fuzzy topologies in crisp approximation spaces [10]. In 2015, Thao et al. introduces the rough picture fuzzy set is the result of approximation of a picture fuzzy set with respect to a crisp approximation space [12].

In this paper, we introduce the concept of standard neutrosophic information system, study the knowledge discovery of standard neutrosophic information system based on rough standard neutrosophic sets. The remaining part of this paper is organized as following: we recall basic notions of rough set, standard neutrosophic set and rough standard neutrosophic set on the crisp approximation space, respectively, in section 2 and section 3. In section 4, we introduce the basic concepts of standard neutrosophic information system. Finally, we investigate the knowledge discovery of standard neutrosophic information system and the knowledge reduction and extension of the standard neutrosophic information system in section 5 and section 6, respectively.

II. BASIC NOTIONS OF SN SET AND ROUGH SET

In this paper, we denote be a nonempty set called the universe of discourse. The class of all subsets of will be denoted by and the class of all fuzzy subsets of will be denoted by

Definition 1. [4]. A standard neutrosophic set (SN - set) on the universe is an object of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) | x \in U\}$$

where $\mu_A(x), \eta_A(x), \gamma_A(x) \in [0,1]$ is called the "degree of positive membership, the degree of neutral membership, the degree of negative membership of x in A", and $\mu_A(x), \eta_A(x), \gamma_A(x) \le 1$.
The family of all standard neutrosophic set in U is denoted by PFS(U). The complement of a picture fuzzy set A is $\sim A = \{(x, \gamma_A(x), \eta_A(x), \mu_A(x)) | x \in U$.

The operators on PFS(U) was introduced [4].

Definition 2. (Lattice $(D^*, \le_*)$. Let $D^* = \{(x_1, x_2, x_3) \in [0,1]^3 : x_1 + x_2 + x_3 < 1\}$. We define a relation $\le_{D^*}$ on D^* as follows: $\forall x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in D^*$ then $x \le_{D^*} y$ iff (or $(x_1 < y_1, x_3 \ge y_3)$ or $(x_1 = y_1, x_3 > y_3)$ or $(x_1 = y_1, x_3 = y_3, x_2 \le y_2)$ and $x = y \Leftrightarrow (x_1 = y_1) \wedge (x_2 = y_2) \wedge (x_3 = y_3)$. We have $(D^*, \le_*)$ is a lattice with $0_{D^*} = (0,0,1)$, $1_{D^*} = (1,0,0)$.

Definition 3.

(i) Negative of $x = (x_1, x_2, x_3) \in D^*$ is $\overline{x} = (x_3, x_2, x_1)$

(ii) $\forall x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in D^*$ we have

$$x \wedge y = (x_1 \wedge y_1, x_2 \vee y_2, x_3 \wedge y_3)$$

$$x \vee y = (x_1 \vee y_1, x_2 \wedge y_2, x_3 \vee y_3)$$

We have some properties of those operators.

Lemma 1.

a) For all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in D^*$ we have $\overline{x \wedge y} = \bar{x} \vee \bar{y}$ and $\overline{x \vee y} = \bar{x} \wedge \bar{y}$

b) For all $x, y, u, v \in D^*$ and $x \le_{D^*} u, y \le_{D^*} v$ we have $x \wedge y \le_{D^*} u \wedge v$ and $x \vee y \le_{D^*} u \vee v$.

Now, we mention the level sets of the SN-sets. Where $(\alpha, \beta, \theta) \in D^*$, we define:

- (α, β, θ) −level cut set of the SN- set $A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) | x \in U\}$ as follows:

$$A_\theta^{\alpha,\beta} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \ge (\alpha, \beta, \theta)\}$$

- strong (α, β, θ) −level cut set of the SN- set A as follows:

$$A_{\theta+}^{\alpha+,\beta+} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \ge (\alpha, \beta, \theta)\}$$

- (α, β, θ) −level cut set of the SN- set A as

$$A_\theta^{\alpha^+,\beta} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) \le \theta\}$$

- $(\alpha, \beta, \theta^+)$ − level cut set of the SN-set A as

$$A_{\theta^+}^{\alpha^+,\beta} = \{x \in U | \mu_A(x) \ge \alpha, \gamma_A(x) < \theta\}$$

- (α^+, θ^+) − level cut set of the SN- set A as

$$A_{\theta^+}^{\alpha^+} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) < \theta\}$$

- α – level cut set of the SN- set A as
$$A^\alpha = \{x \in U | \mu_A(x) \geq \alpha\}$$
- the strong α – level cut set of the SN- set A as
$$A^\alpha = \{x \in U | \mu_A(x) > \alpha\}$$
- θ – level low cut set of the degree of negative membership of x in A as $A_\theta = \{x \in U | \gamma_A \leq \theta\}$
- strong θ – level low cut set of the degree of negative membership of x in A as
$$A_\theta = \{x \in U | \gamma_A < \theta\}$$

Definition 4.[6]. Let (U, R) be a crisp approximation space. For each crisp set $A \subset U$, we define the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows $\overline{R}(A) = \{x \in U: R_s(x) \cap A \neq \emptyset$, $\underline{R}(A) = \{x \in U: R_s(x) \subseteq A$ where $R_s(x) = \{y \in U | (x, y) \in R\}$.

III. ROUGH STANDARD NEUTROSOPHIC SET

A rough SN- set is the approximation of a SN-set w. r. t a crisp approximation space. Here, we consider the upper and lower approximations of a SN-set in the crisp approximation spaces together with their membership functions, respectively.

Definition 5. Let (U, R) be a crisp approximation space. For each $A \in PF(U)$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows:

$$\overline{R}(A) = \{(x, \mu_{\overline{R}(A)}(x), \eta_{\overline{R}(A)}(x), \gamma_{\overline{R}(A)}(x)) | x \in U\}$$

$$\underline{R}(A) = \{(x, \mu_{\underline{R}(A)}(x), \eta_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x) | x \in U\}$$

where $\mu_{\overline{R}(A)}(x) = \vee_{y \in R_s(x)} \mu_A(y)$, $\eta_{\overline{R}(A)}(x) = \wedge_{y \in R_s(x)} \eta_A(y)$, $\theta_{\overline{R}(A)}(x) = \wedge_{y \in R_s(x)} \theta_A(y)$

and $\mu_{\underline{R}(A)}(x) = \wedge_{y \in R_s(x)} \mu_A(y)$, $\eta_{\underline{R}(A)}(x) = \wedge_{y \in R_s(x)} \eta_A(y)$, $\theta_{\underline{R}(A)}(x) = \vee_{y \in R_s(x)} \theta_A(y)$.

Some basic properties of rough SN-set approximation operators represent in [12].

IV. THE STANDARD NEUTROSOPHIC INFORMATION SYSTEMS

In this section, we introduce a new concept: standard neutrosophic information system (SNIS).

Let (U, A, F) be a classical information system. Here U is the (nonempty) set of objects, i.e., $U = \{u_1, u_2, \dots, u_{10}\}$, $A = \{a_1, a_2, \dots, a_m\}$ is the attribute set, and F is the relation set of U and A, i.e., $F = \{f_j: U \to V_j, j = 1,2, \dots, m\}$ where V_j is the domain of the attribute $a_j, j = 1,2, \dots, m$.

We call (U, A, F, D, G) an information system or decision table, where (U, A, F) is the classical information system, A is the condition attribute set and D is the decision attribute set, i.e., $D = \{d_1, d_2, \dots, d_p\}$ and G is the relation set of U and D, i.e., $G = \{g_j: U \to V'_j, j = 1,2, \dots, m\}$ where V'_j is the domain of the attribute $d_j, j = 1,2, \dots, p$.

Let (U, A, F, D, G) be the information system. For $B \subset A \cup D$, we define a relation, denoted $R_B = IND(B)$, as follows, $\forall x, y \in U$: $R_B = \{(x, y) \in U^2: f_j(x) = f_j(y), \forall j \in \{j: a_j \in B\}$.

The equivalence class of $x \in U$ based on R_B is $[x]_B = \{y \in U: yR_Bx\}$.

Here, we consider $R_A = IND(A)$, $R_D = IND(D)$. If $R_A \subseteq R_D$, i.e., for any $[x]_A, x \in U$ there exists $[x]_D$ such that $[x]_A \subseteq [x]_D$, then the information system is called a consistent information system, other called an inconsistent information system.

Let (U, A, F, D, G) be the information system, where (U, A, F) be a classical information system. If $D = \{D_k \mid k = 1, 2, \dots, q\}$, where D_k is a fuzzy subset of U, then (U, A, F, D, G) be the fuzzy information system. If $D = \{D_k \mid k = 1, 2, \dots, q\}$ where D_k is an intutionistic fuzzy subset of U, then (U, A, F, D, G) be an intuitionistic fuzzy information system.

Definition 6. Let (U, A, F, D, G) be the information system or decision table, where (U, A, F) be a classical information system. If $D = \{D_k | k = 1,2, \dots, p\}$ where D_k is a SN- subset of U and G is the relation set of U and D, then (U, A, F, D, G) is called a SN information system.

Example 1. The following table 2 gives a SN information system, where the objects set $\underline{RP}_A(D_2)(x) = (0.15, 0.05, 0.6)$ condition attribute set is $A = \{a_1, a_2, a_3\}$ and the decision attribute set is $D = \{D_1, D_2, D_3\}$, where $D_k | k = 1,2, \dots, p$ is the SN subsets of U.

***Table 1:** A SN- information system*

U	a_1	a_2	a_3	D_1	D_3	D_4
u_1	3	2	1	(0.2,0.3,0.5)	(0.15,0.6,0.2)	(0.4,0.05,0.5)
u_2	1	3	2	(0.3,0.1,0.5)	(0.3,0.3,0.3)	(0.35,0.1,0.4)
u_3	3	2	1	(0.6,0,0.4)	(0.3,0.05,0.6)	(0.1,0.45,0.4)
u_4	3	3	1	(0.15,0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
u_5	2	2	4	(0.05,0.2,0.7)	(0.2,0.4,0.3)	(0.05,0.4,0.5)
u_6	2	3	4	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)
u_7	1	3	2	(0.25,0.3,0.4)	(1,0,0)	(0.3,0.3,0.4)
u_8	2	2	4	(0.1,0.6,0.2)	(0.25,0.3,0.4)	(0.4,0,0.6)
u_9	3	2	1	(0.45,0.1,0.45)	(0.25,0.4,0.3)	(0.2,0.5,0.3)
u_{10}	1	3	2	(0.05,0.05,0.9)	(0.4,0.2,0.3)	(0.05,0.7,0.2)

V. THE KNOWLEDGE DISCOVERY IN THE SNIS

In this section, we will give some results about the knowledge discovery for a SNIS by using the basic theory of rough standard neutrosophic set in section 3. Throughout this paper, let (U, A, F, D, G) be the standard neutrosophic information system and $B \subseteq A$ we denote $\underline{RP_B}(D_j)$ is the lower rough standard neutrosophic approximation of $D_j \in PFS(U)$ on approximation space (U, R_B).

Theorem 1. Let (U, A, F, D, G) be the standard neutrosophic information system and $B \subset A$. If for any $x \in U$: $\left(\mu_{D_i}(x), \eta_{D_i}(x), \gamma_{D_i}(x)\right) \geq (\alpha(x), \beta(x), \theta(x)) = \underline{RP_B}(D_i)(x) > \underline{RP_B}(D_j)(x), i \neq j$ then $[x]_B \cap (D)_{\alpha(x)}^{\theta(x),o} \neq \emptyset$ and $[x]_B \subset (D)_{\theta(x)}^{\alpha(x)\beta(x)}$ where $(\alpha(x), \beta(x), \theta(x)) \in D^*$.

Let (U, A, F, D, G) be the standard neutrosophic information system, R_A is the equivalence classes which induced by the condition attribute set A, and the universe is divided by R_A as following: $U / R_A = \{X_1, X_2 \ldots, X_k\}$. Then the approximation of SN decision denoted as, for all $i = 1, 2, \ldots, k$.

$$\underline{RP_A}(D(X_i)) = \left(\underline{RP_A}(D_1(X_i)), \underline{RP_A}(D_2(X_i)), \ldots, \underline{RP_A}(D_q(X_i))\right)$$

Example 2. We consider the SN information system in Table 1. The equivalent classes

$$U/R_A = \{X_1 = \{u_1, u_3, u_9\}, X_2 = \{u_2, u_7, u_{10}\},$$
$$X_3 = \{u_4\}, X_4 = \{u_5, u_8\}, X_5 = \{u_6\}\}$$

The approximation of the SN decision is as follows:

Table 2: *The approximation of the SN decision*

U / R_A	$\underline{RP_A}(D_1(X_i))$	$\underline{RP_A}(D_2(X_i))$	$\underline{RP_A}(D_3(X_i))$
X_1	(0.2,0,0.5)	(0.15,0.05,0.6)	(0.1,0.05,0.5)
X_2	(0.05,0.05,0.9)	(0.3,0.1,0.3)	(0.05,0.1,0.4)
X_3	(0.15, 0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
X_4	(0.05,0.2,0.7)	(0.2,0.3,0.4)	(0.05,0,0.6)
X_5	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)

VI. THE KNOWLEDGE REDUCTION AND EXTENSION OF SNIS

Definition 7.

(i) Let (U, A, F) be the classical information system and $B \subset A$. B is called the SN reduction of the classical information system (U, A, F), if B is the minimum set which satisfies the following relations: for any $X \in PFS(U), x \in U$.

$$\underline{RP_A}(X) = \underline{RP_B}(X), \overline{RP_A}(X) = \overline{RP_B}(X)$$

(ii) B is called the SN lower approximation reduction of the classical information system (U, A, F), if B is the minimum set which satisfies the following relations: for any $X \in PFS(U), x \in U$ we have $\underline{RP_A}(X) = \underline{RP_B}(X)$.

(iii) B is called the standard neutrosophic upper approximation reduction of the classical information system (U, A, F), if B is the minimum set which satisfies the following relations: for any $X \in PFS(U), x \in U$ $\overline{RP_A}(X) = \overline{RP_B}(X)$.

Where $\underline{RP_A}(X), \underline{RP_B}(X), \overline{RP_A}(X), \overline{RP_B}(X)$ are SN-lower and SN- upper approximation sets of SN- set $X \in PFS(U)$ based on R_A, R_B, respectively.

Now, we express the knowledge of the knowledge reduction of SNIS by introducing the discernibility matrix.

Definition 8. Let (U, A, F, D, G) be the SN- information system. Then $M = \left[D_{ij}\right]_{k \times k}$ where

$$D_{ij} = \begin{cases} \{a_l \in A : f_l(X_i) \neq f_l(X_j)\}; & g_{X_i}(D_t) \neq g_{X_j}(D_t) \\ A & : g_{X_i}(D_t) = g_{X_j}(D_t) \end{cases}$$

is called the discernibility matrix of (U, A, F, D, G) (where $g_{X_i}(D_t)$ is the maximum of $\underline{RP}_A(D(X_i))$ obtained at D_t, i.e., $g_{X_i}(D_t) = \underline{RP}_A(D_t(X_i))$

$= \max\{\underline{RP}_A(D_z(X_i)), z = 1, 2, \ldots, q\}$)

Definition 9. Let (U, A, F, D, G) be the SNIS, for any $B \subset A$, if the following relations holds, for any $x \in B$:

$\underline{RP}_B(D_i)(x) > \underline{RP}_B(D_j)(x) - \underline{RP}_A(D_i)(x) > \underline{RP}_A(D_j)(x)(i \neq j)$

then B is called the consistent set of A.

Theorem 2. Let (U, A, F, D, G) be the standard neutrosophic information system. If there exists a subset $B \subset A$ such that $B \cap D_{ij} \neq \emptyset$, then B is the consistent set of A.

Definition 10. Let (U, A, F, D, G) be the SNIS

$$D_{ij}^C = \begin{cases} \{a_l \in A : f_l(X_i) = f_l(X_j)\}; & g_{X_i}(D_t) \neq g_{X_j}(D_t) \\ \emptyset & : g_{X_i}(D_t) = g_{X_j}(D_t) \end{cases}$$

is called the discernibility matrix of (U, A, F, D, G) (where $g_{X_i}(D_t)$ is the maximum of $\underline{RP}_A(D(X_i))$ obtained at D_t, i.e.,

$g_{X_i}(D_t) = \underline{RP}_A(D_t(X_i)) = \max\{\underline{RP}_A(D_z(X_i)), z = 1, 2, \ldots, q\}$).

Theorem 3. Let (U, A, F, D, G) be the SNIS. If there exists a subset $B \subset A$ such that $B \cap D_{ij}^C = \emptyset$, , then B is the consistent set of A.

The extension of a SNIS present on the following definition:

Definition 11.

(i) Let (U, A, F) be the classical information system and $B \subseteq A$. B is called the SN extension of the classical information system (U, A, F), if B satisfies the following relations: for any $X \in PFS(U)$,

$$\underline{RP}_B(X) = \underline{RP}_A(X), \overline{RP}_B(X) = \overline{RP}_A(X)$$

(i) B is called the SN lower approximation extension of the classical information system (U, A, F), if $A \subset B$ satisfies the following relations: for any $X \in PFS(U)$, $\underline{RP}_B(X) = \underline{RP}_A(X)$

(ii) B is called the SN upper approximation extension of the classical information system (U, A, F), if $A \subset B$ satisfies the following relations: for any $X \in PFS(U)$, $\overline{RP}_B(X) = \overline{RP}_A(X)$

where $\underline{RP}_B(X), \underline{RP}_A(X)$ $\overline{RP}_B(X), \overline{RP}_A(X)$ are SN lower and upper approximation sets of SN set $X \in PFS(U)$ based on R_A, R_B, respectively.

We can be easily obtained the following result.

Definition 12. Let (U, A, F) be the classical information system, for any hyper set B, such that $A \subseteq B$, if A is the SN-reduction of the classical information system (U, B, F), then (U, B, F) is the SN extension of (U, A, F), but not conversely necessary.

Example 3. In the approximation of the SN decision in Table 1, Table 2. Let $B = \{a_1, a_2\}$ then we obtained the family of all equivalent classes of U based on the equivalent relation $R_B = IND(B)$ as follows

$U / R_B = \{X_1 = \{u_1, u_3, u_9\}, X_2 = \{u_2, u_7, u_{10}\}, X_3 = \{u_4\}, X_4 = \{u_5, u_8\}, X_5 = \{u_6\}\}$

We can get the approximation value given in Table 3.

Table 3: *The approximation of the SN decision*

U / R_B	$\underline{RP}_B(D_1(X_i))$	$\underline{RP}_B(D_2(X_i))$	$\underline{RP}_B(D_3(X_i))$
X_1	(0.2,0,0.5)	(0.15,0.05,0.6)	(0.1,0.05,0.5)
X_2	(0.05,0.05,0.9)	(0.3,0.1,0.3)	(0.05,0.1,0.4)
X_3	(0.15, 0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
X_4	(0.05,0.2,0.7)	(0.2,0.3,0.4)	(0.05,0,0.6)
X_5	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)

It is easy to see that B satisfies Definition 7 (ii), i.e., B is the SN lower reduction of the classical information system (U, A, F).

The discernibility matrix of the SN information system (U, A, F, D, G) will be presented in Table 4.

Table 4: *The discernibility matrix of the SNIS*

U/R_B	X_1	X_2	X_3	X_4	X_5
X_1	A				
X_2	A	A			
X_3	$\{a_2\}$	$\{a_1, a_3\}$	A		
X_4	$\{a_1, a_3\}$	A	A	A	
X_5	$\{a_1, a_3\}$	A	A	$\{a_2\}$	A

VII. CONCLUSION

In this paper, we introduce the concept of SN-information system, study the knowledge discovery of standard neutrosophic information system based on rough SN- sets. We investigate some problems of the knowledge discovery of SNIS and the knowledge reduction and extension of the SNIS in section 6. In the future, we introduce the application of this study in the practical problems.

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Shortest Path Problem under Interval Valued Neutrosophic Setting

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ABSTRACT

This paper presents a study of neutrosophic shortest path with interval valued neutrosophic number on a network. A proposed algorithm also gives the shortest path length using ranking function from source node to destination node. Here each arc length is assigned to interval valued neutrosophic number. Finally, a numerical example has been provided for illustrating the proposed approach

1. INTRODUCTION

Neutrosophy was pioneered by Smarandache in 1998. It is a branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Smarandache generalized the concepts of fuzzy sets [28] and intuitionistic fuzzy set [25] by adding an independent indeterminacy-membership. Neutrosophic set is a powerful tool to deal with incomplete, indeterminate and inconsistent information in real world, which have attracted the widespread concerns for researchers. The concept of neutrosophic set is characterized by three independent degrees namely truth-membership degree (T), indeterminacy-membership degree (I), and falsity-membership degree (F). Later on, Smarandache extended the neutrosophic set to neutrosophic overset, underset, and offset [46].

The concept of single valued neutrosophic theory has proven to be useful in many different field such as the decision making problem, medical diagnosis and so on. Later on, the concept of interval valued neutrosophic sets [15] (IVNS for short) appear as a generalization of fuzzy sets, intuitionistic fuzzy set, interval valued fuzzy sets [20], interval valued intuitionistic fuzzy sets [26] and single valued neutrosophic sets. Interval valued neutrosophic set is a model of a neutrosophic set, which can be used to handle uncertainty in fields of scientific, environment and engineering. This concept is characterized by the truth-membership, the indeterminacy-membership and the falsity-membership independently, which is a powerful tool to deal with incomplete, indeterminate and inconsistent information.

3. EXPERIMENTAL

The shortest path problem is a fundamental algorithmic problem, in which a minimum weight path is computed between two nodes of a weighted, directed graph. This problem has been studied for a long time and has attracted researchers from various areas of interests such operation research, computer science, communication network and so on. There are many shortest path problems [2, 3, 4, 12, 31, 45] that have been studied with different types of input data, including fuzzy set, intuitionistic fuzzy sets, trapezoidal intuitionistic fuzzy sets vague set. Till now, few research papers deal with shortest path in neutrosophic environment. Broumi et al.

4. CONCLUSION

In this paper we developed an algorithm for solving shortest path problem on a network with interval valued neutrosophic arc lengths. The process of ranking the path is very useful to make decisions in choosing the best of all possible path alternatives. We have explained the method by an example with the help of a hypothetical data. Further, we plan to extend the following algorithm of interval neutrosophic shortest path problem in an interval valued bipolar neutrosophic environment.

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An Application of Complex Neutrosophic Sets to the Theory of Groups

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Abstract. In this article we introduce the concept of complex neutrosophic subgroups (normal subgroups). We define the notion of alpha-cut of complex neutrosophic set, give examples and study some of its related results. We also define the Cartesian product of complex neutrosophic subgroups. Furthermore, we introduce the concept of image and preimage of complex neutrosophic set and prove some of its properties.

Keywords. Complex fuzzy sets; Complex neutrosophic sets; Neutrosophic subgroups; Complex neutrosophic subgroups; Complex neutrosophic normal subgroups.

1. Introduction

[1], In 1965, Zadeh presented the idea of a fuzzy set. [2], Atanassov's in 1986, initiated the notion of intuitionistic fuzzy set which is the generalization of a fuzzy set. Neutrosophic set was first proposed by Smarandache in 1999 [5], which is the generalization of fuzzy set and intuitionistic fuzzy set. Neutrosophic set is characterized by a truth membership function, an indeterminacy membership function and a falsity membership function. In 2002, the Ramot et al. [8], generalized the concept of fuzzy set and introduced the notion of complex fuzzy set. There are many researchers which have worked on complex fuzzy set for instance, Buckly [6], Nguyen et al. [7] and Zhang et al. [9]. In contrast, Ramot et al. [8] presented an innovative concept that is totally different from other researchers, in which the author extended the range of membership function to the unit circle in the complex plane, unlike the others who limited to. Furthermore to solve enigma they also added an extra term which is called phase term in translating human language to complex valued functions on physical terms and vice versa. Abd Uazeez et al. in 2012 [10], added the non-membership term to the idea of complex fuzzy set which is known as complex intuitionistic fuzzy sets, the range of values are extended to the unit circle in complex plan for both membership and non-membership functions instead of [0, 1]. In 2016, Mumtaz Ali et al. [12], extended the concept of complex fuzzy set, complex intuitionistic fuzzy set, and introduced the concept of complex neutrosophic sets which is a collection of complex-valued truth membership function, complex-valued indeterminacy membership function and complex-valued falsity membership function. Further in 1971, Rosenfeld [3], applied the concept of fuzzy set to groups and introduced the concept of fuzzy groups. The author defined fuzzy subgroups and studied some of its related properties. Vildan and Halis in 2017 [13], extended the concept of fuzzy subgroups on the base of neutrosophic sets and initiated the notion of neutrosophic subgroups.

Due to the motivation and inspiration of the above discussion. In this paper we introduce the concept of a complex neutrosophic subgroups (normal subgroups). We have give examples and study some related results. We also study the concept of Cartesian product of complex neutrosophic subgroups, image and preimage of complex neutrosophic set and alpha-cut of complex neutrosophic set with the help of examples and prove some of its properties.

2. Preliminaries

Here in this part we gathered some basic helping materials.

Definition 2.1. *[1] A function f is defined from a universe $\mathcal{X}$ to a closed interval $[0,1]$ is called a fuzzy set,i.e., a mapping:*

$$f : \mathcal{X} \longrightarrow [0,1].$$

Definition 2.2. *[8] A complex fuzzy set (CFS) $\mathbb{C}$ over the universe $\mathcal{X}$, is defined an object of the form:*

$$\mathbb{C} = \{(x, \mu_{\mathbb{C}}(x)) : x \in \mathcal{X}\}$$

where $\mu_{\mathbb{C}}(x) = r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}$, here the amplitude term $r_{\mathbb{C}}(x)$ and phase term $\omega_{\mathbb{C}}(x)$, are real valued functions, for every $x \in \mathcal{X}$, the amplitude term $\mu_{\mathbb{C}}(x) : \mathcal{X} \to [0,1]$ and phase term $\omega_{\mathbb{C}}(x)$ lying in the interval $[0, 2\pi]$.

Definition 2.3. *[11] Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be any two complex Atanassov's intuitionistic fuzzy sets (CAIFSs) over the universe $\mathcal{X}$, where*

$$\mathbb{C}_1 = \left\{\left\langle x, r_{\mathbb{C}_1}(x) \cdot e^{i\nu_{\mathbb{C}_1}(x)}, k_{\mathbb{C}_1}(x) \cdot e^{i\omega_{\mathbb{C}_1}(x)}\right\rangle : x \in \mathcal{X}\right\}$$

and

$$\mathbb{C}_2 = \left\{\left\langle x, r_{\mathbb{C}_2}(x) \cdot e^{i\nu_{\mathbb{C}_2}(x)}, k_{\mathbb{C}_2}(x) \cdot e^{i\omega_{\mathbb{C}_2}(x)}\right\rangle : x \in \mathcal{X}\right\}.$$

Then

1. Containment:

$$\mathbb{C}_1 \subseteq \mathbb{C}_2 \Leftrightarrow r_{\mathbb{C}_1}(x) \le r_{\mathbb{C}_2}(x), k_{\mathbb{C}_1}(x) \ge k_{\mathbb{C}_2}(x) \text{ and } \nu_{\mathbb{C}_1}(x) \le \nu_{\mathbb{C}_2}(x), \omega_{\mathbb{C}_1}(x) \ge \omega_{\mathbb{C}_2}(x).$$

2. Equal:

$$\mathbb{C}_1 = \mathbb{C}_2 \Leftrightarrow r_{\mathbb{C}_1}(x) = r_{\mathbb{C}_2}(x), k_{\mathbb{C}_1}(x) = k_{\mathbb{C}_2}(x) \text{ and } \nu_{\mathbb{C}_1}(x) = \nu_{\mathbb{C}_2}(x), \omega_{\mathbb{C}_1}(x) = \omega_{\mathbb{C}_2}(x).$$

Definition 2.4. *[12] Let $\mathcal{X}$ be a universe of discourse, and $x \in \mathcal{X}$. A complex neutrosophic set (CNS) $\mathbb{C}$ in $\mathcal{X}$ is characterized by a complex truth membership function $\mathbb{C}_T(x) = p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}$, a complex indeterminacy membership function $\mathbb{C}_I(x) = q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}$ and a complex falsity membership function $\mathbb{C}_F(x) = r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}$. The values $\mathbb{C}_T(x), \mathbb{C}_I(x), \mathbb{C}_F(x)$ may lies all within the unit circle in the complex plane, where $p_{\mathbb{C}}(x)$, $q_{\mathbb{C}}(x)$, $r_{\mathbb{C}}(x)$ and $\mu_{\mathbb{C}}(x)$, $\nu_{\mathbb{C}}(x)$ $\omega_{\mathbb{C}}(x)$ are amplitude terms and phase terms, respectively, and where $p_{\mathbb{C}}(x)$, $q_{\mathbb{C}}(x)$, $r_{\mathbb{C}}(x) \in [0,1]$, such that, $0 \le p_{\mathbb{C}}(x) + q_{\mathbb{C}}(x) + r_{\mathbb{C}}(x) \le 3$ and $\mu_{\mathbb{C}}(x)$, $\nu_{\mathbb{C}}(x)$ $\omega_{\mathbb{C}}(x) \in [0, 2\pi]$.*

The complex neutrosophic set can be represented in the form as:

$$\mathbb{C} = \left\{\left\langle \begin{array}{c} x, \mathbb{C}_T(x) = p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}, \mathbb{C}_I(x) = q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}, \\ \mathbb{C}_F(x) = r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \end{array} \right\rangle : x \in \mathcal{X}\right\}.$$

Example 2.5. *Let $\mathcal{X} = \{x_1, x_2, x_3\}$ be the universe set and $\mathbb{C}$ be a complex neutrosophic set which is given by:*

$$\mathbb{C} = \left\{ \begin{array}{c} \left\langle x_1, 0.2e^{0.5\pi i}, 0.3e^{0.6\pi i}, 0.4e^{0.8\pi i}\right\rangle, \left\langle x_2, 0.4e^{0.6\pi i}, 0.5e^{1.3\pi i}, 0.1e^{0.6\pi i}\right\rangle, \\ \left\langle x_3, 0.1e^{0.6\pi i}, 0.3e^{0.9\pi i}, 0.9e^{0.7\pi i}\right\rangle \end{array} \right\}.$$

Definition 2.6. *[3] Let $\mathcal{G}$ be any group with multiplication and $\mathcal{F}$ be a fuzzy subset of a group $\mathcal{G}$, then $\mathcal{F}$ is called a fuzzy subgroup (FSG) of $\mathcal{G}$, if the following axioms are hold:*

(FSG1): $\mathcal{F}(x \cdot y) \geq \min\{\mathcal{F}(x), \mathcal{F}(y)\}$.

(FSG2): $\mathcal{F}(x^{-1}) \geq \mathcal{F}(x)$, $\forall\, x, y \in \mathcal{G}$.

Definition 2.7. *[13] Let $\mathcal{G}$ be any group with multiplication and $\mathcal{N}$ be a neutrosophic set on a group $\mathcal{G}$. Then $\mathcal{N}$ is called a neutrosophic subgroup (NSG) of $\mathcal{G}$, if its satisfy the following conditions:*

(NSG1): $\mathcal{N}(x \cdot y) \geq \mathcal{N}(x) \wedge \mathcal{N}(y)$, i.e.,

$T_{\mathcal{N}}(x \cdot y) \geq T_{\mathcal{N}}(x) \wedge T_{\mathcal{N}}(y)$, $I_{\mathcal{N}}(x \cdot y) \geq I_{\mathcal{N}}(x) \wedge I_{\mathcal{N}}(y)$ and $F_{\mathcal{N}}(x \cdot y) \leq F_{\mathcal{N}}(x) \vee F_{\mathcal{N}}(y)$.

(NSG2): $\mathcal{N}(x^{-1}) \geq \mathcal{N}(x)$, i.e.,

$T_{\mathcal{N}}(x^{-1}) \geq T_{\mathcal{N}}(x)$, $I_{\mathcal{N}}(x^{-1}) \geq I_{\mathcal{N}}(x)$ and $F_{\mathcal{N}}(x^{-1}) \leq F_{\mathcal{N}}(x)$, for all x and y in $\mathcal{G}$.

3. Complex Neutrosophic Subgroup

Note: It should be noted that through out in this section we use a capital letter $\mathbb{C}$ to denote a complex neutrosophic set:

$$\mathbb{C} = \left\{\left\langle T_{\mathbb{C}} = p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}, I_{\mathbb{C}} = q_{\mathbb{C}} \cdot e^{i\nu_{\mathbb{C}}}, F_{\mathbb{C}} = r_{\mathbb{C}} \cdot e^{i\omega_{\mathbb{C}}}\right\rangle\right\}.$$

Definition 3.1. *A complex neutrosophic set $\mathbb{C} = \left\{\left\langle T_{\mathbb{C}} = p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}, I_{\mathbb{C}} = q_{\mathbb{C}} \cdot e^{i\nu_{\mathbb{C}}}, F_{\mathbb{C}} = r_{\mathbb{C}} \cdot e^{i\omega_{\mathbb{C}}}\right\rangle\right\}$ on a group $(\mathcal{G}, \cdot)$ is known as a complex neutrosophic subgroup (CNSG) of $\mathcal{G}$, if for all elements $x, y \in \mathcal{G}$, the following conditions are satisfied:*

(CNSG1): $\mathbb{C}(xy) \geq \min\{\mathbb{C}(x), \mathbb{C}(y)\}$ i.e.,

(i) $p_{\mathbb{C}}(xy) \cdot e^{i\mu_{\mathbb{C}}(xy)} \geq \min\{p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}, p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)}\}$

(ii) $q_{\mathbb{C}}(xy) \cdot e^{i\nu_{\mathbb{C}}(xy)} \geq \min\{q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}, q_{\mathbb{C}}(y) \cdot e^{i\nu_{\mathbb{C}}(y)}\}$

(iii) $r_{\mathbb{C}}(xy) \cdot e^{i\omega_{\mathbb{C}}(xy)} \leq \max\{r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}, r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)}\}$

(CNSG2): $\mathbb{C}(x^{-1}) \geq \mathbb{C}(x)$ i.e.,

(iv) $p_{\mathbb{C}}(x^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x^{-1})} \geq p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}$

(v) $q_{\mathbb{C}}(x^{-1}) \cdot e^{i\nu_{\mathbb{C}}(x^{-1})} \geq q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}$

(vi) $r_{\mathbb{C}}(x^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x^{-1})} \leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}$.

Example 3.2. *Let $\mathcal{G} = \{1, -1, i, -i\}$ be a group under multiplication, and*

$$\mathbb{C} = \left\{ \begin{array}{c} \left\langle 1, 0.7e^{0.6\pi i}, 0.6e^{0.5\pi i}, 0.5e^{0.2\pi i}\right\rangle, \left\langle -1, 0.6e^{0.5\pi i}, 0.5e^{0.4\pi i}, 0.4e^{0.2\pi i}\right\rangle, \\ \left\langle i, 0.5e^{0.3\pi i}, 0.4e^{0.2\pi i}, 0.1e^{0.2\pi i}\right\rangle, \left\langle -i, 0.5e^{0.3\pi i}, 0.4e^{0.2\pi i}, 0.1e^{0.2\pi i}\right\rangle \end{array} \right\}$$

be a complex neutrosophic set on $\mathcal{G}$. Clearly $\mathbb{C}$ is a complex neutrosophic subgroup of $\mathcal{G}$.

3.1. Cartesian Product of Complex Neutrosophic Subgroups

Definition 3.3. *Let* $\mathbb{C}_1 = \langle \mathbb{C}_{1T}(x), \mathbb{C}_{1I}(x), \mathbb{C}_{1F}(x)\rangle$ *and* $\mathbb{C}_2 = \langle \mathbb{C}_{2T}(x), \mathbb{C}_{2I}(x), \mathbb{C}_{2F}(x)\rangle$ *be any two complex neutrosophic subgroups of the groups* $\mathcal{G}_1$ *and* $\mathcal{G}_2$*, respectively. Then the Cartesian product of* $\mathbb{C}_1$ *and* $\mathbb{C}_2$*, represented by* $\mathbb{C}_1 \times \mathbb{C}_2$ *and define as:*

$$\mathbb{C}_1 \times \mathbb{C}_2 = \left\{ \begin{array}{c} \langle (x,y), (\mathbb{C}_1 \times \mathbb{C}_2)_T(x,y), (\mathbb{C}_1 \times \mathbb{C}_2)_I(x,y), (\mathbb{C}_1 \times \mathbb{C}_2)_F(x,y)\rangle \\ / \ \forall\ x \in \mathcal{G}_1, y \in \mathcal{G}_2 \end{array} \right\}$$

where

$$(\mathbb{C}_1 \times \mathbb{C}_2)_T(x,y) = \min\{\mathbb{C}_{1T}(x), \mathbb{C}_{2T}(y)\},$$
$$(\mathbb{C}_1 \times \mathbb{C}_2)_I(x,y) = \min\{\mathbb{C}_{1I}(x), \mathbb{C}_{2I}(y)\},$$
$$(\mathbb{C}_1 \times \mathbb{C}_2)_F(x,y) = \max\{\mathbb{C}_{1F}(x), \mathbb{C}_{2F}(y)\}.$$

Example 3.4. *Let* $\mathcal{G}_1 = \{1, -1, i, -i\}$ *and* $\mathcal{G}_2 = \{1, \omega, \omega^2\}$ *are two groups under multiplication.*

Consider,

$$\mathbb{C}_1 = \left\{ \begin{array}{c} \left\langle 1, 0.7e^{0.6\pi i}, 0.6e^{0.5\pi i}, 0.5e^{0.2\pi i}\right\rangle, \left\langle -1, 0.6e^{0.5\pi i}, 0.5e^{0.4\pi i}, 0.4e^{0.2\pi i}\right\rangle, \\ \left\langle i, 0.5e^{0.3\pi i}, 0.4e^{0.2\pi i}, 0.1e^{0.2\pi i}\right\rangle, \left\langle -i, 0.5e^{0.3\pi i}, 0.4e^{0.2\pi i}, 0.1e^{0.2\pi i}\right\rangle \end{array} \right\}$$

and

$$\mathbb{C}_2 = \left\{ \begin{array}{c} \left\langle 1, 0.8e^{0.6\pi i}, 0.6e^{0.5\pi i}, 0.3e^{0.2\pi i}\right\rangle, \left\langle \omega, 0.7e^{0.6\pi i}, 0.5e^{0.4\pi i}, 0.3e^{0.2\pi i}\right\rangle, \\ \left\langle \omega^2, 0.7e^{0.6\pi i}, 0.5e^{0.4\pi i}, 0.3e^{0.2\pi i}\right\rangle \end{array} \right\}$$

are two complex neutrosophic subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

Now let $x = 1$ and $y = \omega$, then

$$\begin{aligned} \mathbb{C}_1 \times \mathbb{C}_2 &= \{\langle (\mathbb{C}_1 \times \mathbb{C}_2)_T(1,\omega), (\mathbb{C}_1 \times \mathbb{C}_2)_I(1,\omega), (\mathbb{C}_1 \times \mathbb{C}_2)_F(1,\omega)\rangle, ...\} \\ &= \{\langle \min\{\mathbb{C}_{1T}(1), \mathbb{C}_{2T}(\omega)\}, \min\{\mathbb{C}_{1I}(1), \mathbb{C}_{2I}(\omega)\},\ \max\{\mathbb{C}_{1F}(1), \\ &\quad \mathbb{C}_{2F}(\omega)\}\rangle, ...\} \\ &= \{\left\langle \min\{0.7e^{0.6\pi i}, 0.7e^{0.6\pi i}\}, \min\{0.6e^{0.5\pi i}, 0.5e^{0.4\pi i}\}, \max\{0.5e^{0.2\pi i}, \right. \\ &\quad \left. 0.3e^{0.2\pi i}\}\right\rangle, ...\} \\ &= \{\left\langle 0.7e^{0.6\pi i}, 0.5e^{0.4\pi i}, 0.5e^{0.2\pi i}\right\rangle, ...\}. \end{aligned}$$

Theorem 3.5. *If* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *are any two complex neutrosophic subgroups of the groups* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *respectively, then* $\mathbb{C}_1 \times \mathbb{C}_2$ *is a complex neutrosophic subgroup of* $\mathcal{G}_1 \times \mathcal{G}_2$*.*

Proof: Assume that $\mathbb{C}_1 = \langle \mathbb{C}_{1T}, \mathbb{C}_{1I}, \mathbb{C}_{1F}\rangle$ and $\mathbb{C}_2 = \langle \mathbb{C}_{2T}, \mathbb{C}_{2I}, \mathbb{C}_{2F}\rangle$ be any two complex neutrosophic subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Let any arbitrary elements $x_1, x_2 \in \mathcal{G}_1$ and $y_1, y_2 \in \mathcal{G}_2$, then (x_1, y_1),$(x_2, y_2) \in \mathcal{G}_1 \times \mathcal{G}_2$.

Consider,

$$\begin{aligned} (\mathbb{C}_1 \times \mathbb{C}_2)_T((x_1, y_1), (x_2, y_2)) &= (\mathbb{C}_1 \times \mathbb{C}_2)_T(x_1x_2, y_1y_2) \\ &= \min\{\mathbb{C}_{1T}(x_1x_2), \mathbb{C}_{2T}(y_1y_2)\} \\ &\geq \mathbb{C}_{1T}(x_1) \wedge \mathbb{C}_{1T}(x_2) \wedge \mathbb{C}_{2T}(y_1) \wedge \mathbb{C}_{2T}(y_2) \\ &= \mathbb{C}_{1T}(x_1) \wedge \mathbb{C}_{2T}(y_1) \wedge \mathbb{C}_{1T}(x_2) \wedge \mathbb{C}_{2T}(y_2) \\ &= (\mathbb{C}_1 \times \mathbb{C}_2)_T(x_1, y_1) \wedge (\mathbb{C}_1 \times \mathbb{C}_2)_T(x_2, y_2). \end{aligned}$$

Similarly,

$$(\mathbb{C}_1 \times \mathbb{C}_2)_I((x_1, y_1), (x_2, y_2)) \geq (\mathbb{C}_1 \times \mathbb{C}_2)_I(x_1, y_1) \wedge (\mathbb{C}_1 \times \mathbb{C}_2)_I(x_2, y_2),$$

and

$$\begin{aligned}(\mathbb{C}_1 \times \mathbb{C}_2)_F((x_1, y_1), (x_2, y_2)) &= (\mathbb{C}_1 \times \mathbb{C}_2)_F(x_1x_2, y_1y_2)\\ &= \max\{\mathbb{C}_{1F}(x_1x_2), \mathbb{C}_{2F}(y_1y_2)\}\\ &\leq \mathbb{C}_{1F}(x_1) \vee \mathbb{C}_{1F}(x_2) \vee \mathbb{C}_{2F}(y_1) \vee \mathbb{C}_{2F}(y_2)\\ &= \mathbb{C}_{1F}(x_1) \vee \mathbb{C}_{2F}(y_1) \vee \mathbb{C}_{1F}(x_2) \vee \mathbb{C}_{2F}(y_2)\\ &= (\mathbb{C}_1 \times \mathbb{C}_2)_F(x_1, y_1) \vee (\mathbb{C}_1 \times \mathbb{C}_2)_F(x_2, y_2).\end{aligned}$$

Also,

$$\begin{aligned}(\mathbb{C}_1 \times \mathbb{C}_2)_T(x_1, y_1)^{-1} &= (\mathbb{C}_1 \times \mathbb{C}_2)_T(x_1^{-1}, y_1^{-1})\\ &= \mathbb{C}_{1T}(x_1^{-1}) \wedge \mathbb{C}_{2T}(y_1^{-1})\\ &\geq \mathbb{C}_{1T}(x) \wedge \mathbb{C}_{2T}(y)\\ &= (\mathbb{C}_1 \times \mathbb{C}_2)_T(x, y).\end{aligned}$$

Similarly,

$$(\mathbb{C}_1 \times \mathbb{C}_2)_I(x_1, y_1)^{-1} \geq (\mathbb{C}_1 \times \mathbb{C}_2)_I(x, y).$$

And

$$\begin{aligned}(\mathbb{C}_1 \times \mathbb{C}_2)_F(x_1, y_1)^{-1} &= (\mathbb{C}_1 \times \mathbb{C}_2)_F(x_1^{-1}, y_1^{-1})\\ &= \mathbb{C}_{1F}(x_1^{-1}) \vee \mathbb{C}_{2F}(y_1^{-1})\\ &\leq \mathbb{C}_{1F}(x) \vee \mathbb{C}_{2F}(y)\\ &= (\mathbb{C}_1 \times \mathbb{C}_2)_F(x, y).\end{aligned}$$

Hence $\mathbb{C}_1 \times \mathbb{C}_2$ is a complex neutrosophic subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$. □

Theorem 3.6. *Let $\mathbb{C}$ be a CNSG of a group $\mathcal{G}$. Then the following properties are satisfied:*

(*a*) $\mathbb{C}(\hat{e}) \cdot e^{i\mathbb{C}(\hat{e})} \geq \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)}\ \forall\, x \in \mathcal{G}$, where $\hat{e}$ is the unit element of $\mathcal{G}$.

(*b*) $\mathbb{C}(x^{-1}) \cdot e^{i\mathbb{C}(x^{-1})} = \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)}$ for each $x \in \mathcal{G}$.

Proof: (*a*) Let $\hat{e}$ be the unit element of $\mathcal{G}$ and $x \in \mathcal{G}$ be arbitrary element, then by (CNSG1), (CNSG2) of Definition 3.1,

$$\begin{aligned}p_{\mathbb{C}}(\hat{e}) \cdot e^{i\mu_{\mathbb{C}}(\hat{e})} &= p_{\mathbb{C}}(x \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x \cdot x^{-1})}\\ &\geq p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(x^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x^{-1})}\\ &= p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}\\ &= p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}\\ p_{\mathbb{C}}(\hat{e}) \cdot e^{i\mu_{\mathbb{C}}(\hat{e})} &\geq p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)},\end{aligned}$$

Similarly,

$$q_{\mathbb{C}}(\hat{e}) \cdot e^{i\nu_{\mathbb{C}}(\hat{e})} \geq q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}.$$

And

$$\begin{aligned} r_{\mathbb{C}}(\hat{e}) \cdot e^{i\omega_{\mathbb{C}}(\hat{e})} &= r_{\mathbb{C}}(x \cdot x^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x \cdot x^{-1})} \\ &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(x^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x^{-1})} \\ &= r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \\ &= r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \\ r_{\mathbb{C}}(\hat{e}) \cdot e^{i\omega_{\mathbb{C}}(\hat{e})} &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}. \end{aligned}$$

Hence $\mathbb{C}(\hat{e}) \cdot e^{i\mathbb{C}(\hat{e})} \geq \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)}$ is satisfied, for all $x \in \mathcal{G}$.

(b) Let $x \in \mathcal{G}$. Since $\mathbb{C}$ is a complex neutrosophic subgroup of $\mathcal{G}$,

so $\mathbb{C}(x^{-1}) \cdot e^{i\mathbb{C}(x^{-1})} \geq \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)}$ is clear from (CNSG2) of Definition 3.1.

Again by applying (CNSG2) of Definition 3.1, and using group structure of $\mathcal{G}$, the other side of the inequality is proved as follows;

$$p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} = p_{\mathbb{C}}(x^{-1})^{-1} \cdot e^{i\mu_{\mathbb{C}}(x^{-1})^{-1}} \geq p_{\mathbb{C}}(x^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x^{-1})},$$

$$q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)} = q_{\mathbb{C}}(x^{-1})^{-1} \cdot e^{i\nu_{\mathbb{C}}(x^{-1})^{-1}} \geq q_{\mathbb{C}}(x^{-1}) \cdot e^{i\nu_{\mathbb{C}}(x^{-1})},$$

$$r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} = r_{\mathbb{C}}(x^{-1})^{-1} \cdot e^{i\omega_{\mathbb{C}}(x^{-1})^{-1}} \leq r_{\mathbb{C}}(x^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x^{-1})}.$$

Therefore,

$$\mathbb{C}(x) \cdot e^{i\mathbb{C}(x)} \geq \mathbb{C}(x^{-1}) \cdot e^{i\mathbb{C}(x^{-1})}.$$

Thus,

$$\mathbb{C}(x^{-1}) \cdot e^{i\mathbb{C}(x^{-1})} = \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)}.$$

Hence $\mathbb{C}(x^{-1}) \cdot e^{i\mathbb{C}(x^{-1})} = \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)}$ is satisfied, for all $x \in \mathcal{G}$. □

Theorem 3.7. *Let $\mathbb{C}$ be a complex neutrosophic set on a group $\mathcal{G}$. Then $\mathbb{C}$ is a CNSG of $\mathcal{G}$ if and only if $\mathbb{C}(x \cdot y^{-1}) \cdot e^{i\mathbb{C}(x \cdot y^{-1})} \geq \mathbb{C}(x) \cdot e^{i\mathbb{C}(x)} \wedge \mathbb{C}(y) \cdot e^{i\mathbb{C}(y)}$ for each $x,y \in \mathcal{G}$.*

Proof: Let $\mathbb{C}$ be a complex neutrosophic subgroup of $\mathcal{G}$ and $x, y \in \mathcal{G}$, So, it is clear that,

$$\begin{aligned} p_{\mathbb{C}}(xy^{-1}) \cdot e^{i\mu_{\mathbb{C}}(xy^{-1})} &\geq p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y^{-1}) \cdot e^{i\mu_{\mathbb{C}}(y^{-1})} \\ &\geq p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)}. \end{aligned}$$

Similarly,

$$q_{\mathbb{C}}(xy^{-1}) \cdot e^{i\nu_{\mathbb{C}}(xy^{-1})} \geq q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)} \wedge q_{\mathbb{C}}(y) \cdot e^{i\nu_{\mathbb{C}}(y)}.$$

And

$$\begin{aligned} r_{\mathbb{C}}(xy^{-1}) \cdot e^{i\omega_{\mathbb{C}}(xy^{-1})} &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y^{-1}) \cdot e^{i\omega_{\mathbb{C}}(y^{-1})} \\ &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)}. \end{aligned}$$

Hence

$$\begin{aligned}\mathbb{C}(x\cdot y^{-1})\cdot e^{i\mathbb{C}(x\cdot y^{-1})} &= (p_{\mathbb{C}}(xy^{-1})\cdot e^{i\mu_{\mathbb{C}}(xy^{-1})}, q_{\mathbb{C}}(xy^{-1})\cdot e^{i\nu_{\mathbb{C}}(xy^{-1})},\\ &\quad r_{\mathbb{C}}(xy^{-1})\cdot e^{i\omega_{\mathbb{C}}(xy^{-1})})\\ &\geq (p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)}, q_{\mathbb{C}}(x)\cdot e^{i\nu_{\mathbb{C}}(x)}\\ &\quad \wedge q_{\mathbb{C}}(y)\cdot e^{i\nu_{\mathbb{C}}(y)}, r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y)\cdot e^{i\omega_{\mathbb{C}}(y)})\\ &= (p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}, q_{\mathbb{C}}(x)\cdot e^{i\nu_{\mathbb{C}}(x)}, r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)})\\ &\quad \wedge (p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)}, q_{\mathbb{C}}(y)\cdot e^{i\nu_{\mathbb{C}}(y)}, r_{\mathbb{C}}(y)\cdot e^{i\omega_{\mathbb{C}}(y)})\\ &= \mathbb{C}(x)\cdot e^{i\mathbb{C}(x)} \wedge \mathbb{C}(y)\cdot e^{i\mathbb{C}(y)}.\end{aligned}$$

Thus,

$$\mathbb{C}(x\cdot y^{-1})\cdot e^{i\mathbb{C}(x\cdot y^{-1})} \geq \mathbb{C}(x)\cdot e^{i\mathbb{C}(x)} \wedge \mathbb{C}(y)\cdot e^{i\mathbb{C}(y)}.$$

Conversely, Suppose the condition

$$\mathbb{C}(x\cdot y^{-1})\cdot e^{i\mathbb{C}(x\cdot y^{-1})} \geq \mathbb{C}(x)\cdot e^{i\mathbb{C}(x)} \wedge \mathbb{C}(y)\cdot e^{i\mathbb{C}(y)}$$

is hold.

Let $\hat{e}$ be the unit of $\mathcal{G}$, since $\mathcal{G}$ is a group,

$$\begin{aligned}p_{\mathbb{C}}(x^{-1})\cdot e^{i\mu_{\mathbb{C}}(x^{-1})} &= p_{\mathbb{C}}(\hat{e}\cdot x^{-1})\cdot e^{i\mu_{\mathbb{C}}(\hat{e}\cdot x^{-1})}\\ &\geq p_{\mathbb{C}}(\hat{e})\cdot e^{i\mu_{\mathbb{C}}(\hat{e})} \wedge p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}\\ &= p_{\mathbb{C}}(x\cdot x^{-1})\cdot e^{i\mu_{\mathbb{C}}(x\cdot x^{-1})} \wedge p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}\\ &\geq p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}\\ &= p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}\\ p_{\mathbb{C}}(x^{-1})\cdot e^{i\mu_{\mathbb{C}}(x^{-1})} &\geq p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}.\end{aligned}$$

Similarly,

$$q_{\mathbb{C}}(x^{-1})\cdot e^{i\nu_{\mathbb{C}}(x^{-1})} \geq q_{\mathbb{C}}(x)\cdot e^{i\nu_{\mathbb{C}}(x)}.$$

And

$$\begin{aligned}r_{\mathbb{C}}(x^{-1})\cdot e^{i\omega_{\mathbb{C}}(x^{-1})} &= r_{\mathbb{C}}(\hat{e}\cdot x^{-1})\cdot e^{i\omega_{\mathbb{C}}(\hat{e}\cdot x^{-1})}\\ &\leq r_{\mathbb{C}}(\hat{e})\cdot e^{i\omega_{\mathbb{C}}(\hat{e})} \vee r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)}\\ &= r_{\mathbb{C}}(x\cdot x^{-1})\cdot e^{i\omega_{\mathbb{C}}(x\cdot x^{-1})} \vee r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)}\\ &\leq r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)}\\ &= \vee r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)}.\end{aligned}$$

So, the condition (CNSG2) of Definition 3.1 is satisfied.

Now let us show the condition (CNSG1) of Definition 3.1,

$$\begin{aligned}p_{\mathbb{C}}(x\cdot y)\cdot e^{i\mu_{\mathbb{C}}(x\cdot y)} &= p_{\mathbb{C}}(x\cdot (y^{-1})^{-1})\cdot e^{i\mu_{\mathbb{C}}(x\cdot (y^{-1})^{-1})}\\ &\geq p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y^{-1})\cdot e^{i\mu_{\mathbb{C}}(y^{-1})}\\ &\geq p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)}.\end{aligned}$$

Similarly,

$$q_{\mathbb{C}}(x \cdot y) \cdot e^{i\nu_{\mathbb{C}}(x \cdot y)} \geq q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)} \wedge q_{\mathbb{C}}(y) \cdot e^{i\nu_{\mathbb{C}}(y)}$$

and

$$\begin{aligned} r_{\mathbb{C}}(x \cdot y) \cdot e^{i\omega_{\mathbb{C}}(x \cdot y)} &= r_{\mathbb{C}}(x \cdot (y^{-1})^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x \cdot (y^{-1})^{-1})} \\ &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y^{-1}) \cdot e^{i\omega_{\mathbb{C}}(y^{-1})} \\ &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)}. \end{aligned}$$

Therefore (CNSG1) of Definition 3.1 is also satisfied. Thus $\mathbb{C}$ is a complex neutrosophic subgroup of $\mathcal{G}$. □

★ Based on Theorem 3.7, we define complex neutrosophic subgroup as follows:

Definition 3.8. *Let $\mathcal{G}$ be any group with multiplication. A complex neutrosophic set*

$$\mathbb{C} = \left\{\left\langle T_{\mathbb{C}} = p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}, I_{\mathbb{C}} = q_{\mathbb{C}} \cdot e^{i\nu_{\mathbb{C}}}, F_{\mathbb{C}} = r_{\mathbb{C}} \cdot e^{i\omega_{\mathbb{C}}}\right\rangle\right\}$$

on group $\mathcal{G}$ is known as a complex neutrosophic subgroup (CNSG) of $\mathcal{G}$, if

$\mathbb{C}(x^{-1}y) \geq \min\{\mathbb{C}(x), \mathbb{C}(y)\}$ i.e.,

(*i*) $p_{\mathbb{C}}(x^{-1}y) \cdot e^{i\mu_{\mathbb{C}}(x^{-1}y)} \geq \min\{p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}, p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)}\}$

(*ii*) $q_{\mathbb{C}}(x^{-1}y) \cdot e^{i\nu_{\mathbb{C}}(x^{-1}y)} \geq \min\{q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}, q_{\mathbb{C}}(y) \cdot e^{i\nu_{\mathbb{C}}(y)}\}$

(*iii*) $r_{\mathbb{C}}(x^{-1}y) \cdot e^{i\omega_{\mathbb{C}}(x^{-1}y)} \leq \max\{r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}, r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)}\}, \forall\, x, y \in \mathcal{G}$.

Example 3.9. *Let $\mathcal{G} = \{1, -1, i, -i\}$ be a group under multiplication, and $\mathbb{C} = \langle T_{\mathbb{C}}, I_{\mathbb{C}}, F_{\mathbb{C}}\rangle$ be complex neutrosophic set on $\mathcal{G}$, such that*

$$\begin{aligned} &T_{\mathbb{C}}(1) = 0.8e^{0.6\pi i}, T_{\mathbb{C}}(-1) = 0.7e^{0.5\pi i}, T_{\mathbb{C}}(i) = T_{\mathbb{C}}(-i) = 0.3e^{0.2\pi i} \\ &I_{\mathbb{C}}(1) = 0.7e^{0.5\pi i}, I_{\mathbb{C}}(-1) = 0.6e^{0.4\pi i}, I_{\mathbb{C}}(i) = I_{\mathbb{C}}(-i) = 0.2e^{0.2\pi i} \\ &F_{\mathbb{C}}(1) = 0.5e^{0.4\pi i}, F_{\mathbb{C}}(-1) = 0.1e^{0.2\pi i}, F_{\mathbb{C}}(i) = F_{\mathbb{C}}(-i) = 0.1e^{0.2\pi i}. \end{aligned}$$

Clearly, $\mathbb{C}$ is a complex neutrosophic subgroup of $\mathcal{G}$.

Theorem 3.10. *If $\mathbb{C}_1$ and $\mathbb{C}_2$ are two complex neutrosophic subgroups of a group $\mathcal{G}$, then the intersection $\mathbb{C}_1 \cap \mathbb{C}_2$ is a complex neutrosophic subgroup of $\mathcal{G}$.*

Proof: Let $x, y \in \mathcal{G}$ be any arbitrary elements. By Theorem 3.7, it is enough to show that

$$(\mathbb{C}_1 \cap \mathbb{C}_2)(x \cdot y^{-1}) \geq (\mathbb{C}_1 \cap \mathbb{C}_2)(x) \wedge (\mathbb{C}_1 \cap \mathbb{C}_2)(y).$$

First consider the truth-membership degree of the intersection

$$\begin{aligned} p_{\mathbb{C}_1 \cap \mathbb{C}_2}(x \cdot y^{-1}) \cdot e^{i\mu_{\mathbb{C}_1 \cap \mathbb{C}_2}(x \cdot y^{-1})} &= p_{\mathbb{C}_1}(x \cdot y^{-1}) \cdot e^{i\mu_{\mathbb{C}_1}(x \cdot y^{-1})} \\ &\wedge p_{\mathbb{C}_2}(x \cdot y^{-1}) \cdot e^{i\mu_{\mathbb{C}_2}(x \cdot y^{-1})} \\ &\geq p_{\mathbb{C}_1}(x) \cdot e^{i\mu_{\mathbb{C}_1}(x)} \wedge p_{\mathbb{C}_1}(y) \cdot e^{i\mu_{\mathbb{C}_1}(y)} \\ &\wedge p_{\mathbb{C}_2}(x) \cdot e^{i\mu_{\mathbb{C}_2}(x)} \wedge p_{\mathbb{C}_2}(y) \cdot e^{i\mu_{\mathbb{C}_2}(y)} \\ &= (p_{\mathbb{C}_1}(x) \cdot e^{i\mu_{\mathbb{C}_1}(x)} \wedge p_{\mathbb{C}_2}(x) \cdot e^{i\mu_{\mathbb{C}_2}(x)}) \\ &\wedge (p_{\mathbb{C}_1}(y) \cdot e^{i\mu_{\mathbb{C}_1}(y)} \wedge p_{\mathbb{C}_2}(y) \cdot e^{i\mu_{\mathbb{C}_2}(y)}) \\ &= p_{\mathbb{C}_1 \cap \mathbb{C}_2}(x) \cdot e^{i\mu_{\mathbb{C}_1 \cap \mathbb{C}_2}(x)} \\ &\wedge p_{\mathbb{C}_1 \cap \mathbb{C}_2}(y) \cdot e^{i\mu_{\mathbb{C}_1 \cap \mathbb{C}_2}(y)}. \end{aligned}$$

Similarly,

$$q_{\mathbb{C}_1\cap\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\nu_{\mathbb{C}_1\cap\mathbb{C}_2}(x\cdot y^{-1})} \geq q_{\mathbb{C}_1\cap\mathbb{C}_2}(x)\cdot e^{i\nu_{\mathbb{C}_1\cap\mathbb{C}_2}(x)} \wedge q_{\mathbb{C}_1\cap\mathbb{C}_2}(y)\cdot e^{i\nu_{\mathbb{C}_1\cap\mathbb{C}_2}(y)}.$$

And

$$\begin{aligned} r_{\mathbb{C}_1\cup\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\omega_{\mathbb{C}_1\cup\mathbb{C}_2}(x\cdot y^{-1})} &= r_{\mathbb{C}_1}(x\cdot y^{-1})\cdot e^{i\omega_{\mathbb{C}_1}(x\cdot y^{-1})} \\ &\vee r_{\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\omega_{\mathbb{C}_2}(x\cdot y^{-1})} \\ &\leq r_{\mathbb{C}_1}(x)\cdot e^{i\omega_{\mathbb{C}_1}(x)} \vee r_{\mathbb{C}_1}(y)\cdot e^{i\omega_{\mathbb{C}_1}(y)} \\ &\vee r_{\mathbb{C}_2}(x)\cdot e^{i\omega_{\mathbb{C}_2}(x)} \vee r_{\mathbb{C}_2}(y)\cdot e^{i\omega_{\mathbb{C}_2}(y)} \\ &= r_{\mathbb{C}_1}(x)\cdot e^{i\omega_{\mathbb{C}_1}(x)} \vee r_{\mathbb{C}_2}(x)\cdot e^{i\omega_{\mathbb{C}_2}(x)} \\ &\vee r_{\mathbb{C}_1}(y)\cdot e^{i\omega_{\mathbb{C}_1}(y)} \vee r_{\mathbb{C}_2}(y)\cdot e^{i\omega_{\mathbb{C}_2}(y)} \\ &= r_{\mathbb{C}_1\cup\mathbb{C}_2}(x)\cdot e^{i\omega_{\mathbb{C}_1\cup\mathbb{C}_2}(x)} \\ &\vee r_{\mathbb{C}_1\cup\mathbb{C}_2}(y)\cdot e^{i\omega_{\mathbb{C}_1\cup\mathbb{C}_2}(y)}. \end{aligned}$$

Hence $\mathbb{C}_1 \cap \mathbb{C}_2$ is a complex neutrosophic subgroup of $\mathcal{G}$. □

Theorem 3.11. *If $\mathbb{C}_1$ and $\mathbb{C}_2$ are two complex neutrosophic subgroups of a group $\mathcal{G}$, then the union $\mathbb{C}_1 \cup \mathbb{C}_2$ is a complex neutrosophic subgroup of $\mathcal{G}$.*

Proof: Let $x, y \in \mathcal{G}$ be any arbitrary elements. By Theorem 3.7, it is enough to show that

$$(\mathbb{C}_1 \cup \mathbb{C}_2)(x\cdot y^{-1}) \geq \min\{(\mathbb{C}_1 \cup \mathbb{C}_2)(x), (\mathbb{C}_1 \cup \mathbb{C}_2)(y)\}.$$

Consider,

$$\begin{aligned} p_{\mathbb{C}_1\cup\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\mu_{\mathbb{C}_1\cup\mathbb{C}_2}(x\cdot y^{-1})} &= p_{\mathbb{C}_1}(x\cdot y^{-1})\cdot e^{i\mu_{\mathbb{C}_1}(x\cdot y^{-1})} \\ &\vee p_{\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\mu_{\mathbb{C}_2}(x\cdot y^{-1})} \\ &\geq p_{\mathbb{C}_1}(x)\cdot e^{i\mu_{\mathbb{C}_1}(x)} \wedge p_{\mathbb{C}_1}(y)\cdot e^{i\mu_{\mathbb{C}_1}(y)} \\ &\vee p_{\mathbb{C}_2}(x)\cdot e^{i\mu_{\mathbb{C}_2}(x)} \wedge p_{\mathbb{C}_2}(y)\cdot e^{i\mu_{\mathbb{C}_2}(y)} \\ &= (p_{\mathbb{C}_1}(x)\cdot e^{i\mu_{\mathbb{C}_1}(x)} \vee p_{\mathbb{C}_2}(x)\cdot e^{i\mu_{\mathbb{C}_2}(x)}) \\ &\wedge (p_{\mathbb{C}_1}(y)\cdot e^{i\mu_{\mathbb{C}_1}(y)} \vee p_{\mathbb{C}_2}(y)\cdot e^{i\mu_{\mathbb{C}_2}(y)}) \\ &= \min\{p_{\mathbb{C}_1\cup\mathbb{C}_2}(x)\cdot e^{i\mu_{\mathbb{C}_1\cup\mathbb{C}_2}(x)}, \\ &p_{\mathbb{C}_1\cup\mathbb{C}_2}(y)\cdot e^{i\mu_{\mathbb{C}_1\cup\mathbb{C}_2}(y)}\}. \end{aligned}$$

And

$$\begin{aligned} r_{\mathbb{C}_1\cap\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\omega_{\mathbb{C}_1\cap\mathbb{C}_2}(x\cdot y^{-1})} &= r_{\mathbb{C}_1}(x\cdot y^{-1})\cdot e^{i\omega_{\mathbb{C}_1}(x\cdot y^{-1})} \\ &\wedge r_{\mathbb{C}_2}(x\cdot y^{-1})\cdot e^{i\omega_{\mathbb{C}_2}(x\cdot y^{-1})} \\ &\leq r_{\mathbb{C}_1}(x)\cdot e^{i\omega_{\mathbb{C}_1}(x)} \vee r_{\mathbb{C}_1}(y)\cdot e^{i\omega_{\mathbb{C}_1}(y)} \\ &\wedge r_{\mathbb{C}_2}(x)\cdot e^{i\omega_{\mathbb{C}_2}(x)} \vee r_{\mathbb{C}_2}(y)\cdot e^{i\omega_{\mathbb{C}_2}(y)} \\ &= r_{\mathbb{C}_1}(x)\cdot e^{i\omega_{\mathbb{C}_1}(x)} \wedge r_{\mathbb{C}_2}(x)\cdot e^{i\omega_{\mathbb{C}_2}(x)} \\ &\vee r_{\mathbb{C}_1}(y)\cdot e^{i\omega_{\mathbb{C}_1}(y)} \wedge r_{\mathbb{C}_2}(y)\cdot e^{i\omega_{\mathbb{C}_2}(y)} \\ &= \max\{r_{\mathbb{C}_1\cap\mathbb{C}_2}(x)\cdot e^{i\omega_{\mathbb{C}_1\cap\mathbb{C}_2}(x)}, \\ &r_{\mathbb{C}_1\cap\mathbb{C}_2}(y)\cdot e^{i\omega_{\mathbb{C}_1\cap\mathbb{C}_2}(y)}\}. \end{aligned}$$

Thus, $\mathbb{C}_1 \cup \mathbb{C}_2$ is a complex neutrosophic subgroup of $\mathcal{G}$. □

4. Alpha-Cut of Complex Neutrosophic Set

Definition 4.1. *Let* $\mathbb{C} = \left\langle \mathbb{C}_T = p_{\mathbb{C}} e^{i\mu_{\mathbb{C}}}, \mathbb{C}_I = q_{\mathbb{C}} e^{i\nu_{\mathbb{C}}}, \mathbb{C}_F = r_{\mathbb{C}} e^{i\omega_{\mathbb{C}}} \right\rangle$ *be a complex neutrosophic set on* $\mathcal{X}$ *and* $\alpha = \beta \cdot e^{i\gamma}$, *where* $\beta \in [0,1], \gamma \in [0, 2\pi]$.

Define the α-level set of $\mathbb{C}$ as follows:

$\mathbb{C}_\alpha = \{x \in \mathcal{X} \mid \mathbb{C}(x) \geq \alpha\}$ i.e.,

$$\left(p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}\right)_\alpha = \left\{x \in \mathcal{X} \mid p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \geq \beta \cdot e^{i\gamma}\right\},$$
$$\left(q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}\right)_\alpha = \left\{x \in \mathcal{X} \mid q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)} \geq \beta \cdot e^{i\gamma}\right\},$$
$$\left(r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}\right)^\alpha = \left\{x \in \mathcal{X} \mid r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \leq \beta \cdot e^{i\gamma}\right\}.$$

It is easy to verify that,

(1) If $\mathbb{C}_1 \subseteq \mathbb{C}_2$ and $\alpha = \beta \cdot e^{i\gamma}$, where, $\beta \in [0,1], \gamma \in [0, 2\pi]$, then,

$$\left(p_{\mathbb{C}_1}(x) \cdot e^{i\mu_{\mathbb{C}_1}(x)}\right)_\alpha \subseteq \left(p_{\mathbb{C}_2}(x) \cdot e^{i\mu_{\mathbb{C}_2}(x)}\right)_\alpha$$
$$\left(q_{\mathbb{C}_1}(x) \cdot e^{i\nu_{\mathbb{C}_1}(x)}\right)_\alpha \subseteq \left(q_{\mathbb{C}_2}(x) \cdot e^{i\nu_{\mathbb{C}_2}(x)}\right)_\alpha$$
$$\left(r_{\mathbb{C}_1}(x) \cdot e^{i\omega_{\mathbb{C}_1}(x)}\right)^\alpha \supseteq \left(r_{\mathbb{C}_2}(x) \cdot e^{i\omega_{\mathbb{C}_2}(x)}\right)^\alpha.$$

(2) $\alpha_1 \leq \alpha_2$ where, $\alpha_1 = \beta_1 \cdot e^{i\gamma_1}, \alpha_2 = \beta_2 \cdot e^{i\gamma_2}$ implies that

$$\left(p_{\mathbb{C}_1}(x) \cdot e^{i\mu_{\mathbb{C}_1}(x)}\right)_{\alpha_1} \supseteq \left(p_{\mathbb{C}_1}(x) \cdot e^{i\mu_{\mathbb{C}_1}(x)}\right)_{\alpha_2}$$
$$\left(q_{\mathbb{C}_1}(x) \cdot e^{i\nu_{\mathbb{C}_1}(x)}\right)_{\alpha_1} \supseteq \left(q_{\mathbb{C}_1}(x) \cdot e^{i\nu_{\mathbb{C}_1}(x)}\right)_{\alpha_2}$$
$$\left(r_{\mathbb{C}_1}(x) \cdot e^{i\omega_{\mathbb{C}_1}(x)}\right)^{\alpha_1} \subseteq \left(r_{\mathbb{C}_1}(x) \cdot e^{i\omega_{\mathbb{C}_1}(x)}\right)^{\alpha_2}.$$

Example 4.2. *Let*

$$\mathbb{C} = \left\{ \begin{array}{c} \left\langle x_1, 0.2e^{0.4\pi i}, 0.3e^{0.5\pi i}, 0.7e^{0.1\pi i}\right\rangle, \left\langle x_2, 0.7e^{0.1\pi i}, 0.6e^{0.5\pi i}, 0.7e^{0.4\pi i}\right\rangle, \\ \left\langle x_3, 0.6e^{0.4\pi i}, 0.4e^{0.5\pi i}, 0.1e^{0.4\pi i}\right\rangle \end{array} \right\}$$

be a complex neutrosophic set of $\mathcal{X}$, *and* $\alpha = 0.4e^{0.4\pi i}$. *Then the* α*-level set as:* $\mathbb{C}_\alpha = \{x_3\}$.

Proposition 4.3. $\mathbb{C}$ *is a complex neutrosophic subgroup of* $\mathcal{G}$ *if and only if for all* $\alpha = \beta e^{i\gamma}$ *where,* $\beta \in [0,1], \gamma \in [0, 2\pi]$, α*-level sets of* $\mathbb{C}$, $\left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_\alpha$, $\left(q_{\mathbb{C}} \cdot e^{i\nu_{\mathbb{C}}}\right)_\alpha$ *and* $\left(r_{\mathbb{C}} \cdot e^{i\omega_{\mathbb{C}}}\right)^\alpha$ *are classical subgroups of* $\mathcal{G}$.

Proof: Let $\mathbb{C}$ be a CNSG of $\mathcal{G}$, $\alpha = \beta e^{i\gamma}$ where $\beta \in [0,1]$, $\gamma \in [0, 2\pi]$ and $x, y \in \left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_\alpha$ (similarly $x, y \in \left(q_{\mathbb{C}} \cdot e^{i\nu_{\mathbb{C}}}\right)_\alpha$, $\left(r_{\mathbb{C}} \cdot e^{i\omega_{\mathbb{C}}}\right)^\alpha$).

By the assumption,

$$\begin{aligned} p_{\mathbb{C}}(x \cdot y^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x \cdot y^{-1})} &\geq p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)} \\ &\geq \alpha \wedge \alpha = \alpha. \end{aligned}$$

Similarly,

$$q_{\mathbb{C}}(x \cdot y^{-1}) \cdot e^{i\nu_{\mathbb{C}}(x \cdot y^{-1})} \geq \alpha.$$

And

$$\begin{aligned} r_{\mathbb{C}}(x \cdot y^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x \cdot y^{-1})} &\leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)} \\ &\leq \alpha \vee \alpha = \alpha. \end{aligned}$$

Hence $x \cdot y^{-1} \in \left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_{\alpha}, \left(q_{\mathbb{C}} \cdot e^{i\nu_{\mathbb{C}}}\right)_{\alpha}, \left(r_{\mathbb{C}} \cdot e^{i\omega_{\mathbb{C}}}\right)^{\alpha}$ for each α.

This means that $\left(p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}\right)_{\alpha}, \left(q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}\right)_{\alpha}$ and $\left(r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}\right)^{\alpha}$ is a classical subgroup of $\mathcal{G}$ for each α.

Conversely, let $\left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_{\alpha}$ be a classical subgroup of $\mathcal{G}$, for each $\alpha = \beta e^{i\gamma}$ where $\beta \in [0,1], \gamma \in [0,2\pi]$.
Let $x, y \in \mathcal{G}$, $\alpha = p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)}$ and $\delta = p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}$. Since $\left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_{\alpha}$ and $\left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_{\delta}$ are classical subgroup of $\mathcal{G}$, $x \cdot y \in \left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_{\alpha}$ and $x^{-1} \in \left(p_{\mathbb{C}} \cdot e^{i\mu_{\mathbb{C}}}\right)_{\delta}$. Thus,

$$p_{\mathbb{C}}(x \cdot y) \cdot e^{i\mu_{\mathbb{C}}(x \cdot y)} \geq \alpha = p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)} \wedge p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)},$$

and

$$p_{\mathbb{C}}(x^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x^{-1})} \geq \delta = p_{\mathbb{C}}(x) \cdot e^{i\mu_{\mathbb{C}}(x)}.$$

Similarly,

$$q_{\mathbb{C}}(x \cdot y) \cdot e^{i\nu_{\mathbb{C}}(x \cdot y)} \geq q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)} \wedge q_{\mathbb{C}}(y) \cdot e^{i\nu_{\mathbb{C}}(y)},$$

$$q_{\mathbb{C}}(x^{-1}) \cdot e^{i\nu_{\mathbb{C}}(x^{-1})} \geq q_{\mathbb{C}}(x) \cdot e^{i\nu_{\mathbb{C}}(x)}.$$

And

$$r_{\mathbb{C}}(x \cdot y) \cdot e^{i\omega_{\mathbb{C}}(x \cdot y)} \leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)} \vee r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)},$$

$$r_{\mathbb{C}}(x^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x^{-1})} \leq r_{\mathbb{C}}(x) \cdot e^{i\omega_{\mathbb{C}}(x)}.$$

So, the conditions of Definition 3.1 are satisfied. Hence $\mathcal{G}$ is a complex neutrosophic subgroup. □

5. Image and Preimage of Complex Neutrosophic Set

Definition 5.1. *Let $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a function and $\mathbb{C}_1$ and $\mathbb{C}_2$ be the complex neutrosophic sets of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then the image of a complex neutrosophic set $\mathbb{C}_1$ is a complex neutrosophic set of $\mathcal{G}_2$ and it is defined as follows:*

$$\begin{aligned} f(\mathbb{C}_1)(y) &= \Big(p_f(\mathbb{C}_1)(y) \cdot e^{i\mu_f(\mathbb{C}_1)(y)}, q_f(\mathbb{C}_1)(y) \cdot e^{i\nu_f(\mathbb{C}_1)(y)}, \\ &\quad r_f(\mathbb{C}_1)(y) \cdot e^{i\omega_f(\mathbb{C}_1)(y)}\Big) \\ &= \Big(f(p_{\mathbb{C}_1})(y) \cdot e^{if(\mu_{\mathbb{C}_1})(y)}, f(q_{\mathbb{C}_1})(y) \cdot e^{if(\nu_{\mathbb{C}_1})(y)}, \\ &\quad f(r_{\mathbb{C}_1})(y) \cdot e^{if(\omega_{\mathbb{C}_1})(y)}\Big), \ \forall\, y \in \mathcal{G}_2 \end{aligned}$$

where,

$$f(p_{\mathbb{C}_1})(y) \cdot e^{if(\mu_{\mathbb{C}_1})(y)} = \begin{cases} \bigvee p_{\mathbb{C}_1}(x) \cdot e^{i\mu_{\mathbb{C}_1}(x)}, & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases}$$

$$f(q_{\mathbb{C}_1})(y) \cdot e^{if(\nu_{\mathbb{C}_1})(y)} = \begin{cases} \bigvee q_{\mathbb{C}_1}(x) \cdot e^{i\nu_{\mathbb{C}_1}(x)}, & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases}$$

$$f(r_{\mathbb{C}_1})(y) \cdot e^{if(\omega_{\mathbb{C}_1})(y)} = \begin{cases} \bigwedge r_{\mathbb{C}_1}(x) \cdot e^{i\omega_{\mathbb{C}_1}(x)}, & \text{if } x \in f^{-1}(y) \\ 1 \cdot e^{i2\pi} & \text{otherwise} \end{cases}.$$

And the preimage of a complex neutrosophic set $\mathbb{C}_2$ is a complex neutrosophic set of $\mathcal{G}_1$ and it is defined as follows: for all $x \in \mathcal{G}_1$,

$$\begin{aligned} f^{-1}(\mathbb{C}_2)(x) &= \Big(p_{f^{-1}}(\mathbb{C}_2)(x)\cdot e^{i\mu_{f^{-1}}(\mathbb{C}_2)(x)}, q_{f^{-1}}(\mathbb{C}_2)(x)\cdot e^{i\nu_{f^{-1}}(\mathbb{C}_2)(x)},\\ &\quad r_{f^{-1}}(\mathbb{C}_2)(x)\cdot e^{i\omega_{f^{-1}}(\mathbb{C}_2)(x)}\Big)\\ &= \Big(p_{\mathbb{C}_2}(f(x))\cdot e^{i\mu_{\mathbb{C}_2}(f(x))}, q_{\mathbb{C}_2}(f(x))\cdot e^{i\nu_{\mathbb{C}_2}(f(x))},\\ &\quad r_{\mathbb{C}_2}(f(x))\cdot e^{i\omega_{\mathbb{C}_2}(f(x))}\Big)\\ &= \mathbb{C}_2(f(x)). \end{aligned}$$

Theorem 5.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two groups and $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a group homomorphism. If $\mathbb{C}$ is a complex neutrosophic subgroup of $\mathcal{G}_1$, then the image of $\mathbb{C}$, $f(\mathbb{C})$ is a complex neutrosophic subgroup of $\mathcal{G}_2$.*

Proof: Let $\mathbb{C}$ be a CNSG of $\mathcal{G}_1$ and $y_1, y_2 \in \mathcal{G}_2$. if $f^{-1}(y_1) = \phi$ or $f^{-1}(y_2) = \phi$, then it is obvious that $f(\mathbb{C})$ is a CNSG of $\mathcal{G}_2$. Let us assume that there exist $x_1, x_2 \in \mathcal{G}_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is a group homomorphism,

$$\begin{aligned} f(p_{\mathbb{C}}(y_1\cdot y_2^{-1}))\cdot e^{if(\mu_{\mathbb{C}}(y_1\cdot y_2^{-1}))} &= \bigvee_{y_1\cdot y_2^{-1}=f(x)} p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}\\ &\geq p_{\mathbb{C}}(x_1\cdot x_2^{-1})\cdot e^{i\mu_{\mathbb{C}}(x_1\cdot x_2^{-1})}, \end{aligned}$$

$$\begin{aligned} f(q_{\mathbb{C}}(y_1\cdot y_2^{-1}))\cdot e^{if(\nu_{\mathbb{C}}(y_1\cdot y_2^{-1}))} &= \bigvee_{y_1\cdot y_2^{-1}=f(x)} q_{\mathbb{C}}(x)\cdot e^{i\nu_{\mathbb{C}}(x)}\\ &\geq q_{\mathbb{C}}(x_1\cdot x_2^{-1})\cdot e^{i\nu_{\mathbb{C}}(x_1\cdot x_2^{-1})}, \end{aligned}$$

$$\begin{aligned} f(r_{\mathbb{C}}(y_1\cdot y_2^{-1}))\cdot e^{if(\omega_{\mathbb{C}}(y_1\cdot y_2^{-1}))} &= \bigwedge_{y_1\cdot y_2^{-1}=f(x)} r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)}\\ &\leq r_{\mathbb{C}}(x_1\cdot x_2^{-1})\cdot e^{i\omega_{\mathbb{C}}(x_1\cdot x_2^{-1})}. \end{aligned}$$

By using the above inequalities let us prove that

$$f(\mathbb{C})(y_1\cdot y_2^{-1}) \geq f(\mathbb{C})(y_1) \wedge f(\mathbb{C})(y_2).$$

$$\begin{aligned} f(\mathbb{C})(y_1\cdot y_2^{-1}) &= \Big(f(p_{\mathbb{C}}(y_1\cdot y_2^{-1}))\cdot e^{if(\mu_{\mathbb{C}}(y_1\cdot y_2^{-1}))}, f(q_{\mathbb{C}}(y_1\cdot y_2^{-1}))\cdot e^{if(\nu_{\mathbb{C}}(y_1\cdot y_2^{-1}))},\\ &\quad f(r_{\mathbb{C}}(y_1\cdot y_2^{-1}))\cdot e^{if(\omega_{\mathbb{C}}(y_1\cdot y_2^{-1}))}\Big) \end{aligned}$$

$$= \left(\bigvee_{y_1\cdot y_2^{-1}=f(x)} p_{\mathbb{C}}(x)\cdot e^{i\mu_{\mathbb{C}}(x)}, \bigvee_{y_1\cdot y_2^{-1}=f(x)} q_{\mathbb{C}}(x)\cdot e^{i\nu_{\mathbb{C}}(x)}, \bigwedge_{y_1\cdot y_2^{-1}=f(x)} r_{\mathbb{C}}(x)\cdot e^{i\omega_{\mathbb{C}}(x)}\right)$$

$$\geq \Big(p_{\mathbb{C}}(x_1\cdot x_2^{-1})\cdot e^{i\mu_{\mathbb{C}}(x_1\cdot x_2^{-1})}, q_{\mathbb{C}}(x_1\cdot x_2^{-1})\cdot e^{i\nu_{\mathbb{C}}(x_1\cdot x_2^{-1})}, r_{\mathbb{C}}(x_1\cdot x_2^{-1})\cdot e^{i\omega_{\mathbb{C}}(x_1\cdot x_2^{-1})}\Big)$$

$$\geq \left(p_{\mathbb{C}}(x_1) \cdot e^{i\mu_{\mathbb{C}}(x_1)} \wedge p_{\mathbb{C}}(x_2) \cdot e^{i\mu_{\mathbb{C}}(x_2)}, q_{\mathbb{C}}(x_1) \cdot e^{i\nu_{\mathbb{C}}(x_1)} \wedge q_{\mathbb{C}}(x_2) \cdot e^{i\nu_{\mathbb{C}}(x_2)}, r_{\mathbb{C}}(x_1) \cdot e^{i\omega_{\mathbb{C}}(x_1)} \vee r_{\mathbb{C}}(x_2) \cdot e^{i\omega_{\mathbb{C}}(x_2)}\right)$$

$$= \left(p_{\mathbb{C}}(x_1) \cdot e^{i\mu_{\mathbb{C}}(x_1)}, q_{\mathbb{C}}(x_1) \cdot e^{i\nu_{\mathbb{C}}(x_1)}, r_{\mathbb{C}}(x_1) \cdot e^{i\omega_{\mathbb{C}}(x_1)} \wedge p_{\mathbb{C}}(x_2) \cdot e^{i\mu_{\mathbb{C}}(x_2)}, q_{\mathbb{C}}(x_2) \cdot e^{i\nu_{\mathbb{C}}(x_2)}, r_{\mathbb{C}}(x_2) \cdot e^{i\omega_{\mathbb{C}}(x_2)}\right)$$
$$= f(\mathbb{C})(y_1) \wedge f(\mathbb{C})(y_2).$$

This is satisfied for each $x_1, x_2 \in \mathcal{G}_1$ with $f(x_1) = y_1$ and $f(x_2) = y_2$, then it is obvious that

$$f(\mathbb{C})(y_1 \cdot y_2^{-1}) \geq \left(\bigvee_{y_1=f(x_1)} p_{\mathbb{C}}(x_1) \cdot e^{i\mu_{\mathbb{C}}(x_1)}, \bigvee_{y_1=f(x_1)} q_{\mathbb{C}}(x_1) \cdot e^{i\nu_{\mathbb{C}}(x_1)}, \bigwedge_{y_1=f(x_1)} r_{\mathbb{C}}(x_1) \cdot e^{i\omega_{\mathbb{C}}(x_1)} \right) \wedge \left(\bigvee_{y_2=f(x_2)} p_{\mathbb{C}}(x_2) \cdot e^{i\mu_{\mathbb{C}}(x_2)}, \bigvee_{y_2=f(x_2)} q_{\mathbb{C}}(x_2) \cdot e^{i\nu_{\mathbb{C}}(x_2)}, \bigwedge_{y_2=f(x_2)} r_{\mathbb{C}}(x_2) \cdot e^{i\omega_{\mathbb{C}}(x_2)} \right)$$
$$= \left(f(p_{\mathbb{C}}(y_1)) \cdot e^{if(\mu_{\mathbb{C}}(y_1))}, f(q_{\mathbb{C}}(y_1)) \cdot e^{if(\nu_{\mathbb{C}}(y_1))}, f(r_{\mathbb{C}}(x_1)) \cdot e^{if(\omega_{\mathbb{C}}(x_1))}\right) \wedge \left(f(p_{\mathbb{C}}(y_2)) \cdot e^{if(\mu_{\mathbb{C}}(y_2))}, f(q_{\mathbb{C}}(y_2)) \cdot e^{if(\nu_{\mathbb{C}}(y_2))}, f(r_{\mathbb{C}}(x_2)) \cdot e^{if(\omega_{\mathbb{C}}(x_2))}\right)$$
$$= f(\mathbb{C})(y_1) \wedge f(\mathbb{C})(y_2).$$

Hence the image of a CNSG is also a CNSG. □

Theorem 5.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the two groups and $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a group homomorphism. If $\mathbb{C}_2$ is a complex neutrosophic subgroup of $\mathcal{G}_2$, then the preimage of $f^{-1}(\mathbb{C}_2)$ is a complex neutrosophic subgroup of $\mathcal{G}_1$.*

Proof: Let $\mathbb{C}_2$ be a complex neutrosophic subgroup of $\mathcal{G}_2$, and $x_1, x_2 \in \mathcal{G}_1$. Since f is a group homomorphism, the following inequalities is obtained.

$$f^{-1}(\mathbb{C}_2)(x_1 \cdot x_2^{-1}) = \left(p_{\mathbb{C}_2}(f(x_1 \cdot x_2^{-1})) \cdot e^{i\mu_{\mathbb{C}_2}(f(x_1 \cdot x_2^{-1}))}, q_{\mathbb{C}_2}(f(x_1 \cdot x_2^{-1})) \cdot e^{i\nu_{\mathbb{C}_2}(f(x_1 \cdot x_2^{-1}))}, r_{\mathbb{C}_2}(f(x_1 \cdot x_2^{-1})) \cdot e^{i\omega_{\mathbb{C}_2}(f(x_1 \cdot x_2^{-1}))}\right)$$
$$= \left(p_{\mathbb{C}_2}(f(x_1) \cdot f(x_2)^{-1}) \cdot e^{i\mu_{\mathbb{C}_2}(f(x_1) \cdot f(x_2)^{-1})}, q_{\mathbb{C}_2}(f(x_1) \cdot f(x_2)^{-1}) \cdot e^{i\nu_{\mathbb{C}_2}(f(x_1) \cdot f(x_2)^{-1})}, r_{\mathbb{C}_2}(f(x_1) \cdot f(x_2)^{-1}) \cdot e^{i\omega_{\mathbb{C}_2}(f(x_1) \cdot f(x_2)^{-1})}\right)$$
$$\geq \left(p_{\mathbb{C}_2}(f(x_1) \wedge f(x_2)) \cdot e^{i\mu_{\mathbb{C}_2}(f(x_1) \wedge f(x_2))}, q_{\mathbb{C}_2}(f(x_1) \wedge f(x_2)) \cdot e^{i\nu_{\mathbb{C}_2}(f(x_1) \wedge f(x_2))}, r_{\mathbb{C}_2}(f(x_1) \vee f(x_2)) \cdot e^{i\omega_{\mathbb{C}_2}(f(x_1) \vee f(x_2))}\right)$$
$$= \left(p_{\mathbb{C}_2}(f(x_1)) \cdot e^{i\mu_{\mathbb{C}_2}(f(x_1))}, q_{\mathbb{C}_2}(f(x_1) \cdot e^{i\nu_{\mathbb{C}_2}(f(x_1))}, r_{\mathbb{C}_2}(f(x_1) \cdot e^{i\omega_{\mathbb{C}_2}(f(x_1))}) \wedge (p_{\mathbb{C}_2}(f(x_2)) \cdot e^{i\mu_{\mathbb{C}_2}(f(x_2))}, q_{\mathbb{C}_2}(f(x_2) \cdot e^{i\nu_{\mathbb{C}_2}(f(x_2))}, r_{\mathbb{C}_2}(f(x_2) \cdot e^{i\omega_{\mathbb{C}_2}(f(x_2))}\right)$$
$$= f^{-1}(\mathbb{C}_2)(x_1) \wedge f^{-1}(\mathbb{C}_2)(x_2).$$

Hence $f^{-1}(\mathbb{C}_2)$ is a CNSG of $\mathcal{G}_1$. □

Theorem 5.4. *Let $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a homomorphism of groups, $\mathbb{C}$ is a CNSG of $\mathcal{G}_1$ and define $\mathbb{C}^{-1} : \mathcal{G}_1 \longrightarrow [0,1]\cdot e^{i[0,2\pi]} \times [0,1]\cdot e^{i[0,2\pi]} \times [0,1]\cdot e^{i[0,2\pi]}$ as $\mathbb{C}^{-1}(x) = \mathbb{C}(x^{-1})$ for arbitrary $x \in \mathcal{G}_1$. Then the following properties are valid.*

(1) $\mathbb{C}^{-1}$ is a CNSG of $\mathcal{G}_1$.

(2) $(f(\mathbb{C}))^{-1} = f(\mathbb{C}^{-1})$.

Proof: (1) Let $\mathbb{C}$ is a complex neutrosophic subgroup of $\mathcal{G}_1$.

Since $\mathbb{C}^{-1} : \mathcal{G}_1 \longrightarrow [0,1]\cdot e^{i[0,2\pi]} \times [0,1]\cdot e^{i[0,2\pi]} \times [0,1]\cdot e^{i[0,2\pi]}$.

Let for all $x \in \mathcal{G}_1$, this implies that, $\mathbb{C}^{-1}(x) = (x_T, x_I, x_F)$ where $x_T \in [0,1]\cdot e^{i[0,2\pi]}$, $x_I \in [0,1]\cdot e^{i[0,2\pi]}$ and $x_F \in [0,1]\cdot e^{i[0,2\pi]}$.

So $\mathbb{C}^{-1}$ is a complex neutrosophic subgroup of $\mathcal{G}_1$.

(2) Given that $\mathbb{C}^{-1}(x) = \mathbb{C}(x^{-1})\ \forall\ x \in \mathcal{G}_1$.

Since $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a homomorphism. As $\mathbb{C}$ is a CNSG of $\mathcal{G}_1$ this implies that $\mathbb{C}^{-1}$ is a CNSG of $\mathcal{G}_1$ by part (1), so $f(\mathbb{C}^{-1}) \in \mathcal{G}_2$ and $f(\mathbb{C}) \in \mathcal{G}_2$. Now by (1), $(f(\mathbb{C}))^{-1} \in \mathcal{G}_2$ as $\mathcal{G}_2$ is a group homomorphism.

So $f(\mathbb{C}^{-1}) = (f(\mathbb{C}))^{-1}$ by uniqueness of inverse of an element. □

Corollary 5.5. *Let $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be an isomorphism on of groups, $\mathbb{C}$ is complex neutrosophic subgroup of $\mathcal{G}_1$, then $f^{-1}(f(\mathbb{C})) = \mathbb{C}$.*

Corollary 5.6. *Let $f : \mathcal{G} \longrightarrow \mathcal{G}$ be an isomorphism on a group $\mathcal{G}$, $\mathbb{C}$ is complex neutrosophic subgroup of $\mathcal{G}$, then $f(\mathbb{C}) = \mathbb{C}$ if and only if $f^{-1}(\mathbb{C}) = \mathbb{C}$.*

6. Complex Neutrosophic Normal Subgroup

Definition 6.1. *Let $\mathbb{C}$ be a complex neutrosophic subgroup of a group $\mathcal{G}$ is known as a complex neutrosophic normal subgroup (CNNSG) of $\mathcal{G}$, if*

$\mathbb{C}(xyx^{-1}) \geq \mathbb{C}(y)$ i.e.,

(*i*) $p_{\mathbb{C}}(xyx^{-1})\cdot e^{i\mu_{\mathbb{C}}(xyx^{-1})} \geq p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)}$
(*ii*) $q_{\mathbb{C}}(xyx^{-1})\cdot e^{i\nu_{\mathbb{C}}(xyx^{-1})} \geq q_{\mathbb{C}}(y)\cdot e^{i\nu_{\mathbb{C}}(y)}$
(*iii*) $r_{\mathbb{C}}(xyx^{-1})\cdot e^{i\omega_{\mathbb{C}}(xyx^{-1})} \leq r_{\mathbb{C}}(y)\cdot e^{i\omega_{\mathbb{C}}(y)}$, $\forall\ x,y \in \mathcal{G}$.

Example 6.2. *Let $\mathcal{G} = S_3 = \{1, a, a^2, b, ab, a^2b\}$ be a group and $\mathbb{C} = \langle T_{\mathbb{C}}, I_{\mathbb{C}}, F_{\mathbb{C}}\rangle$ be a complex neutrosophic set of $\mathcal{G}$ such that,*

$$T_{\mathbb{C}}(1) = 0.8e^{0.6\pi i}, T_{\mathbb{C}}(a) = T_{\mathbb{C}}(a^2) = 0.6e^{0.6\pi i}$$

$$T_{\mathbb{C}}(b) = T_{\mathbb{C}}(ab) = T_{\mathbb{C}}(a^2b) = 0.5e^{0.4\pi i}$$

$$I_{\mathbb{C}}(1) = 0.7e^{0.5\pi i}, I_{\mathbb{C}}(a) = I_{\mathbb{C}}(a^2) = 0.6e^{0.5\pi i}$$

$$I_{\mathbb{C}}(b) = I_{\mathbb{C}}(ab) = I_{\mathbb{C}}(a^2b) = 0.4e^{0.3\pi i}$$

$$F_{\mathbb{C}}(1) = 0.5e^{0.4\pi i}, F_{\mathbb{C}}(a) = F_{\mathbb{C}}(a^2) = 0.3e^{0.2\pi i}$$

$$F_{\mathbb{C}}(b) = F_{\mathbb{C}}(ab) = F_{\mathbb{C}}(a^2b) = 0.3e^{0.2\pi i}.$$

Then clearly $\mathbb{C}$ is a complex neutrosophic normal subgroup of $\mathcal{G}$.

Theorem 6.3. *If $\mathbb{C}_1$ and $\mathbb{C}_2$ are any two complex neutrosophic normal subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, then $\mathbb{C}_1 \times \mathbb{C}_2$ is also a complex neutrosophic normal subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$.*

Proof: Similarly to the proof of Theorem 3.5. □

Theorem 6.4. *Let $\mathcal{G}$ be a group, and $\mathbb{C}_1$ and $\mathbb{C}_2$ be two CNNSGs of $\mathcal{G}$, then $\mathbb{C}_1 \cap \mathbb{C}_2$ is also a complex neutrosophic normal subgroup of $\mathcal{G}$.*

Proof: Since $\mathbb{C}_1$ and $\mathbb{C}_2$ are CNNSGs of $\mathcal{G}$, then

$$p_{\mathbb{C}_1}(x \cdot y \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}_1}(x \cdot y \cdot x^{-1})} \geq p_{\mathbb{C}_1}(y) \cdot e^{i\mu_{\mathbb{C}_1}(y)},$$

and

$$p_{\mathbb{C}_2}(x \cdot y \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}_2}(x \cdot y \cdot x^{-1})} \geq p_{\mathbb{C}_2}(y) \cdot e^{i\mu_{\mathbb{C}_2}(y)}.$$

So, by the definition of the intersection,

$$\begin{aligned} p_{\mathbb{C}_1 \cap \mathbb{C}_2}(x \cdot y \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}_1 \cap \mathbb{C}_2}(x \cdot y \cdot x^{-1})} &= p_{\mathbb{C}_1}(x \cdot y \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}_1}(x \cdot y \cdot x^{-1})} \\ &\wedge p_{\mathbb{C}_2}(x \cdot y \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}_2}(x \cdot y \cdot x^{-1})} \\ &\geq p_{\mathbb{C}_1}(y) \cdot e^{i\mu_{\mathbb{C}_1}(y)} \wedge p_{\mathbb{C}_2}(y) \cdot e^{i\mu_{\mathbb{C}_2}(y)} \\ &= p_{\mathbb{C}_1 \cap \mathbb{C}_2}(y) \cdot e^{i\mu_{\mathbb{C}_1 \cap \mathbb{C}_2}(y)}. \end{aligned}$$

By the similar way,

$$q_{\mathbb{C}_1 \cap \mathbb{C}_2}(x \cdot y \cdot x^{-1}) \cdot e^{i\nu_{\mathbb{C}_1 \cap \mathbb{C}_2}(x \cdot y \cdot x^{-1})} \geq q_{\mathbb{C}_1 \cap \mathbb{C}_2}(y) \cdot e^{i\nu_{\mathbb{C}_1 \cap \mathbb{C}_2}(y)}.$$

And

$$\begin{aligned} r_{\mathbb{C}_1 \cup \mathbb{C}_2}(x \cdot y \cdot x^{-1}) \cdot e^{i\omega_{\mathbb{C}_1 \cup \mathbb{C}_2}(x \cdot y \cdot x^{-1})} &= r_{\mathbb{C}_1}(x \cdot y \cdot x^{-1}) \cdot e^{i\omega_{\mathbb{C}_1}(x \cdot y \cdot x^{-1})} \\ &\vee r_{\mathbb{C}_2}(x \cdot y \cdot x^{-1}) \cdot e^{i\omega_{\mathbb{C}_2}(x \cdot y \cdot x^{-1})} \\ &\leq r_{\mathbb{C}_1}(y) \cdot e^{i\omega_{\mathbb{C}_1}(y)} \vee r_{\mathbb{C}_2}(y) \cdot e^{i\omega_{\mathbb{C}_2}(y)} \\ &= r_{\mathbb{C}_1 \cup \mathbb{C}_2}(y) \cdot e^{i\omega_{\mathbb{C}_1 \cup \mathbb{C}_2}(y)}. \end{aligned}$$

Hence the intersection of two CNNSGs is also a CNNSG. □

Theorem 6.5. *If $\mathbb{C}_1$ and $\mathbb{C}_2$ be two CNNSGs of $\mathcal{G}$, then $\mathbb{C}_1 \cup \mathbb{C}_2$ is a complex neutrosophic normal subgroup of $\mathcal{G}$.*

Proof: Similarly to the proof of Theorem 3.11. □

Proposition 6.6. *Let $\mathbb{C}$ be a complex neutrosophic subgroup of a group $\mathcal{G}$. Then the following are correspondent:*

(1) $\mathbb{C}$ is a CNNSG of $\mathcal{G}$.
(2) $\mathbb{C}(x \cdot y \cdot x^{-1}) = \mathbb{C}(y)$, $\forall\ x, y \in \mathcal{G}$.
(3) $\mathbb{C}(x \cdot y) = \mathbb{C}(y \cdot x)$, $\forall\ x, y \in \mathcal{G}$.

Proof: (1) $\Rightarrow$ (2) : Let $\mathbb{C}$ be a complex neutrosophic normal subgroup of $\mathcal{G}$. Take $x, y \in \mathcal{G}$, then by Definition 6.1,

$$p_{\mathbb{C}}(x \cdot y \cdot x^{-1}) \cdot e^{i\mu_{\mathbb{C}}(x \cdot y \cdot x^{-1})} \geq p_{\mathbb{C}}(y) \cdot e^{i\mu_{\mathbb{C}}(y)},$$

$$q_{\mathbb{C}}(x \cdot y \cdot x^{-1}) \cdot e^{i\nu_{\mathbb{C}}(x \cdot y \cdot x^{-1})} \geq q_{\mathbb{C}}(y) \cdot e^{i\nu_{\mathbb{C}}(y)},$$

$$r_{\mathbb{C}}(x \cdot y \cdot x^{-1}) \cdot e^{i\omega_{\mathbb{C}}(x \cdot y \cdot x^{-1})} \leq r_{\mathbb{C}}(y) \cdot e^{i\omega_{\mathbb{C}}(y)}.$$

Thus taking arbitrary element x, the following is got for the truth membership of $\mathbb{C}$,

$$\begin{aligned} p_{\mathbb{C}}(x^{-1}\cdot y\cdot x)\cdot e^{i\mu_{\mathbb{C}}(x^{-1}\cdot y\cdot x)} &= p_{\mathbb{C}}(x^{-1}\cdot y\cdot (x^{-1})^{-1}\cdot e^{i\mu_{\mathbb{C}}(x^{-1}\cdot y\cdot (x^{-1})^{-1})} \\ &\geq p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)}. \end{aligned}$$

Therefore,

$$\begin{aligned} p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)} &= p_{\mathbb{C}}(x^{-1}\cdot(x\cdot y\cdot x^{-1})\cdot x)\cdot e^{i\mu_{\mathbb{C}}(x^{-1}\cdot(x\cdot y\cdot x^{-1})\cdot x)} \\ &\geq p_{\mathbb{C}}(x\cdot y\cdot x^{-1})\cdot e^{i\mu_{\mathbb{C}}(x\cdot y\cdot x^{-1})}. \end{aligned}$$

Thus, $p_{\mathbb{C}}(x\cdot y\cdot x^{-1})\cdot e^{i\mu_{\mathbb{C}}(x\cdot y\cdot x^{-1})} = p_{\mathbb{C}}(y)\cdot e^{i\mu_{\mathbb{C}}(y)}$.

Similarly, $q_{\mathbb{C}}(x\cdot y\cdot x^{-1})\cdot e^{i\nu_{\mathbb{C}}(x\cdot y\cdot x^{-1})} = q_{\mathbb{C}}(y)\cdot e^{i\nu_{\mathbb{C}}(y)}$.

For falsity membership,

$$\begin{aligned} r_{\mathbb{C}}(x^{-1}\cdot y\cdot x)\cdot e^{i\omega_{\mathbb{C}}(x^{-1}\cdot y\cdot x)} &= r_{\mathbb{C}}(x^{-1}\cdot y\cdot (x^{-1})^{-1})\cdot e^{i\omega_{\mathbb{C}}(x^{-1}\cdot y\cdot (x^{-1})^{-1})} \\ &\leq r_{\mathbb{C}}(y)\cdot e^{i\omega_{\mathbb{C}}(y)}. \end{aligned}$$

Therefore,

$$\begin{aligned} r_{\mathbb{C}}(y)\cdot e^{i\omega_{\mathbb{C}}(y)} &= r_{\mathbb{C}}(x^{-1}\cdot(x\cdot y\cdot x^{-1})\cdot x)\cdot e^{i\omega_{\mathbb{C}}(x^{-1}\cdot(x\cdot y\cdot x^{-1})\cdot x)} \\ &\leq r_{\mathbb{C}}(x\cdot y\cdot x^{-1})\cdot e^{i\omega_{\mathbb{C}}(x\cdot y\cdot x^{-1})}. \end{aligned}$$

This implies that

$$r_{\mathbb{C}}(x\cdot y\cdot x^{-1})\cdot e^{i\omega_{\mathbb{C}}(x\cdot y\cdot x^{-1})} = r_{\mathbb{C}}(y)\cdot e^{i\omega_{\mathbb{C}}(y)}.$$

Hence $\mathbb{C}(x\cdot y\cdot x^{-1}) = \mathbb{C}(y)$ for all $x, y\in\mathcal{G}$.

(2) $\Rightarrow$ (3) : Substituting $y = y\cdot x$ in (2), the condition (3) is shown easily.
(3) $\Rightarrow$ (1) : According to $\mathbb{C}(y\cdot x) = \mathbb{C}(x\cdot y)$, the equality

$$\mathbb{C}(x\cdot y\cdot x^{-1}) = \mathbb{C}(y\cdot x\cdot x^{-1}) = \mathbb{C}(y)\geq\mathbb{C}(y)$$

is satisfied. Hence $\mathbb{C}$ is a CNNSG of $\mathcal{G}$. □

Theorem 6.7. *Let $\mathbb{C}$ is a complex neutrosophic subgroup of a group $\mathcal{G}$. Then $\mathbb{C}$ is a complex neutrosophic normal subgroup of $\mathcal{G}$ if and only if for arbitrary $\alpha = \beta e^{i\gamma}$ where $\beta\in[0,1], \gamma\in[0,2\pi]$, if α-level sets of $\mathbb{C}$ are non-empty, then $\left(p_{\mathbb{C}}\cdot e^{i\mu_{\mathbb{C}}}\right)_{\alpha}$, $\left(q_{\mathbb{C}}\cdot e^{i\nu_{\mathbb{C}}}\right)_{\alpha}$ and $\left(r_{\mathbb{C}}\cdot e^{i\omega_{\mathbb{C}}}\right)^{\alpha}$ are classical subgroups of $\mathcal{G}$.*

Proof: Similarly to the proof of Proposition 4.3. □

Theorem 6.8. *Let $\mathbb{C}$ is a complex neutrosophic normal subgroup of a group $\mathcal{G}$. Let $\mathcal{G}_{\mathbb{C}} = \{x\in\mathcal{G}\mid \mathbb{C}(x)e^{i\mathbb{C}(x)} = \mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})}\}$, where $\hat{e}$ is the unit of $\mathcal{G}$. Then the classical subset $\mathcal{G}_{\mathbb{C}}$ of $\mathcal{G}$ is a normal subgroup of $\mathcal{G}$.*

Proof: Let $\mathbb{C}$ be a CNNSG of $\mathcal{G}$. First it is necessary to show that the classical subset $\mathcal{G}_{\mathbb{C}}$ is a subgroup of $\mathcal{G}$. Let us take $x, y\in\mathcal{G}_{\mathbb{C}}$, then by Theorem 3.7,

$$\begin{aligned} \mathbb{C}(x\cdot y^{-1})e^{i\mathbb{C}(x\cdot y^{-1})} &\geq \mathbb{C}(x)e^{i\mathbb{C}(x)}\wedge\mathbb{C}(y)e^{i\mathbb{C}(y)} \\ &= \mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})}\wedge\mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})} \\ &= \mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})} \end{aligned}$$

and always $\mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})} \geq \mathbb{C}(x \cdot y^{-1})e^{i\mathbb{C}(x \cdot y^{-1})}$.

Hence $x \cdot y^{-1} \in \mathcal{G}_{\mathbb{C}}$, i.e., $\mathcal{G}_{\mathbb{C}}$ is a subgroup of $\mathcal{G}$.

Now we will be shown that $\mathcal{G}_{\mathbb{C}}$ is normal. Take arbitrary $x \in \mathcal{G}_{\mathbb{C}}$ and $y \in \mathcal{G}$. Therefore, $\mathbb{C}(x)e^{i\mathbb{C}(x)} = \mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})}$. Since $\mathbb{C}$ $\in$CNNSG($\mathcal{G}$), the following is obtained,

$$\begin{aligned}\mathbb{C}(y \cdot x \cdot y^{-1})e^{i\mathbb{C}(y \cdot x \cdot y^{-1})} &= \mathbb{C}(y^{-1} \cdot y \cdot x)e^{i\mathbb{C}(y^{-1} \cdot y \cdot x)} \\ &= \mathbb{C}(x)e^{i\mathbb{C}(x)} = \mathbb{C}(\hat{e})e^{i\mathbb{C}(\hat{e})}.\end{aligned}$$

Hence, $y \cdot x \cdot y^{-1} \in \mathcal{G}_{\mathbb{C}}$, So $\mathcal{G}_{\mathbb{C}}$ is a normal subgroup of $\mathcal{G}$.

Theorem 6.9. *Let $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a group homomorphism and $\mathbb{C}_2$ is a CNNSG of $\mathcal{G}_2$. Then the preimage $f^{-1}(\mathbb{C}_2)$ is a CNNSG of $\mathcal{G}_1$.*

Proof: From the Theorem 5.3, it is known that $f^{-1}(\mathbb{C}_2)$ is a complex neutrosophic subgroup of $\mathcal{G}_1$. Hence it is sufficient to show that normality property of $f^{-1}(\mathbb{C}_2)$. For arbitrary $x_1, x_2 \in \mathcal{G}_1$, by homomorphism of f and by the normality of $\mathbb{C}_2$,

$$\begin{aligned}f^{-1}(\mathbb{C}_2)(x_1 \cdot x_2)e^{if^{-1}(\mathbb{C}_2)(x_1 \cdot x_2)} &= \mathbb{C}_2(f(x_1 \cdot x_2))e^{i\mathbb{C}_2(f(x_1 \cdot x_2))} \\ &= \mathbb{C}_2(f(x_1) \cdot f(x_2))e^{i\mathbb{C}_2(f(x_1) \cdot f(x_2))} \\ &= \mathbb{C}_2(f(x_2) \cdot f(x_1))e^{i\mathbb{C}_2(f(x_2) \cdot f(x_1))} \\ &= \mathbb{C}_2(f(x_2 \cdot x_1))e^{i\mathbb{C}_2(f(x_2 \cdot x_1))} \\ &= f^{-1}(\mathbb{C}_2)(x_2 \cdot x_1)e^{if^{-1}(\mathbb{C}_2)(x_2 \cdot x_1)}.\end{aligned}$$

Hence, from the Proposition 6.6, $f^{-1}(\mathbb{C}_2)$ is a CNNSG of $\mathcal{G}_1$. □

Theorem 6.10. *Let $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ be a surjective homomorphism of groups $\mathcal{G}_1$ and $\mathcal{G}_2$. if $\mathbb{C}$ is a CNNSG of $\mathcal{G}_1$, then $f(\mathbb{C})$ is a CNNSG of $\mathcal{G}_2$.*

Proof: Since $f(\mathbb{C})$ is a complex neutrosophic subgroup of $\mathcal{G}_2$ is clear from the Theorem 5.2, it is sufficient only to show that the normality condition by using Proposition 6.6 (3). Take $y_1, y_2 \in \mathcal{G}_2$ such that $f^{-1}(y_1) \neq \phi$, $f^{-1}(y_2) \neq \phi$ and $f^{-1}(y_1 \cdot y_2^{-1}) \neq \phi$. So it is inferred that

$$f(p_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))e^{if(\mu_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))} = \bigvee_{l \in f^{-1}(y_1 \cdot y_2 \cdot y_1^{-1})} p_{\mathbb{C}}(l)e^{i\mu_{\mathbb{C}}(l)}$$

and

$$f(p_{\mathbb{C}}(y_2))e^{if(\mu_{\mathbb{C}}(y_2))} = \bigvee_{l \in f^{-1}(y_2)} p_{\mathbb{C}}(l)e^{i\mu_{\mathbb{C}}(l)}.$$

For all $x_2 \in f^{-1}(y_2)$, $x_1 \in f^{-1}(y_1)$ and $x_1^{-1} \in f^{-1}(y_1^{-1})$, since $\mathbb{C}$ is normal,

$$p_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})e^{i\mu_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})} \geq p_{\mathbb{C}}(x_2)e^{i\mu_{\mathbb{C}}(x_2)},$$

$$q_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})e^{i\nu_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})} \geq q_{\mathbb{C}}(x_2)e^{i\nu_{\mathbb{C}}(x_2)},$$

$$r_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})e^{i\omega_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})} \leq r_{\mathbb{C}}(x_2)e^{i\omega_{\mathbb{C}}(x_2)}$$

are obtained.

Since f is a homomorphism , it follows that

$$f(x_1 \cdot x_2 \cdot x_1^{-1}) = f(x_1) \cdot f(x_2) \cdot f(x_1)^{-1} = y_1 \cdot y_2 \cdot y_1^{-1}.$$

So, $x_1 \cdot x_2 \cdot x_1^{-1} \in f^{-1}(y_1 \cdot y_2 \cdot y_1^{-1})$. Hence

$$\bigvee_{l \in f^{-1}(y_1 \cdot y_2 \cdot y_1^{-1})} p_{\mathbb{C}}(l) e^{i\mu_{\mathbb{C}}(l)} \geq \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} p_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1}) e^{i\mu_{\mathbb{C}}(x_1 \cdot x_2 \cdot x_1^{-1})}$$
$$\geq \bigvee_{x_2 \in f^{-1}(y_2)} p_{\mathbb{C}}(x_2) e^{i\mu_{\mathbb{C}}(x_2)}.$$

This means that,

$$f(p_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1})) e^{if(\mu_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))} \geq f(p_{\mathbb{C}}(y_2)) e^{if(\mu_{\mathbb{C}}(y_2))}.$$

On the other hand, the following inequalities are obtained in a similar observation.

$$f(q_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1})) e^{if(\nu_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))} \geq f(q_{\mathbb{C}}(y_2)) e^{if(\nu_{\mathbb{C}}(y_2))},$$

$$f(r_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1})) e^{if(\omega_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))} \geq f(r_{\mathbb{C}}(y_2)) e^{if(\omega_{\mathbb{C}}(y_2))}.$$

So the desired inequality,

$$\begin{aligned} f(\mathbb{C})(y_1 \cdot y_2 \cdot y_1^{-1}) e^{if(\mathbb{C})(y_1 \cdot y_2 \cdot y_1^{-1})} = \Big(& f(p_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1})) e^{if(\mu_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))}, \\ & f(q_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1})) e^{if(\nu_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))}, \\ & f(r_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1})) e^{if(\omega_{\mathbb{C}}(y_1 \cdot y_2 \cdot y_1^{-1}))} \Big) \\ & \geq \Big(f(p_{\mathbb{C}}(y_2)) e^{if(\mu_{\mathbb{C}}(y_2))}, f(q_{\mathbb{C}}(y_2)) e^{if(\nu_{\mathbb{C}}(y_2))}, \\ & f(r_{\mathbb{C}}(y_2)) e^{if(\omega_{\mathbb{C}}(y_2))} \Big) \\ & = \Big(p_{f(\mathbb{C})}(y_2) e^{i\mu_{f(\mathbb{C})}(y_2)}, q_{f(\mathbb{C})}(y_2) e^{i\nu_{f(\mathbb{C})}(y_2)}, \\ & r_{f(\mathbb{C})}(y_2) e^{i\omega_{f(\mathbb{C})}(y_2)} \Big) \\ & = f(\mathbb{C})(y_2) e^{if(\mathbb{C})(y_2)}, \end{aligned}$$

is satisfied. □

7. Conclusion

In this paper we presented the concept of complex neutrosophic subgroups (normal subgroups) and alpha-cut of complex neutrosophic set, and studied some of its motivating results. We have also defined the Cartesian product of complex neutrosophic subgroups and discussed some its related results. Furthermore, we have also defined the concept of image and preimage of complex neutrosophic set and studied some of its properties. In future, we will generalized the study to soft set theory and will initiate the concept of soft complex neutrosophic subgroups (normal subgroups).

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An Extended Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) with Maximizing Deviation Method Based on Integrated Weight Measure for Single-Valued Neutrosophic Sets

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Abstract: A single-valued neutrosophic set (SVNS) is a special case of a neutrosophic set which is characterized by a truth, indeterminacy, and falsity membership function, each of which lies in the standard interval of [0, 1]. This paper presents a modified Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) with maximizing deviation method based on the single-valued neutrosophic set (SVNS) model. An integrated weight measure approach that takes into consideration both the objective and subjective weights of the attributes is used. The maximizing deviation method is used to compute the objective weight of the attributes, and the non-linear weighted comprehensive method is used to determine the combined weights for each attributes. The use of the maximizing deviation method allows our proposed method to handle situations in which information pertaining to the weight coefficients of the attributes are completely unknown or only partially known. The proposed method is then applied to a multi-attribute decision-making (MADM) problem. Lastly, a comprehensive comparative studies is presented, in which the performance of our proposed algorithm is compared and contrasted with other recent approaches involving SVNSs in literature.

Keywords: 2ingle-valued neutrosophic set; Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS); integrated weight; maximizing deviation; multi-attribute decision-making (MADM)

1. Introduction

The study of fuzzy set theory proposed by Zadeh [1] was an important milestone in the study of uncertainty and vagueness. The widespread success of this theory has led to the introduction of many extensions of fuzzy sets such as the intuitionistic fuzzy set (IFS) [2], interval-valued fuzzy set (IV-FS) [3], vague set [4], and hesitant fuzzy set [5]. The most widely used among these models is the IFS model which has also spawned other extensions such as the interval-valued intuitionistic fuzzy set [6] and bipolar intuitionistic fuzzy set [7]. Smarandache [8] then introduced an improvement to IFS theory called neutrosophic set theory which loosely refers to neutral knowledge. The study of the neutrality aspect of knowledge is the main distinguishing criteria between the theory of fuzzy sets, IFSs, and neutrosophic sets. The classical neutrosophic set (NS) is characterized by three membership functions which describe the degree of truth (T), the degree of indeterminacy (I), and the degree of falsity (F), whereby all of these functions assume values in the non-standard interval of $]0^{-}, 1^{+}[$. The truth and falsity membership functions in a NS are analogous to the membership and non-membership functions in an IFS, and expresses the degree of belongingness and non-belongingness of the elements, whereas the indeterminacy membership function expresses the degree of neutrality in the information. This additional indeterminacy membership function gives NSs the ability to handle the neutrality aspects of the information, which fuzzy sets and its extensions are unable to handle. Another distinguishing factor between NSs and other fuzzy-based models is the fact that all the three membership functions in a NS are entirely independent of one another, unlike the membership and non-membership functions in an IFS or other fuzzy-based models in which values of the membership and non-membership functions are dependent on one another. This gives NSs the ability to handle uncertain, imprecise, inconsistent, and indeterminate information, particularly in situations whereby the factors affecting these aspects of the information are independent of one another. This also makes the NS more versatile compared to IFSs and other fuzzy- or IF-based models in literature.

Smarandache [8] and Wang et al. [9] pointed out that the non-standard interval of $]0^{-}, 1^{+}[$ in which the NS is defined in, makes it impractical to be used in real-life problems. Furthermore, values in this non-standard interval are less intuitive and the significance of values in this interval can be difficult to be interpreted. This led to the conceptualization of the single-valued neutrosophic set (SVNS). The SVNS is a straightforward extension of NS which is defined in the standard unit interval of [0, 1]. As values in [0, 1] are compatible with the range of acceptable values in conventional fuzzy set theory and IFS theory, it is better able to capture the intuitiveness of the process of assigning membership values. This makes the SVNS model easier to be applied in modelling real-life problems as the results obtained are a lot easier to be interpreted compared to values in the interval $]0^{-}, 1^{+}[$.

The SVNS model has garnered a lot of attention since its introduction in [9], and has been actively applied in various multi-attribute decision-making (MADM) problems using a myriad of different approaches. Wang et al. [9] introduced some set theoretic operators for SVNSs, and studied some additional properties of the SVNS model. Ye [10,11] introduced a decision-making algorithm based on the correlation coefficients for SVNSs, and applied this algorithm in solving some MADM problems. Ye [12,13] introduced a clustering method and also some decision-making methods that are based on the similarity measures of SVNSs, whereas Huang [14] introduced a new decision-making method for SVNSs and applied this method in clustering analysis and MADM problems. Peng and Liu [15] on the other hand proposed three decision-making methods based on a new similarity measure, the EDAS method and level soft sets for neutrosophic soft sets, and applied this new measure to MADM problems set in a neutrosophic environment. The relations between SVNSs and its properties were first studied by Yang et al. [16], whereas the graph theory of SVNSs and bipolar SVNSs were introduced by Broumi et al. in [17–19] and [20–22], respectively. The aggregation operators of simplified neutrosophic sets (SNSs) were studied by Tian et al. [23] and Wu et al. [24]. Tian et al. [23] introduced a generalized prioritized aggregation operator for SNSs and applied this operator in a MADM problem set in an uncertain linguistic environment, whereas Wu et al. [24] introduced a cross-entropy measure and a prioritized aggregation operator for SNSs and applied these in a MADM problem. Sahin and Kucuk [25] proposed a subsethood measure for SVNSs and applied these to MADM problems.

The fuzzy Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) method for SVNSs were studied by Ye [26] and Biswas et al. [27]. Ye [26] introduced the TOPSIS method for group decision-making (MAGDM) that is based on single-valued neutrosophic linguistic numbers, to deal with linguistic decision-making. This TOPSIS method uses subjective weighting method whereby attribute weights are randomly assigned by the users. Maximizing deviation method or any other objective weighting methods are not used. Biswas et al. [27] proposed a TOPSIS method for group decision-making (MAGDM) based on the SVNS model. This TOPSIS method is based on the original fuzzy TOPSIS method and does not use the maximizing deviation method to calculate the objective weights for each attribute. The subjective weight of each attribute is determined by using the single-valued neutrosophic weighted averaging aggregation operator to calculate the aggregated weights of the attributes using the subjective weights that are assigned by each decision maker.

The process of assigning weights to the attributes is an important phase of decision making. Most research in this area usually use either objective or subjective weights. However, considering the fact that different values for the weights of the attributes has a significant influence on the ranking of the alternatives, it is imperative that both the objective and subjective weights of the attributes are taken into account in the decision-making process. In view of this, we consider the attributes' subjective weights which are assigned by the decision makers, and the objective weights which are computed using the maximizing deviation method. These weights are then combined using the non-linear weighted comprehensive method to obtain the integrated weight of the attributes.

The advantages and drawbacks of the methods that were introduced in the works described above served as the main motivation for the work proposed in this paper, as we seek to introduce an effective SVNS-based decision-making method that is free of all the problems that are inherent in the other existing methods in literature. In addition to these advantages and drawbacks, the works described above have the added disadvantage of not being able to function (i.e., provide reasonable solutions) under all circumstances. In view of this, the objective of this paper is to introduce a novel TOPSIS with maximizing deviation method for SVNSs that is able to provide effective solutions under any circumstances. Our proposed TOPSIS method is designed to handle MADM problems, and uses the maximizing deviation method to calculate the objective weights of attributes, utilizing an integrated weight measure that takes into consideration both the subjective and objective weights of the attributes. The robustness of our TOPSIS method is verified through a comprehensive series of tests which proves that our proposed method is the only method that shows compliance to all the tests, and is able to provide effective solutions under all different types of situations, thus out-performing all of the other considered methods.

The remainder of this paper is organized as follows. In Section 2, we recapitulate some of the fundamental concepts related to neutrosophic sets and SVNSs. In Section 3, we define an SVNS-based TOPSIS and maximizing deviation methods and an accompanying decision-making algorithm. The proposed decision-making method is applied to a supplier selection problem in Section 4. In Section 5, a comprehensive comparative analysis of the results obtained via our proposed method and other recent approaches is presented. The similarities and differences in the performance of the existing algorithms and our algorithm is discussed, and it is proved that our algorithm is effective and provides reliable results in every type of situation. Concluding remarks are given in Section 6, followed by the acknowledgements and list of references.

2. Preliminaries

In this section, we recapitulate some important concepts pertaining to the theory of neutrosophic sets and SVNSs. We refer the readers to [8,9] for further details pertaining to these models.

The neutrosophic set model [8] is a relatively new tool for representing and measuring uncertainty and vagueness of information. It is fast becoming a preferred general framework for the analysis of uncertainty in data sets due to its capability in the handling big data sets, as well as its ability in representing all the different types of uncertainties that exists in data, in an effective and concise manner via a triple membership structure. This triple membership structure captures not only the degree of belongingness and non-belongingness of the objects in a data set, but also the degree of neutrality and indeterminacy that exists in the data set, thereby making it superior to ordinary fuzzy sets [1] and its extensions such as IFSs [2], vague sets [4], and interval-valued fuzzy sets [3]. The formal definition of a neutrosophic set is as given below.

Let U be a universe of discourse, with a class of elements in U denoted by x.

Definition 1. [8] *A neutrosophic set A is an object having the form* $A = \{x,\ T_A(x),\ I_A(x),\ F_A(x) : x \in U\}$, *where the functions* $T,\ I,\ F\ : U \to]^-0,\ 1^+[$ *denote the truth, indeterminacy, and falsity membership functions, respectively, of the element* $x \in U$ *with respect to A. The membership functions must satisfy the condition* $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Definition 2. [8] *A neutrosophic set A is contained in another neutrosophic set B, if* $T_A(x) \le T_B(x),\ I_A(x) \ge I_B(x)$, *and* $F_A(x) \ge F_B(x)$, *for all* $x \in U$. *This relationship is denoted as* $A \subseteq B$.

Wang et al. [9] then introduced a special case of the NS model called the single-valued neutrosophic set (SVNS) model, which is as defined below. This SVNS model is better suited to applied in real-life problems compared to NSs due to the structure of its membership functions which are defined in the standard unit interval of [0, 1].

Definition 3. [9] *A SVNS A is a neutrosophic set that is characterized by a truth-membership function* $T_A(x)$, *an indeterminacy-membership function* $I_A(x)$, *and a falsity-membership function* $F_A(x)$, *where* $T_A(x),\ I_A(x),\ F_A(x) \in [0,\ 1]$. *This set A can thus be written as*

$$A = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle : x \in U\}. \tag{1}$$

The sum of $T_A(x),\ I_A(x)$ and $F_A(x)$ must fulfill the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$. For a SVNS A in U, the triplet $(T_A(x),\ I_A(x),\ F_A(x))$ is called a single-valued neutrosophic number (SVNN). For the sake of convenience, we simply let $x = (T_x,\ I_x,\ F_x)$ to represent a SVNN as an element in the SVNS A.

Next, we present some important results pertaining to the concepts and operations of SVNSs. The subset, equality, complement, union, and intersection of SVNSs, and some additional operations between SVNSs were all defined by Wang et al. [9], and these are presented in Definitions 4 and 5, respectively.

Definition 4. [9] *Let A and B be two SVNSs over a universe U.*

(i) *A is contained in B, if* $T_A(x) \le T_B(x),\ I_A(x) \ge I_B(x)$, *and* $F_A(x) \ge F_B(x)$, *for all* $x \in U$. *This relationship is denoted as* $A \subseteq B$.
(ii) *A and B are said to be equal if* $A \subseteq B$ *and* $B \subseteq A$.
(iii) $A^c = (x,\ (F_A(x),\ 1 - I_A(x),\ T_A(x)))$, *for all* $x \in U$.
(iv) $A \cup B = (x,\ (\max(T_A,\ T_B), \min(I_A,\ I_B), \min(F_A,\ F_B)))$, *for all* $x \in U$.
(v) $A \cap B = (x,\ (\min(T_A,\ T_B), \max(I_A,\ I_B), \max(F_A,\ F_B)))$, *for all* $x \in U$.

Definition 5. [9] *Let* $x = (T_x,\ I_x,\ F_x)$ *and* $y = \left(T_y,\ I_y,\ F_y\right)$ *be two SVNNs. The operations for SVNNs can be defined as follows:*

(i) $x \bigoplus y = \left(T_x + T_y - T_x * T_y,\ I_x * I_y,\ F_x * F_y\right)$
(ii) $x \bigotimes y = \left(T_x * T_y,\ I_x + I_y - I_x * I_y,\ F_x + F_y - F_x * F_y\right)$
(iii) $\lambda x = \left(1 - (1 - T_x)^\lambda,\ (I_x)^\lambda,\ (F_x)^\lambda\right)$, *where* $\lambda > 0$
(iv) $x^\lambda = \left((T_x)^\lambda,\ 1 - (1 - I_x)^\lambda,\ 1 - (1 - F_x)^\lambda\right)$, *where* $\lambda > 0$.

Majumdar and Samanta [28] introduced the information measures of distance, similarity, and entropy for SVNSs. Here we only present the definition of the distance measures between SVNSs as it is the only component that is relevant to this paper.

Definition 6. [28] *Let A and B be two SVNSs over a finite universe* $U = \{x_1,\ x_2, \ldots,\ x_n\}$. *Then the various distance measures between A and B are defined as follows:*

(i) *The Hamming distance between A and B are defined as:*

$$d_H(A,\ B) = \sum_{i=1}^{n}\{|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|\} \tag{2}$$

(ii) *The normalized Hamming distance between A and B are defined as:*

$$d_H^N(A,\ B) = \frac{1}{3n}\sum_{i=1}^{n}\{|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|\} \tag{3}$$

(ii) *The Euclidean distance between A and B are defined as:*

$$d_E(A,\ B) = \sqrt{\sum_{i=1}^{n}\left\{(T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2\right\}} \tag{4}$$

(iv) *The normalized Euclidean distance between A and B are defined as:*

$$d_E^N(A,\ B) = \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\left\{(T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2\right\}} \tag{5}$$

3. A TOPSIS Method for Single-Valued Neutrosophic Sets

In this section, we present the description of the problem that is being studied followed by our proposed TOPSIS method for SVNSs. The accompanying decision-making algorithm which is based on the proposed TOPSIS method is presented. This algorithm uses the maximizing deviation method to systematically determine the objective weight coefficients for the attributes.

3.1. Description of Problem

Let $U = \{u_1, u_2,\ \ldots,\ u_m\}$ denote a finite set of m alternatives, $A = \{e_1,\ e_2,\ \ldots,\ e_n\}$ be a set of n parameters, with the weight parameter w_j of each e_j completely unknown or only partially known, $w_j \in [0,\ 1]$, and $\sum_{j=1}^{n} w_j = 1$.

Let A be an SVNS in which $x_{ij} = \left(T_{ij},\ I_{ij},\ F_{ij}\right)$ represents the SVNN that represents the information pertaining to the ith alternative x_i that satisfies the corresponding jth parameter e_j. The tabular representation of A is as given in Table 1.

Table 1. Tabular representation of the Single Valued Neutrosophic Set (SVNS) A.

U	e_1	e_2	$\ldots$	e_n
x_1	$(T_{11},\ I_{11},\ F_{11})$	$(T_{12},\ I_{12},\ F_{12})$	$\ldots$	$(T_{1n},\ I_{1n},\ F_{1n})$
x_2	$(T_{21},\ I_{21},\ F_{21})$	$(T_{22},\ I_{22},\ F_{22})$	$\ldots$	$(T_{2n},\ I_{2n},\ F_{2n})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$x_m.$	$(T_{m1},\ I_{m1},\ F_{m1})$	$(T_{m2},\ I_{m2},\ F_{m2})$	$\ldots$	$(T_{mn},\ I_{mn},\ F_{mn})$

3.2. The Maximizing Deviation Method for Computing Incomplete or Completely Unknown Attribute Weights

The maximizing deviation method was proposed by Wang [29] with the aim of applying it in MADM problems in which the weights of the attributes are completely unknown or only partially known. This method uses the law of input arguments i.e., it takes into account the magnitude of the membership functions of each alternative for each attribute, and uses this information to obtain exact and reliable evaluation results pertaining to the weight coefficients for each attribute. As such, this method is able to compute the weight coefficients of the attributes without any subjectivity, in a fair and objective manner.

The maximizing deviation method used in this paper is a modification of the original version introduced in Wang [29] that has been made compatible with the structure of the SVNS model. The definitions of the important concepts involved in this method are as given below.

Definition 7. *For the parameter $e_j \in A$, the deviation of the alternative x_i to all the other alternatives is defined as:*

$$D_{ij}(w_j) = \sum_{k=1}^{m} w_j\, d\left(x_{ij},\ x_{kj}\right), \tag{6}$$

where $x_{ij},\ x_{kj}$ are the elements of the SVNS A, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $d\left(x_{ij},\ x_{kj}\right)$ denotes the distance between elements x_{ij} and x_{kj}.

The other deviation values include the deviation value of all alternatives to other alternatives, and the total deviation value of all parameters to all alternatives, both of which are as defined below:

(i) The deviation value of all alternatives to other alternatives for the parameter $e_j \in A$, denoted by $D_j(w_j)$, is defined as:

$$D_j(w_j) = \sum_{i=1}^{m} D_{ij}(w_j) = \sum_{i=1}^{m}\sum_{k=1}^{m} w_j\, d\left(x_{ij},\ x_{kj}\right), \tag{7}$$

where $j = 1, 2, \ldots, n$.

(ii) The total deviation value of all parameters to all alternatives, denoted by $D(w_j)$, is defined as:

$$(w_j) = \sum_{j=1}^{n} D_j(w_j) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} w_j\, d\left(x_{ij},\ x_{kj}\right), \tag{8}$$

where w_j represents the weight of the parameter $e_j \in A$.

(iii) The individual objective weight of each parameter $e_j \in A$, denoted by θ_j, is defined as:

$$\theta_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(x_{ij},\ x_{kj}\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(x_{ij},\ x_{kj}\right)} \tag{9}$$

It should be noted that any valid distance measure between SVNSs can be used in Equations (6)–(9). However, to improve the effective resolution of the decision-making process, in this paper, we use the normalized Euclidean distance measure given in Equation (5) in the computation of Equations (6)–(9).

3.3. TOPSIS Method for MADM Problems with Incomplete Weight Information

The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) was originally introduced by Hwang and Yoon [30], and has since been extended to fuzzy sets, IFSs, and other fuzzy-based models. The TOPSIS method works by ranking the alternatives based on their distance from the positive ideal solution and the negative ideal solution. The basic guiding principle is that the most preferred alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution (Hwang and Yoon [30], Chen and Tzeng [31]). In this section, we present a decision-making algorithm for solving MADM problems in single-valued neutrosophic environments, with incomplete or completely unknown weight information.

3.3.1. The Proposed TOPSIS Method for SVNSs

After obtaining information pertaining to the weight values for each parameter based on the maximizing deviation method, we develop a modified TOPSIS method for the SVNS model. To achieve our goal, we introduce several definitions that are the important components of our proposed TOPSIS method.

Let the relative neutrosophic positive ideal solution (RNPIS) and relative neutrosophic negative ideal solution (RNNIS) be denoted by b^+ and b^-, respectively, where these solutions are as defined below:

$$b^+ = \left\{ \left(\max_i T_{ij},\ \min_i I_{ij},\ \min_i F_{ij} \right) \middle|\ j = 1,\ 2, \ldots,\ n \right\}, \tag{10}$$

and

$$b^- = \left\{ \left(\min_i T_{ij},\ \max_i I_{ij},\ \max_i F_{ij} \right) \middle|\ j = 1,\ 2, \ldots,\ n \right\} \tag{11}$$

The difference between each object and the RNPIS, denoted by D_i^+, and the difference between each object and the RNNIS, denoted by D_i^-, can then be calculated using the normalized Euclidean distance given in Equation (5) and by the formula given in Equations (12) and (13).

$$D_i^+ = \sum_{j=1}^{n} w_j\, d_{NE}\left(b_{ij},\ b_j^+\right), \qquad i = 1,\ 2,\ \ldots,\ m \tag{12}$$

and

$$D_i^- = \sum_{j=1}^{n} w_j\, d_{NE}\left(b_{ij},\ b_j^-\right), \qquad i = 1,\ 2,\ \ldots,\ m \tag{13}$$

Here, w_j denotes the integrated weight for each of the attributes.

The optimal alternative can then be found using the measure of the relative closeness coefficient of each alternative, denoted by C_i, which is as defined below:

$$C_i = \frac{D_i^-}{\max_j D_j^-} - \frac{D_i^+}{\min_j D_j^+}, \quad i,\ j = 1,\ 2,\ \ldots,\ m \tag{14}$$

From the structure of the closeness coefficient in Equation (14), it is obvious that the larger the difference between an alternative and the fuzzy negative ideal object, the larger the value of the closeness coefficient of the said alternative. Therefore, by the principal of maximum similarity between an alternative and the fuzzy positive ideal object, the objective of the algorithm is to determine the alternative with the maximum closeness coefficient. This alternative would then be chosen as the optimal alternative.

3.3.2. Attribute Weight Determination Method: An Integrated WEIGHT MEASure

In any decision-making process, there are two main types of weight coefficients, namely the subjective and objective weights that need to be taken into consideration. Subjective weight refers to the values assigned to each attribute by the decision makers based on their individual preferences and experience, and is very much dependent on the risk attitude of the decision makers. Objective weight refers to the weights of the attributes that are computed mathematically using any appropriate computation method. Objective weighting methods uses the law of input arguments (i.e., the input values of the data) as it determines the attribute weights based on the magnitude of the membership functions that are assigned to each alternative for each attribute.

Therefore, using only subjective weighting in the decision-making process would be inaccurate as it only reflects the opinions of the decision makers while ignoring the importance of each attribute that are reflected by the input values. Using only objective weighting would also be inaccurate as it only reflects the relative importance of the attributes based on the law of input arguments, but fails to take into consideration the preferences and risk attitude of the decision makers.

To overcome this drawback and improve the accuracy and reliability of the decision-making process, we use an integrated weight measure which combines the subjective and objective weights of the attributes. This factor makes our decision-making algorithm more accurate compared to most of the other existing methods in literature that only take into consideration either the objective or subjective weights.

Based on the formula and weighting method given above, we develop a practical and effective decision-making algorithm based on the TOPSIS approach for the SVNS model with incomplete weight information. The proposed Algorithm 1 is as given below.

Algorithm 1. (based on a modified TOPSIS approach).

Step 1. Input the SVNS A which represents the information pertaining to the problem.
Step 2. Input the subjective weight h_j for each of the attributes $e_j \in A$ as given by the decision makers.
Step 3. Compute the objective weight θ_j for each of the attributes $e_j \in A$, using Equation (9).
Step 4. The integrated weight coefficient w_j for each of the attributes $e_j \in A$, is computed using Equation as follow:

$$w_j = \frac{h_j\,\theta_j}{\sum_{j=1}^{n} h_j\,\theta_j}$$

Step 5. The values of RNPIS b^+ and RNNIS b^- are computed using Equations (10) and (11).
Step 6. The difference between each alternative and the RNPIS, D^+ and the RNNIS D^- are computed using Equations (12) and (13), respectively.
Step 7. The relative closeness coefficient C_i for each alternative is calculated using Equation (14).
Step 8. Choose the optimal alternative based on the principal of maximum closeness coefficient.

4. Application of the Topsis Method in a Made Problem

The implementation process and utility of our proposed decision-making algorithm is illustrated via an example related to a supplier selection problem.

4.1. Illustrative Example

In today's extremely competitive business environment, firms must be able to produce good quality products at reasonable prices in order to be successful. Since the quality of the products is directly dependent on the effectiveness and performance of its suppliers, the importance of supplier selection has become increasingly recognized. In recent years, this problem has been handled using various mathematical tools. Some of the recent research in this area can be found in [32–38].

Example 1. *A manufacturing company is looking to select a supplier for one of the products manufactured by the company. The company has shortlisted ten suppliers from an initial list of suppliers. These ten suppliers form the set of alternatives U that are under consideration,*

$$U = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9,\ x_{10}\}.$$

The procurement manager and his team of buyers evaluate the suppliers based on a set of evaluation attributes E which is defined as:

$$E = \{e_1 = \text{service quality},\ e_2 = \text{pricing and cos t structure},\ e_3 = \text{financial stability},\ e_4 = \text{environmental regulation compliance},\ e_5 = \text{reliability},\ e_6 = \text{relevant experience}\}.$$

The firm then evaluates each of the alternatives x_i $(i = 1, 2, \ldots, 10)$, with respect to the attributes e_j $(j = 1, 2, \ldots, 6)$. The evaluation done by the procurement team is expressed in the form of SVNNs in a SVNS A.

Now suppose that the company would like to select one of the five shortlisted suppliers to be their supplier. We apply the proposed Algorithm 1 outlined in Section 3.3 to this problem with the aim of selecting a supplier that best satisfies the specific needs and requirements of the company. The steps involved in the implementation process of this algorithm are outlined below (Algorithm 2).

Algorithm 2. (based on the modified TOPSIS approach).

Step 1. The SVNS A constructed for this problem is given in tabular form in Table 2
Step 2. The subjective weight h_j for each attribute $e_j \in A$ as given by the procurement team (the decision makers) are $h = \{h_1 = 0.15,\ h_2 = 0.15,\ h_3 = 0.22,\ h_4 = 0.25,\ h_5 = 0.14,\ h_6 = 0.09\}$.
Step 3. The objective weight θ_j for each attribute $e_j \in A$ is computed using Equation (9) are as given below:
$\theta = \{\theta_1 = 0.139072,\ \theta_2 = 0.170256,\ \theta_3 = 0.198570,\ \theta_4 = 0.169934, \theta_5 = 0.142685,$
$\theta_6 = 0.179484\}$.
Step 4. The integrated weight w_j for each attribute $e_j \in A$ is computed using Equation (15). The integrated weight coefficent obtained for each attribute is:
$w = \{w_1 = 0.123658,\ w_2 = 0.151386,\ w_3 = 0.258957,\ w_4 = 0.251833,\ w_5 = 0.118412,$
$w_6 = 0.0957547\}$.
Step 5. Use Equations (10) and (11) to compute the values of b^+ and b^- from the neutrosophic numbers given in Table 2. The values are as given below:
$b^+ = \{b_1^+ = [0.7,\ 0.2,\ 0.1],\ b_2^+ = [0.9,\ 0,\ 0.1],\ b_3^+ = [0.8,\ 0,\ 0],\ b_4^+ = [0.9,\ 0.3,\ 0],$
$b_5^+ = [0.7,\ 0.2,\ 0.2],\ b_6^+ = [0.8,\ 0.2\ 0.1\}$
and
$b^- = \{b_1^- = [0.5,\ 0.8,\ 0.5],\ b_2^- = [0.6,\ 0.8,\ 0.5],\ b_3^- = [0.1,\ 0.7,\ 0.5],\ b_4^- = [0.3,\ 0.8,\ 0.7],$
$b_5^- = [0.5,\ 0.8,\ 0.7],\ b_6^- = [0.5,\ 0.8,\ 0.9]\}$.
Step 6. Use Equations (12) and (13) to compute the difference between each alternative and the RNPIS and the RNNIS, respectively. The values of D^+ and D^- are as given below:
$D^+ = \{D_1^+ = 0.262072,\ D_2^+ = 0.306496,\ D_3^+ = 0.340921,\ D_4^+ = 0.276215,\ D_5^+ = 0.292443,$
$D_6^+ = 0.345226,\ D_7^+ = 0.303001,\ D_8^+ = 0.346428,\ D_9^+ = 0.271012,\ D_{10}^+ = 0.339093\}$.
and
$D^- = \{D_1^- = 0.374468,\ D_2^- = 0.307641,\ D_3^- = 0.294889,\ D_4^- = 0.355857,\ D_5^- = 0.323740$
$D_6^- = 0.348903,\ D_7^- = 0.360103,\ D_8^- = 0.338725,\ D_9^- = 0.379516,\ D_{10}^- = 0.349703\}$.
Step 7. Using Equation (14), the closeness coefficient C_i for each alternative is:
$C_1 = -0.0133,\ C_2 = -0.3589,\ C_3 = -0.5239,\ C_4 = -0.1163,\ C_5 = -0.2629,$
$C_6 = -0.3980,\ C_7 = -0.2073,\ C_8 = -0.4294,\ C_9 = -0.0341,\ C_{10} = -0.3725.$
Step 8. The ranking of the alternatives obtained from the closeness coefficient is as given below:

$$x_1 > x_9 > x_4 > x_7 > x_5 > x_2 > x_{10} > x_6 > x_8 > x_3.$$

Therefore the optimal decision is to select supplier x_1.

Table 2. Tabular representation of SVNS A.

U	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
x_1	(0.7, 0.5, 0.1)	(0.7, 0.5, 0.3)	(0.8, 0.6, 0.2)
x_2	(0.6, 0.5, 0.2)	(0.7, 0.5, 0.1)	(0.6, 0.3, 0.5)
x_3	(0.6, 0.2, 0.3)	(0.6, 0.6, 0.4)	(0.7, 0.7, 0.2)
x_4	(0.5, 0.5, 0.4)	(0.6, 0.4, 0.4)	(0.7, 0.7, 0.3)
x_5	(0.7, 0.5, 0.5)	(0.8, 0.3, 0.1)	(0.7, 0.6, 0.2)
U	e_1	e_2	e_3
x_6	(0.5, 0.5, 0.5)	(0.7, 0.8, 0.1)	(0.7, 0.3, 0.5)
x_7	(0.6, 0.8, 0.1)	(0.7, 0.2, 0.1)	(0.6, 0.3, 0.4)
x_8	(0.7, 0.8, 0.3)	(0.6, 0.6, 0.5)	(0.8, 0, 0.5)
x_9	(0.6, 0.7, 0.1)	(0.7, 0, 0.1)	(0.6, 0.7, 0)
x_{10}	(0.5, 0.7, 0.4)	(0.9, 0, 0.3)	(1, 0, 0)

Table 2. *Cont.*

U	e_4	e_5	e_6
x_1	(0.9, 0.4, 0.2)	(0.6, 0.4, 0.7)	(0.6, 0.5, 0.4)
x_2	(0.6, 0.4, 0.3)	(0.7, 0.5, 0.4)	(0.7, 0.8, 0.9)
x_3	(0.5, 0.5, 0.3)	(0.6, 0.8, 0.6)	(0.7, 0.2, 0.5)
x_4	(0.9, 0.4, 0.2)	(0.7, 0.3, 0.5)	(0.6, 0.4, 0.4)
x_5	(0.7, 0.5, 0.2)	(0.7, 0.5, 0.6)	(0.6, 0.7, 0.8)
U	e_4	e_5	e_6
x_6	(0.4, 0.8, 0)	(0.7, 0.4, 0.2)	(0.5, 0.6, 0.3)
x_7	(0.3, 0.5, 0.1)	(0.6, 0.3, 0.6)	(0.5, 0.2, 0.6)
x_8	(0.7, 0.3, 0.6)	(0.6, 0.8, 0.5)	(0.6, 0.2, 0.4)
x_9	(0.7, 0.4, 0.3)	(0.6, 0.6, 0.7)	(0.7, 0.3, 0.2)
x_{10}	(0.5, 0.6, 0.7)	(0.5, 0.2, 0.7)	(0.8, 0.4, 0.1)

4.2. Adaptation of the Algorithm to Non-Integrated Weight Measure

In this section, we present an adaptation of our algorithm introduced in Section 4.1 to cases where only the objective weights or subjective weights of the attributes are taken into consideration. The results obtained via these two new variants are then compared to the results obtained via the original algorithm in Section 4.1. Further, we also compare the results obtained via these two new variants of the algorithm to the results obtained via the other methods in literature that are compared in Section 5.

To adapt our proposed algorithm in Section 3 for these special cases, we hereby represent the objective-only and subjective-only adaptations of the algorithm. This is done by taking only the objective (subjective) weight is to be used, then simply take $w_j = \theta_j$ ($w_j = h_j$). The two adaptations of the algorithm are once again applied to the dataset for SVNS A given in Table 2.

4.2.1. Objective-Only Adaptation of Our Algorithm

All the steps remain the same as the original algorithm; however, only the objective weights of the attributes are used, i.e., we take $w_j = \theta_j$.

The results of applying this variant of the algorithm produces the ranking given below:

$$x_9 > x_1 > x_4 > x_{10} > x_7 > x_6 > x_5 > x_8 > x_3 > x_2.$$

Therefore, if only the objective weight is to be considered, then the optimal decision is to select supplier x_9.

4.2.2. Subjective-Only Adaptation of Our Algorithm

All the steps remain the same as the original algorithm; however, only the subjective weights of the attributes are used, i.e., we take $w_j = h_j$.

The results of applying this variant of the algorithm produces the ranking given below:

$$x_1 > x_9 > x_4 > x_7 > x_5 > x_2 > x_6 > x_{10} > x_8 > x_3$$

Therefore, if only the objective weight is to be considered, then the optimal decision is to select supplier x_1.

From the results obtained above, it can be observed that the ranking of the alternatives are clearly affected by the decision of the decision maker to use only the objective weights, only the subjective weights of the attributes, or an integrated weight measure that takes into consideration both the objective and subjective weights of the attributes.

5. Comparatives Studies

In this section, we present a brief comparative analysis of some of the recent works in this area and our proposed method. These recent approaches are applied to our Example 1, and the limitations that exist in these methods are elaborated, and the advantages of our proposed method are discussed and analyzed. The results obtained are summarized in Table 3.

5.1. Comparison of Results Obtained Through Different Methods

Table 3. The results obtained using different methods for Example 1.

Method	The Final Ranking	The Best Alternative
Ye [39] (i) WAAO * (ii) WGAO **	$x_1 > x_4 > x_9 > x_5 > x_7 > x_2 > x_{10} > x_8 > x_3 > x_6$ $x_{10} > x_9 > x_8 > x_1 > x_5 > x_7 > x_4 > x_2 > x_6 > x_3$	x_1 x_{10}
Ye [10] (i) Weighted correlation coefficient (ii) Weighted cosine similarity measure	$x_1 > x_4 > x_5 > x_9 > x_2 > x_8 > x_7 > x_3 > x_6 > x_{10}$ $x_1 > x_9 > x_4 > x_5 > x_2 > x_{10} > x_8 > x_3 > x_7 > x_6$	x_1 x_1
Ye [11]	$x_1 > x_9 > x_4 > x_7 > x_5 > x_2 > x_8 > x_6 > x_3 > x_{10}$	x_1
Huang [14]	$x_1 > x_9 > x_4 > x_5 > x_2 > x_7 > x_8 > x_6 > x_3 > x_{10}$	x_1
Peng et al. [40] (i) GSNNWA *** (ii) GSNNWG ****	$x_9 > x_{10} > x_8 > x_6 > x_1 > x_7 > x_4 > x_5 > x_2 > x_3$ $x_1 > x_9 > x_4 > x_5 > x_7 > x_2 > x_8 > x_3 > x_6 > x_{10}$	x_9 x_1
Peng & Liu [15] (i) EDAS (ii) Similarity measure	$x_1 > x_4 > x_6 > x_9 > x_{10} > x_3 > x_2 > x_7 > x_5 > x_8$ $x_{10} > x_8 > x_7 > x_4 > x_1 > x_2 > x_5 > x_9 > x_3 > x_6$	x_1 x_{10}
Maji [41]	$x_5 > x_1 > x_9 > x_6 > x_2 > x_4 > x_3 > x_8 > x_7 > x_{10}$	x_5
Karaaslan [42]	$x_1 > x_9 > x_4 > x_5 > x_7 > x_2 > x_8 > x_3 > x_6 > x_{10}$	x_1
Ye [43]	$x_1 > x_9 > x_4 > x_5 > x_7 > x_2 > x_8 > x_3 > x_6 > x_{10}$	x_1
Biswas et al. [44]	$x_{10} > x_9 > x_7 > x_1 > x_4 > x_6 > x_5 > x_8 > x_2 > x_3$	x_{10}
Ye [45]	$x_9 > x_7 > x_1 > x_4 > x_2 > x_{10} > x_5 > x_8 > x_3 > x_6$	x_9
Adaptation of our algorithm (objective weights only)	$x_9 > x_1 > x_4 > x_{10} > x_7 > x_6 > x_5 > x_8 > x_3 > x_2$	x_9
Adaptation of our algorithm (subjective weights only)	$x_1 > x_9 > x_4 > x_7 > x_5 > x_2 > x_6 > x_{10} > x_8 > x_3$	x_1
Our proposed method (using integrated weight measure)	$x_1 > x_9 > x_4 > x_7 > x_5 > x_2 > x_{10} > x_6 > x_8 > x_3$	x_1

* WAAO = weighted arithmetic average operator; ** WGAO = weighted geometric average operator; *** GSNNWA = generalized simplified neutrosophic number weighted averaging operator; **** GSNNWG = generalized simplified neutrosophic number weighted geometric operator.

5.2. Discussion of Results

From the results obtained in Table 3, it can be observed that different rankings and optimal alternatives were obtained from the different methods that were compared. This difference is due to a number of reasons. These are summarized briefly below:

(i) The method proposed in this paper uses an integrated weight measure which considers both the subjective and objective weights of the attributes, as opposed to some of the methods that only consider the subjective weights or objective weights.

(ii) Different operators emphasizes different aspects of the information which ultimately leads to different rankings. For example, in [40], the GSNNWA operator used is based on an arithmetic average which emphasizes the characteristics of the group (i.e., the whole information), whereas the GSNNWG operator is based on a geometric operator which emphasizes the characteristics of each individual alternative and attribute. As our method places more importance on the characteristics of the individual alternatives and attributes, instead of the entire information as a whole, our method produces the same ranking as the GSNNWG operator but different results from the GSNNWA operator.

5.3. Analysis of the Performance and Reliability of Different Methods

The performance of these methods and the reliability of the results obtained via these methods are further investigated in this section.

Analysis

In all of the 11 papers that were compared in this section, the different authors used different types of measurements and parameters to determine the performance of their respective algorithms. However, all of these inputs *always* contain a tensor with at least three degrees. This tensor can refer to different types of neutrosophic sets depending on the context discussed in the respective papers, e.g., simplified neutrosophic sets, single-valued neutrosophic sets, neutrosophic sets, or INSs. For the sake of simplicity, we shall denote them simply as S.

Furthermore, all of these methods consider a weighted approach i.e., the weight of each attribute is taken into account in the decision-making process. The decision-making algorithms proposed in [10,11,14,39,40,43,45] use the subjective weighting method, the algorithms proposed in [42,44] use the objective weighting method, whereas only the decision-making methods proposed in [15] use an integrated weighting method which considers both the subjective and objective weights of the attributes. The method proposed by Maji [41] did not take the attribute weights into consideration in the decision-making process.

In this section, we first apply the inputs of those papers into our own algorithm. We then compare the results obtained via our proposed algorithm with their results, with the aim of justifying the effectiveness of our algorithm. The different methods and their algorithms are analyzed below:

(i) The algorithms in [10,11,39] all use the data given below as inputs

$$S = \left\{ \begin{array}{c} [0.4,\ 0.2,\ 0.3],\ [0.4,\ 0.2,\ 0.3],\ [0.2,\ 0.2,\ 05] \\ [0.6,\ 0.1,\ 0.2],\ [0.6,\ 0.1,\ 0.2],\ [0.5,\ 0.2,\ 0.2] \\ [0.3,\ 0.2,\ 0.3],\ [0.5,\ 0.2,\ 0.3],\ [0.5,\ 0.3,\ 0.2] \\ [0.7,\ 0.0,\ 0.1],\ [0.6,\ 0.1,\ 0.2],\ [0.4,\ 0.3,\ 0.2] \end{array} \right\}$$

The subjective weights w_j of the attributes are given by $w_1 = 0.35$, $w_2 = 0.25$, $w_3 = 0.40$. All the five algorithms from papers [10,11,39] yields either one of the following rankings:

$$A_4 > A_2 > A_3 > A_1 \quad \text{or} \quad A_2 > A_4 > A_3 > A_1$$

Our algorithm yields the ranking $A_4 > A_2 > A_3 > A_1$ which is consistent with the results obtained through the methods given above.

(ii) The method proposed in [44] also uses the data given in S above as inputs but ignores the opinions of the decision makers as it does not take into account the subjective weights of the attributes. The algorithm from this paper yields the ranking of $A_4 > A_2 > A_3 > A_1$. To fit this data into our algorithm, we randomly assigned the subjective weights of the attributes as $w_j = \frac{1}{3}$ for $j = 1, 2, 3$. A ranking of $A_4 > A_2 > A_3 > A_1$ was nonetheless obtained from our algorithm.

(iii) The methods introduced in [14,43,45] all use the data given below as input values:

$$S = \left\{ \begin{array}{c} [0.5,\ 0.1,\ 0.3],\ [0.5,\ 0.1,\ 0.4],\ [0.7,\ 0.1,\ 02],\ [0.3,\ 0.2,\ 0.1] \\ [0.4,\ 0.2,\ 0.3],\ [0.3,\ 0.2,\ 0.4],\ [0.9,\ 0.0,\ 0.1],\ [0.5,\ 0.3,\ 0.2] \\ [0.4,\ 0.3,\ 0.1],\ [0.5,\ 0.1,\ 0.3],\ [0.5,\ 0.0,\ 0.4],\ [0.6,\ 0.2,\ 0.2] \\ [0.6,\ 0.1,\ 0.2],\ [0.2,\ 0.2,\ 0.5],\ [0.4,\ 0.3,\ 0.2],\ [0.7,\ 0.2,\ 0.1] \end{array} \right\}$$

The subjective weights w_j of the attributes are given by $w_1 = 0.30$, $w_2 = 0.25$, $w_3 = 0.25$ and $w_4 = 0.20$.

In this case, all of the three algorithms produces a ranking of $A_1 > A_3 > A_2 > A_4$.

This result is however not very reliable as all of these methods only considered the subjective weights of the attributes and ignored the objective weight which is a vital measurement of the relative importance of an attribute e_j relative to the other attributes in an objective manner i.e., without "prejudice".

When we calculated the objective weights using our own algorithm we have the following objective weights:

$$a_j = [0.203909,\ 0.213627,\ 0.357796,\ 0.224667]$$

In fact, it is indeed <0.9, 0.0, 0.1> that mainly contributes to the largeness of the objective weight of attribute e_3 compared to the other values of e_j. Hence, when we calculate the integrated weight, the weight of attribute e_3 is still the largest.

Since $[0.9,\ 0.0,\ 0.1]$ is in the second row, our algorithm yields a ranking of $A_2 > A_1 > A_3 > A_4$ as a result.

We therefore conclude that our algorithm is more effective and the results obtained via our algorithm is more reliable than the ones obtained in [14,43,45], as we consider both the objective and subjective weights.

(iv) It can be observed that for the methods introduced in [10,11,39,44], we have $0.8 \leq T_{ij} + I_{ij} + F_{ij} \leq 1$ for all the entries. A similar trend can be observed in [14,43,45], where $0.6 \leq T_{ij} + I_{ij} + F_{ij} \leq 1$ for all the entries. Therefore, we are not certain about the results obtained through the decision making algorithms in these papers when the value of $T_{ij} + I_{ij} + F_{ij}$ deviates very far from 1.

Another aspect to be considered is the weighting method that is used in the decision making process. As mentioned above, most of the current decision making methods involving SVNSs use subjective weighting, a few use objective weighting and only two methods introduced in [15] uses an integrated weighting method to arrive at the final decision. In view of this, we proceeded to investigate if all of the algorithms that were compared in this section are able to produce reliable results when both the subjective and objective weights are taken into consideration. Specifically, we investigate if these algorithms are able to perform effectively in situations where the subjective weights clearly prioritize over the objective weights, and vice-versa. To achieve this, we tested all of the algorithms with three sets of inputs as given below:

Test 1: A scenario containing a very small value of $T_{ij} + I_{ij} + F_{ij}$.

$$S_1 = \left\{ \begin{array}{c} A_1 = ([0.5,\ 0.5,\ 0.5],\ [\mathbf{0.9999},\ 0.000\mathbf{1},\ 0.000]) \\ A_2 = ([0.5,\ 0.5,\ 0.5],\ [\mathbf{0.9999},\ 0.000\mathbf{1},\ 0.000\mathbf{1}]) \\ A_3 = ([0.5,\ 0.5,\ 0.5],\ [\mathbf{0.9999},\ 0.0000,\ 0.000\mathbf{1}]) \\ A_4 = ([0.5,\ 0.5,\ 0.5],\ [0.000\mathbf{1},\ 0.0000,\ 0.000]) \end{array} \right\}$$

The subjective weight in this case is assigned as: $a_j = [0.5,\ 0.5]$.

By observation alone, it is possible to tell that an effective algorithm should produce A_4 as the least favoured alternative, and A_2 should be second least-favoured alternative.

Test 2: A scenario where subjective weights *prioritize* over objective weight.

$$S_2 = \left\{ \begin{array}{c} A_1 = ([0.80,\ 0.10,\ 0.10],\ [0.19,\ 0.50,\ 0.50]) \\ A_2 = ([0.20,\ 0.50,\ 0.50],\ [0.81,\ 0.10,\ 0.10]) \end{array} \right\}$$

The subjective weight in this case is assigned as: $a_j = [0.99,\ 0.01]$.

By observation alone, we can tell that an effective algorithm should produce a ranking of $A_1 > A_2$.

Test 3: This test is based on a real-life situation.

Suppose a procurement committee is looking to select the best supplier to supply two raw materials e_1 and e_2. In this context, the triplet $[T,\ I,\ F]$ represents the following:

T : the track record of the suppliers that is approved by the committee
I : the track record of the suppliers that the committee feels is questionable
F : the track record of the suppliers that is rejected by the committee

Based on their experience, the committee is of the opinion that raw material e_1 is slightly more important than raw material e_2, and assigned subjective weights of $w_1^{sub} = 0.5001$ and $w_2^{sub} = 0.4999$.

After an intensive search around the country, the committee shortlisted 20 candidates (A_1 to A_{20}). After checking all of the candidates' track records and analyzing their past performances, the committee assigned the following values for each of the suppliers.

$$S_3 = \left\{ \begin{array}{c} A_1 = ([0.90,\ 0.00,\ 0.10],\ [0.80,\ 0.00,\ 0.10]),\ A_2 = ([0.80,\ 0.00,\ 0.10],\ [0.90,\ 0.00,\ 0.10]) \\ A_3 = ([0.50,\ 0.50,\ 0.50],\ [0.00,\ 0.90,\ 0.90]), A_4 = ([0.50,\ 0.50,\ 0.50],\ [0.10,\ 0.90,\ 0.80]) \\ A_5 = ([0.50,\ 0.50,\ 0.50],\ [0.20,\ 0.90,\ 0.70]), A_6 = ([0.50,\ 0.50,\ 0.50],\ [0.30,\ 0.90,\ 0.60]) \\ A_7 = ([0.50,\ 0.50,\ 0.50],\ [0.40,\ 0.90,\ 0.50]), A_8 = ([0.50,\ 0.50,\ 0.50],\ [0.50,\ 0.90,\ 0.40]) \\ A_9 = ([0.50,\ 0.50,\ 0.50],\ [0.60,\ 0.90,\ 0.30]), A_{10} = ([0.50,\ 0.50,\ 0.50],\ [0.70,\ 0.30,\ 0.90]) \\ A_{11} = ([0.50,\ 0.50,\ 0.50],\ [0.70,\ 0.90,\ 0.30]),\ A_{12} = ([0.50,\ 0.50,\ 0.50],\ [0.00,\ 0.30,\ 0.30]) \\ A_{13} = ([0.50,\ 0.50,\ 0.50],\ [0.70,\ 0.90,\ 0.90]),\ A_{14} = ([0.50,\ 0.50,\ 0.50],\ [0.70,\ 0.30,\ 0.30]) \\ A_{15} = ([0.50,\ 0.50,\ 0.50],\ [0.60,\ 0.40,\ 0.30]),\ A_{16} = ([0.50,\ 0.50,\ 0.50],\ [0.50,\ 0.50,\ 0.30]) \\ A_{17} = ([0.50,\ 0.50,\ 0.50],\ [0.40,\ 0.60,\ 0.30]),\ A_{18} = ([0.50,\ 0.50,\ 0.50],\ [0.30,\ 0.70,\ 0.30]) \\ A_{19} = ([0.50,\ 0.50,\ 0.50],\ [0.20,\ 0.80,\ 0.30]),\ A_{20} = ([0.50,\ 0.50,\ 0.50],\ [0.10,\ 0.90,\ 0.30]) \end{array} \right\}$$

The objective weights for this scenario was calculated based on our algorithm and the values are $w_1^{obj} = 0.1793$ and $w_2^{obj} = 0.8207$.

Now it can be observed that suppliers A_1 and A_2 are the ones that received the best evaluation scores from the committee. Supplier A_1 received better evaluation scores from the committee compared to supplier A_2 for attribute e_1. Attribute e_1 was deemed to be more important than attribute e_2 by the committee, and hence had a higher subjective weight. However, the objective weight of attribute e_2 is much higher than e_1. This resulted in supplier A_2 ultimately being chosen as the best supplier. This is an example of a scenario where the objective weights are prioritized over the subjective weights, and has a greater influence on the decision-making process.

Therefore, in the scenario described above, an effective algorithm should select A_2 as the optimal supplier, followed by A_1. All of the remaining choices have values of $T < 0.8$, $I > 0.0$ and $F > 0.1$. As such, an effective algorithm should rank all of these remaining 18 choices behind A_1.

We applied the three tests mentioned above and the data set for S_3 given above to the decision-making methods introduced in the 11 papers that were compared in the previous section. The results obtained are given in Table 4.

Thus it can be concluded that our proposed algorithm is the most effective algorithm and the one that yields the most reliable results in all the different types of scenario. Hence, our proposed algorithm provides a robust framework that can be used to handle any type of situation and data, and produce accurate and reliable results for any type of situation and data.

Finally, we look at the context of the scenario described in Example 1. The structure of our data (given in Table 2) is more generalized, by theory, having $0 \le T_{ij} + I_{ij} + F_{ij} \le 1$ and $0 \le T_{ij} + I_{ij} + F_{ij} \le 3$, and is similar to the structure of the data used in [15,40–42]. Hence, our choice of input data serves as a more faithful indicator of how each algorithm works under all sorts of possible conditions.

Table 4. Compliance to Tests 1, 2, and 3.

Paper		Test 1 Compliance	Test 2 Compliance	Test 3 Compliance
Ye [39]	WAAO *	Y	Y	N
	WGAO *	N	Y	N
Ye [10]	Weighted correlation coefficient	Y	Y	N
	Weighted cosine similarity measure	N	Y	N
Ye [11]		Y	Y	N
Huang [14]		Y	Y	N
Peng et al. [40]	GSNNWA **	Y	Y	N
	GSNNWG **	Y	Y	N
Peng & Liu [15]	EDAS	Y	Y	N
	Similarity measure	N	Y	Y
Maji [41]		N	N	N
Karaaslan [42]		Y	Y	N
Ye [43]		Y	Y	N
Biswas et al. [44]		Y	N	Y
Ye [45]		Y	Y	N
Adaptation of our proposed algorithm (objective weights only)		Y	N	Y
Adaptation of our proposed algorithm (subjective weights only)		Y	Y	N
Our proposed algorithm		Y	Y	Y

Remarks: Y = Yes (which indicates compliance to Test); N = No (which indicates non-compliance to Test); * WAAO = weighted arithmetic average operator; * WGAO = weighted geometric average operator; ** GSNNWA = generalized simplified neutrosophic number weighted averaging operator; ** GSNNWG = generalized simplified neutrosophic number weighted geometric operator.

6. Conclusions

The concluding remarks and the significant contributions that were made in this paper are expounded below.

(i) A novel TOPSIS method for the SVNS model is introduced, with the maximizing deviation method used to determine the objective weight of the attributes. Through thorough analysis, we have proven that our algorithm is compliant with all of the three tests that were discussed in Section 5.3. This clearly indicates that our proposed decision-making algorithm is not only an effective algorithm but one that produces the most reliable and accurate results in all the different types of situation and data inputs.

(ii) Unlike other methods in the existing literature which reduces the elements from single-valued neutrosophic numbers (SVNNs) to fuzzy numbers, or interval neutrosophic numbers (INNs) to neutrosophic numbers or fuzzy numbers, in our version of the TOPSIS method the input data is in the form of SVNNs and this form is maintained throughout the decision-making process. This prevents information loss and enables the original information to be retained, thereby ensuring a higher level of accuracy for the results that are obtained.

(iii) The objective weighting method (e.g., the ones used in [10,11,14,39,40,43,45]) only takes into consideration the values of the membership functions while ignoring the preferences of the decision makers. Through the subjective weighting method (e.g., the ones used in [42,44]), the attribute weights are given by the decision makers based on their individual preferences and experiences. Very few approaches in the existing literature (e.g., [15]) consider both the objective and subjective weighting methods. Our proposed method uses an integrated weighting model that considers both the objective and subjective weights of the attributes, and this accurately reflects the input values of the alternatives as well as the preferences and risk attitude of the decision makers.

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A Novel Skin Lesion Detection Approach Using Neutrosophic Clustering and Adaptive Region Growing in Dermoscopy Images

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Abstract: This paper proposes novel skin lesion detection based on neutrosophic clustering and adaptive region growing algorithms applied to dermoscopic images, called NCARG. First, the dermoscopic images are mapped into a neutrosophic set domain using the shearlet transform results for the images. The images are described via three memberships: true, indeterminate, and false memberships. An indeterminate filter is then defined in the neutrosophic set for reducing the indeterminacy of the images. A neutrosophic c-means clustering algorithm is applied to segment the dermoscopic images. With the clustering results, skin lesions are identified precisely using an adaptive region growing method. To evaluate the performance of this algorithm, a public data set (ISIC 2017) is employed to train and test the proposed method. Fifty images are randomly selected for training and 500 images for testing. Several metrics are measured for quantitatively evaluating the performance of NCARG. The results establish that the proposed approach has the ability to detect a lesion with high accuracy, 95.3% average value, compared to the obtained average accuracy, 80.6%, found when employing the neutrosophic similarity score and level set (NSSLS) segmentation approach.

Keywords: neutrosophic clustering; image segmentation; neutrosophic c-means clustering; region growing; dermoscopy; skin cancer

1. Introduction

Dermoscopy is an in-vivo and noninvasive technique to assist clinicians in examining pigmented skin lesions and investigating amelanotic lesions. It visualizes structures of the subsurface skin in the superficial dermis, the dermoepidermal junction, and the epidermis [1]. Dermoscopic images are complex and inhomogeneous, but they have a significant role in early identification of skin cancer. Recognizing skin subsurface structures is performed by visually searching for individual features and salient details [2]. However, visual assessment of dermoscopic images is subjective, time-consuming, and prone to errors [3]. Consequently, researchers are interested in developing automated clinical assessment systems for lesion detection to assist dermatologists [4,5]. These systems require efficient image segmentation and detection techniques for further feature extraction and skin cancer lesion classification. However, skin cancer segmentation and detection processes are complex due to dissimilar lesion color, texture, size, shape, and type; as well as the irregular boundaries of various lesions and the low contrast between skin and the lesion. Moreover, the existence of dark hair that covers skin and lesions leads to specular reflections.

Traditional skin cancer detection techniques implicate image feature analysis to outline the cancerous areas of the normal skin. Thresholding techniques use low-level features, including intensity and color to separate the normal skin and cancerous regions. Garnavi et al. [6] applied Otsu's method to identify the core-lesion; nevertheless, such process is disposed to skin tone variations and lighting. Moreover, dermoscopic images include some artifacts due to water bubble, dense hairs, and gel that are a great challenge for accurate detection. Silveira et al. [7] evaluated six skin lesions segmentation techniques in dermoscopic images, including the gradient vector flow (GVF), level set, adaptive snake, adaptive thresholding, fuzzy-based split and merge (FSM), and the expectation–maximization level set (EMLV) methods. The results established that adaptive snake and EMLV were considered the superior semi-supervised techniques, and that FSM achieved the best fully computerized results.

In dermoscopic skin lesion images, Celebi et al. [8] applied an unsupervised method using a modified JSEG algorithm for border detection, where the original JSEG algorithm is an adjusted version of the generalized Lloyd algorithm (GLA) for color quantization. The main idea of this method is to perform the segmentation process using two independent stages, namely color quantization and spatial segmentation. However, one of the main limitations occurs when the bounding box does not entirely include the lesion. This method was evaluated on 100 dermoscopic images, and border detection error was calculated. Dermoscopic images for the initial consultation were analyzed by Argenziano et al. [9] and were compared with images from the last follow-up consultation and the symmetrical/asymmetrical structural changes. Xie and Bovik [10] implemented a dermoscopic image segmentation approach by integrating the genetic algorithm (GA) and self-generating neural network (SGNN). The GA was used to select the optimal samples as initial neuron trees, and then the SGNF was used to train the remaining samples. Accordingly, the number of clusters was determined by adjusting the SD of cluster validity. Thus, the clustering is accomplished by handling each neuron tree as a cluster. A comparative study between this method and other segmentation approaches—namely *k*-means, statistical region merging, Otsu's thresholding, and the fuzzy *c*-means methods—has been conducted revealing that the optimized method provided improved segmentation and more accurate results.

Barata et al. [11] proposed a machine learning based, computer-aided diagnosis system for melanoma using features having medical importance. This system used text labels to detect several significant dermoscopic criteria, where, an image annotation scheme was applied to associate the image regions with the criteria (texture, color, and color structures). Features fusion was then used to combine the lesions' diagnosis and the medical information. The proposed approach achieved 84.6% sensitivity and 74.2% specificity on 804 images of a multi-source data set.

Set theory, such as the fuzzy set method, has been successfully employed into image segmentation. Fuzzy sets have been introduced into image segmentation applications to handle uncertainty. Several researchers have been developing efficient clustering techniques for skin cancer segmentation and other applications based on fuzzy sets. Fuzzy c-means (FCM) uses the membership function to segment the images into one or several regions. Lee and Chen [12] proposed a segmentation technique on different skin cancer types using classical FCM clustering. An optimum threshold-based segmentation technique using type-2 fuzzy sets was applied to outline the skin cancerous areas. The results established the superiority of this method compared to Otsu's algorithm, due its robustness to skin tone variations and shadow effects. Jaisakthi et al. [13] proposed an automated skin lesion segmentation technique in dermoscopic images using a semi-supervised learning algorithm. A *k*-means clustering procedure was employed to cluster the pre-processed skin images, where the skin lesions were identified from these clusters according to the color feature. However, the fuzzy set technique cannot assess the indeterminacy of each element in the set. Zhou et al. [14] introduced the fuzzy c-means (FCM) procedure based on mean shift for detecting regions within the dermoscopic images.

Recently, neutrosophy has provided a prevailing technique, namely the neutrosophic set (NS), to handle indeterminacy during the image processing. Guo and Sengur [15] integrated the NS and FCM frameworks to resolve the inability of FCM for handling uncertain data. A clustering approach called neutrosophic c-means (NCM) clustering was proposed to cluster typical data points. The results proved the efficiency of the NCM for image segmentation and data clustering. Mohan et al. [16] proposed automated brain tumor segmentation based on a neutrosophic and k-means clustering technique. A non-local neutrosophic Wiener filter was used to improve the quality of magnetic resonance images (MRI) before applying the k-means clustering approach. The results found detection rates of 100% with 98.37% accuracy and 99.52% specificity. Sengur and Guo [17] carried out an automated technique using a multiresolution wavelet transform and NS. The color/texture features have been mapped on the NS and wavelet domain. Afterwards, the c-k-means clustering approach was employed for segmentation. Nevertheless, wavelets [18] are sensitive to poor directionality during the analysis of supplementary functions in multi-dimensional applications. Hence, wavelets are relatively ineffectual to represent edges and anisotropic features in the dermoscopic images. Subsequently, enhanced multi-scale procedures have been established, including the curvelets and shearlets to resolve the limitations of wavelet analysis. These methods have the ability to encode directional information for multi-scale analysis. Shearlets provides a sparse representation of the two-dimensional information with edge discontinuities [19]. Shearlet-based techniques were established to be superior to wavelet-based methods [20].

Dermoscopic images include several artifacts such as hair, air bubbles, and other noise factors that are considered indeterminate information. The above-mentioned skin lesion segmentation methods either need a preprocessing to deal with the indeterminate information, or their detection results must be affected by them. To overcome this disadvantage, we introduce the neutrosophic set to deal with indeterminate information in dermoscopic images; we use a shearlet transform and the neutrosophic c-means (NCM) method along with an indeterminacy filter (IF) to eliminate the indeterminacy for accurate skin cancer segmentation. An adaptive region growing method is also employed to identify the lesions accurately.

The rest of the paper is organized as follows. In the second section, the proposed method is presented. Then the experimental results are discussed in the third section. The conclusions are drawn in the final section.

2. Methodology

The current work proposes a skin lesion detection algorithm using neutrosophic clustering and adaptive region growing in dermoscopic images. In this study, the red channel is used to detect the lesion, where healthy skin regions tend to be reddish, while darker pixels often occur in skin lesion regions [21]. First, the shearlet transform is employed on the red channel of dermoscopic image to extract the shearlet features. Then, the red channel of the image is mapped into the neutrosophic set domain, where the map functions are defined using the shearlet features. In the neutrosophic set, an indeterminacy filtering operation is performed to remove indeterminate information, such as noise and hair without using any de-noising or hair removal approaches. Then, the segmentation is performed through the neutrosophic c-means (NCM) clustering algorithm. Finally, the lesions are identified precisely using adaptive region growing on the segmentation results.

2.1. Shearlet Transform

Shearlets are based on a rigorous and simple mathematical framework for the geometric representation of multidimensional data and for multiresolution analysis [22]. The shearlet transform (ST) resolves the limitations of wavelet analysis; where wavelets fail to represent the geometric regularities and yield surface singularities due to their isotropic support. Shearlets include nearly parallel elongated functions to achieve surface anisotropy along the edges. The ST is an innovative two-dimensional wavelet transformation extension using directional and multiscale filter banks to capture smooth contours corresponding to the prevailing features in an image. Typically, the ST is a function with three parameters a, s, and t denoting the scale, shear, and translation parameters, respectively. The shearlet can fix both the locations of singularities and the singularities' curve tracking automatically. For $a > 0, s \in R, t \in R^2$, the ST can be defined using the following expression [23]:

$$ST_\varsigma p(a,s,t) = <\langle p, \varsigma_{a,s,t}\rangle, \tag{1}$$

where $\varsigma_{a,s,t}(f) = |\det N_{a,s}|^{-1/2}\varsigma(N_{a,s}^{-1}(f-t))$ and $N_{a,s} = \begin{bmatrix} a & s \\ 0 & \sqrt{a} \end{bmatrix}$. Each matrix $N_{a,s}$ can be defined as:

$$N_{a,s} = V_s D_a, \tag{2}$$

where the shear matrix is expressed by:

$$V_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \tag{3}$$

and the anisotropic dilation matrix is given by:

$$D_a = \begin{bmatrix} a & s \\ 0 & \sqrt{a} \end{bmatrix}. \tag{4}$$

During the selection of a proper decomposition function for any $\tau = (\tau_1, \tau_2) \in R^2$, and $\tau_2 \neq 0$, ς can be expressed by:

$$\widehat{\varsigma}(\tau) = \widehat{\varsigma}(\tau_1,\tau_2) = \widehat{\varsigma}_1(\tau_1)\widehat{\varsigma}_2\left(\frac{\tau_1}{\tau_2}\right), \tag{5}$$

where $\widehat{\varsigma}_1 \in L^2(R)$ and $\|\varsigma_2\|_{L_2} = 1$.

From the preceding equations, the discrete shearlet transform (DST) is formed by translation, shearing, and scaling to provide the precise orientations and locations of edges in an image. The DST is acquired by sampling the continuous ST. It offers a decent anisotropic feature extraction. Thus, the DST system is properly definite by sampling the continuous ST on a discrete subset of the shearlet group as follows, where $j,k,m \in Z \times Z \times Z^2$ [24]:

$$ST(\varsigma) = \left\{\varsigma_{j,k,m} = a^{-\frac{3}{4}}\varsigma\left(D_a{}^{-1}V_s{}^{-1}(.-t)\right) : (j,k,m) \in \wedge\right\}. \tag{6}$$

The DST can be divided into two steps: multi-scale subdivision and direction localization [25], where the Laplacian pyramid algorithm is first applied to an image in order to obtain the low-and-high-frequency components at any scale j, and then direction localization is achieved with a shear filter on a pseudo polar grid.

2.2. Neutrosophic Images

Neutrosophy has been successfully used for many applications to describe uncertain or indeterminate information. Every event in the neutrosophy set (NS) has a certain degree of truth (T), indeterminacy (I), and falsity (F), which are independent from each other. Previously reported studies have demonstrated the role of NS in image processing [26,27].

A pixel $P(i,j)$ in an image is denoted as $P_{NS}(i,j) = \{T(i,j), I(i,j), F(i,j)\}$ in the NS domain, where $T(i,j)$, $I(i,j)$, and $F(i,j)$ are the membership values belonging to the brightest pixel set, indeterminate set, and non-white set, respectively.

In the proposed method, the red channel of the dermoscopic image is transformed into the NS domain using shearlet feature values as follows:

$$T(x,y) = \frac{ST_L(x,y) - ST_{L\min}}{ST_{L\max} - ST_{L\min}}$$
$$I(x,y) = \frac{ST_H(x,y) - ST_{H\min}}{ST_{H\max} - ST_{H\min}} \tag{7}$$

where T and I are the true and indeterminate membership values in the NS. $ST_L(x,y)$ is the low-frequency component of the shearlet feature at the current pixel $P(x, y)$. In addition, $ST_{L\max}$ and $ST_{L\min}$ are the maximum and minimum of the low-frequency component of the shearlet feature in the whole image, respectively. $ST_H(x,y)$ is the high-frequency component of the shearlet feature at the current pixel $P(x, y)$. Moreover, $ST_{H\max}$ and $ST_{H\min}$ are the maximum and minimum of the high-frequency component of the shearlet feature in the whole image, respectively. In the proposed method, we only use T and I for segmentation because we are only interested in the degree to which a pixel belongs to the high intensity set of the red channel.

2.3. Neutrosophic Indeterminacy Filtering

In an image, noise can be considered as indeterminate information, which can be handled efficiently using NS. Such noise and artifacts include the existence of hair, air bubbles, and blurred boundaries. In addition, NS can be integrated with different clustering approaches for image segmentation [16,28], where the boundary information, as well as the details, may be blurred due to the principal low-pass filter leading to inaccurate segmentation of the boundary pixels. A novel NS based clustering procedure, namely the NCM has been carried out for data clustering [15], which defined the neutrosophic membership subsets using attributes of the data. Nevertheless, when it is applied to the image processing area, it does not account for local spatial information. Several side effects can affect the image when using classical filters in the NS domain, leading to blurred edge information, incorrect boundary segmentation, and an inability to combine the local spatial information with the global intensity distribution.

After the red channel of the dermoscopic image is mapped into the NS domain, an indeterminacy filter (IF) is defined based on the neutrosophic indeterminacy value, and the spatial information is utilized to eliminate the indeterminacy. The IF is defined by using the indeterminate value $I_s(x,y)$, which has the following kernel function [28]:

$$O_I(u,v) = \frac{1}{2\pi\sigma_I^2} e^{-\frac{u^2+v^2}{2\sigma_I^2}} \tag{8}$$

$$\sigma_I(x,y) = f(I(x,y)) = rI(x,y) + q, \tag{9}$$

where σ_I represents the Gaussian distribution's standard deviation, which is defined as a linear function $f(.)$ associated with the indeterminacy degree. Since σ_I becomes large with a high indeterminacy degree, the IF can create a smooth current pixel by using its neighbors. On the other hand, with a low indeterminacy degree, the value of σ_I is small and the IF performs less smoothing on the current pixel with its neighbors.

2.4. Neutrosophic C-Means (NCM)

In the NCM algorithm, an objective function and membership are considered as follows [29]:

$$J(T,I,F,A) = \sum_{i=1}^{N}\sum_{j=1}^{A}(\varpi_1 T_{ij})^m ||x_i - a_j||^2 + \sum_{i=1}^{N}(\varpi_2 I_i)^m ||x_i - \bar{a}_{i\max}||^2 + \sum_{i=1}^{N}\delta^2(\varpi_3 F_i)^m \tag{10}$$

$$\begin{aligned} \overline{a}_{i\max} &= \tfrac{a_{p_i}+a_{q_i}}{2} \\ p_i &= \underset{j=1,2,\cdots,A}{\operatorname{argmax}}\,(T_{ij}) \\ q_i &= \underset{j\neq p_i \cap j=1,2,\cdots,A}{\operatorname{argmax}}\,(T_{ij}) \end{aligned} \tag{11}$$

where m is a constant and usually equal to 2. The value of $\overline{a}_{i\max}$ is calculated, since p_i and q_i are identified as the cluster numbers with the largest and second largest values of T, respectively. The parameter δ is used for controlling the number of objects considered as outliers, and ϖ_i is a weight factor.

In our NS domain, we only defined the membership values of T and I. Therefore, the objective function reduces to:

$$J(T,I,F,A) = \sum_{i=1}^{N}\sum_{j=1}^{A}(\varpi_1 T_{ij})^m ||x_i - a_j||^2 + \sum_{i=1}^{N}(\varpi_2 I_i)^m ||x_i - \overline{a}_{i\max}||^2. \tag{12}$$

To minimize the objective function, three membership values are updated on each iteration as:

$$\begin{aligned} T_{ij} &= \tfrac{K}{\varpi_1}\left(x_i - a_j\right)^{-\frac{2}{m-1}} \\ I_i &= \tfrac{K}{\varpi_2}\left(x_i - \overline{a}_{i\max}\right)^{-\frac{2}{m-1}} \\ K &= \left[\tfrac{1}{\varpi_1}\sum_{j=1}^{A}\left(x_i - a_j\right)^{-\frac{2}{m-1}} + \tfrac{1}{\varpi_2}\left(x_i - \overline{a}_{i\max}\right)^{-\frac{2}{m-1}}\right]^{-1} \end{aligned} \tag{13}$$

where $\overline{a}_{i\max}$ is calculated based on the indexes of the largest and the second largest value of T_{ij}. The iteration does not stop until $\left|T_{ij}^{(k+1)} - T_{ij}^{(k)}\right| < \varepsilon$, where ε is a termination criterion between 0 and 1, and k is the iteration step. In the proposed method, the neutrosophic image after indeterminacy filtering is used as the input for NCM algorithm, and the segmentation procedure is performed using the final clustering results. Since the pixels whose indeterminacy membership values are higher than their true membership values, it is hard to determine which group they belong to. To solve this problem, the indeterminacy filter is employed again on all pixels, and the group is determined according to their biggest true membership values for each cluster after the IF operation.

2.5. Lesion Detection

After segmentation, the pixels in an image are grouped into several groups according to their true membership values. Due to the fact that the lesions have low intensities, especially for the core part inside a lesion, the cluster with lowest true membership value is initially considered as the lesion candidate pixels. Then an adaptive region growing algorithm is employed to precisely detect the lesion boundary parts having higher intensity and lower contrast than the core ones. A contrast ratio is defined adaptively to control the growing speed:

$$DR(t) = \frac{\text{mean}(R_a - R_b)}{\text{mean}(R_b)}, \tag{14}$$

where *DR(t)* is the contrast ratio at the t-th iteration of growing, and R_b and R_a are the regions before and after the t-th iteration of growing, respectively.

A connected component analysis is taken to extract the components' morphological features. Due to the fact that there is only one lesion in a dermoscopic image, the region with the biggest area is identified as the final lesion region. The block diagram of the proposed neutrosophic clustering and adaptive region growing (NCARG) method is illustrated in Figure 1.

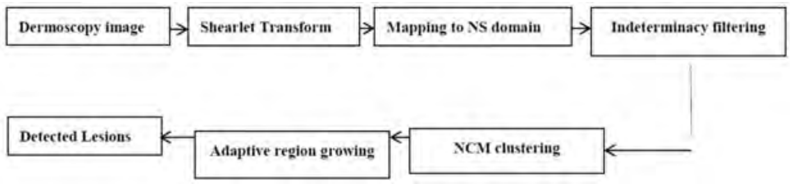

Figure 1. Flowchart of the proposed neutrosophic clustering and adaptive region growing (NCARG) skin lesion detection algorithm.

Figure 1 illustrates the steps of the proposed skin lesion segmentation method (NCARG) using neutrosophic *c*-means and region growing algorithms. Initially, the red channel of the dermoscopic image is transformed using a shearlet transform, and the shearlet features of the image are used to map the image into the NS domain. In the NS domain, an indeterminacy filtering operation is taken to remove the indeterminate information. Afterward, the segmentation is performed through NCM clustering on the filtered image. Finally, the lesion is accurately identified using an adaptive region growing algorithm where the growing speed is controlled by a newly defined contrast ratio.

To illustrate the steps in the proposed method, we use an example to demonstrate the intermediate results in Figure 2. Figure 2a,b are the original image and its ground truth image of segmentation. Figure 2c is its red channel. Figure 2d,e are the results after indeterminacy filtering and the NCM. In Figure 2f, the final detection result is outlined in blue and ground truth in red where the detection result is very close to its ground truth result.

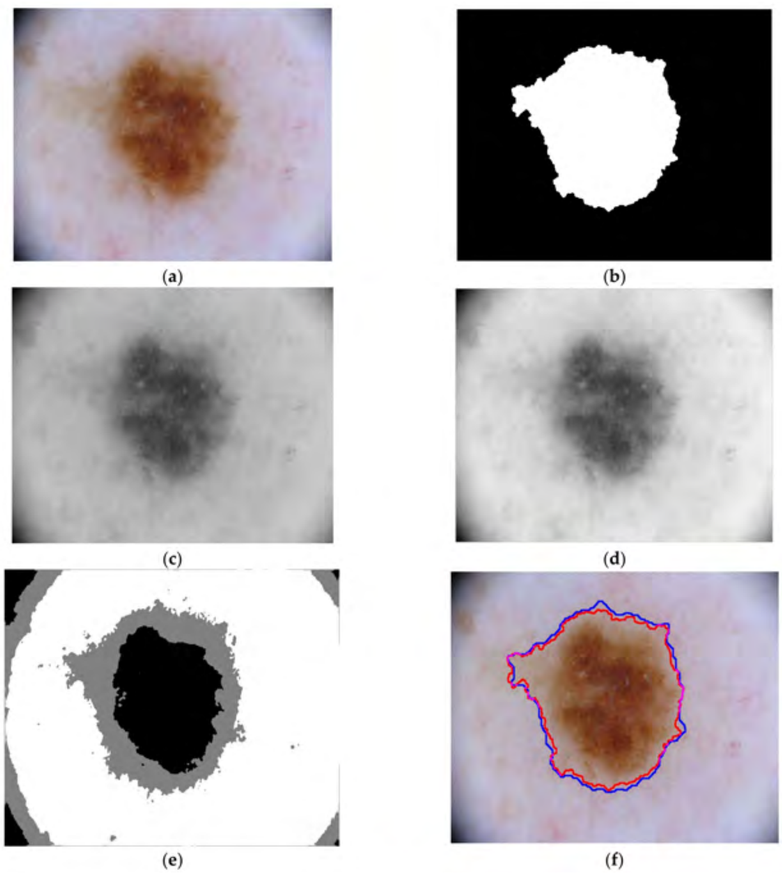

Figure 2. Intermediate results of an example image: ISIC_0000015: (**a**) Original skin lesion image; (**b**) Ground truth image; (**c**) Red channel of the original image; (**d**) Result after indeterminate filtering; (**e**) Result after NCM; (**f**) Detected lesion region after adaptive region growing, where the blue line is for the boundary of the detection result and the red line is the boundary of the ground truth result.

2.6. Evaluation Metrics

Several performance metrics are measured to evaluate the proposed skin cancer segmentation approach, namely the Jaccard index (JAC), Dice coefficient, sensitivity, specificity, and accuracy [30]. Each of these metric is defined in the remainder of this section. JAC is a statistical metric to compare diversity between the sample sets based upon the union and intersection operators as follows:

$$JAC(Y,Q) = \frac{Ar_Y \cap Ar_Q}{Ar_Y \cup Ar_Q}, \tag{15}$$

where $\cap$ and $\cup$ are the intersection and union of two sets, respectively. In addition, Ar_Y and Ar_Q are the automated segmented skin lesion area and the reference golden standard skin lesion area enclosed by the boundaries Y and Q; respectively. Typically, a value of 1 specifies complete similarity, while a JAC value of 0 specifies no similarity.

The Dice index compares the similarity of two sets, which is given as following for two sets X and Y:

$$DSC = \frac{2|X \cap Y|}{|X|+|Y|} \tag{16}$$

Furthermore, the sensitivity, specificity, and accuracy are related to the detection of the lesion region. The sensitivity indicates the true positive rate, showing how well the algorithm successfully predicts the skin lesion region, which is expressed as follows:

$$\text{Sensitivity} = \frac{\text{Number of true positives}}{\text{Number of true positives} + \text{Number of false negatives}}. \tag{17}$$

The specificity indicates the true negative rate, showing how well the algorithm predicts the non-lesion regions, which is expressed as follows:

$$\text{Specificity} = \frac{\text{Number of true negative}}{\text{Number of conditionnegative}}. \tag{18}$$

The accuracy is the proportion of true results (either positive or negative), which measures the reliability degree of a diagnostic test:

$$\text{Accuracy} = \frac{\text{Number of true positive} + \text{Number of true negative}}{\text{Number of total population}}. \tag{19}$$

These metrics are measured to evaluate the proposed NCARG method compared to another efficient segmentation algorithm that is based on the neutrosophic similarity score (NSS) and level set (LS), called NSSLS [31]. In the NSSLS segmentation method, the three membership subsets are used to transfer the input image to the NS domain, and then the NSS is applied to measure the fitting degree to the true tumor region. Finally, the LS method is employed to segment the tumor in the NSS image. In the current work, when the NSSLS is applied to the skin images, the images are interpreted using NSS, and the skin lesion boundary is extracted using the level set algorithm. Moreover, the statistical significance between the evaluated metrics using both segmentation methods is measured by calculating the significant difference value (p-value) to estimate the difference between the two methods. The p-value refers to the probability of error, where the two methods are considered statistically significant when $p \leq 0.05$.

3. Experimental Results and Discussion

3.1. Dataset

The International Skin Imaging Collaboration (ISIC) Archive [32] contains over 13,000 dermoscopic images of skin lesions. Using the images in the ISIC Archive, the 2017 ISBI Challenge on Skin Lesion Analysis

Towards Melanoma Detection was proposed to help participants develop image analysis tools to enable the automated diagnosis of melanoma from dermoscopic images. Image analysis of skin lesions includes lesion segmentation, detection and localization of visual dermoscopic features/patterns, and disease classification. All cases contain training, and binary mask images as ground truth files.

In our experiment, 50 images were selected to tune the parameters in the proposed NCARG algorithm and 500 images were used as the testing dataset. In the experiment, the parameters are set to $r = 1$, $q = 0.05$, $w1 = 0.75$, $w2 = 0.25$, and $\varepsilon = 0.001$.

3.2. Detection Results

Skin lesions are visible by the naked eye; however, early-stage detection of melanomas is complex and difficult to distinguish from benign skin lesions with similar appearances. Detecting and recognizing melanoma at its earliest stages reduces melanoma mortality. Skin lesion digital dermoscopic images are employed in the present study to detect skin lesions for accurate automated diagnosis and clinical decision support. The ISIC images are used to test and to validate the proposed approach of skin imaging. Figure 3 demonstrates the detection results using the proposed NCARG approach compared to the ground truth images. In the Figure 3d, the boundary detection results are marked in blue and the ground truth results are in red. The detection results match the ground truth results, and their boundaries are very close. Figure 3 establishes that the proposed approach accurately detects skin lesion regions, even with lesions of different shapes and sizes.

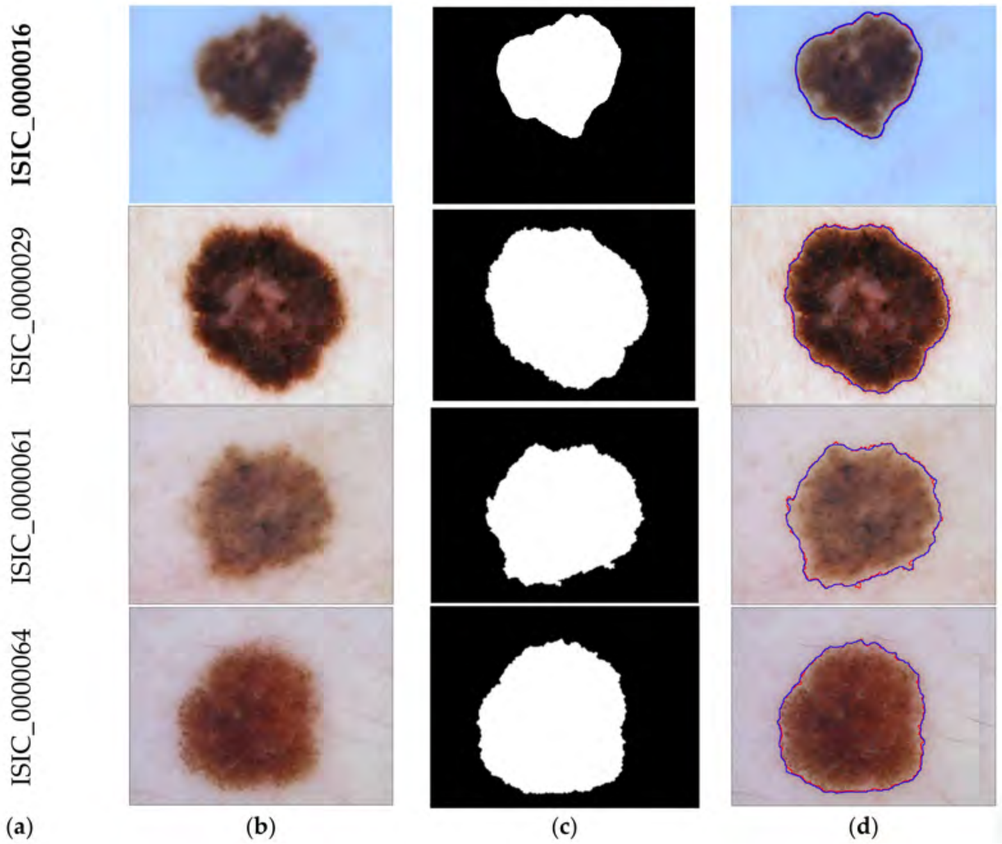

Figure 3. Detection results: (**a**) Skin cancer image number; (**b**) Original skin lesion image; (**c**) Ground truth image; and (**d**) Detected lesion region using the proposed approach.

3.3. Evaluation

Table 1 reports the average values as well as the standard deviations (SD) of the evaluation metrics on the proposed approach's performance over 500 images.

Table 1. The performance of computer segmentation using the proposed NCARG method with reference to ground truth boundaries (Average ± SD).

Metric Value	Accuracy (%)	Dice (%)	JAC (%)	Sensitivity (%)	Specificity (%)
Average	95.3	90.38	83.2	97.5	88.8
Standard deviation	6	7.6	10.5	3.5	11.4

Table 1 establishes that the proposed approach achieved a detection accuracy for the skin lesion regions of 95.3% with a 6% standard deviation, compared to the ground truth images. In addition, the mean values of the Dice index, Jaccard index, sensitivity, and specificity are 90.38%, 83.2%, 97.5%, and 88.8%; respectively, with standard deviations (SD) of 7.6%, 10.5%, 3.5%, and 11.4%; respectively. These reported experimental test results proved that the proposed NCARG approach correctly detects skin lesions of different shapes and sizes with high accuracy. Ten dermoscopic images were randomly selected; their segmentation results are shown in Figure 4, and the evaluation metrics are reported in Table 2.

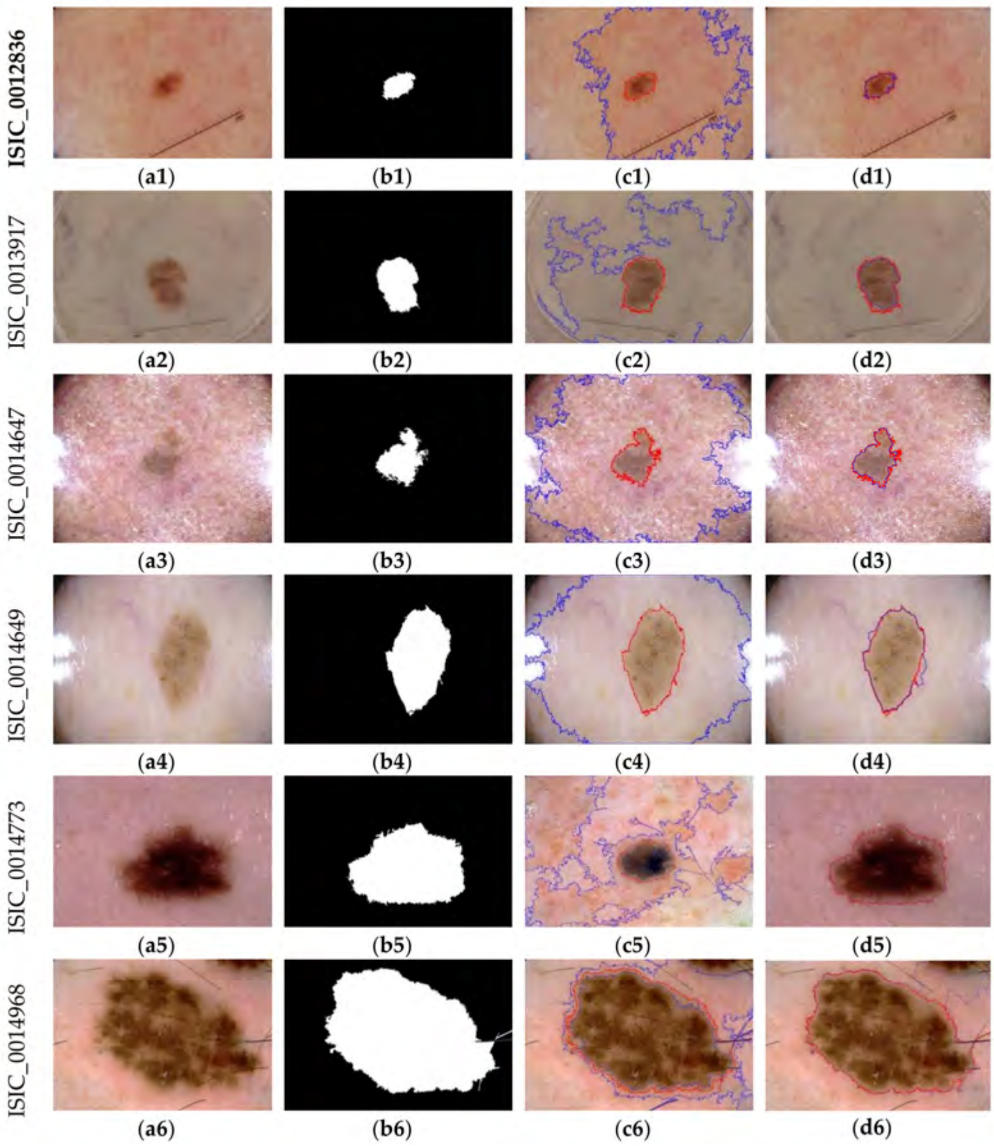

Figure 4. *Cont.*

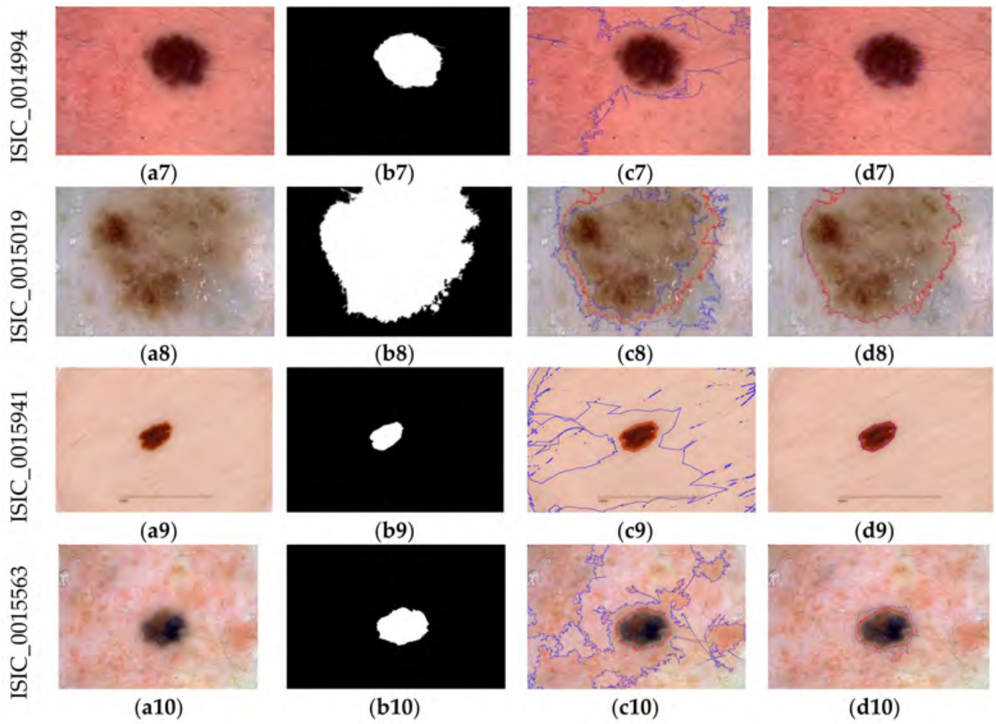

Figure 4. Comparative segmentation results, where (**a1**–**a10**): original dermoscopic test images; (**b1**–**b10**): ground truth images; (**c1**–**c10**): segmented images using the neutrosophic similarity score and level set (NSSLS) algorithm, and (**d1**–**d10**): NCARG proposed approach.

Table 2. The performance of computer segmentation using the proposed method with reference to the ground truth boundaries (Average ± SD) of ten images during the test phase.

Image ID	Accuracy (%)	Dice (%)	JAC (%)	Sensitivity (%)	Specificity (%)
ISIC_0012836	99.7819	93.2747	87.397	99.9909	87.851
ISIC_0013917	99.1485	90.4852	82.6237	1	82.6237
ISIC_0014647	99.4684	92.8643	86.6791	99.7929	91.2339
ISIC_0014649	98.8823	95.2268	90.8886	98.8313	99.2854
ISIC_0014773	98.9017	97.3678	94.8707	98.6294	99.9692
ISIC_0014968	89.5888	89.2267	80.5489	81.7035	99.9913
ISIC_0014994	98.9242	93.0613	87.023	1	87.023
ISIC_0015019	93.8788	93.9689	88.6239	88.6218	99.602
ISIC_0015941	99.7687	94.3589	89.3203	1	89.3203
ISIC_0015563	98.0344	83.939	72.3232	97.928	1
Average (%)	97.63777	92.3774	86.0298	96.54978	93.68998
SD (%)	3.31069	3.7373	6.2549	6.26068	6.76053

3.4. Comparative Study with NSSLS Method

The proposed NCARG approach is compared with the NSSLS algorithm [31] for detecting skin lesions. Figure 4(a1–a10), Figure 4(b1–b10), Figure 4(c1–c10) and Figure 4(d1–d10) include the original dermoscopic images, the ground truth images, the segmented images using the NSSLS algorithm, and the NCARG proposed approach; respectively.

Figure 4 illustrates different samples from the test images with different size, shape, light illumination, skin surface roughness/smoothness, and the existence of hair and/or air bubbles. For these different samples, the segmented image using the proposed NCARG algorithm is matched with the ground truth; while, the NSSLS failed to accurately match the ground truth. Thus, Figure 4 demonstrates that the proposed approach accurately detects the skin lesion under the different cases compared with the NSSLS method. The superiority of the proposed approach is due to the ability of the NCM along with the IF to handle indeterminate information. In addition, shearlet transform achieved the surface anisotropic regularity along the edges leading the algorithm to capture the smooth contours corresponding to the dominant features in the image. For the same images in Figure 4, the comparative results of the previously mentioned evaluation metrics are plotted for the NCARG and NSSLS in Figures 5 and 6; respectively. In both figures, the X-axis denotes the image name under study, and the Y-axis denotes the value of the corresponding metric in the bar graph.

Figure 5 along with Table 2 illustrate the accuracy of the proposed algorithm, which achieves an average accuracy of 97.638% for the segmentation of the different ten skin lesion samples, while Figure 6 illustrates about 44% average accuracy of the NSSLS method. Thus, Figures 5 and 6 establish the superiority of the proposed approach compared with the NSSLS method, owing to the removal the indeterminate information and the efficiency of the shearlet transform. The same results are confirmed by measuring the same metrics using 500 images, as reported in Figure 7.

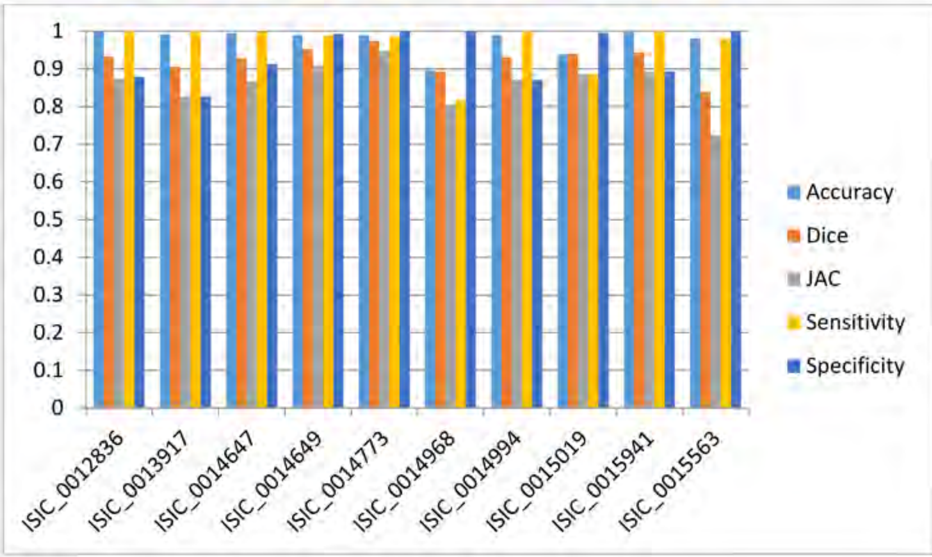

Figure 5. Evaluation metrics of the ten test images using the proposed segmentation NCARG approach.

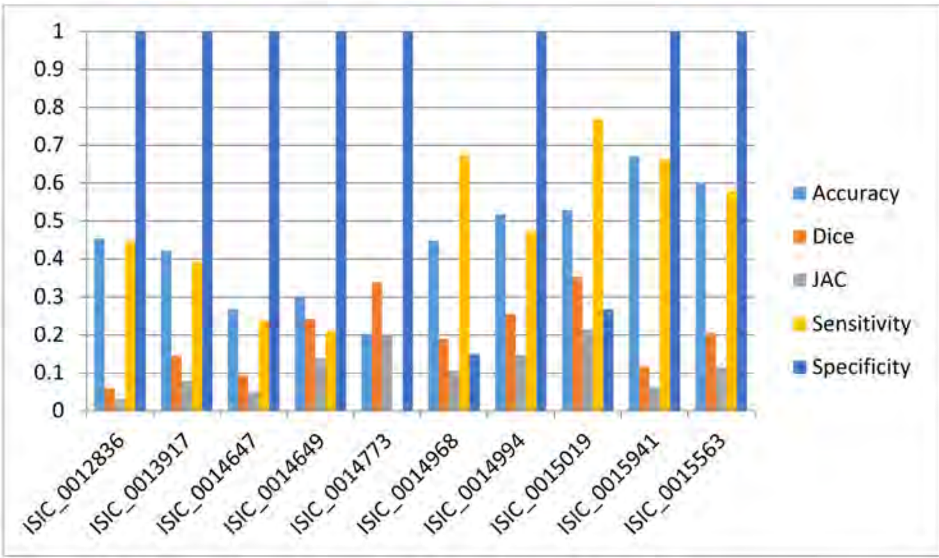

Figure 6. Evaluation metrics of the ten test images using the NSSLS segmentation approach for comparison.

Figure 7 reports that the proposed method achieves about 15% improvement on the accuracy and about 25% improvement in the JAC over the NSSLS method. Generally, Figure 7 proves the superiority of the proposed method compared with the NSSLS method. In addition, Table 3 reports the statistical results on the testing images; it compares the detection performance with reference to the ground truth segmented images for the NSSLS and the proposed NCARG method. The p-values are used to estimate the differences between the metric results of the two methods. The statistical significance was set at a level of 0.05; a p-value of <0.05 refers to the statistically significant relation.

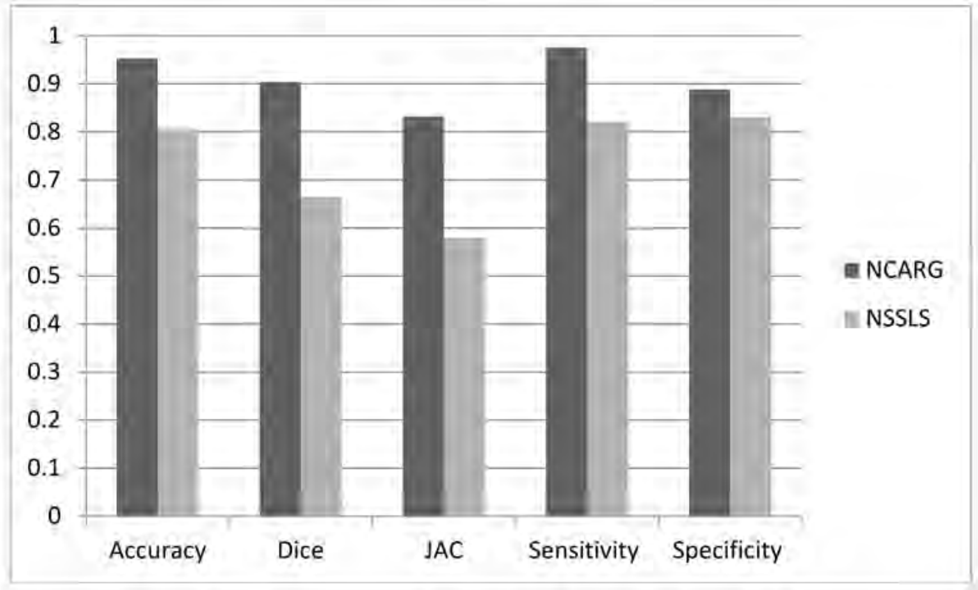

Figure 7. Comparative results of the performance evaluation metrics of the proposed NCARG and NSSLS methods.

The p-values reported in Table 3 establish a significant difference in the performance metric values when using the proposed NCARG and NSSLS methods. The mean and standard deviation of the accuracy, Dice, JAC, sensitivity, and specificity for the NSSLS and NCARG methods, along with the p-values, establish that the proposed NCARG method improved skin lesion segmentation compared with the NSSLS method. Figure 7 along with Table 3 depicts that the NCARG achieved 95.3% average accuracy, which is superior to the 80.6% average accuracy of the NSSLS approach. Furthermore, the proposed algorithm achieved a 90.4% average Dice coefficient value, 83.2% average JAC value, 97.5% average sensitivity value, and 88.8% average specificity value. The segmentation accuracy improved from 80.6 ± 22.1 using the NSSLS to 95.3 ± 6 using the proposed method, which is a significant difference. The skin lesion segmentation improvement is statistically significant ($p < 0.05$) for all measured performances metrics by SPSS software.

Table 3. The average values (mean ± SD) of the evaluation metrics using the NCARG approach compared to the NSSLS approach.

Method	Accuracy (%)	Dice (%)	JAC (%)	Sensitivity (%)	Specificity (%)
NSSLS method	80.6 ± 22.1	66.4 ± 32.6	57.9 ± 33.7	82.1 ± 24	83.1 ± 30.4
Proposed NCARG method	95.3 ± 6	90.4 ± 7.6	83.2 ± 10.5	97.5 ± 6.3	88.8 ± 11.4
p-value	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001

The cumulative percentage is used to measure the percentage of images, which have a metric value less than a threshold value. The cumulative percentage (CP) curves of the measured metrics are plotted for comparing the performance of the NSSLS and NCARG algorithms. Figures 8–12 show the cumulative percentage of images having five measurements less than a certain value; the X-axis represents the different threshold values on the metric and the Y-axis is the percentage of the number of images whose metric values are greater than this threshold value. These figures demonstrate the comparison of performances in terms of the cumulative percentage of the different metrics, namely the accuracy, Dice value, JAC, sensitivity, and specificity; respectively.

Figure 8 illustrates a comparison of performances in terms of the cumulative percentage of the NCARG and NSSLS segmentation accuracy. About 80% of the images have a 95% accuracy for the segmentation using the proposed NCARG, while the achieved cumulative accuracy percentage using the NSSLS is about 65% for 80% of the images.

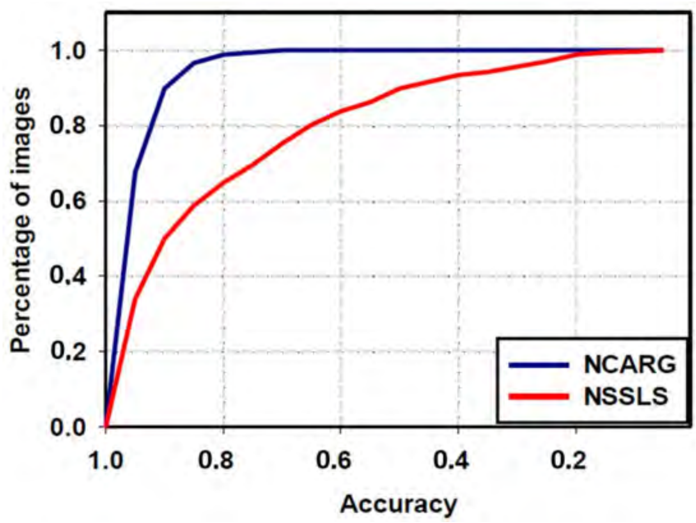

Figure 8. Comparison of performances in terms of the cumulative percentage of the accuracy using the NCARG and NSSLS segmentation methods.

Figure 9 compares the performances, in terms of the cumulative percentage of the Dice index values, of the NCARG and NSSLS segmentations. Figure 9 depicts that 100% of the images have about 82% Dice CP values using the NCARG method, while 58% of the images achieved the same 82% Dice CP values when using the NSSLS method.

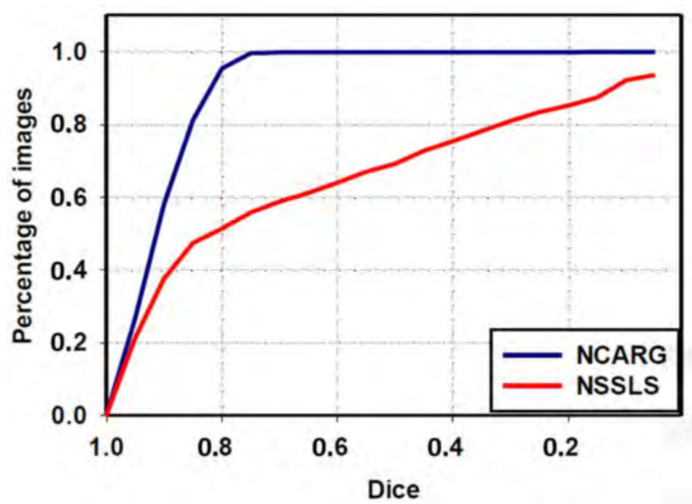

Figure 9. Comparison of performances in terms of the cumulative percentage of the Dice values using the NCARG and NSSLS segmentation methods.

Figure 10 compares the performances, in terms of the cumulative percentage of the JAC values, of the NCARG and NSSLS segmentation. About 50% of the images have 83% CP JAC values using the NCARG method, while the obtained CP JAC using the NSSLS for the same number of images is about 72%.

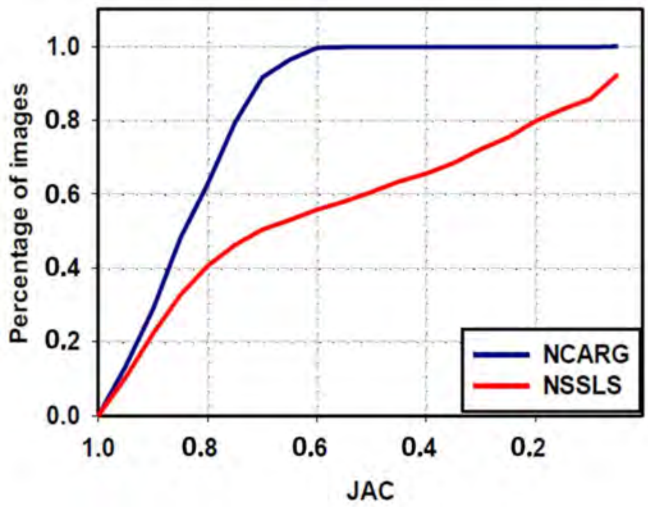

Figure 10. Comparison of performances in terms of the cumulative percentage of the JAC values using NCARG and NSSLS segmentation methods.

Figure 11 compares the performances, in terms of the cumulative percentage of the sensitivity, using the NCARG and NSSLS segmentation methods. About 50% of the images have 97% sensitivity value using the NCARG method, while the NSSLS achieves about 92% sensitivity value.

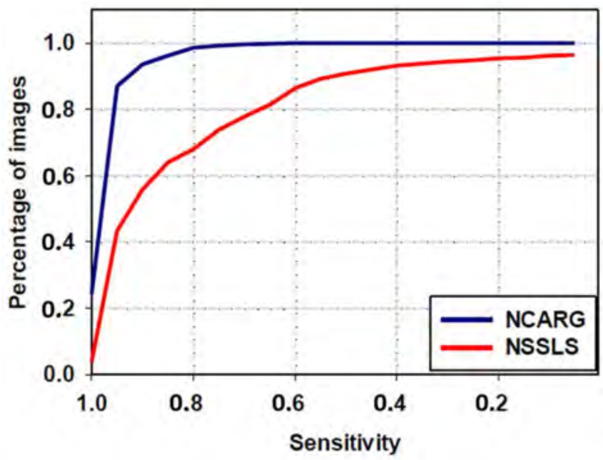

Figure 11. Comparison of performances in terms of the cumulative percentage of the sensitivity using the NCARG and NSSLS segmentation methods.

Figure 12 demonstrates the comparison of performances, in terms of the cumulative percentage of the specificity, using the NCARG and NSSLS segmentation methods. A larger number of images have accuracies in the range of 100% to 85% when using the NSSLS compared to the proposed method. However, about 100% of the images have 63% CP specificity values using the NCARG method, while the NSSLS achieved about 20% cumulative specificity values with 90% of the images. Generally, the cumulative percentage of each metric establishes the superiority of the proposed NCARG method compared with the NSSLS method.

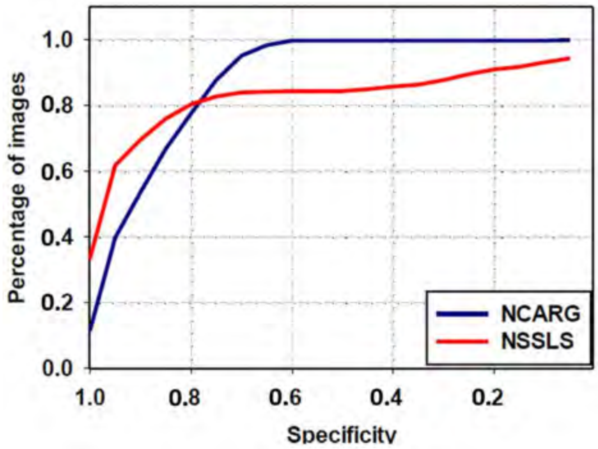

Figure 12. Comparison of performances in terms of the cumulative percentage of the specificity using the NCARG and NSSLS segmentation methods.

3.5. Comparison with Other Segmentation Methods Using the ISIC Archive

In case of lesion segmentation, variability in the images is very high; therefore, performance results highly depend on the data set that is used in the experiments. Several studies and challenges have been conducted to resolve such trials [33]. In order to validate the performance of the proposed NCARG method, a comparison is conducted on the results of previously published studies on the same ISIC dermoscopic image data set. Yu et al. [34] leveraged very deep convolutional neural networks (CNN) for melanoma image recognition using the ISIC data set. The results proved that deeper networks, of more than 50 layers, provided more discriminating features with more accurate recognition. For accurate skin lesion segmentation, fully convolutional residual networks (FCRN) with a multi-scale contextual information integration structure were applied to the further classification stage. The network depth increase achieved enhanced discrimination capability of CNN. The FCRNs of 38 layers achieved 0.929 accuracy, 0.856 Dice, 0.785 JAC, and 0.882 sensitivity. Thus, our proposed NCARG provides superior performance in terms of these metrics. However, with an increased FCRN layer depth of 50, the performance improvement increased compared to our proposed method. However, the complexity also increases. In addition, Yu et al. have compared their study with other studies, namely the fully convolutional VGG-16 network [34,35] and the fully convolutional GoogleNet [34,36] establishing the superiority of our work compared to both of those studies. Table 4 reports a comparative study between the preceding studies, which have used the same ISIC data set, and the proposed NCARG method.

Table 4. Performance metrics comparison of different studies using the ISIC dataset for segmentation.

Method	Accuracy (%)	Dice (%)	JAC (%)	Sensitivity (%)	Specificity (%)
FCRNs of 38 layers [34]	92.9	85.6	78.5	88.2	93.2
FCRNs of 101 layers [34]	93.7	87.2	80.3	90.3	93.5
VGG-16 [34,35]	90.3	79.4	70.7	79.6	94.5
GoogleNet [34,36]	91.6	84.8	77.6	90.1	91.6
Proposed NCARG method	95.3	90.4	83.2	97.5	88.8

The preceding results and the comparative study establish the superiority of the proposed NCARG method compared with other methods. This superiority arises due to the effectiveness of the shearlet transform, the indeterminacy filtering, and the adaptive region growing, yielding an overall accuracy of 95.3%. Moreover, in comparison with previously conducted studies on the same ISIC dermoscopic image data set, the proposed method can be considered an effective method. In addition, the studies in References [37,38] can be improved and compared with the proposed method on the same dataset.

4. Conclusions

In this study, a novel skin lesion detection algorithm is proposed based on neutrosophic *c*-means and adaptive region growing algorithms applied to dermoscopic images. The dermoscopic images are mapped into the neutrosophic domain using the shearlet transform results of the image. An indeterminate filter is used for reducing the indeterminacy on the image, and the image is segmented via a neutrosophic *c*-means clustering algorithm. Finally, the skin lesion is accurately identified using a newly defined adaptive region growing algorithm. A public data set was employed to test the proposed method. Fifty images were selected randomly for tuning, and five hundred images were used to test the process. Several metrics were measured for evaluating the proposed method performance. The evaluation results demonstrate the proposed method achieves better performance to detect the skin lesions when compared to the neutrosophic similarity score and level set (NSSLS) segmentation approach.

The proposed NCARG approach achieved average 95.3% accuracy of 500 dermoscopic images including, ones with different shape, size, color, uniformity, skin surface roughness, light illumination during the image capturing process, and existence of air bubbles. The significant difference in the *p*-values of the measured metrics using the NSSLS and the proposed NCARG proved the superiority of the proposed method. This proposed method determines possible skin lesions in dermoscopic images which can be employed for further accurate automated diagnosis and clinical decision support.

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Interval Complex Neutrosophic Set: Formulation and Applications in Decision-Making

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Abstract Neutrosophic set is a powerful general formal framework which generalizes the concepts of classic set, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, etc. Recent studies have developed systems with complex fuzzy sets, for better designing and modeling real-life applications. The single-valued complex neutrosophic set, which is an extended form of the single-valued complex fuzzy set and of the single-valued complex intuitionistic fuzzy set, presents difficulties to defining a crisp neutrosophic membership degree as in the single-valued neutrosophic set. Therefore, in this paper we propose a new notion, called interval complex neutrosophic set (ICNS), and examine its characteristics. Firstly, we define several set theoretic operations of ICNS, such as union, intersection and complement, and afterward the operational rules. Next, a decision-making procedure in ICNS and its applications to a green supplier selection are investigated. Numerical examples based on real dataset of Thuan Yen JSC, which is a small-size trading service and transportation company, illustrate the efficiency and the applicability of our approach.

Keywords Green supplier selection · Multi-criteria decision-making · Neutrosophic set · Interval complex neutrosophic set · Interval neutrosophic set

Abbreviations

NS	Neutrosophic set
INS	Interval neutrosophic set
CFS	Complex fuzzy set
CIFS	Complex intuitionistic fuzzy set
IVCFS	Interval-valued complex fuzzy set
CNS	Complex neutrosophic set
ICNS	Interval-valued complex neutrosophic set, or interval complex neutrosophic set
SVCNS	Single-valued complex neutrosophic set
MCDM	Multi-criteria decision-making
MCGDM	Multi-criteria group decision-making
$\vee$	Maximum operator (t-conorm)
$\wedge$	Minimum operator (t-norm)

1 Introduction

Smarandache [12] introduced the Neutrosophic Set (NS) as a generalization of classical set, fuzzy set, and intuitionistic fuzzy set. The neutrosophic set handles indeterminate data, whereas the fuzzy set and the intuitionistic fuzzy set fail to work when the relations are indeterminate. Neutrosophic set has been successfully applied in different fields, including decision-making problems [2, 5–8, 11, 14–16, 19–24, 27, 28]. Since the neutrosophic set is difficult to be directly used in real-life applications, Smarandache [12] and Wang et al. [18] proposed the concept of single-valued neutrosophic set and provided its theoretic operations and properties. Nonetheless, in many real-life problems, the degrees of truth, falsehood, and indeterminacy of a certain statement may be suitably presented by interval forms, instead of real numbers [17]. To deal with this situation, Wang et al. [17] proposed the concept of Interval Neutrosophic Set (INS), which is characterized by the degrees of truth, falsehood and indeterminacy, whose values are intervals rather than real numbers. Ye [19] presented the Hamming and Euclidean distances between INSs and the similarity measures between INSs based on the distances. Tian et al. [16] developed a multi-criteria decision-making (MCDM) method based on a cross-entropy with INSs [3, 10, 19, 25].

Recent studies in NS and INS have concentrated on developing systems using complex fuzzy sets [9, 10, 26] for better designing and modeling real-life applications. The functionality of 'complex' is for handling the information of uncertainty and periodicity simultaneously. By adding complex-valued non-membership grade to the definition of complex fuzzy set, Salleh [13] introduced the concept of complex intuitionistic fuzzy set. Ali and Smarandache [1] proposed a complex neutrosophic set (CNS), which is an extension form of complex fuzzy set and of complex intuitionistic fuzzy set. The complex neutrosophic set can handle the redundant nature of uncertainty, incompleteness, indeterminacy, inconsistency, etc., in periodic data. The advantage of CNS over the NS is the fact that, in addition to the membership degree provided by the NS and represented in the CNS by amplitude, the CNS also provides the phase, which is an attribute degree characterizing the amplitude.

Yet, in many real-life applications, it is not easy to find a crisp (exact) neutrosophic membership degree (as in the single-valued neutrosophic set), since we deal with unclear and vague information. To overcome this, we must create a new notion, which uses an *interval neutrosophic membership degree*. This paper aims to introduce a new concept of Interval-Valued Complex Neutrosophic Set or shortly Interval Complex Neutrosophic Set (ICNS), that is more flexible and adaptable to real-life applications than those of SVCNS and INS, due to the fact that many applications require elements to be represented by a more accurate form, such as in the decision-making problems [4, 7, 16, 17, 20, 25]. For example, in the green supplier selection, the linguistic rating set should be encoded by ICNS rather than by INS or by SVCNS, to reflect the hesitancy and indeterminacy of the decision.

This paper is the *first attempt* to define and use the ICNS in decision-making. The contributions and the tidings of this paper are highlighted as follows: First, we define the Interval Complex Neutrosophic Set (Sect. 3.1). Next, we define some set theoretic operations, such as union, intersection and complement (Sect. 3.2). Further, we establish the operational rules of ICNS (Sect. 3.3). Then, we aggregate ratings of alternatives versus criteria, aggregate the importance weights, aggregate the weighted ratings of alternatives versus criteria, and define a score function to rank the alternatives. Last, a decision-making procedure in ICNS and an application to a green supplier selection are presented (Sects. 4, 5).

Green supplier selection is a well-known application of decision-making. One of the most important issues in supply chain to make the company operation efficient is the selection of appropriate suppliers. Due to the concerns over the changes in world climate, green supplier selection is considered as a key element for companies to contribute toward the world environment protection, as well as to maintain their competitive advantages in the global market. In order to select the appropriate green supplier, many potential economic and environmental criteria should be taken into consideration in the selection procedure. Therefore, green supplier selection can be regarded as a multi-criteria decision-making (MCDM) problem. However, the majority of criteria is generally evaluated by personal judgement and thus might suffer from subjectivity. In this situation, ICNS can better express this kind of information.

The *advantages* of the proposal over other possibilities are highlighted as follows:

(a) The complex neutrosophic set is a generalization of interval complex fuzzy set, interval complex intuitionistic fuzzy sets, single-valued complex neutrosophic set and so on. For more detail, we refer to Fig. 1 in Sect. 3.1.

(b) In many real-life applications, it is not easy to find a crisp (exact) neutrosophic membership degree (as in the single-valued neutrosophic set), since we deal with unclear and vague periodic information. To overcome this, the complex interval neutrosophic set is a better representation.

(c) In order to select the appropriate green supplier, many potential economic and environmental criteria should be taken into consideration in the selection procedure. Therefore, green supplier selection can be regarded as a multi-criteria decision-making (MCDM) problem. However, the majority of criteria are generally evaluated by personal judgment, and thus, it might suffer from subjectivity. In this

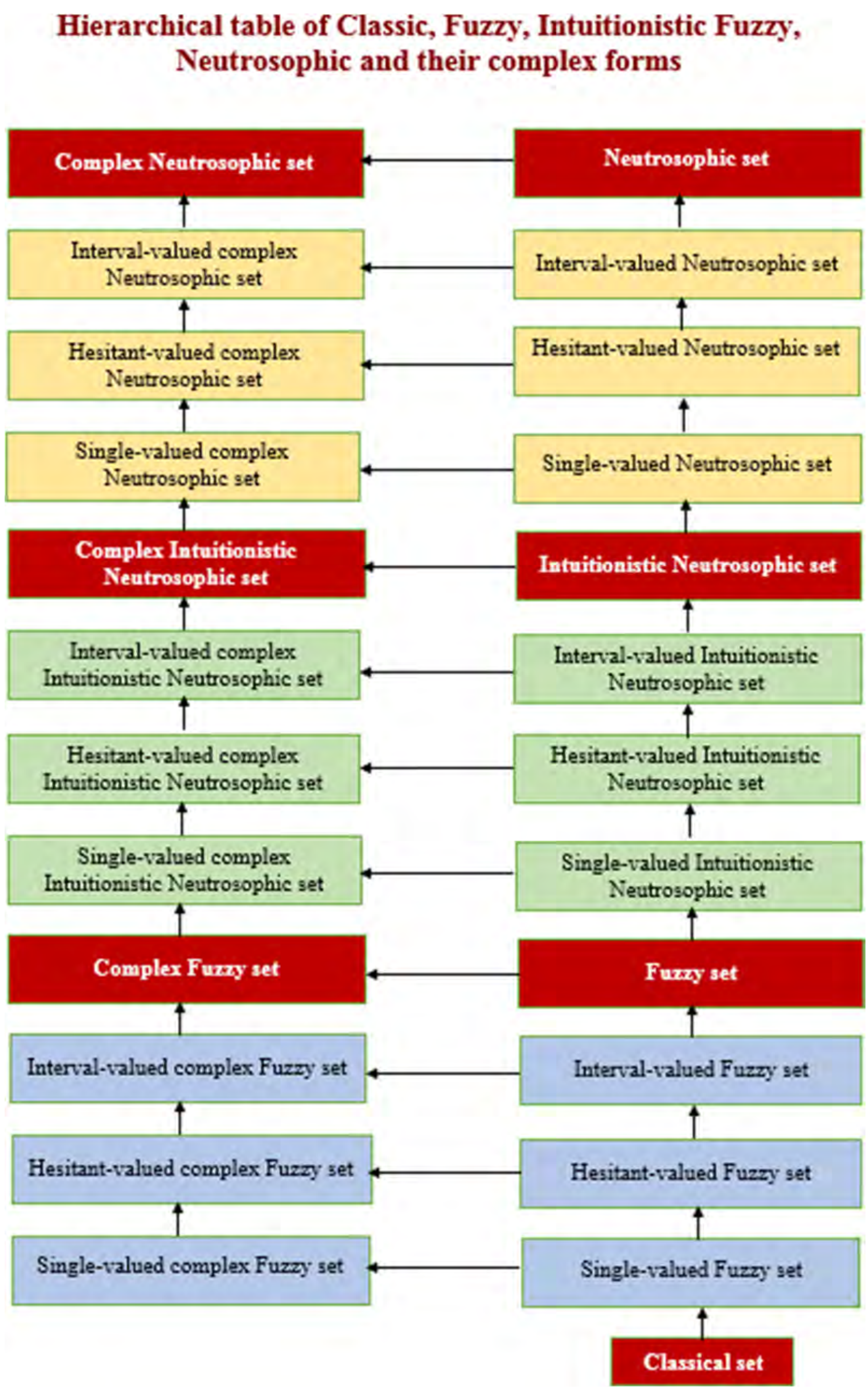

Fig. 1 Relationship of complex neutrosophic set with different types of fuzzy sets

situation, ICNS can better express this kind of information.

(d) The amplitude and phase (attribute) of ICNS have the ability to better catch the unsure values of the membership. Consider an example that we have a car component factory where each worker receives 10 car components per day to polish. The factory needs to have one worker coming in the weekend to work for a day, in order to finish a certain order from a customer. Again, the manager asks for a volunteer worker W. It turns out that the number of car components that will be done over one weekend day is W([0.6, 0.9], [0.1, 0.2], [0.0, 0.2]), which are actually the amplitudes for T, I, F. But what will be their quality? Indeed, their quality will be W([0.6, 0.9] × $e^{[0.6,\ 0.7]}$, [0.1, 0.2] × $e^{[0.4,\ 0.5]}$, [0.0, 0.2] × $e^{[0.0,\ 0.1]}$), by taking the [min, max] for each corresponding phase of T, I, F, respectively, for all workers. The new notion is indeed better in solving the decision-making problem. Unfortunately, other existing approaches cannot handle this type of information.

(e) The modified score function, accuracy function and certainty function of ICNS are more general in nature as compared to classical score, accuracy and certainty functions of existing methods. In modified forms of these functions, we have defined them for both amplitude and phase terms while it is not possible in the traditional case.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts of neutrosophic set, interval neutrosophic set, complex neutrosophic set, and their operations. Section 3 presents the formulation of the interval complex neutrosophic set and its operations. Section 4 proposes a multi-criteria group decision-making model in ICNS. Section 5 demonstrates a numerical example of the procedure for green supplier selection on a real dataset. Section 6 delineates conclusions and suggests further studies.

2 Basic Concepts

Definition 1 [12] Neutrosophic set (NS)

Let X be a space of points and let $x \in X$. A neutrosophic set $\overline{S}$ in X is characterized by a truth membership function $T_{\overline{S}}$, an indeterminacy membership function $I_{\overline{S}}$, and a falsehood membership function $F_{\overline{S}}$. $T_{\overline{S}}$, $I_{\overline{S}}$ and $F_{\overline{S}}$ are real standard or non-standard subsets of $]0^-, 1^+[$. To use neutrosophic set in some real-life applications, such as engineering and scientific problems, it is necessary to consider the interval $[0, 1]$ instead of $]0^-, 1^+[$, for technical applications. The neutrosophic set can be represented as:

$$\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x) \right) : x \in X \right\},$$

where one has that $0 \le \sup T_{\overline{S}}(x) + \sup I_{\overline{S}}(x) + \sup F_{\overline{S}}(x) \le 3$, and $T_{\overline{S}}$, $I_{\overline{S}}$ and $F_{\overline{S}}$ are subsets of the unit interval [0, 1].

Definition 2 [9, 10] Complex fuzzy set (CFS)

A complex fuzzy set $\overline{S}$, defined on a universe of discourse X, is characterized by a membership function $\eta_{\overline{S}}(x)$ that assigns to any element $x \in X$ a complex-valued grade of membership in $\overline{S}$. The values $\eta_{\overline{S}}(x)$ lie within the unit circle in the complex plane, and thus, all forms $p_{\overline{S}}(x) \cdot e^{j \cdot \mu_{\overline{S}}(x)}$ where $p_{\overline{S}}(x)$ and $\mu_{\overline{S}}(x)$ are both real-valued and $p_{\overline{S}}(x) \in [0, 1]$. The term $p_{\overline{S}}(x)$ is termed as amplitude term, and $e^{j \cdot \mu_{\overline{S}}(x)}$ is termed as phase term. The complex fuzzy set can be represented as:

$$\overline{S} = \left\{ \left(x, \eta_{\overline{S}}(x) \right) : x \in X \right\}.$$

Definition 3 [13] Complex intuitionistic fuzzy set (CIFS)

A complex intuitionistic fuzzy set $\overline{S}$, defined on a universe of discourse X, is characterized by a membership function $\eta_{\overline{S}}(x)$ and a non-membership function $\zeta_{\overline{S}}(x)$, respectively, assigning to an element $x \in X$ a complex-valued grade to both membership and non-membership in $\overline{S}$. The values of $\eta_{\overline{S}}(x)$ and $\zeta_{\overline{S}}(x)$ lie within the unit circle in the complex plane and are of the form $\eta_{\overline{S}}(x) = p_{\overline{S}}(x) \cdot e^{j \cdot \mu_{\overline{S}}(x)}$ and $\zeta_{\overline{S}}(x) = r_{\overline{S}}(x) \cdot e^{j \cdot \omega_{\overline{S}}(x)}$ where $p_{\overline{S}}(x), r_{\overline{S}}(x)$, $\mu_{\overline{S}}(x)$ and $\omega_{\overline{S}}(x)$ are all real-valued and $p_{\overline{S}}(x)$, $r_{\overline{S}}(x) \in [0, 1]$ with $j = \sqrt{-1}$. The complex intuitionistic fuzzy set can be represented as:

$$\overline{S} = \left\{ \left(x, \eta_{\overline{S}}(x), \zeta_{\overline{S}}(x) \right) : x \in X \right\}.$$

Definition 4 [4] Interval-valued complex fuzzy set (IVCFS)

An interval-valued complex fuzzy set $\bar{A}$ is defined over a universe of discourse X by a membership function

$$\mu_{\bar{A}} : X \rightarrow \Gamma^{[0,1]} \times R,$$
$$\mu_{\bar{A}}(x) = r_{\bar{A}}(x) \cdot e^{j\omega_{\bar{A}}(x)}$$

In the above equation, $\Gamma^{[0,1]}$ is the collection of interval fuzzy sets and R is the set of real numbers. $r_{\overline{S}}(x)$ is the interval-valued membership function while $e^{j\omega_{\bar{A}}(x)}$ is the phase term, with $j = \sqrt{-1}$.

Definition 5 [1] Single-valued complex neutrosophic set (SVCNS)

A single-valued complex neutrosophic set $\overline{S}$, defined on a universe of discourse X, is expressed by a truth membership function $T_{\overline{S}}(x)$, an indeterminacy membership function $I_{\overline{S}}(x)$ and a falsity membership function $F_{\overline{S}}(x)$, assigning a complex-valued grade of $T_{\overline{S}}(x)$, $I_{\overline{S}}(x)$ and $F_{\overline{S}}(x)$ in $\overline{S}$ for any $x \in X$. The values $T_{\overline{S}}(x)$, $I_{\overline{S}}(x)$, $F_{\overline{S}}(x)$ and their sum may all be within the unit circle in the complex plane, and so it is of the following form:

$$T_{\overline{S}}(x) = p_{\overline{S}}(x) \cdot e^{j\mu_{\overline{S}}(x)},\ I_{\overline{S}}(x) = q_{\overline{S}}(x) \cdot e^{j\nu_{\overline{S}}(x)} \text{ and } F_{\overline{S}}(x) = r_{\overline{S}}(x) \cdot e^{j\omega_{\overline{S}}(x)},$$

where $p_{\overline{S}}(x)$, $q_{\overline{S}}(x)$, $r_{\overline{S}}(x)$ and $\mu_{\overline{S}}(x)$, $\nu_{\overline{S}}(x)$, $\omega_{\overline{S}}(x)$ are, respectively, real values and $p_{\overline{S}}(x), q_{\overline{S}}(x), r_{\overline{S}}(x) \in [0,1]$, such that $0 \le p_{\overline{S}}(x) + q_{\overline{S}}(x) + r_{\overline{S}}(x) \le 3$. The single-valued complex neutrosophic set S can be represented in set form as:

$$\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x)\right) : x \in X \right\}.$$

Definition 6 [1] Complement of single-valued complex neutrosophic set

Let $\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x)\right) : x \in X \right\}$ be a single-valued complex neutrosophic set in X. Then, the complement of a SVCNS $\overline{S}$ is denoted as $\overline{S}^c$ and is defined by:

$$\overline{S}^c = \left\{ \left(x, T_{\overline{S}^c}(x), I_{\overline{S}^c}(x), F_{\overline{S}^c}(x)\right) : x \in X \right\},$$

where $T_{\overline{S}^c}(x) = p_{\overline{S}^c}(x) \cdot e^{j \cdot \mu_{\overline{S}^c}(x)}$ is such that $p_{\overline{S}^c}(x) = r_{\overline{S}}(x)$ and $\mu_{\overline{S}^c}(x) = \mu_{\overline{S}}(x), 2\pi - \mu_{\overline{S}}(x)$ or $\mu_{\overline{S}}(x) + \pi$. Similarly, $I_{\overline{S}^c}(x) = q_{\overline{S}^c}(x) \cdot e^{j \cdot \nu_{\overline{S}^c}(x)}$, where $q_{\overline{S}^c}(x) = 1 - q_{\overline{S}}(x)$ and $\nu_{\overline{S}^c}(x) = \nu_{\overline{S}}(x), 2\pi - \nu_{\overline{S}^c}(x)$ or $\nu_{\overline{S}^c}(x) + \pi$. Finally, $F_{\overline{S}^c}(x) = r_{\overline{S}^c}(x) \cdot e^{j \cdot \omega_{\overline{S}^c}(x)}$, where $r_{\overline{S}^c}(x) = p_{\overline{S}}(x)$ and $\omega_{\overline{S}^c}(x) = \omega_{\overline{S}}(x), 2\pi - \omega_{\overline{S}}(x)$ or $\omega_{\overline{S}}(x) + \pi$

Definition 7 [1] Union of single-valued complex neutrosophic sets

Let $\bar{A}$ and $\bar{B}$ be two SVCNSs in X. Then:

$$\overline{A} \cup \overline{B} = \left\{ \left(x, T_{\overline{A}\cup\overline{B}}(x), I_{\overline{A}\cup\overline{B}}(x), F_{\overline{A}\cup\overline{B}}(X)\right) : x \in X \right\},$$

where

$$T_{\overline{A}\cup\overline{B}}(x) = \left[\left(p_{\overline{A}}(x) \vee p_{\overline{B}}(x)\right)\right] \cdot e^{j \cdot \mu_{T_{\bar{A}\cup\bar{B}}}(x)},$$
$$I_{\overline{A}\cup\overline{B}}(x) = \left[\left(q_{\overline{A}}(x) \wedge q_{\overline{B}}(x)\right)\right] \cdot e^{j \cdot \nu_{I_{\overline{A}\cup\overline{B}}}(x)},$$
$$F_{\overline{A}\cup\overline{B}}(x) = \left[\left(r_{\overline{A}}(x) \wedge r_{\overline{B}}(x)\right)\right] \cdot e^{j \cdot \omega_{F_{\overline{A}\cup\overline{B}}}(x)}$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. To calculate the phase terms $e^{j \cdot \mu_{A\cup B}(x)}$, $e^{j \cdot \nu_{A\cup B}(x)}$ and $e^{j \cdot \omega_{A\cup B}(x)}$, we refer to [1].

Definition 8 [1] Intersection of single-valued complex neutrosophic sets

Let $\bar{A}$ and $\overline{B}$ be two SVCNSs in X. Then:

$$\bar{A} \cap \bar{B} = \{(x, T_{\bar{A}\cap\bar{B}}(x), I_{\bar{A}\cap\bar{B}}(x), F_{\bar{A}\cap\bar{B}}(X)) : x \in X\},$$

where

$$T_{\bar{A}\cap\bar{B}}(x) = [(p_{\bar{A}}(x) \wedge p_{\bar{B}}(x))] \cdot e^{j \cdot \mu_{T_{\bar{A}\cap\bar{B}}}(x)},$$
$$I_{\bar{A}\cap\bar{B}}(x) = [(q_{\bar{A}}(x) \vee q_{\bar{B}}(x))] \cdot e^{j \cdot \nu_{I_{\bar{A}\cap\bar{B}}}(x)},$$
$$F_{\bar{A}\cap\bar{B}}(x) = [(r_{\bar{A}}(x) \vee r_{\bar{B}}(x))] \cdot e^{j \cdot \omega_{F_{\bar{A}\cap\bar{B}}}(x)}$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. To calculate the phase terms $e^{j \cdot \mu_{A\cup B}(x)}$, $e^{j \cdot \nu_{A\cup B}(x)}$ and $e^{j \cdot \omega_{A\cup B}(x)}$, we refer to [1].

3 Interval Complex Neutrosophic Set with Set Theoretic Properties

3.1 Interval Complex Neutrosophic Set

Before we present the definition, let us consider an example below to see the advantages of the new notion ICNS.

Example 1 Suppose we have a car component factory. Each worker from this factory receives 10 car components per day to polish.

- *NS* The best worker, John, successfully polishes 9 car components, 1 car component is not finished, and he wrecks 0 car component. Then, John's neutrosophic work is (0.9, 0.1, 0.0). The worst worker, George, successfully polishes 6, not finishing 2, and wrecking 2. Thus, George's neutrosophic work is (0.6, 0.2, 0.2).
- *INS* The factory needs to have one worker coming in the weekend, to work for a day in order to finish a required order from a customer. Since the factory management cannot impose the weekend overtime to workers, the manager asks for a volunteer. How many car components are to be polished during the weekend? Since the manager does not know which worker (W) will volunteer, he estimates that the work to be done in a weekend day will be: W([0.6, 0.9], [0.1, 0.2], [0.0, 0.2]), i.e., an interval for each T, I, F, respectively, between the minimum and maximum values of all workers.
- *CNS* The factory's quality control unit argues that although many workers correctly/successfully polish their car components, some of the workers do a work of a better quality than the others. Going back to John and George, the factory's quality control unit measures the work quality of each of them and finds out that: John's work is $(0.9 \times e^{0.6}, 0.1 \times e^{0.4}, 0.0 \times e^{0.0})$, and George's work is $(0.6 \times e^{0.7}, 0.2 \times e^{0.5}, 0.2 \times e^{0.1})$. Thus, although John polishes successfully 9 car components, more than George's 6 successfully polished

car components, the quality of John's work (0.6, 0.4, 0.0) is less than the quality of George's work (0.7, 0.5, 0.1).

It is clear from the above example that the amplitude and phase (attribute) of CNS should be represented by intervals, which better catch the unsure values of the membership. Let us come back to Example 1, where the factory needs to have one worker coming in the weekend to work for a day, in order to finish a certain order from a customer. Again, the manager asks for a volunteer worker W. We find out that the number of car components that will be done over one weekend day is W([0.6, 0.9], [0.1, 0.2], [0.0, 0.2]), which are actually the amplitudes for T, I, F. But what will be their quality? Indeed, their quality will be W([0.6, 0.9] × $e^{[0.6,\ 0.7]}$, [0.1, 0.2] × $e^{[0.4,\ 0.5]}$, [0.0, 0.2] × $e^{[0.0,\ 0.1]}$), by taking the [min, max] for each corresponding phases for T, I, F, respectively, for all workers. Therefore, we should propose a new notion for such the cases of decision-making problems.

Definition 9 Interval complex neutrosophic set.

An interval complex neutrosophic set is defined over a universe of discourse X by a truth membership function $T_{\overline{S}}$, an indeterminate membership function $I_{\overline{S}}$, and a falsehood membership function $F_{\overline{S}}$, as follows:

$$\left.\begin{aligned} T_{\overline{S}} &: X \to \Gamma^{[0,1]} \times R, T_{\overline{S}}(x) = t_{\overline{S}}(x) \cdot e^{j\alpha\omega_{\overline{S}}(x)} \\ I_{\overline{S}} &: X \to \Gamma^{[0,1]} \times R, I_{\overline{S}}(x) = i_{\overline{S}}(x) \cdot e^{j\beta\psi_{\overline{S}}(x)} \\ F_{\overline{S}} &: X \to \Gamma^{[0,1]} \times R, F_{\overline{S}}(x) = f_{\overline{S}}(x) \cdot e^{j\gamma\phi_{\overline{S}}(x)} \end{aligned}\right\} \tag{1}$$

In the above Eq. (1), $\Gamma^{[0,1]}$ is the collection of interval neutrosophic sets and R is the set of real numbers, $t_{\overline{S}}(x)$ is the interval truth membership function, $i_{\overline{S}}(x)$ is the interval indeterminate membership and $f_{\overline{S}}(x)$ is the interval falsehood membership function, while $e^{j\alpha\omega_{\overline{S}}(x)}$, $e^{j\beta\psi_{\overline{S}}(x)}$ and $e^{j\gamma\phi_{\overline{S}}(x)}$ are the corresponding interval-valued phase terms, respectively, with $j = \sqrt{-1}$. The scaling factors α, β and γ lie within the interval $(0, 2\pi]$. This study assumes that the values $\alpha, \beta, \gamma = \pi$. In set theoretic form, an interval complex neutrosophic set can be written as:

$$\overline{S} = \left\{\left\langle \frac{T_{\overline{S}}(x) = t_{\overline{S}}(x) \cdot e^{j\alpha\omega_{\overline{S}}(x)}, I_{\overline{S}}(x) = i_{\overline{S}}(x) \cdot e^{j\beta\psi_{\overline{S}}(x)}, F_{\overline{S}}(x) = f_{\overline{S}}(x) \cdot e^{j\gamma\phi_{\overline{S}}(x)}}{x} \right\rangle : x \in X\right\} \tag{2}$$

In (2), the amplitude interval-valued terms $t_{\overline{S}}(x), i_{\overline{S}}(x), f_{\overline{S}}(x)$ can be further split as $t_{\overline{S}}(x) = \left[t_{\overline{S_L}}(x), t_{\overline{S_U}}(x)\right]$, $i_{\overline{S}}(x) = \left[i_{\overline{S_L}}(x), i_{\overline{S_U}}(x)\right]$ and $f_{\overline{S}}(x) = \left[f_{\overline{S_L}}(x), f_{\overline{S_U}}(x)\right]$, where $t_{\overline{S_U}}(x), i_{\overline{S_U}}(x), f_{\overline{S_U}}(x)$ represents the upper bound, while $t_{\overline{S_L}}(x), i_{\overline{S_L}}(x), f_{\overline{S_L}}(x)$ represents the lower bound in each interval, respectively. Similarly, for the phases: $\omega_{\overline{S}}(x) = \left[\omega_{\overline{S_L}}(x), \omega_{\overline{S_U}}(x)\right]$, $\psi_{\overline{S}}(x) = \left[\psi_{\overline{S_L}}(x), \psi_{\overline{S_U}}(x)\right]$, and $\varphi_{\overline{S}}(x) = \left[\varphi_{\overline{S_L}}(x), \varphi_{\overline{S_U}}(x)\right]$.

Example 2 Let $X = \{x_1, x_2, x_3, x_4\}$ be a universe of discourse. Then, an interval complex neutrosophic set $\overline{S}$ can be given as follows:

$$\overline{S} = \left\{\begin{array}{c} \frac{[0.4,0.6]\cdot e^{j\pi[0.5,0.6]}, [0.1,0.7]\cdot e^{j\pi[0.1,0.3]}, [0.3,0.5]\cdot e^{j\pi[0.8,0.9]}}{x_1}, \frac{[0.2,0.4]\cdot e^{j\pi[0.3,0.6]}, [0.1,0.1]\cdot e^{j\pi[0.7,0.9]}, [0.5,0.9]\cdot e^{j\pi[0.2,0.5]}}{x_2}, \\ \frac{[0.3,0.4].e^{j\pi[0.7,0.8]}, [0.6,0.7]\cdot e^{j\pi[0.6,0.7]}, [0.2,0.6]\cdot e^{j\pi[0.6,0.8]}}{x_3}, \frac{[0,0.9]\cdot e^{j\pi[0.9,1]}, [0.2,0.3]\cdot e^{j\pi[0.7,0.8]}, [0.3,0.5]\cdot e^{j\pi[0.4,0.5]}}{x_4} \end{array}\right\}$$

Further on, we present the connections among different types of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, to complex neutrosophic set (in Fig. 1). The arrows (→) refer to the generalization of the preceding term to the next term, e.g., the fuzzy set is the generalization of the classic set, and so on.

3.2 Set Theoretic Operations of Interval Complex Neutrosophic Set

Definition 10 Let $\bar{A}$ and $\bar{B}$ be two interval complex neutrosophic set over X which are defined by $T_{\bar{A}}(x) = t_{\bar{A}}(x) \cdot e^{j\pi\omega_{\bar{A}}(x)}$, $I_{\bar{A}}(x) = i_{\bar{A}}(x) \cdot e^{j\pi\psi_{\bar{A}}(x)}$, $F_{\bar{A}}(x) = f_{\bar{A}}(x) \cdot e^{j\pi\phi_{\bar{A}}(x)}$ and $T_{\bar{B}}(x) = t_{\bar{B}}(x) \cdot e^{j\pi\omega_{\bar{B}}(x)}$, $I_{\bar{B}}(x) = i_{\bar{B}}(x) \cdot e^{j\pi\psi_{\bar{B}}(x)}$, $F_{\overline{S}}(x) = f_{\overline{S}}(x) \cdot e^{j\pi\phi_{\overline{S}}(x)}$, respectively. The *union* of $\bar{A}$ and $\bar{B}$ is denoted as
$\bar{A} \cup \bar{B}$, and it is defined as:

$$T_{\bar{A}\cup\bar{B}}(x) = [\inf t_{\bar{A}\cup\bar{B}}(x), \sup t_{\bar{A}\cup\bar{B}}(x)] \cdot e^{j\pi\omega_{\bar{A}\cup\bar{B}}(x)},$$
$$I_{\bar{A}\cup\bar{B}}(x) = [\inf i_{\bar{A}\cup\bar{B}}(x), \sup i_{\bar{A}\cup\bar{B}}(x)] \cdot e^{j\pi\psi_{\bar{A}\cup\bar{B}}(x)},$$
$$F_{\bar{A}\cup\bar{B}}(x) = [\inf f_{\bar{A}\cup\bar{B}}(x), \sup f_{\bar{A}\cup\bar{B}}(x)] \cdot e^{j\pi\phi_{\bar{A}\cup\bar{B}}(x)},$$

where

$$\inf t_{\bar{A}\cup\bar{B}}(x) = \vee(\inf t_{\bar{A}}(x), \inf t_{\bar{B}}(x)), \sup t_{\bar{A}\cup\bar{B}}(x) = \vee(\sup t_{\bar{A}}(x), \sup t_{\bar{B}}(x));$$
$$\inf i_{\bar{A}\cup\bar{B}}(x) = \wedge(\inf i_{\bar{A}}(x), \inf i_{\bar{B}}(x)), \ \sup i_{\bar{A}\cup\bar{B}}(x) = \wedge(\sup i_{\bar{A}}(x), \sup i_{\bar{B}}(x));$$
$$\inf f_{\bar{A}\cup\bar{B}}(x) = \wedge(\inf f_{\bar{A}}(x), \inf f_{\bar{B}}(x)), \ \sup f_{\bar{A}\cup\bar{B}}(x) = \wedge(\sup f_{\bar{A}}(x), \sup f_{\bar{B}}(x));$$

for all $x \in X$. The union of the phase terms remains the same as defined for single-valued complex neutrosophic set, with the distinction that instead of subtractions and additions of numbers, we now have subtractions and additions of intervals. The symbols $\vee, \wedge$ represent max and min operators.

Example 3 Let $X = \{x_1, x_2, x_3, x_4\}$ be a universe of discourse. Let $\bar{A}$ and $\overline{B}$ be two interval complex neutrosophic sets defined on X as follows:

$$\bar{A} = \left\{\begin{array}{c} \frac{[0.4,0.6]\cdot e^{j\pi[0.5,0.6]}, [0.1,0.7]\cdot e^{j\pi[0.1,0.3]}, [0.3,0.5]\cdot e^{j\pi[0.8,0.9]}}{x_1}, \frac{[0.2,0.4]\cdot e^{j\pi[0.3,0.6]}, [0.1,0.1]\cdot e^{j\pi[0.7,0.9]}, [0.5,0.9]\cdot e^{j\pi[0.2,0.5]}}{x_2}, \\ \frac{[0.3,0.4].e^{j\pi[0.7,0.8]}, [0.6,0.7]\cdot e^{j\pi[0.6,0.7]}, [0.2,0.6]\cdot e^{j\pi[0.6,0.8]}}{x_3}, \frac{[0,0.9]\cdot e^{j\pi[0.9,1]}, [0.2,0.3]\cdot e^{j\pi[0.7,0.8]}, [0.3,0.5]\cdot e^{j\pi[0.4,0.5]}}{x_4} \end{array}\right\}$$

$$\bar{B} = \left\{\begin{array}{c} \frac{[0.3,0.7]\cdot e^{j\pi[0.7,0.8]}, [0.4,0.9]\cdot e^{j\pi[0.3,0.5]}, [0.6,0.8]\cdot e^{j\pi[0.5,0.6]}}{x_1}, \frac{[0.4,0.4]\cdot e^{j\pi[0.6,0.7]}, [0.1,0.9]\cdot e^{j\pi[0.2,0.4]}, [0.3,0.8]\cdot e^{j\pi[0.5,0.6]}}{x_2}, \\ \frac{[0.37,0.64]\cdot e^{j\pi[0.47,0.50}, [0.36,0.57]\cdot e^{j\pi[0.64,0.7]}, [0.28,0.66]\cdot e^{j\pi[0.16,0.2]}}{x_3}, \frac{[0.15,0.52]\cdot e^{j\pi[0.1,0.2]}, [0,0.5]\cdot e^{j\pi[0.6,0.7]}, [0.3,0.3]\cdot e^{j\pi[0.6,0.7]}}{x_4} \end{array}\right\}$$

Then, their union $\bar{A} \cup \bar{B}$ is given by:

$$\bar{A} \cup \bar{B} = \left\{ \begin{array}{c} \frac{[0.4,0.7]\cdot e^{j\pi[0.7,0.8]},[0.1,0.7]\cdot e^{j\pi[0.1,0.3]},[0.3,0.5]\cdot e^{j\pi[0.5,0.6]}}{x_1}, \frac{[0.4,0.4]\cdot e^{j\pi[0.6,0.7]},[0.1,0.1]\cdot e^{j\pi[0.7,0.9]},[0.3,0.8]\cdot e^{j\pi[0.5,0.6]}}{x_2}, \\ \frac{[0.37,0.64]\cdot e^{j\pi[0.7,0.8]},[0.36,0.57]\cdot e^{j\pi[0.6,0.7]},[0.2,0.6]\cdot e^{j\pi[0.16,0.21]}}{x_3}, \frac{[0.15,0.9]\cdot e^{j\pi[0.9,1]},[0,0.3]\cdot e^{j\pi[0.6,,0.7]},[0.3,0.3]\cdot e^{j\pi[0.4,0.5]}}{x_4} \end{array} \right\}$$

Definition 11 Let $\bar{A}$ and $\bar{B}$ be two interval complex neutrosophic set over X which are defined by $T_{\bar{A}}(x) = t_{\bar{A}}(x) \cdot e^{j\pi\omega_{\bar{A}}(x)}$, $I_{\bar{A}}(x) = i_{\bar{A}}(x) \cdot e^{j\pi\psi_{\bar{A}}(x)}$, $F_{\bar{A}}(x) = f_{\bar{A}}(x) \cdot e^{j\pi\phi_{\bar{A}}(x)}$ and $T_{\bar{B}}(x) = t_{\bar{B}}(x) \cdot e^{j\pi\omega_{\bar{B}}(x)}$, $I_{\bar{B}}(x) = i_{\bar{B}}(x) \cdot e^{j\pi\psi_{\bar{B}}(x)}$, $F_{\bar{S}}(x) = f_{\bar{S}}(x) \cdot e^{j\pi\phi_{\bar{S}}(x)}$, respectively. The *intersection* of $\bar{A}$ and $\bar{B}$ is denoted as $\bar{A} \cap \bar{B}$, and it is defined as:

$$T_{\bar{A}\cap\bar{B}}(x) = [\inf t_{\bar{A}\cap\bar{B}}(x), \sup t_{\bar{A}\cap\bar{B}}(x)] \cdot e^{j\pi\omega_{\bar{A}\cap\bar{B}}(x)},$$
$$I_{\bar{A}\cap\bar{B}}(x) = [\inf i_{\bar{A}\cap\bar{B}}(x), \sup i_{\bar{A}\cap\bar{B}}(x)] \cdot e^{j\pi\psi_{\bar{A}\cap\bar{B}}(x)},$$
$$F_{\bar{A}\cap\bar{B}}(x) = [\inf f_{\bar{A}\cap\bar{B}}(x), \sup f_{\bar{A}\cap\bar{B}}(x)] \cdot e^{j\pi\phi_{\bar{A}\cap\bar{B}}(x)},$$

where

$$\inf t_{\bar{A}\cap\bar{B}}(x) = \wedge(\inf t_{\bar{A}}(x), \inf t_{\bar{B}}(x)),\ \sup t_{\bar{A}\cap\bar{B}}(x) = \wedge(\sup t_{\bar{A}}(x), \sup t_{\bar{B}}(x)),$$
$$\inf i_{\bar{A}\cap\bar{B}}(x) = \vee(\inf i_{\bar{A}}(x), \inf i_{\bar{B}}(x)),\ \sup i_{\bar{A}\cap\bar{B}}(x) = \vee(\sup i_{\bar{A}}(x), \sup i_{\bar{B}}(x)),$$
$$\inf f_{\bar{A}\cap\bar{B}}(x) = \vee(\inf f_{\bar{A}}(x), \inf f_{\bar{B}}(x)),\ \sup f_{\bar{A}\cap\bar{B}}(x) = \vee(\sup f_{\bar{A}}(x), \sup f_{\bar{B}}(x)),$$

for all $x \in X$. Similarly, the intersection of the phase terms remains the same as defined for single-valued complex neutrosophic set, with the distinction that instead of subtractions and additions of numbers we now have subtractions and additions of intervals. The symbols $\vee$,$\wedge$ represent max and min operators.

Example 4 Let X, $\bar{A}$ and $\overline{B}$ be as in Example 3. Then, the intersection $\bar{A} \cap \bar{B}$ is given by:

$$\bar{A} \cap \bar{B} = \left\{ \begin{array}{c} \frac{[0.3,0.6]\cdot e^{j\pi[0.5,0.6]},[0.4,0.9]\cdot e^{j\pi[0.3,0.5]},[0.6,0.8]\cdot e^{j\pi[0.8,0.9]}}{x_1}, \frac{[0.2,0.4]\cdot e^{j\pi[0.3,0.6]},[0.1,0.9]\cdot e^{j\pi[0.7.0.9]},[0.5,0.9]\cdot e^{j\pi[0.5,0.6]}}{x_2}, \\ \frac{[0.3,0.4]\cdot e^{j\pi[0.47,0.50]},[0.6,0.7]\cdot e^{j\pi[0.64,0.70]},[0.28,0.6]6\cdot e^{j\pi[0.6,0.8]}}{x_3}, \frac{[0,0.52]\cdot e^{j\pi[0.1,0.2]},[0.2,0.5]\cdot e^{j\pi[0.7,0.8]},[0.3,0.5]\cdot e^{j\pi[0.6,0.7]}}{x_4} \end{array} \right\}$$

Definition 12 Let $\bar{A}$ be an interval complex neutrosophic set over X which is defined by $T_{\bar{A}}(x) = t_{\bar{A}}(x) \cdot e^{j\pi\omega_{\bar{A}}(x)}$, $I_{\bar{A}}(x) = i_{\bar{A}}(x) \cdot e^{j\pi\psi_{\bar{A}}(x)}$, $F_{\bar{A}}(x) = f_{\bar{A}}(x) \cdot e^{j\pi\phi_{\bar{A}}(x)}$. The *complement* of $\bar{A}$ is denoted as $\bar{A}^c$, and it is defined as:

$$\bar{A}^c = \left\{ \left\langle \frac{T_{\bar{A}^c}(x) = t_{\bar{A}^c}(x) \cdot e^{j\pi\omega_{\bar{A}^c}(x)}, I_{\bar{A}^c}(x) = i_{\bar{A}^c}(x) \cdot e^{j\pi\psi_{\bar{A}^c}(x)}, F_{\bar{A}^c}(x) = f_{\bar{A}^c}(x) \cdot e^{j\pi\phi_{\bar{A}^c}(x)}}{x} \right\rangle : x \in X \right\},$$

where $t_{\bar{A}^c}(x) = f_{\bar{A}}(x)$ and $\omega_{\bar{A}^c}(x) = 2\pi - \omega_{\bar{A}}(x)$ or $\omega_{\bar{A}}(x) + \pi$. Similarly, $i_{\bar{A}^c}(x) = (\inf i_{\bar{A}^c}(x), \sup i_{\bar{A}^c}(x))$, where $\inf i_{\bar{A}^c}(x) = 1 - \sup i_{\bar{A}}(x)$ and $\sup i_{\bar{A}^c}(x) = 1 - \inf i_{\bar{A}}(x)$, with phase term $\psi_{\bar{A}^c}(x) = 2\pi - \psi_{\bar{A}}(x)$ or $\psi_{\bar{A}}(x) + \pi$. Also, $f_{\bar{A}^c}(x) = i_{\bar{A}^c}(x)$, while the phase term $\phi_{\bar{A}^c}(x) = 2\pi - \phi_{\bar{A}}(x)$ or $\phi_{\bar{A}}(x) + \pi$.

Proposition 1 *Let $\bar{A}$, $\bar{B}$ and $\overline{C}$ be three interval complex neutrosophic sets over X. Then:*

1. $\bar{A} \cup \bar{B} = \bar{B} \cup \bar{A}$,
2. $\bar{A} \cap \bar{B} = \bar{B} \cap \bar{A}$,
3. $\bar{A} \cup \bar{A} = \bar{A}$,
4. $\bar{A} \cap \bar{A} = \bar{A}$,
5. $\bar{A} \cup (\bar{B} \cup \overline{C}) = (\bar{A} \cup \bar{B}) \cup \overline{C}$,
6. $\bar{A} \cap (\bar{B} \cap \overline{C}) = (\bar{A} \cap \bar{B}) \cap \overline{C}$,
7. $\bar{A} \cup (\bar{B} \cap \overline{C}) = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \overline{C})$,
8. $\bar{A} \cap (\bar{B} \cup \overline{C}) = (\bar{A} \cap \bar{B}) \cup (\bar{A} \cap \overline{C})$,
9. $\bar{A} \cup (\bar{A} \cap \bar{B}) = \bar{A}$,
10. $\bar{A} \cap (\bar{A} \cup \bar{B}) = \bar{A}$,
11. $(\bar{A} \cup \bar{B})^c = \bar{A}^c \cap \bar{B}^c$,
12. $(\bar{A} \cap \bar{B})^c = \bar{A}^c \cup \bar{B}^c$,
13. $(\bar{A}^c)^c = \bar{A}$.

Proof All these assertions can be straightforwardly proven.

Theorem 1 *The interval complex neutrosophic set $\bar{A} \cup \bar{B}$ is the smallest one containing both $\bar{A}$ and $\bar{B}$.*

Proof Straightforwardly.

Theorem 2 *The interval complex neutrosophic set $\bar{A} \cap \bar{B}$ is the largest one contained in both $\bar{A}$ and $\bar{B}$.*

Proof Straightforwardly.

Theorem 3 *Let $\overline{P}$ be the power set of all interval complex neutrosophic set. Then, $(\overline{P}, \cup, \cap)$ forms a distributive lattice.*

Proof Straightforwardly.

Theorem 4 *Let $\bar{A}$ and $\bar{B}$ be two interval complex neutrosophic sets defined on X. Then, $\bar{A} \subseteq \bar{B}$ if and only if $\bar{B}^c \subseteq \bar{A}^c$.*

Proof Straightforwardly.

3.3 Operational Rules of Interval Complex Neutrosophic Sets

Let $\bar{A} = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])$ and $\overline{B} = ([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])$ be two interval complex neutrosophic sets over X which are defined by $[T_A^L, T_A^U] = [t_A^L(x), t_A^U(x)] \cdot e^{j\pi[\omega_A^L(x), \omega_A^U(x)]}$, $[I_A^L, I_A^U] = [i_A^L(x), i_A^U(x)] \cdot e^{j\pi[\psi_A^L(x), \psi_A^U(x)]}$, $[F_A^L, F_A^U] = [f_A^L(x), f_A^U(x)] \cdot e^{j\pi[\phi_A^L(x), \phi_A^U(x)]}$ and $[T_B^L, T_B^U] = [t_B^L(x), t_B^U(x)] \cdot e^{j\pi[\omega_B^L(x), \omega_B^U(x)]}$, $[I_B^L, I_B^U] = [i_B^L(x), i_B^U(x)] \cdot e^{j\pi[\psi_B^L(x), \psi_B^U(x)]}$, $[F_B^L, F_B^U] = [f_B^L(x), f_B^U(x)] \cdot e^{j\pi[\phi_B^L(x), \phi_B^U(x)]}$, respectively. Then, the operational rules of ICNS are defined as follows:

(a) The *product* of $\bar{A}$ and $\bar{B}$, denoted as $\bar{A} \times \bar{B}$, is:

$$T_{\bar{A}*\bar{B}}(x) = \left[t^L_{\bar{A}}(x)t^L_{\bar{B}}(x), t^U_{\bar{A}}(x)t^U_{\bar{B}}(x)\right] \cdot e^{j\pi[\omega^L_{\bar{A}*\bar{B}}(x), \omega^R_{\bar{A}*\bar{B}}(x)]},$$

$$I_{\bar{A}*\bar{B}}(x) = \left[i^L_{\bar{A}}(x) + i^L_{\bar{B}}(x) - i^L_{\bar{A}}(x)i^L_{\bar{B}}(x), i^R_{\bar{A}}(x) + i^R_{\bar{B}}(x) - i^R_{\bar{A}}(x)i^R_{\bar{B}}(x)\right] \cdot e^{j\pi[\psi^L_{\bar{A}*\bar{B}}(x), \psi^R_{\bar{A}*\bar{B}}(x)]},$$

$F_{\bar{A}\times\bar{B}}(x) = \left[f^L_{\bar{A}}(x) + f^L_{\bar{B}}(x) - f^L_{\bar{A}}(x)f^L_{\bar{B}}(x), \quad f^R_{\bar{A}}(x) + f^R_{\bar{B}}(x) - f^R_{\bar{A}}(x)f^R_{\bar{B}}(x)\right] \cdot e^{j\pi[\phi^L_{\bar{A}\times\bar{B}}(x), \phi^R_{\bar{A}\times\bar{B}}(x)]}$ The product of phase terms is defined below:

$$\omega^L_{\bar{A}\times\bar{B}}(x) = \omega^L_{\bar{A}}(x)\omega^L_{\bar{B}}(x), \ \omega^U_{\bar{A}\times\bar{B}}(x) = \omega^U_{\bar{A}}(x)\omega^U_{\bar{B}}(x)$$
$$\psi^L_{\bar{A}\times\bar{B}}(x) = \psi^L_{\bar{A}}(x)\psi^L_{\bar{B}}(x), \ \psi^U_{\bar{A}\times\bar{B}}(x) = \psi^U_{\bar{A}}(x)\psi^U_{\bar{B}}(x)$$
$$\phi^L_{\bar{A}\times\bar{B}}(x) = \phi^L_{\bar{A}}(x)\phi^L_{\bar{B}}(x), \ \phi^U_{\bar{A}\times\bar{B}}(x) = \phi^U_{\bar{A}}(x)\phi^U_{\bar{B}}(x).$$

(b) The *addition* of $\bar{A}$ and $\bar{B}$, denoted as $\bar{A}+\bar{B}$, is defined as:

$$T_{\bar{A}+\bar{B}}(x) = \left[t^L_{\bar{A}}(x) + t^L_{\bar{B}}(x) - t^L_{\bar{A}}(x)t^L_{\bar{B}}(x), t^U_{\bar{A}}(x) + t^U_{\bar{B}}(x) - t^U_{\bar{A}}(x)t^U_{\bar{B}}(x)\right] \cdot e^{j\pi[\omega^L_{\bar{A}+\bar{B}}(x), \omega^L_{\bar{A}+\bar{B}}(x)]},$$

$$I_{\bar{A}+\bar{B}}(x) = \left[i^L_{\bar{A}}(x)i^L_{\bar{B}}(x), i^U_{\bar{A}}(x)i^U_{\bar{B}}(x)\right] \cdot e^{j\pi[\psi^L_{\bar{A}+\bar{B}}(x), \psi^R_{\bar{A}+\bar{B}}(x)]},$$

$$F_{\bar{A}+\bar{B}}(x) = \left[f^L_{\bar{A}}(x)f^L_{\bar{B}}(x), f^R_{\bar{A}}(x)f^R_{\bar{B}}(x)\right] \cdot e^{j\pi[\phi^L_{\bar{A}+\bar{B}}(x), \phi^R_{\bar{A}+\bar{B}}(x)]}$$

The addition of phase terms is defined below:

$$\omega^L_{\bar{A}+\bar{B}}(x) = \omega^L_{\bar{A}}(x) + \omega^L_{\bar{B}}(x), \ \omega^U_{\bar{A}+\bar{B}}(x) = \omega^U_{\bar{A}}(x) + \omega^U_{\bar{B}}(x)$$
$$\psi^L_{\bar{A}+\bar{B}}(x) = \psi^L_{\bar{A}}(x) + \psi^L_{\bar{B}}(x), \ \psi^U_{\bar{A}+\bar{B}}(x) = \psi^U_{\bar{A}}(x) + \psi^U_{\bar{B}}(x)$$
$$\phi^L_{\bar{A}+\bar{B}}(x) = \phi^L_{\bar{A}}(x) + \phi^L_{\bar{B}}(x), \ \phi^U_{\bar{A}+\bar{B}}(x) = \phi^U_{\bar{A}}(x) + \phi^U_{\bar{B}}(x)$$

(c) The *scalar multiplication* of $\bar{A}$ is an interval complex neutrosophic set denoted as $\overline{C} = k\bar{A}$ and defined as:

$$T_{\overline{C}}(x) = \left[1 - (1 - t^L_{\bar{A}}(x))^k, 1 - (1 - t^R_{\bar{A}}(x))^k\right] \cdot e^{j\pi[\omega^L_{\overline{C}}(x), \omega^R_{\overline{C}}(x)]},$$
$$I_{\overline{C}}(x) = [(i^L_{\bar{A}}(x))^k, (i^R_{\bar{A}}(x))^k] \cdot e^{j\pi[\psi^L_{\overline{C}}(x), \psi^R_{\overline{C}}(x)]},$$
$$F_{\overline{C}}(x) = [(f^L_{\bar{A}}(x))^k, (i^R_{\bar{A}}(x))^k] \cdot e^{j\pi[\phi^L_{\overline{C}}(x), \phi^R_{\overline{C}}(x)]}$$

The scalar of phase terms is defined below:

$$\omega^L_{\overline{C}}(x) = \omega^L_{\bar{A}}(x) \cdot k; \quad \omega^R_{\overline{C}}(x) = \omega^R_{\bar{A}}(x) \cdot k,$$
$$\psi^L_{\overline{C}}(x) = \psi^L_{\bar{A}}(x) \cdot k; \quad \psi^R_{\overline{C}}(x) = \psi^R_{\bar{A}}(x) \cdot k,$$
$$\phi^L_{\overline{C}}(x) = \phi^L_{\bar{A}}(x) \cdot k; \quad \phi^R_{\overline{C}}(x) = \phi^R_{\bar{A}}(x) \cdot k$$

4 A Multi-criteria Group Decision-Making Model in ICNS

Definition 13 Let us assume that a committee of h decision-makers $(D_q, q = 1, \ldots, h)$ is responsible for evaluating o alternatives $(A_o, o = 1, \ldots, m)$ under p selection criteria $(C_p, p = 1, \ldots, n)$, where the suitability ratings of alternatives under each criterion, as well as the weights of all criteria, are assessed in IVCNS. The steps of the proposed MCGDM method are as follows:

4.1 Aggregate Ratings of Alternatives Versus Criteria

Let $x_{opq} = ([T^L_{opq}, T^U_{opq}], [I^L_{opq}, I^U_{opq}], [F^L_{opq}, F^U_{opq}])$ be the suitability rating assigned to alternative A_o by decision-maker D_q for criterion C_p, where $[T^L_{opq}, T^U_{opq}] = [t^L_{opq}, t^U_{opq}] \cdot e^{j\pi[\omega^L(x), \omega^U(x)]}$, $[I^L_{opq}, I^U_{opq}] = [i^L_{opq}, i^U_{opq}] \cdot e^{j\pi[\psi^L(x), \psi^U(x)]}$, $[F^L_{opq}, F^U_{opq}] = [f^L_{opq}, f^U_{opq}] \cdot e^{j\pi[\phi^L(x), \phi^U(x)]}$, $o = 1, \ldots, m$; $p = 1, \ldots, n$; $q = 1, \ldots, h$. Using the operational rules of the IVCNS, the averaged suitability rating $x_{op} = ([T^L_{op}, T^U_{op}], [I^L_{op}, I^U_{op}], [F^L_{op}, F^U_{op}])$ can be evaluated as:

$$x_{op} = \frac{1}{h} \otimes (x_{op1} \oplus x_{op2} \oplus \cdots \oplus x_{opq} \oplus \cdots \oplus x_{oph}), \tag{3}$$

where
$$T_{op} = \left[\left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} t^L_{opq}, 1\right), \left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} t^R_{opq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} w^L_q(x), \frac{1}{h}\sum_{q=1}^{h} w^U_q(x)\right]}$$

$$I_{op} = \left[\left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} i^L_{opq}, 1\right), \left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} i^R_{opq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} \psi^L_q(x), \frac{1}{h}\sum_{q=1}^{h} \psi^U_q(x)\right]}$$

$$F_{op} = \left[\left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} f^L_{opq}, 1\right), \left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} f^R_{opq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} \phi^L_q(x), \frac{1}{h}\sum_{q=1}^{h} \phi^U_q(x)\right]}$$

4.2 Aggregate the Importance Weights

Let $w_{pq} = ([T^L_{pq}, T^U_{pq}], [I^L_{pq}, I^U_{pq}], [F^L_{pq}, F^U_{pq}])$ be the weight assigned by decision-maker D_q to criterion C_p, where $[T^L_{pq}, T^U_{pq}] = [t^L_{pq}, t^U_{pq}] \cdot e^{j\pi[\omega^L(x), \omega^U(x)]}$, $[I^L_{pq}, I^U_{pq}] = [i^L_{pq}, i^U_{pq}] \cdot e^{j\pi[\psi^L(x), \psi^U(x)]}$, $[F^L_{pq}, F^U_{pq}] = [f^L_{pq}, f^U_{pq}] \cdot e^{j\pi[\phi^L(x), \phi^U(x)]}$, $F^U_{pq} = f^U_{pq} \cdot e^{j\pi\phi(x)}$, $p = 1, \ldots, n$; $q = 1, \ldots, h$. Using the operational rules of the IVCNS, the average weight $w_p = ([T^L_p, T^U_p], [I^L_p, I^U_p], [F^L_p, F^U_p])$ can be evaluated as:

$$w_p = (\frac{1}{h}) \otimes (w_{p1} \oplus w_{p2} \oplus \cdots \oplus w_{ph}), \tag{4}$$

where
$$T_p = \left[\left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} t^L_{pq}, 1\right), \left(\wedge \ \frac{1}{h}\sum_{q=1}^{h} t^R_{pq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} w^L_q(x), \frac{1}{h}\sum_{q=1}^{h} w^U_q(x)\right]}$$

$$I_p = \left[\wedge\ \frac{1}{h}\sum_{q=1}^{h} i^L_{pq}, 1\right), \wedge\ \frac{1}{h}\sum_{q=1}^{h} i^R_{pq}, 1\Bigg)\Bigg] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h}\psi^L_q(x), \frac{1}{h}\sum_{q=1}^{h}\psi^U_q(x)\right]}$$

$$F_p = \left[\wedge\ \frac{1}{h}\sum_{q=1}^{h} f^L_{pq}, 1\right), \wedge\ \frac{1}{h}\sum_{q=1}^{h} f^R_{pq}, 1\Bigg)\Bigg] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h}\phi^L_q(x), \frac{1}{h}\sum_{q=1}^{h}\phi^U_q(x)\right]}$$

4.3 Aggregate the Weighted Ratings of Alternatives Versus Criteria

The weighted ratings of alternatives can be developed via the operations of interval complex neutrosophic set as follows:

$$V_o = \frac{1}{p}\sum_{p=1}^{h} x_{op} \times w_p, \quad o = 1, \ldots, m; \quad p = 1, \ldots, h. \quad (5)$$

4.4 Ranking the Alternatives

In this section, the modified score function, the accuracy function and the certainty function of an ICNS, i.e., $V_o = ([T^L_o, T^U_o], [I^L_o, I^U_o], [F^L_o, F^U_o])$, $o = 1, \ldots, m$, adopted from Ye [20], are developed for ranking alternatives in decision-making problems, where

$[T^L_o, T^U_o] = [t^L_o, t^U_o]e^{j\pi[\omega^L(x),\omega^U(x)]}$, $[I^L_o, I^U_o] = [i^L_o, i^U_o]e^{j\pi[\psi^L(x),\psi^U(x)]}$, $[F^L_o, F^U_o] = [f^L_o, f^U_o]e^{j\pi[\phi^L(x),\phi^U(x)]\phi^U(x)]}$

The values of these functions for amplitude terms are defined as follows:

$$e^a_{V_o} = \frac{1}{6}(4 + t^L_o - i^L_o - f^L_o + t^U_o - i^U_o - f^U_o),\ h^a_{V_o} = \frac{1}{2}(t^L_o - f^L_o + t^U_o - f^U_o),\ \text{and}\ c^a_{V_o} = \frac{1}{2}(t^L_o + t^U_o)$$

The values of these functions for phase terms are defined below:

$$e^p_{V_o} = \pi[\omega^L(x) - \psi^L(x) - \phi^L(x) + \omega^R(x) - \psi^R(x) - \phi^R(x)],$$
$$h^p_{V_o} = \pi[\omega^L(x) - \phi^L(x) + \omega^R(x) - \phi^R(x)],\ \text{and}\ c^p_{V_o} = \pi[\omega^L(x) + \omega^R(x)]$$

Let V_1 and V_2 be any two ICNSs. Then, the ranking method can be defined as follows:

- If $e^a_{V_1} > e^a_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}$ and $e^p_{V_1} > e^p_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}$ and $h^a_{V_1} > h^a_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}$ and $h^p_{V_1} > h^p_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}, h^p_{V_1} = h^p_{V_2}$ and $c^a_{V_1} > c^a_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}, h^p_{V_1} = h^p_{V_2}, c^a_{V_1} = c^a_{V_2}$ and $c^p_{V_1} > c^p_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}, h^p_{V_1} = h^p_{V_2}, c^a_{V_1} = c^a_{V_2}$ and $c^p_{V_1} = c^p_{V_2}$, then $V_1 = V_2$

5 Application of the Proposed MCGDM Approach

This section applies the proposed MCGDM for green supplier selection in the case study of Thuan Yen JSC, which is a small-size trading service and transportation company. The managers of this company would like to effectively manage the suppliers, due to an increasing number of them. Data were collected by conducting semi-structured interviews with managers and department heads. Three managers (decision-makers), i.e., *D*1–*D*3, were requested to separately proceed to their own evaluation for the importance weights of selection criteria and the ratings of suppliers. According to the survey and the discussions with the managers and department heads, five criteria, namely Price/cost (*C*1), Quality (*C*2), Delivery (*C*3), Relationship Closeness (*C*4) and Environmental Management Systems (*C*5), were selected to evaluate the green suppliers. The entire green supplier selection procedure was characterized by the following steps:

5.1 Aggregation of the Ratings of Suppliers Versus the Criteria

Three managers determined the suitability ratings of three potential suppliers versus the criteria using the linguistic rating set $S = \{$VL, L, F, G, VG$\}$ where VL = Very Low = ([0.1, 0.2]$e^{j\pi[0.7,0.8]}$, [0.7, 0.8]$e^{j\pi[0.9,1.0]}$, [0.6, 0.7]$e^{j\pi[1.0,1.1]}$), L = Low = ([0.3, 0.4]$e^{j\pi[0.8,0.9]}$, [0.6, 0.7]$e^{j\pi[1.0,1.1]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$), F = Fair = ([0.4, 0.5]$e^{j\pi[0.8,0.9]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$,[0.4, 0.5]$e^{j\pi[0.8,0.9]}$), G = Good = ([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$, [0.3, 0.4]$e^{j\pi[0.7,0.8]}$), and VG = Very Good = ([0.7, 0.8] $e^{j\pi[1.1,1.2]}$, [0.2, 0.3]$e^{j\pi[0.8,0.9]}$, [0.1, 0.2]$e^{j\pi[0.6,0.7]}$), to evaluate the suitability of the suppliers under each criteria. Table 1 gives the aggregated ratings of three suppliers (A_1, A_2, A_3) versus five criteria (C_1,…, C_5) from three decision-makers (D_1, D_2, D_3) using Eq. (3).

5.2 Aggregation of the Importance Weights

After determining the green suppliers criteria, the three company managers are asked to determine the level of importance of each criterion using a linguistic weighting set $Q = \{$UI, OI, I, VI, AI$\}$ where UI = Unimportant = ([0.2, 0.3]$e^{j\pi[0.7,0.8]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$, [0.5, 0.6]$e^{j\pi[1.1,1.2]}$), OI = Ordinary Important = ([0.3, 0.4]$e^{j\pi[0.8,0.9]}$, [0.5, 0.6]$e^{j\pi[1.0,1.1]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$), I = Important = ([0.5, 0.6]$e^{j\pi[0.9,1.0]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$, [0.3,

Table 1 Aggregated ratings of suppliers versus the criteria

Criteria	Suppliers	Decision-makers			Aggregated ratings
		D_1	D_2	D_3	
C_1	A_1	G	F	G	$([0.542, 0.644]e^{j\pi[0.867,0.967]}, [0.431, 0.531]e^{j\pi[0.9,1.0]}], [0.33, 0.431]e^{j\pi[0.733,0.833]})$
	A_2	F	F	G	$([0.476, 0.578]e^{j\pi[0.833,0.933]}, [0.464, 0.565]e^{j\pi[0.9,1.0]}, [0.363, 0.464]e^{j\pi[0.767,0.867]})$
	A_3	VG	G	VG	$([0.67, 0.771]e^{j\pi[1.033,1.133]}, [0.252, 0.356]e^{j\pi[0.833,0.933]}, [0.144, 0.252]e^{j\pi[0.633,0.733]})$
C_2	A_1	F	F	F	$([0.4, 0.5]e^{j\pi[0.8,0.9]}, [0.5, 0.6]e^{j\pi[0.9,1.0]}, [0.4, 0.5]e^{j\pi[0.8,0.9]})$
	A_2	VG	G	G	$([0.637, 0.738]e^{j\pi[0.967,1.067]}, [0.317, 0.422]e^{j\pi[0.867,0.967]}, [0.208, 0.317]e^{j\pi[0.667,0.767]})$
	A_3	F	G	G	$([0.542, 0.644]e^{j\pi[0.867,0.967]}, [0.431, 0.531]e^{j\pi[0.9,1.0]}, [0.33, 0.431]e^{j\pi[0.733,0.833]})$
C_3	A_1	L	F	L	$([0.335, 0.435]e^{j\pi[0.8,0.9]}, [0.565, 0.665]e^{j\pi[0.967,1.067]}, [0.464, 0.565]e^{j\pi[0.867,0.967]})$
	A_2	G	G	G	$([0.6, 0.7]e^{j\pi[0.9,1.0]}, [0.4, 0.5]e^{j\pi[0.9,1.0]}, [0.3, 0.4]e^{j\pi[0.7,0.8]})$
	A_3	F	G	F	$([0.476, 0.578]e^{j\pi[0.833,0.933]}, [0.464, 0.565]e^{j\pi[0.9,1.0]}, [0.363, 0.464]e^{j\pi[0.767,0.867]})$
C_4	A_1	G	F	G	$([0.542, 0.644]e^{j\pi[0.867,0.967]}, [0.431, 0.531]e^{j\pi[0.9,1.0]}, [0.33, 0.431]e^{j\pi[0.733,0.833]})$
	A_2	F	F	L	$([0.368, 0.469]e^{j\pi[0.8,0.9]}, [0.531, 0.632]e^{j\pi[0.933,1.033]}, [0.431, 0.531]e^{j\pi[0.833,0.933]})$
	A_3	G	VG	G	$([0.637, 0.738]e^{j\pi[0.967,1.067]}, [0.317, 0.422]e^{j\pi[0.867,0.967]}, [0.208, 0.317]e^{j\pi[0.667,0.767]})$
C_5	A_1	L	F	L	$([0.335, 0.435]e^{j\pi[0.8,0.9]}, [0.565, 0.665]e^{j\pi[0.967,1.067]}, [0.464, 0.565]e^{j\pi[0.867,0.967]})$
	A_2	G	G	VG	$([0.637, 0.738]e^{j\pi[0.967,1.067]}, [0.317, 0.422]e^{j\pi[0.867,0.967]}, [0.208, 0.317]e^{j\pi[0.667,0.767]})$
	A_3	G	F	F	$([0.476, 0.578]e^{j\pi[0.833,0.933]}, [0.464, 0.565]e^{j\pi[0.9,1.0]}, [0.363, 0.464]e^{j\pi[0.767,0.867]})$

Table 2 The importance and aggregated weights of the criteria

Criteria	Decision-makers			Aggregated weights
	D_1	D_2	D_3	
C_1	VI	I	I	$([0.578, 0.683]e^{j\pi[0.9,1.0]}, [0.363, 0.464]e^{j\pi[0.9,1.0]}, [0.262, 0.363]e^{j\pi[0.767,0.867]})$
C_2	AI	VI	VI	$([0.738, 0.841]e^{j\pi[0.933,1.033]}, [0.262, 0.363]e^{j\pi[0.867,0.967]}, [0.159, 0.262]e^{j\pi[0.667,0.767})$
C_3	VI	VI	I	$([0.644, 0.748]e^{j\pi[0.9,1.0]}, [0.33, 0.431]e^{j\pi[0.9,1.0]}, [0.229, 0.33]e^{j\pi[0.733,0.833]})$
C_4	I	I	I	$([0.5, 0.6]e^{j\pi[0.9,1.0]}], [0.4, 0.5]e^{j\pi[0.9,1.0]}, [0.3, 0.4]e^{j\pi[0.8,0.9]})$
C_5	I	OI	OI	$([0.374, 0.476]e^{j\pi[0.833,0.933]}, [0.391, 0.565]e^{j\pi[0.967,1.067]}, [0.363, 0.464]e^{j\pi[0.867,0.967]})$

Table 3 The final fuzzy evaluation values of each supplier

Suppliers	Aggregated weights
A_1	$([0.247, 0.361]e^{j\pi[0.739,0.921]}, [0.673, 0.784]e^{j\pi[0.841,1.034]}, [0.552, 0.679]e^{j\pi[0.614,0.78]})$
A_2	$([0.319, 0.449]e^{j\pi[0.798,0.986]}, [0.607, 0.733]e^{j\pi[0.81,1.0]}, [0.475, 0.617]e^{j\pi[0.558,0.717]})$
A_3	$([0.322, 0.451]e^{j\pi[0.811,1.001]}, [0.6, 0.724]e^{j\pi[0.798,0.987]}, [0.465, 0.606]e^{j\pi[0.547,0.705]})$

$0.4]e^{j\pi[0.8,0.9]})$, VI = Very Important = $([0.7, 0.8]e^{j\pi[0.9,1.0]}, [0.3, 0.4]e^{j\pi[0.9,1.0]}, [0.2, 0.3]e^{j\pi[0.7,0.8]})$, and AI = Absolutely Important = $([0.8, 0.9]e^{j\pi[1.0,1.1]}, [0.2, 0.3]e^{j\pi[0.8,0.9]}, [0.1, 0.2]e^{j\pi[0.6,0.7]})$.

Table 2 displays the importance weights of the five criteria from the three decision-makers. The aggregated weights of criteria obtained by Eq. (4) are shown in the last column of Table 2.

5.3 Compute the Total Value of Each Alternative

Table 3 presents the final fuzzy evaluation values of each supplier using Eq. (5).

Table 4 Modified score function of each alternative

Suppliers	Modified score function		Accuracy function		Certainty function		Ranking
	Amplitude term	Phase term	Amplitude term	Phase term	Amplitude term	Phase term	
A_1	0.320	-1.61π	-0.311	0.265π	0.304	1.659π	3
A_2	0.389	-1.301π	-0.162	0.508π	0.384	1.784π	2
A_3	0.396	-1.225π	-0.149	0.56π	0.387	1.811π	1

Table 5 The importance and aggregated weights of the criteria

Criteria	Decision-makers				Aggregated weights
	D_1	D_2	D_3	D_4	
C_1	AI	AI	AI	VI	([0.269, 0.361]$e^{j\pi[0.194,0.214]}$, [0.115, 0.161]$e^{j\pi[0.156,0.175]}$, [0.066, 0.115]$e^{j\pi[0.117,0.136]}$)
C_2	VI	I	I	VI	([0.157, 0.204]$e^{j\pi[0.175,0.194]}$, [0.191, 0.239]$e^{j\pi[0.175,0.194]}$, [0.144, 0.191]$e^{j\pi[0.148,0.168]}$)
C_3	AI	AI	VI	AI	([0.252, 0.336]$e^{j\pi[0.189,0.208]}$, [0.129, 0.176]$e^{j\pi[0.161,0.18]}$, [0.08, 0.129]$e^{j\pi[0.122,0.141]}$)
C_4	VI	VI	I	OI	([0.186, 0.241]$e^{j\pi[0.175,0.194]}$], [0.176, 0.223]$e^{j\pi[0.175,0.194]}$, [0.129, 0.176]$e^{j\pi[0.141,0.161]}$)
C_5	I	I	AI	AI	([0.168, 0.224]$e^{j\pi[0.18,0.2]}$, [0.17, 0.219]$e^{j\pi[0.175,0.194]}$, [0.12, 0.17]$e^{j\pi[0.145,0.164]}$)

Table 6 Aggregated ratings of suppliers versus the criteria

Criteria	Suppliers	Decision-makers				Aggregated ratings
		D_1	D_2	D_3	D_4	
C_1	A_1	G	F	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
	A_2	G	G	F	F	([0.510, 0.613]$e^{j\pi[0.85,0.95]}$, [0.01, 0.023]$e^{j\pi[0.9,1.0]}$, [0.436, 0.532]$e^{j\pi[0.75,0.85]}$)
	A_3	L	G	F	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_4	G	F	G	F	([0.510, 0.613]$e^{j\pi[0.85,0.95]}$, [0.01, 0.023]$e^{j\pi[0.9,1.0]}$, [0.436, 0.532]$e^{j\pi[0.75,0.85]}$)
	A_5	F	G	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_2	A_1	G	G	F	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.025]}$], [0.495, 0.589]$e^{j\pi[0.725,0.825]}$)
	A_2	G	F	L	F	([0.437, 0.539]$e^{j\pi[0.825,0.925]}$, [0.015, 0.033]$e^{j\pi[0.925,1.025]}$], [0.495, 0.589]$e^{j\pi[0.8,0.9]}$)
	A_3	L	G	G	G	([0.54, 0.643]$e^{j\pi[0.875,0.975]}$, [0.01, 0.023]$e^{j\pi[0.925,1.025]}$], [0.461, 0.557]$e^{j\pi[0.75,0.85]}$)
	A_4	F	L	G	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$], [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_5	G	G	F	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_3	A_1	F	F	L	L	([0.352, 0.452]$e^{j\pi[0.8,0.9]}$, [0.023, 0.047]$e^{j\pi[0.95,1.05]}$], [0.532, 0.622]$e^{j\pi[0.85,0.95]}$)
	A_2	G	G	G	G	([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.006, 0.016]$e^{j\pi[0.9,1.0]}$, [0.405, 0.503]$e^{j\pi[0.7,0.8]}$)
	A_3	L	G	F	F	([0.437, 0.539]$e^{j\pi[0.825,0.925]}$, [0.015, 0.033]$e^{j\pi[0.925,1.025]}$], [0.495, 0.589]$e^{j\pi[0.8,0.9]}$)
	A_4	G	F	G	F	([0.51, 0.613]$e^{j\pi[0.85,0.95]}$, [0.01, 0.023]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.75,0.85]}$)
	A_5	F	G	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_4	A_1	G	L	F	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_2	G	G	L	G	([0.54, 0.643]$e^{j\pi[0.875,0.975]}$, [0.01, 0.023]$e^{j\pi[0.925,1.025]}$, [0.461, 0.557]$e^{j\pi[0.75,0.85]}$)
	A_3	F	F	F	F	([0.4, 0.5]$e^{j\pi[0.8,0.9]}$, [0.016, 0.034]$e^{j\pi[0.9,1.0]}$, [0.503, 0.595]$e^{j\pi[0.8,0.9]}$)
	A_4	L	L	F	G	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_5	F	G	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_5	A_1	L	F	G	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_2	G	L	G	G	([0.54, 0.643]$e^{j\pi[0.875,0.975]}$, [0.01, 0.023]$e^{j\pi[0.925,1.025]}$], [0.461, 0.557]$e^{j\pi[0.75,0.85]}$)
	A_3	G	G	L	F	([0.491, 0.595]$e^{j\pi[0.85,0.95]}$, [0.012, 0.027]$e^{j\pi[0.925,1.025]}$, [0.461, 0.557]$e^{j\pi[0.775,0.875]}$)
	A_4	L	L	F	G	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_5	G	G	G	G	([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.006, 0.016]$e^{j\pi[0.9,1.0]}$, [0.405, 0.503]$e^{j\pi[0.7,0.8]}$)

Table 7 The final fuzzy evaluation values of each supplier

Suppliers	Aggregated weights
A_1	$([0.095, 0.154]e^{j\pi[0.153,0.19]}, [0.166, 0.228]e^{j\pi[0.156,0.192]}, [0.534, 0.639]e^{j\pi[0.106,0.137]})$
A_2	$([0.11, 0.174]e^{j\pi[0.158,0.195]}, [0.162, 0.22]e^{j\pi[0.153,0.189]}, [0.508, 0.616]e^{j\pi[0.101,0.131]})$
A_3	$([0.093, 0.151]e^{j\pi[0.153,0.189]}, [0.166, 0.227]e^{j\pi[0.155,0.191]}, [0.539, 0.643]e^{j\pi[0.106,0.137]})$
A_4	$([0.096, 0.156]e^{j\pi[0.153,0.189]}, [0.165, 0.227]e^{j\pi[0.156,0.192]}, [0.547, 0.651]e^{j\pi[0.107,0.138]})$
A_5	$([0.117, 0.183]e^{j\pi[0.161,0.198]}, [0.16, 0.217]e^{j\pi[0.15,0.187]}, [0.491, 0.6]e^{j\pi[0.097,0.126]})$

Table 8 Modified score function of each alternative

Suppliers	Modified score function		Accuracy function		Certainty function		Ranking
	Amplitude term	Phase term	Amplitude term	Phase term	Amplitude term	Phase term	
A_1	0.447	-0.248π	-0.461	0.100π	0.125	0.344π	3
A_2	0.463	-0.222π	-0.420	0.121π	0.142	0.353π	2
A_3	0.445	-0.247π	-0.469	0.099π	0.122	0.341π	4
A_4	0.444	-0.252π	-0.473	0.096π	0.126	0.342π	5
A_5	0.472	-0.201π	-0.395	0.136π	0.150	0.359π	1

5.4 Ranking the Alternatives

Using the modified ranking method, the final ranking value of each alternative is defined as in Table 4. According to this table, the ranking order of the three suppliers is $A_3 \succ A_2 \succ A_1$.

6 Comparison of the Proposed Method with Another MCGDM Method

6.1 Example 1

This section compares the proposed approach with another MCGDM approach to demonstrate its advantages and applicability by reconsidering the example investigated by Sahin and Yigider [14]. In this example, four decision-makers (D_1,…,D_4) have been appointed to evaluate five suppliers (S_1,…, S_5) based on five performance criteria including delivery (C_1), quality (C_2), flexibility (C_3), service (C_4) and price (C_5).

The information of weights provided to the five criteria by the four decision-makers are presented in Table 5. The aggregated weights of criteria obtained by Eq. (4) are shown in the last column of Table 5.

Table 6 demonstrates the averaged ratings of suppliers versus the criteria based on the data presented in Tables 4, 5, 6, 7 and 8 in the work of Sahin and Yigider [14] and the proposed method.

Table 7 presents the final fuzzy evaluation values of each supplier using Eq. (5).

Using the proposed modified ranking method, the final ranking value of each alternative is defined as in Table 8. According to this table, the ranking order of the five suppliers is $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$. Obviously, the results in Sahin and Yigider [14] conflict with ours in this paper. The reason for the difference is in the proposed method: IVCNS was used to measure the ratings of the suppliers and the importance weights of criteria.

6.2 Example 2

This section uses a numerical example to compare the proposed approach with Ye's method [21] as follows. Consider two ICNS, i.e., $A_1 = ([0.5, 0.6]e^{j\pi[0.9,1.0]}, [0.4, 0.5]e^{j\pi[0.7,0.8]}], [0.3, 0.4]e^{j\pi[0.5,0.6]}$ and $A_2 = ([0.5, 0.6]e^{j\pi[0.8,0.9]}, [0.4, 0.5]e^{j\pi[0.5,0.6]}], [0.3, 0.4]e^{j\pi[0.7,0.8]}$. It is clear that the truth membership, indeterminacy membership and false-membership of A_1 and A_2 have the same amplitude values. Using the Ye's method [21], the similarity measures between ICNS A_1 and A_2 are: $S_1(A_1, A_2) = 1$ and $S_2(A_1, A_2) = 1$. Therefore, the ranking order of A_1 and A_2 is $A_1 = A_2$. This is not reasonable.

However using the proposed ranking method, the modified score, the accuracy and certainty function of A_1 and A_2 are: $e^a_{V_o}(A_1) = e^a_{V_o}(A_2) = 0.583, h^a_{V_o}(A_1) = h^a_{V_o}(A_2) = 0.2$, $c^a_{V_o}(A_1) = c^a_{V_o}(A_2) = 0.55$ and $e^p_{V_o}(A_1) = -0.7\pi, e^p_{V_o}(A_2) = -0.9\pi$; $h^p_{V_o}(A_1) = 0.8\pi, h^p_{V_o}(A_2) = 0.2\pi$ and $c^p_{V_o}(A_1) = 1.9\pi, c^p_{V_o}(A_2) = 1.7\pi$. Accordingly, the ranking order of ICNS A_1 and A_2 is $A_1 > A_2$. Obviously, the proposed ranking method can also rank ICNS other than INS.

7 Conclusion

It is believed that uncertain, ambiguous, indeterminate, inconsistent and incomplete periodic/redundant information can be dealt better with intervals instead of single values. This paper aimed to propose the interval complex neutrosophic set, which is more adaptable and flexible to real-life problems than other types of fuzzy sets. The definitions of interval complex neutrosophic set, accompanied by the set operations, were defined. The relationship of interval complex neutrosophic set with other existing approaches was presented.

A new decision-making procedure in the interval complex neutrosophic set has been presented and applied to a decision-making problem for the green supplier selection. Comparison between the proposed method and the related methods has been made to demonstrate the advantages and applicability. The results are significant to enrich the knowledge of neutrosophic set in the decision-making applications.

Future work plans to use the decision-making procedure to more complex applications, and to advance the interval complex neutrosophic logic system for forecasting problems.

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Interval Valued Neutrosophic Soft Graphs

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ABSTRACT

In this article, we combine the interval valued neutrosophic soft set and graph theory. We introduce the notions of interval valued neutrosophic soft graphs, strong interval valued neutrosophic graphs, complete interval valued neutrosophic graphs, and investigate some of their related properties. We study some operations on interval valued neutrosophic soft graphs. We also give an application of interval valued neutrosophic soft graphs into a decision making problem. We hold forth an algorithm to solve decision making problems by using interval valued neutrosophic soft graphs.

1. INTRODUCTION

The neutrosophic set (NSs), proposed by (Smarandache, 2006, 2011), is a powerful mathematical tool for dealing with incomplete, indeterminate and inconsistent information in real world. Itis a generalization of the theory of fuzzy sets (Zadeh, 1965), intuitionistic fuzzy sets (Atanassov, 1986,1999) and interval-valued intuitionistic fuzzy sets (Atanassov, 1989). The neutrosophic sets are characterized by a truth-membership function (t), an indeterminacy-membership function (i) and a falsity-membership function (f) independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. In order to conveniently employ NS in real life applications, (Wang et al., 2010) introduced the concept of single-valued neutrosophic set (SVNS), a subclass of the neutrosophic sets. The same authors (Wang, Zhang, & Sunderraman, 2005) introduced the concept of interval valued neutrosophic set (IVNS), which is more precise and flexible than single valued neutrosophic set. The IVNS is a generalization of single valued neutrosophic set, in which three membership functions are independent and their value belong to the unit interval [0, 1]. Some more work on single valued neutrosophic set, interval valued neutrosophic set and their applications may be found in (Aydoğdu, 2015; Ansari et a.l, 2012; Ansari et al. 2013; Ansari et al. 2013a; Zhang et al., 2015; Zhang et al., 2015b; Deli et al. ,2015; Ye, 2014, 2014a; Şahin, 2015; Aggarwal et al.,2010; Broumi and Smarandache, 2014; Karaaslan and Davvaz, 2018).

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving a combinatorial problem in different areas, such as geometry, algebra, number theory, topology, optimization and computer science. Most important thing to be noted is that, when we have uncertainty regarding either the set of vertices or edges, or both, the model becomes a fuzzy graph. The extension of fuzzy graph theory (Nagoor and Basheer, 2003; Nagoor & Latha,2012; Bhattacharya,1987) have been developed by several researchers. Intuitionistic fuzzy graphs

(Nagoor & Shajitha, 2010; Akram, 2012) considered the vertex sets and edge sets as intuitionistic fuzzy sets. Interval valued fuzzy graphs (Akram & Dudek, 2011; Akram, 2012a) considered the vertex sets and edge sets as interval valued fuzzy sets. Interval valued intuitionistic fuzzy graphs (Akram, 2014; Hai-Long et.,2016) considered the vertex sets and edge sets as interval valued intuitionistic fuzzy sets. Bipolar fuzzy graphs (Akram, 2011, 2013) considered the vertex sets and edge sets as bipolar fuzzy sets. M-polar fuzzy graphs (Akram, 2016) considered the vertex sets and edge sets as m-polar fuzzy sets. But, when the relations between nodes (or vertices) in problems are indeterminate, the fuzzy graphs and their extensions fail. For this purpose, (Smarandache, 2015,2015a,2015b; Vasantha and Smarandache,2013) defined four main categories of neutrosophic graphs. Two of them are based on literal indeterminacy (I), which are called I-edge neutrosophic graph and I-vertex neutrosophic graph; these concepts are studied deeply and gained popularity among the researchers due to their applications via real world problems (Devadoss et al., 2013, Jiang et al., 2010; Vasantha et al., 2015) The two others graphs arebased on (t, i, f) components and are called:(t, i, f)-edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph; these concepts are not developed at all.

Later on, (Broumi et al., 2016a) introduced a third neutrosophic graph model, and investigated some of its properties. This model allows the attachment of truth-membership (t), indeterminacy–membership (i) and falsity- membership degrees (f) both to vertices and edges. The third neutrosophic graph model is called single valued neutrosophic graph (SVNG for short). The single valued neutrosophic graph is the generalization of fuzzy graph and intuitionistic fuzzy graph. Also, the same authors (Broumi et al., 2016a, 2016e) introduced neighborhood degree of a vertex and closed neighborhood degree of a vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of a vertex in fuzzy graph and intuitionistic fuzzy graph. Also, (Broumi et al., 2016b) introduced the concept of interval valued neutrosophic graph as a generalization of fuzzy graph, intuitionistic fuzzy graph, interval valued fuzzy graph, interval valued intuitionistic fuzzy graph and single valued neutrosophic graph, and have discussed some of their properties with proofs and examples. In addition, (Broumi et al., 2016c) have introduced some operations, such as Cartesian product, composition, union and join on interval valued neutrosophic graphs, and investigate some their properties. On the other hand, (Broumi et al., 2016d) discussed a subclass of interval valued neutrosophic graph, called strong interval valued neutrosophic graph, and introduced some operations such as, Cartesian product, composition and join of two strong interval valued neutrosophic graph with proofs. Interval valued neutrosophic soft sets are the generalization of fuzzy soft sets (Maji, 2001), intuitionistic fuzzy soft sets (Maji, 2001a), interval valued intuitionistic fuzzy soft sets (Jiang, et al., 2010) and (Maji, 2013). (Thumbakara and George,2014) combined the concept of soft set theory with graph theory. (Irfan et al, 2016) proposed a method to represent a graph, which is based on adjacency of vertices and soft set theory and introduced some operations such as restricted intersection, restricted union, extended intersection and extended union for graphs. In addition, the authors defined a metric to find distances between graphs represented by soft sets. Later on, Mohinta (2015) extended the concept of soft graph to the case of fuzzy soft graph. Also, Akram et al. (2015) studied more properties on fuzzy soft graphs and some operations. Shahzadi and Akram (2016) presented different types of new concepts, including intuitionistic fuzzy soft graphs, complete intuitionistic fuzzy soft graph, strong intuitionistic fuzzy soft graph and self-complement of intuitionistic fuzzy soft graph. And described various methods of their construction, and investigated some of their related properties and discussed the applications of intuitionistic fuzzy soft graphs in communication network and decision making.

Recently, the notion of neutrosophic soft set has been extended in the graph theory and the concept of neutrosophic soft graph was provided by (Shah and Hussain, 2016) Later on, Shahzadi and Akram (2016) have applied the concept of neutrosophic soft sets to graphs and discussed various methods of construction of neutrosophic soft graphs. In the literature, the study of interval valued neutrosophic soft graphs (IVNS-graph) is still blank.

In the present paper, interval valued neutrosophic soft sets (Deli, 2015). are employed to study graphs and give rise to a new class of graphs called interval valued neutrosophic soft graphs. We have discussed different operations defined on neutrosophic soft graphs such as Cartesian product, composition, union and join with examples and proofs. The concepts of strong interval valued neutrosophic soft graphs, complete interval valued neutrosophic soft graphs and the complement of strong interval valued neutrosophic soft graphs a real so discussed. Interval valued neutrosophic soft graphs are pictorial representation in which each vertex and each edge is an element of interval valued neutrosophic soft sets.

This paper is organized as follows. In section 2, we give all the basic definitions related to interval valued neutrosophic graphs and interval valued neutrosophic soft sets which will be employed in later sections. In section 3, we introduce certain notions including interval valued neutrosophic soft graphs, strong interval valued neutrosophic soft graphs, complete interval valued neutrosophic soft graphs, the complement of strong interval valued neutrosophic soft graphs, and illustrate these notions by several examples, then we present some operations such as Cartesian product, composition, intersection, union and join on an interval valued neutrosophic soft graphs and investigate some of their related properties. In section 4, we present an application of interval valued neutrosophic soft graphs in decision making.

2. PRELIMINARIES

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, neutrosophic soft sets, interval valued, soft sets, neutrosophic soft sets, single valued neutrosophic graphs, fuzzy graph, intuitionistic fuzzy graph, interval valued intuitionistic fuzzy graphs and interval valued neutrosophic graphs, relevant to the present work. See especially (Mohamed et al, 2014; Nagoor and Basheer, 2003; Nagoor and Shajitha2010; Molodtsov, 1999; Smarandache, 2006; Wang et al., 2005; Wang et al., 2010; Deli, 2015; Broumi et al., 2016a, 2016b) for further details and background.

Definition 2.1 (Smarandache, 2006). Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X→]$^-0,1^+$[define respectively the a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A, with the condition:

$$^-0 \le T_A(x)+ I_A(x)+ F_A(x)\le 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-0,1^+$[.

Since it is difficult to apply NSs to practical problems, (Wang et al., 2010). introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 (Wang et al., 2010). Let Xbe a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X$T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}. \qquad (2)$$

Definition2.3 (Nagoor and Basheer, 2003)A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non-empty set V andμis a symmetric fuzzy relation on σ. i.eσ : V → [0,1] and μ:VxV→[0,1] such thatμ(uv) ≤ σ(u) ∧ σ(v)for all u, v ∈ V, where uv denotes the edge between u and v and σ(u) ∧ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V andμ is called the fuz... ...e set of E.

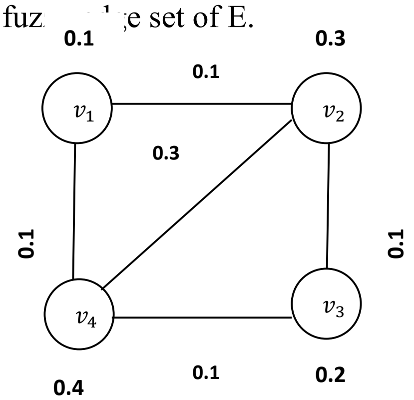


Fig.1:FuzzyGraph

Definition2.4 (Nagoor and Basheer, 2003) The fuzzy subgraph H = (τ, ρ) is called a fuzzy subgraph of G = (σ, μ)
If τ(u) ≤ σ(u) for all u ∈ V and ρ(u, v) ≤μ(u, v)for all u, v ∈ V.

Definition2.5 (Nagoor and Shajitha2010) An Intuitionistic fuzzy graph is of the form G = (V,E) where:

i. V={v_1,v_2,….,v_n} such that μ_1:V→ [0,1] andγ_1:V→ [0,1] denote the degree of membership and non-membership of the element $v_i \in V$, respectively, and $0 \leq \mu_1(v_i)+\gamma_1(v_i)) \leq 1$for every$v_i \in V$,(i=1, 2,……. n),
ii. E⊆ VxV whereμ_2:VxV→[0,1]andγ_2:VxV→ [0,1] are such that $\mu_2(v_i, v_j) \leq \min[\mu_1(v_i),\mu_1(v_j)]$and $\gamma_2(v_i, v_j) \geq \max[\gamma_1(v_i),\gamma_1(v_j)]$and $0 \leq \mu_2(v_i, v_j)+\gamma_2(v_i, v_j) \leq 1$ for every$(v_i, v_j) \in E$,(i,j =1,2,……. n)

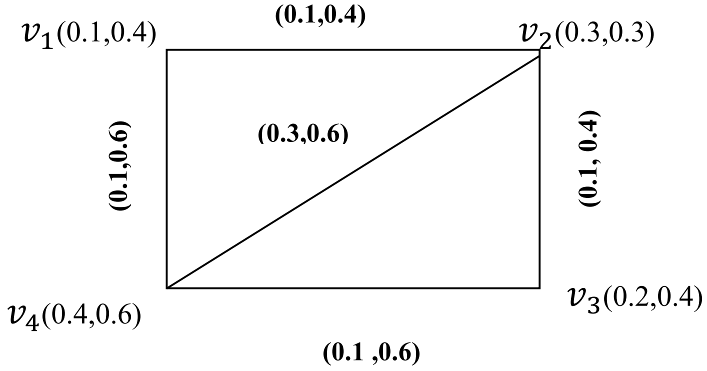


Fig.2: Intuitionistic Fuzzy Graph

Definition 2.6 (Broumi et al., 2016a).Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$be single valued neutrosophic sets on a set *X*. If $A = (T_A, I_A, F_A)$ is a single valued neutrosophic relation on a set *X*, then $A = (T_A, I_A, F_A)$ is called a single valued neutrosophic relation on $B = (T_B, I_B, F_B)$ if

$$T_B(x, y) \le \min(T_A(x), T_A(y))$$
$$I_B(x, y) \ge \max(I_A(x), I_A(y)) \text{ and}$$
$$F_B(x, y) \ge \max(F_A x), F_A(y)) \text{ for all } x, y \in X.$$

A single valued neutrosophic relation *A* on *X* is called symmetric if $T_A(x, y) = T_A(y, x)$, $I_A(x, y) = I_A(y, x)$, $F_A(x, y) = F_A(y, x)$ and $T_B(x, y) = T_B(y, x)$, $I_B(x, y) = I_B(y, x)$ and $F_B(x, y) = F_B(y, x)$, for all $x, y \in X$.

Definition 2.7 (Broumi et al., 2016a). A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G= (A, B) where:
1.The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively,and

$$0 \le T_A(v_i) + I_A(v_i) + F_A(v_i) \le 3 \text{ for all } v_i \in V \text{ (i=1, 2, …,n)}$$

2. The functionsT_B: E ⊆ V x V →[0, 1],I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by

$$T_B(\{v_i, v_j\}) \le \min [T_A(v_i), T_A(v_j)],$$
$$I_B(\{v_i, v_j\}) \ge \max [I_A(v_i), I_A(v_j)], \text{ and}$$
$$F_B(\{v_i, v_j\}) \ge \max [F_A(v_i), F_A(v_j)],$$

Denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where:

$$0 \le T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \le 3 \text{ for all } \{v_i, v_j\} \in E \text{ (i, j = 1, 2,…, n)}$$

We call A the single valued neutrosophic vertex set of V, B the single valued neutrosophic edge set of E, respectively. Note that B is a symmetric single valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E. Thus, G = (A, B) is a single valued neutrosophic graph of G*= (V, E) if:

$$T_B(v_i, v_j) \le \min [T_A(v_i), T_A(v_j)],$$
$$I_B(v_i, v_j) \ge \max [I_A(v_i), I_A(v_j)] \text{ and}$$
$$F_B(v_i, v_j) \ge \max [F_A(v_i), F_A(v_j)], \text{ for all } (v_i, v_j) \in E.$$

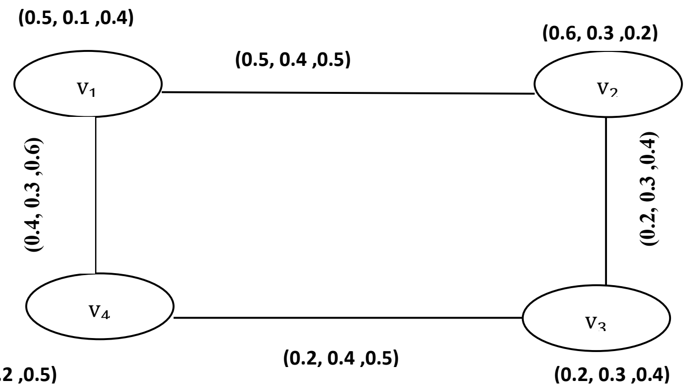


Fig.3: Single valued neutrosophic graph

Definition 2.9 (**Broumi et al., 2016a**).A partial SVN-subgraph of SVN-graph G= (A, B)is a SVN-graph H=(V',E')such that

(i) $\boldsymbol{V' \subseteq V}$, where$\boldsymbol{T'_A(v_i) \le T_A(v_i)}$,$\boldsymbol{I'_A(v_i) \ge I_A(v_i)}$,$\boldsymbol{F'_A(v_i) \ge F_A(v_i)}$,for all$\boldsymbol{v_i \in V}$.

(ii) $\boldsymbol{E' \subseteq E}$, where $\boldsymbol{T'_B(v_i, v_j) \le T_B(v_i, v_j)}$,$\boldsymbol{I'_{Bij} \ge I_B(v_i, v_j)}$, $\boldsymbol{F'_B(v_i, v_j) \ge}$

$\boldsymbol{F_B(v_i, v_j)}$,for all $(\boldsymbol{v_i v_j}) \in \boldsymbol{E}$.

Definition 2.10 (Broumi et al., 2016a). ASVN-subgraph of SVN-graph G= (V, E)is a SVN-graph H=($\boldsymbol{V'}$,$\boldsymbol{E'}$)such that

(i) $\boldsymbol{V' = V}$, where$\boldsymbol{T'_A(v_i) = T_A(v_i)}$,$\boldsymbol{I'_A(v_i) = I_A(v_i)}$,$\boldsymbol{F'_A(v_i) = F_A(v_i)}$for all$\boldsymbol{v_i}$in the vertex set of$\boldsymbol{V'}$.

(ii) $\boldsymbol{E' = E}$, where$\boldsymbol{T'_B(v_i, v_j) = T_B(v_i, v_j)}$,$\boldsymbol{I'_B(v_i, v_j) = I_B(v_i, v_j)}$,$\boldsymbol{F'_B(v_i, v_j) = F_B(v_i, v_j)}$for every $(\boldsymbol{v_i v_j}) \in \boldsymbol{E}$ in the edge set of$\boldsymbol{E'}$.

Definition 2.10 (Broumi et al., 2016a). Let G= (A, B) be a single valued neutrosophic graph. Then the degree of any vertex v is sum of degree of truth-membership, sum of degree of indeterminacy-membership and sum of degree of falsity-membership of all those edges which are incident on vertex v denoted by d(v)= ($d_T(v)$, $d_I(v)$,$d_F(v)$) where:

$d_T(v)$=$\sum_{u \neq v} T_B(u, v)$ denotesdegree of truth-membership vertex.

$d_I(v)$=$\sum_{u \neq v} I_B(u, v)$ denotes degree of indeterminacy-membership vertex.

$d_F(v)$=$\sum_{u \neq v} F_B(u, v)$ denotes degree of falsity-membership vertex.

Definition 2.11(Broumi et al., 2016a). A single valued neutrosophic graph G = (A, B) of G^*= (V, E) is calledstrong single valued neutrosophic graph if:

$T_B(v_i, v_j)$ =min [$T_A(v_i)$,$T_A(v_j)$]

$I_B(v_i, v_j)$ =max [$I_A(v_i)$,$I_A(v_j)$]

$F_B(v_i, v_j)$ =max [$F_A(v_i)$, $F_A(v_j)$], for all $(v_i, v_j) \in$ E.

Definition 2.12(Broumi et al., 2016a). A single valued neutrosophic graph G= (A, B) is called complete if

$T_B(v_i, v_j)$ =min [$T_A(v_i)$,$T_A(v_j)$]

$I_B(v_i, v_j)$ =max [$I_A(v_i)$,$I_A(v_j)$]

$F_B(v_i, v_j)$ =max [$F_A(v_i)$, $F_A(v_j)$], for all $v_i, v_j \in$ V.

Definition 2.13(Broumi et al., 2016a)The complement of a single valued neutrosophic graph G (A, B) onG^* is a single valued neutrosophic graph $\bar{G}$ on G^* where:

1.$\bar{A}$ =A

2.$\overline{T_A}(v_i)$= $T_A(v_i)$,$\overline{I_A}(v_i)$= $I_A(v_i)$,$\overline{F_A}(v_i) = F_A(v_i)$, for all $v_j \in$ V.

3.$\overline{T_B}(v_i, v_j)$= min $[T_A(v_i), T_A(v_j)]$-$T_B(v_i, v_j)$

$\overline{I_B}(v_i, v_j)$= max $[I_A(v_i), I_A(v_j)]$-$I_B(v_i, v_j)$ and

$\overline{F_B}(v_i, v_j)$= max $[F_A(v_i), F_A(v_j)]$-$F_B(v_i, v_j)$,for all $(v_i, v_j) \in$ E.

Definition 2.14 (Mohamed et al, 2014). An interval valued intuitionistic fuzzy graph with underlying set V is defined to be a pair G= (A, B) where

1)The functions M_A : V→ D [0, 1]and N_A : V→ D [0, 1] denote the degree of membership and non-membership of the element x ∈ V, respectively, such that 0 such that0≤M_A(x)+ N_A(x) ≤ 1 for all x ∈ V.

2) The functions M_B : E ⊆ $V \times V$ → D [0, 1]and N_B : : E ⊆ $V \times V$ → D [0, 1] are defined by

$$M_{BL}(x,y))\leq \min (M_{AL}(x), M_{AL}(y)) \text{ and } N_{BL}(x,y)) \geq \max (N_{AL}(x), N_{AL}(y))$$

$$M_{BU}(x,y))\leq \min (M_{AU}(x), M_{AU}(y)) \text{ and } N_{BU}(x,y)) \geq \max (N_{AU}(x), N_{AU}(y)),$$

such that

$$0\leq M_{BU}(x,y))+ N_{BU}(x,y)) \leq 1 \text{ for all } (x,y) \in E.$$

Définition 2.15 (Broumi et al., 2016b). By an interval-valued neutrosophic graph of a graph G* = (V, E) we mean a pair G = (A,B), where A =< $[T_{AL}, T_{AU}]$, $[I_{AL}, I_{AU}]$, $[F_{AL}, F_{AU}]$> is an interval-valued neutrosophic set on V and B =< $[T_{BL}, T_{BU}]$, $[I_{BL}, I_{BU}]$, $[F_{BL}, F_{BU}]$> is an interval-valued neutrosophic relation on E satisfies the following condition:

1. V= $\{v_1,v_2,\ldots,v_n\}$ such that T_{AL}:V→[0, 1],T_{AU}:V→[0, 1], I_{AL}:V→[0, 1],I_{AU}:V→[0, 1]and F_{AL}:V→[0, 1],F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree ofindeterminacy- membership and falsity-membership of the element $y \in$ V, respectively,and0≤ $T_A(v_i)$ + $I_A(v_i)$ +$F_A(v_i)$ ≤3 for allv_i ∈ V (i=1, 2, …,n).
2. The functionsT_{BL}:V x V →[0, 1],T_{BU}:V x V →[0, 1],I_{BL}:V x V →[0, 1],I_{BU}:V x V →[0, 1]and F_{BL}:V x V →[0,1],F_{BU}:V x V →[0, 1] are such that:

$$T_{BL}(\{v_i, v_j\}) \leq \min [T_{AL}(v_i), T_{AL}(v_j)]$$

$$T_{BU}(\{v_i, v_j\}) \leq \min [T_{AU}(v_i), T_{AU}(v_j)]$$

$$I_{BL}(\{v_i, v_j\}) \geq \max[I_{BL}(v_i), I_{BL}(v_j)]$$

$$I_{BU}(\{v_i, v_j\}) \geq \max[I_{BU}(v_i), I_{BU}(v_j)]$$

$$F_{BL}(\{v_i, v_j\}) \geq \max[F_{BL}(v_i), F_{BL}(v_j)]$$

$$F_{BU}(\{v_i, v_j\}) \geq \max[F_{BU}(v_i), F_{BU}(v_j)],$$

Denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i,v_j) ∈ E respectively, where

$$0\leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\})+ F_B(\{v_i, v_j\}) \leq 3 \text{for all} \{v_i, v_j\} \in E \text{ (i, j = 1, 2, ., n).}$$

they call A the interval valued neutrosophic vertex set of V, B the interval valued neutrosophic edge set of E, respectively, Note that B is a symmetric interval valued neutrosophic relation on A. We use the notation (v_i, v_j) for an element of E Thus, G = (A, B) is an interval valued neutrosophic graph of G*= (V, E) if

$$T_{BL}(v_i, v_j) \leq \min [T_{AL}(v_i), T_{AL}(v_j)]$$

$T_{BU}(v_i, v_j) \leq \min\ [T_{AU}(v_i), T_{AU}(v_j)]$

$I_{BL}(v_i, v_j) \geq \max[I_{BL}(v_i), I_{BL}(v_j)]$

$I_{BU}(v_i, v_j) \geq \max[I_{BU}(v_i), I_{BU}(v_j)]$And

$F_{BL}(v_i, v_j) \geq \max[F_{BL}(v_i), F_{BL}(v_j)]$

$F_{BU}(v_i, v_j) \geq \max[F_{BU}(v_i), F_{BU}(v_j)]$,for all$(v_i, v_j) \in$ E.

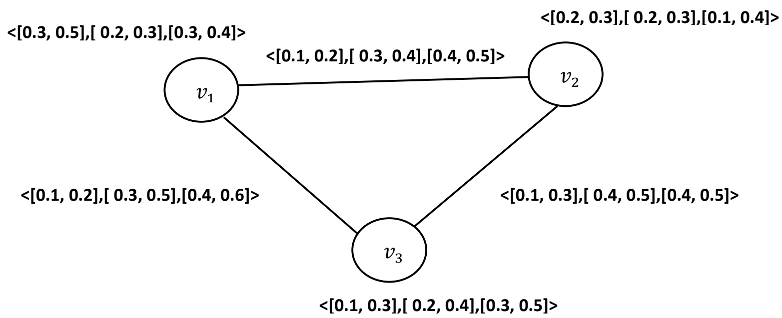


Fig. 4: G: Interval valued neutrosophic graph.

Definition 2.16 (Molodtsov, 1999). Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K: A → P(U).

As an illustration, let us consider the following example.

Example 2. Suppose that U is the set of houses under consideration, say U = $\{h_1, h_2. . ., h_5\}$. Let E be the set of some attributes of such houses, say E = $\{e_1, e_2, . . ., e_5\}$, where $e_1, e_2, . . .,$ e_5 stand for the attributes "beautiful", "costly", "in the green surroundings'", "moderate", respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, say Thomas, may be defined like this:

A=$\{e_1, e_2, e_3, e_4, e_5\}$;

K(e_1) = $\{h_2, h_3, h_5\}$, K(e_2) = $\{h_2, h_4\}$, K(e_3) = $\{h_1\}$, K(e_4) = U, K(e_5) = $\{h_3, h_5\}$.

Definition 2.17 (Wang et al., 2005**)**.Let IVNS(X) denote the family of all the interval valued neutrosophic sets in universe X, assume A, B ∈ IVNS(X) such that

$$A = \{\langle x, [T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)]\rangle : x \in X\}$$
$$B = \{\langle x, [T_B^L(x), T_B^U(x)], [I_B^L(x), I_B^U(x)], [F_B^L(x), F_B^U(x)]\rangle : x \in X\}$$

then some operations can be defined as follows:

(1) $A \cup B = \{\langle x, [\max\{T_A^L(x), T_B^L(x)\}, \max\{T_A^U(x), T_B^U(x)\}],$
$[\min\{I_A^L(x), I_B^L(x)\}, \min\{I_A^U(x), I_B^U(x)\}], [\min\{F_A^L(x), F_B^L(x)\}, \min\{F_A^U(x), F_B^U(x)\}]\rangle : x \in X\};$

(2) $A \cap B = \{\langle x, [\min\{T_A^L(x), T_B^L(x)\}, \min\{T_A^U(x), T_B^U(x)\}],$

$[\max\{I_A^L(x), I_B^L(x)\}, \max\{I_A^U(x), I_B^U(x)\}], [\max\{F_A^L(x), F_B^L(x)\}, \max\{F_A^U(x), F_B^U(x)\}]\rangle: x \in X\};$

(3) $A^c = \{\langle x, [F_A^L(x), F_A^U(x)], [1 - I_A^U(x), 1 - I_A^L(x)], [T_A^L(x), T_A^U(x)]\rangle: x \in X\};$

(4) $A \subseteq B$, iff $T_A^L(x) \le T_B^L(x)$, $T_A^U(x) \le T_B^U(x)$,$I_A^L(x) \ge I_B^L(x)$, $I_A^U(x) \ge I_B^U(x)$ and $F_A^L(x) \ge F_B^L(x)$, $F_A^U(x) \ge F_B^U(x)$ for all $x \in X$.

$A = B$, iff $A \subseteq$ Band B $\subseteq$A.

As an illustration, let us consider the following example.

Example 2.18.Assume that the universe of discourse U= $\{x_1, x_2, x_3, x_4\}$. Then, A is an interval valued neutrosophic set (IVNS) of U such that:

A = {<x_1, [0.1, 0.8], [0.2, 0.6], [0.8, 0.9] >, <x_2, [0.2, 0.5], [0.3, 0.5], [0.6, 0.8]>, <x_3, [0.5, 0.8], [0.4, 0.5], [0.5, 0.6] >, <x_4, [0.1, 0.4], [0.1, 0.5], [0.4, 0.8] >}.

Definition 2.19 (**Deli et al., 2015**).Let U be an initial universe set and A ⊂ E be a set of parameters. Let IVNS (U) denote the set of all interval valued neutrosophic subsets of U. The collection (K, A) is termed to be the soft interval valued neutrosophic set over U, where K is a mapping given by K: A → IVNS(U).

The interval valued neutrosophic soft set defined over a universe is denoted by INSS. Here,

1. ϒ is an ivn-soft subset of Ψ, denoted by ϒ ⋐ Ψ, if K(e) ⊆L(e) for all e∈E.
2. ϒ is an ivn-soft equals toΨ, denoted by ϒ = Ψ, if K(e)=L(e) for all e∈E.
3. The complement of ϒ is denoted by $ϒ^c$, and is defined by $ϒ^c$ = {(x, K^o (x)): x∈E}
4. The union of ϒ and Ψ is denoted by ϒ ∪̈ Ψ, if K(e) ∪L(e) for all e∈E.
5. The intersection of ϒand Ψ is denoted by ϒ ∩̈ Ψ,if K(e) ∪L(e) for all e∈E.

To illustrate let us consider the following example:

Let U be the set of houses under consideration and E is the set of parameters (or qualities). Each parameter is an interval valued neutrosophic word or sentence involving interval valued neutrosophic words. Consider E= {beautiful, costly, in the green surroundings, moderate, expensive}. In this case, to define an interval valued neutrosophic soft set means to point out beautiful houses, costly houses, and so on.

Suppose that there are five houses in the universe U, given by U = {h_1,h_2,h_3,h_4,h_5} and the set of parameters A = {e_1,e_2,e_3,e_4}, where each$e_i$is a specific criterion for houses:

e_1 stands for 'beautiful',
e_2 stands for 'costly',
e_3 stands for 'in the green surroundings',
e_4 stands for 'moderate'.

Suppose that,

K(beautiful)={<h_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,<h_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.3] >, <h3,[0.6, 0.7],[0.2 ,0.3],[0.3, 0.5] >,<h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >,<h5,[0.8, 0.4] ,[0.2 ,0.6],[0.3, 0.4] >}.
K(costly)={<h_1,[0.5, 0.6], [0.3, 0.7], [0.1, 0.4]>,<h_2,[0.3, 0.5], [0.6 ,0.8], [0.1, 0.3] >, <h_3,[0.3, 0.5],[0.2 ,0.6],[0.3, 0.4] >,<h_4,[0.2 ,0.5],[0.1, 0.2],[0.2, 0.4] >,<h_5,[0.2, 0.4] ,[0.1 ,0.5],[0.1,

0.4] >}.
K(in the green surroundings)= {<h_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,<h_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.5]>, <h_3,[0.2, 0.4],[0.2 ,0.3],[0.3, 0.5]>,<h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >,<h_5,[0.8, 0.4] ,[0.2 ,0.6],[0.2, 0.3] >},
K(moderate)={<h_1,[0.1, 0.6], [0.6, 0.7], [0.3, 0.4]>,<h_2,[0.2, 0.5], [0.4,0.8], [0.2, 0.4] >, <h_3,[0.3, 0.7],[0.2 ,0.4],[0.2, 0.5] >,<h_4,[0.7 ,0.8],[0.3, 0.4],[0.1, 0.2] >,<h_5,[0.3, 0.4] ,[0.2,0.6],[0.1, 0.2] >}.

3. INTERVAL VALUED NEUTROSOPHIC SOFT GRAPHS

Let U be an initial universe and P the set of all parameters, P(U) denoting the set of all interval neutrosophic sets of U. Let A be a subset of P. A pair (K, A) is called an interval valued neutrosophic soft set over U. Let P(V) denote the set of all interval valued neutrosophic sets of V and P(E) denote the set of all interval valued neutrosophic sets of E.

Definition3.1 An interval valued neutrosophics of the graph G=(G*,K, M,A) is a 4-tuple such that

a) G^*= (V, E) is a simple graph,
b) A is a nonempty set of parameters,
c) (K, A) is an interval valued neutrosophic soft set over V,
d) (M, A) is an interval valued neutrosophic over E,
e) (K(e), M(e)) is an interval valued neutrosophic (sub)graph of $G^*$for all e∈A.

That is,

$T^L_{M(e)}(xy) \le$min $[T^L_{K(e)}(x), T^L_{K(e)}(y)]$, $T^U_{M(e)}(xy) \le$ min $[T^U_{K(e)}(x), T^U_{K(e)}(y)]$,
$I^L_{M(e)}(xy) \ge$max $[I^L_{K(e)}(x), I^L_{K(e)}(y)]$, $I^U_{M(e)}(xy) \ge$ max $[T^U_{K(e)}(x), T^U_{K(e)}(y)]$
and$F^L_{M(e)}(xy) \ge$max $[F^L_{K(e)}(x), F^L_{K(e)}(y)]$, $F^U_{M(e)}(xy) \ge$ max $[T^U_{K(e)}(x), T^U_{K(e)}(y)]$,

such that

$$0\le T_{M(e)}(xy) + I_{M(e)}(xy) + F(xy) \le 3 \text{ for all } e\in A \text{ and } x, y \in V.$$

The interval valued neutrosophic graph (K(e), M (e)) is denoted by H(e) for convenience. An interval valued neutrosophic graph is a parametrized family of interval valued neutrosophic graphs. The class of all interval valued neutrosophic soft graphs of G^* is denoted by IVN(G^*). Note that

$T^L_{M(e)}(xy)= T^U_{M(e)}(xy) =I^L_{M(e)}(xy)= I^U_{M(e)}(xy) = 0$ and $F^L_{M(e)}(xy)= F^U_{M(e)}(xy) = 0$ for all xy ∈ V− E, e ∉ A.

Definition 3.2 Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G*. Then G_1 is an interval valued neutrosophic soft subgraph of G_2 if

(i) A⊆ B
(ii) H_1(e) is a partial subgraph of H_2(e) for all e ∈ A.

Example 3.3.Consider a simple graphG^*=(V, E) such that V={v_1, v_2,v_3} and E={v_1v_2, v_2v_3,v_3v_1}.

Let A= {e_1, e_2} be a set of parameter and let(K, A)bean interval valued neutrosophic soft set over V with its interval valued neutrosophic approximate function K : A →P(V) defined by

$K(e_1)$={v_1|([0.3, 0.5],[0.2, 0.3], [0.3, 0.4]), v_2|([0.2, 0.3], [0.2, 0.3], [0.1, 0.4]), v_3|([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])},

$K(e_2)$={v_1|([0.1, 0.4], [0.1, 0.3], [0.2, 0.3]), v_2|([0.1, 0.3], [0.1, 0.2], [0.1, 0.4]), v_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}.

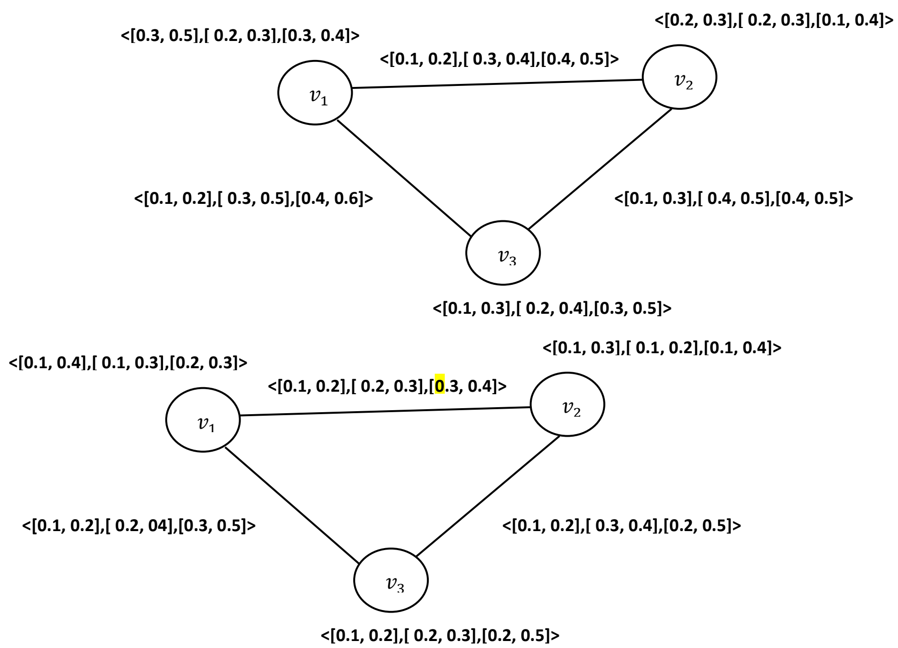
Let (M, A)be an interval valued neutrosophic soft set over E with its interval valued neutrosophic approximate function$M$: A →P(E) defined by

$M(e_1)$={v_1v_2|([0.1, 0.2], [0.3, 0.4], [0.4, 0.5]), v_2v_3|([0.1, 0.3], [0.4, 0.5], [0.4, 0.5]),v_3v_1 |([0.1, 0.2], [0.3, 0.5], [0.5, 0.6])},

$M(e_2)$= {v_1v_2|([0.1, 0.2], [0.2, 0.3], [0.3, 0.4]), v_2v_3|([0.1, 0.2], [0.3, 0.4], [0.2, 0.5]),, v_3v_1 |([0.1, 0.2], [0.2, 0.4], [0.3, 0.5])}.

Thus, $H(e_1)$=($K(e_1)$, $M(e_1)$), $H(e_2)$=($K(e_2)$, $M(e_2)$) are interval valued neutrosophic graphs corresponding to the parameters $e_1$and $e_1$as shown below.

$H(e_1)$



$H(e_2)$

Fig. 3.1:Interval valued neutrosophic soft graph G= {$H(e_1)$, $H(e_2)$}.

Hence G= { $H(e_1)$, $H(e_2)$} is an interval valued neutrosophic soft graph of G^*.

Tabular representation of an interval valued neutrosophic soft graph is given in Table below.

Table 1: Tabular representation of an interval valued neutrosophic soft graph.

K	v_1	v_2	v_3
e_1	<[0.3,0.5],[0.2, 0.3][0.3, 0.4]>	<[0.2,0.3],[0.2, 0.3][0.1, 0.4]>	<[0.1,0.3],[0.2, 0.4][0.3, 0.5]>
e_2	<[0.1,0.4],[0.1, 0.3][0.2, 0.3]>	<[0.1,0.3],[0.1, 0.2][0.1, 0.4]>	<[0.1,0.2],[0.2, 0.3][0.2, 0.5]>

M	(v_1, v_2)	(v_2, v_3)	(v_1, v_3)
e_1	<[0.1,0.2],[0.3, 0.4][0.4, 0.5]>	<[0.1,0.3],[0.4, 0.5][0.4, 0.5]>	<[0.1,0.2],[0.3, 0.5][0.4, 0.6]>
e_2	<[0.1,0.2],[0.2, 0.3][0.3, 0.4]>	<[0.1,0.2],[0.3, 0.4][0.2, 0.5]>	<[0.1,0.2],[0.2, 0.4][0.3, 0.5]>

Definition 3.4 Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) andG_1^* = (V_2, E_2) respectively. The Cartesian product of two graphs$G_1$and G_2 is an interval valued neutrosophic soft graph G= $G_1 \times G_2$ = (K,M, $A \times B$), where (K=$K_1 \times K_2$, $A \times B$) is an interval valued neutrosophic soft set over V= $V_1 \times V_2$, (M=$M_1 \times M_2$, $A \times B$) is an interval valued neutrosophic soft set over E= $\{(x, x_2)\ (x, y_2)\ /x \in V_1,\ x_2y_2 \in E_2\}$ $\cup\{(x_1,z)$ $(y_1, z)\ /z \in V_2,\ x_1y_1 \in E_1\}$and(K, M, $A \times B$) are interval valued neutrosophic soft graphs such that:

1) $(T^L_{K_1(a)} \times T^L_{K_2(b)})\ (x_1, x_2)$ = min $(T^L_{K_1(a)}(x_1), T^L_{K_2(b)}(x_2))$
   $(T^U_{K_1(a)} \times T^U_{K_2(b)})\ (x_1, x_2)$ = min $(T^U_{K_1(a)}(x_1), T^U_{K_2(b)}(x_2))$
   $(I^L_{K_1(a)} \times I^L_{K_2(b)})\ (x_1, x_2)$ = max $(I^L_{K_1(a)}(x_1), I^L_{K_2(b)}(x_2))$
   $(I^U_{K_1(a)} \times I^U_{K_2(b)})\ (x_1, x_2)$ = max $(I^U_{K_1(a)}(x_1), I^U_{K_2(b)}(x_2))$
   $(F^L_{K_1(a)} \times F^L_{K_2(b)})\ (x_1, x_2)$ = max $(F^L_{K_1(a)}(x_1), F^L_{K_2(b)}(x_2))$
   $(F^U_{K_1(a)} \times F^U_{K_2(b)})\ (x_1, x_2)$ = max $(F^U_{K_1(a)}(x_1), F^U_{K_2(b)}(x_2))$ for all $(x_1, x_2) \in A \times B$

2) $(T^L_{M_1(a)} \times T^L_{M_2(b)})\ ((x, x_2)(x, y_2))$ = min $(T^L_{K_1(a)}(x), T^L_{M_2(b)}(x_2y_2))$
   $(T^U_{M_1(a)} \times T^U_{M_2(b)})\ ((x, x_2)(x, y_2))$ = min $(T^U_{K_1(a)}(x), T^U_{M_2(b)}(x_2y_2))$
   $(I^L_{M_1(a)} \times I^L_{M_2(b)})\ ((x, x_2)(x, y_2))$ =max $(I^L_{K_1(a)}(x), I^L_{M_2(b)}(x_2y_2))$
   $(I^U_{M_1(a)} \times I^U_{M_2(b)})\ ((x, x_2)(x, y_2))$ = max $(I^U_{K_1(a)}(x), I^U_{M_2(b)}(x_2y_2))$
   $(F^L_{M_1(a)} \times F^L_{M_2(b)})\ ((x, x_2)\ (x, y_2))$ = max $(F^L_{K_1(a)}(x), F^L_{M_2(b)}(x_2y_2))$
   $(F^U_{M_1(a)} \times F^U_{M_2(b)})\ ((x, x_2)(x, y_2))$ = max$(F^U_{K_1(a)}(x), F^U_{M_2(b)}(x_2y_2))$ $\forall$ x $\in V_1$
   and $\forall x_2y_2 \in E_2$

3) $(T^L_{M_1(a)} \times T^L_{M_2(b)})\ ((x_1, z)\ (y_1, z))$ = min $(T^L_{M_1(a)}(x_1y_1), T^L_{K_2(b)}(z))$
   $(T^U_{M_1(a)} \times T^U_{M_2(b)})\ ((x_1, z)\ (y_1, z))$ = min $(T^U_{M_1(a)}(x_1y_1), T^U_{K_2(b)}(z))$
   $(I^L_{M_1(a)} \times I^L_{M_2(b)})\ ((x_1,z)\ (y_1, z))$ = max $(I^L_{M_1(a)}(x_1y_1), I^L_{K_2(b)}(z))$
   $(I^U_{M_1(a)} \times I^U_{M_2(b)})\ ((x_1,z)\ (y_1,z))$ = max $(I^U_{M_1(a)}(x_1y_1), I^U_{K_2(b)}(z))$
   $(F^L_{M_1(a)} \times F^L_{M_2(b)})((x_1,z)\ (y_1, z))$ = max $(F^L_{M_1(a)}(x_1y_1), F^L_{K_2(b)}(z))$
   $(F^U_{M_1(a)} \times F^U_{M_2(b)})\ ((x_1,z)\ (y_1, z))$ = max $(F^U_{M_1(a)}(x_1y_1), F^U_{K_2(b)}(z))$ $\forall$ z $\in V_2$
   and $\forall x_1y_1 \in E_1$

   H(a, b) = $H_1(a) \times H_2(b)$ for all $(a, b) \in A \times B$ are interval valued neutrosophic graphs of G.

Example 3.5.Let A= $\{e_1, e_2\}$ and B= $\{e_3, e_4\}$ be a set ofparameters. Consider two interval valued neutrosophic soft graphs G_1=(H_1, A) =$\{H(e_1), H(e_2)\}$and G_2=(H_2, B) = $\{H(e_3), H(e_4)\}$such that

$H_1(e_1)$=({u_1|([0.3, 0.5], [0.2, 0.3], [0.3, 0.4]), u_2|([0.6, 0.7], [0.2, 0.4], [0.1, 0.3])}, {u_1u_2|([0.3, 0.6], [0.2, 0.4], [0.2, 0.5])}).

$H_1(e_2)$=({u_1|([0.3, 0.5], [0.2, 0.3], [0.3, 0.4]), u_2|([0.2, 0.3], [0.2, 0.3], [0.1, 0.4]), u_3|([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])}, {u_1u_2|([0.1, 0.2], [0.3, 0.4], [0.4, 0.5]), u_2u_3|([0.1, 0.3], [0.4, 0.5], [0.4, 0.5]), u_3u_1 |([0.1, 0.2], [0.3, 0.5], [0.5, 0.6])}).

$H_2(e_3)$=({v_1|([0.4, 0.6], [0.2, 0.3], [0.1, 0.3]), v_2|([0.4, 0.7], [0.2, 0.4], [0.1, 0.3])}, {v_1v_2|([0.3, 0.5], [0.4, 0.5], [0.3, 0.5])}).

$H_2(e_4)$=({v_1|([0.1, 0.4], [0.1, 0.3], [0.2, 0.3]), v_2|([0.1, 0.3], [0.1, 0.2], [0.1, 0.4]), v_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}, {v_1v_2|([0.1, 0.2], [0.2, 0.3], [0.3, 0.4]), v_2v_3|([0.1, 0.2], [0.3, 0.4], [0.2, 0.5]),v_3v_1 |([0.1, 0.2], [0.2, 0.4], [0.3, 0.5])})

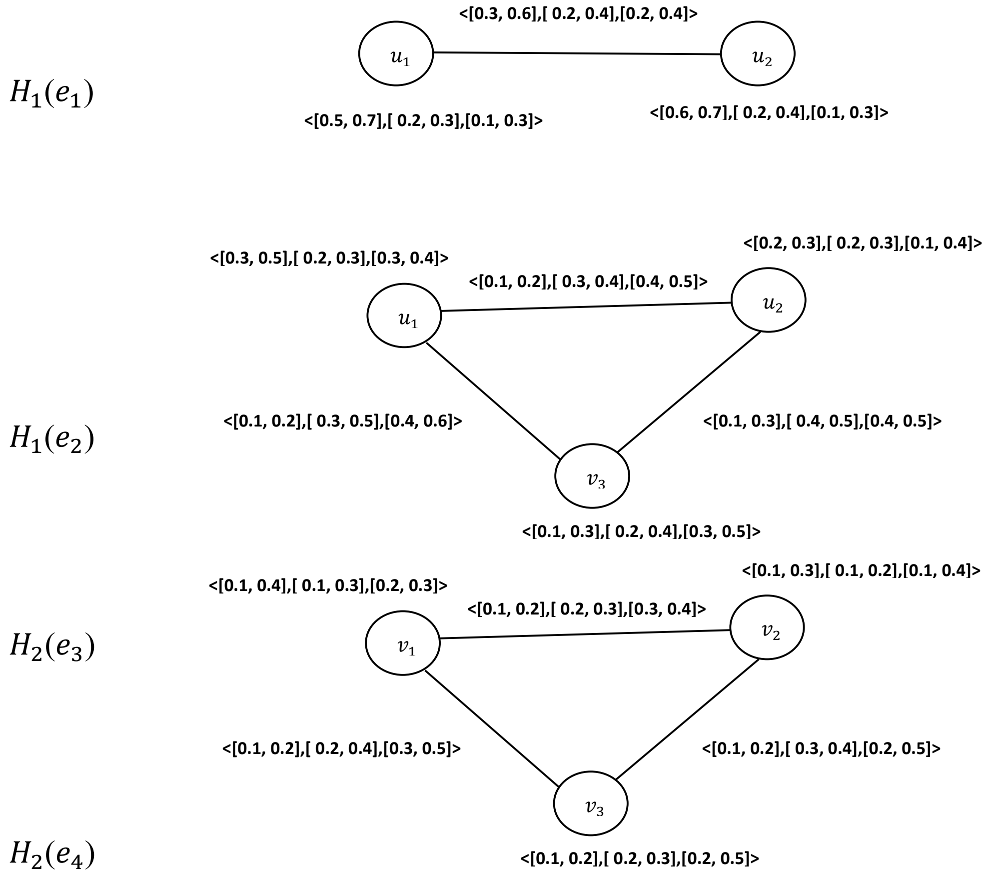


Fig. 3.2: Interval valued neutrosophic soft graph G_1= {$H_1(e_1)$,$H_1(e_2)$} and G_2= {$H_2(e_3)$,$H_2(e_4)$}

The Cartesian product ofG_1 and G_2 is $G_1 \times G_2$ = (H,$A \times B$), where A×B= {(e_1, e_3), (e_1, e_4), (e_2, e_3), (e_2, e_4)}, H(e_1, e_3) = $H_1(e_1)$ ×$H_2(e_3)$, H(e_1, e_4) = $H_1(e_1)$ ×$H_2(e_4)$, H(e_2, e_3) = $H_1(e_2)$ × $H_2(e_3)$ and H(e_2, e_4) = $H_1(e_2)$ ×$H_2(e_4)$ are interval valued neutrosophic graphs of G = $G_1 \times G_2$. H(e_1, e_3) = $H_1(e_1)$ ×$H_2(e_3)$ is shown in Fig. 3.3.

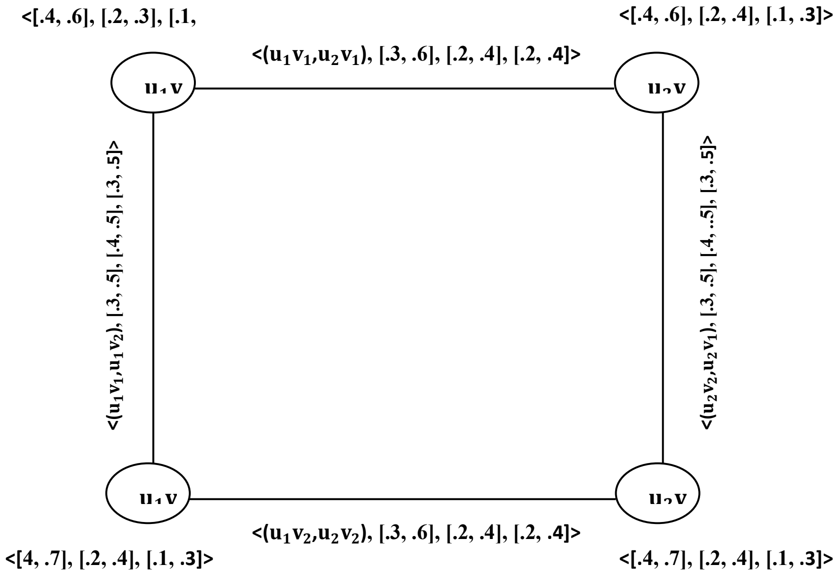


Fig. 3.3: Cartesian product

In the similar way, Cartesian product of H(e_1, e_4) = $H_1(e_1)$ $\times H_2(e_4)$, H(e_2, e_3) = $H_1(e_2)$ $\times$ $H_2(e_3)$ and H(e_2, e_4) = $H_1(e_2)$ $\times H_2(e_4)$ can be drawn.

Hence G =$G_1 \times G_2$= {H(e_1, e_3), H(e_1, e_4), H(e_2, e_3), H(e_2, e_4)} is an interval valued neutrosophic soft graph.

Theorem 3.6. The Cartesian product of two interval valued neutrosophic soft graph is an interval valued neutrosophic soft graph.

Proof. Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) and G_1^* = (V_2, E_2) respectively. Let G= $G_1 \times G_2$ = (K,M, $A \times B$) be the Cartesian product of two graphs$G_1$and G_2. We claim that G= $G_1 \times G_2$ = (K,M, $A \times B$)is an interval valued neutrosophic soft graph G= $G_1 \times G_2$ = (K,M, $A \times B$), where (K=$K_1 \times K_2$, $A \times B$) is an interval valued neutrosophic soft graph and (H, $A \times B$) = { ($K_1 \times K_2$) (a_i, b_j), ($M_1 \times M_2$)(a_i, b_j)} for all a_i ∈ A,b_i ∈ B for i= 1, 2,…, m, j= 1, 2,…,n are interval valued neutrosophic graphs of G.

Consider, (x, x_2) $(x, y_2) \in E$, we have

$T^L_{M(a_i,b_j)}$ $((x, x_2)(x, y_2))$ = min ($T^L_{K_1(a_i)}(x)$, $T^L_{M_2(b_j)}(x_2y_2)$), for i= 1, 2,…, m, j= 1, 2,…,n

$\leq$min $\{T^L_{K_1(a_i)}(x)$, min$\{T^L_{K_2(b_j)}(x_2)$,$T^L_{K_2(b_j)}(y_2)\}\}$

= min {min$\{T^L_{K_1(a_i)}(x)$,$T^L_{K_2(b_j)}(x_2)\}$, min$\{T^L_{K_1(a_i)}(x)$,$T^L_{K_2(b_j)}(y_2)\}\}$

$T^L_{M(a_i,b_j)}$ $((x, x_2)(x, y_2))$ ≤min{($T^L_{K_1(a_i)} \times T^L_{K_2(b_j)}$) (x, x_2), ($T^L_{K_1(a_i)} \times T^L_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, we prove that

$T^U_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≤min{($T^U_{K_1(a_i)} \times T^U_{K_2(b_j)}$) (x, x_2), ($T^U_{K_1(a_i)} \times T^U_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n.

$I^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ = max ($I^L_{K_1(a_i)}(x)$, $I^L_{M_2(b_j)}(x_2 y_2)$), for i= 1, 2,…, m, j= 1, 2,…,n

≥max {$I^L_{K_1(a_i)}(x)$, max{$I^L_{K_2(b_j)}(x_2)$,$I^L_{K_2(b_j)}(y_2)$}}

= max {max{$I^L_{K_1(a_i)}(x)$,$I^L_{K_2(b_j)}(x_2)$}, max{$I^L_{K_1(a_i)}(x)$,$I^L_{K_2(b_j)}(y_2)$}}

$I^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≥max {($I^L_{K_1(a_i)} \times I^L_{K_2(b_j)}$) (x, x_2), ($I^L_{K_1(a_i)} \times I^L_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, we prove that

$I^U_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≥max {($I^U_{K_1(a_i)} \times I^U_{K_2(b_j)}$) (x, x_2), ($I^U_{K_1(a_i)} \times I^U_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

$F^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ = max ($F^L_{K_1(a_i)}(x)$, $F^L_{M_2(b_j)}(x_2 y_2)$), for i= 1, 2,…, m, j= 1, 2,…,n

≥ max {$F^L_{K_1(a_i)}(x)$, max {$F^L_{K_2(b_j)}(x_2)$,$F^L_{K_2(b_j)}(y_2)$}}

= max{ max {$F^L_{K_1(a_i)}(x)$,$F^L_{K_2(b_j)}(x_2)$}, max {$F^L_{K_1(a_i)}(x)$,$F^L_{K_2(b_j)}(y_2)$}}

$F^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≥ max {($F^L_{K_1(a_i)} \times F^L_{K_2(b_j)}$) (x, x_2), ($F^L_{K_1(a_i)} \times F^L_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, we prove that

$F^U_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≥ max {($F^U_{K_1(a_i)} \times F^U_{K_2(b_j)}$) (x, x_2), ($F^U_{K_1(a_i)} \times F^U_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, for (x_1,z) (y_1, z)∈ E, we have

$T^L_{M(a_i,b_j)}$((x_1, z) (y_1, z)) ≤min{($T^L_{K_1(a_i)} \times T^L_{K_2(b_j)}$) ($x_1$, z), ($T^L_{K_1(a_i)} \times T^L_{K_2(b_j)}$) ($y_1$, z),

$T^U_{M(a_i,b_j)}$((x_1, z) (y_1, z)) ≤min{($T^U_{K_1(a_i)} \times T^U_{K_2(b_j)}$) ($x_1$, z), ($T^U_{K_1(a_i)} \times T^U_{K_2(b_j)}$) ($y_1$, z),

$I^L_{M(a_i,b_j)}$((x_1, z) (y_1, z)) ≥ max {($I^L_{K_1(a_i)} \times I^L_{K_2(b_j)}$) ($x_1$, z), ($I^L_{K_1(a_i)} \times I^L_{K_2(b_j)}$) ($y_1$, z),

$I^U_{M(a_i,b_j)}$((x_1, z) (y_1, z)) ≥ max {($I^U_{K_1(a_i)} \times I^U_{K_2(b_j)}$) ($x_1$, z), ($I^U_{K_1(a_i)} \times I^U_{K_2(b_j)}$) ($y_1$, z),

$F^L_{M(a_i,b_j)}$((x_1, z) (y_1, z)) ≥ max {($F^L_{K_1(a_i)} \times F^L_{K_2(b_j)}$) ($x_1$, z), ($F^L_{K_1(a_i)} \times F^L_{K_2(b_j)}$) ($y_1$, z),

$F^U_{M(a_i,b_j)}$((x_1, z) (y_1, z)) ≥ max {($F^U_{K_1(a_i)} \times F^U_{K_2(b_j)}$) ($x_1$, z), ($F^U_{K_1(a_i)} \times F^U_{K_2(b_j)}$) ($y_1$, z),for i= 1, 2,…, m, j= 1, 2,…,n

Hence G = (K, M, A×B) is an interval valued neutrosophic soft graph.

Definition 3.7 Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) and G_1^* = (V_2, E_2) respectively. The strong product of two graphs$G_1$and G_2 is an interval valued neutrosophic soft graph G= $G_1 \otimes G_2$ = (K,M, $A\times B$), where (K=$K_1\times K_2$, $A\times B$) is an interval valued neutrosophic soft set over V= $V_1\times V_2$, (M, $A\times B$) is an interval valued neutrosophic soft set over E= $\{(x, x_2)\,(x, y_2)\,/x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1,z)\,(y_1, z)\,/z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1,x_2)\,(y_1,y_2)\,/x_1y_1 \in E_1, x_2y_2 \in E_2\}$ and(K, M, $A\times B$) are interval valued neutrosophic soft graphs such that:

1) $(T^L_{K_1(a)} \times T^L_{K_2(b)})\,(x_1, x_2) = \min\,(T^L_{K_1(a)}(x_1), T^L_{K_2(b)}(x_2))$
   $(T^U_{K_1(a)} \times T^U_{K_2(b)})\,(x_1, x_2) = \min\,(T^U_{K_1(a)}(x_1), T^U_{K_2(b)}(x_2))$
   $(I^L_{K_1(a)} \times I^L_{K_2(b)})\,(x_1, x_2) = \max\,(I^L_{K_1(a)}(x_1), I^L_{K_2(b)}(x_2))$
   $(I^U_{K_1(a)} \times I^U_{K_2(b)})\,(x_1, x_2) = \max\,(I^U_{K_1(a)}(x_1), I^U_{K_2(b)}(x_2))$
   $(F^L_{K_1(a)} \times F^L_{K_2(b)})\,(x_1, x_2) = \max\,(F^L_{K_1(a)}(x_1), F^L_{K_2(b)}(x_2))$
   $(F^U_{K_1(a)} \times F^U_{K_2(b)})\,(x_1, x_2) = \max\,(F^U_{K_1(a)}(x_1), F^U_{K_2(b)}(x_2))$ for all $(x_1, x_2) \in A\times B$

2) $(T^L_{M_1(a)} \times T^L_{M_2(b)})\,((x, x_2)(x, y_2)) = \min\,(T^L_{K_1(a)}(x), T^L_{M_2(b)}(x_2y_2))$
   $(T^U_{M_1(a)} \times T^U_{M_2(b)})\,((x, x_2)(x, y_2)) = \min\,(T^U_{K_1(a)}(x), T^U_{M_2(b)}(x_2y_2))$
   $(I^L_{M_1(a)} \times I^L_{M_2(b)})\,((x, x_2)(x, y_2)) = \max\,(I^L_{K_1(a)}(x), I^L_{M_2(b)}(x_2y_2))$
   $(I^U_{M_1(a)} \times I^U_{M_2(b)})\,((x, x_2)(x, y_2)) = \max\,(I^U_{K_1(a)}(x), I^U_{M_2(b)}(x_2y_2))$
   $(F^L_{M_1(a)} \times F^L_{M_2(b)})\,((x, x_2)\,(x, y_2)) = \max\,(F^L_{K_1(a)}(x), F^L_{M_2(b)}(x_2y_2))$
   $(F^U_{M_1(a)} \times F^U_{M_2(b)})\,((x,x_2)(x, y_2)) = \max(F^U_{K_1(a)}(x), F^U_{M_2(b)}(x_2y_2))$ $\forall$ x $\in V_1$and $\forall x_2y_2 \in E_2$.

3) $(T^L_{M_1(a)} \times T^L_{M_2(b)})\,((x_1, z)\,(y_1, z)) = \min\,(T^L_{M_1(a)}(x_1y_1), T^L_{K_2(b)}(z))$
   $(T^U_{M_1(a)} \times T^U_{M_2(b)})\,((x_1, z)\,(y_1, z)) = \min\,(T^U_{M_1(a)}(x_1y_1), T^U_{K_2(b)}(z))$
   $(I^L_{M_1(a)} \times I^L_{M_2(b)})\,((x_1, z)\,(y_1, z)) = \max\,(I^L_{M_1(a)}(x_1y_1), I^L_{K_2(b)}(z))$
   $(I^U_{M_1(a)} \times I^U_{M_2(b)})\,((x_1, z)\,(y_1,z)) = \max\,(I^U_{M_1(a)}(x_1y_1), I^U_{K_2(b)}(z))$
   $(F^L_{M_1(a)} \times F^L_{M_2(b)})((x_1, z)\,(y_1, z)) = \max\,(F^L_{M_1(a)}(x_1y_1), F^L_{K_2(b)}(z))$
   $(F^U_{M_1(a)} \times F^U_{M_2(b)})\,((x_1, z)\,(y_1, z)) = \max\,(F^U_{M_1(a)}(x_1y_1), F^U_{K_2(b)}(z))$ $\forall$ z $\in V_2$ and $\forall x_1y_1 \in E_1$.

4) $(T^L_{M_1(a)} \times T^L_{M_2(b)})\,((x_1, x_2), (y_1, y_2)) = \min\,(T^L_{K_1(a)}(x_1y_1), T^L_{K_2(b)}(x_2y_2))$
   $(T^U_{M_1(a)} \times T^U_{M_2(b)})\,((x_1, x_2), (y_1, y_2)) = \min\,(T^U_{K_1(a)}(x_1y_1), T^U_{K_2(b)}(x_2y_2))$
   $(I^L_{M_1(a)} \times I^L_{M_2(b)})\,((x_1, x_2), (y_1, y_2)) = \max\,(I^L_{K_1(a)}(x_1y_1), I^L_{K_2(b)}(x_2y_2))$
   $(I^U_{M_1(a)} \times I^U_{M_2(b)})\,((x_1, x_2), (y_1, y_2)) = \max\,(I^U_{K_1(a)}(x_1y_1), I^U_{K_2(b)}(x_2y_2))$
   $(F^L_{M_1(a)} \times F^L_{M_2(b)})\,((x_1, x_2), (y_1, y_2)) = \max\,(F^L_{K_1(a)}(x_1y_1), F^L_{K_2(b)}(x_2y_2))$
   $(F^U_{M_1(a)} \times F^U_{M_2(b)})\,((x_1, x_2), (y_1, y_2)) = \max\,(F^U_{K_1(a)}(x_1y_1), F^U_{K_2(b)}(x_2y_2))$ for all $(x_1, y_1) \in E_1$, $((x_2, y_2) \in E_2$.

H(a, b) = $H_1(a) \otimes H_2(b)$ for all $(a, b) \in A\times B$ are interval valued neutrosophic graphs of G.

Theorem 3.8. The strong product of two interval valued neutrosophic soft graph is an interval valued neutrosophic soft graph.

Definition 3.9 Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) and G_1^* = (V_2, E_2) respectively. The composition of two graphs$G_1$and G_2 is an interval valued neutrosophic soft graph G= $G_1[G_2]$ = (K,M, $A \circ B$), where (K=$K_1 \circ K_2$, $A \circ B$) is an interval valued neutrosophic soft set over V= $V_1 \times V_2$, (M, $A \circ B$) is an interval valued neutrosophic soft set over E= $\{(x, x_2)\ (x, y_2)\ /x \in V_1, x_2y_2 \in E_2\}$ $\cup\{(x_1,z)\ (y_1, z)\ /z \in V_2, x_1y_1 \in E_1\}$ $\cup\{(x_1,x_2)\ (y_1,y_2)\ /x_1y_1 \in E_1, x_2 \neq y_2\}$and(K, M, $A \circ B$) are interval valued neutrosophic soft graphs such that:

1) $(T^L_{K_1(a)} \circ T^L_{K_2(b)})\ (x_1, x_2)$ = min $(T^L_{K_1(a)}(x_1), T^L_{K_2(b)}(x_2))$
$(T^U_{K_1(a)} \circ T^U_{K_2(b)})\ (x_1, x_2)$ = min $(T^U_{K_1(a)}(x_1), T^U_{K_2(b)}(x_2))$
$(I^L_{K_1(a)} \circ I^L_{K_2(b)})\ (x_1, x_2)$ = max $(I^L_{K_1(a)}(x_1), I^L_{K_2(b)}(x_2))$
$(I^U_{K_1(a)} \circ I^U_{K_2(b)})\ (x_1, x_2)$ = max $(I^U_{K_1(a)}(x_1), I^U_{K_2(b)}(x_2))$
$(F^L_{K_1(a)} \circ F^L_{K_2(b)})\ (x_1, x_2)$ = max $(F^L_{K_1(a)}(x_1), F^L_{K_2(b)}(x_2))$
$(F^U_{K_1(a)} \circ F^U_{K_2(b)})\ (x_1, x_2)$ = max $(F^U_{K_1(a)}(x_1), F^U_{K_2(b)}(x_2))$ for all $(x_1, x_2) \in A \times B$

2) $(T^L_{M_1(a)} \circ T^L_{M_2(b)})\ ((x, x_2)(x, y_2))$ = min $(T^L_{K_1(a)}(x), T^L_{M_2(b)}(x_2y_2))$
$(T^U_{M_1(a)} \circ T^U_{M_2(b)})\ ((x, x_2)(x, y_2))$ = min $(T^U_{K_1(a)}(x), T^U_{M_2(b)}(x_2y_2))$
$(I^L_{M_1(a)} \circ I^L_{M_2(b)})\ ((x, x_2)(x, y_2))$ =max $(I^L_{K_1(a)}(x), I^L_{M_2(b)}(x_2y_2))$
$(I^U_{M_1(a)} \circ I^U_{M_2(b)})\ ((x, x_2)(x, y_2))$ = max $(I^U_{K_1(a)}(x), I^U_{M_2(b)}(x_2y_2))$
$(F^L_{M_1(a)} \circ F^L_{M_2(b)})\ ((x, x_2)\ (x, y_2))$ = max $(F^L_{K_1(a)}(x), F^L_{M_2(b)}(x_2y_2))$
$(F^U_{M_1(a)} \circ F^U_{M_2(b)})\ ((x, x_2)(x, y_2))$= max$(F^U_{K_1(a)}(x), F^U_{M_2(b)}(x_2y_2))$ $\forall$ x $\in V_1$and $\forall x_2y_2 \in E_2$.

3) $(T^L_{M_1(a)} \circ T^L_{M_2(b)})\ ((x_1, z)\ (y_1, z))$ = min $(T^L_{M_1(a)}(x_1y_1), T^L_{K_2(b)}(z))$
$(T^U_{M_1(a)} \circ T^U_{M_2(b)})\ ((x_1, z)\ (y_1, z))$ = min $(T^U_{M_1(a)}(x_1y_1), T^U_{K_2(b)}(z))$
$(I^L_{M_1(a)} \circ I^L_{M_2(b)})\ ((x_1, z)\ (y_1, z))$ = max $(I^L_{M_1(a)}(x_1y_1), I^L_{K_2(b)}(z))$
$(I^U_{M_1(a)} \circ I^U_{M_2(b)})\ ((x_1, z)\ (y_1,z))$ = max $(I^U_{M_1(a)}(x_1y_1), I^U_{K_2(b)}(z))$
$(F^L_{M_1(a)} \circ F^L_{M_2(b)})((x_1, z)\ (y_1, z))$ = max $(F^L_{M_1(a)}(x_1y_1), F^L_{K_2(b)}(z))$
$(F^U_{M_1(a)} \circ F^U_{M_2(b)})\ ((x_1, z)\ (y_1, z))$ = max $(F^U_{M_1(a)}(x_1y_1), F^U_{K_2(b)}(z))$ $\forall$ z $\in V_2$ and $\forall x_1y_1 \in E_1$.

4) $(T^L_{M_1(a)} \circ T^L_{M_2(b)})\ ((x_1, x_2), (y_1, y_2))$ = min $(T^L_{K_1(a)}(x_1y_1), T^L_{K_2(b)}(x_2),T^L_{K_2(b)}(y_2))$
$(T^U_{M_1(a)} \circ T^U_{M_2(b)})\ ((x_1, x_2), (y_1, y_2))$ = min $(T^U_{K_1(a)}(x_1y_1), T^U_{K_2(b)}(x_2), T^U_{K_2(b)}(y_2))$
$(I^L_{M_1(a)} \circ I^L_{M_2(b)})\ ((x_1, x_2), (y_1, y_2))$ = max $(I^L_{K_1(a)}(x_1y_1), I^L_{K_2(b)}(x_2),I^L_{K_2(b)}(y_2))$
$(I^U_{M_1(a)} \circ I^U_{M_2(b)})\ ((x_1, x_2), (y_1, y_2))$ = max $(I^U_{K_1(a)}(x_1y_1), I^U_{K_2(b)}(x_2), I^U_{K_2(b)}(y_2))$
$(F^L_{M_1(a)} \circ F^L_{M_2(b)})\ ((x_1, x_2), (y_1, y_2))$ = max $(F^L_{K_1(a)}(x_1y_1), F^L_{K_2(b)}(x_2),F^L_{K_2(b)}(y_2))$
$(F^U_{M_1(a)} \circ F^U_{M_2(b)})\ ((x_1, x_2), (y_1, y_2))$ = max $(F^U_{K_1(a)}(x_1y_1), F^U_{K_2(b)}(x_2), F^U_{K_2(b)}(y_2))$ for all $(x_1, y_1) \in E_1$, and $x_2 \neq y_2$.

H(a, b) = $H_1(a)[H_2(b)]$ for all $(a, b) \in A \times B$ are interval valued neutrosophic graphs of G.

Example 3.10. Let A= $\{e_1\}$ A= $\{e_2, e_3\}$ be the parameters sets. Consider two interval valued neutrosophic soft graphs G_1=(H_1, A) =$\{H_1(e_1)\}$and G_2=(H_2, B) = $\{H_2(e_2)$,$H_2(e_3)\}$ such that

$H_1(e_1)$=($\{u_1$|([0.5, 0.7], [0.2, 0.3], [0.1, 0.3]), u_2|([0.6, 0.7], [0.2, 0.4], [0.1, 0.3])$\}$,$\{u_1u_2$|([0.3, 0.6], [0.2, 0.4], [0.2, 0.4])$\}$)

$H_2(e_2)$=($\{v_1$|([0.1, 0.4], [0.1, 0.3], [0.2, 0.3]), v_2|([0.1, 0.3], [0.1, 0.2], [0.1, 0.4]), v_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])$\}$, $\{v_1v_2$|([0.1, 0.2], [0.2, 0.3], [0.3, 0.4]), v_2v_3|([0.1, 0.2], [0.3, 0.4], [0.2, 0.5]),v_3v_1 |([0.1, 0.2], [0.2, 0.4], [0.3, 0.5])$\}$)

$H_2(e_3)$=($\{v_1$|([0.4, 0.6], [0.2, 0.3], [0.1, 0.3]), v_2|([0.4, 0.7], [0.2, 0.4], [0.1, 0.3])$\}$,$\{v_1v_2$|([0.3, 0.5], [0.2, 0.5], [0.3, 0.5])$\}$)

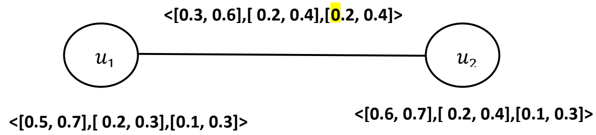


Fig.3.4:Interval valued neutrosophic soft graph corresponding to $H_1(e_1)$

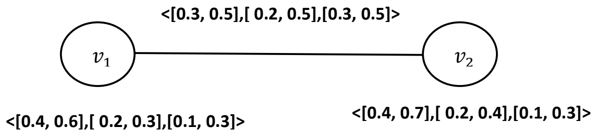


Fig. 3.5: Interval valued neutrosophic soft graph corresponding to$H_2(e_3)$.

The composition ofG_1 and G_2 is $G_1[G_2]$=(H,$A\times B$), where A$\times B$= $\{(e_1, e_2), (e_1, e_3), (e_2, e_3)\}$, H($e_1$, e_2) = $H_1(e_1)$ [$H_2(e_2)$] and H(e_1, e_3) = $H_1(e_1)$ [$H_2(e_3)$] are interval valued neutrosophic graphs of $G_1[G_2]$. $H_1(e_1)$ [$H_2(e_3)$] is shown in Fig. 3.6.

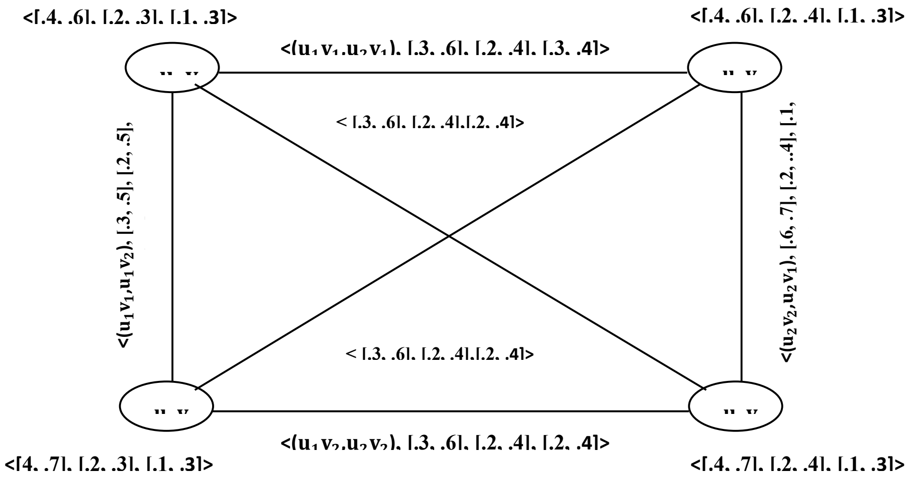


Fig. 3.6:Composition$H_1(e_1)$[$H_2(e_3)$]

Hence G=$G_1[G_2]$=$\{H_1(e_1)$ [$H_2(e_2)$], $H_1(e_1)$ [$H_2(e_3)$]$\}$is an interval valued neutrosophic soft graph.

Theorem 3.11. The composition of two interval valued neutrosophic soft graph is an interval valued neutrosophic soft graph

Proof. Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) and G_1^* = (V_2, E_2) respectively. Let G= $G_1[G_2]$ = (K,M, $A\times B$) be the Cartesian composition of two graphs$G_1$and G_2. We claim that G= $G_1[G_2]$ = (K,M, $A \circ B$)I s an interval valued neutrosophic soft graph and (H, $A \circ B$) = $\{K_1(a_i)[K_2(b_j)]$, $M_1(a_i)[M_2(b_j)]\}$ for all a_i ∈ A,b_i ∈ B for i= 1, 2,…, m, j= 1, 2,…,n are interval valued neutrosophic graphs of G.

Consider, (x, x_2) $(x, y_2) \in E$, we have

$T^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ = min $(T^L_{K_1(a_i)}(x)$, $T^L_{M_2(b_j)}(x_2y_2))$, for i= 1, 2,…, m, j= 1, 2,…,n

≤min $\{T^L_{K_1(a_i)}(x)$, min$\{T^L_{K_2(b_j)}(x_2)$,$T^L_{K_2(b_j)}(y_2)\}\}$

= min { min$\{T^L_{K_1(a_i)}(x)$,$T^L_{K_2(b_j)}(x_2)\}$, min$\{T^L_{K_1(a_i)}(x)$,$T^L_{K_2(b_j)}(y_2)\}\}$

$T^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≤min{($T^L_{K_1(a_i)} \circ T^L_{K_2(b_j)}$) (x, x_2), ($T^L_{K_1(a_i)} \circ T^L_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, we prove that

$T^U_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≤min{($T^U_{K_1(a_i)} \circ T^U_{K_2(b_j)}$) (x, x_2), ($T^U_{K_1(a_i)} \circ T^U_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n.

$I^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ = max $(I^L_{K_1(a_i)}(x)$, $I^L_{M_2(b_j)}(x_2y_2))$, for i= 1, 2,…, m, j= 1, 2,…,n

≥ max $\{I^L_{K_1(a_i)}(x)$, max $\{I^L_{K_2(b_j)}(x_2)$,$I^L_{K_2(b_j)}(y_2)\}\}$

= max{ max $\{I^L_{K_1(a_i)}(x)$,$I^L_{K_2(b_j)}(x_2)\}$, max $\{I^L_{K_1(a_i)}(x)$,$I^L_{K_2(b_j)}(y_2)\}\}$

$I^L_{M(a_i,b_j)}$ $((x, x_2)(x, y_2))$ ≥ max {($I^L_{K_1(a_i)} \circ I^L_{K_2(b_j)}$) (x, x_2), ($I^L_{K_1(a_i)} \circ I^L_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, we prove that

$I^U_{M(a_i,b_j)}$ $((x, x_2)(x, y_2))$ ≥ max {($I^U_{K_1(a_i)} \circ I^U_{K_2(b_j)}$) (x, x_2), ($I^U_{K_1(a_i)} \circ I^U_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

$F^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ = max $(F^L_{K_1(a_i)}(x)$, $F^L_{M_2(b_j)}(x_2y_2))$, for i= 1, 2,…, m, j= 1, 2,…,n

≥ max $\{F^L_{K_1(a_i)}(x)$, max $\{F^L_{K_2(b_j)}(x_2)$,$F^L_{K_2(b_j)}(y_2)\}\}$

= max{ max $\{F^L_{K_1(a_i)}(x)$,$F^L_{K_2(b_j)}(x_2)\}$, max $\{F^L_{K_1(a_i)}(x)$,$F^L_{K_2(b_j)}(y_2)\}\}$

$F^L_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2))$ ≥ max {($F^L_{K_1(a_i)} \circ F^L_{K_2(b_j)}$) (x, x_2), ($F^L_{K_1(a_i)} \circ F^L_{K_2(b_j)}$) (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, we prove that

$F^U_{M(a_i,b_j)}$ $((x,\ x_2)(x,\ y_2)) \geq$ max $\{(F^U_{K_1(a_i)} \circ F^U_{K_2(b_j)})$ (x, x_2), $(F^U_{K_1(a_i)} \circ F^U_{K_2(b_j)})$ (x, y_2),for i= 1, 2,…, m, j= 1, 2,…,n

Similarly, for $(x_1,z)\ (y_1, z) \in E$, we have

$T^L_{M(a_i,b_j)}$ $((x_1, z)\ (y_1, z)) \leq$min$\{(T^L_{K_1(a_i)} \circ T^L_{K_2(b_j)})\ (x_1, z), (T^L_{K_1(a_i)} \circ T^L_{K_2(b_j)})\ (y_1, z)$,

$T^U_{M(a_i,b_j)}$ $((x_1, z)\ (y_1, z)) \leq$min$\{(T^U_{K_1(a_i)} \circ T^U_{K_2(b_j)})\ (x_1, z), (T^U_{K_1(a_i)} \circ T^U_{K_2(b_j)})\ (y_1, z)$,

$I^L_{M(a_i,b_j)}$ $((x_1, z)\ (y_1, z)) \geq$ max $\{(I^L_{K_1(a_i)} \circ I^L_{K_2(b_j)})\ (x_1, z), (I^L_{K_1(a_i)} \circ I^L_{K_2(b_j)})\ (y_1, z)$,

$I^U_{M(a_i,b_j)}$ $((x_1, z)\ (y_1, z)) \geq$ max $\{(I^U_{K_1(a_i)} \circ I^U_{K_2(b_j)})\ (x_1, z), (I^U_{K_1(a_i)} \circ I^U_{K_2(b_j)})\ (y_1, z)$,

$F^L_{M(a_i,b_j)}$ $((x_1, z)\ (y_1, z)) \geq$ max $\{(F^L_{K_1(a_i)} \circ F^L_{K_2(b_j)})\ (x_1, z), (F^L_{K_1(a_i)} \circ F^L_{K_2(b_j)})\ (y_1, z)$,

$F^U_{M(a_i,b_j)}$ $((x_1, z)\ (y_1, z)) \geq$ max $\{(F^U_{K_1(a_i)} \circ F^U_{K_2(b_j)})\ (x_1, z), (F^U_{K_1(a_i)} \circ F^U_{K_2(b_j)})\ (y_1, z)$,for i= 1, 2,…, m, j= 1, 2,…,n

Let $(x_1, x_2)\ (y_1, y_2) \in E$, $(x_1, y_1) \in E_1$and $x_2 \neq y_2$. Then we have

$T^L_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2))$ = min $(T^L_{K_1(a_i)}(x_1y_1), T^L_{K_2(b_j)}(x_2), T^L_{K_2(b_j)}(y_2))$

$\leq$min$\{$min$\{T^L_{K_1(a_i)}(x_1), T^L_{K_1(a_i)}(y_1)\}, T^L_{K_2(b_j)}(x_2), T^L_{K_2(b_j)}(y_2))\}$

=min { min$\{T^L_{K_1(a_i)}(x_1), T^L_{K_2(b_j)}(x_2)\}$, min$\{T^L_{K_1(a_i)}(y_1), T^L_{K_2(b_j)}(y_2)\}\}$

$T^L_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2)) \leq$min$\{T^L_{K(a_i,b_j)}(x_1, x_2), T^L_{K(a_i,b_j)}(y_1, y_2)\}$

We prove also that,

$T^U_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2)) \geq$max$\{T^U_{K(a_i,b_j)}(x_1, x_2), T^U_{K(a_i,b_j)}(y_1, y_2)\}$.

$I^L_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2))$ = max $(I^L_{K_1(a_i)}(x_1y_1), I^L_{K_2(b_j)}(x_2), I^L_{K_2(b_j)}(y_2))$

$\geq$max$\{$max$\{I^L_{K_1(a_i)}(x_1), I^L_{K_1(a_i)}(y_1)\}, I^L_{K_2(b_j)}(x_2), I^L_{K_2(b_j)}(y_2))\}$

=max{ max$\{I^L_{K_1(a_i)}(x_1), I^L_{K_2(b_j)}(x_2)\}$, max$\{I^L_{K_1(a_i)}(y_1), I^L_{K_2(b_j)}(y_2)\}\}$

$I^L_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2)) \geq$max$\{I^L_{K(a_i,b_j)}(x_1, x_2), I^L_{K(a_i,b_j)}(y_1, y_2)\}$

We prove also that,

$I^U_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2)) \geq$max$\{I^U_{K(a_i,b_j)}(x_1, x_2), I^U_{K(a_i,b_j)}(y_1, y_2)\}$

Similarly, we prove also that

$F^L_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2)) \geq$max$\{F^L_{K(a_i,b_j)}(x_1, x_2), F^L_{K(a_i,b_j)}(y_1, y_2)\}$
$F^U_{M(a_i,b_j)}((x_1, x_2), (y_1, y_2)) \geq$max$\{F^U_{K(a_i,b_j)}(x_1, x_2), F^U_{K(a_i,b_j)}(y_1, y_2)\}$

Hence G= (K, M, A∘ B) is an interval valued neutrosophic graph.

Definition 3.12 Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) and G_1^* = (V_2, E_2) respectively. The intersection of two graphs$G_1$and G_2 is an interval valued neutrosophic soft graph G= $G_1 \cap G_2$ = (K,M, $A \cup B$), where (K, $A \cup B$) is an interval valued neutrosophic soft set over V= $V_1 \cap V_2$, (M, $A \cup B$) is an interval valued neutrosophic soft set over E= $E_1 \cap E_2$, truth-membership, indeterminacy–membership, and falsity-membership function of G for all x, z ∈V defined by

1) $$T^L_{K(e)}(x) = \begin{cases} T^L_{K_1(e)}(x) & \text{if } e \in A - B \\ T^L_{K_2(e)}(x) & \text{if } e \in A - B \\ \min(T^L_{K_1(e)}(x), T^L_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$T^U_{K(e)}(x) = \begin{cases} T^U_{K_1(e)}(x) & \text{if } e \in A - B \\ T^U_{K_2(e)}(x) & \text{if } e \in A - B \\ \min(T^U_{K_1(e)}(x), T^U_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$I^L_{K(e)}(x) = \begin{cases} I^L_{K_1(e)}(x) & \text{if } e \in A - B \\ I^L_{K_2(e)}(x) & \text{if } e \in A - B \\ \max(I^L_{K_1(e)}(x), I^L_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$I^U_{K(e)}(x) = \begin{cases} I^U_{K_1(e)}(x) & \text{if } e \in A - B \\ I^U_{K_2(e)}(x) & \text{if } e \in A - B \\ \max(I^L_{K_1(e)}(x), I^U_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$F^L_{K(e)}(x) = \begin{cases} F^L_{K_1(e)}(x) & \text{if } e \in A - B \\ F^L_{K_2(e)}(x) & \text{if } e \in A - B \\ \max(F^L_{K_1(e)}(x), F^L_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$F^U_{K(e)}(x) = \begin{cases} F^U_{K_1(e)}(x) & \text{if } e \in A - B \\ F^U_{K_2(e)}(x) & \text{if } e \in A - B \\ \max(F^L_{K_1(e)}(x), F^U_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

2) $$T^L_{M(e)}(xz) = \begin{cases} T^L_{M_1(e)}(xz) & \text{if } e \in A - B \\ T^L_{M_2(e)}(xz) & \text{if } e \in A - B \\ \min(T^L_{M_1(e)}(xz), T^L_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$T^U_{M(e)}(xz) = \begin{cases} T^U_{M_1(e)}(xz) & \text{if } e \in A - B \\ T^U_{M_2(e)}(xz) & \text{if } e \in A - B \\ \min(T^U_{M_1(e)}(xz), T^U_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$I^L_{M(e)}(xz) = \begin{cases} I^L_{M_1(e)}(xz) & \text{if } e \in A - B \\ I^L_{M_2(e)}(xz) & \text{if } e \in A - B \\ \max(\, I^L_{M_1(e)}(xz), I^L_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$I^U_{M(e)}(xz) = \begin{cases} I^U_{M_1(e)}(xz) & \text{if } e \in A - B \\ I^U_{M_2(e)}(xz) & \text{if } e \in A - B \\ \max(\, I^L_{M_1(e)}(xz), I^U_{K_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$F^L_{M(e)}(x) = \begin{cases} F^L_{M_1(e)}(xz) & \text{if } e \in A - B \\ F^L_{M_2(e)}(xz) & \text{if } e \in A - B \\ \max(\, F^L_{M_1(e)}(xz), F^L_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$F^U_{M(e)}(xz) = \begin{cases} F^U_{M_1(e)}(xz) & \text{if } e \in A - B \\ F^U_{M_2(e)}(xz) & \text{if } e \in A - B \\ \max(\, F^L_{M_1(e)}(xz), F^U_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

Example 3.13. Let A= $\{e_1, e_2\}$ and B= $\{e_1, e_4\}$ be a set ofparameters. Consider two interval valued neutrosophic soft graphs G_1=(H_1, A) =$\{H_1(e_1)$,$H_1(e_2)\}$ and G_2=(H_2, B) = $\{H_2(e_1)$,$H_2(e_4)\}$such that

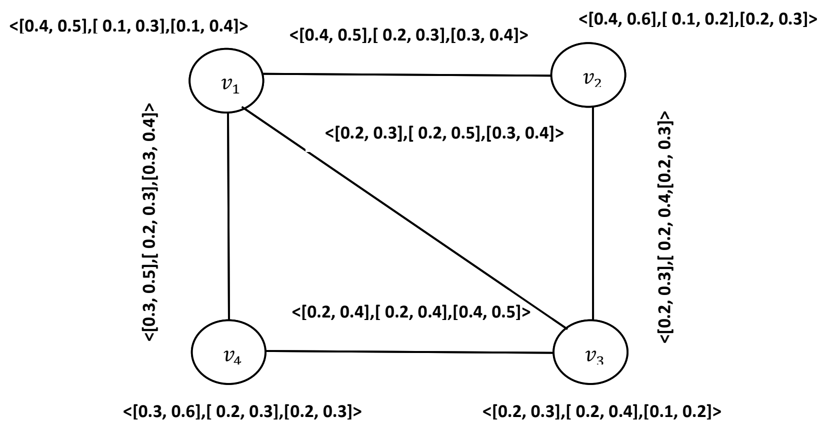
$H_1(e_1)$=($\{v_1$|([0.4, 0.5], [0.1, 0.3], [0.1, 0.4]), v_2|([0.4, 0.6], [0.1, 0.2], [0.2, 0.3]), v_3|([0.2, 0.3], [0.2, 0.4], [0.1, 0.2]), v_4|([0.3, 0.6], [0.2, 0.3], [0.2, 0.3])$\}$, $\{v_1v_2$|([0.4, 0.5], [0.2, 0.3], [0.3, 0.4]), v_2v_3|([0.2, 0.3], [0.2, 0.4], [0.4, 0.5]), v_3v_4 |([0.2, 0.4], [0.2, 0.4], [0.4, 0.5]), v_1v_4 |([0.3, 0.5], [0.2, 0.3], [0.3, 0.4]),v_1v_3 |([0.2, 0.3], [0.2, 0.5], [0.3, 0.4])$\}$).

$H_1(e_2)$=($\{v_1$|([0.4, 0.6], [0.2, 0.3], [0.1, 0.3]), v_2|([0.4, 0.7], [0.2, 0.4], [0.1, 0.3])$\}$, $\{v_1v_2$|([0.3, 0.5], [0.4, 0.5], [0.3, 0.5])$\}$).

$H_2(e_1)$=($\{v_1$|([0.3, 0.5], [0.2, 0.3], [0.3, 0.4]), v_2|([0.2, 0.3], [0.2, 0.3], [0.1, 0.4]), v_3|([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])$\}$, $\{v_1v_2$|([0.1, 0.2], [0.3, 0.4], [0.4, 0.5]), v_2v_3|([0.1, 0.3], [0.4, 0.5], [0.4, 0.5]), v_3v_1 |([0.1, 0.2], [0.3, 0.5], [0.5, 0.6])$\}$).

$H_2(e_4)$=($\{u_1$|([0.4, 0.6], [0.2, 0.3], [0.2, 0.4]), u_2|([0.4, 0.5], [0.1, 0.4], [0.2, 0.3])$\}$, $\{u_1u_2$|([0.3, 0.5], [0.4, 0.5], [0.3, 0.5])$\}$).

$H_1(e_1)$



$H_1(e_2)$

$H_2(e_1)$

$H_2(e_4)$

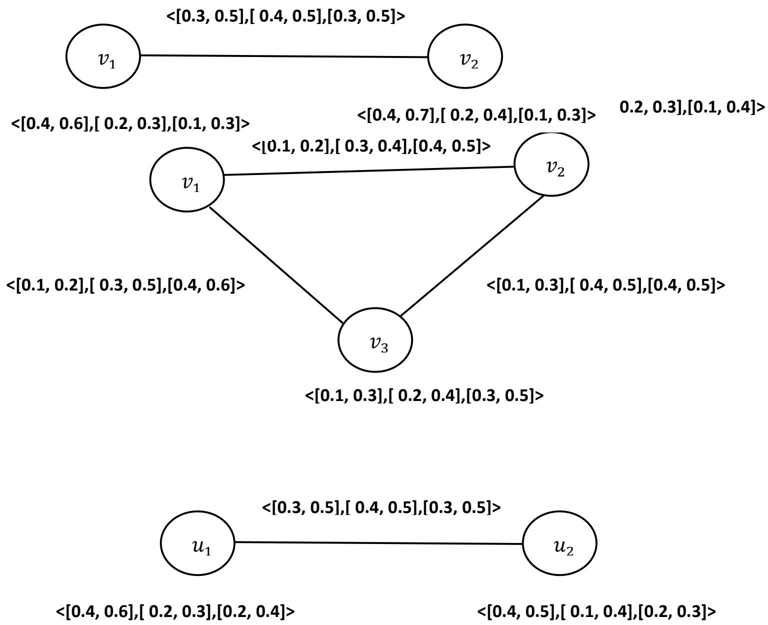


Fig. 3.7: Interval valued neutrosophic soft graph G_1= {$H_1(e_1)$,$H_1(e_2)$} and G_2= {$H_2(e_1)$,$H_2(e_4)$}

The intersection ofG_1 and G_2 is $G_1 \cap G_2$ = (H,$A \cup B$), whereA∪ B= {e_1, e_2, e_3,e_4 }, H(e_1) = $H_1(e_1) \cap H_2(e_1)$, H(e_2) and H(e_4) are interval valued neutrosophic graphs of G = $G_1 \cap G_2$. are shown in Fig. 3.8.

$H(e_1)$

$H(e_2)$

$H(e_4)$

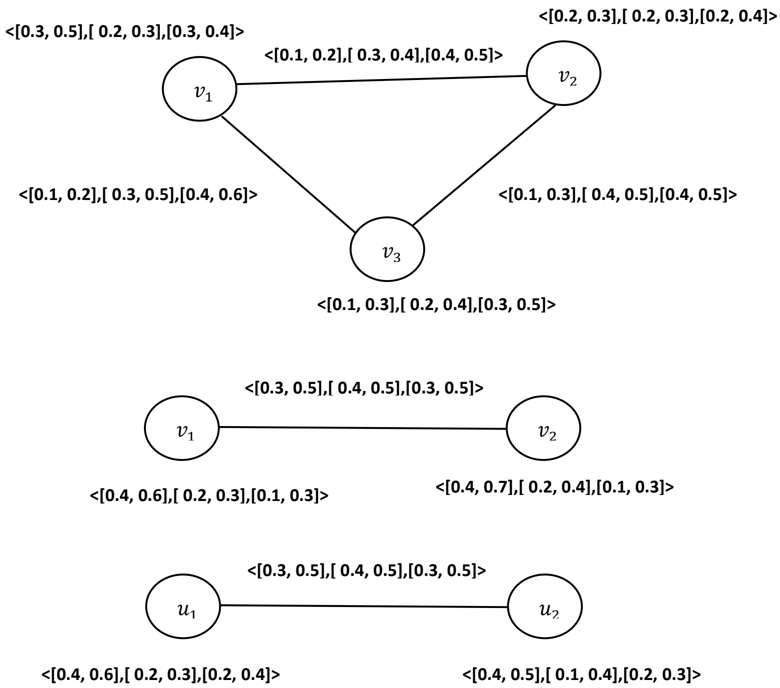


Fig. 3.8: Interval valued neutrosophic soft graph G = $G_1 \cap G_2$.

Definition 3.14 Let G_1=(K_1, M_1, A) and G_2=(K_2, M_2, B) be two interval valued neutrosophic graphs of G_1^* = (V_1, E_1) and G_1^* = (V_2, E_2) respectively. The union of two graphs$G_1$and G_2 is an interval valued neutrosophic soft graph G= $G_1 \cup G_2$ = (K,M, $A \cup B$), where (K, $A \cup B$) is an interval valued neutrosophic soft set over V= $V_1 \cup V_2$, (M, $A \cup B$) is an interval valued neutrosophic soft set over E= $E_1 \cap E_2$, truth-membership, indeterminacy-membership, and falsity-membership function of G for all x, z ∈V defined by:

1) $$T^L_{K(e)}(x) = \begin{cases} T^L_{K_1(e)}(x) & \text{if } e \in A-B \\ T^L_{K_2(e)}(x) & \text{if } e \in A-B \\ \max(T^L_{K_1(e)}(x), T^L_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$T^U_{K(e)}(x) = \begin{cases} T^U_{K_1(e)}(x) & \text{if } e \in A-B \\ T^U_{K_2(e)}(x) & \text{if } e \in A-B \\ \max(T^U_{K_1(e)}(x), T^U_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$I^L_{K(e)}(x) = \begin{cases} I^L_{K_1(e)}(x) & \text{if } e \in A-B \\ I^L_{K_2(e)}(x) & \text{if } e \in A-B \\ \min(I^L_{K_1(e)}(x), I^L_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$I^U_{K(e)}(x) = \begin{cases} I^U_{K_1(e)}(x) & \text{if } e \in A-B \\ I^U_{K_2(e)}(x) & \text{if } e \in A-B \\ \min(I^L_{K_1(e)}(x), I^U_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$F^L_{K(e)}(x) = \begin{cases} F^L_{K_1(e)}(x) & \text{if } e \in A-B \\ F^L_{K_2(e)}(x) & \text{if } e \in A-B \\ \min(F^L_{K_1(e)}(x), F^L_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

$$F^U_{K(e)}(x) = \begin{cases} F^U_{K_1(e)}(x) & \text{if } e \in A-B \\ F^U_{K_2(e)}(x) & \text{if } e \in A-B \\ \min(F^L_{K_1(e)}(x), F^U_{K_2(e)}(x)) & \text{if } e \in A \cap B \end{cases}$$

2) $$T^L_{M(e)}(xz) = \begin{cases} T^L_{M_1(e)}(xz) & \text{if } e \in A-B \\ T^L_{M_2(e)}(xz) & \text{if } e \in A-B \\ \max(T^L_{M_1(e)}(xz), T^L_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$T^U_{M(e)}(xz) = \begin{cases} T^U_{M_1(e)}(xz) & \text{if } e \in A-B \\ T^U_{M_2(e)}(xz) & \text{if } e \in A-B \\ \max(T^U_{M_1(e)}(xz), T^U_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$I^L_{M(e)}(xz) = \begin{cases} I^L_{M_1(e)}(xz) & \text{if } e \in A - B \\ I^L_{M_2(e)}(xz) & \text{if } e \in A - B \\ \min(I^L_{M_1(e)}(xz), I^L_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$I^U_{M(e)}(xz) = \begin{cases} I^U_{M_1(e)}(xz) & \text{if } e \in A - B \\ I^U_{M_2(e)}(xz) & \text{if } e \in A - B \\ \min(I^L_{M_1(e)}(xz), I^U_{K_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$F^L_{M(e)}(x) = \begin{cases} F^L_{M_1(e)}(xz) & \text{if } e \in A - B \\ F^L_{M_2(e)}(xz) & \text{if } e \in A - B \\ \min(F^L_{M_1(e)}(xz), F^L_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

$$F^U_{M(e)}(xz) = \begin{cases} F^U_{M_1(e)}(xz) & \text{if } e \in A - B \\ F^U_{M_2(e)}(xz) & \text{if } e \in A - B \\ \min(F^L_{M_1(e)}(xz), F^U_{M_2(e)}(xz)) & \text{if } e \in A \cap B \end{cases}$$

Definition 3.16. Let G_1 and G_2 be two interval valued neutrosophic soft graphs denoted byG_1 + G_2 =(K_1 + K_2, M_1 + M_2,A ∪B), Where (K_1 + K_2, A ∪ B) is an interval valued neutrosophic soft set overV_1∪ V_2 , (M_1 + M_2 , A ∪ B) is an interval valued neutrosophic soft set overE_1∪E_2∪E' defined by

(K_1 + K_2 , A ∪B) =(K_1, A) ∪ (K_2, B)

(M_1 + M_2 , A ∪B) =(M_1, A) ∪ (M_2, B) if xz ∈ E_1∪E_2,

when e ∈ A ∩ B, xz∈ E', where E' is the set of all edge joining the vertices of V_1 and V_2.

Definition 3.17 The complement of an interval valued neutrosophic soft graph$G$=(K, M, A) denoted by $\bar{G}$=($\overline{K}$, $\overline{M}$, $\bar{A}$).

1. $\bar{A}$ =A
2. $\overline{K(e)}$ =K(e),
3. $T^L_{\overline{M(e)}}$(x, z) = min($T^L_{K(e)}$(x) ,$T^L_{K(e)}$(z)) $-T^L_{M(e)}$(x,z),
   $T^U_{\overline{M(e)}}$(x, z) = min($T^L_{K(e)}$(x) ,$T^L_{K(e)}$(z)) $-T^U_{M(e)}$(x,z),
   $I^L_{\overline{M(e)}}$(x, z) = min($I^L_{K(e)}$(x) ,$I^L_{K(e)}$(z)) $-I^L_{M(e)}$(x,z),
   $I^U_{\overline{M(e)}}$(x, z) = min($T^L_{K(e)}$(x) ,$I^L_{K(e)}$(z)) $-I^U_{M(e)}$(x,z),
   $F^L_{\overline{M(e)}}$(x, z) = min($F^L_{K(e)}$(x) ,$F^L_{K(e)}$(z)) $-F^L_{M(e)}$(x,z),
   $F^U_{\overline{M(e)}}$(x, z) = min($T^L_{K(e)}$(x) ,$F^L_{K(e)}$(z)) $-F^U_{M(e)}$(x,z), for all x, z ∈V,e ∈ A.

Definition 3.18 An interval valued neutrosophic soft graph G is a complete interval valued neutrosophic soft graph if H(e) is a complete interval valued neutrosophic graph of G for all e ∈A, i.e.

$T^L_{M(e)}$(x,z)= min($T^L_{K(e)}$(x) ,$T^L_{K(e)}$(z))
$T^U_{M(e)}$(x,z)= min($T^L_{K(e)}$(x) ,$T^L_{K(e)}$(z))

$I^L_{M(e)}$(x,z)= min($I^L_{K(e)}$(x) ,$I^L_{K(e)}$(z))
$I^U_{M(e)}$(x,z)= min($T^L_{K(e)}$(x) ,$I^L_{K(e)}$(z))
$F^L_{M(e)}$(x,z) = min($F^L_{K(e)}$(x) ,$F^L_{K(e)}$(z))
$F^U_{M(e)}$(x,z) = min($T^L_{K(e)}$(x) ,$F^L_{K(e)}$(z)), For all x, z ∈V, e ∈ A.

Example 3.19.Consider a simple graphG^*=(V, E) such that V=$\{u_1, u_2, u_3, u_4\}$ and E=$\{u_1u_2, u_2u_3, u_3u_1\}$.

Let A= $\{e_1, e_2, e_3\}$be a set ofparameters. Let (K, A) be an interval valued neutrosophic graph soft sets over V with its approximation function. K:A ⟶P(V) defined by

$K(e_1)$=($\{u_1$|([0.1, 0.4], [0.1, 0.3], [0.2, 0.3]), u_2|([0.1, 0.3], [0.1, 0.2], [0.1, 0.4]), u_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])$\}$.

$K(e_2)$=($\{u_1$|([0.3, 0.5], [0.2, 0.3], [0.3, 0.4]), u_2|([0.2, 0.3], [0.2, 0.3], [0.1, 0.4]), u_3|([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])$\}$.

$K(e_3)$=($\{u_1$|([0.4, 0.5], [0.1, 0.3], [0.1, 0.4]), u_2|([0.4, 0.6], [0.1, 0.2], [0.2 0.3]), u_3|([0.2, 0.3], [0.2, 0.4], [0.1, 0.2]), u_4|([0.3, 0.6], [0.2, 0.3], [0.2, 0.3])$\}$.

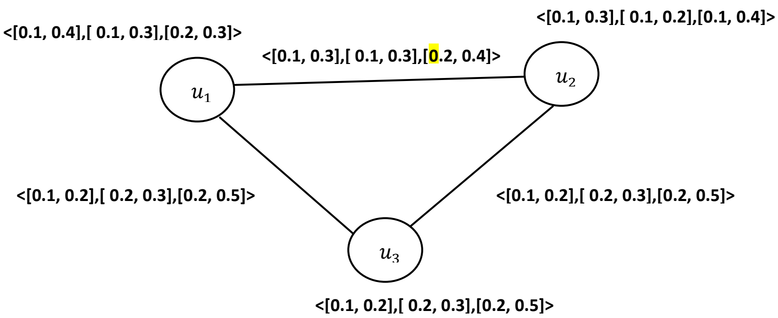
Let (M, A) be an interval valued neutrosophic graph soft sets over E with its approximation function. M:A⟶P(E) defined by

$M(e_1)$=$\{u_1u_2$|([0.1, 0.3], [0.1, 0.3], [0.2, 0.4]), u_2u_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5]), u_3u_1 |([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])$\}$.

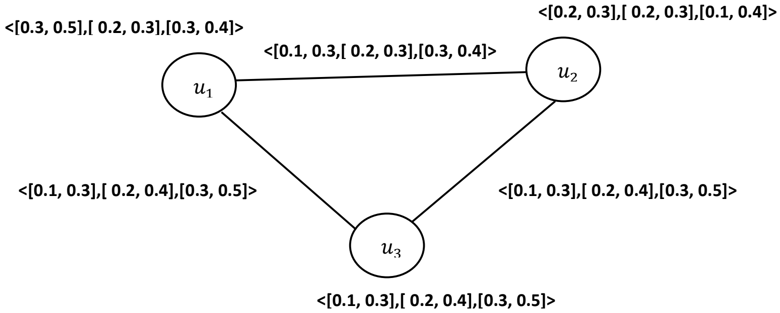
$M(e_2)$=$\{u_1u_2$|([0.1, 0.3], [0.2, 0.3], [0.3, 0.4]), u_2u_3|([0.1, 0.3], [0.2, 0.4], [0.3 0.5]), u_3u_1 |([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])$\}$.

$M(e_3)$=$\{u_1u_2$|([0.4, 0.5], [0.1, 0.3], [0.2, 0.4]), u_2u_3|([0.2, 0.3], [0.2, 0.4], [0.2, 0.3]),u_3u_4|([0.2, 0.3], [0.2, 0.4], [0.2, 0.3]), u_4u_1|([0.3, 0.5], [0.2, 0.3], [0.2, 0.4]),u_1u_3|([0.2, 0.3], [0.2, 0.4], [0.1, 0.4]), u_2u_4|([0.2, 0.6], [0.2, 0.4], [0.2, 0.3])$\}$

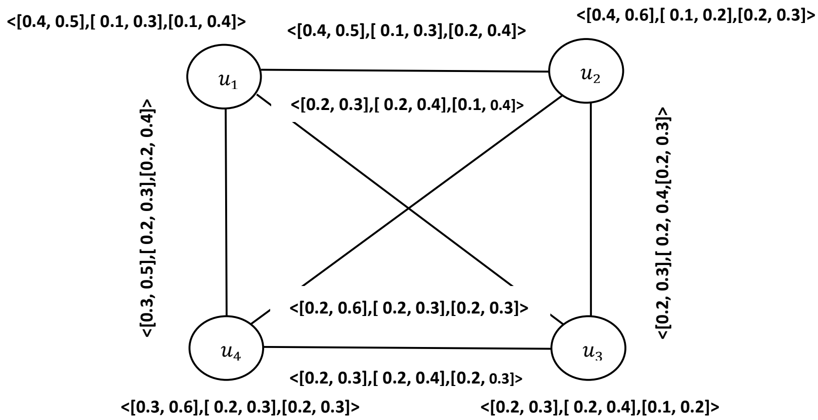
It is easy to see that $H(e_1)$, $H(e_2)$, $H(e_3)$ are complete interval valued neutrosophic graphs of G corresponding to the parameterse_1, e_2, e_3 respectively as shown in Fig. 3.9.



$H(e_1)$



$H(e_2)$



$H(e_3)$

Fig. 3.9: Complete interval valued neutrosophic soft graph G={ $H(e_1)$, $H(e_2)$, $H(e_3)$}.

Definition 3.20: An interval valued neutrosophic soft graph G is a strong interval valued neutrosophic soft graph if H(e) is a strong interval valued neutrosophic graph of G for all e ∈A, i.e.

$T^L_{M(e)}$(x, z)= min($T^L_{K(e)}$(x) ,$T^L_{K(e)}$(z))
$T^U_{M(e)}$(x, z)= min($T^L_{K(e)}$(x) ,$T^L_{K(e)}$(z))
$I^L_{M(e)}$(x, z)= min($I^L_{K(e)}$(x) ,$I^L_{K(e)}$(z))
$I^U_{M(e)}$(x, z)= min($T^L_{K(e)}$(x) ,$I^L_{K(e)}$(z))
$F^L_{M(e)}$(x, z) = min($F^L_{K(e)}$(x) ,$F^L_{K(e)}$(z))
$F^U_{M(e)}$(x, z) = min($T^L_{K(e)}$(x) ,$F^L_{K(e)}$(z)), for all x, z ∈V, e ∈ A.

Example 3.21.Consider a simple graphG^*=(V, E) such that V=$\{u_1$, u_2,u_3, $u_4\}$ and E=$\{u_1u_2$, u_2u_3,$u_3u_1\}$.

Let A= $\{e_1, e_2, e_3\}$be a set ofparameters. Let (K, A) be an interval valued neutrosophic graph soft sets over V with its approximation function. K:A⟶P(V) defined by

$K(e_1)$=($\{u_1$|([0.1, 0.4], [0.1, 0.3], [0.2, 0.3]), u_2|([0.1, 0.3], [0.1, 0.2], [0.1, 0.4]), u_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])$\}$.

$K(e_2)$=($\{u_1$|([0.3, 0.5], [0.2, 0.3], [0.3, 0.4]), u_2|([0.2, 0.3], [0.2, 0.3], [0.1, 0.4]), u_3|([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])$\}$.

$K(e_3)$=({u_1|([0.4, 0.5], [0.1, 0.3], [0.1, 0.4]), u_2|([0.4, 0.6], [0.1, 0.2], [0.2 0.3]), u_3|([0.2, 0.3], [0.2, 0.4], [0.1, 0.2]), u_4|([0.3, 0.6], [0.2, 0.3], [0.2, 0.3])}.

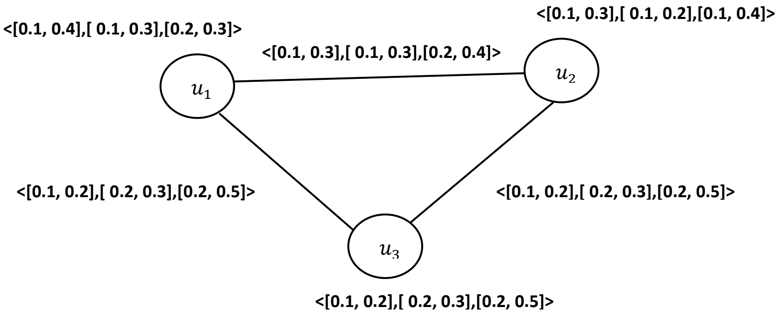
Let (M, A) be an interval valued neutrosophic graph soft sets over E with its approximation function. M:A⟶P(E) defined by

$M(e_1)$={u_1u_2|([0.1, 0.3], [0.1, 0.3], [0.2, 0.4]), u_2u_3|([0.1, 0.2], [0.2, 0.3], [0.2, 0.5]), u_3u_1 |([0.1, 0.2], [0.2, 0.3], [0.2, 0.5])}.

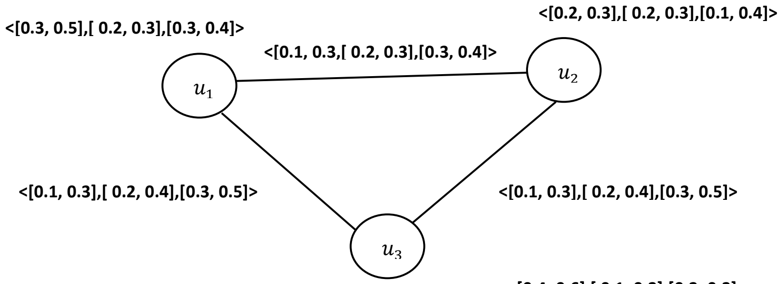
$M(e_2)$={u_1u_2|([0.1, 0.3], [0.2, 0.3], [0.3, 0.4]), u_2u_3|([0.1, 0.3], [0.2, 0.4], [0.3 0.5]), u_3u_1 |([0.1, 0.3], [0.2, 0.4], [0.3, 0.5])}.

$M(e_3)$={u_1u_2|([0.4, 0.6], [0.1, 0.3], [0.2, 0.4]), u_2u_3|([0.2, 0.3], [0.2, 0.4], [0.2, 0.3]),u_3u_4|([0.2, 0.3], [0.2, 0.4], [0.2, 0.3]), u_4u_1|([0.3, 0.5], [0.2, 0.3], [0.2, 0.4])}

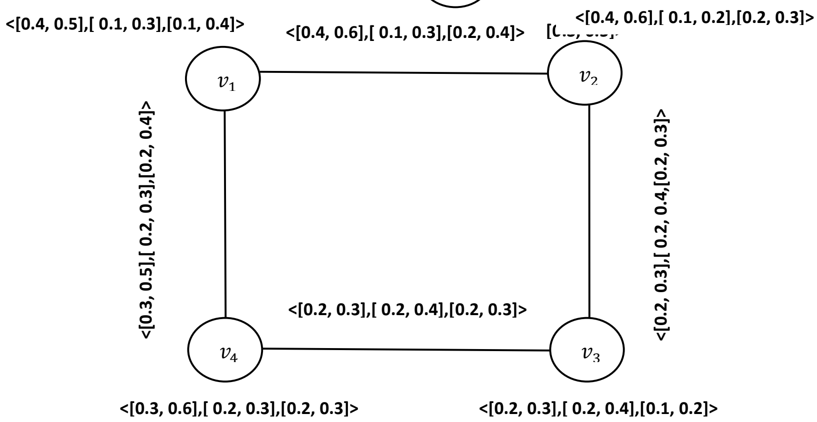
It is easy to see that $H(e_1)$, $H(e_2)$, $H(e_3)$ are strong interval valued neutrosophic graphs of G corresponding to the parameterse_1, e_2, e_3 respectively as shown in Fig. 3.10.



$H(e_1)$



$H(e_2)$



$H(e_3)$

Fig. 3.10: Strong interval valued neutrosophic soft graph G={ $H(e_1)$, $H(e_2)$, $H(e_3)$}.

4. APPLICATION

Interval valued neutrosophic soft set has several applications in decision making problems and can be used to deal with uncertainties from our different daily life problems. In this section, we apply the concept of interval valued neutrosophic soft sets in a decision making problem and then give an algorithm for the selection of optimal object based upon given sets of information.

Suppose that V=$\{h_1,h_2,h_3,h_4,h_5\}$ is the set of five houses under consideration. Mr. X is going to buy one of the houses on the basis of wishing parameters or attributes set A=$\{e_1$= large,e_2= beautiful, e_3= green surrounding$\}$.(K, A) is the interval valued neutrosophic soft set on V which describes the value of the houses based upon the given parameters e_1= large,e_2= beautiful, e_3= green surrounding, respectively.

$K(e_1)$=($\{h_1$|([0.3, 0.4], [0.2, 0.3], [0.3, 0.4]), h_3|([0.2, 0.3], [0.2, 0.3], [0.1, 0.4]), h_4|([0.2, 0.3], [0.2, 0.4], [0.3, 0.5])$\}$.

$K(e_2)$=($\{h_1$|([0.2, 0.5], [0.1, 0.3], [0.1, 0.3]), h_2|([0.3, 0.4], [0.1, 0.2], [0.2, 0.3]), h_3|([0.2, 0.3], [0.2, 0.3], [0.3, 0.4]),h_4|([0.3, 0.4], [0.2, 0.3], [0.1, 0.2]), h_5|([0.3, 0.4], [0.1, 0.2], [0.2, 0.4])$\}$.

$K(e_3)$ =($\{h_1$|([0.4, 0.5], [0.1, 0.3], [0.1, 0.4]), h_2|([0.4, 0.6], [0.1, 0.2], [0.2, 0.3]), h_3|([0.2, 0.3], [0.2, 0.4], [0.1, 0.2]),h_4|([0.3, 0.6], [0.2, 0.3], [0.2, 0.3]), h_5|([0.2, 0.3], [0.2, 0.3], [0.2, 0.4])$\}$.

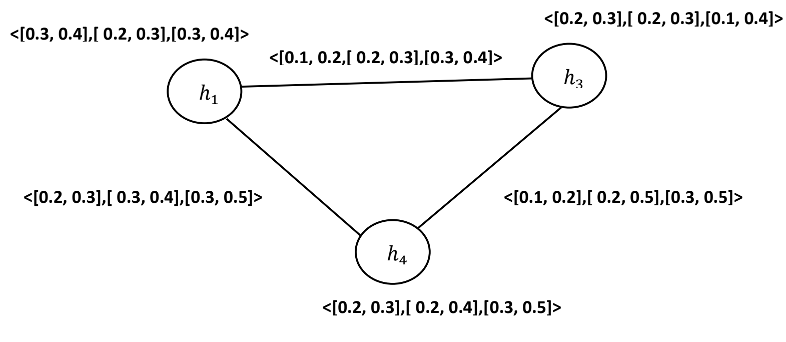
(M, A) is an interval valued neutrosophic soft sets on E= $\{h_1h_2$, h_1h_3, h_1h_4,h_1h_5,h_2h_3,h_2h_4, h_2h_5, h_3h_4,$h_4h_5\}$ which describe the value of two houses corresponding to the given parameters e_1, e_2 and e_3.

$M(e_1)$=$\{h_1h_3$|([0.1, 0.2], [0.2, 0.3], [0.3, 0.4]), h_3h_4|([0.1, 0.2], [0.2, 0.5], [0.3, 0.5]), h_1h_4 |([0.2, 0.3], [0.3, 0.4], [0.3, 0.5])$\}$.

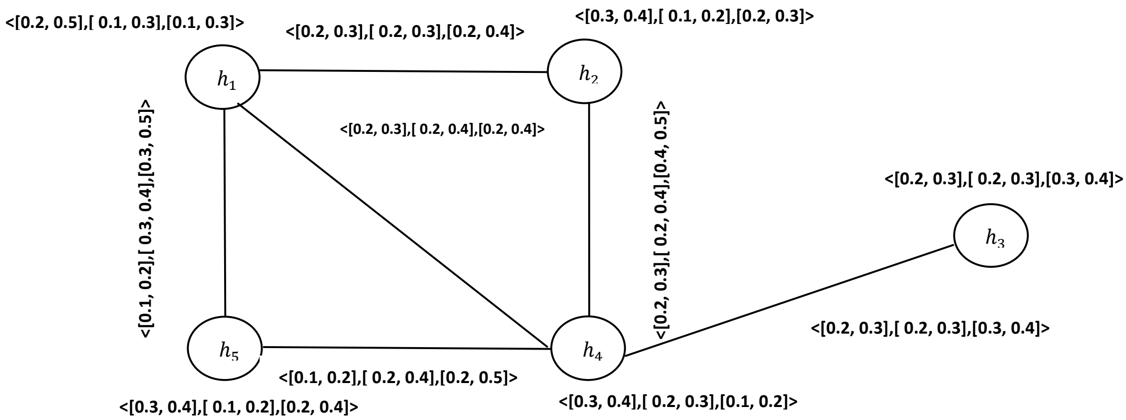
$M(e_2)$=$\{h_1h_2$|([0.2, 0.3], [0.2, 0.3], [0.2, 0.4]), h_1h_4|([0.2, 0.3], [0.2, 0.4], [0.2, 0.4]), h_1h_5 |([0.1, 0.3], [0.3, 0.4], [0.3, 0.5]), h_2h_4|([0.2, 0.3], [0.2, 0.4], [0.4, 0.5]), h_4h_5|([0.1, 0.2], [0.2, 0.4], [0.2, 0.5]), h_4h_3 |([0.2, 0.3], [0.2, 0.3], [0.3, 0.4])$\}$.

$M(e_3)$= $\{h_1h_2$|([0.4, 0.6], [0.2, 0.3], [0.3, 0.4]), h_1h_4|([0.3, 0.5], [0.3, 0.4], [0.2, 0.4]), h_2h_3 |([0.2, 0.3], [0.2, 0.5], [0.3, 0.4]), h_2h_5|([0.1, 0.2], [0.3, 0.4], [0.4, 0.5]), h_2h_4|([0.2, 0.4], [0.3, 0.4], [0.5, 0.6]), h_3h_4 |([0.2, 0.3], [0.4, 0.5], [0.2, 0.3])$\}$.

The interval valued neutrosophic soft sets$H(e_1)$, $H(e_2)$, $H(e_3)$of interval valued neutrosophic graphs of G =(K, M, A) corresponding to the parameterse_1, e_2, e_3 respectively, as shown in Fig. 3.11.



$H(e_1)$



$H(e_2)$

$H(e_3)$

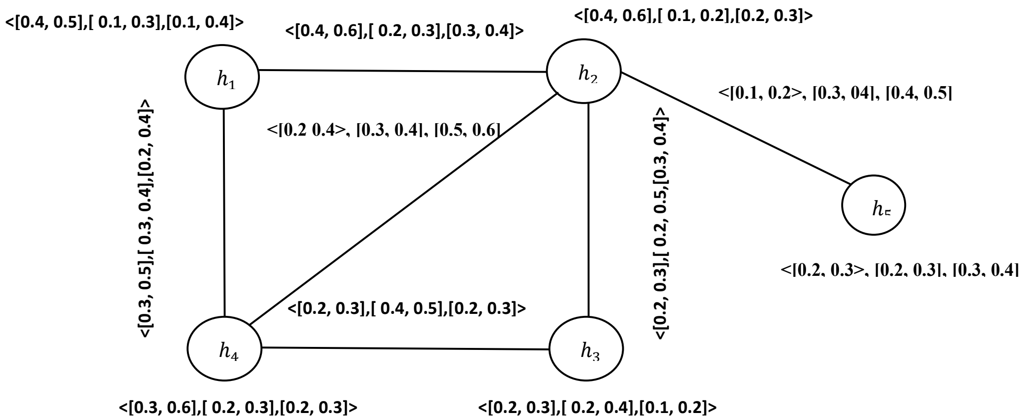


Fig. 3.11: Interval valued neutrosophic soft graph G={ $H(e_1)$, $H(e_2)$, $H(e_3)$}.

The interval valued neutrosophic graphs $H(e_1)$, $H(e_2)$, $H(e_3)$ corresponding to the parameters "large", "beautiful" and "green surrounding", respectively are represented by the following incidence matrix.

$H(e_1)$=

$$\begin{bmatrix} \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0.1,0.2],[0.2,0.3],[0.3,0.4] \rangle & \langle [0.2,0.3],[0.3,0.4],[0.3,0.5] \rangle & \langle [0,0],[0,0],[0,0] \rangle \\ \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle \\ \langle [0.1,0.2],[0.2,0.3],[0.3,0.4] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0.1,0.2],[0.2,0.5],[0.3,0.5] \rangle & \langle [0,0],[0,0],[0,0] \rangle \\ \langle [0.2,0.3],[0.3,0.4],[0.3,0.5] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0.1,0.2],[0.2,0.5],[0.3,0.5] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle \\ \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle & \langle [0,0],[0,0],[0,0] \rangle \end{bmatrix}$$

$H(e_2)$=

$$\begin{bmatrix} <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.3],[0.2,0.4]> & <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.4],[0.2,0.4]> & <[0.1,0.3],[0.3,0.4],[0.3,0.5]> \\ <[0.2,0.3],[0.2,0.3],[0.2,0.4]> & <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.4],[0.4,0.5]> & <[0,0],[0,0],[0,0]> \\ <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.3],[0.3,0.4]> & <[0,0],[0,0],[0,0]> \\ <[0.2,0.3],[0.2,0.4],[0.2,0.4]> & <[0.2,0.3],[0.2,0.4],[0.4,0.5]> & <[0.2,0.3],[0.2,0.3],[0.3,0.4]> & <[0,0],[0,0],[0,0]> & <[0.1,0.2],[0.2,0.4],[0.2,0.5]> \\ <[0.1,0.3],[0.3,0.4],[0.3,0.5]> & <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> & <[0.1,0.2],[0.2,0.4],[0.2,0.5]> & <[0,0],[0,0],[0,0]> \end{bmatrix}$$

And $H(e_3)$=

$$\begin{bmatrix} <[0,0],[0,0],[0,0]> & <[0.4,0.6],[0.2,0.3],[0.3,0.4]> & <[0,0],[0,0],[0,0]> & <[0.3,0.5],[0.3,0.4],[0.2,0.4]> & <[0,0],[0,0],[0,0]> \\ <[0.4,0.6],[0.2,0.3],[0.3,0.4]> & <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.5],[0.3,0.4]> & <[0.2,0.4],[0.3,0.4],[0.5,0.6]> & <[0.1,0.2],[0.3,0.4],[0.4,0.5]> \\ <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.5],[0.3,0.4]> & <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.4,0.5],[0.2,0.3]> & <[0,0],[0,0],[0,0]> \\ <[0.3,0.5],[0.3,0.4],[0.2,0.4]> & <[0.2,0.4],[0.3,0.4],[0.5,0.6]> & <[0.2,0.3],[0.4,0.5],[0.2,0.3]> & <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> \\ <[0,0],[0,0],[0,0]> & <[0.1,0.2],[0.3,0.4],[0.4,0.5]> & <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> & <[0,0],[0,0],[0,0]> \end{bmatrix}$$

After performing some operation (AND or OR); we obtain the resultant interval valued neutrosophic graph H(e), where e= $e_1 \wedge e_2 \wedge e_3$. The incidence matrix of resultant interval neutrosophic soft graph is

$H(e_3)$=

$$\begin{bmatrix} <[0,0],[0,0],[0,0]> & <[0.2,0.3],[0.2,0.3],[0.3,0.4]> & <[0,0],[0.2,0.3],[0.3,0.4]> & <[0.2,0.3],[0.3,0.4],[0.3,0.5]> & <[0,0],[0.3,0.4],[0.4,0.5]> \\ <[0,0],[0.3,0.4],[0.4,0.5]> & <[0,0],[0,0],[0,0]> & <[0,0],[0.2,0.5],[0.3,0.4]> & <[0,0],[0.3,0.4],[0.5,0.6]> & <[0,0],[0.3,0.4],[0.4,0.5]> \\ <[0,0],[0.2,0.3],[0.3,0.4]> & <[0,0],[0.2,0.5],[0.3,0.4]> & <[0,0],[0,0],[0,0]> & <[0.1,0.2],[0.4,0.5],[0.2,0.3]> & <[0,0],[0,0],[0,0]> \\ <[0,0],[0.3,0.4],[0.3,0.5]> & <[0.2,0.3],[0.3,0.4],[0.5,0.6]> & <[0.1,0.2],[0.4,0.5],[0.3,0.5]> & <[0,0],[0,0],[0,0]> & <[0,0],[0.2,0.4],[0.2,0.5]> \\ <[0,0],[0.3,0.4],[0.3,0.5]> & <[0,0],[0.3,0.4],[0.4,0.5]> & <[0,0],[0,0],[0,0]> & <[0,0],[0.2,0.4],[0.2,0.5]> & <[0,0],[0,0],[0,0]> \end{bmatrix}$$

Sahin (2015) defined the average possible membership degree of element x to interval valued neutrosophic set$A = \langle [T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)] \rangle$ as follows:

$$S_k(x) = \frac{1}{3}\left[\frac{T_A^L(x) + T_A^U(x)}{2} + 1 - \frac{I_A^L(x) + I_A^U(x)}{2} + 1 - \frac{F_A^L(x) + Fv_A^U(x)}{2}\right]$$
$$= \frac{T_A^L(x) + T_A^U(x) + 4 - I_A^L(x) - I_A^U(x) - F_A^L(x) - F_A^U(x)}{6}$$

Based on $S_k(x)$ we depictedthe Tabular representation of score value of incidence matrix of resultant interval valued neutrosophic graph H(e)with $S_k$andchoice valuefor each house h_k for k= 1, 2, 3, 4.

Table 2. Tabular representation of score values with choice values.

	h_1	h_2	h_3	h_4	h_5	h'_k
h_1	0.666	0.55	0.466	0.5	0.4	2,582
h_2	0.4	0.666	0.433	0.366	0.4	2,265
h_3	0.466	0.433	0.666	0.483	0.666	2,714
h_4	0.416	0.45	0.433	0.666	0.45	2,415
h_5	0.416	0.383	0.666	0.45	0.666	2,581

Clearly, the maximum score value is 2,714, scored by the h_3 Mr. X, will buy the house h_3.

We present our method as an algorithm that is used in our application.

Algorithm

1. Input the set P of choice of parameters of Mr. X, A is subset of P.
2. Input the interval valued neutrosophic soft sets (K, A) and (M, A).
3. Construct the interval valued neutrosophic soft graph G = (K, M, A).
4. Compute the resultant interval valued neutrosophic soft graph H(e) $=\bigcap_k H(e_k)$ fore $= \bigwedge_k e_k \; \forall$ k.
5. Consider the interval valued neutrosophic graph H(e) and its incidence matrix form.
6. Compute the score S_k of $h_k \forall$ k.
7. The decision is h_k if $h'_k = \max_i h_k$.
8. If k has more than one value then any one of h_k may be chosen.

5. CONCLUSION

Interval valued neutrosophic soft sets is a generalization of fuzzy soft sets, intuitionistic fuzzy soft sets and neutrosophic soft sets. The neutrosophic set model is an important tool for dealing with real scientific and engineering applications; it can handle not only incomplete information, but also the inconsistent information and indeterminate information which exists in real situations. Interval valued neutrosophic models give more precisions, flexibility and compatibility to the system as compared to the classical, fuzzy and/or intuitionistic fuzzy and single valued neutrosophic models. In this paper, we have introduced certain types of interval valued neutrosophic soft graphs, such as strong interval valued neutrosophic soft graph, complete interval valued neutrosophic soft graphs and complement of strong interval valued neutrosophic soft graphs. We introduced some operations such as Cartesian product, composition, intersection, union and join on an interval valued neutrosophic soft graphs. We presented an application of interval valued neutrosophic soft graphs in decision making. In future studies, we plan to extend our research to regular interval valued neutrosophic soft graphs and irregular interval valued neutrosophic soft graphs.

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A Group Decision Making Framework Based on Neutrosophic TOPSIS Approach for Smart Medical Device Selection

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Abstract

Advances in the medical industry has become a major trend because of the new developments in information technologies. This research offers a novel approach for estimating the smart medical devices (SMDs) selection process in a group decision making (GDM) in a vague decision environment. The complexity of the selected decision criteria for the smart medical devices is a significant feature of this analysis. To simulate these processes, a methodology that combines neutrosophics using bipolar numbers with Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) under GDM is suggested. Neutrosophics with TOPSIS approach is applied in the decision making process to deal with the vagueness, incomplete data and the uncertainty, considering the decisions criteria in the data collected by the decision makers (DMs). In this research, the stress is placed upon the choosing of sugar analyzing smart medical devices for diabetics' patients. The main objective is to present the complications of the problem, raising interest among specialists in the healthcare industry and assessing smart medical devices under different evaluation criteria. The problem is formulated as a multi criteria decision type with seven alternatives and seven criteria, and then edited as a multi criteria decision model with seven alternatives and seven criteria. The results of the neutrosophics with TOPSIS model are analyzed, showing that the competence of the acquired results and the rankings are sufficiently stable. The results of the suggested method are also compared with the neutrosophic extensions AHP and MOORA models in order to validate and prove the acquired results. In addition, we used the SPSS program to check the stability of the variations in the rankings by the Spearman coefficient of correlation. The selection methodology is applied on a numerical case, to prove the validity of the suggested approach.

Introduction

In the light of emerging digital technologies and their applications in medical systems, a rapid development is noticed in an extensive number of medical devices. The healthcare manufacture is revolutionizing how patients are cured by using technological advances. The leading factor of this transformation is based on evolutions in actuator and sensor technology, becoming more qualified in merging with electrical and chemical elements. Developments in Nano and micro technology make it easier for communicating with extrinsic systems, better data collecting, creating tools, devices and apparatuses helping medical staff as well as patients, more and better substances that can be vaccinated immediately into human body. These innovative ways of remediation help curing health cases outside of a healthcare facility. Minimally invasive or noninvasive small scale medical tools offer significant challenges to inspiring new smart and powerful devices. Portable devices today can accurately measure the percentage of diabetes in the blood. Different medical tasks and abilities are performed by medical robots. Digitalization is revolutionizing the submission of healthcare services, both at hospitals and at home, by using surgical robots for help during more complex procedures or for simpler tasks, such as use a diabetes analyzer, management of medicines to patients. In the future, developed medical devices can possibly enable patients to connect healthcare services without the necessity of physically attend the hospitals. In addition, smartphone applications help patients to interact with medical devices connected to the patients, and help patients to remotely access these services. Instead of patients visiting medical staff and hospitals to measure percentage of diabetes in the blood, patients can use the portable devices to measure the percentage of diabetes in the blood at home, without discomfort and fatigue, especially after a breakthrough in the manufacture of portable a diabetes analyzer. Also, a diabetes analyzer can be connected to external smartphones for analyzing the results more accurately. As a response to this market growth of various devices of broad availability, the healthcare industry is actively following new ways on how to select those devices that best address the requirements of patients. Occasionally, the requirements can be uncertain, ambiguous and vague, as they are related with the expectations and demands of human beings. Thus, MDs can be selected based on decision criteria such as their accuracy, precision, and reliability. This study aims to suggest a set of valuation criteria for the healthcare industry in relationship to the selection and valuation of portable diabetes analyzer devices and their results. There are many resources that can be used for collecting the evaluation criteria, such as the judgments of academic experts, industrial and decision makers, the current scientific literature or available regulations. Decision making is mostly about choosing the preferable choice between a set of alternatives by considering the influence of many criteria altogether. In the last five decades, the multi criteria decision making (MCDM) methodology became one of the most important key in solving complicated and complex decision problems in the existence of multiple criteria and alternatives [1]. The MCDM methodology can be used to resolve multi valuation and ordering problems that combine a number of inconsistent criteria. After this progress, several types of MCDM methods are suggested to successfully solve various types of decision making problems. This powerful methodology often needs qualitative and quantitative data, which are used in the measurement of obtainable alternatives. In multi MCDM problems, interdependency, mutuality and interactivity features between decision criteria are of a vague nature, which obscures the task of a membership [2]. However, most methods proved inadequate and inappropriate in solving and explaining real life problems, mostly because they rely on crisp values. Many MCDM methods use the fuzzy or the intuitionistic fuzzy set theories to overcome this obstacle. Nevertheless, F and IF numbers are also not always appropriate. Classes of F and IF sets proved to be efficient in some implementations. Nevertheless, in our opinion that is a compromise, since the Neutrosophic set offers major and better possibilities [3, 4]. The notion / concept of neutrosophic set provides a substitute approach where there is a lack of accuracy to the determinations imposed by the crisp sets or traditional fuzzy sets, and in situations where the presented information is not suitable to locate its inaccuracy. Neutrosophic sets are very powerful and successful in overcoming situations and cases in incomplete information environment, uncertainty, vagueness and imprecision, and it is described by a membership degree, an indeterminacy degree and a nonmembership degree [5]. Therefore, neutrosophic sets introduce a qualified tool for expressing DMs' preferences and priorities, completely determining the membership function in situations where DM opinions are subject to indeterminacy or lack of information. DMs use linguistic variables expressed in two parts, where the first part is employed to voice their preferences and the other part is used to convey the confirmation degree of linguistic variable according to each DM [6]. Neutrosophic set is becoming a scientific key tool, receiving attention from many DMs and academic researchers for developing and improving the neutrosophic methodology [7–9]. Many decision problems faced in life require the contribution of more than one DM in the decision making processes. Thus, most of MCDM methods are also extended to GDM. The key advantage of Neutrosophic sets over the crisp or fuzzy and IFs is their capability to present the positive and the negative designation of an element's value on membership, indeterminacy membership and nonmembership in the sets. When DMs express their views and opinions, they generally rely on information about more criteria and more alternatives that become more complicated. To overcome these situations, a widely accepted MCDM method is the TOPSIS method with a major advantage due to its simplicity and ability to consider a non-limited number of alternatives and criteria in the decision making process [10]. Hence, using TOPSIS is very effective in finding the expected utility of an uncertain situation, incomplete information and vagueness. TOPSIS method defines a solution at the shortest distance to the ideal solution and the greatest distance from the negative-ideal solution, but it does not reflect the proportional significance of these distances, as indicated in Fig. 1.

The main accomplishments of this research are:

- The characterization and preparation of an effective evaluation framework to lead the medical industry towards the suitable smart medical device selection.
- It also contributes to the literature by providing a novel Neutrosophic with TOPSIS method under GDM setting, by considering the interactions among medical device selection criteria in a vague environment.

The structure of this research is summarized as follows. In section 2, a review of related publications is given. Section 3 provides an introduction to the bipolar neutrosophic numbers

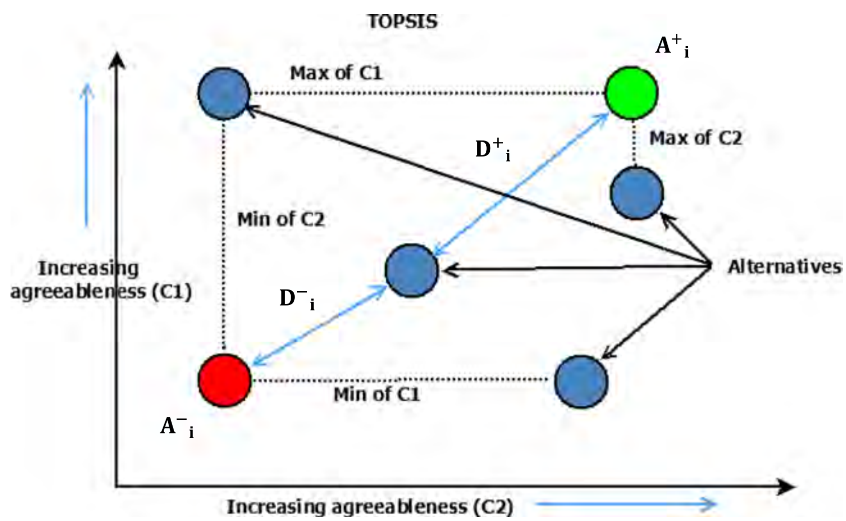

Fig. 1 The ideal solution of TOPSIS method

and to the steps of the suggested method. Section 4 gives a detailed commentary of the alternatives and evaluation criteria in a numerical experiment, in which diabetes analyzer devices are selected to present the execution of the applied method. Finally, we close our research with some remarks.

Literature Review

Decision making in real life situations is the means of selecting the best candidate from several options. DMs need to consider multiple criteria in order to evaluate the best

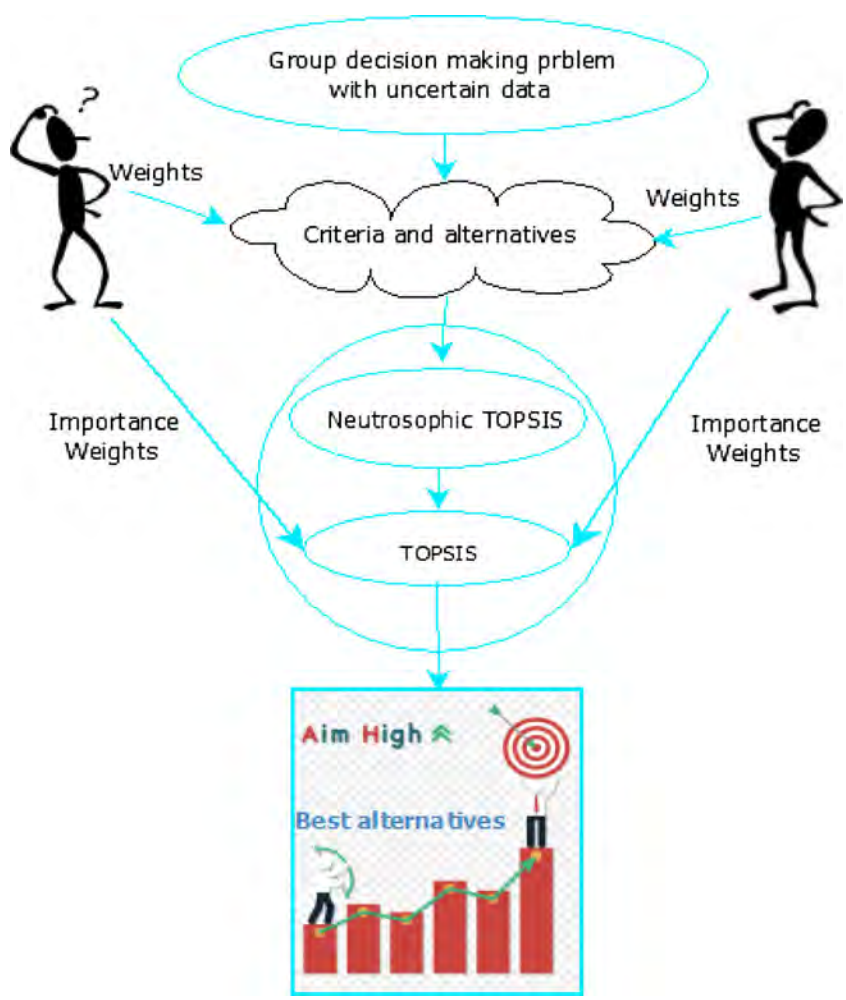

Fig. 2 The general conceptualization of the suggested method

candidate. MCDM methods are recognized by scholars since the early 1970s. In situation of multiple criteria or goals, MCDM methods include essential area of research to transact with complex problems. Many MCDM methods with characteristic features have been suggested in the literature, such as Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [11], Analytic Hierarchy Process (AHP) [12], Multi Objective Optimization on the basis of Ratio Analysis (MOORA) [13, 14]. Many types of MCDM approaches have been successfully implemented to various types of decision making problems. As these methods mostly work with crisp sets, they have been seen imperfect to deal with many decisions problems. Also, the task of identifying the best alternative becomes more challenging for a DM, as decision making gets more complex. Many such mechanisms are successfully extended to other environments. In the last two

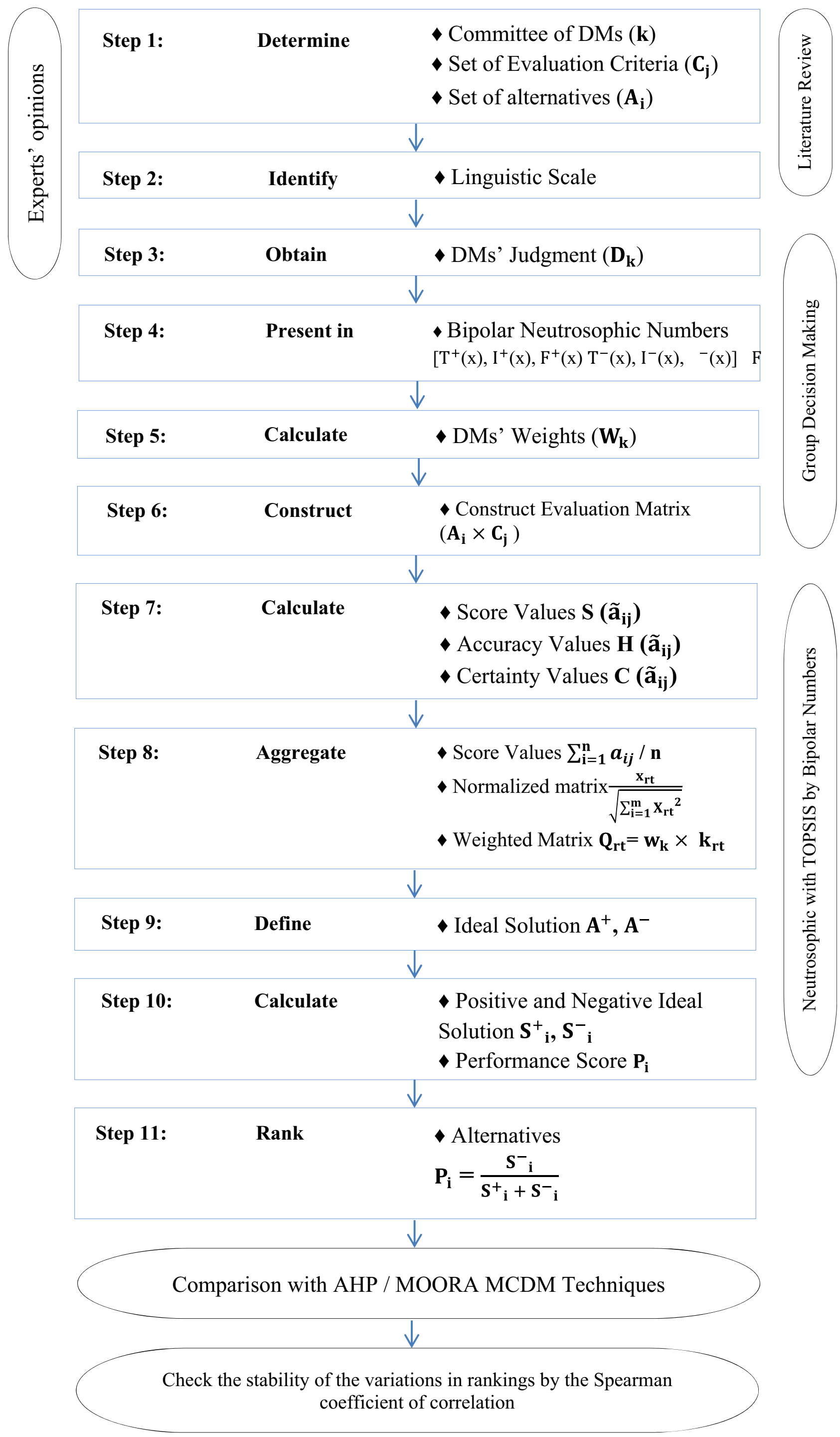

Fig. 3 Diagram of the Neutrosophic with TOPSIS method

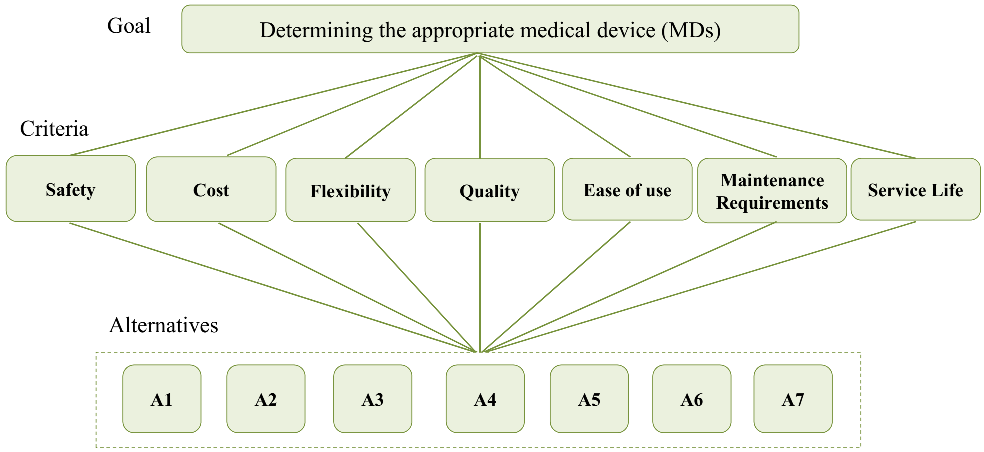

Fig. 4 The hierarchy for selecting the appropriate medical device

decades, most studies in literature apply the fuzzy and IFs theory due to its similarity to human reasoning [15, 16]. Following that, Smarandache introduced the concept of neutrosophic set, which is the generalization of Atanassov IFs, where to each element of the set is attributed a membership value, an indeterminacy value and a membership value [17]. Various types of MCDM approaches are integrated by neutrosophic set. When compared to IFs, neutrosophic sets have many advantages. Consequently, it is extensively studied by many academics [18–27]. The specific method in this study is presented in detail in the next section.

Methodology

The purpose of the suggested technique is to incubate a conceptual framework for valuation of sugar analyzing smart medical devices for diabetics' patients with consideration to predefined objectives. The following subsection comments on neutrosophic and TOPSIS, respectively. Then, the suggested method is presented.

Preliminaries

In this subsection, we give the basic definitions of neutrosophic set and bipolar neutrosophic numbers (BNNs).

Bipolar Neutrosophic Set (BNS)

We give the definition of bipolar neutrosophic set (BNS), and discuss some of its properties, including certainty, score and accuracy functions [28–32].

Definition 2.1.1.1 A bipolar neutrosophic set A in X is defined as an object of the form A = $\{\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\rangle : x \in X\}$, where $T^+, I^+, F^+: X \to [1,0]$ and $T^-, I^-, F^-: X \to [-1,0]$. The positive membership degree $T^+(x), I^+(x), F^+(x)$ denotes the truth membership, the indeterminate membership and the false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A, and the negative membership degree $T^-(x), I^-(x), F^-(x)$ denotes the truth membership, the indeterminate membership and the false membership of an element $x \in X$ to some implicit counter property corresponding to a bipolar neutrosophic set A.

Definition 2.1.1.2 Let $A_1 = \{\langle x, T_1^+(x), I_1^+(x), F_1^+(x), T_1^-(x), I_1^-(x), F_1^-(x)\rangle$ and $A_2 = \{\langle x, T_2^+(x), I_2^+(x), F_2^+(x), T_2^-(x), I_2^-(x), F_2^-(x)\rangle$ be two bipolar neutrosophic sets. Then, their union is defined as: $(A_1 \cup A_2)(x) = (\max(T_1^+(x), T_2^+(x)), \frac{I_1^+(x)+I_2^+(x)}{2}, \min((F_1^+(x), F_2^+(x)), \min(T_1^-(x), = T_2^+(x)), \frac{I_1^-(x)+I_2^-(x)}{2}, \max((F_1^-(x), F_2^-(x))),$ for all $x \in X$.

Definition 2.1.1.3 Let $\tilde{a}_1 = (T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-)$ and $\tilde{a}_2 = (T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^-)$ be two bipolar neutrosophic numbers. Then, the operations for NNs are defined as below:

i. $\lambda \tilde{a}_1 = \left(1-\left(1-T_1^+\right)^\lambda, \left(I_1^+\right)^\lambda, \left(F_1^+\right)^\lambda, -\left(-T_1^-\right)^\lambda, -\left(-I_1^-\right)^\lambda, -\left(1-\left(1-F_1^-\right)\right)^\lambda\right))$

ii. $\tilde{a}_1^\lambda = \left(\left(T_1^+\right)^\lambda, 1-\left(1-I_1^+\right)^\lambda, 1-\left(1-F_1^+\right)^\lambda, -\left(1-\left(1-T_1^-\right)\right)^\lambda\right), -\left(I_1^-\right)^\lambda\right), -\left(-F_1^-\right)^\lambda\right)$

iii. $\tilde{a}_1 + \tilde{a}_2 = \left(T_1^+ + T_2^+ - T_1^+ T_2^+, I_1^+ I_2^+, F_1^+ F_2^+, -T_1^- T_2^-, -\left(-I_1^- - I_2^- - I_1^- I_2^-\right), -\left(-F_1^- - F_2^- - F_1^- F_2^-\right)\right)$

Table 1 Criteria weights according to all decision makers

DMs	C_1	C_2	C_3	C_4	C_5	C_6	C_7
DM_1	⟨VG⟩	⟨EB⟩	⟨P⟩	⟨EG⟩	⟨VG⟩	⟨B⟩	⟨AS⟩
DM_2	⟨P⟩	⟨AS⟩	⟨MB⟩	⟨AS⟩	⟨MG⟩	⟨AS⟩	⟨EB⟩
DM_3	⟨MB⟩	⟨B⟩	⟨VB⟩	⟨P⟩	⟨VB⟩	⟨MG⟩	⟨P⟩
DM_4	⟨EG⟩	⟨MG⟩	⟨AS⟩	⟨VG⟩	⟨MB⟩	⟨EG⟩	⟨EG⟩

Table 2 Criteria weights according to all decision makers by bipolar neutrosophic numbers

Criteria	DM_1	DM_2	DM_3	DM_4
C_1	[1.0,0.0,0.1, −0.3, −0.8, −0.9]	[0.7,0.6,0.5, −0.2, −0.5, −0.6]	[0.3, 0.1, 0.9, −0.4, −0.2, −0.1]	[0.9, 0.1, 0.0, 0.0, −0.8, −0.9]
C_2	[0.1, 0.9, 0.8, −0.9, −0.2, −0.1]	[0.5, 0.2, 0.3, −0.3, −0.1, −0.3]	[0.4, 0.4, 0.3, −0.5, −0.2, −0.1]	[0.8, 0.5, 0.6, −0.1, −0.8, −0.9]
C_3	[0.7,0.6,0.5, −0.2, −0.5, −0.6]	[0.3, 0.1, 0.9, −0.4, −0.2, −0.1]	[0.2, 0.3, 0.4, −0.8, −0.6, −0.4]	[0.5, 0.2, 0.3, −0.3, −0.1, −0.3]
C_4	[0.9, 0.1, 0.0, 0.0, −0.8, −0.9]	[0.5, 0.2, 0.3, −0.3, −0.1, −0.3]	[0.7,0.6,0.5, −0.2, −0.5, −0.6]	[1.0,0.0,0.1, −0.3, −0.8, −0.9]
C_5	[1.0,0.0,0.1, −0.3, −0.8, −0.9]	[0.8, 0.5, 0.6, −0.1, −0.8, −0.9]	[0.2, 0.3, 0.4, −0.8, −0.6, −0.4]	[0.3, 0.1, 0.9, −0.4, −0.2, −0.1]
C_6	[0.4, 0.4, 0.3, −0.5, −0.2, −0.1]	[0.5, 0.2, 0.3, −0.3, −0.1, −0.3]	[0.8, 0.5, 0.6, −0.1, −0.8, −0.9]	[0.9, 0.1, 0.0, 0.0, −0.8, −0.9]
C_7	[0.5, 0.2, 0.3, −0.3, −0.1, −0.3]	[0.1, 0.9, 0.8, −0.9, −0.2, −0.1]	[0.7,0.6,0.5, −0.2, −0.5, −0.6]	[0.9, 0.1, 0.0, 0.0, −0.8, −0.9]

iv. $\tilde{a}_1.\tilde{a}_2 = \left(T_1^+ T_2^+,\ I_1^+ + I_2^+ - I_1^+ I_2^+ + F_1^+ + F_2^+ - F_1^+ F_2^+, -\left(-T_1^- - T_2^- - T_1^- T_2^-\right), -I_1^- I_2^-, -F_1^- F_2^-\right)$, where $\lambda > 0$.

Definition 2.1.1.4 Let $\tilde{a}_1 = (T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-)$ be a bipolar neutrosophic number. Then, the score function s ($\tilde{a}_1$), accuracy function a ($\tilde{a}_1$) and certainty function c ($\tilde{a}_1$) of an NBN are defined as follows:

$$\tilde{S}\left(\tilde{a}_1\right) = \left(T_1^+ + 1 - I_1^+ + 1 - F_1^+ + 1 + T_1^- - I_1^- - F_1^-\right)/6 \quad (1)$$

$$\tilde{a}\left(\tilde{a}_1\right) = T_1^+ - F_1^+ + T_1^- - F_1^- \quad (2)$$

$$\tilde{c}\left(\tilde{a}_1\right) = T_1^+ - F_1^- \quad (3)$$

Definition 2.1.1.5 Let $\tilde{a}_1 = (T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^-)$ and $\tilde{a}_2 = (T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^-)$ be two bipolar neutrosophic numbers. The comparison method can be defined as follows:

i. if $\tilde{s}(\tilde{a}_1) > \tilde{s}(\tilde{a}_2)$, then $\tilde{a}_1$ is greater than $\tilde{a}_2$, that is, $\tilde{a}_1$ is superior to $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$
ii. $\tilde{s}(\tilde{a}_1) = \tilde{s}(\tilde{a}_2)$ and $\tilde{a}(\tilde{a}_1) > \tilde{a}(\tilde{a}_2)$, then $\tilde{a}_1$ is greater than $\tilde{a}_2$, that is, $\tilde{a}_1$ is superior to $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
iii. if $\tilde{s}(\tilde{a}_1) = \tilde{s}(\tilde{a}_2)$, $\tilde{a}(\tilde{a}_1) = \tilde{a}(\tilde{a}_2)$) and $\tilde{c}(\tilde{a}_1) > \tilde{c}(\tilde{a}_2)$, then $\tilde{a}_1$ is greater than $\tilde{a}_2$, that is, $\tilde{a}_1$ is superior to $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;
iv. if $\tilde{s}(\tilde{a}_1) = \tilde{s}(\tilde{a}_2)$, $\tilde{a}(\tilde{a}_1) = \tilde{a}(\tilde{a}_2)$) and $\tilde{c}(\tilde{a}_1) = \tilde{c}(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$, that is, $\tilde{a}_1$ is indifferent to $\tilde{a}_2$, denoted by $\tilde{a}_1 = \tilde{a}_2$.

Definition 2.1.1.6 Let $\tilde{a}_j = (T_j^+, I_j^+, F_j^+, T_j^-, I_j^-, F_j^-)$ (j = 1, 2,…, n) be a family of bipolar neutrosophic numbers. A mapping A_ω: $Q_n \to Q$ is called bipolar neutrosophic weighted average operator if it satisfies the condition:

$$A_w\left(\tilde{a}_1, \tilde{a}_2, \ldots.., \tilde{a}_n\right) = \sum_{j=1}^{n} \omega_j \tilde{a}_j = \left(1 - \prod_{j=1}^{n} \left(1 - T_j^+\right)^{\omega j}, \prod_{j=1}^{n} I_j^{+\omega j}, \prod_{j=1}^{n} F_j^{+\omega j}, -\prod_{j=1}^{n} \left(-T_j^-\right)^{\omega j}, -1\left(\prod_{j=1}^{n} \left(1-\left(-I_j^-\right)\right)^{\omega j}\right), -\left(1-\prod_{j=1}^{n}\left(1-\left(-F_j^-\right)\right)^{\omega j}\right)\right),$$

where ω_j is the weight of $\tilde{a}_j$ (j = 1,2, …, n), $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

The Suggested Method Procedure

In this section, the steps of the suggested bipolar neutrosophic with TOPSIS framework are presented in details, and the general conceptualization of the framework is exposed in Fig. 2.

The suggested framework consists of many steps, see Fig. 3 below.

Step 1. Organize a committee of DMs and determine the goal, the alternatives and the valuation criteria.

Suppose that DMs want to appreciate the collection of n criteria and m alternatives. DMs are symbolized by D_E = {DM_1, DM_2, DM_3, DM_4}, where E = 1, 2, ..., E, and alterna-

Table 3 The normalized criteria weights

Weight $\tilde{w}_n$	Aggregation weights in BNNs	crisp	Normalized Weight
C_1	[0.725, 0.2, 0.375, −0.225, −0.575, −0.625]	0.6875	0.17
C_2	[0.450, 0.50, 0.500, −0.45, −0.325, −0.35]	0.4458	0.09
C_3	[0.425, 0.3, 0.525, –0.425, –0.350, –0.350]	0.4792	0.11
C_4	[0.775, 0.225, 0.225, −0.20, −0.55, −0.675]	0.7250	0.21
C_5	[0.575, 0.225, 0.500, −0.4, −0.600, −0.575]	0.6042	0.14
C_6	[0.650, 0.300, 0.3, −0.225, −0.475, −0.550]	0.6417	0.15
C_7	[0.550, 0.450, 0.40, −0.35, −0.400, −0.475]	0.5375	0.13

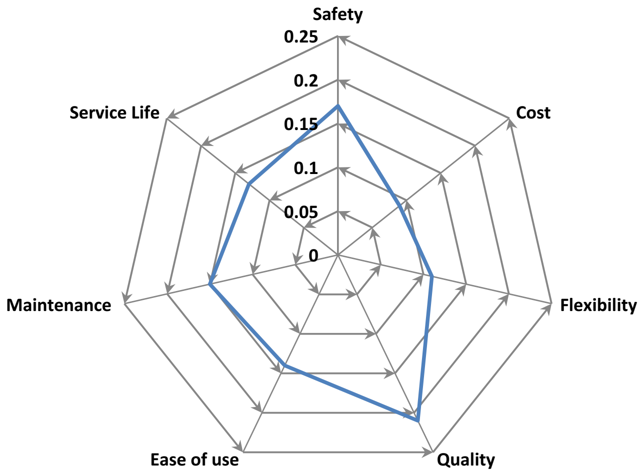

Fig. 5 Criteria weights according to all decision makers

tives by $A_i = \{A_1, A_2, ..., A_m\}$, where $i = 1, 2, ..., m$, assessed on n criteria $c_j = \{c_1, c_2, .., c_n\}$, $j = 1, 2, ..., n$.

Table 5 The aggregated crisp values of decision matrix

C_n/ A_n	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ϕ_1	0.48	0.69	0.5	0.64	0.55	0.51	0.82
ϕ_2	0.53	0.73	0.55	0.67	0.51	0.84	0.69
ϕ_3	0.85	0.48	0.63	0.54	0.61	0.63	0.76
ϕ_4	0.47	0.38	0.66	0.60	0.64	0.64	0.29
ϕ_5	0.65	0.69	0.78	0.57	0.42	0.68	0.75
ϕ_6	0.76	0.44	0.44	0.56	0.77	0.39	0.22
ϕ_7	0.53	0.51	0.77	0.49	0.39	0.68	0.64

Step 2. Depict and design the linguistic scales to describe DMs, and set the alternatives.

Step 3. Obtain DMs' judgments on each element.

Based on previously knowledge and experience, DMs are demanded to convey their judgments. Every DM gives his / her judgment on every of these elements.

Table 4 Ratings of alternatives and criteria by DMs

DMs	Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7
DM_1	ϕ_1	⟨P⟩	⟨MB⟩	⟨EG⟩	⟨EB⟩	⟨AS⟩	⟨VB⟩	⟨VG⟩
	ϕ_2	⟨EB⟩	⟨VG⟩	⟨P⟩	⟨B⟩	⟨VG⟩	⟨P⟩	⟨MG⟩
	ϕ_3	⟨EG⟩	⟨AS⟩	⟨MB⟩	⟨VG⟩	⟨P⟩	⟨MG⟩	⟨EG⟩
	ϕ_4	⟨AS⟩	⟨B⟩	⟨MG⟩	⟨VB⟩	⟨MB⟩	⟨EG⟩	⟨EB⟩
	ϕ_5	⟨EG⟩	⟨P⟩	⟨EG⟩	⟨EB⟩	⟨AS⟩	⟨VG⟩	⟨B⟩
	ϕ_6	⟨VG⟩	⟨B⟩	⟨AS⟩	⟨EG⟩	⟨EG⟩	⟨MB⟩	⟨VB⟩
	ϕ_7	⟨EB⟩	⟨MB⟩	⟨VG⟩	⟨MB⟩	⟨EB⟩	⟨EG⟩	⟨VG⟩
DM_2	ϕ_1	⟨MB⟩	⟨VG⟩	⟨VB⟩	⟨P⟩	⟨MG⟩	⟨EB⟩	⟨EG⟩
	ϕ_2	⟨AS⟩	⟨P⟩	⟨MB⟩	⟨EG⟩	⟨EB⟩	⟨EG⟩	⟨AS⟩
	ϕ_3	⟨MG⟩	⟨EG⟩	⟨AS⟩	⟨MG⟩	⟨MG⟩	⟨MG⟩	⟨VG⟩
	ϕ_4	⟨EB⟩	⟨VB⟩	⟨EB⟩	⟨EG⟩	⟨VG⟩	⟨EG⟩	⟨MB⟩
	ϕ_5	⟨MB⟩	⟨EG⟩	⟨VG⟩	⟨MB⟩	⟨VB⟩	⟨B⟩	⟨MG⟩
	ϕ_6	⟨VG⟩	⟨MG⟩	⟨MG⟩	⟨P⟩	⟨AS⟩	⟨EB⟩	⟨VB⟩
	ϕ_7	⟨P⟩	⟨AS⟩	⟨MG⟩	⟨EG⟩	⟨MG⟩	⟨MG⟩	⟨MG⟩
DM_3	ϕ_1	⟨B⟩	⟨EG⟩	⟨P⟩	⟨EG⟩	⟨VB⟩	⟨EG⟩	⟨P⟩
	ϕ_2	⟨MG⟩	⟨EG⟩	⟨MB⟩	⟨B⟩	⟨VG⟩	⟨EG⟩	⟨EG⟩
	ϕ_3	⟨VG⟩	⟨EB⟩	⟨EG⟩	⟨B⟩	⟨VB⟩	⟨MB⟩	⟨VG⟩
	ϕ_4	⟨AS⟩	⟨MG⟩	⟨VG⟩	⟨MG⟩	⟨MB⟩	⟨EB⟩	⟨AS⟩
	ϕ_5	⟨MB⟩	⟨AS⟩	⟨MG⟩	⟨EG⟩	⟨AS⟩	⟨VG⟩	⟨EG⟩
	ϕ_6	⟨EG⟩	⟨VB⟩	⟨MB⟩	⟨AS⟩	⟨EG⟩	⟨EG⟩	⟨EB⟩
	ϕ_7	⟨VG⟩	⟨B⟩	⟨P⟩	⟨EB⟩	⟨EB⟩	⟨MB⟩	⟨P⟩
DM_4	ϕ_1	⟨AS⟩	⟨P⟩	⟨VB⟩	⟨EG⟩	⟨MG⟩	⟨MG⟩	⟨VG⟩
	ϕ_2	⟨MG⟩	⟨AS⟩	⟨VG⟩	⟨EG⟩	⟨EB⟩	⟨EG⟩	⟨P⟩
	ϕ_3	⟨VG⟩	⟨MB⟩	⟨MG⟩	⟨EB⟩	⟨VG⟩	⟨MG⟩	⟨MB⟩
	ϕ_4	⟨MG⟩	⟨EB⟩	⟨EG⟩	⟨AS⟩	⟨EG⟩	⟨P⟩	⟨EB⟩
	ϕ_5	⟨EG⟩	⟨MG⟩	⟨P⟩	⟨VG⟩	⟨MB⟩	⟨AS⟩	⟨EG⟩
	ϕ_6	⟨MB⟩	⟨MB⟩	⟨EB⟩	⟨VB⟩	⟨MG⟩	⟨EB⟩	⟨VB⟩
	ϕ_7	⟨AS⟩	⟨MG⟩	⟨VG⟩	⟨AS⟩	⟨P⟩	⟨MG⟩	⟨MB⟩

Table 6 The normalized decision matrix

C_n/ A_n	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ϕ_1	0.30	0.43	0.31	0.40	0.34	0.32	0.51
ϕ_2	0.31	0.42	0.32	0.39	0.29	0.48	0.40
ϕ_3	0.49	0.28	0.36	0.31	0.35	0.36	0.44
ϕ_4	0.33	0.26	0.46	0.42	0.45	0.45	0.20
ϕ_5	0.37	0.40	0.45	0.33	0.24	0.39	0.43
ϕ_6	0.53	0.31	0.31	0.39	0.53	0.27	0.15
ϕ_7	0.34	0.33	0.50	0.32	0.25	0.44	0.41

Table 8 The TOPSIS result and ranking of alternatives

C_n/ A_n	S^+_i	S^-_i	p_i	Rank
ϕ_1	0.057	0.065	0.53	3
ϕ_2	0.050	0.058	0.54	2
ϕ_3	0.040	0.053	0.57	1
ϕ_4	0.060	0.049	0.44	6
ϕ_5	0.054	0.041	0.43	4
ϕ_6	0.069	0.070	0.50	5
ϕ_7	0.063	0.035	0.36	7

a. Collect the judgments of DMs about each other and from the viewpoint of the Kth DM
b. Gather the judgments on all the alternatives for every criterion from the viewpoint of the Kth DM.

Step 4. Obtain the conversion of (BNNs) bipolar neutrosophic numbers.

When all DMs give their valuations on each element, the bipolar neutrosophic values preference scale in subsection 3.2 is used.

a. Transforming DMs' linguistic valuations into bipolar neutrosophic numbers for every DM provides judgment with assistance of the linguistic weighting terms as shown in subsection 3.2.
b. Building the preference relation matrix with the assistance of BNNS to determine weights of criteria. DMs use the linguistic terms shown in subsection 3.2 to evaluate their opinions with respect to each criterion. Let R^k_{ij} be a (BN) decision matrix of the Kth DMs for calculating weights of criteria by opinions of DMs, then:

$$R^k_{ij} = \begin{bmatrix} r^k_{11} & \cdots & r^k_{1n} \\ \vdots & \ddots & \vdots \\ r^k_{m1} & \cdots & r^k_{mn} \end{bmatrix}, k \in K \quad (4)$$

where $r^k_{ij} = [T^+(x), I^+(x), F^+(x)\ T^-(x), I^-(x), F^-(x)]$, $k = 1, 2, \ldots, K$, $i = 1, 2, \ldots, m$, $j = 1,2, \ldots, n$.

Step 5. Calculating the weights of DMs.

DMs' judgments are collected by using the following equation:

$$r^k_{ij} = \frac{\left[T^+(x)_{n1}, I^+(x)_{n1}, F^+(x)_{n1},\quad T^-(x)_{n1}, I^-(x)_{n1}, F^-(x)_{n1}\right]}{n} \quad (5)$$

Then, the score value after aggregating the opinions of DMs for each criteria using Eq. (1) is calculated, and the obtaining weights are normalized.

Step 6. Construct the evaluation matrix.

Build the evaluation matrix $A_i \times C_j$ with the assistance of BNNS to evaluate the ratings of alternatives with respect to each criterion. DMs use the linguistic terms shown in subsection 3.2. Let R^k_{ij} be a (BN) decision matrix of the Kth DMs, then:

$$R^k_{ij} = \begin{bmatrix} r^k_{11} & \cdots & r^k_{1n} \\ \vdots & \ddots & \vdots \\ r^k_{m1} & \cdots & r^k_{mn} \end{bmatrix}, k \in K \quad (6)$$

where $r^k_{ij} = [T^+(x), I^+(x), F^+(x)\ T^-(x), I^-(x), F^-(x)]$, $k = 1, 2, \ldots, K$, $i = 1, 2, \ldots, m$, $j = 1,2, \ldots, n$.

Step 7. Calculate the crisp value of matrix.

Use the de-neutrosophication Eq. (1) for transforming bipolar neutrosophic numbers into crisp values for each factor

Table 7 The weighted matrix

C_n/ A_n	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Weight	0.17	0.09	0.11	0.21	0.14	0.15	0.13
ϕ_1	0.051	0.039	0.034	0.084	0.048	0.048	0.066
ϕ_2	0.053	0.038	0.035	0.081	0.041	0.072	0.052
ϕ_3	0.083	0.025	0.040	0.065	0.049	0.054	0.057
ϕ_4	0.056	0.023	0.051	0.088	0.063	0.068	0.026
ϕ_5	0.063	0.036	0.050	0.069	0.034	0.059	0.056
ϕ_6	0.090	0.028	0.034	0.081	0.074	0.041	0.019
ϕ_7	0.058	0.030	0.055	0.067	0.035	0.066	0.053

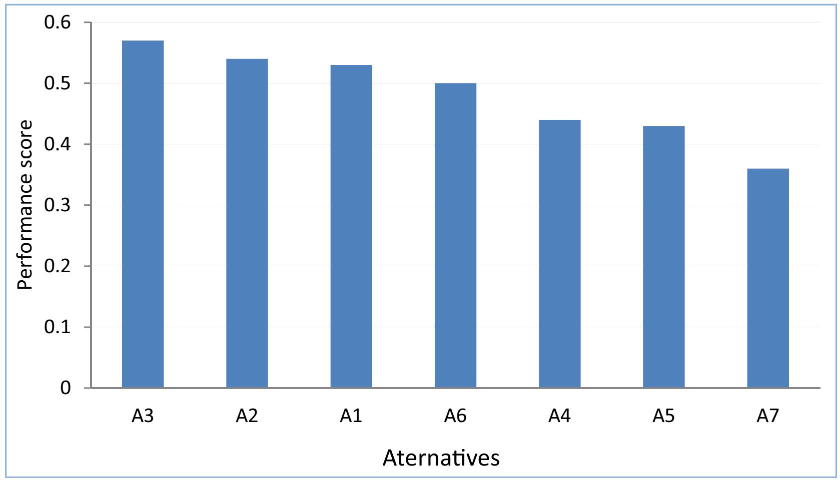

Fig. 6 The ranking of alternatives by the suggested method

r^k_{ij} and compare the score values according to Definition 2.1.1.5.

Step 8. Aggregate the final evaluation matrix.

Using Eq. (7), aggregate the crisp values of evaluation matrices into a final matrix.

$$\tilde{a}_{ij} = \frac{\tilde{a}_{ij}^{\;1} + \ldots\ldots\ldots\ldots\ldots + \tilde{a}_{ij}^{\;n}}{n} \tag{7}$$

Then, normalize the obtained matrix by Eq. (8).

$$H_{rt} = \frac{x_{rt}}{\sqrt{\sum_{i=1}^{m} X_{rt}^{\;2}}}; r = 1, 2 \ldots m; t = 1, 2 \ldots n. \tag{8}$$

After that, calculate the weight matrix by Eq. (9).

$$Q_{rt} = w_z \times H_{rt} \tag{9}$$

or using Eq. (10)

$$A_w\left(\tilde{a}_1, \tilde{a}_2, \ldots.., \tilde{a}_n\right) = \sum_{j=1}^{n} \omega_j \tilde{a}_j$$
$$= \left(1-\prod_{j=1}^{n} \left(1-T_j^+\right)^{\omega j}, \prod_{j=1}^{n} \; I_j^{+\omega j}, \prod_{j=1}^{n} F_j^{+\omega j}, -\prod_{j=1}^{n} \left(-T_j^-\right)^{\omega j}, -1\left(\prod_{j=1}^{n} \left(1-\left(-I_j^-\right)\right)^{\omega j}\right), -\left(1-\prod_{j=1}^{n} \left(1-\left(-F_j^-\right)\right)^{\omega j}\right)\right) \tag{10}$$

Step 9. Define Ideal Solution A^+, A^-.

Calculate the positive and negative ideal solution using Eqs. (11, 12).

$$A^+ = \{< \max \left(\delta_{ij} | i = 1, 2, \ldots, m\right) | \; j \in J^+ >, < \min \left(\delta_{ij} | i = 1, 2, \ldots, m\right) | \; j \in J^- >\} \tag{11}$$

$$A^- = \{< \min \left(\delta_{ij} | i = 1, 2, \ldots, m\right) | \; j \in J^+ >, < \max \left(\delta_{ij} | i = 1, 2, \ldots, m\right) | \; j \in J^- >\} \tag{12}$$

Step 10. Positive and Negative Ideal Solution S^+_i, S^-_i.

Calculate the Euclidean distance between positive solution (S^+_i) and negative ideal solution (S^-_i) using Eqs. (13, 14).

$$S^+_i = \sqrt{\sum_{j=1}^{n} \left(\delta_{ij} - \delta_j^{\;+}\right)^2} \quad i = 1, 2, \ldots, m, \tag{13}$$

$$S^-_i = \sqrt{\sum_{j=1}^{n} \left(\delta_{ij} - \delta_j^{\;-}\right)^2} \quad i = 1, 2, \ldots, m \tag{14}$$

Step 11. Rank the alternatives based on closeness coefficient.

$$P_i = \frac{S^-_i}{S^+_i + S^-_i} \qquad i = 1, 2, \ldots, m \tag{15}$$

Step 12. Comparing the obtained results with other methods.
Step 13. Check the stability of variations in rankings by Spearman coefficient of correlation.

Numerical Experiment

We introduced in this section a numerical case, which requires methods and data analysis to test the competence and

Table 9 The aggregated crisp values of decision matrix using by AHP method

C_n/ A_n	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ϕ1	0.48	0.69	0.5	0.64	0.55	0.51	0.82
ϕ2	0.53	0.73	0.55	0.67	0.51	0.84	0.69
ϕ3	0.85	0.48	0.63	0.54	0.61	0.63	0.76
ϕ4	0.47	0.38	0.66	0.60	0.64	0.64	0.29
ϕ5	0.65	0.69	0.78	0.57	0.42	0.68	0.75
ϕ6	0.76	0.44	0.44	0.56	0.77	0.39	0.22
ϕ7	0.53	0.51	0.77	0.49	0.39	0.68	0.64

efficiency of suggested framework for selection of appropriate medical devices (MDs).

Case Study

Companies seek developing sugar analyzing devices for diabetics. Therefore, we introduce a practical case to select a sugar analyzing device. There are four decision makers; DM_1, DM_2, DM_3 and DM_4, and seven alternatives A_1, A_2, A_3,A_4,A_5, A_6 and A_7. For evaluating the SMDs alternatives, seven criteria are considered as selection factors: C_1(Safety), C_2 (Cost), (Flexibility), C_4 (Quality), C_5 (Ease of use), C_6(Maintenance Requirements) and C_7(Service Life), as listed in Fig. 4.

The Calculation Process of the Neutrosophic with TOPSIS Technique

Step 1. Organize a committee of DMs and determine the goal, alternatives and valuation criteria.

A committee consisting of four DMs is constructed to select the best alternatives of sugar analyzing smart medical devices for diabetics, $A_i = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$, offered by different medical device producers . These alternatives are estimated based on seven criteria $c_j = \{c_1, c_2, c_3, c_4$,-c5 ,c_6 ,$c_7\}$, which are collected from comprehensive commentaries and DMs' opinions.

Table 10 The ranking of alternatives under AHP method

Alternatives	V_i	Ranking
ϕ1	0.59	3
ϕ2	0.64	2
ϕ3	0.65	5
ϕ4	0.54	1
ϕ5	0.63	7
ϕ6	0.52	4
ϕ7	0.56	6

Table 11 The weighted matrix under MOORA method

C_n/ A_n	C_1	C_2	C_3	C_4	C_5	C_6	C_7
ϕ1	0.051	0.039	0.034	0.084	0.048	0.048	0.066
ϕ2	0.053	0.038	0.035	0.081	0.041	0.072	0.052
ϕ3	0.083	0.025	0.040	0.065	0.049	0.054	0.057
ϕ4	0.056	0.023	0.051	0.088	0.063	0.068	0.026
ϕ5	0.063	0.036	0.050	0.069	0.034	0.059	0.056
ϕ6	0.090	0.028	0.034	0.081	0.074	0.041	0.019
ϕ7	0.058	0.030	0.055	0.067	0.035	0.066	0.053

Step 2, 3, 4. Determine the appropriate (LVs) linguistic variables for weights (W_n) of criteria (C_n) and alternatives (A_n) with regard to each criterion. Each linguistic variable is a bipolar neutrosophic number (BNN). For criteria weights and for compilation alternatives, the linguistic variables are as follows: Excessively Good (EG) = $\langle 0.9, 0.1, 0.0, 0.0, -0.8, -0.9 \rangle$; where the first three numbers present the positive membership degree $T^+(x)$, $I^+(x)$, $F^+(x)$ 0.9, 0.1 and 0.0 respectively, $T^+(x)$ the truth degree in positive membership, $I^+(x)$ the indetermininancy degree and finally $F^+(x)$ the falsity degree. The last three numbers present the negative membership degree $T^-(x)$, $I^-(x)$, $F^-(x)$ 0.0, −0.8 and − 0.9 respectively, where $T^-(x)$ the truth degree in negative membership, $I^-(x)$ the indetermininancy degree and finally $F^-(x)$ the falsity degree. Very Good (VG) = $\langle 1.0, 0.0, 0.1, -0.3, -0.8, -0.9 \rangle$, Midst Good (MG) = $\langle 0.8, 0.5, 0.6, -0.1, -0.8, -0.9 \rangle$, Perfect (P) = $\langle 0.7, 0.6, 0.5, -0.2, -0.5, -0.6 \rangle$, Approximately Similar (AS) = $\langle 0.5, 0.2, 0.3, -0.3, -0.1, -0.3 \rangle$, Bad (B) = $\langle 0.4, 0.4, 0.3, -0.5, -0.2, -0.1 \rangle$, Midst Bad (MB) = $\langle 0.3, 0.1, 0.9, -0.4, -0.2, -0.1 \rangle$, Very Bad (VB) = $\langle 0.2, 0.3, 0.4, -0.8, -0.6, -0.4 \rangle$, Excessively Bad (EB) = $\langle 0.1, 0.9, 0.8, -0.9, -0.2, -0.1 \rangle$.

Step 5. Calculating the weights of DMs

The prior (LVs) linguistic variables are used by experts and (DMs) decision makers to clarify their priorities, preferences and the confirmation degree of linguistic variable according to

Table 12 The ranking of alternatives under MOORA method

Alternatives	P^*_i	Ranking
ϕ1	1.44	3
ϕ2	1.83	1
ϕ3	1.39	2
ϕ4	2.35	7
ϕ5	2.22	5
ϕ6	2.76	4
ϕ7	2.17	6

Table 13 The ranking of alternatives by methods

Alternatives	Pros. Method (1)	AHP (2)	MOORA(3)
ϕ_1	3	3	3
ϕ_2	2	2	1
ϕ_3	1	5	2
ϕ_4	6	1	7
ϕ_5	4	7	5
ϕ_6	5	4	4
ϕ_7	7	6	6

each (DM) decision maker or expert. The Table 1 presents the criteria weights according to all decision makers, after deciding (LVs) linguistic variables to each decision maker or expert.

Convert the linguistic variables into bipolar neutrosophic numbers as in Table 2. Use Eq. (5) to aggregate weights in BNNs. Then, employ Eq. (1) to calculate the crisp weight values. After that, make a normalization procedure on the previous values, as in Table 3. The Fig. 5 shows the values of weights that equals: $w_1 = 0.17$, $w_2 = 0.09$, $w_3 = 0.11$, $w_4 = 0.21$, $w_5 = 0.14$, $w_6 = 0.15$, $w_7 = 0.13$.

Step 6. Construct the evaluation matrix.

Obtain the final decision matrix by making the aggregation procedure of decision makers' priorities and preferences, as in Table 4.

Step 7, 8. Calculate the crisp values of matrices and insert them into the aggregated matrix.

Let each decision maker construct the matrix by comparing the five alternatives against each criterion, by utilizing the bipolar neutrosophic scale, previously presented in Step 2 of this section. Use Eq. (1) to progress towards de-neutrosophication in order to transform the bipolar neutrosophic numbers into their crisp forms. Then, aggregate the matrices and get the last evaluation matrix, pertinent to decision makers' committee. Employ Eq. (7) to aggregate crisp values of evaluation matrices into a final matrix, as in Table 5.

Apply the normalization process by using Eq. (8) to obtain the normalized evaluation matrix, as presented in Table 6.

Build the weighted matrix by multiplying the normalized evaluation matrix by the weights of criteria using Eq. (9), as in Table 7.

Step 9. Define Ideal Solution A^+, A^-.

Define the ideal solutions using Eqs. (11) and (12) as follows:

$$A^+ = \{0.090, 0.039, 0.055, 0.088, 0.074, 0.072 \text{ and } 0.066\},$$
$$A^- = \{0.051, 0.023, 0.034, 0.065, 0.034, 0.041 \text{ and } 0.019\}.$$

Step 10. Positive and Negative Ideal Solution S^+_i, S^-_i.

Calculate the Euclidean distance between positive solution (S^+_i) and negative ideal solution (S^-_i) using Eqs. (13) and (14) as follows:

$$S^+_1 = \{0.057\},\ S^+_2 = \{0.050\}, S^+_3 = \{0.040\}, S^+_4 = \{0.060\}, S^+_5 = \{0.054\}, S^+_6 = \{0.069\}, S^+_7 = \{0.063\}.$$

$$S^-_1 = \{0.065\},\ S^-_2 = \{0.058\}, S^-_3 = \{0.053\}, S^-_4 = \{0.049\}, S^-_5 = \{0.041\}, S^-_6 = \{0.070\}, S^-_7 = \{0.035\}.$$

Step 11. Rank the alternatives based on closeness coefficient.

Calculate the performance score using Eq. (15), and make the last ranking of alternatives as presented in Table 8.

The order for the optimal alternatives of smart medical devices is Alternative 3, Alternative 2, Alternative 1, Alternative 6, Alternative 4, Alternative 5 and Alternative 7, as drawn in Fig. 6.

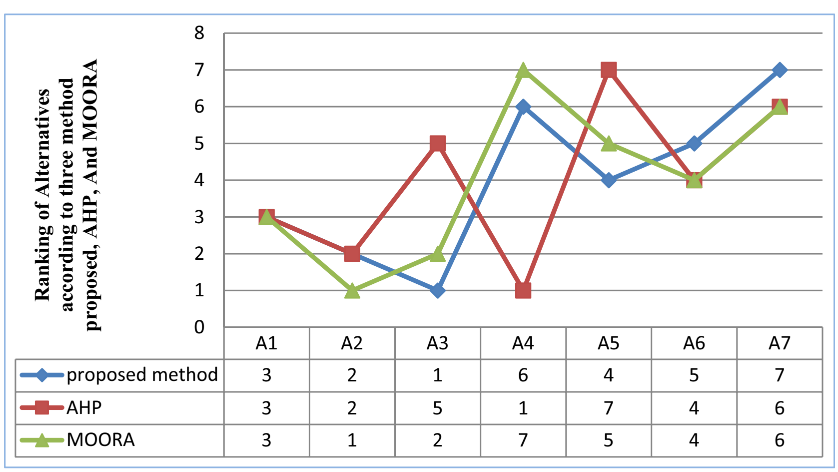

Fig. 7 The ranking of alternatives according to the three methods: our method, AHP and MOORA

Table 14 The correlation coefficient between the proposed model and other methods

Correlate (1, 2)	Correlate (1, 3)
0.071	0.893

Step 12. Comparing the obtained results with other methods.

In this step, compare the results of suggested method with the results obtained by other existing methods, such as analytic hierarchy process (AHP) or a method from multi objective decision-making techniques, as MOORA, to validate our model. It is known that AHP does not consider feedback and interdependency among elements of problem. The comparison matrix of alternatives relevant to each sub-criterion is presented in Table 9. The final ranking of alternatives by AHP method is listed in Table 10.

In addition, we used MOORA technique to validate our proposed approach that is the multi objective decision making (MODM) techniques submitted by Esra and Işık [33]. The equations that are applied in our computation of MOORA method are founded in [13]. The MOORA normalized matrix and ranking of alternatives are listed in Tables 11 and 12.

The proposed method and the other two methods used for the ranking of alternatives are aggregated in Table 13 and presented in Fig. 7. We used SPSS program to calculate the correlation coefficient among the different techniques and the proposed approach, as shown in Table 14.

It is clear that alternative 3 is the best alternative according to the results of the three methods applied, including the proposed method.

Concluding Remarks

Better health attention can be possible by tracking medical requirements of patients. Nowadays, patients tend to measure themselves their activity, and consequently the medical devices are not solely designed for healthcare specialists. Medical tools, such as cardiac monitors or sugar analysis, are getting smarter and started to be incorporated into numerous devices, e.g. smart watches or smartphones. This research introduces the Neutrosophic TOPSIS for an MCDM problem method, namely the selection of sugar analyzing device for diabetics. Furthermore, the suggested method is applied to a practical case to compare seven smart medical devices using seven evaluation criteria to validate the suggested approach, employing experts' opinion and extensive literature review. The suggested method produces more realistic and accurate results than other MCDM techniques, because TOPSIS can capture, implement and model interactions between selection criteria. In addition, a collection of experts is often more beneficial than a single one, in order to reduce partiality and bias of individual opinions and judgments, and the use of Neutrosophic values enhances the transaction of selection. The Neutrosophic theory can help in preventing the loss of data, present linguistic declarations into analytical models and help in including of nonnumeric statements. To complete the ranking of alternatives based on the information collected, we employ the neutrosophic with TOPSIS method. The verification and effectiveness of the suggested method are compared with other methods. Eventually, the Spearman's coefficient of correlation is applied to determine the relation among the results obtained by comparison. Although the presented methodology is used for the selection of sugar analyzing devices for diabetics' patients, it can also be applied for other SMD valuations. Additional research could extend the suggested methodology to other types of SMDs selection procedures.

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A Hybrid Neutrosophic Multiple Criteria Group Decision Making Approach for Project Selection

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Abstract

The project selection is one of the most important phases of a project life cycle. The project selection is considered as a Multi-Criteria Decision-Making (MCDM) problem. This research aims to study the integration between Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) into Decision-Making Trial and Evaluation Laboratory (DEMATEL) under neutrosophic environment to provide a new technique for making a decision regarding the choice of appropriate project (project selection) as one of the most important phases of the project life cycle. Projects are selected by comparing them against many criteria. Criteria are evaluated based on expert's opinion. Sometimes experts cannot give reliable information due to the non-deterministic environment. The neutrosophic set theory will be used to handle and overcome the ambiguity or lack of confirmation of information. The criteria are weighted by DEMATEL, then the best project alternative is selected using TOPSIS. In the proposed model, each pairwise comparison judgments is symbolized as a trapezoidal neutrosophic number. Experts will focus only on (n − 1) judgments for n alternatives to overcome the difficulties of [(n * (n − 1))/2] consistence judgments in case of increasing number of alternatives. A numerical example is developed to show the validation of the suggested model in the neutrosophic environment.

Keywords: Project life cycle; Project selection; TOPSIS; DEMATEL; Neutrosophic set; Trapezoidal neutrosophic number

1. Introduction and related works

A project is a set of related activities that are employed to accomplish some goals. Any project has a life cycle. It has been widely recognized that the selection of a project is a critical phase of project life cycle. A life cycle of a project consists of four stages, as shown in Fig. 1. The fastest and most important stage in the life cycle of a project is the project selection after the identification and evaluation of the project. Project life cycle always starts with the client by choosing the appropriate project from a set of available alternatives (projects) for investment or for any other purposes. Once the project is selected, the second stage is the planning of the project by defining and determining the scope of the work, basic schedule, time tradeoffs, and resource consideration in a project. The third stage is project implementation, and finally, the project completion.

In this research, we focus on the fastest and most important stage of the project life cycle, i.e. the project selection phase. Project selection is considered as a multi-criteria decision-making (MCDM) problem, where the choice of the preferred project among several projects depends on the differentiation between projects based on certain criteria. There are many studies (Aragonés-Beltrán, Chaparro-González, Pastor-Ferrando, & Pla-Rubio,

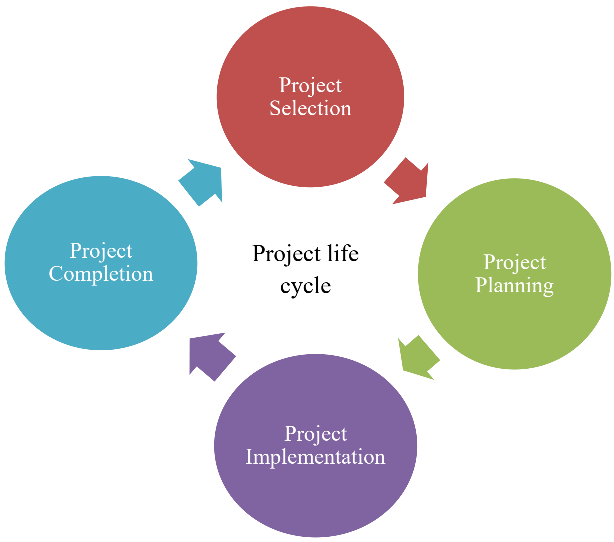

Fig. 1. Project life cycle.

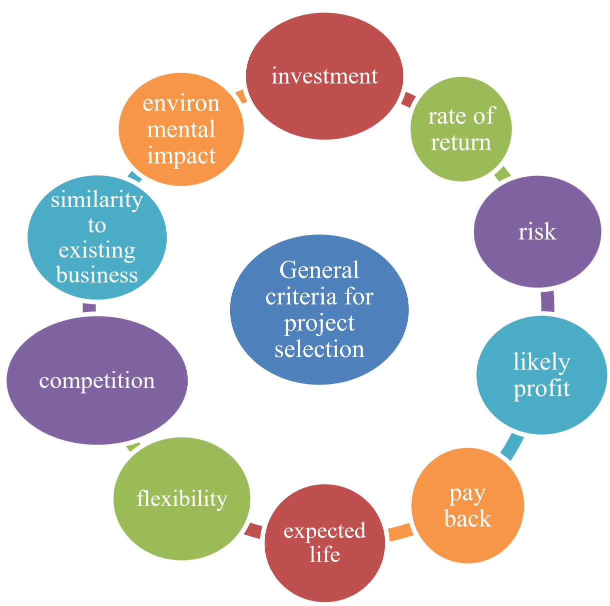

Fig. 2. Main and general criteria for project selection.

2014; Greco, Figueira, & Ehrgott, 2005; Lee & Kim, 2000, 2001; Meredith & Mantel, 2011; Pohekar & Ramachandran, 2004; San Cristóbal, 2011; Santhanam & Kyparisis, 1995; Schwalbe, 2015; Zavadskas, Turskis, Tamošaitiene, & Marina, 2008) that discussed the most important criteria on which to choose the best project among several projects. There are several important criteria related to the project selection. These criteria are investment, rate of return, risk, likely profit, pay back, similarity to existing businesses, expected life, flexibility, environmental impact, and competition, as shown in Fig. 2. Multi-criteria evaluation for project selection is a comparison between several alternatives of the project against some of criteria, as shown in Fig. 3, where rarely would one

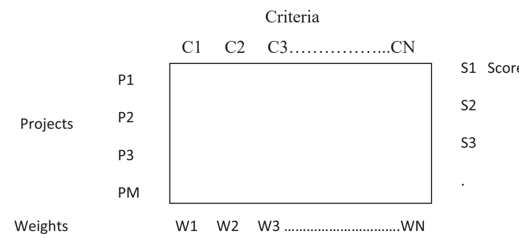

Fig. 3. Multi-criteria (MCDM) evaluation.

project emerge as the best on all chosen criteria. If that happens, it is a dominant project and it should be clearly chosen, but if this is not the case, as it happens in most of the real situations, we should compare the different alternatives on different sets of chosen criteria. The criteria (Fig. 4) are divided into tangible criteria and intangible criteria. The tangible criteria are the measurable criteria in units (e.g., payback period criterion measured in years and investments measured in millions of dollars, and so on). The intangible criteria are non-measurable criteria (such as risk measured not in a unit that may be expressed by very high, high, medium, low or very low). In case of intangible criteria, we should develop a scale. In this paper, we use the scale of (0–1) instead of (1–9). There are many techniques used for evaluated the criteria and selecting the best alternative among several ones considering several criteria such as AHP, ANP, Delphi, MOORA, and so on. In this research, we weighted the criteria using the neutrosophic DEMATEL and then select the best project alternative using neutrosophic TOPSIS. Multi-criteria decision-making problem (MCDM) is a formal and systematic way of decision-making on complex problems (Daneshvar Rouyendegh, 2011). Hwang and Yoon (Wang and Yoon, 1981) proposed one of the most used methods for MCDM; this method is TOPSIS (Technique for order preference by similarity to an ideal solution). Then the proposed set theories have provided the different multi-criteria decision-making methods. TOPSIS method is used to weight and compare set of alternatives against a set of criteria and select the best one. The alternatives are compared by the distance between alternatives and the optimal solution, where the best alternative is of the shortest distance from the optimal solution and the worst alternative is of the largest distance from the optimal solution. Many research focus on MCDM methods used fuzzy data (Bayrak, Celebi, & Taşkin, 2007; Carlsson and Fullér, 1996; Chan, Kumar, Tiwari, Lau, & Choy, 2008; Chen, 2000; Chu, 2002; Haq & Kannan, 2006; Izadikhah, 2009; Jahanshahloo, Lotfi, & Izadikhah, 2006a, 2006b; Önüt, Kara, & Işik, 2009; Tsaur, Chang, & Yen, 2002). Fuzzy sets focus only on the membership value and don't aware about non membership functions and indeterminacy value. Fuzzy sets unable to deal with ambiguity and non deterministic conditions. So we used neutrosophic set to deal and overcome the lack of certain information and uncertainty conditions. Boran, Genç, Kurt, and Akay (2009) suggested TOPSIS method under intuitionistic fuzzy

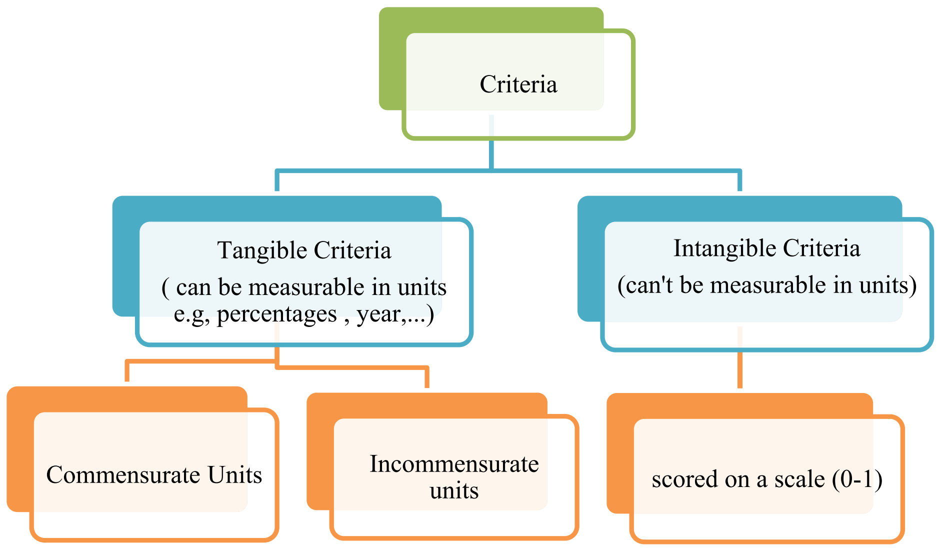

Fig. 4. Types of criteria based on their measurable.

environment. Ye (2010) extended the TOPSIS technique in interval-valued intuitionistic fuzzy sets. Science and Human Affairs Program of the Battelle Memorial Institute of Geneva founded DEMATEL method in the period from 1972 to 1979. Today it's become one of the most widely used tool for evaluating and weighting different criteria related to specific problem Chiu, Chen, Tzeng, & Shyu, 2006; Liou, Tzeng, & Chang, 2007; Tzeng, Chiang, & Li, 2007; Wu and Lee, 2007; Lin and Tzeng, 2009). Yang, Shieh, Leu, and Tzeng (2008) applied DEMATEL to study and analyze the relationship of reasons and effect among weighted criteria or to conclude interrelationship among factors (Broumi, Bakali, Talea, & Smarandache, 2016). In this research, we combine the TOPSIS into DEMATEL under neutrosophic set to solve the project selection problem.

2. Preliminaries

Neutrosophic theory was developed by Florentin Smarandache in 1998. In this section, we present definitions involving neutrosophic sets, single-valued neutrosophic sets, trapezoidal neutrosophic numbers, and operations on trapezoidal neutrosophic numbers.

Definition 1 El-Hefenawy, Metwally, Ahmed, & El-Henawy, 2016.. Let X be a space of points and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]{-0}, 1{+}[$. That is $T_A(x){:}X\to]{-0}, 1{+}[$ $I_A(x){:}X\to]$-0, 1+[*and* $F_A(x){:}X\to]$-0, 1+[. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, *so* $0- \le \sup(x) + \sup x \le 3+$.

Definition 2 (*Abdel-Baset, Hezam, & Smarandache, 2016; El-Hefenawy et al., 2016; Hezam, Abdel-Baset, & Smarandache, 2015; Saaty, 2006.*). Let X be a universe of discourse. A single valued neutrosophic set A over X is an object taking the form $A = \{\langle x, T_A(x), I_A(x), F_A(x),\rangle{:}x \in X\}$, where $T_A(x){:}X\to[0, 1]$, $I_A(x){:}X\to[0, 1]$ and $F_A(x){:}X\to[0, 1]$ with $0\le T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x *to* A, respectively. For convenience, a Single Valued Neutrosophic (SVN) number is represented by $A= (a, b, c)$, *where* $a, b, c\in [0, 1]$ and $a + b + c \le 3$.

Definition 3 Mahdi, Riley, Fereig, & Alex, 2002.. Suppose $\alpha_a, \theta_a, \beta_a \in [0, 1]$ and $a_1, a_2, a_3, a_4 \in$ R, where $a_1 \le a_2 \le a_3 \le a_4$. Then, a single valued trapezoidal neutrosophic number $a=\langle(a_1, a_2, a_3, a_4); \alpha_a, \theta_a, \beta_a\rangle$ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_a(x) = \begin{cases} \alpha_a\left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \le x \le a_2) \\ \alpha_a & (a_2 \le x \le a_3) \\ \alpha_a\left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \le x \le a_4) \\ 0 & \textit{otherwise} \end{cases} \quad (1)$$

$$I_a(x) = \begin{cases} \frac{(a_2-x+\theta_a(x-a_1))}{(a_2-a_1)} & (a_1 \le x \le a_2) \\ \alpha_a & (a_2 \le x \le a_3) \\ \frac{(x-a_3+\theta_a(a4-x))}{(a_4-a_3)} & (a_3 \le x \le a_4) \\ 1 & \textit{otherwise} \end{cases} \quad (2)$$

$$F_a(x) = \begin{cases} \frac{(a_2 - x + \beta_a(x - a_1))}{(a_2 - a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_a & (a_2 \leq x \leq a_3) \\ \frac{(x - a_3 + \beta_a(a4 - x))}{(a_4 - a_3)} & (a_3 \leq x \leq a_4) \\ 1 & otherwise, \end{cases} \quad (3)$$

where α_a, θ_a and β_a typify the maximum truth-membership degree, the minimum indeterminacy-membership degree and the minimum falsity-membership degree, respectively. A single valued trapezoidal neutrosophic number $a = \langle(a_1, a_2, a_3, a_4); \alpha_a, \theta_a, \beta_a\rangle$ may express an ill-defined quantity of the range, which is approximately equal to the interval $[a_2, a_3]$.

Definition 4 (*Izadikhah*, 2009; *Liou et al.*, 2007.). Let $a = \langle(a_1, a_2, a_3, a_4); \alpha_a, \theta_a, \beta_a\rangle$ and $b = \langle(b_1, b_2, b_3, b_4); \alpha_b, \theta_b, \beta_b\rangle$ be two single valued trapezoidal neutrosophic numbers, and $Y \neq 0$ be any real number. Then:

1. Addition of two trapezoidal neutrosophic numbers:

$$a + b = \langle(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_a \acute{a}\acute{\alpha}_b, \theta_a \acute{a}\acute{}\theta_b, \beta_a \acute{a}\acute{}\beta_b\rangle$$

2. Subtraction of two trapezoidal neutrosophic numbers:

$$a - b = \langle(a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_a \acute{a}\acute{\alpha}_b, \theta_a \acute{a}\acute{}\theta_b, \beta_a \acute{a}\acute{}\beta_b\rangle$$

3. Inverse of trapezoidal neutrosophic numbers:

$$a^{-1} = (\left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}\right); \alpha_a, \theta_a, \beta_a\rangle \quad \text{where} \quad (a \neq 0)$$

4. Multiplication of trapezoidal neutrosophic numbers by constant value:

$$Y\,a = \begin{cases} \langle(Ya_1, Ya_2, Ya_3, Ya_4); \alpha_a, \theta_a, \beta_a\rangle & \text{if} \quad (Y > 0) \\ \langle(Ya_4, Ya_3, Ya_2, Ya_1); \alpha_a, \theta_a, \beta_a\rangle & \text{if} \quad (Y < 0) \end{cases}$$

5. Division of two trapezoidal neutrosophic numbers:

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \left\langle \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_2}{b_1}\right); \ \alpha_{\tilde{a}} \Lambda \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} V \beta_{\tilde{b}} \right\rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \left\langle \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); \ \alpha_{\tilde{a}} \Lambda \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} V \beta_{\tilde{b}} \right\rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \left\langle \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}\right); \ \alpha_{\tilde{a}} \Lambda \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} V \beta_{\tilde{b}} \right\rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

6. Multiplication of trapezoidal neutrosophic numbers:

$$\tilde{a}\tilde{b} = \begin{cases} \langle(a_1b_1, a_2b_2, a_3b_3, a_4b_4); \ \alpha_{\tilde{a}} \Lambda \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} V \beta_{\tilde{b}}\rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \langle(a_1b_4, a_2b_3, a_3b_2, a_4b_1); \ \alpha_{\tilde{a}} \Lambda \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} V \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \langle(a_4b_4, a_3b_3, a_2b_2, a_1b_1); \ \alpha_{\tilde{a}} \Lambda \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} V \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

3. Methodology

Fuzzy set theory was applied in many studies, but it focuses only on membership value. The intuitionistic fuzzy set theory developed by Atanassov, deals with membership and non-membership value. The neutrosophic set theory is developed by Smarandache, and it treats the uncertainty and ambiguity by adding the indeterminacy besides truthiness and falsity values. In this section, the framework of the proposed model is shown in Fig. 5, we present the proposed TOPSIS - DEMATEL based on the neutrosophic set model as follows:

3.1. The neutrosophic DEMATEL technique

Step1: We start with neutrosophic DEMATEL method for evaluating and weighting the important criteria affecting the project selection problem. To weight the criteria, we should do the following:

1. Select those experts who have great experiences in project management.

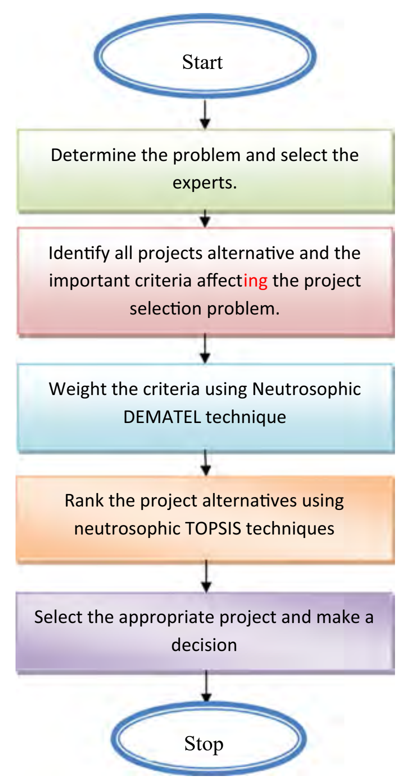

Fig. 5. The framework of the proposed model.

Table 1
Pairwise comparison among criteria with the degree of (α, β, and θ).

C	Y1	Y2	…	Yn
Y1	$(L_{11}, m_{11l}, m_{11u}, u_{11;}\ \alpha, \beta, \theta)$	$(L_{12}, m_{12l}, m_{12u}, u_{12;}\ \alpha, \beta, \theta)$	…	$(L_{1n}, m_{1nl}, m_{1nu}, u_{1n;}\ \alpha, \beta, \theta)$
Y2	$(L_{21}, m_{21l}, m_{21u}, u_{21;}\ \alpha, \beta, \theta)$	$(L_{22}, m_{22l}, m_{22u}, u_{22;}\ \alpha, \beta, \theta)$	…	$(L_{2n}, m_{2nl}, m_{2nu}, u_{2n;}\ \alpha, \beta, \theta)$
…	………	………	…	………
Yn	$(L_{n1}, m_{n1l}, m_{n1u}, u_{n1;}\ \alpha, \beta, \theta)$	$(L_{n2}, m_{n2l}, m_{n2u}, u_{n2;}\ \alpha, \beta, \theta)$	…	$(L_{nn}, m_{nnl}, m_{nnu}, u_{nn;}\ \alpha, \beta, \theta)$

Table 2
Crisp value relative to each expert.

Criteria	Y1	Y2	…	Yn
Y1	CV_{11}	CV_{12}	…	CV_{1n}
Y2	CV_{21}	CV_{22}	…	CV_{2n}
…	………	………	…	………
Yn	CV_{n1}	CV_{n2}	…	CV_{nn}

2. List and identify the most important criteria affecting the project selection problem.
3. Each expert makes a pairwise comparison among the important related criteria (Y1, Y2, …, Yn) in a trapezoidal neutrosophic number (l_{nm}, m_{nml}, m_{nmu}, u_{nm}), and also express the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (β), and the minimum falsity membership degree (θ) of single valued neutrosophic numbers (l_{nm}, m_{nml}, m_{nmu}, $u_{nm;}$ α, β, θ), using a scale form(0–1) and focusing only on (n−1) consensus judgments (Abdel-Basset, Mohamed, & Sangaiah, 2017), as shown in Table 1.
4. Calculate the crisp value of each expert's opinion, as shown in Table 2, using the following equations:

$$S(a_{ij}) = 1/16[a1 + b1 + c1 + d1] \times (2 + \alpha\,a - \theta\,a - \beta\,a) \tag{4}$$

$$A(a_{ij}) = 1/16[a1 + b1 + c1 + d1] \times (2 + \alpha\,a - \theta\,a + \beta\,a) \tag{5}$$

5. Combine all experts' opinions in one integration matrix and calculate the average of expert's opinions by dividing all experts' opinion for each criterion by the number of experts (n) considered in the problem. Calculate average value for each value for each expert by dividing each value by the number of experts (n) as shown in Eq. (6), and then combine all averaged values of the all of expert's opinion in one matrix called the initial directed relation matrix A, where a is n × n matrix of pairwise comparisons by all expert, $S = [S_{ij}]_{n*n}$, where S is the degree of each criterion i on criterion j.

$$CV_{11} = \frac{CV11n1 + CV11n2 + \cdots + CV11nm}{n} \tag{6}$$

6. Normalizing the initial direct relation matrix (A) using Eqs. (7) and (8).

$$K = \frac{1}{\text{Max}\ \ (1 \le i \le n)\sum_{j=1}^{n} aij} \tag{7}$$

$$S = K \times A \tag{8}$$

7. Obtaining the total relation matrix (T) by applying Eq. (9), where I is the identity matrix of the same size of S matrix obtained in the previous step.

$$T = S \times (I - S)^{-1} \tag{9}$$

8. Calculate the sum of rows (D) and the sum of columns (R), then calculate (R + D) and (R − D), furthermore make a causal diagram between (R + D) and (R − D), and arrange the criteria relative to their importance by weighting them.

Step 2: After weighting the criteria, we apply the neutrosophic TOPSIS method to compare between the set of projects alternatives against set weighted criteria obtained from step 1. To select the best project among several projects using neutrosophic TOPSIS, we should do the following:

1. Obtain the decision matrix between different project alternatives(Pi) and criteria (Yj) based on the opinion

Table 4
Crisp value of pairwise comparison relative to each expert.

	Y1	Y2	…	Yn
P1	CV_{11}	CV_{12}	…	CV_{1n}
P2	CV_{21}	CV_{22}	…	CV_{2n}
…	………	………	…	………
Pm	CV_{m1}	CV_{m2}	…	CV_{mn}

Table 3
Decision matrix of pairwise comparisons based for each expert.

	Y1	Y2	…	Yn
P1	$(l_{11}, m_{11l}, m_{11u}, u_{11;}\ \alpha, \beta, \theta)$	$(l_{12}, m_{12l}, m_{12u}, u_{12;}\ \alpha, \beta, \theta)$	…	$(l_{1n}, m_{1nl}, m_{1nu}, u_{1n;}\ \alpha, \beta, \theta)$
P2	$(l_{21}, m_{21l}, m_{21u}, u_{21;}\ \alpha, \beta, \theta)$	$(l_{22}, m_{22l}, m_{22u}, u_{22;}\ \alpha, \beta, \theta)$	…	$(l_{2n}, m_{2nl}, m_{2nu}, u_{2n;}\ \alpha, \beta, \theta)$
…	………	………	…	………
Pm	$(l_{m1}, m_{m1l}, m_{m1u}, u_{m1;}\ \alpha, \beta, \theta)$	$(l_{m2}, m_{m2l}, m_{m2u}, u_{m2;}\ \alpha, \beta, \theta)$	…	$(l_{mn}, m_{mnl}, m_{mnu}, u_{mn;}\ \alpha, \beta, \theta)$

Fig. 6. Main criteria for fighter aircraft selection problem example.

of decision makers based on trapezoidal neutrosophic single values with (α, β, θ)and after using the numerical scale (0–1) for intangible criteria, as shown in Fig. 3 and expressed in Table 3.

2. Determine the crisp value of the decision matrix obtained in the previous matrix by Eqs. (4) and (5), to obtain the following Table 4.
3. In Table 4, we evaluated each project alternative (Pi) by a set of criteria (Yj) because the criteria have not the same measuring units, and some of them are tangible and some are not tangible, as shown in the introduction. The next step is getting the normalized decision matrix, R, using the equation (10). The elements of normalized decision matrix are fractions between 0 and 1.

$$r_{ij} = Y_{ij}/\text{sqroot}\ \ (\text{sum}, i = 1, \cdots .., n\ \ \text{of}\ \ Y_{ij}^2) \qquad (10)$$

4. Obtain the weighted decision matrix V taking into consideration the fact that the individual criteria have a certain weight (obtained from step 1 neutrosophic DEMATEL). We get V by multiplying each column of R by the corresponding weights, where W_{1*n} is the result of step 1.

$$V_{m*n} = R_{m*n} * W_{1*n} \qquad (11)$$

5. Obtain IDEAL (A^*) and Negative IDEAL (A^-) solutions from the weighted decision matrix V. where (A^*) is the best possibilities for each criterion among all alternatives in V and it's the largest value if profit and the smallest value in case of cost criterion measures. And (A^-) is the worst possibilities for each criterion among all alternatives in V and it's the smallest value if profit criterion measure and largest if cost measurable criterion.

Table 5
Pairwise comparisons among six criteria for the first expert.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	(0.5,0.5,0.5,0.5)	(0.5,0.6,0.9,0.2;0.4,0.2,0.3)	(0.8,0.7,0.9,1.0;0.7,0.2,0.5)	(0.5,0.8,1.0,0.0;0.7,0.2,0.5)	(0.8,0.4,1.0,0.9;0.4,0.2,0.3)	(0.7,0.5,0.4,0.2;0.5,0.4,0.3)
Y2	(0.4,0.3,0.9,0.8;0.9,0.7,0.4)	(0.5,0.5,0.5,0.5)	(0.0,0.1,0.3,0.4;0.8,0.2,0.6)	(0.5,0.2,0.8,1.0;0.7,0.6,0.5)	(0.4,0.3,0.2,0.5;0.9,0.7,0.3)	(0.3,0.2,0.4,0.9;0.4,0.3,0.7
Y3	(0.7,0.3,0.5,0.8;0.2,0.4,0.3)	(0.6,0.1,0.7,1.0;0.3,0.1,0.5)	(0.5,0.5,0.5,0.5)	(0.3,0.5,0.9,0.2;0.5,0.1,0.2)	(0.7,0.6,0.5,0.4;0.4,0.3,0.5)	(0.4,0.7,0.3,0.4;0.5,0.4,0.2)
Y4	(0.8,0.9,0.4,0.9;0.9,0.5,0.4)	(0.2,0.4,0.5,0.6;0.7,0.6,0.3)	(0.1,0.5,0.3,0.7;0.3,0.4,0.6)	(0.5,0.5,0.5,0.5)	(0.2,0.5,0.6,0.8;0.7,0.3,0.4)	(0.8,0.9,0.8,0.2;0.8,0.2,0.4)
Y5	(0.3,0.4,0.1,0.2;0.7,0.6,0.5)	(0.7,0.4,0.3,0.1;0.7,0.4,0.3)	(0.8,0.4,0.2,0.5;0.7,0.3,0.2)	(0.9,0.2,0.6,0.3;0.5,0.2,0.9)	(0.5,0.5,0.5,0.5)	(0.4,0.3,0.2,0.5;0.9,0.7,0.4)
Y6	(0.3,0.4,0.5,0.8;0.2,0.4,0.3)	(0.2,0.4,0.5,0.6;0.7,0.6,0.3)	(0.4,0.7,0.5,0.1;0.1,0.3,0.4)	(0.2,0.3,0.5,0.1;0.7,0.1,0.3)	(0.7,0.6,0.5,0.6;0.6,0.5,0.4)	(0.5,0.5,0.5,0.5)

Table 6
The crisp matrix for expert 1.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	0.5	0.261	0.425	0.375	0.368	0.203
Y2	0.27	0.5	0.1	0.25	0.166	0.158
Y3	0.216	0.255	0.5	0.261	0.22	0.214
Y4	0.375	0.191	0.13	0.5	0.263	0.371
Y5	0.1	0.169	0.261	0.219	0.5	0.158
Y6	0.188	0.191	0.191	0.244	0.255	0.5

6. Calculate the separation measures from ideal (S_i^*) and negative ideal (S_i^-) Eqs. (12) and (13) solution for all alternatives i = 1, ...,m. where:

$$S_i^* = [\text{sqroot (sum of squares for j} = 1,\cdots..,n \text{ of } (v_{ij} - v_j^*))] \quad (12)$$

$$S_i^- = [\text{sqroot (sum of squares for j} = 1,\cdots..,n \text{ of } (v_{ij} - v_j^-))] \quad (13)$$

7. Determine the relative closeness ideal solution; for each alternative calculate the relative closeness to the ideal solution (C_i^*, i = 1,...., m) by Eq. (14). The closeness rating is a number between 0 and 1with 0 being the worst possible alternative and 1 being the best possible alternative.

$$C_i^* = \frac{S_i^-}{(S_i^* + S_i^-)} \quad (14)$$

1. Make a decision for selecting the preference alternative project and determine the preference order by arranging alternatives in descending order, based on the relative closeness value for each alternative.

4. Illustrative example

This example illustrates the process of evaluating several projects and selects the best project using neutrosophic TOPSIS-DEMATEL, which is employed for weighting the different criteria affecting the process of projects evaluation. Then, a comparison is performed between the alternative projects and the weighted criteria. In this example, we consider four projects under the fighter aircraft selection. We consider six important criteria affecting the fighter aircraft selection. The six important criteria and their measurable units are presented in Fig. 6.

First, we apply the neutrosophic DEMATEL technique for weighting the main six criteria (in Fig. 6) for this problem, and then we apply the TOPSIS technique in the neutrosophic environment to select the best project. For more details, we follow the next steps:

Step 1: Start with neutrosophic DEMATEL by implementing the following:

1. Select the experts in project management field; we select three experts in this example.

Table 7
Pairwise comparisons among six criteria for the second expert.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	(0.5,0.5,0.5,0.5)	(0.3,0.4,0.7,0.5;0.6,0.2,0.1)	(0.9,0.6,0.8,0.9;0.8,0.4,0.3)	(0.4,0.9,1.0,0.0;0.7,0.4,0.3)	(0.3,0.2,0.4,0.9;0.5,0.3,0.1)	(0.8,0.4,1.0,0.9;0.4,0.2,0.3)
Y2	(0.6,0.2,0.8,0.9;0.9,0.7,0.4)	(0.5,0.5,0.5,0.5)	(0.3,0.1,0.6,0.4;0.6,0.2,0.4)	(0.7,0.2,0.6,1.0;0.7,0.6,0.5)	(0.3,0.0,0.3,0.4;0.8,0.2,0.6)	(0.3,0.2,0.4,0.9;0.4,0.3,0.7)
Y3	(0.6,0.3,0.5,0.9;0.2,0.4,0.1)	(0.9,0.1,0.7,1.0;0.5,0.1,0.5)	(0.5,0.5,0.5,0.5)	(0.8,0.5,0.9,0.2;0.5,0.4,0.2)	(0.3,0.6,0.5,0.4;0.4,0.3,0.2)	(0.3,0.7,0.3,0.4;0.4,0.5,0.2)
Y4	(0.6,0.9,0.4,0.6;0.6,0.5,0.4)	(0.7,0.4,0.5,0.6;0.7,0.4,0.3)	(0.1,0.5,0.5,0.7;0.3,0.4,0.2)	(0.5,0.5,0.5,0.5)	(0.6,0.5,0.6,0.8;0.7,0.3,0.1)	(0.9,0.7,0.8,0.2;0.8,0.2,0.4)
Y5	(0.6,0.4,0.1,0.2;0.7,0.6,0.5)	(0.3,0.4,0.3,0.1;0.7,0.4,0.4)	(0.2,0.1,0.3,0.4;0.8,0.2,0.4)	(0.9,0.6,0.9,0.2;0.4,0.2,0.1)	(0.5,0.5,0.5,0.5)	(0.4,0.6,0.2,0.5;0.9,0.6,0.4)
Y6	(0.6,0.4,0.5,0.8;0.2,0.4,0.3)	(0.3,0.4,0.5,0.6;0.7,0.5,0.3)	(0.3,0.7,0.5,0.1;0.1,0.5,0.4)	(0.4,0.3,0.5,0.1;0.2,0.4,0.3)	(0.8,0.6,0.5,0.6;0.6,0.5,0.3)	(0.5,0.5,0.5,0.5)

Table 8
The crisp matrix for the second expert 2.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	0.5	0.273	0.42	0.263	0.236	0.368
Y2	0.281	0.5	0.175	0.25	0.125	0.158
Y3	0.244	0.321	0.5	0.285	0.214	0.181
Y4	0.266	0.275	0.191	0.5	0.359	0.358
Y5	0.13	0.131	0.138	0.341	0.5	0.202
Y6	0.216	0.214	0.12	0.122	0.281	0.5

2. Identify the main criteria affecting the fighter aircraft selection problem, as presented in Fig. 6.
3. Make pairwise comparison matrix for each expert based on the trapezoidal neutrosophic number to evaluate each criterion against the others, as shown in Tables 5, 7, and 9.
4. Calculate the crisp value for each pairwise comparison matrix (for each expert opinion) using Eqs. (4), and (5). These crisp values are presented in Tables 6, 8, and 10.
5. Generate the initial directed matrix (s) by integrating the three matrices of expert's opinion using Eq. (6). The initial directed matrix is displayed in Table 11.
6. Generate the generalized direct relation matrix by normalizing the initial directed matrix using Eq. (7) to get the value of K, and then apply Eq. (8) to get the generalized direct relation matrix, as carried forth in Table 12.

Row 1	2.077
Row 2	1.451
Row 3	1.704
Row 4	1.944
Row 5	1.462
Row 6	1.494

$$K = \frac{1}{2.077}$$

7. Calculate the total relation matrix using Eq. (9), as introduced in Table 13, where (I) is the identity matrix.
8. Calculate the sum of each row and column in the total relation matrix (T), then draw causal diagram between the summation of rows and columns as a horizontal line and the differences between rows and column as vertical axes, as pictured in Fig. 7.

Sum of rows and columns

Col 1	4.1226	Row 1	5.4187
Col 2	4.2583	Row 2	3.695
Col 3	4.049	Row 3	4.3606
Col 4	4.5037	Row 4	5.0038
Col 5	4.4394	Row 5	3.6215
Col 6	4.4172	Row6	3.6906

Table 9
Pairwise comparisons among six criteria for the third expert.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	(0.5,0.5,0.5,0.5)	(0.6,0.4,0.7,0.6;0.6,0.2,0.1)	(0.7,0.6,0.8,0.8;0.8,0.4,0.3)	(0.3,0.9,1.0,0.8;0.3,0.4,0.4)	(0.4,0.2,0.4,0.8;0.5,0.3,0.1)	(0.7,0.4,1.0,0.5;0.4,0.2,0.3)
Y2	(0.3,0.2,0.8,0.9;0.9,0.7,0.4)	(0.1,0.5,0.5,0.5)	(0.6,0.1,0.6,0.3;0.6,0.2,0.4)	(0.6,0.2,0.6,1.0;0.7,0.6,0.6)	(0.4,0.0,0.3,0.3;0.8,0.2,0.6)	(0.2,0.2,0.4,0.6;0.4,0.3,0.7)
Y3	(0.3,0.3,0.5,0.9;0.2,0.4,0.1)	(0.5,0.1,0.7,0.9;0.5,0.1,0.5)	(0.5,0.5,0.5,0.5)	(0.7,0.5,0.9,0.2;0.5,0.4,0.1)	(0.4,0.6,0.5,0.5;0.4,0.3,0.2)	(0.5,0.7,0.3,0.4;0.4,0.5,0.2)
Y4	(0.5,0.9,0.4,0.6;0.5,0.3,0.4)	(0.5,0.4,0.5,0.7;0.7,0.4,0.3)	(0.6,0.5,0.5,0.6;0.3,0.4,0.2)	(0.5,0.5,0.5,0.5)	(0.5,0.5,0.6,0.9;0.7,0.3,0.1)	(0.9,0.7,0.8,0.7;0.8,0.2,0.4)
Y5	(0.6,0.4,0.1,0.2;0.7,0.6,0.5)	(0.6,0.4,0.3,0.3;0.7,0.4,0.4)	(0.7,0.1,0.3,0.5;0.8,0.2,0.4)	(0.4,0.6,0.9,0.2;0.4,0.2,0.2)	(0.5,0.5,0.5,0.5)	(0.4,0.6,0.2,0.8;0.9,0.6,0.4)
Y6	(0.8,0.4,0.5,0.8;0.2,0.4,0.3)	(0.4,0.4,0.5,0.5;0.7,0.5,0.3)	(0.5,0.7,0.5,0.3;0.1,0.5,0.4)	(0.2,0.3,0.5,0.1;0.2,0.4,0.1)	(0.7,0.6,0.5,0.4;0.6,0.5,0.3)	(0.5,0.5,0.5,0.5)

Table 10
The crisp matrix for the third expert.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	0.5	0.331	0.381	0.282	0.236	0.309
Y2	0.248	0.5	0.2	0.225	0.125	0.123
Y3	0.213	0.261	0.5	0.288	0.238	0.202
Y4	0.27	0.263	0.234	0.5	0.359	0.426
Y5	0.13	0.19	0.22	0.263	0.5	0.238
Y6	0.234	0.214	0.15	0.117	0.248	0.5

Table 11
The initial directed matrix.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	0.5	0.288	0.409	0.307	0.28	0.293
Y2	0.266	0.5	0.158	0.242	0.139	0.146
Y3	0.224	0.279	0.5	0.278	0.224	0.199
Y4	0.304	0.243	0.185	0.5	0.327	0.385
Y5	0.12	0.163	0.206	0.274	0.5	0.199
Y6	0.213	0.206	0.153	0.161	0.261	0.5

Table 12
The generalized direct relation matrix X.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	0.2405	0.138528	0.196729	0.147667	0.13468	0.140933
Y2	0.127946	0.2405	0.075998	0.116402	0.066859	0.070226
Y3	0.107744	0.134199	0.2405	0.133718	0.107744	0.095719
Y4	0.146224	0.116883	0.088985	0.2405	0.157287	0.185185
Y5	0.05772	0.078403	0.099086	0.131794	0.2405	0.095719
Y6	0.102453	0.099086	0.073593	0.077441	0.125541	0.2405

	Col + Row	Col-Row
1	9.5413	−1.2961
2	7.9533	0.5633
3	8.4096	−0.3116
4	9.5075	−0.5001
5	8.0609	0.8179
6	8.1078	0.7266

9. Weight the six criteria based on the causal diagram. The importance of all criteria is evaluated and ranked based on the expert's opinion and introduced in the causal diagram as follows: the reliability criterion is the most important criterion for project selection (Y5), and the least important criterion is the ferry range (Y1).
Based on the expert's opinion and neutrosophic DEMATEL method, the weights of considered criteria relative to their importance are (0.1, 0.2, 0.1, 0.1, 0.3, and 0.2).

Table 13
The total relation matrix T.

	Y1	Y2	Y3	Y4	Y5	Y6
Y1	0.9547	0.8651	0.9025	0.9166	0.8880	0.8918
Y2	0.6164	0.7566	0.5401	0.6411	0.5680	0.5728
Y3	0.6711	0.7220	0.8108	0.7538	0.7101	0.6928
Y4	0.7956	0.7783	0.7128	0.9574	0.8641	0.8956
Y5	0.5102	0.5502	0.5513	0.6478	0.7664	0.5956
Y6	0.5746	0.5861	0.5315	0.5870	0.6428	0.7686

Step 2: Apply the Neutrosophic TOPSIS for ranking the four projects and select the best one, by performing the following:

1. Obtain the decision matrix between the four project alternatives (P1-P4) and the six criteria (Y1-Y6) [Tables 14–17]. These values are crisp values, based on the opinion of decision-makers expressed by trapezoidal neutrosophic single values with (α, β, θ), using the numerical scale (0–1) for intangible criteria [Fig. 8].

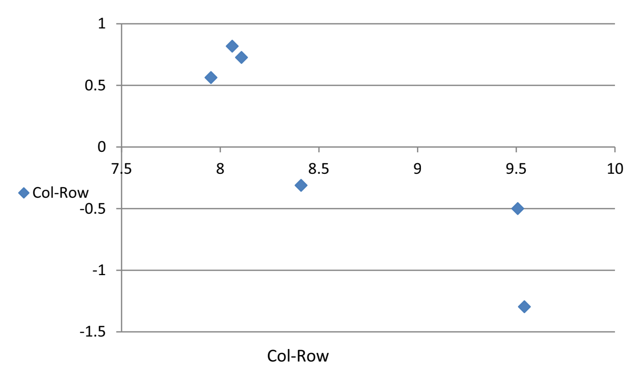

Fig. 7. The causal diagram for the six criteria.

Table 14
The decision matrix of the fighter aircraft selection.

	Y1	Y2	Y3	Y4	Y5	Y6
P1	1500~	5.5~	2~	20,000~	Avg~	V.high~
P2	2700~	6.5~	2.5~	18,000~	Low~	Avg~
P3	200~	4.5~	1.8~	21,000~	High~	High~
P4	1800~	5~	2.2~	20,000~	Avg~	Avg~

Table 15
The decision matrix of the fighter aircraft selection with the numerical scale of intangible criteria.

	Y1	Y2	Y3	Y4	Y5	Y6
P1	1500~	5.5~	2~	20,000~	0.5~	0.9~
P2	2700~	6.5~	2.5~	18,000~	0.3~	0.5~
P3	200~	4.5~	1.8~	21,000~	0.7~	0.7~
P4	1800~	5~	2.2~	20,000~	0.5~	0.5~

Table 16
The decision matrix in trapezoidal values based on trapezoidal neutrosophic single values with (α, β, θ).

	Y1	Y2	Y3	Y4	Y5	Y6
P1	(800,1200,1500,1800;0.6,0.5,0.3)	(3.5,4,5.5,5.7;0.5,0.3,0.4)	(1,1.5,2,2.5;0.5,0.4,0.3)	(18000,19000,20000,20500;0.5,0.3,0.4)	(0.3,0.4,0.5,0.6;0.5,0.6,0.2)	0.7,0.8,0.9,0.95;0.8,0.5,0.3)
P2	(1700,2100,2700,2009;0.9,0.5,0.4)	(5,5.5,6.5,7;0.9,0.5,0.3)	(1.5,2,2.5,3;0.2,0.3,0.1)	17500,17800,18,000,20000;0.8,0.6,0.4)	(0.2,0.25,0.3,0.4;0.9,0.5,0.3)	(0.4,0.54,0.5,0.6;0.5,0.7,0.3)
P3	(90,150,200,220;0.8,0.6,0.4)	(4,4.2,4.5,5.5;0.8,0.7,0.6)	(1.3,1.5,1.8,1.9;0.5,0.6,0.2)	(20000,20800,21000,21500;0.9,0.8,0.7)	(0.5,0.6,0.7,0.75;0.5,0.6,0.4)	(0.5,0.6,0.7,0.8;0.7,0.6,0.5)
P4	(1200,1500,1800,2000;0.5,0.6,0.2)	(4,4.5,5,5.2;0.6,0.5,0.4)	(1.8,2,2.2,2.5;0.7,0.5,0.4)	(18500,19100,20,000,20500;0.6,0.3,0.1)	(0.3,0.4,0.5,0.6;0.6,0.2,0.1)	(0.3,0.4,0.5,0.6;0.3,0.5,0.4)

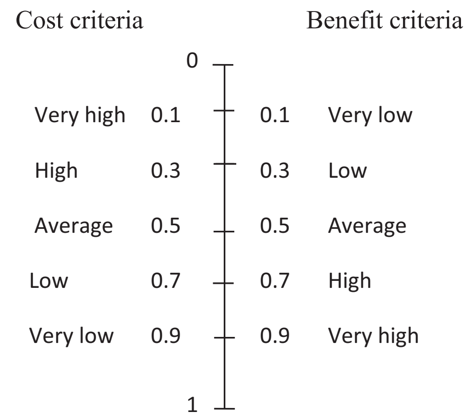

Fig. 8. The scale for intangible criteria.

2. Generate the normalized decision matrix (R) using Eq. (10), as presented in Table 18; notice that all r_{ij} is between 0 and 1.

   W= (0.1, 0.2, 0.1, 0.1, 0.3, 0.2)

3. Obtain the weighted decision matrix V by multiplying each column of R with the corresponding criterion weight (the output of step 1), using Eq. (11), as presented in Table 19.
4. In the weighted matrix (Table 19), we determine for each criterion the best value (the largest value) and the worst value (the smallest value). This is done for all benefits criteria such as (Y1, Y3, Y4, Y5, and Y6), but in case of cost criteria we select the smallest value as the best value, and the largest value as the worst value, such as criterion Y2 in our example, where Y2 represents the acquisition cost. Obtain the ideal (the best possible solution) and negative (the worst possible solution) ideal solution A^*, and A^-.
   $A^* =$ (0.078886, 0.072335, 0.05809, 0.063816, 0.18124, 0.152606)
   $A^- =$ (0.004985, 0.137315, 0.039623, 0.03697, 0.105724, 0.057398)
5. Calculate the separation measures from ideal and negative ideal solution Si^*, Si^- using Eqs. (12) and (13), as shown in Table 20.
6. Compute the relative closeness to the ideal solution for each alternative by using Eq. (14); the relative closeness values are expressed in Table 21:
7. Finally, rank four alternatives based on their relative closeness value. Determine the preference order by arranging the alternatives of the relative closeness values for alternatives in the descending order of Ci^*, $i = 1, 2, 3, 4$. Thus, the rank of alternatives in the fighter aircraft selection problem using neutrosophic TOPSIS-DEMATEL emerges as A1, A4, A3, and A2.

Table 17
The equivalent crisp values of the decision matrix.

	Y1	Y2	Y3	Y4	Y5	Y6
P1	596.25	2.10375	0.7875	8718.75	0.19125	0.41875
P2	1175	3.15	1.0125	6221.25	0.150938	0.182813
P3	74.25	1.659375	0.690625	7288.75	0.239063	0.26
P4	690.625	1.986875	0.95625	10738.75	0.25875	0.1575

Table 18
The normalized decision matrix.

	Y1	Y2	Y3	Y4	Y5	Y6
P1	0.400302	0.458533	0.451807	0.518117	0.446533	0.763028
P2	0.788855	0.686574	0.580895	0.369702	0.352412	0.333114
P3	0.049849	0.361677	0.396228	0.433139	0.558167	0.473761
P4	0.463662	0.433059	0.548623	0.638157	0.604133	0.28699

Table 19
The weighted decision matrix V, with best and worst values.

	Y1	Y2	Y3	Y4	Y5	Y6
P1	0.04003	0.091707	0.045181	0.051812	0.13396	**0.152606**
P2	**0.078886**	**0.137315**	**0.05809**	**0.03697**	**0.105724**	0.066623
P3	**0.004985**	**0.072335**	**0.039623**	0.043314	0.16745	0.094752
P4	0.046366	0.086612	0.054862	**0.063816**	**0.18124**	**0.057398**

Table 20
The separation measures from ideal and negative ideal solution Si^*, and Si^-.

Separation measures from	
Ideal solution	Negative ideal solution
$S1^* = 0.0666$	$S1^- = 0.1158$
$S2^* = 0.1343$	$S2^- = 0.0767$
$S3^* = 0.0988$	$S3^- = 0.0973$
$S4^* = 0.1017$	$S4^- = 0.1046$

Table 21
The relative closeness to the ideal solution for each alternative.

Alternatives	Relative closeness value
1	$C1^* = 0.634868$
2	$C2^* = 0.363507$
3	$C3^* = 0.496175$
4	$C4^* = 0.507029$

5. Conclusion and future work

Neutrosophic set is the most comprehensive set, which includes both fuzzy set and intuitionistic fuzzy set, as it considers the indeterminacy function in addition to truth-membership and falsity membership, being suitable in analyzing real situations. Also, in real life situations, accurate judgments are rarely since ambiguity and uncertainty surround the decision-making process. To solve the problem of project selection, the important criteria should be identified well, and then the selection process should be performed among several alternative projects. In this research, we considered parameters of TOPSIS-DEMATEL comparison matrices as trapezoidal neutrosophic numbers. TOPSIS is combined with the DEMATEL for more powerful and accurate weighted criteria, helping the selection of the best project alternative. Neutrosophic TOPSIS-DEMATEL model presented here is used for assisting the decision of project selection phase of project life cycle. We consider only (n-1) consensus judgment for each expert, for n numbers of alternatives. As well, we consider the (0–1) scale for intangible criteria. The project selection is a very important phase of any project life cycle after identification and appraisal of projects. In the future, we enhance the proposed model to solve the different phases of a project's life cycle. Moreover, we plan to solve the selection project problem with more complex techniques dealing with Multi-Criteria Decision-Making problems.

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An approach of TOPSIS Technique for Developing Supplier Selection with Group Decision Making under Type-2 Neutrosophic Number

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A B S T R A C T

This paper proposes an advanced type of neutrosophic technique, called type 2 neutrosophic numbers, and defines some of its operational rules. The type 2 neutrosophic number weighted averaging operator is determined in order to collective the type 2 neutrosophic number set, inferring some properties of the suggested operator. The operator is employed in a MADM problem to collect the type 2 neutrosophic numbers based classification values of each alternative over the features. The convergent classification values of every alternative are arranged with the assistance of score and accuracy values with the aim to detect the superior alternative. We introduce an illuminating example to confirm the suggested approach for multi attribute decision making issues, ordering the alternatives based on the accuracy function. Selecting an appropriate alternative among the selection options is a difficult activity for decision makers, since it is complicated to express attributes as crisp numbers. To tackle the problem, type 2 neutrosophic numbers can be efficiently used to estimate information in the decision making process. The type 2 neutrosophic numbers can accurately describe real cognitive information. We propose a novel T2NN-TOPSIS strategy combining type 2 neutrosophic numbers and TOPSIS under group decision making as application of T2NN, suggesting a type 2 neutrosophic number expression for linguistic terms. Finally, we provide a real case dealing with a decision making problem based on the proposed T2NN-TOPSIS methodology to prove the efficiency and the applicability of the type 2 neutrosophic number.

1. Introduction

Fuzzy theory was established on the notion of membership function to take linguistic variables into consideration. The theory seeks to deem uncertain data which can be related with existent fuzziness of peoples' observations and perceptions. The results indicate that they are strongly affected by self-regulation in such circumstances. Fuzzy has confirmed functionality in dealing with vagueness and ambiguity of human intellect and expression while decision making. Hence, the Neutrosophic is an extension of the fuzzy theory and intuitionistic fuzzy set (IFS). Smarandache proposed the neutrosophic sets in [1,2], attracting the attention of many scholars. The neutrosophic sets proved to be a valid workspace in describing incompatible and indefinite information. z(T, I, F) is a Type-1 Neutrosophic Number. But $z((T_T, T_I, T_F), (I_T, I_I, I_F), (F_T, F_I, F_F))$ is a Type-2 Neutrosophic Number, which means that each neutrosophic component T, I, and F is split into its truth, indeterminacy, and falsehood subparts. The procedure of splitting may be executed recurrently, as many times as needed, obtaining

Abbreviations: T2NN, Type 2 neutrosophic number; T2NNWA, Type 2 neutrosophic number weighted averaging; NPIS, Neutrosophic positive ideal solution; NNIS, Neutrosophic negative ideal solution; GDM, Group decision making; TOPSIS, Technique for order preference by similarity to ideal solution; SVN, Single valued neutrosophic; ANP, Analytical network process; MDM, Multi decision making; MADM, Multi attribute decision making; MAGDM, Multi attribute group decision making; MCDM, Multicriteria decision making a general Type-n Neutrosophic Number, for any integer $n \geq 1$. Here, we use the type 2 neutrosophic number as advancement of neutrosophic number to solve MCDM problems.

A neutrosophic set has more possibility strength than other forming mathematical apparatus, such as fuzzy set [3], interval valued intuitionistic fuzzy set (IVIFS) [4] or IFS [5]. Smarandache combined the degree of indeterminacy as independent element in IFS and defined the neutrosophic set [6] as a generalization of IFSs. Georgiev [7] questioned that the neutrosophic logic is qualified for preserving formal operators, since there is no standardization base for the elements T, I and F. However, fuzzy sets and IFSs cannot deal with certain types of uncertain information, such as incompatible, indefinite, or incomplete information. Smarandache [8] recognized neutrosophic set as a generalization of IFS, which performs a significant role to transact unclear, unpredictable and indeterminate information in the real world. The truth, indeterminacy and falsity degrees exist in the non-standard item interval suitable for each element of the universe [8]. In these days, Neutrosophic received attentions from many researches were proceed to develop, improve and expand the neutrosophic theory [9–16]. The neutrosophic set expanded to many branches, such as topology, image conversion or social science. We used single valued neutrosophic set [17] (SVNS), a subclass of neutrosophic set, in which every component of the universe is described by the truth, indeterminacy and falsity memberships existing in the actual unit interval. Liu and Liu [18] introduced neutrosophic number generalized weighted power averaging operator (NNGWPAO) and suggested a MAGDM strategy in neutrosophic number environment. Peng et al. [19] suggested a MAGDM strategy constructed on neutrosophic number generalized hybrid weighted averaging operator. Ye [20] introduced weighted arithmetic average operator for simplified neutrosophic sets. Hence, we will refer to TOPSIS methodology that is a widespread strategy to transact MAGDM. TOPSIS [21] helps choosing the best selection, which is the nearest to the quixotic solution and the farthest from the negative quixotic solution. Information of attributes that aggregated from experts and decision maker/makers is the base of the TOPSIS strategy. In crisp setting, an extended TOPSIS strategy for MAGDM under GDM was established by Shih [22]. A TOPSIS strategy for group decision making was suggested by Hatami [23]. Ravasan et al. [24] developed a fuzzy TOPSIS strategy for an e-banking outsourcing strategy selection in fuzzy environment. Banaeian et al. [25] introduced a fuzzy TOPSIS for GDM for green supplier selection for an actual company from the agri-food sector. In intuitionistic fuzzy environment, Büyüközkan et al. [26] elaborated an MAGDM for supplier election with TOPSIS strategy. Gupta et al. [27] established a protracted TOPSIS method under interval-valued intuitionistic fuzzy environment. Wang et al. [28] suggested a TOPSIS strategy for MAGDM in single valued neutrosophic environment. Ju et al. [29] propounded a TOPSIS strategy for MAGDM established on SVN linguistic numbers. A TOPSIS strategy was presented in neutrosophic cubic set environment by Pramanik et al. [30]. A TOPSIS strategy for MADM in bipolar neutrosophic set environment was put forward by Dey et al. [31]. Abdel-Basset et al. [32] suggested an ANP-TOPSIS strategy for supplier selection problems with interval valued neutrosophic. TOPSIS strategy is yet to approach T2NN environment. To fill the research gap, we improve a MAGDM strategy built on TOPSIS in type 2 neutrosophic number environment, namely T2NN-TOPSIS strategy to solve MAGDM issues.

Contribution of this paper:

- We state a T2NN, score function and accuracy function of T2NN, and prove its basic properties.
- We define T2NNWA to aggregate T2NN decision matrices.
- We propose linguistic terms to present T2NN.
- We suggest a tangential function to locate unidentified weights of attributes in T2NN setting.
- We develop a T2NN-TOPSIS strategy to solve MAGDM problems in T2NN environment.
- The proposed T2NN-TOPSIS is comprehensive, presenting all vague and incomplete information about all elements.
- We present an illustrative model of a MADM problem.

Table 1 below provides a literature review. Section 2 introduces several basic concepts of T2NN, operations on T2NN, applications of T2NNWA operator to MADM, two properties on T2NNWA and a numerical example. Section 3 clarifies the procedure for TOPSIS-T2NN methodology for the evaluation suppliers. Section 4 provides a real example based on the proposed T2NN-TOPSIS strategy. Section 5 concludes the research.

2. Preliminaries

We introduce several basic concepts of T2NN, operations on T2NN, applications of T2NNWA operator to MADM, and two properties on T2NNWA.

Definition 1. Let Z be the limited universe of discourse and F [0, 1] be the set of all triangular neutrosophic numbers on F [0, 1]. A type 2 neutrosophic number set (T2NNS) $\tilde{U}$ in Z is represented by $\tilde{U} = \left\{\left\langle z, \tilde{T}_{\tilde{U}}(z), \tilde{I}_{\tilde{U}}(z), \tilde{F}_{\tilde{U}}(z) \mid z \in Z\right\rangle\right\}$, where $\tilde{T}_{\tilde{U}}(z): Z \to F[0,1]$, $\tilde{I}_{\tilde{U}}(z): Z \to F[0,1]$, $\tilde{F}_{\tilde{U}}(z): Z \to F[0,1]$. A T2NNS $\tilde{T}_{\tilde{A}}(z) = \left(T_{T_{\tilde{U}}}(z), T_{I_{\tilde{U}}}(z), T_{F_{\tilde{U}}}(z)\right)$, $\tilde{I}_{\tilde{U}}(z) = \left(I_{T_{\tilde{U}}}(z), I_{I_{\tilde{U}}}(z), I_{F_{\tilde{U}}}(z)\right)$, $\tilde{F}_{\tilde{U}}(z) = \left(F_{T_{\tilde{U}}}(z), F_{I_{\tilde{U}}}(z), F_{F_{\tilde{U}}}(z)\right)$, respectively, denote the truth, indeterminacy, and falsity memberships of z in $\tilde{U}$ and for every $z \in Z: 0 \leq \tilde{T}_{\tilde{U}}(z)^3 + \tilde{I}_{\tilde{U}}(z)^3 + \tilde{F}_{\tilde{U}}(z)^3 \leq 3$; for convenience, we consider that $\tilde{U} = \left\langle\left(T_{T_{\tilde{U}}}(z), T_{I_{\tilde{U}}}(z), T_{F_{\tilde{U}}}(z)\right), \left(I_{T_{\tilde{U}}}(z), I_{I_{\tilde{U}}}(z), I_{F_{\tilde{U}}}(z)\right), \left(F_{T_{\tilde{U}}}(z), F_{I_{\tilde{U}}}(z), F_{F_{\tilde{U}}}(z)\right)\right\rangle$ as a type 2 neutrosophic number.

Definition 2. Suppose $\tilde{U}_1 = \left\langle\left(T_{T_{\tilde{U}_1}}(z), T_{I_{\tilde{U}_1}}(z), T_{F_{\tilde{U}_1}}(z)\right), \left(I_{T_{\tilde{U}_1}}(z), I_{I_{\tilde{U}_1}}(z), I_{F_{\tilde{U}_1}}(z)\right), \left(F_{T_{\tilde{U}_1}}(z), F_{I_{\tilde{U}_1}}(z), F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$ and $\tilde{U}_2 = \left\langle\left(T_{T_{\tilde{U}_2}}(z), T_{I_{\tilde{U}_2}}(z), T_{F_{\tilde{U}_2}}(z)\right), \left(I_{T_{\tilde{U}_2}}(z), I_{I_{\tilde{U}_2}}(z), I_{F_{\tilde{U}_2}}(z)\right), \left(F_{T_{\tilde{U}_2}}(z), F_{I_{\tilde{U}_2}}(z), F_{F_{\tilde{U}_2}}(z)\right)\right\rangle$ are two T2NNS in the set real numbers. Then the procedures are defined as Eqs. (1)–(4) in Box I.

The procedures defined in Definition 2 satisfy the following properties:

1. $\tilde{U}_1 \oplus \tilde{U}_2 = \tilde{U}_2 \oplus \tilde{U}_1$, $\tilde{U}_1 \otimes \tilde{U}_2 = \tilde{U}_2 \otimes \tilde{U}_1$;
2. $\delta(\tilde{U}_1 \oplus \tilde{U}_2) = \delta\tilde{U}_1 \oplus \delta\tilde{U}_2$, $(\tilde{U}_1 \otimes \tilde{U}_2)^{\delta} = \tilde{U}_1^{\delta} \otimes \tilde{U}_2^{\delta}$ for $\delta > 0$, and
3. $\delta_1\tilde{U}_1 \oplus \delta_2\tilde{U}_1 = (\delta_1 + \delta_2)\tilde{U}_1$, $\tilde{U}_1^{\delta_1} \oplus \tilde{U}_1^{\delta_2} = \tilde{U}_1^{(\delta_1+\delta_2)}$ for $\delta_1, \delta_2 > 0$.

Definition 3. Suppose that $\tilde{U}_1 = \left\langle\left(T_{T_{\tilde{U}_1}}(z), T_{I_{\tilde{U}_1}}(z), T_{F_{\tilde{U}_1}}(z)\right), \left(I_{T_{\tilde{U}_1}}(z), I_{I_{\tilde{U}_1}}(z), I_{F_{\tilde{U}_1}}(z)\right), \left(F_{T_{\tilde{U}_1}}(z), F_{I_{\tilde{U}_1}}(z), F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$ are T2NNS in the set of real numbers, the score function $S(\tilde{U}_1)$ of $\tilde{U}_1$ is defined

Table 1
Literature review.

References	Methods	GDM	Application type	Objective of the study
Peng, X. and J. Dai [33]	SVN–TOPSIS	–	Methodology proposal	A new axiomatic definition of single-valued neutrosophic distance measure and similarity measure
Pouresmaeil, H., et al. [34]	TOPSIS and SVN	X	Methodology proposal	Multiple attribute decision making
Selvachandran, G., et al. [35]	TOPSIS-MDM–SVN Sets	–	Methodology proposal	New aggregation operator proposal
Biswas, P., et al. [36]	Neutrosophic TOPSIS	X	Methodology proposal	New aggregation operator proposal
Biswas, P., et al. [37]	TOPSIS for MAGDM under SVN	X	Methodology proposal	A new strategy for MAGDM problems
Broumi, S., et al. [38]	TOPSIS method for MADM based on interval neutrosophic	–	Methodology proposal	TOPSIS solve the MADM
Smarandache, F. and S. Pramanik [39]	Neutrosophic under bi-polar neutrosophic	–	Methodology proposal	Select the most desirable alternative

1. $\tilde{U}_1 \oplus \tilde{U}_2$

$$= \left\langle \begin{array}{c} \left(\begin{array}{c} \left(T_{T_{\tilde{U}_1}}(z) + T_{T_{\tilde{U}_2}}(z) - T_{T_{\tilde{U}_1}}(z).T_{T_{\tilde{U}_2}}(z)\right), \left(T_{I_{\tilde{U}_1}}(z) + T_{I_{\tilde{U}_2}}(z) - T_{I_{\tilde{U}_1}}(z).T_{I_{\tilde{U}_2}}(z)\right), \\ \left(T_{F_{\tilde{U}_1}}(z) + T_{F_{\tilde{U}_2}}(z) - T_{F_{\tilde{U}_1}}(z).T_{F_{\tilde{U}_2}}(z)\right) \end{array} \right), \\ \left(I_{T_{\tilde{U}_1}}(z).I_{T_{\tilde{U}_2}}(z),\ I_{I_{\tilde{U}_1}}(z).I_{I_{\tilde{U}_2}}(z),\ I_{F_{\tilde{U}_1}}(z).I_{F_{\tilde{U}_2}}(z)\right), \left(F_{T_{\tilde{U}_1}}(z).F_{T_{\tilde{U}_2}}(z),\ F_{I_{\tilde{U}_1}}(z).F_{I_{\tilde{U}_2}}(z),\ F_{F_{\tilde{U}_1}}(z).F_{F_{\tilde{U}_2}}(z)\right) \end{array} \right\rangle \tag{1}$$

2. $\tilde{U}_1 \otimes \tilde{U}_2$

$$= \left\langle \begin{array}{c} \left(\left(T_{T_{\tilde{U}_1}}(z).T_{T_{\tilde{U}_2}}(z),\ T_{I_{\tilde{U}_1}}(z).T_{I_{\tilde{U}_2}}(z),\ T_{F_{\tilde{U}_1}}(z).T_{F_{\tilde{U}_2}}(z)\right)\right), \\ \left(\begin{array}{c} \left(I_{T_{\tilde{U}_1}}(z) + I_{T_{\tilde{U}_2}}(z) - I_{T_{\tilde{U}_1}}(z).I_{T_{\tilde{U}_2}}(z)\right), \left(I_{I_{\tilde{U}_1}}(z) + I_{I_{\tilde{U}_2}}(z) - I_{I_{\tilde{U}_1}}(z).I_{I_{\tilde{U}_2}}(z)\right), \\ \left(I_{F_{\tilde{U}_1}}(z) + I_{F_{\tilde{U}_2}}(z) - I_{F_{\tilde{U}_1}}(z).I_{F_{\tilde{U}_2}}(z)\right) \end{array} \right) \\ \left(\begin{array}{c} \left(F_{T_{\tilde{U}_1}}(z) + F_{T_{\tilde{U}_2}}(z) - F_{T_{\tilde{U}_1}}(z).F_{T_{\tilde{U}_2}}(z)\right), \left(F_{I_{\tilde{U}_1}}(z) + F_{I_{\tilde{U}_2}}(z) - F_{I_{\tilde{U}_1}}(z).F_{I_{\tilde{U}_2}}(z)\right), \\ \left(F_{F_{\tilde{U}_1}}(z) + F_{F_{\tilde{U}_2}}(z) - F_{F_{\tilde{U}_1}}(z).F_{F_{\tilde{U}_2}}(z)\right) \end{array} \right) \end{array} \right\rangle \tag{2}$$

3. $\delta\tilde{U}$

$$= \left\langle \begin{array}{c} \left(1-(1-T_{T_{\tilde{U}_1}}(z))^{\delta},\ 1-(1-T_{I_{\tilde{U}_1}}(z))^{\delta},\ 1-(1-T_{F_{\tilde{U}_1}}(z))^{\delta}\right), \\ \left(\left(I_{T_{\tilde{U}_1}}(z)\right)^{\delta}, \left(I_{I_{\tilde{U}_1}}(z)\right)^{\delta}, \left(I_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right), \\ \left(\left(F_{T_{\tilde{U}_1}}(z)\right)^{\delta}, \left(F_{I_{\tilde{U}_1}}(z)\right)^{\delta}, \left(F_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right) \end{array} \right\rangle \text{ for } \delta > 0 \tag{3}$$

4. $\tilde{U}^{\delta}$

$$= \left\langle \begin{array}{c} \left(\left(T_{T_{\tilde{U}_1}}(z)\right)^{\delta}, \left(T_{I_{\tilde{U}_1}}(z)\right)^{\delta}, \left(T_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right), \\ \left(1-\left(1-I_{T_{\tilde{U}_1}}(z)\right)^{\delta},\ 1-\left(1-I_{I_{\tilde{U}_1}}(z)\right)^{\delta},\ 1-\left(1-I_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right), \\ \left(1-\left(1-F_{T_{\tilde{U}_1}}(z)\right)^{\delta},\ 1-\left(1-F_{I_{\tilde{U}_1}}(z)\right)^{\delta},\ 1-\left(1-F_{F_{\tilde{U}_1}}(z)\right)^{\delta}\right) \end{array} \right\rangle \text{ for } \delta > 0 \tag{4}$$

Box I.

as follows:

$$S(\tilde{U}_1) = \frac{1}{12}\Big\langle 8 + \left(T_{T_{\tilde{U}_1}}(z) + 2\left(T_{I_{\tilde{U}_1}}(z)\right) + T_{F_{\tilde{U}_1}}(z)\right) - \left(I_{T_{\tilde{U}_1}}(z) + 2\left(I_{I_{\tilde{U}_1}}(z)\right) + I_{F_{\tilde{U}_1}}(z)\right) - \left(F_{T_{\tilde{U}_1}}(z) + 2\left(F_{I_{\tilde{U}_1}}(z)\right) + F_{F_{\tilde{U}_1}}(z)\right)\Big\rangle \tag{5}$$

$$A(\tilde{U}_1) = \frac{1}{4}\Big\langle \left(T_{T_{\tilde{U}_1}}(z) + 2\left(T_{I_{\tilde{U}_1}}(z)\right) + T_{F_{\tilde{U}_1}}(z)\right) - \left(F_{T_{\tilde{U}_1}}(z) + 2\left(F_{I_{\tilde{U}_1}}(z)\right) + F_{F_{\tilde{U}_1}}(z)\right)\Big\rangle \tag{6}$$

$$\mathrm{T2NNWA}_{\omega}(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_n) = \omega_1\tilde{U}_1 \oplus \omega_2\tilde{U}_2 \oplus \ldots \omega_n\tilde{U}_n = \oplus_{p=1}^{n}(\omega_p\tilde{U}_p)$$
$$= \left\langle \begin{array}{c} \left(1-\prod_{p=1}^{n}\left(1-T_{T_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{n}\left(1-T_{I_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{n}\left(1-T_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{n}\left(I_{T_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{n}\left(I_{I_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{n}\left(I_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{n}\left(F_{T_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{n}\left(F_{I_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{n}\left(F_{F_p}(z)\right)^{\omega_p}\right) \end{array} \right\rangle \quad (11)$$

Box II.

Definition 4. Suppose that $\tilde{U}_1 = \left\langle\left(T_{T_{\tilde{U}_1}}(z), T_{I_{\tilde{U}_1}}(z), T_{F_{\tilde{U}_1}}(z)\right), \left(I_{T_{\tilde{U}_1}}(z), I_{I_{\tilde{U}_1}}(z), I_{F_{\tilde{U}_1}}(z)\right), \left(F_{T_{\tilde{U}_1}}(z), F_{I_{\tilde{U}_1}}(z), F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$ and $\tilde{U}_2 = \left\langle\left(T_{T_{\tilde{U}_2}}(z), T_{I_{\tilde{U}_2}}(z), T_{F_{\tilde{U}_2}}(z)\right), \left(I_{T_{\tilde{U}_2}}(z), I_{I_{\tilde{U}_2}}(z), I_{F_{\tilde{U}_2}}(z)\right), \left(F_{T_{\tilde{U}_2}}(z), F_{I_{\tilde{U}_2}}(z), F_{F_{\tilde{U}_2}}(z)\right)\right\rangle$ are two T2NNS in the set of real numbers. Suppose that $S(\tilde{U}_i)$ and $A(\tilde{U}_i)$ are the score and accuracy functions of T2NNS $\tilde{U}_i (i = 1, 2)$, then the order relations are defined as follows:

1. If $\tilde{S}(\tilde{U}_1) > \tilde{S}(\tilde{U}_2)$, then $\tilde{U}_1$ is greater than $\tilde{U}_2$, that is $\tilde{U}_1$ is superior to $\tilde{U}_2$, denoted by $\tilde{U}_1 > \tilde{U}_2$;
2. If $\tilde{S}(\tilde{U}_1) = \tilde{S}(\tilde{U}_2)$, $\tilde{A}(\tilde{U}_1) > \tilde{A}(\tilde{U}_2)$)then $\tilde{U}_1$ is superior than $\tilde{U}_2$, that is $\tilde{U}_1$ is superior to $\tilde{U}_2$, denoted by $\tilde{U}_1 > \tilde{U}_2$;
3. If $\tilde{S}(\tilde{U}_1) = \tilde{S}(\tilde{U}_2)$, $\tilde{A}(\tilde{U}_1) = \tilde{A}(\tilde{U}_2)$) then $\tilde{U}_1$ is equal to $\tilde{U}_2$, that is $\tilde{U}_1$ is indifferent to $\tilde{U}_2$, denoted by $\tilde{U}_1 = \tilde{U}_2$;

Example 1. Consider two T2NNS in the group of real numbers: $\tilde{U}_1 = \left\langle\left(T_{T_{\tilde{U}_1}}(z), T_{I_{\tilde{U}_1}}(z), T_{F_{\tilde{U}_1}}(z)\right), \left(I_{T_{\tilde{U}_1}}(z), I_{I_{\tilde{U}_1}}(z), I_{F_{\tilde{U}_1}}(z)\right), \left(F_{T_{\tilde{U}_1}}(z), F_{I_{\tilde{U}_1}}(z), F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$ and $\tilde{U}_2 = \left\langle\left(T_{T_{\tilde{U}_2}}(z), T_{I_{\tilde{U}_2}}(z), T_{F_{\tilde{U}_2}}(z)\right), \left(I_{T_{\tilde{U}_2}}(z), I_{I_{\tilde{U}_2}}(z), I_{F_{\tilde{U}_2}}(z)\right), \left(F_{T_{\tilde{U}_2}}(z), F_{I_{\tilde{U}_2}}(z), F_{F_{\tilde{U}_2}}(z)\right)\right\rangle$ $\tilde{U}_1 = \langle(0.65, 0.70, 0.75), (0.20, 0.15, 0.30), (0.15, 0.20, 0.10)\rangle$, $\tilde{U}_2 = \langle(0.45, 0.40, 0.55), (0.35, 0.45, 0.30), (0.25, 0.35, 0.40)\rangle$. From Eqs. (5) and (6), we get the following outcomes:

1. Score value of $\tilde{S}(\tilde{U}_1) = (8 + (2.8 - 0.8 - .065))/12 = 0.78$, and $\tilde{S}(\tilde{U}_2) = (8 + (1.8 - 1.55 - 1.35))/12 = 0.58$;
2. Accuracy value of $A(\tilde{U}_1) = (2.8 - 0.65)/4 = 0.54$, and $A(\tilde{U}_2) = (1.8 - 1.35)/4 = 0.11$; it is obvious that $A_1 > A_2$.

Example 2. Consider two T2NNS in the set of real numbers: $\tilde{U}_1 = \langle(0.50, 0.20, 0.35), (0.30, 0.45, 0.30), (0.10, 0.25, 0.35)\rangle$, $\tilde{U}_2 = \langle(0.15, 0.60, 0.20), (0.35, 0.20, 0.30), (0.45, 0.35, 0.20)\rangle$. From Eqs. (5) and (6), we obtain the following results:

1. Score value of $\tilde{S}(\tilde{U}_1) = (8 + (1.25 - 1.5 - 0.95))/12 = 0.57$, and $\tilde{S}(\tilde{U}_2) = (8 + (1.55 - 1.05 - 1.35))/12 = 0.60$;
2. Accuracy value of $A(\tilde{U}_1) = (1.25 - 0.95)/4 = 0.075$, and $A(\tilde{U}_2) = (1.55 - 1.35)/12 = 0.05$; it is obvious that $A_2 > A_1$.

2.1. Aggregation of type 2 neutrosophic number

In this part, we recall some basic descriptions of aggregation operators for real numbers.

Definition 5 ([40]). Suppose that $\omega: (Z)^n \to Z$, and $\alpha_p(= 1, 2, \ldots, n) = 1$ are a group of numbers. The weighted averaging operator ωA_ω is defined as:

$$\omega A_\omega(\alpha_1,\ \alpha_2,\ \ldots,\alpha_n) = \sum_{p=1}^{n}\omega_p\alpha_p, \quad (7)$$

where Z is the set of numbers, and $\omega = (\omega_1,\ \omega_2,\ \ldots,\omega_n)^T$ is the weight vector of $\alpha_p(p = 1, 2, \ldots,\ n)$ such that $\omega_p \in [0, 1]$ $(p = 1, 2, \ldots,\ n)$ and $\sum_{p=1}^{n}\omega_p = 1$.

Definition 6 ([40]). Suppose that $\omega: (Z)^n \to Z$ and $\alpha_p(p = 1, 2, \ldots, n)$ are a group of numbers. The weighted averaging operator ωA_ω is defined as:

$$\omega A_\omega(\alpha_1,\ \alpha_2,\ \ldots,\alpha_n) = \prod_{p=1}^{n}\alpha_p^{\omega_p}, \quad (8)$$

Where Z is the set of number, and $\omega = (\omega_1,\ \omega_2,\ \ldots,\omega_n)^T$ is the weight vector of $\alpha_p(p = 1, 2, \ldots,\ n)$ such that $\omega_p \in [0, 1]$ $(p = 1, 2, \ldots,\ n)$ and $\sum_{p=1}^{n}\omega_p = 1$. Based on Definitions 5 and 6, we suggest the next aggregation operator of T2NNS to be used in decision making.

Definition 7. Suppose that $\tilde{U}_p = \left\langle\left(T_{T_{\tilde{U}_p}}(z), T_{I_{\tilde{U}_p}}(z), T_{F_{\tilde{U}_p}}(z)\right), \left(I_{T_{\tilde{U}_p}}(z), I_{I_{\tilde{U}_p}}(z), I_{F_{\tilde{U}_p}}(z)\right), \left(F_{T_{\tilde{U}_p}}(z), F_{I_{\tilde{U}_p}}(z), F_{F_{\tilde{U}_p}}(z)\right)\right\rangle$ $(p = 1, 2, \ldots,\ n)$ is a collection T2NNS in the set of numbers and let us have T2NNWA: $\Theta^n \to \Theta$. A type 2 neutrosophic number weighted averaging (T2NNWA) operator denoted by T2NNWA $(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_n)$ is defined as $\mathrm{T2NNWA}_\omega$

$$(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_n) = \omega_1\tilde{U}_1 \oplus \omega_2\tilde{U}_2 \oplus \ldots \omega_n\tilde{U}_n = \oplus_{p=1}^{n}(\omega_p\tilde{U}_p), \quad (9)$$

Where $\omega_p \in [0, 1]$ is the weight vector of $U_p(p = 1, 2, \ldots,\ n)$ such that $\sum_{p=1}^{n}\omega_p = 1$. If $\omega = (1/n,\ 1/n,\ \ldots,1/n)^T$, then the T2NNWA $(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_n)$ operator decrease to type 2 neutrosophic number averaging (T2NNA) operator: T2NNA

$$(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_n) = \frac{1}{n}(\tilde{U}_1 \oplus \tilde{U}_2 \oplus \cdots \oplus \tilde{U}_n) \quad (10)$$

Now, we can enunciate the following theorem by using the basic procedures of T2NNVs expressed in Definition 2.

Theorem 1. *Let* $\tilde{U}_p = \left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle(p = 1, 2, \ldots,\ n)$ *be a group T2NNS in the set of numbers. Then the combined value obtained by T2NNWA is also a T2NNV, and* $\mathrm{T2NNWA}_\omega(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_n)$ *is given as Eq.* (11) *in* Box II, *where* $\omega_p \in [0, 1]$ *is the weight vector of* $U_p(p = 1, 2, \ldots,\ n)$ *such that* $\sum_{p=1}^{n}\omega_p = 1$.

$$=\left\langle \begin{matrix} \left(1-\left(1-T_{T_1}(z)\right)^{\omega_1},\ 1-\left(1-T_{I_1}(z)\right)^{\omega_1},1-\left(1-T_{F_1}(z)\right)^{\omega_1}\right), \\ \left(\left(I_{T_1}(z)\right)^{\omega_1},\left(I_{I_1}(z)\right)^{\omega_1},\left(I_{F_1}(z)\right)^{\omega_1}\right),\left(\left(F_{T_1}(z)\right)^{\omega_1},\left(F_{I_1}(z)\right)^{\omega_1},\left(F_{F_1}(z)\right)^{\omega_1}\right) \end{matrix} \right\rangle \tag{12}$$

$$=\left\langle \begin{matrix} \left(1-\prod_{p=1}^{1}\left(1-T_{T_p}(z)\right)^{\omega_p},1-\prod_{p=1}^{1}\left(1-T_{I_p}(z)\right)^{\omega_p},1-\prod_{p=1}^{1}\left(1-T_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{1}\left(I_{T_p}(z)\right)^{\omega_p},\prod_{p=1}^{1}\left(I_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{1}\left(I_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{1}\left(F_{T_p}(z)\right)^{\omega_p},\prod_{p=1}^{1}\left(F_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{1}\left(F_{F_p}(z)\right)^{\omega_p}\right) \end{matrix} \right\rangle \tag{13}$$

Box III.

$$\oplus_{p=1}^{2}\left(\omega_p\tilde{U}_p\right)=\omega_1\tilde{U}_1\oplus\omega_2\tilde{U}_2$$

$$=\left\langle \begin{matrix} \left\langle \begin{matrix} \left(1-\left(1-T_{T_1}(z)\right)^{\omega_1},\ 1-\left(1-T_{I_1}(z)\right)^{\omega_1},1-\left(1-T_{F_1}(z)\right)^{\omega_1}\right), \\ \left(\left(I_{T_1}(z)\right)^{\omega_1},\left(I_{I_1}(z)\right)^{\omega_1},\left(I_{F_1}(z)\right)^{\omega_1}\right),\left(\left(F_{T_1}(z)\right)^{\omega_1},\left(F_{I_1}(z)\right)^{\omega_1},\left(F_{F_1}(z)\right)^{\omega_1}\right) \end{matrix} \right\rangle \\ \oplus \\ \left\langle \begin{matrix} \left(1-\left(1-T_{T_2}(z)\right)^{\omega_2},\ 1-\left(1-T_{I_2}(z)\right)^{\omega_2},1-\left(1-T_{F_2}(z)\right)^{\omega_2}\right), \\ \left(\left(I_{T_2}(z)\right)^{\omega_2},\left(I_{I_2}(z)\right)^{\omega_2},\left(I_{F_2}(z)\right)^{\omega_2}\right),\left(\left(F_{T_2}(z)\right)^{\omega_2},\left(F_{I_2}(z)\right)^{\omega_2},\left(F_{F_2}(z)\right)^{\omega_2}\right) \end{matrix} \right\rangle \end{matrix} \right\rangle \tag{14}$$

$$=\left\langle \begin{matrix} \left(\begin{matrix} \left(\begin{matrix} \left(1-\left(1-T_{T_1}(z)\right)^{\omega_1}\right)+\left(1-\left(1-T_{T_2}(z)\right)^{\omega_2}\right) \\ -\left(1-\left(1-T_{T_1}(z)\right)^{\omega_1}\right).\left(1-\left(1-T_{T_2}(z)\right)^{\omega_2}\right) \end{matrix} \right), \\ \left(\begin{matrix} \left(1-\left(1-T_{I_1}(z)\right)^{\omega_1}\right)+\left(1-\left(1-T_{I_2}(z)\right)^{\omega_2}\right) \\ -\left(1-\left(1-T_{I_1}(z)\right)^{\omega_1}\right).\left(1-\left(1-T_{I_2}(z)\right)^{\omega_2}\right) \end{matrix} \right), \\ \left(\begin{matrix} \left(1-\left(1-T_{F_1}(z)\right)^{\omega_1}\right)+\left(1-\left(1-T_{F_2}(z)\right)^{\omega_2}\right) \\ -\left(1-\left(1-T_{F_1}(z)\right)^{\omega_1}\right).\left(1-\left(1-T_{F_2}(z)\right)^{\omega_2}\right) \end{matrix} \right) \end{matrix} \right), \\ \left(\left(I_{T_1}(z)\right)^{\omega_1}\left(I_{T_2}(z)\right)^{\omega_2},\left(I_{I_1}(z)\right)^{\omega_1}\left(I_{I_2}(z)\right)^{\omega_2},\left(I_{F_1}(z)\right)^{\omega_1}\left(I_{F_2}(z)\right)^{\omega_2}\right), \\ \left(\left(F_{T_1}(z)\right)^{\omega_1}\left(F_{T_2}(z)\right)^{\omega_2},\left(F_{I_1}(z)\right)^{\omega_1}\left(F_{I_2}(z)\right)^{\omega_2},\left(F_{F_1}(z)\right)^{\omega_1}\left(F_{F_2}(z)\right)^{\omega_2}\right) \end{matrix} \right\rangle$$

$$=\left\langle \begin{matrix} \left(1-\prod_{p=1}^{2}\left(1-T_{T_p}(z)\right)^{\omega_p},1-\prod_{p=1}^{2}\left(1-T_{I_p}(z)\right)^{\omega_p},1-\prod_{p=1}^{2}\left(1-T_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{2}\left(I_{T_p}(z)\right)^{\omega_p},\prod_{p=1}^{2}\left(I_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{2}\left(I_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{2}\left(F_{T_p}(z)\right)^{\omega_p},\prod_{p=1}^{2}\left(F_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{2}\left(F_{F_p}(z)\right)^{\omega_p}\right) \end{matrix} \right\rangle \tag{15}$$

Box IV.

Proof. We verify the theorem by mathematical induction.

1. When $n = 1$, it is a normal case. We mention it here for clarification only (see Eqs. (12) and (13) in Box III).

Consequently, the theorem is true for $n = 1$.

2. When $n = 2$, we have Eqs. (14) and (15) in Box IV. Consequently, the theorem is true for $n = 2$.

3. When $n = k$, we suppose that Eq. (11) is also true.

Then, T2NNWA $(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_k)$ is given as Eq. (16) in Box V.

4. When $n = k+1$, we have $T2NNWA\,(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_{k+1})$ given as Eq. (17) in Box VI.

We notice that the theorem is true for $n = k + 1$. So, by mathematical induction, we can say that Eq. (11) holds for all values of n. As the components of all three membership functions of $\tilde{U}_p$ belong to [0, 1], the following relations are valid:

$$0\le\left(1-\prod_{p=1}^{k}\left(1-T_{F_p}(z)\right)^{\omega_p}\right)\le 1,\ 0\le\left(\prod_{p=1}^{k}\left(I_{F_p}(z)\right)^{\omega_p}\right)\le 1,$$

$$0\le\left(\prod_{p=1}^{k}\left(F_{F_p}(z)\right)^{\omega_p}\right)\le 1. \tag{18}$$

It follows that this relation completes the proof of Theorem 1.

$$0\le\left\langle\left(1-\prod_{p=1}^{k}\left(1-T_{F_p}(z)\right)^{\omega_p}\right)+\left(\prod_{p=1}^{k}\left(I_{F_p}(z)\right)^{\omega_p}\right)+\left(\prod_{p=1}^{k}\left(F_{F_p}(z)\right)^{\omega_p}\right)\right\rangle\le 3.$$

$$\text{T2NNWA}(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_k) = \omega_1\tilde{U}_1 \oplus \omega_2\tilde{U} \oplus \ldots \omega_n\tilde{U}_n = \oplus_{p=1}^{k}(\omega_p\tilde{U}_p)$$

$$= \left\langle \begin{matrix} \left(1-\prod_{p=1}^{k}\left(1-T_{T_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{k}\left(1-T_{I_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{k}\left(1-T_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{k}\left(I_{T_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{k}\left(I_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{k}\left(I_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{k}\left(F_{T_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{k}\left(F_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{k}\left(F_{F_p}(z)\right)^{\omega_p}\right) \end{matrix} \right\rangle \tag{16}$$

Box V.

$$\text{T2NNWA}(\tilde{U}_1,\ \tilde{U}_2,\ \ldots,\tilde{U}_{k+1}) = \oplus_{p=1}^{n}(\omega_p\tilde{U}_p) \oplus (\omega_{k+1}\tilde{U}_{k+1})$$

$$= \left\langle \begin{matrix} \left(\begin{matrix} \left(\begin{matrix} 1-\prod_{p=1}^{k}\left(1-T_{T_p}(z)\right)^{\omega_p} + 1-\prod_{p=1}^{k}\left(1-T_{T_{k+1}}(z)\right)^{\omega_{k+1}} \\ -1-\prod_{p=1}^{k}\left(1-T_{T_p}(z)\right)^{\omega_p}\ 1-\prod_{p=1}^{k}\left(1-T_{T_{k+1}}(z)\right)^{\omega_{k+1}} \end{matrix}\right), \\ \left(\begin{matrix} 1-\prod_{p=1}^{k}\left(1-T_{I_p}(z)\right)^{\omega_p} + 1-\prod_{p=1}^{k}\left(1-T_{I_{k+1}}(z)\right)^{\omega_{k+1}} \\ -1-\prod_{p=1}^{k}\left(1-T_{I_p}(z)\right)^{\omega_p}\ 1-\prod_{p=1}^{k}\left(1-T_{I_{k+1}}(z)\right)^{\omega_{k+1}} \end{matrix}\right), \\ \left(\begin{matrix} 1-\prod_{p=1}^{k}\left(1-T_{F_p}(z)\right)^{\omega_p} + 1-\prod_{p=1}^{k}\left(1-T_{F_{k+1}}(z)\right)^{\omega_{k+1}} \\ -1-\prod_{p=1}^{k}\left(1-T_{F_p}(z)\right)^{\omega_p}\ 1-\prod_{p=1}^{k}\left(1-T_{F_{k+1}}(z)\right)^{\omega_{k+1}} \end{matrix}\right) \end{matrix} \right), \\ \left(\prod_{p=1}^{k}\left(I_{T_p}(z)\right)^{\omega_p} \cdot \left(I_{T_p}(z)\right)_{k+1}^{\omega_{p+1}},\ \prod_{p=1}^{k}\left(I_{I_p}(z)\right)^{\omega_p} \cdot \left(I_{I_p}(z)\right)_{k+1}^{\omega_{p+1}},\ \prod_{p=1}^{k}\left(I_{F_p}(z)\right)^{\omega_p} \cdot \left(I_{F_p}(z)\right)_{k+1}^{\omega_{p+1}}\right), \\ \left(\prod_{p=1}^{k}\left(F_{T_p}(z)\right)^{\omega_p} \cdot \left(F_{T_p}(z)\right)_{k+1}^{\omega_{p+1}},\ \prod_{p=1}^{k}\left(F_{I_p}(z)\right)^{\omega_p} \cdot \left(F_{I_p}(z)\right)_{k+1}^{\omega_{p+1}},\ \prod_{p=1}^{k}\left(F_{F_p}(z)\right)^{\omega_p} \cdot \left(F_{F_p}(z)\right)_{k+1}^{\omega_{p+1}}\right) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \left(1-\prod_{p=1}^{k+1}\left(1-T_{T_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{k+1}\left(1-T_{I_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{k+1}\left(1-T_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{k+1}\left(I_{T_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{k+1}\left(I_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{k+1}\left(I_{F_p}(z)\right)^{\omega_p}\right), \\ \left(\prod_{p=1}^{k+1}\left(F_{T_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{k+1}\left(F_{I_p}(z)\right)^{\omega_p},\prod_{p=1}^{k+1}\left(F_{F_p}(z)\right)^{\omega_p}\right) \end{matrix} \right\rangle \tag{17}$$

Box VI.

2.2. Now, we will refer to one property to confirm the T2NNWA operator

Property 1 (*Boundedness*)**.** *if all* $\tilde{U}_p(p = 1, 2, \ldots, \eta)$ *are equal* $\tilde{U}_p = \tilde{U} = \left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle$*, for all p, then T2NNWA* $\tilde{U}_p(p = 1, 2, \ldots, n) = \tilde{U}$.

Suppose $\tilde{U}^{+}$

$$= \left\langle \begin{matrix} \left(\max_p\left(T_{T_p}(z)\right),\ \max_p\left(T_{I_p}(z)\right),\ \max_p\left(T_{F_p}(z)\right)\right), \\ \left(\min_p\left(I_{T_p}(z)\right),\ \min_p\left(I_{I_p}(z)\right),\ \min_p\left(I_{F_p}(z)\right)\right), \\ \left(\min_p\left(F_{T_p}(z)\right),\ \min_p\left(F_{I_p}(z)\right),\ \min_p\left(F_{F_p}(z)\right)\right) \end{matrix} \right\rangle$$

$$S(\tilde{U}) \le \frac{1}{12}\left\langle \begin{array}{c} 8+\left(\max\limits_p\left(T_{T_p}(z)\right)+2.\max\limits_p\left(T_{I_p}(z)\right)+\max\limits_p\left(T_{F_p}(z)\right)\right)\\ -\left(\min\limits_p\left(I_{T_p}(z)\right)+2.\min\limits_p\left(I_{I_p}(z)\right)+\min\limits_p\left(I_{F_p}(z)\right)\right)\\ -\left(\min\limits_p\left(F_{T_p}(z)\right)+2.\min\limits_p\left(F_{I_p}(z)\right)+\min\limits_p\left(F_{F_p}(z)\right)\right)\end{array}\right\rangle = S(\tilde{U}^+) \qquad (22)$$

Box VII.

Suppose $\tilde{U}^-$

$$= \left\langle \begin{array}{c} \left(\min\limits_p\left(T_{T_p}(z)\right),\ \min\limits_p\left(T_{I_p}(z)\right),\ \min\limits_p\left(T_{F_p}(z)\right)\right),\\ \left(\max\limits_p\left(I_{T_p}(z)\right),\ \max\limits_p\left(I_{I_p}(z)\right),\ \max\limits_p\left(I_{F_p}(z)\right)\right),\\ \left(\max\limits_p\left(F_{T_p}(z)\right),\ \max\limits_p\left(F_{I_p}(z)\right),\ \max\limits_p\left(F_{F_p}(z)\right)\right)\end{array}\right\rangle$$

For all $p = 1, 2, \ldots, n$. *Then,* $\tilde{U}^- \le T2NNWA\tilde{U}_p(p = 1, 2, \ldots, n) \le \tilde{U}^+$. *Now, we demonstrate that:*

$$\begin{aligned}&\min_p\left(T_{F_p}(z)\right) \le \left(T_{F_p}(z)\right) \le \max_p\left(T_{F_p}(z)\right),\\ &\min_p\left(I_{F_p}(z)\right) \le \left(I_{F_p}(z)\right) \le \max_p\left(I_{F_p}(z)\right),\\ &\min_p\left(F_{F_p}(z)\right) \le \left(F_{F_p}(z)\right) \le \max_p\left(F_{F_p}(z)\right),\ \textit{for all } p = 1, 2, \ldots, n.\end{aligned} \qquad (19)$$

Then, $1-\prod_{p=1}^n\left(1-\min_p\left(T_{F_p}(z)\right)\right)^{\omega_p} \le 1-\prod_{p=1}^n\left(1-\left(T_{F_p}(z)\right)\right)^{\omega_p} \le 1-\prod_{p=1}^n\left(1-\max_p\left(T_{F_p}(z)\right)\right)^{\omega_p} = 1-\left(1-\min_p\left(T_{F_p}(z)\right)\right)^{\sum_{p=1}^n\omega_p} \le 1-\prod_{p=1}^n\left(1-\left(T_{F_p}(z)\right)\right)^{\omega_p} \le 1-\left(1-\max_p\left(T_{F_p}(z)\right)\right)^{\sum_{p=1}^n\omega_p} = \min_p\left(T_{F_p}(z)\right) \le 1-\prod_{p=1}^n\left(1-\left(T_{F_p}(z)\right)\right)^{\omega_p} \le \max_p\left(T_{F_p}(z)\right)$.

Then, from Eq. (19), we have for $p = 1, 2, \ldots, n$.

$$\begin{aligned}&\prod_{p=1}^n \min_p\left(I_{F_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n\left(I_{F_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n \max_p\left(I_{F_p}(z)\right)^{\omega_p}\\ &= \prod_{p=1}^n \min_p\left(I_{F_p}(z)\right)^{\sum_{p=1}^n\omega_p}\\ &\le \prod_{p=1}^n\left(I_{F_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n \max_p\left(I_{F_p}(z)\right)^{\sum_{p=1}^n\omega_p}\\ &= \min_p\left(I_{F_p}(z)\right) \le \prod_{p=1}^n\left(I_{F_p}(z)\right)^{\omega_p} \le \max_p\left(I_{F_p}(z)\right)\ \textit{and}\\ &\prod_{p=1}^n \min_p\left(F_{F_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n\left(F_{F_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n \max_p\left(F_{F_p}(z)\right)^{\omega_p}\\ &= \prod_{p=1}^n \min_p\left(F_{F_p}(z)\right)^{\sum_{p=1}^n\omega_p} \le \prod_{p=1}^n\left(F_{F_p}(z)\right)^{\omega_p}\\ &\le \prod_{p=1}^n \max_p\left(F_{F_p}(z)\right)^{\sum_{p=1}^n\omega_p}\\ &= \min_p\left(F_{F_p}(z)\right) \le \prod_{p=1}^n\left(F_{F_p}(z)\right)^{\omega_p} \le \max_p\left(F_{F_p}(z)\right).\end{aligned} \qquad (20)$$

Then, for $\min_p\left(T_{T_p}(z)\right) \le \left(T_{T_p}(z)\right) \le \max_p\left(T_{T_p}(z)\right)$, *we prove the following:*

$$\begin{aligned}&\prod_{p=1}^n \min_p\left(T_{T_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n\left(T_{T_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n \max_p\left(T_{T_p}(z)\right)^{\omega_p}\\ &= \prod_{p=1}^n \min_p\left(T_{T_p}(z)\right)^{\sum_{p=1}^n\omega_p} \le \prod_{p=1}^n\left(T_{T_p}(z)\right)^{\omega_p}\\ &\le \prod_{p=1}^n \max_p\left(T_{T_p}(z)\right)^{\sum_{p=1}^n\omega_p}\\ &= \min_p\left(T_{T_p}(z)\right) \le \prod_{p=1}^n\left(T_{T_p}(z)\right)^{\omega_p} \le \max_p\left(T_{T_p}(z)\right)\ \textit{and}\\ &\prod_{p=1}^n \min_p\left(T_{T_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n\left(T_{T_p}(z)\right)^{\omega_p} \le \prod_{p=1}^n \max_p\left(T_{T_p}(z)\right)^{\omega_p}\\ &= \prod_{p=1}^n \min_p\left(T_{T_p}(z)\right)^{\sum_{p=1}^n\omega_p} \le \prod_{p=1}^n\left(T_{T_p}(z)\right)^{\omega_p}\\ &\le \prod_{p=1}^n \max_p\left(T_{T_p}(z)\right)^{\sum_{p=1}^n\omega_p}\\ &= \min_p\left(T_{T_p}(z)\right) \le \prod_{p=1}^n\left(T_{T_p}(z)\right)^{\omega_p} \le \max_p\left(T_{T_p}(z)\right).\end{aligned} \qquad (21)$$

Likewise, from previous Eqs. (19)–(21).

Then, for $\left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle$ $(p = 1, 2, \ldots, n)$. *Similarly, we have:* $\min_p\left(T_{I_p}(z)\right) \le \left(T_{I_p}(z)\right) \le \max_p\left(T_{I_p}(z)\right)$, $\min_p\left(F_{T_p}(z)\right) \le \left(F_{T_p}(z)\right) \le \max_p\left(F_{T_p}(z)\right)$, $\min_p\left(F_{I_p}(z)\right) \le \left(F_{I_p}(z)\right) \le \max_p\left(F_{I_p}(z)\right)$, $\min_p\left(I_{T_p}(z)\right) \le \left(I_{T_p}(z)\right) \le \max_p\left(I_{T_p}(z)\right)$, $\min_p\left(I_{I_p}(z)\right) \le \left(I_{I_p}(z)\right) \le \max_p\left(I_{I_p}(z)\right)$, *for* $p = 1, 2, \ldots, n$.

Then, suppose that T2NNW $A_\omega\ \tilde{U}_p(p = 1, 2, \ldots, n) = \tilde{U} = \left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle$, *and the score function of* $\tilde{U} = S(\tilde{U}) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\tilde{U}_1}}(z) + 2\left(T_{I_{\tilde{U}_1}}(z)\right) + T_{F_{\tilde{U}_1}}(z)\right) - \left(I_{T_{\tilde{U}_1}}(z) + 2\left(I_{I_{\tilde{U}_1}}(z)\right) + I_{F_{\tilde{U}_1}}(z)\right) - \left(F_{T_{\tilde{U}_1}}(z) + 2\left(F_{I_{\tilde{U}_1}}(z)\right) + F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$ *from this, we have Eq. (22) given in Box VII.*

Also, $S(\tilde{U}) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\tilde{U}_1}}(z) + 2\left(T_{I_{\tilde{U}_1}}(z)\right) + T_{F_{\tilde{U}_1}}(z)\right) - \left(I_{T_{\tilde{U}_1}}(z) + 2\left(I_{I_{\tilde{U}_1}}(z)\right) + I_{F_{\tilde{U}_1}}(z)\right) - \left(F_{T_{\tilde{U}_1}}(z) + 2\left(F_{I_{\tilde{U}_1}}(z)\right) + F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$. *From this, we have Eq. (23) given in Box VIII. Also,* $S(\tilde{U}) = \frac{1}{12}\left\langle 8 + \left(T_{T_{\tilde{U}_1}}(z) + 2\left(T_{I_{\tilde{U}_1}}(z)\right) + T_{F_{\tilde{U}_1}}(z)\right) - \left(I_{T_{\tilde{U}_1}}(z) + 2\left(I_{I_{\tilde{U}_1}}(z)\right) + I_{F_{\tilde{U}_1}}(z)\right) - \left(F_{T_{\tilde{U}_1}}(z) + 2\left(F_{I_{\tilde{U}_1}}(z)\right) + F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$. *From this, we have Eq. (24) given in Box IX.*

$$S(\tilde{U}) \geq \frac{1}{12}\left\langle \begin{array}{c} 8+\left(\max_p\left(T_{T_p}(z)\right)+2.\max_p\left(T_{I_p}(z)\right)+\max_p\left(T_{F_p}(z)\right)\right) \\ -\left(\min_p\left(I_{T_p}(z)\right)+2.\min_p\left(I_{I_p}(z)\right)+\min_p\left(I_{F_p}(z)\right)\right) \\ -\left(\min_p\left(F_{T_p}(z)\right)+2.\min_p\left(F_{I_p}(z)\right)+\min_p\left(F_{F_p}(z)\right)\right) \end{array}\right\rangle = S(\tilde{U}^-) \quad (23)$$

Box VIII.

$$S(\tilde{U}) = \frac{1}{12}\left\langle \begin{array}{c} 8+\left(\max_p\left(T_{T_p}(z)\right)+2\max_p\left(T_{I_p}(z)\right)+\max_p\left(T_{F_p}(z)\right)\right) \\ -\left(\min_p\left(I_{T_p}(z)\right)+2\min_p\left(I_{I_p}(z)\right)+\min_p\left(I_{F_p}(z)\right)\right) \\ -\left(\min_p\left(F_{T_p}(z)\right)+2\min_p\left(F_{I_p}(z)\right)+\min_p\left(F_{F_p}(z)\right)\right) \end{array}\right\rangle = S(\tilde{U}^+) \quad (24)$$

Box IX.

Therefore, we found the following cases: $S(\tilde{U}) < S(\tilde{U}^+)$, $S(\tilde{U}) < S(\tilde{U}^-)$ *and* $S(\tilde{U}) = S(\tilde{U}^+)$, *hence*

$$\tilde{U}^- < T2NNWA\ \tilde{U}_p(p = 1, 2, \ldots, n) < \tilde{U}^+ \quad (25)$$

By using the previous equations and by proving the score value, we can prove in the same way the accuracy value using this equation: $A(\tilde{U}_1) = \frac{1}{4}\left\langle\left(T_{T_{\tilde{U}_1}}(z) + 2\left(T_{I_{\tilde{U}_1}}(z)\right) + T_{F_{\tilde{U}_1}}(z)\right) - \left(F_{T_{\tilde{U}_1}}(z) + 2\left(F_{I_{\tilde{U}_1}}(z)\right) + F_{F_{\tilde{U}_1}}(z)\right)\right\rangle$.

Property 2 (*Idempotency*). *if all* $\tilde{U}_p(p = 1, 2, \ldots, n)$ *are equal* $\tilde{U}_p = \tilde{U} = \left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle$, *for all p, then T2NNWA* $\tilde{U}_p(p = 1, 2, \ldots, n) = \tilde{U}$. *From Eq. (11), we have T2NNWA* $\tilde{U}_p(p = 1, 2, \ldots, n)$ *given in Box X).*

Consequently,

$$\left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle = \tilde{U}.$$

This proves Property 2.

Example 3. Consider the following four T2NN values. Using the T2NNWA operator defined in Eq. (11), we can aggregate ($\tilde{U}_1$, $\tilde{U}_2$, $\tilde{U}_3$, and $\tilde{U}_4$) with weight vector $\omega = (0.25, 0.20, 0.35, 0.20)$ as $\tilde{U} = T2NNWA(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \text{and} \tilde{U}_4) = \omega_1\tilde{U}_1 \oplus \omega_2\tilde{U}_2 \oplus \omega_3\tilde{U}_3 \oplus \omega_4\tilde{U}_4$. $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$ and $\tilde{U}_4$ are given in Box XI. After aggregation, we find that $\tilde{U}_{all}$

$$= \left\langle\left(T_{T_p}(z), T_{I_p}(z), T_{F_p}(z)\right), \left(I_{T_p}(z), I_{I_p}(z), I_{F_p}(z)\right), \left(F_{T_p}(z), F_{I_p}(z), F_{F_p}(z)\right)\right\rangle$$
$$= \langle(0.881, 0.710, 0.768), (0.2851, 0.0872, 0.2093), (0.0941, 0.2163, 0.2268)\rangle$$

2.3. Application of T2NNWA operator to MADM

Consider a MADM issue in which we have the collection of $\phi_i = \{\phi_1, \phi_2, \ldots, \phi_n\}$ suitable alternatives, where i = 1, 2, …, m, assessed on n criteria $Œi_p = \{Œi_1, Œi_2, .., Œi_n\}, p = 1, 2, \ldots, n$. Assume that $\omega_p = \{\omega_1, \omega_2, \ldots, \omega_p\}$ is the weight vector of attributes, where $\omega_p > 0$ and sum $\sum_{p=1}^{n} \omega_p = 1$ for $p = 1, 2, \ldots, n$. The standing of all alternatives $\phi_i = \{\phi_1, \phi_2, \ldots, \phi_n\}$ with regard to the attributes $Œi_p = \{Œi_1, Œi_2, \ldots, Œi_n\}$, $p = 1, 2, \ldots, n$ have been supposed in T2NN values based relation matrix $R = (k_{ip})_{m \times n}$, as in Table 2. Furthermore, in the relation matrix $R = (k_{ip})_{m \times n}$, the standing $\tilde{A}_{ip} = \left\langle\left(T_{T_{ip}}(z), T_{I_{ip}}(z), T_{F_{ip}}(z)\right), \left(I_{T_{ip}}(z), I_{I_{ip}}(z), I_{F_{ip}}(z)\right), \left(F_{T_{ip}}(z), F_{I_{ip}}(z), F_{F_{ip}}(z)\right)\right\rangle$ represents a T2NN value, where the type 2 neutrosophic number $\left(T_{T_{ip}}(z), T_{I_{ip}}(z), T_{F_{ip}}(z)\right)$ signifies the degree an alternative satisfies the attribute $Œi_p = Œi_1, Œi_2, \ldots, Œi_n$, $p = 1, 2, \ldots, \eta$, with three degrees of truth (truth, indeterminacy, and falsity). Also, $\left(I_{T_{ip}}(z), I_{I_{ip}}(z), I_{F_{ip}}(z)\right)$ signifies the degree an alternative is undefined about the attribute $Œi_p = Œi_1, Œi_2, \ldots, Œi_n$, $p = 1, 2, \ldots, n$, where the uncertain degree contains three degrees of indeterminacy (truth, indeterminacy, and falsity). Also, $\left(F_{T_{ip}}(z), F_{I_{ip}}(z), F_{F_{ip}}(z)\right)$ introduces the degree an alternative does not satisfy the attribute $Œi_p = \{Œi_1, Œi_2, .., Œi_n\}$, $p = 1, 2, \ldots$, n, where the unsatisfied degree contains three degrees of dissatisfaction (truth, indeterminacy, and falsity). We improve a functional approach for solving MADM problems based on the T2NNWA, in which we rank the alternatives over the attributes. The graphical schema of the developed technique for MADM is shown in Fig. 1.

2.4. Numerical case

In this section, a mathematical example of data and methods is presented to check the competence and efficiency of submitted framework for selection the best alternative. Currently in Egypt, people seek for choosing the best bank to operate banking transactions such as deposit their money, withdraw financial loans, transfer of money, change currencies, etc. This section presents a numerical case to select the best bank for citizens and investors. There are four evaluation alternatives ϕ_1, ϕ_2, ϕ_3 and ϕ_4, five criteria are considered as selection factors $Œi_1$ (Reputation and elegance), $Œi_2$ (Customer service), $Œi_3$ (Place of the bank and its branches), $Œi_4$ (Fees), $Œi_5$ (Offers). The classification of alternatives ϕ_i (i = 1, 2, …, 4) with regard to $Œi_i$(i = 1, 2, …, 5) are expressed with T2NN values, as presented in Table 3. We suppose that $\omega = (0.20, 0.25, 0.30, 0.15, 0.10)^T$ is the proportional weight for criteria $Œi_i$ (i = 1, 2, …, 5).

$$
\begin{aligned}
&\mathrm{T2NNWA}\tilde{\mathrm{U}}_p(p=1,2,\ldots,n)=\mathrm{T2NNWA}(\tilde{\mathrm{U}}_1,\ \tilde{\mathrm{U}}_2,\ \ldots,\tilde{\mathrm{U}}_{\mathrm{k+1}})=\oplus_{p=1}^{n}(\omega_p\tilde{\mathrm{U}}_p)\\
&=\left\langle
\begin{array}{c}
\left(1-\prod_{p=1}^{n}\left(1-\mathrm{T}_{\mathrm{T}_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{n}\left(1-\mathrm{T}_{\mathrm{I}_p}(z)\right)^{\omega_p},\ 1-\prod_{p=1}^{n}\left(1-\mathrm{T}_{\mathrm{F}_p}(z)\right)^{\omega_p}\right),\\
\left(\prod_{p=1}^{n}\left(\mathrm{I}_{\mathrm{T}_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{n}\left(\mathrm{I}_{\mathrm{I}_p}(z)\right)^{\omega_p},\prod_{p=1}^{n}\left(\mathrm{I}_{\mathrm{F}_p}(z)\right)^{\omega_p}\right),\\
\left(\prod_{p=1}^{n}\left(\mathrm{F}_{\mathrm{T}_p}(z)\right)^{\omega_p},\ \prod_{p=1}^{n}\left(\mathrm{F}_{\mathrm{I}_p}(z)\right)^{\omega_p},\prod_{p=1}^{n}\left(\mathrm{F}_{\mathrm{F}_p}(z)\right)^{\omega_p}\right)
\end{array}
\right\rangle\\
&=\left\langle
\begin{array}{c}
\left(1-\left(1-\mathrm{T}_{\mathrm{T}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p},1-\left(1-\mathrm{T}_{\mathrm{I}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p},1-\left(1-\mathrm{T}_{\mathrm{F}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p}\right),\\
\left(\left(\mathrm{I}_{\mathrm{T}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p},\ \left(\mathrm{I}_{\mathrm{I}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p},\left(\mathrm{I}_{\mathrm{F}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p}\right),\\
\left(\left(\mathrm{F}_{\mathrm{T}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p},\ \left(\mathrm{F}_{\mathrm{I}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p},\left(\mathrm{F}_{\mathrm{F}_p}(z)\right)^{\sum_{p=1}^{n}\omega_p}\right)
\end{array}
\right\rangle
\end{aligned}
$$

Box X.

$$
\begin{aligned}
&\tilde{\mathrm{U}}_1=\langle(0.75,0.65,0.95),(0.30,0.15,0.20),(0.15,0.25,0.20)\rangle,\\
&\tilde{\mathrm{U}}_2=\langle(0.85,0.75,0.65),(0.20,0.10,0.25),(0.10,0.30,0.25)\rangle,\\
&\tilde{\mathrm{U}}_3=\langle(0.90,0.70,0.65),(0.30,0.05,0.20),(0.05,0.25,0.20)\rangle,\\
&\tilde{\mathrm{U}}_4=\langle(0.95,0.70,0.60),(0.35,0.10,0.20),(0.15,0.10,0.30)\rangle\\
&=\left\langle
\begin{array}{c}
\left(\begin{array}{c}
\left(1-(1-0.75)^{0.25}\ \ (1-0.85)^{0.20}\,(1-0.90)^{0.35}\ \ (1-0.95)^{0.20}\right),\\
\left(1-(1-0.65)^{0.25}\ \ (1-0.75)^{0.20}\,(1-0.70)^{0.35}\ \ (1-0.75)^{0.20}\right),\\
\left(1-(1-0.95)^{0.25}\ \ (1-0.65)^{0.20}\,(1-0.60)^{0.35}\ \ (1-0.60)^{0.20}\right)
\end{array}\right),\\
\left(\begin{array}{c}
\left((0.30)^{0.25}\,(0.20)^{0.20}\,x\,(0.30)^{0.35}\,(0.35)^{0.20}\right),\\
\left((0.15)^{0.25}\,(0.10)^{0.20}\,(0.05)^{0.35}\,(0.10)^{0.20}\right),\\
\left((0.20)^{0.25}\,(0.25)^{0.20}\,(0.20)^{0.35}\,(0.20)^{0.20}\right)
\end{array}\right),\\
\left(\begin{array}{c}
\left((0.15)^{0.25}\,(0.10)^{0.20}\,(0.05)^{0.35}\,(0.15)^{0.20}\right),\\
\left((0.25)^{0.25}\,(0.30)^{0.20}\,(0.25)^{0.35}\,(0.10)^{0.20}\right),\\
\left((0.20)^{0.25}\,(0.25)^{0.20}\,(0.20)^{0.35}\,(0.30)^{0.20}\right)
\end{array}\right)
\end{array}
\right\rangle\\
&=\left\langle
\begin{array}{c}
\left(\begin{array}{c}
(1-0.707\times0.684\times0.447\times0.549),(1-0.769\times0.758\times0.656\times0.758),\\
(1-0.473\times0.811\times0.726\times0.833)
\end{array}\right)\\
\left(\begin{array}{c}
(0.740\times0.725\times0.656\times0.811),(0.622\times0.631\times0.350\times0.631),\\
(0.669\times0.758\times0.569\times0.725)
\end{array}\right)\\
\left(\begin{array}{c}
(0.622\times0.631\times0.350\times0.684),(0.707\times0.786\times0.616\times0.631),\\
(0.669\times0.758\times0.569\times0.786)
\end{array}\right)
\end{array}
\right\rangle
\end{aligned}
$$

Box XI.

Table 2
Type 2 neutrosophic number value based relation matrix.

	Œi$_1$	Œi$_2$	...	Œi$_n$
ϕ_1	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{11}}(z),\mathrm{T}_{\mathrm{I}_{11}}(z),\mathrm{T}_{\mathrm{F}_{11}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{11}}(z),\mathrm{I}_{\mathrm{I}_{11}}(z),\mathrm{I}_{\mathrm{F}_{11}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{11}}(z),\mathrm{F}_{\mathrm{I}_{11}}(z),\mathrm{F}_{\mathrm{F}_{11}}(z)\right)\end{array}\right\rangle$	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{12}}(z),\mathrm{T}_{\mathrm{I}_{12}}(z),\mathrm{T}_{\mathrm{F}_{12}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{12}}(z),\mathrm{I}_{\mathrm{I}_{12}}(z),\mathrm{I}_{\mathrm{F}_{12}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{12}}(z),\mathrm{F}_{\mathrm{I}_{12}}(z),\mathrm{F}_{\mathrm{F}_{12}}(z)\right)\end{array}\right\rangle$	...	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{1n}}(z),\mathrm{T}_{\mathrm{I}_{1n}}(z),\mathrm{T}_{\mathrm{F}_{1n}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{1n}}(z),\mathrm{I}_{\mathrm{I}_{1n}}(z),\mathrm{I}_{\mathrm{F}_{1n}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{1n}}(z),\mathrm{F}_{\mathrm{I}_{1n}}(z),\mathrm{F}_{\mathrm{F}_{1n}}(z)\right)\end{array}\right\rangle$
ϕ_2	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{21}}(z),\mathrm{T}_{\mathrm{I}_{21}}(z),\mathrm{T}_{\mathrm{F}_{21}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{21}}(z),\mathrm{I}_{\mathrm{I}_{21}}(z),\mathrm{I}_{\mathrm{F}_{21}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{21}}(z),\mathrm{F}_{\mathrm{I}_{21}}(z),\mathrm{F}_{\mathrm{F}_{21}}(z)\right)\end{array}\right\rangle$	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{22}}(z),\mathrm{T}_{\mathrm{I}_{22}}(z),\mathrm{T}_{\mathrm{F}_{22}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{22}}(z),\mathrm{I}_{\mathrm{I}_{22}}(z),\mathrm{I}_{\mathrm{F}_{22}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{22}}(z),\mathrm{F}_{\mathrm{I}_{22}}(z),\mathrm{F}_{\mathrm{F}_{22}}(z)\right)\end{array}\right\rangle$	...	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{2n}}(z),\mathrm{T}_{\mathrm{I}_{2n}}(z),\mathrm{T}_{\mathrm{F}_{2n}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{2n}}(z),\mathrm{I}_{\mathrm{I}_{2n}}(z),\mathrm{I}_{\mathrm{F}_{2n}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{2n}}(z),\mathrm{F}_{\mathrm{I}_{2n}}(z),\mathrm{F}_{\mathrm{F}_{2n}}(z)\right)\end{array}\right\rangle$
...	...	...	...	...
ϕ_{m}	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{\mathrm{m}1}}(z),\mathrm{T}_{\mathrm{I}_{\mathrm{m}1}}(z),\mathrm{T}_{\mathrm{F}_{\mathrm{m}1}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{\mathrm{m}1}}(z),\mathrm{I}_{\mathrm{I}_{\mathrm{m}1}}(z),\mathrm{I}_{\mathrm{F}_{\mathrm{m}1}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{\mathrm{m}1}}(z),\mathrm{F}_{\mathrm{I}_{\mathrm{m}1}}(z),\mathrm{F}_{\mathrm{F}_{\mathrm{m}1}}(z)\right)\end{array}\right\rangle$	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{\mathrm{m}2}}(z),\mathrm{T}_{\mathrm{I}_{\mathrm{m}2}}(z),\mathrm{T}_{\mathrm{F}_{\mathrm{m}2}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{\mathrm{m}2}}(z),\mathrm{I}_{\mathrm{I}_{\mathrm{m}2}}(z),\mathrm{I}_{\mathrm{F}_{\mathrm{m}2}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{\mathrm{m}2}}(z),\mathrm{F}_{\mathrm{I}_{\mathrm{m}2}}(z),\mathrm{F}_{\mathrm{F}_{\mathrm{m}2}}(z)\right)\end{array}\right\rangle$	...	$\left\langle\begin{array}{c}\left(\mathrm{T}_{\mathrm{T}_{\mathrm{mn}}}(z),\mathrm{T}_{\mathrm{I}_{\mathrm{mn}}}(z),\mathrm{T}_{\mathrm{F}_{\mathrm{mn}}}(z)\right),\\\left(\mathrm{I}_{\mathrm{T}_{\mathrm{mn}}}(z),\mathrm{I}_{\mathrm{I}_{\mathrm{mn}}}(z),\mathrm{I}_{\mathrm{F}_{\mathrm{mn}}}(z)\right),\\\left(\mathrm{F}_{\mathrm{T}_{\mathrm{mn}}}(z),\mathrm{F}_{\mathrm{I}_{\mathrm{mn}}}(z),\mathrm{F}_{\mathrm{F}_{\mathrm{mn}}}(z)\right)\end{array}\right\rangle$

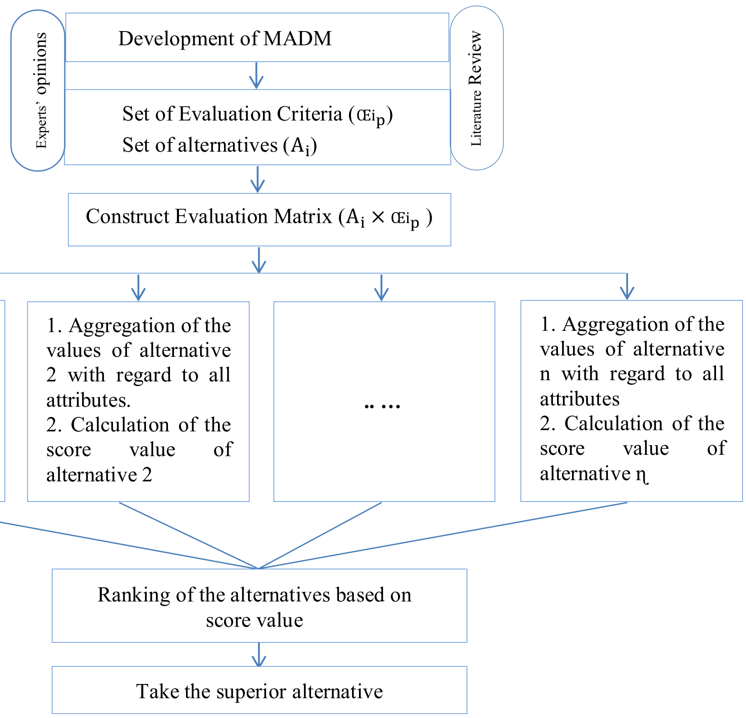

Fig. 1. The general framework of the submitted method.

Table 3
Decision matrix between alternatives and criteria using T2NN values.

	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$	Œi$_5$
ϕ_1	⟨(0.75, 0.80, 0.85), (0.20, 0.15, 0.30), (0.10, 0.15, 0.20)⟩	⟨(0.65, 0.70, 0.75), (0.40, 0.45, 0.50), (0.35, 0.40, 0.35)⟩	⟨(0.85, 0.90, 0.95), (0.30, 0.35, 0.40), (0.25, 0.40, 0.35)⟩	⟨(0.50, 0.40, 0.55), (0.10, 0.15, 0.30), (0.10, 0.20, 0.20)⟩	⟨(0.30, 0.45, 0.25), (0.20, 0.10, 0.30), (0.10, 0.25, 0.20)⟩
ϕ_2	⟨(0.60, 0.50, 0.65), (0.30, 0.25, 0.30), (0.20, 0.30, 0.25)⟩	⟨(0.65, 0.70, 0.75), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15)⟩	⟨(0.45, 0.35, 0.50), (0.15, 0.10, 0.10), (0.20, 0.30, 0.25)⟩	⟨(0.45, 0.50, 0.60), (0.30, 0.20, 0.30), (0.25, 0.30, 0.25)⟩	⟨(0.50, 0.45, 0.35), (0.30, 0.25, 0.30), (0.20, 0.30, 0.25)⟩
ϕ_3	⟨(0.45, 0.50, 0.80), (0.15, 0.30, 0.55), (0.55, 0.20, 0.25)⟩	⟨(0.40, 0.45, 0.50), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20)⟩	⟨(0.40, 0.45, 0.60), (0.05, 0.20, 0.25), (0.40, 0.20, 0.25)⟩	⟨(0.45, 0.80, 0.90), (0.40, 0.70, 0.55), (0.55, 0.20, 0.40)⟩	⟨(0.80, 0.50, 0.80), (0.45, 0.30, 0.55), (0.55, 0.20, 0.25)⟩
ϕ_4	⟨(0.85, 0.70, 0.95), (0.60, 0.50, 0.65), (0.45, 0.15, 0.35)⟩	⟨(0.60, 0.65, 0.70), (0.35, 0.40, 0.45), (0.30, 0.40, 0.45)⟩	⟨(0.95, 0.70, 0.80), (0.15, 0.10, 0.30), (0.30, 0.35, 0.30)⟩	⟨(0.90, 0.70, 0.95), (0.60, 0.40, 0.65), (0.45, 0.15, 0.35)⟩	⟨(0.65, 0.70, 0.80), (0.40, 0.35, 0.25), (0.15, 0.15, 0.20)⟩

Table 4
Aggregated T2NN values based classification.

	Aggregating values
ϕ_1	⟨(0.7131, 0.7654, 0.8302), (0.2420, 0.2444, 0.3716), (0.1801, 0.2827, 0.2721)⟩
ϕ_2	⟨(0.5434, 0.5193, 0.6113), (0.1852, 0.1616, 0.1950), (0.1462, 0.2280, 0.2200)⟩
ϕ_3	⟨(0.4785, 0.5408, 0.7210), (0.1395, 0.2726, 0.3565), (0.3264, 0.1861, 0.2537)⟩
ϕ_4	⟨(0.8588, 0.6882, 0.8638), (0.3322, 0.2723, 0.4273), (0.3226, 0.2472, 0.3365)⟩

We apply the proposed aggregation operator T2NNWA to solve the best bank selection issue by using the next procedures.

Step 1. Collect the classification values of the alternatives $\phi_i(i = 1, 2, 3, 4)$ defined in the previous matrix with T2NNWA operator that is located by Eq. (11) and the values introduced in Table 4.

Step 2. Compute the score value and the accuracy value of alternatives $\phi_i(i = 1, 2, 3, 4)$ by applying Eq. (5) and Eq. (6), as shown in Table 5.

Step 3. Ranking the alternatives based on score values, we found that alternative ϕ_1 is the best alternative, and the classification of alternatives is : $\phi_1 > \phi_4 > \phi_2 > \phi_3$.

Table 5
The score and accuracy values of alternatives.

	Score values	Accuracy values
ϕ_1	0.8382	0.5141
ϕ_2	0.7809	0.3428
ϕ_3	0.7775	0.3322
ϕ_4	0.8288	0.4864

3. The proposed method procedure

We now suggest an orderly approach to TOPSIS technique to the neutrosophic environment under type 2 of neutrosophic number. We found that the GDM problem can be easily solved by this

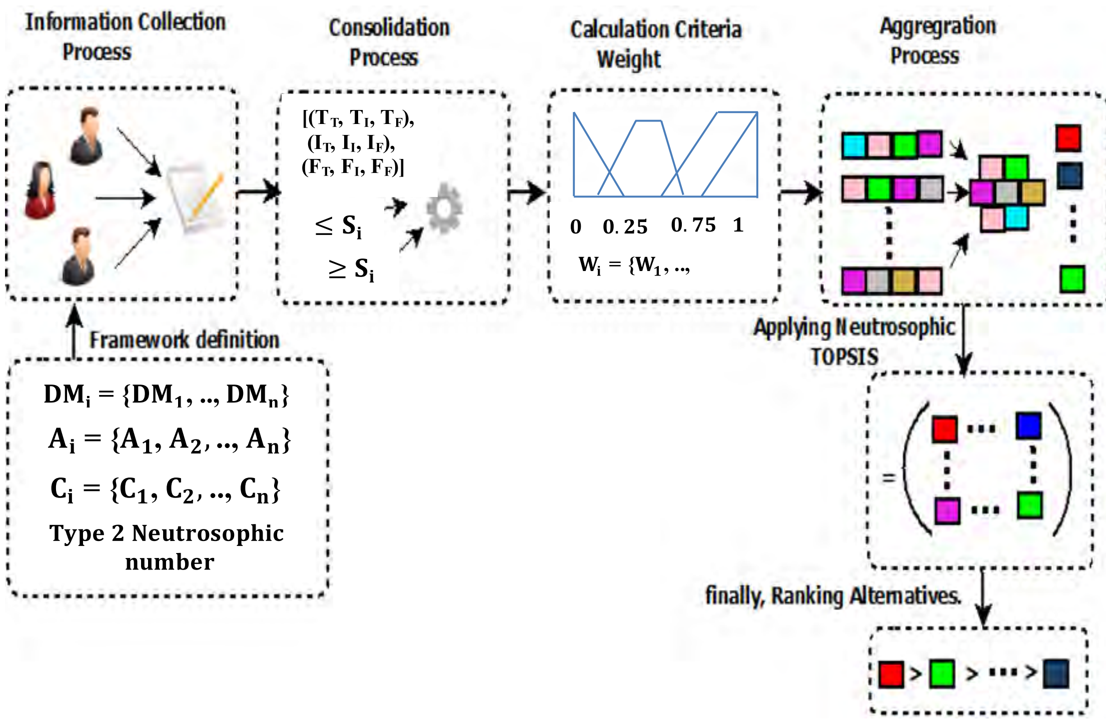

Fig. 2. The general framework for applying TOPSIS using type 2 neutrosophic number.

Table 6
Semantic terms for the significance weight of each criteria.

Linguistic variables	The type 2 neutrosophic number scale for relative importance of each criteria $[(T_T, T_I, T_F), (I_T, I_I, I_F), (F_T, F_I, F_F)]$
Weakly important (WI)	$\langle(0.20, 0.30, 0.20), (0.60, 0.70, 0.80), (0.45, 0.75, 0.75)\rangle$
Equal important (EI)	$\langle(0.40, 0.30, 0.25), (0.45, 0.55, 0.40), (0.45, 0.60, 0.55)\rangle$
Strong important (SI)	$\langle(0.65, 0.55, 0.55), (0.40, 0.45, 0.55), (0.35, 0.40, 0.35)\rangle$
Very strongly important (VSI)	$\langle(0.80, 0.75, 0.70), (0.20, 0.15, 0.30), (0.15, 0.10, 0.20)\rangle$
Absolutely important (AI)	$\langle(0.90, 0.85, 0.95), (0.10, 0.15, 0.10), (0.05, 0.05, 0.10)\rangle$

method under advanced neutrosophic environment. The general conceptualization of framework is displayed in Fig. 2.

The suggested framework consists of many phases, as presented in Fig. 2.

Phase 1. Establish a group of Exs and decide the goal, alternatives and criteria.

• Assume that EXs want to estimate the combination of n criteria and m alternatives EXs are symbolized by $Ex_E = \{Ex_1, Ex_2, Ex_3\}$, where $E = 1, 2, \ldots, E$, and alternatives by $Alt_i = \{Alt_1, Alt_2, \ldots, Alt_m\}$, where $i = 1, 2, \ldots, m$, assessed on n criteria $Œi_p = \{Œi_1, Œi_2, .., Œi_n\}$, $p = 1, 2, \ldots, n$.

Phase 2. Depict and design the linguistic scales.

- Obtain Exs' judgments on each element. Based on previous knowledge and experience on the topic, Exs are wanted to convey their judgments. Every Ex gives his/her judgment linguistically on all of these elements.
- Transform EXs' linguistic evaluations into type 2 neutrosophic numbers for every Ex providing his judgment with assistance of the linguistic terms.
- The significance weights of different criteria and the ordering of specific criteria are deemed as linguistic terms. These linguistic terms can be presented in type 2 neutrosophic number as in Tables 6 and 7. The significance weight of each criterion can be obtained either by direct allocation or indirectly by pairwise comparisons [41]. Herein, we propose that the experts and decision makers use the linguistic terms presented in Tables 6 and 7 to evaluate the weight of the criteria and the classification of alternatives with account to different criteria.
- Build the preference relation matrix to locate the weights of criteria. Exs use the linguistic terms presented in Table 6 to assess the opinions of Exs with regard to each criterion.
- A neutrosophic multicriteria GDM problem can be briefly expressed in matrix:

$$\text{Format as}\quad \tilde{A} = \begin{array}{c} \\ Ex_1 \\ \vdots \\ Ex_m \end{array}\begin{array}{c} \begin{array}{ccc} Œi_p & \ldots & Œi_n \end{array} \\ \begin{bmatrix} \tilde{z}_{11} & \ldots & \tilde{z}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{z}_{m1} & \ldots & \tilde{z}_{mn} \end{bmatrix} \end{array} \tag{26}$$

$$\tilde{\omega} = [\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n] \tag{27}$$

Where $\tilde{z}_{ip} = \left\langle \left(T_{T_{ip}}(z), T_{I_{ip}}(z), T_{F_{ip}}(z)\right), \left(I_{T_{ip}}(z), I_{I_{ip}}(z), I_{F_{ip}}(z)\right), \left(F_{T_{ip}}(z), F_{I_{ip}}(z), F_{F_{ip}}(z)\right)\right\rangle$, $i = 1, 2 \ldots, m$, $p = 1, 2, \ldots, n$, where $\tilde{z}_{ip}$, $\forall_{i,p}$ and $\tilde{\omega}_p$, $p = 1, 2 \ldots, n$ are linguistic terms. These linguistic terms can be described by type 2 neutrosophic number.

- Calculating the weights of Exs. Exs' judgments are collected by using the equation given in Box XII.
- Calculate the score value after aggregating the opinions of Exs for each criteria using Eq. (5). Then, normalize the obtained weights.

Phase 3. Construct the evaluation matrix.

$$\tilde{z}_{ip} = \frac{\left[T_{T_{ip}}(z), T_{I_{ip}}(z), T_{F_{ip}}(z), I_{T_{ip}}(z), I_{I_{ip}}(z), I_{F_{ip}}(z), F_{T_{ip}}(z), F_{I_{ip}}(z), F_{F_{ip}}(z)\right]}{n} \quad (28)$$

Box XII.

Table 7
Linguistic variables for the classification.

Linguistic variables	The type - 2 neutrosophic number scale for relative importance of comparison matrix $[(T_T, T_I, T_F), (I_T, I_I, I_F), (F_T, F_I, F_F)]$
Very Bad (VB)	⟨(0.20, 0.20, 0.10) , (0.65, 0.80, 0.85) , (0.45, 0.80, 0.70)⟩
Bad (B)	⟨(0.35, 0.35, 0.10) , (0.50, 0.75, 0.80) , (0.50, 0.75, 0.65)⟩
Medium Bad (MB)	⟨(0.50, 0.30, 0.50) , (0.50, 0.35, 0.45) , (0.45, 0.30, 0.60)⟩
Medium (M)	⟨(0.40, 0.45, 0.50) , (0.40, 0.45, 0.50) , (0.35, 0.40, 0.45)⟩
Medium Good (MG)	⟨(0.60, 0.45, 0.50) , (0.20, 0.15, 0.25) , (0.10, 0.25, 0.15)⟩
Good (G)	⟨(0.70, 0.75, 0.80) , (0.15, 0.20, 0.25) , (0.10, 0.15, 0.20)⟩
Very Good (VG)	⟨(0.95, 0.90, 0.95) , (0.10, 0.10, 0.05) , (0.05, 0.05, 0.05)⟩

- Build the evaluation matrix $A_i \times Œi_p$ to assess the classification of alternatives with respect to each criterion. Exs use the linguistic terms shown in Table 7.

Format as
$$\tilde{R} = \begin{array}{c} \\ Alt_1 \\ \vdots \\ Alt_m \end{array} \begin{array}{c} \begin{array}{ccc} Œi_p & \dots & Œi_n \end{array} \\ \begin{bmatrix} \tilde{z}_{11} & \dots & \tilde{z}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{z}_{m1} & \dots & \tilde{z}_{mn} \end{bmatrix} \end{array} \quad (29)$$

- Aggregate the final evaluation matrix using Eq. (1) divided by 3.
- Use the de-neutrosophication Eq. (5) for transforming type 2 neutrosophic number to the crisp value for each factor $\tilde{z}_{ip}$.
- Then, normalize the obtained matrix by Eq. (30)

$$\tilde{y}_{ip} = \frac{\tilde{z}_{ip}}{\sqrt{\sum_{i=1}^{m} \tilde{Z}_{ip}^2}}; \; i = 1, 2, \dots, m; \; p = 1, 2, \dots, n. \quad (30)$$

- Compute the weighted matrix by multiplying Eq. (27) by the normalized matrix as in Eq. (31).

$$Z_{ip} = \omega_p \times NM_{ip} \quad (31)$$

Phase 4. Rank the alternatives

- We can describe the neutrosophic positive ideal solution (NPIS, A^*) and Neutrosophic negative ideal solution (NNIS, A^-)

$$A^* = \{< \max(\delta_{ip}|i = 1, 2, \dots, m)|p\epsilon p^+ >, < \min(\delta_{ip}|i = 1, 2, \dots, m)|p\epsilon p^- >\} \quad (32)$$

$$A^- = \{< \min(\delta_{ip}|i = 1, 2, \dots, m)|p\epsilon p^+ >, < \max(\delta_{ip}|i = 1, 2, \dots, m)|p\epsilon p^- >\} \quad (33)$$

Where p^+ related with the criteria that have a profitable effect and p^- related with the criteria that have a non-beneficial effect.

- The dimension of each alternative from A^* and A^- can be currently computed as:

$$d_i^* = \sqrt{\sum_{p=1}^{n} (\tilde{A}_{ip} - A_p^*)^2}, \; i = 1, 2, \dots, m, \quad (34)$$

$$d_i^- = \sqrt{\sum_{p=1}^{n} (\tilde{A}_{ip} - A_p^-)^2}, \; i = 1, 2, \dots, m, \quad (35)$$

- A proximity factor is defined to locate the classification system of all available alternatives once the d_i^* and d_i^- of each alternative $A_i = (1, 2, \dots, m)$ have been computed. The proximity coefficient of every available alternative is computed as:

$$CC_i = \frac{d_i^-}{d_i^+ + d_i^-} \quad i = 1, 2, \dots, m \quad (36)$$

Clearly, an alternative A_i is closer to the (NPIS, A^*) and further from (NNIS, A^-). Thus, according to the closeness coefficient, we can decide the classification order of all alternatives and select the superior one from a set of available alternatives.

4. Real case study

We introduce a numerical case which implicates methods and data analysis to test the competence and the efficiency of suggested framework for selection of the best supplier to import cars, performed on an importing company in Egypt, Ghabbour Company, founded in 1960 and based in Cairo. Egypt the Corporation seeks to increase the numbers of suppliers. For this purpose, the executive managers suggested some alternatives such as Alt_1 India, Alt_2 Japan, Alt_3 China, Alt_4 USA and Alt_5 Germany. Consequently, the organization must evaluate suppliers and their sustainability. For this study, the corporation determined the most important criteria as being $Œi_1$ competency, $Œi_2$ capacity, $Œi_3$ commitment, $Œi_4$ control, $Œi_5$ cash, $Œi_6$ cost, $Œi_7$ consistency and $Œi_8$ communication for comparing alternatives and select the best alternative. These criteria are considered by three experts. The experts are: strategic expert, marketing expert and manufacturing expert, all with more than ten years of experience in this field. The hierarchical construction of this decision problem is presented in Fig. 3. The suggested technique is employed to solve this issue and the computational steps are as follows:

Phase 1. Organize a group of Exs and determine goals, alternatives and criteria.

- A group consisting of three Exs, symbolized by $Ex_E = (Ex_1, Ex_2, Ex_3)$, is constructed to select the best supplier which the Ghabbour Company can deal with it for importing motors. Alternatives are introduced as $A_i = (Alt_1, Alt_2, Alt_3, Alt_4, Alt_5)$. These alternatives are estimated based on eight criteria $Œi_p = (Œi_1, Œi_2, Œi_3, Œi_4, Œi_5, Œi_6, Œi_7, Œi_8)$, which are collected from a comprehensive literature and EXs' opinions.

Phase 2. Depict and design the linguistic scales.

- Obtain Exs' judgments on each element. Based on the previously knowledge and experience on the topics, Exs are demanded to convey their judgments. Every Ex gives his judgment linguistically on every of these elements. Then,

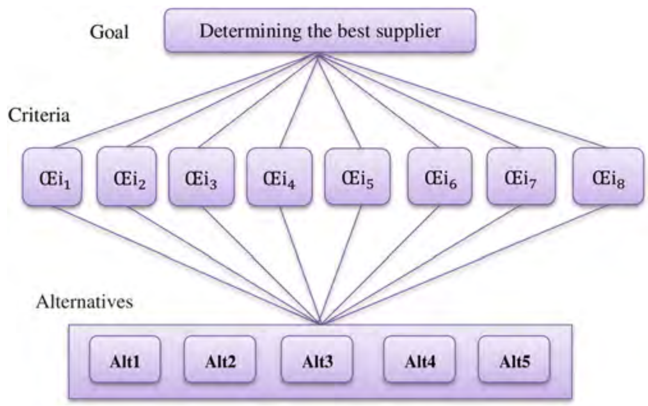

Fig. 3. The hierarchy of the problem.

Table 8
The weight of criteria by experts.

EXs	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$	Œi$_5$	Œi$_6$	Œi$_7$	Œi$_8$
Ex$_1$	⟨SI⟩	⟨WI⟩	⟨SI⟩	⟨SI⟩	⟨AI⟩	⟨EI⟩	⟨EI⟩	⟨EI⟩
Ex$_2$	⟨VSI⟩	⟨EI⟩	⟨VSI⟩	⟨AI⟩	⟨SI⟩	⟨SI⟩	⟨WI⟩	⟨SI⟩
Ex$_3$	⟨AI⟩	⟨EI⟩	⟨VSI⟩	⟨AI⟩	⟨SI⟩	⟨EI⟩	⟨EI⟩	⟨VSI⟩

transform EXs' linguistic evaluations into type 2 neutrosophic numbers as in Tables 6 and 7.

- Build the preference relation matrix to locate the weights of criteria using Eq. (26) as presented in Table 8. EXs employ the semantic terms displayed in Table 6 to assess the opinions of EXs with consideration to every criterion.
- Calculate the weights of Exs; Exs' judgments are collected by using Eq. (28). Then, calculate the score value after aggregating the opinions of Exs for each criteria using Eq. (5). Then, normalize the obtaining weights as presented in Table 9.

Phase 3. Create the valuation matrix.

- Form the valuation matrix $A_i \times Œi_j$ using Eq. (29) to assess the ratings of alternatives with esteem to every criterion, as in Table 10. Exs use the linguistic terms presented in Table 7.
- Aggregate the final evaluation matrix using Eq. (1) as in Table 11.

• Use the de-neutrosophication Eq. (5) for transforming type 2 neutrosophic numbers to the crisp values, as shown in Table 12.
• Then, construct the normalized decision matrix by Eq. (30), as presented in Table 13.
• Compute the weighted matrix by multiplying Eq. (27) by the normalized matrix as in Eq. (31), as shown in Table 14.

Phase 4. Rank the alternatives

- We can define the neutrosophic positive ideal solution (NPIS, A^*) and the Neutrosophic negative ideal solution (NNIS, A^-) by Eqs. (32) and (33).

Table 10
Classification of alternatives and criteria by EXs.

Exs	Alt$_n$	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$	Œi$_5$	Œi$_6$	Œi$_7$	Œi$_8$
Ex$_1$	Alt$_1$	⟨MG⟩	⟨G⟩	⟨VG⟩	⟨MG⟩	⟨B⟩	⟨VG⟩	⟨VB⟩	⟨VG⟩
	Alt$_2$	⟨VB⟩	⟨VG⟩	⟨G⟩	⟨B⟩	⟨MG⟩	⟨G⟩	⟨G⟩	⟨G⟩
	Alt$_3$	⟨MG⟩	⟨MG⟩	⟨MG⟩	⟨M⟩	⟨B⟩	⟨MG⟩	⟨G⟩	⟨MG⟩
	Alt$_4$	⟨G⟩	⟨MB⟩	⟨VG⟩	⟨MG⟩	⟨VG⟩	⟨VG⟩	⟨MG⟩	⟨VG⟩
	Alt$_5$	⟨VB⟩	⟨B⟩	⟨B⟩	⟨VG⟩	⟨VB⟩	⟨MG⟩	⟨MG⟩	⟨M⟩
Ex$_2$	Alt$_1$	⟨G⟩	⟨MB⟩	⟨MB⟩	⟨VG⟩	⟨VG⟩	⟨VG⟩	⟨MG⟩	⟨G⟩
	Alt$_2$	⟨MB⟩	⟨VB⟩	⟨G⟩	⟨M⟩	⟨M⟩	⟨G⟩	⟨VB⟩	⟨MG⟩
	Alt$_3$	⟨VG⟩	⟨MG⟩	⟨VG⟩	⟨MG⟩	⟨VB⟩	⟨MG⟩	⟨G⟩	⟨VG⟩
	Alt$_4$	⟨VG⟩	⟨VB⟩	⟨VG⟩	⟨VG⟩	⟨MB⟩	⟨VB⟩	⟨MB⟩	⟨VG⟩
	Alt$_5$	⟨MB⟩	⟨B⟩	⟨VG⟩	⟨VG⟩	⟨VB⟩	⟨VG⟩	⟨MB⟩	⟨M⟩
Ex$_3$	Alt$_1$	⟨VG⟩	⟨B⟩	⟨VG⟩	⟨VG⟩	⟨G⟩	⟨VG⟩	⟨VG⟩	⟨VG⟩
	Alt$_2$	⟨M⟩	⟨VG⟩	⟨MB⟩	⟨MB⟩	⟨MG⟩	⟨M⟩	⟨M⟩	⟨M⟩
	Alt$_3$	⟨G⟩	⟨B⟩	⟨MG⟩	⟨MG⟩	⟨VB⟩	⟨B⟩	⟨MG⟩	⟨G⟩
	Alt$_4$	⟨B⟩	⟨MB⟩	⟨VG⟩	⟨MB⟩	⟨MG⟩	⟨M⟩	⟨G⟩	⟨VG⟩
	Alt$_5$	⟨MB⟩	⟨VG⟩	⟨M⟩	⟨MB⟩	⟨MG⟩	⟨VG⟩	⟨MB⟩	⟨MB⟩

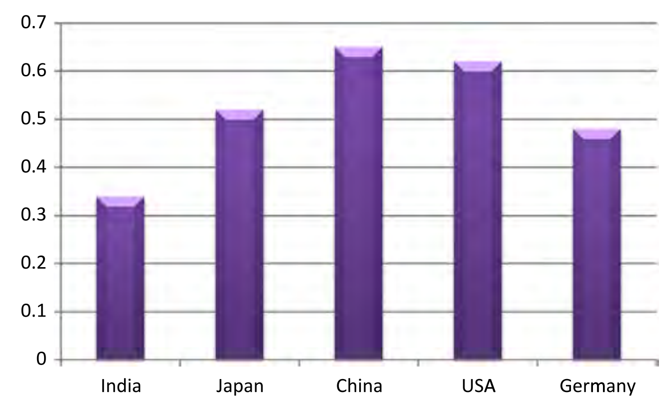

Fig. 4. Ranking the alternatives according to the best supplier.

- The distance of each alternative from A^* and A^- can be currently calculated by Eqs. (34) and (35) as: $d^* = (0.021, 0.016, 0.012, 0.012, \text{and } 0.018)$, $d^- = \{0.011, 0.017, 0.019, 0.022, \text{and } 0.017)$.
- The proximity coefficient of each available alternative is computed by Eq. (36) as in Table 15.

• The ordering for the optimal alternatives of selecting the best supplier is: Alt$_3$, Alt$_4$, Alt$_2$, Alt$_5$, and Alt$_1$, as presented in Fig. 4.

5. Concluding remarks

MADM issues generally occur in difficult environments related to uncertainty and imprecise data. The type 2 neutrosophic number is an efficient tool to deal with expert's impreciseness or incompleteness, and the decision maker's appreciations and assessments over alternative with esteem to attribute. In the first part of the article, we present the proposed method, introducing the type 2 neutrosophic number and defining its operations, properties and

Table 9
The final results of normalized criteria weights.

Weight $\tilde{\omega}_n$	Aggregation weight by T2NN	Crisp	Normalized weight
Œi$_1$	$\langle(0.78, 0.72, 0.73), (0.23, 0.25, 0.32), (0.18, 0.18, 0.22)\rangle$	0.7617	0.16
Œi$_2$	$\langle(0.33, 0.30, 0.23), (0.50, 0.60, 0.53), (0.45, 0.65, 0.62)\rangle$	0.3800	0.08
Œi$_3$	$\langle(0.75, 0.68, 0.65), (0.27, 0.25, 0.38), (0.22, 0.20, 0.25)\rangle$	0.7283	0.15
Œi$_4$	$\langle(0.82, 0.75, 0.82), (0.20, 0.25, 0.25), (0.15, 0.17, 0.18)\rangle$	0.7933	0.17
Œi$_5$	$\langle(0.73, 0.65, 0.68), (0.30, 0.35, 0.40), (0.25, 0.28, 0.27)\rangle$	0.6858	0.14
Œi$_6$	$\langle(0.48, 0.38, 0.35), (0.43, 0.52, 0.45), (0.42, 0.53, 0.48)\rangle$	0.4758	0.10
Œi$_7$	$\langle(0.33, 0.30, 0.23), (0.50, 0.60, 0.53), (0.45, 0.65, 0.62)\rangle$	0.3800	0.08
Œi$_8$	$\langle(0.62, 0.53, 0.50), (0.35, 0.38, 0.42), (0.32, 0.37, 0.37)\rangle$	0.6017	0.12

Table 11
The consolidated decision matrix.

EXs	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$
Alt$_1$	$\langle (0.617, 0.599, 0.623), (0.013, 0.001, 0.014), (0.003, 0.005, 0.005) \rangle$	$\langle (0.476, 0.440, 0.453), (0.013, 0.018, 0.030), (0.008, 0.011, 0.026) \rangle$	$\langle (0.650, 0.619, 0.650), (0.003, 0.004, 0.004), (0.001, 0.008, 0.005) \rangle$	$\langle (0.653, 0.629, 0.650), (0.006, 0.005, 0.006), (0.008, 0.006, 0.001) \rangle$
Alt$_2$	$\langle (0.353, 0.308, 0.358), (0.043, 0.042, 0.064), (0.024, 0.032, 0.063) \rangle$	$\langle (0.640, 0.613, 0.637), (0.002, 0.003, 0.007), (0.004, 0.007, 0.006) \rangle$	$\langle (0.552, 0.544, 0.593), (0.004, 0.005, 0.009), (0.002, 0.002, 0.008) \rangle$	$\langle (0.393, 0.351, 0.358), (0.033, 0.039, 0.060), (0.026, 0.030, 0.059) \rangle$
Alt$_3$	$\langle (0.617, 0.599, 0.623), (0.013, 0.001, 0.014), (0.003, 0.005, 0.005) \rangle$	$\langle (0.475, 0.393, 0.358), (0.006, 0.006, 0.017), (0.002, 0.016, 0.005) \rangle$	$\langle (0.603, 0.539, 0.571), (0.001, 0.008, 0.001), (0.001, 0.001, 0.004) \rangle$	$\langle (0.485, 0.420, 0.458), (0.005, 0.003, 0.011), (0.001, 0.008, 0.003) \rangle$
Alt$_4$	$\langle (0.589, 0.588, 0.591), (0.006, 0.005, 0.003), (0.008, 0.005, 0.002) \rangle$	$\langle (0.383, 0.261, 0.358), (0.054, 0.003, 0.057), (0.030, 0.024, 0.084) \rangle$	$\langle (0.664, 0.657, 0.664), (0.003, 0.003, 0.004), (0.004, 0.004, 0.004) \rangle$	$\langle (0.588, 0.510, 0.571), (0.003, 0.001, 0.002), (0.008, 0.001, 0.002) \rangle$
Alt$_5$	$\langle (0.383, 0.261, 0.358), (0.054, 0.003, 0.057), (0.030, 0.024, 0.084) \rangle$	$\langle (0.511, 0.497, 0.380), (0.008, 0.019, 0.011), (0.004, 0.009, 0.007) \rangle$	$\langle (0.522, 0.519, 0.501), (0.007, 0.011, 0.007), (0.003, 0.005, 0.005) \rangle$	$\langle (0.650, 0.619, 0.650), (0.002, 0.001, 0.004), (0.004, 0.003, 0.005) \rangle$
EXs	Œi$_5$	Œi$_6$	Œi$_7$	Œi$_8$
Alt$_1$	$\langle (0.589, 0.588, 0.591), (0.006, 0.005, 0.003), (0.008, 0.005, 0.002) \rangle$	$\langle (0.664, 0.657, 0.664), (0.003, 0.003, 0.004), (0.004, 0.004, 0.004) \rangle$	$\langle (0.545, 0.490, 0.501), (0.004, 0.004, 0.004), (0.003, 0.001, 0.002) \rangle$	$\langle (0.656, 0.648, 0.659), (0.005, 0.007, 0.003), (0.008, 0.001, 0.002) \rangle$
Alt$_2$	$\langle (0.485, 0.420, 0.458), (0.005, 0.003, 0.011), (0.001, 0.008, 0.003) \rangle$	$\langle (0.535, 0.566, 0.593), (0.003, 0.006, 0.021), (0.001, 0.003, 0.006) \rangle$	$\langle (0.415, 0.444, 0.453), (0.013, 0.024, 0.035), (0.005, 0.016, 0.021) \rangle$	$\langle (0.511, 0.499, 0.533), (0.004, 0.005, 0.010), (0.001, 0.005, 0.005) \rangle$
Alt$_3$	$\langle (0.245, 0.245, 0.099), (0.070, 0.160, 0.193), (0.034, 0.160, 0.106) \rangle$	$\langle (0.408, 0.365, 0.232), (0.017, 0.028, 0.053), (0.008, 0.047, 0.021) \rangle$	$\langle (0.535, 0.566, 0.593), (0.003, 0.006, 0.021), (0.001, 0.003, 0.006) \rangle$	$\langle (0.617, 0.599, 0.623), (0.013, 0.001, 0.014), (0.003, 0.005, 0.005) \rangle$
Alt$_4$	$\langle (0.588, 0.510, 0.571), (0.003, 0.001, 0.002), (0.008, 0.001, 0.002) \rangle$	$\langle (0.491, 0.490, 0.501), (0.008, 0.012, 0.007), (0.003, 0.005, 0.005) \rangle$	$\langle (0.530, 0.466, 0.533), (0.005, 0.004, 0.009), (0.002, 0.004, 0.006) \rangle$	$\langle (0.664, 0.657, 0.664), (0.003, 0.003, 0.004), (0.004, 0.004, 0.004) \rangle$
Alt$_5$	$\langle (0.325, 0.277, 0.232), (0.028, 0.032, 0.060), (0.007, 0.053, 0.025) \rangle$	$\langle (0.653, 0.629, 0.650), (0.006, 0.005, 0.006), (0.008, 0.006, 0.001) \rangle$	$\langle (0.483, 0.337, 0.458), (0.017, 0.006, 0.017), (0.007, 0.008, 0.018) \rangle$	$\langle (0.407, 0.380, 0.458), (0.027, 0.024, 0.038), (0.018, 0.016, 0.041) \rangle$

Table 12
The final aggregated matrix.

Œi$_n$/Alt$_n$	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$	Œi$_5$	Œi$_6$	Œi$_7$	Œi$_8$
Alt$_1$	0.8659	0.8061	0.8751	0.8765	0.8598	0.8844	0.8336	0.8814
Alt$_2$	0.7488	0.8720	0.8497	0.7614	0.8118	0.8509	0.8002	0.8385
Alt$_3$	0.8659	0.7954	0.8523	0.8118	0.6493	0.7600	0.8509	0.8759
Alt$_4$	0.8597	0.7487	0.8844	0.8467	0.8467	0.8263	0.8298	0.8844
Alt$_5$	0.7487	0.8166	0.8339	0.8763	0.7351	0.8765	0.7940	0.7851

Table 13
The normalized decision matrix.

Œi$_n$/Alt$_n$	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$	Œi$_5$	Œi$_6$	Œi$_7$	Œi$_8$
Alt$_1$	0.34	0.31	0.33	0.34	0.33	0.34	0.32	0.35
Alt$_2$	0.32	0.38	0.37	0.32	0.35	0.36	0.34	0.36
Alt$_3$	0.38	0.35	0.37	0.36	0.28	0.33	0.37	0.39
Alt$_4$	0.36	0.31	0.37	0.35	0.35	0.35	0.35	0.37
Alt$_5$	0.32	0.35	0.36	0.38	0.32	0.38	0.34	0.33

functioning rules. Then, we suggest an aggregation operator, called T2NNWA operator, the score function and the accuracy function, and apply them to solve a MADM problem under neutrosophic environment using type 2 neutrosophic numbers. We discuss two properties of the T2NNWA operator. Finally, the competence, the performance and the applicability of the suggested technique is illustrated with the best bank selection problem to do some banking transactions. In the second part, we present a powerful application of the proposed method under GDM in neutrosophic environment and employ the TOPSIS method in the neutrosophic environment by the type 2 neutrosophic numbers. We apply the proposed method in a problem of selection of the best supplier for importing cars. The method can be easily used to compute and rank the alternatives under group decision making process. The suggested technique can be as well employed in other decision making issues, such as pattern recognition, medical diagnosis, personnel selection, information project selection, material selection and other management decision problems.

Table 14
The weighted matrix.

Œi$_n$/Alt$_n$	Œi$_1$	Œi$_2$	Œi$_3$	Œi$_4$	Œi$_5$	Œi$_6$	Œi$_7$	Œi$_8$
Alt$_1$	0.0544	0.0248	0.0495	0.0578	0.0462	0.0340	0.0256	0.0420
Alt$_2$	0.0512	0.0304	0.0555	0.0544	0.0490	0.0360	0.0272	0.0432
Alt$_3$	0.0608	0.0280	0.0555	0.0612	0.0392	0.0330	0.0296	0.0468
Alt$_4$	0.0576	0.0248	0.0555	0.0595	0.0490	0.0350	0.0280	0.0444
Alt$_5$	0.0512	0.0280	0.0540	0.0664	0.0448	0.0380	0.0272	0.0396

Table 15
The final result of ranking.

$Œi_n$/Alt_n	D_i^+	D_i^-	$Œi_i$	Arranging
Alt_1	0.021	0.011	0.34	5
Alt_2	0.016	0.017	0.52	3
Alt_3	0.012	0.022	0.65	1
Alt_4	0.012	0.019	0.62	2
Alt_5	0.018	0.017	0.48	4

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An Integrated Neutrosophic ANP and VIKOR Method for Achieving Sustainable Supplier Selection: A Case Study in Importing Field

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A B S T R A C T

Sustainable supplier selections have been improved by an increased number of multi criteria group decision making (MCGDM) methods and techniques. This paper provides a multi criteria group decision making (MCGDM) proposed technique of the ANP (analytical network process) method and the VIKOR (ViseKriterijumska Optimizacija I Kompromisno Resenje) method under neutrosophic environment for dealing with incomplete information and high order imprecision. This is done by using of the triangular neutrosophic numbers (TriNs) to present the linguistic variables based on opinions of experts and decision makers. The aim is to solve the problem of supplier selection in sustainable supplier chain management (SSCM). The suggested technique consists of two phases. First, we use the ANP method to calculate the weights of criteria and sub criteria. Second, with the aid of VIKOR method and with obtained weights of the criteria and sub criteria from step one, we can find the solution. A case study is used to present the decision process in detail. Our proposed method is compared directly with the entropy method to justify our approach. We also use genetic algorithm to compute predicted values for five selected cities while varying economic, environmental and social criteria. Explanations of forecasted outputs and limitation for research have been presented. Our objective is to demonstrate that our proposal can calculate key measurement for major import and export cities, as well as to provide fair and reliable forecasted outcomes.

1. Introduction

Due to the competitions for economic growth, many countries and cities have exploited more natural resources. In the process of doing so, it has caused environmental issues and hazards, such as air pollution and water contamination. Therefore, companies, corporations and any members in the community of manufacturing have the growing social responsibilities, due to the depletion of natural resources, climate change and environmental hazards. To make a balance, the recommendation is to have green suppliers who are able to contribute to economic development and maintain good business ethics. The type of work and the systems green suppliers do, are known as sustainable supplier chain management (SSCM). There are three factors determining the extents of success of SSCM, namely social, economic and environment factors, which can be used to evaluate sustainable suppliers systematically [1,2].

The majority of businesses and investors takes risk all the times. This is particularly true if foreign investors have no much knowledge about a city or a country, and they tend to rely on information given to them by their networks and local governments. To help businesses and investors avoid medium and high risk cases, the multi criteria group decision making (MCGDM) methods and techniques can be used to evaluate sustainable suppliers as follows. First, researchers could use mathematical operations and fuzzy predilection relations. One specific example is the fuzzy multiple criteria hierarchical group decision-making problems presented by Chen and Lee [3,4]. Second, an integrated methodology based on analytic network process (ANP) and VIKOR (ViseKriterijumska Optimizacija I Kompromisno Resenje) can be used to evaluate supplier selection [5,6]. Third, TODIM (an acronym in Portuguese of interactive and multi criteria decision making) approach is developed for supplier evaluation based on interval type 2 fuzzy sets by Qin [7].

It has become a trend for researchers and scientists to perform a multiplicity of studies, particularly in the relative area of evaluation and supplier selection, together with the fuzzy research [8]. The following methods can be used to analyze supplier selection, such as Decision-making Trial and Evaluation Laboratory (DEMATEL), TOPSIS, TODIM, analytic hierarchy process (AHP), analytic network process (ANP), DEA and their subsets. There are three ways in the mainstream research as follows. First, the process of selecting the suppliers can be adopted by combining different methods, such Fuzzy Delphi and AHP-DEMATEL method [9], and combining AHP and TOPSIS under the fuzzy environment for evaluation suppliers [10]. Second, another alternative can be relied on one method, such as structured MCGDM under the intuitionistic fuzzy environment. Researchers in this area focus on mathematical operations and intuitionistic fuzzy predilection relations. Third, VIKOR method can be used under the intuitionistic fuzzy to find suitable suppliers with their locations [11]. Among these three mainstream, one thing in common is that the intuitionistic fuzzy environment for the evaluation of suppliers [12,13]. The majority of researchers used traditional fuzzy and intuitionistic fuzzy set to handle the incomplete information, vague data and the ambiguity of expert's judgment in order to solve MCGDM problems.

For the common approaches, we can locate the weights of criteria for resolving MCGDM problems which have unknown weights. To gain the weights in many MCDM problems, the AHP method is often excessively used. The reason is that AHP method assumes that the criteria are mutually independent, since there are no interactions between sub-criteria. Additionally, a usual problem is that much decision information is unclear and vague within the operation of decision making. It is commonly known that fuzzy and IFs can model these uncertainty and vague information well. Meanwhile the results of decision making should be more rigorous to be useful for the sustainable expansion of the company. Obviously, fuzzy and IFs cannot illustrate the linguistic imprecision and ambiguity of experts' opinions.

Due to intense market competition, increased consumer demand, and faster replication of the product, there are more factors which cannot be revealed easily in the sustainable supplier valuation process. Therefore, we plan to prepare to develop a method which can be more effective in sustainable supplier evaluation. The aim is to include a fair, step-by-step and logical measurement on important factors. This is our motivation for this research. To fulfill our motivation, we can adopt neutrosophic research in our proposed method to integrate ANP and VIKOR methods together. Neutrosophic is very effective in dealing with incomplete information, as well as unclear and vague data. Hence, experts and decision makers use neutrosophic to denote information in an uncertain environment [14]. The notion of neutrosophic set was suggested by Smarandache [15–17] to make the concept of IFs general. Many researchers head for solving MCGDM problems under the neutrosophic environment because the accuracy of the results [18–20].

The main achievements of this research are:

- Considering the significance of integrating of ANP method and VIKOR method under the environment of neutrosophic.
- Recognizing the most effective and detailed criteria for supplier's selection.
- Demonstrating the case study of analyzing social, economic and environmental factors to select the best suppliers for importing, and its predictive analysis.

The model can be closer to the actual decision making problems because being different from AHP method, the ANP depicts relations among elements and interdependencies and feedback. The ANP produces more accurate weights and more reasonable outputs. Additionally, the solution offered by VIKOR method is a feasible solution closest to the ideal solution. Besides, we should take the accessibility of the proposed method into consideration, so that we can use (TriNs) to depict the uncertain information instead of real numbers. Thus, to overcome these disadvantages, this paper seeks to develop an integrated ANP (Saaty 1980) and VIKOR (Opricovic 1998) method under neutrosophic environment to transact with sustainable supplier selections problems. The structure of this paper is as follows: Section 2 reviews the ANP method and the difference between ANP and AHP methods. Section 3 presents the VIKOR method. Section 4 clarifies the

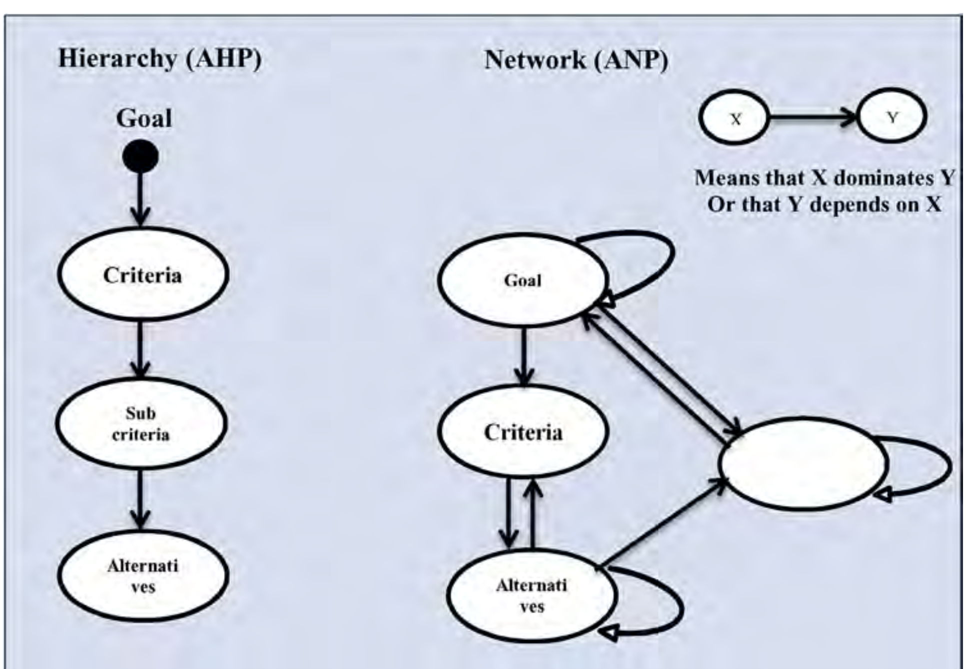

Fig. 1. Model structure for AHP and ANP.

preliminaries of neutrosophic. Section 5 clarifies the procedure for the proposed ANP-VIKOR method for the evaluation of the suppliers. Section 6 describe a case study to certify the practicality of the ANP-VIKOR method. Section 7 shows our evaluation based on predicted outcomes and Section 8 present discussion about research contributions and limitations. Section 9 concludes this paper with our contributions and future work.

2. The ANP method

There are a lot of multi-criteria decision making (MCDM) techniques such as the analytic network process (ANP) [21] that is modified based on the (AHP) method [22]. The (ANP) consider the dependency and feedback between elements of the problem that it sophisticated by Saaty in 1996. The (ANP) makes models of the decision making problems as networks not as hierarchies. From disadvantages of the analytic hierarchy it's not assumed the effect of criteria on the alternatives. But the feedback and dependencies are considered in the (ANP) method. In Fig. 1, we present the difference between AHP and ANP.

The Fig. 1 shows how the hierarchy of the AHP is presented and the higher element depends on the lower element but in the network there exist dependencies between elements of the problem that can be inner or outer dependencies. So, the analytic network process is appropriate for complex problem. We present in Fig. 2 the main process in ANP method.

3. The VIKOR method

Opricovic has developed the VIKOR method [23] and later sophisticated it for multi criteria optimization of difficult systems and complex problems [24–26]. When solving MADM problems to get compromise solutions, we use the VIKOR method which is considered an effective tool for solving problems that contain a set of clashing criteria. The next form M_P metric is considered the original formula for developing the VIKOR technique.

$$M_{Pj} = \left\{ \sum_{i=1}^{n} \left[\frac{w_i(A^* - A_{ij})}{(A^*_i - A^-_i)} \right] \right\}^{\frac{1}{p}}, 1 \le M \le \infty, j = 1, 2 \ldots J \quad (1)$$

In the previous M_P metric, A^-_i = min A_{ij}, A^*_i = max A_{ij}, respectively, clarify the best and worst values w_i = (i = 1, 2, . . . ,I) are the conformable weights of the attributes. The dimension between alternative A_i to the perfect solution are noted by M_{Pj}. The VIKOR method is interested in finding a compromise solution satisfying the maximized utility of the entire group and Fig. 3 shows that compromise solution.

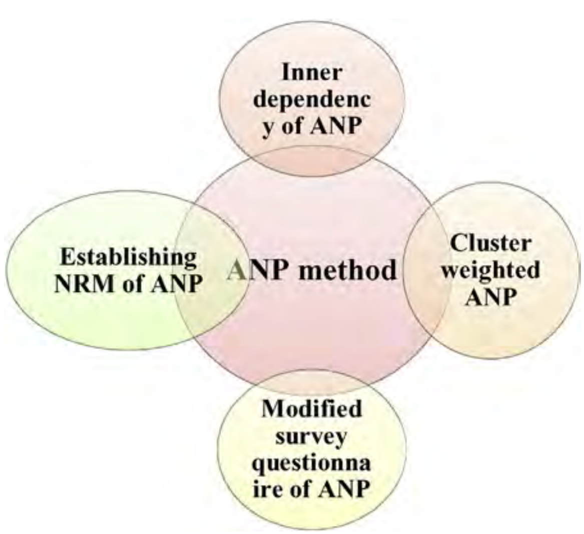

Fig. 2. Main process in ANP.

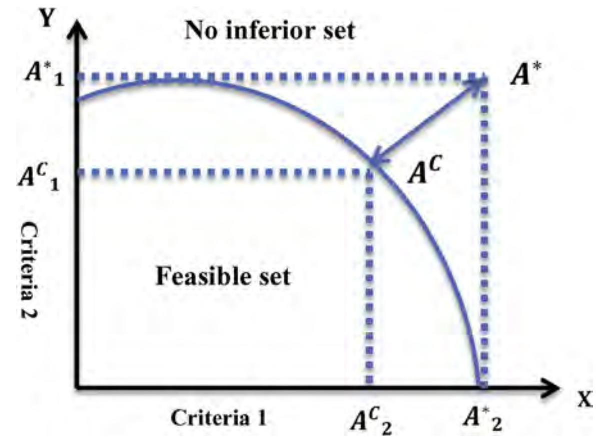

Fig. 3. Ideal and compromise solutions.

4. Basic and fundamental concepts of neutrosophic

This section shows the basic definitions of neutrosophic set.

Definition 1. [27,28] Any neutrosophic set N in X, has a truth-membership function $T_N(x)$, an indeterminacy-membership function $I_N(x)$ and a falsity-membership function $F_N(x)$. Where X is a space of points and $x\in$. $T_N(x)$, $I_N(x)$ and $F_N(x)$ are real subsets of$]^-0, 1^+[$. The sum of $T_N(x)$, $I_N(x)$ and $F_N(x)$ has no constraints so $0^- \le$ sup $T_N(x)$ + sup $I_N(x)$ + sup $F_N(x)$ $\le 3^+$.

Definition 2. [27,29] A single valued neutrosophic set N over X taking the form N={⟨x, $T_N(x)$, $I_N(x)$, $F_N(x)$ ⟩:$x\in X$}, where X be a universe of discourse, $T_N(x)$:$X\to[0,1]$, $I_N(x)$:$X\to[0,1]$ and $F_N(x)$: $X\to[0,1]$ with $0\le T_N(x)+ I_N(x) + F_N(x) \le 3$ for all $x\in X$. For convenience, a single valued neutrosophic number is represented by N= (n_1, n_2, n_3), where n_1, n_2, n_3 ∈ [0,1] and $n_1 + n_2 + n_3 \le 3$.

Definition 3. [30,31] Suppose that $\alpha_{\tilde{n}}$, $\theta_{\tilde{n}}$, $\beta_{\tilde{n}}\, \varepsilon\, [0,1]$ and $n_1, n_2, n_3 \varepsilon R$ where $n_1 \le n_2 \le n_3$.Then a single valued triangular neutrosophic number, $\tilde{n} = \langle\langle(n_1, n_2, n_3); \alpha_{\tilde{n}}, \theta_{\tilde{n}}, \beta_{\tilde{n}}\rangle\rangle$ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_{\tilde{n}}(x) = \begin{cases} \alpha_{\tilde{n}}\left(\frac{x-n_1}{n_2-n_1}\right) (n_1 \le x \le n_2) \\ \alpha_{\tilde{n}}\ (x = n_2) \\ \alpha_{\tilde{n}}\left(\frac{n_3-x}{n_3-n_2}\right) (n_2 < x \le n_3) \\ 0\ otherwise, \end{cases} \quad (2)$$

$$I_{\tilde{n}}(x) = \begin{cases} \frac{(n_2 - x + \theta_{\tilde{n}}\,(x - n_1))}{(n_2 - n_1)} (n_1 \le x \le n_2) \\ \theta_{\tilde{n}}\ (x = n_2) \\ \frac{(x - n_2 + \theta_{\tilde{n}}(n_3 - x))}{(n_3 - n_2)} (n_2 < x \le n_3) \\ 1\ otherwise \end{cases} \quad (3)$$

$$F_{\tilde{n}}(x) = \begin{cases} \frac{(n_2 - x + \beta_{\tilde{n}}(x - n_1))}{(n_2 - n_1)} (n_1 \le x \le n_2) \\ \beta_{\tilde{n}}\ (x = n_2) \\ \frac{(x - n_2 + \beta_{\tilde{n}}\,(n_3 - x))}{(n_3 - n_2)} (n_2 < x \le n_3) \\ 1\ otherwise. \end{cases} \quad (4)$$

$\alpha_{\tilde{n}}$, $\theta_{\tilde{n}}$ And $\beta_{\tilde{n}}$ exemplify the superior degree of truth-membership, lower indeterminacy and falsity membership degree. A single valued triangular neutrosophic number $\tilde{n} = \langle\langle(n_1, n_2, n_3); \alpha_{\tilde{n}}, \theta_{\tilde{n}}, \beta_{\tilde{n}}\rangle\rangle$ may express an ill-defined quantity about n, which is approximately equal to n.

Definition 4. [28,30] Let $\tilde{a} = \langle\langle(a_1, a_2, a_3); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle\rangle$ and $\tilde{b} = \langle\langle(b_1, b_2, b_3); \alpha_{\tilde{b}}, \theta_{\tilde{b}}, \beta_{\tilde{b}}\rangle\rangle$ be two single valued triangular neutrosophic numbers and $\gamma \neq 0$ be any real number. Then,

1 The addition of two numbers is as follows:

$$a\tilde{+}\tilde{b} = \langle\langle(a_1 + b_1, a_2 + b_2, a_3 + b_3); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle\rangle$$

2 The subtraction of two numbers is as follows:

$$\tilde{a} - \tilde{b} = \langle\langle(a_1 - b_3, a_2 - b_2, a_3 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle\rangle$$

3 Inverse of a neutrosophic number is as follows:

$$\tilde{a}-1 = \left\langle\left\langle\left(\frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle, \text{Where } (\tilde{a} \neq 0)$$

4 The multiplication of a neutrosophic number by a fixed value is as follows:

$$\gamma\tilde{a} = \begin{cases} \left\langle\left\langle\left(\frac{a_3}{\gamma}, \frac{a_2}{\gamma}, \frac{a_1}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma < 0) \\ \left\langle\left\langle\left(\frac{a_3}{\gamma}, \frac{a_2}{\gamma}, \frac{a_1}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma < 0) \end{cases}$$

5 The division of a triangular neutrosophic number by fixed value

$$\frac{\tilde{a}}{\gamma} = \begin{cases} \left\langle\left\langle\left(\frac{a_1}{\gamma}, \frac{a_2}{\gamma}, \frac{a_3}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma > 0) \\ \left\langle\left\langle\left(\frac{a_3}{\gamma}, \frac{a_2}{\gamma}, \frac{a_1}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma < 0) \end{cases}$$

6 Division of two numbers is as follows:

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \left\langle\left\langle\left(\frac{a_3}{\gamma}, \frac{a_2}{\gamma}, \frac{a_1}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma < 0) \\ \left\langle\left\langle\left(\frac{a_3}{\gamma}, \frac{a_2}{\gamma}, \frac{a_1}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma < 0) \\ \left\langle\left\langle\left(\frac{a_3}{\gamma}, \frac{a_2}{\gamma}, \frac{a_1}{\gamma}\right); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\right\rangle\right\rangle \text{ if } (\gamma < 0) \end{cases}$$

7 Multiplication of two numbers is as follows:

$$\tilde{a}\tilde{b} = \begin{cases} \langle\langle(a_1b_1, a_2b_2, a_3b_3); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle\rangle & \text{if } (a_3 > 0, b_3 > 0) \\ \langle\langle(a_1b_3, a_2b_2, a_3b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle\rangle & \text{if } (a_3 < 0, b_3 > 0) \\ \langle\langle(a_3b_3, a_2b_2, a_1b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle\rangle & \text{if } (a_3 < 0, b_3 < 0) \end{cases}$$

5. The ANP and VIKOIR methods

5.1. The functionality of linguistic variables

Words have more extent to describe the semantic and sentimental expressions compared with numbers. This research chooses triangular neutrosophic numbers which includes nine parameters to model linguistic variables. The (TriN) scales that are used in this proposed research are exhibited in Table 1.

5.2. The suggested method

In this section, the steps of the suggested triangular neutrosophic ANP-VIKOR framework are presented with detail. The suggested framework consists of five phases which contain many stages as follows:

Phase 1: Build a representative structure of ANP model to define the goal.

Before the process of decision making starts establish a panel of experts, e = [e_1, e_2, ..., e_n] for any MCGDM problem. It should reach a convention on what the goal is before the process starts. In any evaluation of a sustainable supplier, the goal is to find out the best suppliers for the corporation. So, determine the criteria of the problem from experts' opinions, target group survey and literature reviews and surveys to confirm these criteria. Then, determine the alternatives of the problem by introducing the best suppliers and choosing the best alternative. Before that, all the problems were presented by AHP which assumes that these criteria affect goal and alternatives and depend on criteria but in the real problem may be interdependency between elements of the problem criteria, sub criteria and alternatives. Concisely to overcome this drawback of AHP we used the Analytical network process that presents the problem in the network model to show the interdependency between elements such as feedback, interaction and circular relationships as exhibited in Fig. 4.

Table 1
The triangular neutrosophic scale for comprise matrix.

Linguistic term	Triangular Neutrosophic Scale
Low influence (LF)	$\langle\langle(0.1, 0.2, 0.3); 0.5, 0.1, 0.3\rangle\rangle$
	$\langle\langle(0.2, 0.3, 0.4); 0.8, 0.2, 0.3\rangle\rangle$
Fairly low influence (FLF)	$\langle\langle(0.3, 0.4, 0.5); 1.0, 0.1, 0.1\rangle\rangle$
Medium influence (MF)	$\langle\langle(0.4, 0.5, 0.6); 0.7, 0.3, 0.2\rangle\rangle$
Fairly high influence (FHF)	$\langle\langle(0.5, 0.6, 0.7); 0.9, 0.2, 0.1\rangle\rangle$
High influence (HF)	$\langle\langle(0.6, 0.7, 0.8); 0.8, 0.3, 0.5\rangle\rangle$
Strong influence (SF)	$\langle\langle(0.7, 0.8, 0.9); 0.8, 0.3, 0.5\rangle\rangle$
	$\langle\langle(0.8, 0.9, 1.0); 0.9, 0.2, 0.3\rangle\rangle$
	$\langle\langle(0.9, 1.0, 1.0); 0.1, 0.2, 0.2\rangle\rangle$

Phase 2: Compute the weights of the criteria and sub criteria of the problem.

This phase is considered the main phase in the solution of the problem and weighting the criteria.

Step 1. Each expert structure comparisons matrices on the same problem and aggregate matrices which are on the same problem element.

Step 2. In this step, experts compare all overall objectives criteria with sub criteria. Also criteria with alternatives and the interdependencies are considered in the comparison matrices between all elements.

$$\begin{array}{c} \\ C_1 \\ C_2 \\ \cdots \\ \cdots \\ \cdots \\ Cn \end{array} \begin{array}{c} C_1 \quad C_2 \;\; \ldots . C_n \quad \text{weights} \\ \begin{bmatrix} C_1 & C_{12} & \ldots & C_{1m} & w_1 \\ C_{21} & C_{22} & \ldots & C_{2m} & w_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ C_{n1} & C_{n2} & \ldots & C_{nm} & w_n \end{bmatrix} \end{array} \quad (5)$$

The previous matrix shows the relationships between the criteria and calculating of weights and the following shows how present weights of sub criteria relevant to each criteria and calculating the local weight by Eq. (8). We obtain the global weight by multiplying Eq. (5) by the weights in Eq. (8).

In the suggested method, the triangular neutrosophic numbers are used to present the pairwise comparisons matrices as exhibited in Table 1. On the contrary, the ANP in traditional using of Saaty [32] scale of a nine point to represent the comparisons of matrices.

Step 3. Transform the comparisons matrices of the triangular neutrosophic numbers into crisp values by using the following Eqs. (5) and (6). Then, check the (CR) for each matrix which should be less than 0.1 [33].

Score function:

$$S\left(\tilde{a}_{ij}\right) = \frac{1}{8}[a_1 + b_1 + c_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}) \quad (6)$$

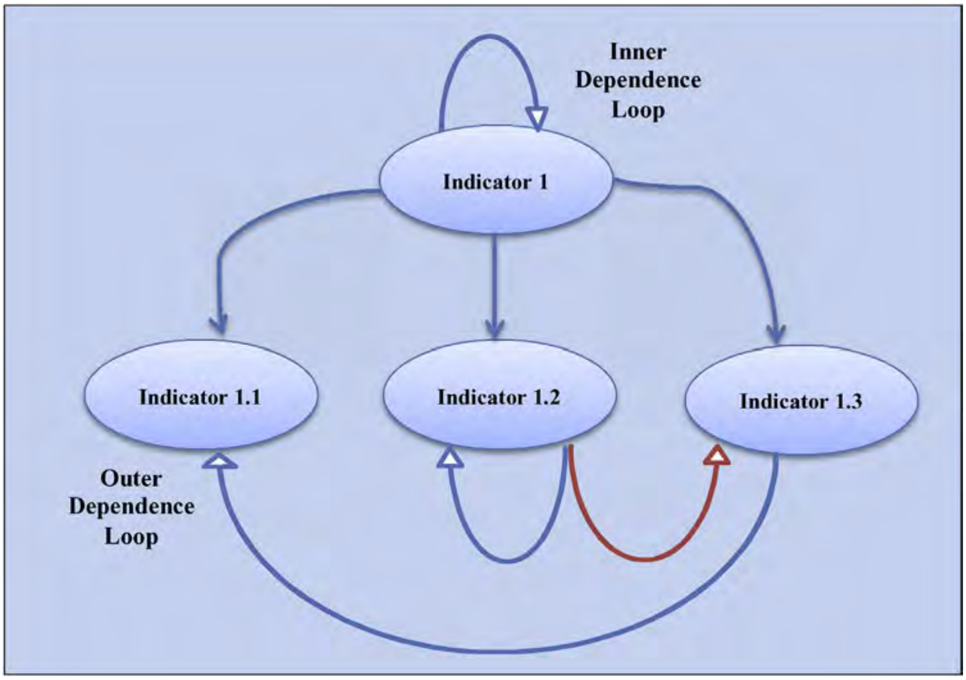

Fig. 4. Interdependencies in ANP.

Accuracy function:

$$A\left(\tilde{a}_{ij}\right) = \frac{1}{8}[a_1 + b_1 + c_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}) \quad (7)$$

Step 4. Determine weight by calculating the eigenvector of matrices which will be used in the constructing of the super matrix of interdependencies.

$$\begin{array}{c} \\ C_{11} \\ C_{21} \\ \cdots \\ \cdots \\ \cdots \\ C_{n1} \end{array} \begin{array}{c} C_1 \quad C_2 \quad \cdots \quad C_n \\ \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} & w_{11} & w_{12} & \cdots & w_{1m} & w_{11} & w_{12} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2m} & w_{21} & w_{22} & \cdots & w_{2m} & w_{21} & w_{22} & \cdots & w_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & w_{nm} & w_{n1} & w_{n2} & \cdots & w_{nm} & w_{n1} & w_{n2} & \cdots & w_{nm} \end{bmatrix} \end{array} \quad (8)$$

$$W_{local} = [W_{C_{11}}, W_{C_{12}} \ldots W_{C_{1n}}, W_{C_{21}}, W_{C_{22}} \ldots W_{C_{2n}}, W_{C_{31}}, W_{C_{32}} \ldots W_{C_{3n}}]^T \quad (9)$$

Step 5. In this step, we calculate the weights for the criteria. Firstly, to obtain the local weight for the sub criteria multiplying the weight of interdependence of criteria by the local weight obtained from experts' comparison matrices of criteria relevant to objective. Secondly, the global weight is calculated by multiplying the inner interdependent weight of the criterion to which it belongs by the local weight.

Phase 3: Rank the alternatives of the problem by using the VIKOR technique

Step 1. Every expert from three experts makes the separated evaluation matrix which consists of alternatives compared to criteria. Then, aggregate the three separated evaluation matrices by each expert into one matrix by using Eq. (10).

$$\tilde{X}_{ij} = \tilde{X}_{ij}{}^{1} + \ldots\ldots\ldots\ldots\ldots + \frac{\tilde{X}_{ij}{}^{n}}{n} \quad (10)$$

Step 2. Determine the cost attributes and the benefits attributes of the sub criteria in the problem.

Step 3. Make indexes value being dimensionless, set decision-making matrix by using Eqs. (11) and (12).

The cost type indicators are calculated as follow:

$$Z_{ij} = \left(\frac{\text{Min}\, X_{ij}}{X_{ij}}\right) \quad (11)$$

The benefit type indicators are calculated as follow:

$$Z_{ij} = \left(\frac{X_{ij}}{\text{Max}\, X_{ij}}\right) \quad (12)$$

Step 5. Calculating the positive and negative ideal solutions using the Eqs. (13) and
(14).

Calculate the best and worst values which are:

For all cost criteria, i.e. $i\ \varepsilon\ c_c$

$$A^*{}_i = \min X_{ij} \text{ and } A^-{}_i = \max X_{ij} \quad (13)$$

For all benefit criteria, i.e. $i\ \varepsilon\ c_b$

$$A^*{}_i = X_{ij} \text{ and } A^-{}_i = \min X_{ij} \quad (14)$$

The adjustment solution A^c is the practical solution that is the "relative" to the ideal A^* and adjustment means a convention determined by mutual renunciations by $\Delta A_1 = A^*{}_1 - A^c{}_1$ and $\Delta A_2 = A^*{}_2 - A^c{}_2$.

Step 6. From here, we start using the weighted which are obtained from the ANP method.

$$W_{Global} = [W_{C_{11}}, W_{C_{12}} \ldots W_{C_{1n}}, W_{C_{21}}, W_{C_{22}} \ldots W_{C_{2n}}, W_{C_{31}}, W_{C_{32}} \ldots W_{C_{3n}}]^T \quad (15)$$

And the evaluation matrix for alternatives relevant to sub criteria after applies the two equations (13, 14) for cost criteria and benefit criteria as follows:

$C_1 \quad C_2 \quad \ldots. \quad C_n$

$$\begin{matrix} A_1 \\ A_2 \\ \cdots \\ \cdots \\ \cdots \\ An \end{matrix}\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1m} & C_{11} & C_{12} & \cdots & C_{1m} & C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2m} & C_{21} & C_{22} & \cdots & C_{2m} & C_{21} & C_{22} & \cdots & C_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nm} & C_{n1} & C_{n2} & \cdots & C_{nm} & C_{n1} & C_{n2} & \cdots & C_{nm} \end{bmatrix} \quad (16)$$

Step 7. Calculate the value of S_i, R_i and Q_i

From the previous two Eqs. (11) and (12), we can obtain the following S_i and R_i which are segregation of i^{th} with the better value and the segregation of i^{th} with the worst value by two equations exhibited as the following:

$$S_i = \sum_{x=1}^{N} \sum_{y=1}^{n_N} w_{c_{xy}} \frac{d(A^*_{xy}, X_{ij})}{d(A^*_{xy}, A^-_{xy})}\Bigg) \quad (17)$$

$$R_i = \max_{xy}\left\{ w_{c_{xy}} \frac{d(A^*_{xy}, X_{ij})}{d(A^*_{xy}, A^-_{xy})}\right\} \quad (18)$$

$$Q_i = \mu \frac{S_i - S_i^-}{S_i^* - S_i^-} + (1-\mu)\frac{R_i - R_i^-}{R_i^* - R_i^-} \quad (19)$$

In Eq. (19) μ mean the weight of the strategy of the maximum group utility that equal 0.5 where S^*, S^-, R^*, R^- calculating as follows:

$$S^*_q = \max_q\{S_q\} \text{ and } S^-_q = \min_q\{S_q\} \quad (20)$$

$$R^*_q = \max_q\{R_q\} \text{ and } R^-_q = \min_q\{R_q\} \quad (21)$$

Step 8. Contemplate the suppliers

Respectively, rank the S_i, R_i and Q_i and there are two conditions to satisfy before the alternative in the first position in Q_i ranking suggested as the adjustment's solutions. There are two conditions that should be satisfied:

Case 1

$$Q\left(S^2\right) - Q\left(S^1\right) \geq \frac{1}{M-1} \quad (22)$$

In which, M is the number of alternative suppliers in the problem and S^2 in the Q ranking list means the alternative with the second position.

Case 2: agreeable persistence

In the ranking list of Q the alternative S^1 should be the superior in the S or R. Go to the extra phase to get the compromises solution, if either condition is not satisfied. When case 1 is not satisfied, the maximum values of M need to be searched with the following relationship:

$$Q\left(S^N\right) - Q\left(S^1\right) < \frac{1}{M-1} \quad (23)$$

And when case 2 is not satisfied, then both S^1 and S^2 are adjustment solutions.

Phase 4: Sorting the alternatives again using of VIKOR method combined with entropy Method.

Numerical illustration for the previous supplier selection problem:

Step 1. Calculate the T_{ij} for the matrix

$$T_{ij} = \frac{v_{ij}}{\sum_{j=1}^{m} v_{ij}} \quad (24)$$

Step 2. Calculate the entropy value t_{ij} for the matrix

$$t_{ij} = -k\sum_{i=1}^{m} t_{ij}\ln(t_{ij}) \quad (25)$$

$$K = \frac{1}{\ln(m)},\ m = \text{number of alternatives} \quad (26)$$

Step 3. Calculate the weights

Calculate the weights as the following:

$$W_j = \frac{(1-t_{ij})}{\sum_{j=1}^{n}(1-t_{ij})} \quad (27)$$

Step 4. Calculate the value of S_i, R_i and Q_i

In this step, follow step 3 to step 8 as in the previous illustrations. Then, start to sort the alternatives again after entropy method.

Step 5. Ending

Finally, the diagrammatic clarification of the offered framework is exhibited in Fig. 5.

6. The case study: results and analysis

In this section, the results are analyzed and presented as follows. The suggested structure has been applied to a real sustainable supplier selection problem of an import company.

This study has been conducted on a large importing company in Egypt. The United corporation for importing and exporting was founded in 2005 and is based in Port Said, Egypt. The corporation imports a lot of products such as hardware, electrical devices, toys, housewares et cetera from different countries. The corporation is seeking to increase the import rate and sales. The corporation is seeking to import from one of the largest countries in East Asia that is called the Asian Tigers and choose the best supplier. It must be considered the values of the society and citizenship and religious values in the products they import and be satisfaction of all citizens in terms of the social factor and be environmentally friendly products and a low financial cost affordable to citizens. Therefore, the corporation must evaluate available suppliers and their sustainability to select the best supplier to import from it. So, for this study the corporation collected information about the factors (economic, environment and social) that are considered by three experts. The experts are: strategic expert, marketing expert and Manufacturing expert with more than seven years of experience in this field. The suppliers are five and denoted by five cities respectively: A_1 Qingdao City (mainland China's base for green suppliers), A_2 Singapore, A_3 Johor Bahru (Malaysia), A_4 Taipei City and A_5 Hong Kong City. These five are considered as part of "Active Asian Economic Cities" (AAECs) due to its developed economy and active import and export businesses, particularly the establishment of green suppliers and recycling businesses.

Phase 1: Understanding of the problem.

Identify the criteria, the sub criteria and the alternatives of the available suppliers of the problem as exhibited in Fig. 6. In addition, we can determine how to apply the ANP and how criteria and sub criteria influence each other.

Phase 2: Compute the weights of the elements of the problem.

In this phase, opinions of experts presented in the comparisons matrices between criteria relevant to sub criteria and alternatives relevant to sub criteria and present the comparison matrices using of the (TRiNs) to deal with vague and incomplete information using of scales exhibited in Table 1. All the following tables from 1 to 21 presented how to calculate the weight between elements of the case study we studied (Table 2).

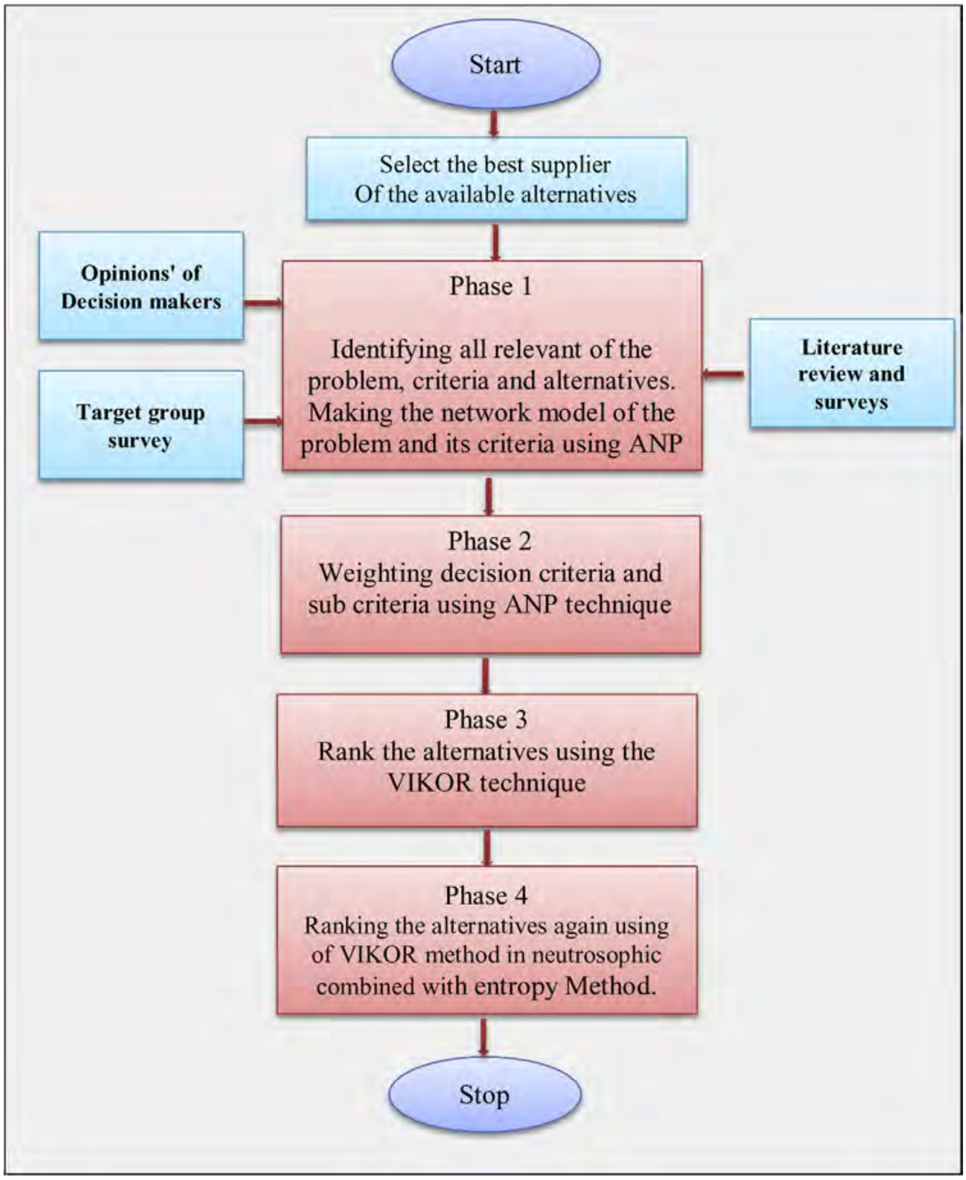

Fig. 5. The proposed framework of the model phases.

Adding the membership to values (truthiness, falsity, indeterminacy) to values that is presented in Table 3.

The next step is to convert the neutrosophic values to the crisp values in the matrix. Results are presented in Table 3 by Eq. (6). This is called the deneutrosophication function (Tables 4–10).

All the following matrices from 2 to 22 are consistent by checking the consistency ratio and the consistency ratio (CR) less than 0.1 in all matrices.

Hence, we calculated the comparative importance of the criteria on the base of their interdependence, which was calculated by using the matrix in Table 11 and the preferences of Table 4 as follows:

$$w_{factor} = \begin{bmatrix} economic\,(EC) \\ social\,(SO) \\ environment\,(EN) \end{bmatrix}$$

$$= \begin{bmatrix} 0.37 & 0.35 & 0.34 \\ 0.31 & 0.34 & 0.30 \\ 0.32 & 0.31 & 0.36 \end{bmatrix} \times \begin{bmatrix} 0.26 \\ 0.36 \\ 0.38 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0.31 \\ 0.33 \end{bmatrix}$$

From the previous matrix, it is clear that the inner interdependencies of criteria effect on its weights. It's obvious that, the weights of main criteria changed from (0.26, 0.36, and 0.38) to (0.36, 0.31, and 0.33). Therefore, when evaluating and selecting the suppliers the most significant factor is the economic (EC) criteria followed by social (SO) criteria and environmental (EN) criteria according to decision makers and experts.

Let's start the comparison matrices to calculate the local weights of sub criteria relevant to their clusters (criteria), showed in Tables 12–14 (Tables 15–20).

Economic factors (EC):

- $\mathbf{c}_{11}$ (cost of product "CP")
- $\mathbf{c}_{12}$ (Revenue on product "RP")
- $\mathbf{c}_{13}$ (Transportation cost "CO")

Social factors (SO):

- $\mathbf{c}_{21}$ (Vocational health and safety systems "VS")
- $\mathbf{c}_{22}$ (Information revelation "IR")

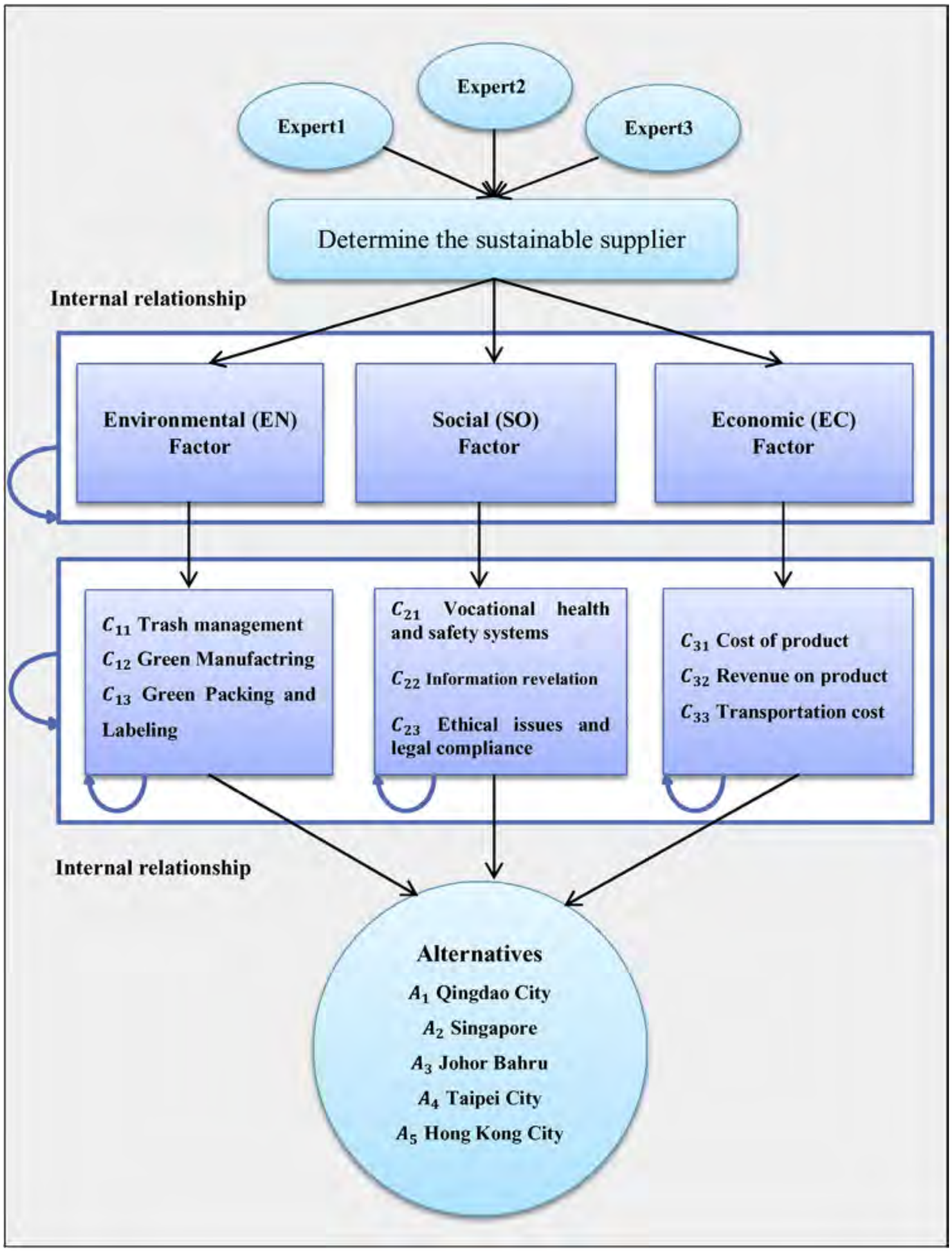

Fig. 6. Typical structure for determining the weights of sub criteria and selecting the best alternatives.

- $\mathbf{c}_{23}$ (Ethical issues and legal compliance "EL")

Environmental factors (EN):

- $\mathbf{c}_{31}$ (Trash management "TM")
- $\mathbf{c}_{32}$ (Green manufacturing "GM")
- $\mathbf{c}_{33}$ (Green packing and labeling "GL")

Concisely, the previous matrices showed the local weight of sub-criteria by inner interdependency to each criterion. So, we obtain the global weight of sub-criteria by multiplying local weight by the inner interdependent weight of the criterion showed in Table 21.

We obtain the global weight of sub criteria as follow $\boldsymbol{W_{Global}} = [0.13, 0.09, 0.14, 0.06, 0.15, 0.10, 0.16, 0.07, 0.11]^{\mathbf{T}}$. Fig. 7 shows the local and global weights of sub criteria (Tables 22 and 23).

Phase 3: Sorting alternatives of problems.

Every expert from the set of experts makes the evaluation matrix via the comparison between the five alternatives relative to each sub criteria by using the (TriNs) scale in Table 1. We can then convert every matrix into crisp value then aggregate the matrices into one matrix using Eq. (10) then obtain the matrix as exhibited in Table 24.

Establishing decision- making matrix by making indicators value being dimensionless where the following are benefits attributes the greater value being better.

- $\mathbf{c}_{11}$ (cost of product "CP")
- $\mathbf{c}_{13}$ (Transportation cost "CO")

Table 2
Pairwise comparison of factors and local weight.

Factors	Economic (EC)	Social (SC)	Environmental (EN)
Economic (EC)	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.4, 0.5, 0.6⟩⟩	⟨⟨0.9, 1.0, 1.0⟩⟩
Social (SC)	⟨⟨0.6, 0.7, 0.8⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.7, 0.8, 0.9⟩⟩
Environmental (EN)	⟨⟨⟨0.8, 0.9, 1.0⟩⟩	⟨⟨⟨0.3, 0.4, 0.5⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 3
Pairwise comparison matrix with the memberships.

Factors	Economic (EC)	Social (SC)	Environmental (EN)
Economic (EC)	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.4, 0.5, 0.6⟩; 0.7, 0.3, 0.2⟩⟩	⟨⟨⟨0.9, 1.0, 1.0⟩; 0.1, 0.2, 0.2⟩⟩
Social (SC)	⟨⟨⟨0.6, 0.7, 0.8⟩; 0.8, 0.3, 0.5⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.7, 0.8, 0.9⟩; 0.8, 0.3, 0.5⟩⟩
Environmental (EN)	⟨⟨⟨0.8, 0.9, 1.0⟩; 0.9, 0.2, 0.3⟩⟩	⟨⟨⟨0.3, 0.4, 0.5⟩; 1.0, 0.1, 0.1⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 4
Pairwise comparison of factors and local weight.

Factors	Economic (EC)	Social (SC)	Environmental (EN)	Weights
Economic (EC)	0.5	0.41	0.62	0.26
Social (SC)	0.58	0.5	0.61	0.36
Environmental (EN)	0.81	0.42	0.5	0.38

Table 5
Calculating the membership and crisp value of economic (EC) factor matrix relative to other factors.

Economic (EC) factor	Economic (EC)	Social (SC)	Environmental (EN)
Economic (EC)	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.3, 0.4, 0.5⟩; 1.0, 0.1, 0.1⟩⟩	⟨⟨⟨0.6, 0.7, 0.8⟩; 0.8, 0.3, 0.5⟩⟩
Social (SC)	⟨⟨⟨0.2, 0.3, 0.4⟩; 0.8, 0.2, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.7, 0.8, 0.9⟩; 0.8, 0.3, 0.5⟩⟩
Environmental (EN)	⟨⟨⟨0.1, 0.2, 0.3⟩; 0.5, 0.1, 0.3⟩⟩	⟨⟨⟨0.8, 0.9, 1.0⟩; 0.9, 0.2, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 6
Interdependency matrix of the main factors relative to economic (EC) factor.

Economic Factor	Economic (EC)	Social (SC)	Environmental (EN)	Weights
Economic (EC)	0.5	0.42	0.55	0.37
Social (SC)	0.26	0.5	0.60	0.31
Environmental (EN)	0.16	0.81	0.5	0.32

Table 7
Calculating the memberships and crisp value of Social (SC) factor matrix relative to other factors.

Social (SC) Factor	Economic (EC)	Social (SC)	Environmental (EN)
Economic (EC)	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.3, 0.4, 0.5⟩; 1.0, 0.1, 0.1⟩⟩	⟨⟨⟨0.8, 0.9, 1.0⟩; 09, 0.2, 0.3⟩⟩
Social (SC)	⟨⟨⟨0.6, 0.7, 0.8⟩; 0.8, 0.3, 0.5⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.7, 0.8, 0.9⟩; 0.8, 0.3, 0.5⟩⟩
Environmental (EN)	⟨⟨⟨0.4, 0.5, 0.6⟩; 0.7, 0.3, 0.2⟩⟩	⟨⟨⟨0.9, 1.0, 1.0⟩; 0.1, 0.2, 0.2⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 8
Interdependency matrix of the main factors relative to social (SC) factor.

Social (SC) Factor	Economic (EC)	Social (SC)	Environmental (EN)	Weights
Economic (EC)	0.5	0.42	0.81	0.35
Social (SC)	0.55	0.5	0.60	0.34
Environmental (EN)	0.41	0.61	0.5	0.30

Table 9
Calculating the memberships and crisp value of Environmental (EN) factor matrix relative to other factors.

Environmental (EN)Factor	Economic (EC)	Social (SC)	Environmental (EN)
Economic (EC)	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.4, 0.5, 0.6⟩; 0.7, 0.3, 0.2⟩⟩	⟨⟨⟨0.8, 0.9, 1.0⟩; 09, 0.2, 0.3⟩⟩
Social (SC)	⟨⟨⟨0.2, 0.3, 0.4⟩; 0.8, 0.2, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.8, 0.9, 1.0⟩; 0.9, 0.2, 0.3⟩⟩
Environmental (EN)	⟨⟨⟨0.7, 0.8, 0.9⟩; 0.8, 0.3, 0.5⟩⟩	⟨⟨⟨0.5, 0.6, 0.7⟩; 0.9, 0.2, 0.1⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 10
Interdependency matrix of the main factors relative to Environmental (EN) factor.

Environmental (EN)	Economic (EC)	Social (SC)	Environmental (EN)	Weights
Economic (EC)	0.5	0.41	0.81	0.34
Social (SC)	0.26	0.5	0.81	0.30
Environmental (EN)	0.60	0.59	0.5	0.36

Table 11
The comparative influence of decision criteria, EC, SC, EN.

criteria	Economic (EC)	Social (SC)	Environmental (EN)
Economic (EC)	0.37	0.35	0.34
Social (SC)	0.31	0.34	0.30
Environmental (EN)	0.32	0.30	0.36

Table 12
The comparison matrix between sub criteria of environmental (EN) criteria.

Economic (EC)Factor	C_{11} / **CP**	C_{12} / RP	C_{13} / CO
C_{11} / **CP**	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.6, 0.7, 0.8⟩⟩	⟨⟨0.9, 1.0, 1.0⟩⟩
C_{12} / RP	⟨⟨0.4, 0.5, 0.6⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.8, 0.9, 1.0⟩⟩
C_{13} / CO	⟨⟨0.7, 0.8, 0.9⟩⟩	⟨⟨0.1, 0.2, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

- c_{31} (Trash management "TM")
- c_{32} (Green manufacturing "GM")
- c_{33} (Green packing and labeling "GL")

The following are cost attributes the smaller value being better.

- c_{21} (Vocational health and safety systems "VS")
- c_{22} (Information revelation "IR")
- c_{23} (Ethical issues and legal compliance "EL")
- c_{12} (Revenue on product "RP")

Calculating the positive and negative ideal solutions

The next step is to calculate the positive and negative ideal solutions (A^*, A^-) from the previous matrix using Eqs. (13) and (14) as follows: The positive ideal solutions are A^* = (1, 1, 1, 1, 1, 1, 1, 1, and 1) and the negative ideal solutions are A^- = (0.63, 0.73, 0.57, 0.51, 0.60, 0.62, 0.62, 0.40, and 0.60) (Tables 25 and 26).

Step 6: calculating the S, R, Q for each alternative

Calculating the S, R, and Q for each alternative using Eqs. (17)–(21) as shown in the following Table 27.

In the previous Table 27 it is showed a list of alternatives Q after using the equations. From the results showed follows that the two conditions are satisfied. The first condition: Q (S^2) - Q $(S^1) \geq \frac{1}{M-1}$ that presented 0.334 – 0.051 $\geq \frac{1}{5-1}$. Then, the second condition that the first order value of Q is the first order value of R also of S as showed in Table 28.

Ranking of S for values are: $A_2 > A_5 > A_1 > A_3 > A_4$

Ranking of R for values are: $A_2 > A_3 > A_4 > A_5 > A_1$

Ranking of Q for values are: $A_2 > A_5 > A_3 > A_1 > A_4$

Hence, the final ranking of the alternatives $A_2 > A_5 > A_3 > A_1 > A_4$ **is** presented. In other words, Taipei City is considered the best supplier for importing and dealing with company compared to other competitors. The worst choice is the city of Singapore as exhibited in Fig. 8. This is identical to the reality since Singapore has no natural resources and heavily relies on

Phase 4: Sorting the alternatives again using of VIKOR method combined with entropy method.

Numerical illustration for the previous supplier selection problem are as follows:

Table 13
The comparison matrix between sub criteria of environmental (EN) criteria via using the memberships.

Economic (EC)Factor	C_{11} / **CP**	C_{12} / RP	C_{13} / CO
C_{11} / **CP**	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.6, 0.7, 0.8);0.8,0.4, 0.3⟩⟩	⟨⟨⟨0.9, 1.0, 1.0);0.1,0.2, 0.2⟩⟩
C_{12} / RP	⟨⟨⟨0.4, 0.5, 0.6);0.7,0.3, 0.2⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.8, 0.9, 1.0);0.9,0.2, 0.3⟩⟩
C_{13} / CO	⟨⟨⟨0.7, 0.8, 0.9);0.8,0.3, 0.5⟩⟩	⟨⟨⟨0.1, 0.2, 0.3);0.5,0.1, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 14
The local weight of sub criteria of environmental (EN) criteria.

Economic (EC)Factor	C_{11} / **CP**	C_{12} / RP	C_{13} / CO	Weights
C_{11} / **CP**	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.4, 0.5, 0.6);0.7,0.3, 0.2⟩⟩	⟨⟨⟨0.8, 0.9, 1.0);09,0.2, 0.3⟩⟩	0.35
C_{12} / RP	⟨⟨⟨0.2, 0.3, 0.4);0.8,0.2, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.8, 0.9, 1.0);0.9,0.2, 0.3⟩⟩	0.26
C_{13} / CO	⟨⟨⟨0.7, 0.8, 0.9);0.8,0.3, 0.5⟩⟩	⟨⟨⟨0.5, 0.6, 0.7);0.9,0.2, 0.1⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	0.39

Table 15
The comparison matrix between sub criteria of social (SO) criteria.

Social (SO)Factor	C_{21}/ VS	C_{22} / IR	C_{23} / EL
C_{21}/ VS	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.1, 0.2, 0.3⟩⟩	⟨⟨0.2, 0.3, 0.4⟩⟩
C_{22} / IR	⟨⟨0.9, 1.0, 1.0⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.8, 0.9, 1.0⟩⟩
C_{23} / EL	⟨⟨0.4, 0.5, 0.6⟩⟩	⟨⟨0.3, 0.4, 0.5⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 16
The comparison matrix between sub criteria of social (SO) criteria via using the memberships.

Social (SO)Factor	C_{21}/ VS	C_{22} / IR	C_{23} / EL
C_{21}/ VS	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.1, 0.2, 0.3);0.5,0.1, 0.3⟩⟩	⟨⟨⟨0.2, 0.3, 0.4);0.8,0.2, 0.3⟩⟩
C_{22} / IR	⟨⟨⟨0.9, 1.0, 1.0);0.1,0.2, 0.2⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.8, 0.9, 1.0);0.9,0.2, 0.3⟩⟩
C_{23} / EL	⟨⟨⟨0.4, 0.5, 0.6);0.7,0.3, 0.2⟩⟩	⟨⟨⟨03, 0.4, 0.5);1.0,0.0, 0.2⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 17
The local weight of sub criteria of social (SO) criteria.

Social (SO)Factor	C_{21}/ VS	C_{22} / IR	C_{23} / EL	Weights
C_{21}/ VS	0.5	0.16	0.26	0.22
C_{22} / IR	0.61	0.5	0.81	0.47
C_{23} / EL	0.41	0.42	0.5	0.31

Table 18
The comparison matrix between sub criteria of environmental (EN) criteria.

Environmental (EN)Factor	C_{31} / TM	C_{32} / GM	C_{33} / GL
C_{31} / TM	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.8, 0.9, 1.0⟩⟩	⟨⟨0.8, 0.9, 1.0⟩⟩
C_{32} / GM	⟨⟨0.1, 0.2, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨0.5, 0.6, 0.7⟩⟩
C_{33} / GL	⟨⟨0.6, 0.7, 0.8⟩⟩	⟨⟨04, 0.5, 0.6⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Step 1. Calculate the T_{ij} for the matrix

Calculate the T_{ij} by the using of Eq. (24) as in the following Table 29.

Step 2. Calculate the entropy value t_{ij} for the matrix

Calculate the entropy value t_{ij} for the matrix using the two Eqs. (25) and (26)

Let m = 5, then k = $\frac{1}{\ln 5}$, k = 0.621,

Then, we calculate the value of $t_{ij} = \sum_{i=1}^{m} t_{ij}\ \ln(t_{ij})$ as the following for each column as the following:

$$\sum_{i=1}^{m} t_{i1}\ln(t_{i1}) = -1.603,\ \sum_{i=1}^{m} t_{i2}\ln(t_{i2}) = -1.602,$$

$$\sum_{i=1}^{m} t_{i3}\ln(t_{i3}) = -1.589,\ \sum_{i=1}^{m} t_{i4}\ln(t_{i4}) = -1.580,$$

$$\sum_{i=1}^{m} t_{i5}\ln(t_{i5}) = -1.591,\ \sum_{i=1}^{m} t_{i6}\ln(t_{i6}) = -1.596,$$

$$\sum_{i=1}^{m} t_{i7}\ln(t_{i7}) = -1.594,\ \sum_{i=1}^{m} t_{i8}\ln(t_{i8}) = -1.564,$$

$$\sum_{i=1}^{m} t_{i9}\ln(t_{i9}) = -1.593,$$

Then, we calculate the entropy value by using Eq. (25)
$t_1 = (-0.621)(-1.603) = 0.995$, $t_2 = (-0.621)(-1.602) = 0.994$,
$t_3 = (-0.621)(-1.589) = 0.986$, $t_4 = (-0.621)(-1.580) = 0.981$,
$t_5 = (-0.621)(-1.591) = 0.988$, $t_6 = (-0.621)(-1.596) = 0.991$,
$t_7 = (-0.621)(-1.594) = 0.989$, $t_8 = (-0.621)(-1.564) = 0.971$,
$t_9 = (-0.621)(-1.593) = 0.989$,

Step 3. Calculate the weights

Calculate the weights using Eq. (27)

$w_1 = 0.043$, $w_2 = 0.052$, $w_3 = 0.121$, $w_4 = 0.164$, $w_5 = 0.103$, $w_6 = 0.078$,

$w_7 = 0.095$, $w_8 = 0.250$, $w_9 = 0.095$.

W = [0.043, 0.052 0.121, 0.164, 0.103, 0.078, 0.095, 0.250, 0.095]

Step 4. Calculate the value of S_i, R_i and Q_i

In this step, follow step 3 to step 8 as in the previous illustrations. Then, we can start to sort the alternatives again after adopting entropy method (Table 30).

Calculating the S, R, and Q for each alternative using Eqs. (17)–(21) as shown in the following Table 31 and Fig. 9 present the ranking of alternatives using of entropy method.

The previous Table 27 showed a list of alternatives Q after using the equations. From the results showed follows that the first condition is not satisfied $Q(S^2) - Q(S^1) \geq \frac{1}{M-1}$ that presented 0.191 – 0.075 $\geq \frac{1}{5-1}$. So, we achieve this condition by this Eq. $Q(S^r) - Q(S^1)$

Table 19
The comparison matrix between sub criteria of environmental (EN) criteria via using the memberships.

Environmental (EN)Factor	C_{31} / TM	C_{32} / GM	C_{33} / GL
C_{31} / TM	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.8, 0.9, 1.0⟩; 0.9 , 0.2, 0.3⟩⟩	⟨⟨⟨0.8, 0.9, 1.0⟩; 0.9 , 0.2, 0.3⟩⟩
C_{32} / GM	⟨⟨⟨0.1, 0.2, 0.3⟩; 0.5 , 0.1, 0.3⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩	⟨⟨⟨0.5, 0.6, 0.7⟩; 0.9 , 0.2, 0.1⟩⟩
C_{33} / GL	⟨⟨⟨0.6, 0.7, 0.8⟩; 0.8 , 0.4, 0.3⟩⟩	⟨⟨⟨04, 0.5, 0.6⟩; 0.7 , 0.3, 0.2⟩⟩	⟨⟨0.5, 0.5, 0.5⟩⟩

Table 20
The local weight of sub criteria of environmental (EN) criteria.

Environmental (EN)Factor	C_{31} / TM	C_{32} / GM	C_{33} / GL	Weights
C_{31} / TM	0.5	0.82	0.81	0.45
C_{32} / GM	0.16	0.5	0.59	0.24
C_{33} / GL	0.55	0.40	0.5	0.31

Table 21
The global weights of all sub-criteria.

Criteria and local weight	Sub-criteria	Local weight	Global weight
Economic (EC) criteria (0.36)	c_{11} (cost of product "CP")	0.35	0.13
	c_{12} (Revenue on product "RP")	0.26	0.09
	c_{13} (Transportation cost "CO")	0.39	0.14
Social (SO) criteria (0.31)	c_{21} (Vocational health and safety systems "VS")	0.22	0.06
	c_{22} (Information revelation "IR")	0.47	0.15
	c_{23} (Ethical issues and legal compliance "EL")	0.31	0.10
Environmental (EN) criteria (0.33)	c_{31} (Trash management "TM")	0.45	0.16
	c_{32} (Green manufacturing "GM")	0.24	0.07
	c_{33} (Green packing and labeling "GL")	0.31	0.11

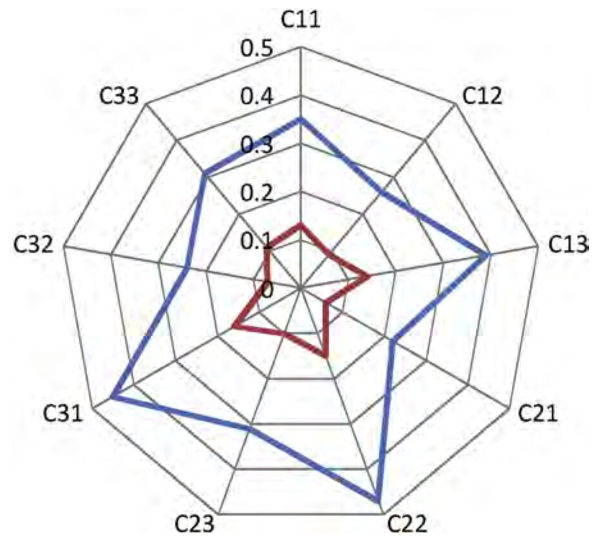

Fig. 7. Presenting the local and global weights of sub criteria.

Table 22
The evaluation matrix for alternatives with sub-criteria.

Experts	Alternatives	Sub-criteria								
		C_{11} / **CP**	C_{12} / **RP**	C_{13} / **CO**	C_{21} / **VS**	C_{22} / **IR**	C_{23} / **EL**	C_{31} / **TM**	C_{32} / **GM**	C_{33} / **GL**
Expert 1	A_1	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**SF**⟩⟩
	A_2	⟨⟨**LF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩
	A_3	⟨⟨**FHF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**HF**⟩⟩
	A_4	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**MF**⟩⟩
	A_5	⟨⟨**LF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩
Expert 2	A_1	⟨⟨**SF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**MF**⟩⟩
	A_2	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩
	A_3	⟨⟨**FHF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**LF**⟩⟩
	A_4	⟨⟨**LF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**FHF**⟩⟩
	A_5	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩
Expert 3	A_1	⟨⟨**FLF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩
	A_2	⟨⟨**SF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩
	A_3	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**FLF**⟩⟩	⟨⟨**FHF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩
	A_4	⟨⟨**SF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩
	A_5	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**MF**⟩⟩	⟨⟨**HF**⟩⟩	⟨⟨**LF**⟩⟩	⟨⟨**SF**⟩⟩	⟨⟨**SF**⟩⟩

Table 23
The crisp value for the evaluation matrix for alternatives with sub-criteria Aggregation of the evaluation matrix for the alternatives with sub criteria of the three experts using Eq. (10).

Experts	Alternatives	Sub-criteria								
		C_{11} / **CP**	C_{12} / **RP**	C_{13} / **CO**	C_{21} / **VS**	C_{22} / **IR**	C_{23} / **EL**	C_{31} / **TM**	C_{32} / **GM**	C_{33} / **GL**
Expert1	A_1	0.61	0.62	0.61	0.43	0.26	0.18	0.44	0.61	0.60
	A_2	0.18	0.26	0.44	0.43	0.61	0.57	0.60	0.81	0.62
	A_3	0.61	0.57	0.44	0.57	0.81	0.43	0.18	0.62	0.57
	A_4	0.81	0.81	0.62	0.18	0.57	0.81	0.81	0.18	0.43
	A_5	0.26	0.43	0.57	0.81	0.43	0.62	0.62	0.43	0.81
Expert 2	A_1	0.62	0.44	0.61	0.18	0.62	0.43	0.18	0.61	0.43
	A_2	0.43	0.62	0.81	0.43	0.44	0.61	0.63	0.62	0.62
	A_3	0.61	0.18	0.26	0.62	0.57	0.57	0.62	0.26	0.18
	A_4	0.26	0.43	0.57	0.61	0.81	0.62	0.43	0.44	0.61
	A_5	0.62	0.62	0.57	0.62	0.57	0.26	0.44	0.81	0.62
Expert 3	A_1	0.44	0.26	0.18	0.61	0.62	0.61	0.61	0.43	0.60
	A_2	0.60	0.61	0.57	0.81	0.26	0.44	0.18	0.43	0.81
	A_3	0.18	0.81	0.43	0.62	0.57	0.44	0.61	0.57	0.60
	A_4	0.81	0.57	0.81	0.18	0.81	0.62	0.81	0.18	0.60
	A_5	0.62	0.43	0.62	0.43	0.43	0.57	0.26	0.81	0.81

Table 24
The aggregation matrix of experts' opinions for alternatives with sub-criteria.

Aggregation matrix	Alternatives	Sub-criteria								
		C_{11} / **CP**	C_{12} / **RP**	C_{13} / **CO**	C_{21}/ **VS**	C_{22} / **IR**	C_{23} / **EL**	C_{31} / **TM**	C_{32} / **GM**	C_{33} / **GL**
Experts	A_1	0.55	0.44	0.47	0.41	0.5	0.41	0.42	0.56	0.54
	A_2	0.40	0.49	0.61	0.56	0.44	0.51	0.47	0.62	0.68
	A_3	0.47	0.52	0.38	0.60	0.65	0.48	0.47	0.48	0.45
	A_4	0.63	0.60	0.67	0.32	0.73	0.66	0.68	0.27	0.55
	A_5	0.5	0.48	0.59	0.62	0.48	0.48	0.44	0.67	0.75

Table 25
The value of the cost and the benefit attributes.

The cost and benefit attributes	Alternatives	Sub-criteria								
		C_{11} / **CP**	C_{12} / RP	C_{13} / CO	C_{21}/ VS	C_{22} / IR	C_{23} / EL	C_{31} / TM	C_{32} / GM	C_{33} / GL
	A_1	0.87	1.0	0.70	0.78	0.88	1.0	0.62	0.83	0.72
	A_2	0.63	0.89	0.91	0.57	1.0	0.80	0.69	0.93	0.90
	A_3	0.75	0.84	0.57	0.53	0.67	0.85	0.69	0.71	0.60
	A_4	1.0	0.73	1.0	1.0	0.60	0.62	1.0	0.40	0.73
	A_5	0.79	0.91	0.88	0.51	0.91	0.85	0.65	1.0	1.0

Table 26
The value of the cost and the benefit attributes and the global weight.

Sub criteria and weights	C_{11} / **CP** 0.13	C_{12} / RP 0.09	C_{13} / CO 0.14	C_{21}/ VS 0.06	C_{22} / IR 0.15	C_{23} / EL 0.10	C_{31} / TM 0.16	C_{32} / GM 0.07	C_{33} / GL 0.11
A_1	0.87	1.0	0.70	0.78	0.88	1.0	0.62	0.83	0.72
A_2	0.63	0.89	0.91	0.57	1.0	0.80	0.69	0.93	0.90
A_3	0.75	0.84	0.57	0.53	0.67	0.85	0.69	0.71	0.60
A_4	1.0	0.73	1.0	1.0	0.60	0.62	1.0	0.40	0.73
A_5	0.79	0.91	0.88	0.51	0.91	0.85	0.65	1.0	1.0

Table 27
The evaluation value of each alternative S, R, Q.

	A_1	A_2	A_3	A_4	A_5
S	0.472	0.424	0.772	0.934	0.467
R	0.16	0.13	0.14	0.15	0.15
Q	0.547	0.051	0.507	0.834	0.334

Table 28
Ranking of alternatives.

Alternatives	S	R	Q
A_1	A_2	A_2	A_2
A_2	A_5	A_3	A_5
A_3	A_1	A_4	A_3
A_4	A_3	A_5	A_1
A_5	A_4	A_1	A_4

$\geq \frac{1}{M-1}$ that presented 0.727- 0.075 $\geq \frac{1}{5-1}$. Then, the second condition that the first order value of Q is the first order value of R as showed in Table 31. Therefore, the final ranking of the alternatives $\boldsymbol{A_1} > \boldsymbol{A_2} > \boldsymbol{A_5} > \boldsymbol{A_3} > \boldsymbol{A_4}$ as exhibited in Table 32 and the two final ranking presented in Table 33.

7. Evaluation based on forecasted analysis

In this section, the objective is to forecast the future perspectives of the five "Active Asian Economic Cities" (AAECs) in places, particularly when we have changed the emphasis for the economic, social and environmental criteria. This is similar to the scenario-based prediction, since it is important for the decision-makers and policy-makers to know all possible consequences, so that the alternative recommendations can be given as soon as possible to minimize any potential loss.

The method of the predicted value is based on the development of the genetic algorithm, which can be useful to blend with deep learning, machine learning and artificial intelligence [34]. Genetic algorithm is very effective to predict the outcomes as follows.

There are inputs required: 1) the outputs of phase 1–4 in Fig. 5; 2) the mean scores of experts for all scenarios in Fig. 6; and 3) the poll of the general public with at least 300 samples per participating city. For the third category, it was taken and measured based on different websites reflecting the poll opinions of the people in the participating city. The general public reflected their opinions based on specific questions related to our research. The availability of the government data can be even more useful. However, since each city's forecasted data can vary extensively, the opinions polls on the populations of the city will be a more neutral and effective. Big data methods can analyze all the data analysis very quickly and accurately.

The term "Everythingworks" means if the algorithm can calculate the forecasted results automatically. It needs to be initialized (false) before starting. The term "results" contain the forecasted values of the five AAECs. The term "Call" is to start the forecasting process. When the forecasted simulation starts smoothly, then "results" equals to one, the algorithm begins to perform calculations, known as "Compute()", based on all input data. Sometimes errors may happen during the forecasting process. "Check()" is used to identify any errors. If no errors are found, it returns to the results. The algorithm will stop when all forecasting work has been analyzed, and update all status. Eventually, the final forecasted outputs are given.

Fig. 8. Ranking the alternatives using the ANP and VIKOR methods.

Table 29
The value of T_{ij} by entropy method.

Alternatives	Sub-criteria								
	C_{11}/ CP	**C_{12} / RP**	**C_{13} / CO**	**C_{21}/ VS**	**C_{22} / IR**	**C_{23} / EL**	**C_{31} / TM**	**C_{32} / GM**	**C_{33} / GL**
A_1	0.22	0.17	0.17	0.16	0.18	0.16	0.17	0.22	0.18
A_2	0.16	0.19	0.22	0.22	0.16	0.20	0.19	0.24	0.23
A_3	0.18	0.21	0.14	0.24	0.23	0.19	0.19	0.18	0.15
A_4	0.25	0.24	0.25	0.13	0.26	0.26	0.27	0.10	0.19
A_5	0.20	0.19	0.22	0.25	0.17	0.19	0.18	0.26	0.25

Table 30
The value of the cost and the benefit attributes and weight obtained by entropy method.

Sub criteria and weights	**C_{11}/ CP** 0.043	**C_{12}** / RP 0.052	**C_{13}** / CO 0.121	**C_{21}**/ VS 0.164	**C_{22}** / IR 0.103	**C_{23}** / EL 0.078	**C_{31}** / TM 0.095	**C_{32}** / GM 0.250	**C_{33}** / GL 0.095
A_1	0.87	1.0	0.70	0.78	0.88	1.0	0.62	0.83	0.72
A_2	0.63	0.89	0.91	0.57	1.0	0.80	0.69	0.93	0.90
A_3	0.75	0.84	0.57	0.53	0.67	0.85	0.69	0.71	0.60
A_4	1.0	0.73	1.0	1.0	0.60	0.62	1.0	0.40	0.73
A_5	0.79	0.91	0.88	0.51	0.91	0.85	0.65	1.0	1.0

Table 31
The evaluation value of each alternative S, R, Q.

	A_1	A_2	A_3	A_4	A_5
S	0.436	0.405	0.747	0.547	0.381
R	0.095	0.144	0.157	0.250	0.164
Q	0.075	0.191	0.700	0.727	0.223

For the parameters, values in Fig. 9 are used as the input values, since they can be used as the starting points for forecasting. Three factors, Economic (EC), Social (S) and Environment (EV), are used by the analysis. This paper's focus is not to demonstrate neural network training with many simulations, but the use of predictive method to identify the forecasted results, and understand the rationale and explanations behind. Three scenarios will be presented as follows.

1) Scenario 1: Economic: 30%; Social: 30% and Environment: 40%

Environmental hazards can post threats to the health of the population and increase medical burden. Therefore, more governments have spent more resources and efforts on the environments. Economic development will not go ahead if they do not pass environmental checks and evaluations. Fig. 10 shows the predicted ranking of these five AAERs. Compared to Fig. 9, Qingdao City,

Fig. 9. Ranking the alternatives using entropy method under the neutrosophic environment.

Table 32
Ranking of alternatives by entropy method.

Alternatives	S	R	Q
A_1	A_5	A_1	A_1
A_2	A_2	A_2	A_2
A_3	A_1	A_3	A_5
A_4	A_4	A_5	A_3
A_5	A_3	A_4	A_4

Table 33
Comparison of the two methods.

Method	Ranking of Alternatives
Method 1: Using of ANP and VIKOR	$\boldsymbol{A}_2 > \boldsymbol{A}_5 > \boldsymbol{A}_3 > \boldsymbol{A}_1 > \boldsymbol{A}_4$
Method 2: Using of ANP and VIKOR with entropy	$\boldsymbol{A}_1 > \boldsymbol{A}_2 > \boldsymbol{A}_5 > \boldsymbol{A}_3 > \boldsymbol{A}_4$

Singapore and Hong Kong City can perform better. Johor Bahru and Taipei City can perform lower than values in Fig. 9. The possible reasons include the facts that in some AAECs, economic improvements are based on destructing the environments. When the environmental agenda has been the center of attention, it can affect the import and export businesses. For example, the lower scores experienced in Qingdao City can be due to the trade wars with USA, and the prohibitions from the Chinese government dealing with suppliers involved in certain sectors, such as recycling businesses. As a result, some of these businesses diverted to Taipei City.

The predicted value of Taipei City stays high while taking more recycling business suppliers. However, due to the environmental awareness and restrictions, scores are lower than in Fig. 9.

Hong Kong City stays higher than in Fig. 9 because more people from mainland China and abroad visit it and it has more demands and expectations on supplier management, particularly for recycling businesses. On the other hand, cases between Johor Bahru and Singapore are different. Johor Bahru has lower taxes, legal requirements and costs to process green wastes and recycling businesses. It is also 10 km away from Singapore. Thus, it is common for Singaporean businesses to operate in Johor Bahru.

2) Scenario 2: Economic: 40%; Social: 30% and Environment: 30%

In this scenario, the emphasis is on economic development over social and environmental criteria. This is common in Asia that many AAECs focus on economic development over all other factors. Fig. 11 shows the predicted results. Comparing with Fig. 10, the values of all AAECs go up, particularly Qingdao City, with significant values exceeding 0.35. Fig. 11 is close to the situations in the economy of East Asia and Southeast Asia, since some AAECs have better economic performance due to their government's policies and focus. Scores between Hong Kong City and Qingdao City are very close. After returning to China as the motherland, Hong Kong City has better economic performances (Fig. 12).

Singapore's focus is not on supply chain management. Some Singaporean businesses man do that in Johor Bahru due to lower tax, labor fees and more relaxed laws. Hence, Singapore has a lower score and Johor Bahru has a higher score. This observation is consistent with Fig. 10. It is common for businesses to have their offices in a more economic area and their main business operations in a developing region due to the benefits of lower taxes, lower labor and material costs and less restricted laws.

Fig. 10. The predicted ranking the alternatives using entropy method under the neutrosophic environment, with environmental emphasis.

Fig. 11. The predicted ranking the alternatives using entropy method under the neutrosophic environment, with economic emphasis.

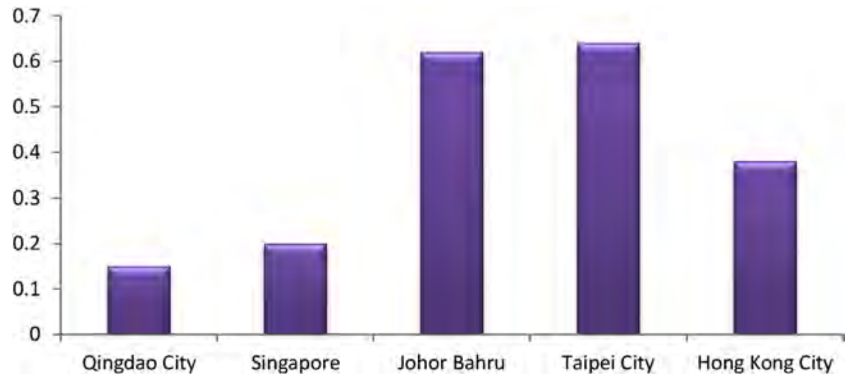

Fig. 12. The predicted ranking the alternatives using entropy method under the neutrosophic environment, with social emphasis.

3) Scenario 3: Social: 40%; Economic: 30% and Environment: 30%

The third scenario is focused on the social emphasis. It means the general public has more influences on the economic development. Fig. 11 shows Qingdao City has the lowest score, since a lot of policies are dependent on the government. Any decisions from the government can influence and even overturn the market demands and response. Singapore has the second lowest values. After the post-Lee Kuan Yew era, Singapore has become more liberal and allows free trade without government interference. However, there are still rooms for development. These may include lowering some taxes for smart manufacturing sector, since they may need green inputs and generate environmental-friendly outputs for recycling purposes.

Taipei City has the highest score due to the freedom to do free-trade with any cities. Since the decision from Qingdao City is not to take in more wastes, Taipei City has taken over recycling businesses. This is the same as Johor Bahru, the second best performer. However, despite of their higher scores, it is also partly because they have lower labor fees and transaction fees. Some international businesses seem to take more advantages of that. Additionally, Hong Kong City performs better in this aspect, as it has accelerated economic development and has strengthened its recycling businesses.

8. Discussion

8.1. Research contributions

This paper presents our neutrosophic analysis. It starts with the theory and the systematic steps to analyze. We use both the ANP-VIKOR framework and expert review, and apply social, economic and environmental factors in our analysis. We then demonstrate scores in each factor and apply it to macroeconomic cases. Furthermore, we develop a Genetic algorithm to predict the scores for five selected AAECs. By varying social, economic and environmental factors, we get different predicted outputs, which can help decision-makers to make better and more accurate decisions. We provide our rationale and explanations for scores in our predictive analysis.

8.2. Limitation of this research

While no research work can be perfect, the limitation of this research is as follows. First, our neutrosophic analysis requires the inputs from the experts before getting the required scores by hands. It is not easy to find experts meeting our requirements. Second, the forecasted results may improve its validity by having more input from the general public's polls, which tend to be harder to collect. Even if we use the web crawling method collecting and analyzing data from different sources, it may not present the best outputs.

Third, this only shows one aspects of analysis for those five AAECs. Qingdao City can perform much better in other aspects of economic development. One obvious reason for its low score is due to the trade war with USA and the prohibition of Chinese government on certain goods and regulations. If those policy-related criteria are not included, they may have different performance measurement, however, that is not the focus of this research. Similarly, even though Taipei City and Johor Bahru perform well, it is also partly because of the cheaper labor and transaction fees than their competing cities. The average salary in first-tier cities in mainland China has already surpassed the mean average salary in Taipei City. Johor Bahru also offers cheaper alternative with more options not bound by regulations than Singapore. However, the geo-economic research is not our focus. Our objective is to demonstrate that our proposed method can calculate key measurement for major import cities and provide fair and reliable forecasted outcomes.

9. Conclusion and future work

Among several problems in MCGDM, the most concerning problem is the selection process of the sustainable suppliers. As a result, we proposed a new framework involving with four phases for solving this problem. Our framework could integrate two techniques ANP and VIKOR in neutrosophic environment by using triangular neutrosophic numbers to present the linguist variables. Neutrosophic number could consider all aspects of making a decision (i.e. agree, not sure and falsity). The ANP method used to weight the elements of the problem as it considered the feedback and interdependencies. We used the VIKOR to rank alternatives to avoid comparisons in ANP. The proposed framework is suitable for implementing in real cases. respectively, we listed phases where the first phase included how to select the experts and shows that it is not an easy process, presented criteria, sub criteria and alternatives that must be identified, the inner and outer interdependencies that should be determined and the feedback. The second phase is the calculating of the weights of criteria by using of the ANP method because it considered the interdependencies and feedback between elements. The third phase, ranks the alternatives using the VIKOR method. In the last phase, comparing the result of the suggested ANP and VIKOR technique by other method, such as entropy method, to notice the difference in the results of ranking of the alternatives. Finally, the suggested framework of the ANP and VIKOR technique was used to solve a real case study about an importing corporation and select the best supplier, which depend on accurate information integrated by experts. The final ranking of alternatives showed that Taipei City has the best suppliers for imports. According to the results, Taipei City is considered the best in the manufacturing process of its products, based on the three factors: economic, environment and social. Singapore is considered the worst alternative. We also use genetic algorithm to compute predicted values for those five AAECs while varying economic, environmental and social criteria. Forecasted results show Taipei city was still considered the best option, followed by Johr Bahru. Qingdao city and Singapore are considered the worst performer. However, this research only shows one aspect of each participating city's development and supplier strategy. In summary, we successfully demonstrated our proposed work as a valid and useful method for importing cities and provide forecasted outcomes.

Future work will include the development of a more robust Genetic algorithm to predict more AAECs including Tokyo, Shanghai, Seoul and other cities specializing in imports and exports. More cases will be analyzed and discussed to gain a deeper understanding on social, economic and environmental developments in AAECs. We plan to develop our work so that it can be applied to more economically active cities and offer more accurate analysis and forecasted outputs.

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Entropy of Polysemantic Words for the Same Part of Speech

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ABSTRACT In this paper, a special type of polysemantic words, that is, words with multiple meanings for the same part of speech, are analyzed under the name of neutrosophic words. These words represent the most difficult cases for the disambiguation algorithms as they represent the most ambiguous natural language utterances. For approximate their meanings, we developed a semantic representation framework made by means of concepts from neutrosophic theory and entropy measure in which we incorporate sense related data. We show the advantages of the proposed framework in a sentiment classification task.

INDEX TERMS Neutrosophic sets, semantic word representation, sentiment classification.

I. INTRODUCTION

Every natural language word can have multiple realisations from the part-of-speech point of view, and for each of its possible parts-of-speech, it can have multiple meanings (especially the English words). Each sense creates a ‘‘subdimension’’ in the word's space determined by the part-of-speech to which it belongs in the given statement. The polysemantic words (words with multiple senses) can be described by several spaces (one space for each possible part-of-speech) and each space can include several subspaces determined by the meanings the word can have. In this manner, every dimension describes a certain facet of the analysed word. It is also true that certain senses are more frequent than others and in this manner they can force a certain facet to be more prominent than others.

We need a comprehensive and unitary study for natural language words formulated as a Multicriteria Decision Making problem [1] in which uncertainty is inevitably involved due to the subjectivity of humans [2]. It has been shown that different senses of the same word usually imply different sentiment orientations for the word under analysis. For instance, the word ‘‘good’’: in ‘‘good man’’ produces a positive utterance while in ‘‘good fight’’ indicates a negative statement. As a direct consequence we need studies that address both the interaction between word sense disambiguation and sentiment analysis. These are quite new studies in the literature as the researchers in this area must be intrigued by the usability of sense level information in sentiment analysis. Some researchers take this approach and compute the polarity score for each word sense [3], [4].

The present paper proposes a novel approach for word sentiment classification by extracting a set of semantic data from the SentiWordNet in order to compute a final estimation of the word polarity. SentiWordNet [3], [5] is a well-known freely available lexical resource for sentiment analysis which annotates each sense of a word with three polarity scores. These polarity scores represent the positivity, objectivity and negativity degrees of the annotated word sense ranging from 0 to 1 with their sum up to one. SentiWordNet (SWN) was built on the semantically-oriented WordNet [6], [7], which in its primary form, that is for English language, comprises 155287 words and 117659 senses.

There are two main approaches for sentiment analysis: machine learning and knowledge-based. From the machine learning perspective, the Support Vector Machines (SVM) classification (see, for example, [8], [9]) has the best classification performance for sentiment analysis [10], [11] outperforming both the Naïve Bayes and Maximum entropy classification methods. The knowledge-based methods usually make use of the most common sense of the words and in this manner an improvement of accuracy over the baseline was observed [12]. Also, the overall polarities of different senses in each part-of-speech tag categories are also determined [13]. However, the commonly used n-gram features are not robust enough and show widely varying behaviour across different domains [14].

The method we propose in this paper offers a knowledge-based solution for semantic word representation which targets sentiment classification and makes use of the general concepts of neutrosophic theory and entropy measure. A previous study that applies neutrosophic theory in sentiment analysis domain is given in [15]. In this paper we concentrate our approach by keeping in mind only the most difficult cases for sentiment classification. They are represented by a special class of polysemantic words with different meanings for the same part-of-speech realisation. In the present paper these words are named *neutrosophic words* because their representation involves the core concepts of neutrosophic theory.

With this article we are in line with the neutrosphic word representations firstly proposed in [16] and then refined in [17] in which the SentiWordNet (shortly SWN) sentiment scores are interpreted as truth-fullness degrees. The study proposed in this paper also makes use of the SWN polarity scores of each word's sense, this time in order to determine the overall sentiment score value. The involved computations apply entropy on the words' sentiment scores as a measure of disorder for the words' polarities.

The paper is organised as follows: the Related Works section overviews the most commonly used multi-space representation techniques in neutrosophy. Section III presents the proposed semantic-level representation which treats the words as union of neutrosophic sets. In Section IV we show how this type of representation can be used in conjunction with a sentiment analysis study. Section V exemplifies all the involved theoretical concepts on a study case also providing the obtained results and the last section is dedicated to the conclusions and our future directions.

II. RELATED WORKS

The concept of multi-space was introduced by Smarandache in 1969 [18] by following the idea of hybrid mathematics - especially hybrid geometry [19], [20] for combining different fields into a unifying field [21]–[24].

Let Ω be a universe of discourse and a subset $S \subseteq \Omega$. Let $[0, 1]$ be a closed interval and three subsets $T, I, F \subseteq [0, 1]$. Then, a relationship of an element $x \in S$ with respect to the subset S is $x(T, I, F)$, which means the following: the *confidence set* of x is T, the *indefinite set* of x is I, and the *failing set* of x is F. A set S, together with the corresponding three subsets T, I, F for each element x in S, is said to be a *neutrosophic set* [19], [25].

Let Σ be a set and $A_1, A_2, \ldots, A_k \subseteq \Sigma$. Define $3k$ functions $f_1^z, f_2^z, \ldots, f_k^z$ by $f_i^z : A_i \to [0, 1]$, $1 \leq i \leq k$, where $z \in \{T, I, F\}$. If we denote by $(A_i; f_i^T, f_i^I, f_i^F)$ the subset A_i together with three functions f_i^T, f_i^I, f_i^F, $1 \leq i \leq k$, then [19]:

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F)$$

is a union of neutrosophic sets which are generalizations of classical sets.

Indeed, if we take $f_i^T = 1, f_i^I = f_i^F = 0$ for $i = \overline{1, k}$ we obtain [19]:

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F) = \bigcup_{i=1}^{k} A_i$$

and correspondingly, for $f_i^T = f_i^I = 0, f_i^F = 1$, $i = \overline{1, k}$ we obtain the complementary sets [19]:

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F) = \overline{\bigcup_{i=1}^{k} A_i}$$

The appurtenance and non-appurtenance is obtained if there is an integer s such that $f_i^T = 1, f_i^I = f_i^F = 0$, $1 \leq i \leq s$, but $f_j^T = f_j^I = 0, f_j^F = 1$, $s + 1 \leq j \leq k$.

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F) = \bigcup_{i=1}^{s} A_i \cup \overline{\bigcup_{i=s+1}^{k} A_i}$$

The general neutrosophic set is obtained if there is an integer l such that $f_l^T \neq 1$ for $1 \leq l \leq s$, or $f_l^F \neq 1$ for $s < l \leq n$. The resulted union cannot be represented by abstract sets.

III. SEMANTIC-LEVEL REPRESENTATION FOR WORDS

As we have already pointed out in the Introduction section, a word is not a simple data, it can have several (syntactic) attributes and can support more than one semantic interpretations. Metaphorically speaking a word is like a diamond: it can brighter a life or, by contrary, it can cut and destroy. But, from our study point of view, a word is just an entity that can have multiple semantic facets.

As we have already pointed out, a word can have more than one part-of-speech, like the word "good" which can be adjective, noun or adverb and to which we dedicate an extensive study in the Section V. There are programs that can automatically identify the part-of-speech of a certain word in a given context. These programs are named Part-Of-Speech Taggers and for most of the languages their accuracy is quite high (more than 90%).

On contrary, determining the meaning of a polysemous word in a specific context - that is, performing a disambiguation on the word's senses, can be a laborious task. In spite of the great number of existing disambiguation algorithms, the problem of word sense disambiguation remains an open one [26]. For some languages like English the accuracy of the disambiguation algorithms does not overcome 75%.

It is obviously that we need to model indeterminacy in the semantic word representations. This is the reason why, in the present study we choose to model word representations using neutrosophic theory as, in contrast to intuitionistic fuzzy sets and also interval valued intuitionistic fuzzy sets, indeterminacy degree of an element is explicitly expressed by the neutrosophic sets [27]. Moreover, in [29] the authors state that single valued neutrosophic (SVN) set is a better tool to deal with incomplete, inconsistent and indeterminate information than fuzzy set (FS) and intuitionistic FS (IFS). With the present study we are in line with these assumptions continuing also our previous works in which the natural language words are modelled as single-valued neutrosophic sets in order to approximate their ambiguous meaning [16], [17].

In the representation we propose in this paper a word can have multiple dimensions organised on several plans:

- the POS plans are determined by the possible part of speech data of the word
- each POS plan can have several sense units, determined by the possible word's senses under that POS data
- finally, each sense unit is made of some components (sentiment scores) which describe the sense meaning polarity

A. WORDS AS UNION OF NEUTROSOPHIC SETS

The first step in creating a semantic representation is to decide what features to use and how to encode these features. From the features set a word can have, in this study we consider the part-of-speech as the syntactic feature and the word's sense(s) as its semantic interpretation(s).

In what follows, we name semantic facets or simply facets - all the word's data based on which the semantic interpretation can be defined. Using concepts from neutrosophic sets theory [30] we propose the following semantic representation of a word.

Definition 1: The semantic representation of a word by means neutrosophic theory concepts is defined as:

$$w = \bigcup_{i=1}^{k} (sense_i; f_i^T, f_i^I, f_i^F)$$

where:

- k represents the number of senses the word can have
- $f_i^T, f_i^I f_i^F$: $Facets \rightarrow [0, 1]$ are the membership functions for the $sense_i$, $i = \overline{1, k}$, such that:
  - f_i^T represents the membership degree,
  - f_i^I represents the indeterminacy degree and
  - f_i^F is the degree of nonmembership degree
- *Facets* set includes all the data that characterise the word from the semantic point of view.

In this assertion, a word becomes a union of neutrosophic sets. For the ith sense of the word w, the membership functions of the word's semantic facets fulfil the following properties:

$$\forall x \in Facets : f_i^T(x) + f_i^I(x) + f_i^F(x) = 1 \tag{1}$$

and if $Facets = \{x_1, \ldots, x_m\}$ then:

$$\sum_{j=1}^{m} f_i^T(x_j) + f_i^I(x_j) + f_i^F(x_j) = m \tag{2}$$

In order to include the information about the part-of-speech data (shortly POS data) we need to refine the representation given in Definition 1. We consider the general case in which a word can have n possible parts of speech $POS_1, \ldots, POS_n$, with $n \geq 1$, and for each part of speech POS_j the word can have k_j senses, $k_j \geq 1$. The representation given in Definition 1 becomes:

$$w = \bigcup_{i=1}^{k_1} (sense_{i;POS_1}; f_{i;POS_1}^T, f_{i;POS_1}^I, f_{i;POS_1}^F) \cup \ldots$$
$$\ldots \cup \bigcup_{i=1}^{k_n} (sense_{i;POS_n}; f_{i;POS_n}^T, f_{i;POS_n}^I, f_{i;POS_n}^F) \tag{3}$$

Using the representation given in Equation 3, the senses corresponding to a certain part of speech POS_j with $j \in \{1, \ldots n\}$, can be obtained as follows:

$$(w)_{POS_j} = w \cap (w)_{POS_j} = \bigcup_{i=1}^{k_j} (sense_{i;POS_j}; f_{i;POS_j}^T, f_{i;POS_j}^I, f_{i;POS_j}^F) \tag{4}$$

Furthermore, we can apply another filtering on word representation given in Equation 4 if we consider the case in which a specific sense of the word w results to be realised in a given context. Let us note this sense with $sense_{m;POS_j}$ with $m \in \{1, k_j\}$. By applying concepts from neutrosophic sets theory we obtain:

$f_{m;POS_j}^T = 1$, $f_{m;POS_j}^I = f_{m;POS_j}^F = 0$ and $f_{l;POS_j}^T = f_{l;POS_j}^I = 0$, $f_{l;POS_j}^F = 1$ for $l \neq m$, $l, m = \overline{1, k_j}$

which implies:

$$\begin{aligned}(w)_{POS_j} &= \bigcup_{i=1}^{k_j} (sense_{i;POS_j}; f_{i;POS_j}^T, f_{i;POS_j}^I, f_{i;POS_j}^F) \\ &= (sense_{m;POS_j}; 1, 0, 0) \cup \bigcup_{l \neq m} (sense_{l;POS_j}; 0, 0, 1) \\ &= sense_{m;POS_j} \cup \overline{\bigcup_{l \neq m} sense_{l;POS_j}} \\ &= sense_{m;POS_j} = (m\text{-}th\ sense\ of\ w)_{POS_j}\end{aligned} \tag{5}$$

The representation given in Equation 5 corresponds to the most unambiguous case, more precisely to the situation in which we know both the word's part of speech (noted here with POS_j) and the word sense (noted with $sense_{m;POS_j}$).

But, the problems with natural language processing comes from ambiguity - when we could not identify (using automatic tools) which sense is realised in the given context from the set of the word's possible senses (noted here with $\cup_{i=1}^{k_j} sense_{i;POS_j}$). This ambiguity case is depicted by the general case given in Equation 3.

In what follows we will use a simplified form of Equation 3 in which POS_j data is removed from the annotations sequences corresponding to the senses and membership functions. Thus, Equation 3 becomes:

$$(w)_{POS_j} = \bigcup_{i=1}^{k_j} (sense_i; f_i^T, f_i^I, f_i^F) \tag{6}$$

In the next section we present a method by means of which we can eliminate the "noises" from an ambiguous semantic word representation, more precisely, a representation that includes more than one possible sense. We resolve these issues using Neutrosophic Theory and Entropy measure. Our proposal is described in conjunction with a sentiment analysis study in which the semantic word representation has the form of a three sentiment scores tuple.

IV. WORD SEMANTIC REPRESENTATION WITH SENTIMENT SCORES

Sense discrimination addresses words with multiple senses and is done in conjunction with a particular context in which only one sense is realised. This analysis has a semantic nature and is quite difficult to perform it using automatic tools, especially if the realisation context is poor in information that could filter the correct word meaning from the set of possible ones. In order to exemplify our proposal we choose to interpret the word semantics from a sentiment analysis the point of view. Thus, each sense of a word will be represented using its sentiment scores.

In what follows, let us consider the approach firstly proposed in [16] and then extended in [17] in which a word w is interpreted as a single-value neutrosophic set constructed upon its sentiment scores which describe the word's sense-level polarity information being denoted in what follows with (sc_+, sc_0, sc_-), where:

- sc_+ denotes the word positive score,
- sc_0 represents the word neutral score and
- sc_- stands for the word negative score.

As in [16] and [17] we use SentiWordNet lexical resource for providing the required information for the sentiment scores of the English words.

For a word w with k_j senses under a POS_j part-of-speech realisation, the semantic representation is defined as a union of the tuples: $sense_i = (sc_{+_i}, sc_{0_i}, sc_{-_i})$ with $i \in \{1, \ldots, k_j\}$. The Equation 6 becomes:

$$(w)_{POS_j} = \bigcup_{i=1}^{k_j}((sc_{+_i}, sc_{0_i}, sc_{-_i}); f_i^T, f_i^I, f_i^F) \quad (7)$$

with $sc_{+_i}, sc_{0_i}, sc_{-_i} \in [0, 1]$. The semantic representation given in Equation 7 implies that each word's sense will include three facets: the positive, the neutral and the negative one. By preserving the notation where $+$ stands for positive, 0 for neutral and $-$ for negative facet, we take $Facets = \{+, 0, -\}$.

The representation given in Equation 7 can be rewritten as:

$$(w)_{POS_j} = \bigcup_{i=1}^{k_j}((sc_{+_i}, sc_{0_i}, sc_{-_i}); (\{f_i^T(x)\}_{x \in Facets}), (\{f_i^I(x)\}_{x \in Facets}), (\{f_i^F(x)\}_{x \in Facets})) \quad (8)$$

where $f_i^T(x)$, $f_i^I(x)$ and $f_i^F(x)$ represents the membership functions corresponding to the facet x of the ith sense, $x \in Facets$ and $(\{f_i^M(x)\}_{x \in Facets})$ briefly notes the membership functions $\begin{pmatrix} f_i^M(+) \\ f_i^M(0) \\ f_i^M(-) \end{pmatrix}$, $M \in \{T, I, F\}$.

Remark: For the representation given in Equation 8, the default case corresponds to the maximum certainty case where no imprecision occurs which, in terms of membership function is depicted by $f_i^T(\{+ \mid 0 \mid -\}_i) = 1, f_i^I(\{+ \mid 0 \mid -\}_i) = 0, f_i^F(\{+ \mid 0 \mid -\}_i) = 0$, $i = \overline{1, k_j}$.

We preface the study that addresses the multi-facets words by enumerating the form in which the *one facet words* are represented in our proposal. These words are the extreme cases of our study and every neutrosophic study provides them.

Case 1: If $sc_{+_i} = 1$, $sc_{0_i} = sc_{-_i} = 0$ and $f_i^T(\{+ \mid 0 \mid -\}_i) = 1, f_i^I(\{+ \mid 0 \mid -\}_i) = 0, f_i^F(\{+ \mid 0 \mid -\}_i) = 0$ for every $i = \overline{1, k_j}$ then:

$$(w)_{POS_j} = \bigcup_{i=1}^{k_j}\left((1, 0, 0); \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) = (1, 0, 0)$$

The interpretation of Case 1 is: for all the senses corresponding to the POS_j part-of-speech the word w is *pure positive*.

Case 2: If $sc_{+_i} = sc_{0_i} = 0$, $sc_{-_i} = 1$ and $f_i^T(\{+ \mid 0 \mid -\}_i) = 1, f_i^I(\{+ \mid 0 \mid -\}_i) = 0, f_i^F(\{+ \mid 0 \mid -\}_i) = 0$ for every $i = \overline{1, k_j}$ then:

$$(w)_{POS_j} = \bigcup_{i=1}^{k_j}\left((0, 0, 1); \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) = (0, 0, 1)$$

The interpretation of Case 2 is: for all the senses corresponding to the POS_j part-of-speech the word w is *pure negative*.

Case 3: If $sc_{+_i} = sc_{-_i} = 0$, $sc_{0_i} = 1$ and $f_i^T(\{+ \mid 0 \mid -\}_i) = 1, f_i^I(\{+ \mid 0 \mid -\}_i) = 0, f_i^F(\{+ \mid 0 \mid -\}_i) = 0$ for every $i = \overline{1, k_j}$ then:

$$(w)_{POS_j} = \bigcup_{i=1}^{k_j}\left((0, 1, 0); \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) = (0, 1, 0)$$

The interpretation of Case 3 is: for all the senses corresponding to the POS_j part-of-speech the word w is *pure neutral*.

These three cases correspond to the non-ambiguous words, that is, words with a unique sense (one semantic representation) or similar semantic representations for all of their possible senses.

Since in a natural language there are many words (especially in English) with multiple senses - the *polysemantic words*, in what follows we will concentrate our study only on these words. For the polysemantic words we get different semantic representations that must be resolved by dealing with many degrees of uncertainties. In this case, the simple reunion of their semantic dimensions is a general neutrosophic set that cannot be formalised using abstract set theories. For this reason, in the next definition we introduce the concept of *neutrosophic word* in conjunction with a sentiment analysis.

Definition 2: A *neutrosophic word* is a polysemantic word that under the same part of speech realization has at least two different sentiment polarities which means:

$(\exists(w)_{POS_j}$ with $k_j > 1$ senses) $\wedge$ $(\exists i_1, i_2 \in \{1, \ldots, k_j\}, i_1 \neq i_2$: $sense_{i_1} \neq sense_{i_2})$

Different sense tuples imply different sentiment scores and we obtain:

$(\exists(w)_{POS_j}$ with $k_j > 1$ senses) $\wedge$ $[\exists i_1, i_2 \in \{1, \ldots, k_j\}, i_1 \neq i_2$: $(sc_{+_{i_1}}, sc_{0_{i_1}}, sc_{-_{i_1}}) \neq (sc_{+_{i_2}}, sc_{0_{i_2}}, sc_{-_{i_2}})]$

As a direct consequence, the semantic representation of neutrosophic words is:

$$(w)_{POS_j} = \bigcup_{i \in \{i_1, i_2, \ldots\}} ((sc_{+_i}, sc_{0_i}, sc_{-_i}); (\{f_i^T(x)\}_{x \in Facets}), (\{f_i^I(x)\}_{x \in Facets}), (\{f_i^F(x)\}_{x \in Facets}))$$

with $sc_{+_{i_1}} \neq sc_{+_{i_2}}$ or $sc_{0_{i_1}} \neq sc_{0_{i_2}}$ or $sc_{-_{i_1}} \neq sc_{-_{i_2}}$ and $f_{i_1}^T(\{+ \mid 0 \mid -\}), f_{i_2}^T(\{+ \mid 0 \mid -\}) > 0$, $i_1 \neq i_2$. By the fact that the membership degrees are greater than 0, we obtain for a neutrosophic word *w the necessity of having (at least) two different sentiment representations* for the same $(w)_{POS_j}$.

The neutrosophic theory means from the very beginning dealing with uncertainty. This is also true for the neutrosophic words. These words can be evidenced in case of an imprecise disambiguation mechanism which fails in recognising what sense is realised in the given context even if the part-of-speech data is correctly provided.

In our approach, a neutrosophic word is synonym with a word that has different sense facets and for which we cannot establish a unique semantic representation. For the chosen sentiment analysis exemplification, different sense facets for a word means different sentiment scores tuples.

In the next section we exemplify how the proposed method works. We show that using the neutrosophic sets theory and applying the entropy measure on the word representations we can identify the word's sentiment facet with respect to the given part-of-speech.

A. ENTROPY AS A MEASURE OF UNCERTAINTY FOR THE NEUTROSOPHIC WORDS REPRESENTATIONS

Fuzzy entropy, distance measure and similarity measure are three basic concepts used in fuzzy sets theory [27]. Among them, Entropy is an efficient tool to model uncertainty [28] or, in layman terms, Entropy is a measure of disorder. It can be used in order to measure how disorganised an input values set is by calculating the entropy of their values/labels. Entropy is high if the input values are highly varied and low if many input data have the same value. In mathematical terms, Entropy is defined as the sum of the probability of each input values or labels times the log probability of that label:

$$E(labels) = -\sum_{l \in labels} P(l) log_2 P(l) \tag{9}$$

where $P(l)$ is the frequency probability of the *label* item in the considered data and *labels* denotes the set of possible labels.

From this definition we obtain that labels with low frequency do not affect much the entropy (because $P(l)$ is small). The same result for labels with high frequency as in their case, $log_2 P(l)$ is small. Only when the inputs have wide varieties of labels (and as a direct consequence, these many labels have a medium frequency) the entropy is high because neither $P(l)$ nor $log_2 P(l)$ is small.

Entropy has values between 0 and 1 and high entropy values stand for high levels of disorder or ''low level of purity''. Following this property, we can qualify the uncertainty of the words' semantic nature by applying the entropy measure on their sense representation labels: *the higher the values for entropy measure the higher the level of uncertainty for the analyzed word representations*.

The neutrosophic word is a concept with more than one possible sense for at least one of its possible part-of-speech data. On the other hand, entropy is a measure of uncertainty. Between the possible senses we can have certain similarity degrees and the entropy measure can be used in order to determine how similar or dissimilar these senses are.

The most common manner to unify a set of possible representations into a single one is to consider only the maximum (or the minimum) value or to average the values (in our case, the sentiment scores) as in the following formula:

$$Avg\left(\bigcup_{i=1}^{k_j}(sc_{+_i}, sc_{0_i}, sc_{-_i})\right) = \left(\frac{1}{k_j}\sum_{i=1}^{k_j} sc_{+_i}, \frac{1}{k_j}\sum_{i=1}^{k_j} sc_{0_i}, \frac{1}{k_j}\sum_{i=1}^{k_j} sc_{-_i}\right) \tag{10}$$

where k_j notes the number of senses for the analysed word. But this method of unifying different representation can be trustful only if the averaged values are not very dissimilar with the initial ones.

Example 1: Let us consider a word *w* with two extreme sentiment scores tuples: (0, 0, 1) and (1, 0, 0). Overall, we obtain two different facets: in the first representation we have a *pure positive word* while in the second we get a *pure negative word*. If we merge these two representation by averaging their sentiment scores values we get (0.5, 0, 0.5) - a representation that could be interpreted as a *neutral word*. Definitely this would be a wrong classification for a strong sentiment word.

We define a bijective mapping for labelling the sentiment score values to a set of three strength degrees, $SD = \{low, medium, high\}$. We obtain $sd : [0, 1] \rightarrow SD$ with:

$$sd(score) = \begin{cases} low, & \text{if } score < 0.4 \\ medium, & \text{if } score \in [0.4, 0.6] \\ high, & \text{if } score > 0.6 \end{cases}$$

Mapping the score values to the *SD* labels we get ''low'' label for a small score, ''medium'' for not a small but also not a high score and ''high'' for a big score. Using these strength degrees we can qualify by means of the entropy measure calculated as in Equation 9 how disorganised the scores values are from the point of view of the sentiment strength. All the involved operations are given in Algorithm 1.

Algorithm 1 Merging Multiple Semantic Representations of a Neutrosophic Word $(w)_{POS_j}$

INPUT: $\cup_{i=1}^{k_j}(sc_{+_i}, sc_{0_i}, sc_{-_i})$
for each x in Facets:
  $Entropy(x) \leftarrow E(\cup_{i=1}^{k_j} sd(sc_{x_i}))$
  $Avg(x) \leftarrow Avg(\cup_{i=1}^{k_j} sc_{x_i}) \leftarrow \frac{1}{k_j}\sum_{i=1}^{k_j} sc_{x_i}$
  $f^T(x) \leftarrow 1 - Entropy(x)$
  $f^I(x) \leftarrow Entropy(x)$
  $f^F(x) \leftarrow 0$
endfor
OUTPUT: $\cup_{x\in Facets} Avg(x), f^T(x), f^I(x), f^F(x)$

We can now give the manner in which the multiple representations of a neutrosophic word $(w)_{POS_j}$ can be unified into a unique sentiment representation $Avg(w)_{POS_j}$ based on the values provided by Algorithm 1:

$$Avg((w)_{POS_j}) = Avg\left(\bigcup_{i=1}^{k_j}\left((sc_{+_i}, sc_{0_i}, sc_{-_i}); \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right)\right)$$
$$= \left(\left(Avg(\cup_{i=1}^{k_j} sc_{+_i}), Avg(\cup_{i=1}^{k_j} sc_{0_i}), Avg(\cup_{i=1}^{k_j} sc_{-_i})\right); (\{f^T(x)\}_{x\in Facets}), (\{f^I(x)\}_{x\in Facets}), (\{f^F(x)\}_{x\in Facets})\right) \quad (11)$$

In Algorithm 1 we model the degrees of trustfulness for the resulted average scores representation by means of the membership functions, such that $\forall x \in Facets$:

- If the entropy $Entropy(x)$ is small (the minimum value is 0) then the average value $Avg(x)$ can approximate with high degree of certainty the initial word's sentiment scores; in this case the membership function for the facet x is set to a big value (almost 1) as $f^T(x) \leftarrow 1 - Entropy(x)$.
- If the entropy $Entropy(x)$ is high (the maximum value is 1) then the membership function is set to a small value (almost 0) while the indeterminacy degree $f^I(x)$ is set to be equal with the entropy function value.
- For preserving the sum of the membership functions to value 1 (see Equation 1), the nonmembership degree $f^F(x)$ for the facet x is always 0.

For the case given in **Example 1** we obtain that the entropy corresponding to the positive and negative scores is equal to its maximum value: $E(+) = E(-) = 1$, while the entropy for the neutral scores is zero. The resulted average representation can be written as follows:

$$Avg(w) = ((Avg(\cup_{i=1}^{2} sc_{+_i}), Avg(\cup_{i=1}^{2} sc_{0_i}), Avg(\cup_{i=1}^{2} sc_{-_i})); f^T, f^I, f^F)$$
$$= ((0.5, 0, 0.5); f^T, f^I, f^F)$$
$$= \left((0.5, 0, 0.5); \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \quad (12)$$

The representation given in Equation 12 tells more about *what the word is not* than about the type the word *is* as we consider **Example 1** only for showing why the simple unification of multiple representations by averaging their values is not always enough. As one can observe, the representation given in Equation 12 tells with maximum certainty that the word is not a neutral word. For the obtained positive and negative scores the indeterminacy membership functions have maximum values, illustrating in this way a maximum indeterminacy degree. This extreme case is quite rarely, being presented only for its theoretical purpose.

In the next section we apply the proposed method on a real data: a neutrosophic word in its all possible parts of speech. With this complex case we show that the method described in this article succeeds in merging multiple and diverse semantic word representations.

V. STUDY CASE

The word "good" appears in WordNet with three different parts of speech (noun, adjective, and adverb) and with many senses for each of its syntactic labels. We consider this word represents a perfect example for the neutrosophic word concept introduced in this paper and for this reason we dedicate the study case to it.

In Table 1 are given all the senses the word "good" can have, grouped upon the part-of-speech data. Each sense is given together with the sentiment scores extracted from SentiWordNet and also with its definition and some examples (as they are given in WordNet).

In Table 2 we gather all the data extracted from SentiWordNet: the word's parts of speech, the three facets given by the corresponding sentiment scores and the distributions among the senses of the sentiment scores. We also give the entropy measures for each word's facet in all the three parts of speech and also the average values of the sentiment scores.

By applying Algorithm 1 on the SentiWordNet scores of the word "good" we obtain the following representations (see also Table 2):

$$Avg((good)_{ADJ}) = \left(\left(Avg(\cup_{i=1}^{21} sc_{+_i}), Avg(\cup_{i=1}^{21} sc_{0_i}), Avg(\cup_{i=1}^{21} sc_{-_i})\right); f^T_{ADJ}, f^I_{ADJ}, f^F_{ADJ}\right)$$
$$= \left((0.61, 0.38, 0); f^T_{ADJ}, f^I_{ADJ}, f^F_{ADJ}\right)$$
$$= \left((0.61, 0.38, 0); \begin{pmatrix}0.59\\0.59\\1\end{pmatrix}, \begin{pmatrix}0.41\\0.41\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \quad (13)$$

$$Avg((good)_{NOUN}) = \left(\left(Avg(\cup_{i=1}^{4} sc_{+_i}), Avg(\cup_{i=1}^{4} sc_{0_i}), Avg(\cup_{i=1}^{4} sc_{-_i})\right); f^T_{NOUN}, f^I_{NOUN}, f^F_{NOUN}\right)$$
$$= \left((0.5, 0.5, 0); f^T_{ADJ}, f^I_{ADJ}, f^F_{ADJ}\right)$$
$$= \left((0.5, 0.5, 0); \begin{pmatrix}0.25\\0.25\\1\end{pmatrix}, \begin{pmatrix}0.75\\0.75\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \quad (14)$$

TABLE 1. The SentiWordNet data for the word "good".

POS	Sentiment Scores	Example
Noun	(0.5, 0.5, 0)	benefit; "for your own good"; "what's the good of worrying?"
	(0.875, 0.125, 0)	moral excellence or admirableness; "there is much good to be found in people"
	(0.625, 0.375, 0)	that which is pleasing or valuable or useful; "weigh the good against the bad"; "among the highest goods of all are happiness and self-realization"
	(0, 1, 0)	articles of commerce
ADV	(0.375, 0.625, 0)	in a good or proper or satisfactory manner or to a high standard; "the baby can walk pretty good"
	(0, 1, 0)	completely and absolutely; "he was soundly defeated"; "we beat him good"
ADJ	(0.75, 0.25, 0)	having desirable or positive qualities especially those suitable for a thing specified; "good news from the hospital"; "a good report card"
	(0, 1, 0)	having the normally expected amount; "gives full measure"; "gives good measure"
	(1, 0, 0)	morally admirable
	(1, 0, 0)	deserving of esteem and respect; "ruined the family's good name"
	(0.625, 0.375, 0)	promoting or enhancing well-being; "the experience was good for her"
	(1, 0, 0)	agreeable or pleasing; "we all had a good time"; "good manners"
	(0.75, 0.25, 0)	of moral excellence; "a genuinely good person"
	(0.625, 0.375, 0)	having or showing knowledge and skill and aptitude; "a good mechanic"
	(0.625, 0.375, 0)	thorough; "had a good workout"; "gave the house a good cleaning"
	(0.5, 0.5, 0)	with or in a close or intimate relationship; "a good friend"
	(0.5, 0.5, 0)	financially sound; "a good investment"
	(0.375, 0.625, 0)	most suitable or right for a particular purpose; "a good time to plant tomatoes"
	(0.625, 0.375, 0)	resulting favorably; "it's a good thing that I wasn't there"; "it is good that you stayed"
	(0, 1, 0)	exerting force or influence; "a warranty good for two years"
	(0.625, 0.375, 0)	capable of pleasing; "good looks"
	(0.75, 0.25, 0)	appealing to the mind; "good music"
	(0.75, 0.25, 0)	in excellent physical condition; "good teeth"; "I still have one good leg"
	(0.875, 0.125, 0)	tending to promote physical well-being; beneficial to health; "a good night's sleep"
	(0.5, 0.5, 0)	not forged; "a good dollar bill"
	(0.375, 0.5, 0.125)	not left to spoil; "the meat is still good"
	(0.75, 0.25, 0)	generally admired; "good taste"

$$Avg((good)_{ADV}) = \left(\left(Avg(\cup_{i=1}^{2} sc_{+_i}), Avg(\cup_{i=1}^{2} sc_{0_i}), Avg(\cup_{i=1}^{2} sc_{-_i})\right); f_{ADV}^{T}, f_{ADV}^{I}, f_{ADV}^{F}\right) = \left((0.18, 0.81, 0); f_{ADV}^{T}, f_{ADV}^{I}, f_{ADV}^{F}\right) = \left((0.18, 0.81, 0); \begin{pmatrix}0.59\\0.59\\1\end{pmatrix}, \begin{pmatrix}0.41\\0.41\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right) \quad (15)$$

These results can be interpreted as follows: no matter its part of speech realisation, we can precisely say that the word "good" is NOT a negative word. Two possible facets remain:

TABLE 2. The semantic representations of the word "good". The negative scores, being not representative (the greatest value is 0.12), are omitted in the listing.

	Adjective			Noun			Adverb		
	sc_+	sc_0	#senses	sc_+	sc_0	#senses	sc_+	sc_0	#senses
Sent. Scores	0.75	0.25	5	0.5	0.5	1			
	0	1	2				0.37	0.62	1
	1	0	3	0.87	0.12	1			
	0.62	0.37	5						
	0.5	0.5	3	0.62	0.37	1			
	0.37	0.62	1				0	1	1
	0.87	0.12	1	0	1	1			
	0.37	0.5	1						
Entropy	0.41	0.41		0.75	0.75		0	0	
Avg Scores	0.61	0.38		0.5	0.5		0.18	0.81	

the positive and the neutral. From the results obtained in Equations 13 and 15 we can conclude:

- the word "good" as adverb is a neutral word because its neutral average score is 0.81 with $f_{ADV}^{T}(0) = 0.59$, a value that exceeds by far its positive average score (0.18 with $f_{ADV}^{T}(+) = 0.59$)
- the word "good" as adjective is a positive word because its positive average score is 0.61 with $f_{ADJ}^{T}(+) = 0.59$ while the neutral average score is only 0.38, with $f_{ADJ}^{T}(0) = 0.59$

As a noun, we can consider it positive or neutral word, in both cases with high indeterminate degrees: $f_{NOUN}^{I}(+) = f_{NOUN}^{I}(0) = 0.75$, its average positive and neutral scores equal with 0.5 (see Equation 14). This is the case when additional filters taken from the context in which the word occurs must be applied in order to establish the word semantic facet.

VI. CONCLUSION AND FUTURE WORK

As pointed out in [31] each object has a corresponding (fuzzy, intuitionistic fuzzy, or neutrosophic) degree of appurtenance to a set of classification classes, with respect to its attributes' values.

In the present paper we propose a method that determines the appurtenance degrees of the semantic facets of a natural language word based on the entropy measure. We apply the proposed method on a real data: a polysemantic word in its all possible parts of speech. We prove with this complex study case that the method succeeds in merging multiple and diverse semantic word representations by filtering the "noises" through the entropy function values. The proposed method can be improved in case of high entropy values when additional filters must be applied by taken into account the word contextual data. The developing of these additional filters represents the trigger of our future studies.

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Neutrosophic Triplet Group (revisited)

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Abstract. We have introduced for the first time the notion of neutrosophic triplet since 2014, which has the form (x, neut(x), anti(x)) with respect to a given binary well-defined law, where neut(x) is the neutral of x, and anti(x) is the opposite of x. Then we define the neutrosophic triplet group (2016), prove several theorems about it, and give some examples. This paper is an improvement and a development of our 2016 published paper.
Groups are the most fundamental and rich algebraic structure with respect to some binary operation in the study of algebra. In this paper, for the first time, we introduced the notion of neutrosophic triplet, which is a collection of three elements that satisfy certain axioms with respect to a binary operation. These neutrosophic triplets highly depend on the defined binary operation. Further, in this paper, we used these neutrosophic triplets to introduce the innovative notion of neutrosophic triplet group, which is a completely different from the classical group in the structural properties. A big advantage of neutrosophic triplet is that it gives a new group (neutrosophic triplet group) structure to those algebraic structures, which are not group with respect to some binary operation in the classical group theory. In neutrosophic triplet group, we apply the fundamental law of Neutrosophy that for an idea A, we have the neutral of A denoted as neut(a) and the opposite of A dented as anti(A) to capture this beautiful picture of neutrosophic triplet group in algebraic structures. We also studied some interesting properties of this newly born structure. We further defined neutro-homomorphisms for neutrosophic triplet groups. A neutro-homomorphism is the generalization of the classical homomorphism with two extra conditions. As a further generalization, we gave rise to a new field or research called Neutrosophic Triplet Structures (such as neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, etc.). In the end, we gave main distinctions and comparison of neutrosophic triplet group with the Molaei's generalized group as well as the possible application areas of the neutrosophic triplet groups. In this paper we improve our [13] results on neutrosophic triplet groups.

Keywords: Groups, homomorphism, neutrosophic triplet, neutrosophic triplet group, neutro-homomorphism t.

1 Introduction

Neutrosophy is a new branch of philosophy that studies the nature, origin and scope of neutralities as well as their interaction with ideational spectra. Florentin Smarandache [8] in 1995, first introduced the concept of neutrosophic logic and neutrosophic set where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic especially of the intuitionistic fuzzy logic. In fact neutrosophic set is the generalization of classical sets[9], fuzzy set[12], intuitionistic fuzzy set[1,9], and interval valued fuzzy set[9] etc. This mathematical tool is used to handle problems consisting uncertainty, imprecision, indeterminacy, inconsistency, incompleteness and falsity. By utilizing the idea of neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache studied neutrosophic algebraic structures in [4,5,6] by inserting an indeterminate element "I" in the algebraic structure and then combine "I" with each element of the structure with respect to corresponding binary operation *. They call it neutrosophic number { $a + bI$, with a, b real numbers, and I = literal indeterminacy, $I^2 = I$ } and the generated algebraic structure is then termed as neutrosophic algebraic structure. They further study several neutrosophic algebraic structures such as neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

Groups [2,3,11] are very important in algebraic structures because they play the role of a backbone in almost all algebraic structures theory. Groups are thought as old algebra due to its rich structure than any other notion. In many algebraic structures, groups provide concrete foundation such as, rings, fields, vector spaces, etc. Groups are also important in many other areas like physics, chemistry, combinatorics, biology etc. to study the symmetries and other behavior among their elements. The most important aspect of a group is group action. There are many types of groups, such as: permutation groups, matrix groups, transformation groups, Lie-groups etc. that are highly used as a practical perspective in our daily life. Generalized groups [7] are important in this aspect.

In this paper, for the first time, we introduced the idea of neutrosophic triplet. The newly born neutrosophic triplets are highly dependable on the proposed binary operation. These neutrosophic triplets have been discussed by Smarandache and Ali in Physics [10]. Moreover, we used these neutrosophic triplets to introduce neutrosophic triplet group, which is different from the classical group both in structural and foundational properties from all aspects. Furthermore, we gave some interesting and fundamental properties and notions with illustrative examples. We also introduced a new type of homomorphism called as neutro-homomorphism, which is in fact a generalization of the classical homomorphism under some conditions. We also study neutro-homomorphism for neutrosophic triplet groups. The rest of the paper is organized as follows. After the literature review in section 1, we introduced neutrosophic triplets in section 2. Section 3 is dedicated to the introduction of neutrosophic triplet groups with some of its interesting properties. In section 4, we developed neutro-homomorphism and in section 5, we gave distinction and comparison of neutrosophic triplet group with the Molaei's generalized group. We also draw a brief sketch of the possible applications of neutrosophic triplet group in other research areas. Conclusion is given in section 6.

2 Neutrosophic Triplet

Remark 2.1. All below theorems and propositions in a Neutrosophic Triplet Set (NTS) and Neutrosophic Triplet Group (NTG) are true when the multipliers are non-zero and cancellable multipliers.

An element $a \in (S, *)$, where * is a binary law, *is cancellable to the left* if:

$\forall\, b, c \in S$, from $a*b = a*c$ one gets only $b = c$.

The element a is *cancellable to the right* if:

$\forall\, b, c \in S$, from $b*a = c*a$ one gets only $b = c$.

And, the element a is *cancellable (in general)* if the element a is both cancelable to the left and to the right.

Definition 2.1.1. Let N be a set together with a binary operation *. Then N is called a *neutrosophic triplet set* if for any $a \in N$, there is a neutral of "a" called $neut(a)$, different from the classical algebraic unitary element, and an opposite of "a" called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a$$

and

$$a * anti(a) = anti(a) * a =$$

The elements a, $neut(a)$, and $anti(a)$ are collectively called as neutrosophic triplet and we denote it by (a, $neut(a)$, $anti(a)$). By $neut(a)$, we mean $neutral$ of a and apparently, a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet.

For the same element *a* in *N*, there may be more neutrals to it *neut(a)* and more opposites of it *anti(a).*

Remark 2.2

If a well-defined binary law * on the set *N* has a classical algebraic unitary element *e* in *N*, then no other triplet of the form *(e, b, c)* can be formed, except the *(e, e, e),* i.e. when *b* = *c* = *e*, which is not accepted as neutrosophic triplet.

Consequently, the set *(N, *)* with a classical unitary element cannot be a neutrosophic triplet set.

Remark 2.2.

It is important that there are at least two different neutral elements with respect to all set elements into a neutrosophic triplet set.

Definition 2.1.3. A *Zero Neutrosophic Triplet* on the neutrosophic triplet set N, is a neutrosophic triplet of the form *(0, 0, a),* where *0, a* ∈ *N* {of course, the triplet *(0, 0, a)* must satisfy the axioms of the neutrosophic triplet}.

Example 2.1.3.1. Let N=be a set with respect to multiplication $\times$ modulo 10 in $a \in N, 6 \times a = a \times 6 = a(mod\ 10)$.
It should be remarked, to this example, that *6* is a classical algebraic unitary element on *N,* with respect to the multiplication $\times$ modulo *10*, because for any $a \in N$, $6 \times a = a \times 6 = a$ *(mod 10)*.
But *6* cannot be a neutral for the element $0 \in N$, because *(0, 6, ?)* cannot form a neutrosophic triplet since there is no *anti(0)* such that:
$0 \times anti(0) = anti(0) \times 0 = 6$.
Therefore, the neutrosophic triplets of *0* [called *Zero Neutrosophic Triplets*] are
(0, 0, 0), (0, 0, 2), (0, 0, 4), (0, 0, 6), (0, 0, 8).
N is not a neutrosophic triplet set since, except element *0*, the other elements *2, 4, 6,* and *8* do not have neutral elements different from the classical unitary element *6.*
Theorem 2.1. Let *N* be a set endowed with the binary law *, which is well-defined and has the classical algebraic unitary element $e \in N$,

$$\forall x \in N,\ e * x = x * e = x.$$

If *(e, b, c)* is a neutrosophic triplet, with $b, c \in N$, then $b = c = e$.
{In other words, if a set *N* has a classical algebraic unitary element *e,* with respect to the binary well-defined law *, then the only neutrosophic triplet of *e* is *(e, e, e),* which is mutually called trivial neutrosophic triplet, the only triplet that makes exception from the definition of neutrosophic triplets.}
***Proof*.**
Let *(e, b, c)* be a neutrosophic triplet. Since *neut(e)* = *b,* one has:
$e*neut(e) = neut(e)*e = e$,
but $e*b = b$ and $b*e = b$ too (since *e* is the classical algebraic unitary element on the set *N*),
whence $b = e$.
And, because *anti(e)* = *c,* one has:
$e*c = c*e = e$,
but $e*c = c$ and $c*e = c$ too (since *e* is the classical algebraic unitary element on the set *N*),
whence $c = e$.
Therefore, the only triplet of the classical algebraic unitary (identity) element is
(e, e, e), but it cannot be considered a neutrosophic triplet.
Definition 2.2: The element b in (N, *) is the second component , denoted as $neut(\cdot)$ of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c) .
Definition 2.3: The element c in $(N,*)$ is the third component, denoted as $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c).
Example 2.2. Consider Z under multiplication modulo 6 , where

$$Z_6 = \{0,1,2,3,4,5\}$$

The classical unitary element is *e = 1.*
Then 2 gives rise to a neutrosophic triplet because $neut(2) = 4$, as $2 \times 4 = 8$. Also $anti(2) = 2$ because $2 \times 4 = 4$. Thus $(2,4,2)$ is a neutrosophic triplet. Similarly 4 gives rise to a neutrosophic triplet because $neut(4) = anti(4) = 4$. So $(4,4,4)$ is a neutrosophic triplet. 3 has two neutrals, $neut(3) = \{3, 5\}$, and forms one neutrosophic triplet (3, 3, 3), but 3 does not give rise to a neutrosophic triplet for $neut(3) = 5$ since $anti(3)$ does not exist in Z_6 for this neutral,
5 has no *neut(5)* so no neutrosophic triplet related to *5*, and last but not the least 0 gives rise to a trivial neutrosophic triplet as $neut(0) = anti(0) = 0$. The zero neutrosophic triplets are denoted by(0,0,0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (0, 0, 4), (0, 0, 5).
Z_6 is not a neutrosophic set, since *1* and *5* have no corresponding neutrosophic triplets, but
$M_6 = \{0, 2, 3, 4\} \subset Z_6$ *is a commutative neutrosophic group* [whose definition will be provided below].

Theorem 2.3. If $(a, neut(a), anti(0))$ form a neutrosophic triplet, then
1. $(anti(a), neut(a), a)$ also form a neutrosophic triplet, and similarly

2. $(neut(a), neut(a), neut(a))$ form a neutrosophic triplet.

Proof. We prove both 1 and 2.
1. Of course, $anti(a) * a = neut(a)$.

We need to prove that:

$$anti(a) * neut(a) = anti(a)$$

Multiply by a to the left and we get:
$a * anti(a) * neut(a) = a * anti(a)$ Or
$[a * anti(a)] * neut(a) = neut(a)$Or

$$neut(a) * neut(a) = neut(a)$$

Again multiply by a to the left and we get:

$$a * neut(a) * neut(a) = a * neut(a)$$

Or

$$[a * neut(a)] * neut(a) = a$$

Or

$$a * neut(a) = a$$

2. To show that $(neut(a), neut(a), neut(a))$ is a neutrosophic triplet, it results from the fact that

$$neut(a) * neut(a) = neut(a).$$

3 Neutrosophic Triplet Group

Definition 3.1: Let (N,*) be a neutrosophic triplet set (which includes the trivial neutrosophic triplet too, if any). Then N is called a neutrosophic triplet group, if the following conditions are satisfied.
1) If (N,*) is well-defined, i.e. for any $a, b \in N$, one has $a * b \in N$.
2) If (N,*) is associative, i.e. (a*b)*c= a*(b*c) for all $a, b, c \in N$.

The neutrosophic triplet group, in general, is not a group in the classical algebraic way.

We consider, as the neutrosophic neutrals replacing the classical unitary element, and the neutrosophic opposites as replacing the classical inverse elements.

Example 3.2. *Consider*$(Z_{10}, \#)$*, where # is defined as* $a\#b = 3ab$

Let $M_{10} = \{0, 2, 4, 5, 6, 8\} \subset Z_{10}$. *Then* $(M_{10}, \#)$ *is a neutrosophic triplet group under the binary #.*
It is also associative, i.e.

(a#b)#c=a#(b#c).

Now take L. H. S to prove the R. H. S, so

a#(b#c)=3ab#c.

,

3(3ab)c=9abc,
3a(3bc)=3a(b#c),
a#(b#c).

The classical unitary element on Z_{10} with respect to the law # is $e = 7$, *since:*

$a \# e = e \# a = 3ae = 3a(7) = 21a = a \pmod{10}$ for any $a \in Z_{10}$.

Therefore, we choose all triplets whose neutral elements are different from *7*, and we get the following neutrosophic triplets:

(0, 0, 0), (0, 0, 2), (0, 0, 4), (0, 0, 5), (0, 0, 6), (0, 0, 8), (2, 2, 2), (4, 2, 6), (5, 5, 5), (6, 2, 4), and *(8, 2, 8).*

All above neutrals *neut(.) = 0, 2,* and *5* are different from the classical unitary element *7.*

Z_{10} is not a neutrosophic triplet group, nor even a neutrosophic triplet set.

But its subset $M_{10} = \{0, 2, 4, 5, 6, 8\}$ is a commutative neutrosophic triplet group, since the law # is well-defined, commutative, associative, and each element belonging to *M* has a corresponding neutrosophic triplet.

Definition 3.3: Let $(N,*)$ be a neutrosophic triplet group. Then N is called a commutative neutrosophic triplet group if for all $a, b \in N$ we have $a * b = b * a$.

Example 3.4. Consider *(M, *)*, where $M = \{0, 1\}$, and the binary law * is defined as

$a*b = a + b - ab \ (mod\ 4)$ for all $a, b \in M$.

Then *(M, *)* is a not a neutrosophic triplet group, not even a neutrosophic triplet set.

Proof.

The law * has a classical algebraic unitary element $e = 0$, since:

For any $a \in M$, $a*0 = 0*a = a + 0 - a \times 0 = a \ (mod\ 4)$.

Therefore, (M, *) cannot be a neutrosophic triplet group.

Theorem 3.5. Every idempotent element gives rise to a neutrosophic triplet.

Proof. Let a be an idempotent element. Then by definition $a^2 = a$. Since $a^2 = a$, which clearly implies that $neut(a) = a$ and $anti(a) = a$. Hence a gives rise to a neutrosophic triplet (a, a, a) .

Theorem 3.6. There are no neutrosophic triplets in Z_n with respect to multiplication modulo n if n is a prime, except the zero neutrosophic triplets $(0, 0, 0), (0, 0, 1), \ldots, (0, 0, n\text{-}1)$.

Proof. It is obvious. The multiplication modulo n is well-defined, associative, and commutative.

For $n = 2$ (even prime), $Z_2 = \{0, 1\}$ has the classical algebraic unitary element, with respect to multiplication modulo 2, $e = 1$, and Z_2 has the zero neutrosophic triplets $(0, 0, 0), (0, 0, 1)$.

Whence Z_2 is a not a neutrosophic triplet group, not even a neutrosophic triplet set.

Let $Z_n = \{0, 1, 2, \ldots, n\text{-}1\}$, for n odd prime. The classical algebraic unitary element of Z_n is 1, and the zero neutrosophic triplets are $(0, 0, 0), (0, 0, 1), \ldots, (0, 0, n\text{-}1)$.

Let's compute the neutral of $2 \le p \le p - 1$, if any, let $neut(p) = x$. We need to find x.

$px = p \ (mod\ n)$, or $px - p = 0 \ (mod\ n)$, or $p(x\text{-}1) = 0 \ (mod\ n)$,

whence $x\text{-}1 = 0 \ (mod\ n)$ since n is an odd prime, and n and p are relatively prime numbers,

or $x = 1 \ (mod\ n)$, therefore there is no neutral of the elements $p \in \{2, 3, \ldots, n\text{-}1\}$, since 1 is excluded as classical algebraic unitary element. Thus no neutrosophic triplets corresponding to the elements $2, 3, \ldots, n\text{-}1 \in Z_n$.

Remark 3.6.1. Let $(N,*)$ be a neutrosophic triplet group under $*$ and let $a \in N$. Then $neut(a)$ is not the same for all elements in N (as in classical group), but $neut(a)$ depends on the a and on the operation $*$.

(In example 3.8, $neut(0) = 0$, $neut(4) = 4$, $neut(8) = 4$ and $neut(9) = 9$, so we have three different neutral elements: 0, 4, and 9.)

Theorem number 3.6.2. Let $(N,*)$ be a neutrosophic triplet group under $*$ that satisfies the cancellation law for all its elements. Then, for any a in N, the $neut(a)$ and $anti(a)$ are unique and depend on a.

Proof: Suppose $neut^{(1)}(a)$ and $neut^{(2)}(a)$ be two neutrals of a.

Then since $neut^{(1)}(a)*a = neut^{(2)}(a)*a$, and by cancellation law to the right-hand side, we have $neut^{(1)}(a) = neut^{(2)}(a)$.

Similarly, if $a*neut^{(1)}(a) = a*neut^{(2)}(a)$, by cancellation law to the left-hand side, we have $neut^{(1)}(a) = neut^{(2)}(a)$.

In the same way, since $anti^{(1)}(a)*a = anti^{(2)}(a)*a$, we get $anti^{(1)}(a) = anti^{(2)}(a)$, by cancellation law to the right-hand side.

Again, since $a*anti^{(1)}(a) = a*anti^{(2)}(a)$, we get $anti^{(1)}(a) = anti^{(2)}(a)$, by cancellation law to the left-hand side.

Theorem 3.6.3. If the elements a of NTG do not satisfy the cancellation law, then still for each a in NTG the $neut(a)$ is unique and depending on a, but the $anti(a)$ may not be unique.

Proof.

Let's suppose that $neut^{(1)}(a)$ and $neut^{(2)}(a)$ are two neutrals of a, we have

$$\begin{aligned}
neut^{(1)}(a) &= a*anti^{(1)}(a) = (neut^{(2)}(a)*a)*anti^{(1)}(a)\\
&= neut^{(2)}(a)*(a*anti^{(1)}(a))\\
&= neut^{(2)}(a)*neut^{(1)}(a)\\
&= (anti^{(2)}(a)*a)*neut^{(1)}(a)\\
&= anti^{(2)}(a)*(a*neut^{(1)}(a))\\
&= anti^{(2)}(a)*a\\
&= neut^{(2)}(a).
\end{aligned}$$

Yet, $anti(a)$ is not unique.

To prove this, let's take a look at the following example.

Example 3.8. Let N=(0,4,8) be a commutative neutrosophic triplet group under multiplication modulo 12 in

(Z_{12}, N). N does not have a classical unitary element.
Then $neut(4) = 4$, $neut(8) = 4$ and $anti(0) ==$ {0,4,8,9}, neut(0) = 0 and *anti(0) = {0, 4, 8, 9}*. This shows that $neut(a)$ is not the same for all elements as in classical group theory; further, each element has only one neutral. Also, the element *0* has four *anti(0)'*s.
The neutrosophic triplets are: *(0, 0, 0), (0, 0, 4), (0, 0, 8), (0, 0, 9), (4, 4, 4), (8, 4, 8), (9, 9, 9).*
Remark 3.9. Let $(N,*)$ be a neutrosophic triplet group with respect to * and let $a \in N$. Then $anti(a)$ is not the same for all elements in N and also anti(N) depends on the element a and on the operation *, and some elements may have many anti's unlike classical group and generalized group.
To prove the above remark, let's take a look to the following example.
Example 3.10. Let N be the commutative neutrosophic triplet group in the above Example 3.8. Then *anti(0) = {0, 4, 8, 9}, anti(4)=4* , anti(8)=8 and anti(9)=9. Therefore, the element 0 has four anti's, and the anti's of the elements are different from each other: 4, 8, 9, and 0.
Proposition 3.11. Let (N, *) be a neutrosophic triplet group with respect to * and let
A
$a, b, c \in N$. Then
1) a*b=a*c if and only if neut(a)*b=neut(a)*c.
2) b*a = c*a if and only if b*neut(a)=c*neut(a).
Proof. 1. Suppose that a*b=a*c. Since N is a neutrosophic triplet group, so $anti(a) \in N$. Multiply anti(a) to the left side of a*b=a*c.

anti(a)*a*b=anti(a)*a*c
[anti(a)*a]*b=[anti(a)*a]*c
neut(a)*b=neut(a)*c

Conversely suppose that neut(a)*b=neut(a)*c.
Multiply a to the left side, we get:

$$[a * neut(a)] * b = [a * neut(a)] * c.$$
$$a * b = a * c$$

2. The proof is similar to 1.
Proposition 3.12. Let (N,*) be a neutrosophic triplet group with respect to * and let
$a, b, c \in N$.
1) If anti(a)*b=anti(b)*c, then neut(a)*b=neut(a)*c.
2) If b*anti(a), then c*anti(a), then b*neut(a)=c*neut(a).
Proof. 1-. Suppose that anti(a)*b=anti(a)*c. Since N is a neutrosophic triplet group with respect to *, so $a \in N$. Multiply a to the left side of anti(a)*b=anti(a)*c, we get:

a*anti(a)*b=a*anti(a)*c
[a*anti(a)]*b=[a*anti(a)]*c
neut(a)*b=neut(a)*c.

2. The proof is the same as (1).
Theorem 3.13. Let (N,*) be a commutative neutrosophic triplet group with respect to * and $a, b, n \in N$. Then

neut(a)*neut(b)=neut(a*b).

Proof. Consider left hand side,neut(a)*neut(b)=neut(a*b).
Now multiply to the left with a and to the right with b , we get:

a*neut(a*b)*b=[a*b]*[neut(a*b)], as * in asociative
=a*b.

Now consider right hand side, we have neut(a*b).
Again multiply to the left with a and to the right with b, we get:

a*neut(a*b)*b , as is associative,
=a*b.

This completes the proof.
Theorem 3.14. Let (N,*) be a commutative neutrosophic triplet group with respect to * and $a, b \in N$. Then

anti(a)*anti(b)=anti(a*b).

Proof. Consider left hand side, anti(a)*anti(b).
Multiply to the left with a and to the right with b, we get:

a*anti(a)*anti(b)*b=[a*anti(a)]*[anti(b)*b]
=neut(a)*neut(b)
=neut(a*b), from the above theorem.

Now consider right hand side, which is anti(a*b).
Multiply to the left with a and to the right with b, we get:

a*anti(a*b), since * is associative.

neut(a*b).

This shows that anti(a)*anti(b) is true for all $a, b \in N$.

Theorem 3.15. Let (N,*) be a commutative neutrosophic triplet group under * and $a, b \in N$. Then

1) neut(a)*neut(b)=neut(b)*neut(a).
2) anti(a)*anti(b)=anti(b)*anti(a).

Proof 1. Consider right hand side neut(b)*neut(a). By Theorem 3, we have

neut(b)*neut(a)=neut(b)*neut(a), as N is commutative,

neut(a)*neut(b), again by Theorem 3.

Hence neut(a)*neut(b)=neut(b)*neut(a).

2) On similar lines, one can easily obtained the proof of (2).

{Actually, both proofs could also result straightforwardly from the commutative property of the neutrosophic triplet group.}

Definition 3.16. Let (N,*) be a neutrosophic triplet group under * and let H be a subset of N. Then H is called a neutrosophic triplet subgroup of N if H itself is a neutrosophic triplet group with respect to*.

Proposition 3.18. Let (N,*) be a neutrosophic triplet group and let $a \in N$ be a subset ofN. Then H is a neutrosophic triplet subgroup of N if and only if the following conditions hold.

1) $a * b \in H$ for all $a, b \in H$.
2) $neut(a) \in H$ for all $a \in H$.
3) $anti(a) \in H$ for all $a \in H$.

Proof. The proof is straightforward.

Definition 3.19. Let N be a neutrosophic triplet group and let $a \in N$. A smallest positive integer $n \geq 1$ such that is called neutrosophic triplet order {with respect to a given *neut(a)*, when the case when there are many neutrals of a}. It is denoted by nto(a).

Theorem 3.21. Let (N,*) be a neutrosophic triplet group with respect to * and let $a \in N$. Then

1) neut(a)*neut(a)=neut(a).

In general $(neut(a))^2 = neut(a)$, where n is a non-zero positive integer.

2) neut(a)*anti(a)=anti(a)*neut(a)=anti(a).

(Theorem 3.21 (1) and (2) were proven in Theorem 2.3. (1) and (2), except $(neut(a))^n = neut(a)$.)

Proof. Consider neut(a)*neut(a)=neut(a).
Multiply a to the left side, we get;

a* neut(a)*neut(a)=a*neut(a)

[a* neut(a)]*neut(a)=[a*neut(a)]

a*neut(a)=a

a=a.

On the same lines, we can see that $(neut(a))^2 = neut(a)$ for a non-zero positive integer n.

2) Consider neut(a)*anti(a)=anti(a).
Multiply to the left with a, we get;

a* neut(a)*anti(a)=a*anti(a)

a* anti(a)=neut(a)

neut(a)=neut(a)

a=a.

Similarly anti(a)*neut(a)=anti(a).

Definition 3.22. Let N be a NTG. If *N* = <*a*> for some *a* ∈ *N*, then *N* is called a neutro-cyclic triplet group" is better for the definition.

We say that a is a generator part of the neutrosophic triplet group.

Theorem 3.24. Let N be a neutro-cyclic triplet group and let a be a generator part of the neutrosophic triplet. Then

1) $\langle neut(a)\rangle$ generates neutro-cyclic triplet subgroup of N.
2) $\langle anti(a)\rangle$ generates neutro-cyclic triplet subgroup of N.

Proof. Straightforward.

4 Neutro-Homomorphism

In this section, we introduce the neutro-homomorphism for the neutrosophic triplet groups. We also study some of their properties. Further, we defined neutro-isomorphisms.

Definition 4.1. Let $(N_1,*_1)$ and $(N_2,*_2)$be two neutrosophic triplet groups. Let

$$f: N_1 \to N_2$$

be a mapping. Then f is called neutro-homomorphism if for all $a,b \in N_1$we have

1) $f(a *_1 b) = f(a) *_1 f(b)$,
2) $f(neut_2(a)) = neut_2(f(a))$, and
3) $f(anti_{*_1}{}^{(k_1)}(a)) = anti_{*_2}{}^{(k_2)} f(a),$

where $k_1 = 1, 2, \ldots$ is the order of the neutrosophic triplet $(a, neut_{*1}(a), anti_{*1}(a))$ in the case when the element a has more opposites, and one uses the notations:

$$(a, neut_{*_1}(a), anti_{*1}{}^{(1)}(a)), (a, neut_{*_1}(a), anti_{*1}{}^{(2)}(a)), (a, neut_{*_1}(a), anti_{*1}{}^{(3)}(a)), \text{ etc.}$$

and similar notations for the second law $*_2$:

$$(b, neut_{*2}(b), anti_{*2}{}^{(1)}(b)), (b, neut_{*_2}(b), anti_{*2}{}^{(2)}(b)), (b, neut_{*2}(b), anti_{*2}{}^{(3)}(b)), \text{ etc.}$$

Theorem 4.1.

Axioms (1), (2), and (3) are equivalent to extending the one-variable (neutro-homomorphism) function *f(x)* to three-variable (neutro-homomorphism) function *F(x, y, z),* defined as follows:

$$F: N_1^3 \to N_2^3$$

if $(a, b, c) \in N_1^3$ is a neutrosophic triplet, then

$$F(a, b, c) = (f(a), f(b), f(c)) \in N_2^3$$

is also a neutrosophic triplet.

Hence in general, for $k_1, k_2 = 1, 2, \ldots,$ one has:

$$F(a, neut_{*_1}(a), anti_{*_1}{}^{(k_1)}(a)) = (f(a), f(neut_{*_1}(a)), f(anti_{*_1}{}^{(k_1)}(a)))$$

$$= (f(a), neut_{*_2} f(a), anti_{*_2}{}^{(k_2)} f(a)).$$

Proof. Almost straightforwardly.

We construct a well-defined law of neutrosophic triplets $\#_1$ on N_1^3 as follows:

for any two neutrosophic triplets (a, b, c) and (α, β, γ) from N_1^3, one has:

$(a, b, c)\ \#_1\ (\alpha, \beta, \gamma) = (a *_1 \alpha,\ b *_1 \beta,\ c *_1 \gamma)$,

and a well-defined law of neutrosophic triplets $\#_2$ on N_2^3 as follows:

for any two neutrosophic triplets (u, v, w) and $(\delta, \varepsilon, \zeta)$ from N_2^3, one has:

$(u, v, w)\ \#_2\ (\delta, \varepsilon, \zeta) = (u *_2 \delta,\ v *_2 \varepsilon,\ w *_2 \zeta)$.

Whence,

$F((a, b, c)\ \#_1\ (\alpha, \beta, \gamma)) = F(a *_1 \alpha,\ b *_1 \beta,\ c *_1 \gamma) = (f(a *_1 \alpha), f(b *_1 \beta), f(c *_1 \gamma))$

$= (f(a) *_2 f(\alpha), f(b) *_2 f(\beta), f(c) *_2 f(\gamma))$.

And further, for $b = neut_{*1}(a)$, $c = anti_{*1}{}^{(k11)}(a)$, and respectively $\beta = neut_{*1}(\alpha)$, $\gamma = anti_{*2}{}^{(k12)}(\alpha)$, one gets:

$F(\ a, neut_{*1}(a), anti_{*1}{}^{(k11)}(a))\ \#_1\ (\alpha, neut_{*2}(\alpha), anti_{*2}{}^{(k12)}(\alpha))\ ) =$

$F(\ (a *_1 \alpha), (neut_{*1}(a) *_1 neut_{*1}(\alpha)), (anti_{*1}{}^{(k11)}(a)) *_1 (anti_{*1}{}^{(k12)}(\alpha))\ ) =$

$(\ f(a *_1 \alpha), f(neut_{*1}(a) *_1 neut_{*1}(\alpha)), f(anti_{*1}{}^{(k11)}(a) *_1 anti_{*1}{}^{(k12)}(\alpha))\ ) =$

$(\ f(a) *_2 f(\alpha), f(neut_{*1}(a)) *_2 f(neut_{*1}(\alpha)), f(anti_{*1}{}^{(k11)}(a)) *_2 f(anti_{*1}{}^{(k12)}(\alpha))\ ) =$

$(\ f(a) *_2 f(\alpha), neut_{*2}(f(a)) *_2 neut_{*2}(f(\alpha)), anti_{*2}{}^{(k21)}(f(a)) *_2 anti_{*2}{}^{(k22)}(f(\alpha))\ ) =$

$F(\ a, neut_{*1}(a), anti_{*1}{}^{(k11)}(a)\ )\ \#_2\ F(\ \alpha, neut_{*1}(\alpha), anti_{*1}{}^{(k12)}(\alpha)\ )$.

Therefore *F(x, y, z), for (x, y, z)* neutrosophic triplets in N_1^3, is its self a neutro-homomorphism.

Proposition 4.3. Every neutro-homomorphism is a classical homomorphism by neglecting the classical unitary element in classical homomorphism.

Proof. First, we neglect the classical unitary element that classical homomorphism maps unitary element to the corresponding unitary element. Now suppose that f is a neutro-homomorphism from a neutrosophic triplet group N_1 to a neutrosophic triplet group N_2. Then by condition (1), it follows that f is a classical homomorphism.

Definition 4.4. A neutro-homomorphism is called neutro-isomorphism if it is one-to-one and onto.

5 Distinctions and Comparison

The distinctions between Molaei's Generalized Group [7] and Neutrosophic Triplet Group are:

I. - in MGG for each element there exists a unique neutral element, which can be the classical group unitary element; while in NTG each element may have a unique neutral element but which is different form the classical element;

II. - in MGG there exists a unique inverse of an element, while in NTG there may be many inverses for the same given element;

III. - MGG has a weaker structure than NTG.

IV. - Smarandache (2016-2017) has generalized the NTG to Neutrosophic Extended Triplet Group (NETG), where the *neut(x)* is allowed to be equal to the classical unitary algebraic element of the group theory [14-16].

So far the applications of neutrosophic triplet sets are in Z, modulo n, n ≥ 2.

But new applications can be found, for example in social science:

One person <A> that has an enemy $\langle anti(A_{d_1}\rangle$(enemy in a degree d_1 of enemy-city), and a neutral person $\langle neut(A_{d_1}\rangle$ with respect to$\langle anti(A_{d_1}\rangle$. Then another enemy $\langle anti(A_{d_2}\rangle$ in a different degree of enemy-city, and a neutral $\langle neut(A_{d_2}\rangle$, and so on. Hence one has the neutrosophic triplets:

$$\langle A, \langle neut\ (A_{d_1})\rangle, \langle anti\ (A_{d_1})\rangle\ \rangle,$$

$$\langle A, \langle neut\ (A_{d_2})\rangle, \langle anti\ (A_{d_2})\rangle\ \rangle \text{, and so on.}$$

Then we take another person B in the same way...

$$\langle A, \langle neut\ (B_{d_1})\rangle, \langle anti\ (B_{d_1})\rangle\ \rangle,$$

$$\langle A, \langle neut\ (B_{d_2})\rangle, \langle anti\ (B_{d_2})\rangle\ \rangle \text{ etc.}$$

More applications may be found, if we deeply think about cases where we have neutrosophic triplets $\langle A, \langle neut\ (A)\rangle, \langle anti\ (A)\rangle\ \rangle$ in technology and in science.

Conclusion

Inspired on the Neutrosophic philosophy, we defined for the first time the neutrosophic triplet. Basically, a neutrosophic triplet is a triad of certain elements, which satisfy certain axioms, which highly depend upon the proposed binary operation. The main theme of this paper is first to introduce the neutrosophic triplets, which are completely new notions, and then apply these neutrosophic triplets to introduce the neutrosophic triplet groups. This neutrosophic triplet group has several extra-ordinary properties as compared to the classical group. We also studied some interesting properties of this newly born structure. We further defined neutro-homomorphisms for neutrosophic triplet groups. A neutro-homomorphism is the generalization of the classical homomorphism with two extra conditions. As a further generalization, we gave rise to a new field or research called Neutrosophic Triplet Structures (such as neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, etc.). In the end, we offered main distinctions and comparison of neutrosophic triplet group with the Molaei's generalized group as well as the possible application areas for the neutrosophic triplet groups.

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m-Polar Neutrosophic Topology with Applications to Multicriteria Decision-Making in Medical Diagnosis and Clustering Analysis

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Abstract In this paper, we first introduce novel concepts of *m*-polar neutrosophic set (MPNS) and topological structure on *m*-polar neutrosophic set by combining the *m*-polar fuzzy set (MPFS) and neutrosophic set. Then, we investigate several characterizations of *m*-polar neutrosophic set and establish its various operations with the help of examples. We propose score functions for the comparison of *m*-polar neutrosophic numbers (MPNNs). We establish *m*-polar neutrosophic topology and define interior, closure, exterior, and frontier for *m*-polar neutrosophic sets (MPNSs) with illustrative examples. We discuss some results with counter examples, which hold for classical set theory, but do not hold for *m*-polar neutrosophic set theory. We introduce a cosine similarity measure and a set theoretic similarity measure for *m*-polar neutrosophic sets (MPNSs). Furthermore, we present three algorithms for multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis under uncertainty by using *m*-polar neutrosophic sets (MPNSs) and *m*-polar neutrosophic topology. Lastly, we present advantages, validity, flexibility, and comparison of our proposed algorithms with the existing techniques.

Keywords *m*-Polar neutrosphic set · Score functions for MPNNs · *m*-Polar neutrosphic topological space · Similarity measures for MPNSs · Multi-criteria decision-making for medical diagnosis · Multi-criteria decision-making for clustering analysis

1 Introduction and Background

Multi-criteria decision-making (MCDM) is a process that explicitly evaluates best alternative(s) among the feasible options. In archaic times, decisions were framed without handling the uncertainties in the data, which may lead to inadequate results to the real-life operational situations. If we amass the data and deduce the result without handling hesitations, then given results will be ambivalent, indefinite, or equivocal. MCDM is an integral part in modern management, business, medical diagnosis, and many other real-world problems. Essentially, rational or sound decision is necessary for a decision-maker. Every decision-maker takes hundreds of decisions subconsciously or consciously making it as the central part of his execution. Medical diagnosis with MCDM provides solutions for the doctors to determine symptoms of disease and kind of illness. MCDM is used in solving problems that contain complex and multiple criteria. In MCDM, we have to identify the problem by determining the possible alternatives, evaluate each alternative based upon the criteria given by the decision-maker or group of decision-makers and lastly select the best alternative. MCDM problems under fuzzy environment were first introduced by Bellman and Zadeh in (1970) [4]. A number of useful mathematical tools such as fuzzy sets, *m*-polar fuzzy sets, neutrosophic sets, and soft sets have been developed to deal with uncertainties and ambiguities for multi-criteria decision-making problems.

Zadeh introduced fuzzy set [48] as a significant mathematical model to characterize and assembling of the objects whose boundary is ambiguous. A fuzzy set $\mathfrak{F}$ in the reference set $\mathcal{Q}$ is represented by a mapping $\sigma : \mathcal{Q} \rightarrow [0, 1]$. In real-life problems, we face various situations including uncertainties and ambiguities. For instance, if we speak about the "beautiful cities of a country" then the exact decision is ambiguous. Some cities are very beautiful, some of them are medium beautiful, and some are less beautiful. The criteria of being "beautiful" can be changed according to the decision-maker's choice. In these situations, the classical set theory fails and we use fuzzy set theory to treat these type of hesitations in the decision-making problems. We use linguistic terms to relate a real-world situation to the fuzzy numeric value and accumulate the input in the form of fuzzy numbers or fuzzy sets.

After Zadeh, many extensions of fuzzy sets have been presented and investigated such as, intuitionistic fuzzy sets (IFSs) [3], single valued neutrosophic sets (SVNSs) [28–30, 35], picture fuzzy sets [8], bipolar fuzzy sets (BPFSs) [50–52], m-polar fuzzy sets (MPFSs) [5], interval-valued fuzzy sets (IVFSs) [49], and Pythagorean fuzzy sets (PFSs) [42–44]. A neutrosophic set $\mathfrak{N}$ is defined by $\mathfrak{N} = \{\langle \varsigma, \mathfrak{A}(\varsigma), \mathfrak{S}(\varsigma), \mathfrak{Y}(\varsigma)\rangle, \varsigma \in \mathcal{Q}\}$, where $\mathfrak{A}, \mathfrak{S}, \mathfrak{Y} : \mathcal{Q} \rightarrow]^{-}0, 1^{+}[$ and $^{-}0 \leq \mathfrak{A}(\varsigma) + \mathfrak{S}(\varsigma) + \mathfrak{Y}(\varsigma) \leq 3^{+}$. The neutrosophic set yields the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. It is difficult to utilize these values in daily life science and technology problems. Consequently, the neutrosophic set which takes the value from the subset of [0, 1] is to be regarded here. An abstraction of bipolar fuzzy set was inaugurated by Chen [5] named as MPFS. An MPFS $\mathfrak{C}$ in a non-empty universal set $\mathcal{Q}$ is a function $\mathfrak{C} : \mathcal{Q} \rightarrow [0, 1]^{m}$, symbolized by $\mathfrak{C} = \{\langle \varsigma, P_i o \Lambda(\varsigma)\rangle : \varsigma \in \mathcal{Q}; i = 1, 2, 3, \ldots, m\}$ where $P_i : [0, 1]^{m} \rightarrow [0, 1]$ is the ith projection mathematical function $(i \in m)$. $\mathfrak{C}_{\phi}(\varsigma) = (0, 0, \ldots, 0)$ is the smallest value in $[0, 1]^{m}$, and $\mathfrak{C}_{\widetilde{X}}(\varsigma) = (1, 1, \ldots, 1)$ is the greatest value in $[0, 1]^{m}$.

In the last few decades, many mathematicians worked on similarity measures, correlation coefficients, topological spaces, aggregation operators, and decision-making applications. These structures have different formulae according to the different sets and give better solutions to decision-making problems. It has numerous applications in the field of pattern recognition, medical diagnosis, artificial intelligence, social sciences, business, and multi-attribute decision-making problems.

Akram et al. [1] presented certain applications of m-polar fuzzy sets in the decision-making problems. Ali et al. [2] presented various properties of soft sets and rough sets with fuzzy soft sets. Garg [10] introduced new generalized Pythagorean fuzzy information aggregation using Einstein operations and established its application to decision-making problems. Garg [11] introduced generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision-making. Karaaslan [15] introduced neutrosophic soft sets with its applications in decision-making. Xu et al. [41] established clustering algorithm for intuitionistic fuzzy sets and presented its applications for clustering. Jose and Kuriaskose [14] investigated aggregation operators with the corresponding score function for MCDM in the context of IFNs. Mahmood et al. [19] established generalized aggregation operators for cubic hesitant fuzzy numbers (CHFNs) and use it into MCDM problems. In 1968, Chang [7] introduced fuzzy topology on fuzzy sets. After fuzzy topology, many researchers have been introduced topologies and their properties on different hybrid structures of fuzzy sets. Pao-Ming and Ying-Ming [20, 21] introduced the structure of neighborhood of fuzzy-point. They provided the concept of fuzzy quasi-coincident and Q-neighborhood. They also discussed important properties of fuzzy topological space by using fuzzy Q-neighborhood. Shabir and Naz [31] established soft topological spaces. Deli et al. [9] introduced bipolar neutrosophic sets and their application based on multi-criteria decision-making problems. Riaz and Hashmi [23–25] developed fixed point theorems of fuzzy neutrosophic soft (FNS) mapping with its decision-making. They established multi-attribute group decision-making (MAGDM) for agribusiness by using various cubic m-polar fuzzy averaging aggregation operators. They introduced a novel structure of linear Diophantine fuzzy set as a generalization of intuitionistic fuzzy set, Pythagorean fuzzy set, and q-rung orthopair fuzzy set with its applications in multi-attribute decision-making problems. Riaz et al. [26, 27] introduced N-soft topology and its applications to multi-criteria group decision-making (MCGDM). They established cubic bipolar fuzzy ordered weighted geometric aggregation operators and presented their applications by using internal and external bipolar fuzzy information.

Feng et al. [12, 13] introduced properties of soft sets combined with fuzzy soft sets and multi-attribute decision-making (MADM) models in the environment of generalized intuitionistic fuzzy soft sets and fuzzy soft sets. Liu et al. [16] established hesitant intuitionistic fuzzy linguistic operators and presented its MAGDM problems. Wei et al. [36] invented hesitant triangular fuzzy operators in MADM problems. Wei et al. [37, 38] worked on similarity measures on picture fuzzy sets and correlation coefficient to the interval-valued intuitionistic fuzzy sets with application in decision-making problems. Ye [45–47] introduced prioritized aggregation operators in the context of interval-valued hesitant fuzzy numbers (IVHFNs) and established it on MAGDM algorithms. He also established MCDM methods for interval neutrosophic sets and correlation coefficient under single-value neutrosophic environment. He established cosine similarity measures for intuitionistic fuzzy sets with application in decision-making problems. Zhang et al. [53] introduced aggregation operators with MCDM by using interval-valued fuzzy neutrosophic sets (IVFNSs). An extended TOPSIS method for decision-making was developed by Chi and Lui [6] on IVFNSs. Zhao et al. [55] introduced generalized aggregation operators in the context of intuitionistic fuzzy sets. Zhang et al. [54] established various results on clustering approach to intuitionistic fuzzy sets. Peng et al. [22] introduced Pythagorean fuzzy information measures and established interesting results on Pythagorean fuzzy sets. They introduced clustering algorithm for Pythagorean fuzzy sets and presented numerous applications on Pythagorean fuzzy input data. Li and Cheng [17] established new similarity measures of IFSs and its applications to pattern recognition. Lin et al. [18] studied hesitant fuzzy linguistic information and presented its application to models of selecting an ERP system. Salton and McGill [32] introduced modern information retrieval. Singh [33] established correlation coefficients of picture fuzzy sets. Son [34] inaugurated a novel distributed picture fuzzy clustering method on picture fuzzy sets. Xu and Chen [39, 40] established correlation, distance, and similarity measures on intuitionistic fuzzy sets.

In this era, experts think that the universe is moving towards multi-polarity. Therefore, it comes as no surprise that multi-polarity in data and information plays a vital role in various fields of science and technology. In neurobiology, multi-polar neurons in brain gather a great deal of information from other neurons. In information technology, multi-polar technology can be exploited to operate large-scale systems. In some real-life situations, we have to deal with the dissatisfaction and indeterminacy grades for the alternatives of the reference set. For instance, in the operation of throwing up a ballot, there exist some people who vote in favor, some of them vote against, and some abstain. In the area of electrical engineering, we deal with the conductors and non-conductors, but there also exist some substances which are insulators. These types of situations can easily handled by using neutrosophic set theory. In some real-life applications, we have to deal with multi-polarity, truth values, indeterminacy, and falsity grades of alternatives. To deal with these type of hesitations and uncertainties, we establish the idea of m-polar neutrosophic set (MPNS).

The motivation and objectives of this extended and hybrid work are given step by step in the whole manuscript. We establish that other hybrid structures of fuzzy sets become special cases of MPNS under some suitable conditions. We discuss about the robustness, flexibility, simplicity, and superiority of our suggested model and algorithms. This model is most generalized form and use to collect data at a large scale and applicable in medical, engineering, artificial intelligence, agriculture, and other daily life problems. In future, this work can be gone easily for other approaches and different types of hybrid structures.

The scheme of this manuscript is organized as follows. Section 2, implies a novel idea of m-polar neutrosophic set (MPNS). We establish some of its operations, score function, and improved score function. In Sect. 3, we use MPNS to establish m-polar neutrosophic topological space (MPNTS). We define various topological structures such as interior, closure, exterior, and frontier for MPNSs with the help of illustrations. We establish various results with their counter examples, which holds for classical set theory, but do not hold for m-polar neutrosophic set theory. We introduce cosine similarity measure and set theoretic similarity measure for MPNSs. In Sect. 4, we establish some methods for the solution of MCDM problems based on medical diagnosis and clustering analysis using MPNTS and MPNSs. We propose three algorithms with linguistic information based on m-polar neutrosophic data using MPNTS, similarity measures, and clustering analysis. It is interesting to note that first two algorithms for medical diagnosis yield the same result. Furthermore, we present advantages, simplicity, flexibility, and validity of the proposed algorithms. We give a brief discussion and comparative analysis of our proposed approach with some existing methodologies. In the end, the conclusion of this work is summarized in Sect. 5.

2 m-Polar Neutrosophic Set (MPNS)

Chen et al. [5] have proposed the concept of m-polar fuzzy set (MPFS) in 2014, which have the capability to deal with the data having vagueness and uncertainty under multi-criteria, multi-source, multi-sensor, and multi-polar information. Smarandache [30] extended the neutrosophic set, respectively, to neutrosophic overset (when some neutrosophic component is >1), neutrosophic underset (when some neutrosophic component is <0), and to neutrosophic offset (when some neutrosophic components are off the interval [0, 1], i.e., some neutrosophic component >1 and other neutrosophic component <0). In 2016, Smarandache introduced the neutrosophic tripolar set and neutrosophic multi-polar set, also the neutrosophic tripolar graph and neutrosophic multi-polar graph [30].

The membership grades of m-polar fuzzy sets range over the interval $[0,1]^m$, which represent m criteria of the object, but it cannot deal with the falsity and indeterminacy part of the object.

Neutrosophic set (NS) deals with truth, falsity, and indeterminacy for one criteria of the alternative, but cannot deal with the multi-criteria, multi-source, multi-polar information fusion of the alternatives. To overcome this problem, we introduce a new model of m-polar neutrosophic set (MPNS) by combining the concepts of m-polar fuzzy set (MPFS) and neutrosophic set (NS). MPNS has the ability to deal with the m criteria and to deal with the truth, falsity, and indeterminacy grades for each alternative. In fact, m-polar neutrosophic set is an extension of bipolar neutrosophic set introduced by Deli et al. [9]. We establish various properties and operations on m-polar neutrosophic sets. We propose score functions for the comparison of m-polar neutrosophic numbers (MPNNs). In the whole manuscript, we use $\mathcal{Q}$ as a fixed sample space and Δ as an indexing set. We use $\mathfrak{A}, \mathfrak{S}$ and $\mathfrak{Y}$ as membership, indeterminacy, and non-membership grades, respectively.

Definition 2.1 An object $\mathcal{M}_{\mathfrak{R}}$ in the reference set $\mathcal{Q}$ is called m-polar neutrosophic set (MPNS), if it can be expressed as

$$\mathcal{M}_{\mathfrak{R}} = \{(\varsigma, \langle \mathfrak{A}_\alpha(\varsigma), \mathfrak{S}_\alpha(\varsigma), \mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1, 2, 3, \ldots, m\}$$

where $\mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha : \mathcal{Q} \to [0, 1]$ and $0 \leq \mathfrak{A}_\alpha(\varsigma) + \mathfrak{S}_\alpha(\varsigma) + \mathfrak{Y}_\alpha(\varsigma) \leq 3; \alpha = 1, 2, 3, \ldots, m$. This condition shows that all the three grades $\mathfrak{A}_\alpha, \mathfrak{S}_\alpha$ and $\mathfrak{Y}_\alpha$; $(\alpha = 1, 2, 3, \ldots, m)$ are independent and represents the truth, indeterminacy, and falsity of the considered object or alternative for multiple criteria, respectively. Simply an m-polar neutrosophic number (MPNN) can be represented as $\mathfrak{I} = (\langle \mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha \rangle)$, where $0 \leq \mathfrak{A}_\alpha + \mathfrak{S}_\alpha + \mathfrak{Y}_\alpha \leq 3; \alpha = 1, 2, 3, \ldots, m$. In tabular form, the MPNS can be represented as Table 1.

Example 2.2 Let $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \varsigma_3\}$ be the collection of some well-known smart phones. Then 4-polar neutrosophic set in $\mathcal{Q}$ can be written as

$$\mathcal{M}_{\mathfrak{R}} = \Big\{(\varsigma_1, \langle 0.512, 0.231, 0.321\rangle, \langle 0.653, 0.223, 0.116\rangle, \langle 0.875, 0.114, 0.243\rangle, \langle 0.961, 0.115, 0.431\rangle), (\varsigma_2, \langle 0.657, 0.114, 0.226\rangle, \langle 0.765, 0.224, 0.245\rangle, \langle 0.875, 0.465, 0.213\rangle, \langle 0.961, 0.141, 0.212\rangle), (\varsigma_3, \langle 0.876, 0.221, 0.321\rangle, \langle 0.657, 0.115, 0.116\rangle, \langle 0.987, 0.114, 0.322\rangle, \langle 0.675, 0.221, 0.423\rangle)\Big\}.$$

In this set, multi-polarity ($m = 1$,2,3,4) of each allternative ς shows its characteristic or qualities according to the considered criteria such as

$$\alpha_1 = \text{affordable}, \alpha_2 = \text{longlastingbattery},$$
$$\alpha_3 = \text{extrastorage}, \alpha_4 = \text{goodcameraquality}.$$

For each ς and each of its criteria, we have neutrosophic values to represent the truth, indeterminacy, and falsity of corresponding alternative according to the considered criteria under the influence of expert's opinion. In the set $\mathcal{M}_{\mathfrak{R}}$ for ς_1 the first triplet $\langle 0.512, 0.231, 0.321\rangle$ shows that the smart phone ς_1 has 51.2% truth value, 23.1% indeterminacy, and 32.1% falsity value for the criteria "affordable." Similarly, we can see the values for all alternatives corresponding to the other criteria.

There is a relationship between MPNS and other hybrid structures of fuzzy set. This relationship can be elaborated in the given flow chart diagram of Fig. 1, where $\alpha = 1, 2, 3, \ldots, m$.

Definition 2.3 An MPNS $\mathcal{M}_{\mathfrak{R}}$ is said to be an empty MPNS, if $\mathfrak{A}_\alpha(\varsigma) = 0, \mathfrak{S}_\alpha(\varsigma) = 1$ and $\mathfrak{Y}_\alpha(\varsigma) = 1, \forall \alpha = 1, 2, 3, \ldots, m$ and it can be written as

$${}^0\mathcal{M}_{\mathfrak{R}} = \{\varsigma, (\langle 0, 1, 1\rangle, \langle 0, 1, 1\rangle, \cdots, \langle 0, 1, 1\rangle) : \varsigma \in \mathcal{Q}\}$$

and for absolute MPNS we have $\mathfrak{A}_\alpha(\varsigma) = 1, \mathfrak{S}_\alpha(\varsigma) = 0$ and $\mathfrak{Y}_\alpha(\varsigma) = 0, \forall \alpha = 1, 2, 3, \ldots, m$ and it can be written as

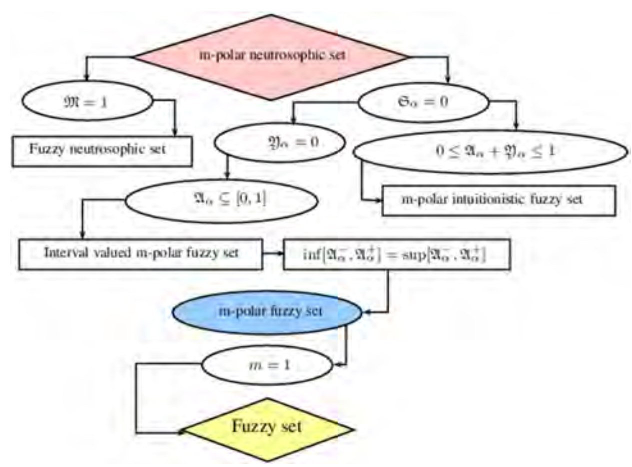

Fig. 1 Relationship between MPNS and other hybrid fuzzy sets

Table 1 Tabular representation of m-polar neutrosophic set

$\mathcal{M}_{\mathfrak{R}}$	*MPNS*
ς_1	$(\langle \mathfrak{A}_1(\varsigma_1), \mathfrak{S}_1(\varsigma_1), \mathfrak{Y}_1(\varsigma_1)\rangle, \langle \mathfrak{A}_2(\varsigma_1), \mathfrak{S}_2(\varsigma_1), \mathfrak{Y}_2(\varsigma_1)\rangle, \cdots, \langle \mathfrak{A}_m(\varsigma_1), \mathfrak{S}_m(\varsigma_1), \mathfrak{Y}_m(\varsigma_1)\rangle)$
ς_2	$(\langle \mathfrak{A}_1(\varsigma_2), \mathfrak{S}_1(\varsigma_2), \mathfrak{Y}_1(\varsigma_2)\rangle, \langle \mathfrak{A}_2(\varsigma_2), \mathfrak{S}_2(\varsigma_2), \mathfrak{Y}_2(\varsigma_2)\rangle, \cdots, \langle \mathfrak{A}_m(\varsigma_2), \mathfrak{S}_m(\varsigma_2), \mathfrak{Y}_m(\varsigma_2)\rangle)$
...	....
$\varsigma_{\mathfrak{R}}$	$(\langle \mathfrak{A}_1(\varsigma_{\mathfrak{R}}), \mathfrak{S}_1(\varsigma_{\mathfrak{R}}), \mathfrak{Y}_1(\varsigma_{\mathfrak{R}})\rangle, \langle \mathfrak{A}_2(\varsigma_{\mathfrak{R}}), \mathfrak{S}_2(\varsigma_{\mathfrak{R}}), \mathfrak{Y}_2(\varsigma_{\mathfrak{R}})\rangle, \cdots, \langle \mathfrak{A}_m(\varsigma_{\mathfrak{R}}), \mathfrak{S}_m(\varsigma_{\mathfrak{R}}), \mathfrak{Y}_m(\varsigma_{\mathfrak{R}})\rangle)$

$^{1}\mathcal{M}_{\mathfrak{R}} = \{\varsigma, (\langle 1,0,0\rangle, \langle 1,0,0\rangle, \cdots, \langle 1,0,0\rangle) : \varsigma \in \mathcal{Q}\}$

The assembling of all MPNSs in $\mathcal{Q}$ is represented as $\mathfrak{mpn}(\mathcal{Q})$.

Now we define some operations for MPNSs.

Definition 2.4 Let $\mathcal{M}_{\mathfrak{R}}, \mathcal{M}_{\mathfrak{R}_\wp} \in \mathfrak{mpn}(\mathcal{Q})$, where

$$\mathcal{M}_{\mathfrak{R}} = \{(\varsigma, \langle \mathfrak{A}_\alpha(\varsigma), \mathfrak{S}_\alpha(\varsigma), \mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}$$
$$\mathcal{M}_{\mathfrak{R}_\wp} = \{(\varsigma, \langle {}^\wp\mathfrak{A}_\alpha(\varsigma), {}^\wp\mathfrak{S}_\alpha(\varsigma), {}^\wp\mathfrak{Y}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}, \wp \in \Delta$$

then:

(i) $\mathcal{M}^c_{\mathfrak{R}} = \{(\varsigma, \langle \mathfrak{Y}_\alpha(\varsigma), 1-\mathfrak{S}_\alpha(\varsigma), \mathfrak{A}_\alpha(\varsigma)\rangle) : \varsigma \in \mathcal{Q}, \quad \alpha = 1,2,3,\ldots,m\}$

(ii) $\mathcal{M}_{\mathfrak{R}_1} = \mathcal{M}_{\mathfrak{R}_2} \Leftrightarrow \langle {}^1\mathfrak{A}_\alpha(\varsigma), {}^1\mathfrak{S}_\alpha(\varsigma), {}^1\mathfrak{Y}_\alpha(\varsigma)\rangle = \langle {}^2\mathfrak{A}_\alpha(\varsigma), {}^2\mathfrak{S}_\alpha(\varsigma), {}^2\mathfrak{Y}_\alpha(\varsigma)\rangle; \varsigma \in \mathcal{Q}, \alpha = 1,2,3,\ldots,m$

(iii) $\mathcal{M}_{\mathfrak{R}_1} \subseteq \mathcal{M}_{\mathfrak{R}_2} \Leftrightarrow {}^1\mathfrak{A}_\alpha(\varsigma) \le {}^2\mathfrak{A}_\alpha(\varsigma), {}^1\mathfrak{S}_\alpha(\varsigma) \ge {}^2\mathfrak{S}_\alpha(\varsigma), {}^1\mathfrak{Y}_\alpha(\varsigma) \ge {}^2\mathfrak{Y}_\alpha(\varsigma); \varsigma \in \mathcal{Q}, \alpha = 1,2,3,\ldots,m$

(iv) $\bigcup_\wp \mathcal{M}_{\mathfrak{R}_\wp} = \{(\varsigma, \langle \sup_\wp {}^\wp\mathfrak{A}_\alpha(\varsigma), \inf_\wp {}^\wp\mathfrak{S}_\alpha(\varsigma), \inf_\wp {}^\wp\mathfrak{Y}_\alpha(\varsigma)\rangle); \varsigma \in \mathcal{Q}, \wp \in \Delta, \alpha = 1,2,3,\ldots,m\}$

(v) $\bigcap_\wp \mathcal{M}_{\mathfrak{R}_\wp} = \{(\varsigma, \langle \inf_\wp {}^\wp\mathfrak{A}_\alpha(\varsigma), \sup_\wp {}^\wp\mathfrak{S}_\alpha(\varsigma), \sup_\wp {}^\wp\mathfrak{Y}_\alpha(\varsigma)\rangle); \varsigma \in \mathcal{Q}, \wp \in \Delta, \quad \alpha = 1,2,3,\ldots,m\}$

Example 2.5 Consider two 4-polar neutrosophic sets $\mathcal{M}_{\mathfrak{R}_1}$ and $\mathcal{M}_{\mathfrak{R}_2}$ given in tabular form as Table 2.

Now we calculate complement, union, and intersection by using Definition 2.4 and results can be seen in tabular form as Table 3.

In order to deal with multi-criteria decision-making problems with m-polar neutrosophic numbers (MPNNs), we define some score functions for the ranking of MPNNs.

Definition 2.6 Let $\mathfrak{I} = (\langle \mathfrak{A}_\alpha, \mathfrak{S}_\alpha, \mathfrak{Y}_\alpha\rangle; \alpha = 1,2,3,\ldots,m)$ be an MPNN, then its score functions are given as:

$$£_1(\mathfrak{I}) = \frac{1}{2m}\left(m + \sum_{\alpha=1}^{m}(\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha)\right); £_1(\mathfrak{I}) \in [0,1]$$

$$£_2(\mathfrak{I}) = \frac{1}{m}\sum_{\alpha=1}^{m}(\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha); £_2(\mathfrak{I}) \in [-1,1]$$

In the case, when score value of two MPNNs is same, we define an improved score function for the ranking of MPNNs given as

$$£_3(\mathfrak{I}) = \frac{1}{2m}\left(m + \sum_{\alpha=1}^{m}((\mathfrak{A}_\alpha - 2\mathfrak{S}_\alpha - \mathfrak{Y}_\alpha)(2 - \mathfrak{A}_\alpha - \mathfrak{Y}_\alpha))\right);$$
$£_3(\mathfrak{I}) \in [-1,1]$.

In the case, when $\mathfrak{A}_\alpha + \mathfrak{Y}_\alpha = 1;\ \forall\ \alpha = 1,2,\ldots,m$, then $£_3(\mathfrak{I})$ reduces to $£_1(\mathfrak{I})$.

Definition 2.7 Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be two MPNNs, then the following order relation between the score values of MPNNs hold:

(a) If $£_1(\mathfrak{I}_1) \succ £_1(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \succ \mathfrak{I}_2$.
(b) If $£_1(\mathfrak{I}_1) = £_1(\mathfrak{I}_2)$ then
(1) If $£_2(\mathfrak{I}_1) \succ £_2(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \succ \mathfrak{I}_2$.
(2) If $£_2(\mathfrak{I}_1) = £_2(\mathfrak{I}_2)$ then
(i) If $£_3(\mathfrak{I}_1) \succ £_3(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \succ \mathfrak{I}_2$.
(ii) If $£_3(\mathfrak{I}_1) \prec £_3(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \prec \mathfrak{I}_2$.
(iii) If $£_3(\mathfrak{I}_1) = £_3(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \sim \mathfrak{I}_2$.

Example 2.8 Consider two 2-polar neutrosophic numbers $\mathfrak{I}_1$ and $\mathfrak{I}_2$ given in tabular form as Table 4.

Then by using Definition 2.6 $£_1(\mathfrak{I}_1) = \frac{1}{2(2)}[2 + 0.5 - 2(0.3) - 0.4 + 0.5 - 2(0.1) - 0.8] = 0.25.$ Similarly, $£_1(\mathfrak{I}_2) = 0.25$. This shows that $£_1$ fails to give the ranking between both 2PNNs. Now we will use second score function $£_2$. By using Definition 2.6, we obtain the score values $£_2(\mathfrak{I}_1) = -0.5 = £_2(\mathfrak{I}_2)$. This shows that $£_2$ also fails to evaluate the ranking. Now we will use improved score function for the ranking of 2PNNs. After calcula-

Table 4 2-polar neutrosophic numbers $\mathfrak{I}_1$ and $\mathfrak{I}_2$

$\mathcal{Q}$	2*PNNs*
$\mathfrak{I}_1$	$(\langle 0.5,0.3,0.4\rangle, \langle 0.5,0.1,0.8\rangle)$
$\mathfrak{I}_2$	$(\langle 0.2,0.3,0.1\rangle, \langle 0.2,0.1,0.5\rangle)$

Table 2 4-polar neutrosophic sets $\mathcal{M}_{\mathfrak{R}_1}$ and $\mathcal{M}_{\mathfrak{R}_2}$

$\mathcal{Q}$	4*PNSs*
$\mathcal{M}_{\mathfrak{R}_1}$	$(\langle 0.611,0.111,0.251\rangle, \langle 0.821,0.631,0.111\rangle, \langle 0.721,0.381,0.591\rangle, \langle 0.211,0.321,0.411\rangle)$
$\mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.321,0.621,0.511\rangle, \langle 0.831,0.111,0.921\rangle, \langle 0.521,0.431,0.391\rangle, \langle 0.181,0.931,0.821\rangle)$

Table 3 Complement, union, and intersection of 4-polar neutrosophic sets

$\mathcal{Q}$	4*PNSs*
$\mathcal{M}^c_{\mathfrak{R}}$	$(\langle 0.251,0.889,0.611\rangle, \langle 0.111,0.369,0.821\rangle, \langle 0.591,0.619,0.721\rangle, \langle 0.411,0.679,0.211\rangle)$
$\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.611,0.111,0.251\rangle, \langle 0.831,0.111,0.111\rangle, \langle 0.721,0.381,0.391\rangle, \langle 0.211,0.321,0.411\rangle)$
$\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.321,0.621,0.511\rangle, \langle 0.821,0.631,0.921\rangle, \langle 0.521,0.431,0.591\rangle, \langle 0.181,0.931,0.821\rangle)$

tions, we get $\pounds_3(\mathfrak{I}_1) = 0.275$ and $\pounds_3(\mathfrak{I}_2) = 0.125$. Hence $\pounds_3(\mathfrak{I}_1) \succ \pounds_3(\mathfrak{I}_2)$, so $\mathfrak{I}_1 \succ \mathfrak{I}_2$.

Remark

- For null MPNN ${}^0\mathfrak{I}$ we have $\pounds_3({}^0\mathfrak{I}) = -1$.
- For absolute MPNN ${}^1\mathfrak{I}$ we have $\pounds_3({}^1\mathfrak{I}) = 1$.

Proposition 2.9 *Let $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}(\mathcal{Q})$, and ${}^0\mathcal{M}_{\mathfrak{R}}$ and ${}^1\mathcal{M}_{\mathfrak{R}}$ be null and absolute MPNSs. Then the following axioms hold:*

(i) $\mathcal{M}_{\mathfrak{R}} \subseteq \mathcal{M}_{\mathfrak{R}} \cup \mathcal{M}_{\mathfrak{R}}$,
(ii) $\mathcal{M}_{\mathfrak{R}} \cap \mathcal{M}_{\mathfrak{R}} \subseteq \mathcal{M}_{\mathfrak{R}}$,
(iii) $\mathcal{M}_{\mathfrak{R}} \cup {}^0\mathcal{M}_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$,
(iv) $\mathcal{M}_{\mathfrak{R}} \cap {}^0\mathcal{M}_{\mathfrak{R}} = {}^0\mathcal{M}_{\mathfrak{R}}$,
(v) $\mathcal{M}_{\mathfrak{R}} \cup {}^1\mathcal{M}_{\mathfrak{R}} = {}^1\mathcal{M}_{\mathfrak{R}}$,
(vi) $\mathcal{M}_{\mathfrak{R}} \cap {}^1\mathcal{M}_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$

Proof The proof is obvious and can be proved by Definition 2.4. □

Proposition 2.10 *Let $\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_3} \in \mathfrak{mpn}(\mathcal{Q})$, then the following results hold:*

(i) $\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2} = \mathcal{M}_{\mathfrak{R}_2} \cup \mathcal{M}_{\mathfrak{R}_1}$,
(ii) $\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2} = \mathcal{M}_{\mathfrak{R}_2} \cap \mathcal{M}_{\mathfrak{R}_1}$,
(iii) $\mathcal{M}_{\mathfrak{R}_1} \cup (\mathcal{M}_{\mathfrak{R}_2} \cup \mathcal{M}_{\mathfrak{R}_3}) = (\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2}) \cup \mathcal{M}_{\mathfrak{R}_3}$,
(iv) $\mathcal{M}_{\mathfrak{R}_1} \cap (\mathcal{M}_{\mathfrak{R}_2} \cap \mathcal{M}_{\mathfrak{R}_3}) = (\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2}) \cap \mathcal{M}_{\mathfrak{R}_3}$,
(v) $(\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2})^c = \mathcal{M}^c_{\mathfrak{R}_1} \cap \mathcal{M}^c_{\mathfrak{R}_2}$,
(vi) $(\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2})^c = \mathcal{M}^c_{\mathfrak{R}_1} \cup \mathcal{M}^c_{\mathfrak{R}_2}$

Proof The proof is obvious and can be proved by Definition 2.4. □

3 *m*-Polar Neutrosophic Topology

In this section, we introduce the *m*-polar neutrosophic topology on *m*-polar neutrosophic set and discuss interior, closure, exterior, and frontier of MPNSs with the help of illustrations. We introduce various results which hold for classical set theory, but do not hold for MPN data. We present a cosine similarity measure and set theoretic similarity measure to find the similarity between MPNSs.

3.1 *m*-Polar Neutrosophic Topological Space

In mathematics, topology is concerned with the alternatives of a geometric object that are kept under continuous deformations, such as stretching, twisting, crumpling, and bending, but not tearing or gluing. "A topological space is a set endowed with a structure, called a topology, which allows defining continuous deformation of subspaces and more broadly, all kinds of continuity." The concept of topology can be defined by using sets, continuous functions, manifolds, algebra, differentiable functions, differential geometry, etc. It has numerous applications in biology, medical diagnosis, physics, computer science, robotics, game theory, and fiber art.

The question arises here that why we use *m*-polar neutrosophic topological space? Crisp topological space cannot deal with the uncertainties and imprecision in the decision-making problems. To handle these ambiguities, Chang [7] introduced fuzzy topological spaces in 1968. After that, many mathematicians established topological spaces on other hybrid structures of fuzzy sets. Every topological space has its own boundaries, e.g., neutrosophic topological space cannot deals with the multiple criteria or multi-polarity of alternatives. *m*-polar topological space cannot deal with the indeterminacy part and dissatisfaction part of alternatives in decision-making problems. To remove these restrictions, we introduce *m*-polar neutrosophic topological space (MPNTS) by combining the *m*-polar fuzzy sets and neutrosophic sets. MPNTS handle these hesitations in the input data by treating with the multi-polarity, membership, non-membership, and indeterminacy grades for the decision-making problems. The motivation of our projected model is given step by step in the whole manuscript, especially in Sect. 4.

Definition 3.1 Let $\mathcal{Q}$ be the non-empty reference set and $\mathfrak{mpn}(\mathcal{Q})$ be the collection of all MPNSs in $\mathcal{Q}$. Then the collection $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$ containing MPNSs is called *m*-polar neutrosophic topology (MPNT) if it satisfies the following properties:

(i) ${}^0\mathcal{M}_{\mathfrak{R}}, {}^1\mathcal{M}_{\mathfrak{R}} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.
(ii) If $(\mathcal{M}_{\mathfrak{R}})_{\wp} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}, \forall \wp \in \Delta$, then $\bigcup_{\wp \in \Delta} (\mathcal{M}_{\mathfrak{R}})_{\wp} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.
(iii) If $\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$, then $\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.

Then the pair $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ is called MPNTS. The members of $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$ are called open MPNSs and their complements are called closed MPNSs.

Theorem 3.2 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be an MPNTS. Then the following conditions are satisfied:*

(i) *${}^0\mathcal{M}_{\mathfrak{R}}$ and ${}^1\mathcal{M}_{\mathfrak{R}}$ are open MPNSs.*
(ii) Union of any number of open MPNSs is open.

(iii) Intersection of finite number of closed MPNSs is closed.

Proof The proof is obvious. □

Example 3.3 Let $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be an assembling of books. Then $\mathfrak{mpn}(\mathcal{Q})$ be the collection of all MPNSs in $\mathcal{Q}$. We consider two 3-polar neutrosophic subsets of $\mathfrak{mpn}(\mathcal{Q})$ given as

$$\mathcal{M}_{\mathfrak{R}_1} = \Big\{(\varsigma_1, \langle 0.871, 0.451, 0.412\rangle, \langle 0.317, 0.412, 0.321\rangle, \langle 0.187, 0.213, 0.118\rangle), (\varsigma_2, \langle 0.547, 0.158, 0.413\rangle, \langle 0.518, 0.152, 0.118\rangle, \langle 0.618, 0.418, 0.321\rangle), (\varsigma_3, \langle 0.618, 0.341, 0.231\rangle, \langle 0.815, 0.118, 0.527\rangle, \langle 0.511, 0.431, 0.215\rangle), (\varsigma_4, \langle 0.518, 0.391, 0.812\rangle, \langle 0.815, 0.321, 0.415\rangle, \langle 0.911, 0.321, 0.512\rangle)\Big\}$$

$$\mathcal{M}_{\mathfrak{R}_2} = \Big\{(\varsigma_1, \langle 0.611, 0.512, 0.611\rangle, \langle 0.218, 0.531, 0.415\rangle, \langle 0.035, 0.311, 0.211\rangle), (\varsigma_2, \langle 0.212, 0.218, 0.513\rangle, \langle 0.435, 0.218, 0.315\rangle, \langle 0.519, 0.511, 0.438\rangle), (\varsigma_3, \langle 0.418, 0.432, 0.321\rangle, \langle 0.639, 0.218, 0.357\rangle, \langle 0.211, 0.531, 0.316\rangle), (\varsigma_4, \langle 0.219, 0.491, 0.815\rangle, \langle 0.716, 0.421, 0.518\rangle, \langle 0.712, 0.421, 0.618\rangle)\Big\}$$

Then clearly the collection $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} = \{{}^0\mathcal{M}_{\mathfrak{R}}, {}^1\mathcal{M}_{\mathfrak{R}}, \mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2}\}$ is 3-polar neutrosophic topological space.

Definition 3.4 Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ and $(\mathcal{Q}, \mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}})$ be two MPNTSs in $\mathcal{Q}$. Two MPNTSs are said to be comparable if $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} \subseteq \mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$ or $\mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}} \subseteq \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.

If $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} \subseteq \mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$, then $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$ is courser or weaker than $\mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$ and $\mathcal{T}'_{\mathcal{M}_{\mathfrak{R}}}$ is stronger and finer than $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}$.

Theorem 3.5 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be an MPNTS. Then the following conditions are satisfied:*

(i) *${}^0\mathcal{M}_{\mathfrak{R}}$ and ${}^1\mathcal{M}_{\mathfrak{R}}$ are closed MPNSs.*
(ii) *Intersection of any number of closed MPNSs is closed.*
(iii) *Union of finite number of closed MPNSs is closed.*

Proof

(i) $({}^1\mathcal{M}_{\mathfrak{R}})^c = {}^0\mathcal{M}_{\mathfrak{R}}$ and $({}^0\mathcal{M}_{\mathfrak{R}})^c = {}^1\mathcal{M}_{\mathfrak{R}}$ are both open and closed MPNSs.
(ii) If $\{\mathcal{M}_{\mathfrak{R}_\alpha} : \mathcal{M}^c_{\mathfrak{R}_\alpha} \in \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}}, \alpha \in \Delta\}$ is an assembling of closed MPNSs then $(\bigcap_{\alpha\in\Delta} \mathcal{M}_{\mathfrak{R}_\alpha})^c = \bigcup_{\alpha\in\Delta} \mathcal{M}^c_{\mathfrak{R}_\alpha}$ is open. This shows that $\bigcap_{\alpha\in\Delta} \mathcal{M}_{\mathfrak{R}_\alpha}$ is closed MPNS.
(iii) Since $\mathcal{M}_{\mathfrak{R}_\beta}$ is closed for $\beta = 1, 2, \ldots, z$, then $(\bigcup_{\beta=1}^{z} \mathcal{M}_{\mathfrak{R}_\beta})^c = \bigcap_{\beta=1}^{z} \mathcal{M}^c_{\mathfrak{R}_\beta}$ is open MPNS. Thus $\bigcup_{\beta=1}^{z} \mathcal{M}_{\mathfrak{R}_\beta}$ is closed MPNS.

□

Definition 3.6 Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{R}})$, then interior of $\mathcal{M}_{\mathfrak{R}}$ is denoted as $\mathcal{M}^o_{\mathfrak{R}}$ and defined as the union of all open MPN subsets contained in $\mathcal{M}_{\mathfrak{R}}$. It is the greatest open MPNS contained in $\mathcal{M}_{\mathfrak{R}}$.

Example 3.7 We consider the 3-polar neutrosophic topological space constructed in Example 3.3 and let $\mathcal{M}_{\mathfrak{R}_3} \in \mathfrak{mpn}(\mathcal{Q})$ given as

$$\mathcal{M}_{\mathfrak{R}_3} = \{(\varsigma_1, \langle 0.713, 0.412, 0.311\rangle, \langle 0.318, 0.418, 0.311\rangle, \langle 0.451, 0.211, 0.218\rangle), (\varsigma_2, \langle 0.312, 0.117, 0.418\rangle, \langle 0.513, 0.212, 0.218\rangle, \langle 0.613, 0.411, 0.438\rangle), (\varsigma_3, \langle 0.518, 0.321, 0.311\rangle, \langle 0.718, 0.118, 0.257\rangle, \langle 0.317, 0.461, 0.217\rangle), (\varsigma_1, \langle 0.319, 0.219, 0.615\rangle, \langle 0.719, 0.321, 0.418\rangle, \langle 0.811, 0.321, 0.417\rangle)\}$$

Then $\mathcal{M}^o_{\mathfrak{R}_3} = {}^o\mathcal{M}_{\mathfrak{R}} \cup \mathcal{M}_{\mathfrak{R}_2} = \mathcal{M}_{\mathfrak{R}_2}$ is open MPNS.

Theorem 3.8 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}(\mathcal{Q})$. Then $\mathcal{M}_{\mathfrak{R}}$ is open MPNS $\Leftrightarrow \mathcal{M}^o_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$.*

Proof If $\mathcal{M}_{\mathfrak{R}}$ is open MPNS then greatest open MPNS contained in $\mathcal{M}_{\mathfrak{R}}$ is itself $\mathcal{M}_{\mathfrak{R}}$. Thus $\mathcal{M}^o_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$.

Conversely, if $\mathcal{M}^o_{\mathfrak{R}} = \mathcal{M}_{\mathfrak{R}}$ then $\mathcal{M}^o_{\mathfrak{R}}$ is open MPNS. This implies that $\mathcal{M}_{\mathfrak{R}}$ is open MPNS. □

Theorem 3.9 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2} \in \mathfrak{mpn}({}^1\mathcal{M}_{\mathfrak{R}})$, then*

(i) $(\mathcal{M}^o_{\mathfrak{R}_1})^o = \mathcal{M}^o_{\mathfrak{R}_1}$,
(ii) $\mathcal{M}_{\mathfrak{R}_1} \subseteq \mathcal{M}_{\mathfrak{R}_2} \Rightarrow \mathcal{M}^o_{\mathfrak{R}_1} \subseteq \mathcal{M}^o_{\mathfrak{R}_2}$,
(iii) $(\mathcal{M}_{\mathfrak{R}_1} \cap \mathcal{M}_{\mathfrak{R}_2})^o = \mathcal{M}^o_{\mathfrak{R}_1} \cap \mathcal{M}^o_{\mathfrak{R}_2}$,
(iv) $(\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}_{\mathfrak{R}_2})^o \supseteq \mathcal{M}^o_{\mathfrak{R}_1} \cup \mathcal{M}^o_{\mathfrak{R}_2}$.

Proof The proof is obvious. □

Definition 3.10 Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}(\mathcal{Q})$, then the closure of $\mathcal{M}_{\mathfrak{R}}$ is denoted by $\overline{\mathcal{M}_{\mathfrak{R}}}$ and defined by intersection of all closed-MPN supersets of $\mathcal{M}_{\mathfrak{R}}$. It is the smallest closed-MPN superset of $\mathcal{M}_{\mathfrak{R}}$.

Example 3.11 We consider the 3-polar neutrosophic topological space constructed in Example 3.3, then closed MPNSs are given as,

$$^{o}\mathcal{M}^{c}_{\mathfrak{R}} = {}^{1}\mathcal{M}_{\mathfrak{R}}, {}^{1}\mathcal{M}^{c}_{\mathfrak{R}} = {}^{o}\mathcal{M}_{\mathfrak{R}},$$

$$\begin{aligned}\mathcal{M}^{c}_{\mathfrak{R}_1} = \{&(\varsigma_1, \langle 0.412, 0.549, 0.871\rangle, \langle 0.321, 0.588, 0.317\rangle, \\ &\langle 0.118, 0.787, 0.187\rangle), (\varsigma_2, \langle 0.413, 0.842, 0.547\rangle, \\ &\langle 0.118, 0.848, 0.518\rangle, \langle 0.321, 0.582, 0.618\rangle), \\ &(\varsigma_3, \langle 0.231, 0.659, 0.618\rangle, \langle 0.257, 0.882, 0.815\rangle, \\ &\langle 0.215, 0.569, 0.511\rangle), (\varsigma_4, \langle 0.812, 0.609, 0.518\rangle, \\ &\langle 0.415, 0.679, 0.815\rangle, \langle 0.512, 0.679, 0.911\rangle)\}\end{aligned}$$

$$\begin{aligned}\mathcal{M}^{c}_{\mathfrak{R}_2} = \{&(\varsigma_1, \langle 0.611, 0.488, 0.611\rangle, \langle 0.415, 0.487, 0.218\rangle, \\ &\langle 0.211, 0.689, 0.035\rangle), (\varsigma_2, \langle 0.513, 0.782, 0.212\rangle, \\ &\langle 0.315, 0.782, 0.435\rangle, \langle 0.438, 0.489, 0.519\rangle), \\ &(\varsigma_3, \langle 0.321, 0.568, 0.418\rangle, \langle 0.357, 0.782, 0.639\rangle, \\ &\langle 0.316, 0.469, 0.211\rangle), (\varsigma_4, \langle 0.815, 0.509, 0.219\rangle, \\ &\langle 0.518, 0.579, 0.716\rangle, \langle 0.618, 0.579, 0.712\rangle)\}\end{aligned}$$

Let $\mathcal{M}_{\mathfrak{R}_4} \in \mathfrak{mpn}({}^{1}\mathcal{M}_{\mathfrak{R}})$ given as

$$\begin{aligned}\mathcal{M}_{\mathfrak{R}_4} = \{&(\varsigma_1, \langle 0.319, 0.615, 0.888\rangle, \langle 0.217, 0.618, 0.411\rangle, \\ &\langle 0.115, 0.817, 0.345\rangle), (\varsigma_2, \langle 0.312, 0.888, 0.617\rangle, \\ &\langle 0.113, 0.878, 0.678\rangle, \langle 0.231, 0.598, 0.765\rangle), \\ &(\varsigma_3, \langle 0.112, 0.767, 0.887\rangle, \langle 0.213, 0.889, 0.889\rangle, \\ &\langle 0.114, 0.667, 0.665\rangle), (\varsigma_4, \langle 0.319, 0.768, 0.615\rangle, \\ &\langle 0.321, 0.778, 0.898\rangle, \langle 0.435, 0.767, 0.987\rangle)\}\end{aligned}$$

Then $\overline{\mathcal{M}_{\mathfrak{R}_4}} = {}^{1}\mathcal{M}_{\mathfrak{R}} \cap \mathcal{M}^{c}_{\mathfrak{R}_1} \cap \mathcal{M}^{c}_{\mathfrak{R}_2} = \mathcal{M}^{c}_{\mathfrak{R}_1}$ is closed MPNS.

Theorem 3.12 *Let $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$ be MPNTS and $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}(\mathcal{Q})$. $\mathcal{M}_{\mathfrak{R}}$ is closed MPNS $\Leftrightarrow \overline{\mathcal{M}_{\mathfrak{R}}} = \mathcal{M}_{\mathfrak{R}}$.*

Proof The proof is obvious. □

Definition 3.13 Let $\mathcal{M}_{\mathfrak{R}}$ be an MPN-subset of $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$, then its frontier or boundary is defined by $F_r(\mathcal{M}_{\mathfrak{R}}) = \overline{\mathcal{M}_{\mathfrak{R}}} \cap \overline{\mathcal{M}^{c}_{\mathfrak{R}}}$.

Definition 3.14 Let $\mathcal{M}_{\mathfrak{R}}$ be an MPN-subset of $(\mathcal{Q}, \mathcal{T}_{\mathcal{M}_{\mathfrak{R}}})$, then its exterior can be represented as $Ext(\mathcal{M}_{\mathfrak{R}})$ and defined as $Ext(\mathcal{M}_{\mathfrak{R}}) = (\overline{\mathcal{M}_{\mathfrak{R}}})^{c} = (\mathcal{M}^{c}_{\mathfrak{R}})^{o}$.

Example 3.15 We consider the MPNTS constructed in Example 3.3 and consider the MPNSs $\mathcal{M}_{\mathfrak{R}_3}$ and $\mathcal{M}_{\mathfrak{R}_4}$ given in Examples 3.7 and 3.11. Then by using previous definitions we can write that

$$\begin{aligned}\mathcal{M}^{o}_{\mathfrak{R}_3} &= \mathcal{M}_{\mathfrak{R}_2}, \overline{\mathcal{M}_{\mathfrak{R}_3}} = {}^{1}\mathcal{M}_{\mathfrak{R}}, \\ F_r(\mathcal{M}_{\mathfrak{R}_3}) &= {}^{1}\mathcal{M}_{\mathfrak{R}}, Ext(\mathcal{M}_{\mathfrak{R}_3}) = {}^{0}\mathcal{M}_{\mathfrak{R}}, \\ \mathcal{M}^{o}_{\mathfrak{R}_4} &= {}^{0}\mathcal{M}_{\mathfrak{R}}, \overline{\mathcal{M}_{\mathfrak{R}_4}} = \mathcal{M}^{c}_{\mathfrak{R}_1}, \\ F_r(\mathcal{M}_{\mathfrak{R}_4}) &= \mathcal{M}^{c}_{\mathfrak{R}_1}, Ext(\mathcal{M}_{\mathfrak{R}_4}) = \mathcal{M}_{\mathfrak{R}_1}.\end{aligned}$$

Now, we present some results which do not hold in MPNTS but hold in crisp set theory due to the law of contradiction and law of excluded middle.

Remark

(i) In MPNTS, the members of discrete topology are infinite due to the infinite subsets of an arbitrary MPNS.

(ii) In MPNTS law of contradiction $\mathcal{M}_{\mathfrak{R}} \cap \mathcal{M}^{c}_{\mathfrak{R}} = {}^{0}\mathcal{M}_{\mathfrak{R}}$ and law of excluded middle $\mathcal{M}_{\mathfrak{R}} \cup \mathcal{M}^{c}_{\mathfrak{R}} = {}^{1}\mathcal{M}_{\mathfrak{R}}$ do not hold in general. In Example 3.15, we can observe that $\mathcal{M}_{\mathfrak{R}_3} \cap \mathcal{M}^{c}_{\mathfrak{R}_3} \neq {}^{0}\mathcal{M}_{\mathfrak{R}}$ and $\mathcal{M}_{\mathfrak{R}_3} \cup \mathcal{M}^{c}_{\mathfrak{R}_3} \neq {}^{1}\mathcal{M}_{\mathfrak{R}}$.

(iii) In *m*-polar neutrosophic set theory, an assembling $\mathcal{T}_{\mathcal{M}_{\mathfrak{R}}} = \{{}^{0}\mathcal{M}_{\mathfrak{R}}, {}^{1}\mathcal{M}_{\mathfrak{R}}, \mathcal{M}_{\mathfrak{R}}, \mathcal{M}^{c}_{\mathfrak{R}}\}$ is not an MPNTS in general. But this result hold in classical set theory. This result can be easily seen by using Example 3.15.

Theorem 3.16 *Let $\mathcal{M}_{\mathfrak{R}} \in \mathfrak{mpn}({}^{1}\mathcal{M}_{\mathfrak{R}})$, then*

(1) $(\mathcal{M}^{o}_{\mathfrak{R}})^{c} = \overline{(\mathcal{M}^{c}_{\mathfrak{R}})}$,
(2) $(\overline{\mathcal{M}}_{\mathfrak{R}})^{c} = (\mathcal{M}^{c}_{\mathfrak{R}})^{o}$,
(3) $Ext(\mathcal{M}^{c}_{\mathfrak{R}}) = \mathcal{M}^{o}_{\mathfrak{R}}$,
(4) $Ext(\mathcal{M}_{\mathfrak{R}}) = (\mathcal{M}^{c}_{\mathfrak{R}})^{o}$,
(5) $Ext(\mathcal{M}_{\mathfrak{R}}) \cup F_r(\mathcal{M}_{\mathfrak{R}}) \cup \mathcal{M}^{o}_{\mathfrak{R}} \neq {}^{1}\mathcal{M}_{\mathfrak{R}}$,
(6) $F_r(\mathcal{M}_{\mathfrak{R}}) = F_r(\mathcal{M}^{c}_{\mathfrak{R}})$,
(7) $\mathcal{M}^{o}_{\mathfrak{R}} \cap F_r(\mathcal{M}_{\mathfrak{R}}) \neq {}^{0}\mathcal{M}_{\mathfrak{R}}$.

Proof

(1) and (2): are obvious.
(3) $Ext(\mathcal{M}^{c}_{\mathfrak{R}}) = (\overline{\mathcal{M}}^{c}_{\mathfrak{R}})^{c}$
$\Rightarrow Ext(\mathcal{M}^{c}_{\mathfrak{R}}) = [(\mathcal{M}^{c}_{\mathfrak{R}})^{c}]^{o}$
$\Rightarrow Ext(\mathcal{M}^{c}_{\mathfrak{R}}) = \mathcal{M}^{o}_{\mathfrak{R}}$.
(4) $Ext(\mathcal{M}_{\mathfrak{R}}) = (\overline{\mathcal{M}}_{\mathfrak{R}})^{c}$
$\Rightarrow Ext(\mathcal{M}_{\mathfrak{R}}) = (\mathcal{M}^{c}_{\mathfrak{R}})^{o}$.
(5) $Ext(\mathcal{M}_{\mathfrak{R}}) \cup F_r(\mathcal{M}_{\mathfrak{R}}) \cup \mathcal{M}^{o}_{\mathfrak{R}} \neq {}^{1}\mathcal{M}_{\mathfrak{R}}$. By Example 3.15, we can see that $\mathcal{M}_{\mathfrak{R}_1} \cup \mathcal{M}^{c}_{\mathfrak{R}_1} \cup {}^{0}\mathcal{M}_{\mathfrak{R}} \neq {}^{1}\mathcal{M}_{\mathfrak{R}}$.
(6) $F_r(\mathcal{M}^{c}_{\mathfrak{R}}) = \overline{(\mathcal{M}^{c}_{\mathfrak{R}})} \cap \overline{[(\mathcal{M}^{c}_{\mathfrak{R}})]^{c}}$

$\Rightarrow F_r(\mathcal{M}_{\mathfrak{N}}^c) = \overline{(\mathcal{M}_{\mathfrak{N}}^c)} \cap \overline{(\mathcal{M}_{\mathfrak{N}})} = F_r(\mathcal{M}_{\mathfrak{N}})$.

(7) $\mathcal{M}_{\mathfrak{N}}^o \cap F_r(\mathcal{M}_{\mathfrak{N}}) \neq {}^0\mathcal{M}_{\mathfrak{N}}$. Example 3.15 shows that $\mathcal{M}_{\mathfrak{N}_2} \cap {}^1\mathcal{M}_{\mathfrak{N}} \neq {}^0\mathcal{M}_{\mathfrak{N}}$.

□

3.2 Similarity Measures

In this part, we present two different formulae for similarity measures between MPNSs. This concept will help us in the forthcoming section of multi-criteria decision-making.

Definition 3.17 (**Cosine similarity measure for MPNSs**) We define the cosine similarity measure for m-polar neutrosophic sets based on Bhattacharyas distance [32, 47]. Suppose that $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}(\mathcal{M}_{\mathfrak{N}})$, in $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_l\}$. A cosine similarity measure between $\mathcal{M}_{\mathfrak{N}_1}$ $\mathcal{M}_{\mathfrak{N}_2}$ is given as

$$\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \frac{1}{ml}\sum_{\eta=1}^{l}\sum_{\alpha=1}^{m} \frac{{}^1\mathfrak{A}_\alpha(\varsigma_\eta)^2\mathfrak{A}_\alpha(\varsigma_\eta) + {}^1\mathfrak{S}_\alpha(\varsigma_\eta)^2\mathfrak{S}_\alpha(\varsigma_\eta) + {}^1\mathfrak{Y}_\alpha(\varsigma_\eta)^2\mathfrak{Y}_\alpha(\varsigma_\eta)}{\sqrt{({}^1\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{Y}_\alpha(\varsigma_\eta))^2}\sqrt{({}^2\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{Y}_\alpha(\varsigma_\eta))^2}}.$$

$\mathfrak{C}^1_{MPNS}$ satisfies the following properties,

(1) $0 \leq \mathfrak{C}^1_{MPNS} \leq 1$,
(2) $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_1})$,
(3) $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = 1$ if $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2}$,
(4) If $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \subseteq \mathcal{M}_{\mathfrak{N}_3}$ then $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2})$ and $\mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^1_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_3})$. The proof of these properties can be easily done by using the above definition.

Definition 3.18 (*Set theoretic similarity measure of MPNSs*) We define the set theoretic similarity measure for m-polar neutrosophic sets based on set theoretic viewpoint [40]. Suppose that $\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2} \in \mathfrak{mpn}(\mathcal{M}_{\mathfrak{N}})$, in $\mathcal{Q} = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_l\}$. A set theoretic similarity measure between $\mathcal{M}_{\mathfrak{N}_1}$ $\mathcal{M}_{\mathfrak{N}_2}$ is given as

$$\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \frac{1}{ml}\sum_{\eta=1}^{l}\sum_{\alpha=1}^{m} \frac{{}^1\mathfrak{A}_\alpha(\varsigma_\eta)^2\mathfrak{A}_\alpha(\varsigma_\eta) + {}^1\mathfrak{S}_\alpha(\varsigma_\eta)^2\mathfrak{S}_\alpha(\varsigma_\eta) + {}^1\mathfrak{Y}_\alpha(\varsigma_\eta)^2\mathfrak{Y}_\alpha(\varsigma_\eta)}{\max[({}^1\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^1\mathfrak{Y}_\alpha(\varsigma_\eta))^2, ({}^2\mathfrak{A}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{S}_\alpha(\varsigma_\eta))^2 + ({}^2\mathfrak{Y}_\alpha(\varsigma_\eta))^2]}.$$

$\mathfrak{C}^2_{MPNS}$ satisfies the following properties,

(1) $0 \leq \mathfrak{C}^2_{MPNS} \leq 1$,
(2) $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_2}, \mathcal{M}_{\mathfrak{N}_1})$,
(3) $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}) = 1$ if $\mathcal{M}_{\mathfrak{N}_1} = \mathcal{M}_{\mathfrak{N}_2}$,
(4) If $\mathcal{M}_{\mathfrak{N}_1} \subseteq \mathcal{M}_{\mathfrak{N}_2} \subseteq \mathcal{M}_{\mathfrak{N}_3}$ then $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2})$ and $\mathfrak{C}^2_{MPNS}(\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_3}) \leq \mathfrak{C}_{\mathfrak{MPNS}}(\mathcal{M}_{\mathfrak{N}}, \mathcal{M}_{\mathfrak{N}})$. The proof of these properties can be easily done by using the above definition.

4 Multi-criteria Decision-Making Under *m*-Polar Neutrosophic Data

Multi-criteria decision-making (MCDM) is a process to find an optimal alternative that has the highest degree of satisfaction from a set of feasible alternatives characterized by multiple criteria, and these kinds of MCDM problems arise in many real-world situations. In this section, we discuss two applications of medical diagnosis and clustering analysis of students with the help of m-polar fuzzy neutrosophic data. We present three novel algorithms for multi-criteria decision-making (MCDM) with linguistic information based on the MPNTS and MPFNSs for medical diagnosis and clustering analysis.

In each algorithm, we use m-polar neutrosophic input data. Firstly, we collect input information for every algorithm in the form of linguistic variables and then convert them into m-polar neutrosophic numbers (MPNNs) by using fuzzy logics. When our data set is covered into proposed m-polar neutrosophic numeric values, then we apply each algorithm one by one. At last, we get better results for medical diagnosis and clustering analysis.

4.1 MCDM for Medical Diagnosis

In this part of our manuscript, we establish two different techniques based on MPNTS and on similarity measures to investigate the disease with m-polar neutrosophic information.

Proposed Technique of Algorithm 1

Algorithm 1 (Algorithm for m-polar neutrosophic topological space)

Input:

Step 1: Input the set $\mathfrak{P}$ for a patient according to his doctor, corresponding to the "m" number of symptoms appearing to the patient. All the input data leads to those "p" diseases which will be possible outcome according to the appearing symptoms in the form of m-polar neutrosophic set.

Step 2: Input the sets $\Im_\xi; \xi = 1, 2, ..., z$, for "p" diseases $\eth_\delta; \delta = 1, 2, ..., p$, according to "z" number of experts, corresponding to the "m" number of symptoms in the form of m-polar neutrosophic sets (MPNSs).

Calculations:

Step 3: Construct m-polar neutrosophic topological space (MPNTS) $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$ using MPNSs $\Im_\xi; \xi = 1, 2, ..., z$ given by "z" number of experts.

Step 4: Find interior $\mathfrak{P}^o$ of $\mathfrak{P}$ by using Definition 3.6 under the constructed $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. $\mathfrak{P}^o$ shows the actual condition of the patient according to the "z" number of experts and give better decision to diagnosis.

Step 5: Calculate scores of each disease corresponding to "m" number of symptoms by using Definition 2.6.

Output:

Step 6: We rank the alternative (disease) on the basis of score values according to the Definition 2.7.

Step 7: Alternative (disease) with the higher score has the maximum rank according to the given numerical example. This implies that patient is suffering from that disease.

In this algorithm, rating of each criteria according to the corresponding alternative is constructed by using m-polar neutrosophic information for MCDM and given in input matrix (can be taken in tabular form by using m-polar neutrosophic numbers) as

$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = [\langle \mathfrak{A}^\alpha_{\eta\xi}, \mathfrak{S}^\alpha_{\eta\xi}, \mathfrak{Y}^\alpha_{\eta\xi}\rangle]_{r\times s}; \alpha = 1, 2, 3, \ldots, m$$

$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = \begin{pmatrix} (\langle \mathfrak{A}^\alpha_{11}, \mathfrak{S}^\alpha_{11}, \mathfrak{Y}^\alpha_{11}\rangle) & (\langle \mathfrak{A}^\alpha_{12}, \mathfrak{S}^\alpha_{12}, \mathfrak{Y}^\alpha_{12}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{1s}, \mathfrak{S}^\alpha_{1s}, \mathfrak{Y}^\alpha_{1s}\rangle) \\ (\langle \mathfrak{A}^\alpha_{21}, \mathfrak{S}^\alpha_{21}, \mathfrak{Y}^\alpha_{21}\rangle) & (\langle \mathfrak{A}^\alpha_{22}, \mathfrak{S}^\alpha_{22}, \mathfrak{Y}^\alpha_{22}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{2s}, \mathfrak{S}^\alpha_{2s}, \mathfrak{Y}^\alpha_{2s}\rangle) \\ \vdots & \vdots & \ddots & \vdots \\ (\langle \mathfrak{A}^\alpha_{r1}, \mathfrak{S}^\alpha_{r1}, \mathfrak{Y}^\alpha_{r1}\rangle) & (\langle \mathfrak{A}^\alpha_{r2}, \mathfrak{S}^\alpha_{r2}, \mathfrak{Y}^\alpha_{r2}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{rs}, \mathfrak{S}^\alpha_{rs}, \mathfrak{Y}^\alpha_{rs}\rangle) \end{pmatrix} \quad (1)$$

In matrix $\mathfrak{P}$, the entries $\mathfrak{A}^\alpha_{\eta\xi}$, $\mathfrak{S}^\alpha_{\eta\xi}$, and $\mathfrak{Y}^\alpha_{\eta\xi}$ represent truth, indeterminacy, and falsity membership grades for alternative $\eth_\eta$ corresponding to the criteria $\mathfrak{C}_\xi$, where $\eta = 1, 2, 3, \ldots, r; \xi = 1, 2, 3, \ldots, s$. These grades satisfies the following properties under MPN environment.

(1) $0 \le \mathfrak{A}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{S}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{Y}^\alpha_{\eta\xi} \le 1$.

(2) $0 \le \mathfrak{A}^\alpha_{\eta\xi} + \mathfrak{S}^\alpha_{\eta\xi} + \mathfrak{Y}^\alpha_{\eta\xi} \le 3$, for $\eta = 1, 2, 3, \ldots, r; \xi = 1, 2, 3, \ldots, s; \alpha = 1, 2, 3 \ldots, m$.

The rating of each criteria corresponding to the alternative for m-triplets is illustrated in this work. The input decision matrices $\Im_\xi; \xi = 1, 2, 3, \ldots, z$ for z number of experts can be written by using m-polar neutrosophic data same as Equation 2. Then we construct an m-polar neutrosophic topological space $\mathcal{T}_{\mathcal{M}N}$ by using experts data $\Im_\xi; \xi = 1, 2, 3, \ldots, z$. Find interior $\mathfrak{P}^o$ of MPN-matrix $\mathfrak{P}$ under the constructed $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. Then we calculate score values of all the alternatives in $\mathfrak{P}^o$. We rank these fuzzy values and choose alternative having maximum fuzzy value as an optimal decision. The step-wise description of this proposed technique is given as Algorithm 1.

4.1.1 Proposed Technique of Algorithm 2:

In this algorithm, rating of each criteria according to the corresponding alternative is constructed by using m-polar neutrosophic information for MCDM and given in input matrix (can be taken in tabular form by using m-polar neutrosophic numbers) as

$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = [\langle \mathfrak{A}^\alpha_{\eta\xi}, \mathfrak{S}^\alpha_{\eta\xi}, \mathfrak{Y}^\alpha_{\eta\xi}\rangle]_{r\times s}; \alpha = 1, 2, 3, \ldots, m$$

$$\mathfrak{P} = [\Im^\alpha_{\eta\xi}]_{r\times s} = \begin{pmatrix} (\langle \mathfrak{A}^\alpha_{11}, \mathfrak{S}^\alpha_{11}, \mathfrak{Y}^\alpha_{11}\rangle) & (\langle \mathfrak{A}^\alpha_{12}, \mathfrak{S}^\alpha_{12}, \mathfrak{Y}^\alpha_{12}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{1s}, \mathfrak{S}^\alpha_{1s}, \mathfrak{Y}^\alpha_{1s}\rangle) \\ (\langle \mathfrak{A}^\alpha_{21}, \mathfrak{S}^\alpha_{21}, \mathfrak{Y}^\alpha_{21}\rangle) & (\langle \mathfrak{A}^\alpha_{22}, \mathfrak{S}^\alpha_{22}, \mathfrak{Y}^\alpha_{22}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{2s}, \mathfrak{S}^\alpha_{2s}, \mathfrak{Y}^\alpha_{2s}\rangle) \\ \vdots & \vdots & \ddots & \vdots \\ (\langle \mathfrak{A}^\alpha_{r1}, \mathfrak{S}^\alpha_{r1}, \mathfrak{Y}^\alpha_{r1}\rangle) & (\langle \mathfrak{A}^\alpha_{r2}, \mathfrak{S}^\alpha_{r2}, \mathfrak{Y}^\alpha_{r2}\rangle) & \cdots & (\langle \mathfrak{A}^\alpha_{rs}, \mathfrak{S}^\alpha_{rs}, \mathfrak{Y}^\alpha_{rs}\rangle) \end{pmatrix} \quad (2)$$

In matrix $\mathfrak{P}$, the entries $\mathfrak{A}^\alpha_{\eta\xi}$, $\mathfrak{S}^\alpha_{\eta\xi}$, and $\mathfrak{Y}^\alpha_{\eta\xi}$ represents truth, indeterminacy, and falsity membership grades for alternative $\eth_\eta$ corresponding to the criteria $\mathfrak{C}_\xi$, where $\eta = 1, 2, 3, \ldots, r; \xi = 1, 2, 3, \ldots, s$. These grades satisfies the following properties under MPN environment.

(1) $0 \le \mathfrak{A}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{S}^\alpha_{\eta\xi} \le 1; 0 \le \mathfrak{Y}^\alpha_{\eta\xi} \le 1$.

(2) $0 \le \mathfrak{A}^\alpha_{\eta\xi} + \mathfrak{S}^\alpha_{\eta\xi} + \mathfrak{Y}^\alpha_{\eta\xi} \le 3$, for $\eta = 1, 2, 3, \ldots, r; \xi = 1, 2, 3, \ldots, s; \alpha = 1, 2, 3 \ldots, m$.

The rating of each criteria corresponding to the alternative for m-triplets is illustrated in this work. The input decision matrices $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$ for z number of experts can be written by using m-polar neutrosophic data same as Equation 2. We calculate cosine similarity measure and set theoretic similarity measure between $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$ and $\mathfrak{P}$. We choose the m-polar neutrosophic sets from $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$ having highest cosine similarity measure and highest set theoretic similarity measure. Then we calculate score values of all the alternatives in the selected sets from $\Im_{\xi}; \xi = 1, 2, 3, \ldots, z$. We rank these fuzzy values and choose alternative having maximum fuzzy value as an optimal decision. The step-wise description of this proposed technique is given as Algorithm 2.

Algorithm 2 (Algorithm for m-polar neutrosophic sets using similarity measures)

Input:

Step 1: Input the set $\mathfrak{P}$ for a patient according to his doctor, corresponding to the "m" number of symptoms appearing to the patient. All the input data leads to those "p" diseases which will be possible outcome according to the appearing symptoms in the form of m-polar neutrosophic set.

Step 2: Input the sets $\Im_{\xi}; \xi = 1, 2, ..., z$, for "p" diseases $\eth_{\delta}; \delta = 1, 2, ..., p$, according to "z" number of experts, corresponding to the "m" number of symptoms in the form of m-polar neutrosophic sets (MPNSs).

Calculations:

Step 3: calculate cosine similarity measure using Definition 3.17 between $\Im_{\xi}; \xi = 1, 2, ..., z$ and $\mathfrak{P}$.

Step 3': calculate set theoretic similarity measure using Definition 3.18 between $\Im_{\xi}; \xi = 1, 2, ..., z$ and $\mathfrak{P}$.

Step 4: Choose the MPNS from $\Im_{\xi}; \xi = 1, 2, ..., z$ having highest cosine similarity measure with $\mathfrak{P}$. That $\Im_{\xi}$ gives the best decision for diagnosis of patient.

Step 4': Choose the MPNS from $\Im_{\xi}; \xi = 1, 2, ..., z$ having highest set theoretic similarity measure with $\mathfrak{P}$. That $\Im_{\xi}$ gives the best decision for diagnosis of patient.

Step 5: Calculate scores of each disease $\eth_{\delta}$ of selected $\Im_{\xi}$ after finding cosine and set theoretic similarity measures corresponding to "m" number of symptoms by using Definition 2.6. From this method we get two different results (rankings) according to two different similarity measures.

Output:

Step 6: We rank the alternative (disease) on the basis of score values according to the Definition 2.7.

Step 7: Alternative (disease) with the higher score has the maximum rank according to the given numerical example. This implies that patient is suffering from that disease.

The flow chart diagram of proposed algorithms can be seen in Fig. 2.

4.1.2 Numerical example

Suppose that a patient is facing some health issues and the symptoms are temperature, headache, fatigue, loss of appetite, stomach pain, inadequate immune system, muscle, and joint pain. According to the doctor's opinion, all these symptoms lead to the following diseases Tuberculosis, Hepatitis C, and Typhoid fever. Let us consider the set $\mathcal{Q} = \{\eth_1, \eth_2, \eth_3\}$ of the alternatives consisting of three diseases and the set $\mathfrak{Z} = \{\mathcal{J}, \mathcal{J}, \mathcal{J}, \mathcal{J}\}$ of symptoms, where

$\eth_1 =$ Tuberculosis, $\eth_2 =$ Hepatitis C, $\eth_3 =$ Typhoid fever,
$\mathcal{J}_1 =$ Fever, $\mathcal{J}_2 =$ Poor immune system
$\mathcal{J}_3 =$ Muscle and joint pain, fatigue,
$\mathcal{J}_4 =$ Unintentional weight loss, loss of appetite

We input the data of patient according to his doctor in the form of 4-polar neutrosophic set for each disease corresponding to every symptom. In this data, the numeric values corresponding to each symptom show that how many chances he have to be suffered from the considered disease. In Table 5 for disease $\eth_1 =$Tuberculosis, the first triplet $\langle 0.635, 0.115, 0.114 \rangle$ shows that according to his symptom "$\mathcal{J}_1 =$fever" patient has $63,5\%$ truth chances, 11.5% indeterminacy, and 11.4% falsity chances to have tuberculosis. Similarly, we can observe all values of patient according to his symptoms for all diseases.

We consider that we have "z=3" highly qualified experts, then according to these experts the data of each disease corresponding to each symptom is given in tabular form of 4-polar neutrosophic sets as Tables 6, 7, and 8. Each $\Im_{\xi}; \xi = 1, 2, 3$ representing the data of each disease corresponding to each symptom according to 3 experts. This means that for expert $\Im_1$ and disease $\eth_1 =$tuberculosis

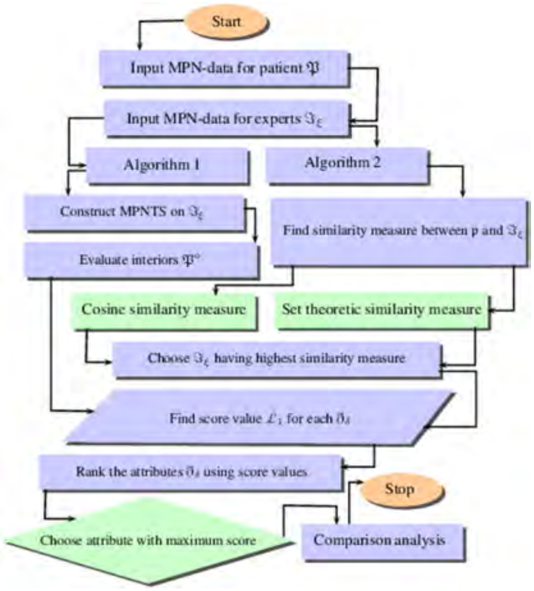

Fig. 2 Flowchart diagram of proposed algorithm 1 and algorithm 2

the first triplet $\langle 0.511, 0.311, 0.213\rangle$ shows that according to symptom "$\mathcal{J}_1$ = fever" there are 63, 5% truth chances, 11.5% indeterminacy, and 11.4% falsity chances to have tuberculosis. On the same pattern, we can observe all values of diseases according to the corresponding symptoms for each expert.

4.1.3 Solution by using Algorithm 1

Now we construct 4-polar neutrosophic topological space $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$ on $\mathfrak{I}_\xi; \xi = 1, 2, 3$ given as $\mathcal{T}_{\mathcal{MN}} = \{\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, {}^0\mathcal{M}_\mathfrak{N}, {}^1\mathcal{M}_\mathfrak{N}\}$. We find the interior $\mathfrak{P}^o$ of $\mathfrak{P}$ by using Definition 3.6 under the 4PNTS $\mathcal{T}_{\mathcal{M}_\mathfrak{N}}$. Thus $\mathfrak{P}^o = {}^0\mathcal{M}_\mathfrak{N} \cup \mathfrak{I}_1 \cup \mathfrak{I}_2 = \mathfrak{I}_2$. Now we use Definition 2.6 on $\mathfrak{I}_2$ to find scores of all the diseases $\eth_\delta, \delta = 1, 2, 3$.

$$\begin{aligned}\pounds_1(\mathfrak{I}_{2\eth_1}) = &\frac{1}{2\times 4}(4 + (0.611 - 2(0.213) - 0.118)\\ &+ (0.711 - 2(0.321) - 0.118)\\ &+ (0.412 - 2(0.511) - 0.611)\\ &+ (0.813 - 2(0.211) - 0.341)) = 0.3558.\end{aligned}$$

Similarly, we can find $\pounds_1(\mathfrak{I}_{2\eth_2}) = 0.662$ and $\pounds_1(\mathfrak{I}_{2\eth_3}) = 0.3691$. By Definition 2.7 we can write that $\eth_2 \succ \eth_3 \succ \eth_1$. Hence, patient is suffering from Hepatitis C. Graphically results can be seen as Fig. 3.

Table 5 4-Polar neutrosophic data of patient $\mathfrak{P}$

$\mathfrak{P}$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.635, 0.115, 0.114\rangle, \langle 0.813, 0.239, 0.115\rangle, \langle 0.513, 0.431, 0.513\rangle \langle 0.911, 0.119, 0.238\rangle)$
$\eth_2$	$(\langle 0.739, 0.119, 0.115\rangle, \langle 0.923, 0.111, 0.108\rangle, \langle 0.889, 0.108, 0.117\rangle, \langle 0.835, 0.113, 0.218\rangle)$
$\eth_3$	$(\langle 0.919, 0.113, 0.122\rangle, \langle 0.818, 0.112, 0.211\rangle, \langle 0.611, 0.513, 0.618\rangle, \langle 0.713, 0.218, 0.319\rangle)$

Table 6 4-polar neutrosophic data for expert $\mathfrak{I}_1$

$\mathfrak{I}_1$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.511, 0.311, 0.213\rangle, \langle 0.631, 0.431, 0.211\rangle, \langle 0.328, 0.611, 0.782\rangle \langle 0.713, 0.348, 0.411\rangle)$
$\eth_2$	$(\langle 0.638, 0.324, 0.237\rangle, \langle 0.816, 0.118, 0.119\rangle, \langle 0.717, 0.115, 0.218\rangle, \langle 0.719, 0.222, 0.249\rangle)$
$\eth_3$	$(\langle 0.889, 0.212, 0.213\rangle, \langle 0.699, 0.189, 0.232\rangle, \langle 0.413, 0.718, 0.818\rangle, \langle 0.518, 0.421, 0.518\rangle)$

Table 7 4-polar neutrosophic data for expert $\mathfrak{I}_2$

$\mathfrak{I}_2$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.611, 0.213, 0.118\rangle, \langle 0.711, 0.321, 0.118\rangle, \langle 0.412, 0.511, 0.611\rangle \langle 0.813, 0.211, 0.341\rangle)$
$\eth_2$	$(\langle 0.718, 0.211, 0.117\rangle, \langle 0.916, 0.113, 0.112\rangle, \langle 0.817, 0.113, 0.211\rangle, \langle 0.815, 0.211, 0.234\rangle)$
$\eth_3$	$(\langle 0.918, 0.116, 0.132\rangle, \langle 0.713, 0.116, 0.213\rangle, \langle 0.511, 0.611, 0.713\rangle, \langle 0.613, 0.321, 0.416\rangle)$

Table 8 4-polar neutrosophic data for expert $\mathfrak{I}_3$

$\mathfrak{I}_3$	4-polar neutrosophic sets
$\eth_1$	$(\langle 0.711, 0.118, 0.108\rangle, \langle 0.811, 0.213, 0.108\rangle, \langle 0.512, 0.421, 0.521\rangle \langle 0.815, 0.118, 0.213\rangle)$
$\eth_2$	$(\langle 0.723, 0.119, 0.111\rangle, \langle 0.928, 0.112, 0.110\rangle, \langle 0.888, 0.111, 0.119\rangle, \langle 0.889, 0.181, 0.201\rangle)$
$\eth_3$	$(\langle 0.929, 0.115, 0.128\rangle, \langle 0.813, 0.112, 0.211\rangle, \langle 0.611, 0.511, 0.613\rangle, \langle 0.718, 0.213, 0.325\rangle)$

Fig. 3 Ranking of alternatives under MPNTS

4.1.4 Solution by using Algorithm 2

Now by using Tables 5, 6, 7, and 8, we find cosine similarity measures between $(\mathfrak{I}_1, \mathfrak{P}), (\mathfrak{I}_2, \mathfrak{P})$ and $(\mathfrak{I}_3, \mathfrak{P})$ by using Definition 3.17 given as

$$\mathfrak{C}^1_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{1}{3\times 4}\Bigg(\frac{(0.611)(0.635)+(0.213)(0.115)+(0.118)(0.114)}{\sqrt{(0.611)^2+(0.213)^2+(0.118)^2}\sqrt{(0.635)^2+(0.115)^2+(0.114)^2}} + \frac{(0.711)(0.813)+(0.321)(0.329)+(0.118)(0.115)}{\sqrt{(0.711)^2+(0.321)^2+(0.118)^2}\sqrt{(0.813)^2+(0.329)^2+(0.115)^2}} + \cdots + \frac{(0.613)(0.713)+(0.321)(0.218)+(0.416)(0.319)}{\sqrt{(0.613)^2+(0.321)^2+(0.416)^2}\sqrt{(0.713)^2+(0.218)^2+(0.319)^2}}\Bigg).$$

$\mathfrak{C}^1_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{11.89053}{12} = 0.990878$. Similarly, we can find similarity between other MPNSs given as $\mathfrak{C}^1_{MPNS}(\mathfrak{I}_1, \mathfrak{P}) = \frac{11.50807}{12} = 0.95900$, $\mathfrak{C}^1_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) = \frac{11.996}{12} = 0.99966$. This shows that $\mathfrak{C}^1_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) \succ \mathfrak{C}^1_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) \succ \mathfrak{C}^1_{MPNS}(\mathfrak{I}_1, \mathfrak{P})$. From this ranking it is clear to see that opinion of expert $\mathfrak{I}_3$ is most related and similar to the condition of patient $\mathfrak{P}$. So, we select $\mathfrak{I}_3$ and calculate score values of all diseases $\mathfrak{d}_\delta; \delta = 1, 2, 3$ by using Definition 2.6. This implies that $\pounds_1(\mathfrak{I}_{3\mathfrak{d}_1}) = 0.5198$, $\pounds_1(\mathfrak{I}_{3\mathfrak{d}_2}) = 0.7301$, $\pounds_1(\mathfrak{I}_{3\mathfrak{d}_3}) = 0.4977$. By Definition 2.7 we can write that $\mathfrak{d}_2 \succ \mathfrak{d}_1 \succ \mathfrak{d}_3$. Hence patient is suffering from Hepatitis C.

Now, we use set theoretic similarity measure $\mathfrak{C}^2_{MPNS}$ to find similarity between $(\mathfrak{I}_1, \mathfrak{P}), (\mathfrak{I}_2, \mathfrak{P})$ and $(\mathfrak{I}_3, \mathfrak{P})$ by using Definition 3.18 given as

$$\mathfrak{C}^2_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{1}{3\times 4}\Bigg(\frac{(0.611)(0.635)+(0.213)(0.115)+(0.118)(0.114)}{\max((0.611)^2+(0.213)^2+(0.118)^2, (0.635)^2+(0.115)^2+(0.114)^2)} + \frac{(0.711)(0.813)+(0.321)(0.329)+(0.118)(0.115)}{\max((0.711)^2+(0.321)^2+(0.118)^2, (0.813)^2+(0.329)^2+(0.115)^2)} + \cdots + \frac{(0.613)(0.713)+(0.321)(0.218)+(0.416)(0.319)}{\max((0.613)^2+(0.321)^2+(0.416)^2, (0.713)^2+(0.218)^2+(0.319)^2)}\Bigg).$$

$\mathfrak{C}^2_{MPNS}(\mathfrak{I}_2, \mathfrak{P}) = \frac{10.44972}{12} = 0.87081$. Similarly, we can find similarity between other MPNSs given as $\mathfrak{C}^2_{MPNS}(\mathfrak{I}_1, \mathfrak{P}) = \frac{10.51971}{12} = 0.87664$, $\mathfrak{C}^2_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) = \frac{11.2283}{12} = 0.9355$. This shows that $\mathfrak{C}^2_{MPNS}(\mathfrak{I}_3, \mathfrak{P}) \succ \mathfrak{C}^2_{MPNS}(\mathfrak{I}_1, \mathfrak{P}) \succ \mathfrak{C}^2_{MPNS}(\mathfrak{I}_2, \mathfrak{P})$. From this ranking it is clear to see that opinion of expert $\mathfrak{I}_3$ is most related and similar to the condition of patient $\mathfrak{P}$. So, we select $\mathfrak{I}_3$ and calculate score values of all diseases $\mathfrak{d}_\delta; \delta = 1, 2, 3$ by using Definition 2.6. This implies that $\pounds_1(\mathfrak{I}_{3\mathfrak{d}_1}) = 0.5198$, $\pounds_1(\mathfrak{I}_{3\mathfrak{d}_2}) = 0.7301$, $\pounds_1(\mathfrak{I}_{3\mathfrak{d}_3}) = 0.4977$. By Definition 2.7 we can write that $\mathfrak{d}_2 \succ \mathfrak{d}_1 \succ \mathfrak{d}_3$. Hence patient is suffering from Hepatitis C. Graphically results can be seen as Fig 4.

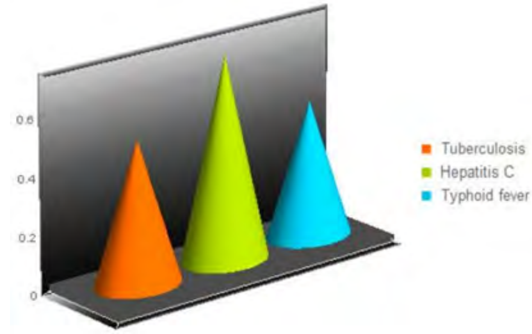

Fig. 4 Ranking of attributes under similarity measures

4.1.5 Discussion and Comparison Analysis:

In this section, we discuss advantages validity, simplicity, flexibility, and superiority of our proposed approach and algorithms. We also give a brief comparison analysis of proposed method with existing approaches.

Advantages of Proposed Approach

Now we discuss some advantages of the proposed techniques based on MPNSs.

(i) Validity of the Method

The suggested method is valid and suitable for all types of input data. we present two novel algorithms in this manuscript one for MPNTS and other for similarity measures. We introduced two similarity measures between MPNSs. It is interesting to note that both algorithms and both formulas of similarity gives the same result (see Table 9). In this work, both algorithms have their own importance and can be used according to the requirement of decision-maker. Both algorithms are valid and give best decision in multi-criteria decision-making (MCDM) problems.

(ii) Simplicity and Flexibility Dealing with Different Criteria

In MCDM problems, we experience different types of criteria and input data according to the given situations. The proposed algorithms are simple and easy to understand which can be applied easily on whatever type of alternatives and measures. Both algorithms are flexible and easily altered according to the different situations, inputs, and outputs. There is a slightly difference between the ranking of the proposed approaches because different formulae have

Table 9 Score values for optimal choice under both algorithms

Algorithm	Method	$\eth_1$	$\eth_2$	$\eth_3$	Ranking of alternatives
Algorithm1	*m*-Polar neutrosophic topological space	0.3558	0.622	0.3691	$\eth_2 \succ \eth_3 \succ \eth_1$
Algorithm2	Cosine similarity on *m*-polar neutrosophic sets	0.5198	0.7301	0.4977	$\eth_2 \succ \eth_1 \succ \eth_3$
Algorithm2	Set theoretic similarity on *m*-polar neutrosophic sets	0.5198	0.7301	0.4977	$\eth_2 \succ \eth_1 \succ \eth_3$

Table 10 Comparison of proposed algorithms with some existing approaches

Methods	Similarity measures on sets	Ranking of alternatives
Wei [37]	Picture fuzzy set	$\eth_2 \succ \eth_1 \succ \eth_3$
Xu and Chen [39, 40]	Intuitionistic fuzzy set and correlation measures	$\eth_2 \succ \eth_1 \succ \eth_3$
Ye [45]	Correlation coefficient of neutrosophic set	$\eth_2 \succ \eth_1 \succ \eth_3$
Ye [47]	Intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Li and Cheng [17]	Intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Lin [18]	Hesitant fuzzy linguistic information	$\eth_2 \succ \eth_1 \succ \eth_3$
Wei [38]	Interval-valued intuitionistic fuzzy set	$\eth_2 \succ \eth_3 \succ \eth_1$
Proposed algorithm1	*m*-Polar neutrosophic topological space	$\eth_2 \succ \eth_3 \succ \eth_1$
Proposed algorithm2	Cosine similarity on *m*-polar neutrosophic sets	$\eth_2 \succ \eth_1 \succ \eth_3$
Proposed algorithm2	Set theooretic similarity on *m*-polar neutrosophic sets	$\eth_2 \succ \eth_1 \succ \eth_3$

different ordering strategies, so they can afford the slightly different effect according to their deliberations.

(iii) Superiority of Proposed Method

From all above discussion, we observe that our proposed models of *m*-polar neutrosophic set and *m*-polar neutrosophic topological space are superior to existing approaches including fuzzy neutrosophic sets, *m*-polar intuitionistic fuzzy sets, interval-valued *m*-polar fuzzy sets and *m*-polar fuzzy sets. Moreover, many hybrid structures of fuzzy sets become the special cases of *m*-polar neutrosophic set with the addition of some suitable conditions (see Fig. 1). So our proposed approach is valid, flexible, simple, and superior to other hybrid structures of fuzzy sets.

Comparison Analysis

(1) In our proposed method, we define *m*-polar neutrosophic topological space and two algorithms based on MPN input data. The impressive point of this model is that we can use it for mathematical modeling at a large scale or "*m*" numbers of criteria with its truth, falsity, and indeterminacy part. These *m*-degrees basically show the corresponding properties or any set criteria about the alternatives. As in giving numerical example, we use $m = 4$ to analyze the data for four symptoms appearing to the patient. The value of "m" can be taken as large as possible, which is not possible for other approaches. Moreover, many hybrid structures of fuzzy set become the special cases of *m*-polar neutrosophic set with the addition of some suitable conditions (see Fig. 1).

(2) Table 10 as given above listing the results of the comparison in the final ranking of top 3 alternatives (diseases). As it could be observed in the comparison Table 10, the best selection made by the proposed methods is comparable to already established methods which is expressive in itself and approves the reliability and validity of the proposed method. Now the question turns out that why we need to specify a novel algorithm based on this novel structure? There are many arguments which show that proposed operator is more suitable than other existing methods. As we know that intuitionistic fuzzy sets, picture fuzzy sets, fuzzy sets, hesitant fuzzy sets, neutrosophic sets, and other existing hybrid structures of fuzzy sets have some limitations and not able to present full information about the situation. But our proposed model of *m*-polar neutrosophic set is most suitable for MCDM methods and deals with multi-criteria having truth, indeterminacy, and falsity values. Due to the addition of neutrosophic nature in multi-polarity, these three grades go independent of each other and give a lot of information about the multiple criteria for the alternatives.

(3) The similarity measures for other existing hybrid structures of fuzzy set become special cases of similarity measures of m-polar neutrosophic set. So, this model is more generalized and can easily deal with the problems involving intuitionistic, neutrosophy, hesitant, picture, and fuzziness of alternatives. The constructed topological space on MPNS becomes superior to existing topological spaces and easily deals with the problems in MCDM methods.

4.2 Clustering Analysis in Multi-criteria Decision-Making

We introduce a novel clustering algorithm under m-polar neutrosophic environment to solve multi-criteria decision-making problem. Before this, we revise some basic concepts.

Definition 4.1 [41] Let $\mathcal{M}_{\mathfrak{N}_\zeta}$ be "q" m-polar neutrosophic sets (MPNSs), then $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ is said to be similarity matrix, where $g_{\beta\zeta} = \mathfrak{C}(\mathcal{M}_{\mathfrak{N}_\beta}, \mathcal{M}_{\mathfrak{N}_\zeta})$ represents the similarity measure of MPNSs $\mathcal{M}_{\mathfrak{N}_\beta}$ and $\mathcal{M}_{\mathfrak{N}_\zeta}$ and satisfy the following:

(1) $0 \leq g_{\beta\zeta} \leq 1; \beta, \zeta = 1, 2, 3, \ldots, q,$
(2) $g_{\beta\beta} = 1; \beta = 1, 2, 3, \ldots, q,$
(3) $g_{\beta\zeta} = g_{\zeta\beta}; \beta, \zeta = 1, 2, 3, \ldots, q.$

Definition 4.2 [41] Let $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ be the similarity matrix. Then $\mathcal{G}^2 = \mathcal{G} \circ \mathcal{G} = (\overline{g_{\beta\zeta}})_{q\times q}$ is said to be a composition matrix of $\mathcal{G}$, where

$$\overline{g_{\beta\zeta}} = \max_{\delta}\{\min\{g_{\beta\delta}, g_{\delta\zeta}\}\}; \quad \beta, \zeta = 1, 2, 3, \ldots, q$$

Theorem 4.3 [41] *Let $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ be a similarity matrix, then after a finite compositions ($\mathcal{G} \to \mathcal{G}^2 \to \mathcal{G}^4 \to \cdots \to \mathcal{G}^{2^\delta} \to \cdots$), $\exists$ a positive integer δ such that $\mathcal{G}^{2^\delta} = \mathcal{G}^{2^{(\delta+1)}}$. $\mathcal{G}^{2^\delta}$ is an equivalence similarity matrix.*

Definition 4.4 [41] Let $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$ be an equivalence similarity matrix. Then $\mathcal{G}_{\eth} = (g^{\eth}_{\beta\zeta})_{q\times q}$ is said to be ð-cutting matrix of $\mathcal{G}$, where

$$g^{\eth}_{\beta\zeta} = \begin{cases} 0 & \text{if} g_{\beta\zeta} < \eth \\ 1 & \text{if} g_{\beta\zeta} \geq \eth \end{cases}$$

$\beta, \zeta = 1, 2, 3, \ldots, q$ and ð is confidence level with $\eth \in [0, 1]$.

Now, we use these basic ideas for the construction of a novel clustering algorithm based on MPNSs given as algorithm 3. In the constructed numerical example of clustering analysis, we discuss algorithm 3 with more detail and clarity.

Algorithm 3 (Algorithm for clustering analysis using m-polar neutrosophic sets)

Input:

Step 1: Let $\{\mathcal{M}_{\mathfrak{N}_1}, \mathcal{M}_{\mathfrak{N}_2}, ..., \mathcal{M}_{\mathfrak{N}_q}\}$ be an assembling of MPNSs over $\mathcal{Q}$ and $\{\eta_1, \eta_2, ..., \eta_r\}$ be the collection of attributes. Input MPN-data in tabular form to see the relationship between sets and attributes.

Calculations:

Step 2: Construct similarity matrix $\mathcal{G} = (g_{\beta\zeta})_{q\times q}$, where $g_{\beta\zeta} = \mathfrak{C}(\mathcal{M}_{\mathfrak{N}_\beta}, \mathcal{M}_{\mathfrak{N}_\zeta})$ and can be calculated as

$$\mathfrak{C}(\mathcal{M}_{\mathfrak{N}_\beta}, \mathcal{M}_{\mathfrak{N}_\zeta}) = 1 - \frac{1}{3m}\sum_{\alpha=1}^{m}\sum_{t=1}^{r}\wp_t(|^{\beta t}\mathfrak{A}_\alpha - {}^{\zeta t}\mathfrak{A}_\alpha| + |^{\beta t}\mathfrak{S}_\alpha - {}^{\zeta t}\mathfrak{S}_\alpha| + |^{\beta t}\mathfrak{Y}_\alpha - {}^{\zeta t}\mathfrak{Y}_\alpha|).$$

Step 3: Find $\mathcal{G}^2$ and check whether the similarity matrix satisfy $\mathcal{G}^2 \subseteq \mathcal{G}$. If it does not hold, then find the equivalence similarity matrix $\mathcal{G}^{2^\delta}$:

$$(\mathcal{G} \to \mathcal{G}^2 \to \mathcal{G}^4 \to ... \to \mathcal{G}^{2^\delta} \to ...) \quad \text{until,} \quad \mathcal{G}^{2^\delta} = \mathcal{G}^{2^{(\delta+1)}}.$$

Step 4: Find confidence level ð and construct a ð-cutting matrix $\mathcal{G}_{\eth} = (g^{\eth}_{\beta\zeta})_{q\times q}$ by using Definition 4.4.

Output:

Step 5: Classify the MPNSs by using the following argument:
If all the members of βth line (column) in $\mathcal{G}_{\eth}$ are same as the corresponding elements of ζth line (column) in $\mathcal{G}_{\eth}$, then MPNSs $\mathcal{M}_{\mathfrak{N}_\beta}$ and $\mathcal{M}_{\mathfrak{N}_\zeta}$ are of the same type, otherwise not.

Table 11 Characteristics of decision variables

Decision variables	Characteristics for 2-polar neutrosophic soft set
Intellectually curious	$\langle \text{creative}, \text{originality} \rangle$
Obedient and punctual	$\langle \text{hard} - \text{working}, \text{honest} \rangle$
Experience	$\langle \text{high}, \text{mediumhigh} \rangle$

Table 12 Linguistic terms for rating criteria for weight vector

Linguistic terms (LTs)	Fuzzy numbers
Good/G	$0.60 \le x \le 1$
Medium good/MG	$0.20 \le x < 0.60$
Medium/M	$0.10 \le x < 0.20$
Medium bad/MB	$0.05 \le x < 0.10$
Bad/B	$0 \le x < 0.05$

4.2.1 Numerical Example

Suppose that $\mathcal{Q} = \{\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2}, \mathcal{M}_{\mathfrak{R}_3}, \mathcal{M}_{\mathfrak{R}_4}, \mathcal{M}_{\mathfrak{R}_5}, \mathcal{M}_{\mathfrak{R}_6}, \mathcal{M}_{\mathfrak{R}_7}\}$ be the collection of seven students. They take an admission in a Science project learning academy for the preparation of a national competition on Science projects. Every student is evaluated on the basis of some important educational parameters, which are set according to the experts of that academy. To get fair assessment of these students, the evaluation committee establish the set of decision variables given as $\mathfrak{Z} = \{\eta_1, \eta_2, \eta_3\}$, where

$\eta_1 =$ Intellectually curious, $\eta_2 =$ Obedient and punctual,
$\eta_3 =$ Experience

Experts need to categorize the students according to these parameters and create their clustering corresponding to different sections of that academy. We subdivide these parameters into further criteria given as

- "Intellectually curious" student may be creative and give his original ideas.
- "Obedient and punctual" may be hard-working and honest.
- "Experience" means that some students have high or medium high experience.

In tabular form, this information can be seen as Table 11.

Some linguistic terms are defined to convert verbal description of experts about $\mathfrak{Z}$ into mathematical language given in Table 12.

Experts select the weight vector "$\wp$" for the strength of established decision variables as $\wp = (0.60, 0.25, 0.15)^T$. To clarify the differences of the opinion of experts and to cover the input data, we construct 2-polar neutrosophic sets given in Table 13. The flow chart diagram of proposed algorithm is given in Fig. 5.

Now, we calculate similarity measure $\mathfrak{C}$ between elements of Table 13 and construct similarity matrix.

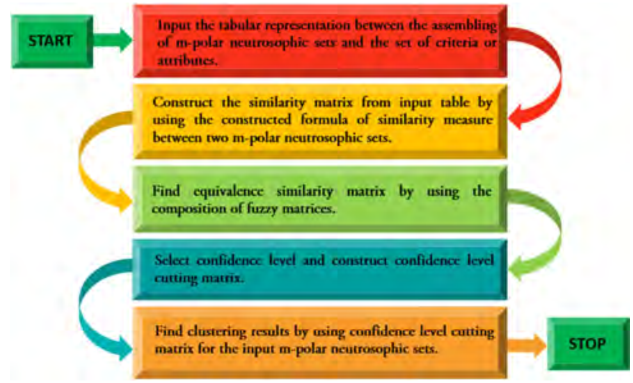

Fig. 5 Flow chart diagram of proposed algorithm 3 for clustering

Table 13 2-Polar neutrosophic input table

Students	η_1	η_2	η_3
$\mathcal{M}_{\mathfrak{R}_1}$	$(\langle 0.81, 0.21, 0.11 \rangle, \langle 0.89, 0.23, 0.38 \rangle)$	$(\langle 0.78, 0.32, 0.17 \rangle, \langle 0.83, 0.21, 0.11 \rangle)$	$(\langle 0.61, 0.42, 0.31 \rangle, \langle 0.71, 0.31, 0.41 \rangle)$
$\mathcal{M}_{\mathfrak{R}_2}$	$(\langle 0.73, 0.23, 0.18 \rangle, \langle 0.79, 0.21, 0.31 \rangle)$	$(\langle 0.79, 0.23, 0.14 \rangle, \langle 0.81, 0.31, 0.21 \rangle)$	$(\langle 0.83, 0.31, 0.18 \rangle, \langle 0.73, 0.41, 0.37 \rangle)$
$\mathcal{M}_{\mathfrak{R}_3}$	$(\langle 0.91, 0.11, 0.15 \rangle, \langle 0.86, 0.31, 0.24 \rangle)$	$(\langle 0.83, 0.21, 0.43 \rangle, \langle 0.89, 0.21, 0.41 \rangle)$	$(\langle 0.72, 0.43, 0.39 \rangle, \langle 0.69, 0.41, 0.43 \rangle)$
$\mathcal{M}_{\mathfrak{R}_4}$	$(\langle 0.74, 0.31, 0.44 \rangle, \langle 0.79, 0.37, 0.28 \rangle)$	$(\langle 0.79, 0.28, 0.32 \rangle, \langle 0.73, 0.41, 0.28 \rangle)$	$(\langle 0.81, 0.31, 0.21 \rangle, \langle 0.83, 0.19, 0.22 \rangle)$
$\mathcal{M}_{\mathfrak{R}_5}$	$(\langle 0.93, 0.11, 0.18 \rangle, \langle 0.91, 0.12, 0.15 \rangle)$	$(\langle 0.91, 0.21, 0.31 \rangle, \langle 0.89, 0.15, 0.19 \rangle)$	$(\langle 0.89, 0.21, 0.23 \rangle, \langle 0.87, 0.23, 0.24 \rangle)$
$\mathcal{M}_{\mathfrak{R}_6}$	$(\langle 0.78, 0.21, 0.37 \rangle, \langle 0.75, 0.21, 0.41 \rangle)$	$(\langle 0.82, 0.31, 0.34 \rangle, \langle 0.79, 0.25, 0.42 \rangle)$	$(\langle 0.88, 0.28, 0.23 \rangle, \langle 0.75, 0.21, 0.15 \rangle)$
$\mathcal{M}_{\mathfrak{R}_7}$	$(\langle 0.79, 0.28, 0.15 \rangle, \langle 0.83, 0.15, 0.19 \rangle)$	$(\langle 0.86, 0.23, 0.31 \rangle, \langle 0.87, 0.13, 0.31 \rangle)$	$(\langle 0.89, 0.31, 0.24 \rangle, \langle 0.79, 0.28, 0.24 \rangle)$

$$\mathcal{G}=\begin{pmatrix} 0.1000 & 0.9339 & 0.9100 & 0.8670 & 0.8863 & 0.9055 & 0.9092\\ 0.9339 & 0.1000 & 0.8860 & 0.9130 & 0.8903 & 0.9207 & 0.9388\\ 0.9100 & 0.8860 & 0.1000 & 0.8634 & 0.9145 & 0.8771 & 0.9100\\ 0.8670 & 0.9130 & 0.8634 & 0.1000 & 0.8535 & 0.9204 & 0.8973\\ 0.8863 & 0.8903 & 0.9145 & 0.8535 & 0.1000 & 0.8701 & 0.9354\\ 0.9055 & 0.9207 & 0.8771 & 0.9204 & 0.8701 & 0.1000 & 0.9085\\ 0.9092 & 0.9388 & 0.9100 & 0.8973 & 0.9354 & 0.9085 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^2=\begin{pmatrix} 0.1000 & 0.9339 & 0.9100 & 0.9130 & 0.9100 & 0.9207 & 0.9339\\ 0.9339 & 0.1000 & 0.9100 & 0.9204 & 0.9354 & 0.9207 & 0.9388\\ 0.9100 & 0.9100 & 0.1000 & 0.8973 & 0.9145 & 0.9100 & 0.9145\\ 0.9130 & 0.9204 & 0.8973 & 0.1000 & 0.8973 & 0.9204 & 0.9130\\ 0.9100 & 0.9354 & 0.9145 & 0.8973 & 0.1000 & 0.9085 & 0.9354\\ 0.9207 & 0.9207 & 0.9100 & 0.9204 & 0.9085 & 0.1000 & 0.9207\\ 0.9339 & 0.9388 & 0.9145 & 0.9130 & 0.9354 & 0.9207 & 0.1000 \end{pmatrix}$$

As $\mathcal{G}^2 \not\subseteq \mathcal{G}$, so we move towards the further calculations.

$$\mathcal{G}^4=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9207 & 0.9339\\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9207 & 0.9388\\ 0.9145 & 0.9145 & 0.1000 & 0.9130 & 0.9145 & 0.9145 & 0.9145\\ 0.9204 & 0.9204 & 0.9130 & 0.1000 & 0.9204 & 0.9204 & 0.9204\\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9207 & 0.9354\\ 0.9207 & 0.9207 & 0.9145 & 0.9204 & 0.9207 & 0.1000 & 0.9388\\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^8=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9339 & 0.9339\\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9339 & 0.9388\\ 0.9145 & 0.9145 & 0.1000 & 0.9145 & 0.9145 & 0.9145 & 0.9145\\ 0.9204 & 0.9204 & 0.9145 & 0.1000 & 0.9204 & 0.9204 & 0.9204\\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9354 & 0.9354\\ 0.9339 & 0.9339 & 0.9145 & 0.9204 & 0.9354 & 0.1000 & 0.9388\\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^{16}=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9339 & 0.9339\\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9339 & 0.9388\\ 0.9145 & 0.9145 & 0.1000 & 0.9145 & 0.9145 & 0.9145 & 0.9145\\ 0.9204 & 0.9204 & 0.9145 & 0.1000 & 0.9204 & 0.9204 & 0.9204\\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9354 & 0.9354\\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.1000 & 0.9388\\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

$$\mathcal{G}^{32}=\begin{pmatrix} 0.1000 & 0.9339 & 0.9145 & 0.9204 & 0.9339 & 0.9339 & 0.9339\\ 0.9339 & 0.1000 & 0.9145 & 0.9204 & 0.9354 & 0.9339 & 0.9388\\ 0.9145 & 0.9145 & 0.1000 & 0.9145 & 0.9145 & 0.9145 & 0.9145\\ 0.9204 & 0.9204 & 0.9145 & 0.1000 & 0.9204 & 0.9204 & 0.9204\\ 0.9339 & 0.9354 & 0.9145 & 0.9204 & 0.1000 & 0.9354 & 0.9354\\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.1000 & 0.9388\\ 0.9339 & 0.9388 & 0.9145 & 0.9204 & 0.9354 & 0.9388 & 0.1000 \end{pmatrix}$$

It is clear that $\mathcal{G}^{32} = \mathcal{G}^{16} \circ \mathcal{G}^{16} = \mathcal{G}^{16}$ is an equivalence similarity matrix. Since the confidence level ð has a strong connection with the elements of the equivalence similarity matrix. For ð we construct ð-cutting matrix $\mathcal{G}_ð$. Different ð produces different $\mathcal{G}_ð$ and different clustering for the universal set $\mathcal{Q}$. For different values of ð different clustering results are given in Table 14.

The clustering effect diagram for different ð-cutting of seven students can be seen in Fig. 6. This means that by utilizing this novel algorithm experts of academy can easily classify the students corresponding to different sections of the academy according to their ability. All the clustering depend upon the parameter ð, which is confidence level and selected according to the opinions and suggestions of experts.

4.2.2 Comparison

Now, we compare our proposed method with some existing approaches and we see that our proposed approach has the following advantages.

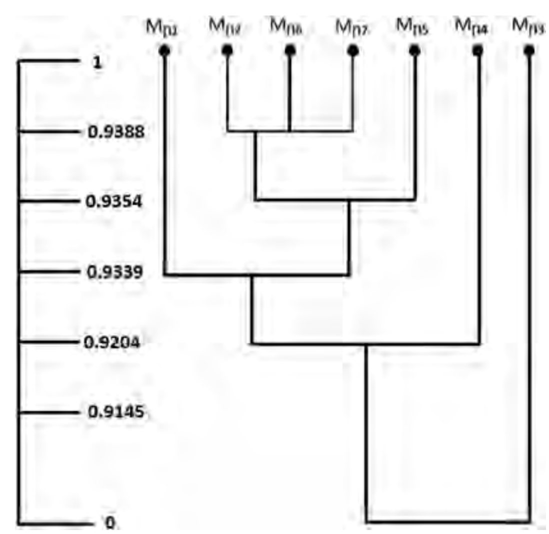

Fig. 6 The clustering effect diagram of seven students

Table 14 The clustering results of seven students

Confidence level ð	Clusters
$0.9388 < ð \leq 1$	$\{\mathcal{M}_{\mathfrak{R}_1}\}, \{\mathcal{M}_{\mathfrak{R}_2}\}, \{\mathcal{M}_{\mathfrak{R}_3}\}, \{\mathcal{M}_{\mathfrak{R}_4}\}, \{\mathcal{M}_{\mathfrak{R}_5}\}, \{\mathcal{M}_{\mathfrak{R}_6}\}, \{\mathcal{M}_{\mathfrak{R}_7}\}$
$0.9354 < ð \leq 0.9388$	$\{\mathcal{M}_{\mathfrak{R}_1}\}, \{\mathcal{M}_{\mathfrak{R}_2}, \mathcal{M}_{\mathfrak{R}_6}, \mathcal{M}_{\mathfrak{R}_7}\}, \{\mathcal{M}_{\mathfrak{R}_3}\}, \{\mathcal{M}_{\mathfrak{R}_4}\}, \{\mathcal{M}_{\mathfrak{R}_5}\}$
$0.9339 < ð \leq 0.9354$	$\{\mathcal{M}_{\mathfrak{R}_1}\}, \{\mathcal{M}_{\mathfrak{R}_2}, \mathcal{M}_{\mathfrak{R}_5}, \mathcal{M}_{\mathfrak{R}_6}, \mathcal{M}_{\mathfrak{R}_7}\}, \{\mathcal{M}_{\mathfrak{R}_3}\}, \{\mathcal{M}_{\mathfrak{R}_4}\}$
$0.9204 < ð \leq 0.9339$	$\{\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2}, \mathcal{M}_{\mathfrak{R}_5}, \mathcal{M}_{\mathfrak{R}_6}, \mathcal{M}_{\mathfrak{R}_7}\}, \{\mathcal{M}_{\mathfrak{R}_3}\}, \{\mathcal{M}_{\mathfrak{R}_4}\}$
$0.9145 < ð \leq 0.9204$	$\{\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2}, \mathcal{M}_{\mathfrak{R}_4}, \mathcal{M}_{\mathfrak{R}_5}, \mathcal{M}_{\mathfrak{R}_6}, \mathcal{M}_{\mathfrak{R}_7}\}, \{\mathcal{M}_{\mathfrak{R}_3}\}$
$0 \leq ð \leq 0.9145$	$\{\mathcal{M}_{\mathfrak{R}_1}, \mathcal{M}_{\mathfrak{R}_2}, \mathcal{M}_{\mathfrak{R}_3}, \mathcal{M}_{\mathfrak{R}_4}, \mathcal{M}_{\mathfrak{R}_5}, \mathcal{M}_{\mathfrak{R}_6}, \mathcal{M}_{\mathfrak{R}_7}\},$

Table 15 Comparison of proposed approach with the existing methodologies

Authors	Set	Truth grade	Indeterminacy grade	Falsity grade	Multi-polarity	Loss of information
Xu et al. [41]	IFS	✓	×	✓	×	×
Zhang et al. [54]	IFS	✓	×	✓	×	✓
Peng et al. [22]	PFS	✓	×	✓	×	×
Proposed approach	MPNS	✓	✓	✓	✓	×

(1) By using the methods of Xu et al. [41] and Zhang et al. [54], we cannot handle the multi-polar input data and cannot deal with the indeterminacy part of the alternatives. They used intuitionistic fuzzy sets (IFSs) for the clustering of input data. In our proposed approach, we deal our clustering with the multiple data with the truth, indeterminacy, and falsity part of the alternatives. So, our method is more efficient and deal with numerous applications having multiple data.

(2) Peng et al. [22] presented the clustering idea on Pythagorean fuzzy sets (PFSs). They increased the domain of Xu et al. [41] and Zhang et al. [54] approaches, but they cannot handle the multi-polar input data and cannot deal with the indeterminacy part of the alternatives. Our proposed method removes these restrictions and can easily handle multi-criteria decision-making problems.

(3) According to Peng et al. [22] research idea, Zhang et al. [54] produced the loss of too much information in the data during the calculation by using intuitionistic fuzzy similarity degrees. This loss effects upon the final result of clustering. Our proposed approach does not lose any input data during the calculations and produces accurate and appropriate results. This comparison is given in tabular form in Table 15.

5 Conclusion

Decision analysis has been intensively examined by numerous scholars and researchers. The techniques developed for this task mainly depend on the type of decision problem under consideration. Most of its relating issues are associated with uncertain, imprecise and multi-polar information, which cannot be tackled properly through fuzzy set. To overcome this particular deficiency of fuzzy sets, Chen et al. [5] have proposed the concept of m-polar fuzzy set (MPFS) in 2014, which has the capability to deal with the data having vagueness and uncertainty under multi-polar information. Neutrosophic set deals with the MCDM methods having truth, falsity, and indeterminacy grades for the corresponding alternatives. In this manuscript, we have established the idea of m-polar neutrosophic set (MPNS) by combining the two independent concepts of m-polar fuzzy set and neutrosophic set. We have established the notion of m-polar neutrosophic topology and defined interior, closure, exterior, and frontier in the context of MPNSs with the help of illustrations. We have presented cosine similarity measure and set theoretic similarity measure to find the similarity between MPNSs. Three novel algorithms for multi-criteria decision-making (MCDM) with linguistic information have been developed on the basis of MPNTS, similarity measures, and clustering analysis. Furthermore, we have presented advantages, simplicity, flexibility, and validity of the proposed algorithms. We have discussed and compared our results with some existing methodologies.

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An Innovative Approach to Evaluation of the Quality of Websites in the Tourism Industry: a Novel MCDM Approach Based on Bipolar Neutrosophic Numbers and the Hamming Distance

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ABSTRACT*. Nowadays, the performance and business operations of organizations are closely linked to the quality of their websites compared to the competition. With growing market competition, the quality of websites becomes a significant component and is increasingly being explored and identified as the main factor of comparative advantage over the competition and the maintenance of good customer relationships. A multiple criteria decision-making approach based on the use of bipolar neutrosophic numbers and the Hamming distance is proposed in this paper. The main aim of this article is to emphasize the fact that MCDM models with a smaller number of criteria can be formed without a loss of precision by applying bipolar neutrosophic numbers. In addition to this, the three variants for ranging bipolar neutrosophic numbers based on the Hamming distance and a distance from the ideal point are proposed. The applicability of the proposed approach is considered in the case of website evaluation.

KEYWORDS: bipolar neutrosophic set, Hamming distance, MCDM.

Introduction

The beginnings of the Internet use in tourism are considered to be the revolutionary changes that have completely transformed the tourism sector. Today, an increasing number of customers avoid traditional intermediaries when buying products and services in tourism – customers first obtain information on products and services and then buy them online.

The rapid developments of the Internet, the expansion of its availability and its integration with other technologies have led to significant changes in consumer behaviour when buying products and services in tourism (Verma, 2010).

Nowadays, the performance and business operations of organizations are closely linked to the quality of their websites compared to the competition. With growing market competition, the quality of websites becomes a significant component and is increasingly being explored and identified as the main factor of comparative advantage over the competition. Websites could be also important for maintenance of good customer relationships.

Therefore, the measuring of the quality of an organization's website is of particular importance for the organization. Based on the evaluation of the website quality, organizations may receive feedback on the segment which they need to improve in order to be ahead of the competition. The significance of the quality evaluation, especially when websites are concerned, are highlighted by Hsu *et al*., (2018), Abbasi *et al*., (2018), Chen *et al*., (2017), Tian, Wang (2017), Wang *et al*. (2015), Al-Qeisi *et al*. (2014), Parasuraman *et al*. (1985) and so on.

A Multiple Criteria Evaluation (MCE), often referred to as Multiple Criteria Decision Analysis (MCDA), refers to the evaluation of alternatives in relation to several our often mutually conflicting criteria of a larger number.

Compared to Multiple Criteria Decision Making (MCDM), which is usually carried out with the aim of selecting one out of a set of available alternatives, the primary goal of the MCE is more often the ranking or determination of the relative importance of alternatives. Such an approach can be very useful in a competitive environment, especially when taking into consideration the fact that the entry of new players may affect the positions of the existing players in the market.

In the MCE, as well as in the MCDM, the selected set of evaluation criteria and their relative significance have a significant impact on the results of the evaluation. It is also known that a more accurate evaluation can be made by using a greater number of evaluation criteria. However, an increase in the number of the evaluation criteria can affect the increasing complexity of a proposed decision-making model, which can have a negative impact on the effectiveness and real usage of proposed MCE models.

Certain possibilities of forming the decision-making models based on the use of a smaller number of evaluation criteria, without losing precision, can be obtained based on the use of grey, fuzzy or neutrosophic numbers. In the decision-making models formed in such a manner, certain types of grey, fuzzy or neurotrophic numbers can be used to collect the ratings obtained from respondents.

Therefore, the rest of this article is structured as follows: in the first section, some significant elements of the neutrosophic sets theory are considered, with a special emphasis on the bipolar neutrosophic sets, whereas in the second section, certain approaches to the evaluation of websites are considered, with the aim of defining an effective set of evaluation criteria containing as small a number of evaluation criteria as possible. In Section Three, a framework for the evaluation of the quality of websites is proposed, and in Section Four, its use is illustrated with the aim to demonstrate its practical usability. Finally, conclusions are given.

1. The Basic Concepts of a Bipolar Neutrosophic Set

As is previously mentioned, Zadeh (1965) proposed fuzzy set theory and introduced the membership function.

Definition 1. **A Fuzzy Set** (Zadeh, 1965). Let *X* be a nonempty set. Then, a fuzzy set *A* in *X* is a set of ordered pairs:

$$A = \left\{ \langle x, \mu_A(x) \rangle \middle| x \in X \right\}, \tag{1}$$

where the membership function $\mu_A^+(x)$ denotes the degree of the membership of an element *x* to the set *A*, and $\mu_A(x) \in [0,1]$.

Atanassov (1986) extended the concept of fuzzy set theory and introduced intuitionistic fuzzy sets, which are characterized by using the membership and non-membership functions.

Definition 2. **An Intuitionistic Fuzzy Set** (Atanassov, 1986). Let *X* be a nonempty set. Then, an intuitionistic fuzzy set is defined as follows:

$$A = \left\{ \langle x, \mu_A(x), \nu_A(x) \rangle \middle| x \in X \right\}, \tag{2}$$

where: $\mu_A(x)$ and $\nu_A(x)$ represent the degree of the membership and the degree of the non-membership of the element *x* to the set *A*, respectively; $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$, where $\mu_{A(x)}$ and $\nu_{A(x)}$ satisfy the following condition $0 \le \mu_A(x) + \nu_A(x) \le 1$.

In Intuitionistic Set Theory, Atanassov (1986) also implicitly introduced the indeterminacy-membership function $\pi_A(x)$, which is defined as $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Lee (2000) introduced the notion of bipolar fuzzy sets by extending the concept of fuzzy sets, where the degree of the membership is expanded from [0, 1] to [-1, 1].

Definition 3. **A Bipolar Fuzzy Set** (Lee, 2000). Let *X* be a nonempty set. Then, a bipolar fuzzy set is defined as follows:

$$A = \left\{ \langle x, \mu_A^+(x), \nu_A^-(x) \rangle \middle| x \in X \right\}, \tag{3}$$

where: the positive membership function $\mu_A^+(x)$ denotes the satisfaction degree of the element *x* to the property corresponding to a bipolar-valued fuzzy set, and the negative membership function $\nu_A^-(x)$, denotes the degree of the satisfaction degree of the element *x* to a corresponding complementary bipolar-valued fuzzy set, respectively; $\mu_A^+ : X \to [0,1]$ and $\nu_A^- : X \to [-1,0]$.

Smarandache (1999) introduced the neutrosophic sets theory, as the generalization of fuzzy sets and intuitionistic fuzzy sets.

Definition 4. **Neutrosophic Sets** (Smarandache, 1999). Let *X* be a nonempty set. Then, Neutrosophic Set (NS) *A* in *X* is defined as:

$$A=\left\{\left\langle x,T_A(x),I_A(x),F_A(x)\right\rangle \middle| x\in X\right\}, \quad (4)$$

where: $T_A(x)$, $I_A(x)$ and $F_A(x)$, denote the truth-membership $T_A(x)$, the indeterminacy-membership $I_A(x)$ and the falsity-membership functions $F_A(x)$, and $T_A, I_A, F_A : X \to]^-0,1^+[$.

In contrast to intuitionistic sets, the restriction regarding to the sum of the membership functions is eliminated, so that $^-0 \le T_A(x)+I_A(x)+U_A(x) \le 3^+$.

In 2015, Deli *et al.* (2015) introduced Bipolar Neutrosophic Sets (BNS) by generalizing the concept of bipolar fuzzy sets. Deli *et al.* (2015) also defined the Score, Certainty and Accuracy functions, as well as the Bipolar Neutrosophic Weighted Average and the Bipolar Neutrosophic Weighted Geometric operators for the BNS.

Definition 5. **Bipolar Neutrosophic Sets** (Deli *et al.,* 2015). Let X be a nonempty set. Then, a BNS A in X is as follows:

$$A=\left\{\left\langle x,T_A^+(x),I_A^+(x),F_A^+(x),T_A^-(x),I_A^-(x),F_A^-(x)\right\rangle \middle| x\in X\right\}, \quad (5)$$

where: $T^+(x), I^+(x), F^+(x)$ denote the membership, the indeterminate membership and the falsity membership of x to the BNS A, and $T^-(x), I^-(x), F^-(x)$ denote the membership, the indeterminate membership and the falsity membership of x to a complemenry BNS; $T^+, I^+, F^+ : X \to [1,0]$ and $T^-, I^-, F^- : X \to [-1,0]$.

Deli *et al.* (2015) also introduced the Bipolar Neutrosophic Number (BNN), which can be denoted as follows $a =< t^+, i^+, f^+, t^-, i^-, f^- >$ for convenience.

Definition 6. (Deli *et al.,* 2015) Let $a_1 =< t_1^+, i_1^+, f_1^+, t_1^-, i_1^-, f_1^- >$ and $a_1 =< t_2^+, i_2^+, f_2^+, t_2^-, i_2^-, f_2^- >$ be two BNNs and $\lambda > 0$. The basic operations for these numbers are as follows:

$$a_1 + a_2 =< t_1^+ + t_2^+ - t_1^+t_2^+, i_1^+i_2^+, f_1^+f_2^+, -t_1^-t_2^-, -(-i_1^+ - i_2^+ - i_1^+i_2^+), -(-f_1^+ - f_2^+ - f_1^+f_2^+) > \quad (6)$$

$$a_1 \cdot a_2 =< t_1^+t_2^+, i_1^+ + i_2^+ - i_1^+i_2^+, f_1^+ + f_2^+ - f_1^+f_2^+, -(-t_1^- - t_2^- - t_1^-t_2^-), -i_1^-i_2^-, -f_1^-f_2^- > \quad (7)$$

$$\lambda a_1 =< 1-(1-t_1^+)^\lambda, (i_1^+)^\lambda, (f_1^+)^\lambda, -(-t_1^-)^\lambda, -(-i_1^-)^\lambda, -(1-(1-(-f_1^-))^\lambda) > \quad (8)$$

$$a_1^\lambda =< (t_1^+)^\lambda, 1-(1-i_1^+)^\lambda, 1-(1-f_1^+)^\lambda, -(1-(1-(-t_1^-))^\lambda), -(-i_1^-)^\lambda, -(-f_1^-)^\lambda > \quad (9)$$

Definition 7. (Deli *et al.,* 2015) Let $a =< t^+, i^+, f^+, t^-, i^-, f^- >$ be a BNN. The score function $s_{(a)}$ of an BNN is as follows:

$$s_{(a)} = (t^+ + 1 - i^+ + 1 - f^+ + 1 + t^- - i^- - f^+)/6. \quad (10)$$

Definition 8. (Deli *et al.,* 2015) Let $a_j =< t_j^+, i_j^+, f_j^+, t_j^-, i_j^-, f_j^- >$ be a collection of BNNs. The Bipolar Neutrosophic Weighted Average Operator (A_w) of the n dimensions is a mapping $A_w : Q_n \to Q$ as follows:

$$A_w(a_1, a_2, ..., a_n) = \sum_{j=1}^n w_j a_j$$
$$= \left(1-\prod_{j=1}^n (1-t_j^+)^{w_j}, \prod_{j=1}^n (i_j^+)^{w_j}, \prod_{j=1}^n (f_j^+)^{w_j}, -\prod_{j=1}^n (-t_j^-)^{w_j}, -(1-\prod_{j=1}^n (1-(-i_j^-))^{w_j}), -(1-\prod_{j=1}^n (1-(-f_j^-))^{w_j})\right) \quad ,(11)$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 8. Let $a_1 = < t_1^+, i_1^+, f_1^+, t_1^-, i_1^-, f_1^- >$ and $a_2 = < t_2^+, i_2^+, f_2^+, t_2^-, i_2^-, f_2^- >$ be two BNNs. The Hamming distance between a_1 and a_2 is as follows:

$$d_H(a_1, a_2) = \frac{1}{6}\left(|t_1^+ - t_2^+| + |i_1^+ - i_2^+| + |f_1^+ - f_2^+| + |t_1^- - t_2^-| + |i_1^- - i_2^-| + |f_1^- - f_2^-| \right) \quad (12)$$

2. Choosing Criteria for Evaluating Websites

In a business environment, websites can have different purposes. Moreover, a website must often play multiple roles, such as: providing information to customers, acquiring and retaining new customers, and so on. In addition to the said, the fact that customers cannot be treated as homogeneous groups and that specific customer groups can have their own specific needs and requirements should not be ignored.

Therefore, designing, developing and maintaining an adequate website is not an easy task to do at all. After using a website for the very first time, many useful pieces of information about its functionality can be obtained by using the website's analytics tools, as well as visitors' comments.

Additionally, based on the Service Quality Model, i.e. the SERVQUAL Model, proposed by Parasuraman *et al.* (1998), several specialized models for the evaluation of websites were proposed, such as: WebQyal (Barnes, Vidgen 2000), SITEQUAL (Yoo, Donthu, 2001), eTailQ (Wolfinbarger, Gilly, 2001) and E-S-SERVQUAL (Parasuraman *et al.,* 2005).

The alleged models, as well as the other models developed based on them, are successfully used to evaluate numerous websites, particularly so e-commerce, e-marketing and e-banking websites.

As a significant characteristic of the above-mentioned models, it can be emphasized that they use several dimensions and sub-dimensions for determining customer satisfaction. In the MCA and/or MCDM terminology, this means that evaluation is based on the use of multiple criteria, which have their own sub-criteria.

The evaluation models based on the use of MCDM methods can also be emphasized as a significant approach to the determination of the quality of websites. For example, Sun, Lin (2009) evaluated shopping websites by used fuzzy TOPSIS method, whereas Tsai (2010) evaluated a national park website by using the ANP and VIKOR methods.

These are not isolated research studies related to the use of the MCDM methods for evaluating websites. The following can be mentioned as some of earlier studies: Lee, Kozar (2006) and Bilsel *et al.* (2006), who used the AHP and PROMETHEE II for websites ranking.

There are also a number of recent research studies, such as those by: Abdel-Basset *et al.* (2018), who used the VIKOR method and neutrosophic numbers for evaluating e-government websites; and Stanujkic *et al.* (2017), who proposed a group multiple-criteria approach for evaluating hotels' websites, based on the use of triangular intuitionistic fuzzy numbers. Stanujkic *et al.* (2016) also proposed an approach for evaluating websites quality, based on the use of single-valued neutrosophic numbers.

As has been mentioned earlier, the selected set of evaluation criteria can significantly affect the characteristics of a proposed MCA/MCDM model. Therefore, the problem of selecting an adequate set of evaluation criteria for evaluating website has been considered in many previous studies. Kapoun (1998) and Lydia (2009) can be mentioned as some of such studies.

According to Kapoun (1998), the following criteria can be used for evaluating a website: Accuracy, Authority, Objectivity, Currency, and Coverage. Kapoun's set of criteria is often used, and based on it, similar sets of criteria are proposed. For example, Lydia (2009) adds the sixth criterion, Appearance, while the CRAAP test is proposed at the California State University of Chico, which suggests the use of the following criteria: Currency, Relevance, Authority, Accuracy, and Purpose.

However, there are also studies where appropriate, or specialized, sets of criteria are proposed for the evaluation of different types of websites. For example, Chung, Law (2003) proposed the following six criteria for evaluating websites in the hotel industry: Facilities Information, Customer Contact Information, Reservation Information, Surrounding Area Information, and Website Management, and for each criterion, appropriate sub-criteria are defined. Contrary to this, Herrero, San Martin (2012) suggested that only three criteria, namely: Information, Interactivity, and Navigability should be used.

A set of criteria proposed by the Webby Awards[1], can also be listed as a significant set of criteria for evaluating websites. This set of criteria includes the following criteria: Content, Structure and Navigation, Visual Design, Interactivity, Functionality, Innovation, and Overall Experience.

As a result, there are different approaches to selecting the criteria for websites evaluation: the use of a number of criteria and sub-criteria contrary to the use of a smaller number of criteria; the use of a standard set of criteria against the use of specialized sets of criteria, etc.

The choice of an appropriate set of evaluation selection criteria is very important for the successful solving of each MCA/MCDM problem. The use of a larger number of criteria usually leads to the formation of more precise models; on the one hand, a larger number of criteria can be less desirable if certain data should be collected through a survey.

In contrast to the said, a smaller number of criteria can be much more efficient when certain data should be collected through a survey, on the one hand, whereas on the other, the usage of a smaller number of criteria may require the use of significantly more complex criteria.

Neutrosophic numbers, particularly bipolar neutrosophic numbers, contain more information than crisp numbers, or fuzzy numbers, for which reason their application can be very beneficial when a small number of evaluation criteria are used.

Therefore, in this approach, the following three criteria are selected out of the set proposed by the Webby Awards: Structure and Navigation, Content and Visual Design.

3. The Alternative Procedure for Ranking Alternatives Based on the Hamming Distance

Deli *et al.* (2015) proposed a MCDM approach to the selection of the best alternative based on the use of the score, certainty and accuracy functions, as well the A_w and G_w operators.

In this paper, an approach based on the use of the Hamming distance is proposed. The detailed step-by-step procedure of the proposed approach can be described through the

[1] http://webbyawards.com/judging-criteria/

following steps:

Step 1. ***Identify available alternatives and select a set of evaluation criteria.*** In this step, a team of experts identifies a set of available alternatives and defines the criteria for their evaluation.

Step 2. ***Determine the relative importance of evaluation criteria.*** In the literature, many techniques are proposed for determining the weights of criteria, such as pair-wise comparisons (Saaty, 1980), SWARA (Kersuliene *et al.,* 2010), the Best-Worst Method (Rezaei, 2015), R-SWARA (Zavadskas *et al.,* 2018) and PIPRECIA (Stanujkic *et al.*, 2017). In this approach, any of the mentioned techniques can be used for determining the weights of the criteria.

Step 3. ***Construct a bipolar neutrosophic decision-making matrix*****,** and do it for each decision-maker. In this step, each decision-maker forms his/her evaluation matrix, in which matrix alternatives are evaluated by using BNNs. As a result of these activities, each decision-maker forms his/her evaluation matrix, whose elements are BNNs.

The specificity of the BNSs is used in this step to perform a two-phase evaluation of the alternatives in relation to each criterion, where satisfaction is measured in the first phase and dissatisfaction in the second.

By using such an approach, respondents are enabled to carry out a sufficiently precise evaluation based on a smaller number of evaluation criteria.

Step 4. ***Construct a group bipolar neutrosophic decision-making matrix.*** The integration of the individual evaluation matrices into a group decision-making matrix can be carried out by using an aggregation operator. In this approach, the use of the A_W aggregation operator is proposed for aggregating individual evaluation matrices into a group decision-making matrix.

After this step, the most appropriate alternative can be determined in several ways. As one of the most commonly used approaches, the approach based on the use of the score function can be specified. In such an approach, the value of the score function of each of the considered alternatives could be determined by applying Eq. (10). After that, the alternative with the highest value of the score function is the most acceptable one.

In addition to this, an increasing use of fuzzy sets theory, as well as its previously mentioned extensions, has had a significant impact on proposing the numerous extensions of the TOPSIS and VIKOR methods, as well as the extensions of the other MCDM methods. As a result, some other approaches are often proposed, out of which the approaches based on the distance from the ideal point can be especially emphasized. Therefore, as an alternative to applying the score function for ranking alternatives, the three variants of the distance-based approaches are considered in the remaining part of the paper, where all of the three variants are based on the Hamming distance.

The first variant. In the first of the three proposed variants, the ideal point is formed as follows: $a^+ =<1,0,0,0,0,1>$. After that, the Hamming distances of the alternatives to the ideal point are determined by applying Eq. (12).

In this approach, the alternative with the smallest Hamming distance is the most preferable one.

The second variant. In the second variant, the ideal point is determined much more realistically, i.e. in the following manner:

$$a^+ =< \max_i t_{ij}^+, \min_i i_{ij}^+, \min_i f_{ij}^+, \max_i t_{ij}^-, \max_i i_{ij}^-, \min_i f_{ij}^- > \quad (13)$$

After that, similarly as in the first variant, the Hamming distance is determined for each alternative, and the best alternative is that with the smallest distance from the ideal point.

The third variant. Unlike the previous two variants, the third variant is based on the use of the well-known approach proposed in the TOPSIS method, i.e. the determination of the distances of the alternatives from the ideal and the anti-ideal points, and the determination of the relative closeness C_i of each such alternative, as follows:

$$c_i = \frac{d_i^-}{d_i^+ + d_i^-} \tag{14}$$

In the proposed approach, d_i^+ and d_i^- denote the Hamming distance of the alternative *i* from the ideal and the anti-deal points, respectively, the ideal point being determined as in the previous variant, and the anti-ideal point being determined as follows:

$$a^+ =< \min_i t_{ij}^+, \max_i i_{ij}^+, \max_i f_{ij}^+, \min_i t_{ij}^-, \min_i i_{ij}^-, \max_i f_{ij}^- > \tag{15}$$

Finally, the most acceptable alternative based on the third variant is the alternative that has the highest C_i.

4. A Numerical Illustration

In this numerical illustration, the proposed approach is used to evaluate the websites of the four regional tourism organizations, at the following web addresses:

- http://tookladovo.rs,
- http://www.toom.rs,
- http://tobor.rs, and
- http://toon.org.rs.

The evaluation was made in order to compare the quality of the website of one of the mentioned tourism organizations in relation to the others, whereby the respondents were not awarded with the main goal of the evaluation in advance. For the same reason, the order of the alternatives in the remaining segment of the numerical example is not identical with the appearance of the aforementioned alternatives.

In order to create the conditions for conducting this study, several potential respondents were introduced by applying bipolar intuitionist sets and the SWARA method.

For the purpose of this consideration, the responses obtained from the three selected respondents are chosen. The opinions related to the weights of the criteria, the weights of criteria and the ratings obtained from the first of the three respondents are presented in *Table 1* and *Table 2*.

Table 1. The opinions and the weights of the criteria obtained from the first of the three respondents

Criteria		s_j	k_j	q_j	w_j
Structure and Navigation	C_1		1.00	1.00	0.30
Content	C_2	1.20	0.80	1.25	0.37
Visual Design	C_3	0.90	1.10	1.14	0.34
			2.90	3.39	

Source: own calculations.

Table 2. The ratings obtained from the first of the three respondents

	C_1	C_2	C_3
w_j	0.30	0.37	0.34
A_1	<0.7, 0.2, 0.3, -0.3, 0, 0>	<0.7, 0, 0.1, -0.2, 0, 0>	<0.7, 0.1, 0, -0.3, 0, 0>
A_2	<0.7, 0, 0.2, -0.2, 0, -0.1>	<0.6, 0, 0.1, -0.3, 0, 0>	<0.4, 0, 0.2, -0.2, 0, 0>
A_3	<0.6, 0, 0, -0.7, 0, 0>	<0.7, 0, 0, -0.4, 0, 0>	<0.4, 0, 0.2, -0.2, 0, 0>
A_4	<0.9, 0, 0, -0.7, 0, 0>	<0.3, 0.2, 0, -0.1, 0, 0>	<0.6, 0, 0, 0, 0, 0>

Source: own calculations.

The opinions obtained from the three surveys, as well as the appropriate weights, are accounted for in *Table 3*.

Table 3. The opinions and the weights of criteria obtained from the three respondents

	E_1		E_1		E_1	
	s_j	w_j	s_j	w_j	s_j	w_j
C_1		0.30		0.34		0.38
C_2	1.20	0.37	1.00	0.34	1.10	0.34
C_3	0.90	0.34	0.90	0.31	1.20	0.28

Source: own calculations.

The group criteria weights calculated as the average value of the criteria weight from Table 3 are shown in *Table 4*.

Table 4. The group criteria weights

	w_j
C_1	0.34
C_2	0.35
C_3	0.31

Source: own calculations.

The ratings of the alternatives expressed in terms of the BNNs obtained from the second and the third respondents are given in *Table 5* and *Table 6*.

Table 5. The ratings obtained from the second respondent

	C_1	C_2	C_3
A_1	<0.7, 0.2, 0.3, -0.3, 0, 0>	<0.7, 0, 0.1, -0.2, 0, 0>	<0.7, 0.1, 0, -0.3, 0, 0>
A_2	<0.7, 0, 0.2, -0.2, 0, -0.1>	<0.6, 0, 0.1, -0.3, 0, 0>	<0.4, 0, 0.2, -0.2, 0, 0>
A_3	<0.6, 0, 0, -0.7, 0, 0>	<0.7, 0, 0, -0.4, 0, 0>	<0.4, 0, 0.2, -0.2, 0, 0>
A_4	<0.9, 0, 0, -0.7, 0, 0>	<0.3, 0.2, 0, -0.1, 0, 0>	<0.6, 0, 0, 0, 0, 0>

Source: own calculations.

Table 6. The ratings obtained from the third respondent

	C_1	C_2	C_3
A_1	<0.7, 0.2, 0.3, -0.3, 0, 0>	<0.9, 0, 0, 0, 0, 0>	<0.5, 0, 0.1, -0.2, 0, 0>
A_2	<0.7, 0, 0.2, -0.2, 0, -0.1>	<0.9, 0, 0.3, -0.1, 0, 0>	<0.5, 0, 0, -0.3, 0, 0>
A_3	<0.6, 0, 0, -0.7, 0, 0>	<0.9, 0, 0.2, -0.5, 0, -0.3>	<0.5, 0, 0, -0.7, 0, -0.2>
A_4	<0.9, 0, 0, -0.7, 0, 0>	<0.3, 0.2, 0, 0, 0, 0>	<0.5, 0, 0, 0, 0, 0>

Source: own calculations.

The group ratings calculated by applying Eq. (11) are accounted for in *Table 7*. In this calculation, the following weights are assigned to the respondents: w_{E1}=0.35, w_{E2}=0.33, and w_{E3}=0.32.

Table 7. The group ratings

	C_1	C_2	C_3
A_1	<0.7, 0.2, 0.3, -0.3, 0, 0>	<0.79, 0, 0, 0, 0, 0>	<0.65, 0, 0, -0.26, 0, 0>
A_2	<0.7, 0, 0.2, -0.2, 0, -0.85>	<0.74, 0, 0.14, -0.21, 0, 0>	<0.43, 0, 0, -0.23, 0, 0>
A_3	<0.6, 0, 0, -0.7, 0, 0>	<0.79, 0, 0, -0.43, 0, -0.68>	<0.43, 0, 0, -0.3, 0, -0.6>
A_4	<0.9, 0, 0, -0.7, 0, 0>	<0.3, 0.2, 0, 0, 0, 0>	<0.57,0,0,0,0,0>

Source: own calculations.

The overall ratings calculated by applying Eq. (11), as well as the ranking order of the alternatives, are presented in *Table 8*.

Table 8. The overall ratings, the score and the ranking order of the considered alternatives

	Overall ratings	S_i	Rank
A_1	<0.72, 0, 0, 0, 0, 0>	3.72	3
A_2	<0.65, 0, 0, -0.21, 0, -0.95>	4.39	1
A_3	<0.64, 0, 0, -0.45, 0, -0.98>	4.17	2
A_4	<0.69, 0, 0, 0, 0, 0>	3.69	4

Source: own calculations.

As can be seen from *Table 8*, the most acceptable alternative based on the Score Function is the alternative denoted as A_2.

The results achieved by using the Hamming distance and the three proposed variants are considered in the rest of this section. The results obtained by using the first of the three considered variants are demonstrated in *Table 9*.

Table 9. The overall ratings, the Hamming distances and the ranking order of the considered alternatives

	Overall ratings	d_H	Rank
a^+	<1, 0, 0, 0, 0, 0>		
A_1	<0.72, 0, 0, 0, 0, 0>	0.21	3
A_2	<0.65, 0, 0, -0.21, 0, -0.95>	0.10	1
A_3	<0.64, 0, 0, -0.45, 0, -0.98>	0.14	2
A_4	<0.69, 0, 0, 0, 0, 0>	0.22	4

Source: own calculations.

As can be seen from *Table 9*, the ranking orders obtained by using the Score Function and the first of the three proposed variants based on the Hamming distance are identical.

The results obtained by using the second of the three considered variants are shown in *Table 10*.

Table 10. The ranking of the alternatives based on the second of the three proposed variants

	Overall ratings	d_H	Rank
a^+	<0.72, 0, 0, 0, 0, -0.98>		
A_1	<0.72, 0, 0, 0, 0, 0>	0.16	3
A_2	<0.65, 0, 0, -0.21, 0, -0.95>	0.05	1
A_3	<0.64, 0, 0, -0.45, 0, -0.98>	0.09	2
A_4	<0.69, 0, 0, 0, 0, 0>	0.17	4

Source: own calculations.

As can be seen from *Table 10*, the ranking orders obtained by using the second variant of the three proposed variants based on the Hamming distance is the same as in the previously considered cases. However, we should be careful because Stanujkic (2013) indicates that, in some cases, the ideal point may have an effect on the ranking order of alternatives.

Ultimately, the results obtained by applying the third proposed variant are shown in *Table 11*.

Table 11. The ranking of the alternatives based on the third of the three proposed variants

	Overall ratings	d_i^+	d_i^- -	C_i	Rank
a^+	<0.72, 0, 0, 0, 0, -0.98>				
a^-	<0.64, 0, 0, -0.45, 0, 0>				
A_1	<0.72, 0, 0, 0, 0, 0>	0.16	0.09	0.35	3
A_2	<0.65, 0, 0, -0.21, 0, -0.95>	0.05	0.20	0.79	1
A_3	<0.64, 0, 0, -0.45, 0, -0.98>	0.09	0.16	0.65	2
A_4	<0.69, 0, 0, 0, 0, 0>	0.17	0.08	0.33	4

Source: own calculations.

The results shown in *Table 11* also confirm the fact that the ranking results obtained by using the third variant based on the Hamming distance are identical with the results obtained by using the procedure for ranking BNNs, proposed by Deli *et al.* (2015).

Conclusion

Bipolar neutrosophic numbers contain more information than the other types of fuzzy or crisp numbers. In addition, these numbers can be used to carry out a two-phase evaluation of the alternative in relation to the selected criteria, where satisfaction is measured in the first phase and dissatisfaction in the second.

By applying such an approach, respondents are enabled to perform a sufficiently precise evaluation, based on a smaller number of criteria.

However, it should be emphasized that the use of bipolar neutrosophic numbers is not so simple in the case of pre-unmanaged subjects.

This paper also proposes a group multiple criteria approach based on the Hamming distance application. The numerical illustration shows that the application of this approach generates the same ranking results as is the case with the application of the Score, which confirms the applicability of the proposed approach.

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A Note on Neutrosophic Chaotic Continuous Functions

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T. Madhumathi, F. Nirmala Irudayam, Florentin Smarandache (2019). A note on Neutrosophic Chaotic Continuous Functions. *Neutrosophic Sets and Systems*, 25, 76-84

Abstract. Many real time problems are based on uncertainity and chaotic environment. To demonstrate this ambiguous suituation more precisely we intend to amalgamate the ideas of chaos theory and neutrosophy. Neutrosophy is a flourishing arena which conceptualizes the notions of true, falsity and indeterminacy attributes of an event. Chaos theory is another branch which brings out the concepts of periodic point, orbit and sensitive of a set. Hence in this paper we focus on the introducing the idea of chaotic periodic points, orbit sets, sensitive functions under neutrosophic settings. We start with defining a neutrosophic chaotic space and enlist its properties, As a futher extension we coin neutrosophic chaotic continuous functions and discuss its charaterizations and their interrelationships. We have also illustrated the above said concepts with suitable examples.

Keywords: Neutrosophic periodic points, neutrosophic orbit sets, neutrosophic chaotic sets, neutrosophic sensitive functions, neutrosophic orbit extremally disconnected spaces.

1 Introduction

The introduction of the idea of fuzzy set was introduced in the year 1965 by Zadeh[16]. He proposed that each element in a fuzzy set has a degree of membership. Following this concept K.Atanassov[1,2,3] in 1983 introduced the idea of intuitionistic fuzzy set on a universe X as a generalization of fuzzy set. Here besides the degree of membership a degree of non-membership for each element is also defined. Smarandache[11,12] originally gave the definition of a neutrosophic set and neutrosophic logic. The neutrosophic logic is a formal frame trying to measure the truth, indeterminacy and falsehood. The significance of neutrosophy is that it finds and indispensible place in decision making. Several authors[7, 8, 9, 10] have done remarkable achievements in this area. One of the prime discoveries of the 20th century which has been widely investigated with significant progress and achievements is the theory of Chaos and fractals.It has become an exciting emerging interdisciplinary area in which a broad spectrum of technologies and methodologies have emerged to deal with large-scale, complex and dynamical systems and problems. In 1989, R.L. Deveney[4] defined chaotic function in general metric space. A breakthrough in the conventional general topology was intiated by T. Thrivikraman and P.B. Vinod Kumar[15] by defining Chaos and fractals in general topological spaces. M. Kousalyaparasakthi, E. Roja, M.K. Uma[6] introduced the above said idea to intuitionistic chaotic continuous functions. Tethering around this concept we introduce neutrosophic periodic points, neutrosophic orbit sets, neutrosophic sensitive functions, neutrosophic clopen chaotic sets and neutrossophic chaos spaces. The concepts of neutrosophic chaotic continuous functions, neutrosophic chaotic* continuous functions, neutrosophic chaotic** continuous functions, neutrosophic chaotic*** continuous functions are introduced and studied. Some interrelation are discussed with suitable examples. Also the concept of neutrosophic orbit extremally disconnected spaces, neutrosophic chaotic extremally disconnected spaces, neutrosophic orbit irresolute function are discussed.

2 Preliminaries

2.1 Definition [12]

Let X be a non empty set. A neutrosophic set (NS for short) V is an object having the form V = $<x, V^1, V^2, V^3>$ where V^1, V^2, V^3 represent the degree of membership, the degree of indeterminacy and the degree of non-membership respectively of each element x ∈ X to the set V.

2.2 Definition [12]

Let X be a non empty set, U = $<x, U^1, U^2, U^3>$ and V = $<x, V^1, V^2, V^3>$ be neutrosophic sets on X, and let $\{V_i: i \in J\}$ be an arbitrary family of neutrosophic sets in X, where $V^i = <x, V^1, V^2, V^3>$

(i) $U \subseteq V \Leftrightarrow U^1 \subseteq V^1, U^2 \supseteq V^2$ and $U^3 \supseteq V^3$

(ii) $U = V \Leftrightarrow U \subseteq V$ and $V \subseteq U$.
(iii) $\overline{V} = <x, V^3, V^2, V^1>$
(iv) $U \cap V = <x, U^1 \cap V^1, U^2 \cup V^2, U^3 \cup V^3>$
(v) $U \cup V = <x, U^1 \cup V^1, U^2 \cap V^2, U^3 \cap V^3>$
(vi) $\cup V_i = <x, \cup V_i^1, \cap V_i^2, \cap V_i^3>$
(vii) $\cap V_i = <x, \cap V_i^1, \cup V_i^2, \cup V_i^3>$
(viii) $U - V = U \cap \overline{V}$.
(ix) $\varphi_N = <x, \varphi, X, X>$; $X_N = <x, X, \varphi, \varphi>$.

2.3 Definition [14]

A neutrosophic topology (NT for short) on a nonempty set X is a family τ of neutrosophic set in X satisfying the following axioms:
(i) $\varphi_N, X_N \in \tau$.
(ii) $T_1 \cap T_2 \in \tau$ for any $T_1, T_2 \in \tau$.
(iii) $\cup T_i \in \tau$ for any arbitrary family $\{T_i : i \in J\} \subseteq \tau$.
In this case the pair (X, τ) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ is called a neutrosophic open set (NOS for short) in X. The complement V of a neutrosophic open set V is called a neutrosophic closed set (NCS for short) in X.

2.4 Definition [14]

Let (X, τ) be a neutrosophic topological space and $V = <X, V_1, V_2, V_3>$ be a set in X. Then the closure and interior of V are defined by
$Ncl(V) = \cap\{M : M$ is a neutrosophic closed set in X and $V \subseteq M\}$,
$Nint(V) = \cup\{N : N$ is a neutrosophic open set in X and $N \subseteq V\}$.
It can be also shown that Ncl(V) is a neutrosophic closed set and Nint(V) is a neutrosophic open set in X, and V is a neutrosophic closed set in X iff Ncl(V) = V; and V is a neutrosophic open set in X iff Nint(V) = V.
Where Ncl - neutrosophic closure and Nint – neutrosophic interior

2.5 Definition [5]

(a) If $V = <y,V^1,V^2,V^3>$ is a neutrosophic set in Y , then the preimage of V under f, denoted by $f^{-1}(V)$, is the neutrosophic set in X defined by $f^{-1}(V) = <x,f^{-1}(V^1),f^{-1}(V^2),f^{-1}(V^3)>$.
(b) If $U = <x,U^1,U^2,U^3>$ is a neutrosophic set in X, then the image of U under f, denoted by f(U), is the neutrosophic set in Y defined by $f(U) = <y,f(U^1),f(U^2),Y-f(X-U^3)>$ where

$$f(U^1)=\begin{cases}\sup_{x\in f^{-1}(y)} U^1 & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

$$f(U^2)=\begin{cases}\sup_{x\in f^{-1}(y)} U^2 & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{otherwise}\end{cases}$$

$$Y-f(X-U^3)=\begin{cases}\inf_{x\in f^{-1}(y)} U^3 & \text{if } f^{-1}(y)\neq\phi\\ 1 & \text{otherwise}\end{cases}$$

2.6 Definition [13]

Let (X, τ) and (Y,σ) be any two neutrosophic topological spaces and let $f : X \to Y$ be a function. Then f is said to be continuous if and only if the preimage of each neutrosophic set in σ is a neutrosophic set in τ.

2.7 Definition [13]

Let (X, τ) and (Y,σ) be two neutrosophic topological spaces and let $f : (X, \tau) \to (Y,\sigma)$ be a function. Then f is said to be open iff the image of each neutrosophic set in τ is a neutrosophic set in σ.

2.8 Definition [4]

Orbit of a point x in X under the mapping f is $O_f(x)=\{x, f(x), f^2(x),...\}$

2.9 Definition [4]

x in X is called a periodic point of f if $f^n(x) = x$, for some $n \in Z^+$. Smallest of these n is called period of x.

2.10 Definition [4]

f is sensitive if for each $\delta > 0$ ∃ (a) $\varepsilon > 0$ (b) $y \in X$ and (c) $n \in Z_+$ ∋ $d(x,y) < \delta$ and $d(f^n(x),f^n(y)) > \varepsilon$.

2.11 Definition [4]
f is chaotic on (X,d) if (i) Periodic points of f are dense in X (ii) Orbit of x is dense in X for some x in X and (iii) f is sensitive.
2.12 Definition [15]
Let (X, τ) be a topological space and f : (X, τ) → (X, τ) be continuous map. Then f is sensitive at x ∈ X if given any open set U containing x ∃ (i) y ∈ U (ii) n ∈ Z^+ and (iii) an open set V ∋ $f^n(x) \in V$, $f^n(y) \notin cl(V)$. We say that f is sensitive on a F if f|F is sensitive at every point of F.
2.13 Definition [15]
Let (X, τ) be a topological space and F ∈ K(X). Let f : F → F be a continuous. Then f is chaotic on F if
(i) $cl(O_f(x)) = F$ for some x ∈ F.
(ii) periodic points of f are dense in F.
(iii) f ∈ S(F).

2.14 Definition [15]

(i) C(F) = {f : F → F | f is chaotic on F} and (ii) CH(X) = {F ∈ NK(X) | C(F) ≠ φ}.

2.15 Definition [15]

A topological space (X, τ) is called a chaos space if CH(X) ≠ φ. The members of CH(X) are called chaotic sets.

3 Characterizations of neutrosophic chaotic continuous functions

3.1 Definition
Let (X, τ) be a neutrosophic topological space and $V = \langle X, V^1, V^2, V^3 \rangle$ be a neutrosophic set of X.
(i) Ncl(V) denotes neutrosophic closure of V.
(ii) Nint(V) denotes neutrosophic interior of V.
(iii) NK(X) denotes the collection of all non empty neutrosophic compact sets of X.
(iv) clopen denotes closed and open
3.2 Definition
Let (X, τ) be a neutrosophic topological space. An orbit of a point x in X under the function f : (X, τ) → (X, τ) is denoted and defined as $O_f(x) = \{x, f^1(x), f^2(x), ... f^n(x)\}$ for x ∈ X and n ∈ Z^+.
3.3 Example
Let X = {p,q,r}. Let f : X → X be a function defined by f(p) = q, f(q) = r, and f(r) = p. If n = 1, then the orbit points $O_f(p)$ = {p,q}, $O_f(q)$ = {q,r} and $O_f(r)$ = {p,r}. If n = 2, then the orbit points $O_f(p)$ = X, $O_f(q)$ = X and $O_f(r)$ = X.
3.4 Definition
Let (X, τ) be a neutrosophic topological space. A neutrosophic orbit set in X under the function f : (X, τ) → (X, τ) is denoted and defined as $NO_f(x) = \langle x, O_{fT}(x), O_{fI}(x), O_{fF}(x) \rangle$ for x ∈ X.
3.5 Example
Let X = {p,q,r,s}. Let f : X → X be a function defined by f(p) = <q,s,q>, f(q) = <s,p.r>, f(r) = <p,q,s> and f(s) = <r,r,p>. If n = 1, then the neutrosophic orbit sets $NO_f(p)$ = <x,{p,q},{p,s},{p,q}>, $NO_f(q)$ = <x,{q,s},{q,p},{q,r}>, $NO_f(r)$ = <x,{p,r},{q,r},{r,s}> and $NO_f(s)$ = <x,{r,s},{r,s},{p,s}>. If n = 2, then the neutrosophic orbit sets $NO_f(p)$ = <x,{p,q,s},{p,r,s},{p,q,r}>, $NO_f(q)$ = <x,{q,r,s},{p,q,s},{q,r,s}>, $NO_f(r)$ = <x,{p,q,r},{p,q,r},{p,r,s}> and $NO_f(s)$ =<x,{p,r,s},{q,r,s},{p,q,s}>. If n = 3, then the neutrosophic orbit sets $NO_f(a)$ = <x,X,X,X>, $NO_f(b)$ =<x,X,X,X>, $NO_f(c)$ = <x,X,X,X>and $NO_f(d)$ =<x,X,X,X>.
3.6 Definition
Let (X, τ) be a neutrosophic topological space and f : (X, τ) → (X, τ) be a neutrosophic continuous function. Then f is said to be neutrosophic sensitive at x ∈ X if given any neutrosophic open set $U = \langle x, U^1, U^2, U^3 \rangle$ containing x ∃ a neutrosophic open set $V = \langle x, V^1, V^2, V^3 \rangle$ ∋ $f^n(x) \in V$, $f^n(y) \notin Ncl(V)$ and y ∈ U, n ∈ Z^+. We say that f is neutrosophic sensitive on a neutrosophic compact set $F = \langle x, F^1, F^2. F^3 \rangle$ if f|F is neutrosophic sensitive at every point of F.
3.7 Example
Let X = {p,q,r,s}. Then the neutrosophic sets P, Q, R and S are defined by P = <x,{p,r,s},{p,q,r},{p,r,s}>, Q = <x,{r,s},{p,r},{p,s}>, R = <x,{r,s},{p,q,r},{p,r,s}> and S = <x,{p,r,s},{p,r},{p,s}>. Then the family τ = {X_N,$φ_N$,P,Q,R,S} is neutrosophic topology on X. Clearly, (X, τ) is an neutrosophic topological space. Let f : (X,

τ) → (X, τ) be a function defined by f(p) = <r,q,s> f(q) = <s,s,r>, f(r) =< q,p,p> and f(s) = <p,r,q>. Let x = p and y = r. If n = 1,3,5, then the neutrosophic open set P = <x,{p,r,s},{p,q,r},{p,r,s}> containing x there exists an neutrosophic open set R = <x,{r,s},{p,q,r},{p,r,s}> such that $f^n(x) \in R, f^n(y) \notin Ncl(R)$ and $y \in P$. Hence the function f is called neutrosophic sensitive.

3.8 Notation

Let (X, τ) be a neutrosophic topological space. Let $F = <x,F^1,F^2,F^3> \subseteq X_N$ then $S(F) = <x,S(F)^1,S(F)^2,S(F)^3>$ where $S(F)^1$ = {f | f is neutrosophic sensitive on F}, $S(F)^2$ = {f | f is indeterminacy neutrosophic sensitive on F} and $S(F)^3$ = {f | f is not neutrosophic sensitive on F}.

3.9 Definition

Let (X, τ) be a two neutrosophic topological space. Let f : (X, τ) → (X, τ) be a function. A neutrosophic periodic set is denoted and defined as $NP_f(x) = <x,\{x \in X \mid f^n{}_T(x) = x\},\{x \in X \mid f^n{}_I(x) = x\},\{x \in X \mid f^n{}_F(x) = x\}>$

3.10 Example

Let X = {p,q,r}. Let f : X → X be a function defined by f(p) = <p,q,r>, f(q) = <r,p,q> and f(r) = <q,r,p>. If n = 1, then the neutrosophic periodic set $NP_f(p)$ = <x,{p},{q},{r}>, $NP_f(q)$ = <x,{r},{p},{q}>and $NP_f(q)$ = <x,{q},{r},{p}>. If n = 2, then the neutrosophic periodic sets $NP_f(p)$ = <x,{p},{p},{p}>, $NP_f(q)$ = <x,{q},{q},{q}> and $NP_f(r)$ = <x,{r},{r},{r}>.

3.11 Definition

Let (X, τ) be a neutrosophic topological space. A neutrosophic set $V = <X,V^1,V^2,V^3>$ of X is said to be a neutrosophic dense in X, if Ncl(V) = X.

3.12 Definition

Let (X, τ) be a neutrosophic topological space and $F = <x,F^1,F^2,F^3> \in NK(X)$. Let f : F → F be a neutrosophic continuous function. Then f is said to be neutrosophic chaotic on F if

(i) $Ncl(NO_f(x)) = F$ for some x ∈ F.
(ii) neutrosophic periodic points of f are neutrosophic dense in F. That is, $Ncl(NP_f(x)) = F$.
(iii) f ∈ S(F).

3.13 Notation

Let (X, τ) be a neutrosophic topological space then $C(F) = <x,C(F)^1,C(F)^2,C(F)^3>$ where $C(F)^1$ = {f : F → F | f is neutrosophic chaotic on F}, $C(F)^2$ = {f : F → F | f is indeterminacy neutrosophic chaotic on F},and $C(F)^3$ = {f : F → F | f is not neutrosophic chaotic on F}.

3.14 Notation

Let (X, τ) be a neutrosophic topological space then $CH(X) = \{F = <x,F^1,F^2,F^3> \in NK(X) \mid C(F) \neq \varphi\}$.

3.15 Definition

A neutrosophic topological space (X, τ) is called a neutrosophic chaos space if CH(X) ≠ φ. The members of CH(X) are called neutrosophic chaotic sets.

3.16 Definition

Let (X, τ) be a neutrosophic topological space. A neutrosophic set $V = <x,V^1,V^2,V^3>$ is neutrosophic clopen if it is both neutrosophic open and neutrosophic closed.

3.17 Definition

Let (X, τ) be a neutrosophic topological space.

(i) A neutrosophic open orbit set is a neutrosophic set which is both neutrosophic open and neutrosophic orbit.
(ii) A neutrosophic closed orbit set is a neutrosophic set which is both neutrosophic closed and neutrosophic orbit.
(iii) A neutrosophic clopen orbit set is a neutrosophic set which is both neutrosophic clopen and neutrosophic orbit.

3.18 Definition

Let (X, τ) be a neutrosophic topological space.

(i) A neutrosophic open chaotic set is a neutrosophic set which is both neutrosophic open and neutrosophic chaotic.
(ii) A neutrosophic closed chaotic set is a neutrosophic set which is both neutrosophic closed and neutrosophic chaotic.
(iii) A neutrosophic clopen chaotic set is a neutrosophic set which is both neutrosophic clopen and neutrosophic chaotic.

3.19 Definition

Let (X, τ) and (X,σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X,σ) is said to be neutrosophic chaotic continuous if for each periodic point x ∈ X and each neutrosophic clopen chaotic set F = <x,F^1,F^2,F^3> of f(x) ∃ a neutrosophic open orbit set $NO_f(x)$ of the periodic point x ∍ $f(NO_f(x)) \subseteq F$.

3.20 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q and R are defined by M=<x,{q,r},{r},{p,r}>, N=<x,{p},{p,q},{p,s}>, O=<x,{p,q,r},φ,{p}>, P=<x,φ,{p,q,r},{p,r,s}>, Q=<x,{p,q,r},{r},{p}>, R=<x,{p},{r},{p,q,r}>. Let τ={X_N,$φ_N$,M,N,O,P} and σ = {X_N,$φ_N$,Q,R} be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by f(p) = <p,q,s> f(q) = <r,s,r>, f(r) =< q,r,p> and f(s) = <s,p,q>. Now the function f is called neutrosophic chaotic continuous.

3.21 Theorem

Let (X, τ) and (X,σ) be any two neutrosophic chaos spaces. Let f : (X, τ) → (X,σ) be a function. Then the following statements are equivalent:

(i) f is neutrosophic chaotic continuous.

(ii) Inverse image of every neutrosophic clopen chaotic set of (X,σ) is a neutrosophic open orbit set of (X, τ).

(iii) Inverse image of every neutrosophic clopen chaotic set of (X,σ) is a neutrosophic clopen orbit set of (X, τ).

Proof

(i)⇒ (ii) Let F = <x,F^1,F^2,F^3> be a neutrosophic clopen chaotic set of (X, σ) and the periodic point x ∈ $f^{-1}(F)$. Then f(x) ∈ F. Since f is neutrosophic chaotic continuous, ∃ a neutrosophic open orbit set $NO_f(x)$ of (X, τ) ∍ x ∈ $NO_f(x)$, $f(NO_f(x)) \subseteq F$. That is, x ∈ $NO_f(x) \subseteq f^{-1}(F)$. Now, $f^{-1}(F) = \cup\{NO_f(x) : x \in f^{-1}(F)\}$. Since $f^{-1}(F)$ is union of neutrosophic open orbit sets. Therefore, $f^{-1}(F)$ is an neutrosophic open orbit set.

(ii) ⇒ (iii) Let F be a neutrosophic clopen chaotic set of (X, σ). Then X − F is also a neutrosophic clopen chaotic set, By (ii) $f^{-1}(X - F)$ is neutrosophic open orbit in (X, τ). So $X - f^{-1}(F)$ is a neutrosophic open orbit set in (X, τ). Hence, $f^{-1}(F)$ is neutrosophic closed orbit in (X, τ). By (ii), $f^{-1}(F)$ is a neutrosophic open orbit set of (X, τ). Therefore, $f^{-1}(F)$ is both neutrosophic open orbit and neutrosophic closed orbit in (X, τ). Hence, $f^{-1}(F)$ is a neutrosophic clopen orbit set of (X, τ).

(iii) ⇒ (i) Let x be a periodic point, x ∈ X and F be a neutrosophic clopen chaotic set containing f(x) then $f^{-1}(F)$ is a neutrosophic open orbit set of (X, τ) containing x and $f(f^{-1}(F)) \subseteq F$. Hence, f is neutrosophic chaotic continuous.

3.22 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X, σ) is said to be neutrosophic chaotic* continuous if for each periodic point x ∈ X and each neutrosophic closed chaotic set F containing f(x), ∃ neutrosophic open orbit set $NO_f(x)$ containing x ∍ $f(Ncl(NO_f(x))) \subseteq F$.

3.23 Theorem

A neutrosophic chaotic continuous function is a neutrosophic chaotic* continuous function.

Proof Since f is a neutrosophic chaotic continuous function, F is a neutrosophic clopen chaotic set containing f(x), ∃ a neutrosophic open orbit set $NO_f(x)$ containing x ∍ $f(NO_f(x)) \subseteq F$. Then $f^{-1}(F)$ is a neutrosophic clopen chaotic set of (X, σ). By (iii) of Theorem 3.21., $f^{-1}(F)$ is a neutrosophic clopen orbit set in (X, τ). Therefore, F is a neutrosophic closed chaotic set containing f(x) and $f^{-1}(F)$ is a neutrosophic open orbit set ∍ $f(f^{-1}(F)) \subseteq F$. Since $f^{-1}(F)$ is neutrosophic closed orbit set, $Ncl(f^{-1}(F)) = f^{-1}(F)$. This implies that, $f(Ncl(f^{-1}(F))) \subseteq F$. Hence, f is a neutrosophic chaotic* continuous function.

3.24 Remark

The converse of Theorem 3.23. need not be true as shown in Example 3.25.

3.25 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q,R,S and T are defined by M=<x,{p,r},{q,r},{r}>, N=<x,{r},{q},{p,q,r}>, O=<x,{r},{q,r},{p,q,r}>, P=<x,{p,r},{q},{r}>, Q=<x,{p,q,s},{q,s},{p,r}>, R=<x,{q,s},{p,q},{q,r}>, S=<x,{q,s},{p,q,s},{p,q,r}> and T=<x,{p,q,s},{q},{r}>. Let τ={X_N,$φ_N$,M,N,O,P} and σ = {X_N,$φ_N$,Q,R,S,T} be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by f(p) = <q,p,s> f(q) = <s,r,p>, f(r) =< p,q,r> and f(s) = <r,s,q>. Now the function f is neutrosophic chaotic* continuous but not neutrosophic chaotic continuous. Hence, neutrosophic chaotic* continuous function need not be neutrosophic chaotic continuous function.

3.26 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X, σ) is said to be neutrosophic chaotic** continuous if for each periodic point x ∈ X and each neutrosophic closed chaotic set F of f(x), ∃ a neutrosophic open orbit set $NO_f(x)$ of the periodic point x ∍ $f(NO_f(x)) \subseteq Nint(F)$.

3.27 Theorem

A neutrosophic chaotic continuous function is a neutrosophic chaotic** continuous function.

Proof Since f is a neutrosophic chaotic continuous function, F is a neutrosophic clopen chaotic set containing f(x), $\exists$ a neutrosophic open orbit set $NO_f(x)$ containing x $\ni$ $f(NO_f(x)) \subseteq F$. Since F is a neutrosophic open orbit set in (X, σ), F = Nint(F). This implies that, $f(NO_f(x)) \subseteq Nint(F)$. Hence, f is an neutrosophic chaotic** continuous function.

3.28 Remark

The converse of Theorem 3.27 need not be true as shown in the Example 3.29.

3.29 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q,R,S and T are defined by $M=\langle x,\{q,r\},\{r\},\{p,r\}\rangle$, $N=\langle x,\{p,s\},\{p,q\},\{p,q\}\rangle$, $O=\langle x,\varphi,\{p,q,r\},\{p,q,r\}\rangle$, $P=\langle x,X,\varphi,\{p\}\rangle$, $Q=\langle x,\{p,q,r\},\{r\},\{p,s\}\rangle$, $R=\langle x,\{q\},\{q,r\},\{p,r\}\rangle$, $S=\langle x,\{p,q,r\},\{r\},\{p\}\rangle$ and $T=\langle x,\{q\},\{r\},\{p,r,s\}\rangle$. Let $\tau=\{X_N,\varphi_N,M,N,O,P\}$ and $\sigma = \{X_N,\varphi_N,Q,R,S,T\}$ be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by $f(p) = \langle p,q,s\rangle$ $f(q) = \langle r,s,r\rangle$, $f(r) =\langle q,r,p\rangle$ and $f(s) = \langle s,p,q\rangle$. Now the function f is neutrosophic chaotic** continuous but not neutrosophic chaotic continuous. Hence, neutrosophic chaotic** continuous function need not be neutrosophic chaotic continuous function.

3.30 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function f : (X, τ) → (X, σ) is said to be a neutrosophic chaotic*** continuous if for each periodic point x ∈ X and each neutrosophic closed chaotic set F of f(x) $\exists$ a neutrosophic clopen orbit set $NO_f(x)$ of the periodic point x $\ni$ $f(Nint(NO_f(x))) \subseteq F$.

3.31 Theorem

A neutrosophic chaotic continuous function is a neutrosophic chaotic*** continuous function.

Proof Since f is a neutrosophic chaotic continuous function, F is a neutrosophic clopen chaotic set containing f(x), $\exists$ a neutrosophic open orbit set $NO_f(x)$ containing x $\ni$ $f(NO_f(x)) \subseteq F$. This implies that, $NO_f(x) \subseteq f^{-1}(F)$. Then, $f^{-1}(F)$ is a neutrosophic clopen chaotic set of (X, σ). By (iii) of Theorem 3.21, $f^{-1}(F)$ is a neutrosophic clopen orbit set in (X, τ). Therefore, F is a neutrosophic closed chaotic set containing f(x) and $f^{-1}(F)$ is a neutrosophic open orbit set $\ni$ $f(f^{-1}(F)) \subseteq F$. Since $f^{-1}(F)$ is neutrosophic open orbit set, $Nint(f^{-1}(F)) = f^{-1}(F)$. This implies that, $f(Nint(f^{-1}(F))) \subseteq F$. Hence, f is a neutrosophic chaotic*** continuous function.

3.32 Remark

The converse of Theorem 3.31 need not be true as shown in the Example 3.33.

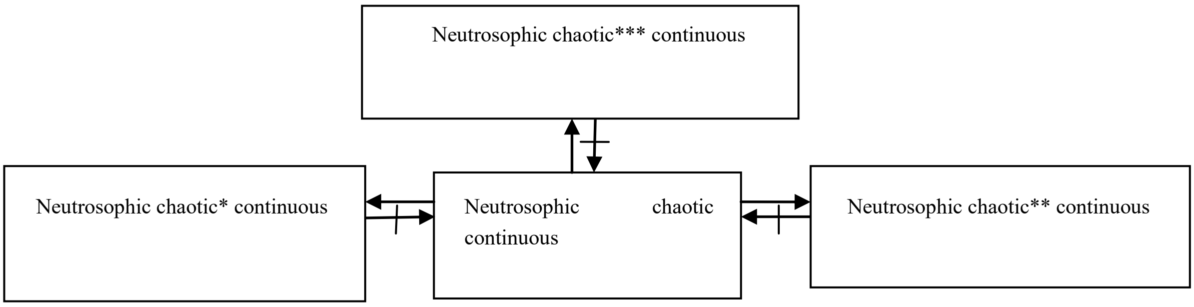
3.33 Example

Let X={p,q,r,s}. Then the neutrosophic sets M,N,O,P,Q,R,S and T are defined by $M=\langle x,\{q,r\},\{r\},\{p,r\}\rangle$, $N=\langle x,\{p,r\},\{r\},\{q,r\}\rangle$, $O=\langle x,\{p,q,r\},\{r\},\{r\}\rangle$, $P=\langle x,\{r\},\{r\},\{p,q,r\}\rangle$, $Q=\langle x,\{p,q,r\},\{q,r\},\{p,s\}\rangle$, $R=\langle x,\{q,r\},\{p,q\},\{r,s\}\rangle$, $S=\langle x, \{p,q,r\},\{r\},\{p\}\rangle$ and $T=\langle x,\{q\},\{r\},\{p,r,s\}\rangle$. Let $\tau=\{X_N,\varphi_N,M,N,O,P\}$ and $\sigma = \{X_N,\varphi_N,Q,R,S,T\}$ be a neutrosophic topologies on X. Clearly (X, τ) and (X,σ) be any two neutrosophic chaos spaces. The function f : (X, τ) → (X, σ) is defined by $f(p) = \langle p,q,s\rangle$ $f(q) = \langle r,s,r\rangle$, $f(r) =\langle q,r,p\rangle$ and $f(s) = \langle s,p,q\rangle$. Now the function f is neutrosophic chaotic*** continuous but not neutrosophic chaotic continuous. Hence, neutrosophic chaotic*** continuous function need not be neutrosophic chaotic continuous function.

3.34 Remark

The interrelation among the functions introduced are given clearly in the following diagram.

Figure 1:



4 Properties of neutrosophic chaotic continuous functions

4.1 Definition

A neutrosophic chaos space (X, τ) is said to be a neutrosophic orbit extremally disconnected space if .the neutrosophic closure of every neutrosophic open orbit set is neutrosophic open orbit.

4.2 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. If f : (X, τ) → (X, σ) is a neutrosophic chaotic continuous function and (X, τ) is a neutrosophic orbit extremally disconnected space then f is a neutrosophic chaotic* continuous function.

Proof Let x be a periodic point and x ∈ X. Since f is neutrosophic chaotic continuous, F = $<x,F^1,F^2,F^3>$ is a neutrosophic clopen chaotic set of (X, σ), ∃ a neutrosophic open orbit set $NO_f(x)$ of (X, τ) containing x ∍ $f(NO_f(x)) \subseteq F$. Therefore, $NO_f(x)$ is a neutrosophic open orbit set $NO_f(x)$ of (X, τ). Since (X, τ) is neutrosophic orbit extremally disconnected, $Ncl(NO_f(x))$ is a neutrosophic open orbit set. Therefore, F is a neutrosophic closed chaotic set containing f(x) ∃ a neutrosophic open orbit set $Ncl(NO_f(x))$ ∍ $f(Ncl(NO_f(x))) \subseteq F$. Hence, f is neutrosophic chaotic* continuous.

4.3 Definition

A neutrosophic chaos space (X, τ) is said to be neutrosophic chaotic 0- dimensional if it has a neutrosophic base consisting of neutrosophic clopen chaotic sets.

4.4 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. Let f : (X, τ) → (X, σ) be a neutrosophic chaotic*** continuous function. If (X, σ) is neutrosophic chaotic 0-dimensional then f is a neutrosophic chaotic continuous function.

Proof Let the periodic point x ∈ X. Since (X, σ) is neutrosophic chaotic 0-dimensional, ∃ a neutrosophic clopen chaotic set F = $<x,F^1,F^2,F^3>$ in (X, σ). Since f is a neutrosophic chaotic*** continuous function, ∃ a neutrosophic clopen orbit set $NO_f(x)$ ∍ $f(Nint(NO_f(x))) \subseteq F$. Since $NO_f(x)$ is a neutrosophic open orbit set, $Nint(NO_f(x) = NO_f(x)$. This implies that, $f(NO_f(x)) \subseteq F$. Therefore, f is neutrosophic chaotic continuous.

4.5 Definition

A neutrosophic chaos space (X, τ) is said to be a neutrosophic orbit connected space if X_N cannot be expressed as the union of two neutrosophic open orbit sets $NO_f(x)$ and $NO_f(y)$, x,y ∈ X of (X, τ) with $NO_f(x) \cap NO_f(y) \neq \varphi_N$.

4.6 Definition

A neutrosophic chaos space (X, τ) is said to be a neutrosophic chaotic connected space if X_N cannot be expressed as the union of two neutrosophic open chaotic sets U = $<x,U^1,U^2,U^3>$ and V = $<x,V^1,V^2,V^3>$ of (X, τ) with $U \cap V \neq \varphi_N$.

4.7 Theorem

A neutrosophic chaotic continuous image of a neutrosophic orbit connected space is a neutrosophic chaotic connected space.

Proof Let (X, σ) be neutrosophic chaotic disconnected. Let $F_1 = <x, F_1^1, F_1^2, F_1^3>$ and $F_2 = <x, F_2^1, F_2^2, F_2^3>$ be a neutrosophic chaotic disconnected sets of (X, σ). Then $F_1 \neq \varphi_N$ and $F_2 \neq \varphi_N$ are neutrosophic clopen chaotic sets in (X, σ) and $Y_N = F_1 \cup F_2$ where $F_1 \cap F_2 = \varphi_N$. Now, $X_N = f^{-1}(Y_N) = f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2)$.Since f is neutrosophic chaotic continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are neutrosophic open orbit sets in (X, τ). Also $f^{-1}(F_1) \cap f^{-1}(F_2) = \varphi_N$. Therefore, (X, τ) is not neutrosophic orbit connected. Which is a contradiction. Hence, (X, σ) is neutrosophic chaotic connected.

4.8 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. If f : (X, τ) → (X, σ) is a neutrosophic chaotic continuous function and $NO_f(x)$ is neutrosophic open orbit set then the restriction $f|NO_f(x) : NO_f(x) \to (X, \sigma)$ is neutrosophic chaotic continuous.

Proof Let F = $<x,F^1,F^2,F^3>$ be a neutrosophic clopen chaotic set in (X, σ). Then, $(f|NO_f(x))^{-1}(F) = f^{-1}(F) \cap NO_f(x)$. Since f is neutrosophic chaotic continuous, $f^{-1}(F)$ is neutrosophic open orbit in (X, τ) and $NO_f(x)$ is a neutrosophic open orbit set. This implies that, $f^{-1}(F) \cap NO_f(x)$ is a neutrosophic open orbit set. Therefore, $(f|NO_f(x))^{-1}(F)$ is neutrosophic open orbit in (X, τ). Hence, $f|NO_f(x)$ is neutrosophic chaotic continuous.

4.9 Definition

Let (X, τ) be a neutrosophic chaos space. If a family $\{NO_f(x_i) : i \in J\}$ of neutrosophic open orbit set in (X, τ) satisfies the condition $\cup NO_f(x_i) = X_N$, then it is called a neutrosophic open orbit cover of (X, τ).

4.10 Theorem

Let $\{NO_f(x)_\gamma : \gamma \in \Gamma\}$ be any neutrosophic open orbit cover of a neutrosophic chaos space (X, τ). A function f : (X, τ) → (X, σ) is a neutrosophic chaotic continuous function if and only if the restriction $f|NO_f(x)_\gamma : NO_f(x)_\gamma \to (X, \sigma)$ is neutrosophic chaotic continuous for each γ ∈ Γ.

Proof Let γ be an arbitrarily fixed index and $NO_f(x)_\gamma$ be a neutrosophic open orbit set of (X, τ). Let the periodic point $x \in NO_f(x)_\gamma$ and F = $<x,F^1,F^2,F^3>$ is neutrosophic clopen chaotic set containing $(f|NO_f(x)_\gamma)(x) = f(x)$. Since f is neutrosophic chaotic continuous there exists a neutrosophic open orbit set $NO_f(x)$ containing x such that

$f(NO_f(x)) \subseteq F$. Since $(NO_f(x)_\gamma)$ is neutrosophic open orbit cover in (X, τ), $x \in NO_f(x) \cap NO_f(x)_\gamma$ and $(f|NO_f(x)_\gamma)(NO_f(x) \cap (NO_f(x)_\gamma) = f(NO_f(x) \cap (NO_f(x)_\gamma)) \subset f(NO_f(x) \subset F$. Hence $f|NO_f(x)_\gamma$ is a neutrosophic chaotic continuous function. Conversely, let the periodic point $x \in X$ and F be a neutrosophic chaotic set containing f(x). There exists an $\gamma \in \Gamma$ such that $x \in NO_f(x)_\gamma$. Since $(f|NO_f(x)_\gamma) : NO_f(x)_\gamma \to (X, \sigma)$ is neutrosophic chaotic continuous, there exists a $NO_f(x) \in NO_f(x)_\gamma$ containing x such that $(f|NO_f(x)_\gamma)(NO_f(x)) \subseteq F$. Since $NO_f(x)$ is neutrosophic open orbit in (X, τ), $f(NO_f(x)) \subseteq F$. Hence, f is neutrosophic chaotic continuous.

4.11 Theorem

If a function $f : (X, \tau) \to \Pi\ (X, \sigma)_\lambda$ is neutrosophic chaotic continuous then $P_\lambda \circ f : (X, \tau) \to (X, \sigma)_\lambda$ is neutrosophic chaotic continuous for each $\lambda \in \Lambda$, where P_λ is the projection of $\Pi\ (X, \sigma)_\lambda$ onto $(X, \sigma)_\lambda$.

Proof Let $F_\lambda = \langle x, F_\lambda^1, F_\lambda^2, F_\lambda^3 \rangle$ be any neutrosophic clopen chaotic set of $(X, \sigma)_\lambda$. Then $P_\lambda^{-1}(F_\lambda)$ is a neutrosophic clopen chaotic set in $\Pi\ (X, \sigma)_\lambda$ and hence $(P_\lambda \circ f)^{-1}(F_\lambda) = f^{-1}(P_\lambda^{-1}(F_\lambda))$ is a neutrosophic open orbit set in (X, τ). Therefore, $P_\lambda \circ f$ is neutrosophic chaotic continuous.

4.12 Theorem

If a function $f : \Pi\ (X, \tau)_\lambda \to \Pi\ (X, \sigma)_\lambda$ is neutrosophic chaotic continuous then $f_\lambda : (X, \tau)_\lambda \to (X, \sigma)_\lambda$ is a neutrosophic chaotic continuous function for each $\lambda \in \Lambda$.

Proof Let $F_\lambda = \langle x, F_\lambda^1, F_\lambda^2, F_\lambda^3 \rangle$ be any neutrosophic clopen chaotic set of $(X, \sigma)_\lambda$. Then $P_\lambda^{-1}(F_\lambda)$ is neutrosophic clopen chaotic in $\Pi\ (X, \sigma)_\lambda$ and $f^{-1}(P_\lambda^{-1}(F_\lambda)) = f_\lambda^{-1}(F_\lambda) \times \Pi\ \{(X, \tau)_\alpha : \alpha \in \Lambda - \{\lambda\}\}$. Since f is neutrosophic chaotic continuous, $f^{-1}(P_\lambda^{-1}(F_\lambda))$ is a neutrosophic open orbit set in $\Pi\ (X, \tau)_\lambda$. Since the projection P_λ of $\Pi\ (X, \tau)_\lambda$ onto $(X, \tau)_\lambda$ is a neutrosophic open function, $f_\lambda^{-1}(F_\lambda)$ is neutrosophic open orbit in $(X, \tau)_\lambda$. Hence, f_λ is neutrosophic chaotic continuous.

4.13 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function $f : (X, \tau) \to (X, \sigma)$ is said to be neutrosophic chaotic irresolute if for each neutrosophic clopen chaotic set $F = \langle x, F^1, F^2, F^3 \rangle$ in (X, σ), $f^{-1}(A)$ is a neutrosophic clopen chaotic set of (X, τ).

4.14 Theorem

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. If $f : (X, \tau) \to (X, \sigma)$ is a neutrosophic chaotic continuous function and $g : (X, \sigma) \to (X, \eta)$ is a neutrosophic chaotic irresolute function, then $g \circ f : (X, \tau) \to (X, \eta)$ is neutrosophic chaotic continuous.

Proof Let $F = \langle x, F^1, F^2, F^3 \rangle$ be a neutrosophic clopen set of (X, η). Since g is neutrosophic chaotic irresolute, $g^{-1}(F)$ is neutrosophic clopen chaotic set of (X, σ). Since f is neutrosophic chaotic continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is a neutrosophic open orbit set of (X, τ) such that $f^{-1}(g^{-1}(F)) \subseteq F$. Hence $g \circ f$ is neutrosophic chaotic continuous.

4.15 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. A function $f : (X, \tau) \to (X, \sigma)$ is said to be neutrosophic orbit irresolute if for each neutrosophic open orbit set $NO_f(x)$ in (X, σ), $f^{-1}(NO_f(x))$ is a neutrosophic open orbit set of (X, τ).

4.16 Definition

Let (X, τ) and (X, σ) be any two neutrosophic chaos spaces. Let $f : (X, \tau) \to (X, \sigma)$ be a function. Then f is said to be a neutrosophic open orbit function if the image of every neutrosophic open orbit set in (X, τ) is neutrosophic open orbit in (X, σ).

4.17 Theorem

Let $f : (X, \tau) \to (X, \sigma)$ be neutrosophic orbit irresolute, surjective and neutrosophic open orbit function. Then $g \circ f : (X, \tau) \to (X, \eta)$ is neutrosophic chaotic continuous iff $g : (X, \sigma) \to (X, \eta)$ is neutrosophic chaotic continuous.

Proof Let $F_\lambda = \langle x, F_\lambda^1, F_\lambda^2, F_\lambda^3 \rangle$ be a neutrosophic clopen chaotic set of (X, η). Since g is neutrosophic chaotic continuous, $g^{-1}(F)$ is neutrosophic open orbit in (X, σ). Since f is neutrosophic orbit irresolute, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is neutrosophic open orbit in (X, τ). Hence $g \circ f$ is neutrosophic chaotic continuous. Conversely, let $g \circ f : (X, \tau) \to (X, \eta)$ be neutrosophic chaotic continuous function. Let F be a neutrosophic clopen chaotic set of (X, η), then $(g \circ f)^{-1}(F)$ is a neutrosophic open orbit set of (X, τ). Since f is neutrosophic open orbit and surjective, $f(f^{-1}(g^{-1}(F))$ is a neutrosophic open orbit set of (X, σ). Therefore, $g^{-1}(F)$ is a neutrosophic open orbit set in (X, σ). Hence, g is neutrosophic chaotic continuous.

Conclusion

In this paper, characterization of neutrosophic chaotic continuous functions are studied. Some interrelations are discussed with suitable examples. Also, neutrosophic orbit, extremally disconnected spaces and neutrosophic chaotic zero-dimensional spaces has been discussed with some interesting properties. This paper paves way in future to introduce and study the notions of neutrosophic orbit Co-kernal spaces, neutrosophic hardly open orbit spaces, neutrosophic orbit quasi regular spaces and neutrosophic orbit strongly complete spaces, neutrosophic orbit Co-kernal function, neutrosophic hardly open orbit function for which the above discussed set form the basis.

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Neutrosophic Cubic Einstein Geometric Aggregation Operators with Application to Multi-Criteria Decision Making Method

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Abstract: Neutrosophic cubic sets (NCs) are amore generalized version of neutrosophic sets(Ns) and interval neutrosophic sets (INs). Neutrosophic cubic setsare better placed to express consistent, indeterminate and inconsistent information, which provides a better platform to deal with incomplete, inconsistent and vague data. Aggregation operators play a key role in daily life, and in relation to science and engineering problems. In this paper we defined the algebraic and Einstein sum, multiplication and scalar multiplication, score and accuracy functions. Using these operations we defined geometric aggregation operators and Einstein geometric aggregation operators. First, we defined the algebraic and Einstein operators of addition, multiplication and scalar multiplication. We defined score and accuracy function to compare neutrosophic cubic values. Then we definedthe neutrosophic cubic weighted geometric operator (NCWG), neutrosophic cubic ordered weighted geometric operator (NCOWG), neutrosophic cubic Einstein weighted geometric operator (NCEWG), and neutrosophic cubic Einstein ordered weighted geometric operator (NCEOWG) over neutrosophic cubic sets. A multi-criteria decision making method is developed as an application to these operators. This method is then applied to a daily life problem.

Keywords: neutrosophic cubic weighted geometric operator (NCWG); neutrosophic cubic ordered weighted geometric operator (NCOWG); neutrosophic cubic Einstein weighted geometric operator (NCEWG); neutrosophic cubic Einstein ordered weighted geometric operator (NCEOWG)

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [1].Soon after, it attracted experts of sciences and engineering due to its possibilistic behavior. The applicability of fuzzy sets extended it to interval valued fuzzy sets(IVFs) [2,3]. In 1986, K. Atnassov developed the theory of intuitionistic fuzzy sets [4], which were further extended to interval valued intuitionistic fuzzy sets in 1989 [5]. In 2012, Y.B. Jun generalized the idea of fuzzy sets and intuitionistic fuzzy sets to form cubic sets [6]. Smarandache presented his theory regarding the inconsistent and indeterminate behavior of data in 1999, and named it the neutrosophic set [7]. Neutrosophic sets consist of three components:Truth, indeterminate and falsehood, which provides a more general platform to deal with vague and insufficient data. In 2005, Wang et al. [8] presented the idea of interval valued neutrosophic sets. Interval valued neutrosophic sets provide a range to experts which makes them more comfortable with making the choice. Jun et al. defined the neutrosophic cubic set [9,10]. Neutrosophic cubic sets are a generalization of neutrosophic sets and interval neutrosophic sets. They enable us to choose both interval values and single value membership. This characteristic of neutrosophic cubic sets enables us to deal with uncertain and vague data more efficiently.

Decision making is one of the most important factors in scienceand day-to-day life as well. Aggregation operators are an imperative part of modern decision making. A lack of data or information makes it difficult for decision makers to take an appropriatedecision. This uncertain situation can be minimized using the vague nature neutrosophic cubic set and its extensions. Neutrosophic cubic set (NCs) are a more generalized version of neutrosophic sets (Ns) and interval neutrosophic sets (INs). Neutrosophic cubic setsare better placed to express consistent, indeterminate, and inconsistent information, which provides a better platform to deal with incomplete, inconsistent, and vague data. Aggregation operators have a key role in daily life, science and engineering problems. Zhan et al. [11] in their workapplications of neutrosophic cubic sets in multi-criteria decision making in 2017. Banerjee et al. [12] usedgrey rational analysis in their workGRA for multi attribute decision making in neutrosophic cubic set environment in 2017.Lu and Ye [13] definedcosine measure for neutrosophic cubic sets for multiple attribte decision making in 2017. Pramanik et al. [14] defined neutrosophic cubic MCGDM method based on similarity measurein 2017. Shi and Ye [15] defined Dombi aggregation operators of neutrosophic cubic set for multiple attribute deicision makingin 2018. Baolin et al. [16] applied Einstein aggregations onneutrosophic sets in a novel generalized simplified neutrosophic number Einstein aggregation operator 2018. Alot of work has been done and is being done by different researchers in decision making using neutrosophic cubic sets.

In this paper, we define algebraic and Einstein sum, multiplication and scalar multiplication, score and accuracy functions. Using these operations, we define geometric aggregation operators and Einstein geometric aggregation operators. First, we define algebraic and Einstein operators of addition, multiplication and scalar multiplication. We then define score and accuracy functions to compare neutrosophic cubic values. Following this, we propose a neutrosophic cubic ordered weighted geometric operator (NCOWG), neutrosophic cubic Einstein weighted geometric operator (NCEWG), and a neutrosophic cubic Einstein ordered weighted geometric operator (NCEOWG) over neutrosophic cubic sets. A multi-criteria decision making method is then developed as an application for these operators. This method is then applied to a daily life problem.

2. Preliminaries

This section consists of two parts: Notations, which consists of notations with their descriptions and some previous definitions; and results. We recommend the reader to see [1–3,6–9,16].

2.1. Notations

This section consists of some notations with their descriptions, as shown in Table 1.

Table 1. Some notations with their descriptions.

S. No	Notation	Description
1	$\boldsymbol{U}$	Ground set
2	$\boldsymbol{u}$	Element of ground set (U).
3	ψ	Fuzzy set
4	$\widetilde{\Psi} = \left[\Psi^L, \Psi^U\right]$	Interval valued fuzzy set which is an interval of [0,1]. The left extreme ψ^L is referred as lower fuzzy and right extreme ψ^U is referred as upper fuzzy function.
5	(T_N, I_N, F_N)	components of neutrosophic sets each one is fuzzy sets.
6	$\left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N\right)$	The components of interval neutrosophic each one is an interval valued fuzzy set.
7	$\left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$	The components of neutrosophic cubic set. Referred to 5 and 6.
8	Γ^*, Γ	t-conorm, t-norm
9	$\oplus, \otimes$	Algebraic sum, product
10	$\oplus_E, \otimes_E$	Einstein sum, product

2.2. Pre-Defined Definitions

This section consists of some predefined definitions and results.

Definition 1 [1]. *A mapping $\psi: U \to [0,1]$ is called a fuzzy set, and $\psi(u)$ is called a membership function, simply denoted by ψ.*

Definition 2 [2,3]. *A mapping $\widetilde{\Psi}: U \to D[0,1]$, where $D[0,1]$ is the interval value of $[0,1]$, called the interval valued fuzzy set(IVF). For all $u \in U$ $\widetilde{\Psi}(u) = \left\{\left[\psi^L(u), \psi^U(u)\right] \middle| \psi^L(u), \psi^U(u) \in [0,1] \text{ and } \psi^L(u) \leq \psi^U(u)\right\}$ is membership degree of u in $\widetilde{\Psi}$. This is simply denoted by $\widetilde{\Psi} = \left[\Psi^L, \Psi^U\right]$.*

Definition 3 [6]. *A structure $C = \left\{\left(u, \widetilde{\Psi}(u), \Psi(u)\right) \middle| u \in U\right\}$ is a cubic set in U in which $\widetilde{\Psi}(u)$ is IVF in U, that is, $\widetilde{\Psi} = \left[\Psi^L, \Psi^U\right]$ and Ψ is a fuzzy set in U. This can be simply denoted by $C = \left(\widetilde{\Psi}, \Psi\right)$. C^U denotes the collection of cubic sets in U.*

Definition 4 [7]. *A structure $N = \{(T_N(u), I_N(u), F_N(u)) | u \in U\}$ is a neutrosophic set(Ns), where $\{T_N(u), I_N(u), F_N(u) \in [0,1]\}$ are called truth, indeterminacy and falsity functions, respectively.This can be simply denoted by $N = (T_N, I_N, F_N)$.*

Definition 5 [8]. *An interval neutrosophic set (INs) in U is a structure $N = \left\{\left(\widetilde{T}_N(u), \widetilde{I}_N(u), \widetilde{F}_N(u)\right) \middle| u \in U\right\}$, where $\left\{\widetilde{T}_N(u), \widetilde{I}_N(u), \widetilde{F}_N(u) \in D[0,1]\right\}$ is calledtruth, indeterminacy an falsity functionin U, respectively. This can be simply denoted by $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N\right)$. For convenience, we denote $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N\right)$ by $N = \left(\widetilde{T}_N = [T_N^L, T_N^U], \widetilde{I}_N = [I_N^L, I_N^U], \widetilde{F}_N = [F_N^L, F_N^U]\right)$.*

Definition 6 [9]. *A structure $N = \left\{\left(u, \widetilde{T}_N(u), \widetilde{I}_N(u), \widetilde{F}_N(u), T_N(u), I_N(u), F_N(u)\right) \middle| u \in U\right\}$ is neutrosophic cubic set(NCs) in U, in which $\left(\widetilde{T}_N = [T_N^L, T_N^U], \widetilde{I}_N = [I_N^L, I_N^U], \widetilde{F}_N = [F_N^L, F_N^U]\right)$ is an interval neutrosophic set and (T_N, I_N, F_N) is a neutrosophic set in U. Simply denoted by $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$, $[0,0] \leq \widetilde{T}_N + \widetilde{I}_N + \widetilde{F}_N \leq [3,3]$ and $0 \leq T_N + I_N + F_N \leq 3$. N^U denotes the collection of neutrosophic cubic sets in U. Simply denoted by $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$.*

Definition 7 [16]. *The t-operators are basically union and intersection operators in the theory of fuzzy sets, which are denoted by t-conorm (Γ^*) and t-norm (Γ), respectively. The role of t-operators is very important in fuzzy theory and its applications.*

Definition 8 [16]. $\Gamma^* : [0,1] \times [0,1] \to [0,1]$ *is called t-conorm if it satisfies the following axioms:*

Axiom 1. $\Gamma^*(1,u) = 1$ *and* $\Gamma^*(0,u) = 0$;
Axiom 2. $\Gamma^*(u,v) = \Gamma^*(v,u)$ *for all a and b;*
Axiom 3. $\Gamma^*(u,\Gamma^*(v,w)) = \Gamma^*(\Gamma^*(u,v),w)$ *for all a, b and c;*
Axiom 4. *If* $u \leq u'$ *and* $v \leq v'$, *then* $\Gamma^*(u,v) \leq \Gamma^*(u',v')$.

Most known t-conorms are as follows:

1. The default t-conorm: $\Gamma^*_{\max}(u,v) = \max(u,v)$.
2. The bounded t-conorm: $\Gamma^*_{bounded}(u,v) = \min(1,u+v)$.
3. The algebraic t-conorm: $\Gamma^*_{algebraic}(u,v) = u + v - uv$.

Definition 9 [16]. $\Gamma : [0,1] \times [0,1] \to [0,1]$ *is called t-norm if it satisfies the following axioms:*

Axiom 5. $\Gamma(1,u) = u$ *and* $\Gamma(0,u) = 0$;
Axiom 6. $\Gamma(u,v) = \Gamma(v,u)$ *for all a and b;*
Axiom 7. $\Gamma(u,\Gamma(v,w)) = \Gamma(\Gamma(u,v),w)$ *for all a, b and c;*
Axiom 8. *If* $u \leq u'$ *and* $v \leq v'$, *then* $\Gamma(u,v) \leq \Gamma(u',v')$.

Most well known t-norms are as follows:

1. The default t-norm: $\Gamma_{\min}(u,v) = \min(u,v)$.
2. The bounded t-norm: $\Gamma_{bounded}(u,v) = \max(0,u+v-1)$.
3. The algebraic t-norm: $\Gamma_{algebraic}(u,v) = uv$.

If $\Gamma^*(u,v)$, $\Gamma(u,v)$ are continuous and $\Gamma^*(u,u) > u$, $\Gamma(u,u) < u$, then Γ^* and Γ are said to be Archimedes t-conorm and t-norm, respectively. Any pair of dual t-conorm (Γ^*) and t-norm (Γ) is used. It is known that t-norms and t-conorms operators satisfy the condition of conjunction and disjunction operators, respectively. However, the algebraic operations, like algebraic sum and product, are not unique and may correspond to union and intersection. The t-conorms and t-norms families have a vast range, which corresponds to unions and intersections. Among these, the Einstein sum and Einstein product are good choices since they give the smooth approximation like algebraic sum and algebraic product, respectively. Einstein sum $\oplus_E$ and Einstein product $\otimes_E$ are examples of t-conorm and t-norm, respectively:

$$\Gamma^*_E(u,v) = \frac{u+v}{1+uv}$$

$$\Gamma_E(u,v) = \frac{uv}{1+(1-u)(1-v)}$$

Group decision making is an important aspect of decision making theory. We are often in situationsin which we have to deal with more then one expert, attribute and alternative. Motivated by such situations, a multi-attribute decision making method for more then one expert is proposed on neutrosophic cubic aggregation operators.This whole work consisted of six sections. In Section 3, we define some algebraicEinstein operations and score and accuracy functions, along with some important results and examples. On the basis of these definitions and results, we define geometric and Einstein geometric aggregation operators on neutrosophic cubic sets in Section 4. In Section 5, an algorithm is proposed based on neutrosophic cubic geometric and Einstein geometric aggregation operators to deal with multi-attribute decision making problems. In the final section, a numerical example from daily life is presented as an application of the work.

3. Operations on Neutrosophic Cubic Sets

In this section, we introduce some new operations on neutrosophic cubic sets which are further used in the article.

3.1. Algebraic Addition, Multiplication and Scalar Multiplication

We introduce the algebraic addition, multiplication, and scalar multiplication on neutrosophic cubic sets(NCs). An important result of exponential multlipliction is established on the basis of these defintions, which provides the basis to define neutrosophic cubic geometric aggregation operators.

Definition 10. *The sum of two neutrosophic cubic sets(NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = \left[T_A^L, T_A^U\right], \widetilde{I}_A = \left[I_A^L, I_A^U\right], \widetilde{F}_A = \left[F_A^L, F_A^U\right]$, *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = \left[T_B^L, T_B^U\right], \widetilde{I}_B = \left[I_B^L, I_B^U\right], \widetilde{F}_B = \left[F_B^L, F_B^U\right]$ *is defined as*

$$A \oplus B = \begin{pmatrix} \left[T_A^L + T_B^L - T_A^L T_B^L, T_A^U + T_B^U - T_A^U T_B^U\right], \\ \left[I_A^L + I_B^L - I_A^L I_B^L, I_A^U + I_B^U - I_A^U I_B^U\right], \\ \left[F_A^L F_B^L, F_A^U F_B^U\right], \\ T_A T_B, I_A I_B, F_A + F_B - F_A F_B \end{pmatrix}$$

Definition 11. *The product between two neutrosophic cubic sets (NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = \left[T_A^L, T_A^U\right], \widetilde{I}_A = \left[I_A^L, I_A^U\right], \widetilde{F}_A = \left[F_A^L, F_A^U\right]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = \left[T_B^L, T_B^U\right], \widetilde{I}_B = \left[I_B^L, I_B^U\right], \widetilde{F}_B = \left[F_B^L, F_B^U\right]$ *is defined as*

$$A \otimes B = \begin{pmatrix} \left[T_A^L T_B^L, T_A^U T_B^U\right], \\ \left[I_A^L I_B^L, I_A^U I_B^U\right], \\ \left[F_A^L + F_B^L - F_A^L F_B^L, F_A^U + F_B^U - F_A^U F_B^U\right], \\ T_A + T_B - T_A T_B, I_A + I_B - I_A I_B, F_A F_B \end{pmatrix}$$

Definition 12. *The scalar multiplication on a neutrosophic cubic set (NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = \left[T_A^L, T_A^U\right], \widetilde{I}_A = \left[I_A^L, I_A^U\right], \widetilde{F}_A = \left[F_A^L, F_A^U\right]$, *and a Scalar* k is defined as

$$kA = \begin{pmatrix} \left[1 - (1 - T_A^L)^k, 1 - (1 - T_A^U)^k\right], \\ \left[1 - (1 - I_A^L)^k, 1 - (1 - I_A^U)^k\right], \\ \left[(F_A^L)^k, (F_A^U)^k\right], \\ (T_A)^k, (I_A)^k, 1 - (1 - F_A)^k \end{pmatrix}$$

The following result is established to deal with the exponential multiplication on neutrosophic cubic values. This result enables us to define geometric aggregation operators along some important results on neutrosophic cubic sets.

Theorem 1. *Let* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = \left[T_A^L, T_A^U\right], \widetilde{I}_A = \left[I_A^L, I_A^U\right], \widetilde{F}_A = \left[F_A^L, F_A^U\right]$, *be a neutrosophic cubic value, then the exponential operation can be defined by*

$$A^k = \begin{pmatrix} \left[(T_A^L)^k, (T_A^U)^k\right], \\ \left[(I_A^L)^k, (I_A^U)^k\right], \\ \left[1 - (1 - F_A^L)^k, 1 - (1 - F_A^U)^k\right], \\ 1 - (1 - T_A)^k, 1 - (1 - I_A)^k, (F_A)^k \end{pmatrix}$$

where $A^k = A \otimes A \otimes, \ldots \otimes A(k - times)$, *and* A^k *is a neutrosophic cubic value for every positive value of* k.

Proof. We prove the theorem by mathematical induction, as the $k = 1$, $A^1 = A$ result holds.We assume that for $k = m$ the result is true:

$$A^m = \begin{pmatrix} \left[(T_A^L)^m, (T_A^U)^m\right], \\ \left[(I_A^L)^m, (I_A^U)^m\right], \\ \left[1-(1-F_A^L)^m, 1-(1-F_A^U)^m\right], \\ 1-(1-T_A)^m, 1-(1-I_A)^m, (F_A)^m \end{pmatrix}$$

That is A^m is neutrosophic cubic value. We prove that for $k = m+1$ is also neutrosophic cubic value.

Since

$$A^m \otimes A = \begin{pmatrix} \left[(T_A^L)^m, (T_A^U)^m\right], \\ \left[(I_A^L)^m, (I_A^U)^m\right], \\ \left[1-(1-F_A^L)^m, 1-(1-F_A^U)^m\right], \\ 1-(1-T_A)^m, 1-(1-I_A)^m, (F_A)^m \end{pmatrix} \otimes \begin{pmatrix} \left[(T_A^L), (T_A^U)\right], \\ \left[(I_A^L), (I_A^U)\right], \\ \left[F_A^L, F_A^U\right], \\ T_A, I_A, F_A \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1}\right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1}\right], \\ \left[1-(1-F_A^L)^m + F_A^L - \left(1-(1-F_A^L)^m\right)F_A^L, 1-(1-F_A^U)^m + F_A^U - \left(1-(1-F_A^U)^m\right)F_A^U\right], \\ 1-(1-T_A)^m + T_A - (1-(1-T_A)^m)T_A, 1-(1-I_A)^m + I_A - (1-(1-I_A)^m I_A), (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1}\right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1}\right], \\ \left[1-(1-F_A^L)^m + F_A^L - F_A^L + (1-F_A^L)^m F_A^L, 1-(1-F_A^U)^m + F_A^U - F_A^U + (1-F_A^U)^m F_A^U\right], \\ 1-(1-T_A)^m + T_A - T_A + (1-T_A)^m T_A, 1-(1-I_A)^m + I_A - I_A + (1-I_A)^m I_A, (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1}\right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1}\right], \\ \left[1-(1-F_A^L)^m + (1-F_A^L)^m F_A^L, 1-(1-F_A^U)^m + (1-F_A^U)^m F_A^U\right], \\ 1-(1-T_A)^m + (1-T_A)^m T_A, 1-(1-I_A)^m + (1-I_A)^m I_A, (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1}\right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1}\right], \\ \left[1-(1-F_A^L)^m(1-F_A^L), 1-(1-F_A^U)^m(1-F_A^U)\right], \\ 1-(1-T_A)^m(1-T_A), 1-(1-I_A)^m(1-I_A), (F_A)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left[(T_A^L)^{m+1}, (T_A^U)^{m+1}\right], \\ \left[(I_A^L)^{m+1}, (I_A^U)^{m+1}\right], \\ \left[1-(1-F_A^L)^{m+1}, 1-(1-F_A^U)^{m+1}\right], \\ 1-(1-T_A)^{m+1}, 1-(1-I_A)^{m+1}, (F_A)^{m+1} \end{pmatrix} = A^{m+1}.$$

□

3.2. Einstein Addition, Multiplication and Scalar Multiplication

Taking into account the dual t-conorm (Γ^*) and t-norm (Γ), the Einstein operations of union, intersection, addition, multiplication and scalar multiplication are defined on the neutrosophic cubic sets.An important result of Einstein exponential multlipliction is established on the basis of these defintions, which provides the base with which to define neutrosophic cubic Einstein geometric aggregation operators.

Definition 13. *The Einstein union between two neutrosophic cubic sets (NCs),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$ *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$, *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$ *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \vee B = \left(\Gamma\left\{\widetilde{T}_A, \widetilde{T}_B\right\}, \Gamma\left\{\widetilde{I}_A, \widetilde{I}_B\right\}, \Gamma^*\left\{\widetilde{F}_A, \widetilde{F}_B\right\}, \Gamma^*\{T_A, T_B\}, \Gamma^*\{I_A, I_B\}, \Gamma\{F_A, F_B\}\right)$$

Definition 14. *The Einstein intersection between two neutrosophic cubic sets(NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \wedge B = \left(\Gamma^*\left\{\widetilde{T}_A, \widetilde{T}_B\right\}, \Gamma^*\left\{\widetilde{I}_A, \widetilde{I}_B\right\}, \Gamma\left\{\widetilde{F}_A, \widetilde{F}_B\right\}, \Gamma\{T_A, T_B\}, \Gamma\{I_A, I_B\}, \Gamma^*\{F_A, F_B\}\right).$$

On the basis of Einstein union and intersection the Einstein sum and product is defined over neutrosophic cubic values.

Definition 15. *The Einstein sum between two neutrosophic cubic sets (NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \oplus_E B = \begin{pmatrix} \left[\frac{T_A^L+T_B^L}{1+T_A^L T_B^L}, \frac{T_A^U+T_B^U}{1+T_A^U T_B^U}\right], \\ \left[\frac{I_A^L+I_B^L}{1+I_A^L I_B^L}, \frac{I_A^U+I_B^U}{1+I_A^U I_B^U}\right], \\ \left[\frac{F_A^L F_B^L}{1+(1-F_A^L)(1-F_B^L)}, \frac{F_A^U F_B^U}{1+(1-F_A^U)(1-F_B^U)}\right] \\ \frac{T_A T_B}{1+(1-T_A)(1-T_B)}, \frac{I_A I_B}{1+(1-I_A)(1-I_B)}, \frac{F_A+F_B}{1+F_A F_B} \end{pmatrix}$$

Definition 16. *The Einstein product between two neutrosophic cubic sets (NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$ *and* $B = \left(\widetilde{T}_B, \widetilde{I}_B, \widetilde{F}_B, T_B, I_B, F_B\right)$, *where* $\widetilde{T}_B = [T_B^L, T_B^U], \widetilde{I}_B = [I_B^L, I_B^U], \widetilde{F}_B = [F_B^L, F_B^U]$ *is defined as*

$$A \otimes_E B = \begin{pmatrix} \left[\frac{T_A^L T_B^L}{1+(1-T_A^L)(1-T_B^L)}, \frac{T_A^U T_B^U}{1+(1-T_A^U)(1-T_B^U)}\right], \\ \left[\frac{I_A^L I_B^L}{1+(1-I_A^L)(1-I_B^L)}, \frac{I_A^U I_B^U}{1+(1-I_A^U)(1-I_B^U)}\right], \\ \left[\frac{F_A^L+F_B^L}{1+F_A^L F_B^L}, \frac{F_A^U+F_B^U}{1+F_A^U F_B^U}\right] \\ \frac{T_A+T_B}{1+T_A T_B}, \frac{I_A+I_B}{1+I_A I_B}, \frac{F_A F_B}{1+(1-F_A)(1-F_B)} \end{pmatrix}$$

Definition 17. *The scalar multiplication on a neutrosophic cubic set(NCS),* $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, *where* $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$, *and scalar k is defined as*

$$k_E A = \begin{pmatrix} \left[\frac{(1+T_A^L)^k-(1-T_A^L)^k}{(1+T_A^L)^k+(1-T_A^L)^k}, \frac{(1+T_A^U)^k-(1-T_A^U)^k}{(1+T_A^U)^k+(1-T_A^U)^k}\right], \\ \left[\frac{(1+I_A^L)^k-(1-I_A^L)^k}{(1+I_A^L)^k+(1-I_A^L)^k}, \frac{(1+I_A^U)^k-(1-I_A^U)^k}{(1+I_A^U)^k+(1-I_A^U)^k}\right], \\ \left[\frac{2(F_A^L)^k}{(2-F_A^L)^k+(F_A^L)^k}, \frac{2(F_A^U)^k}{(2-F_A^U)^k+(F_A^U)^k}\right], \\ \frac{2(T_A)^k}{(2-T_A)^k+(T_A)^k}, \frac{2(I_A)^k}{(2-I_A)^k+(I_A)^k}, \frac{(1+F_A)^k-(1-F_A)^k}{(1+F_A)^k+(1-F_A)^k} \end{pmatrix}$$

After defining the scalar multiplication over the neutrosophic cubic set, we established the following result, which deals with the Einstein exponential multiplication on neutrosophic cubic values. This result enabled us to define Einstein geometric aggregation operators along with some important results on neutrosophic cubic sets.

Theorem 2. *Let $A = \left(\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A, T_A, I_A, F_A\right)$, where $\widetilde{T}_A = [T_A^L, T_A^U], \widetilde{I}_A = [I_A^L, I_A^U], \widetilde{F}_A = [F_A^L, F_A^U]$, be a neutrosophic cubic value, then the exponential operation defined by*

$$A^{E^k} = \begin{pmatrix} \left[\frac{2(T_A^L)^k}{2(-T_A^L)^k+(T_A^L)^k}, \frac{2(T_A^U)^k}{(2-T_A^U)^k+(T_A^U)^k}\right], \\ \left[\frac{2(I_A^L)^k}{(2-I_A^L)^k+(I_A^L)^k}, \frac{2(I_A^U)^k}{(2-I_A^U)^k+(I_A^U)^k}\right], \\ \left[\frac{(1+F_A^L)^k-(1-F_A^L)^k}{(1+F_A^L)^k+(1-F_A^L)^k}, \frac{(1+F_A^U)^k-(1-F_A^U)^k}{(1+F_A^U)^k+(1-F_A^U)^k}\right], \\ \frac{(1+T_A)^k-(1-T_A)^k}{(1+T_A)^k+(1-T_A)^k}, \frac{(1+I_A)^k-(1-I_A)^k}{(1+I_A)^k+(1-I_A)^k}, \frac{2(F_A)^k}{(2-F_A)^k+(F_A)^k} \end{pmatrix}$$

where $A^{E^k} = A \otimes_E A \otimes_E \ldots \otimes_E A(k - times)$, moreover A^{E^k} is a neutrosophic cubic value for every positive value of k.

Proof. We prove the theorem by mathematical induction. For $k = 1$

$$A^E = \begin{pmatrix} \left[\frac{2(T_A^L)}{(2-T_A^L)+(T_A^L)}, \frac{2(T_A^U)}{(2-T_A^U)+(T_A^U)}\right], \\ \left[\frac{2(I_A^L)}{(2-I_A^L)+(I_A^L)}, \frac{2(I_A^U)}{(2-I_A^U)+(I_A^U)}\right], \\ \left[\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}, \frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)}\right], \\ \frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}, \frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}, \frac{2(F_A)}{(2-F_A)+(F_A)} \end{pmatrix}$$

We observe that the components $T_A^L, T_A^U, I_A^L, I_A^U, F_A$ are of the form $\frac{2x}{(2-x)+x}$, and F_A^L, F_A^U, T_A, I_A are of the form $\frac{(1+y)-(1-y)}{(1+y)+(1-y)}$,

For all $x, y \in [0,1]$, clearly $x = \frac{2x}{(2-x)+x}$ and $y = \frac{(1+y)-(1-y)}{(1+y)+(1-y)}$

Hence A^E is neutrosophic cubic value.

Assuming $k = m$ is a neutrosophic cubic value i.e.,

$$A^{E^m} = \begin{pmatrix} \left[\frac{2(T_A^L)^m}{(2-T_A^L)^m+(T_A^L)^m}, \frac{2(T_A^U)^m}{(2-T_A^U)^m+(T_A^U)^m}\right], \\ \left[\frac{2(I_A^L)^m}{(2-I_A^L)^m+(I_A^L)^m}, \frac{2(I_A^U)^m}{(2-I_A^U)^m+(I_A^U)^m}\right], \\ \left[\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}, \frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m}\right], \\ \frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}, \frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}, \frac{2(F_A)^m}{(2-F_A)^m+(F_A)^m} \end{pmatrix}$$

is a neutrosophic cubic value. Then we prove $A^{E^{k+1}}$ is neutrosophic cubic value.

Consider,

$$A^{E^m} \otimes_E A^E = \begin{pmatrix} \left[\frac{2(T_A^L)^m}{(2-T_A^L)^m+(T_A^L)^m}, \frac{2(T_A^U)^m}{(2-T_A^U)^m+(T_A^U)^m}\right], \\ \left[\frac{2(I_A^L)^m}{(2-I_A^L)^m+(I_A^L)^m}, \frac{2(I_A^U)^m}{(2-I_A^U)^m+(I_A^U)^m}\right], \\ \left[\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}, \frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m}\right], \\ \frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}, \frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}, \frac{2(F_A)^m}{(2-F_A)^m+(F_A)^m} \end{pmatrix} \otimes_E \begin{pmatrix} \left[\frac{2(T_A^L)}{(2-T_A^L)+(T_A^L)}, \frac{2(T_A^U)}{(2-T_A^U)+(T_A^U)}\right], \\ \left[\frac{2(I_A^L)^1}{(2-I_A^L)+(I_A^L)}, \frac{2(I_A^U)^1}{(2-I_A^U)+(I_A^U)}\right], \\ \left[\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}, \frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)}\right], \\ \frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}, \frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}, \frac{2(F_A)}{(2-F_A)+(F_A)} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\frac{\frac{4(T_A^L)^{m+1}}{((2-T_A^L)^m+(T_A^L)^m)((2-T_A^L)+T_A^L)}}{1+\left(1-\frac{2(T_A^L)^m}{(2-T_A^L)^m+(T_A^L)^m}\right)\left(1-\frac{2T_A^L}{(2-T_A^L)+T_A^L}\right)}, \frac{\frac{4(T_A^U)^{m+1}}{((2-T_A^U)^m+(T_A^U)^m)((2-T_A^U)+T_A^U)}}{1+\left(1-\frac{2(T_A^U)^m}{(2-T_A^U)^m+(T_A^U)^m}\right)\left(1-\frac{2T_A^U}{(2-T_A^U)+T_A^U}\right)}\right], \\ \left[\frac{\frac{4(I_A^L)^{m+1}}{((2-I_A^L)^m+(I_A^L)^m)((2-I_A^L)+I_A^L)}}{1+\left(1-\frac{2(I_A^L)^m}{(2-I_A^L)^m+(I_A^L)^m}\right)\left(1-\frac{2I_A^L}{(2-I_A^L)+I_A^L}\right)}, \frac{\frac{4(I_A^U)^{m+1}}{((2-I_A^U)^m+(I_A^U)^m)((2-I_A^U)+I_A^U)}}{1+\left(1-\frac{2(I_A^U)^m}{(2-I_A^U)^m+(I_A^U)^m}\right)\left(1-\frac{2I_A^U}{(2-I_A^U)+I_A^U}\right)}\right], \\ \left[\frac{\left(\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}\right)+\left(\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}\right)}{1+\left(\frac{(1+F_A^L)^m-(1-F_A^L)^m}{(1+F_A^L)^m+(1-F_A^L)^m}\right)\left(\frac{(1+F_A^L)-(1-F_A^L)}{(1+F_A^L)+(1-F_A^L)}\right)}, \frac{\left(\frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m}\right)+\left(\frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)}\right)}{1+\left(\frac{(1+F_A^U)^m-(1-F_A^U)^m}{(1+F_A^U)^m+(1-F_A^U)^m}\right)\left(\frac{(1+F_A^U)-(1-F_A^U)}{(1+F_A^U)+(1-F_A^U)}\right)}\right], \\ \frac{\left(\frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}\right)+\left(\frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}\right)}{1+\left(\frac{(1+T_A)^m-(1-T_A)^m}{(1+T_A)^m+(1-T_A)^m}\right)\left(\frac{(1+T_A)-(1-T_A)}{(1+T_A)+(1-T_A)}\right)}, \frac{\left(\frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}\right)+\left(\frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}\right)}{1+\left(\frac{(1+I_A)^m-(1-I_A)^m}{(1+I_A)^m+(1-I_A)^m}\right)\left(\frac{(1+I_A)-(1-I_A)}{(1+I_A)+(1-I_A)}\right)}, \\ \frac{\frac{4(F_A)^{m+1}}{((2-F_A)^m+(F_A)^m)((2-F_A)+F_A)}}{1+\left(1-\frac{2(F_A)^m}{(2-F_A)^m+(F_A)^m}\right)\left(1-\frac{2F_A}{(2-F_A)+F_A}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\frac{\frac{4(T_A^L)^{m+1}}{((2-T_A^L)^m+(T_A^L)^m)((2-T_A^L)+T_A^L)}}{1+\left(\frac{(2-T_A^L)^m+(T_A^L)^m-2(T_A^L)^m}{(2-T_A^L)^m+(T_A^L)^m}\right)\left(\frac{(2-T_A^L)+T_A^L-2T_A^L}{(2-T_A^L)+T_A^L}\right)}, \frac{\frac{4(T_A^U)^{m+1}}{((2-T_A^U)^m+(T_A^U)^m)((2-T_A^U)+T_A^U)}}{1+\left(\frac{(2-T_A^U)^m+(T_A^U)^m-2(T_A^U)^m}{(2-T_A^U)^m+(T_A^U)^m}\right)\left(\frac{(2-T_A^U)+T_A^U-2T_A^U}{(2-T_A^U)+T_A^U}\right)}\right], \\ \left[\frac{\frac{4(I_A^L)^{m+1}}{((2-I_A^L)^m+(I_A^L)^m)((2-I_A^L)+I_A^L)}}{1+\left(\frac{(2-I_A^L)^m+(I_A^L)^m-2(I_A^L)^m}{(2-I_A^L)^m+(I_A^L)^m}\right)\left(\frac{(2-I_A^L)+I_A^L-2I_A^L}{(2-I_A^L)+I_A^L}\right)}, \frac{\frac{4(I_A^U)^{m+1}}{((2-I_A^U)^m+(I_A^U)^m)((2-I_A^U)+I_A^U)}}{1+\left(\frac{(2-I_A^U)^m+(I_A^U)^m-2(I_A^U)^m}{(2-I_A^U)^m+(I_A^U)^m}\right)\left(\frac{(2-I_A^U)+I_A^U-2I_A^U}{(2-I_A^U)+I_A^U}\right)}\right], \\ \left[\begin{array}{c} \frac{\frac{((1+F_A^L)^m-(1-F_A^L)^m)((1+F_A^L)+(1-F_A^L))+((1+F_A^L)^m+(1-F_A^L)^m)((1+F_A^L)-(1-F_A^L))}{((1+F_A^L)^m+(1-F_A^L)^m)((1+F_A^L)+(1-F_A^L))}}{\frac{((1+F_A^L)^m+(1-F_A^L)^m)((1+F_A^L)+(1-F_A^L))+(1+F_A^L)^{m+1}-(1+F_A^L)^m(1-F_A^L)-(1-F_A^L)^m(1+F_A^L)+(1-F_A^L)^{m+1}}{((1+F_A^L)^m+(1-F_A^L)^m)((1+F_A^L)+(1-F_A^L))}}, \\ \frac{\frac{((1+F_A^U)^m-(1-F_A^U)^m)((1+F_A^U)+(1-F_A^U))+((1+F_A^U)^m+(1-F_A^U)^m)((1+F_A^U)-(1-F_A^U))}{((1+F_A^U)^m+(1-F_A^U)^m)((1+F_A^U)+(1-F_A^U))}}{\frac{((1+F_A^U)^m+(1-F_A^U)^m)((1+F_A^U)+(1-F_A^U))+(1+F_A^U)^{m+1}-(1+F_A^U)^m(1-F_A^U)-(1-F_A^U)^m(1+F_A^U)+(1-F_A^U)^{m+1}}{((1+F_A^U)^m+(1-F_A^U)^m)((1+F_A^U)+(1-F_A^U))}} \end{array}\right], \\ \frac{\frac{((1+T_A)^m-(1-T_A)^m)((1+T_A)+(1-T_A))+((1+T_A)^m+(1-T_A)^m)((1+T_A)-(1-T_A))}{((1+T_A)^m+(1-T_A)^m)((1+T_A)+(1-T_A))}}{\frac{((1+T_A)^m+(1-T_A)^m)((1+T_A)+(1-T_A))+(1+T_A)^{m+1}-(1+T_A)^m(1-T_A)-(1-T_A)^m(1+T_A)+(1-T_A)^{m+1}}{((1+T_A)^m+(1-T_A)^m)((1+T_A)+(1-T_A))}}, \\ \frac{\frac{((1+I_A)^m-(1-I_A)^m)((1+I_A)+(1-I_A))+((1+I_A)^m+(1-I_A)^m)((1+I_A)-(1-I_A))}{((1+I_A)^m+(1-I_A)^m)((1+I_A)+(1-I_A))}}{\frac{((1+I_A)^m+(1-I_A)^m)((1+I_A)+(1-I_A))+(1+I_A)^{m+1}-(1+I_A)^m(1-I_A)-(1-I_A)^m(1+I_A)+(1-I_A)^{m+1}}{((1+I_A)^m+(1-I_A)^m)((1+I_A)+(1-I_A))}}, \\ \frac{\frac{4(F_A)^{m+1}}{((2-F_A)^m+(F_A)^m)((2-F_A)+F_A)}}{1+\left(\frac{(2-F_A)^m+(F_A)^m-2(F_A)^m}{(2-F_A)^m+(F_A)^m}\right)\left(\frac{(2-F_A)+F_A-2F_A}{(2-F_A)+F_A}\right)} \end{pmatrix}$$

$$
= \left(\begin{array}{c}
\left[\frac{\frac{4\left(T_A^L\right)^{m+1}}{\left((2-T_A^L)^m+\left(T_A^L\right)^m\right)\left((2-T_A^L)+T_A^L\right)}}{\frac{\left(\left((2-T_A^L)^m+\left(T_A^L\right)^m\right)\left((2-T_A^L)+T_A^L\right)\right)+\left(\left((2-T_A^L)^m-\left(T_A^L\right)^m\right)\left((2-T_A^L)-T_A^L\right)\right)}{\left((2-T_A^L)^m+\left(T_A^L\right)^m\right)\left((2-T_A^L)+T_A^L\right)}}, \frac{\frac{4\left(T_A^U\right)^{m+1}}{\left((2-T_A^U)^m+\left(T_A^U\right)^m\right)\left((2-T_A^U)+T_A^U\right)}}{\frac{\left(\left((2-T_A^U)^m+\left(T_A^U\right)^m\right)\left((2-T_A^U)+T_A^U\right)\right)+\left(\left((2-T_A^U)^m-\left(T_A^U\right)^m\right)\left((2-T_A^U)-T_A^U\right)\right)}{\left((2-T_A^U)^m+\left(T_A^U\right)^m\right)\left((2-T_A^U)+T_A^U\right)}} \right], \\
\left[\frac{\frac{4\left(I_A^L\right)^{m+1}}{\left((2-I_A^L)^m+\left(I_A^L\right)^m\right)\left((2-I_A^L)+I_A^L\right)}}{\frac{\left(\left((2-I_A^L)^m+\left(I_A^L\right)^m\right)\left((2-I_A^L)+I_A^L\right)\right)+\left(\left((2-I_A^L)^m-\left(I_A^L\right)^m\right)\left((2-I_A^L)-I_A^L\right)\right)}{\left((2-I_A^L)^m+\left(I_A^L\right)^m\right)\left((2-I_A^L)+I_A^L\right)}}, \frac{\frac{4\left(I_A^U\right)^{m+1}}{\left((2-I_A^U)^m+\left(I_A^U\right)^m\right)\left((2-I_A^U)+I_A^U\right)}}{\frac{\left(\left((2-I_A^U)^m+\left(I_A^U\right)^m\right)\left((2-I_A^U)+I_A^U\right)\right)+\left(\left((2-I_A^U)^m-\left(I_A^U\right)^m\right)\left((2-I_A^U)-I_A^U\right)\right)}{\left((2-I_A^U)^m+\left(I_A^U\right)^m\right)\left((2-I_A^U)+I_A^U\right)}} \right], \\
\left[\frac{\begin{array}{c}(1+F_A^L)^{m+1}+(1+F_A^L)^m(1-F_A^L)-(1-F_A^L)^m(1+F_A^L)-(1-F_A^L)^{m+1}+ \\ (1+F_A^L)^{m+1}-(1+F_A^L)^m(1-F_A^L)+(1-F_A^L)^m(1+F_A^L)-(1-F_A^L)^{m+1}\end{array}}{\begin{array}{c}(1+F_A^L)^{m+1}+(1+F_A^L)^m(1-F_A^L)-(1-F_A^L)^m(1+F_A^L)+(1-F_A^L)^{m+1}+ \\ (1+F_A^L)^{m+1}-(1+F_A^L)^m(1-F_A^L)+(1-F_A^L)^m(1+F_A^L)+(1-F_A^L)^{m+1}\end{array}}, \frac{\begin{array}{c}(1+F_A^U)^{m+1}+(1+F_A^U)^m(1-F_A^U)-(1-F_A^U)^m(1+F_A^U)-(1-F_A^U)^{m+1}+ \\ (1+F_A^U)^{m+1}-(1+F_A^U)^m(1-F_A^U)+(1-F_A^U)^m(1+F_A^U)-(1-F_A^U)^{m+1}\end{array}}{\begin{array}{c}(1+F_A^U)^{m+1}+(1+F_A^U)^m(1-F_A^U)-(1-F_A^U)^m(1+F_A^U)+(1-F_A^U)^{m+1}+ \\ (1+F_A^U)^{m+1}-(1+F_A^U)^m(1-F_A^U)+(1-F_A^U)^m(1+F_A^U)+(1-F_A^U)^{m+1}\end{array}} \right], \\
\frac{\begin{array}{c}(1+T_A)^{m+1}+(1+T_A)^m(1-T_A)-(1-T_A)^m(1+T_A)-(1-T_A)^{m+1}+ \\ (1+T_A)^{m+1}-(1+T_A)^m(1-T_A)+(1-T_A)^m(1+T_A)-(1-T_A)^{m+1}\end{array}}{\begin{array}{c}(1+T_A)^{m+1}+(1+T_A)^m(1-T_A)-(1-T_A)^m(1+T_A)+(1-T_A)^{m+1}+ \\ (1+T_A)^{m+1}-(1+T_A)^m(1-T_A)+(1-T_A)^m(1+T_A)+(1-T_A)^{m+1}\end{array}}, \\
\frac{\begin{array}{c}(1+I_A)^{m+1}+(1+I_A)^m(1-I_A)-(1-I_A)^m(1+I_A)-(1-I_A)^{m+1}+ \\ (1+I_A)^{m+1}-(1+I_A)^m(1-I_A)+(1-I_A)^m(1+I_A)-(1-I_A)^{m+1}\end{array}}{\begin{array}{c}(1+I_A)^{m+1}+(1+I_A)^m(1-I_A)-(1-I_A)^m(1+I_A)+(1-I_A)^{m+1}+ \\ (1+I_A)^{m+1}-(1+I_A)^m(1-I_A)+(1-I_A)^m(1+I_A)+(1-I_A)^{m+1}\end{array}}, \\
\frac{\frac{4(F_A)^{m+1}}{\left((2-F_A)^m+(F_A)^m\right)((2-F_A)+F_A)}}{\frac{\left(\left((2-F_A)^m+(F_A)^m\right)((2-F_A)+F_A)\right)+\left(\left((2-F_A)^m-(F_A)^m\right)((2-F_A)-F_A)\right)}{\left((2-F_A)^m+(F_A)^m\right)((2-F_A)+F_A)}}
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left[\frac{4\left(T_A^L\right)^{m+1}}{(2-T_A^L)^{m+1}+T_A^L(2-T_A^L)^m+\left(T_A^L\right)^{m+1}+T_A^{Lm}(2-T_A^L)+\left((2-T_A^L)^{m+1}-T_A^L(2-T_A^L)^m+\left(T_A^L\right)^{m+1}-\left(T_A^L\right)^m(2-T_A^L)\right)}, \frac{4\left(T_A^U\right)^{m+1}}{(2-T_A^U)^{m+1}+T_A^U(2-T_A^U)^m+\left(T_A^U\right)^{m+1}+T_A^{Um}(2-T_A^U)+\left((2-T_A^U)^{m+1}-T_A^U(2-T_A^U)^m+\left(T_A^U\right)^{m+1}-\left(T_A^U\right)^m(2-T_A^U)\right)} \right], \\
\left[\frac{4\left(I_A^L\right)^{m+1}}{(2-I_A^L)^{m+1}+I_A^L(2-I_A^L)^m+\left(I_A^L\right)^{m+1}+I_A^{Lm}(2-I_A^L)+\left((2-I_A^L)^{m+1}-I_A^L(2-I_A^L)^m+\left(I_A^L\right)^{m+1}-\left(I_A^L\right)^m(2-I_A^L)\right)}, \frac{4\left(I_A^U\right)^{m+1}}{(2-I_A^U)^{m+1}+I_A^U(2-I_A^U)^m+\left(I_A^U\right)^{m+1}+I_A^{Um}(2-I_A^U)+\left((2-I_A^U)^{m+1}-I_A^U(2-I_A^U)^m+\left(I_A^U\right)^{m+1}-\left(I_A^U\right)^m(2-I_A^U)\right)} \right], \\
\left[\frac{2\left(\left(1+F_A^L\right)^{m+1}-\left(1-F_A^L\right)^{m+1}\right)}{2\left(\left(1+F_A^L\right)^{m+1}+\left(1-F_A^L\right)^{m+1}\right)}, \frac{2\left(\left(1+F_A^U\right)^{m+1}-\left(1-F_A^U\right)^{m+1}\right)}{2\left(\left(1+F_A^U\right)^{m+1}+\left(1-F_A^U\right)^{m+1}\right)} \right], \\
\frac{2\left((1+T_A)^{m+1}-(1-T_A)^{m+1}\right)}{2\left((1+T_A)^{m+1}+(1-T_A)^{m+1}\right)}, \\
\frac{2\left((1+I_A)^{m+1}-(1-I_A)^{m+1}\right)}{2\left((1+I_A)^{m+1}+(1-I_A)^{m+1}\right)}, \\
\frac{4(F_A)^{m+1}}{(2-F_A)^{m+1}+F_A(2-F_A)^m+(F_A)^{m+1}+F_A^m(2-F_A)+\left((2-F_A)^{m+1}-F_A(2-F_A)^m+(F_A)^{m+1}-(F_A)^m(2-F_A)\right)}
\end{array} \right)
$$

$$= \begin{pmatrix} \left[\frac{4\left(T_A^L\right)^{m+1}}{2\left((2-T_A^L)^{m+1}+\left(T_A^L\right)^{m+1}\right)}, \frac{4\left(T_A^U\right)^{m+1}}{2\left((2-T_A^U)^{m+1}+\left(T_A^U\right)^{m+1}\right)}\right], \\ \left[\frac{4\left(I_A^L\right)^{m+1}}{2\left((2-I_A^L)^{m+1}+\left(I_A^L\right)^{m+1}\right)}, \frac{4\left(I_A^U\right)^{m+1}}{2\left((2-I_A^U)^{m+1}+\left(I_A^U\right)^{m+1}\right)}\right], \\ \left[\frac{\left((1+F_A^L)^{m+1}-\left(1-F_A^L\right)^{m+1}\right)}{\left((1+F_A^L)^{m+1}+\left(1-F_A^L\right)^{m+1}\right)}, \frac{\left((1+F_A^U)^{m+1}-\left(1-F_A^U\right)^{m+1}\right)}{\left((1+F_A^U)^{m+1}+\left(1-F_A^U\right)^{m+1}\right)}\right], \\ \frac{\left((1+T_A)^{m+1}-(1-T_A)^{m+1}\right)}{\left((1+T_A)^{m+1}+(1-T_A)^{m+1}\right)}, \\ \frac{\left((1+I_A)^{m+1}-(1-I_A)^{m+1}\right)}{\left((1+I_A)^{m+1}+(1-I_A)^{m+1}\right)}, \\ \frac{4(F_A)^{m+1}}{2\left((2-F_A)^{m+1}+(F_A)^{m+1}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\frac{2\left(T_A^L\right)^{m+1}}{\left((2-T_A^L)^{m+1}+\left(T_A^L\right)^{m+1}\right)}, \frac{2\left(T_A^U\right)^{m+1}}{\left((2-T_A^U)^{m+1}+\left(T_A^U\right)^{m+1}\right)}\right], \\ \left[\frac{2\left(I_A^L\right)^{m+1}}{\left((2-I_A^L)^{m+1}+\left(I_A^L\right)^{m+1}\right)}, \frac{2\left(I_A^U\right)^{m+1}}{\left((2-I_A^U)^{m+1}+\left(I_A^U\right)^{m+1}\right)}\right], \\ \left[\frac{\left((1+F_A^L)^{m+1}-\left(1-F_A^L\right)^{m+1}\right)}{\left((1+F_A^L)^{m+1}+\left(1-F_A^L\right)^{m+1}\right)}, \frac{\left((1+F_A^U)^{m+1}-\left(1-F_A^U\right)^{m+1}\right)}{\left((1+F_A^U)^{m+1}+\left(1-F_A^U\right)^{m+1}\right)}\right], \\ \frac{\left((1+T_A)^{m+1}-(1-T_A)^{m+1}\right)}{\left((1+T_A)^{m+1}+(1-T_A)^{m+1}\right)}, \\ \frac{\left((1+I_A)^{m+1}-(1-I_A)^{m+1}\right)}{\left((1+I_A)^{m+1}+(1-I_A)^{m+1}\right)}, \\ \frac{2(F_A)^{m+1}}{\left((2-F_A)^{m+1}+(F_A)^{m+1}\right)} \end{pmatrix}$$

Which shows that $k = m+1$ is a neutrosophic cubic value. □

3.3. Score and Accuracy Function of Neutrosophic Cubic Set

For the comparison of two neutrosophic values, the score and accuracy function are defined. The score function is used tocompare two neutrosophic cubic values; sometimes the score of two neutrosophic cubic values becomes equal, although they have different components of truth, indeterminancy and falsity functions. This situation can be overcome by the help of an accuracy function. The following definition, along with examples, providesa better view of understanding to the reader.

Definition 18. *Let* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$, *where* $\widetilde{T}_N = [T_N^L, T_N^U], \widetilde{I}_N = [I_N^L, I_N^U], \widetilde{F}_N = [F_N^L, F_N^U]$, *be a neutrosophic cubic value and we define the score function as*

$$S(N) = \left[T_N^L - F_N^L + T_N^U - F_N^U + T_N - F_N\right]$$

Sometimes the situation arises that the score of two neutrosophic cubic values are equal. In such a situation, a comparison is made on the basis of an accuracy function.

Definition 19. *Let* $N = \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$, *where* $\widetilde{T}_N = [T_N^L, T_N^U], \widetilde{I}_N = [I_N^L, I_N^U], \widetilde{F}_N = [F_N^L, F_N^U]$, *be a neutrosophic cubic value, the accuracy function is defined as*

$$H(u) = \frac{1}{9}\left\{T_N^L + I_N^L + F_N^L + T_N^U + I_N^U + F_N^U + T_N + I_N + F_N\right\}$$

The following definition is accomplished for the comparison relation of the neutrosophic cubic values.

Definition 20. *Let N_1 and N_2 be two neutrosophic cubic values, where S_{N_1} and S_{N_2} are scores and H_{N_1} and H_{N_2} are accuracy functions of N_1 and N_2, respectively.*

1. *If* $S_{N_1} > S_{N_2} \Rightarrow N_1 > N_2$
2. *If* $S_{N_1} = S_{N_2}$ *and* $H_{N_1} > H_{N_2} \Rightarrow N_1 > N_2$ $H_{N_1} = H_{N_2} \Rightarrow N_1 = N_2$

Example 1. *Let* $N_1 = ([0.5, 0.9][0.6, 0.9][0.1, 0.4], 0.3, 0.4, 0.4)$ *and* $N_2 = ([0.2, 0.8][0.5, 0.9][0.4, 0.8], 0.4, 0.45, 0.8)$ *be two neutrosophic sets.*

Then

$$S_{N_1} = 0.8, \text{and } S_{N_2} = -0.6$$

$$S_{N_1} > S_{N_2} \Rightarrow N_1 > N_2$$

In the following example the score funtions are equal, so accuracy functions are used to compare neutrosophic cubic values.

Example 2. *Let* $N_1 = ([0.4, 0.9][0.5, 0.8][0.1, 0.7], 0.4, 0.5, 0.8)$ *and* $N_2 = ([0.4, 0.6][0.5, 0.9][0.6, 0.7], 0.7, 0.5, 0.3)$ *be two neutrosophic sets.*

$$S_{N_1} = 0.1, S_{N_2} = 0.1$$

$$S_{N_1} = S_{N_2} \Rightarrow N_1 = N_2$$

$$H_{N_1} = 0.566,\ H_{N_2} = 0.577$$

$$H_{N_1} < H_{N_2} \Rightarrow N_1 < N_2$$

4. Neutrosophic Cubic Geometric and Einstein Geometric Aggregation Operators

In this section, we introduce the concept of neutrosophic cubic geometric aggregation operators and neutrosophic cubic Einstein geometric aggregation operators.

This section consists of two sub-sections: In Section 4.1, the neutrosophic cubic geometric aggregation operators are defined on the basis of Section 3.1; and in Section 4.2, the neutrosophic cubic Einstein geometric aggregation operators are defined on the basis of Section 3.2.

4.1. Neutrosophic Cubic Weighted Geometric Aggregation Operator

We define neutrosophic cubic geometric aggregation operators using Section 3.1.

Definition 21. *We define the neutrosophic cubic weighted geometric operator(NCWG) as*

$$NCWG : R^m \to R \text{ defined by } NCWG_w(N_1, N_2, \ldots, N_m) = \bigotimes_{j=1}^{m} N_j^{w_j}$$

where the weight $W = (w_1, w_2, \ldots, w_m)^T$ of corresponding neutrosophic cubic values is such that each $w_j \in [0, 1]$ and $\sum\limits_{j=1}^{m} w_j = 1$.

In NCWG, the neutrosophic cubic values are first weighted then aggregated.

Definition 22. *We define the neutrosophic cubic ordered weighted geometric operator(NCOWG) as*

$$NCOWG : R^m \to R \text{ defined by } NCOWG_w(N_1, N_2, \ldots, N_m) = \overset{m}{\underset{j=1}{\otimes}} N_{(\gamma)_j}^{w_j}$$

where $N_{(\gamma)_j}$ *are descending ordered neutrosophic cubic values, and the weight* $W = (w_1, w_2, \ldots, w_m)^T$ *of corresponding neutrosophic cubic values* $N_j (j = 1, 2, 3, \ldots, m)$ *is such that each* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

In NCOWG, the neutrosophic cubic values are first arranged in decending order, weighted and then aggregated.

Theorem 3. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T^L_{N_j}, T^U_{N_j}\right], \widetilde{I}_{N_j} = \left[I^L_{N_j}, I^U_{N_j}\right], \widetilde{F}_{N_j} = \left[F^L_{N_j}, F^U_{N_j}\right] (j = 1, 2, \ldots, n)$ *are a collection of neutrosophic cubic values, then neutrosophic cubic weighted geometric(NCWG) operator of* N_j *is also a neutrosophic cubic value and*

$$NCWG(N_j) = \begin{pmatrix} \left[\prod_{j=1}^{m} \left(T^L_{N_j}\right)^{w_j}, \prod_{j=1}^{m} \left(T^U_{N_j}\right)^{w_j}\right], \\ \left[\prod_{j=1}^{m} \left(I^L_{N_j}\right)^{w_j}, \prod_{j=1}^{m} \left(I^U_{N_j}\right)^{w_j}\right] \\ \left[1 - \prod_{j=1}^{m} (1 - F^L_{N_j})^{w_j}, 1 - \prod_{j=1}^{m} (1 - F^U_{N_j})^{w_j}\right] \\ 1 - \prod_{j=1}^{m} (1 - \left(T_{N_j}\right))^{w_j}, 1 - \prod_{j=1}^{m} (1 - \left(I_{N_j}\right))^{w_j}, \prod_{j=1}^{m} \left(F_{N_j}\right)^{w_j} \end{pmatrix}$$

where the weight $W = (w_1, w_2, \ldots, w_m)^T$ *of* $N_j (j = 1, 2, 3, \ldots, m)$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

Proof. By mathematical induction for $m = 2$, using

$$\overset{2}{\underset{j=1}{\otimes}} N_j^{w_j} = N_1^{w_1} \otimes N_2^{w_2}$$

$$= \begin{pmatrix} \left[(T^L_{N_j})^{w_1}, (T^U_{N_j})^{w_1}\right], \\ \left[(I^L_{N_j})^{w_1}, (I^U_{N_j})^{w_1}\right], \\ \left[1 - \left(1 - F^L_{N_j}\right)^{w_1}, 1 - \left(1 - F^U_{N_j}\right)^{w_1}\right], \\ 1 - \left(1 - \left(T_{N_j}\right)\right)^{w_1}, 1 - \left(1 - \left(I_{N_j}\right)\right)^{w_1}, \left(F_{N_j}\right)^{w_1} \end{pmatrix} \otimes \begin{pmatrix} \left[(T^L_{N_j})^{w_2}, (T^U_{N_j})^{w_2}\right], \\ \left[(I^L_{N_j})^{w_2}, (I^U_{N_j})^{w_2}\right], \\ \left[1 - \left(1 - F^L_{N_j}\right)^{w_2}, 1 - \left(1 - F^U_{N_j}\right)^{w_2}\right], \\ 1 - \left(1 - \left(T_{N_j}\right)\right)^{w_2}, 1 - \left(1 - \left(I_{N_j}\right)\right)^{w_2}, (F_{N_j})^{w_2} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\prod_{j=1}^{2} (T^L_{N_j})^{w_j}, \prod_{j=1}^{2} (T^U_{N_j})^{w_j}\right], \\ \left[\prod_{j=1}^{2} (I^L_{N_j})^{w_j}, \prod_{j=1}^{2} (I^U_{N_j})^{w_j}\right], \\ \left[1 - \prod_{j=1}^{2} \left(1 - F^L_{N_j}\right)^{w_j}, \prod_{j=1}^{2} \left(F^U_{N_j}\right)^{w_j}\right], \\ 1 - \prod_{j=1}^{2} \left(1 - T_{N_j}\right)^{w_j}, 1 - \prod_{j=1}^{2} \left(1 - I_{N_j}\right)^{w_j}, \prod_{j=1}^{2} \left(F_{N_j}\right)^{w_j} \end{pmatrix}$$

For $m = n$, we have

$$\overset{n}{\underset{j=1}{\otimes}} N_j^{w_j} = \left(\begin{array}{c} \left[\prod_{j=1}^{n} (T_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (T_{N_j}^U)^{w_j} \right], \\ \left[\prod_{j=1}^{n} (I_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (I_{N_j}^U)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} \left(1 - F_{N_j}^L\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}^U\right)^{w_j} \right], \\ 1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}\right)^{w_j} \end{array} \right)$$

We prove the result holds for $m = n + 1$,

$$N_{n+1}^{w_{n+1}} = \left(\begin{array}{c} \left[(T_{N_{n+1}}^L)^{w_{n+1}}, (T_{N_{n+1}}^U)^{w_{n+1}} \right], \\ \left[(I_{N_{j+1}}^L)^{w_{j+1}}, (I_{N_{j+1}}^U)^{w_{j+1}} \right], \\ \left[1 - \left(1 - F_{N_{n+1}}^L\right)^{w_{n+1}}, 1 - \left(1 - F_{N_{n+1}}^U\right)^{w_{n+1}} \right], \\ 1 - (1 - T_{N_{n+1}})^{w_{n+1}}, 1 - (1 - I_{N_{n+1}})^{w_{n+1}}, (F_{N_{n+1}})^{w_{n+1}} \end{array} \right)$$

$$\overset{n}{\underset{j=1}{\otimes}} N_j^{w_j} \oplus N_{n+1}^{w_{n+1}}$$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{n} (T_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (T_{N_j}^U)^{w_j} \right], \\ \left[\prod_{j=1}^{n} (I_{N_j}^L)^{w_j}, \prod_{j=1}^{n} (I_{N_j}^U)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} \left(1 - F_{N_j}^L\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}^U\right)^{w_j} \right], \\ 1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j}, \prod_{j=1}^{n} \left(F_{N_j}\right)^{w_j} \end{array} \right) \oplus \left(\begin{array}{c} \left[(T_{N_{n+1}}^L)^{w_{n+1}}, (T_{N_{n+1}}^U)^{w_{n+1}} \right], \\ \left[(I_{N_{j+1}}^L)^{w_{j+1}}, (I_{N_{j+1}}^U)^{w_{j+1}} \right], \\ \left[1 - \left(1 - F_{N_{n+1}}^L\right)^{w_{n+1}}, 1 - \left(1 - F_{N_{n+1}}^U\right)^{w_{n+1}} \right], \\ 1 - (1 - T_{N_{n+1}})^{w_{n+1}}, 1 - (1 - I_{N_{n+1}})^{w_{n+1}}, (F_{N_{n+1}})^{w_{n+1}} \end{array} \right)$$

$$\overset{n+1}{\underset{j=1}{\otimes}} N_j^{w_j} = \left(\begin{array}{c} \left[\prod_{j=1}^{n} \left(T_{N_j}^L\right)^{w_j} \left(T_{N_{m+1}}^L\right)^{w_{m+1}}, \prod_{j=1}^{n} \left(T_{N_j}^U\right)^{w_j} \left(T_{N_{m+1}}^U\right)^{w_{m+1}} \right], \\ \left[\prod_{j=1}^{n} \left(I_{N_j}^L\right)^{w_j} \left(I_{N_{m+1}}^L\right)^{w_{m+1}}, \prod_{j=1}^{n} \left(I_{N_j}^U\right)^{w_j} \left(I_{N_{m+1}}^U\right)^{w_{m+1}} \right], \\ \left[\begin{array}{c} 1 - \prod_{j=1}^{n} (1 - F_{N_j}^L)^{w_j} + 1 - (1 - F_{N_{m+1}}^L)^{w_{m+1}} - \\ \left(1 - \prod_{j=1}^{n} (1 - F_{N_j}^L)^{w_j}\right) \left(1 - (1 - F_{N_{m+1}}^L)^{w_{m+1}}\right), \\ 1 - \prod_{j=1}^{n} (1 - F_{N_j}^U)^{w_j} + 1 - (1 - F_{N_{m+1}}^U)^{w_{m+1}} - \\ \left(1 - \prod_{j=1}^{n} (1 - F_{N_j}^U)^{w_j}\right) \left(1 - (1 - F_{N_{m+1}}^U)^{w_{m+1}}\right), \end{array} \right], \\ 1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j} + 1 - (1 - T_{N_{m+1}})^{w_{m+1}} - \\ \left(1 - \prod_{j=1}^{n} \left(1 - T_{N_j}\right)^{w_j}\right) \left(1 - (1 - T_{N_{m+1}})^{w_{m+1}}\right) \\ 1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j} + 1 - (1 - I_{N_{m+1}})^{w_{m+1}} - \\ \left(1 - \prod_{j=1}^{n} \left(1 - I_{N_j}\right)^{w_j}\right) \left(1 - (1 - I_{N_{m+1}})^{w_{m+1}}\right) \\ \prod_{j=1}^{n} \left(F_{N_j}\right)^{w_j} (F_{N_{m+1}})^{w_{m+1}} \end{array} \right)$$

$$= \begin{pmatrix} \left[\prod_{j=1}^{n+1}\left(T^L_{N_j}\right)^{w_j}, \prod_{j=1}^{n+1}\left(T^U_{N_j}\right)^{w_j}\right] \\ \left[\prod_{j=1}^{n+1}\left(I^L_{N_j}\right)^{w_j}, \prod_{j=1}^{n+1}\left(I^U_{N_j}\right)^{w_j}\right], \\ \left[\begin{array}{c} 2-\prod_{j=1}^{n+1}(1-F^L_{N_j})^{w_j} - 1 + \prod_{j=1}^{n}(1-F^L_{N_j})^{w_j} + (1-F^L_{N_{m+1}})^{w_{m+1}} \\ -\left(\prod_{j=1}^{n}(1-F^L_{N_j})^{w_j}\right)(1-F^L_{N_{m+1}})^{w_{m+1}}, \\ 2-\prod_{j=1}^{n+1}(1-F^U_{N_j})^{w_j} - 1 + \prod_{j=1}^{n+1}(1-F^U_{N_j})^{w_j} + (1-F^U_{N_{m+1}})^{w_{m+1}} \\ -\left(\prod_{j=1}^{n}(1-F^U_{N_j})^{w_j}\right)(1-F^U_{N_{m+1}})^{w_{m+1}} \end{array}\right], \\ 2-\prod_{j=1}^{n+1}(1-T_{N_j})^{w_j} - 1 + \prod_{j=1}^{n}(1-T_{N_j})^{w_j} + (1-T_{N_{m+1}})^{w_{m+1}} \\ -\left(\prod_{j=1}^{n}(1-T_{N_j})^{w_j}\right)(1-T_{N_{m+1}})^{w_{m+1}}, \\ 2-\prod_{j=1}^{n+1}(1-I_{N_j})^{w_j} - 1 + \prod_{j=1}^{n}(1-I_{N_j})^{w_j} + (1-I_{N_{m+1}})^{w_{m+1}} \\ -\left(\prod_{j=1}^{n}(1-I_{N_j})^{w_j}\right)(1-I_{N_{m+1}})^{w_{m+1}} \\ , \prod_{j=1}^{n+1}\left(F_{N_j}\right)^{w_j} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\prod_{J=1}^{n+1}\left(T^L_{N_j}\right)^{w_j}, \prod_{J=1}^{n+1}\left(T^U_{N_j}\right)^{w_j}\right] \\ \left[\prod_{J=1}^{n+1}\left(I^L_{N_j}\right)^{w_j}, \prod_{J=1}^{n+1}\left(I^U_{N_j}\right)^{w_j}\right], \\ \left[1-\prod_{J=1}^{n+1}\left(1-F^L_{N_j}\right)^{w_j}, 1-\prod_{J=1}^{n+1}\left(1-F^U_{N_j}\right)^{w_j}\right], \\ , 1-\prod_{J=1}^{n+1}\left(1-T_{N_j}\right)^{w_j}, 1-\prod_{J=1}^{n+1}\left(1-I_{N_j}\right)^{w_j}, \\ \prod_{J=1}^{n+1}\left(1-F_{N_j}\right)^{w_j} \end{pmatrix}$$

□

Theorem 4. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$*, where* $\widetilde{T}_{N_j} = \left[T^L_{N_j}, T^U_{N_j}\right], \widetilde{I}_{N_j} = \left[I^L_{N_j}, I^U_{N_j}\right], \widetilde{F}_{N_j} = \left[F^L_{N_j}, F^U_{N_j}\right]$*,* $(j = 1, 2, \ldots, m)$ *is a collection of neutrosophic cubic values The weight* $W = (w_1, w_2, \ldots, w_m)^T$ *of* $N_j(j = 1, 2, 3, \ldots, m)$*, be such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$*.*

1. ***Idempotency:*** *If for all* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$*, where* $\widetilde{T}_{N_j} = \left[T^L_{N_j}, T^U_{N_j}\right]$*,* $\widetilde{I}_{N_j} = \left[I^L_{N_j}, I^U_{N_j}\right], \widetilde{F}_{N_j} = \left[F^L_{N_j}, F^U_{N_j}\right]$*,* $(j = 1, 2, \ldots, m)$ *are equal, that is,* $N_j = N$ *for all k, then NCW* $G_w(N_1, N_2, \ldots, N_m) = N$
2. ***Monotonicity:*** *Let* $B_j = \left(\widetilde{T}_{B_j}, \widetilde{I}_{B_j}, \widetilde{F}_{B_j}, T_{B_j}, I_{B_j}, F_{B_j}\right)$ *where* $\widetilde{T}_{B_j} = \left[T^L_{B_j}, T^U_{B_j}\right]$*,* $\widetilde{I}_{B_j} = \left[I^L_{B_j}, I^U_{B_j}\right], \widetilde{F}_{B_j} = \left[F^L_{B_j}, F^U_{B_j}\right]$ $(j = 1, 2, \ldots, m)$ *is the collection of neutrosophic cubic values. If* $S_{B_j}(u) \geq S_{N_j}(u)$ *and* $B_j(u) \geq N_j(u)$ *then* $NCWG_w(N_1, N_2, \ldots, N_m) \leq NCWG_w(B_1, B_2, \ldots, B_m)$*.*

3. ***Boundary:*** $N^- \le NCWG_w\{(N_1)_T, (N_2)_T, \ldots, (N_m)_T\} \le N^+$, *where*

$$N^- = \left\{\min_j T^L_{N_j}, \min_j I^L_{N_j}, 1-\max_j F^L_{N_j}, \min_j T_{N_j}, \min_j I_{N_j}, 1-\max_j F^L_{N_j}, \min_j T_{N_j}, \min_j I_{N_j}, 1-\max_j F^L_{N_j}\right\},$$
$$N^+ = \left\{\max_j T^U_{N_j}, \max_j I^U_{N_j}, 1-\min_j F^U_{N_j}, \max_j T_{N_j}, \max_j I_{N_j}, 1-\min_j F_{N_j}, \max_j T_{N_j}, \max_j I_{N_j}, 1-\min_j F_{N_j}\right\}$$

Proof.

1. **Idempotent**: Since $N_j = N$, so

$$NCWG(N_j) = \begin{pmatrix} \left[\prod_{j=1}^{m}\left(T^L_N\right)^{w_j}, \prod_{j=1}^{m}\left(T^U_N\right)^{w_j}\right], \\ \left[\prod_{j=1}^{m}\left(I^L_N\right)^{w_j}, \prod_{j=1}^{m}\left(I^U_N\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{m}\left(1-F^L_N\right)^{w_j}, 1-\prod_{j=1}^{m}\left(1-F^L_N\right)^{w_j}\right], \\ 1-\prod_{j=1}^{m}(1-T_N)^{w_j}, 1-\prod_{j=1}^{m}(1-I_N)^{w_j}, \prod_{j=1}^{m}(F_N)^{w_j} \end{pmatrix}$$
$$= \begin{pmatrix} \left[(T^L_N)^{\sum_{j=1}^{m} w_j}, (T^U_N)^{\sum_{j=1}^{m} w_j}\right], \\ \left[(I^L_N)^{\sum_{j=1}^{m} w_j}, (I^U_N)^{\sum_{j=1}^{m} w_j}\right], \\ \left[1-(1-F^L_N)^{\sum_{j=1}^{m} w_j}, 1-(1-F^U_N)^{\sum_{j=1}^{m} w_j}\right], \\ 1-(1-T_N)^{\sum_{j=1}^{m} w_j}, 1-(1-I_N)^{\sum_{j=1}^{m} w_j}, (F_N)^{\sum_{j=1}^{m} w_j} \end{pmatrix}$$
$$= \left(\widetilde{T}_N, \widetilde{I}_N, \widetilde{F}_N, T_N, I_N, F_N\right)$$

2. **Monotonicity:** Since NCOWG is strictly monotone function.
3. **Boundary:** Let $u = \min N^-$ and $y = \max N^+$, then by monotonicity we have $u \le NCOWA(N_j) \le y \Rightarrow N^- \le NCOWG(N_j) \le N^+$.

□

Theorem 5. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T^L_{N_j}, T^U_{N_j}\right], \widetilde{I}_{N_j} = \left[I^L_{N_j}, I^U_{N_j}\right], \widetilde{F}_{N_j} = \left[F^L_{N_j}, F^U_{N_j}\right]$, $(j = 1, 2, \ldots, n)$ *be the collection of neutrosophic cubic values and* $W = (w_1, w_2, \ldots, w_n)^T$ *is the weight of the NCOWG, with* $w_j \in [0,1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

1. *If* $W = (1, 0, \ldots, 0)^T$, *then* $NCOWG(N_1, N_2, \ldots, N_n) = \max N_j$
2. *If* $W = (0, 0, \ldots, 1)^T$, *then* $NCOWG(N_1, N_2, \ldots, N_n) = \min N_j$
3. *If* $w_j = 1$, $w_l = 0$, *and* $j \neq l$, *then* $NCOWG(N_1, N_2, \ldots, N_n) = N_j$

where N_j *is the jth largest of* $(N_1, N_2, \ldots, N_n)$.

Proof. Since in NCOWG the neutrosophic values are ordered in descending order. □

4.2. Neutrosophic Cubic Einstein Weighted Geometric Aggregation Operator

We define neutrosophic cubic Einstein geometric aggregation operators using Section 3.2.

Definition 23. *The neutrosophic cubic Einstein weighted geometric operator(NCEWA) is defined as* $NCEWG\colon R^m \to R$, *defined by* $NCEWG_w(N_1, N_2, \ldots, N_m) = \overset{m}{\underset{j=1}{\otimes}} \left(N_j^E\right)^{w_j}$

where, $W = (w_1, w_2, \ldots, w_m)^T$ *is the weight of* $N_j (j = 1, 2, 3, \ldots, m)$, *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

That is, first all the neutrosophic values are weighted then aggregated using Einstein operations.

Definition 24. *Order neutrosophic cubic Einstein weighted geometric operator(NCEOWG) is defined as* $NCEOWG : R^m \to R$ *by* $NCEOWG_w(N_1, N_2, \ldots, N_m) = \overset{m}{\underset{j=1}{\otimes}} \left(B_j^E\right)^{w_j}$

where B_j *is the jth largest,* $W = (w_1, w_2, \ldots, w_m)^T$ *is the weight of* $N_j (j = 1, 2, 3, \ldots, m)$, *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

That is, first all the neutrosophic values are ordered and then weighted, after ordering weighted values are aggregated using Einstein operations.The fundamental concept of ordered weighted operators is to rearrange the neutrosophic cubic values in descending order.

Theorem 6. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T^L_{N_j}, T^U_{N_j}\right], \widetilde{I}_{N_j} = \left[I^L_{N_j}, I^U_{N_j}\right]$, $\widetilde{F}_{N_j} = \left[F^L_{N_j}, F^U_{N_j}\right]$, $(j = 1, 2, \ldots, m)$ *is a collection of neutrosophic cubic values, then their Einstein weighted geometric aggregated value by NCEWG operator is also a neutrosophic cubic value, and*

$$NCEWG(N_j) = \left(\begin{array}{c} \left[\frac{2\prod_{j=1}^{m}\left(T^L_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(2-T^L_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(T^L_{N_j}\right)^{w_j}}, \frac{2\prod_{j=1}^{m}\left(T^U_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(2-T^U_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(T^U_{N_j}\right)^{w_j}} \right], \\ \left[\frac{2\prod_{j=1}^{m}\left(I^L_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(2-I^L_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(I^L_{N_j}\right)^{w_j}}, \frac{2\prod_{j=1}^{m}\left(I^U_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(2-I^U_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(I^U_{N_j}\right)^{w_j}} \right], \\ \left[\frac{\prod_{j=1}^{m}\left(1+F^L_{N_j}\right)^{w_j}-\prod_{j=1}^{m}\left(1-F^L_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+F^L_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(1-F^L_{N_j}\right)^{w_j}}, \frac{\prod_{j=1}^{m}\left(1+F^U_{N_j}\right)^{w_j}-\prod_{j=1}^{m}\left(1-F^U_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+F^U_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(1-F^U_{N_j}\right)^{w_j}} \right], \\ \frac{\prod_{j=1}^{m}\left(1+T_{N_j}\right)^{w_j}-\prod_{j=1}^{m}\left(1-T_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+T_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(1-T_{N_j}\right)^{w_j}}, \\ \frac{\prod_{j=1}^{m}\left(1+I_{N_j}\right)^{w_j}-\prod_{j=1}^{m}\left(1-I_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+I_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(1-I_{N_j}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{m}\left(F_{N_j}\right)^{w_j}}{\prod_{j=1}^{m}\left(2-F_{N_j}\right)^{w_j}+\prod_{j=1}^{m}\left(F_{N_j}\right)^{w_j}} \end{array} \right)$$

where $W = (w_1, w_2, \ldots, w_m)^T$ *is the weight vector of* $N_j (j = 1, 2, 3, \ldots, m)$, *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

Proof. We use mathematical induction to prove this result, for $m = 2$, using definition (Einstein sum and Einstein scalar multiplication).

$$\left(N_1^E\right)^{w_1} = \left(\begin{array}{c} \left[\frac{2\left(T^L_{N_1}\right)^{w_1}}{\left(2-T^L_{N_1}\right)^{w_1}+T^L_{N_1}}, \frac{2\left(T^U_{N_1}\right)^{w_1}}{\left(2-T^U_{N_1}\right)^{w_1}+T^U_{N_1}}\right], \\ \left[\frac{2\left(I^L_{N_1}\right)^{w_1}}{\left(2-I^L_{N_1}\right)^{w_1}+I^L_{N_1}}, \frac{2\left(I^U_{N_1}\right)^{w_1}}{\left(2-I^U_{N_1}\right)^{w_1}+I^U_{N_1}}\right], \\ \left[\frac{(1+F^L_{N_1})^{w_1}-(1-F^L_{N_1})^{w_1}}{(1+F^L_{N_1})^{w_1}+(1-F^L_{N_1})^{w_1}}, \frac{(1+F^U_{N_1})^{w_1}-(1-F^U_{N_1})^{w_1}}{(1+F^U_{N_1})^{w_1}+(1-F^U_{N_1})^{w_1}}\right], \\ \frac{(1+T_{N_1})^{w_1}-(1-T_{N_1})^{w_1}}{(1+T_{N_1})^{w_1}+(1-T_{N_1})^{w_1}}, \\ \frac{(1+I_{N_1})^{w_1}-(1-I_{N_1})^{w_1}}{(1+I_{N_1})^{w_1}+(1-I_{N_1})^{w_1}}, \\ \frac{2\left(F_{N_1}\right)^{w_1}}{\left(2-F_{N_1}\right)^{w_1}+F_{N_1}} \end{array}\right)$$

$$\left(N_2^E\right)^{w_2} = \left(\begin{array}{c} \left[\frac{2\left(T^L_{N_2}\right)^{w_2}}{\left(2-T^L_{N_2}\right)^{w_2}+T^L_{N_2}}, \frac{2\left(T^U_{N_2}\right)^{w_2}}{\left(2-T^U_{N_2}\right)^{w_2}+T^U_{N_2}}\right], \\ \left[\frac{2\left(I^L_{N_2}\right)^{w_2}}{\left(2-I^L_{N_2}\right)^{w_2}+I^L_{N_2}}, \frac{2\left(I^U_{N_2}\right)^{w_2}}{\left(2-I^U_{N_2}\right)^{w_2}+I^U_{N_2}}\right], \\ \left[\frac{(1+F^L_{N_2})^{w_2}-(1-F^L_{N_2})^{w_2}}{(1+F^L_{N_2})^{w_2}+(1-F^L_{N_2})^{w_2}}, \frac{(1+F^U_{N_2})^{w_2}-(1-F^U_{N_2})^{w_2}}{(1+F^U_{N_2})^{w_2}+(1-F^U_{N_2})^{w_2}}\right], \\ \frac{(1+T_{N_2})^{w_2}-(1-T_{N_2})^{w_2}}{(1+T_{N_2})^{w_2}+(1-T_{N_2})^{w_2}}, \frac{(1+I_{N_2})^{w_2}-(1-I_{N_2})^{w_2}}{(1+I_{N_2})^{w_2}+(1-I_{N_2})^{w_2}}, \\ \frac{2\left(F_{N_2}\right)^{w_2}}{\left(2-F_{N_2}\right)^{w_2}+F_{N_2}} \end{array}\right)$$

$$\overset{2}{\underset{j=1}{\otimes}} \left(N_j^E\right)^{w_j} = \left(\begin{array}{c} \left[\frac{2\prod_{j=1}^{2}\left(T^L_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(2-T^L_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(T^L_{N_j}\right)^{w_j}}, \frac{2\prod_{j=1}^{2}\left(T^U_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(2-T^U_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(T^U_{N_j}\right)^{w_j}}\right], \\ \left[\frac{2\prod_{j=1}^{2}\left(I^L_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(2-I^L_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(I^L_{N_j}\right)^{w_j}}, \frac{2\prod_{j=1}^{2}\left(I^U_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(2-I^U_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(I^U_{N_j}\right)^{w_j}}\right], \\ \left[\frac{\prod_{j=1}^{2}\left(1+F^L_{N_j}\right)^{w_j}-\prod_{j=1}^{2}\left(1-F^L_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+F^L_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(1-F^L_{N_j}\right)^{w_j}}, \frac{\prod_{j=1}^{2}\left(1+F^U_{N_j}\right)^{w_j}-\prod_{j=1}^{2}\left(1-F^U_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+F^U_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(1-F^U_{N_j}\right)^{w_j}}\right], \\ \frac{\prod_{j=1}^{2}\left(1+T_{N_j}\right)^{w_j}-\prod_{j=1}^{2}\left(1-T_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+T_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(1-T_{N_j}\right)^{w_j}}, \frac{\prod_{j=1}^{2}\left(1+I_{N_j}\right)^{w_j}-\prod_{j=1}^{2}\left(1-I_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+I_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(1-I_{N_j}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{2}\left(F_{N_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(2-F_{N_j}\right)^{w_j}+\prod_{j=1}^{2}\left(F_{N_j}\right)^{w_j}} \end{array}\right)$$

for $m = n$

$$\overset{n}{\underset{j=1}{\otimes}}\left(N_j^E\right)^{w_j} = \left(\begin{array}{c} \left[\frac{2\prod_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n}\left(2-T_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}}, \frac{2\prod_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n}\left(2-T_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}\right], \\ \left[\frac{2\prod_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}}, \frac{2\prod_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}\right], \\ \left[\frac{\prod_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}-\prod_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}-\prod_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}\right], \\ \frac{\prod_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(F_{N_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-F_{N_j}\right)^{w_j}+\prod_{j=1}^{n}\left(F_{N_j}\right)^{w_j}} \end{array}\right)$$

We prove the result holds for $m = n + 1$

$$as\left(N_{n+1}^E\right)^{w_{n+1}} = \left(\begin{array}{c} \left[\frac{2\left(T_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^L\right)^{w_{n+1}}+\left(T_{N_{n+1}}^L\right)^{w_{n+1}}}, \frac{2\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^U\right)^{w_{n+1}}+\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}\right], \\ \left[\frac{2\left(I_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^L\right)^{w_{n+1}}+\left(I_{N_{n+1}}^L\right)^{w_{n+1}}}, \frac{2\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^U\right)^{w_{n+1}}+\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}\right], \\ \left[\frac{(1+F_{N_{n+1}}^L)^{w_{n+1}}-(1-F_{N_{n+1}}^L)^{w_{n+1}}}{(1+F_{N_{n+1}}^L)^{w_{n+1}}+(1-F_{N_{n+1}}^L)^{w_{n+1}}}, \frac{(1+F_{N_{n+1}}^U)^{w_{n+1}}-(1-F_{N_{n+1}}^U)^{w_{n+1}}}{(1+F_{N_{n+1}}^U)^{w_{n+1}}+(1-F_{N_{n+1}}^U)^{w_{n+1}}}\right], \\ \frac{(1+T_{N_{n+1}})^{w_{n+1}}-(1-T_{N_{n+1}})^{w_{n+1}}}{(1+T_{N_{n+1}})^{w_{n+1}}+(1-T_{N_{n+1}})^{w_{n+1}}}, \frac{(1+I_{N_{n+1}})^{w_{n+1}}-(1-I_{N_{n+1}})^{w_{n+1}}}{(1+I_{N_{n+1}})^{w_{n+1}}+(1-I_{N_{n+1}})^{w_{n+1}}}, \\ \frac{2\left(F_{N_{n+1}}\right)^{w_{n+1}}}{\left(2-F_{N_{n+1}}\right)^{w_{n+1}}+\left(F_{N_{n+1}}\right)^{w_{n+1}}} \end{array}\right)$$

$$so\ \overset{n}{\underset{j=1}{\otimes}}\left(N_j^E\right)^{w_j}\otimes_E\left(N_{m+1}^E\right)^{w_{m+1}} =$$

$$\left(\begin{array}{c} \left[\frac{2\prod_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n}\left(2-T_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n}\left(T_{N_j}^L\right)^{w_j}}, \frac{2\prod_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n}\left(2-T_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n}\left(T_{N_j}^U\right)^{w_j}}\right], \\ \left[\frac{2\prod_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n}\left(I_{N_j}^L\right)^{w_j}}, \frac{2\prod_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n}\left(2-I_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n}\left(I_{N_j}^U\right)^{w_j}}\right], \\ \left[\frac{\prod_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}-\prod_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n}\left(1+F_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n}\left(1-F_{N_j}^L\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}-\prod_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n}\left(1+F_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n}\left(1-F_{N_j}^U\right)^{w_j}}\right], \\ \frac{\prod_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+T_{N_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-T_{N_j}\right)^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}-\prod_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+I_{N_j}\right)^{w_j}+\prod_{j=1}^{n}\left(1-I_{N_j}\right)^{w_j}}, \\ \frac{2\prod_{j=1}^{n}\left(F_{N_j}\right)^{w_j}}{\prod_{j=1}^{n}\left(2-F_{N_j}\right)^{w_j}+\prod_{j=1}^{n}\left(F_{N_j}\right)^{w_j}} \end{array}\right) \oplus_E \left(\begin{array}{c} \left[\frac{2\left(T_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^L\right)^{w_{n+1}}+\left(T_{N_{n+1}}^L\right)^{w_{n+1}}}, \frac{2\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-T_{N_{n+1}}^U\right)^{w_{n+1}}+\left(T_{N_{n+1}}^U\right)^{w_{n+1}}}\right], \\ \left[\frac{2\left(I_{N_{n+1}}^L\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^L\right)^{w_{n+1}}+\left(I_{N_{n+1}}^L\right)^{w_{n+1}}}, \frac{2\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}{\left(2-I_{N_{n+1}}^U\right)^{w_{n+1}}+\left(I_{N_{n+1}}^U\right)^{w_{n+1}}}\right], \\ \left[\frac{(1+F_{N_{n+1}}^L)^{w_{n+1}}-(1-F_{N_{n+1}}^L)^{w_{n+1}}}{(1+F_{N_{n+1}}^L)^{w_{n+1}}+(1-F_{N_{n+1}}^L)^{w_{n+1}}}, \frac{(1+F_{N_{n+1}}^U)^{w_{n+1}}-(1-F_{N_{n+1}}^U)^{w_{n+1}}}{(1+F_{N_{n+1}}^U)^{w_{n+1}}+(1-F_{N_{n+1}}^U)^{w_{n+1}}}\right], \\ \frac{(1+T_{N_{n+1}})^{w_{n+1}}-(1-T_{N_{n+1}})^{w_{n+1}}}{(1+T_{N_{n+1}})^{w_{n+1}}+(1-T_{N_{n+1}})^{w_{n+1}}}, \\ \frac{(1+I_{N_{n+1}})^{w_{n+1}}-(1-I_{N_{n+1}})^{w_{n+1}}}{(1+I_{N_{n+1}})^{w_{n+1}}+(1-I_{N_{n+1}})^{w_{n+1}}}, \\ \frac{2\left(F_{N_{n+1}}\right)^{w_{n+1}}}{\left(2-F_{N_{n+1}}\right)^{w_{n+1}}+\left(F_{N_{n+1}}\right)^{w_{n+1}}} \end{array}\right)$$

$$\overset{n+1}{\underset{j=1}{\otimes}}\left(N_j^E\right)^{w_j} = \left(\begin{array}{c}
\left[\frac{2\prod_{j=1}^{n+1}\left(T_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n+1}\left(2-T_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n+1}\left(T_{N_j}^L\right)^{w_j}}, \frac{2\prod_{j=1}^{n+1}\left(T_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n+1}\left(2-T_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n+1}\left(T_{N_j}^U\right)^{w_j}}\right], \\
\left[\frac{2\prod_{j=1}^{n+1}\left(I_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n+1}\left(2-I_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n+1}\left(I_{N_j}^L\right)^{w_j}}, \frac{2\prod_{j=1}^{n+1}\left(I_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n+1}\left(2-I_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n+1}\left(I_{N_j}^U\right)^{w_j}}\right], \\
\left[\frac{\prod_{j=1}^{n+1}\left(1+F_{N_j}^L\right)^{w_j}-\prod_{j=1}^{n+1}\left(1-F_{N_j}^L\right)^{w_j}}{\prod_{j=1}^{n+1}\left(1+F_{N_j}^L\right)^{w_j}+\prod_{j=1}^{n+1}\left(1-F_{N_j}^L\right)^{w_j}}, \frac{\prod_{j=1}^{n+1}\left(1+F_{N_j}^U\right)^{w_j}-\prod_{j=1}^{n+1}\left(1-F_{N_j}^U\right)^{w_j}}{\prod_{j=1}^{n+1}\left(1+F_{N_j}^U\right)^{w_j}+\prod_{j=1}^{n+1}\left(1-F_{N_j}^U\right)^{w_j}}\right], \\
\frac{\prod_{j=1}^{n+1}\left(1+T_{N_j}\right)^{w_j}-\prod_{j=1}^{n+1}\left(1-T_{N_j}\right)^{w_j}}{\prod_{j=1}^{n+1}\left(1+T_{N_j}\right)^{w_j}+\prod_{j=1}^{n+1}\left(1-T_{N_j}\right)^{w_j}}, \frac{\prod_{j=1}^{n+1}\left(1+I_{N_j}\right)^{w_j}-\prod_{j=1}^{n+1}\left(1-I_{N_j}\right)^{w_j}}{\prod_{j=1}^{n+1}\left(1+I_{N_j}\right)^{w_j}+\prod_{j=1}^{n+1}\left(1-I_{N_j}\right)^{w_j}}, \\
\frac{2\prod_{j=1}^{n+1}\left(F_{N_j}\right)^{w_j}}{\prod_{j=1}^{n+1}\left(2-F_{N_j}\right)^{w_j}+\prod_{j=1}^{n+1}\left(F_{N_j}\right)^{w_j}}
\end{array}\right)$$

so result holds for all values of m. □

Theorem 7. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U\right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U\right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U\right]$, $(j = 1, 2, \ldots, m)$ *is a collection of neutrosophic cubic values and* $W = (w_1, w_2, \ldots, w_m)^T$ *is a weight vector of* $N_j (j = 1, 2, 3, \ldots, m)$, *with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

1. ***Idempotency**: If for all* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U\right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U\right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U\right]$, $(j = 1, 2, \ldots, m)$ *are equal, that is,* $N_j = N$ *for all k, then* $NCEWG_w(N_1, N_2, \ldots, N_m) = N$
2. ***Monotonicity:** Let* $B_j = \left(\widetilde{T}_{B_j}, \widetilde{I}_{B_j}, \widetilde{F}_{B_j}, T_{B_j}, I_{B_j}, F_{B_j}\right)$, *where* $\widetilde{T}_{B_j} = \left[T_{B_j}^L, T_{B_j}^U\right]$, $\widetilde{I}_{B_j} = \left[I_{B_j}^L, I_{B_j}^U\right], \widetilde{F}_{B_j} = \left[F_{B_j}^L, F_{B_j}^U\right]$ $(j = 1, 2, \ldots, m)$ *be the collection of cubic values. If* $S_B(u) \geq S_N(u)$ *and* $B_j(u) \geq N_j(u)$ *then* $NCW\ G_w(N_1, N_2, \ldots, N_m) \leq NCWG_w(B_1, B_2, \ldots, B_m)$
3. ***Boundary:*** $N^- \leq NCWG_w\{(N_1)_T, (N_2)_T, \ldots, (N_m)_T\} \leq N^+$, *where*

$$N^- = \left\{\min_j T_{N_j}^L, \min_j I_{N_j}^L, 1 - \max_j F_{N_j}^L, \min_j T_{N_j}, \min_j I_{N_j}, 1 - \max_j F_{N_j}^L\right\},$$
$$N^+ = \left\{\max_j T_{N_j}^U, \max_j I_{N_j}^U, 1 - \min_j F_{N_j}^U, \max_j T_{N_j}, \max_j I_{N_j}, 1 - \min_j F_{N_j}\right\}$$

Proof. Followed by Theorem 2. □

Theorem 8. *Let* $N_j = \left(\widetilde{T}_{N_j}, \widetilde{I}_{N_j}, \widetilde{F}_{N_j}, T_{N_j}, I_{N_j}, F_{N_j}\right)$, *where* $\widetilde{T}_{N_j} = \left[T_{N_j}^L, T_{N_j}^U\right]$, $\widetilde{I}_{N_j} = \left[I_{N_j}^L, I_{N_j}^U\right], \widetilde{F}_{N_j} = \left[F_{N_j}^L, F_{N_j}^U\right]$, $(j = 1, 2, \ldots, m)$ *be a collection of neutrosophic cubic values and* $W = (w_1, w_2, \ldots, w_m)^T$ *is a weight vector of the NCOWA, with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{m} w_j = 1$.

1. *If* $w = (1, 0, \ldots, 0)^T$, *then NCEOWG* $(N_1, N_2, \ldots, N_m) = max\, N_j$
2. *If* $w(0, 0, \ldots, 1)^T$, *then NCEOWG* $(N_1, N_2, \ldots, N_m) = \min N_j$
3. *If* $w_j = 1$, $w_j = 0$, *and* $j \neq j$, *then NCEOWG* $(N_1, N_2, \ldots, N_m) = N_j$

where N_j *is the jth largest of* $(N_1, N_2, \ldots, N_m)$.

Proof. Followed by Theorem 3. □

5. An Application of Neutrosophic cubic Geometric and Einstein Geometric Aggregation Operator to Group Decision Making Problems

Group decision making is an important factor of decision making theory. We are often in a situation with more then one expert, attribute and alternative to deal with. Motivated by such situations, a multi-attribute decision making method for more then one expert is proposed in this section.

In this section, we develop an algorithm for group decision making problems using the geometric and Einstein geometric aggregations (NCWG and NCEWG) under the neutrosophic cubic environment.

Algorithm. *Let* $F = \{F_1, F_2, \ldots, F_n\}$ *be the set of n alternatives,* $H = \{H_1, H_2, \ldots, H_m\}$ *be the m attributes subject to their corresponding weight* $W = \{w_1, w_2, \ldots, w_m\}$ *such that* $w_j \in [0,1]$ *and* $\sum_{j=1}^{m} w_j = 1$, *and* $D = \{D_1, D_2, ...D_r\}$ *be the r decision makers with their corresponding weight* $V = \{v_1, v_2, \ldots, v_r\}$. *such that* $v_j \in [0,1]$ *and* $\sum_{j=1}^{r} v_j = 1$ The method has the following steps:

Step1. *First, we construct neutrosophic cubic decision matrices for each decision maker* $D^{(s)} = \left[N_{ij}^{(s)}\right]_{n \times m} (s = 1, 2, \ldots, r)$.

Step2. *All decision matrices are aggregated to a single matrix consisting of m attributes, by NCWG and NCEWG corresponding to the weight assigned to the decision maker.*

Step3. *By using aggregation operators like NCWG and NCEWG, the decision matrix is aggregated by the weight assigned to the m attributes.*

Step4. *The n alternatives are ranked according to their scores and arranged in descending order to select the alternative with highest score.*

6. Application

Mobile companies play a vital role in Pakistan's stock market. The performance of these companies affects resources of capital market and have become a common concern of shareholders, government authorities, creditors and other stakeholders. In this example, an investor company wants to invest his capital levy in listed companies. They acquire two types of experts: Attorney and market maker. The attorney is acquired to look at the legal matters and the market maker is aquired to provide his expertise in capital market matters. Data are collected on the basis of stock market analysis and growth in different areas. Let the listed mobile companies be (x_1) Zong, (x_2) Jazz, (x_3) Telenor and (x_4) Ufone, which have higher ratios of earnings than the others available in the market, from the three alternatives of (A_1) stock market trends, (A_2) policy directions and (A_3) the annual performance. The two experts evaluated the mobile companies $\left(x_j,\ j = 1, 2, 3, 4\right)$ with respect to the corresponding attributes $(A_i, i = 1, 2, 3)$, and proposed their decision making matrices consisting of neutrosophic cubic values in Equation (1) and Equation (2). The Equation (3) represents the single matrix as the aggregation of Equtiona1 and Equation (2) by NCWG or NCEWG. The Equation (4) is obtained by applying NCWG or NCEWG on attributes. The decision matrices are aggregated to a single decision matrix. At the end we rank the alternatives according to their score to get the desirable alternative(s).

Step 1. We construct the decision maker matrices in Equations (1) and (2).

Equation (1): Decision making matrix for the first expert(attorney) D_a is

$$\begin{pmatrix} & A_1 & A_2 & A_3 \\ X_1 & \begin{pmatrix}[0.2,0.6],[0.4,0.6],\\ [0.5,0.8],0.7,0.4,0.3\end{pmatrix} & \begin{pmatrix}[0.1,0.4],[0.5,0.8],\\ [0.4,0.8],0.6,0.7,0.5\end{pmatrix} & \begin{pmatrix}[0.4,0.6],[0.2,0.7],\\ [0.5,0.9],0.4,0.5,0.3\end{pmatrix} \\ X_2 & \begin{pmatrix}[0.3,0.5],[0.6,0.9],\\ [0.3,0.6],0.3,0.6,0.7\end{pmatrix} & \begin{pmatrix}[0.5,0.9],[0.1,0.3],\\ [0.4,0.8],0.8,0.3,0.6\end{pmatrix} & \begin{pmatrix}[0.2,0.7],[0.1,0.6],\\ [0.4,0.7],0.5,0.4,0.7\end{pmatrix} \\ X_3 & \begin{pmatrix}[0.6,0.9],[0.2,0.7],\\ [0.4,0.9],0.5,0.5,0.6\end{pmatrix} & \begin{pmatrix}[0.2,0.6],[0.7,0.3],\\ [0.3,0.8],0.4,0.6,0.5\end{pmatrix} & \begin{pmatrix}[0.5,0.9],[0.7,0.9],\\ [0.1,0.5],0.5,0.6,0.4\end{pmatrix} \\ X_4 & \begin{pmatrix}[0.4,0.8],[0.5,0.9],\\ [0.3,0.8],0.5,0.8,0.5\end{pmatrix} & \begin{pmatrix}[0.2,0.7],[0.4,0.9],\\ [0.5,0.7],0.6,0.4,0.5\end{pmatrix} & \begin{pmatrix}[0.3,0.5],[0.5,0.9],\\ [0.7,0.3],0.3,0.3,0.8\end{pmatrix} \end{pmatrix} \tag{1}$$

Equation (2): Decision making matrix for the second expert(market maker) D_m is

$$\begin{pmatrix} & A_1 & A_2 & A_3 \\ X_1 & \begin{pmatrix}[0.3,0.6],[0.2,0.6],\\ [0.2,0.6],0.8,0.7,0.2\end{pmatrix} & \begin{pmatrix}[0.3,0.8],[0.4,0.8],\\ [0.3,0.8],0.6,0.7,0.4\end{pmatrix} & \begin{pmatrix}[0.2,0.7],[0.2,0.6],\\ [0.3,0.8],0.5,0.3,0.5\end{pmatrix} \\ X_2 & \begin{pmatrix}[0.2,0.5],[0.6,0.9],\\ [0.7,0.3],0.4,0.8,0.7\end{pmatrix} & \begin{pmatrix}[0.4,0.9],[0.1,0.4],\\ [0.5,0.8],0.6,0.5,0.7\end{pmatrix} & \begin{pmatrix}[0.4,0.9],[0.1,0.4],\\ [0.5,0.8],0.6,0.5,0.7\end{pmatrix} \\ X_3 & \begin{pmatrix}[0.5,0.9],[0.2,0.6],\\ [0.3,0.8],0.7,0.7,0.8\end{pmatrix} & \begin{pmatrix}[0.2,0.5],[0.2,0.7],\\ [0.5,0.8],0.6,0.7,0.2\end{pmatrix} & \begin{pmatrix}[0.3,0.5],[0.3,0.9],\\ [0.2,0.5],0.6,0.5,0.4\end{pmatrix} \\ X_4 & \begin{pmatrix}[0.3,0.5],[0.3,0.9],\\ [0.2,0.5],0.6,0.5,0.4\end{pmatrix} & \begin{pmatrix}[0.4,0.7],[0.2,0.8],\\ [0.7,0.3],0.6,0.7,0.7\end{pmatrix} & \begin{pmatrix}[0.2,0.6],[0.5,0.9],\\ [0.2,0.8],0.4,0.4,0.8\end{pmatrix} \end{pmatrix} \tag{2}$$

Step2. Let $W = (0.4, 0.6)^T$, then the single matrix corresponding to weight W by use of NCWG operator is

Equation (3): The single decision matrix.

$$\begin{pmatrix} & A_1 & A_2 & A_3 \\ X_1 & \begin{pmatrix}[0.2551,0.6000],\\ [0.2885,0.6732],\\ [0.3371,0.6968],\\ 0.7647,0.6041,0.2352\end{pmatrix} & \begin{pmatrix}[0.1933,0.6062],\\ [0.4430,0.8001],\\ [0.3418,0.8680],\\ 0.6000,0.7000,0.4772\end{pmatrix} & \begin{pmatrix}[0.2638,0.6581],\\ [0.1999,0.6381],\\ [0.3881,0.8484],\\ 0.4621,0.3881,0.2223\end{pmatrix} \\ X_2 & \begin{pmatrix}[0.2352,0.5577],\\ [0.6000,0.9000],\\ [0.3000,0.6634],\\ 0.3618,0.7360,0.7000\end{pmatrix} & \begin{pmatrix}[0.2352,0.5577],\\ [0.6000,0.9000],\\ [0.3000,0.6634],\\ 0.3618,0.7360,0.7000\end{pmatrix} & \begin{pmatrix}[0.5253,0.8670],\\ [0.1515,0.6000],\\ [0.4621,0.8448],\\ 0.3371,0.4621,0.3301\end{pmatrix} \\ X_3 & \begin{pmatrix}[0.5378,0.9000],\\ [0.3565,0.8385],\\ [0.3418,0.8484],\\ 0.6319,0.6319,0.7130\end{pmatrix} & \begin{pmatrix}[0.2000,0.5378],\\ [0.2352,0.7000],\\ [0.4279,0.8000],\\ 0.5295,0.6634,0.2885\end{pmatrix} & \begin{pmatrix}[0.3680,0.6325],\\ [0.4210,0.9000],\\ [0.1614,0.5000],\\ 0.5626,0.5426,0.4000\end{pmatrix} \\ X_4 & \begin{pmatrix}[0.5101,0.8000],\\ [0.3465,0.6325],\\ [0.2416,0.7449],\\ 0.5000,0.6133,0.3807\end{pmatrix} & \begin{pmatrix}[0.3031,0.7000],\\ [0.2639,0.8385],\\ [0.3881,0.7000],\\ 0.6000,0.6041,0.6118\end{pmatrix} & \begin{pmatrix}[0.2352,0.5578],\\ [0.5000,0.9000],\\ [0.2416,0.7647],\\ 0.3618,0.3618,0.8000\end{pmatrix} \end{pmatrix} \tag{3}$$

Step3. Let the weight of attributes are $W = \{0.35, 0.30, 0.35\}$, using NCWG operators on attributes A's we get Equation (4),

$$NCWG = \begin{pmatrix} X_1 & \begin{pmatrix} [0.2375, 0.6195], \\ [0.2885, 0.7916], \\ [0.3567, 0.8146], \\ 0.6315, 0.5757, 0.2851 \end{pmatrix} \\ X_2 & \begin{pmatrix} [0.4426, 0.7657], \\ [0.2165, 0.5915], \\ [0.5382, 0.7804], \\ 0.4827, 0.5729, 0.5282 \end{pmatrix} \\ X_3 & \begin{pmatrix} [0.3500, 0.6616], \\ [0.3335, 0.8142], \\ [0.3131, 0.7498], \\ 0.5791, 0.6133, 0.4439 \end{pmatrix} \\ X_4 & \begin{pmatrix} [0.3327, 0.6774], \\ [0.3630, 0.7787], \\ [0.2888, 0.7396], \\ 0.4906, 0.5359, 0.5692 \end{pmatrix} \end{pmatrix} \quad (4)$$

Step4. Using the score function we rank the alternatives as:
$S(X_1) = 0.0321$, $S(X_2) = 0.0548$, $S(X_3) = 0.0839$ and $S(X_4) = -0.0969$, $X_3 > X_2 > X_1 > X_4$
The most desirable alternative is X_3.

7. Conclusions

Dealing with real life problems, decision makers encounter incomplete and vague data. The characteristics of neutrosophic cubic sets enablesdecision makers to deal with such a situation. Consequently, for each situation we defined the algebraic and Einstein sum, product and scalar multiplication. It is often difficult to compare two or more neutrosophic cubic values. The score and accuracy functions are defined to compare the neutrosophic cubic values values. Using these operations we defined neutrosophic cubic geometric, neutrosophic cubic weighted geometric, neutrosophic cubic Einstein geometric, and neutrosophic cubic Einstein weighted geometric aggregation operators with some useful properties. In the next section, a multi-criteria decision making algorithm was constructed. In the last section, a daily life problem was solved usingmulti-criteria decision making method (MCDM). This paper is based on some basic definitions and aggregation operators, which can be further extended to new horizons, like neutrosophic cubic hybrid geometric and neutrosophic cubic Einstein hybrid geometric aggregation operators.

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Neutrosophic Image Segmentation with Dice Coefficients

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A B S T R A C T

This paper explores various properties of Neutrosophic sets (NS) and proposes a novel idea on Image Segmentation using NS. A theoretical Neutrosophic model is proposed to reduce uncertainty from missing data. Besides, we also tackle the problem of image segmentation with fewer assumptions. Min-Max Normalization is used to reduce any uncertain noise in an image due to a number of factors during image capturing. Next, we apply activation functions to resolve the non-linearity in the image followed by the computed membership functions. These sets are then transformed and compared with others to find similarities and dissimilarities. Neutrosophic Sets and Dice's Coefficients are fused to ensure proper evaluation of uncertainty of the missing data and their indeterminacy for image segmentation. The proposed method is experimentally validated.

Abbreviations: NS, Neutrosophic Sets; D_{Score}, Dice's Coefficients; NL, Neutrosophic Logic; NM, Neutrosophic Measure; NI, Neutrosophic Integral; NP, Neutrosophic Probability; FS, Fuzzy Set; IFS, Intuitionistic Fuzzy Set; RS, Rough Set; INS, Interval Valued NS; GPU, Graphics processing units; ABC, Artificial Bee Colony; FDB, Factorized Directional Bandpass; NSC, Neutrosophic Similarity Clustering; SVM, Support vector machine; FOM, Figure of Merit; ROC, Receiver operating characteristic; T, E, C, R, P, D, t, I, f, F, X, Threshold based, Edge-based, Cluster-based, Region-based, PDE-based, Deep-learning based, truth, numerical indeterminacy, falsehood, falsehood membership function, universe of discourse; SNS, Soft Neutrosophic Sets.

1. Introduction

An image incorporates information which is needed to analyze through the process of Image Segmentation. Partitioning of an image into several of these segments (pixels or super-pixels) which is confined to a particular region bound by some characteristics in the forms of texture, intensity or color. If two pixels belong to adjacent regions, their characteristics differ [48]. Therefore, image segmentation is performed in order to locate objects and boundaries. It leads to the assignment of a specific label to each pixel in an image. Regional based segmentation selects a seed pixel and then merges similar pixels around it. Then, there are segmentation techniques based on clustering like K-Means. However, they have their own limitations in the form of overlapping images, computational cost, difficulty in estimating, etc. Hence, more sophisticated methods are used such as image segmentation using fuzzy algorithms, pattern recognition [42] and machine learning [43]. However, these advanced methods have limitations like lack of robustness and variability. Hence, there is a need to present a method of image segmentation that is contemporary and a step ahead. We present neutrosophic image segmentation as a result [47].

Neutrosophic science incorporates neutrosophic logic (NL) and its applications in many fields. It is conceivable to characterize the neutrosophic measure (NM), neutrosophic integral (NI), neutrosophic probability (NP) in light of the fact that there are different kinds of indeterminacies we have to highlight [50–54]. NM is a speculation of the established measure for the situation where the space containing some indeterminacy whereas NP is a speculation of the established and uncertain probabilities. A few traditional probability rules are balanced as NP rules. Moreover, they can be shown by different particular methodologies along with the likelihood theory, fuzzy set (FS) [1], rough set (RS) [2], intuitionistic fuzzy set (IFS) [3], and neutrosophic set (NS) [3]. Molodtsov [4] successfully proposed a novel delicate set theory by utilizing traditional sets since it has been brought up that delicate sets are not appropriate to manage dubious and fuzzy parameters. IFSs can just deal with inadequate data in light of the fact that the whole of degree genuine, indeterminacy and false is one in IFSs. However, NSs can deal with the uncertain and contrasting data which exist regularly in conviction frameworks in NS since indeterminacy is evaluated with free truth-membership, indeterminacy-membership and lie membership [5]. NSs can handle inadequate data, yet not the uncertain and contrasting data which exist normally in genuine circumstances. Several other related research papers featured neutrosophic science [49,50]. Broumi and Smarandache [6] presented the idea of correlation coefficients of interval valued NS (INS). Solis and Panoutsos [7] exhibited another system for making Granular Computing, Neural-Fuzzy displaying structures by means of Neutrosophic Logic to sort-out the problem of vulnerability amid the information granulation process.

The previous works focused on specific characteristics like texture, intensity or color. However, they have limitations in the form of overlapping images, computational cost, difficulty in estimating, etc. Hence, we use Dice's Coefficients (DScore) to resolve earlier sophisticated methods and to ensure proper evaluation of uncertainty of the missing data for image segmentation. Specifically, we propose new definitions regarding various features of an image using membership functions, activating them and then applying fitness functions. A universe of discourse has been defined, and subsequently the subsets are used with pixels as the most important parameters. We use fuzzy set partially and neutrosophic set along with rigorous mathematical operations on real numbers with real standard subsets. They involve crisping an image into smaller non-overlapping subsets so that the characteristics of the image can be explored at a molecular level for better analysis. Mathematical modeling is done for better accuracy for identifying an object. This is our contribution in this paper.

The rest of the paper is organized as follows: Section 2 presents the background of the paper. Section 3 shows the proposed method. Sections 4 and 5 give experiments and conclusions.

2. Literature review

In this section, we list out a few of the existing works that target the process of image segmentation. Taha and Hanbury [31] proposed an efficient evaluation tool for 3D medical image segmentation. Some of the metrics considered for this research are sensitivity, specificity, rand index, Jaccard index, average distance, probabilistic distance, etc. Thai et al. [32] presented a filter design and performance evaluation for fingerprint image segmentation using the factorized directional bandpass (FDB) segmentation method. A systematic performance comparison was conducted between the FDB method and other fingerprint image segmentation algorithms. For evaluation, the metrics considered are a number of orientations in Angular pass filter, Order of Butterworth bandpass filter, constant for selecting morphology threshold, the number of neighboring blocks, etc. Several aspects of fingerprint image quality may affect segmentation (dryness, ghost fingerprint, small scale noise, image artifacts, scars and creases). Accurate verification may become difficult due to distortions or overlapping of images. Bose and Mali [33] proposed an image segmentation algorithm based on Fuzzy Based Artificial Bee Colony and Fuzzy C means. It takes randomized characters and performs better in terms of convergence, time complexity, robustness and accuracy. The images considered for the research are synthetic, medical and texture images whose segmentation are difficult due to noise and ambiguous. Validity index and time complexity have been used to validate the superiority of the proposed work. The proposed work is well supported by the experimental results, but there could be several limitations of using the Artificial Bee Colony method. It does not take any secondary information and may require new fitness tests on new algorithm parameters. Since it performs a higher number of object evaluations, it may be very slow during sequential processing.

Moftah et al. [34] introduced adaptive k-means clustering algorithm for image segmentation. The idea is to perform image segmentation based on identifying target objects by virtue of optimizations so as to maintain optimum results during iterations. It is an extension to the traditional k-means clustering algorithm so as to increase effectiveness and efficiency. The experimental results exhibit the overall performance of the adaptive clustering algorithm in terms of entropy, standard deviation, mean, circularity, orientation and solidity. The major drawback is the fact that there were three samples considered, out of which for only one sample the proposed algorithm showed significant increase. For the other two samples, there was not much improvement. Liu et al. [35] presented a modified particle swarm optimization technique for image segmentation. The aim is to address the issue of computational expense by applying strategies to improve the performance of the initial particle swarm optimization technique. 16 standard test images have been considered in the experimental analysis, which validated that the modified technique is much more superior than the original method in terms of performance and quality. The parameters considered are twelve benchmark functions like quadric, rosenbrock, step, quadric noise, ratrigin, noncontinuous restrain, etc. The issue with the proposed method is the small dataset an only 30 independent runs that have been considered for validating the research, which questions the robustness and application in real world scenarios.

Ayyoub et al. [36] proposed a GPU based implementation of the fuzzy C-means algorithm for image segmentation to address the issue of large data set and slow processing. The idea is to introduce a parallel processing unit to validate the same. A faster variant of fuzzy-c means has been implemented on different GPU cards i.e., Tesla M2070 and Tesla K20m. Experimental analysis reveals that the proposed technique is significantly fast. Speed up, execution time, performance and memory are some parameters which validate the experimental analysis. Due to parallel processing, the process may be computationally expensive. Kloster et al. [37] suggested an image segmentation and outline feature extraction tool for microscopic analysis. The tool SHERPA (SHapE Recognition, Processing and Analysis) could identify and measure objects, and incorporate functions like object identification and feature extraction. It could also perform full image analysis, multiple segmentation methods, matching an object against templates, object scoring and processing large batch of images. Several parameters were taken into consideration for the same, some of which are area, parameter, width, height, optimization method, standard deviation, ellipticity, roundness, compactness etc. The issue with the proposed technique is that it cannot deal with texture and structural features; thereby questioning its versatility and identification specificity.

Chen et al. [38] suggested an interactive image segmentation method in hand gesture recognition so as to recognize the rate of hand gestures effectively. The Gaussian mixture model has been used for image modelling, whereas Gibbs random field is associated with image segmentation and minimization of Gibbs energy for optimal segmentation. The result has been tested on an image dataset and compared with others. The parameters considered are region accuracy and boundary accuracy. Five hand gestures have been relied on for experimental analysis. The limitation of the proposed research work is that it cannot handle issues like highlights, shadows and image distortions. Yu et al. [39] introduced a Semantic Image Segmentation Method with Multiple Adjacency Trees and Multiscale Features. A segment-based classifier and conditional random field are deployed in order to generate large scale regions, whose features have been used for training a region-based classifier. For capturing context, a multiple adjacency tree model has been suggested where each tree denotes a relevant region which can be further generated graphs. Relying on a few assumptions, some inference can be made. MSRC-21 and Stanford background datasets have been used for experimentations. The accuracy is determined by Support Vector Machines. The limitation of this research work lies in the assumptions made in order to make inferences. Further, using SVM has its own limitations like slow processing time.

Guo et al. [40] suggested a novel image segmentation approach based on neutrosophic c-means clustering and indeterminacy filtering. The idea is to transfer the image into neutrosophic domain and then the indeterminacy value of the neutrosophic image, devise an indeterminacy filter. Neutrosophic c-means clustering then clusters the pixels into several groups to find intensity. After the indeterminacy filtering operation, segmentation results are produced. The neutrosophic similarity clustering (NSC) segmentation algorithm has been compared to the proposed method quantitatively. Signal to noise ratio and misclassification error measure are some parameters considered for this research. Figure of Merit (FOM) has been used to measure the difference between the real results with the ideal segmentation result and the difference is not significant. Other works can be found in [23–29,41,46,55–67].

3. Methodology

3.1. Ideas

A universe of discourse is defined, and subsequently the subsets have been used using pixels as the pixels are the most important parameters for any image segmentation. We use Neutrosophic set along with DS_{core} with rigorous mathematical operations on real numbers with real standard subsets. It involves crisping an image into smaller non-overlapping subsets for better analysis. A new definition of DS_{core} is shown as:

$$D_{score} = \frac{S \cap T}{S \cup T}$$

where S is the area of segmentation of the object using our method and T is the manual or original area of segmentation of the object or the ground truth value. To calculate DS_{core}, we have assumed the following parameters which are applicable to all images:

Threshold based = "T",
Edge-based = "E",
Cluster-based = "C",
Region-based = "R",
PDE-based = "P",
Deep-learning based = "D".

Here the (t, I, f) -NS is referred as t = truth, I = numerical indeterminacy, f = falsehood. The (t, I, f) [3] are non-identical from the Neutrosophic Algebraic Structures (NAS) defined in the form of A + bI, where I = literal Indeterminacy. We render the image as I-NAS i.e., this is an algebraic structure established on indeterminacy "I" only. However, we can merge them and get the (t, I, f) -INAS. This means that the algebraic structures based on Neutrosophic Ontology (NoU) in the form a + bI where a and b are the real numbers, a is the determinant part on N, bI is the in-determinant part of N, bI ⊆ mI + nI = (m + n) I, 0•I = 0, I^n = I for integer n ≥ 1, I/I = undefined. When a, b are real numbers, then a + bI gives real numbers as results. If at least one of a, b is a complex number, then a + bI is known as a N complex number. These structures, in any field of learning, are considered from a NL perspective, i.e., from the truth-indeterminacy-falsehood (t, i, f) values [7,25,27,28].

3.2. Support Neutrosophic set (SNS)

Let X be a nonempty set, where x ∈ X, called the universe of discourse. First, let us define some terms about fuzzy set and Neutrosophic set. Here, we use mathematical operations on real numbers. Let A1 and A2 be two real standard or non-standard subsets, then we can apply some basic set operations such as [8–24]:

$$A1 + A2 = \{x|x = a1 + a2, a1 \in A1, a2 \in A2\}$$

$$A1 - A2 = \{x|x = a1 - a2, a1 \in A1, a2 \in A2\}$$

Now, we perform complementary operations and compute the Cross Product of the two sets, A1 and A2:

$$A2^{-} = \{1^{+}\} - A2 = \{x|x = 1 - a2, a2 \in A2\}$$

$$A1 \times A2 = \{x|x = a1 \times a2, a1 \in A1, a2 \in A2\}$$

Given a subset Y of a partially ordered set X, the Infimum, represented as in (Y), is the is greatest element in X, that is, X ({all elements in Y}, Conversely, The Suprema of Z on a partially ordered set X, represented as sup (Z), is the smallest element in X that is, X ({all elements in Z}, Therefore, we can define the logical operations in terms of Infimum and supreme as follows:

Infimum: A1 ∨ A2 = [max {Inf (A1), Inf (A2)}, max {sup (A1), sup (A2)}].

Suprema: A1 ∧ A2 = [min{inf(A1) , inf(A2)} , min{sup(A1) , sup (A2)}].

Observation: Applying Demorgan's Laws, let us consider two cases:

1. If inf(A1) ≤ inf(A2) and sup(A1) ≤ sup(A2).

Case 1 above implies that complement of inf(A2) is less than complement of inf(A1) and same for the suprema. From above definitions, we prove that

$$A1 \wedge A2 = A1,\ A1 \vee A2 = A3$$

2. If inf(A1) ≤ inf(A2) ≤ sup(A1) ≤ sup(A2).

Then, the logical operations can be expressed as the set of infimums and suprema.

Definition 1: A fuzzy set A on the universe X is a function which maps each element in the universe with a truth value $[0,1]^{+}$, also called the degree of membership of an element.

Definition 2: A Neutrosophic set A on the universe X is a function which maps each element with various set membership functions, such as truth membership function T, indeterminacy-membership function I and falsehood membership function F, each representing a truth value. Combining a Neutrosophic set with a fuzzy set leads to a new concept called support-Neutrosophic set (SNS). In which, there are four membership functions of each element in a given set.

Definition 3: A support Neutrosophic set (SNS) in the universe X is a function of four membership functions each corresponding to truth values of either 0 or 1. We denote support Neutrosophic set (SNS) as:

$$A = \{(x, T(x), I(x), F(x), S(x))|x \in X\}.$$

If universe X is continuous then the SNS is the integration of the mapping between each membership function divided by the elements over the entire universe X

$$A = \int < T(x), I(x), F(x), S(x) > /x.$$

If universe X is discrete, then, SNS can be written as the sum of each membership divided by the elements of the universe. If T(x) = I(x) = F(x) = S(x) = 0, *then x is called the worst element.* If T(x) = I(x) = F(x) = S(x) = 1, *then x is called the best element.*

Observations:

1. If the support membership function S(x) attains a constant value c, in a universe of truth labels [0,1] $^+$ then the support Neutrosophic set reduces to a Neutrosophic set.
2. A support-Neutrosophic set is called a standard Neutrosophic set if all the membership functions belong from the set [0,1] and the sum of the functions is less than 1 always.
3. A support-Neutrosophic set is called an intuitionistic fuzzy set if the truth T(x) and falsity FA(x) membership functions belong to [0,1] with their sum less than 1 and the Indeterminacy function I (x) is zero.
4. A constant SNS set can be represented using four symbols having a value between [0,1].

Definition 4: The complement of a SNS set A is denoted by c(A). Here, the truth membership function of c(A) is equal to the Falsity membership function, and vice versa, which is obvious as we are taking the complement of the SNS set. The indeterminacy set of the complementary SNS set is equal to the complement of each element of the original indeterminacy function. The same goes for the support membership function.

Definition 5: A SNS set A is a subset SNS set B if and only if the following conditions are satisfied:

1. The infimum and suprema of T(x) for set A is less than the infimum and suprema of T(x) for set B.
2. The infimum and suprema of F(x) of set A is greater than the infimum and suprema for F(x) of set B.
3. The infimum and suprema of S(x) for set A is less than the infimum and suprema of S(x) of set B.

Definition 6: The intersection of two SNS sets A and B is $D = A \cap B$, defined as follows:

1. The T (x), I (x) and S (x) of D is defined as the corresponding AND functions of the sets A and B.
2. The F (x) of D is defined as the corresponding OR functions of sets A and B.

Example 1: Let U = {x1, x2, x3, x4} be the universe of discourse. Then the support Neutrosophic set A is defined as the sum of all membership functions divided over each element of the universe. Let A = <[0.5,0.8]. [0.4,0.6], [0.2,0.7], [0.7,0.9] >/x1 +…., where, T (x) = [0.5,0.8], I(x) = [0.4,0.6], F (x) = [0.2,0.7] and S (x) = [0.7,0.9]. Then the complement of SNS set A, c (A), is given as c (A) = < [0.2,0.7], [0.4,0.6], [0.5,0.8], [0.1,0.3]>, Here, we see that T(x) of c (A) = F (x) of A, and F (x) of A = T(x) of c (A)

Definition 7 (Distance between support-Neutrosophic sets): Let X = {x1, x2, x3…., x (n)} be the universe set. We define; two support Neutrosophic sets A and B over X which is a universe of discourse.

1. The Hamming distance – It is calculated as the sum of the difference between each corresponding membership function value averaged over all the elements in the universe set.
2. The Euclidean distance – It is calculated as the sum of squares of the difference of the corresponding membership functions averaged over all the elements in the universe U.

3.3. The proposed method

Neutrosophic Sets can be applied on images to acquire understanding on indeterminate and missing data. That is, we are able to apply Neutrosophic sets on missing pixels and still be able to extract information about them. Our method uses activation functions to extract features in a from set of pixels such as edges and circles and then use membership functions to derive useful properties on shades and gradients. A Neutrosophic image I_a can be defined as a set of membership functions, T_s, I_s and F_s. Each image consists of pixel coordinates P (x, y) defined on arbitrary axes. Therefore, each pixel can be assigned membership value with T_s representing the foreground I_s representing the pixel intensity and F_s representing the background or the channels. Given an image I_a, subjected to various kinds of noise, which can be handled by normalization to standardize the pixels. We used non-linear normalization to account for missing and indeterminate data.

$$I(n) = Max(a) - Min(a) * 1/exp\left(1 + \frac{I(a) - \beta}{\alpha}\right) + Min(a) \quad (1)$$

where I (n) is the normalized image with reduced noise, Max (a) and Min (a) are the maximum and minimum pixel intensities in the image, β is the range of pixel intensity values around which image I (a) is centered, α is the width of the input image which is same as the total number of pixels in the image with noise reduction. We apply activation functions to find non-linearity on the segments of the image and found patterns by applying them sequentially on each row. Using Eq. (1), we use a filter size of 2 * 2 with stride 1 and apply a non-linear sigmoid activation function, which serves two purposes:

1. Firstly, it captures shapes such as lines and edges, which are crucial for object detection.
2. Secondly, it squashes each pixel value in the range [0,1] to be used by membership functions. Thus, we apply a non-linear sigmoid activation function *S (z)* as below:

$$S(z) = 1/(1 + exp(-z)) \quad (2)$$

where z is the mean pixel intensity value of the filter which is being activated. From Eq. (2), it is clear that the filters are sets of pixel values based on indeterminacy used to deal with indeterminate and missing data. But for better and smoother segmentation, we used a Gaussian function to smooth out the curves for standardization which is depicted in Eq. (3).

$$G(I(x,y)) = \frac{1}{2\pi\sigma^2} exp\left(-\frac{(x^2 + y^2)}{2\sigma^2}\right) \quad (3)$$

From this equation, we see that

1. If the indeterminacy intensities are low, the variance around the neighborhood is low as well.
2. This results in lowering down the value of σwhich make a less smooth transition around the edges.
3. If σ is large, the current filter neighborhood pixels become smoother.

Therefore, in order to reduce the lowering value of σ, the following equation is used. This is a linear variance function used to transform the filter values to parameter values (Crisping)

$$\sigma = f(I(x,y)) = m * I(x,y) + n \quad (4)$$

where m and n are parameters of the linear function which are used to transform the indeterminacy level to parameter level. A 2×2 filter is a square filter consisting of 4 pixels. We take the mean of these four-pixel values and apply the Gaussian standardization and the activation function. Using the activated values on pixel intensities, we define the truth and indeterminacy membership functionality on the local neighborhood as:

$$T(x,y) = \frac{i(x,y) - i(min)}{i(max) - i(min)} \tag{5}$$

$$I(x,y) = \frac{gd(x,y) - gd(min)}{gd(max) - gd(min)} \tag{6}$$

where I(x,y) corresponds to the intensity of pixel P(x,y) and gd(x,y) corresponds to the gradient magnitude of pixel P(x,y).

Proposed Algorithm: The entire algorithm can be summarized in the following steps:

Step 1: Normalize the Image using Min-Max Method.
Step 2: Apply activation function on successive pixels over the entire image using Gaussian filtering.
Step 3: Find the regions of interest by capturing pixels with higher scores after activation.
Step 4: Compute the membership functions T, I and F by Eqs. ((2), (5), (6)).
Step 5: From the Neutrosophic sets for each object to be identified.
Step 6: Perform De-Neutrosophication and Contrast Reduction to re-construct the objects of interest in the image.
Step 7: Present the results in a Tabular form of fitness scores or values.

The above algorithm can be used for any number of images. For large datasets, we can perform automated using various programs and check the algorithm's accuracy. We can determine the accuracy of our model using Dice's coefficient.

3.4. An illustrative example

In the grayscale image (Fig. 1), it is evident that the background is distorted and not visible. This corresponds to our falsehood membership function to be nearly undefined [30]. The foreground of the image is well defined around the region of the crow, which is the main focus on segmentation. Our goal is to segment each feature such as, the distorted or indeterminate bushes in the background, the features of the crow such as its beak, feathers and its tail, etc. We start by normalizing the image to standardize it and reduce any unnecessary noise in the background. We apply non-linear normalization function in each row and column pixels by using the sigmoid function to transform them to deal with indeterminate pixel values such as the bush near the crow. After normalization, we see that the indeterminacy still prevails, but has improved significantly accordingly with the foreground image. Now, we apply the sigmoid activation functions by taking 2×2 filters to account for the various shapes and edges. We start by applying them from the top-left corner and increment each filter by a stride of 1.

For understanding how this works, we take a small filter segment near the head of the crow. Applying the activation function, near the neighborhood, we find that outside the region of the head, the value of the activation function is significantly low compared to its corresponding filter value after that. We also find that this pattern persists till the end. From this, we can find the edges significantly easier for segmentation.

After applying the activation function, the pixel intensities are squashed between [0,1]. We prepare Neutrosophic sets by defining the truth, indeterminacy and falsehood membership functions. We take each row of the image as a set of Neutrosophics values (Fig. 2). The truth membership function T accounts for the intensity of the pixels in the foreground. We normalize the foreground by using the Min-Max method. In our example, the foreground consists of the crow. The truth membership function defines the patterns in the crow.

The indeterminacy membership function I use the gradients or shades of the neighboring pixels into account (Fig. 3). Therefore, it is a function that defines the lower saturation regions in our image such as the bushes in our example.

The falsehood membership function F operates in the background as opposed to the truth membership function. It would mainly serve by applying the filters on a colored image which has more than one channel. In our image, we have three channels, the falsehood membership function is same as the truth membership function T for each channel. We then define a Neutrosophic set A by combining these three functions for each pixel x in the pixel-space X. Here the universe of discourse X is the set of all pixels in the image, also called as the pixel-space of the image.

$$A = \{(x, T(x), I(x), F(x)) | x \in X\} \tag{7}$$

Now, using this Neutrosophic set, we define a fitness function L, also called Loss function, which will determine the quality of our output generation. This involves the calculation of average Neutrosophic values of each membership function. This analysis shows better insights such as segmentation score etc.

$$L = \left\{ \frac{T(x) + I(x) + F(x)}{3} \right\} \textit{for all } x \in X \tag{8}$$

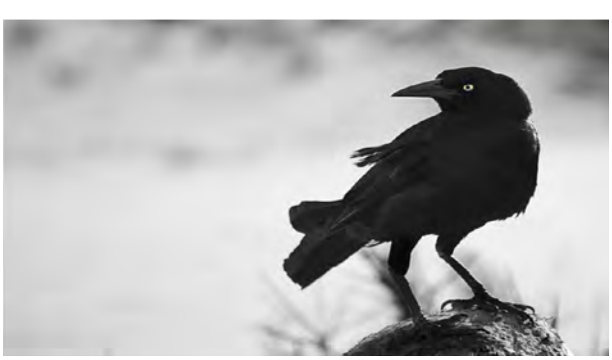

Fig. 1. Original image to be used for segmentation.

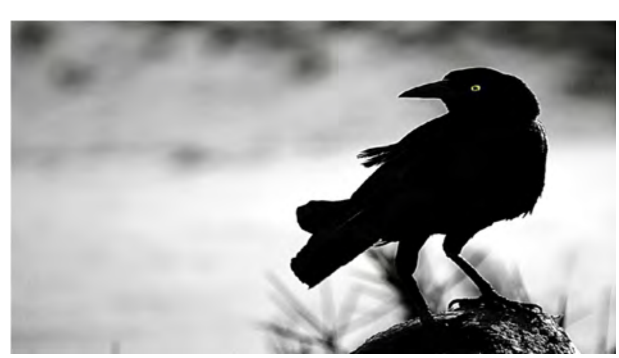

Fig. 2. Truth values after normalization in image.

Fig. 3. Indeterminate values in image.

As we apply the activation functions to downgrade our image, it may look distorted, due to which we need to reconstruct the image for better clarity. Here we use, de-Neutrosophication and contrast reduction to backtrack to a better image clarification. A Neutrosophic set N can be transformed to a de-Neutrosophic set by the following transformation

$$H(x) = \alpha * T(x) + \beta * \frac{F(x)}{4} + \gamma * \frac{I(x)}{2} \tag{9}$$

$$den(H(x)) = \frac{\int_a^b H(x) * x dx}{\int_a^b H(x) dx} \tag{10}$$

Here, $\alpha, \beta and \gamma$ are parameters where $0 \le \alpha, \beta, \gamma \le 1$ and $\alpha + \beta + \gamma = 1$, den (H (x)) is the de-Neutrosophic set which is calculated using the center of gravity method. The de-Neutrosophic set consists of the pixel values corresponding to the objects we want to segment. Hence, they can be transformed and compared with other images for better understanding of the model. Let N and M be two Neutrosophic sets, we can find similarities with them using set theory. Using intersect, we can find similarities between two sets.

$$A = N \cap M | T(N) \wedge T(M); I(N) \wedge I(M); F(N) \vee F(M) \tag{11}$$

Using Union, it might be possible to combine two pixel sets as well.

$$B = N \cup M | T(N) \vee T(M); I(N) \vee I(M); F(N) \wedge F(M) \tag{12}$$

Using complement, we can get the negative of an image.

$$C = N\,| T(N) = F_c(N); F(N) = T_c(N); I_c(N) = 1^+ - I(N) \tag{13}$$

We now use the contrast reduction methods from the Eqs. ((11)–(13)) to further clarify the image. We found that by using maximum clarity, the model had accurately identified the object in the frame.

4. Result and discussions

In this section, we focus on extracting the image of the crow [30] to discard other relevant features so that there are no external noises. The Python library OpenCV is a scientific library for solving problems in the computer vision domain. OpenCV takes an image as input and produces an array representing its pixel values as output. The pixel values are in the range (0–255). The original image is converted into its corresponding pixel formats by using OpenCV 2 function imread () converting it to a numpy array. The pseudocode for it can be given as below:

```
Import cv2
im = imread('file.jpg')
The image can be converted to grayscale format
im_gray = cv2.cvtColor(im, cv2.BGR2GRAY)
In order to flip an image vertically, we can use OpenCV flip()
   function which helps us in rotating the image by certain
   degree. The image can be rotated vertically by 180 degrees. The
   pseudocode is given as follows:
im_flip = cv2.flip(im, 1)
The image can be rescaled to a certain degree using OpenCV
   rescale() function. We rescaled the image to 1.5x to better
   analyze our images using the proposed method. The
   followingpseudocode can be used to rescale any image.
r = 1.5 * im.shape[1]
Dim = (100, int(im.shape[0] * r)
resized = cv2.resize(im, dim, interpolation = cv2.INTER_AREA)
```

Firstly, we use normalization to reduce any kind of internal noise. This gives us a better understanding of the objects in the frame, such as the blurred bushes in the background. We use the Min-Max Scaler normalization technique, which is a non-linear normalization process to reduce noise. Let I(X) be the image where X is the pixel-space set or the universe of discourse. Let X = {$x_1, x_2, x_3 \ldots x_n$} where x_i represents pixels of the flattened image. Then, for the above image we have:

$$N(x(i)) = (255 - 0) * 1/\exp(1 + (x(i) - 1001)/1395) + 0 \tag{16}$$

Using the Min-Max Normalization technique (Fig. 4), it becomes easier to interpret data and reduce noise and other factors which hinder Image processing in general. Therefore, using Normalization is a good start on reducing complexity. For the above image, we get the following normalized image:

We then apply the sigmoid activation function on each Gaussian filter as weights successively on each pixel. Then we compare each output value by the adjacent one. If the adjacent value is less

Fig. 4. Normalized image using non-linear Min-Max Method.

Table 1
Observations of bush, head and tail from original image 1.

Type	Parameters of various types of observations		
Bush	Length (0.23,0.82,0.23)	Width (0.1,0.5,0.1)	Tip (0.02,0.09,0.02)
Head	Break (0.86, 0.78,0.86)	Crown (0.96,0.54,0.96)	Eyes (0.23,0.67,0.23)
Tail	Hand (0.36,0.25,0.36)	Legs (0.24,0.35,0.24)	Fingers (0.48,0.89,0.48)

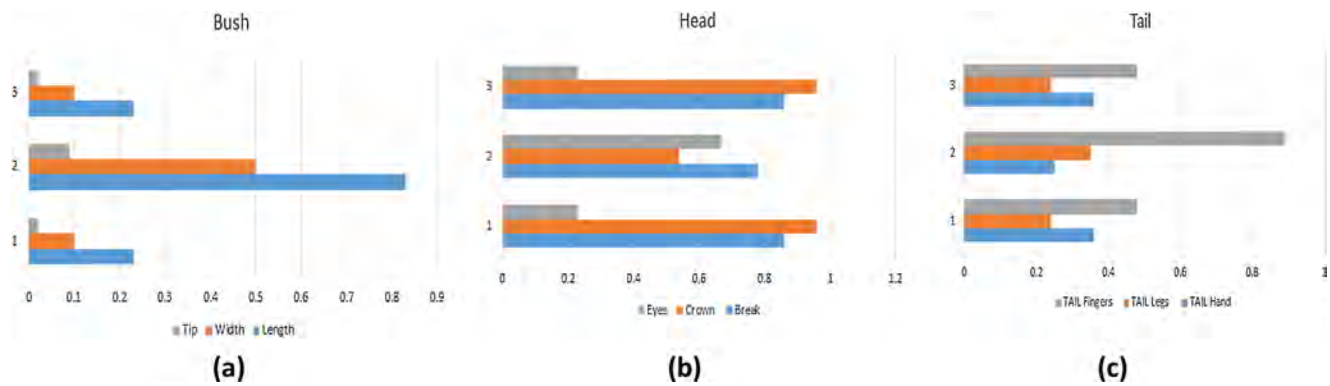

Fig. 5. (a–c): The observational result following Table 2.

Table 2
Observations of bush, head and tail by flipping the original image 1 upside down.

Type	Parameters of various types of observations		
Bush	Length (0.56,0.78,0.0.56)	Width (0.21,0.4,0.21)	Tip (0.07, 0.12,0.07)
Head	Break (0.87, 0.65,0.87)	Crown (0.75,0.65,0.75)	Eyes (0.25,0.78,0.25)
Tail	Hand (0.40,0.89,0.40)	Legs (0.27,0.37,0.27)	Fingers (0.50,0.87,0.50)

Table 3
Observations of bush, head and tail by rescaling the original image by 1.5 × times.

Type	Parameters of various types of observations		
Bush	Length (0.89,0.74,0.0.89)	Width (0.30,0.47,0.30)	Tip (0.89, 0.88,0.89)
Head	Break (0.56, 0.23,0.0.56)	Crown (0.78,0.68,0.78)	Eyes (0.52,0.34,0.52)
Tail	Hand (0.60,0.64,0.60)	Legs (0.67,0.75,0.67)	Fingers (0.50,0.78,0.50)

compared to the next pixel, we start to prepare our Neutrosophic set from the next pixel till the end of the pattern. Given a pixel $x(i) \in X$, the sigmoid activation function is calculated for each Gaussian filter separately.

$$G(I(x,y)) = \frac{1}{2\pi\sigma^2} exp\left(-\frac{(x^2+y^2)}{2\sigma^2}\right) \quad (17)$$

$$S(x(i)) = 1/(1+exp(x(i))) \quad (18)$$

$$R(I(x,y)) = G(I(x,y)) * S(x(i)) \quad (19)$$

Hare, R (I (x, y)) represents the transformed Image after applying the activation function. The Gaussian function keeps track whether the pixels are changing or not. According to those observations, we can pick the pixels which we are interested in and from their corresponding Neutrosophic sets. We form Neutrosophic sets for three sections named, Bush, Head and Tail, each corresponding to the bush, the crow's head, and tail. We then transform the image by rescaling and flipping the image and computing the Neutrosophic values for each transformed image. For the original image, we find the following observations (Table 1):

Referring to Fig. 5(a)–(c), it can be inferred that the orientation of length is in proportion with that of Tip and Width whereas the orientation for Head are almost same. But if we infer to the orientation of Tail, the Tail is highly proportional with respect to Tail Legs and tail Heads (Table 2).

Referring to Fig. 6(a)–(c), it can be inferred that the orientation of length for head is almost consistent whereas the length is highest for "Tail". The reason is due to upside down position of the original image. Likewise, the orientation for Tip gradually increases from Bush to Tail. If we infer to the orientation of Head, it is very clear that the Head is highly proportional with respect to Tail and Bush. We now rescale the image by 1.5 times i.e., the original image × 1.5 times. The observations are tabulated as below (Table 3).

Referring to Fig. 7(a)–(c), it can be inferred that the orientation of Tip, Width and length are almost consistent for all the three i.e., Bush, Head and Tail. Thus, this result infers to the fact that even if we change and rescale our image, the Neutrosophic sets do not

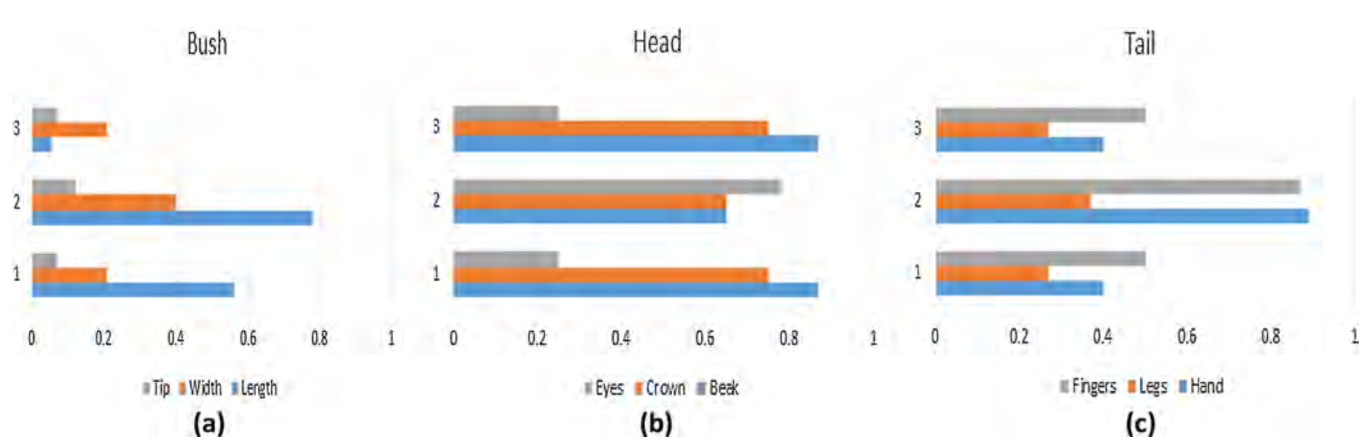

Fig. 6. (a–c): The Observational result following Table 2.

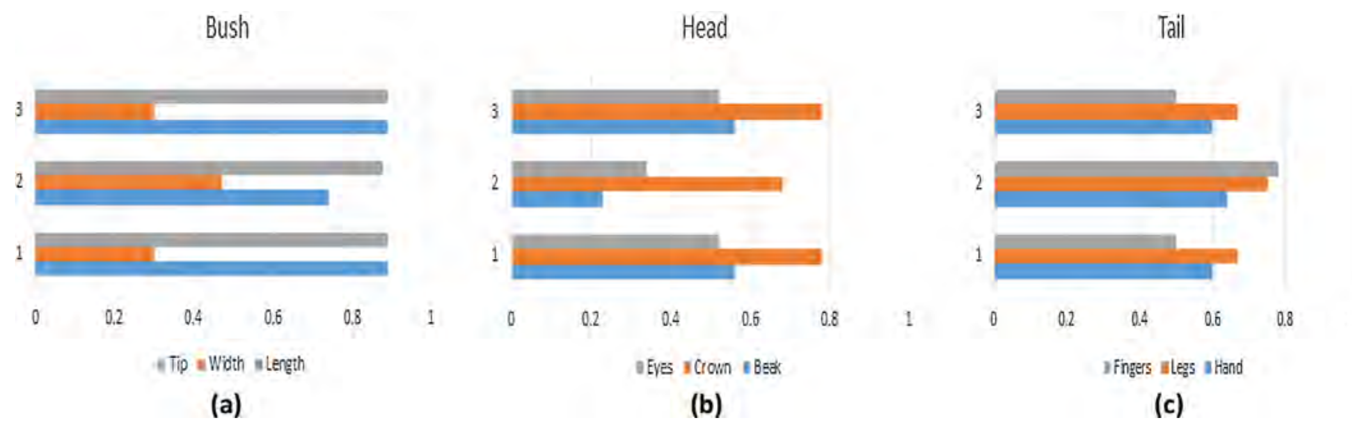

Fig. 7. (a–c): The Observational result following Table 4.

Fig. 8. Final image showing object of interest (crow) in a frame.

Table 4
Comparison for different image orientations using Ds_{core} with neutrosophic sets.

	Parameters of various types of observations taken using the DS_{core}		
Orientation	Bush	Head	Tail
Normal	Length (0.21,0.75,0.19) Width (0.09,0.3,0.03) Tip (0.01, 0.04,0.03)	Beak (0.66,0.68,0.68) Crown (0.86,0.24,0.76) Eyes (0.13,0.37,0.32)	Hand (0.21,0.12,0.13) Legs (0.16,0.23,0.14) Fingers (0.24,0.67,0.29)
Upside Down	Length (0.46,0.38, 0.0.46) Width (0.11,0.3,0.16), Tip (0.04, 0.08,0.04)	Beak (0.75, 0.63,0.76), Crown (0.64,0.46,0.64), Eyes (0.16,0.68,0.15)	Hand (0.31,0.69,0.20), Legs (0.21,0.27,0.19), Fingers (0.43,0.65,0.42)
Rescaling (1.5x)	Length (0.67,0.67, 0.0.71) Width (0.20,0.37,0.21), Tip (0.79, 0.78,0.69)	Beak (0.46, 0.13, 0.0.46), Crown (0.63,0.62,0.65), Eyes (0.42,0.30,0.42)	Hand (0.44,0.44,0.48), Legs (0.61,0.67,0.48), Fingers (0.49,0.65,0.46)

change much, i.e., whatever be the pixel data presented in the same image, the final sets will not alter much. Hence, our observations closely resemble with that of the theoretical observation. Now, the above data can be normalized and cumulatively tabulated as below (Fig. 8).

From the above observations (Table 4), we find that even if we change and rescale our image, the Neutrosophic sets do not change much. This means that however, be the data presented, the final sets will not change much at all and we will get roughly the same results always. This can be proved easily using properties of Neutrosophic sets. Hence, our observations closely resemble the theoretical observations. The below image is the final image after performing all the steps, which shows us the object crow in the frame which is of best interest to us. The comparison between the various segmentation methods with the descriptions as well as advantages and disadvantages are indicated in Table 5.

The Segmentation Method for Threshold based using DS_{core} is observed to be 0.56 whereas using Neutrosophic sets, we obtained 0.78 which is much better accurate value in comparison all other segmentation methods (Table 6).

Now we have shown this analysis with the help of bar graph visualization (Fig. 9).

The proposed method performs better than others as it requires less computation power and time to find the results (Table 7). The results were verified and validated by humans and the method works fine. It is important to note that this method takes very few assumptions about the data provided. The data can be presented with indeterminate form and in different orientations and sizes. The model works on these types of scenarios as well, which makes it unique from other previous works done on Image Segmentation using Neutrosophic sets. In medical diagnosis as an example, most of the data is indeterminate and come in various orientations as well. Sometimes, we need to use sonar projections of certain organs of the body which are captured better at certain

Table 5
Comparison between the various types of image segmentation methods.

Segmentation Methods	Description	Advantages	Disadvantages
Threshold based	Find particular threshold values of the image	No need to hold the previous image related information	Highly dependent on peaks, simplest method
Edge-based	Discontinuity detection	Relevant for images having good contrast	Not suitable a large
Cluster-based	Homogeneous clusters	Use of membership function to address the real-life problems	The evaluating membership function is not an easy task
Region-based	Based on splitting image into consistent regions	More protective for noise	Expensive in terms of memory and time
PDE-based	Used differential equations	Fastest method	Computational complexity is higher than previous methods
Deep-learning based	Replication of learning process for decision making	Simple programs	Training data time is too high
Proposed Work	Finding the regions of interest (ROI) by capturing pixels with higher scores after activation to compute the membership functions T, I and F	This method can be applied to any number of images and any type of typical problem (blurred images)	No need to training, so it is less time consuming than deep learning approaches

Table 6
Comparison between the various types of image segmentation methods and our proposed model in terms of DS_{core} value.

Segmentation methods	Average D-score
Threshold based	0.56
Edge-based	0.62
Cluster-based	0.64
Region-based	0.67
PDE-based	0.70
Deep-learning based	0.73
Neutrosophic sets-based (Proposed work)	0.78

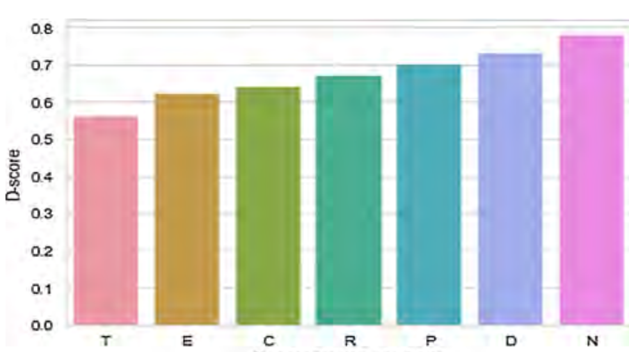

Fig. 9. Graphical representation of methods in terms of $DS_{core.}$

Table 7
Comparison of the proposed work with existing ones.

No.	Authors	Existing methods	Proposed method
1.	Liang-Chieh Chen et al. [44]	- Used benchmarking of pre-release Cityscapes dataset - Performance is 63.1%	- We have used three orientations of sample crow images and the datasets were generated by ourselves (Section 4) - Performance is 78% as depicted in Table 7
2.	Ma et al. [13]	- Used a generalized interval of Neutrosophic set values (higher number of assumptions) - Aggregated the interval Neutrosophic linguistic information. - Performance is 71%	- Very less assumptions made in our paper, due to which the results are more absolute than their work - The model works on different types of scenarios as well, which makes it unique from them on Image Segmentation using Neutrosophic sets because priority has been given to each Neutrosophic Set values (pixels) generated - Performance is 78%
3.	Irfan Deli et al. [45]	- Define the concepts of cut sets of SVN-numbers - Applied to single valued trapezoidal neutrosophic numbers - Developed a ranking method by using the concept of values and ambiguities	- Our paper defines basic concepts of Neutrosophic Set theory and - Applicable in various membership functions and fitness functions - Applied to multiple sets of values - Multiple set of values experimented for different image orientations - No ambiguities - We concluded with the absolute performance value of 78%

orientations only. The same is true for X-ray images as well. In these types of scenarios, Neutrosophic sets can be used effectively to analyze the problems. In the future, we can optimize this algorithm further to be applied to various other domains and surpass the current state-of-the-art deep neural networks.

5. Conclusion

This paper explores the idea of applying Neutrosophic sets to the domain of Image Segmentation. We firstly discussed various properties of Neutrosophic sets and then lay out to tackle the problem of image segmentation with fewer assumptions. To accomplish this, we first used Min-Max Normalization to reduce any uncertain noise in the image that may be caused due to a number of factors during image capturing. Then, we applied activation functions to account for non-linearities in the image. We then computed the membership functions on different regions and formed the Neutrosophic sets. These sets are then transformed and compared with other sets to find similarities and dissimilarities.

Throughout the entire paper, we used images of a crow and presented our findings. It is worth noting that Neutrosophic sets can be applied to datasets with missing data with different orientations. This calls for a better understanding of Neutrosophic systems and their further research on solving complex problems simply and replace the current state-of-the-art methodologies. Using Neutrosophic Sets and using Dice's Coefficients (D_{Score}), this paper has resolved earlier sophisticated methods and ensured the proper evaluation of the uncertainty of the missing data and their indeterminacy with various results to prove effectiveness for the image processing and segmentation.

The proposed work could be refined further in order to achieve better results. Several other parameters may be considered for the same:

1. Neutrosophic sets (NS) can be remarkably used along with neural networks to get in depth of various fields. Natural Language Processing, Image Captioning etc are few of them.
2. Digital Communication have used lots of research to reduce internal noise in an image. They have also used many filters apart from quantization and sampling to reduce the noise and errors in an image. However, the Neutrosophic Set theory can be extensively used to predict indeterminacy and normalize them using various membership functions.
3. NS can be used for noise detection and minimization using various factors, such as poor lighting, dust particles blockage, etc. by normalization.
4. In short, the NS concept is best suitable in working conditions, i.e., the real time problems. For example, in medical diagnosis, it is very important to reduce noise before we perform any Image Processing as it is impossible to find best. We used an existing method for reducing noise which has proved effective in X-ray images significantly high.

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Ordinary Single Valued Neutrosophic Topological Spaces

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Abstract: We define an ordinary single valued neutrosophic topology and obtain some of its basic properties. In addition, we introduce the concept of an ordinary single valued neutrosophic subspace. Next, we define the ordinary single valued neutrosophic neighborhood system and we show that an ordinary single valued neutrosophic neighborhood system has the same properties in a classical neighborhood system. Finally, we introduce the concepts of an ordinary single valued neutrosophic base and an ordinary single valued neutrosophic subbase, and obtain two characterizations of an ordinary single valued neutrosophic base and one characterization of an ordinary single valued neutrosophic subbase.

Keywords: ordinary single valued neutrosophic (co)topology; ordinary single valued neutrosophic subspace; α-level; ordinary single valued neutrosophic neighborhood system; ordinary single valued neutrosophic base; ordinary single valued neutrosophic subbase

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1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy sets as the generalization of an ordinary set. In 1986, Chang [2] was the first to introduce the notion of a fuzzy topology by using fuzzy sets. After that, many researchers [3–13] have investigated several properties in fuzzy topological spaces.

However, in their definitions of fuzzy topology, fuzziness in the notion of openness of a fuzzy set was absent. In 1992, Samanta et al. [14,15] introduced the concept of gradation of openness (closedness) of fuzzy sets in X in two different ways, and gave definitions of a smooth topology and a smooth co-topology on X satisfying some axioms of gradation of openness and some axioms of gradation of closedness of fuzzy sets in X, respectively. After then, Ramadan [16] defined level sets of a smooth topology and smooth continuity, and studied some of their properties. Demirci [17] defined a smooth neighborhood system and a smooth Q-neighborhood system, and investigated their properties. Chattopadhyay and Samanta [18] introduced a fuzzy closure operator in smooth topological spaces. In addition, they defined smooth compactness in the sense of Lowen [8,9], and obtained its properties. Peters [19] gave the concept of initial smooth fuzzy structures and found its properties. He [20] also introduced a smooth topology in the sense of Lowen [8] and proved that the collection of smooth topologies forms a complete lattice. Al Tahan et al. [21] defined a topology such that the hyperoperation is pseudocontinuous, and showed that there is no relation in general between pseudotopological and strongly pseudotopological hypergroupoids. In addition, Onassanya and Hošková-Mayerová [22] investigated some topological properties of α-level subsets' topology of a fuzzy subset. Moreover, Çoker and Demirci [23], and Samanta and Mondal [24,25] defined intuitionistic gradation of openness (in short IGO) of fuzzy sets in Šostak's sense [26] by using intuitionistic fuzzy sets introduced by Atanassov [27]. They mainly dealt with intuitionistic gradation of openness of fuzzy sets in the sense of Chang. However, in 2010, Lim et al. [28] investigated intuitionistic smooth topological spaces in Lowen's sense. Recently, Kim et al. [29] studied continuities and neighborhood systems in intuitionistic smooth topological spaces. In addition, Choi et al. [30] studied an interval-valued smooth topology by gradation of openness of interval-valued fuzzy sets introduced by Gorzalczany [31] and Zadeh [32], respectively. In particular, Ying [33] introduced the concept of the topology (called a fuzzifying topology) considering the degree of openness of an ordinary subset of a set. In 2012, Lim et al. [34] studied general properties in ordinary smooth topological spaces. In addition, they [35–37] investigated closures, interiors and compactness in ordinary smooth topological spaces.

In 1998, Smarandache [38] defined the concept of a neutrusophic set as the generalization of an intuitionistic fuzzy set. Salama et al. [39] introduced the concept of a neutrosophic crisp set and neutrosophic crisp relation (see [40] for a neutrosophic crisp set theory). After that, Hur et al. [41,42] introduced categories $\mathbf{NSet}(H)$ and $\mathbf{NCSet}$ consisting of neutrosophic sets and neutrosophic crisp sets, respectively, and investigated them in a topological universe view-point. Smarandache [43] defined the notion of neutrosophic topology on the non-standard interval and Lupiáñez proved that Smarandache's definitions of neutrsophic topology are not suitable as extensions of the intuitionistic fuzzy topology (see Proposition 3 in [44,45]). In addition, Salama and Alblowi [46] defined a neutrosophic topology and obtained some of its properties. Salama et al. [47] defined a neutrosophic crisp topology and studied some of its properties. Wang et al. [48] introduced the notion of a single valued neutrosophic set. Recently, Kim et al. [49] studied a single valued neutrosophic relation, a single valued neutrosophic equivalence relation and a single valued neutrosophic partition.

In this paper, we define an ordinary single valued neutrosophic topology and obtain some of its basic properties. In addition, we introduce the concept of an ordinary single valued neutrosophic subspace. Next, we define the ordinary single valued neutrosophic neighborhood system and we show that an ordinary single valued neutrosophic neighborhood system has the same properties in a classical neighborhood system. Finally, we introduce the concepts of an ordinary single valued neutrosophic base and an ordinary single valued neutrosophic subbase, and obtain two characterizations of an ordinary single valued neutrosophic base and one characterization of an ordinary single valued neutrosophic subbase.

2. Preliminaries

In this section, we introduce the concepts of single valued neutrosophic set, the complement of a single valued neutrosophic set, the inclusion between two single valued neutrosophic sets, the union and the intersection of them.

Definition 1 ([43])**.** *Let X be a non-empty set. Then, A is called a neutrosophic set (in sort, NS) in X, if A has the form $A = (T_A, I_A, F_A)$, where*

$$T_A : X \to]^-0, 1^+[, \quad I_A : X \to]^-0, 1^+[, \quad F_A : X \to]^-0, 1^+[.$$

Since there is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, for each $x \in X$,

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

Moreover, for each $x \in X$, $T_A(x)$ (resp., $I_A(x)$ and $F_A(x)$) represent the degree of membership (resp., indeterminacy and non-membership) of x to A.

From Example 2.1.1 in [17], we can see that every IFS (intutionistic fuzzy set) A in a non-empty set X is an NS in X having the form

$$A = (T_A, 1 - (T_A + F_A), F_A),$$

where $(1 - (T_A + F_A))(x) = 1 - (T_A(x) + F_A(x))$.

Definition 2 ([43]). *Let A and B be two NSs in X. Then, we say that A is contained in B, denoted by $A \subset B$, if, for each $x \in X$, $inf\ T_A(x) \leq inf\ T_B(x)$, $sup\ T_A(x) \leq sup\ T_B(x)$, $inf\ I_A(x) \geq inf\ I_B(x)$, $sup\ I_A(x) \geq sup\ I_B(x)$, $inf\ F_A(x) \geq inf\ F_B(x)$ and $sup\ F_A(x) \geq sup\ F_B(x)$.*

Definition 3 ([48]). *Let X be a space of points (objects) with a generic element in X denoted by x. Then, A is called a single valued neutrosophic set (in short, SVNS) in X, if A has the form $A = (T_A, I_A, F_A)$, where $T_A,\ I_A,\ F_A : X \to [0,1]$.*

In this case, $T_A,\ I_A,\ F_A$ are called truth-membership function, indeterminacy-membership function, falsity-membership function, respectively, and we will denote the set of all SVNSs in X as $SVNS(X)$.

Furthermore, we will denote the empty SVNS (resp. the whole SVNS] in X as 0_N (resp. 1_N) and define by $0_N(x) = (0,1,1)$ (resp. $1_N = (1,0,0)$), for each $x \in X$.

Definition 4 ([48]). *Let $A \in SVNS(X)$. Then, the complement of A, denoted by A^c, is an SVNS in X defined as follows: for each $x \in X$,*

$$T_{A^c}(x) = F_A(x),\ I_{A^c}(x) = 1 - I_A(x) \text{ and } F_{A^c}(x) = T_A(x).$$

Definition 5 ([50]). *Let $A,\ B \in SVNS(X)$. Then,*

(i) A is said to be contained in B, denoted by $A \subset B$, if, for each $x \in X$,

$$T_A(x) \leq T_B(x),\ I_A(x) \geq I_B(x) \text{ and } F_A(x) \geq F_B(x),$$

(ii) A is said to be equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

Definition 6 ([51]). *Let $A,\ B \in SVNS(X)$. Then,*

(i) the intersection of A and B, denoted by $A \cap B$, is a SVNS in X defined as:

$$A \cap B = (T_A \wedge T_B, I_A \vee I_B, F_A \vee F_B),$$

where $(T_A \wedge T_B)(x) = T_A(x) \wedge T_B(x)$, $(F_A \vee F_B) = F_A(x) \vee F_B(x)$, for each $x \in X$,

(ii) the union of A and B, denoted by $A \cup B$, is an SVNS in X defined as:

$$A \cup B = (T_A \vee T_B, I_A \wedge I_B, F_A \wedge F_B).$$

Remark 1. *Definitions 5 and 6 are different from the corresponding definitions in [48].*

Result 1 ([51], Proposition 2.1). *Let $A,\ B \in SVNS(X)$. Then,*

(1) *$A \subset A \cup B$ and $B \subset A \cup B$,*

(2) *$A \cap B \subset A$ and $A \cap B \subset B$,*

(3) *$(A^c)^c = A$,*

(4) *$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.*

The following are immediate results of Definitions 5 and 6.

Proposition 1. *Let $A,\ B,\ C \in SVNS(X)$. Then,*

(1) (Commutativity) *$A \cup B = B \cup A$, $A \cap B = B \cap A$,*

(2) (Associativity) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,
(3) (Distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(4) (Idempotency) $A \cup A = A$, $A \cap A = A$,
(5) (Absorption) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$,
(5) (DeMorgan's laws) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$,
(7) $A \cap 0_N = 0_N$, $A \cup 1_N = 1_N$,
(8) $A \cup 0_N = A$, $A \cap 1_N = A$.

Definition 7 (see [46]). *Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset SVNS(X)$. Then,*

(i) the union of $\{A_\alpha\}_{\alpha\in\Gamma}$, denoted by $\bigcup_{\alpha\in\Gamma} A_\alpha$, is a single valued neutrosophic set in X defined as follows: for each $x \in X$,

$$(\bigcup_{\alpha\in\Gamma} A_\alpha)(x) = (\bigvee_{\alpha\in\Gamma} T_{A_\alpha}(x), \bigwedge_{\alpha\in\Gamma} I_{A_\alpha}(x), \bigwedge_{\alpha\in\Gamma} F_{A_\alpha}(x)),$$

(ii) the intersection of $\{A_\alpha\}_{\alpha\in\Gamma}$, denoted by $\bigcap_{\alpha\in\Gamma} A_\alpha$, is a single valued neutrosophic set in X defined as follows: for each $x \in X$,

$$(\bigcap_{\alpha\in\Gamma} A_\alpha)(x) = (\bigwedge_{\alpha\in\Gamma} T_{A_\alpha}(x), \bigvee_{\alpha\in\Gamma} I_{A_\alpha}(x), \bigvee_{\alpha\in\Gamma} F_{A_\alpha}(x)).$$

The following are immediate results of the above definition.

Proposition 2. *Let $A \in SVNS(X)$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset SVNS(X)$. Then,*

(1) (Generalized Distributivity)

$$A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha), \quad A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha),$$

(2) (Generalized DeMorgan's laws)

$$(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A_\alpha^c, \quad (\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A_\alpha^c.$$

3. Ordinary Single Valued Neutrosophic Topology

In this section, we define an ordinary single valued neutrosophic topological space and obtain some of its properties. Throughout this paper, we denote the set of all subsets (resp. fuzzy subsets) of a set X as 2^X (resp. I^X).

For $T_\alpha, I_\alpha, F_\alpha \in I$, $\alpha = (T_\alpha, I_\alpha, F_\alpha) \in I \times I \times I$ is called a single valued neutrosophic value. For two single valued neutrosophic values α and β,

(i) $\alpha \le \beta$ iff $T_\alpha \le T_\beta$, $I_\alpha \ge I_\beta$ and $F_\alpha \ge F_\beta$,
(ii) $\alpha < \beta$ iff $T_\alpha < T_\beta$, $I_\alpha > I_\beta$ and $F_\alpha > F_\beta$.

In particular, the form $\alpha^* = (\alpha, 1-\alpha, 1-\alpha)$ is called a single valued neutrosophic constant.

We denote the set of all single valued neutrosophic values (resp. constant) as **SVNV** (resp. **SVNC**) (see [49]).

Definition 8. *Let X be a nonempty set. Then, a mapping $\tau = (T_\tau, I_\tau, F_\tau) : 2^X \to I \times I \times I$ is called an ordinary single valued neutrosophic topology (in short, osvnt) on X if it satisfies the following axioms: for any $A, B \in 2^X$ and each $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$,*

(OSVNT1) $\tau(\phi) = \tau(X) = (1, 0, 0)$,
(OSVNT2) $T_\tau(A \cap B) \ge T_\tau(A) \wedge T_\tau(B)$, $I_\tau(A \cap B) \le I_\tau(A) \vee I_\tau(B)$, $F_\tau(A \cap B) \le F_\tau(A) \vee F_\tau(B)$,
(OSVNT3) $T_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \ge \bigwedge_{\alpha\in\Gamma} T_\tau(A_\alpha)$, $I_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \le \bigvee_{\alpha\in\Gamma} I_\tau(A_\alpha)$, $F_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \le \bigvee_{\alpha\in\Gamma} F_\tau(A_\alpha)$.

The pair (X, τ) is called an ordinary single valued neutrosophic topological space (in short, osvnts). We denote the set of all ordinary single valued neutrosophic topologies on X as $OSVNT(X)$.

Let $2 = \{0, 1\}$ and let $\tau : 2^X \to 2 \times 2 \times 2$ satisfy the axioms in Definition 8. Since we can consider as $(1, 0, 0) = 1$ and $(0, 1, 1) = 0$, $\tau \in T(X)$, where $T(X)$ denotes the set of all classical topologies on X. Thus, we can see that $T(X) \subset OSVNT(X)$.

Example 1. *(1) Let $X = \{a, b, c\}$. Then, $2^X = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. We define the mapping $\tau : 2^X \to I \times I \times I$ as follows:*

$\tau(\phi) = \tau(X) = (1, 0, 0)$,
$\tau(\{a\}) = (0.7, 0.3, 0.4), \tau(\{b\}) = (0.6, 0.2, 0.3), \tau(\{c\}) = (0.8, 0.1, 0.2)$,
$\tau(\{a, b\}) = (0.6, 0.3, 0.4), \tau(\{b, c\}) = (0.7, 0.1, 0.2), \tau(\{a, c\}) = (0.8, 0.2, 0.3)$.

Then, we can easily see that $\tau \in OSVNT(X)$.

(2) Let X be a nonempty set. We define the mapping $\tau_\phi : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,

$$\tau_\phi(A) = \begin{cases} (1, 0, 0) & \textit{if} \text{ either } A = \phi \text{ or } A = X, \\ (0, 1, 1) & \textit{otherwise}. \end{cases}$$

Then, clearly, $\tau_\phi \in OSVT(X)$.

In this case, τ_ϕ (resp. (X, τ_ϕ)) is called the ordinary single valued neutrosophic indiscrete topology on X (resp. the ordinary single valued neutrosophic indiscrete space].

(3) Let X be a nonempty set. We define the mapping $\tau_X : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,

$$\tau_X(A) = (1, 0, 0).$$

Then, clearly, $\tau_X \in OSVNT(X)$.

In this case, τ_X (resp. (X, τ_X)) is called the ordinary single valued neutrosophic discrete topology on X (resp. the ordinary single valued neutrosophic discrete space].

*(4) Let X be a set and let $\alpha = (T_\alpha, I_\alpha, F_\alpha) \in$ **SVNV** be fixed, where $T_\alpha \in I_1$ and I_α, $F_\alpha \in I_0$. We define the mapping $\tau : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,*

$$\tau(A) = \begin{cases} (1, 0, 0) & \textit{if} \text{ either } A = \phi \text{ or } A^c \text{ is finite}, \\ \alpha & \textit{otherwise}. \end{cases}$$

Then, we can easily see that $\tau \in OSVNT(X)$.

In this case, τ is called the α-ordinary single valued neutrosophic finite complement topology on X and will be denoted by $OSVNCof(X)$. $OSVNCof(X)$ is of interest only when X is an infinite set because if X is finite, then $OSVNCof(X) = \tau_\phi$.

*(5) Let X be an infinite set and let $\alpha = (T_\alpha, I_\alpha, F_\alpha) \in$ **SVNV** be fixed, where $T_\alpha \in I_1$ and I_α, $F_\alpha \in I_0$. We define the mapping $\tau : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,*

$$\tau(A) = \begin{cases} (1, 0, 0) & \textit{if} \text{ either } A = \phi \text{ or } A^c \text{ is countable}, \\ \alpha & \textit{otherwise}. \end{cases}$$

Then, clearly, $\tau \in OSVNT(X)$.

In this case, τ is called the α-ordinary single valued neutrosophic countable complement topology on X and is denoted by $OSVNCoc(X)$.

*(6) Let T be the topology generated by $\mathcal{S} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$ as a subbase, let T_0 be the family of all open sets of $\mathbb{R}$ with respect to the usual topology on $\mathbb{R}$ and let $\alpha = (T_\alpha, I_\alpha, F_\alpha) \in$ **SVNV** be fixed, where $T_\alpha \in I_1$ and $I_\alpha,\ F_\alpha \in I_0$. We define the mapping $\tau : 2^{\mathbb{R}} \to I \times I \times I$ as follows: for each $A \in I^{\mathbb{R}}$,*

$$\tau(A) = \begin{cases} (1,0,0) & \text{if } A \in T_0, \\ \alpha & \text{if } A \in T \setminus T_0, \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Then, we can easily see that $\tau \in OSVNT(X)$.

(7) Let $T \in T(X)$. We define the mapping $\tau_T : 2^X \to I \times I \times I$ as follows : for each $A \in 2^X$,

$$\tau_T(A) = \begin{cases} (1,0,0) & \text{if } A \in T, \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Then, it is easily seen that $\tau_T \in OSVNT(X)$. Moreover, we can see that if T is the classical indiscrete topology, then $\tau_T = \tau_\phi$ and if T is the classical discrete topology, then $\tau_T = \tau_X$.

Remark 2. *(1) If $I = 2$, then we can think that Definition 8 also coincides with the known definition of classical topology.*

(2) Let (X, τ) be an osvnsts. We define two mappings $[\ ]\tau,\ <\ \ > \tau : 2^X \to I \times I \times I$, respectively, as follows : for each $A \in 2^X$,

$$([\ ]\tau)(A) = (T_\tau(A), I_\tau(A), 1 - T_\tau(A)),\ (<\ \ > \tau)(A) = (1 - F_\tau(A), I_\tau(A), F_\tau(A)).$$

Then, we can easily see that $[\ ]\tau,\ <\ \ > \tau \in OSVNT(X)$.

Definition 9. *Let X be a nonempty set. Then, a mapping $\mathcal{C} = (\mu_{\mathcal{C}}, \nu_{\mathcal{C}}) : 2^X \to I \times I \times I$ is called an ordinary single valued neutrosophic cotopology (in short, osvnct) on X if it satisfies the following conditions: for any $A, B \in 2^X$ and each $\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X$,*

(OSVNCT1) $\mathcal{C}(\phi) = \mathcal{C}(X) = (1,0,0)$,

(OSVNCT2) $T_{\mathcal{C}}(A \cup B) \geq T_{\mathcal{C}}(A) \wedge T_{\mathcal{C}}(B)$, $\quad I_{\mathcal{C}}(A \cup B) \leq I_{\mathcal{C}}(A) \vee I_{\mathcal{C}}(B)$,
$F_{\mathcal{C}}(A \cup B) \leq F_{\mathcal{C}}(A) \vee F_{\mathcal{C}}(B)$,

(OSVNCT3) $T_{\mathcal{C}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} T_{\mathcal{C}}(A_\alpha)$, $\quad I_{\mathcal{C}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \leq \bigvee_{\alpha \in \Gamma} I_{\mathcal{C}}(A_\alpha)$,
$F_{\mathcal{C}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \leq \bigvee_{\alpha \in \Gamma} F_{\mathcal{C}}(A_\alpha)$.

The pair $(X, \mathcal{C})$ is called an ordinary single valued neutrosophic cotopological space (in short, osvncts).

The following is an immediate result of Definitions 8 and 9.

Proposition 3. *We define two mappings $f : OSVNT(X) \to OSVNCT(X)$ and $g : OSVNCT(X) \to OSVNT(X)$ respectively as follows:*

$$[f(\tau)](A) = \tau(A^c) \quad \text{for any } \tau \in OSVNT(X) \text{ and any } A \in 2^X$$

and

$$[g(\mathcal{C})](A) = \mathcal{C}(A^c) \quad \text{for any } \mathcal{C} \in OSVNCT(X) \text{ and any } A \in 2^X.$$

Then, f and g are well-defined. Moreover, $g \circ f = 1_{OSVNT(X)}$ and $f \circ g = 1_{OSVNCT(X)}$.

Remark 3. *(1) For each $\tau \in OSVNT(X)$ and each $\mathcal{C} \in OSVNCT(X)$, let $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_{\mathcal{C}}$. Then, from Proposition 3, we can see that $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_{\mathcal{C}}} = \mathcal{C}$.*

(2) Let $(X, \mathcal{C})$ be an osvncts. We define two mappings $[\ ]\mathcal{C},\ <\ \ >\mathcal{C} : 2^X \to I \times I \times I$, respectively, as follows: for each $A \in 2^X$,

$$([\ ]\mathcal{C})(A) = (T_{\mathcal{C}}(A), I_{\mathcal{C}}(A), 1 - T_{\mathcal{C}}(A)),\ (<\ \ >\mathcal{C})(A) = (1 - F_{\mathcal{C}}(A), I_{\mathcal{C}}(A), F_{\mathcal{C}}(A)).$$

Then, we can easily see that $[\ ]\mathcal{C},\ <\ \ >\mathcal{C} \in OSVNCT(X)$.

Definition 10. *Let $\tau_1, \tau_2 \in OSVNT(X)$ and let $\mathcal{C}_1, \mathcal{C}_2 \in OSVNCT(X)$.*

(i) We say that τ_1 is finer than τ_2 or τ_2 is coarser than τ_1, denoted by $\tau_2 \preceq \tau_1$, if $\tau_2(A) \leq \tau_1(A)$, i.e., for each $A \in 2^X$,

$$T_{\tau_2}(A) \leq T_{\tau_1}(A),\ I_{\tau_2}(A) \geq I_{\tau_1}(A),\ F_{\tau_2}(A) \geq F_{\tau_1}(A).$$

(ii) We say that $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ or $\mathcal{C}_2$ is coarser than $\mathcal{C}_1$, denoted by $\mathcal{C}_2 \preceq \mathcal{C}_1$, if $\mathcal{C}_2(A) \leq \mathcal{C}_1(A)$, i.e., for each $A \in 2^X$,

$$T_{\mathcal{C}_2}(A) \leq T_{\mathcal{C}_1}(A),\ I_{\mathcal{C}_2}(A) \geq I_{\mathcal{C}_1}(A),\ F_{\mathcal{C}_2}(A) \geq F_{\mathcal{C}_1}(A).$$

We can easily see that τ_1 is finer than τ_2 if and only if $\mathcal{C}_{\tau_1}$ is finer than $\mathcal{C}_{\tau_2}$, and $(OSVNT(X), \preceq)$ and $(OSVNCT(X), \preceq)$ are posets, respectively.

From Example 1 (2) and (3), it is obvious that τ_ϕ is the coarsest ordinary single valued neutrosophic topology on X and τ_X is the finest ordinary single valued neutrosophic topology on X.

Proposition 4. *If $\{\tau_\alpha\}_{\alpha \in \Gamma} \subset OSVNT(X)$, then $\bigcap_{\alpha \in \Gamma} \tau_\alpha \in OSVNT(X)$, where $[\bigcap_{\alpha \in \Gamma} \tau_\alpha](A) = (\bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(A), \bigvee_{\alpha \in \Gamma} I_{\tau_\alpha}(A), \bigvee_{\alpha \in \Gamma} F_{\tau_\alpha}(A)),\ \forall\ A \in 2^X$.*

Proof. Let $\tau = \bigcap_{\alpha \in \Gamma} \tau_\alpha$ and let $\alpha \in \Gamma$. Since $\tau_\alpha \in OSVNT(X)$, $\tau_\alpha(X) = \tau_\alpha(\phi) = (1, 0, 0)$, i.e.,

$$T_{\tau_\alpha}(X) = T_{\tau_\alpha}(\phi) = 1, \quad I_{\tau_\alpha}(X) = I_{\tau_\alpha}(\phi) = 0, \quad F_{\tau_\alpha}(X) = F_{\tau_\alpha}(\phi) = 0.$$

Then, $T_\tau(X) = \bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(X) = 1$, $I_\tau(X) = \bigvee_{\alpha \in \Gamma} I_{\tau_\alpha}(X) = 0 = F_\tau(X)$. Similarly, we have $T_\tau(\phi) = 1$, $I_\tau(\phi) = 0 = F_\tau(\phi)$. Thus, the condition (OSVNT1) holds.

Let $A, B \in 2^X$. Then,

$$\begin{aligned} T_\tau(A \cap B) &= \textstyle\bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(A \cap B) && \text{[By the definition of } \tau] \\ &\geq \textstyle\bigwedge_{\alpha \in \Gamma}(T_{\tau_\alpha}(A) \wedge T_{\tau_\alpha}(B)) && [\text{Since } \tau_\alpha \in OSVNT(X)] \\ &= (\textstyle\bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(A)) \wedge (\textstyle\bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(B)) \\ &= T_\tau(A) \wedge T_\tau(B) && \text{[By the definition of } \tau] \end{aligned}$$

and

$$\begin{aligned} I_\tau(A \cap B) &= \textstyle\bigvee_{\alpha \in \Gamma} I_{\tau_\alpha}(A \cap B) && \text{[By the definition of } \tau] \\ &\leq \textstyle\bigvee_{\alpha \in \Gamma}(I_{\tau_\alpha}(A) \vee I_{\tau_\alpha}(B)) && [\text{Since } \tau_\alpha \in OSVNT(X)] \\ &= (\textstyle\bigvee_{\alpha \in \Gamma} I_{\tau_\alpha}(A)) \vee (\textstyle\bigvee_{\alpha \in \Gamma} I_{\tau_\alpha}(B)) \\ &= I_\tau(A) \vee I_\tau(B). && \text{[By the definition of } \tau] \end{aligned}$$

Similarly, we have $F_\tau(A \cap B) \leq F_\tau(A) \vee F_\tau(B)$. Thus, the condition (OSVNT2) holds:

Now, let $\{A_j\}_{j \in J} \subset 2^X$. Then,

$$\begin{aligned} T_\tau(\textstyle\bigcup_{j \in J} A_j) &= \textstyle\bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(\textstyle\bigcup_{j \in J} A_j) && \text{[By the definition of } \tau] \\ &\geq \textstyle\bigwedge_{\alpha \in \Gamma}(\textstyle\bigwedge_{j \in J} T_{\tau_\alpha}(A_j)) && [\text{Since } \tau_\alpha \in OSVNT(X)] \\ &= \textstyle\bigwedge_{j \in J}(\textstyle\bigwedge_{\alpha \in \Gamma} T_{\tau_\alpha}(A_j)) \\ &= \textstyle\bigwedge_{j \in J}[\textstyle\bigcap_{\alpha \in \Gamma} T_{\tau_\alpha}](A_j) && \text{[By the definition of } \tau] \\ &= \textstyle\bigvee_{j \in J} T_\tau(A_j) \end{aligned}$$

and

$$\begin{aligned}
I_\tau(\bigcup_{j\in J} A_j) &= \bigvee_{\alpha\in\Gamma} I_{\tau_\alpha}(\bigcup_{j\in J} A_j) && \text{[By the definition of } \tau]\\
&\le \bigvee_{\alpha\in\Gamma}(\bigvee_{j\in J} I_{\tau_\alpha}(A_j)) && [\text{Since } \tau_\alpha \in OSVNT(X)]\\
&= \bigvee_{j\in J}(\bigvee_{\alpha\in\Gamma} I_{\tau_\alpha}(A_j))\\
&= \bigvee_{j\in J}[\bigcup_{\alpha\in\Gamma} I_{\tau_\alpha}](A_j) && \text{[By the definition of } \tau]\\
&= \bigvee_{j\in J} I_\tau(A_j).
\end{aligned}$$

Similarly, we have $F_\tau(\bigcup_{j\in J} A_j) \le \bigvee_{j\in J} F_\tau(A_j)$. Thus, the condition (OSVNT3) holds. This completes the proof. □

From Definition 10 and Proposition 4, we have the following.

Proposition 5. *$(OSVNT(X), \preceq)$ is a meet complete lattice with the least element τ_ϕ and the greatest element τ_X.*

Definition 11. *Let (X, τ) be an osvnts and let $\alpha \in$ **SVNV**. We define two sets $[\tau]_\alpha$ and $[\tau]^*_\alpha$ as follows, respectively:*

(i) $[\tau]_\alpha = \{A \in 2^X : T_\tau(A) \ge T_\alpha,\ I_\tau(A) \le I_\alpha,\ I_\tau(A) \le F_\alpha\}$,

(ii) $[\tau]^*_\alpha = \{A \in 2^X : T_\tau(A) > T_\alpha,\ I_\tau(A) < I_\alpha,\ F_\tau(A) < F_\alpha\}$.

In this case, $[\tau]_\alpha$ (resp. $[\tau]^*_\alpha$) is called the α-level (resp. strong α-level] of τ. If $\alpha = (0, 1, 1)$, then $[\tau]_{(0,1,1)} = 2^X$, i.e., $[\tau]_{(0,1,1)}$ is the classical discrete topology on X and if $\alpha = (1, 0, 0)$, then $[\tau]^*_{(1,0,0)} = \phi$. Moreover, we can easily see that for any $\alpha \in$ **SVNV**, $[\tau]^*_\alpha \subset [\tau]_\alpha$.

Lemma 1. *Let $\tau \in OSVNT(X)$ and let $\alpha, \beta \in$ **SVNV**. Then,*

(1) $[\tau]_\alpha \in T(X)$,

(2) *if* $\alpha \le \beta$, *then* $[\tau]_\beta \subset [\tau]_\alpha$,

(3) $[\tau]_\alpha = \bigcap_{\beta<\alpha} [\tau]_\beta$, *where* $\alpha \in I_0 \times I_1 \times I_1$,

$(1)'$ $[\tau]^*_\alpha \in T(X)$, *where* $\alpha \in I_1 \times I_0 \times I_0$,

$(2)'$ *if* $\alpha \le \beta$, *then* $[\tau]^*_\beta \subset [\tau]^*_\alpha$,

$(3)'$ $[\tau]^*_\alpha = \bigcup_{\beta>(\alpha} [\tau]^*_\beta$, *where* $\alpha \in I_1 \times I_0 \times I_0$.

Proof. The proofs of (1), $(1)'$, (2) and $(2)'$ are obvious from Definitions 8 and 11.

(3) From (2), $\{[\tau]_\alpha\}_{\alpha\in I_0\times I_1\times I_1}$ is a descending family of classical topologies on X. Then, clearly, $[\tau]_\alpha \subset \bigcap_{\beta<\alpha}[\tau]_\beta$, for each $\alpha \in I_0 \times I_1 \times I_1$.

Suppose $A \notin [\tau]_\alpha$. Then, $T_\tau(A) < T_\alpha$ or $I_\tau(A) > I_\alpha$ or $F_\tau(A) > F_\alpha$. Thus,

there exists $T_\beta \in I_0$ such that $T_\tau(A) < T_\beta < T_\alpha$

or

there exists $I_\beta \in I_1$ such that $I_\tau(A) > I_\beta > I_\alpha$

or

there exists $F_\beta \in I_1$ such that $F_\tau(A) > F_\beta > F_\alpha$.

Thus, $A \notin [\tau]_\beta$, for some $\beta \in$ **SVNV** such that $\beta < \alpha$, i.e., $A \notin \bigcap_{\beta<\alpha} [\tau]_\beta$. Hence, $\bigcap_{\beta<\alpha} [\tau]_\beta \subset [\tau]_\alpha$.

Therefore, $[\tau]_\alpha = \bigcap_{\beta<\alpha} [\tau]_\beta$.

$(3)'$ The proof is similar to (3). □

Remark 4. *From (1) and (2) in Lemma 1, we can see that, for each* $\tau \in OSVNT(X)$, $\{[\tau]_\alpha\}_{\alpha \in \mathbf{SVNV}}$ *is a family of descending classical topologies called the α-level classical topologies on X with respect to τ.*

The following is an immediate result of Lemma 1.

Corollary 1. *Let* (X, τ) *be an osvnts. Then,* $[\tau]_{\alpha^*} = \bigcap_{\beta<\alpha} [\tau]_{\beta^*}$ *for each* $\alpha^* \in \mathbf{SVNC}$*, where* $\alpha \in I_0$.

Lemma 2. (1) *Let* $\{\tau_\alpha\}_{\alpha \in \mathbf{SVNV}}$ *be a descending family of classical topologies on X such that* $\tau_{(0,1,1)}$ *is the classical discrete topology on X. We define the mapping* $\tau : 2^X \to I \times I \times I$ *as follows: for each* $A \in 2^X$,

$$\tau(A) = (\bigvee_{A\in\tau_\alpha} T_\alpha, \bigwedge_{A\in\tau_\alpha} I_\alpha, \bigwedge_{A\in\tau_\alpha} F_\alpha).$$

Then, $\tau \in OSVNT(X)$.
(2) *If* $\tau_\alpha = \bigcap_{\beta<\alpha} \tau_\alpha$*, for each* $\alpha \in \mathbf{SVNV}$ $(\alpha \in I_0 \times I_1 \times I_1)$*, then* $[\tau]_\alpha = \tau_\alpha$.
(3) *If* $\tau_\alpha = \bigcup_{\beta>\alpha} \tau_\beta$*, for each* $\alpha \in \mathbf{SVNV}$ $(\alpha \in I_1 \times I_0 \times I_0)$*, then* $[\tau]^*_\alpha = \tau_\alpha$.

Proof. The proof is similar to Lemma 3.9 in [28]. □

The following is an immediate result of Lemma 2.

Corollary 2. *Let* $\{\tau_{\alpha^*}\}_{\alpha\in I_0}$ *be a descending family of classical topologies on X such that* $\tau_{(0,1,1)}$ *is the classical discrete topology on X. We define the mapping* $\tau : 2^X \to I \times I \times I$ *as follows: for each* $A \in 2^X$,

$$\tau(A) = (\bigvee_{A\in\tau_{\alpha^*}} \alpha, \bigwedge_{A\in\tau_{\alpha^*}} (1-\alpha), \bigwedge_{A\in\tau_{\alpha^*}} (1-\alpha)).$$

Then, $\tau \in OSVNT(X)$ *and* $[\tau]_{\alpha^*} = \bigcap_{\beta<\alpha} \tau_{\beta^*} = \tau_{\alpha^*}\ \forall\, \alpha \in I_0$.

From Lemmas 1 and 2, we have the following result.

Proposition 6. *Let* $\tau \in OSVNT(X)$ *and let* $[\tau]_\alpha$ *be the α-level classical topology on X with respect to τ. We define the mapping* $\eta : 2^X \to I \times I \times I$ *as follows: for each* $A \in 2^X$,

$$\eta(A) = (\bigvee_{A\in[\tau]_\alpha} T_\alpha, \bigwedge_{A\in[\tau]_\alpha} I_\alpha, \bigwedge_{A\in[\tau]_\alpha} F_\alpha).$$

Then, $\eta = \tau$.

The fact that an ordinary single valued neutrosophic topological space fully determined by its decomposition in classical topologies is restated in the following theorem.

Theorem 1. *Let* $\tau_1, \tau_2 \in OSVNT(X)$*. Then,* $\tau_1 = \tau_2$ *if and only if* $[\tau_1]_\alpha = [\tau_2]_\alpha$ *for each* $\alpha \in \mathbf{SVNV}$*, or alternatively, if and only if* $[\tau_1]^*_\alpha = [\tau_2]^*_\alpha$ *for each* $\alpha \in \mathbf{SVNV}$.

Remark 5. *In a similar way, we can construct an ordinary single valued neutrosophic cotopology* $\mathcal{C}$ *on a set X, by using the α-levels,*

$$[\mathcal{C}]_\alpha = \{A \in I^X : T_\mathcal{C}(A) \geq T_\alpha,\ I_\mathcal{C}(A) \leq I_\alpha,\ F_\mathcal{C}(A) \leq F_\alpha\}$$

and

$$[\mathcal{C}]^*_\alpha = \{A \in I^X : T_\mathcal{C}(A) > T_\alpha,\ I_\mathcal{C}(A) < I_\alpha,\ F_\mathcal{C}(A) < F_\alpha\},$$

for each $\alpha \in \mathbf{SVNV}$.

Definition 12. *Let $T \in T(X)$ and let $\tau \in OSVNT(X)$. Then, τ is said to be compatible with T if $T = S(\tau)$, where $S(\tau) = \{A \in 2^X : T_\tau(A) > 0, I_\tau(A) < 1,\ F_\tau(A) < 1\}$.*

Example 2. *(1) Let τ_ϕ be the ordinary single valued neutrosophic indiscrete topology on a nonempty set X and let T_0 be the classical indiscrete topology on X. Then, clearly,*

$$S(\tau_\phi) = \{A \in 2^X : T_{\tau_\phi}(A) > 0, I_{\tau_\phi}(A) < 1,\ F_{\tau_\phi}(A) < 1\} = \{\phi, X\} = T_0.$$

Thus, τ_ϕ is compatible with T_0.

(2) Let τ_X be the ordinary single valued neutrosophic discrete topology on a nonempty set X and let T_1 be the classical discrete topology on X. Then, clearly,

$$S(\tau_X) = \{A \in 2^X : T_{\tau_X}(A) > 0, I_{\tau_X}(A) < 1,\ F_{\tau_X}(A) < 1\} = 2^X = T_1.$$

Thus, τ_X is compatible with T_1.

*(3) Let X be a nonempty set and let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_0 \times I_1 \times I_1$. We define the mapping $\tau : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,*

$$\tau(A) = \begin{cases} (1,0,0) & \textit{if} \text{ either } A = \phi \text{ or } A = X, \\ \alpha & \text{otherwise.} \end{cases}$$

Then, clearly, $\tau \in OSVNT(X)$ and τ is compatible with T_1.

Furthermore, every classical topology can be considered as an ordinary single valued neutrosophic topology in the sense of the following result.

Proposition 7. *Let (X, τ) be a classical topological space and and let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_0 \times I_1 \times I_1$. Then, there exists $\tau^\alpha \in OSVNT(X)$ such that τ^α is compatible with T. Moreover, $[\tau^\alpha]_\alpha = \tau$.*

In this case, τ^α is called the α-th ordinary single valued neutrosophic topology on X and (X, τ^α) is called the α-th ordinary single valued neutrosophic topological space.

Proof. Let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_0 \times I_1 \times I_1$ and we define the mapping $\tau^\alpha : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,

$$\tau^\alpha(A) = \begin{cases} (1,0,0) & \text{if either } A = \phi \text{ or } A = X, \\ \alpha & \text{if } A \in \tau \setminus \{\phi, X\}, \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Then, we can easily see that $\tau^\alpha \in OSVNT(X)$ and $[\tau^\alpha]_\alpha = \tau$. Moreover, by the definition of τ^α,

$$S(\tau^\alpha) = \{A \in 2^X : T_{\tau^\alpha}(A) > 0,\ I_{\tau^\alpha}(A) < 1,\ F_{\tau^\alpha}(A) < 1\} = \tau.$$

Thus, τ^α is compatible with τ. □

Proposition 8. *Let (X, T) be a classical topological space, let $C(T)$ be the set of all osvnts on X compatible with T, let $\widetilde{T} = T \setminus \{\phi, X\}$ and let $(I \times I \times I)^{\widetilde{T}}_{(0,1,1)}$ be the set of all mappings $f : \widetilde{T} \to I \times I \times I$ satisfying the following conditions: for any $A,\ B \in \widetilde{T}$ and each $(A_j)_{j \in J} \subset \widetilde{T}$,*

(1) $f(A) \neq (0,1,1)$,

(2) $T_f(A \cap B) \geq T_f(A) \wedge T_f(B)$, $\quad I_f(A \cap B) \leq I_f(A) \vee T_f(B)$,
$F_f(A \cap B) \leq F_f(A) \vee F_f(B)$,

(3) $T_f(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} T_f(A_j)$, $\quad I_f(\bigcup_{j \in J} A_j) \leq \bigvee_{j \in J} I_f(A_j)$,
$F_f(\bigcup_{j \in J} A_j) \leq \bigvee_{j \in J} F_f(A_j)$.

Then, there is a one-to-one correspondence between $C(T)$ and $(I \times I \times I)^{\widetilde{T}}_{(0,1,1)}$.

Proof. We define the mapping $F : (I \times I \times I)^{\widetilde{T}}_{(0,1,1)} \to C(T)$ as follows: for each $f \in (I \times I \times I)^{\widetilde{T}}_{(0,1,1)}$,

$$F(f) = \tau_f,$$

where $\tau_f : 2^X \to I \times I \times I$ is the mapping defined by: for each $A \in 2^X$,

$$\tau_f(A) = \begin{cases} (1,0,0) & \text{if either } A = \phi \text{ or } A = X, \\ f(A) & \text{if } A \in \widetilde{T}, \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Then, we easily see that $\tau_f \in C(T)$.

Now, we define the mapping $G : C(T) \to (I \times I \times I)^{\widetilde{T}}_{(0,1,1)}$ as follows: for each $\tau \in C(T)$,

$$G(\tau) = f_\tau,$$

where $f_\tau : \widetilde{T} \to I \times I \times I$ is the mapping defined by: for each $A \in \widetilde{T}$,

$$f_\tau(A) = \tau(A).$$

Then, clearly, $f_\tau \in (I \times I \times I)^{\widetilde{T}}_{(0,1,1)}$. Furthermore, we can see that $F \circ G = id_{C(T)}$ and $G \circ F = id_{(I \times I \times I)^{\widetilde{T}}_{(0,1,1)}}$. Thus, $C(T)$ is equipotent to $I \times I \times I)^{\widetilde{T}}_{(0,1,1)}$. This completes the proof. □

Proposition 9. *Let (X, τ) be an osvnts and let $Y \subset X$. We define the mapping $\tau_Y : 2^Y \to I \times I \times I$ as follows: for each $A \in 2^Y$,*

$$\tau_Y(A) = \Big(\bigvee_{B \in 2^X,\ A = B \cap Y} T_\tau(B), \bigwedge_{B \in 2^X,\ A = B \cap Y} I_\tau(B), \bigwedge_{B \in 2^X,\ A = B \cap Y} F_\tau(B)\Big).$$

Then, $\tau_Y \in OSVNT(Y)$ and for each $A \in 2^Y$,

$$T_{\tau_Y}(A) \geq T_\tau(A),\ I_{\tau_Y}(A) \leq I_\tau(A),\ F_{\tau_Y}(A) \leq F_\tau(A).$$

In this case, (Y, τ_Y) is called an ordinary single valued neutrosophic subspace of (X, τ) and τ_Y is called the induced ordinary single valued neutrosophic topology on A by τ.

Proof. It is obvious that the condition (OSVNT1) holds, i.e., $\tau_Y(\phi) = \tau_Y(Y) = (1,0,0)$.

Let $A, B \in 2^Y$. Then, by proof of Proposition 5.1 in [34], $T_{\tau_Y}(A \cap B) \geq T_{\tau_Y}(A) \wedge T_{\tau_Y}(B)$.

Let us show that $I_{\tau_Y}(A \cap B) \leq I_{\tau_Y}(A) \vee I_{\tau_Y}(B)$. Then,

$$\begin{aligned} I_{\tau_Y}(A) \vee I_{\tau_Y}(B) &= (\textstyle\bigwedge_{C_1 \in 2^X,\ A = Y \cap C_1} I_\tau(C_1)) \vee (\bigwedge_{C_2 \in 2^X,\ B = Y \cap C_2} I_\tau(C_2)) \\ &= \textstyle\bigwedge_{C_1,\ C_1 \in 2^X,\ A \cap B = Y \cap (C_1 \cap C_2)} [I_\tau(C_1) \vee I_\tau(C_2)] \\ &\geq \textstyle\bigwedge_{C_1,\ C_1 \in 2^X,\ A \cap B = Y \cap (C_1 \cap C_2)} I_\tau(C_1 \cap C_2) \\ &= I_{\tau_Y}(A \cap B). \end{aligned}$$

Similarly, we have $F_{\tau_Y}(A \cap B) \leq F_{\tau_Y}(A) \vee F_{\tau_Y}(B)$. Thus, the condition (OSVNT2) holds.

Now, let $\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^Y$. Then, by the proof of Proposition 5.1 in [34], $T_{\tau_Y}(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} T_{\tau_Y}(A_\alpha)$. On the other hand,

$$\begin{aligned} I_{\tau_Y}(\textstyle\bigcup_{\alpha \in \Gamma} A_\alpha) &= \textstyle\bigwedge_{B_\alpha \in 2^X,\ (\bigcup_{\alpha \in \Gamma} B_\alpha) \cap Y = \bigcup_{\alpha \in \Gamma} A_\alpha} I_\tau(\bigcup_{\alpha \in \Gamma} B_\alpha) \\ &\leq \textstyle\bigwedge_{B_\alpha \in 2^X,\ (\bigcup_{\alpha \in \Gamma} B_\alpha) \cap Y = \bigcup_{\alpha \in \Gamma} A_\alpha} [\bigwedge_{\alpha \in \Gamma} I_\tau(B_\alpha)] \\ &= \textstyle\bigwedge_{\alpha \in \Gamma} [\bigwedge_{B_\alpha \in 2^X,\ (\bigcup_{\alpha \in \Gamma} B_\alpha) \cap Y = \bigcup_{\alpha \in \Gamma} A_\alpha} I_\tau(B_\alpha)] \end{aligned}$$

$= \bigwedge_{\alpha\in\Gamma} I_{\tau_Y}(A_\alpha)$.

Similarly, we have $F_{\tau_Y}(\bigcup_{\alpha\in\Gamma} A_\alpha) \leq \bigwedge_{\alpha\in\Gamma} F_{\tau_Y}(A_\alpha)$. Thus, the condition (OSVNT3) holds. Thus, $\tau_Y \in OSVNT(Y)$.

Furthermore, we can easily see that for each $A \in 2^Y$,

$$T_{\tau_Y}(A) \geq T_\tau(A), \quad I_{\tau_Y}(A) \leq I_\tau(A), \quad F_{\tau_Y}(A) \leq F_\tau(A).$$

This completes the proof. □

The following is an immediate result of Proposition 9.

Corollary 3. *Let (Y, τ_Y) be an ordinary single valued neutrosaophic subspace of (X, τ) and let $A \in 2^Y$.*

(1) $\mathcal{C}_Y(A) = (\bigvee_{B\in 2^X, A=B\cap Y} T_{\mathcal{C}}(B), \bigwedge_{B\in 2^X, A=B\cap Y} I_{\mathcal{C}}(B), \bigwedge_{B\in 2^X, A=B\cap Y} F_{\mathcal{C}}(B))$, *where* $\mathcal{C}_Y(A) = \tau_Y(Y - A)$.

(2) *If $Z \subset Y \subset X$, then $\tau_Z = (\tau_Y)_Z$.*

4. Ordinary Single Valued Neutrosophic Neighborhood Structures of a Point

In this section, we define an ordinary single valued neutrosophic neighborhood system of a point, and prove that it has the same properties in a classical neighborhood system.

Definition 13. *Let (X, τ) be an osvnts and let $x \in X$. Then, a mapping $\mathcal{N}_x : 2^X \to I \times I \times I$ is called the ordinary single valued neutrosophic neighborhood system of x if, for each $A \in 2^X$,*

$$A \in \mathcal{N}_x := \exists B(B \in \tau) \wedge (x \in B \subset A)),$$

i.e.,

$$[A \in \mathcal{N}_x] = \mathcal{N}_x(A) = (\bigvee_{x\in B\subset A} T_\tau(B), \bigwedge_{x\in B\subset A} I_\tau(B), \bigwedge_{x\in B\subset A} F_\tau(B)).$$

Lemma 3. *Let (X, τ) be an osvnts and let $A \in 2^X$. Then,*

$$\bigwedge_{x\in A} \bigvee_{x\in B\subset A} T_\tau(B) = T_\tau(A),$$

$$\bigvee_{x\in A} \bigwedge_{x\in B\subset A} I_\tau(B) = I_\tau(A)$$

and

$$\bigvee_{x\in A} \bigwedge_{x\in B\subset A} F_\tau(B) = F_\tau(A).$$

Proof. By Theorem 3.1 in [33], it is obvious that $\bigwedge_{x\in A} \bigvee_{x\in B\subset A} T_\tau(B) = T_\tau(A)$.

On the other hand, it is clear that $\bigvee_{x\in A} \bigwedge_{x\in B\subset A} I_\tau(B) \geq I_\tau(A)$. Now, let $\mathcal{B}_x = \{B \in 2^X : x \in B \subset A\}$ and let $f \in \Pi_{x\in A}\mathcal{B}_x$. Then, clearly, $\bigcup_{x\in A} f(x) = A$. Thus,

$$\bigvee_{x\in A} I_\tau(f(x)) \leq I_\tau(\bigcup_{x\in A} f(x)) = I_\tau(A).$$

Thus,

$$\bigvee_{x\in A} \bigwedge_{x\in B\subset A} I_\tau(B) = \bigwedge_{f\in\Pi_{x\in A}} \bigvee_{x\in A} I_\tau(f(x)) \leq I_\tau(A).$$

Hence, $\bigvee_{x\in A} \bigwedge_{x\in B\subset A} I_\tau(B) = I_\tau(A)$. Similarly, we have

$$\bigvee_{x\in A} \bigwedge_{x\in B\subset A} F_\tau(B) = F_\tau(A).$$

□

Theorem 2. *Let (X,τ) be an osvnts, let $A \in 2^X$ and let $x \in X$. Then,*

$$\models (A \in \tau) \leftrightarrow \forall x(x \in A \to \exists B(B \in \mathcal{N}_x) \wedge (B \subset A)),$$

i.e.,

$$[A \in \tau] = [\forall x(x \in A \to \exists B(B \in \mathcal{N}_x) \wedge (B \subset A))],$$

i.e.,

$$[A \in \tau] = (\bigwedge_{x\in A}\bigvee_{B\subset A} T_{\mathcal{N}_x}(B), \bigvee_{x\in A}\bigwedge_{B\subset A} I_{\mathcal{N}_x}(B), \bigvee_{x\in A}\bigwedge_{B\subset A} F_{\mathcal{N}_x}(B)).$$

Proof. From Theorem 3.1 in [33], it is clear that $T_\tau(A) = \bigwedge_{x\in A}\bigvee_{B\subset A} T_{\mathcal{N}_x}(B)$.

On the other hand,

$$\begin{aligned} I_\tau(A) &= \textstyle\bigvee_{x\in A}\bigwedge_{x\in C\subset A} I_\tau(C) && \text{[By Lemma 3]} \\ &= \textstyle\bigvee_{x\in A}\bigwedge_{B\subset A}\bigwedge_{x\in C\subset B} I_\tau(C) \\ &= \textstyle\bigvee_{x\in A}\bigwedge_{B\subset A} I_{\mathcal{N}_x}(B). && \text{[By Definition 13]} \end{aligned}$$

Similarly, we have $F_\tau(A) = \bigvee_{x\in A}\bigwedge_{B\subset A} F_{\mathcal{N}_x}(B)$. This completes the proof. □

Definition 14. *Let $\mathcal{A}$ be a single valued neutrosophic set in a set 2^X. Then, $\mathcal{A}$ is said to be normal if there is $A_0 \in 2^X$ such that $\mathcal{A}(A_0) = (1,0,0)$.*

We will denote the set of all normal single valued neutrosophic sets in 2^X as $(I \times I \times I)_N^{2^X}$.

From the following result, we can see that an ordinary single valued neutrosophic neighborhood system has the same properties in a classical neighborhood system.

Theorem 3. *Let (X,τ) be an osvnts and let $\mathcal{N} : X \to (I \times I \times I)_N^{2^X}$ be the mapping given by $\mathcal{N}(x) = \mathcal{N}_x$, for each $x \in X$. Then, $\mathcal{N}$ has the following properties:*

(1) *for any $x \in X$ and $A \in 2^X$, $\models A \in \mathcal{N}_x \to x \in A$,*

(2) *for any $x \in X$ and $A, B \in 2^X$, $\models (A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x) \to A \cap B \in \mathcal{N}_x$,*

(3) *for any $x \in X$ and $A, B \in 2^X$, $\models (A \subset B) \to (A \in \mathcal{N}_x \to B \in \mathcal{N}_x)$,*

(4) *for any $x \in X$, $\models (A \in \mathcal{N}_x) \to \exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))$.*

Conversely, if a mapping $\mathcal{N} : X \to (I \times I \times I)_N^{2^X}$ satisfies the above properties (2) and (3), then there is an ordinary single valued neutrosophic topology $\tau : 2^X \to I \times I \times I$ on X defined as follows: for each $A \in 2^X$,

$$A \in \tau := \forall x(x \in A \to A \in \mathcal{N}_x),$$

i.e.,

$$[A \in \tau] = \tau(A) = (\bigwedge_{x\in A} T_{\mathcal{N}_x}(A), \bigvee_{x\in A} I_{\mathcal{N}_x}(A), \bigvee_{x\in A} F_{\mathcal{N}_x}(A)).$$

In particular, if $\mathcal{N}$ also satisfies the above properties (1) and (4), then, for each $x \in X$, $\mathcal{N}_x$ is an ordinary single valued neutrosophic neighborhood system of x with respect to τ.

Proof. (1) Since $A \in 2^X$, we can consider A as a special single valued neutrosophic set in x represented by $A = (\chi_A, \chi_{A^c}, \chi_{A^c})$. Then,

$$[x \in A] = A(x) = (\chi_A(x), \chi_{A^c}(x), \chi_{A^c}(x)) = (1,0,0).$$

On the other hand,

$$[A \in \mathcal{N}_x] = (\bigvee_{x\in C\subset A} T_\tau(C), \bigwedge_{x\in C\subset A} I_\tau(C), \bigwedge_{x\in C\subset A} F_\tau(C)) \le (1,0,0).$$

Thus, $[A \in \mathcal{N}_x] \le [x \in A]$.
(2) By the definition of $\mathcal{N}_x$,

$$[A\cap B \in \mathcal{N}_x] = (\bigvee_{x\in C\subset A\cap B} T_\tau(C), \bigwedge_{x\in C\subset A\cap B} I_\tau(C)), \bigwedge_{x\in C\subset A\cap B} F_\tau(C)).$$

From the proof of Theorem 3.2 (2) in [33], it is obvious that

$$T_{\mathcal{N}_x}(A\cap B) \ge T_{\mathcal{N}_x}(A) \wedge T_{\mathcal{N}_x}(B).$$

Thus, it is sufficient to show that $I_{\mathcal{N}_x}(A\cap B) \le I_{\mathcal{N}_x}(A) \vee I_{\mathcal{N}_x}(B)$:

$$\begin{aligned} I_{\mathcal{N}_x}(A\cap B) &= \textstyle\bigwedge_{x\in C\subset A\cap B} I_\tau(C) = \bigwedge_{x\in C_1\subset A,\ x\in C_2\subset B} I_\tau(C_1\cap C_2) \\ &\le \textstyle\bigwedge_{x\in C_1\subset A,\ x\in C_2\subset B}(I_\tau(C_1) \vee I_\tau(C_2)) \\ &= \textstyle\bigwedge_{x\in C_1\subset A} I_\tau(C_1) \vee \bigwedge_{x\in C_2\subset B} I_\tau(C_2) \\ &= I_{\mathcal{N}_x}(A) \vee I_{\mathcal{N}_x}(B). \end{aligned}$$

Similarly, we have $F_{\mathcal{N}_x}(A\cap B) \le F_{\mathcal{N}_x}(A) \vee F_{\mathcal{N}_x}(B)$. On the other hand,

$$[(A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x)] = (T_{\mathcal{N}_x}(A) \wedge T_{\mathcal{N}_x}(B), I_{\mathcal{N}_x}(A) \vee I_{\mathcal{N}_x}(B), F_{\mathcal{N}_x}(A) \vee F_{\mathcal{N}_x}(B)).$$

Thus, $[A\cap B \in \mathcal{N}_x] \ge [(A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x)]$.
(3) From the definition of $\mathcal{N}_x$, we can easily show that $[A \in \mathcal{N}_x] \le [B \in \mathcal{N}_x]$.
(4) It is clear that

$$\begin{aligned} &[\exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))] \\ &= (\textstyle\bigvee_{C\subset A}[T_{\mathcal{N}_x}(C) \wedge \bigwedge_{y\in C} T_{\mathcal{N}_y}(C)],\ \bigwedge_{C\subset A}[I_{\mathcal{N}_x}(C) \vee \bigvee_{y\in C} I_{\mathcal{N}_y}(C)], \\ &\qquad \textstyle\bigwedge_{C\subset A}[F_{\mathcal{N}_x}(C) \vee \bigvee_{y\in C} F_{\mathcal{N}_y}(C)]). \end{aligned}$$

Then, by the proof of Theorem 3.2 (4) in [33], it is obvious that

$$\bigvee_{C\subset A} [T_{\mathcal{N}_x}(C) \wedge \bigwedge_{y\in C} T_{\mathcal{N}_y}(C)] \ge T_{\mathcal{N}_x}(A).$$

From Lemma 3, $\bigvee_{y\in C} I_{\mathcal{N}_y}(C) = \bigvee_{y\in C} \bigwedge_{y\in D\subset C} I_\tau(D) = I_\tau(C)$. Thus,

$$\begin{aligned} \textstyle\bigwedge_{C\subset A}[I_{\mathcal{N}_x}(C) \vee \bigvee_{y\in C} I_{\mathcal{N}_y}(C)] &= \textstyle\bigwedge_{C\subset A}[I_{\mathcal{N}_x}(C) \vee I_\tau(C)] = \bigwedge_{C\subset A} I_\tau(C) \\ &\le \textstyle\bigwedge_{x\in C\subset A} I_\tau(C) = I_{\mathcal{N}_x}(A). \end{aligned}$$

Similarly, we have $\bigwedge_{C\subset A}[F_{\mathcal{N}_x}(C) \vee \bigvee_{y\in C} F_{\mathcal{N}_y}(C)] \le \bigwedge_{x\in C\subset A} F_\tau(C) = F_{\mathcal{N}_x}(A)$. Thus,

$$[\exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))] \ge [A \in \mathcal{N}_x].$$

Conversely, suppose $\mathcal{N}$ satisfies the above properties (2) and (3) and let $\tau : 2^X \to I \times I \times I$ be the mapping defined as follows: for each $A \in 2^X$,

$$\tau(A) = (\bigwedge_{x\in A} T_{\mathcal{N}_x}(A), \bigvee_{x\in A} I_{\mathcal{N}_x}(A), \bigvee_{x\in A} F_{\mathcal{N}_x}(A)).$$

Then, clearly, $\tau(\phi) = (1,0,0)$. Since $\mathcal{N}_x$ is single valued neutrosophic normal, there is $A_0 \in 2^X$ such that $\mathcal{N}_x(A_0) = (1,0,0)$. Thus, $\mathcal{N}_x(X) = (1,0,0)$. Thus,

$$\tau(X) = (\bigwedge_{x\in X} T_{\mathcal{N}_x}(X), \bigvee_{x\in X} I_{\mathcal{N}_x}(X), \bigvee_{x\in X} F_{\mathcal{N}_x}(X)) = (1,0,0).$$

Hence, τ satisfies the axiom (OSVNT1).

From the proof of Theorem 3.2 in [33], it is clear that $T_\tau(A \cap B) \geq T_\tau(A) \wedge T_\tau(B)$.

On the other hand,

$$\begin{aligned} I_\tau(A \cap B) &= \bigvee_{x \in A \cap B} I_{\mathcal{N}_x}(A \cap B) \leq \bigvee_{x \in A \cap B}(I_{\mathcal{N}_x}(A) \vee I_{\mathcal{N}_x}(B)) \\ &= \bigvee_{x \in A \cap B} I_{\mathcal{N}_x}(A) \vee \bigvee_{x \in A \cap B} I_{\mathcal{N}_x}(B) \\ &\leq \bigvee_{x \in A} I_{\mathcal{N}_x}(A) \vee \bigvee_{x \in B} I_{\mathcal{N}_x}(B) \\ &= I_\tau(A) \vee I_\tau(B). \end{aligned}$$

Similarly, we have $F_\tau(A \cap B) \leq F_\tau(A) \vee F_\tau(B)$. Then, τ satisfies the axiom (OSVNT2). Moreover, we can easily see that τ satisfies the axiom (OSVNT3). Thus, $\tau \in OSVNT(X)$.

Now, suppose $\mathcal{N}$ satisfies additionally the above properties (1) and (4). Then, from the proof of Theorem 3.2 in [33], we have $T_{\mathcal{N}_x}(A) = \bigvee_{x \in B \subset A} T_\tau(B)$ for each $x \in X$ and each $A \in 2^X$.

Let $x \in X$ and let $A \in 2^X$. Then, by property (4),

$$I_{\mathcal{N}_x}(A) \geq \bigwedge_{C \subset A} [I_{\mathcal{N}_x}(C) \vee \bigvee_{y \in C} I_{\mathcal{N}_y}(C)].$$

From the property (1), $I_{\mathcal{N}_x}(C) = 1$ for any $x \notin C$. Thus,

$$\begin{aligned} I_{\mathcal{N}_x}(A) &\geq \bigwedge_{x \in C \subset A}[I_{\mathcal{N}_x}(C) \vee \bigvee_{y \in C} I_{\mathcal{N}_y}(C)] \\ &\geq \bigwedge_{x \in C \subset A} \bigvee_{y \in C} I_{\mathcal{N}_y}(C) \\ &= \bigwedge_{x \in B \subset A} I_\tau(B). \end{aligned}$$

Now, suppose $x \in C \subset A$. Then, clearly, $\bigvee_{y \in C} I_{\mathcal{N}_y}(C) \geq I_{\mathcal{N}_x}(C) \geq I_{\mathcal{N}_x}(A)$.

Thus,

$$\bigwedge_{x \in B \subset A} I_\tau(B) = \bigwedge_{x \in C \subset A} \bigvee_{y \in C} I_{\mathcal{N}_y}(C) \geq I_{\mathcal{N}_x}(A).$$

Thus, $I_{\mathcal{N}_x}(A) = \bigwedge_{x \in B \subset A} I_\tau(B)$. Similarly, we have $F_{\mathcal{N}_x}(A) = \bigwedge_{x \in B \subset A} F_\tau(B)$. This completes the proof. □

5. Ordinary Single Valued Neutrosophic Bases and Subbases

In this section, we define an ordinary single valued neutrosophic base and subbase for an ordinary single valued neutrosophic topological space, and investigated general properties. Moreover, we obtain two characterizations of an ordinary single valued neutrosophic base and one characterization of an ordinary single valued neutrosophic subbase.

Definition 15. *Let (X, τ) be an osvnts and let $\mathcal{B} : 2^X \to I \times I \times I$ be a mapping such that $\mathcal{B} \leq \tau$, i.e., $T_\mathcal{B} \leq T_\tau$, $I_\mathcal{B} \geq I_\tau$, $F_\mathcal{B} \geq F_\tau$. Then, $\mathcal{B}$ is called an ordinary single valued neutrosophic base for τ if, for each $A \in 2^X$,*

$$T_\tau(A) = \bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ A = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} T_\mathcal{B}(B_\alpha),$$

$$I_\tau(A) = \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ A = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_\mathcal{B}(B_\alpha),$$

$$F_\tau(A) = \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ A = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} F_\mathcal{B}(B_\alpha).$$

Example 3. *(1) Let X be a set and let $\mathcal{B} : 2^X \to I \times I \times I$ be the mapping defined by:*

$$\mathcal{B}(\{x\}) = (1, 0, 0) \ \ \forall x \in X.$$

Then, $\mathcal{B}$ is an ordinary single valued neutrosophic base for τ_X.

*(2) Let $X = \{a, b, c\}$, let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_1 \times I_0 \times I_0$ and let $\mathcal{B} : 2^X \to I \times I \times I$ be the mapping as follows: for each $A \in 2^X$,*

$$\mathcal{B}(A) = \begin{cases} (1,0,0) & \textit{if } \text{either } A = \{a,b\} \text{ or } \{b,c\} \text{ or } X, \\ \alpha & \text{otherwise.} \end{cases}$$

Then, $\mathcal{B}$ is not an ordinary single valued neutrosophic base for an osvnt on X.

Suppose that $\mathcal{B}$ is an ordinary single valued neutrosophic base for an osvnt τ on X. Then, clearly, $\mathcal{B} \leq \tau$. Moreover, $\tau(\{a,b\}) = \tau(\{b,c\}) = (1,0,0)$. Thus,

$$T_\tau(\{b\}) = T_\tau(\{a,b\} \cap \tau(\{b,c\}) \geq T_\tau(\{a,b\} \wedge T_\tau(\{b,c\} = 1$$

and

$$I_\tau(\{b\}) = I_\tau(\{a,b\} \cap \tau(\{b,c\}) \leq I_\tau(\{a,b\} \wedge I_\tau(\{b,c\} = 0.$$

Similarly, we have $F_\tau(\{b\}) = 0$. Thus, $\tau(\{b\}) = (1,0,0)$. On the other hand, by the definition of $\mathcal{B}$,

$$T_\tau(\{b\}) = \bigvee_{\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ \{b\}=\bigcup_{\alpha\in\Gamma} A_\alpha} \bigwedge_{\alpha\in\Gamma} T_{\mathcal{B}}(A_\alpha) = T_\alpha$$

and

$$I_\tau(\{b\}) = \bigwedge_{\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ \{b\}=\bigcup_{\alpha\in\Gamma} A_\alpha} \bigvee_{\alpha\in\Gamma} I_{\mathcal{B}}(A_\alpha) = I_\alpha.$$

Similarly, we have $F_\tau(\{b\}) = F_\alpha$. This is a contradiction. Hence, $\mathcal{B}$ is not an ordinary single valued neutrosophic base for an osvnt on X

Theorem 4. *Let (X, τ) be an osvnts and let $\mathcal{B} : 2^X \to I \times I \times I$ be a mapping such that $\mathcal{B} \leq \tau$. Then, $\mathcal{B}$ is an ordinary single valued neutrosophic base for τ if and only if for each $x \in X$ and each $A \in 2^X$,*

$$T_{\mathcal{N}_x}(A) \leq \bigvee_{x\in B\subset A} T_{\mathcal{B}}(B),$$

$$I_{\mathcal{N}_x}(A) \geq \bigwedge_{x\in B\subset A} I_{\mathcal{B}}(B),$$

$$F_{\mathcal{N}_x}(A) \geq \bigwedge_{x\in B\subset A} F_{\mathcal{B}}(B).$$

Proof. ($\Rightarrow$): Suppose $\mathcal{B}$ is an ordinary single valued neutrosophic base for τ. Let $x \in X$ and let $A \in 2^X$. Then, by Theorem 4.4 in [34], it is obvious that $T_{\mathcal{N}_x}(A) \leq \bigvee_{x\in B\subset A} T_{\mathcal{B}}(B)$. On the other hand,

$$\begin{aligned} I_{\mathcal{N}_x}(A) &= \textstyle\bigwedge_{x\in B\subset A} I_\tau(B) && \text{[By Definition 13]} \\ &= \textstyle\bigwedge_{x\in B\subset A} \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ B=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_{\mathcal{B}}(B_\alpha). && \text{[By Definition 15]} \end{aligned}$$

If $x \in B \subset A$ and $B = \bigcup_{\alpha\in\Gamma} B_\alpha$, then there is $\alpha_0 \in \Gamma$ such that $x \in B_{\alpha_0}$. Thus,

$$\bigvee_{\alpha\in\Gamma} I_{\mathcal{B}}(B_\alpha) \geq I_{\mathcal{B}}(B_{\alpha_0}) \geq \bigwedge_{x\in B\subset A} I_{\mathcal{B}}(B).$$

Thus, $I_{\mathcal{N}_x}(A) \geq \bigwedge_{x\in B\subset A} I_{\mathcal{B}}(B)$. Similarly, we have $F_{\mathcal{N}_x}(A) \geq \bigwedge_{x\in B\subset A} F_{\mathcal{B}}(B)$. Hence, the necessary condition holds.

($\Leftarrow$): Suppose the necessary condition holds. Then, by Theorem 4.4 in [34], it is clear that

$$T_\tau(A) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} T_{\mathcal{B}}(B_\alpha).$$

Let $A \in 2^X$. Suppose $A = \bigcup_{\alpha\in\Gamma} B_\alpha$ and $\{B_\alpha\} \subset 2^X$. Then,

$$\begin{aligned} I_\tau(A) &\leq \bigvee_{\alpha\in\Gamma} I_\tau(B_\alpha) && \text{[By the axiom (OSVNT3)]} \\ &\leq \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha). && \text{[Since } \mathcal{B} \leq \tau] \end{aligned}$$

Thus,

$$I_\tau(A) \leq \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha). \tag{1}$$

On the other hand,

$$\begin{aligned} I_\tau(A) &= \bigvee_{x\in A} \bigwedge_{x\in B\subset A} I_\tau(B) && \text{[By Lemma 3]} \\ &= \bigvee_{x\in A} I_{\mathcal{N}_x}(A) && \text{[By Definition 13]} \\ &= \bigvee_{x\in A} \bigwedge_{x\in B\subset A} I_\mathcal{B}(B) && \text{[By the hypothesis]} \\ &= \bigwedge_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigvee_{x\in A} I_\mathcal{B}(f(x)), \end{aligned}$$

where $\mathcal{B}_x = \{B \in 2^X : x \in B \subset A\}$. Furthermore, $A = \bigcup_{x\in A} f(x)$ for each $f \in \Pi_{x\in A}\mathcal{B}_x$. Thus,

$$\bigwedge_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigvee_{x\in A} I_\mathcal{B}(f(x)) = \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha).$$

Hence,

$$I_\tau(A) \geq \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha). \tag{2}$$

By (1) and (2), $I_\tau(A) = \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha)$. Similarly, we have $F_\tau(A) = \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} F_\mathcal{B}(B_\alpha)$. Therefore, $\mathcal{B}$ is an ordinary single valued neutrosophic base for τ. □

Theorem 5. *Let $\mathcal{B} : 2^X \to I \times I \times I$ be a mapping. Then, $\mathcal{B}$ is an ordinary single valued neutrosophic base for some oist τ on X if and only if it has the following conditions:*

(1) $\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} T_\mathcal{B}(B_\alpha) = 1$,
$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha) = 0$,
$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} F_\mathcal{B}(B_\alpha) = 0$,

(2) *for any $A_1,\ A_2 \in 2^X$ and each $x \in A_1 \cap A_2$,*

$$T_\mathcal{B}(A_1) \wedge T_\mathcal{B}(A_2) \leq \bigvee_{x\in A\subset A_1\cap A_2} T_\mathcal{B}(A),$$

$$I_\mathcal{B}(A_1) \vee I_\mathcal{B}(A_2) \geq \bigwedge_{x\in A\subset A_1\cap A_2} I_\mathcal{B}(A),$$

$$F_\mathcal{B}(A_1) \vee F_\mathcal{B}(A_2) \geq \bigwedge_{x\in A\subset A_1\cap A_2} F_\mathcal{B}(A).$$

In fact, $\tau : 2^X \to I \times I \times I$ is the mapping defined as follows: for each $A \in 2^X$,

$$T_\tau(A) = \begin{cases} 1 & \text{if } A = \phi \\ \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} T_\mathcal{B}(B_\alpha) & \text{otherwise,} \end{cases}$$

$$I_\tau(A) = \begin{cases} 0 & \text{if } A = \phi \\ \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_\mathcal{B}(B_\alpha) & \text{otherwise,} \end{cases}$$

$$F_\tau(A) = \begin{cases} 0 & \text{if } A = \phi \\ \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} F_\mathcal{B}(B_\alpha) & \text{otherwise.} \end{cases}$$

In this case, τ is called an ordinary single valued neutrosophic topology on X induced by $\mathcal{B}$.

Proof. (⇒): Suppose $\mathcal{B}$ is an ordinary single valued neutrosophic base for some *osvnt* τ on X. Then, by Definition 15 and the axiom (OSVNT1),

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma}B_\alpha}\ \bigwedge_{\alpha\in\Gamma} T_{\mathcal{B}}(B_\alpha)=T_\tau(X)=1,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma}B_\alpha}\ \bigvee_{\alpha\in\Gamma} I_{\mathcal{B}}(B_\alpha))=I_\tau(X)=0,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma}B_\alpha}\ \bigvee_{\alpha\in\Gamma} F_{\mathcal{B}}(B_\alpha))=F_\tau(X)=0.$$

Thus, condition (1) holds.

Let $A_1,\ A_2\in 2^X$ and let $x\in A_1\cap A_2$. Then, by the proof of Theorem 4.2 in [33], it is obvious that $T_{\mathcal{B}}(A_1)\wedge T_{\mathcal{B}}(A_2)\le\bigvee_{x\in A\subset A_1\cap A_2}T_{\mathcal{B}}(A)$. On the other hand,

$$I_{\mathcal{B}}(A_1)\vee I_{\mathcal{B}}(A_2)\ge I_\tau(A_1)\vee I_\tau(A_2)\ge I_\tau(A_1\cap A_2)\ge I_{\mathcal{N}_x}(A_1\cap A_2)\ge\bigwedge_{x\in A\subset A_1\cap A_2}I_{\mathcal{B}}(A).$$

Thus,

$$I_{\mathcal{B}}(A_1)\vee I_{\mathcal{B}}(A_2)\ge\bigwedge_{x\in A\subset A_1\cap A_2}I_{\mathcal{B}}(A).$$

Similarly, we have

$$F_{\mathcal{B}}(A_1)\vee F_{\mathcal{B}}(A_2)\ge\bigwedge_{x\in A\subset A_1\cap A_2}F_{\mathcal{B}}(A).$$

Thus, condition (2) holds.

(⇐): Suppose the necessary conditions (1) and (2) are satisfied. Then, by the proof of Theorem 4.2 in [33], we can see that the following hold:

$T_\tau(X)=T_\tau(\phi)=1$,
$T_\tau(A\cap B)\ge T_\tau(A)\wedge T_\tau(B)$ for any $A,\ B\in 2^X$

and

$T_\tau(\bigcup_{\alpha\in\Gamma}A_\alpha)\ge\bigwedge_{\alpha\in\Gamma}T_\tau(A_\alpha)$ for each $\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X$.

From the definition of τ, it is obvious that $I_\tau(X)=I_\tau(\phi)=0$. Similarly, we have $F_\tau(X)=F_\tau(\phi)=0$. Thus, τ satisfies the axiom (OSVNT1).

Let $\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X$ and let $\mathcal{B}_\alpha=\{\{B_{\delta_\alpha}:\delta_\alpha\in\Gamma_\alpha\}:\bigcup_{\delta_\alpha\in\Gamma_\alpha}B_{\delta_\alpha}=A_\alpha\}$. Let $f\in\Pi_{\alpha\in\Gamma}\mathcal{B}_\alpha$. Then, clearly, $\bigcup_{\alpha\in\Gamma}\bigcup_{B_{\delta_\alpha}\in f(\alpha)}B_{\delta_\alpha}=\bigcup_{\alpha\in\Gamma}A_\alpha$. Thus,

$$\begin{aligned}I_\tau(\textstyle\bigcup_{\alpha\in\Gamma}A_\alpha)&=\textstyle\bigwedge_{\bigcup_{\delta\in\Gamma}B_\delta=\bigcup_{\alpha\in\Gamma}A_\alpha}\bigvee_{\delta\in\Gamma}I_{\mathcal{B}}(B_\delta)\\&\le\textstyle\bigwedge_{f\in\Pi_{\alpha\in\Gamma}\mathcal{B}_\alpha}\bigvee_{\alpha\in\Gamma}\bigvee_{B_{\delta_\alpha}\in f(\alpha)}I_{\mathcal{B}}(B_{\delta_\alpha})\\&=\textstyle\bigvee_{\alpha\in\Gamma}\bigwedge_{\{B_{\delta_\alpha}:\delta_\alpha\in\Gamma_\alpha\}\in\mathcal{B}_\alpha}\bigvee_{\delta_\alpha\in\Gamma_\alpha}I_{\mathcal{B}}(B_{\delta_\alpha})\\&=\textstyle\bigvee_{\alpha\in\Gamma}I_\tau(A_\alpha).\end{aligned}$$

Similarly, we have $F_\tau(\bigcup_{\alpha\in\Gamma}A_\alpha)\le\bigvee_{\alpha\in\Gamma}F_\tau(A_\alpha)$. Thus, τ satisfies the axiom (OSVNT3).

Now, let $A,\ B\in 2^X$ and suppose $I_\tau(A)<I_\alpha$ and $I_\tau(B)<I_\alpha$ for $\alpha\in\mathbf{SVNV}$. Then, there are $\{A_{\alpha_1}:\alpha_1\in\Gamma_1\}$ and $\{B_{\alpha_2}:\alpha_2\in\Gamma_2\}$ such that $\bigcup_{\alpha_1\in\Gamma_1}A_{\alpha_1}=A$, $\bigcup_{\alpha_2\in\Gamma_2}B_{\alpha_2}=B$ and $I_{\mathcal{B}}(A_{\alpha_1})<I_\alpha$ for each $\alpha_1\in\Gamma_1$, $I_{\mathcal{B}}(B_{\alpha_2})<I_\alpha$ for each $\alpha_2\in\Gamma_2$. Let $x\in A\cap B$. Then, there are $\alpha_{1x}\in\Gamma_1$ and $\alpha_{2x}\in\Gamma_2$ such that $x\in A_{\alpha_{1x}}\cap B_{\alpha_{2x}}$. Thus, from the assumption,

$$I_\alpha>I_{\mathcal{B}}(A_{\alpha_{1x}})\vee I_{\mathcal{B}}(B_{\alpha_{2x}})\ge\bigwedge_{x\in C\subset A_{\alpha_{1x}}\cap B_{\alpha_{2x}}}I_{\mathcal{B}}(C).$$

Moreover, there is C_x such that $x \in C_x \subset A_{\alpha_{1x}} \cap B_{\alpha_{2x}} \subset A \cap B$ and $I_{\mathcal{B}}(C_x) < I_\alpha$. Since $\bigcup_{x \in A\cap B} C_x = A \cap B$, we obtain

$$I_\alpha \geq \bigvee_{x \in A\cap B} I_{\mathcal{B}}(C_x) \geq \bigwedge_{\bigcup_{\alpha\in\Gamma} B_\alpha = A\cap B} \bigvee_{\alpha\in\Gamma} I_{\mathcal{B}}(B_\alpha) = I_\tau(A \cap B).$$

Now, let $I_\beta = I_\tau(A) \vee I_\tau(B)$ and let n be any natural number, where $I_\beta \in I$. Then, $I_\tau(A) < I_\beta + 1/n$ and $I_\tau(B) < I_\beta + 1/n$. Thus, $I_\tau(A \cap B) \leq I_\beta + 1/n$. Thus, $I_\tau(A \cap B) \leq I_\beta = I_\tau(A) \vee I_\tau(B)$. Similarly, we have $F_\tau(A \cap B) \leq F_\tau(A) \vee F_\tau(B)$. Hence, τ satisfies the axiom (OSVNT2). This completes the proof. □

Example 4. *(1) Let $X = \{a, b, c\}$ and let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_1 \times I_0 \times I_0$. We define the mapping $\mathcal{B} : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,*

$$T_{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = \{b\} \text{ or } \{a,b\} \text{ or } \{b,c\} \\ T_\alpha & \text{otherwise,} \end{cases}$$

$$I_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = \{b\} \text{ or } \{a,b\} \text{ or } \{b,c\} \\ I_\alpha & \textit{otherwise,} \end{cases}$$

$$F_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = \{b\} \text{ or } \{a,b\} \text{ or } \{b,c\} \\ F_\alpha & \textit{otherwise.} \end{cases}$$

Then, we can easily see that $\mathcal{B}$ satisfies conditions (1) and (2) in Theorem 5. Thus, $\mathcal{B}$ is an ordinary single valued neutrosophic base for an osvnt τ on X. In fact, $\tau : 2^X \to I \times I \times I$ is defined as follows: for each $A \in 2^X$,

$$T_\tau(A) = \begin{cases} 1 & \text{if } A \in \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\} \\ T_\alpha & \text{otherwise,} \end{cases}$$

$$I_\tau(A) = \begin{cases} 0 & \text{if } A \in \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\} \\ I_\alpha & \text{otherwise,} \end{cases}$$

$$F_\tau(A) = \begin{cases} 0 & \text{if } A \in \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\} \\ F_\alpha & \text{otherwise.} \end{cases}$$

*(2) Let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_1 \times I_0 \times I_0$. We define the mapping $\mathcal{B} : 2^{\mathbb{R}} \to I \times I \times I$ as follows: for each $A \in 2^{\mathbb{R}}$,*

$$T_{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = (a,b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ T_\alpha & \text{otherwise,} \end{cases}$$

$$I_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = (a,b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ I_\alpha & \text{otherwise,} \end{cases}$$

$$F_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = (a,b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ F_\alpha & \text{otherwise.} \end{cases}$$

Then, it can be easily seen that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5. Thus, $\mathcal{B}$ is an ordinary single valued neutrosophic base for an osvnt τ_α on $\mathbb{R}$.

In this case, τ_α is called the α-ordinary single valued neutrosophic usual topology on $\mathbb{R}$.

*(3) Let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_1 \times I_0 \times I_0$. We define the mapping $\mathcal{B} : 2^{\mathbb{R}} \to I \times I \times I$ as follows: for each $A \in 2^{\mathbb{R}}$,*

$$T_{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = [a,b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ T_\alpha & \text{otherwise,} \end{cases}$$

$$I_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = [a, b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ I_\alpha & \text{otherwise,} \end{cases}$$

$$F_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = [a, b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ F_\alpha & \text{otherwise.} \end{cases}$$

Then, we can easily see that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5. Thus, $\mathcal{B}$ is an ordinary single valued neutrosophic base for an osvnt τ_l on $\mathbb{R}$.

In this case, τ_l is called the α-ordinary single valued neutrosophic lower-limit topology on $\mathbb{R}$.

Definition 16. *Let $\tau_1,\ \tau_2 \in OSVNT(X)$, and let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary single valued neutrosophic bases for τ_1 and τ_2, respectively. Then, $\mathcal{B}_1$ and $\mathcal{B}_1$ are said to be equivalent if $\tau_1 = \tau_2$.*

Theorem 6. *Let $\tau_1,\ \tau_2 \in OSVNT(X)$, and let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary single valued neutrosophic bases for τ_1 and τ_2 respectively. Then, τ_1 is coarser than τ_2, i.e.,*

$$T_{\tau_1} \leq T_{\tau_2},\ I_{\tau_1} \geq I_{\tau_2},\ F_{\tau_1} \geq F_{\tau_2}$$

if and only if for each $A \in 2^X$ and each $x \in A$,

$$T_{\mathcal{B}_1}(A) \leq \bigvee_{x \in B \subset A} T_{\mathcal{B}_2}(B), \quad I_{\mathcal{B}_1}(A) \geq \bigwedge_{x \in B \subset A} I_{\mathcal{B}_2}(B), \quad F_{\mathcal{B}_1}(A) \geq \bigwedge_{x \in B \subset A} F_{\mathcal{B}_2}(B).$$

Proof. ($\Rightarrow$): Suppose τ_1 is coarser than τ_2. For each $x \in X$, let $x \in A \in 2^X$. Then, by Theorem 4.8 in [34], $T_{\mathcal{B}_1}(A) \leq \bigvee_{x \in B \subset A} T_{\mathcal{B}_2}(B)$. On the other hand,

$$\begin{aligned} I_{\mathcal{B}_1}(A) &\geq I_{\tau_1}(A) && \text{[since } \mathcal{B}_1 \text{ is an ordinary single valued neutrosophic base for } \tau_1] \\ &\geq I_{\tau_2}(A) && \text{[By the hypothesis]} \\ &= \bigwedge_{\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ A = \bigcup_{\alpha \in \Gamma} A_\alpha} \bigvee_{\alpha \in \Gamma} I_{\mathcal{B}_2}(A_\alpha). \\ & && \text{[Since } \mathcal{B}_2 \text{ is an ordinary single valued neutrosophic base for } \tau_2] \end{aligned}$$

Since $x \in A$ and $A = \bigcup_{\alpha \in \Gamma} A_\alpha$, there is $\alpha_0 \in \Gamma$ such that $x \in A_{\alpha_0}$. Thus,

$$\bigwedge_{\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ A = \bigcup_{\alpha \in \Gamma} A_\alpha} \bigvee_{\alpha \in \Gamma} I_{\mathcal{B}_2}(A_\alpha) \geq I_{\mathcal{B}_2}(A_{\alpha_0}) \geq \bigwedge_{x \in B \subset A} I_{\mathcal{B}_2}(B).$$

Thus, $I_{\mathcal{B}_1}(A) \geq \bigwedge_{x \in B \subset A} I_{\mathcal{B}_2}(B)$. Similarly, we have $F_{\mathcal{B}_1}(A) \geq \bigwedge_{x \in B \subset A} F_{\mathcal{B}_2}(B)$.

($\Leftarrow$): Suppose the necessary condition holds. Then, by Theorem 4.8 in [34], $T_{\tau_1} \leq T_{\tau_2}$. Let $A \in 2^X$. Then,

$$\begin{aligned} I_{\tau_1}(A) &= \bigvee_{x \in A} \bigwedge_{x \in B \subset A} I_{\mathcal{B}_1}(B) && \text{[By Lemma 3]} \\ &\geq \bigvee_{x \in A} \bigwedge_{x \in B \subset A} \bigwedge_{x \in C \subset B} I_{\mathcal{B}_2}(C) && \text{[By the hypothesis]} \\ &= \bigwedge_{x \in C \subset A} \bigvee_{x \in A} I_{\mathcal{B}_2}(C) \\ &= \bigwedge_{\{C_x\}_{x \in A} \subset 2^X,\ A = \bigcup_{x \in A} C_x} \bigvee_{x \in A} I_{\mathcal{B}_2}(C_x) \\ &= I_{\tau_2}(A). \end{aligned}$$

Thus, $I_{\tau_1} \geq I_{\tau_2}$. Similarly, we have $F_{\tau_1} \geq F_{\tau_2}$. Thus, τ_1 is coarser than τ_2. This completes the proof. □

The following is an immediate result of Definition 16 and Theorem 6.

Corollary 4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be ordinary single valued neutrosophic bases for two ordinary single valued neutrosophic topologies on a set X, respectively. Then,*

$\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent if and only if the following two conditions hold:

(1) *for each* $B_1 \in 2^X$ *and each* $x \in B_1$,

$$T_{\mathcal{B}_1}(B_1) \leq \bigvee_{x \in B_2 \subset B_1} T_{\mathcal{B}_2}(B_2),$$

$$I_{\mathcal{B}_1}(B_1) \geq \bigwedge_{x \in B_2 \subset B_1} I_{\mathcal{B}_2}(B_2),$$

$$F_{\mathcal{B}_1}(B_1) \geq \bigwedge_{x \in B_2 \subset B_1} F_{\mathcal{B}_2}(B_2),$$

(2) *for each* $B_2 \in 2^X$ *and each* $x \in B_2$,

$$T_{\mathcal{B}_2}(B_2) \leq \bigvee_{x \in B_1 \subset B_2} T_{\mathcal{B}_1}(B_1),$$

$$I_{\mathcal{B}_2}(B_2) \geq \bigwedge_{x \in B_1 \subset B_2} I_{\mathcal{B}_1}(B_1),$$

$$F_{\mathcal{B}_2}(B_2) \geq \bigwedge_{x \in B_1 \subset B_2} F_{\mathcal{B}_1}(B_1).$$

It is obvious that every ordinary single valued neutrosophic topology itself forms an ordinary single valued neutrosophic base. Then, the following provides a sufficient condition for one to see if a mapping $\mathcal{B} : 2^X \to I \times I \times I$ such that $T_{\mathcal{B}} \leq T_\tau$, $I_{\mathcal{B}} \geq I_\tau$ and $F_{\mathcal{B}} \geq F_\tau$ is an ordinary single valued neutrosophic base for $\tau \in OSVNT(X)$.

Proposition 10. *Let* (X, τ) *be an osvnts and let* $\mathcal{B} : 2^X \to I \times I \times I$ *be a mapping such that* $T_{\mathcal{B}} \leq T_\tau$, $I_{\mathcal{B}} \geq I_\tau$ *and* $F_{\mathcal{B}} \geq F_\tau$. *For each* $A \in 2^X$ *and each* $x \in A$, *suppose* $T_\tau(A) \leq \bigvee_{x \in B \subset A} T_{\mathcal{B}}(B)$, $I_\tau(A) \geq \bigwedge_{x \in B \subset A} I_{\mathcal{B}}(B)$ *and* $F_\tau(A) \geq \bigwedge_{x \in B \subset A} F_{\mathcal{B}}(B)$. *Then,* $\mathcal{B}$ *is an ordinary single valued neutrosophic base for* τ.

Proof. From the proof of Proposition 4.10 in [34], it is clear that the first part of the condition (1) of Theorem 5 holds, i.e., $\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} T_{\mathcal{B}}(B_\alpha) = 1$. On the other hand,

$$\begin{aligned}
& \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_{\mathcal{B}}(B_\alpha) \\
\geq\ & \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_\tau(B_\alpha) && \text{[since } I_{\mathcal{B}} \geq I_\tau\text{]} \\
\geq\ & \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} I_\tau(\textstyle\bigcup_{\alpha \in \Gamma} B_\alpha) && \text{[by the axiom (OSVNT3)]} \\
=\ & I_\tau(X) \\
=\ & \bigvee_{x \in X} \bigwedge_{x \in B \subset X} I_\tau(B) && \text{[By Lemma 3]} \\
\geq\ & \bigvee_{x \in X} \bigwedge_{x \in B \subset X} \bigwedge_{x \in C \subset B} I_{\mathcal{B}}(C) && \text{[By the hypothesis]} \\
=\ & \bigwedge_{x \in C \subset X} \bigvee_{x \in X} I_{\mathcal{B}}(C) \\
=\ & \bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_{\mathcal{B}}(B_\alpha).
\end{aligned}$$

Since $\tau \in OSVNT(X)$, $I_\tau(X) = 0$. Thus, $\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_{\mathcal{B}}(B_\alpha) = 0$. Similarly, we have $\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} F_{\mathcal{B}}(B_\alpha) = 0$. Thus, condition (1) of Theorem 5 holds.

Now, let $A_1,\ A_2 \in 2^X$ and let $x \in A_1 \cap A_2$. Then, by the proof of Proposition 4.10 in [34], it is obvious that $T_{\mathcal{B}}(A_1) \wedge T_{\mathcal{B}}(A_2) \leq \bigvee_{x \in A \subset A_1 \cap A_2} T_{\mathcal{B}}(A)$. On the other hand,

$$\begin{aligned}
I_{\mathcal{B}}(A_1) \vee I_{\mathcal{B}}(A_2) &\geq I_\tau(A_1) \vee I_\tau(A_2) && \text{[Since } I_{\mathcal{B}} \geq I_\tau\text{]} \\
&\geq I_\tau(A_1 \cap A_2) && \text{[by the axiom (OSVNT2)]} \\
&\geq \bigwedge_{x \in A \subset A_1 \cap A_2} I_{\mathcal{B}}(A). && \text{[by the hypothesis]}
\end{aligned}$$

Similarly, we have $F_{\mathcal{B}}(A_1) \vee F_{\mathcal{B}}(A_2) \geq \bigwedge_{x \in A \subset A_1 \cap A_2} F_{\mathcal{B}}(A)$. Thus, condition (2) of Theorem 5 holds. Thus, by Theorem 5, $\mathcal{B}$ is an ordinary single valued neutrosophic base for τ. This completes the proof. □

Definition 17. *Let (X,τ) be an osvnts and let $\simeq : 2^X \to I \times I \times I$ be a mapping. Then, φ is called an ordinary single valued neutrosophic subbase for τ, if $\varphi^{\sqcap}$ is an ordinary single valued neutrosophic base for τ, where $\varphi^{\sqcap} : 2^X \to I \times I \times I$ is the mapping defined as follows: for each $A \in 2^X$,*

$$T_{\varphi^{\sqcap}}(A) = \bigvee_{\{B_\alpha\} \sqsubset 2^X,\ A=\bigcap_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} T_{\simeq}(B_\alpha),$$

$$I_{\varphi^{\sqcap}}(A) = \bigwedge_{\{B_\alpha\} \sqsubset 2^X,\ A=\bigcap_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_{\simeq}(B_\alpha),$$

$$F_{\varphi^{\sqcap}}(A) = \bigwedge_{\{B_\alpha\} \sqsubset 2^X,\ A=\bigcap_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} F_{\simeq}(B_\alpha),$$

where $\sqsubset$ stands for "a finite subset of".

Example 5. *Let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_1 \times I_0 \times I_0$. We define the mapping $\simeq : 2^{\mathbb{R}} \to I \times I \times I$ as follows: for each $A \in 2^{\mathbb{R}}$,*

$$T_{\simeq}(A) = \begin{cases} 1 & \text{if } A = (a,\infty) \text{ or } (-\infty,b) \text{ or } (a,b) \\ T_\alpha & \text{otherwise,} \end{cases}$$

$$I_{\simeq}(A) = \begin{cases} 0 & \text{if } A = (a,\infty) \text{ or } (-\infty,b) \text{ or } (a,b) \\ I_\alpha & \text{otherwise,} \end{cases}$$

$$F_{\simeq}(A) = \begin{cases} 0 & \text{if } A = (a,\infty) \text{ or } (-\infty,b) \text{ or } (a,b) \\ F_\alpha & \text{otherwise,} \end{cases}$$

where $a,\ b \in \mathbb{R}$ such that $a < b$. Then, we can easily see that $\simeq$ is an ordinary single valued neutrosophic subbase for the α-ordinary single valued neutrosophic usual topology $\mathcal{U}_\alpha$ on $\mathbb{R}$.

Theorem 7. *Let $\simeq : 2^X \to I \times I \times I$ be a mapping. Then, $\simeq$ is an ordinary single valued neutrosophic subbase for some osvnt if and only if*

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} T_{\simeq}(B_\alpha) = 1,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} I_{\simeq}(B_\alpha) = 0,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} F_{\simeq}(B_\alpha) = 0.$$

Proof. ($\Rightarrow$): Suppose $\simeq$ is an ordinary single valued neutrosophic subbase for some *osvnt*. Then, by Definition 17, it is clear that the necessary condition holds.

($\Leftarrow$): Suppose the necessary condition holds. We only show that $\varphi^{\sqcap}$ satisfies the condition (2) in Theorem 5. Let $A,\ B \in 2^X$ and $x \in A \cap B$. Then, by the proof of Theorem 4.3 in [33], it is obvious that $T_{\varphi^{\sqcap}}(A) \wedge T_{\varphi^{\sqcap}}(B) \leq \bigvee_{x\in C\subset A\cap B} T_{\varphi^{\sqcap}}(C)$. On the other hand,

$$\begin{aligned} & I_{\varphi^{\sqcap}}(A) \vee I_{\varphi^{\sqcap}}(B) \\ &= (\textstyle\bigwedge_{\bigcap_{\alpha_1\in\Gamma_1} B_{\alpha_1}=A} \bigvee_{\alpha_1\in\Gamma_1} I_{\simeq}(B_{\alpha_1})) \vee (\bigwedge_{\bigcap_{\alpha_2\in\Gamma_2} B_{\alpha_2}=B} \bigvee_{\alpha_2\in\Gamma_2} I_{\simeq}(B_{\alpha_2})) \\ &= \textstyle\bigwedge_{\bigcap_{\alpha_1\in\Gamma_1} B_{\alpha_1}=A} \bigwedge_{\bigcap_{\alpha_2\in\Gamma_2} B_{\alpha_2}=B} (\bigvee_{\alpha_1\in\Gamma_1} I_{\simeq}(B_{\alpha_1}) \vee \bigvee_{\alpha_2\in\Gamma_2} I_{\simeq}(B_{\alpha_2})) \\ &\geq \textstyle\bigwedge_{\bigcap_{\alpha\in\Gamma} B_\alpha=A\cap B} \bigvee_{\alpha\in\Gamma} I_{\simeq}(B_\alpha) \\ &= I_{\varphi^{\sqcap}}(A\cap B). \end{aligned}$$

Since $x \in A \cap B$, $I_{\varphi^{\sqcap}}(A) \vee I_{\varphi^{\sqcap}}(B) \geq I_{\varphi^{\sqcap}}(A \cap B) \geq \bigwedge_{x \in C \subset A \cap B} I_{\varphi^{\sqcap}}(C)$. Similarly, we have $F_{\varphi^{\sqcap}}(A) \vee F_{\varphi^{\sqcap}}(B) \geq F_{\varphi^{\sqcap}}(A \cap B) \geq \bigwedge_{x \in C \subset A \cap B} F_{\varphi^{\sqcap}}(C)$. Thus, $\varphi^{\sqcap}$ satisfies the condition (2) in Theorem 5. This completes the proof. □

Example 6. *Let $X = \{a, b, c, d, e\}$ and let $\alpha \in$ **SVNV** be fixed, where $\alpha \in I_1 \times I_0 \times I_0$. We define the mapping $\simeq : 2^X \to I \times I \times I$ as follows: for each $A \in 2^X$,*

$$T_{\simeq}(A) = \begin{cases} 1 & \text{if } A \in \{\{a\}, \{a,b,c\}, \{b,c,d\}, \{c,e\}\} \\ T_\alpha & \text{otherwise,} \end{cases}$$

$$I_{\simeq}(A) = \begin{cases} 0 & \text{if } A \in \{\{a\}, \{a,b,c\}, \{b,c,d\}, \{c,e\}\} \\ I_\alpha & \text{otherwise,} \end{cases}$$

$$F_{\simeq}(A) = \begin{cases} 0 & \text{if } A \in \{\{a\}, \{a,b,c\}, \{b,c,d\}, \{c,e\}\} \\ F_\alpha & \text{otherwise.} \end{cases}$$

Then, $X = \{a\} \cup \{b,c,d\} \cup \{c,e\}$,

$$T_{\varphi^{\sqcap}}(\{a\}) = T_{\varphi^{\sqcap}}(\{b,c,d\}) = T_{\varphi^{\sqcap}}(\{c,e\}) = 1,$$
$$I_{\varphi^{\sqcap}}(\{a\}) = I_{\varphi^{\sqcap}}(\{b,c,d\}) = I_{\varphi^{\sqcap}}(\{c,e\}) = 0.$$
$$F_{\varphi^{\sqcap}}(\{a\}) = F_{\varphi^{\sqcap}}(\{b,c,d\}) = F_{\varphi^{\sqcap}}(\{c,e\}) = 0.$$

Thus,

$$\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} T_{\simeq}(B_\alpha) = 1,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_{\simeq}(B_\alpha) = 0,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} F_{\simeq}(B_\alpha) = 0.$$

Thus, by Theorem 7, $\simeq$ is an ordinary single valued neutrosophic subbase for some osvnt.

The following is an immediate result of Corollary 4 and Theorem 7.

Proposition 11. *$\simeq_1$, $\simeq_2 : 2^X \to I \times I \times I$ be two mappings such that*

$$\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} T_{\simeq_1}(B_\alpha) = 1,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_{\simeq_1}(B_\alpha) = 0,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} F_{\simeq_1}(B_\alpha) = 0$$

and

$$\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} T_{\simeq_2}(B_\alpha) = 1,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} I_{\simeq_2}(B_\alpha) = 0,$$

$$\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} F_{\simeq_2}(B_\alpha) = 0.$$

Suppose the two conditions hold:

(1) *for each* $S_1 \in 2^X$ *and each* $x \in S_1$,

$$T_{\simeq_1}(S_1) \leq \bigvee_{x \in S_2 \subset S_1} T_{\simeq_2}(S_2),\ I_{\simeq_1}(S_1) \geq \bigwedge_{x \in S_2 \subset S_1} I_{\simeq_2}(S_2),\ F_{\simeq_1}(S_1) \geq \bigwedge_{x \in S_2 \subset S_1} F_{\simeq_2}(S_2),$$

(2) *for each* $S_2 \in 2^X$ *and each* $x \in S_2$,

$$T_{\simeq_2}(S_2) \leq \bigvee_{x \in S_1 \subset S_2} T_{\simeq_1}(S_1),\ I_{\simeq_2}(S_2) \geq \bigwedge_{x \in S_1 \subset S_2} I_{\simeq_1}(S_1),\ f_{\simeq_2}(S_2) \geq \bigwedge_{x \in S_1 \subset S_2} f_{\simeq_1}(S_1).$$

Then, $\simeq_1$ *and* $\simeq_2$ *are ordinary single valued neutrosophic subbases for the same ordinary single valued neutrosophic topology on* X.

6. Conclusions

In this paper, we defined an ordinary single valued neutrosophic topology and level set of an *osvnst* to study some topological characteristics of neutrosophic sets and obtained some their basic properties. In addition, we defined an ordinary single valued neutrosophic subspace. Next, the concepts of an ordinary single valued neutrosophic neighborhood system and an ordinary single valued neutrosophic base (or subbase) were introduced and studied. Their results are summarized as follows:

First, an ordinary single valued neutrosophic neighborhood system has the same properties in a classical neighborhood system (see Theorem 3).

Second, we found two characterizations of an ordinary single valued neutrosophic base (see Theorems 4 and 5).

Third, we obtained one characterization of an ordinary single valued neutrosophic subbase (see Theorem 7).

Finally, we expect that this paper can be a guidance for the research of separation axioms, compactness, connectedness, etc. in ordinary single valued neutrosophic topological spaces. In addition, one can deal with single valued neutrosophic topology from the viewpoint of lattices.

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The Shortest Path Problem in Interval Valued Trapezoidal and Triangular Neutrosophic Environment

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Abstract
Real-life decision-making problem has been demonstrated to cover the indeterminacy through single valued neutrosophic set. It is the extension of interval valued neutrosophic set. Most of the problems of real life involve some sort of uncertainty in it among which, one of the famous problem is finding a shortest path of the network. In this paper, a new score function is proposed for interval valued neutrosophic numbers and SPP is solved using interval valued neutrosophic numbers. Additionally, novel algorithms are proposed to find the neutrosophic shortest path by considering interval valued neutrosophic number, trapezoidal and triangular interval valued neutrosophic numbers for the length of the path in a network with illustrative example. Further, comparative analysis has been done for the proposed algorithm with the existing method with the shortcoming and advantage of the proposed method and it shows the effectiveness of the proposed algorithm.

Keywords Interval valued triangular neutrosophic number · Interval valued trapezoidal neutrosophic number · Ranking methods · Deneutrosophication · Neutrosophic shortest path problem · Network

Introduction and literature of review

In this part, introduction to the objective of the paper is given by presenting basic concepts and procedure of the shortest path problem (SPP) and the literature of review have been collected to know the recent work related to the presented concept which shows the novelty of the presented work

Introduction

SPP is the ultimate and popular problem in the different areas also it is the heart of the network flows. In conventional problem, the distance between the nodes is considered to be certain and for the uncertain environment fuzzy numbers can be adopted to get an optimized result. Computing the minimum cost of the path from every vertex is called single source SPP. Especially in the process of finding shortest path, finding the path which has minimum number of bends is very important and will give the most optimized result. And the cost is the mapping of length and bends. The conventional SPP is to catch the minimum cost path from initial to end node and the cost is the addition of the costs of the curves on the path [1, 2, 4].

While applying in real time situations the vertices and the edges will be considered as follows. In transmission networks, telephone exchange, communication proficiency, satellites, work stations terminals and computers will be considered as the vertices and cables, wires and fiber optics will be treated as the arcs or paths and it is expected to meet transmission requirements at the minimum cost whereas in traffic control management the cost is due to only the paths with heavy traffic [8]. In the established network every path has a weight which will extend the flow in a recurrence fashion. The fusion of costs and weights proposes different ways of cost minimizing cycles. There may be cycles with negative cost which allow raise to perpetual instances and cost of minimum infinity and weight minimizing cycles which permits rise to a sink in such a way that it is inexpensive to consume a flow in an infinite cycle rather than transit to the station.

SPP plays an essential role in combinatorial optimization due to its elemental aspects and a broad range of applications. Investigating shortest paths is an essential thing in communication, computer networks, manufacturing systems and transportation. The weight of the path will represent the transportation timing from one end to other, i.e., the traveling time from the source to the destination. The efficiency of the transmission can be improved by speed up some of the routes to reduce the traveling time between some of the pairs of sources and terminals by minimizing the weights of the links. One needs some amount to reduce the traveling time by improving the road conditions for the faster traveling and the total cost supposed to be less to face the needs of the speedup [9].

In all the SPP, the source and terminal nodes should satisfy a set of conditions defined over a set of resources which associates to a quantity like the time, pickup of load by the vehicle or the duration of the break. The constraint of the resource will be given in the form of intervals which regulate the values that can be considered by the resources at each node on the path. SPP using complete graph can be encrypted as an assignment problem and is equivalent to an exceptional case of the assignment problem. Providing the shortest path is a necessary thing to the system of transport management, from a particular source node to the terminal node. The arc lengths are stimulated to represent time or cost of the transportation rather the geographical distances [10, 11].

The technique of using fuzzy numbers can be adopted for the environment with uncertainty. Crisp number is obtained from fuzzy number using defuzzification function and it is widely used in an optimization methods. SPP is not restricted to the geometric distance. Even though it is fixed, the traveling time within the cities may be represented by fuzzy variable. Since the weight of the arcs is uncertain in almost all the communication and transportation networks, it cannot be designed into crisp graphs. Dubois and Prade solved fuzzy shortest path problem for the first time. The most crucial combinatorial optimization problem is to find the SP to the directed graph and its primary format unable to represent the situations where the value of the detached function should be found not only by the preference of each single arc [15–19].

Shortest path of the network can be found using neutrosophic set (NS) by considering edge weight as neutrosophic numbers (NNs) and that may be single and interval valued, and bipolar as well [21, 22]. Samarandache described about neutrosophic for the first time in the year 1995 and proposed an important mathematical mechanism called neutrosophic set theory to handle imprecise, uncertain and indeterminate problems which cannot be dealt by fuzzy and its various type. NS is obtained by three autonomous mapping such as truth (T), indeterminacy (I) and falsity (F) and takes values from $]0^{-}, 1^{+}[$. It is very difficult to utilize NS directly.

While getting uncertainty in the set of vertices and edge then fuzzy graph can be adopted for SPP, but if there is indeterminacy exist between the relation of nodes and vertices then neutrosophic will be the appropriate concept to deal the real life problems [23]. Since indeterminacy is also treated seriously, NSs can be able to handle uncertainty in a better way [35]. The model of the NS is an important mechanism to deal with real scientific and engineering as it is able to deal uncertain, inconsistent and also indeterminate information [36]. Route maintenance or supply with uncertainty is playing a primary role in intelligent transport systems.

Due to inadequate data, as the stochastic shortest path needs accurate probability distributions, it is unable give the optimized result. Due to accuracy, adoptability and rapport to a system, single valued neutrosophic graph (SVNG) gets more attention and produce optimized solution than other types of fuzzy sets. Application of probabilities in machine learning is done by the score function. These functions play an essential role to find the minimum cost path in SPP and minimum spanning tree (MST) to UIVNGs (undirected interval valued neutrosophic graphs). When the data are in the form of intervals then that can dealt effectively by considering interval valued neutrosophic setting [40, 41]. Many group decision making methods including hybrid methods have been proposed to solve decision making problems such as supplier selection, project selection under triangular and trapezoidal neutrosophic environment [55–64].

The rest of the paper is arranged as follows. In Sect. 1.2, literature of review has been collected. In Sect. 2, over view of interval valued neutrosophic set is given. In Sect. 3, novel algorithms are proposed to find the neutrosophic shortest path under interval valued neutrosophic environment and interval valued triangular and trapezoidal neutrosophic environments with the help of proposed score function. In Sect. 4, shortcoming of the existing methods, advantages of the proposed method and comparative analysis are presented for the proposed method with the existing method. In Sect. 5, conclusion of the presented work is given.

Literature of review

The authors of, Ahuja et al. [1] proposed a different model redistributive heap as a rapid algorithm to find SP of the network. Yang et al. [2] presented a graph-theoretic strategy of rectilinear paths on bends and lengths. Ibarra and Zheng [3] proved that the single-origin shortest path problem for permutation graphs can be determined by order of the logarithmic of n. Arsham [4] examined the robustness of the shortest path problem. Tzoreff [5] examined the disconnected SPP with group path lengths. Batagelj et al. [6] proposed generalized SPP.

Zhang and Lin [7] introduced the calculation of the reverse SPP. Vasantha and Samaranadache [8] proposed primary neutrosophic algebraic framework. Also their utilization to fuzzy and NEUTROSOPHIC models as well. Roditty and Zwick [9] acquired some results associated with effective forms of the SPP. Irnich and Desaulniers [10] proposed SPP with support force. Buckley and Jowers [11] introduced SPP using the concept of fuzzy logic. Wastlund [12] analyzed the relationship between random assignment and SPP problem on the complete graph. Turner [13] attained strongly polynomial algorithms for a collection of SPP on acyclic and normal digraphs. Deng et al. [14] proposed fuzzy Dijkstra algorithm for SPP for imprecise environment.

Biswas et al. [15] introduced an algorithm for deriving shortest path in intuitionistic fuzzy environment. Arnautovic et al. [16] obtained the complement of the ant colony development for the SPP using open MP and CUDA. Gabrel and Murat [17] presented different models, methods and principle for the stability of the SPP. Grigoryan and Harutyunyan [18] proposed SPP in the Knodel graph. Rostami et al. [19] proposed quadratic SPP. Randour et al. [20] presented algorithms to incorporate the approaches with various securities on the length allocation of the paths instead of decreasing its normal value. Broumi et al. [21] solved SPP under neutrosophic setting using Dijkstra algorithm. Broumi et al. [22] introduced SPP based on triangular fuzzy neutrosophic environment.

Broumi et al. [23] proposed assertive types of SVNGs and examination of properties with validation and examples. Nancy and Harish [24] proposed an improved score function and applied in decision making process. Sahin and Liu [25] maximized method of deviation for neutrosophic decision making problem with a support of incomplete weight. Broumi et al. [26] proposed the measurements for SPP using SV-triangular neutrosophic numbers. Broumi et al. [27] calculated MST in interval valued bipolar neutrosophic (IVBN) setting. Hu and Sotirov [28] proposed amenity of semi definite programming for the quadratic SPP and performed some arithmetic operations to solve the QSPP using branch and bound algorithm. Dragan and Leitert [29] solved SPP on minimal peculiarity. Zhang et al. [30] proposed stable SPP with circulated uncertainty.

Broumi et al. [31] solved SPP using SVNG. Broumi et al. [32] solved SSP under bipolar neutrosophic environment. Peng and Dai [33] proposed interval-based algorithms based on neutrosophic environment for decision making process. Liu and You [34] proposed muirhead mean operators and employed them in decision making problem. Smarandache [35] solved SPP using trapezoidal neutrosophic knowledge. Wang et al. [36] applied SV-trapezoidal neutrosophic preference in decision making problem. Deli and Subas [37] proposed a ranking method of SVNNs and applied in decision making problem. Broumi et al. [38] proposed matrix algorithm for MST in undirected IVNG. Enayattabar et al. [39] applied Dijkstra algorithm to find the shortest path under IV Pythagorean fuzzy setting. Broumi et al. [40] proposed IVN soft graphs. Broumi et al. [41] proposed some notion with respect to neutrosophic set with triangular and trapezoidal concept and primary operations as well. Also done a contingent analysis with the existing concepts and proposed neutrosophic numbers.

Broumi et al. [42] proposed an innovative system and technique for the planning of telephone network using NG. Broumi et al. [43] proposed SPP under interval valued neutrosophic setting. Bolturk and Kahraman [44] presented a novel IVN AHP with cosine similarity measure. Wang et al. [45] proposed interval neutrosophic set and logic in detail. Biswas et al. [46] proposed distance measure using interval trapezoidal neutrosophic numbers. Deli [47] given detailed work on expansion and contraction on conventional neutrosophic soft set. Deli [48] solved a decision making problem using interval valued neutrosophic soft numbers.

Deli [49] proposed theory of npn-soft set and its application. Deli [50] proposed single valued trapezoidal neutrosophic operators and applied them in a decision making problem. Deli and Subas [51] proposed weighted geometric operators under single valued triangular neutrosophic numbers and applied in a decision making problem. Deli et al. [52] solved a decision making problem using neutrosophic soft sets. Basset et al. [53] proposed framework of hybrid neutrosophic group AND-TOPSIS for supplier selection. Chang et al. [54] experimented in detail about framework for the pattern of reuse necessary decision from theoretical perspective to practices.

Basset et al. [55] proposed a hybrid method of neutrosophic sets and method of DEMATEL to develop criteria for supplier selection. Basset et al. [56] proposed a structure based on VIKOR technique for e-government website evaluation. Basset et al. [57] Introduced a framework to evaluate cloud computing services. Basset et al. [58] proposed a hybrid method for project selection under neutrosophic environment. Basset et al. [59] proposed a new method for a neutrosophic linear programming problem.

Basset et al. [60] proposed an economic tool for risk quantification for supply chain. Basset et al. [61] proposed a framework for AHP-QFD to solve a supplier selection. Basset et al. [62] proposed neutrosophic AHP-Delphi group decision model under trapezoidal neutrosophic numbers. Basset et al. [63] solved a group decision making problem using neutrosophic analytic hierarchy process. Basset et al. [64] proposed a group decision making problem using triangular neutrosophic numbers. Kumar et al. [65] proposed an algorithm to solve neutrosophic shortest path problem under triangular and trapezoidal neutrosophic environment.

From this literature review, to the best of our knowledge, there is no contribution of research for SPP using interval neutrosophic numbers under triangular and trapezoidal environments. Additionally, this is the first study that SPP is solved by considering interval valued triangular and trapezoidal neutrosophic numbers for the length of the arc for a given network.

Overview on interval valued neutrosophic set

Here, a brief description of some basic concepts on NSs, SVNSs, IVNSs and some existing ranking functions for IVNNs are given.

Definition 2.1 [35] NS is constructed by $N = \{< x; T_N(x), I_N(x), F_N(x) >, x \in X\}$, where X be an universal set of elements x and $T_N(x), I_N(x), F_N(x) : X \to]^-0, 1^+[$ are the truth, indeterminacy and also falsity membership functions and satisfies the criterion,

$$^-0 \le T_N(x) + I_N(x) + F_N(x) \le 3^+. \tag{1}$$

Definition 2.2 [36] SVNS is defined by $\dot{N} = \{< x; T_{\dot{N}}(x), I_{\dot{N}}(x), F_{\dot{N}}(x) >, x \in X\}$ and for every

$$x \in X, \quad T_{\dot{N}}(x), I_{\dot{N}}(x), F_{\dot{N}}(x) \in [0, 1], \tag{2}$$

and the sum of these three is less than or equal to 3.

Definition 2.3 [45] An interval valued NS is defined by $\dot{N} = \left\{< x : \left[T^L_{\dot{N}}(x), T^U_{\dot{N}}(x)\right], \left[I^L_{\dot{N}}(x), I^U_{\dot{N}}(x)\right], \left[F^L_{\dot{N}}(x), F^U_{\dot{N}}(x)\right] >, x \in X\right\}$, where $T_{\dot{N}}(x) = \left[T^L_{\dot{N}}(x), T^U_{\dot{N}}(x)\right] \subseteq [0, 1]$,

$$I_{\dot{N}}(x) = \left[I^L_{\dot{N}}(x), I^U_{\dot{N}}(x)\right] \subseteq [0, 1], \quad F_{\dot{N}}(x) = \left[F^L_{\dot{N}}(x), F^U_{\dot{N}}(x)\right] \subseteq [0, 1] \text{ and} \tag{3}$$

$$0 \le \sup T_{\dot{N}}(x) + \sup I_{\dot{N}}(x) + \sup F_{\dot{N}}(x) \le 3. \tag{4}$$

Now we assume some mathematical operations on IVNNs (interval valued neutrosophic numbers).

Definition 2.4 [45] Let $\dot{N}_1 = \left\{< x : \left[T^L_{\dot{N}_1}, T^U_{\dot{N}_1}\right], \left[I^L_{\dot{N}_1}, I^U_{\dot{N}_1}\right], \left[F^L_{\dot{N}_1}, F^U_{\dot{N}_1}\right] >, x \in X\right\}$ and $\dot{N}_2 = \left\{< x : \left[T^L_{\dot{N}_2}, T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_2}, I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_2}, F^U_{\dot{N}_2}\right] >, x \in X\right\}$ be two IVNNs and $\delta > 0$ then we have the following operational laws.

$$\dot{N}_1 \oplus \dot{N}_2 = \left\langle \left[T^L_{\dot{N}_1} + T^L_{\dot{N}_2} - T^L_{\dot{N}_1} T^L_{\dot{N}_2}, T^U_{\dot{N}_1} + T^U_{\dot{N}_2} - T^U_{\dot{N}_1} T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_1} I^L_{\dot{N}_2}, I^U_{\dot{N}_1} I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_1} F^L_{\dot{N}_2}, F^U_{\dot{N}_1} F^U_{\dot{N}_2}\right] \right\rangle \tag{5}$$

$$\dot{N}_1 \otimes \dot{N}_2 = \left\langle \left[T^L_{\dot{N}_1} T^L_{\dot{N}_2}, T^U_{\dot{N}_1} T^U_{\dot{N}_2}\right], \left[I^L_{\dot{N}_1} + I^L_{\dot{N}_2} - I^L_{\dot{N}_1} I^L_{\dot{N}_2}, I^U_{\dot{N}_1} + I^U_{\dot{N}_2} - I^U_{\dot{N}_1} I^U_{\dot{N}_2}\right], \left[F^L_{\dot{N}_1} + F^L_{\dot{N}_2} - F^L_{\dot{N}_1} F^L_{\dot{N}_2}, F^U_{\dot{N}_1} + F^U_{\dot{N}_2} - F^U_{\dot{N}_1} F^U_{\dot{N}_2}\right] \right\rangle \tag{6}$$

$$\delta \dot{N} = \left\langle \left[1 - \left(1 - T^L_{\dot{N}}\right)^\delta, 1 - \left(1 - T^U_{\dot{N}}\right)^\delta\right], \left[\left(T^L_{\dot{N}}\right)^\delta, \left(T^U_{\dot{N}}\right)^\delta\right], \left[\left(F^L_{\dot{N}}\right)^\delta, \left(F^U_{\dot{N}}\right)^\delta\right] \right\rangle \tag{7}$$

$$\dot{N}^\delta = \left\langle \left[\left(T^L_{\dot{N}}\right)^\delta, \left(T^U_{\dot{N}}\right)^\delta\right], \left[1 - \left(1 - I^L_{\dot{N}}\right)^\delta, 1 - \left(1 - I^U_{\dot{N}}\right)^\delta\right], \left[1 - \left(1 - F^L_{\dot{N}}\right)^\delta, 1 - \left(1 - F^U_{\dot{N}}\right)^\delta\right] \right\rangle. \tag{8}$$

Deneutrosophication formulas for IVNNs: To compare two IVNNs $\dot{N}_1$ and $\dot{N}_2$. We use the score function (SF) which represents a map from $[N(R)]$ into the real line. In the literature there are some deneutrosophication formulas, here paper, we focus on some of types [24, 25, 33, 34, 44] defined as follows:

$$S_{\text{Bolturk}}\left(\dot{N}_1\right) = \left(\frac{\left(T^L_x + T^U_x\right)}{2} + \left(1 - \frac{\left(I^L_x + I^U_x\right)}{2}\right) * \left(I^U_x\right) - \left(\frac{\left(F^L_x + F^U_x\right)}{2}\right) * \left(1 - F^U_x\right)\right) \tag{9}$$

$$S_{\text{Ridvan}}\left(\dot{N}_1\right) = \left(\frac{1}{4}\right) \times \left(2 + T^L_x + T^U_x - 2I^L_x - 2I^U_x - F^L_x - F^U_x\right) \tag{10}$$

$$S_{\text{Peng}}\left(\dot{N}_1\right) = \left[\frac{2}{3} + \frac{\left(T_x^L + T_x^U\right)}{6} - \frac{\left(I_x^L + I_x^U\right)}{6} - \frac{\left(F_x^L + F_x^U\right)}{6}\right] \tag{11}$$

$$S_{\text{Liu}}\left(\dot{N}_1\right) = \left[2 + \frac{\left(T_x^L + T_x^U\right)}{2} - \frac{\left(I_x^L + I_x^U\right)}{2} - \frac{\left(F_x^L + F_x^U\right)}{2}\right] \tag{12}$$

$$S_{\text{Harish}}\left(\dot{N}_1\right) = \left(\frac{1}{8}\right) \times \left[4 + \left(T_x^L + T_x^U - F_x^L - F_x^U - 2I_x^L - 2I_x^U\right)\left(4 - T_x^L - T_x^U - F_x^L - F_x^U\right)\right]. \tag{13}$$

The ranking of $\dot{N}_1$ and $\dot{N}_2$ by SF is defined as follows:

(i) $\dot{N}_1 \prec \dot{N}_2$ if $\mathbb{S}\left(\dot{N}_1\right) \prec \mathbb{S}\left(\dot{N}_2\right)$

(ii) $\dot{N}_1 \succ \dot{N}_2$ if $\mathbb{S}\left(\dot{N}_1\right) \succ \mathbb{S}\left(\dot{N}_2\right)$

(iii) $\dot{N}_1 = \dot{N}_2$ if $\mathbb{S}\left(\dot{N}_1\right) = \mathbb{S}\left(\dot{N}_2\right)$

Definition 2.5 [36] Let $R_N = \left\langle \left[R_T, R_I, R_M, R_E\right], \left(T_R, I_R, F_R\right)\right\rangle$ and $S_N = \left\langle \left[S_T, S_I, S_M, S_E\right], \left(T_S, I_S, F_S\right)\right\rangle$ be two trapezoidal neutrosophic numbers (TpNNs) and $\theta \geq 0$, then

$$R_N \oplus S_N = \left\langle \left[R_T + S_T, R_I + S_I, R_M + S_M, R_E + S_E\right], \left(T_R + T_S - T_R T_S, I_R I_S, F_R F_S\right)\right\rangle \tag{14}$$

$$R_N \otimes S_N = \left\langle \left[R_T \cdot S_T, R_I \cdot S_I, R_M \cdot S_M, R_E \cdot S_E\right], \left(T_R \cdot T_S, I_R + I_S - I_R \cdot I_S, F_R + F_S - F_R \cdot F_S\right)\right\rangle \tag{15}$$

$$\theta R_N = \left\langle \left[\theta R_T, \theta R_I, \theta R_M, \theta R_E\right], \left(1 - \left(1 - T_R\right)^\theta, \left(I_R\right)^\theta, \left(F_R\right)^\theta\right)\right\rangle. \tag{16}$$

Definition 2.6 [36] Let $R = \left[R_T, R_I, R_M, R_E\right]$ and $R_T \leq R_I \leq R_M \leq R_E$ then the centre of gravity (COG) in R is

$$\text{COG}\,(R) = \begin{cases} R \text{ if } & R_T = R_I = R_M = R_E \\ \frac{1}{3}\left[R_T + R_I + R_M + R_E - \frac{R_E R_M - R_I R_T}{R_E + R_M - R_I - R_T}\right] & \text{otherwise} \end{cases}. \tag{17}$$

Definition 2.7 [36] Let $S_N = \left\langle \left[S_T, S_I, S_M, S_E\right], \left(T_S, I_S, F_S\right)\right\rangle$ be a TpNN then the score, accuracy and certainty functions are as follows

$$\mathbb{S}\left(S_N\right) = \text{COG}(R) \times \frac{\left(2 + T_S - I_S - F_S\right)}{3} \tag{18}$$

$$a\left(S_N\right) = \text{COG}(R) \times \left(T_S - F_S\right) \tag{19}$$

$$C\left(S_N\right) = \text{COG}(R) \times \left(T_S\right). \tag{20}$$

Definition 2.8 [36] Let $R_N = \left\langle \left[R_T, R_I, R_P\right], \left(T_R, I_R, F_R\right)\right\rangle$ be a triangular neutrosophic number then the score and accuracy function are,

$$\mathbb{S}\left(R_N\right) = \frac{1}{12}\left[R_T + 2 \cdot R_T + R_P\right] \times \left[2 + T_R - I_R - F_R\right] \tag{21}$$

$$a\left(R_N\right) = \frac{1}{12}\left[R_T + 2 \cdot R_T + R_P\right] \times \left[2 + T_R - I_R + F_R\right]. \tag{22}$$

Definition 2.9 [46] Let N be a trapezoidal neutrosophic number in the set of real numbers with the truth, indeterminacy and falsity membership functions are defined by

$$T_N(x) = \begin{cases} \frac{(x-a)t_N}{b-a} & , a \leq x < b \\ t_N & , b \leq x \leq c \\ \frac{(d-x)t_N}{d-c} & , c < x \leq d \\ 0 & , \text{otherwise} \end{cases} \tag{23}$$

$$I_N(x) = \begin{cases} \frac{b-x+(x-a)t_N}{b-a} & , a \leq x < b \\ i_N & , b \leq x \leq c \\ \frac{x-c+(d-x)i_N}{d-c} & , c < x \leq d \\ 0 & , \text{otherwise} \end{cases} \tag{24}$$

$$F_N(x) = \begin{cases} \frac{b-x+(x-a)f_N}{b-a} & , a \leq x < b \\ f_N & , b \leq x \leq c \\ \frac{x-c+(d-x)f_N}{d-c} & , c < x \leq d \\ 0 & , \text{otherwise} \end{cases}, \tag{25}$$

where $t_N = [t^L, t^U] \subset [0, 1], i_N = [i^L, i^U] \subset [0, 1]$ and $f_N = [f^L, f^U] \subset [0, 1]$ are interval numbers. Then the number N can be denoted by $\left([a, b, c, d]; [t^L, t^U], [i^L, i^U], [f^L, f^U]\right)$ and is called interval valued trapezoidal neutrosophic number.

- If $b = c$ in interval valued trapezoidal neutrosophic number then it becomes interval valued triangular neutrosophic number.

Proposed improved algorithm and score function

To find the length of the arc, the following algorithm and score function are proposed as follows.

Improved algorithm to solve SPP under interval valued neutrosophic number

Step 1: Determine the source node (SN) arc length $l_1 = \langle [1,1],[0,0],[0,0] \rangle$ and classify SN, node 1 by

$$\left[l_1 = \langle [1,1],[0,0],[0,0] \rangle, -\right]$$

Step 2: Find the minimum of the length of n_1 with its acquaintance node using $l_i = \min\{l_i \oplus l_{ij}\}, \quad j = 2,3,\ldots,r$.

Step 3: If there is a minimum in the node and equating to the singular measure of i (i.e., $i = k$), then classify that node j as $[l_j, k]$.

Step 4: If the minimum value exists in the node matching to more values from i then it can be concluded that there are more IVN paths between SN (i) and DN (j) and select any value of i.

Step 5: Classify the destination node (DN) (node r) by $[l_r, 1]$. Then the interval valued neutrosophic distance (IVND) among SN l_r.

Step 6: Find the IVNSP between initial and terminal node according to $[l_r, 1]$ and check the label of n_1 and is denoted by $[l_a, d]$. Classify node a and so on. Rerun the process until get n_1.

Step 7: By connecting all the nodes acquired by repeating the process in step 4, IVNSP can be found.

Note: If $\mathbb{S}(N_i) < \mathbb{S}(N_p)$ then the interval valued neutrosophic number (IVNN) is the minimum of N_p, where $N_i, i = 1,2,\ldots,r$ is the set of IVNN and $\mathbb{S}$ is the score function.

Proposed score function

The novel SF for finding the minimum cost path under interval valued neutrosophic shortest path (IVNSP) problem is provided as follows

$$\mathbb{S}_{\text{Nagarajan}}\left(\dot{N}_1\right) = \frac{1}{2}\left[\left(T_x^L + T_x^U\right) - \left(I_x^L.I_x^U\right) + \left(I_x^U - 1\right)^2 + \left(F_x^U\right)\right]. \tag{26}$$

Numerical example:

For the edge 1–2: $S_{\text{Nagarajan}}(\ddot{A}_1) = \frac{1}{2}[(0.1 + 0.2) - (0.2)(0.3) + (0.3 - 1)^2 + (0.5)] = 0.125$

For the edge 1–3: $S_{\text{Nagarajan}}(\ddot{A}_1) = \frac{1}{2}[(0.2 + 0.4) - (0.3)(0.5) + (0.5 - 1)^2 + (0.2)] = 0.2.$

Similarly for other edges.

Note: Formulas used in the proposed algorithms.

Score function used in the proposed algorithm under IVN environment and COG for TFN are

$$\mathbb{S}(\theta) = \text{COG}(R) \times \frac{1}{2}\left[T^L + T^U - \left(I^L \cdot I^U\right) + \left(I^U - 1\right)^2 + F^U\right] \tag{27}$$

$$\text{COG for TFN is } \frac{1}{3}\left[R_T + 2R_M + R_E - \frac{R_M\left(R_E - R_I\right)}{\left(R_E - R_I\right)}\right]. \tag{28}$$

Computation of shortest path using IVNNs

Illustrate to the basic process of the improved algorithm, one simple example is shown.

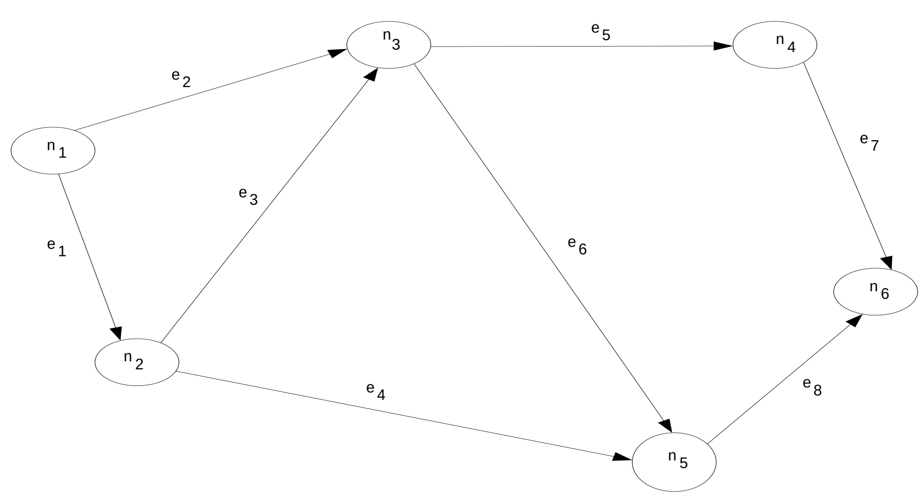

Fig. 1 Interval-valued neutrosophic network

Table 1 The details of edges information in term of IVNNs

Edges	Interval valued neutrosophic distance	Edges	Interval valued neutrosophic distance
1–2 (e_1)	([0.1, 0.2], [0.2, 0.3], [0.4, 0.5])	3–4 (e_5)	([0.2, 0.3], [0.2, 0.5], [0.4, 0.5])
1–3 (e_2)	([0.2, 0.4], [0.3, 0.5], [0.1, 0.2])	3–5 (e_6)	([0.3, 0.6], [0.1, 0.2], [0.1, 0.4])
2–3 (e_3)	([0.3, 0.4], [0.1, 0.2], [0.3, 0.5])	4–6 (e_7)	([0.4, 0.6], [0.2, 0.4], [0.1, 0.3])
2–5 (e_4)	([0.1, 0.3], [0.3, 0.4], [0.2, 0.3])	5–6 (e_8)	([0.2, 0.3], [0.3, 0.4], [0.1, 0.5])

Illustrative example

This section is based on a numerical problem adapted from Broumi et al. [40] to show the potential application of the proposed algorithm and score function.

Consider a network Fig. 1 with six nodes and eight edges with SN, node 1 and DN, node 6. The interval valued neutrosophic distance is given in Table 1.

In this situation, we need to evaluate the shortest distance from SN, i.e., node 1 to DN, i.e., node 6.

Calculating the shortest path using proposed algorithm of interval valued neutrosophic path problem is given as follows.

Here $r = 6$, since there are totally 6 nodes.

Let, $l_1 = \langle [1,1],[0,0],[0,0]\rangle$ and classify the SN $n_1 = \left[\langle [1,1],[0,0],[0,0]\rangle, -\right]$.

To find the value of l_j, $\quad j = 2,3,4,5,6$.

Iteration no. 1:

Since n_2 has only n_1 as the predecessor, let $i = 1, \quad j = 2$ in step 2.

To find l_2:

$$l_2 = \min\left\{l_1 \oplus l_{12}\right\}$$

$$= \min\{\langle [1,1],[0,0],[0,0]\rangle \oplus \langle [0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle\}$$
$$= \langle [0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle.$$

Since, minimum occurs for $i = 1$, classify the node $n_2 = \left[\langle [0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle, 1\right]$.

Iteration no. 2:

Since n_3 has two predecessors n_1 and n_2, let $i = 1,2 \,\&\, j = 3$ in step 2.

To find l_3:

$$l_3 = \min\left\{l_1 \oplus l_{13}, l_2 \oplus l_{23}\right\}$$

$$= \min\{\langle [1,1],[0,0],[0,0]\rangle \oplus \langle [0.2,0.4],[0.3,0.5],[0.1,0.2]\rangle,$$
$$\langle [0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle \oplus \langle [0.3,0.4],[0.1,0.2],[0.3,0.5]\rangle\}$$

$$= \min\{\langle [1+0.2-1(0.2), 1+0.4-1(0.4)],$$
$$[0(0.3), 0(0.5)], [0(0.1), 0(0.2)]\rangle,$$
$$\langle [0.1+0.3-(0.1)(0.3), 0.2+0.4-(0.2)(0.4)],$$
$$[(0.2)(0.1), (0.3)(0.2)], [(0.4)(0.5), (0.5)(0.5)]\rangle\}$$

$$= \min\{\langle [1,1],[0,0],[0,0]\rangle, \langle [0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle\}$$
$$= \langle [0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle.$$

Since the score function values are,

$$\mathbb{S}(\langle [1,1],[0,0],[0,0]\rangle$$
$$= \frac{1}{2}\left[(1+1)-(0\times 0)+(0-1)^2+0\right] = 1.5$$

$$\mathbb{S}(\langle [0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle)$$
$$= \frac{1}{2}\left[(0.37+0.52)-(0.02\times 0.06)+(0.06-1)^2+0.25\right]$$
$$= 0.9$$

and the minimum occurs for $i = 2$, then classify the node $n_3 = \left[\langle [0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle, 2\right]$.

Iteration no. 3:

Since n_4 has one predecessors n_3, let $i = 3 \,\&\, j = 4$ in step 2.

To find the value of l_4:

$$l_4 = \min\left\{l_3 \oplus l_{34}\right\}$$
$$= \min\{\langle [0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle$$
$$\oplus \langle [0.2,0.3],[0.2,0.5],[0.4,0.5]\rangle\}$$
$$= \langle [0.6,0.67],[0.004,0.018],[0.048,0.125]\rangle.$$

Since, minimum occurs for $i = 3$, hence classify the node $n_4 = \left[\langle [0.6,0.67],[0.004,0.018],[0.048,0.125]\rangle, 3\right]$.

Iteration no. 4:

Since n_5 has two predecessors n_2 and n_3, let $i = 2,3 \& j = 5$ in step 2.

To find the value of l_5:

$$l_5 = \min\left\{l_2 \oplus l_{25}, l_3 \oplus l_{35}\right\}$$

$$= \min\{\langle [0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle$$
$$\oplus \langle [0.1,0.3],[0.3,0.4],[0.2,0.3]\rangle,$$
$$\langle [0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle$$
$$\oplus \langle [0.3,0.6],[0.1,0.2],[0.1,0.4]\rangle\}$$

$$= \min\{\langle [0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle,$$
$$\langle [0.56,0.81],[0.002,0.012],[0.012,0.1]\rangle\}$$
$$= \langle [0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle.$$

Since the score function values are,

$$\mathbb{S}(\langle [0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle) = 0.75$$

$\mathbb{S}(\langle[0.56,0.81],[0.002,0.012],[0.012,0.1]\rangle)=1$

and the minimum occurs for $i=2$, hence classify the node $n_5=\left[\langle[0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle,2\right]$

Iteration no. 5:

Since n_6 has two predecessors n_4 and n_5, let $i=4,5$ & $j=6$ in step 2.

To find the value of l_6:

$$l_6=\min\left\{l_4\oplus l_{46},l_5\oplus l_{56}\right\}$$

$$\begin{aligned}&=\min\{\langle[0.6,0.67],[0.004,0.018],[0.048,0.125]\rangle\\&\quad\oplus\langle[0.4,0.6],[0.2,0.4],[0.1,0.3]\rangle,\\&\quad\langle[0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle\\&\quad\oplus\langle[0.2,0.3],[0.3,0.4],[0.1,0.5]\rangle\}\end{aligned}$$

$$\begin{aligned}&=\min\{\langle[0.76,0.87],[0.008,0.0018],[0.0048,0.0375]\rangle,\\&\quad\langle[0.352,0.63],[0.018,0.048],[0.008,0.075]\rangle\}\\&=\langle[0.35,0.63],[0.018,0.048],[0.008,0.075]\rangle.\end{aligned}$$

Since the score function values are,

$\mathbb{S}(\langle[0.76,0.87],[0.008,0.0018],[0.0048,0.0375]\rangle)=1$

$\mathbb{S}(\langle\langle[0.352,0.63],[0.018,0.048],[0.008,0.075]\rangle\rangle)=0.82$

and the minimum occurs for $i=5$ hence classify $n_6=\left[\langle[0.35,0.63],[0.018,0.048],[0.008,0.075]\rangle,5\right]$.

Since n_6 is the DN of the given network, IVNSP between n_1 and n_6 is $\langle[0.35,0.63],[0.018,0.048],[0.008,0.075]\rangle$.

Now, IVNSP from n_1 and n_6 is obtained as follows.

Since, $n_6=\left[\langle[0.35,0.63],[0.018,0.048],[0.008,0.075]\rangle,5\right]\Rightarrow$ a person is coming from $5\to 6$ $n_5=\left[\langle[0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle,2\right]\Rightarrow$ a person is coming from $2\to 5$ $n_2=\left[\langle[0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle,1\right]\Rightarrow$ a person is coming from $1\to 2$.

By joining all the acquired nodes, interval valued neutrosophic shortest path from n_1 and n_6 is obtained.

Hence IVNSP of the given network is $1\to 2\to 5\to 6$.

The IVNS distance and IVNSP of all the nodes from SN node 1 in the below Table 2 and the classification of all the nodes are shown in Fig. 2.

The following table is formed using different deneutrosophic functions called score functions for all the possible edges and using proposed improved score function in the last column (Table 3).

According to the improved score function proposed in Sect. 3, the shortest path from node one to node six can be computed as follows (Table 4).

Therefore, the path $P:1\to 2\to 5\to 6.$ is identified as the neutrosophic shortest path.

Algorithm: a new approach to find SPP using TpIVNN and TIVNN

Consider a directed and noncyclic graph, where the length of the arcs is represented by IVNN. The introduced algorithm

Table 2 Interval valued neutrosophic shortest path

Node number (j)	l_i	IVNSP between jth and node 1
2	$\langle[0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle$	$1\to 2$
3	$\langle[0.37,0.52],[0.02,0.06],[0.12,0.25]\rangle$	$1\to 2\to 3$
4	$\langle[0.6,0.67],[0.004,0.018],[0.048,0.125]\rangle$	$1\to 2\to 3\to 4$
5	$\langle[0.19,0.47],[0.06,0.12],[0.08,0.15]\rangle$	$1\to 2\to 5$
6	$\langle[0.35,0.63],[0.018,0.048],[0.008,0.075]\rangle$	$1\to 2\to 5\to 6$

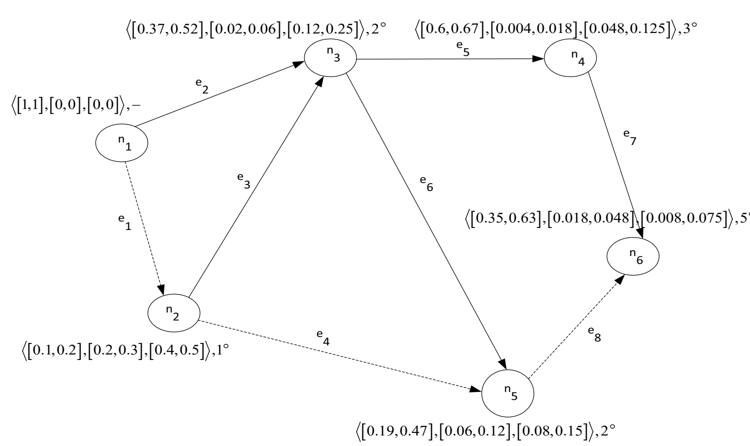

Fig. 2 Interval-valued neutrosophic shortest path

Table 3 Different deneutrosophication value of edge (i, j)

Edges	S_{Ridvan} [43]	$\mathbb{S}_{\text{Nagarajan}}$
1–2	0.1	**0.125**
1–3	0.175	**0.2**
2–3	0.325	**0.17**
2–5	0.125	**0.11**
3–4	0.05	**0.325**
3–5	0.45	**0.32**
4–6	0.35	**0.43**
5–6	0.125	**0.26**

Table 4 Crisp path length for proposed algorithm

The proposed algorithm based $\mathbb{S}_{\text{Nagarajan}}$	Crisp path length	Ranking
$1 \to 2 \to 5 \to 6$	**0.485**	**1**
$\mathbf{1 \to 3 \to 5 \to 6}$	0.78	2
$\mathbf{1 \to 2 \to 3 \to 5 \to 6}$	0.875	3
$\mathbf{1 \to 3 \to 4 \to 6}$	0.955	4
$\mathbf{1 \to 2 \to 3 \to 4 \to 6}$	1.05	5

determines the shortest path from the initial node to the terminal node. The algorithm is described as follows.

Step 1: Let n be the total number of paths from the initial node to terminal one. Find the score function of every arc length for the given network using Eqs. (18), (19) and (24), (25).

Step 2: Find all the available paths P_i, $i = 1, 2, \ldots, n$ and the corresponding path length. Also every n paths can be dealt as an arc which are represented by IVNN.

Step 3: Find the sum of all score functions $\mathbb{S}(\theta_i)$ of each available path.

Step 4: The path which have minimum score value will represent an optimized interval valued shortest path by ranking in ascending order.

End

Note: TpIVNN-Trapezoidal interval valued neutrosophic number.

TIVNN-Triangular interval valued neutrosophic number.

Table 6 Available paths and its score value

Available path	$\mathbb{S}(\theta_i)$	Ranking
$P_1 : 1 \to 2 \to 5 \to 6$	4.18	1
$P_2 : 1 \to 3 \to 5 \to 6$	8.25	2
$P_4 : 1 \to 3 \to 4 \to 6$	12.43	3
$P_3 : 1 \to 2 \to 3 \to 5 \to 6$	13.31	4
$P_5 : 1 \to 2 \to 3 \to 4 \to 6$	17.5	5

Illustrative example to find the shortest path using TpIVNN

For the validation of the proposed algorithm, a network is adopted from Broumi et al. [43] and Kumar et al. [65].

Consider a network with six nodes and eight edges. The TpIVN cost is given below (Tables 5, 6).

Applying steps 1–4 of the proposed algorithm, it if found that $1 \to 2 \to 5 \to 6$ is IVNP with lowest cost 4.18 and the IVNP is $\langle(4, 11, 15, 20); [0.35, 0.608], [0.018, 0.048], [0.008, 0.075]\rangle$.

Illustrative example to find the shortest path using TIVNN

For the validation of the proposed algorithm, an example network is adopted from Broumi et al. [26, 35].

Consider a network with six nodes and eight edges. The TIVN cost is given below (Tables 7, 8).

Applying steps 1–4 of the proposed algorithm, it if found that $1 \to 2 \to 5 \to 6$ is IVNP with lowest cost 4.18 and the IVNP is $\langle(4, 11, 15); [0.35, 0.61], [0.02, 0.05], [0.01, 0.08]\rangle$.

Comparative study of the proposed algorithm

In this section, a comparative study is carried out with the shortcomings and advantage of the proposed algorithm and it shows the effectiveness of the proposed algorithm

Shortcoming of the existing method

The compared existing method is unable to handle the interval-based information about the length of the arc and

Table 5 Trapezoidal interval valued neutrosophic distance

Edges	Trapezoidal interval valued neutrosophic distance	Edges	Trapezoidal interval valued neutrosophic distance
1–2 (e_1)	$\langle(1, 2, 3, 4); [0.1, 0.2], [0.2, 0.3], [0.4, 0.5]\rangle$	3–4 (e_5)	$\langle(2, 4, 8, 9); [0.2, 0.3], [0.2, 0.5], [0.4, 0.5]\rangle$
1–3 (e_2)	$\langle(2, 5, 7, 8); [0.2, 0.4], [0.3, 0.5], [0.1, 0.2]\rangle$	3–5 (e_6)	$\langle(3, 4, 5, 10); [0.3, 0.6], [0.1, 0.2], [0.1, 0.4]\rangle$
2–3 (e_3)	$\langle(3, 7, 8, 9); [0.3, 0.4], [0.1, 0.2], [0.3, 0.5]\rangle$	4–6 (e_7)	$\langle(7, 8, 9, 10); [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]\rangle$
2–5 (e_4)	$\langle(1, 5, 7, 9); [0.1, 0.3], [0.3, 0.4], [0.2, 0.3]\rangle$	5–6 (e_8)	$\langle(2, 4, 5, 7); [0.2, 0.3], [0.3, 0.4], [0.1, 0.5]\rangle$

Table 7 Triangular interval valued neutrosophic distance

Edges	Triangular interval valued neutrosophic distance	Edges	Triangular interval valued neutrosophic distance
1–2 (e_1)	$\langle(1,2,3);[0.1,0.2],[0.2,0.3],[0.4,0.5]\rangle$	3–4 (e_5)	$\langle(2,4,8);[0.2,0.3],[0.2,0.5],[0.4,0.5]\rangle$
1–3 (e_2)	$\langle(2,5,7);[0.2,0.4],[0.3,0.5],[0.1,0.2]\rangle$	3–5 (e_6)	$\langle(3,4,5);[0.3,0.6],[0.1,0.2],[0.1,0.4]\rangle$
2–3 (e_3)	$\langle(3,7,8);[0.3,0.4],[0.1,0.2],[0.3,0.5]\rangle$	4–6 (e_7)	$\langle(7,8,9);[0.4,0.6],[0.2,0.4],[0.1,0.3]\rangle$
2–5 (e_4)	$\langle(1,5,7);[0.1,0.3],[0.3,0.4],[0.2,0.3]\rangle$	5–6 (e_8)	$\langle(2,4,5);[0.2,0.3],[0.3,0.4],[0.1,0.5]\rangle$

Table 8 Available paths and its score value

Available path	$\mathbb{S}(\theta_i)$	Ranking
$P_1: 1 \to 2 \to 5 \to 6$	4.9	1
$P_2: 1 \to 3 \to 5 \to 6$	8.27	2
$P_4: 1 \to 3 \to 4 \to 6$	11.1	3
$P_3: 1 \to 2 \to 3 \to 5 \to 6$	12.86	4
$P_5: 1 \to 2 \to 3 \to 4 \to 6$	15.69	5

shortest path cannot be obtained for interval-based neutrosophic network.

Advantage of the proposed algorithm

If the length of the path is interval-based one then the shortest path of the given network can be obtained by interval valued neutrosophic numbers for an optimized path. Since triangular and trapezoidal numbers are widely used in many of the real world applications for their simplicity of computation, interval valued triangular and trapezoidal neutrosophic numbers have been used to find the neutrosophic shortest path. This is the advantage of the proposed algorithm.

Comparative study of algorithm

This section provides a comparative study of the proposed algorithm with the existing method of for neutrosophic shortest path problems.

A comparison of the results between existing and new techniques is shown in Table 9.

The result shows that the proposed algorithm provides sequence of visited nodes which shown to be similar with neutrosophic shortest path.

Table 9 Comparison of sequence of nodes using neutrosophic shortest path and our proposed algorithm

Algorithm of Broumi	Path	Crisp path length
S_{Ridvan} [43]	$1 \to 2 \to 5 \to 6$	0.35
$S_{\text{Nagarajan}}$	$1 \to 2 \to 5 \to 6$	0.485

The neutrosophic shortest path (abbr.NSP) remains the same namely $1 \to 2 \to 5 \to 6$, but the crisp shortest path length (CSPL) differs namely $\langle[0.35,0.60],[0.01,0.04],[0.008,0.075]\rangle$, respectively. From here we come to the conclusion that there exists no unique method for comparing neutrosophic numbers and different methods may satisfy different desirable criteria (Table 10).

Conclusion and future implication

The heart of the network community is nothing but the SPP. The objective of this problem is finding the minimum cost path among all other paths. This issue has been solved using many methods starts from conventional SPP with crisp weights. As many of the real world applications have uncertain vertices and edges fuzzy environment was useful to handle this problem. But still fuzzy setting cannot handle indeterminacy of the information, neutrosophic sets are found to be the best choice to handle this issue and has applied successfully. In this paper, neutrosophic SPP has been solved under interval valued neutrosophic, trapezoidal and triangular interval valued neutrosophic environments as it handles interval values. Also an improved score function and center of gravity has been proposed and applied to find the minimum cost of the path. Our proposed score function is without having the lower membership of falsity and which saves the time naturally. Further comparative analysis is done for Broumi's algorithm with different

Table 10 Sequence of nodes with shortest path length

Possible path	Sequence of nodes	Neutrosophic shortest path length
Neutrosophic shortest path with interval valued neutrosophic numbers [43]	$1 \to 2 \to 5 \to 6$	$\langle[0.35,0.60],[0.01,0.04],[0.008,0.075]\rangle$
Proposed algorithm on $S_{\text{Nagarajan}}$	$1 \to 2 \to 5 \to 6$	$\langle[0.35,0.60],[0.01,0.04],[0.008,0.075]\rangle$

deneutrosophication function and proposed one. It is found that minimum cost is less compare than other existing method using proposed algorithms and score function. Also the proposed algorithm and improved score function have less computational complexity and saves the time. In future, the SPP would be extended to neutrosophic soft and rough set environments for interval-based path lengths. Also the proposed concept will be extended to complex neutrosophic environment.

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Word-level neutrosophic sentiment similarity

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H I G H L I G H T S

- A new word-level similarity measure defined by means of the words' sentiment scores.
- The similarity measure is defined without considering the words' lexical category.
- The resulted scores correctly distance the neutral words from the sentiment words.

Keywords:
Word-level similarity
Neutrosophic sets
Sentiwordnet
Sentiment relatedness

A B S T R A C T

In the specialized literature, there are many approaches developed for capturing textual measures: textual similarity, textual readability and textual sentiment. This paper proposes a new sentiment similarity measures between pairs of words using a fuzzy-based approach in which words are considered *single-valued neutrosophic sets*. We build our study with the aid of the lexical resource *SentiWordNet* 3.0 as our intended scope is to design a new word-level similarity measure calculated by means of the sentiment scores of the involved words. Our study pays attention to the polysemous words because these words are a real challenge for any application that processes natural language data. After our knowledge, this approach is quite new in the literature and the obtained results give us hope for further investigations.

1. Introduction

Semantic textual similarity is a measure of the degree of semantic equivalence between some pieces of texts [1]. This measure is exploited in many natural language processing (NLP) tasks, very actual at the present moment, such as paraphrase recognition [2], tweets search [3], image retrieval by caption [4,5], query reformulation [6] or automatic machine translation evaluation [7]. In information retrieval (IR) the user's query is usually expressed by means of a short sequence of words based on which the most similar documents related to the query must be returned to the user.

On the other hand, textual sentiment analysis consists of measuring the attitude or emotional affect of the text. Using this kind of data very actual research fields such as affective computing or sentiment analysis can understand and predict human emotions [8] as their basic tasks are emotion recognition [9,10] and polarity detection [11–14]. Emotion recognition means to find a set of emotion triggers while polarity detection is usually designed as a binary classifier with "positive" and "negative" outputs [15,16].

In a world full of indeterminacy [17] the reality cannot be drawn only using two colours: "white" and "black" or "positive" and "negative" or "true" and "false" because uncertainty plays a determinant role. Fuzzy set theory has been used in many studies where uncertainty plays a determinant role. Natural language texts contain large amount of uncertain information [18] mainly caused by: 1. the polysemy of same words (for example, the English word "line" has more than 20 distinct senses); 2. the fact that different words can have the same mining (for example "stomach pain" and "belly ache"); 3. the ambiguities of natural language construction which can happen at many levels of analysis, both syntactic and semantic, which imply different interpretations for the same words or phrases. If we consider also the natural diversity in subjectivity of any natural language utterance, we can conclude that this domain can be regarded as uncertain one.

To deal with large amount of uncertain knowledge, many fuzzy based systems have been developed, but they still remained weak explored in the domain of identifying the sentiment orientation of sentences. The detection of the polarity or subjectivity predictors in written text usually implies to compute the terms grade membership in various pre-defined or computed categories [19,20]. These studies usually require a pre-defined sentiment lexicon for detecting the sentiment words. If this step ends successfully, they have to compute the distance between the identified words and the class centroid in order to measure the fuzzy membership [21–23]. Each membership function is interpreted as the appurtenance degree of the analysed piece of text to a certain sentiment class [24].

These systems could benefit from on a robust word-level similarity component. Most of the existing approaches for determining the semantic similarity between words do not incorporate the words' sentiment information. The present study focuses on the task of measuring the sentiment similarity at a word-level.

Sentiment similarity indicates the similarity of word pairs from their underlying sentiments. In the linguistic literature, sentiment similarity has not received enough attention. In fact, the majority of previous works employed semantic similarity as a measure to also compute the sentiment similarity of word pairs [25,26]. Nevertheless, some works stated that sentiment similarity can reflect better the similarity between sentiment words than semantic similarity measures [27].

Following [28] we consider that the sentiment information is crucial in finding the similarity between two concepts, in particular, between two words. In this assumption, in this study we propose a new sentiment similarity measure between pairs of words using a neutrosophic approach [29–33] and with the aid of the SentiWordNet 3.0 [34] lexical resource. Our intended scope is to suggest a new measure for the sentiment similarity degree of two words which takes into account not only the "positive" and "negative" sentiment labels but also their more refined derivates such as: "objective", "weak positive", "weak negative", "strong positive" and "strong negative".

1.1. Justification

An important number of word-level similarity measures were defined using lexico-semantic information. Based on the syntactic category of the involved words we can have a *similarity measures* or a *relatedness measures*. Most similarity measures are computed for words within the same category, usually for nouns and verbs. Still, many similarity approaches consider the semantics and not the lexical category in the process of similarity findings as in the case when the verb "mary" should be found semantically equivalent with nouns such as "wife" or "husband" [1] and not necessarily with another verb.

Corresponding, the relatedness measures are used to compute the similarity degree between words with different categories, e.g. between a noun and a verb such as "tears" and "to cry" [35]. Nevertheless, this restriction is not always obey, as many word similarity measures are developed without paying attention to the syntactic category of the involved words [36]. When defining our proposal we do not differentiate words upon their part of speech as we consider the *sentiment similarity* just as the inverse difference value between the sentiment polarities of two words. Thus, in what follows, the terms similarity and relatedness will be considered equivalent.

There is another important aspect of the proposed measure: it has a symmetric dimension, following thus the key assumption of the most similarity models even if this idea is not universally true, especially when it comes to model human similarity judgments [37]. "Asymmetrical similarity occurs when an object with many features is judged as less similar to a sparser object than vice versa" [38] such as, for example, when comparing a very frequent word with an infrequent word as "boat" with "dinghy" [37].

The reason we choose a *symmetric* measure to model the proposed word-level similarity measure is determined by two aspects of the study:

1. it treats the words as independent entities, defined only by their SentiWordNet scores and therefore, additional information such as word frequency are not considered
2. by following a neutrosophic approach, the proposed method aggregates all the scores corresponding to all the senses a word can have in a *single-valued neutrosophic set* representation and thus, information about a particular sense are not computed and the words are treated as entities with a single facet

1.2. WordNet

WordNet thesaurus is a collection of nouns, verbs, adjectives and adverbs, being a graph-formed dictionary with a unique organization based on word sense and synonyms [39]. Graph-based structures are widely used in natural language processing applications such as [40,41]. In WordNet structure there are two main forms of word representations: lemma and synset [42]. The synsets are considered "logical groups of cognitive synonyms" or "logical groups of word forms" which are inter-connected by "semantic pointers" with the purpose of describing the semantic relatedness between the connected synsets. These relations were used to find similarity measures between word senses based on the lengths of the relationships between them.

The "net" structure of the WordNet is constructed by means of the lexical or conceptual links differentiated upon the part of speech of the words from the connected synsets. The noun synsets are connected through the "hyperonymy" (and its inverse, "hyponymy") and the "meronymy" (and its inverse, "holonymy") relations. The verbs are linked through the "troponym", "hypernym" and "entailment" relations. Adjectives point to their antonyms or to the related nouns while adverbs are linked to adjectives through the "pertainym" relation.

1.3. SentiWordNet as a sentiment lexicon

SentiWordNet extends the usability of WordNet to another dimension, by mapping a large number of WordNet synsets to sentiment scores indicating their "positivity", "negativity" and "objectivity" [42]. Always, the sum of these three values is 1.0.

Because SentiWordNet is built upon the WordNet data, the common problem that is observed at WordNet appears also at SentiWordNet senses: the too fine-grained synsets make hard the distinguishing between the senses of a word. As a direct consequence, the scoring of synsets are even more difficult to predict. The main problem is how much the related synsets and glosses or even the terms of the same synset share or not the same sentiment.

Table 1 presents some sentiment scores examples of the most positive and the most negative words' senses in SentiWordNet [43]. It is important to mention that all the SentiWordNet scores were obtained after weighting 8 classifiers and averaging their classifications [44].

With the construction of this lexical resource, a wide category of tasks, usually in the domain of Opinion Mining (or Sentiment Analysis) started to take shape. Here are three categories of tasks that can be implemented by making usage of the synsets sentiment scores [44]:

Table 1
Example of scores in SentiWordNet [43].

Synsets & sentiment score	Positive score	Negative score	Neutral score
good#1 (0.75, 0, 0.25)	0.75	0	0.25
superb#1 (0.875, 0, 0.125)	0.875	0	0.125
abject#1 (0, 1, 0)	0	1	0
bad#1 (0, 0.625, 0.325)	0	0.625	0.325
unfortunate#1 (0, 0.125, 0.875)	0	0.125	0.875

– *subjectivity–objectivity polarity*: its scope is to determine whether the given text is subjective or objective [11,45];
– *positivity–negativity polarity*: its scope is to determine whether the text is positive or negative on its subject matter [11,46];
– *strength of the positivity–negativity polarity*: its scope is to determine how positive or negative the given text is. More precisely, these tasks have to decide if the opinion expressed by a text is weakly or strongly positive/negative [12,29];
– *extracting opinions from a text*, which firstly implies to determine if the given text includes an opinion or not, and (if it is the case) to determine the author of the opinion, the opinion subject and/or the opinion type [26].

Sentiment analysis was defined for textual content analysis but recent studies perform this kind of analysis on visual content such as images and videos [4]. Performing sentiment analysis on visual content implies to identify the "visual concepts that are strongly related to sentiments" and to label these concepts with few lexical terms (for example, in [4] the authors propose a visual labelling mechanism by means of adjective-noun pairs as usually opinion detection is based on the examination of adjectives in sentences [19]).

This paper is dedicated to the problem of sentiment similarity between pairs of words using a neutrosophic approach in which a word is interpreted as a *single-valued neutrosophic set* [47,48]. At our knowledge, this is the second study that addresses the problem of words sentiment data using neutrosophic concepts. With the intended scope of filling the gap concerning the objectivity aspect of some words, the previous study [49] addresses the problem of the so-called "neutral words" with the aid of neutrosophic measures applied on the words' sentiment scores.

The study presented in this paper includes and extends the work initiated in [49] as it addresses all types of words, whether sentiment words or objective words. The proposed formalism can be used in any sentiment analysis task as it determines the sentiment polarity of a word by computing its similarity with some seed words (words whose sentiment labels are known or provided). The considered similarity measures can be of great help also for the text similarity techniques that pair the words of the involved texts in order to quantify the degree to which the analysed texts are semantically related [1,50]. In these techniques, pairs of text sequences are aligned based on the similarity measures of their component words.

The remainder of the paper is organized as follows: in the following section we summarize the most recent studies in the domain of similarity measures with focus on the investigated neutrosophic concepts. Section 3 describes the method we designed for constructing a new word-level similarity measure using the sentiment scores of the involved words and applying the neutrosophic theory. In Section 4 the evaluation results are given. The final section sketches the conclusions and the future plan directions.

2. Similarity measures. Related works

There is an important number of works concerning the semantic similarity with different levels of granularity starting from the word-to-word similarity to the document-to-document similarity (important issue for any search engine) [1,35].

Many approaches have been proposed with the intended scope of capturing the semantic similarity between words: Latent Semantic Analysis (LSA) [51], Point-wise Mutual Information (PMI) [52] (for estimate the sentiment orientation) or numerous WordNet based similarity measures. Much attention has recently been given to calculating the similarity of word senses, in support of various natural language learning and processing tasks. One can use the shortest path or the Least Common Subsumer (LCS) depth length algorithm to calculate the distance between the nodes (words) as a measure of similarity between word senses [36,42]. One difficulty here is that some words have different meanings (senses) in different contexts, and thus different scores for each sense.

Such techniques can be applied within a semantic hierarchy, or ontology, such as WordNet. WordNet acts as a thesaurus, in that it groups words together based on their meanings. The semantic distance between words can be estimated as the number of vertices that connect the two words. Another approach makes usage of a large corpus (e.g. Wikipedia) to count the terms that appear close to the words being analysed in order to construct two vectors and compute a distance (e.g. cosine). In this method, the similarity degree between the two entities is given by the cosine value of the angle determined by their vectors representation [53].

The similarity problems are also modelled using concepts from fuzzy set theory and it is our belief (which will be further proved) that neutrosophic theory, that was defined in order to generalize the concepts of classic set and fuzzy set, offers more appropriate tools. Indeed, in a *Neutrosophic Set* the indeterminacy, which is so often encountered in real-life problems such as decision support [54], is quantified explicitly [30,31] as it will be shown in what follows.

2.1. Fuzzy and neutrosophic sets

A *fuzzy set* is built from a reference set called universe of discourse which is never fuzzy. Let us consider U - the universe of discourse. A fuzzy set A over U is defined as:

$$A = \{(x_i, \mu_A(x_i)) \mid x_i \in U\}$$

where $\mu_A(x_i) \in [0, 1]$ represents the membership degree of the element $x_i \in U$ in the set A [55,56].

Now, if we take A be a *intuitionistic fuzzy set* (IFS) in the universe of discourse U, then the set A is defined as [57]:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in U\}$$

where $\mu_A(x) : U \to [0, 1]$ is the membership degree and $\nu_A(x) : U \to [0, 1]$ represents the non-membership degree of the element $x \in U$ in A, with $0 \le \mu_A(x) + \nu_A(x) \le 1$.

The concept of *neutrosophic set A* in the universe of discourse U is defined as an object having the form [47]:

$$A = \{< x : t_A(x), i_A(x), f_A(x) >, x \in U\}$$

where the functions $t_A(x), i_A(x), f_A(x) : U \to [0, 1]$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of a generic element $x \in U$ to the set A.

If on a neutrosophic set A we impose the following condition on the membership functions $t_A, i_A, f_A : U \to [0, 1]$:

$$0 \le t_A + i_A + f_A \le 3, x \in A$$

then the resulted set $A \subset U$ is called a *single-valued neutrosophic set* [58]. We can also write $x(t_A, i_A, f_A) \in A$.

Corresponding to the notions of neutrosophic set and single-valued neutrosophic set, similar works have been done on graph-theory resulting the notions of neutrosophic graphs [59] and single-valued neutrosophic graphs [60] and on number-theory resulting the concept of neutrosophic numbers and single valued trapezoidal neutrosophic number [61,62].

2.2. Neutrosophic similarity measures

Neutrosophic distance and similarity measures were applied in many scientific fields such as decision making [63,64], pattern recognition [65,66], medical diagnosis [67,68] or market prediction [69].

In this section we enumerate the similarity measures together with their complements — the distance measures, that are applied and then compared in the proposed neutrosophic method for words similarity (see Section 3).

Intuitionistic fuzzy similarity measure between two IFSs A and B satisfies the following properties [70]:

(1) $0 \leq S(A, B) \leq 1$
(2) $S(A, B) = 1$ if $A = B$
(3) $S(A, B) = S(B, A)$
(4) $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$ if $A \subseteq B \subseteq C$ for any A, B, C - intuitionistic fuzzy sets.

We have that similarity and distance (dissimilarity) measures are complementary, which implies $S(A, B) = 1 - d(A, B)$.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in U\}$, $B = \{(x, \mu_B(x), \nu_B(x)) \mid x \in U\}$ be two IFSs in the universe $U = \{x_1, \ldots, x_n\}$. Several distance measures between A and B were proposed in the literature, from which we consider here only the *Normalized Euclidean distance* for two IFSs [71]:

$$d_{IE}(A, B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}((\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)} \quad (1)$$

which will be called in what follows as *Intuitionistic Euclidean distance measure*.

In general a *similarity measure* between two single-value neutrosophic sets A and B is a function defined as [33,72,73]:

$S : NS(X)^2 \rightarrow [0, 1]$

where NS denotes the *Neutrosophic Set* concept.

The *Euclidean distance* or the *Euclidean dissimilarity measure* between two single-value neutrosophic elements $x_1(t_A^1, i_A^1, f_A^1)$, $x_2(t_A^2, i_A^2, f_A^2) \in A$ is defined as [72,73]:

$$d_E(x_1, x_2) = \sqrt{\frac{1}{3}[(t_A^1 - t_A^2)^2 + (i_A^1 - i_A^2)^2 + (f_A^1 - f_A^2)^2]} \quad (2)$$

Properties of the Euclidean distance. If x_1 and x_2 are two neutrosophic elements and $d_E(x_1, x_2)$ denotes the *Euclidean distance* as in definition (2), then the following properties are fulfilled:

1. $d_E(x_1, x_2) \in [0, 1]$
2. $d_E(x_1, x_2) = 0$ if and only if $x_1 = x_2$ (or $t_A^1 = t_A^2$, $i_A^1 = i_A^2$ and $f_A^1 = f_A^2$)
3. $d_E(x_1, x_2) = 1$ if and only if $\mid t_A^1 - t_A^2 \mid = \mid i_A^1 - i_A^2 \mid = \mid f_A^1 - f_A^2 \mid = 1$
   For examples: $x_1(1, 1, 1)$ and $x_2(0, 0, 0)$; or $x_1(1, 0, 0)$ and $x_2(0, 1, 1)$; or $x_1(0, 1, 0)$ and $x_2(1, 0, 1)$, etc.

The *Euclidean similarity measure* or the complement of the *Euclidean distance* between two neutrosophic elements $x_1(t_A^1, i_A^1, f_A^1)$, $x_2(t_A^2, i_A^2, f_A^2) \in A$ is defined as [72,73]:

$$s_E(x_1, x_2) = 1 - d_E(x_1, x_2)$$

$$= 1 - \sqrt{\frac{1}{3}[(t_A^1 - t_A^2)^2 + (i_A^1 - i_A^2)^2 + (f_A^1 - f_A^2)^2]} \quad (3)$$

Properties of the Euclidean similarity measure. If x_1 and x_2 are two neutrosophic elements and $s_E(x_1, x_2)$ denotes the *Euclidean similarity measure* as in definition (3), then the following properties are fulfilled:

1. $s_E(x_1, x_2) \in [0, 1]$
2. $s_E(x_1, x_2) = 0$ if and only if $x_1 = x_2$ (or $t_A^1 = t_A^2$, $i_A^1 = i_A^2$ and $f_A^1 = f_A^2$)
3. $s_E(x_1, x_2) = 1$ if and only if $\mid t_A^1 - t_A^2 \mid = \mid i_A^1 - i_A^2 \mid = \mid f_A^1 - f_A^2 \mid = 1$
   For examples: $x_1(1, 1, 1)$ and $x_2(0, 0, 0)$; or $x_1(1, 0, 0)$ and $x_2(0, 1, 1)$; or $x_1(0, 1, 0)$ and $x_2(1, 0, 1)$, etc.

The *Euclidean distance* between two neutrosophic elements can be extended to the *Normalized Euclidean distance* or *Normalized Euclidean dissimilarity measure* as follows.

Let A and B be two *single-valued neutrosophic sets* from the universe of discourse U,

$A = \{x_i \in U, where\ t_A(x_i), i_A(x_i), f_A(x_i) \in [0, 1], for\ 1 \leq i \leq n\ and\ n \geq 1\}$,

and

$B = \{x_i \in U, where\ t_B(x_i), i_B(x_i), f_B(x_i) \in [0, 1], for\ 1 \leq i \leq n\ and\ n \geq 1\}$

The *Normalized Euclidean distance* between the two single-valued neutrosophic sets A and B is defined as [72–75]:

$$d_{nE}(A, B) = \left\{ \frac{1}{3n}\sum_{i=1}^{n}(t_A(x_i) - t_B(x_i))^2 + (i_A(x_i) - i_B(x_i))^2 + (f_A(x_i) - f_B(x_i))^2 \right\}^{\frac{1}{2}} \quad (4)$$

Properties of the Normalized Euclidean distance between two Neutrosophic Sets. If A and B are two single-valued neutrosophic sets then the *Normalized Euclidean distance* between A and B follows the distance measures properties:

1. $d_{nE}(A, B) \in [0, 1]$
2. $d_{nE}(A, B) = 0$ if and only if $A = B$ or for all $i \in \{1, 2, \ldots, n\}$, $t_A(x_i) = t_B(x_i)$, $i_A(x_i) = i_B(x_i)$ and $f_A(x_i) = f_B(x_i)$
3. $d_{nE}(A, B) = 1$ if and only if for all $i \in \{1, 2, \ldots, n\}$, $\mid t_A(x_i) - t_B(x_i) \mid = \mid i_A(x_i) - i_B(x_i) \mid = \mid f_A(x_i) - f_B(x_i) \mid = 1$

The *Normalized Euclidean similarity measure* or the *complement of the Normalized Euclidean distance* between two single-valued neutrosophic sets A and B is defined as [30,72–75]:

$$s_{nE}(A, B) = 1 - d_{nE}(A, B) \quad (5)$$

which implies

$$s_{nE}(A, B) = 1 - \left\{ \frac{1}{3n}\sum_{i=1}^{n}(t_A(x_i) - t_B(x_i))^2 + (i_A(x_i) - i_B(x_i))^2 + (f_A(x_i) - f_B(x_i))^2 \right\}^{\frac{1}{2}} \quad (6)$$

Properties of the Normalized Euclidean Similarity Measure between two Neutrosophic Sets If A and B are two single-valued neutrosophic sets then the Normalized Euclidean Similarity Measure between A and B follows the similarity measures properties:

1. $s_{nE}(A, B) \in [0, 1]$
2. $s_{nE}(A, B) = 0$ if and only if $A = B$ or for all $i \in \{1, 2, \ldots, n\}$, $t_A(x_i) = t_B(x_i)$, $i_A(x_i) = i_B(x_i)$ and $f_A(x_i) = f_B(x_i)$
3. $s_{nE}(A, B) = 1$ if and only if for all $i \in \{1, 2, \ldots, n\}$, $\mid t_A(x_i) - t_B(x_i) \mid = \mid i_A(x_i) - i_B(x_i) \mid = \mid f_A(x_i) - f_B(x_i) \mid = 1$

Another commonly used distance measure for two single-valued neutrosophic sets A and B is *Normalized Hamming distance measure* defined as [76]:

$$d_{nH}(A,B) = \frac{1}{3n}\sum_{i=1}^{n}(\mid t_A(x_i) - t_B(x_i) \mid + \mid i_A(x_i) - i_B(x_i) \mid + \mid f_A(x_i) - f_B(x_i) \mid) \quad (7)$$

3. Proposed approach

In this section we present a method designed for determining the semantic distance between pairs of words using a neutrosophic approach in which a word is interpreted as a *single-valued neutrosophic set* [47,48]. The semantic distances are determined without taking into account the part of speech data of the involved words. In our approach, the words are internally represented as vectors of three values, their corresponding SentiWordNet scores (shortly, SWN scores). Thus, any lexical and syntactical information about words is discarded.

In what follows we describe all the involved data, the theoretical concepts and the representations used in the implementation of the proposed similarity method.

3.1. Word-level neutrosophic sentiment similarity

In this study we address the problem of sentiment similarity between pairs of words by following the neutrosophic approach firstly proposed in [49] in which a word w is interpreted as a *single-valued neutrosophic set* [47,48] having the representation:

$$w = (\mu_{truth}(w), \mu_{indeterminacy}(w), \mu_{false}(w)) \quad (8)$$

where $\mu_{truth}(w)$ denotes the *truth membership degree* of w, $\mu_{indeterminacy}(w)$ represents the *indeterminacy membership degree* of w and $\mu_{false}(w)$ represents the *false membership degree* of the word w, with $\mu_{truth}(w), \mu_{indeterminacy}(w), \mu_{false}(w) \in [0, 1]$.

Similar with [49] we use the SentiWordNet lexical resource (shortly, SWN) in order to fuel the proposed approach with data. More precisely, the three membership degrees of the words representation (see Eq. (8)) are the positive, neutral and, correspondingly, the negative scores provided by SentiWordNet.

Problem definition. We propose and evaluate a method for the problem of determining the sentiment class of a word w by measuring its distance from several *seed words*, one seed word for each sentiment class. In this assumption, we propose the usage of three semantic distances: *Intuitionistic Euclidean distance, Euclidean distance* and *Hamming distance*. We work with 7 seed words, each seed word being a representative sentiment word for each of the seventh sentiment degrees: *strong positive*, *positive*, *weak positive*, *neutral*, *weak negative*, *negative* and *strong negative*. We prove that all the considered theoretical concepts work very well as we apply and evaluate them on all the SentiWordNet words (that is, 155 287 words).

If w_1 and w_2 are highly similar, we expect the semantic distance value to be closer to 0, otherwise semantic relatedness value should be closer to 1. We consider SentiWordNet sentiment scores as the only features of the words.

As we have already pointed out, in this approach, a word internal representation consists of its SWN scores. In this assumption, a word w can be considered a *single-valued neutrosophic set* and thus, all the properties involving this concept can be used and applied.

In order to exemplify this assumption, let us consider the verb "scam". In the SWN dataset this word has a single entry, that is it has a single SWN score triplet:

$$scam = (0, 0.125, 0.875)$$

By following the neutrosophic assumption in which a word is considered a single-value neutrosophic set, the representation of the word w becomes:

$$w(t_w, i_w, f_w)$$

where:

- the degree of membership, t_w, is the word positive score,
- the degree of indeterminate-membership, i_w, is the word neutral score,
- the degree of non-membership, f_w, is the word negative score.

Obviously the conditions imposed on these degree values are preserved: $t_w, i_w, f_w \in [0, 1]$ and $0 \leq t_w + i_w + f_w = 1 \leq 3$.

For the considered example we have: $t_{scam} = 0$, $i_{scam} = 0.125$ and $f_{scam} = 0.875$, which implies $scam(0, 0.125, 0.875)$.

Let us now consider the general case in which a word w can appear in more than one synset in the SentiWordNet lexicon, meaning that the word has more than one sense. In this case we have n SWN score triplets for a single word w, with $n \geq 1$.

In order to construct the neutrosophic word representation, a single scores triplet must be provided. For this reason, for every word w with n senses, $n \geq 1$, we implemented the *weighted average formula* (after [77]) over all its positive, negative and, respectively, neutral scores obtaining in this manner three sentiment scores for all the three facets of a word sentiment polarity:

- the overall positive score of the word w:

$$t_w = \frac{t_{w^1} + \frac{1}{2}t_{w^2} + \cdots + \frac{1}{n}t_{w^n}}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} \quad (9)$$

- the overall neutral score of the word w:

$$i_w = \frac{i_{w^1} + \frac{1}{2}i_{w^2} + \cdots + \frac{1}{n}i_{w^n}}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} \quad (10)$$

- the overall negative score of the word w:

$$f_w = \frac{f_{w^1} + \frac{1}{2}f_{w^2} + \cdots + \frac{1}{n}f_{w^n}}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} \quad (11)$$

where w^1 denotes the first sense of the word w, w^2 represents the second sense of the word w, etc.

In order to calculate the overall scores of a word w we use the weighted average formula because it considers frequencies of the words' senses: the score of the first sense (which is the most frequent) is preserved entirely, while the rest of the scores, which correspond to the less used senses, appear divided accordingly (by 1/2, 1/3, etc.)

The sentiment class of a word is determined by computing a single score upon these overall scores. This unique score will represent the average of the differences between the positivity and negativity scores calculated per each sense.

More precisely, for a word w with n senses, the single sentiment score is determined by following the already defined mechanism for words' scores calculus based on SentiWordNet triplets (see [42]) which implies to determine the average weighted difference between their positive and negative scores such as:

$$\frac{1}{n}\sum_{i=1}^{n}\omega_i(pos_i - neg_i)$$

where the weights ω_i are chosen taking into account several word characteristics which can carry different levels of importance in conveying the described sentiment [42] (such as part of speech) and n represents the number of synsets in which the word w

```
function sent_class(score)
    sent_class <- "neutral"
    IF (score > 0.5) THEN sent_class <- "strong positive" ELSE
    IF (0.25 < score <= 0.5) THEN sent_class <- "positive" ELSE
    IF (0 < score <= 0.25) THEN sent_class <- "weak positive" ELSE
    IF (-0.25 <= score < 0) THEN sent_clas <- "weak negative" ELSE
    IF (-0.5 <= score < -0.25) THEN sent_class <- "negative" ELSE
    IF (score < -0.5) THEN sent_class <- "strong negative"

    return sent_class
endfunction
```

Fig. 1. The *sent_class* function.

```
function distance(dist, sent_class_w1, sent_class_w2)
    IF (dist is between Table2(sent_class_w1, sent_class_w2))
        return true
    return false
endfunction
```

Fig. 2. The *evaluate* function.

appears, that is the number of its senses. The average is used in order to ensure that the resulted scores are ranging between −1 and 1 [42].

Let us consider a word w with n senses, $w_1, w_2, \ldots, w_n$. In this study the overall score of the word w is determined using the formula [42,77]:

$$score = \frac{(t_{w_1} - f_{w_1}) + \frac{1}{2}(t_{w_2} - f_{w_2}) + \cdots + \frac{1}{n}(t_{w_n} - f_{w_n})}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} \quad (12)$$

As we have already pointed out, the values of *score* vary between −1 (meaning that the word w is a “strong negative” word) and 1 (the word w is a “strong positive” word).

Usually sentiment analysis applications deal with binary (positive vs. negative) or ternary (positive vs. negative vs. objective) classifications which normally leads to very good state-of-the-art accuracy (more then 70%) [42]. In this study, using the sentiment scores defined for the SentiWordNet synsets, we consider all the degrees of sentiments referred in the literature:

- *strong positive/negative word*: great difference between the positive/ negative scores and the negative/positive scores of the word (usually, above 0.5)
- *positive/negative word*: the positive/negative scores are greater than the negative/positive ones (the difference is smaller than 0.5 but greater than 0.25)
- *weak positive/negative word*: small difference between the positive/ negative scores and the negative/positive ones
- *neutral word*: the neutral scores subsume the positive and negative scores.

We defined a set of rules in order to uniquely map the general score of a word to one of the following sentiment classes: “*strong positive*”, “*positive*”, “*weak positive*”, “*neutral*”, “*weak negative*”, “*negative*”, “*strong negative*”. The rules are given in an algorithmic form under the *sent_class* function in Fig. 1.

If w_1 and w_2 are two words: $w_1(t_{w_1}, i_{w_1}, f_{w_1})$, $w_2(t_{w_2}, i_{w_2}, f_{w_2})$, the distance measures between w_1 and w_2 are as follows:

1. Intuitionistic Euclidean distance:

$$d_{IE}(w_1, w_2) = \sqrt{\frac{1}{2}[(t_{w_1} - t_{w_2})^2 + (f_{w_1} - f_{w_2})^2]} \quad (13)$$

2. Euclidean distance:

$$d_E(w_1, w_2) = \sqrt{\frac{1}{3}[(t_{w_1} - t_{w_2})^2 + (i_{w_1} - i_{w_2})^2 + (f_{w_1} - f_{w_2})^2]} \quad (14)$$

3. Hamming distance:

$$d_H(w_1, w_2) = \frac{1}{3}\left[| t_{w_1} - t_{w_2} | + | i_{w_1} - i_{w_2} | + | f_{w_1} - f_{w_2} |\right] \quad (15)$$

4. Experimental setup

We evaluate the accuracy of the considered mechanism by implementing the Normalized Euclidean and, in order to give terms of comparison, we also evaluate the Normalized Hamming distance and Intuitionistic Euclidean distance in the same scenario.

In Table 2 we give the values we impose on the distance measures with respect to the sentiment classes of the involved two words. The values of Table 2 are symmetrical and for this reason only the values under the main diagonal are given.

Obviously, we considered the smallest distance values in cases of words having the same sentiment class (these cases are given on the diagonal). A strong value for distance value means that the two words are completely dissimilar from the sentiment polarity point of view. For example, a word having “*negative*” sentiment class (or shortly, a negative word) and a word with “*positive*” sentiment class (a positive word) must have the distance value d bigger than 0.65, where d cannot be greater than 1.

Based on Table 2 values, the evaluation of the distance values with respect to the sentiment classes of the involved words is depicted in Fig. 2.

For the evaluation scenario we chose seven “seed words”, one for each sentiment class and we iterate through the lexical resource and calculate the distance measures between each of the seven seed words and all the words that appear in SentiWordNet (155287 words in total).

Resuming, the algorithmic form of the evaluation scenario for the proposed word-level sentiment similarity method is given in Fig. 3.

4.1. Evaluation scores

In Table 3 we present the selected seed words together with the results obtained by implementing and evaluating all the three distance measures proposed for this study: Normalized Euclidean

```
foreach seed_w <- seed word from sentiment classes: {strong positive, positive,
                                                     weak positive, neutral,
                                                     weak negative, negative,
                                                     strong negative}
    seed_sent_class <- sentiment class of seed_w

    foreach w <- word from SentiWordNet
        score <- overall score of w
        w_sent_class <- sentiment_class(score)

        foreach d <- distance from {dIF, dE, dH}
            evaluate(d, w_sent_class, seed_sent_class)
        endfor
    endfor
endfor
```

Fig. 3. The evaluation scenario.

Table 2
The used distance measure values with respect to the words sentiment classes.

Sent. classes	Strong positive	Positive	Weak positive	Neutral	Weak negative	Negative	Strong negative
Strong positive	[0, 0.2)						
Positive	[0, 0.3)	[0, 0.2)					
Weak positive	[0.25, 0.5)	[0, 0.3)	[0, 0.2)				
Neutral	[0.3, 0.65)	[0.3, 0.65)	[0, 0.3)	[0, 0.2)			
Weak negative	(0.65, 1]	(0.65, 1]	[0.25, 0.5)	[0, 0.3)	[0, 0.2)		
Negative	(0.65, 1]	(0.65, 1]	(0.65, 1]	[0.3, 0.65)	[0, 0.3)	[0, 0.2)	
Strong negative	(0.65, 1]	(0.65, 1]	(0.65, 1]	[0.3, 0.65)	[0.25, 0.5)	[0, 0.3)	[0, 0.2)

Table 3
Evaluation scores.

Seed word	Similarity distance precision		
	Euclidean distance	Hamming distance	Intuitionistic Euclidean distance
Sent. class: Strong positive Word: singable#a Overall scores: (0.75, 0.0, 0.25)	0.8411	0.8580	0.8808
Sent. class: Positive Word: spunky#a Overall scores: (0.5416, 0.2083, 0.25)	0.7714	0.7725	0.8059
Sent. class: Weak positive Word: immunized#a Overall scores: (0.5, 0.375, 0.125)	0.0392	0.0608	0.1219
Sent. class: Neutral Word: hydrostatic#a Overall scores: (0.0, 0.0, 1.0)	0.9676	0.9489	0.9570
Sent. class: Weak negative Word: misguided#a Overall scores: (0.25, 0.4583, 0.2916)	0.0973	0.1070	0.1279
Sent. class: Negative Word: reformable#a Overall scores: (0.125, 0.5, 0.375)	0.8259	0.8260	0.8573
Sent. class: Strong negative Word: unworkmanlike#a Overall scores: (0.0, 0.75, 0.25)	0.8542	0.8764	0.8875

distance, Normalized Hamming distance and Intuitionistic Euclidean distance measure.

The obtained accuracy results are mainly influenced by the way in which the considered seed words can be distinguished from the most preponderant words of this lexical resource, that is from the *neutral words* as they are the most frequent words of the SentiWordNet resource.

As it can be seen in Table 3 and Fig. 4 the considered distance measures have a similar behaviour: all the distance measures have more than 77% precision for the most of the considered seed words, which is above the average precision (70%) recognized in the specialized literature for the sentiment classifiers accuracy.

The highest precision (more than 74%) is achieved by applying the distance measures between the *neutral seed word* and all the SentiWordNet's words. Also very good scores (more than 82%) were achieved by applying the distances between the *negative seed word* and SentiWordNet words, then we have the scores corresponding to the *strong positive seed word* (more than 0.84 as precision) and finally the scores corresponding to the *positive seed word* (more than 77% precision).

But these very good results were not achieved for the *weak positive seed word* and *weak negative seed word* where the precision is almost zero. This failure can be caused by the fact that these particular sentiment words cannot be distinguished very well from the most preponderant words of SentiWordNet, that is from the *neutral words*.

We can therefore conclude that all the considered distance measures can distinguish very well the words of the most important sentiment classes from the point of view of a sentiment classifier: the (strong) positive or negative words and the neutral words. Still, the proposed measures are not capable for measuring the similarity of *weak sentiment words* with the rest of the sentiment words.

The most important conclusion that comes from the performed experiment is that the behaviour of all the considered distance measures is very similar — almost identical (see Fig. 4). We interpret this result as a proof for the robustness of the considered theory.

5. Conclusions and future work

In the latest years there has been developed a relatively large number of word-to-word similarity studies that can be grouped in two main categories: distance-oriented measures applied on structured representations and metrics based on distributional similarity learned from large text collections [50].

In this paper we propose a sentiment similarity method that fits in the first category of similarity studies and which takes into account only the sentiment aspects of the words and not their lexical category. We follow here recent text similarity approaches such as [1,28] defined around the same hypothesis which postulates that knowing the sentiment is beneficial in measuring the similarity.

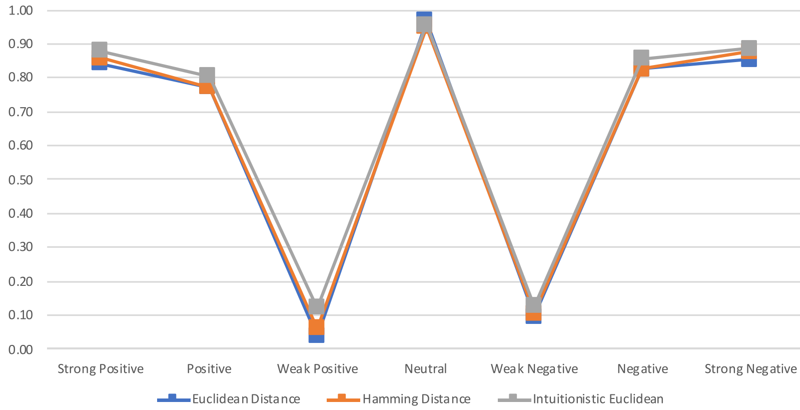

Fig. 4. The graphical visualization of the similarity distances precision.

Our proposal is formalized in a domain that was never used before for this kind of task — the neutrosophic theory, as it uses neutrosophic sets for representing the sentiment aspects of the words. The neutrosophic set is a generalization of the intuitionistic fuzzy set concept, and thus our proposal is in line with the recent fuzzy based studies that started to emerge for text processing tasks [20,78,79]. Indeed, fuzzy logic is capable of dealing with linguistic uncertainty as it considers the classification problem to be a "degree of grey" problem rather than a "black and white" problem [20] (the last one is the most used approach in sentiment analysis tasks).

For this first approach we obtained very promising results. Indeed, by applying distance measures on the neutrosophic words representations we shown that we can thus obtain a similarity method as we manage very clear to distinguish the words of the most important sentiment classes from the rest of the considered words: the SentiWordNet entries, that is, 155 287 words of all possible sentiment classes.

We also plan to extend our study to sequences of words with the intended scope of designing a method that can be applied for measuring documents similarity.

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An Approach to Determining Customer Satisfaction in Traditional Serbian Restaurants

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Abstract. The aim of this paper is to make a proposal for an easy–to–use approach to the evaluation of customer satisfaction in restaurants. In order to provide a reliable way to collect respondents' real attitudes, an approach based on the use of smaller number of evaluation criteria and interactive questionnaire created in a spreadsheet file is proposed in this paper, whereby an easy-to-understand and simple-to-use procedure is proposed for determining weights of criteria. In addition to the said, the proposed approach applies the simplified SERVQUAL-based approach, for which reason a simplified version of the Weighted Sum Method based on the decision maker's Preferred Levels of Performances is used for the final ranking of the alternatives. The usability of the proposed approach is considered in the case study intended for the evaluation of traditional restaurants in the city of Zajecar.

Keywords: hospitality, restaurant industry, customer satisfaction, PIPRECIA, WS PLP approach

1. Introduction

The Serbian word “kafana” originates from the Turkish word “kahvehane”, which means “a place for drinking coffee”. Such places have emerged in the Balkan region under the influence of the Ottoman Empire in the 16th century.

Under the influence of different cultures, kafana generated its specificity on the Balkan Peninsula, so that it also became a place where food was consumed and later a place where alcoholic drinks were served. Over time, kafanas have increasingly become and have found their place in the social and cultural life, as well as in business. Nowadays, kafanas continue to be a place where you meet your friends, a place for celebrations, talking about and discussing things and so on. Therefore, kafanas could be denoted as traditional Serbian restaurants. Compared with the other types of restaurants, kafanas have similarities to taverns and pubs, as places of a pleasant ambience.

Certain new trends in the restaurant and food industry, as well as the growing presence of various cuisines, have had an impact on traditional Serbian restaurants. Fortunately, in some parts of Serbia, traditional Serbian restaurants somehow still resist unfortunately unstoppable trends.

In the city of Zajecar, located in eastern Serbia, traditional restaurants are successfully resisting the actual trends and it is still possible for you to find good restaurants, such as: “Dva brata” (“The Two Brothers”), “Gradska Mehana” (The City Meyhane”), “Meda” (“The Bear”), “Roko” (“The Roko”) and so forth.

The factors influencing the satisfaction of restaurants’ customers have been considered in many previous studies. Based on these studies, an approach to the determining of the significance of the relevant factors that influence customer satisfaction is proposed.

The proposed approach also uses the concept of measuring the difference between expectations and perceptions, so it provides an easy identification of the criteria against which customer expectations are not met. Beside all of the above-said, the proposed model can also be used to determine the overall ratings of the considered alternatives, thus making a comparison with competitors.

Based on all of the above-mentioned reasons that have been taken into account, the remaining part of this paper is organized as follows: In Section 2, a review of the relevant research studies is given. After that, in Section 3 and Section 4, the PIPRECIA and the WS PLP methods are considered. In Section 5, an empirical illustration of the evaluation of Serbian traditional restaurants, based on the integrated use of the PIPRECIA and the WS PLP methods, is presented in detail. Finally, the conclusions are given at the end of the paper.

2. Literature Research

Measuring customer satisfaction could be very important in a competitive environment (e.g. Stepaniuk 2018; Raudeliūnienė et al. 2018). For the purpose of determining that, Parasuraman et al. (1988) proposed the Service Quality and Customer Satisfaction (SERVQUAL) model. On the basis of that model, many others more specialized models have been proposed later, such as: WebQual (Loiacono et al. 2002; Parasuraman et al. 2005), eTailQ Wolfinbarger and Gilly (2003), E-RecS-QUAL (Parasuraman et al. 2005), and eTransQual (Bauer et al. 2006).

The SERVQUAL model was used for determining the levels of customer satisfaction in many different areas. As one of these areas, tourism and hospitality can be mentioned. For example: Saleh and Ryan (1991) used

SERVQUAL to determine the gap between clients' and the management's perceptions in the hotel industry, whereas Devi Juwaheer (2004) explore the tourists' perceptions about hotels in Mauritius by using an adapted SQRVQUAL approach. Further, on the basis of the SERVQUAL model, Tribe and Snaith (1998) proposed the HOLSAT model, adapted for determining tourists' satisfaction with their holidays.

Besides, a number of other approaches have also been used to determine customer satisfaction in tourism and hospitality industry, such as: Chaturvedi (2017), Lee and Severt (2017), Engeset and Elvekrok (2015), Albayrak and Caber (2015), Chan et al. (2015), Bernini and Cagnone (2014), Battour et al. (2014).

The SERVQUAL model has also been used in the restaurant industry for determining customer satisfaction. As some examples of these studies, the following can be mentioned: Liu at al (2017), Kurian and Muzumdar, (2017), Hanks et al (2017); Bufquin, et al. (2017), Saad Andaleeb and Conway (2006), Heung, et al. (2000), Lee and Hing (1995).

Some other studies have also been dedicated to the restaurant industry. For example: Adam et al. (2015) investigates tourist satisfaction with Ghanaian restaurants based on a factor analysis, and Jung and Yoon (2013) investigate the relationship between employees' satisfaction and customers' satisfaction in a family restaurant.

Dobrovolskienė et al. (2017) state that decision making is crucial to every aspect of business. Multiple-criteria decision-making (MCDM) is a scientific field that has undergone extremely rapid development over the last two decades. Multiple-criteria decision-making considers situations in which the decision-maker must choose one of the alternatives from a set of available alternatives and which are judged on the basis of a number of criteria. This is why MCDM contributes to easier decision-making and adoption of long-term and lasting solutions.

MCDM has also been successfully applied in the hospitality industry. Chou et al. (2008) and Tzeng (2008) used MCDM models for selecting the restaurant location. Yildiz and Yildiz (2015) proposed a model for evaluating customer satisfaction in restaurants, based on the use of the AHP and TOPSIS methods. In their studies: Duarte Alonso et al. (2013), Chi et al. (2013), Kim et al. (2007), Yuksel and Yuksel (2003) and Jack Kivela (1997) investigate the criteria that have an impact on customer preferences and satisfaction.

3. The PIPRECIA Method

The Step-wise Weight Assessment Ratio Analysis (SWARA) method was proposed by Kersuliene et al. (2010). The usability of the SWARA method has been proven in solving many MCDM problems, of which only several are mentioned: Zolfani et al. (2013), Zolfani and Saparauskas (2013), Stanujkic et al. (2017; 2015), Karabasevic et al. (2017), Mardani et al. (2017) and Juodagalviene et al. (2017).

The SWARA method has a certain similarity with the prominent AHP method. The first similarity is that both methods can be used to completely solve MCDM problems or to only determine the weight of the criteria; the second is that both methods are based on the use of pairwise comparisons.

However, the computational procedures of the SWARA and the AHP methods significantly differ from one another. Because of that, the SWARA method has some advantages, as well as some disadvantages, in comparison with the AHP method.

As the main disadvantage of the SWARA method, the fact that its computational procedure does not include a procedure for determining the consistency of pairwise comparisons made can be mentioned. Contrary to that, a significantly lower number of pairwise comparisons required for solving an MCDM problem and for determining criteria weights, too, can be mentioned as an advantage of the SWARA method.

Its requirement that evaluation criteria should be sorted in descending order according to their expected significances, which can prove to be inadequate in some survey cases, can also be mentioned as the weakness of the SWARA method. Therefore, with the aim of extending the use of the SWARA method in the cases where a consensus on the expected significance of the criteria is not easy to reach, Stanujkic et al. (2017) proposed the use of the following equation for the purpose of determining the importance of criteria as follows:

$$s_j = \begin{cases} >1 & \text{when } C_j \succ C_{j-1} \\ 1 & \text{when } C_j = C_{j-1} \\ <1 & \text{when } C_j \prec C_{j-1} \end{cases}. \tag{1}$$

where: s_j denotes the comparative importance of the criterion *j*, and $C_j \,\Theta\, C_{j-1}$ denotes the significance of the criterion *j* in relation to the *j*-1 criterion.

In an extension of the SWARA method, proposed under the name of PIPRECIA, Stanujkic et al. (2017) also mention that a lack an integrated procedure for checking the consistency in the ordinary SWARA method can successfully be compensated for by using Kendall's Tau or Spearman's Rank Correlation Coefficient.

Because of all the foregoing, the PIPRECIA method has been chosen to be used in this approach.

3.1. The Computational Procedure of the PIPRECIA Method

The computational procedure of the PIPRECIA method can be shown as follows:

***Step* 1**. Choose the criteria on the basis of which an evaluation of alternatives will be carried out.

***Step* 2**. Set the value of the relative importance of the criteria by using Eq. (1), starting from the second criterion.

***Step* 3**. Calculate the coefficient k_j for the criterion *j* as follows:

$$k_j = 2 - s_j. \tag{2}$$

***Step* 4**. Calculate the recalculated weight q_j for the criterion *j* as follows:

$$q_j = \begin{cases} 1 & \text{if } j = 1 \\ \dfrac{q_{j-1}}{k_j} & \text{when } j > 1 \end{cases}. \tag{3}$$

***Step* 5**. Calculate the weights of the criteria as follows:

$$w_j = \frac{q_j}{\sum_{k=1}^{n} q_k}. \tag{4}$$

where w_j denotes the weight of the criterion *j*.

4. The WS PLP Approach

Based on the Weighted Sum Method (Churchman and Ackoff, 1954, MacCrimon, 1968), Stanujkic and Zavadskas (2015) proposed the Weighted Sum Preferred Levels of Performances (WS PLP) approach.

The simplified computational procedure of the WS PLP approach for solving an MCDM problem that contains the *m* alternatives that are evaluated based on the *n* beneficial criteria (a higher value of the performance rating is desirable) can be shown as follows:

***Step* 1**. Evaluate the alternatives in relation to the selected criteria.

***Step* 2**. Set the preferred performance ratings for each criterion.

***Step* 3**. Calculate the normalized performance ratings of the alternatives as follows:

$$r_{ij} = \frac{x_{ij} - x_{0j}}{x_j^+ - x_j^-}, \qquad (5)$$

where: x_{ij} and r_{ij} denote the performance rating and the normalized performance rating of the alternative *i* in relation to the criterion *j*, respectively; x_{0j} denotes the preferred performance rating of the criterion *j*; $x_j^+ = \max_i x_{ij}$, and $x_j^- = \min_i x_{ij}$.

***Step* 4**. Calculate the overall performance rating of the alternatives as follows:

$$S_i = \sum_{j=1}^{n} w_j \, r_{ij}, \qquad (6)$$

where S_i denotes the overall performance rating of the alternative *i*, $S_i \in [-1, 1]$; w_j is the weight of the criterion *j*.

In the proposed approach, the alternatives whose S_i is greater than or equal to zero make a set of the most appropriate alternatives, out of which one should be selected.

5. A Case Study

In order to determine the preferences of the passionate visitors of Serbian traditional restaurants, a supervised survey has been performed in the city of Zajecar, located in Serbia, near the Romanian and the Bulgarian borders.

In this study, the five previously mentioned restaurants have been evaluated on the basis of the six criteria adopted from Stanujkic et al. (2016):

- C_1 - the interior of the building and the friendly atmosphere,
- C_2 - the helpfulness and friendliness of the staff,
- C_3 - the variety of traditional food and drinks,
- C_4 - the quality and taste of the food and drinks, including the manner of serving,
- C_5 - the appropriate price for the quality of the services provided, and
- C_6 - other.

In the proposed approach the criterion “other” is used to enable personalization.

The survey presented in this study was conducted by e-mail, or more precisely by using an interactive questionnaire created in a spreadsheet file. By using such an approach, the respondents can see the calculated weights of the criteria and can also modify his/her responses if he or she is not satisfied with the obtained results. In addition, by using such an approach, the obtained results can also be presented graphically, which can make easier to understand the procedure used for determining weights of criteria, and thus lead to obtaining more realistic views of the respondents.

The interactive questionnaire was sent to the selected respondents known as the “bohemians” and/or frequent visitors of traditional Serbian restaurants. Out of the approximately 80 sent questionnaires, the 42 of them were returned, out of which only 30 questionnaires were selected as those properly filled in.

The weights of the criteria calculated on the basis of the responses obtained from the two selected respondents are accounted for in Table 1 and Table 2.

Table 1. The weights of the criteria obtained from the first respondent

Criteria		s_j	w_j
C_1	The interior of the building and friendly atmosphere		0.13
C_2	The helpfulness and friendliness of the staff	1.10	0.15
C_3	The variety of traditional food and drinks	1.20	0.19
C_4	The quality and taste of the food and drinks, including the manner of serving	1.05	0.20
C_5	The appropriate price for the quality of the services provided	0.95	0.19
C_6	Other	0.70	0.14

Source: Own calculations

Table 2. The weights of the criteria obtained from the second respondent

Criteria		s_j	w_j
C_1	The interior of the building and friendly atmosphere		0.15
C_2	The helpfulness and friendliness of the staff	1.10	0.17
C_3	The variety of traditional food and drinks	0.90	0.16
C_4	The quality and taste of the food and drinks, including the manner of serving	1.15	0.18
C_5	The appropriate price for the quality of the services provided	0.95	0.17
C_6	Other	0.90	0.16

Source: Own calculations

Some significant descriptive statistical parameters related to the weights of the criteria obtained by the conducted survey are presented in Table 3.

Table 3. The descriptive statistics for the weights of the criteria

Criteria	Min	Max	Range	Mean	Standard Deviation	Variance	Screw	Kurtosis
C_1	0.01	0.17	0.17	0.12	0.05	0.002	-0.84	-0.12
C_2	0.05	0.19	0.15	0.15	0.05	0.002	-0.77	-0.81
C_3	0.03	0.19	0.15	0.14	0.05	0.003	-0.52	-1.13
C_4	0.17	0.37	0.19	**0.23**	0.06	0.003	0.91	-0.27
C_5	0.17	0.35	0.18	**0.22**	0.06	0.003	0.76	-0.65
C_6	0.11	0.23	0.12	0.16	0.03	0.001	0.41	-0.77

Source: Own calculations

According to Table 3, the criteria C_4 and C_5 have significantly higher importance related to the other criteria, i.e. the quality and the taste of the food and the appropriate price are identified as the most significant criteria.

The obtained correlation coefficient between the responses obtained from the respondents and the mean ranges between 0.44 and 0.98.

Criterion C_6 - "other" also has a high weight, which can be interpreted as follows:

- in addition to the criteria C_1 - C_5 there are other criteria that affect satisfaction of restaurant customers, which can be applied in much more sophisticated models, and
- criterion C_6 can successfully substitute many less significant criteria and such enable forming an efficient MCDM models based on the use of a smaller number of criteria.

In addition to the conducted research, the respondents also evaluated the five preselected traditional restaurants by using the five-point Likert Scale. The results obtained from the two of the above-mentioned respondents are accounted for in Tables 4 and 5.

Table 4. The ratings obtained from the first respondent

Criteria \ Alternatives	Expected	Meda	Dva brata	MS	Roko	Nasa kafana	S_i	Rank
C_1	3	4	5	3	4	4	0.61	2
C_2	4	5	5	3	3	3	0.85	1
C_3	4	4	5	3	3	4	-0.09	5
C_4	3	5	5	3	4	3	0.20	3
C_5	3	5	4	2	4	4	0.20	4
C_6	2	3	4	3	3	3	0.61	2

Source: Own calculations

Table 5. The ratings obtained from the second respondent

Criteria \ Alternatives	Expected	Meda	Dva brata	MS	Roko	Nasa kafana	S_i	Rank
C_1	4	4	4	2	2	4	0.32	1
C_2	5	5	4	3	3	4	-0.02	2
C_3	3	4	4	3	3	4	-0.42	4
C_4	5	5	4	4	3	4	-0.51	5
C_5	4	4	4	4	4	3	-0.04	3
C_6	3	4	3	3	3	4	0.32	1

Source: Own calculations

Ranges between the maximum and minimum weights of criteria are also not negligible, as previously shown in Table 3. Therefore, the separate ranking list of considered alternatives has been formed for each respondent, in this approach, by using the WS PLP approach.

In this way, the attitudes of the respondents do not drown into the group attitudes, obtained on the basis of the average weight of and average ratings, and remain clear until the end of the evaluation, where the final ranking of the considered alternative was made based on dominance theory.

The results achieved based on all properly filled questionnaires are shown in Table 6. The appearance of the considered alternative in the first position is given in Column I of Table 6. The appearance of the considered alternatives in the second and the third positions is given in Columns II and III of Table 6.

Table 6. The number of the appearances of the alternatives in different positions

	Number of appearances at positions		
Alternatives	I	II	III
A_1	15	7	3
A_2	12	6	6
A_3	1	1	9
A_4	4	10	7
A_5	0	4	5

Source: Own calculations

According to Column I of Table 5, based on the dominance theory, the best-placed alternative is the alternative labelled as A_1.

In this approach, only the appearances on the first position are used for the determination of the best alternative, or more precisely, the most popular traditional restaurant. The appearances in the second, the third, as well as the other positions, could be used for a further analysis.

The overall ratings, obtained by using WS PLP approach, can also be used for various analysis, especially when it is known that WS PLP approach $S_i < 0$ indicates an alternative where expected customers' satisfaction has not been reached yet.

Conclusions

The main objective of this paper is to determine the most significant criteria that have an influence on customers' satisfaction in traditional Serbian restaurants, as well as weights of these criteria, and propose an easy–to–use approach for the evaluation of customers' satisfaction in restaurants.

For that reason, the newly proposed PIPRECIA method, that is an extension of the SWARA method, is proposed for determining the weight of criteria in order to provide an effective and simple-to-use procedure for gathering the attitudes of the examined respondents that will be as realistic as possible.

The gaps between the expected and the achieved satisfaction obtained based on a set of criteria are used to determine the overall performance of any of the considered alternatives, which is done by applying the WS PLP approach. The final ranking of the alternatives is made by referring to dominance theory.

The approach proposed in this paper has significant similarities to the proven SERVQUAL model or models like that one. However, it is based on the use of a significantly smaller number of evaluation criteria, which could allow the forming of the simplest questionnaires that could be more appropriate when preferences and ratings are collected through conducting surveys with ordinary respondents, i.e. those unprepared in advice for surveying.

The usability of the proposed approach has been verified in the case study on the evaluation of traditional Serbian restaurants. The achieved results confirm the efficiency and usability of the proposed approach for solving similar, as well as numerous other, decision-making problems.

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A New Algorithm for Finding Minimum Spanning Trees with Undirected Neutrosophic Graphs

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Abstract

In this paper, we discuss the minimum spanning tree (MST) problem of an undirected neutrosophic weighted connected graph in which a single-valued neutrosophic number, instead of a real number/fuzzy number, is assigned to each arc as its arc length.We define this type of MST as neutrosophic minimum spanning tree (NMST). We describe the utility of neutrosophic numbers as arc lengths and its application in different real world MST problems. Here, a new algorithm for designing the MST of a neutrosophic graph is introduced. In the proposed algorithm, we incorporate the uncertainty in Kruskal algorithm for designing MST using neutrosophic number as arc length. A score function is used to compare different NMSTs whose weights are computed using the addition operation of neutrosophic numbers. We compare this weight of the NMST with that of an equivalent classical MST with real numbers as arc lengths. Compared with the existing algorithms for NMST, the proposed algorithm is more efficient due to the fact that the addition operation and the ranking of neutrosophic number can be done in straightforward manners. The proposed algorithm is illustrated by numerical examples.

Keywords Neutrosophic sets · Neutrosophic graph · Score function · Spanning tree problem

1 Introduction

Zadeh (1965) introduced the theory of fuzzy sets, which capture natural phenomenon of imprecision and uncertainty. The characteristic of fuzzy set, namely the membership function, is a function whose range is between 0 and 1. Since then, fuzzy sets has been used to model many real life problems in various fields. Classical fuzzy set (type-1 fuzzy set), whose membership degrees/functions are single values/classical sets, is not able to handle different kinds of uncertainty that appears in real life scenarios. Turksen (1986) introduced the idea of interval valued fuzzy sets and Atanassov (1986) proposed the intuitionistic fuzzy sets to handle with the lack of non-membership degrees of fuzzy sets. Intuitionistic fuzzy sets have been widely used in optimization problems, decision making, neural network, medical diagnosis, and so forth. An intuitionistic fuzzy set is an extension of standard fuzzy set that assigns not only to each element a membership degree and but also a non-membership degree. It is more flexible to handle the uncertainty of real life scenarios than the standard fuzzy set. The membership degree, non-membership degree and hesitation degree of an element in the intuitionistic fuzzy set may not be a real number. Atanassov and Gargov (1989) extended the concept of intuitionistic fuzzy sets to the interval valued intuitionistic fuzzy sets to handle more uncertainty than intuitionistic fuzzy sets. To handle the uncertainty due to the hesitance in representing their preference over elements in an optimization, hesitant fuzzy sets were proposed by Torra (2010) and Torra and Narukawa (2009).

Although the fuzzy set and fuzzy logic have been applied to solve many real life problems, it cannot represent many type of uncertainties properly. For example, uncertainties in the inconsistent information and indeterminate information cannot be expressed by fuzzy set. If we want to know the opinion of a decision maker about a statement, decision maker may say that the possibility of truthfulness in the statement is 0.6, the possibility of false in the statement is 0.7 and the possibility of not sure is 0.4. This type of real life scenarios cannot be represented by fuzzy set. Therefore, we need some a new concept to handle this scenarios.

Smarandache (1998) proposed the concept of neutrosophic set (NS) from the philosophical point of view, to represent uncertain, imprecise, incomplete, inconsistent, and indeterminate information that exist in the real world problems.NS is characterized by a truth-membership function (t), an indeterminate-membership function (i) and a false-membership function (f) independently, which are within the real standard or non-standard unit interval]0, 1+[. The neutrosophic set model is an important tool for dealing with real scientific and engineering applications because it can handle not only incomplete information but also the inconsistent information and indeterminate information as shown in Broumi et al. (2016), Ngan et al. (2016), Wijayanto et al. (2016), Thanh et al. (2017), Phong and Son (2017), Ali et al. (2017, 2018a, b, c).

Minimum spanning tree (MST) is a fundamental and well-known optimization problem in graph theory (Chen and Chang 2001; Dey et al. 2015; Dey and Pal 2013). It aims to find the minimum weighted spanning tree of a weighted connected graph. MST has many real life applications, including communications, transportation, image processing, logistics, wireless telecommunication networks, cluster analysis, data storage and speech recognition. In the classical MST problem, the arc lengths are assumed to be fixed and decision maker uses crisp values to represent the arc lengths of a connected weighted graph. However, in real world scenarios, the arc length of a graph may represent a parameter which may not have a precise value, e.g., demand, cost, time, traffic frequencies, capacities, etc. (Pedrycz and Chen 2011, 2014, 2015; Dey et al. 2016). As an example on road networks, even though the geometric distance is fixed, arc length representing the vehicle travel time may fluctuate due to different weather condition, traffic flow or some other unexpected factors (Dey et al. 2016). Therefore, it becomes hard for decision makers to estimate proper edge cost in crisp values.

In general, decision makers use possible values of arc lengths in linguistic terms, approximate intervals, etc. In such real time scenarios, the arc lengths can be expressed as about 30 minute, around 30–90 min, nearly 90 min, between 90 and 110 min, etc. Fuzzy set is one of the most important mathematical tools to handle the uncertainty of the model (Chen 1996; Chen et al. 1997, 2001, 2006; Chen and Hsiao 2000; Wang and Chen 2008; Horng et al. 2005; Chen and Hong 2014). Most of the researchers have used type-1 fuzzy set to express those uncertain arc weights. Type-1 fuzzy set is unable to directly model properly such uncertainties because their membership values are totally crisp. Neutrosophic graph can be introduced as an alternative to fuzzy graph to deal with this uncertain situation.

The MST problem has received researchers attention over last decades and several approaches have been proposed to solve MST problem in deterministic graphs. [See Kruskal (1956), Prim (1957), Bondy and Murty (1976), Dijkstra (1959) and Harel and Tarjan (1984)]. Kruskal (1956) algorithm is one of the simple and effective algorithm to find the MST. We can calculate the MST using Kruskal algorithm if the arc costs of a graph are fixed. Uncertainty exists in almost every real life applications of MST problems. Uncertainty of parameters come from two different sources: randomness and vagueness or incomplete information. The randomness of a stochastic problem can be handled by probability theory. Due to this reason, some researches represent the arc of a MST problem as a random variables.

Ishii et al. (1981) described the MST problem with random arc costs. Ishii and Matsutomi (1995) proposed a polynomial time algorithm to solve the MST problem. In this algorithm, the parameters of the probability distributions of the arc costs are unknown and the parameters are estimated by applying a confidence region from stochastic data. However, in many real world problems, the parameter values are vague or incomplete in nature. Itoh and Ishii (1996) first proposed the MST problem with fuzzy arc costs as a chance-constrained programming. Their idea was based on necessity measure. Chang and Lee (1999) introduced the MST problem whose arc costs are fuzzy. They used three approaches based on the overall existence ranking index for ranking fuzzy arc costs. Based on fuzzy set and probability theory, de Almeida et al. (2005) introduced a genetic algorithm for MST problem with fuzzy parameters. Janiak and Kasperski (2008) proposed the MST where the arc costs are represented by fuzzy intervals. They applied the possibility theory to characterize and chose the arcs of a MST for a graph. Zhao et al. (2012) used intuitionistic fuzzy variables to represent the arc length of a intuitionistic fuzzy graph and developed an algorithm to solve this problem. Zhang and Xu (2012) described the MST problem with hesitant fuzzy variables and introduced an algorithmic approach to solve it.

Recently, few researchers have used neutrosophic methods to find minimum spanning tree in neutrosophic environment. Ye (2014) presented a method to find minimum spanning tree of a graph where nodes were samples are represented in the form of NSs and distance between two nodes represents the dissimilarity between the corresponding samples. Kandasamy (2016) proposed a double-valued neutrosophic minimum spanning tree (DVN-MST) clustering algorithm to cluster the data represented by double-valued neutrosophic information. Mandal and Basu (2016) proposed a new approach of optimum spanning tree problems considering the inconsistency, incompleteness and indeterminacy of the information. They considered a network problem with multiple criteria represented by weight of each edge in neutrosophic sets. It should be noted that the triangular fuzzy numbers and single-valued neutrosophic numbers are similar in mathematical notations, but totally different. To the best of our knowledge, no algorithm exists for MST with neutrosophic arc lengths.

The minimum spanning tree (MST) problem is one of the most well-known optimization problems in graph theory due to its importance to various applications. The uncertainty in the application of MST makes it difficult to find the edge weights exactly. Neutrosophic set and neutrosophic logic are renowned theories, with which one can handle and capture the natural phenomenon of the imprecision and uncertainty in the edge weights of the spanning tree. Neutrosophicness is explored as an alternative to fuzziness for describing uncertainty. The motivation of this work is to find an algorithm for the minimum spanning tree of undirected neutrosophic graph which will be simple enough and efficient in real world scenarios or real life problems. In the past years, there were few methods in Ye (2014), Kandasamy (2016) and Mandal and Basu (2016), to find the MST of a neutrosophic graph. In these algorithms, they can obtain either the cost or the MST of the neutrosophic graph. It is the purpose of this paper to propose a new algorithm that can obtain both of them.

In this paper, we work on MST problem of undirected weighted graph whose arc lengths are represented by neutrosophic number. This work is unique of its kind as there is no such work in the literature done before. An undirected connected neutrosophic graph is considered whose arc length is represented by neutrosophic number. We define this problem as the neutrosophic minimum spanning tree (NMST) problem. The NMST problem, involving addition and comparison operation of neutrosophic numbers, is quite different from the classical MST problem, which involves real numbers only. In an NMST problem, the weights of MST are neutrosophic numbers, and the task is to determine a spanning tree which is smaller than the others. It is not easy, as the comparison of neutrosophic numbers as an operation can be described in a wide variety of ways. We have introduced a modified Kruskal algorithm to solve the NMST problem. The proposed algorithm is used to compute the MST of the neutrosophic graph and its cost. We use the score-based ranking method to choose the minimum arc associated with the lowest value of score. Compared with existing algorithms for NMST, the proposed algorithm is more efficient due to the fact that the addition operation and the ranking of neutrosophic number can be done is a easy and straight manner. A numerical example illustrates the proposed algorithm.

The rest of the paper is organized as follows. Section 2 briefly introduces the concepts of neutrosophic sets, single-valued neutrosophic sets and the score function of single-valued neutrosophic number. A mathematical formulation of the NMST problem is given in Sect. 3. Section 4 proposes a novel approach for finding the minimum spanning tree of neutrosophic undirected graph. In Sect. 5, an illustrative example is presented to illustrate the proposed method. Finally, Sect. 6 concludes the paper.

2 Preliminary

Definition 1 Let ξ be an universal set. The neutrosophic set A on the universal set ξ categorized in to three membership functions called the true $T_A(x)$, indeterminate $I_A(x)$ and false $F_A(x)$ contained in real standard or non-standard subset of $]^-0, 1^+[$, respectively (Smarandache 1998).

$$^-0 \le \sup\ T_A(x) + \sup\ I_A(x) + \sup\ F_A(x) \le 3^+ \quad (1)$$

Definition 2 Let ξ be a universal set. The single-valued neutrosophic sets (SVNs) A on the universal ξ is denoted as following (Wang et al. 2010),

$$A = \{\langle x : T_A(x), I_A(x), F_A(x) | x \in \xi \rangle\} \quad (2)$$

The functions $T_A(x) \in [0, 1]$, $I_A(x) \in [0, 1]$ and $F_A(x) \in [0, 1]$ are named as degree of truth, indeterminacy and falsity membership of x in A, satisfy the following condition:

$$^-0 \le \sup\ T_A(x) + \sup\ I_A(x) + \sup\ F_A(x) \le 3^+ \quad (3)$$

Definition 3 Let $A = (T, I, F)$ be a SVNs, a score function S, based on the truth-membership degree (T), an indeterminacy membership degree (I) and a falsity membership degree (F) is defined as follows (Garg 2016):

$$S(A) = \frac{(1 + (T - 2I - F)(2 - T - F))}{2} \quad (4)$$

3 Problem formulation for NMST

A spanning tree of a connected graph G is a connected acyclic maximum sub-graph which includes all the nodes of G. Every spanning tree has exactly $n - 1$ arcs, where n is the number of nodes of graph G. A MST problem is to find a spanning tree such that the sum of all its arc length is minimum. The classical MST problem considers the exact weights associated with the arcs of the graph. However, in real world scenarios the arc lengths may be imprecise due to lack of evidence or incompleteness. The effective way to handle with these imprecision is to consider a neutrosophic graph. Consider a neutrosophic graph G, consisting of n number of nodes $V = \{v_1, v_2, \dots, v_n\}$ and a finite set of m number of arcs $E \subseteq V \times V$. Each arc of the graph is denoted by e, which is an order pair (i, j), where $i, j \in V$ and $i \neq j$. If the arc e is present in the NMST then $x_e = 1$, otherwise $x_e = 0$. The cost of all the arcs of graph G is represented by number. We defined the MST of this neutrosophic graph as neutrosophic minimum spanning tree (NMST). The NMST is expressed as the following linear programming problem.

$$\min \sum_{e \in E} A_e x_e \tag{5}$$

Subject to

$$\sum_{e \in E} x_e = n - 1 \tag{6}$$

$$\sum_{e \in \delta(s)} x_e \geq 1 \ \ \forall s \subset V, \ \ \emptyset \neq s \neq V \tag{7}$$

$$x_e \in \{0, 1\} \ \ \forall e \in E \tag{8}$$

Here, A_e is a neutrosophic set that represents the length of the arc $e \in E$ and $\sum$ in Eq. (5) is the sum of a neutrosophic sets. Equation (6) ensures that the number of edges in the NMST is $n - 1$. In Eq. (7), $\delta(s) = \{(i, j | i \in s, j \notin s)\}$ is used for the cutset of a subset of vertices s, i.e., the arcs that have one node in the set s and the other one outside the s. Thus, a spanning tree must have at least one arc in the cutset of any subset of the nodes.

4 Proposed algorithm for NMST

The proposed algorithm is an extension of the Kruskal algorithm for MST problem. We have incorporated the concept of uncertainty in Kruskal algorithm using neutrosophic number as an edge weight. The classical Kruskal algorithm is a MST algorithm which determines an arc of the minimum cost that connects any two trees in the forest. This algorithm is a type of greedy algorithm in graph theory as it determines a MST for a connected weighted graph adding increasing cost arcs at each step. Two key matters are needed to address to modify the Kruskal algorithm to solve the NMST problem. The first is determining the addition operation of two edges to find the cost of the spanning and the second is to compare the costs of the spanning trees of two different spanning trees. Based on the score function of neutrosophic number, the classical Kruskal algorithm can be easily modified to a neutrosophic Kruskal algorithm as follows.

In this algorithm, the variable T is used to represent the NMST and A is the set of all unvisited arc that are to be removed. n is the total number of vertices in the neutrosophic graph G. The main steps of the neutrosophic Kruskal algorithm are shown as follows:

Step 1 The score value are computed for all the arc $e \in E$ of G, using Eq. (4).

Step 2 Arrange all the arcs $e \in E$ of G by their corresponding score values: least score value first and largest score value last. The score values are used as the edge weights of the neutrosophic graph G.

Step 3 Choose the not-examined edge from the graph G. Add this chosen arc to the NMST if this will not make a cycle.

Step 4 Stop the process whenever $n - 1$ arcs have been added to the NMST.

Algorithm 1 Pseudocode of the proposed algorithm for designing NMST

```
Input: A connected undirected weighted neutrosophic graph
Output: The resultant NMST
Begin
T ← {∅}            ▷ T describes a set of edges, which is the NMST.
for each arc e ∈ G do
    Find the score value of each arc e using (4)
    insert e into A
end for
while |T| ≤ n − 1 do
    Choose an arc e form A with minimum score value.
    if T ∪ e has no cycle then
        T ← T ∪ e
    end if
    Remove e from A
end while
return T
End
```

The pseudocode of the proposed algorithm is given in Algorithm 1. We use an adjacency matrix to describe the neutrosophic graph. The linear searching method is used to determine the minimum weight edge based on the concept of score of neutrosophic number. The computational complexity of the proposed algorithm is $O(V^2)$.

5 Numerical example

We demonstrate our modified Kruskal algorithm step by step considering an example graph, shown in Fig. 2.

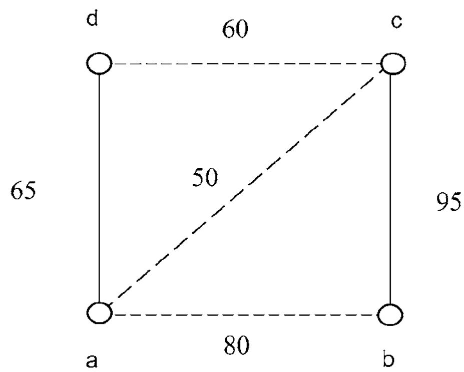

Fig. 1 An undirected weighted connected classical graph

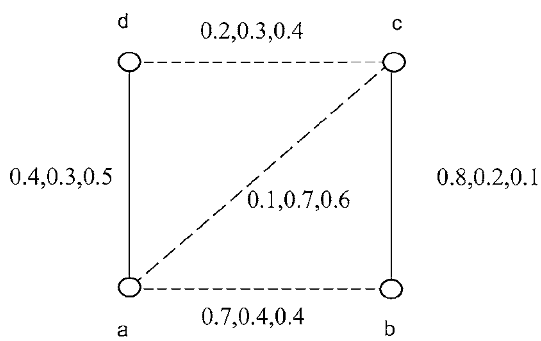

Fig. 2 An undirected neutrosophic graph

First, we consider an undirected weighted graph shown in Fig. 1 and apply Kruskal algorithm to find the MST of G. The Kruskal algorithm finds the MST= $\{(a, c), (c, d), (b, d)\}$ and the effective weight is 190.

The graph G_1 consists of four nodes and five edges. The costs of all edges of the network are in the form of neutrosophic number. We do not find any graph in the literature whose edge costs are given in terms of neutrosophic number. For the graph, shown in Fig. 2, we have generated the values of neutrosophic number and assigned those values to the arcs of the graph randomly. We have to find the MST of the graph G.

Step 1. There are five possible arcs in the graph G. They are, respectively:

i. (a, b)
ii. (a, c)
iii. (a, d)
iv. (b, d)
v. (c, d)

Now we compute the score of every arc based on score function, as defined in Eq. (4). The ranks are, respectively, 0.27, −0.73, 0.115, −0.06 and 0.66 for the arcs (a, b), (a, c), (a, d), (c, d) and (b, d). A stores this five arcs with their corresponding costs.

Step 2. The arc (a, c) is the smallest arc as its score −0.73 is lowest among all the values of score of all the arcs in A. The arc (a, c) is inserted in T and the (a, c) is removed from A. Now, the NMST T is $\{(a, c)\}$.

Step 3. The arc (c, d) is the smallest arc as its score −0.06 is lowest among all the values of score of all the arcs in A. The arc (c, d) is inserted in T and the (c, d) is removed from A. Now, the NMST T is $\{(a, c), (c, d)\}$.

Step 4. The arc (a, d) is the smallest arc as its score 0.115 is lowest among all the values of score of all the arcs in A. The arc (a, d) is not inserted in T because it creates a cycle. The arc (a, d) is removed from A. The arc (a, b) is the next smallest arc as its score 0.27 is lowest among all the values of score of all the arcs in A and the (a, d) is removed from A. Now, the NMST T is $\{(a, c), (c, d), (c, d)\}$. The cost of the NMST is −0.52.

We have also computed the MST of neutrosophic graph using binary programming. The result for same graph using binary programming with the same arc cost is computed. Our computed MST is same as the binary programming. It shows the effectiveness of the proposed approach. Here, we consider binary programming as we did not find, best of our knowledge, any existing algorithm for MST on neutrosophic graph.

Compared with existing algorithms for NMST, the proposed algorithm is more efficient due to the fact that the addition operation and the ranking of neutrosophic number can be done is a easy and straight manner. In the existing literature, the works on neutrosophic graph were done by Ye (2014) and Mandal and Basu (2016) where similarity approaches were used to compare the path in a graph with neutrosophic set. The similarity measure of neutrosophic set approach has the limitation of time consumption. In these algorithms, they can obtain either the cost or the MST of the neutrosophic graph. However, the proposed method in this research can obtain both of them. This shows the advantages of the proposal.

6 Conclusion

This paper investigated the minimum spanning tree problem whose edges weights are represented by neutrosophic numbers. The main contribution of this study is to provide an algorithmic approach to find the minimum spanning tree in uncertain environment using neutrosophic numbers as arc lengths. We have incorporated the concept of uncertainty in Kruskal algorithm using neutrosophic number as an edge weight. The proposed algorithm finds the MST under neutrosophic edge weights based on the score values of neutrosophic numbers. A numerical example was presented to illustrate the mechanism of the proposed algorithm. The proposed algorithm for minimum spanning tree is simple enough and effective for real world scenarios. This work can be extended to the case of directed neutrosophic graphs and other types of neutrosophic graphs such as bipolar neutrosophic graphs, interval valued neutrosophic graphs.

In the future, the proposed algorithm can be applied to real world problems such as in supply chain management, transportation, etc. (Tsai et al. 2008; Chen and Chien 2011; Tsai et al. 2012; Chen and Kao 2013). It should be noted that the uncertainty in the arc length of a MST problem is not limited to the geometrical distance. For example, due to the several reason, the travel cost between two cities may be expressed as a neutrosophic numbers, even if the geometrical distance is fixed. This observation further gives the light for possible expansion.

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An Integrated Neutrosophic-TOPSIS Approach and its Application to Personnel Selection: A New Trend in Brain Processing and Analysis

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Nada A. Nabeeh, Florentin Smarandache, Mohamed Abdel-Basset, Haitham A. El-Ghareeb, Ahmed Aboelfetouh (2019). An Integrated Neutrosophic-TOPSIS Approach and its Application to Personnel Selection: A New Trend in Brain Processing and Analysis. *New Trends in Brain Signal Processing and Analysis*, 7, 29734;
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ABSTRACT Personnel selection is a critical obstacle that influences the success of the enterprise. The complexity of personnel selection is to determine efficiently the proper applicantion to fulfill enterprise requirements. The decision makers do their best to match enterprise requirements with the most suitable applicant. Unfortunately, the numerous criterions, alternatives, and goals make the process of choosing among several applicants is very complex and confusing to decision making. The environment of decision making is a multi-criteria decision making surrounded by inconsistency and uncertainty. This paper contributes to support personnel selection process by integrating neutrosophic analytical hierarchy process (AHP) with the technique for order preference by similarity to an ideal solution (TOPSIS) to illustrate an ideal solution amongst different alternatives. A case study on smart village Cairo Egypt is developed based on decision maker's judgments recommendations. The proposed study applies neutrosophic AHP and TOPSIS to enhance the traditional methods of personnel selection to achieve the ideal solutions. By reaching the ideal solutions, the smart village will enhance the resource management for attaining the goals to be a successful enterprise. The proposed method demonstrates a great impact on the personnel selection process rather than the traditional decision-making methods.

INDEX TERMS Personnel selection, multi-criteria decision making (MCDM), neutrosophic sets, analytic hierarchy process (AHP), topsis.

I. INTRODUCTION

Human resources are considered to be the real wealth for any organization. Personnel selection is a partial sector of human resources that aimed to recommend the ideal candidate to the right position on enterprise [1]. Indeed, the power of personnel selection process manages the input quality in such a way to improve human resource management. To keep going on with globalization and competition, the personnel selection processes need to be improved. Due to many enterprises have not enough capabilities for funding personnel selection, the enterprises used to choose the candidates with traditional and quickly methods [2]. Nowadays enterprises must adequate with the business environmental factors and organization responses which make the necessary to enhance the methods for personnel selection. The researchers mention three factors for the resources of IT which are human, business, and technology resources. The strategies of choosing persons altered according to enterprises police and funds. Researchers perceive that IT infrastructure do not lead to distinct important benefits, due to the complexity of mobility and imitation [3]. However, human resources sector have direct influence on the performance of enterprise. The association between IT skills and enterprise performance has been divided into three groups human, business, and technology resources. Such that, researchers conducted in US retail, human resources combined with IT would improve the enterprises productivity and efficiency [4]. Hence, the enterprise that strength the human resource with efficient personnel selection methods can handle internal communications between enterprise partners either internal or external sides, in addition, can efficiently predict the requirements of the competitive requirements [5], [6]. Fig.1 models the process for traditional personnel selection. Therefore there is a need to enhance the methods of personnel selection among numerous alternatives by owning different technical and commercial scope [7]–[10].

FIGURE 1. Ontology for traditional methods for personnel selection.

In Research, numerous methods used for personnel selection such as interview, examination, mastery tests, work sample tests, predictive index tests, and personality trait quizzes, however there is shortage of the use of MCDM techniques [3]. The enhancements methods of personnel selection problems propose the use of MCDM methods [2], [7], [11]–[13]. The MCDM can handle complex problems and select the appropriate solution among numerous of alternatives solutions with respect to enterprise's goals [14]. Sometimes not all candidates match enterprise's purposes, for this reason [15], mentions different patterns for MCDM problems:

1) Identify problem statements.
2) Sort the problem: Classify candidates into related groups.
3) Rank problem's candidates.
4) Indicate problem candidates' features.

Due to the complexity of human cognition, MCDM methods of selection ideal candidates are surrounding with vague, impression, inconsistency, and uncertainty [16].

Mostly decision makers do not have a clear consciousness about all criterions in order to make the proper decisions, which leads for challenges of MCDM methods.The major categories of MCDM are Multi-Criteria Decision Analysis (MCDA) and the one of Multi-Objective Mathematical Programming (MOMP) [17]. First category, MCDA is a method used to detect the relations between different alternatives. The decision is taken based on the surrounding criterions and alternatives. The criterions have some characteristics such that, they can be measured and their output can be clearly computed. The consequences of outcome afford the ability of observations and facilities of final decisions. Second category, MOMP deals with numerous and conflict objectives and applies optimization techniques to obtain possible solutions of decisions [18]. The alternatives have been formed using mathematical methods.

AHP is a method to structure complex problems into hierarchical structure to display relationship of goals, alternatives, and criteria to aid decision maker to judge the performance of decisions in such efficient manner [19]. However classical methods of AHP cannot handle the conditions of vague, impression, and uncertainty. The fuzzy AHP is proposed to handle the conditions of vague and impression, however fuzzy is working with membership function which is very difficult for decision maker to detect in real situations [20].

Neutrosophy is a new field in philosophy, which studies the scope and origin of neutralities [21], [22]. Neutrosophic is used to resolve numerous applications' challenges such as critical path problem in project management [23], [14]. Regularly, the preferences between criteria cannot be clearly determined by decision makers in real life situations. The contributions of the use of neutrosophic sets are to overcome the conditions of uncertainty and inconsistency that surrounding environment and affecting on decision maker's judgments. The degree of importance and weakness within criteria and alternatives should be evaluated. The neutrosophic method has the ability to model the relationships and dependents between criteria and alternatives. The neutrosophic theory can explicitly show decision maker's knowledge, reference, and judgments [24]. Neutrosophic are an expansion of Intuitionistic Fuzzy Sets (IFS) that illustrate accurate perspectives and enhancing interpretation of uncertainty [20]. The neutrosophic set is moving forward by the use of membership of truth, indeterminacy, and non-membership in a given set. The neutrosophic set illustrates the cases of indeterminacy that exists in real life situations to aid experts of decision makers to make accurate and efficient judgments.

TOPSIS has been first proposed in [25], the method depends on synthesizing the criteria like in AHP. The TOPSIS is rely on dividing alternatives into two groups positive and negative solutions, the best solution has the shortest distance from positive group of solutions and the longest distance from negative group of solutions [26]. Considering the personnel selection is an MCDM problem, the proposed model integrates the AHP with TOPSIS by the use of neutrosophic sets. The decision maker provides the basis judgments for the selection problem. The judgments are obtained in environment of inconsistency, and uncertainty. The proposed framework handles the current challenges and recommends the ideal solutions with respect to constraint of environment criteria.

Section 2 reviews the literatures for the problem of personnel selections neutrosophic AHP, and TOPSIS. Section 3 presents the proposed methodology to aid decision makers for selecting the appropriate applicant for achieving enterprise goals. Section 4 provides an empirical application to validate of proposed model. Section 5 summarizes the research key point and assigns the research future work.

II. LITERATURE REVIEW

Many issues can affect the process of personnel selection including changing in work, government, behavior, regulations, the evolution technology, and others [8], [9], [27], [28]. The traditional methods of interview are validity, reliability, interviewer differences, employment opportunity issues, and decision-making processes. In addition to, the interview can be used as an approach to help personnel and organization effectiveness [29]. The personality judgments measures are used to efficiently enhance the personnel selections process [30]. Inside enterprise the managers identifies many defects in the business relationships of IT [31]. The focus on the deficiency of information technology managers effectiveness, is leading to partial failures in enterprise business relations [32]. In [33], demonstrates a problem in the department which is the lack for the super-manager for communication skills that make negative impact on the organization success. To overcome the negative impacts, the technical capabilities should be considered. In [5], demonstrates the effectiveness of Information technology to enterprise. The enterprise IT resources are divided into infrastructure, human resources, and enabled intangibles. The IT skills are classified into soft skills such as interpersonal skills, creativity, and time management, and technical skills such as marketing, accounting, and emerging information technologies [34]. An model based for efficiency analysis are proposed for ranking alternatives using the ordered weighted average (OWA) aggregation operators for the purpose of improving decision making processes [35].

Towards enhancing the decision maker's judgments, decision support system tools are proposed to enhance the personnel selection process [36], [37]. In [38], uses the MCDM methods for personnel selection. The MCDM methods are relying on an aggregating function which represents ''closeness to the ideal'' solution [39]. AHP decomposes obstacles into hierarchal structure in order to obtain priorities and weights to enhance decision maker's judgments [40]. Due to vagueness and impression, the fuzzy methods are provided to enhance the decision maker's judgments in the process of personnel selection [41]. The fuzzy methods combined with AHP to solve information systems problems for the personnel selections [42]. A fuzzy model based on a two-level personnel selection, is proposed to minimize subjective judgments in the methods of choosing between proper candidates to be hired to enterprise [43]. An approach based on ranking fuzzy numbers by metric distance and comparing the proposed method with other methods. In addition, proposing a computer-based group decision support system used three ranking methods for improving personnel selection problem [44]. Fuzzy multiple objective methods are illustrated in order to solve personnel selection problems [45]. Fuzzy multi-objective Boolean linear programming formulation is presented to show the degree of importance for each alternative. A system for personnel selection is based on fuzzy analytic hierarchy for ranking the candidates to achieve the most appropriate candidates [42]. In [46] focuses on the analytical thinking approaches, an analytical network process is proposed to handle the impression and ambiguity in the pairwise comparison matrix to reduce the personnel selection biasness.

TOPSIS is proposed as the best solution that has the shortest distance to positive solutions [33]. Traditionally the weights of TOPSIS presented as crisp numbers which cannot be applicable in real environment [47]. References [3] and [26] proposes TOPSIS methods for enhancing personnel selection problems. A new TOPSIS based on multi-criteria methods for ranking alternatives according to veto threshold [3]. According to Karnik–Mendel (KM), a fuzzy TOPSIS is proposed to obtain an accurate fuzzy relative closeness instead of crisp value in order to prevent loss of information and efficiency [26]. In [2] and [48] illustrate the method of TOPSIS with a fuzzy multi-criteria decision making algorithms to allow managers to assess information using linguistic and numerical scales with different data sources for solving the personnel selection problems. The AHP TOPSIS combined with fuzzy methods are mentioned in the case of education committee, an integrated method of fuzzy AHP and TOPSIS in an MCDM environment are used to enhance the personnel selection of the training and education staff [49]. For uncertainty and inconsistency conditions, the AHP TOPSIS combined with neutrosophic are illustrated in different fields like risks and supplier selections for aiding decision makers to achieve to ideal decisions [50], [51]. We propose to be the first to integrate the neutrosophic environment to AHP and TOPSIS techniques in personnel selection.

III. METHODOLOGY

In our study, the integrating of TOPSIS with neutrosophic is regard as a new contribution to make an ideal selection of applicants in the personnel selection problem. As mentioned in the literature review section, TOPSIS method is used to solve the problem of personnel selection. The recent

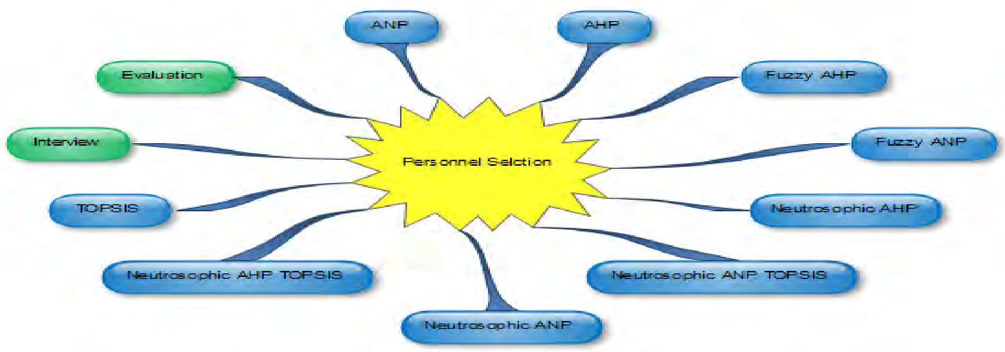

FIGURE 2. Mind map of personnel selection problem methods.

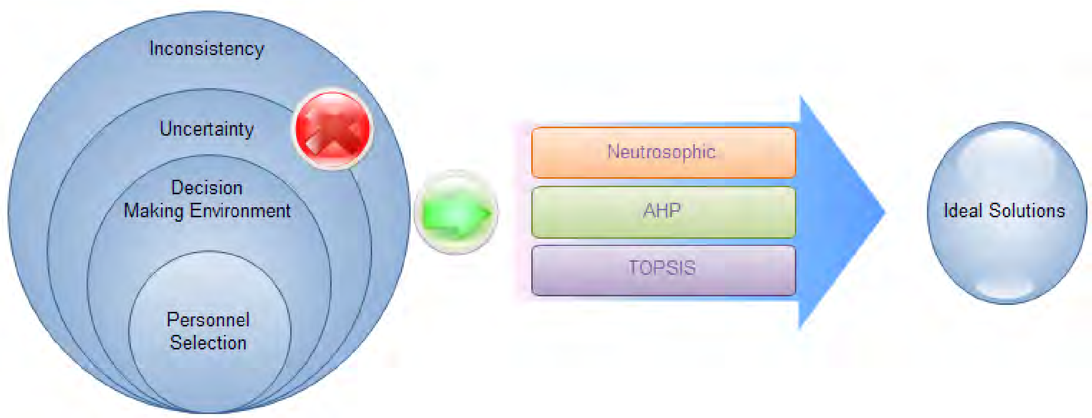

FIGURE 3. Conceptual flow to personnel selection problem.

researches as mentioned in [3] use TOPSIS in solving classical business problems as follows:

- **Manufacturing:** the supplier selection problems are handled using fuzzy positive ideal solution (PIS) and negative ideal solution (NIS) [52], [53]. The neutrosophic environment proposed to overcome the personnel selection problems of uncertainty and inconsistency [54], [55]. The hierarchical fuzzy TOPSIS is used as a recent method for recommending the most appropriate business process [56]. The AHP combined with TOPSIS are used to select the ideal maintenance strategy [57].
- **Marketing**: the evaluation of new products, service quality of services, and tourism management, to enhance hotel services by the use of fuzzy methods with AHP [58]–[60].

The traditional personnel selection process steps are divided into two phases. First a group of expertise makes the appraisal methods to evaluate the applicants. The reason to take more than one decision makers are to overcome any personnel biasness perspectives for the committee, and to focus on the success of enterprise factors. Second a final decision is proposed based on the committee judgments. Unfortunately, the conditions of uncertainty and inconsistency cannot be detected by human, due to decision maker's confusion or less experience.

Mind map is modeled to show the possible methods either traditional or non-traditional that can be used to handle personnel selection problems as mentioned in Fig.2. For the sake of uncertainty and inconsistency, we combine the neutrosophic AHP with TOPSIS techniques in order to handle the personnel selection environment problems as mentioned in Fig.3, to achieve ideal solutions for such a successful organization. The proposed methods steps are mentioned in Fig.4. The conceptual flow is presented in three stages. The first stage is to determine the objectives, criterions and alternatives are considered to insure that the candidate applicant will fulfill the enterprise needs. The second stage depicts the neutrosophic scales methods to evaluate the surrounding criteria of candidate's applicants. The third Stage is TOPSIS methods have been applied to choose the ideal candidates by establishing positive and negative areas of candidates. Finally, choose the ideal solution by using the relative closeness centric methods. For more details revise [61].

The explanation for the conceptual steps of combining the neutrosophic AHP with TOPSIS techniques are mentioned in the next steps:

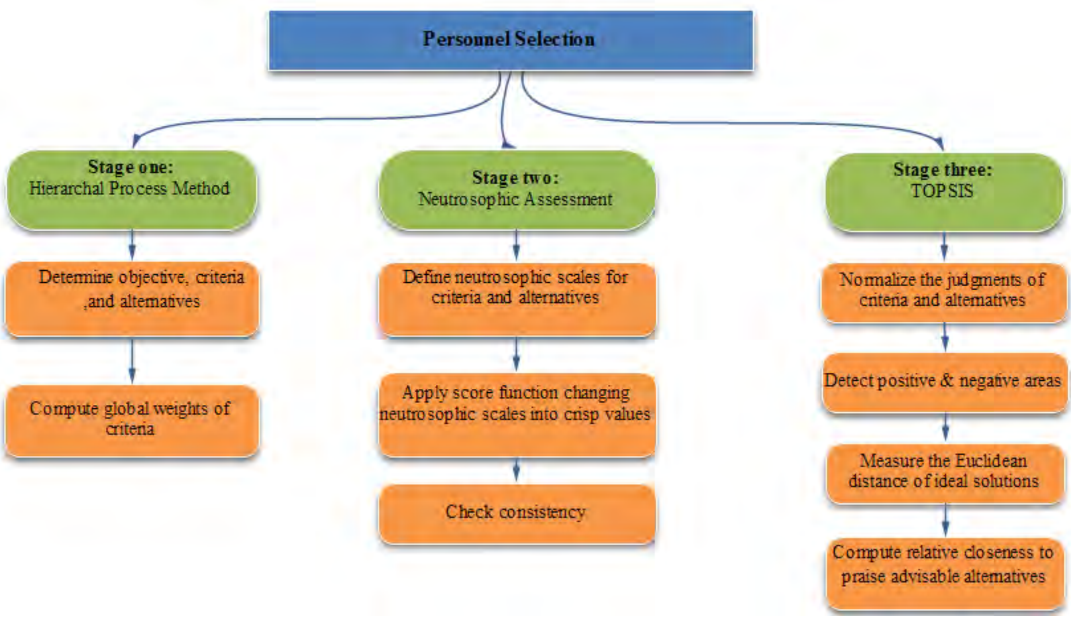

FIGURE 4. **The conceptual view steps for the proposed method.**

Step 1: Determine objectives and criteria by the model of AHP.

TABLE 1. **The triangular neutrosophic scale of AHP.**

Saaty scale	*Explanation*	*Neutrosophic Triangular Scale*
1	Equally significant	$1 = \langle\langle 1,1,1\rangle; 0.50, 0.50, 0.50\rangle$
3	Slightly significant	$3 = \langle\langle 2,3,4\rangle; 0.30, 0.75, 0.70\rangle$
5	Strongly significant	$5 = \langle\langle 4,5,6\rangle; \langle 0.80, 0.15, 0.20\rangle$
7	very strongly significant	$7 = \langle\langle 6,7,8\rangle, 0.90, 0.10, 0.10\rangle$
9	Absolutely significant	$9 = \langle\langle 9,9,9\rangle; 1.00, 0.00, 0.00\rangle$
2	sporadic values between two close scales	$2 = \langle\langle 1,2,3\rangle; 0.40, 0.60, 0.65\rangle$
4		$4 = \langle\langle 3,4,5\rangle; 0.35, 0.60, 0.40\rangle$
6		$6 = \langle\langle 5,6,7\rangle; 0.70, 0.25, 0.30\rangle$
8		$8 = \langle\langle 7,8,9\rangle; 0.85, 0.10, 0.15\rangle$

Step 2: Structure a committee from expertise decision makers to assign their judgments about the proposed alternatives and criteria. Aggregate the committee judgments using neutrosophic scales mentioned in table 1. The criteria is represented in comparison matrix, in the case of criteria 1 is strongly significant than criteria 2, the neutrosophic scale value is written as $\langle 4, 5, 6\rangle$.Conversely, the neutrosophic scale of criteria 2 to criteria 1 is the inverse of $\langle 4, 5, 6\rangle$ which is denoted as $\left\langle \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\rangle$.In addition, the neutrosophic scale value will be attached with sureness degree for truth, indeterminacy, and false degree that represents decision maker's perspectives. The sureness degree will be used further in the research computations. Giving an example, the preceding decision maker's perspective presents the structure of neutrosophic triangular as $\langle\langle 4, 5, 6\rangle; \langle 0.80, 0.15, 0.20\rangle$. The neutrosophic triangular scale values are represented as$\langle 4, 5, 6\rangle$, respectively to lower, median, and upper values. The sureness degree of decision maker point of view is mentioned as $\langle 0.80, 0.15 , 0.20\rangle$. In addition, the sureness degree of truth, indeterminacy, and falsity are regarded to be independent.

Step 3: Convert the neutrosophic scales 1 to crisp values by apply score functions of a_{ij} as mentioned:

$$s(a_{ij}) = \left| l_{r_{ij}} \times m_{r_{ij}} \times u_{r_{ij}}) \frac{T_{r_{ij}} + I_{r_{ij}} + F_{r_{ij}}}{9} \right| \tag{1}$$

where l, m, u denotes lower, median, upper of the scale neutrosophic numbers, T, I, F are the truth-membership, indeterminacy, and falsity membership functions respectively of triangular neutrosophic number.

After the conversion of neutrosophic scales into crisp values, the perspectives of decision makers should be aggregated. The aggregation should reflect the real preferences within relations as mentioned:

$$x_{ij} = \frac{\sum_{z=1}^{z} (a_{ij}^{z})}{z} \tag{2}$$

The aggregated pair-wise comparison matrix represents the estimation between preferences has been formed as mentioned:

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{ij} \\ \vdots & \vdots & \vdots & \\ x_{i1} & x_{i2} & \cdots & x_{ji} \end{bmatrix} \quad (3)$$

Step 4: Check consistency using the following equation:

$$CR = \frac{CI}{RI} \quad (4)$$

where CR is consistency rate, CI is consistency Index, and RI is a random consistency index. The detailed steps to measure consistency are mentioned in [61].

Step 5: Tabulate the weight for each criteria with respect to decision maker's judgments.

1) Calculate the total of row averages:

$$w_i = \frac{\sum_{j=1}^{n}(x_{ij})}{n}; \; i = 1, 2, 3, \ldots .m; \quad j = 1, 2, 3, \ldots, n \quad (5)$$

2) Normalize w_i using the following equation:

$$w_i^m = \frac{w_i}{\sum_{i=1}^{m} w_i}; \quad i = 1, 2, 3, \ldots, m. \quad (6)$$

Step 6: In order to achieve efficient personnel selection apply TOPSIS methods:

- *Step 6.1:* Create judgments of decision matrix according to perspectives and expertise of decision makers. Aggregate the decision makers judgments matrices in the case of the existence of more than one decision maker:
- *Step 6.2:* Convert the aggregated decision matrix to crisp values using equation (1). In case of multiple decision makers, the aggregation of pairwise comparison matrix is calculated using equation (2) and formed in form (3).
- *Step 6.3:* Afterwards the de-neutrosophic process, the crisp value of x_{ij} should be normalized which are in the form of decision matrix by applying the mentioned equation:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i-1}^{m} x_{ij}^2}}; \; i = 1, 2, 3 \ldots m; \quad j = 1, 2, 3 \ldots n \quad (7)$$

- *Step 6.4:* Multiply the weights $w_j$of criteria produced from neutrosophic AHP by the normalized decision matrix to produce the weighted matrix as mentioned:

$$z_{ij} = w_j \times r_{ij} \quad (8)$$

- *Step 6.5:* Compute the positive and negative areas by the use equation (9), and (10):

$$A^+ = \left\{ \begin{matrix} < \max(z_{ij}|i = 1, 2, \ldots, m)|j \in j^+ >, \\ < \min(z_{ij}|i = 1, 2, \ldots, m)|j \in j^- > \end{matrix} \right\} \quad (9)$$

$$A^- = \left\{ \begin{matrix} < \min(z_{ij}|i = 1, 2, \ldots, m)|j \in j^+ >, \\ < \max(z_{ij}|i = 1, 2, \ldots, m)|j \in j^- > \end{matrix} \right\} \quad (10)$$

Such that j^+ refers to profitable impact while j^-indicates non profitable impact.

- *Step 6.6:* Compute the euclidean distance between positive (d_i^+) and negative ideal solution (d_i^-) to the proposed alternatives as mentioned:

$$d_i^+ = \sqrt{\sum_{i=1}^{n}(z_{ij} - z_j^+)^2}, \; i = 1, 2, \ldots, m \quad (11)$$

$$d_i^- = \sqrt{\sum_{i=1}^{n}(z_{ij} - z_j^-)^2}, \; i = 1, 2, \ldots, m \quad (12)$$

- *Step 6.7:* Compute the relative closeness to choose the most appropriate and efficient decision by ranking the alternatives:

$$c_i = \frac{d_i^-}{d_i^+ + d_i^-}; \quad 1 = 1, 2, \ldots, m \quad (13)$$

Step 7: Based on alternative's rank, choose the best decision.

IV. AN EMPIRICAL APPLICATION

We illustrate an empirical application to present the proposed methodology in real world problems. The case study is applied on smart village Cairo Egypt. The customer service department need to hire new manager, because of the current manager is transferred to another branch outside the country. The judgments committee consists of four decision makers, they recommends five applicants to be the ideal from all the available applicants. After the meeting for decision makers, the general criteria's for selections are mentioned:

- C1: Professional Knowledge Edge and Expertise.
- C2: Previous Professional Career.
- C3: Personnelality and Potential

The proposed model neutrosophic AHP with TOPSIS applied on case study as follows in the next steps

Step1: In this phase, there are four experts decision makers:

1) Chief executive.
2) chief operating officer.
3) Non-executive director.
4) Social entrepreneur.

The vital criteria's according to decision judgments and applicants are represented in Fig.5.

FIGURE 5. The AHP structure for the proposed criteria and alternatives.

Step 2: The proposed case study includes four experts; each decision maker is representing his/her judgment

TABLE 2. The aggregated perspectives of decision makers for criteria.

Criteria	*C1*	*C2*	*C3*
C1	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨3,4,5⟩;0.60,0.35,0.40⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩
C2	1/⟨⟨3,4,5⟩;0.60,0.35,0.40⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩
C3	1/⟨⟨5,6,7⟩;0.70,0.25,0.430⟩	1/⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩

TABLE 3. The crisp comparison matrix of criteria according to objective with respect to manager's opinion.

Criteria	*C1*	*C2*	*C3*
C1	1	1.8481	2.1015
C2	0.541	1	1.3889
C3	0.4758	0.719	1

TABLE 4. The wieghts of criteria.

Criteria	*Weights*
C1	0.493
C2	0.285
C3	0.219

TABLE 5. Decision matrix for judgments committee with respect to criteria and alternative.

		C1	*C2*	*C3*
D.M.1	*A1*	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨3,4,5⟩;0.60,0.35,0.40⟩
	A2	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩
	A3	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩
	A4	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩
	A5	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩
D.M.2	*A1*	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩
	A2	⟨⟨3,4,5⟩;0.60,0.35,0.40⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩
	A3	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩
	A4	⟨⟨3,4,5⟩;0.60,0.35,0.40⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩
	A5	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩	⟨⟨3,4,5⟩;0.60,0.35,0.40⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩
D.M.3	*A1*	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩
	A2	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩
	A3	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨5,6,7⟩;0.70,0.25,0.30⟩	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩
	A4	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩
	A5	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩
D.M.4	*A1*	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩
	A2	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩
	A3	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩
	A4	⟨⟨1,1,1⟩;0.50,0.50,0.50⟩	⟨⟨1,2,3⟩;0.40,0.65,0.60⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩
	A5	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩	⟨⟨2,3,4⟩;0.30,0.75,0.70⟩	⟨⟨4,5,6⟩;0.80,0.15,0.20⟩

using table1. In order to regard a final judgment, a session has been performed with decision makers. The average preferences have been illustrated in table 2, where c1, c2, and c3 corresponds to, professional knowledge edge and expertise, previous professional career, and personality and potential

TABLE 6. De-neutrosophic crisp values for decision maker's judgments committee.

		C1	*C2*	*C3*
D.M.1	*A1*	1	1.3889	1.8481
	A2	1.85	1	1.85
	A3	1.85	2.1015	1
	A4	1.836	1.3889	1
	A5	1.3889	2.1015	2.1015
D.M.2	*A1*	1	1.3889	1.85
	A2	1.8481	1.3889	1
	A3	1.836	1	1
	A4	1.8481	2.1015	2.1015
	A5	2.1015	1.8481	2.1015
D.M.3	*A1*	1	1.3889	1.3889
	A2	1	1.836	1.836
	A3	1.3889	2.1015	1.836
	A4	1.836	1.3889	1.836
	A5	1.836	1.85	1.85
D.M.4	*A1*	1	1.3889	1
	A2	1	1.3889	1
	A3	1.3889	1.85	1.836
	A4	1	1.3889	1.85
	A5	1.836	1.85	1.836

TABLE 7. The normalization of decision matrix.

	C1	*C2*	*C3*
A1	0.08	0.10	0.11
A2	0.12	0.10	0.10
A3	0.14	0.13	0.10
A4	0.14	0.11	0.12
A5	0.15	0.14	0.15

Step 3: The perspectives of decision makers have been converted to crisp values by applying score value of equation (1), the results is presented in table 3.

Step 4: The consistency rate is computed. The consistency rate is accepted which is 1%.

Step 5: The weights of criteria are computed, and represented in table 4 and modeled in Fig.6.

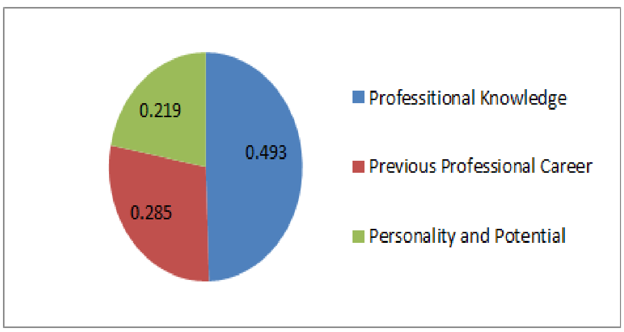

FIGURE 6. The pie chart of personnel selection criteria.

Step 6: The judgments for decision committee for the proposed alternatives and criteria presented in table 5 for more details in steps. Then use equation 1 to change neutrosophic scales into crisp values as shown in table 6.

- The aggregated results for committee judgments are calculated using equation (2). The equation (7), obtains the normalization of alternatives and criteria, and the normalization results are represented in table 7.
- Multiply the weights w_j of criteria from table 4 by the normalized decision matrix in table 7,in order to produce the weighted matrix by the use of equation (8) and results mentioned in table 8.
  - Compute the positive and negative areas by the use of equation (9), and (10)

$$A^+ = \{0.073, 0.039, 0.032\}.$$
$$A^- = \{0.039, 0.028, 0.021\}.$$

- Compute the Euclidean distance between positive (d_i^+) and negative ideal solution (d_i^- as mentioned in equation (11), and (12). After that, compute the relative closeness to choose the most appropriate and efficient decision by ranking the alternatives using equation (13). The results of d_i^+,d_i^-,c_i, and final ranking are presented in table 9.

Step 7: The applicants are sorted by the rank of neutrosophic AHP and TOPSIS methods. The applicant four is considered

TABLE 8. The wieghted decision matrix.

	C1	*C2*	*C3*
A1	0.039	0.028	0.024
A2	0.059	0.028	0.021
A3	0.069	0.037	0.021
A4	0.069	0.031	0.026
A5	0.073	0.039	0.032

TABLE 9. Ranking of applicants.

	d_i^+	d_i^-	c_i	*Rank*
A1	0.036	0.003	0.0769	5
A2	0.02	0.02	0.5	3
A3	0.011	0.031	0.219	4
A4	0.010	0.305	0.968	1
A5	0.007	0.0373	0.841	2

to be the best one to be hired to smart village. The A4 applicant meets the judgments of decision makers committee and criteria to achieve the success of the smart village in Cairo Egypt. However A1 is considered to be the worst choice that cannot meet the specified judgments and criteria to meet the enterprise goals.

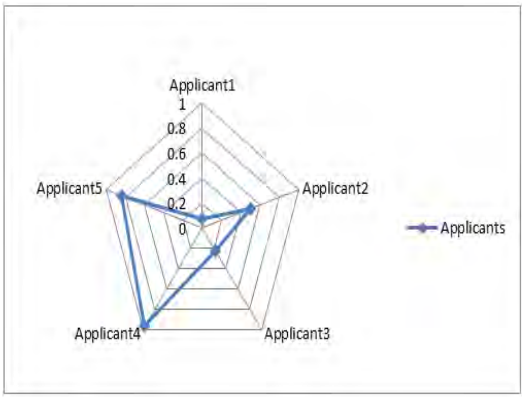

FIGURE 7. The relative closeness for applicants.

- The applicants ranking are modeled to show the results of relative closeness between applicants as mentioned in Fig.7. In addition, the sorting of alternatives with respect to ranking results shown that applicant 4 should be hired to the proper position. The proposed case study shows that human resources can be improved by the use of non-traditional methods of neutrosophic AHP with TOPSIS to achieve the best alternatives in personnel selection problem.

V. CONCLUSION AND FUTURE WORK

Personnel selection is a vital problem that impact on the quality of management and enterprises. Many studies try to aid decision makers to improve the decision making. But the use of non-traditional methods to assist decision makers to choose the right person in the right position becomes an obligatory condition. So the weight of our study evolved to overcome the inconsistency and uncertainty conditions that found in MCDM environment. The proposed study integrates neutrosophic AHP with TOPSIS methods to improve the decision committee judgments by considering the constraint of the environmental criterions. The case study is applied on smart village Cairo, Egypt, shows the efficiency for the proposed method and provides final decision to hire applicant four to be in the right position for achieving success organization.

Since, the personnel selection problem is an important issue for gaining true achievements in enterprises, the future work will focus on enhancing of personnel selection criteria. The enhancements are applied by the use of evolutionary algorithms to choose the most effective criteria. Another important discipline is to improve the TOPSIS methods.

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A Refined Approach for Forecasting Based on Neutrosophic Time Series

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Abstract: This research introduces a neutrosophic forecasting approach based on neutrosophic time series (NTS). Historical data can be transformed into neutrosophic time series data to determine their truth, indeterminacy and falsity functions. The basis for the neutrosophication process is the score and accuracy functions of historical data. In addition, neutrosophic logical relationship groups (NLRGs) are determined and a deneutrosophication method for NTS is presented. The objective of this research is to suggest an idea of first-and high-order NTS. By comparing our approach with other approaches, we conclude that the suggested approach of forecasting gets better results compared to the other existing approaches of fuzzy, intuitionistic fuzzy, and neutrosophic time series.

Keywords: neutrosophic time series; triangular neutrosophic number; neutrosophic logical relationship; neutrosophic logical relationship groups

1. Introduction

There are different methods in the literature on fuzzy and intuitionistic fuzzy time series methods to forecast future values. The major difference between traditional and fuzzy time series is that the values of traditional time series are presented in numbers, whereas the values in fuzzy time series are fuzzy sets or linguistic values with real meanings. In intuitionistic fuzzy time series, the values are intuitionistic fuzzy sets or linguistic values. The first method in literature for forecasting future values based on fuzzy time series was introduced by Song and Chissom [1]. They also applied time-variant and time-invariant models for forecasting the enrollment data at the University of Alabama [1,2]. The identification of fuzzy relationship and the defuzzification process in both models were the main steps for calculating forecasted values. In time variant fuzzy time series it is proposed that autocorrelation is dependent due to the time, while in time invariant it is proposed that autocorrelation is independent due to the time.

The term “fuzzy relationship” means a collection of fuzzy sets which are caused only by other sets. In addition, the “defuzzification” process means converting the fuzzy values into crisp ones. Furthermore, a straightforward approach for time series forecasting was presented by Chen [3] by using uncomplicated arithmetic computations. To enhance the accuracy of forecasted outputs, some papers suggested various methods on fuzzy time series (FTS) forecasting [4–7]. A high-order FTS method was also presented by Chen [8] and Singh [9], and a method of bivariate fuzzy time series analysis for the forecasting of a stock index was introduced by Hsu et al. [10]. Furthermore, a framework developed for evaluation and forecasting based on the fuzzy NEAT F-PROMETHEE method was presented by Ziemba and Becker [11] for taking into account the uncertainty of input data, which is particularly burdened with the forecast values of the information and communication technologies development indicators.

The concept of fuzzy set was introduced by Zadeh [12], and it was generalized by Atanassov [13] to intuitionistic fuzzy set (IFS) to make it more suitable to handle ambiguity. The IFS considers both the membership (truth) and non-membership (falsity) degrees. However, the fuzzy set considers only the membership degree. Recently, the IFS was used for handling the fuzzy time series forecasting by Gangwar and Kumar [14] and Wang et al. [15]. In addition, the notion of intuitionistic fuzzy time series (IFTS)was employed in forecasting, as in [16–18]. Several researchers [19,20]proposed forecasting models using a genetic algorithm, or suggested a method of forecasting based on aggregated FTS and particle swarm optimization [21]. A novel method of forecasting based on hesitant fuzzy set was proposed by Bisht and Kumar [22], and fuzzy descriptor models for earthquakes was introduced by Bahrami and Shafiee [23]. A heuristic adaptive-order IFTS forecasting model was presented by Wang et al. [24]. Subsequently, Abhishekh et al. [25,26] presented a weighted type 2 FTS and score function-based IFTS forecasting approach. Moreover**,** Abhishekh and Kumar [27] suggested an approach for forecasting rice production in the area of FTS.

Since the accuracy rates of forecasting in the previous approaches are not good enough in the field of fuzzy and intuitionistic fuzzy time series, we introduce the notion of first- and high-order neutrosophic time series data for this research. Additionally, with the growing need to represent vague and random information, neutrosophic set (NS) theory [28] is an effective extension of fuzzy and intuitionistic fuzzy set theories. Smarandache [29] suggested NSs, which consist of truth membership function, indeterminacy membership function, and falsity membership function, as a better representation of reality. Neutrosophic sets received wide attention, as well as benefitting from various practical applications in diverse fields [30–39]. However, there are only two recent research papers published in the forecasting field (e.g., for stock market analysis). Guan et al. [40] proposed a new forecasting model based on multi-valued neutrosophic sets and two-factor third-order fuzzy logical relationships to forecast the stock market. Subsequently, Guan et al. [41] proposed a new forecasting method based on high-order fluctuation trends and information entropy.

The aim of this research is to enhance accuracy rates of forecasting in the area of fuzzy, intuitionistic fuzzy, and neutrosophic time series (NTS). In this research, we present the notion of forecasting based on first-and high-order NTS data by determining the suitable length of neutrosophic numbers that influence on expected values. We also suggest a neutrosophication of historical time series data, based on the biggest score function (i.e., the maximum value of score function), and define neutrosophic logical relationship groups (NLRGs) for obtaining forecasted outputs. The suggested approach of neutrosophic time series forecasting has been validated and compared with different existing models for showing its superiority.

The remaining parts of this research are organized as follows. The essential concepts of neutrosophic set and neutrosophic time series are briefly presented in Section 2. Section3presents the proposed neutrosophic time series method for the forecasting process. Section 4validates the proposed method by applying it to two numerical examples for showing its effectiveness; a comparison with other existing methods is presented. Finally, Section 5 concludes the research and determines future trends.

2. Some Basic Definitions of Neutrosophic Set and Neutrosophic Time Series

Neutrosophic time series is a concept for solving forecasting problems using neutrosophic concepts. In this section, we present the basic concepts of the neutrosophic set and of the neutrosophic time series (NTS).

Definition 1. *Let X be a finite universal set. A neutrosophic set N in X is an object having the following form:* $N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle \mid x \in X\}$*, where* $T_N(x): X \to [0,1]$ *determines the degree of truth membership function,* $I_N(x): X \to [0,1]$ *determines the degree of indeterminacy, and function* $F_N(x): X \to [0,1]$

determines the degree of non-membership or falsity function. For every $\boldsymbol{x} \in \boldsymbol{X}, \boldsymbol{0^-} \leq \boldsymbol{T_N}(\boldsymbol{x}) + \boldsymbol{I_N}(\boldsymbol{x}) + \boldsymbol{F_N}(\boldsymbol{x}) \leq \boldsymbol{3^+}$ [29].

Definition 2. *A single valued triangular neutrosophic number* $\widetilde{\boldsymbol{N}} = \langle(\boldsymbol{n_1}, \boldsymbol{n_2}, \boldsymbol{n_3}); \boldsymbol{T_{\widetilde{N}}}, \boldsymbol{I_{\widetilde{N}}}, \boldsymbol{F_{\widetilde{N}}}\rangle$ *is a special neutrosophic set on the real number set R whose truth (membership), indeterminacy, and falsity (non-membership) degrees are as follows* [29]:

$$T_{\widetilde{N}}(x) = \begin{cases} \boldsymbol{T_{\widetilde{N}}}\left(\dfrac{\boldsymbol{x}-\boldsymbol{n_1}}{\boldsymbol{n_2}-\boldsymbol{n_1}}\right) & (\boldsymbol{n_1} \leq \boldsymbol{x} \leq \boldsymbol{n_2}) \\ \boldsymbol{T_{\widetilde{N}}} & (\boldsymbol{x} = \boldsymbol{n_2}) \\ \boldsymbol{T_{\widetilde{N}}}\left(\dfrac{\boldsymbol{n_3}-\boldsymbol{x}}{\boldsymbol{n_3}-\boldsymbol{n_2}}\right) & (\boldsymbol{n_2} < \boldsymbol{x} \leq \boldsymbol{n_3}) \\ \boldsymbol{0} & \textbf{otherwise}, \end{cases} \tag{1}$$

$$I_{\widetilde{N}}(x) = \begin{cases} \dfrac{\left(n_2 - x + I_{\widetilde{N}}(x - n_1)\right)}{(n_2 - n_1)} & (n_1 \leq x \leq n_2) \\ I_{\widetilde{N}} & (x = n_2) \\ \dfrac{(x - n_2 + I_{\widetilde{N}}(n_3 - x))}{(n_3 - n_2)} & (n_2 < x \leq n_3) \\ 1 & \text{otherwise}, \end{cases} \tag{2}$$

$$F_{\widetilde{N}}(x) = \begin{cases} \dfrac{\left(n_2 - x + F_{\widetilde{N}}(x - n_1)\right)}{(n_2 - n_1)} & (n_1 \leq x \leq n_2) \\ F_{\widetilde{N}} & (x = n_2) \\ \dfrac{(x - n_2 + F_{\widetilde{N}}(n_3 - x))}{(n_3 - n_2)} & (n_2 < x \leq n_3) \\ 1 & \text{otherwise}, \end{cases} \tag{3}$$

where $0 \leq T_{\widetilde{N}} \leq 1$, $0 \leq I_{\widetilde{N}} \leq 1$, $0 \leq F_{\widetilde{N}} \leq 1$, $0 \leq T_{\widetilde{N}} + I_{\widetilde{N}} + F_{\widetilde{N}} \leq 3$, $n_1, n_2, n_3 \in R$, and being the lower, median, and upper values of the triangular neutrosophic number.

Definition 3. *Let X and Y be two finite universal sets. A neutrosophic relation R from X to Y is a neutrosophic set in the direct product space X to Y:*

$$R = \{\langle(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{T_N}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{I_N}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{F_N}(\boldsymbol{x})\rangle | (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{Y}\}$$

where $0^- \leq T_N(x,y) + I_N(x,y) + F_N(x,y) \leq 3^+$, $\forall (x,y) \in X \times Y$ for $T_N(x,y) \to [0,1]$, $I_N(x,y) \to [0,1]$, and $F_N(x,y) \to [0,1]: X \times Y \to [0,1]$.

Definition 4. *Let* $\boldsymbol{X}(\boldsymbol{t})(\boldsymbol{t} = \boldsymbol{1}, \boldsymbol{2}, \dots,)$*, a subset of* $\boldsymbol{R}$*, be the universe of discourse on which neutrosophic sets* $\boldsymbol{f_i}(\boldsymbol{t}) = \langle \boldsymbol{T_N}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{I_N}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{F_N}(\boldsymbol{x})\rangle(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots)$ *are defined.* $\boldsymbol{F}(\boldsymbol{t}) = \{\boldsymbol{f_1}(\boldsymbol{x}), \boldsymbol{f_2}(\boldsymbol{x}), \dots\}$ *is a collection of* $\boldsymbol{f_i}(\boldsymbol{t})$ *and it defines a neutrosophic time series on* $\boldsymbol{X}(\boldsymbol{t})(\boldsymbol{t} = \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{2}, \dots)$.

Definition 5. *If there exists a neutrosophic relationship* $\boldsymbol{R}(\boldsymbol{t}-\boldsymbol{1}, \boldsymbol{t})$*, such that* $\boldsymbol{F}(\boldsymbol{t}) = \boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1}) \times \boldsymbol{R}(\boldsymbol{t}-\boldsymbol{1}, \boldsymbol{t})$*, where '×' represents an operator, then* $\boldsymbol{F}(\boldsymbol{t})$ *is said to be caused by* $\boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1})$*. The relationship between* $\boldsymbol{F}(\boldsymbol{t})$ *and* $\boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1})$ *is symbolized by* $\boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1}) \to \boldsymbol{F}(\boldsymbol{t})$.

Definition 6. *Let* $\boldsymbol{F}(\boldsymbol{t})$ *caused by* $\boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1})$ *only and symbolized by* $\boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1}) \to \boldsymbol{F}(\boldsymbol{t})$*; consequently, a neutrosophic relationship exists between* $\boldsymbol{F}(\boldsymbol{t})$ *and* $\boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1})$ *that is denoted as* $\boldsymbol{F}(\boldsymbol{t}) = \boldsymbol{F}(\boldsymbol{t}-\boldsymbol{1}) \times \boldsymbol{R}(\boldsymbol{t}-\boldsymbol{1}, \boldsymbol{t})$, *since* $\boldsymbol{R}$ *is a first-order model of* $\boldsymbol{F(t)}$. *The* $\boldsymbol{F(t)}$ *is a time-invariant neutrosophic time series if* $\boldsymbol{R(t-1,t)}$ *is independent of time* $\boldsymbol{t}$, $\boldsymbol{R(t,t-1) = R(t-1,t-2)}\ \forall\ \boldsymbol{t}$. *Otherwise,* $\boldsymbol{F(t)}$ *is called a time-variant neutrosophic time series.*

Definition 7. *Let* $\boldsymbol{F(t-1)} = \widetilde{\boldsymbol{N}}_{\boldsymbol{i}}$ *and* $(\boldsymbol{t}) = \widetilde{\boldsymbol{N}}_{\boldsymbol{j}}$; *a neutrosophic logical relationship (NLR) can be defined as* $\widetilde{\boldsymbol{N}}_{\boldsymbol{i}} \rightarrow \widetilde{\boldsymbol{N}}_{\boldsymbol{j}}$, *where* $\widetilde{\boldsymbol{N}}_{\boldsymbol{i}}$, $\widetilde{\boldsymbol{N}}_{\boldsymbol{j}}$ *are the current and next state of NLR. Since* $\boldsymbol{F(t)}$ *is occurred by more than one neutrosophic set* $\boldsymbol{F(t-n)}$, $\boldsymbol{F(t-n+1)}, \ldots \boldsymbol{F(t-1)}$, *then the neutrosophic relationship is represented by* $\widetilde{\boldsymbol{N}}_{\boldsymbol{i1}}, \widetilde{\boldsymbol{N}}_{\boldsymbol{i2}}, \ldots, \widetilde{\boldsymbol{N}}_{\boldsymbol{in}} \rightarrow \widetilde{\boldsymbol{N}}_{\boldsymbol{j}}$, *where* $\boldsymbol{F(t-n)} = \widetilde{\boldsymbol{N}}_{\boldsymbol{i1}}$, $\boldsymbol{F(t-n+1)} = \widetilde{\boldsymbol{N}}_{\boldsymbol{i2}}$. *The relationship is called high-order neutrosophic time series model.*

3. Neutrosophic Time Series Forecasting Algorithm

Because a neutrosophic set plays a significant role in decision-making and data analysis problems by handling vague, inconsistent, and incomplete information [30–39], we propose in this section an enhanced approach of forecasting using the concept of neutrosophic time series (NTS).

The stepwise method of the suggested algorithm of neutrosophic time series forecasting is dependent on historical time series data.

3.1. The Proposed Method of Forecasting Based on First-Order NTS Data

Step 1: By depending on the range of the existing data set, determine the universe of discourse U as follows:

- Select the largest D_l and the smallest D_s from all available data D_v, then

$$U = [D_s - D_1,\ D_l + D_2] \tag{4}$$

where D_1 and D_2 are two proper positive numbers assigned by experts in the problem domain. So, we can define D_1, D_2 as the values by which the range of the universe of discourse is less than the specified value of D_s for the first (i.e., D_1) or greater than the specified value of D_l for the latter (i.e., D_2).

Step 2: Create a partition of the universe of discourse, to m triangular neutrosophic numbers as follows:

- Decide the suitable length (Le) of available time series data:
    - o Among the value D_{v-1}, D_v, calculate all absolute differences and take the average of these differences.
    - o Consider half the average as the initial length.
    - o According to the obtained result, use the base mapping table [42] to determine the base for the length of intervals.
    - o Round the result to determine the appropriate length of neutrosophic numbers.
    - o For example: if we have these time series data $30,50,80,120,100,70$, then the absolute differences will be 20,30,40,20,30, and the average of these values = 28. Then, half of the average will be 14 and this is the initial value of length. By using the base mapping table [42], the base for length = 10 because 14 locates in the range $[11-100]$ and by rounding the length 14 by the base ten, the result will equal 10. Here, the appropriate length of neutrosophic numbers equals 10.
- Compute the number of triangular neutrosophic numbers (m) as follows:

$$m = \frac{D_l + D_2 - D_s + D_1}{le} \tag{5}$$

Step 3: According to the numbers of triangular neutrosophic numbers on the universe of discourse and determined length (le), begin to construct the triangular neutrosophic numbers. The triangular neutrosophic numbers are $\widetilde{N}_1, \widetilde{N}_2, \ldots, \widetilde{N}_m$.

As we illustrated in Definition 2, each triangular neutrosophic number consists of two parts which are the value of the triangular neutrosophic number (lower, median, upper) and the degree of confirmation (truth/membership degree T, indeterminacy degree I, falsity/non-membership degree F). The initial value of T, I, F must be determined by experts according to the existing problem.

Step 4: Make a neutrosophication process of the existing data:
For $i, j = 1,2, \ldots, v$ (the end of data):

Rule 1: Use this equation to calculate the score degree, and if the score degree of two neutrosophic numbers is not equal for any data, then choose the maximum value of the score degree:

$$SC_{\tilde{N}_j}(x_i) = 2 + T_{\tilde{N}_j}(x_i) - I_{\tilde{N}_j}(x_i) - F(x_i) \quad (6)$$

Then, select $SC_{\tilde{N}_k} = max\ (SC_{\tilde{N}_k}, SC_{\tilde{N}_k}, \ldots, SC_{\tilde{N}_k})$ for $x_i, i = 1,2 \ldots., n,\ 1 \leq k \leq n$, and assign the neutrosophic number $\tilde{N}_k$ to x_i.

Rule 2: If two neutrosophic numbers have the same score degree, then use the following equation to calculate the score degree, and select the minimum accuracy degree:

$$AC_{\tilde{N}_j}(x_i) = 2 + T_{\tilde{N}_j}(x_i) - I_{\tilde{N}_j}(x_i) + F(x_i) \quad (7)$$

Furthermore, $AC_{\tilde{N}_k} = min\ (AC_{\tilde{N}_k}, AC_{\tilde{N}_k}, \ldots, AC_{\tilde{N}_k})$ for $x_i, i = 1,2 \ldots., n,\ 1 \leq k \leq n$; assign the neutrosophic number $\tilde{N}_k$ to x_i.

Step 5: Construct the neutrosophic logical relationships (NLRs) as follows:
If $\tilde{N}_j, \tilde{N}_k$ are the neutrosophication values of year $n$and year $n+1$, respectively, then the NLR is symbolized as $\tilde{N}_j \rightarrow \tilde{N}_k$.

Step 6: Based on the NLR, begin to establish the neutrosophic logical relationship groups (NLRGs).

Step 7: Calculate the forecasted values as follows:

Rule 1: If the neutrosophication value of $data_i$ is $\tilde{N}_k$ and it is not caused by any other neutrosophication values and, by looking at the NLRG of this value, you cannot find the value which it depends on (i.e., $\neq\rightarrow \tilde{N}_k$), then the forecasted value in this case will equal—(i.e., leave it empty). The ≠ symbol means no value.

Rule 2: If the neutrosophication value of $data_i$ is $\tilde{N}_k$ and it is caused by $\tilde{N}_j$ ($\tilde{N}_j \rightarrow \tilde{N}_k$), then look at NLRG of $\tilde{N}_j$, and

- If NLRG of $\tilde{N}_j$ is empty (i.e., $\tilde{N}_j \rightarrow \emptyset$, or $\tilde{N}_j \rightarrow \tilde{N}_j$), then the forecasted value is the middle value of $\tilde{N}_j$.
- If NLRG of $\tilde{N}_j$ is one-to-one (i.e., $\tilde{N}_j \rightarrow \tilde{N}_k$), then the forecasted value is the middle value of $\tilde{N}_k$.
- If NLRG of $\tilde{N}_j$ is one-to-many (i.e., $\tilde{N}_j \rightarrow \tilde{N}_{k1}, \tilde{N}_{k2}, \ldots, \tilde{N}_{kn}$), then the forecasted value is the average of the middle values of $\tilde{N}_{k1}, \tilde{N}_{k2}, \ldots, \tilde{N}_{kn}$.

Step 8: Use the following equations to calculate the forecasting error:

$$\text{Root mean square error (RMSE)} = \sqrt{\frac{\sum_{i=1}^{n}(\text{Forecast}_i - \text{Actual}_i)^2}{n}}, \quad (8)$$

$$\text{Forecasting error} = \frac{|\text{ Forecast} - \text{Actual}|}{Actual} \times 100, \quad (9)$$

$$\text{Average forecasting error (AFE) (\%)} = \frac{\text{Sum of forecasting error}}{number\ of\ errors} \times 100. \quad (10)$$

3.2. The Proposed Method of Forecasting Based on High-Order NTS Data

We can also apply the proposed method of forecasting based on high-order NTS data:

- All steps from 1 to 4 are the same as previously, but in step 5 we begin to construct the neutrosophic logical relationships (NLRs) of the nth order NTS, where $n \geq 2$.
- Based on the NLR of the nth order, NTS begin to establish the neutrosophic logical relationship groups (NLRGs).
- Calculate the forecasted values as follows:

- Rule 1: If the neutrosophication values of $data_i$ is $\tilde{N}_l$ and it is not caused by any other neutrosophication values and, by looking at the NLRG of this value, you cannot find the values which it depends on (i.e., $\neq\rightarrow \tilde{N}_l$), then the forecasted value in this case will equal— (i.e., leave it empty). The ≠ symbol means no value.
- Rule 2: If the neutrosophication value of $data_i$ is $\tilde{N}_l$ and it is caused by $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik}$ (i.e., $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik} \rightarrow \tilde{N}_l$), then look at the NLRG of $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik}$, and

  - If $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik} \rightarrow \emptyset$, then the forecasted value at this year is the average of the middle value of $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik}$.
  - If $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik} \rightarrow \tilde{N}_j$, then the forecasted value at this year is the middle value of $\tilde{N}_j$.
  - If $\tilde{N}_{in}$, $\tilde{N}_{i(n-1)}$, …, $\tilde{N}_{ik} \rightarrow \tilde{N}_j, \tilde{N}_{j1}, \tilde{N}_{j2}$, then the forecasted value at this year is the average of the middle value of $\tilde{N}_j, \tilde{N}_{j1}, \tilde{N}_{j2}$.

4. Numerical Examples

In this section, we solve two numerical examples and compare outputs with other existing methods for verifying the applicability and superiority of the suggested method.

4.1. Numerical Example 1

In this example, the suggested approach is implemented on the benchmarking time series data of student enrollments at the University of Alabama from year 1971 to 1992 adopted from [26]. The steps are as follows:

Step 1: Let the two proper positive numbers D_1 and D_2 be 5 and 13, determined by the expert. By selecting the largest and the smallest observation from all available data which are presented in Table 1, then $D_l = 19{,}337$ and $D_s = 13{,}055$, respectively. Consequently, the universe of discourse $U = [13{,}055 - 5, 19{,}337 + 13] = [13{,}050, 19{,}350]$.

Step 2: Create a partition of the universe of discourse, to m triangular neutrosophic numbers, as follows:

- Determine the suitable length (Le) of available time series data:
  - From Table 1, the average of absolute differences = 510.3.
  - The initial length = $\frac{510.3}{2} = 255.15$.
  - By using the base mapping table [42], the base for length of intervals = 100, since it is located in the range [101,1000].
  - By rounding 255.15 with regard to base 100, then the appropriate length of neutrosophic numbers = 300.
- Compute the number of triangular neutrosophic numbers (m) as follows:

$$m = \frac{19350 - 13050}{300} = 21.$$

Then, we can partition U into 21 triangular neutrosophic numbers with length = 300.

Step 3: According to the number of triangular neutrosophic numbers on the universe of discourse and determined length (le), begin to construct the triangular neutrosophic numbers as follows:

$$\tilde{N}_1 = \langle 13050,13350,13650; 0.90,0.10,0.10\rangle,$$

$$\tilde{N}_2 = \langle 13350,13650,13950; 0.80,0.20,0.10\rangle,$$

$$\tilde{N}_3 = \langle 13650,13950,14250; 0.90,0.20,0.10\rangle,$$

$$\tilde{N}_4 = \langle 13950,14250,14550; 0.85,0.15,0.10\rangle,$$

$$\tilde{N}_5 = \langle 14250,14550,14850; 0.75,0.10,0.30\rangle,$$

$$\tilde{N}_6 = \langle 14550,14850,15150; 0.90,0.10,0.10\rangle,$$

$\widetilde{N}_7 = \langle 14850,15150,15450; 0.60,0.30,0.40\rangle$,

$\widetilde{N}_8 = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$,

$\widetilde{N}_9 = \langle 15450,15750,16050; 0.70,0.20,0.30\rangle$,

$\widetilde{N}_{10} = \langle 15750,16050,16350; 0.90,0.10,0.30\rangle$,

$\widetilde{N}_{11} = \langle 16050,16350,16650; 0.85,0.10,0.15\rangle$,

$\widetilde{N}_{12} = \langle 16350,16650,16950; 0.80,0.20,0.20\rangle$,

$\widetilde{N}_{13} = \langle 16650,16950,17250; 0.90,0.10,0.30\rangle$,

$\widetilde{N}_{14} = \langle 16950,17250,17550; 0.90,0.10,0.30\rangle$,

$\widetilde{N}_{15} = \langle 17250,17550,17850; 0.75,0.10,0.30\rangle$,

$\widetilde{N}_{16} = \langle 17550,17850,18150; 0.65,0.20,0.35\rangle$,

$\widetilde{N}_{17} = \langle 17850,18150,18450; 0.90,0.10,0.10\rangle$,

$\widetilde{N}_{18} = \langle 18150,18450,18750; 0.90,0.10,0.10\rangle$,

$\widetilde{N}_{19} = \langle 18450,18750,19050; 0.60,0.20,0.30\rangle$,

$\widetilde{N}_{20} = \langle 18750,19050,19350; 0.90,0.10,0.10\rangle$,

$\widetilde{N}_{21} = \langle 19050,19350,19350; 0.90,0.10,0.10\rangle$.

Step 4: Make a neutrosophication of the available time series data:

The first value of actual enrollments is 13,055 which is located only in the range of triangular neutrosophic number $\widetilde{N}_1$, then the neutrosophication value of 13,055 is $\widetilde{N}_1$ as in Table 1.

Also, the second value of actual enrollments (i.e., 13,563) locates in the range of triangular neutrosophic numbers $\widetilde{N}_1 = \langle 13050,13350,13650; 0.90,0.10,0.10\rangle$ and $\widetilde{N}_2 = \langle 13350,13650,13950; 0.80,0.20,0.10\rangle$.

Then, we must select the highest score degree of 13,563 as follows:

The membership, indeterminacy, and non-membership degrees of this value are calculated by using Equations (1)–(3) as follows:

$$T_{\widetilde{N}_1}(13563) = 0.261, \mathrm{I}_{\widetilde{N}_1}(13563) = 0.739, \mathrm{F}_{\widetilde{N}_1}(13563) = 0.739.$$

We must also calculate membership, indeterminacy, and non-membership degrees of 13,563 according to $\widetilde{N}_2 = \langle 13350,13650,13950; 0.80,0.20,0.10\rangle$ as follows:

$$T_{\widetilde{N}_2}(13563) = 0.568, \mathrm{I}_{\widetilde{N}_2}(13563) = 0.432, \mathrm{F}_{\widetilde{N}_2}(13563) = 0.361.$$

In this case, we must calculate the score degree of 13563 in both $\widetilde{\mathrm{N}}_1$ and $\widetilde{\mathrm{N}}_2$ and select the maximum value.

$$SC_{\widetilde{N}_1}(13563) = 2 + 0.262 - 0.739 - 0.739 = 0.783,$$
$$\text{and } SC_{\widetilde{N}_2}(13563) = 2 + 0.568 - 0.432 - 0.361 = 1.775.$$

Since the score degree of 13563 in $\widetilde{N}_2$ is greater than $\widetilde{N}_1$, then the neutrosophication value of 13563 is $\widetilde{N}_2$, as in Table 1.

We will apply the previous steps on the remaining data as follows:

The value 13867 locates in the range of $\widetilde{N}_2 = \langle 13350,13650,13950; 0.80,0.20,0.10\rangle$, and $\widetilde{N}_3 = \langle 13650,13950,14250; 0.90,0.20,0.10\rangle$.

Then

$$T_{\widetilde{N}_2}(13867) = 0.221, I_{\widetilde{N}_2}(13867) = 0.156, F_{\widetilde{N}_2}(13867) = 0.751.$$
$$T_{\widetilde{N}_3}(13867) = 0.651, I_{\widetilde{N}_3}(13867) = 0.421, F_{\widetilde{N}_3}(13867) = 0.349,$$
$$SC_{\widetilde{N}_2}(13867) = 2 + 0.221 - 0.156 - 0.751 = 1.314,$$
$$\text{and } SC_{\widetilde{N}_3}(13867) = 2 + 0.651 - 0.421 - 0.349 = 1.881.$$

So, the neutrosophication value of 13867 is $\widetilde{N}_3$.

Also, the value of 14,696 locates in the range of $\widetilde{N}_5 = \langle 14250,14550,14850; 0.75,0.10,0.30\rangle$, $\widetilde{N}_6 = \langle 14550,14850,15150; 0.90,0.10,0.10\rangle$, then

$$T_{\tilde{N}_5}(14696) = 0.385, I_{\tilde{N}_5}(14696) = 0.538, F_{\tilde{N}_5}(14696) = 0.641.$$

$$T_{\tilde{N}_6}(14696) = 0.438, I_{\tilde{N}_6}(14696) = 0.562, F_{\tilde{N}_6}(14696) = 0.562.$$

$$SC_{\tilde{N}_5}(14696) = 2 + 0.385 - 0.538 - 0.641 = 1.206,$$

$$\text{and } SC_{\tilde{N}_6}(14696) = 2 + 0.438 - 0.562 - 0.562 = 1.314.$$

So, the neutrosophication value of 14,696 is $\tilde{N}_6$.

The value 15,460 locates in the range of $\tilde{N}_8 = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$, and $\tilde{N}_9 = \langle 15450,15750,16050; 0.70,0.20,0.30\rangle$, then

$$T_{\tilde{N}_8}(15460) = 0.773, I_{\tilde{N}_8}(15460) = 0.226, F_{\tilde{N}_8}(15460) = 0.226.$$

$$T_{\tilde{N}_9}(15460) = 0.023, I_{\tilde{N}_9}(15460) = 0.973, F_{\tilde{N}_9}(15460) = 0.973.$$

$$SC_{\tilde{N}_8}(15460) = 2 + 0.773 - 0.226 - 0.226 = 2.321,$$

$$\text{and } SC_{\tilde{N}_9}(15460) = 2 + 0.023 - 0.973 - 0.976 = 0.074.$$

So, the neutrosophication value of 15,460 is $\tilde{N}_8$.

The value of 15,311 locates in the range of $\tilde{N}_7 = \langle 14850,15150,15450; 0.60,0.30,0.40\rangle$ and $\tilde{N}_8 = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$, then

$$T_{\tilde{N}_7}(15311) = 0.278, I_{\tilde{N}_7}(15311) = 0.675, F_{\tilde{N}_7}(15311) = 0.722.$$

$$SC_{\tilde{N}_7}(15311) = 2 + 0.278 - 0.675 - 0.722 = 0.881.$$

$$T_{\tilde{N}_8}(15311) = 0.429, I_{\tilde{N}_8}(15311) = 0.570, F_{\tilde{N}_8}(15311) = 0.570.$$

$$SC_{\tilde{N}_8}(15311) = 2 + 0.429 - 0.570 - 0.570 = 1.289.$$

So, the neutrosophication value of 15,311 is $\tilde{N}_8$.

The value of 15,603 locates in the range of $\tilde{N}_8 = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$ and $\tilde{N}_9 = \langle 15450,15750,16050; 0.70,0.20,0.30\rangle$, then

$$T_{\tilde{N}_8}(15603) = 0.392, I_{\tilde{N}_8}(15603) = 0.608, F_{\tilde{N}_8}(15603) = 0.608.$$

$$SC_{\tilde{N}_8}(15603) = 2 + 0.392 - 0.608 - 0.608 = 1.176.$$

$$T_{\tilde{N}_9}(15603) = 0.357, I_{\tilde{N}_9}(15603) = 0.592, F_{\tilde{N}_9}(15603) = 0.643.$$

$$SC_{\tilde{N}_9}(15603) = 2 + 0.357 - 0.592 - 0.643 = 1.122.$$

So, the neutrosophication value of 15,603 is $\tilde{N}_8$.

The value of 15,861 locates in the range of $\tilde{N}_9 = \langle 15450,15750,16050; 0.70,0.20,0.30\rangle$, and $\tilde{N}_{10} = \langle 15750,16050,16350; 0.90,0.10,0.30\rangle$, then

$$T_{\tilde{N}_9}(15861) = 0.441, I_{\tilde{N}_9}(15861) = 0.496, F_{\tilde{N}_9}(15861) = 0.559.$$

$$SC_{\tilde{N}_9}(15861) = 2 + 0.441 - 0.496 - 0.559 = 1.386.$$

$$T_{\tilde{N}_{10}}(15861) = 0.333, I_{\tilde{N}_{10}}(15861) = 0.667, F_{\tilde{N}_{10}}(15861) = 0.741.$$

$$SC_{\tilde{N}_{10}}(15861) = 2 + 0.333 - 0.667 - 0.741 = 0.925.$$

So, the neutrosophication value of 15861 is $\tilde{N}_9$.

The value of 16,807 locates in the range of $\tilde{N}_{12} = \langle 16350,16650,16950; 0.80,0.20,0.20\rangle$, $\tilde{N}_{13} = \langle 16650,16950,17250; 0.90,0.10,0.30\rangle$ then,

$$T_{\tilde{N}_{12}}(16807) = 0.381, I_{\tilde{N}_{12}}(16807) = 0.618, F_{\tilde{N}_{12}}(16807) = 0.618.$$

$$SC_{\tilde{N}_{12}}(16807) = 2 + 0.381 - 0.618 - 0.618 = 1.145.$$

$$T_{\tilde{N}_{13}}(16807) = 0.471, I_{\tilde{N}_{13}}(16807) = 0.529, F_{\tilde{N}_{13}}(16807) = 0.634.$$

$$SC_{\tilde{N}_{13}}(16807) = 2 + 0.471 - 0.529 - 0.634 = 1.308.$$

So, the neutrosophication value of 16807 is $\widetilde{N}_{13}$.

The value of 16919 locates in the range of $\widetilde{N}_{12} = \langle 16350,16650,16950; 0.80,0.20,0.20\rangle$, $\widetilde{N}_{13} = \langle 16650,16950,17250; 0.90,0.10,0.30\rangle$, then

$$T_{\widetilde{N}_{12}}(16919) = 0.063, I_{\widetilde{N}_{12}}(16919) = 0.917, F_{\widetilde{N}_{12}}(16919) = 0.917.$$

$$SC_{\widetilde{N}_{12}}(16919) = 2 + 0.063 - 0.917 - 0.917 = 0.229.$$

$$T_{\widetilde{N}_{13}}(16919) = 0.807, I_{\widetilde{N}_{13}}(16919) = 0.193, F_{\widetilde{N}_{13}}(16919) = 0.372.$$

$$SC_{\widetilde{N}_{13}}(16919) = 2 + 0.807 - 0.193 - 0.372 = 2.24.$$

So, the neutrosophication value of 16919 is $\widetilde{N}_{13}$.

The value of 16388 locates in the range of $\widetilde{N}_{11} = \langle 16050,16350,16650; 0.85,0.10,0.15\rangle$, $\widetilde{N}_{12} = \langle 16350,16650,16950; 0.80,0.20,0.20\rangle$, then

$$T_{\widetilde{N}_{11}}(16388) = 0.742, I_{\widetilde{N}_{11}}(16388) = 0.214, F_{\widetilde{N}_{11}}(16388) = 0.257.$$

$$SC_{\widetilde{N}_{11}}(16388) = 2 + 0.742 - 0.214 - 0.257 = 2.271.$$

$$T_{\widetilde{N}_{12}}(16388) = 0.101, I_{\widetilde{N}_{12}}(16388) = 0.898, F_{\widetilde{N}_{12}}(16388) = 0.898.$$

$$SC_{\widetilde{N}_{12}}(16388) = 2 + 0.101 - 0.898 - 0.898 = 0.305.$$

So, the neutrosophication value of 16388 is $\widetilde{N}_{11}$.

The value of 15433 locates in the range of $\widetilde{N}_{7} = \langle 14850,15150,15450; 0.60,0.30,0.40\rangle$, and $\widetilde{N}_{8} = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$, then

$$T_{\widetilde{N}_{7}}(15433) = 0.034, I_{\widetilde{N}_{7}}(15433) = 0.960, F_{\widetilde{N}_{7}}(15433) = 0.966.$$

$$SC_{\widetilde{N}_{7}}(15433) = 2 + 0.034 - 0.960 - 0.966 = 0.108.$$

$$T_{\widetilde{N}_{8}}(15433) = 0.754, I_{\widetilde{N}_{8}}(15433) = 0.245, F_{\widetilde{N}_{8}}(15433) = 0.245.$$

$$SC_{\widetilde{N}_{8}}(15433) = 2 + 0.754 - 0.245 - 0.245 = 2.264.$$

So, the neutrosophication value of 15433 is $\widetilde{N}_{8}$.

The value of 15497 locates in the range of $\widetilde{N}_{8} = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$ and $\widetilde{N}_{9} = \langle 15450,15750,16050; 0.70,0.20,0.30\rangle$ then,

$$T_{\widetilde{N}_{8}}(15497) = 0.674, I_{\widetilde{N}_{8}}(15497) = 0.325, F_{\widetilde{N}_{8}}(15497) = 0.325.$$

$$SC_{\widetilde{N}_{8}}(15497) = 2.024.$$

Also,

$$T_{\widetilde{N}_{9}}(15497) = 0.109, I_{\widetilde{N}_{9}}(15497) = 0.874, F_{\widetilde{N}_{9}}(15497) = 0.890.$$

$$SC_{\widetilde{N}_{9}}(15497) = 0.345.$$

So, the neutrosophication value of 15,433 is $\widetilde{N}_{8}$.

The value of 15,145 locates in the range of $\widetilde{N}_{6} = \langle 14550,14850,15150; 0.90,0.10,0.10\rangle$, and $\widetilde{N}_{7} = \langle 14850,15150,15450; 0.60,0.30,0.40\rangle$, then

$$T_{\widetilde{N}_{6}}(15145) = 0.015, I_{\widetilde{N}_{6}}(15145) = 0.985, F_{\widetilde{N}_{6}}(15145) = 0.985.$$

$$SC_{\widetilde{N}_{6}}(15145) = 0.045.$$

Also,

$$T_{\widetilde{N}_{7}}(15145) = 0.59, I_{\widetilde{N}_{7}}(15145) = 0.311, F_{\widetilde{N}_{7}}(15145) = 0.41.$$

$$SC_{\widetilde{N}_{7}}(15145) = 1.869.$$

So, the neutrosophication value of 15,145 is $\widetilde{N}_{7}$.

The value of 15,163 locates in the range of $\widetilde{N}_{7} = \langle 14850,15150,15450; 0.60,0.30,0.40\rangle$, $\widetilde{N}_{8} = \langle 15150,15450,15750; 0.80,0.20,0.20\rangle$, then

$$T_{\widetilde{N}_7}(15163) = 0.6,\ I_{\widetilde{N}_7}(15163) = 0.330,\ F_{\widetilde{N}_7}(15163) = 0.426.$$

$$SC_{\widetilde{N}_7}(15163) = 1.844.$$

Also

$$T_{\widetilde{N}_8}(15163) = 0.034, I_{\widetilde{N}_8}(15163) = 0.965, F_{\widetilde{N}_8}(15163) = 0.965.$$

$$SC_{\widetilde{N}_8}(15163) = 0.104.$$

So, the neutrosophication value of 15,163 is $\widetilde{N}_7$.

The value of 15,984 locates in the range of $\widetilde{N}_9 = \langle 15450,15750,16050; 0.70,0.20,0.30\rangle$, $\widetilde{N}_{10} = \langle 15750,16050,16350; 0.90,0.10,0.30\rangle$, then

$$T_{\widetilde{N}_9}(15984) = 0.154,\quad I_{\widetilde{N}_9}(15984) = 0.824,\quad F_{\widetilde{N}_9}(15984) = 0.846.$$

$$SC_{\widetilde{N}_9}(15984) = 0.484.$$

Also,

$$T_{\widetilde{N}_{10}}(15984) = 0.702,\ I_{\widetilde{N}_{10}}(15984) = 0.298,\ F_{\widetilde{N}_{10}}(15984) = 0.454,$$

$$SC_{\widetilde{N}_{10}}(15984) = 1.95.$$

So, the neutrosophication value of 15984 is $\widetilde{N}_{10}$.

The value of 16859 locates in the range of $\widetilde{N}_{12} = \langle 16350,16650,16950; 0.80,0.20,0.20\rangle$, $\widetilde{N}_{13} = \langle 16650,16950,17250; 0.90,0.10,0.30\rangle$, then

$$T_{\widetilde{N}_{12}}(16859) = 0.242,\ I_{\widetilde{N}_{12}}(16859) = 0.757,\quad F_{\widetilde{N}_{12}}(16859) = 0.757,$$

$$SC_{\widetilde{N}_{12}}(16859) = 0.728.$$

Also,

$$T_{\widetilde{N}_{13}}(16859) = 0.627, I_{\widetilde{N}_{13}}(16859) = 0.373,\quad F_{\widetilde{N}_{13}}(16859) = 0.512,$$

$$SC_{\widetilde{N}_{13}}(16859) = 1.442.$$

So, the neutrosophication value of 16859 is $\widetilde{N}_{13}$.

The value of 18150 locates in the range of $\widetilde{N}_{16} = \langle 17550,17850,18150; 0.65,0.20,0.35\rangle$, $\widetilde{N}_{17} = \langle 17850,18150,18450; 0.90,0.10,0.10\rangle$, then

$$T_{\widetilde{N}_{16}}(18150) = 0, I_{\widetilde{N}_{16}}(18150) = 1,\quad F_{\widetilde{N}_{16}}(18150) = 1,$$

$$SC_{\widetilde{N}_{16}}(18150) = 0.$$

Also,

$$T_{\widetilde{N}_{17}}(18150) = 0.90, I_{\widetilde{N}_{17}}(18150) = 0.1,\ F_{\widetilde{N}_{17}}(18150) = 0.1,$$

$$SC_{\widetilde{N}_{17}}(18150) = 2.7.$$

So, the neutrosophication value of 18150 is $\widetilde{N}_{17}$.

The value of 18970 locates in the range of $\widetilde{N}_{19} = \langle 18450,18750,19050; 0.60,0.20,0.30\rangle$, $\widetilde{N}_{20} = \langle 18750,19050,19350; 0.90,0.10,0.10\rangle$, then

$$T_{\widetilde{N}_{19}}(18970) = 0.16, I_{\widetilde{N}_{19}}(18970) = 0.786,\quad F_{\widetilde{N}_{19}}(18970) = 0.813.$$

$$SC_{\widetilde{N}_{19}}(18970) = 0.561.$$

Also,

$$T_{\widetilde{N}_{20}}(18970) = 0.66, I_{\widetilde{N}_{20}}(18970) = 0.34,\quad F_{\widetilde{N}_{20}}(18970) = 0.34.$$

$$SC_{\widetilde{N}_{20}}(18970) = 1.98.$$

So, the neutrosophication value of 18,970 is $\widetilde{N}_{20}$.

The value of 19,328 locates in the range of $\widetilde{N}_{20} = \langle 18750,19050,19350; 0.90,0.10,0.10\rangle$, $\widetilde{N}_{21} = \langle 19050,19350,19; 0.90,0.10,0.10\rangle$, then

$$T_{\widetilde{N}_{20}}(19328) = 0.066,\ I_{\widetilde{N}_{20}}(19328) = 0.992,\quad F_{\widetilde{N}_{20}}(19328) = 0.992.$$

$$SC_{\widetilde{N}_{20}}(19328) = 0.082.$$

Also,

$$T_{\widetilde{N}_{21}}(19328) = 0.834, I_{\widetilde{N}_{21}}(19328) = 0.166,\quad F_{\widetilde{N}_{21}}(19328) = 0.166.$$

$$SC_{\widetilde{N}_{21}}(19328) = 2.502.$$

So, the neutrosophication value of 19,328 is $\widetilde{N}_{21}$.

The value of 19,337 locates in the range of $\widetilde{N}_{20} = \langle 18750,19050,19350; 0.90,0.10,0.10\rangle$,

$\widetilde{N}_{21} = \langle 19050,19350,19; 0.90,0.10,0.10\rangle$, then

$$T_{\widetilde{N}_{20}}(19337) = 0.039, I_{\widetilde{N}_{20}}(19337) = 0.961,\quad F_{\widetilde{N}_{20}}(19337) = 0.961.$$

$$SC_{\widetilde{N}_{20}}(19337) = 0.117.$$

Also,

$$T_{\widetilde{N}_{21}}(19337) = 0.861, I_{\widetilde{N}_{21}}(19337) = 0.139,\quad F_{\widetilde{N}_{21}}(19337) = 0.139.$$

$$SC_{\widetilde{N}_{21}}(19337) = 2.583.$$

So, the neutrosophication value of 19,337 is $\widetilde{N}_{21}$.

Finally, the value of 18,876 locates in the range of $\widetilde{N}_{19} = \langle 18450,18750,19050; 0.60,0.20,0.30\rangle$, $\widetilde{N}_{20} = \langle 18750,19050,19350; 0.90,0.10,0.10\rangle$, then

$$T_{\widetilde{N}_{19}}(18876) = 0.348, I_{\widetilde{N}_{19}}(18876) = 0.536,\quad F_{\widetilde{N}_{19}}(18876) = 0.594.$$

$$SC_{\widetilde{N}_{19}}(18876) = 1.218.$$

Also,

$$T_{\widetilde{N}_{20}}(18876) = 0.378, I_{\widetilde{N}_{20}}(18876) = 0.622,\quad F_{\widetilde{N}_{20}}(18876) = 0.622.$$

$$SC_{\widetilde{N}_{20}}(18876) = 1.134.$$

So, the neutrosophication value of 18,876 is $\widetilde{N}_{19}$.

Table 1. Actual and neutrosophication values of student enrollments.

Years	Actual Enrollments	Neutrosophication Values of Enrollments $\widetilde{N}$
1971	13,055	$\widetilde{N}_1$
1972	13,563	$\widetilde{N}_2$
1973	13,867	$\widetilde{N}_3$
1974	14,696	$\widetilde{N}_6$
1975	15,460	$\widetilde{N}_8$
1976	15,311	$\widetilde{N}_8$
1977	15,603	$\widetilde{N}_8$
1978	15,861	$\widetilde{N}_9$
1979	16,807	$\widetilde{N}_{13}$
1980	16,919	$\widetilde{N}_{13}$
1981	16,388	$\widetilde{N}_{11}$
1982	15,433	$\widetilde{N}_8$
1983	15,497	$\widetilde{N}_8$
1984	15,145	$\widetilde{N}_7$
1985	15,163	$\widetilde{N}_7$
1986	15,984	$\widetilde{N}_{10}$

1987	16,859	$\widetilde{N}_{13}$
1988	18,150	$\widetilde{N}_{17}$
1989	18,970	$\widetilde{N}_{20}$
1990	19,328	$\widetilde{N}_{21}$
1991	19,337	$\widetilde{N}_{21}$
1992	18,876	$\widetilde{N}_{19}$

Step 5: Construct the neutrosophic logical relationships (NLRs) as in Table 2:

Table 2. Neutrosophic logical relationships.

$\widetilde{N}_1 \rightarrow \widetilde{N}_2$	$\widetilde{N}_2 \rightarrow \widetilde{N}_3$	$\widetilde{N}_3 \rightarrow \widetilde{N}_6$	$\widetilde{N}_6 \rightarrow \widetilde{N}_8$	$\widetilde{N}_8 \rightarrow \widetilde{N}_8$
$\widetilde{N}_8 \rightarrow \widetilde{N}_9$	$\widetilde{N}_9 \rightarrow \widetilde{N}_{13}$	$\widetilde{N}_{13} \rightarrow \widetilde{N}_{13}$	$\widetilde{N}_{13} \rightarrow \widetilde{N}_{11}$	$\widetilde{N}_{11} \rightarrow \widetilde{N}_8$
$\widetilde{N}_8 \rightarrow \widetilde{N}_7$	$\widetilde{N}_7 \rightarrow \widetilde{N}_7$	$\widetilde{N}_7 \rightarrow \widetilde{N}_{10}$	$\widetilde{N}_{10} \rightarrow \widetilde{N}_{13}$	$\widetilde{N}_{13} \rightarrow \widetilde{N}_{17}$
$\widetilde{N}_{17} \rightarrow \widetilde{N}_{20}$	$\widetilde{N}_{20} \rightarrow \widetilde{N}_{21}$	$\widetilde{N}_{21} \rightarrow \widetilde{N}_{21}$	$\widetilde{N}_{21} \rightarrow \widetilde{N}_{19}$	

Step 6: Based on NLR, begin to establish the neutrosophic logical relationship groups (NLRGs) as in Table 3.

Table 3. Neutrosophic logical relationship groups (NLRGs) of enrollments.

$\widetilde{N}_1 \rightarrow \widetilde{N}_2$		
$\widetilde{N}_2 \rightarrow \widetilde{N}_3$		
$\widetilde{N}_3 \rightarrow \widetilde{N}_6$		
$\widetilde{N}_6 \rightarrow \widetilde{N}_8$		
$\widetilde{N}_7 \rightarrow \widetilde{N}_7$	$\widetilde{N}_7 \rightarrow \widetilde{N}_{10}$	
$\widetilde{N}_8 \rightarrow \widetilde{N}_7$	$\widetilde{N}_8 \rightarrow \widetilde{N}_8$	$\widetilde{N}_8 \rightarrow \widetilde{N}_9$
$\widetilde{N}_9 \rightarrow \widetilde{N}_{13}$		
$\widetilde{N}_{10} \rightarrow \widetilde{N}_{13}$		
$\widetilde{N}_{11} \rightarrow \widetilde{N}_8$		
$\widetilde{N}_{13} \rightarrow \widetilde{N}_{11}$	$\widetilde{N}_{13} \rightarrow \widetilde{N}_{13}$	$\widetilde{N}_{13} \rightarrow \widetilde{N}_{17}$
$\widetilde{N}_{17} \rightarrow \widetilde{N}_{20}$		
$\widetilde{N}_{20} \rightarrow \widetilde{N}_{21}$		
$\widetilde{N}_{21} \rightarrow \widetilde{N}_{19}$	$\widetilde{N}_{21} \rightarrow \widetilde{N}_{21}$	

Step 7: Calculate the forecasted values as in Table 4:

To calculate the forecasted value of 13,055 in year 1971, do the following:

- Look at the neutrosophication value of 13055 in year 1971 which is $\widetilde{N}_1$ as it appears in Table 1.
- Go to NLRG which is presented in Table 3, and because $\widetilde{N}_1$is the first neutrosophication value of data, then it is not caused by any other value (i.e., $\neq\rightarrow \widetilde{N}_1$) as in Table 3.

Therefore, the forecasted value of 13,055 is—Which means leaving it empty, as we illustrated in Step 7, Rule 1 of the proposed algorithm.

Also, to calculate the forecasted value of 13,563 in year 1972, do the following:

- Look at the neutrosophication value of 13,563 in year 1972 which is $\widetilde{N}_2$ as it appears in Table 1, and because $\widetilde{N}_2$ is caused by $\widetilde{N}_1$ (i.e., $\widetilde{N}_1 \rightarrow \widetilde{N}_2$), then
- Go to Table 3, and look at the NLRG which starts with $\widetilde{N}_1$, and we noted that it is $\widetilde{N}_1 \rightarrow \widetilde{N}_2$. Then the forecasted value of 13,563 is the middle value of $\widetilde{N}_2$.

Another illustrating example for calculating the forecasted value of 18,876 in year 1992:

- Look at the neutrosophication value of 18,876 in year 1992 which is $\widetilde{N}_{19}$ as it appears in Table 1. Since $\widetilde{N}_{19}$ is caused by $\widetilde{N}_{21}$, then
- Go to Table 3, and look at the NLRG which starts with $\widetilde{N}_{21}$ (i.e., $\widetilde{N}_{21} \rightarrow \widetilde{N}_{19}, \widetilde{N}_{21} \rightarrow \widetilde{N}_{21}$). Then the forecasted value of 18876 is the average of the middle values of $\widetilde{N}_{19}$, $\widetilde{N}_{21}$, and it will equal 19,050.

The other forecasted values are calculated in the same manner.

Table 4. Actual and forecasted values of enrollments.

Years	Actual Enrollments	Forecasted Values of Enrollments
1971	13,055	–
1972	13,563	13,650
1973	13,867	13,950
1974	14,696	14,850
1975	15,460	15,450
1976	15,311	15,450
1977	15,603	15,450
1978	15,861	15,450
1979	16,807	16,950
1980	16,919	17,150
1981	16,388	17,150
1982	15,433	15,450
1983	15,497	15,450
1984	15,145	15,450
1985	15,163	15,600
1986	15,984	15,600
1987	16,859	16,950
1988	18,150	17,150
1989	18,970	19,050
1990	19,328	19,350
1991	19,337	19,050
1992	18,876	19,050

The actual and forecasted values of enrollments appear in Figure 1.

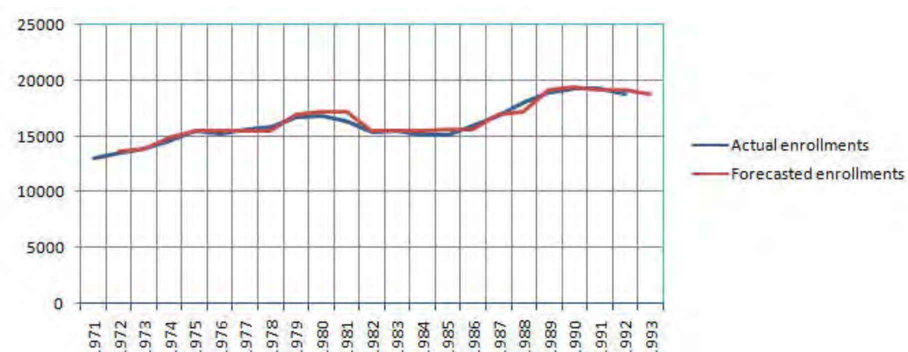

Figure 1. Forecasted and actual enrollments.

The forecasted enrollment data obtained with the suggested method, along with the forecasted data obtained with the models in [14,17,43–46], are presented in Table 5.

Table 5. Forecasted values by suggested method and other methods.

Years	Actual Values	Forecasted Values						
		Proposed	[43]	[44]	[45]	[46]	[14]	[17]
1971	13,055	–	–	–	–	–	–	–
1972	13,563	13,650	14,242.0	14,025	13,250	14,031.35	14,586	13,693
1973	13,867	13,950	14,242.0	14,568	13,750	14,795.36	14,586	13,693
1974	14,696	14,850	14,242.0	14,568	13,750	14,795.36	15,363	14,867
1975	15,460	15,450	15,774.3	15,654	14,500	14,795.36	15,363	15,287
1976	15,311	15,450	15,774.3	15,654	15,375	16,406.57	15,442	15,376
1977	15,603	15,450	15,774.3	15,654	15,375	16,406.57	15,442	15,376
1978	15,861	15,450	15,774.3	15,654	15,625	16,406.57	15,442	15,376
1979	16,807	16,950	16,146.5	16,197	15,875	16,406.57	15,442	16,523
1980	16,919	17,150	16,988.3	17,283	16,833	17,315.29	17,064	16,606
1981	16,388	17,150	16,988.3	17,283	16,833	17,315.29	17,064	17,519
1982	15,433	15,450	16,146.5	16,197	16,500	17,315.29	15,438	16,606
1983	15,497	15,450	15,474.3	15,654	15,500	16,406.57	15,442	15,376
1984	15,145	15,450	15,474.3	15,654	15,500	16,406.57	15,442	15,376
1985	15,163	15,600	15,474.3	15,654	15,125	16,406.57	15,363	15,287
1986	15,984	15,600	15,474.3	15,654	15,125	16,406.57	15,363	15,287
1987	16,859	16,950	16,146.5	15,654	16,833	16,406.57	15,438	16,523
1988	18,150	17,150	16,988.3	16,197	16,667	17,315.29	17,064	17,519
1989	18,970	19,050	19,144.0	17,283	18,125	19,132.79	19,356	19,500
1990	19,328	19,350	19,144.0	18,369	18,750	19,132.79	19,356	19,000
1991	19,337	19,050	19,144.0	19,454	19,500	19,132.79	19,356	19,500
1992	18,876	19,050	19,144.0	19,454	19,500	19,132.79	19,356	19,500

By comparing the proposed method with other existing methods in Table 5, the RMSE and AFE tools confirm that the suggested method is better than others, as shown in Table 6.

Table 6. Error measures.

Tool	Proposed	[43]	[44]	[45]	[46]	[14]	[17]
RMSE	342.68	478.45	781.47	646.67	805.17	642.68	493.56
AFE (%)	1.44	2.39	3.61	2.98	4.28	2.96	2.33

We combined forecasted values with respect to all methods in Figure 2.

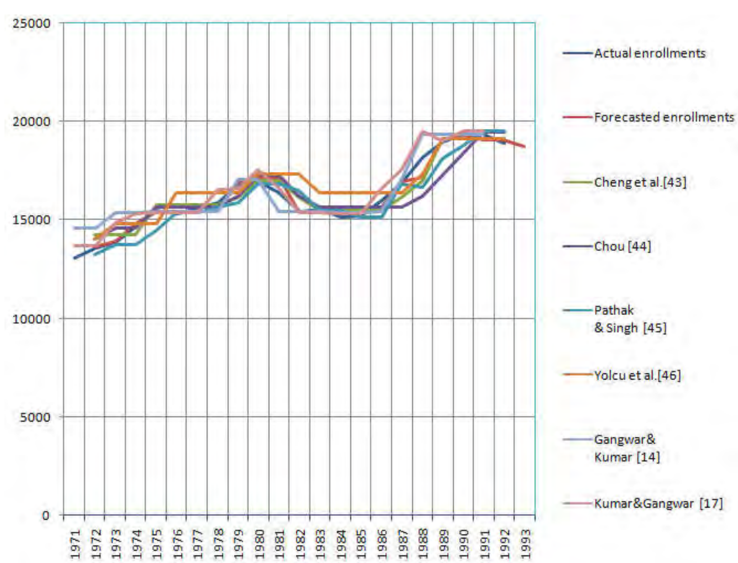

Figure 2. Comparison figures between all forecasted values.

If we plan to find the second-order neutrosophic logical relationships of the previous example by applying the proposed method of forecasting based on the second-order NTS, they are as shown in Table 7.

Table 7. Second-order NLR.

$\widetilde{N}_1, \widetilde{N}_2 \to \widetilde{N}_3$
$\widetilde{N}_2, \widetilde{N}_3 \to \widetilde{N}_6$
$\widetilde{N}_3, \widetilde{N}_6 \to \widetilde{N}_8$
$\widetilde{N}_6, \widetilde{N}_8 \to \widetilde{N}_8$
$\widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_8$
$\widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_9$
$\widetilde{N}_8, \widetilde{N}_9 \to \widetilde{N}_{13}$
$\widetilde{N}_9, \widetilde{N}_{13} \to \widetilde{N}_{13}$
$\widetilde{N}_{13}, \widetilde{N}_{13} \to \widetilde{N}_{11}$ $\widetilde{N}_{13}, \widetilde{N}_{11} \to \widetilde{N}_8$
$\widetilde{N}_{11}, \widetilde{N}_8 \to \widetilde{N}_8$
$\widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_7$
$\widetilde{N}_8, \widetilde{N}_7 \to \widetilde{N}_7$
$\widetilde{N}_7, \widetilde{N}_7 \to \widetilde{N}_{10}$
$\widetilde{N}_7, \widetilde{N}_{10} \to \widetilde{N}_{13}$
$\widetilde{N}_{10}, \widetilde{N}_{13} \to \widetilde{N}_{17}$
$\widetilde{N}_{13}, \widetilde{N}_{17} \to \widetilde{N}_{20}$
$\widetilde{N}_{17}, \widetilde{N}_{20} \to \widetilde{N}_{21}$
$\widetilde{N}_{20}, \widetilde{N}_{21} \to \widetilde{N}_{21}$
$\widetilde{N}_{21}, \widetilde{N}_{21} \to \widetilde{N}_{19}$

The second-order neutrosophic logical relationship groups of the previous example are as shown in Table 8.

Table 8. Second-order NLRGs.

$\widetilde{N}_1, \widetilde{N}_2 \to \widetilde{N}_3$		
$\widetilde{N}_2, \widetilde{N}_3 \to \widetilde{N}_6$		
$\widetilde{N}_3, \widetilde{N}_6 \to \widetilde{N}_8$		
$\widetilde{N}_6, \widetilde{N}_8 \to \widetilde{N}_8$		
$\widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_8$	$\widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_9$	$\widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_7$
$\widetilde{N}_8, \widetilde{N}_9 \to \widetilde{N}_{13}$		
$\widetilde{N}_9, \widetilde{N}_{13} \to \widetilde{N}_{13}$		
$\widetilde{N}_{13}, \widetilde{N}_{13} \to \widetilde{N}_{11}$		
$\widetilde{N}_{13}, \widetilde{N}_{11} \to \widetilde{N}_8$		
$\widetilde{N}_{11}, \widetilde{N}_8 \to \widetilde{N}_8$		
$\widetilde{N}_8, \widetilde{N}_7 \to \widetilde{N}_7$		
$\widetilde{N}_7, \widetilde{N}_7 \to \widetilde{N}_{10}$		
$\widetilde{N}_7, \widetilde{N}_{10} \to \widetilde{N}_{13}$		
$\widetilde{N}_{10}, \widetilde{N}_{13} \to \widetilde{N}_{17}$		
$\widetilde{N}_{13}, \widetilde{N}_{17} \to \widetilde{N}_{20}$		
$\widetilde{N}_{17}, \widetilde{N}_{20} \to \widetilde{N}_{21}$		
$\widetilde{N}_{20}, \widetilde{N}_{21} \to \widetilde{N}_{21}$		
$\widetilde{N}_{21}, \widetilde{N}_{21} \to \widetilde{N}_{19}$		

We compared forecasted values of enrollments based on the second order of neutrosophic logical relationship groups of the proposed method with the method of second order presented by Gautam and Singh [47]. The results are shown in Table 9.

Table 9. Actual and forecasted values of enrollments based on the second order of the proposed method vs. the Gautam and Singh [47] method.

Years	Actual Enrollments	Second-Order Forecasted Values of the Proposed Method	Forecasted Values in [47]
1971	13,055	–	–
1972	13,563	–	–
1973	13,867	13,950	13,800
1974	14,696	14,850	14,400
1975	15,460	15,450	15,300
1976	15,311	15,450	15,300
1977	15,603	15,450	15,600
1978	15,861	15,450	15,600
1979	16,807	16,950	16,800
1980	16,919	16,950	16,800
1981	16,388	16,350	16,200
1982	15,433	15,450	15,300
1983	15,497	15,450	15,300
1984	15,145	15,450	15,000
1985	15,163	15,150	15,000
1986	15,984	16,050	15,900
1987	16,859	16,950	16,800
1988	18,150	18,150	18,000
1989	18,970	19,050	18,900
1990	19,328	19,350	19,200
1991	19,337	19,350	19,200
1992	18,876	18,750	18,600

The MSE and AFE of the two methods are presented in Table 10.

Table 10. Error measures of the proposed method and the Gautam and Singh method [47].

Tool	Proposed	[47]
MSE	19,823.4	24,443.4
AFE (%)	0.60	0.81

From Table 10, it appears that our proposed method of second order is also better than the proposed method of second order presented by Gautam and Singh [47].

In addition, the third-order neutrosophic logical relationship groups of the previous example are constructed and shown in Table 11.

Table 11. Third-order NLRGs.

$\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3 \to \widetilde{N}_6$
$\widetilde{N}_2, \widetilde{N}_3, \widetilde{N}_6 \to \widetilde{N}_8$
$\widetilde{N}_3, \widetilde{N}_6, \widetilde{N}_8 \to \widetilde{N}_8$
$\widetilde{N}_6, \widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_8$
$\widetilde{N}_8, \widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_9$
$\widetilde{N}_8, \widetilde{N}_8, \widetilde{N}_9 \to \widetilde{N}_{13}$
$\widetilde{N}_8, \widetilde{N}_9, \widetilde{N}_{13} \to \widetilde{N}_{13}$
$\widetilde{N}_9, \widetilde{N}_{13}, \widetilde{N}_{13} \to \widetilde{N}_{11}$
$\widetilde{N}_{13}, \widetilde{N}_{13}, \widetilde{N}_{11} \to \widetilde{N}_8$
$\widetilde{N}_{13}, \widetilde{N}_{11}, \widetilde{N}_8 \to \widetilde{N}_8$
$\widetilde{N}_{11}, \widetilde{N}_8, \widetilde{N}_8 \to \widetilde{N}_7$
$\widetilde{N}_8, \widetilde{N}_8, \widetilde{N}_7 \to \widetilde{N}_7$
$\widetilde{N}_8, \widetilde{N}_7, \widetilde{N}_7 \to \widetilde{N}_{10}$
$\widetilde{N}_7, \widetilde{N}_7, \widetilde{N}_{10} \to \widetilde{N}_{13}$
$\widetilde{N}_7, \widetilde{N}_{10}, \widetilde{N}_{13} \to \widetilde{N}_{17}$
$\widetilde{N}_{10}, \widetilde{N}_{13}, \widetilde{N}_{17} \to \widetilde{N}_{20}$
$\widetilde{N}_{13}, \widetilde{N}_{17}, \widetilde{N}_{20} \to \widetilde{N}_{21}$
$\widetilde{N}_{17}, \widetilde{N}_{20}, \widetilde{N}_{21} \to \widetilde{N}_{21}$
$\widetilde{N}_{20}, \widetilde{N}_{21}, \widetilde{N}_{21} \to \widetilde{N}_{19}$

We also compared the forecasted values of enrollments based on the third order of neutrosophic logical relationship groups of the proposed method with the proposed methods of third order presented by [8,9,47], and the results are shown in Table 12.

Table 12. Actual and forecasted values of enrollments based on the third order of the proposed method vs. the methods presented by [8,9,47].

Years	Actual Enrollments	Third-Order Forecasted Values of the Proposed Method	Forecasted Values in [47]	Forecasted Values in [8]	Forecasted Values in [9]
1971	13,055	–	–	–	–
1972	13,563	–	–	–	–
1973	13,867	–	–	–	–
1974	14,696	14,850	14,400	14,500	14,750
1975	15,460	15,450	15,300	15,500	15,750
1976	15,311	15,450	15,300	15,500	15,500
1977	15,603	15,450	15,600	15,500	15,500
1978	15,861	15,750	15,600	15,500	15,500
1979	16,807	16,950	16,800	16,500	16,500
1980	16,919	16,950	16,800	16,500	16,500
1981	16,388	16,350	16,200	16,500	16,500
1982	15,433	15,450	15,300	15,500	15,500
1983	15,497	15,450	15,300	15,500	15,500
1984	15,145	15,150	15,000	15,500	15,250
1985	15,163	15,150	15,000	15,500	15,500
1986	15,984	16,050	15,900	15,500	15,500
1987	16,859	16,950	16,800	16,500	16,500
1988	18,150	18,150	18,000	18,500	18,500
1989	18,970	19,050	18,900	18,500	18,500
1990	19,328	19,350	19,200	19,500	19,500
1991	19,337	19,350	19,200	19,500	19,500
1992	18,876	18,750	18,600	18,500	18,750

The MSE and AFE of the methods are presented in Table 13.

Table 13. Error measures of the proposed method and the [8,9,47] methods.

Tool	Proposed	[47]	[8]	[9]
MSE	7367.316	25,493.6	86,694	76,509
AFE (%)	0.40	0.82	1.52	1.40

4.2. Numerical example 2

We verified the proposed method by solving the TAIEX2004 example [40], and by putting D_1 and D_2 equal 56 and 61, respectively, then U = $[5600.17, 6200.69]$. Also we calculated the suitable length as we illustrated previously and found that it is equal to 40. Therefore, the number of triangular neutrosophic numbers is equal to 12. For these neutrosophic numbers, the decision makers determined the truth, indeterminacy, and falsity degrees equal to 0.9,0.1,0.1, respectively. The actual and forecasted values of the TAIEX2004 example are presented in Table 14 and Figure 3.

Table 14. Actual and forecasted values of TAIEX2004.

Dates	Actual Values	Forecasted Values of the Proposed Method
01/11/2004	5656.17	–
02/11/2004	5759.61	5760.17
03/11/2004	5862.85	5813.5
04/11/2004	5860.73	5900.17
05/11/2004	5931.31	5900.17
08/11/2004	5937.46	5903.02
09/11/2004	5945.2	5903.02
10/11/2004	5948.49	5940.17

11/11/2004	5874.52	5940.17
12/11/2004	5917.16	5903.02
15/11/2004	5906.69	5903.02
16/11/2004	5910.85	5903.02
17/11/2004	6028.68	5940.17
18/11/2004	6049.49	5940.17
19/11/2004	6026.55	5940.17
22/11/2004	5838.42	5830.17
23/11/2004	5851.1	5830.17
24/11/2004	5911.31	5903.02
25/11/2004	5855.24	5830.17
26/11/2004	5778.65	5813.5
29/11/2004	5785.26	5813.5
30/11/2004	5844.76	5860.17
1/12/2004	5798.62	5830.17
02/12/2004	5867.95	5860.17
03/12/2004	5893.27	5900.17
06/12/2004	5919.17	5900.17
07/12/2004	5925.28	5903.02
08/12/2004	5892.51	5903.02
09/12/2004	5913.97	5900.17
10/12/2004	5911.63	5903.02
13/12/2004	5878.89	5903.02
14/12/2004	5909.65	5900.17
15/12/2004	6002.58	5903.02
16/12/2004	6019.23	6040.17
17/12/2004	6009.32	6040.17
20/12/2004	5985.94	6040.17
21/12/2004	5987.85	6040.17
22/12/2004	6001.52	6040.17
23/12/2004	5997.67	6040.17
24/12/2004	6019.42	6040.17
27/12/2004	5985.94	6040.17
28/12/2004	6000.57	6040.17
29/12/2004	6088.49	6040.17
30/12/2004	6100.86	6080.17
31/12/2004	6139.69	6080.17

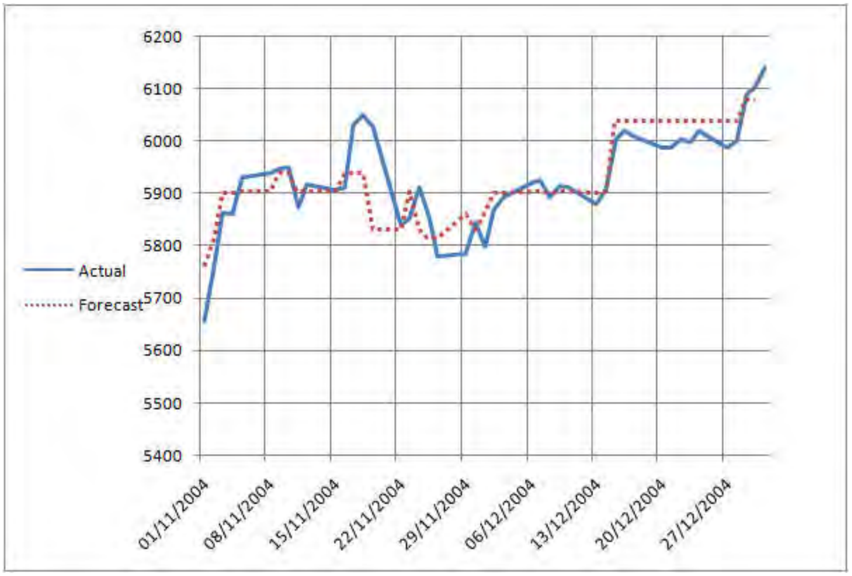

Figure 3. Actual and forecasted values of TAIEX2004.

The RMSE and AFE of the proposed method are presented in Table 15.

Table 15. Error measures of the proposed method.

Tool	Proposed
RMSE	42.05
AFE (%)	0.005

To confirm the performance of the suggested method, we compared it with other existing methods and the results are shown in Table 16 and Figure 4.

Table 16. Error measures of the proposed method and other existing methods which solved the TAIEX2004 example.

Methods	RMSE
Guan et al.'s method [40]	53.01
Huarng et al.'s method [48]	73.57
Chen and Kao's method [49]	58.17
Cheng et al.'s method [50]	54.24
Chen et al.'s method [51]	56.16
Chen and Chang's method [52]	60.48
Chen and Chen's method [53]	61.94
Yu and Huarng's method [54]	55.91
Proposed method	42.05

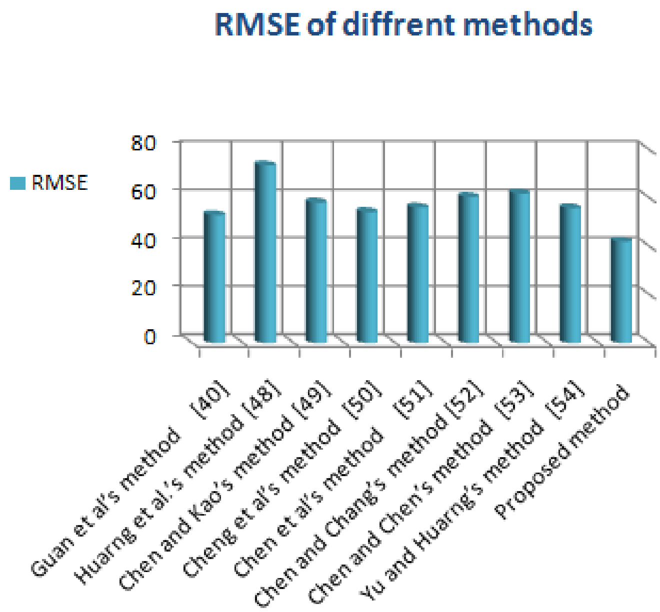

Figure 4. The RMSE of different methods that solved the TAIEX2004 example.

TAIEX2004 is used as a baseline to compare our method with other competitive methods, to compare and identify how all the methods can manage error reduction. The RMSE is a common approach used in financial analysis [55]. Compared with the existing methods as shown in Table 16, our proposed method can offer the least presence of errors since it has the most minimized RMSE. In other words, our method appears to be performing the best in reducing errors and ensuring all our analyses are accurate with insights. This may provide a new insight for business intelligence with artificial intelligence, cloud computing, and neutrosophic research.

5. Conclusion and future directions

The objective of this research was to enhance the accuracy rates of forecasting, since the forecasting accuracy rates in the existing approaches of fuzzy and intuitionistic fuzzy time series were not accurate enough. Thus, in this research we introduced the notion of first-and high-order neutrosophic time series data by defining the fitting length of intervals and proposing a novel method for calculating forecasted values. In order to obtain truth, indeterminacy, and falsity membership degrees of historical data, we defined triangular neutrosophic numbers. The neutrosophication process of historical time series data depends on the biggest score function of the triangular neutrosophic numbers. For the deneutrosophication process of first- and high-order NTS, we used simple arithmetic computations. The suggested approach of first- and high-order neutrosophic time series proved its superiority against other existing methods in the field of fuzzy, intuitionistic fuzzy, and neutrosophic time series. In the future, we plan to apply meta-heuristic optimization techniques for improving the accuracy of the suggested method. We will apply this model for predicting other time series, such as demand forecasting, electricity consumption, etc. Furthermore**,** we may consider using other approaches for comparing similarities of historical data, like information entropy.

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Extended Nonstandard Neutrosophic Logic, Set, and Probability Based on Extended Nonstandard Analysis

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Abstract: We extend for the second time the nonstandard analysis by adding the left monad closed to the right, and right monad closed to the left, while besides the pierced binad (we introduced in 1998) we add now the unpierced binad—all these in order to close the newly extended nonstandard space under nonstandard addition, nonstandard subtraction, nonstandard multiplication, nonstandard division, and nonstandard power operations. Then, we extend the Nonstandard Neutrosophic Logic, Nonstandard Neutrosophic Set, and Nonstandard Probability on this Extended Nonstandard Analysis space, and we prove that it is a nonstandard neutrosophic lattice of first type (endowed with a nonstandard neutrosophic partial order) as well as a nonstandard neutrosophic lattice of second type (as algebraic structure, endowed with two binary neutrosophic laws: inf_N and sup_N). Many theorems, new terms introduced, better notations for monads and binads, and examples of nonstandard neutrosophic operations are given.

Keywords: nonstandard analysis; extended nonstandard analysis; open and closed monads to the left/right; pierced and unpierced binads; MoBiNad set; infinitesimals; infinities; nonstandard reals; standard reals; nonstandard neutrosophic lattices of first type (as poset) and second type (as algebraic structure)

1. Short Introduction

In order to more accurately situate and fit the neutrosophic logic into the framework of extended nonstandard analysis [1–3], we present the nonstandard neutrosophic inequalities, nonstandard neutrosophic equality, nonstandard neutrosophic infimum and supremum, and nonstandard neutrosophic intervals, including the cases when the neutrosophic logic standard and nonstandard components T, I, F get values outside of the classical unit interval [0, 1], and a brief evolution of neutrosophic operators [4].

2. Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason we have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between Relative Truth (which is truth in some Worlds, according to Leibniz) and Absolute Truth (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than 0.8 (or 0.8^{+}), or infinitesimally smaller than 0.8 (or $^{-}0.8$). But these can easily be overcome by roughly using interval neutrosophic values, for example (0.80, 0.81) and (0.79, 0.80), respectively.

3. Why the Sum of Neutrosophic Components Is Up to 3

We was more prudent when we presented the sum of single valued standard neutrosophic components [5–9], saying

$$\text{Let } T, I, F \text{ be single valued numbers, } T, I, F \in [0, 1], \text{ such that } 0 \le T + I + F \le 3. \quad (1)$$

The sum of the single-valued neutrosophic components, $T + I + F$ is up to 3 since they are considered completely (100%) independent of each other. But if the two components T and F are completely (100%) dependent, then $T + F \le 1$ (as in fuzzy and intuitionistic fuzzy logics), and let us assume the neutrosophic middle component I is completely (100%) independent from T and F, then $I \le 1$, whence $T + F + I \le 1 + 1 = 2$.

But the degree of dependence/independence [10] between T, I, F all together, or taken two by two, may be, in general, any number between 0 and 1.

4. Neutrosophic Components outside the Unit Interval [0, 1]

Thinking out of box, inspired from the real world, was the first intent, i.e., allowing neutrosophic components (truth/indeterminacy/falsehood) values be outside of the classical (standard) unit real interval [0, 1] used in all previous (Boolean, multivalued, etc.) logics if needed in applications, so neutrosophic component values < 0 and > 1 had to occurs due to the Relative/Absolute stuff, with

$$^{-}0 <_N 0 \text{ and } 1^{+} >_N 1 \quad (2)$$

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic, Set, and Probability), and it was thus possible the extension of the neutrosophic set to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e., some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics, etc., [11–14], extending the unit interval [0, 1] to

$$[\Psi, \Omega], \text{ with } \Psi \le 0 < 1 \le \Omega, \quad (3)$$

where Ψ, Ω are standard (or nonstandard) real numbers.

5. Refined Neutrosophic Logic, Set, and Probability

We wanted to get the neutrosophic logic as general as possible [15], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, and dialethism), and to have it able to deal with all kind of logical propositions (including paradoxes, nonsensical propositions, etc.).

That is why in 2013 we extended the Neutrosophic Logic to Refined Neutrosophic Logic / Set / Probability (from generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz's and Bochvar's 3-symbol valued logics or Belnap's 4-symbol valued logic, to the most general n-symbol or n-numerical valued refined neutrosophic logic, for any integer $n \ge 1$), the largest ever so far, when some or all neutrosophic components T, I, F were split/refined into neutrosophic subcomponents $T_1, T_2, \ldots ; I_1, I_2, \ldots ; F_1, F_2, \ldots$, which were deduced from our everyday life [16].

6. From Paradoxism Movement to Neutrosophy Branch of Philosophy and then to Neutrosophic Logic

We started first from Paradoxism (that we founded in the 1980s in Romania as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then we introduced the Neutrosophy (as generalization of Dialectics of Hegel and Marx, which is actually the ancient YinYang Chinese philosophy), neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form

$$\langle A\rangle, \text{ its opposite } \langle antiA\rangle, \text{ and their neutrals } \langle neutA\rangle, \tag{4}$$

where <*A*> is any item or entity [17]. (Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as *1*, while the Absolute Truth neutrosophic value was marked as 1^+ (a tinny bigger than the Relative Truth's value): $1^+ >_N 1$, where $>_N$ is a neutrosophic inequality, meaning 1^+ is neutrosophically bigger than *1*.

Similarly for Relative Falsehood/Indeterminacy (which is falsehood/indeterminacy in some Worlds) and Absolute Falsehood/Indeterminacy (which is falsehood/indeterminacy in all possible worlds).

7. Introduction to Nonstandard Analysis

An *infinitesimal* (or infinitesimal number) (ε) is a number ε, such that $|\varepsilon| < 1/n$, for any non-null positive integer n. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus.

An *infinite* (or infinite number) (ω) is a number greater than anything:

$$1 + 1 + 1 + \ldots + 1 \text{ (for any finite number terms)} \tag{5}$$

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (or *nonstandard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*:

$$1/\varepsilon = \omega/1. \tag{6}$$

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

8. First Extension of Nonstandard Analysis

Let us denote by R_+^* the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and the (*pierced*) *binad* that we have introduced as extension in 1998 [5]:

Left Monad {that we denote, for simplicity, by (^-a) or only ^-a} is defined as:

$$\mu(^-a) = (^-a) = {}^-a = \overset{-}{a} = \{a - x,\ x \in R_+{}^* \mid x \text{ is infinitesimal}\}. \tag{7}$$

Right Monad {that we denote, for simplicity, by (a^+) or only by a^+} is defined as:

$$\mu(a^+) = (a^+) = a^+ = \overset{+}{a} = \{a + x,\ x \in R_+{}^* \mid x \text{ is infinitesimal}\}. \tag{8}$$

Pierced Binad {that we denote, for simplicity, by $(^-a^+)$ or only $^-a^+$} is defined as:

$$\begin{aligned}\mu(^-a^+) = (^-a^+) = {}^-a^+ = \overset{-+}{a} &= \{a - x,\ x \in R_+{}^* \mid x \text{ is infinitesimal}\} \cup \{a + x,\ x \in R_+{}^* \mid x \text{ is infinitesimal}\} \\ &= \{a \pm x,\ x \in R_+{}^* \mid x \text{ is infinitesimal}\}.\end{aligned} \tag{9}$$

The left monad, right monad, and the pierced binad are subsets of R^*.

9. Second Extension of Nonstandard Analysis

For the necessity of doing calculations that will be used in nonstandard neutrosophic logic in order to calculate the nonstandard neutrosophic logic operators (conjunction, disjunction, negation, implication, and equivalence) and in order to have the Nonstandard Real MoBiNad Set closed under arithmetic operations, we extend, for the time being, the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left, and the Pierced Binad to the Unpierced Binad, defined as follows [18–21].

Left Monad Closed to the Right

$$\mu\left(\overset{-0}{a}\right) = \left(\overset{-0}{a}\right) = \overset{-0}{a} = \{a - x \mid x = 0,\ or\ x \in R_+^*\ and\ x\ is\ infinitesimal\} = \mu({}^{-}a) \cup \{a\} = ({}^{-}a)\ 0\cup \{a\} = {}^{-}a \cup \{a\}. \tag{10}$$

Right Monad Closed to the Left

$$\mu\left(\overset{0+}{a}\right) = \left(\overset{0+}{a}\right) = \overset{0+}{a} = \{a + x \mid x = 0,\ or\ x \in R_+^*\ and\ x\ is\ infinitesimal\} = \mu(a^{+}) \cup \{a\} = (a^{+})\ 0\cup \{a\} = a^{+} \cup \{a\}. \tag{11}$$

Unpierced Binad

$$\mu\left(\overset{-0+}{a}\right) = \left(\overset{-0+}{a}\right) = \overset{-0+}{a} = \{a - x | x \in R_+^* and x is infinitesimal\} \cup \{a + x | x \in R_+^* and x is infinitesimal\} \cup \{a\} = \{a \pm x | x = 0, or x \in R_+^* and x is infinitesimal\} = \mu({}^{-}a^{+}) \cup \{a\} = ({}^{-}a^{+}) \cup \{a\} = {}^{-}a^{+} \cup \{a\} \tag{12}$$

The element $\{a\}$ has been included into the left monad, right monad, and pierced binad respectively.

10. Nonstandard Neutrosophic Function

In order to be able to define equalities and inequalities in the sets of monads, and in the sets of binads, we construct a *nonstandard neutrosophic function* that approximates the monads and binads to tiny open (or half open and half closed respectively) standard real intervals as below. It is called 'neutrosophic' since it deals with indeterminacy: unclear, vague monads and binads, and the function approximates them with some tiny real subsets.

Taking an arbitrary infinitesimal

$$\varepsilon_1 > 0, \text{ and writing } {}^{-}a = a - \varepsilon_1,\ a^{+} = a + \varepsilon_1, \text{ and } {}^{-}a^{+} = a \pm \varepsilon_1, \tag{13}$$

or taking an arbitrary infinitesimal $\varepsilon_2 \geq 0$, and writing

$$\overset{-0}{a} = (a - \varepsilon_2, a],\ \overset{0+}{a} = [a, a + \varepsilon_2),\ \overset{-0+}{a} = (a - \varepsilon_2, a + \varepsilon_2) \tag{14}$$

We meant to actually pick up a representative from each class of the monads and of the binads.

Representations of the monads and binads by intervals is not quite accurate from a classical point of view, but it is an approximation that helps in finding a partial order and computing nonstandard arithmetic operations on the elements of the nonstandard set NR_{MB}.

Let ε be a generic positive infinitesimal, while a be a generic standard real number.

Let $P(R)$ be the power set of the real number set R.

$$\mu_N : NR_{MB} \to P(R) \tag{15}$$

For any $a \in R$, the set of real numbers, one has

$$\mu_N(({}^{-}a)) =_N (a-\varepsilon, a), \tag{16}$$

$$\mu_N((a^{+})) =_N (a, a+\varepsilon), \tag{17}$$

$$\mu_N(({}^{-}a^{+})) =_N (a-\varepsilon, a) \cup (a, a+\varepsilon), \tag{18}$$

$$\mu_N\left(\left(\overset{-0}{a}\right)\right) =_N (a-\varepsilon, a], \tag{19}$$

$$\mu_N\left(\left(\overset{0+}{a}\right)\right) =_N [a, a+\varepsilon), \tag{20}$$

$$\mu_N\left(\left(\overset{-0+}{a}\right)\right) =_N (a-\varepsilon, a+\varepsilon), \tag{21}$$

$$\mu_N\left(\left(\overset{0}{a}\right)\right) =_N \mu_N(a) =_N a = [a, a], \tag{22}$$

in order to set it as real interval too.

11. General Notations for Monads and Binads

Let $a \in R$ be a standard real number. We use the following general notation for monads and binads.

$$\overset{m}{a} \in \left\{a, \overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a}\right\} \text{ and by convention } \overset{0}{a} = a; \tag{23}$$

or

$$m \in \{\,, {}^{-}, {}^{-0}, {}^{+}, {}^{+0}, {}^{-+}, {}^{-0+}\} = \{{}^{0}, {}^{-}, {}^{-0}, {}^{+}, {}^{+0}, {}^{-+}, {}^{-0+}\}; \tag{24}$$

therefore "m" above a standard real number "a" may mean anything: a standard real number (^{0}, or nothing above), a left monad ($^{-}$), a left monad closed to the right ($^{-0}$), a right monad ($^{+}$), a right monad closed to the left ($^{0+}$), a pierced binad ($^{-+}$), or a unpierced binad ($^{-0+}$), respectively.

The notations of monad's and binad's diacritics above (not laterally) the number a as

$$\overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a} \tag{25}$$

are the best, since they also are designed to avoid confusion for the case when the real number a is negative.

For example, if $a = {}^{-}2$, then the corresponding monads and binads are respectively represented as:

$$\overset{-}{{}^{-}2}, \overset{-0}{{}^{-}2}, \overset{+}{{}^{-}2}, \overset{0+}{{}^{-}2}, \overset{-+}{{}^{-}2}, \overset{-0+}{{}^{-}2} \tag{26}$$

Classical and Neutrosophic Notations

Classical notations on the set of real numbers:

$$<, \leq, >, \geq, \wedge, \vee, \rightarrow, \leftrightarrow, \cap, \cup, \subset, \supset, \subseteq, \supseteq, =, \in, \\ +, -, \times, \div, \hat{}, * \tag{27}$$

Operations with real subsets:

$$\circledast \tag{28}$$

Neutrosophic notations on nonstandard sets (that involve indeterminacies, approximations, and vague boundaries):

$$<_N, \leq_N, >_N, \geq_N, \wedge_N, \vee_N, \rightarrow_N, \leftrightarrow_N, \cap_N, \cup_N, \subset_N, \supset_N, \subseteq_N, \supseteq_N, =_N, \in_N +_N, -_N, \times_N, \div_N, \hat{}_N, *_N \tag{29}$$

12. Neutrosophic Strict Inequalities

We recall the neutrosophic strict inequality which is needed for the inequalities of nonstandard numbers.

Let α and β be elements in a partially ordered set M.

We have defined the neutrosophic strict inequality

$$\alpha >_N \beta \tag{30}$$

and read as

"α is neutrosophically greater than β"

if α in general is greater than β, or α is approximately greater than β, or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction* (situation when α is smaller than or equal to β) *α is greater than β*.

It means that in most of the cases, on the set M, *α is greater than β*.

And similarly for the opposite neutrosophic strict inequality

$$\alpha <_N \beta \tag{31}$$

13. Neutrosophic Equality

We have defined the neutrosophic inequality

$$\alpha =_N \beta \tag{32}$$

and read as

"α is neutrosophically equal to β"

if α in general is equal to β, or α is approximately equal to β, or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction* (situation when α is not equal to β) *α is equal to β*.

It means that in most of the cases, on the set M, *α is equal to β*.

14. Neutrosophic (Nonstrict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the $\geq_N$ and $\leq_N$ neutrosophic inequalities.

Let α and β be elements in a partially ordered set M.

The neutrosophic (nonstrict) inequality

$$\alpha \geq_N \beta \tag{33}$$

and read as

"α is neutrosophically greater than or equal to β"

if α in general is greater than or equal to β, or α is approximately greater than or equal to β, or subject to some indeterminacy (unknown or unclear ordering relationship between α and β) or subject to some contradiction (situation when α is smaller than β) α is greater than or equal to β.

It means that in most of the cases, on the set M, *α is greater than or equal to β*.

And similarly for the opposite neutrosophic (nonstrict) inequality

$$\alpha \leq_N \beta. \tag{34}$$

15. Neutrosophically Ordered Set

Let M be a set. $(M, <_N)$ is called a neutrosophically ordered set if

$$\forall \alpha, \beta \in M, \text{onehas : either} \alpha <_N \beta, \text{or} \alpha =_N \beta, \text{or} \alpha >_N \beta. \quad (35)$$

16. Neutrosophic Infimum and Neutrosophic Supremum

As an extension of the classical infimum and classical supremum, and using the neutrosophic inequalities and neutrosophic equalities, we define the neutrosophic infimum (denoted as inf_N) and the neutrosophic supremum (denoted as sup_N).

Neutrosophic Infimum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S.

The neutrosophic infimum of M, denoted as $inf_N(M)$ is the neutrosophically greatest element in S that is neutrosophically less than or equal to all elements of M:

Neutrosophic Supremum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered and M a subset of S.

The neutrosophic supremum of M, denoted as $sup_N(M)$ is the neutrosophically smallest element in S that is neutrosophically greater than or equal to all elements of M.

17. Definition of Nonstandard Real MoBiNad Set

Let $\mathbb{R}$ be the set of standard real numbers, and $\mathbb{R}^*$ be the set of hyper-reals (or nonstandard reals) that consists of infinitesimals and infinites.

The Nonstandard Real MoBiNad Set is now defined for the first time as follows

$$NR_{MB} =_N \left\{ \begin{array}{c} \varepsilon,\ \omega,\ a, ({}^{-}a)\ , \left({}^{-}a^{0}\right),\ (a^{+}), \left({}^{0}a^{+}\right), ({}^{-}a^{+}), \left({}^{-}a^{0+}\right) | \text{where } \varepsilon \text{ are infinitesimals,} \\ \text{with } \varepsilon \in \mathbb{R}^*;\ \omega = 1/\varepsilon \text{ are infinites, with } \omega \in \mathbb{R}^*; \text{ and } a \text{ are real numbers, with } a \in \mathbb{R} \end{array} \right\} \quad (36)$$

Therefore

$$NR_{MB} =_N \mathbb{R}^* \cup \mathbb{R} \cup \mu({}^{-}\mathbb{R}) \cup \mu\left({}^{-}\mathbb{R}^{0}\right) \cup \mu\left(\mathbb{R}^{+}\right) \cup \mu\left({}^{0}\mathbb{R}^{+}\right) \cup \mu\left({}^{-}\mathbb{R}^{+}\right) \cup \mu\left({}^{-}\mathbb{R}^{0\,+}\right), \quad (37)$$

where

$\mu({}^{-}\mathbb{R})$ is the set of all real left monads,
$\mu\left({}^{-}\mathbb{R}^{0}\right)$ is the set of all real left monads closed to the right,
$\mu(\mathbb{R}^{+})$ is the set of all real right monads,
$\mu\left({}^{0}\mathbb{R}^{+}\right)$ is the set of all real right monads closed to the left,
$\mu({}^{-}\mathbb{R}^{+})$ is the set of all real pierced binads,
and $\mu\left({}^{-}\mathbb{R}^{0\,+}\right)$ is the set of all real unpierced binads.

Also,

$$NR_{MB} =_N \left\{ \varepsilon,\ \omega,\ \overset{m}{a} \middle| \begin{array}{c} \text{where } \varepsilon,\ \omega \in \mathbb{R}^*, \varepsilon \text{ are infinitesimals, } \omega = \frac{1}{\varepsilon} \text{ are infinities;} \\ a \in \mathbb{R}; \text{and } m \in \left\{ ,^{-}, ^{-\,0}, ^{+}, ^{+\,0}, ^{-\,+}, ^{-\,0\,+} \right\} \end{array} \right\} \quad (38)$$

NR_{MB} is closed under addition, subtraction, multiplication, division (except division by $\overset{m}{a}$, with $a = 0$ and $m \in \left\{ ,^{-}, ^{-\,0}, ^{+}, ^{0+}, ^{-\,+}, ^{-\,0\,+} \right\}$), and power

$\{\left(\overset{m_1}{a}\right)^{\left(\overset{m_2}{b}\right)}$ with: either $a > 0$, or $a = 0$ and $m \in \left\{ ,^{+}, ^{0+} \right\}$ and $b > 0$, or $a < 0$ but $b = \frac{p}{r}$ (irreducible fraction) and p, r are integers with r an odd positive integer $r \in \{1, 3, 5, \ldots\}\}$.

These mobinad (nonstandard) above operations are reduced to set operations, using Set Analysis and Neutrosophic Analysis (both introduced by the author [22] (page 11), which are generalizations of Interval Analysis), and they deal with sets that have indeterminacies.

18. Etymology of MoBiNad

MoBiNad comes from **mo***nad* + **bi***nad*, introduced now for the first time.

19. Definition of Nonstandard Complex MoBiNad Set

The Nonstandard Complex MoBiNad Set, introduced here for the first time, is defined as

$$NC_{MB} =_N \left\{\alpha + \beta i \middle| \text{ where } i = \sqrt{-1};\ \alpha,\ \beta \in NR_{MB}\right\} \tag{39}$$

20. Definition of Nonstandard Neutrosophic Real MoBiNad Set

The Nonstandard Neutrosophic Real MoBiNad Set, introduced now for the first time, is defined as

$$NNR_{MB} =_N \left\{\alpha + \beta I \middle| \text{ where } I = \text{literal indeterminacy},\ I^2 = I;\ \alpha,\ \beta \in NR_{MB}\right\}. \tag{40}$$

21. Definition of Nonstandard Neutrosophic Complex MoBiNad Set

The Nonstandard Neutrosophic Complex MoBiNad Set, introduced now for the first time, is defined as

$$NNC_{MB} =_N \left\{\alpha + \beta I \middle| \text{ where } I = \text{literal indeterminacy},\ I^2 = I;\ \alpha,\ \beta \in NC_{MB}\right\} \tag{41}$$

22. Properties of the Nonstandard Neutrosophic Real Mobinad Set

Since in nonstandard neutrosophic logic we use only the nonstandard neutrosophic real mobinad set, we study some properties of it.

Theorem 1. *The nonstandard real mobinad set (NR_{MB}, $\leq_N$), endowed with the nonstandard neutrosophic inequality is a lattice of first type [as partially ordered set (poset)].*

Proof. The set NR_{MB} is partially ordered, because (except the two-element subsets of the form $\left\{a, \overset{-+}{a}\right\}$, and $\left\{a, \overset{-0+}{a}\right\}$, with $a \in \mathbb{R}$, beetwen which there is no order) all other elements are ordered:

If $a < b$, where $a,\ b \in \mathbb{R}$, then: $\overset{m_1}{a} <_N \overset{m_2}{b}$, for any monads or binads

$$m_1,\ m_2 \in_N \left\{ ,^{-} ,^{-0} ,^{+} ,^{0+} ,^{-+} ,^{-0+} \right\}. \tag{42}$$

If $a = b$, one has:

$$^{-}a <_N a, \tag{43}$$

$$a^{-} <_N a^{+} \tag{44}$$

$$a <_N a^{+} \tag{45}$$

$$^{-}a \leq_N {}^{-}a^{+}, \tag{46}$$

$$^{-}a \leq_N {}^{-}a^{+}, \tag{47}$$

and there is no neutrosophic ordering relationship between a and $^{-}a^{+}$,

$$\text{nor between } a \text{ and } \overset{-0+}{a} \text{ (that is why } \leq_N \text{ on } NR_{MB} \text{ is a partial ordering set).} \tag{48}$$

$$\text{If } a > b, \text{ then : } \overset{m_1}{a} >_N \overset{m_2}{b}, \text{ for any monads or binads } m_1, m_2. \tag{49}$$

□

Any two-element set $\{\alpha, \beta\} \subset_N NR_{MB}$ has a neutrosophic nonstandard infimum (meet, or greatest lower bound) that we denote by $\inf_N$, and a neutrosophic nonstandard supremum (joint, or least upper bound) that we denote by $\sup_N$, where both

$$\inf_N\{\alpha, \beta\} \text{ and } \sup_N\{\alpha, \beta\} \in NR_{MB}. \tag{50}$$

For the nonordered elements a and ${}^-a^+$:

$$\inf_N\left\{a, {}^-a^+\right\} =_N {}^-a \in_N NR_{MB}, \tag{51}$$

$$\sup_N\left\{a, {}^-a^+\right\} =_N a^+ \in_N NR_{MB} \tag{52}$$

And similarly for nonordered elements a and ${}^-a^{0\,+}$:

$$\inf_N\left\{a, {}^-a^{0\,+}\right\} =_N {}^-a \in_N NR_{MB}, \tag{53}$$

$$\sup_N\left\{a, {}^-a^{0\,+}\right\} =_N a^+ \in_N NR_{MB}. \tag{54}$$

Dealing with monads and binads which neutrosophically are real subsets with indeterminate borders, and similarly $a = [a, a]$ can be treated as a subset, we may compute $\inf_N$ and $\sup_N$ of each of them.

$$\inf_N({}^-a) =_N {}^-a \text{ and } \sup_N({}^-a) =_N {}^-a \tag{55}$$

$$\inf_N\left(a^+\right) =_N a^+ \text{ and } \sup_N\left(a^+\right) =_N a^+; \tag{56}$$

$$\inf_N\left({}^-a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a^+\right) =_N a^+; \tag{57}$$

$$\inf_N\left({}^-a^{0\,+}\right) =_N {}^-a \text{ and } \sup_N\left({}^-a^{0\,+}\right) =_N a^+. \tag{58}$$

Also,

$$\inf_N(a) =_N a \text{ and } \sup_N(a) =_N a. \tag{59}$$

If $a < b$, then $\overset{m_1}{a} <_N \overset{m_2}{b}$, hence

$$\inf_N\left\{\overset{m_1}{a}, \overset{m_2}{b}\right\} =_N \inf_N\left(\overset{m_1}{a}\right) \text{ and } \sup_N\left\{\overset{m_1}{a}, \overset{m_2}{b}\right\} =_N \sup_N \overset{m_2}{b}, \tag{60}$$

which are computed as above.

Similarly, if

$$a > b, \text{ with } \overset{m_1}{a} <_N \overset{m_2}{b}. \tag{61}$$

If $a = b$, then: $\inf_N\left\{\overset{m_1}{a}, \overset{m_2}{a}\right\} =_N$ the neutrosophically smallest ($<_N$) element among

$$\inf_N\left\{\overset{m_1}{a}\right\} \text{ and } \inf_N\left\{\overset{m_2}{a}\right\}. \tag{62}$$

While $\sup_N\left\{\overset{m_1}{a}, \overset{m_2}{a}\right\} =_N$ the neutrosophically greatest ($>_N$) element among

$$\sup_N\left\{\overset{m_1}{a}\right\} \text{ and } \sup_N\left\{\overset{m_2}{a}\right\}. \tag{63}$$

Examples:

$$\inf_N\left({}^-a,\ a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a,\ a^+\right) =_N a^+; \tag{64}$$

$$\inf_N\left({}^-a,{}^-a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a,{}^-a^+\right) =_N a^+; \tag{65}$$

$$\inf_N\left({}^-a^+,\ a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a^+,\ a^+\right) =_N a^+. \tag{66}$$

Therefore, $(NR_{MB},\ \leq_N)$ is a nonstandard real mobinad lattice of first type (as partially ordered set).

Consequence

If we remove all pierced and unpierced binads from NR_{MB} and we denote the new set by $NR_M = \left\{\varepsilon,\ \omega,\ a,{}^-a,{}^-a^0,\ a^+,{}^0a^+,\text{ where } \varepsilon \text{ are infinitesimals, } \omega \text{ are infinites, and } a \in \mathbb{R}\right\}$ we obtain a totally neutrosophically ordered set.

Theorem 2. *Any finite non-empty subset L of* $(NR_{MB}, \leq_N)$ *is also a sublattice of first type.*

Proof. It is a consequence of any classical lattice of first order (as partially ordered set). □

Theorem 3. *$(NR_{MB},\ \leq_N)$ is bounded neither to the left nor to the right, since it does not have a minimum (bottom, or least element), or a maximum (top, or greatest element).*

Proof. Straightforward, since NR_{MB} includes the set of real number $R = (-\infty,\ +\infty)$ which is clearly unbounded to the left and right-hand sides. □

Theorem 4. $(NR_{MB},\ inf_N, sup_N)$*, where* inf_N *and* sup_N *are two binary operations, dual to each other, defined before as a lattice of second type (as an algebraic structure).*

Proof. We have to show that the two laws inf_N and sup_N are commutative, associative, and verify the absorption laws.

Let $\alpha,\ \beta,\ \gamma \in NR_{MB}$ be two arbitrary elements.

Commutativity Laws

(i)

$$\inf_N\{\alpha,\ \beta\} =_N \inf_N\{\beta,\ \alpha\} \tag{67}$$

(ii)

$$\sup_N\{\alpha,\ \beta\} =_N \sup_N\{\beta,\ \alpha\} \tag{68}$$

Their proofs are straightforward.

Associativity Laws

(i)

$$\inf_N\{\alpha,\ \inf_N\{\beta,\gamma\}\} =_N \inf_N\{\inf_N\{\alpha,\ \beta\},\ \gamma\}. \tag{69}$$

□

Proof.

$$\inf_N\{\alpha,\ \inf_N\{\beta,\gamma\}\} =_N \inf_N\{\alpha,\ \beta,\ \gamma\}, \tag{70}$$

and

$$\inf_N\{\inf_N\{\alpha,\ \beta\},\ \gamma\} =_N \inf_N\{\alpha,\ \beta,\ \gamma\}, \tag{71}$$

where we have extended the binary operation $\inf_N$ to a trinary operation $\inf_N$.

(ii)

$$\sup_N\left\{\alpha,\ \sup_N\{\beta,\gamma\}\right\} =_N \sup_N\left\{\sup_N\{\alpha,\ \beta\},\ \gamma\right\} \tag{72}$$

□

Proof.

$$\sup_N\left\{\alpha,\ \sup_N\{\beta, \gamma\}\right\} =_N \sup_N\{\alpha,\ \beta,\ \gamma\}, \tag{73}$$

and

$$\sup_N\left\{\alpha,\ \sup_N\{\beta, \gamma\}\right\} =_N \sup_N\{\alpha,\ \beta,\ \gamma\}, \tag{74}$$

where similarly we have extended the binary operation $\sup_N$ to a trinary operation $\sup_N$.

Absorption Laws (as peculiar axioms to the theory of lattice)

(i) We need to prove that

$$\inf_N\left\{\alpha,\ \sup_N\{\alpha,\ \beta\}\right\} =_N \alpha. \tag{75}$$

Let $\alpha \leq_N \beta$, then

$$\inf_N\left\{\alpha,\ \sup_N\{\alpha,\ \beta\}\right\} =_N \inf_N\{\alpha, \alpha\} =_N \alpha. \tag{76}$$

Let $\alpha >_N \beta$, then

$$\inf_N\left\{\alpha,\ \sup_N\{\alpha,\ \beta\}\right\} =_N \inf_N\{\alpha, \alpha\} =_N \alpha. \tag{77}$$

(ii) Now, we need to prove that

$$\sup_N\{\alpha,\ \inf_N\{\alpha,\ \beta\}\} =_N \alpha. \tag{78}$$

Let $\alpha \leq_N \beta$, then

$$\sup_N\{\alpha,\ \inf_N\{\alpha,\ \beta\}\} =_N \sup_N\{\alpha,\ \alpha\} =_N \alpha. \tag{79}$$

Let $\alpha >_N \beta$, then

$$\sup_N\{\alpha,\ \inf_N\{\alpha,\ \beta\}\} =_N \sup_N\{\alpha, \beta\} =_N \alpha. \tag{80}$$

Consequence

The binary operations $\inf_N$ and $\sup_N$ also satisfy the idempotent laws:

$$\inf_N\{\alpha,\ \alpha\} =_N \alpha, \tag{81}$$

$$\sup_N\{\alpha,\ \alpha\} =_N \alpha. \tag{82}$$

□

Proof. The axioms of idempotency follow directly from the axioms of absorption proved above. □

Thus, we have proved that (NR_{MB}, inf_N, sup_N) is a lattice of second type (as algebraic structure).

23. Definition of General Nonstandard Real MoBiNad Interval

Let $a,\ b \in \mathbb{R}$, with

$$-\infty < a \leq b < \infty, \tag{83}$$

$$]^-a,\ b^+[_{MB} = \left\{x \in NR_{MB}, {}^-a \leq_N x \leq_N b^+\right\}. \tag{84}$$

As particular edge cases:

$$]^-a,\ a^+[_{MB} =_N \left\{^-a,\ a, {}^-a^+, a^+\right\}, \tag{85}$$

a discrete nonstandard real set of cardinality 4.

$$]^-a,\ ^-a[_{MB} =_N \{^-a\}; \tag{86}$$

$$]a^+,\ a^+[_{MB} =_N \left\{a^+\right\} \tag{87}$$

$$]a,\ a^+[_{MB} =_N \left\{a,\ a^+\right\} \tag{88}$$

$$]^-a,\ a[_{MB} =_N \{^-a,\ a\} \tag{89}$$

$$]^{-}a,\ {}^{-}a^{+}[_{MB} =_N \left\{{}^{-}a, {}^{-}a^{+},\ a^{+}\right\}, \tag{90}$$

where $a \notin]^{-}a, {}^{-}a^{+}[_{MB}$ since $a \not\leq_N {}^{-}a^{+}$ (there is no relation of order between a and ${}^{-}a^{+}$);

$$]^{-}a^{+},\ a^{+}[_{MB} =_N \left\{{}^{-}a^{+},\ a^{+}\right\}. \tag{91}$$

Theorem 5.

$$\left(\ ]^{-}a,\ b^{+}[,\ \leq_N\right) \textit{ is a nonstandard real mobinad sublattice of first type (poset).} \tag{92}$$

Proof. Straightforward since $]^{-}a,\ b^{+}[$ is a sublattice of the lattice of first type NR_{MB}. □

Theorem 6.

$$\left(\ ]^{-}a,\ b^{+}[,\ \inf_N,\ \sup_N, {}^{-}a,\ b^{+}\right) \textit{ is a nonstandard bounded real mobinad sublattice of second type (as algebraic structure).} \tag{93}$$

Proof. $]^{-}a,\ b^{+}[_{MB}$ as a nonstandard subset of NR_{MB} is also a poset, and for any two-element subset

$$\{\alpha,\ \beta\} \subset_N\ ]^{-}0,\ 1^{+}[_{MB} \tag{94}$$

one obviously has the triple neutrosophic nonstandard inequality:

$${}^{-}a \leq_N \inf_N\{\alpha,\ \beta\} \leq_N \sup_N\{\alpha,\ \beta\} \leq_N b^{+} \tag{95}$$

hence $(\ ]^{-}a,\ b^{+}[_{MB} \leq_N)$ is a nonstandard real mobinad sublattice of first type (poset), or sublattice of NR_{MB}.

Further on, $]^{-}a,\ b^{+}[$, endowed with two binary operations $\inf_N$ and $\sup_N$, is also a sublattice of the lattice NR_{MB}, since the lattice axioms (Commutative Laws, Associative Laws, Absorption Laws, and Idempotent Laws) are clearly verified on $]^{-}a,\ b^{+}[$.

The nonstandard neutrosophic modinad Identity Join Element (Bottom) is ${}^{-}a$, and the nonstandard neutrosophic modinad Identity Meet Element (Top) is b^{+}, or

$$\inf_N]^{-}a,\ b^{+}[=_N{}^{-} \text{ and } \sup_N]^{-}a,\ b^{+}[=_N b^{+}. \tag{96}$$

The sublattice Identity Laws are verified below.

$$\text{Let } \alpha \in_N]^{-}a,\ b^{+}[, \text{ whence } {}^{-}a \leq_N \alpha \leq_N b^{+}. \tag{97}$$

Then:

$$\inf_N\left\{\alpha,\ b^{+}\right\} =_N \alpha, \text{ and } \sup_N\{\alpha, {}^{-}a\} =_N \alpha. \tag{98}$$

□

24. Definition of Nonstandard Real MoBiNad Unit Interval

$$\begin{aligned} &]^{-}0,\ 1^{+}[_{MB} =_N \{x \in NR_{MB}, {}^{-}0 \leq_N x \leq_N 1^{+}\} \\ &=_N \left\{ \begin{array}{c} \varepsilon, a, \overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a} \,\middle|\, \textit{where } \varepsilon \textit{ are infinitesimals}, \\ \varepsilon \in \mathbb{R}^*,\ \textit{with } \varepsilon > 0, \textit{ and } a \in [0,\ 1] \end{array} \right\} \end{aligned} \tag{99}$$

This is an extension of the previous definition (1998) [5] of nonstandard unit interval

$$]^{-}0,\ 1^{+}[=_N ({}^{-}0) \cup [0,\ 1] \cup \left(1^{+}\right) \quad (100)$$

Associated to the first published definitions of neutrosophic set, logic, and probability was used. One has

$$]^{-}0,\ 1^{+}[\subset_N\,]^{-}0,\ 1^{+}[_{MB} \quad (101)$$

where the index $_{MB}$ means: all monads and binads included in $]^{-}0,\ 1^{+}[$, for example,

$$({}^{-}0.2),\ ({}^{-}0.3^{0}),\ (0.5^{+}),\ ({}^{-}0.7^{+}),\ ({}^{-}0.8^{0+}) \text{ etc.} \quad (102)$$

or, using the top diacritics notation, respectively,

$$\overset{-}{0.2},\ \overset{-0}{0.3},\ \overset{+}{0.5},\ \overset{-+}{0.7},\ \overset{-0+}{0.8} \text{ etc.} \quad (103)$$

Theorem 7. *The Nonstandard Real MoBiNad Unit Interval $]^{-}0,\ 1^{+}[_{MB}$ is a partially ordered set (poset) with respect to $\leq_N$, and any of its two elements have an inf_N and sup_N hence $]^{-}0,\ 1^{+}[_{MB}$ is a nonstandard neutrosophic lattice of first type (as poset).*

Proof. Straightforward. □

Theorem 8. *The Nonstandard Real MoBiNad Unit Interval $]^{-}0,\ 1^{+}[_{MB}$, endowed with two binary operations inf_N and sup_N, is also a nonstandard neutrosophic lattice of second type (as an algebraic structure).*

Proof. Replace $a = 0$ and $b = 1$ into the general nonstandard real mobinad interval $]^{-}a,\ b^{+}[$. □

25. Definition of Extended General Neutrosophic Logic

We extend and present in a clearer way our 1995 definition (published in 1998) of neutrosophic logic.

Let $\mathcal{U}$ be a universe of discourse of propositions and $P \in \mathcal{U}$ be a generic proposition.

A General Neutrosophic Logic is a multivalued logic in which each proposition P has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F), and where T, I, and F are standard real subsets or nonstandard real mobinad subsets of the nonstandard real mobinat unit interval $]^{-}0,\ 1^{+}[_{MB}$,

With

$$T,\ I,\ F \subseteq_N]^{-}0,\ 1^{+}[_{MB} \quad (104)$$

where

$${}^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}. \quad (105)$$

26. Definition of Standard Neutrosophic Logic

If in the above definition of general neutrosophic logic all neutrosophic components, T, I, and F are standard real subsets, included in or equal to the standard real unit interval, $T,\ I,\ F \subseteq [0,\ 1]$, where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3 \quad (106)$$

we have a standard neutrosophic logic.

27. Definition of Extended Nonstandard Neutrosophic Logic

If in the above definition of general neutrosophic logic at least one of the neutrosophic components T, I, or F is a nonstandard real mobinad subset, neutrosophically included in or equal to the nonstandard real mobinad unit interval $]^{-}0,\ 1^{+}[_{MB}$, where

$$^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}, \quad (107)$$

we have an extended nonstandard neutrosophic logic.

Theorem 9. *If M is a standard real set, $M \subset \mathbb{R}$, then*

$$inf_N(M) = inf(M) \text{ and } sup_N(M) = sup(M). \quad (108)$$

Proof. The neutrosophic infimum and supremum coincide with the classical infimum and supremum since there is no indeterminacy on the set M, meaning M contains no nonstandard numbers. □

28. Definition of Extended General Neutrosophic Set

We extend and present in a clearer way our 1995 definition of neutrosophic set.

Let $\mathcal{U}$ be a universe of discourse of elements and $S \in \mathcal{U}$ a subset.

A Neutrosophic Set is a set such that each element x from S has a degree of membership (T), a degree of indeterminacy (I), and a degree of nonmembership (F), where T, I, and F are standard real subsets or nonstandard real mobinad subsets, neutrosophically included in or equal to the nonstandard real mobinat unit interval

$$]^{-}0,\ 1^{+}[_{MB}, \text{with } T,\ I,\ F \subseteq_N]^{-}0,\ 1^{+}[_{MB}, \quad (109)$$

where

$$^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}. \quad (110)$$

29. Definition of Standard Neutrosophic Set

If in the above general definition of neutrosophic set all neutrosophic components, T, I, and F, are standard real subsets included in or equal to the classical real unit interval, $T,\ I,\ F \subseteq [0,\ 1]$, where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3, \quad (111)$$

we have a standard neutrosophic set.

30. Definition of Extended Nonstandard Neutrosophic Set

If in the above general definition of neutrosophic set at least one of the neutrosophic components T, I, or F is a nonstandard real mobinad subsets, neutrosophically included in or equal to $]^{-}0,\ 1^{+}[_{MB}$, where

$$^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}, \quad (112)$$

we have a nonstandard neutrosophic set.

31. Definition of Extended General Neutrosophic Probability

We extend and present in a clearer way our 1995 definition of neutrosophic probability.

Let $\mathcal{U}$ be a universe of discourse of events, and $E \in \mathcal{U}$ be an event.

A Neutrosophic Probability is a multivalued probability such that each event E has a chance of occuring (T), an indeterminate (unclear) chance of occuring or not occuring (I), and a chance of not occuring (F), and where T, I, and F are standard or nonstandard real mobinad subsets, neutrosophically included in or equal to the nonstandard real mobinat unit interval

$$]^{-}0,\ 1^{+}[_{MB},\ T,\ I,\ F \subseteq_N]^{-}0,\ 1^{+}[_{MB},\ \text{where } {}^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}. \tag{113}$$

32. Definition of Standard Neutrosophic Probability

If in the above general definition of neutrosophic probability all neutrosophic components, T, I, and F are standard real subsets, included in or equal to the standard unit interval $T,\ I,\ F \subseteq [0,\ 1]$, where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3, \tag{114}$$

we have a standard neutrosophic probability.

33. Definition of Extended Nonstandard Neutrosophic Probability

If in the above general definition of neutrosophic probability at least one of the neutrosophic components T, I, F is a nonstandard real mobinad subsets, neutrosophically included in or equal to $]^{-}0,\ 1^{+}[_{MB}$, where

$${}^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}, \tag{115}$$

we have a nonstandard neutrosophic probability.

34. Classical Operations with Real Sets

Let $A,\ B \subseteq \mathbb{R}$ be two real subsets. Let $\circledast$ and * denote any of the real subset classical operations and real number classical operations respectively: addition (+), subtraction (−), multiplication (×), division (÷), and power (^).

Then,

$$A \circledast B = \{a * b,\ \text{where } a \in A \text{ and } b \in B\} \tag{116}$$

Thus

$$A \oplus B = \{a + b | a \in A, b \in B\} \tag{117}$$

$$A \ominus B = \{a - b | a \in A, b \in B\} \tag{118}$$

$$A \otimes B = \{a \times b | a \in A, b \in B\} \tag{119}$$

$$A \oslash B = \{a \div b | a \in A; b \in B, b \neq 0\} \tag{120}$$

$$A^B = \left\{a^b \middle| a \in A, a > 0; b \in B\right\} \tag{121}$$

For the division (÷), of course, we consider $b \neq 0$. While for the power (^), we consider $a > 0$.

35. Operations on the Nonstandard Real MoBiNad Set (NR_{MB})

For all nonstandard (addition, subtraction, multiplication, division, and power) operations

$$\alpha, \beta \in_N NR_{MB}, \alpha *_N \beta =_N \mu_N(\alpha) \circledast \mu_N(\beta) \tag{122}$$

where $*_N$ is any neutrosophic arithmetic operations with neutrosophic numbers ($+_N$, $-_N$, $\times_N$, $\div_N$, $\hat{}_N$), while the corresponding $\circledast$ is an arithmetic operation with real subsets.

So, we approximate the nonstandard operations by standard operations of real subsets.

We sink the nonstandard neutrosophic real mobinad operations into the standard real subset operations, then we resurface the last ones back to the nonstandard neutrosophic real mobinad set.

Let ε_1 and ε_2 be two non-null positive infinitesimals. We present below some particular cases, all others should be deduced analogously.

Nonstandard Addition

First Method

$$({}^{-}a) + ({}^{-}b) =_N (a - \varepsilon_1,\ a) + (b - \varepsilon_2,\ b) =_N (a + b - \varepsilon_1 - \varepsilon_2,\ a + b) =_N (a + b - \varepsilon,\ a + b) =_N {}^{-}(a + b), \tag{123}$$

where we denoted $\varepsilon_1 + \varepsilon_2 = \varepsilon$ (the addition of two infinitesimals is also an infinitesimal).

Second Method

$$({}^{-}a) + ({}^{-}b) =_N (a - \varepsilon_1) + (b - \varepsilon_2) =_N (a + b - \varepsilon_1 - \varepsilon_2) =_N {}^{-}(a + b) \tag{124}$$

Adding two left monads, one also gets a left monad.

Nonstandard Subtraction

First Method

$$\begin{aligned} ({}^{-}a) - ({}^{-}b) =_N \quad & (a - \varepsilon_1, a) \\ & -(b - \varepsilon_2, b) =_N (a - \varepsilon_1 - b,\ a - b + \varepsilon_2) =_N (a - b - \varepsilon_1,\ a - b \\ & +\varepsilon_2) =_N \begin{pmatrix} - & 0 & + \\ & a - b & \end{pmatrix} \end{aligned} \tag{125}$$

Second Method

$$({}^{-}a) - ({}^{-}b) =_N (a - \varepsilon_1) - (b - \varepsilon_2) =_N a - b - \varepsilon_1 + \varepsilon_2, \tag{126}$$

since ε_1 and ε_2 may be any positive infinitesimals,

$$=_N \begin{cases} {}^{-}(a - b), \text{ when } \varepsilon_1 > \varepsilon_2; \\ \begin{pmatrix} 0 \\ a - b \end{pmatrix}, \text{ when } \varepsilon_1 = \varepsilon_2 \quad =_N \begin{pmatrix} 0 \\ a - b \end{pmatrix} =_N a - b; \\ (a - b)^{+}, \text{ when } \varepsilon_1 < \varepsilon_2. \end{cases} \tag{127}$$

Subtracting two left monads, one obtains an unpierced binad (that is why the unpierced binad had to be introduced).

Nonstandard Division

Let $a, b > 0$.

$$({}^{-}a) \div ({}^{-}b) =_N (a - \varepsilon_1,\ a) \div (b - \varepsilon_2,\ b) =_N \left(\frac{a - \varepsilon_1}{b}, \frac{a}{b - \varepsilon_2}\right) \tag{128}$$

Since

$$\varepsilon_1 > 0 \text{ and } \varepsilon_2 > 0,\ \frac{a - \varepsilon_1}{b} < \frac{a}{b} \text{ and } \frac{a}{b - \varepsilon_2} > \frac{a}{b}, \tag{129}$$

while between $\frac{a-\varepsilon_1}{b}$ and $\frac{a}{b-\varepsilon_2}$ there is a continuum whence there are some infinitesimals ε_1^0 and ε_2^0 such that $\frac{a-\varepsilon_1^0}{b-\varepsilon_2^0} = \frac{a}{b}$, or $ab - b\varepsilon_1^0 = ab - a\varepsilon_2^0$, and for a given ε_1^0 there exists an

$$\varepsilon_2^0 = \varepsilon_1^0 \cdot \frac{b}{a} \tag{130}$$

Hence

$$\frac{({}^{-}a)}{({}^{-}b)} =_N \begin{pmatrix} - & 0 & + \\ & \frac{a}{b} & \end{pmatrix} \tag{131}$$

For a or/and b negative numbers, it is similar but it is needed to compute the inf_N and sup_N of the products of intervals.

Dividing two left monads, one obtains an unpierced binad.

Nonstandard Multiplication

Let $a,\ b \geq 0$.

$$\left({}^{-}a^{0}\right)\times\left({}^{-}b^{0\,+}\right) \quad =_N (a-\varepsilon_1,\ a] \times(b-\varepsilon_2,\ b+\varepsilon_2) =_N ((a-\varepsilon_1)\cdot(b-\varepsilon_2), a\cdot(b+\varepsilon_2)) =_N \left({}^{-}ab^{0\,+}\right) \tag{132}$$

Since

$$(a-\varepsilon_1)\cdot(b-\varepsilon_2) < a\cdot b \text{ and } a\cdot(b+\varepsilon_2) > a\cdot b. \tag{133}$$

For a or/and b negative numbers, it is similar but it is needed to compute the inf_N and sup_N of the products of intervals.

Multiplying a positive left monad closed to the right, with a positive unpierced binad, one obtains an unpierced binad.

Nonstandard Power

Let $a,\ b > 1$.

$$\left({}^{0}a^{+}\right)^{\left({}^{-}b^{0}\right)} =_N [a,\ a+\varepsilon_1)^{(b-\varepsilon_2,\ b]} =_N \left(a^{b-\varepsilon_2},\ (a+\varepsilon_1)^{b}\right) =_N \begin{pmatrix} - & 0 & + \\ & a^{b} & \end{pmatrix} \tag{134}$$

$$\text{since } a^{b-\varepsilon_1} < a^{b} \text{ and } (a+\varepsilon_1)^{b} > a^{b}. \tag{135}$$

Raising a right monad closed to the left to a power equal to a left monad closed to the right, for both monads above 1, the result is an unpierced binad.

Consequence

In general, when doing arithmetic operations on nonstandard real monads and binads, the result may be a different type of monad or binad.

That is why is was imperious to extend the monads to closed monads, and the pierced binad to unpierced binad, in order to have the whole nonstandard neutrosophic real mobinad set closed under arithmetic operations.

36. Conditions of Neutrosophic Nonstandard Inequalities

Let NR_{MB} be the Nonstandard Real MoBiNad. Let's endow (NR_{MB}, $<_N$) with a neutrosophic inequality.

Let $\alpha, \beta \in NR_{MB}$, where α, β may be real numbers, monads, or binads.

And let

$$\begin{gathered}\left(\overset{-}{a}\right),\left(\overset{-0}{a}\right),\left(\overset{+}{a}\right),\left(\overset{0+}{a}\right),\left(\overset{-+}{a}\right),\left(\overset{-0+}{a}\right) \in NR_{MB},\ \text{and} \\ \left(\overset{-}{b}\right),\left(\overset{-0}{b}\right),\left(\overset{+}{b}\right),\left(\overset{0+}{b}\right),\left(\overset{-+}{b}\right),\left(\overset{-0+}{b}\right) \in NR_{MB},\end{gathered} \tag{136}$$

be the left monads, left monads closed to the right, right monads, right monads closed to the left, and binads, and binads nor pierced of the elements (standard real numbers) a and b, respectively. Since all monads and binads are real subsets, we may treat the single real numbers

$$a = [a, a] \text{ and } b = [b, b] \text{ as real subsets too} \tag{137}$$

as real subsets too.

NR_{MB} is a set of subsets, and thus we deal with neutrosophic inequalities between subsets.

(i) If the subset α has many of its elements above all elements of the subset β,

(ii) then $\alpha >_N \beta$ (partially).

(iii) If the subset α has many of its elements below all elements of the subset β,

(iv) then $\alpha <_N \beta$ (partially).

(v) If the subset α has many of its elements equal with elements of the subset β,
(vi) then $\alpha =_N \beta$ (partially).

If the subset α verifies (*i*) and (*iii*) with respect to subset β, then $\alpha \geq_N \beta$.
If the subset α verifies (*ii*) and (*iii*) with respect to subset β, then $\alpha \leq_N \beta$.
If the subset α verifies (*i*) and (*ii*) with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

For example, between a and $(^-a^+)$ there is no neutrosophic order, similarly between a and $\overset{-0+}{a}$.

Similarly, if the subset α verifies (*i*), (*ii*) and (*iii*) with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

37. Open Neutrosophic Research

The quantity or measure of "many of its elements" of the above (*i*), (*ii*), or (*iii*) conditions depends on each *neutrosophic application* and on its *neutrosophic experts*.

An approach would be to employ the *Neutrosophic Measure* [23,24], that handles indeterminacy, which may be adjusted and used in these cases.

In general, we do not try in purpose to validate or invalidate an existing scientific result, but to investigate how an existing scientific result behaves in a new environment (that may contain indeterminacy), or in a new application, or in a new interpretation.

38. Nonstandard Neutrosophic Inequalities

For the neutrosophic nonstandard inequalities, we propose, based on the previous six neutrosophic equalities, the following.

$$(^-\mathrm{a}) <_N \mathrm{a} <_N (a^+) \tag{138}$$

Since the standard real interval $(a-\varepsilon, a)$ is below a, and a is below the standard real interval $(a, a+\varepsilon)$ by using the approximation provided by the nonstandard neutrosophic function μ, or because

$$\forall x \in R_+^*, a - x < a < a + x \tag{139}$$

where x is of course a (nonzero) positive infinitesimal (the above double neutrosophic inequality actually becomes a double classical standard real inequality for each fixed positive infinitesimal).

The converse double neutrosophic inequality is also neutrosophically true:

$$(a^+) >_N \mathrm{a} >_N (^-a) \tag{140}$$

Another nonstandard neutrosophic double inequality:

$$(^-\mathrm{a}) \leq_N (^-a^+) \leq_N (a^+) \tag{141}$$

This double neutrosophic inequality may be justified since $(^-a^+) = (^-a) \cup (a^+)$ and, geometrically, on the Real Number Line, the number a is in between the subsets $^-a = (a-\varepsilon, a)$ and $a^+ = (a, a+\varepsilon)$, so

$$(^-a) \leq_N (^-a) \cup (a^+) \leq_N (a^+) \tag{142}$$

Hence the left side of the inequality's middle term coincides with the inequality first term, while the right side of the inequality middle term coincides with the third inequality term.

Conversely, it is neutrosophically true as well:

$$(a^+) \geq_N (^-a) \cup (a^+) \geq_N (^-a) \tag{143}$$

Also,

$$\overset{-}{a} \leq_N \overset{-0}{a} \leq_N a \leq_N \overset{0+}{a} \leq_N \overset{+}{a} \text{ and } \overset{-}{a} \leq_N \overset{-+}{a} \leq_N \overset{-0+}{a} \leq_N \overset{+}{a} \tag{144}$$

Conversely, they are also neutrosophically true:

$$\overset{+}{a} \geq_N \overset{0+}{a} \geq_N a \geq_N \overset{-0}{a} \geq_N \overset{-}{a} \text{ and } \overset{+}{a} \geq_N \overset{-0+}{a} \geq_N \overset{-+}{a} \geq_N \overset{-}{a} \text{ respectively.} \tag{145}$$

If $a > b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities.

$$a >_N ({}^{-}b), a >_N \left(b^{+}\right), a >_N \left({}^{-}b^{+}\right), a >_N \overset{-0}{b}, a >_N \overset{0+}{b}, a >_N \overset{-0+}{b}; \tag{146}$$

$$({}^{-}a) >_N b, ({}^{-}a) >_N ({}^{-}b), ({}^{-}a) >_N \left(b^{+}\right), ({}^{-}a) >_N \left({}^{-}b^{+}\right), \overset{-}{a} >_N \overset{-0}{b}, \overset{-}{a} >_N \overset{0+}{b}, \overset{-}{a} >_N \overset{-0+}{b}; \tag{147}$$

$$\left(a^{+}\right) >_N b, \left(a^{+}\right) >_N ({}^{-}b), \left(a^{+}\right) >_N \left(b^{+}\right), \left(a^{+}\right) >_N \left({}^{-}b^{+}\right), \overset{+}{a} >_N \overset{-0}{b}, \overset{+}{a} >_N \overset{0+}{b}, \overset{+}{a} >_N \overset{-0+}{b}; \tag{148}$$

$$({}^{-}a^{+}) >_N b, ({}^{-}a^{+}) >_N ({}^{-}b), ({}^{-}a^{+}) >_N (b^{+}), ({}^{-}a^{+}) >_N ({}^{-}b^{+}), \text{ etc.} \tag{149}$$

No Ordering Relationships

For any standard real number a, there is no relationship of order between the elements a and $({}^{-}a^{+})$, or between the elements a and

$$\left(\overset{-0+}{a}\right) \tag{150}$$

Therefore, NR_{MB} is a neutrosophically partially order set.

If one removes all binads from NR_{MB}, then $(NR_{MB}, \leq_N)$ is neutrosophically totally ordered. (151)

Theorem 10. *Using the nonstandard general notation one has:*

If $a > b$, which is a (standard) classical real inequality, then

$$\overset{m_1}{a} >_N \overset{m_2}{b} \text{ for any } m_1,\ m_2 \in \left\{,\ ^{-},\ ^{-0},\ ^{+},\ ^{+0},\ ^{-0},\ ^{-0+}\right\}. \tag{152}$$

Conversely, if $a < b$, which is a (standard) classical real inequality, then

$$\overset{m_1}{a} <_N \overset{m_2}{b} \text{ for any } m_1,\ m_2 \in \left\{,\ ^{-},\ ^{-0},\ ^{+},\ ^{+0},\ ^{-0},\ ^{-0+}\right\}. \tag{153}$$

39. Nonstandard Neutrosophic Equalities

Let a, b be standard real numbers; if $a = b$ that is a (classical) standard equality, then

$$({}^{-}a) =_N ({}^{-}b), (a^{+}) =_N (b^{+}), ({}^{-}a^{+}) =_N ({}^{-}b^{+}), \tag{154}$$

$$\left(\overset{-0}{a}\right) =_N \left(\overset{-0}{b}\right), \left(\overset{0+}{a}\right) =_N \left(\overset{0+}{b}\right), \left(\overset{-0+}{a}\right) =_N \left(\overset{-0+}{b}\right) \tag{155}$$

40. Nonstandard Neutrosophic Belongingness

On the nonstandard real set NR_{MB}, we say that

$$\overset{m}{c} \in_N \left]\overset{m_1}{a}, \overset{m_2}{b}\right[\text{ iff } \overset{m_1}{a} \leq_N \overset{m}{c} \leq_N \overset{m_2}{b}, \tag{156}$$

where

$$m_1, m_2, m \in \{, ^{-}, ^{-0}, ^{+}, ^{+0}, ^{-+}, ^{-0+}\}. \quad (157)$$

We use the previous nonstandard neutrosophic inequalities.

41. Nonstandard Hesitant Sets

Nonstandard Hesitant sets are sets of the form:

$$A = \{a_1, a_2, \ldots, a_n\}, 2 \leq n < \infty, A \subset_N NR_{MB}, \quad (158)$$

where at least one element a_{i_0}, $1 \leq i_0 \leq n$, is an infinitesimal, a monad, or a binad (of any type); while other elements may be standard real numbers, infinitesimals, or also monads or binads (of any type).

If the neutrosophic components T, I, and F are nonstandard hesitant sets, then one has a Nonstandard Hesitant Neutrosophic Logic/Set/Probability.

42. Nonstandard Neutrosophic Strict Interval Inclusion

On the nonstandard real set NR_{MB},

$$]\overset{m_1}{a}, \overset{m_2}{b}[\subset_N]\overset{m_3}{c}, \overset{m_4}{d}[\quad (159)$$

iff

$$\overset{m_3}{c} \leq_N \overset{m_1}{a} <_N \overset{m_2}{b} <_N \overset{m_4}{d} \text{ or } \overset{m_3}{c} <_N \overset{m_1}{a} <_N \overset{m_2}{b} \leq_N \overset{m_4}{d} \text{ or } \overset{m_3}{c} <_N \overset{m_1}{a} <_N \overset{m_2}{b} <_N \overset{m_4}{d} \quad (160)$$

43. Nonstandard Neutrosophic (Nonstrict) Interval Inclusion

On the nonstandard real set NR_{MB},

$$]\overset{m_1}{a}, \overset{m_2}{b}[\subseteq_N]\overset{m_3}{c}, \overset{m_4}{d}[\text{ iff} \quad (161)$$

$$\overset{m_3}{c} \leq_N \overset{m_1}{a} <_N \overset{m_2}{b} \leq_N \overset{m_4}{d}. \quad (162)$$

44. Nonstandard Neutrosophic Strict Set Inclusion

The nonstandard set A is neutrosophically strictly included in the nonstandard set B, $A \subset_N B$, if:

$$\forall x \in_N A, x \in_N B, \text{ and } \exists y \in_N B : y \notin_N A. \quad (163)$$

45. Nonstandard Neutrosophic (Nonstrict) Set Inclusion

The nonstandard set A is neutrosophically not strictly included in the nonstandard set B,

$$A \subseteq_N B, \text{ iff:} \quad (164)$$

$$\forall x \in_N A, x \in_N B. \quad (165)$$

46. Nonstandard Neutrosophic Set Equality

The nonstandard sets A and B are neutrosophically equal,

$$A =_N B, \text{ iff:} \quad (166)$$

$$A \subseteq_N B \text{ and } B \subseteq_N A. \quad (167)$$

47. The Fuzzy, Neutrosophic, and Plithogenic Logical Connectives $\wedge$, $\vee$, $\rightarrow$

All fuzzy, intuitionistic fuzzy, and neutrosophic logic operators are *inferential approximations*, not written in stone. They are improved from application to application.

Let's denote:

$$\wedge_F, \wedge_N, \wedge_P \text{ representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction;} \tag{168}$$

similarly

$$\vee_F, \vee_N, \vee_P \text{ representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,} \tag{169}$$

and

$$\rightarrow_F, \rightarrow_N, \rightarrow_P \text{ representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication.} \tag{170}$$

I agree that my beginning neutrosophic operators (when I applied the same *fuzzy t-norm*, or the same *fuzzy t-conorm*, to all neutrosophic components T, I, F) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out in 2002 by Ashbacher [25] and confirmed in 2008 by Rivieccio [26]. They observed that if on T_1 and T_2 one applies a *fuzzy t-norm*, for their opposites F_1 and F_2, one needs to apply the *fuzzy t-conorm* (the opposite of fuzzy t-norm), and reciprocally.

About inferring I_1 and I_2, some researchers combined them in the same directions as T_1 and T_2.

Then,

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2), \tag{171}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2), \tag{172}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2). \tag{173}$$

others combined I_1 and I_2 in the same direction as F_1 and F_2 (since both I and F are negatively qualitative neutrosophic components, while F is qualitatively positive neutrosophic component), the most used one is as follows.

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2), \tag{174}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2), \tag{175}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2). \tag{176}$$

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [27] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction* $c(,)$ between the neutrosophic components T, I, and F have been defined (in order to facilitate the design of the aggregation operators), as follows:

$$c(T, F) = 1 \text{ (or 100\%, because they are totally opposite)}, c(T, I) = c(F, I) = 0.5 \text{ (or 50\%, because they are only half opposite).} \tag{177}$$

Then,

$$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) = (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2), \tag{178}$$

$$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) = (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2), \tag{179}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2). \tag{180}$$

48. Fuzzy t-norms and Fuzzy t-conorms

The most used $\wedge_F$ (Fuzzy t-norms), and $\vee_F$ (Fuzzy t-conorms) are as follows.
Let

$$a, b \in [0, 1]. \tag{181}$$

Fuzzy t-norms (fuzzy conjunctions, or fuzzy intersections):

$$a \wedge_F b = \min\{a, b\}; \tag{182}$$

$$a \wedge_F b = ab; \tag{183}$$

$$a \wedge_F b = \max\{a + b - 1, 0\}. \tag{184}$$

Fuzzy t-conorms (fuzzy disjunctions, or fuzzy unions):

$$a \vee_F b = \max\{a, b\}; \tag{185}$$

$$a \vee_F b = a + b - ab; \tag{186}$$

$$a \vee_F b = \min\{a + b, 1\} \tag{187}$$

49. Nonstandard Neutrosophic Operators

Nonstandard Neutrosophic Conjunctions

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) = (inf_N(T_1, T_2), sup_N(I_1, I_2), sup_N(F_1, F_2)) \tag{188}$$

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) = (T_1 \times_N T_2, I_1 +_N I_2 -_N I_1 \times_N I_2, F_1 +_N F_2 -_N F_1 \times_N F_2) \tag{189}$$

Nonstandard Neutrosophic Disjunctions

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) = (sup_N(T_1, T_2), inf_N(I_1, I_2), inf_N(F_1, F_2)) \tag{190}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) = (T_1 +_N T_2 -_N T_1 \times_N T_2, I_1 \times_N I_2, F_1 \times_N F_2) \tag{191}$$

Nonstandard Neutrosophic Negations

$$\neg(T_1, I_1, F_1) = (F_1, I_1, T_1) \tag{192}$$

$$\neg(T_1, I_1, F_1) = (F_1, (1^+) -_N I_1, T_1) \tag{193}$$

Nonstandard Neutrosophic Implications

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2) = (F_1 +_N T_2 -_N F_1 \times_N T_2, I_1 \times_N I_2, T_1 \times_N F_2) \tag{194}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, (1^+) -_N I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, ((1^+) -_N I_1) \wedge_F I_2, T_1 \wedge_F F_2) = (F_1 +_N T_2 -_N F_1 \times_N T_2, ((1^+) -_N I_1) \times_N I_2, T_1 \times_N F_2) \tag{195}$$

Let $P_1(T_1, I_1, F_1)$ and $P_2(T_2, I_2, F_2)$ be two nonstandard neutrosophic logical propositions, whose nonstandard neutrosophic components are, respectively,

$$T_1, I_1, F_1, T_2, I_2, F_2 \in_N NR_{MB}. \tag{196}$$

50. Numerical Examples of Nonstandard Neutrosophic Operators

Let us take a particular numeric example, where

$$P_1 =_N (\overset{0+}{0.3}, \overset{-+}{0.2}, 0.4), P_2 =_N (\overset{-0}{0.6}, \overset{-0+}{0.1}, \overset{+}{0.5}) \tag{197}$$

are two nonstandard neutrosophic logical propositions.

We use the nonstandard arithmetic operations previously defined *Numerical Example of Nonstandard Neutrosophic Conjunction*

$$\overset{0+}{0.3} \times \overset{-0}{0.6} =_N [0.3, 0.3+\varepsilon_1) \times (0.6-\varepsilon_2, 0.6) = (0.18-0.3\varepsilon_2, 0.18+0.6\varepsilon_1) =_N \overset{-0+}{0.18} \tag{198}$$

$$\begin{aligned} &\overset{-+}{0.2} +_N \overset{-0+}{0.1} -_N \overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N [(0.2-\varepsilon_1, 0.2) \cup (0.2, 0.2+\varepsilon_1)] + (0.1-\varepsilon_2, 0.1+\varepsilon_2) \\ &-[(0.2-\varepsilon_1, 0.2) \cup (0.2, 0.2+\varepsilon_1)] \times (0.1-\varepsilon_2, 0.1+\varepsilon_2) \\ &= [(0.3-\varepsilon_1-\varepsilon_2, 0.3+\varepsilon_2) \cup (0.3-\varepsilon_2, 0.3+\varepsilon_1+\varepsilon_2)] \\ &-[(0.2-\varepsilon_1)\times(0.1-\varepsilon_2), (0.02+0.2\varepsilon_2)] \cup [(0.02-0.2\varepsilon_2), (0.2+\varepsilon_1)\times(0.1+\varepsilon_2)] \\ &= [\overset{-0+}{0.3} \cup \overset{-0+}{0.3}] - [\overset{-0+}{0.02} \cup \overset{-0+}{0.02}] = [\overset{-0+}{0.3}] - [\overset{-0+}{0.02}] = 0.3 \overset{-0+}{-} 0.02 =_N \overset{-0+}{0.28} \end{aligned} \tag{199}$$

$$\begin{aligned} &0.4 +_N \overset{+}{0.5} =_N [0.4, 0.4] + (0.5, 0.5+\varepsilon_1) - [0.4, 0.4] \times (0.5, 0.5+\varepsilon_1) \\ &= (0.4+0.5, 0.4+0.5+\varepsilon_1) - (0.4 \times 0.5, 0.4 \times 0.5 + 0.4\varepsilon_1) \\ &= (0.9, 0.9+\varepsilon_1) - (0.2, 0.2+0.4\varepsilon_1) \\ &= (0.9-0.2-0.4\varepsilon_1, 0.9+\varepsilon_1-0.2) = (0.7-0.4\varepsilon_1, 0.7+\varepsilon_1) =_N \overset{-0+}{0.70} \end{aligned} \tag{200}$$

Hence

$$P_1 \wedge P_2 =_N (\overset{-0+}{0.18}, \overset{-0+}{0.28}, \overset{-0+}{0.70}) \tag{201}$$

Numerical Example of Nonstandard Neutrosophic Disjunction

$$\begin{aligned} &\overset{0+}{0.3} +_N \overset{-0}{0.6} - \overset{0+}{0.3} \times_N \overset{-0}{0.6} =_N \{[0.3, 0.3+\varepsilon_1) + (0.6-\varepsilon_1, 0.6]\} - \{[0.3, 0.3+\varepsilon_1) \times (0.6-\varepsilon_1, 0.6]\} \\ &= (0.9-\varepsilon_1, 0.9+\varepsilon_1) - (0.18-0.3\varepsilon_1, 0.18+0.6\varepsilon_1) = (0.72-1.6\varepsilon_1, 0.72+1.3\varepsilon_1) =_N \overset{-0+}{0.72} \end{aligned} \tag{202}$$

$$\overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N \left(0.2 \overset{-0+}{\times} 0.1\right) =_N \overset{-0+}{0.02} \tag{203}$$

$$0.4 \times_N \overset{+}{0.5} =_N \left(0.4 \overset{+}{\times} 0.5\right) =_N \overset{+}{0.20} \tag{204}$$

Hence

$$P_1 \vee_N P_2 =_N (\overset{-0+}{0.72}, \overset{-0+}{0.02}, \overset{+}{0.20}) \tag{205}$$

Numerical Example of Nonstandard Neutrosophic Negation

$$\neg_N P_1 =_N \neg_N (\overset{0+}{0.3}, \overset{-+}{0.2}, 0.4) =_N (0.4, \overset{-+}{0.2}, \overset{0+}{0.3}) \tag{206}$$

Numerical Example of Nonstandard Neutrosophic Implication

$$(P_1 \rightarrow_N P_2) \Leftrightarrow_N (\neg_N P_1 \vee_N P_2) =_N (0.4, \overset{-+}{0.2}, \overset{0+}{0.3}) \vee_N (\overset{-0}{0.6}, \overset{-0+}{0.1}, \overset{+}{0.5}) \tag{207}$$

Afterwards,

$$0.4 +_N \overset{-0}{0.6} - 0.4 \times_N \overset{-0}{0.6} =_N \left(0.4 \overset{-0}{+} 0.6\right) -_N \left(0.4 \overset{-0}{\times} 0.6\right) =_N \overset{-0}{1.0} -_N \overset{-0}{0.24} =_N \overset{-0+}{0.76} \tag{208}$$

$$\overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N \overset{-0+}{0.02} \tag{209}$$

$$\overset{0+}{0.3} \times \overset{+}{0.5} =_N \overset{+}{0.15} \tag{210}$$

whence

$$\neg_N P_1 =_N (\overset{-0+}{0.76}, \overset{-0+}{0.02}, \overset{+}{0.15}) \tag{211}$$

Therefore, we have showed above how to do nonstandard neutrosophic arithmetic operations on some concrete examples.

51. Conclusions

In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes, and others.

That's why we have extended in 1998 for the first time the monads to pierced binad, and then in 2019 for the second time we extended the left monad to left monad closed to the right, the right monad to right monad closed to the left, and the pierced binad to unpierced binad. These were necessary in order to construct a general nonstandard neutrosophic real mobinad space, which is closed under the nonstandard neutrosophic arithmetic operations (such as addition, subtraction, multiplication, division, and power), which are needed in order to be able to define the nonstandard neutrosophic operators (such as conjunction, disjunction, negation, implication, and equivalence) on this space, and to transform the newly constructed nonstandard neutrosophic real mobinad space into a lattice of first order (as partially ordered nonstandard set, under the neutrosophic inequality $\leq_N$) and a lattice of second type (as algebraic structure, endowed with two binary laws: neutrosophic infimum (*infN*) and neutrosophic supremum (sup_N)).

As a consequence of extending the nonstandard analysis, we also extended the nonstandard neutrosophic logic, set, measure, probability and statistics.

As future research it would be to introduce the nonstandard neutrosophic measure, and to find applications of extended nonstandard neutrosophic logic, set, probability into calculus, since in calculus one deals with infinitesimals and their aggregation operators, due to the tremendous number of applications of the neutrosophic theories [28].

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Neutrosophic Optimization Model and Computational Algorithm for Optimal Shale Gas Water Management under Uncertainty

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Abstract: Shale gas energy is the most prominent and dominating source of power across the globe. The processes for the extraction of shale gas from shale rocks are very complex. In this study, a multiobjective optimization framework is presented for an overall water management system that includes the allocation of freshwater for hydraulic fracturing and optimal management of the resulting wastewater with different techniques. The generated wastewater from the shale fracking process contains highly toxic chemicals. The optimal control of a massive amount of contaminated water is quite a challenging task. Therefore, an on-site treatment plant, underground disposal facility, and treatment plant with expansion capacity were designed to overcome environmental issues. A multiobjective trade-off between socio-economic and environmental concerns was established under a set of conflicting constraints. A solution method—the neutrosophic goal programming approach—is suggested, inspired by independent, neutral/indeterminacy thoughts of the decision-maker(s). A theoretical computational study is presented to show the validity and applicability of the proposed multiobjective shale gas water management optimization model and solution procedure. The obtained results and conclusions, along with the significant contributions, are discussed in the context of shale gas supply chain planning policies over different time horizons.

Keywords: intuitionistic fuzzy parameters; uncertainty modeling; neutrosophic goal programming approach; shale gas water management system

1. Introduction

Energy sources play a dynamic role in the development, nourishment, and enrichment reputation of any country. Presently, many conventional sources of energy are being used for energy production, but shale gas energy is booming among different energy sources [1–3]. Apart from conventional sources of energy, shale gas—which is located within shale rocks—is the most promising source of natural gas. Recently, shale gas has become an emerging source of natural gas across the world [4,5]. The United States is the second-richest country after China in terms of the abundance of shale gas resources. Since the start of this century, significant interest has been shown in the potential extraction of shale gas across the world [6–8]. In 2000, only 1% of the US natural gas production was contributed by shale gas energy; by 2010, it was more than 20%, and according to predictions of the US government's Energy Information Administration (EIA), by 2035 more than 46% of the US' natural gas supply will be from shale gas [9]. The first extraction of shale gas from shale rocks was done in Fredonia (New York) in 1821 by using shallow and low-pressure fractures. However, horizontal drilling started in the

1930s, and the first well was fractured in the US in 1947. Presently, shale gas potential extraction and enriched abundance in many nations are being investigated. According to Sieminski et al. [10], in 2013, only a few countries (e.g., the US, Canada, and China) have sufficient shale gas enrichment, and future production is planned at commercial scale [10]. China has an apparent strategy to dramatically grow its shale gas production and investment, which has been restricted by its insufficient approach to water, land, and the latest technology. Shale hosts rocks trapping potential shale gas quantities that have numerous common properties, namely, being composed of organic material, a mature petroleum source, containing a high amount of natural gas in the thermogenic gas window spread inside the Earth's crust where there is high heat and pressure being applied to convert petroleum into natural gas. Most commonly, hydraulic fracturing (also known as *fracking*) and horizontal drilling are two dominant methods that are being used in the process of shale gas extraction across the world. The high concentration of released toxic and contaminated wastewater from the extraction and use of shale gas affects the environment. A challenge in the shale gas extraction process is preventing environmental pollution. This depends on drilling wells and their capacity, which varies with shale use. Water cannot be reused until a well is fractured and the water starts to withdraw from the well. A study was published by Kerr [11] in May 2011 that strongly suggested shale gas wells contain a rigorous abundance of toxic surface groundwater flows with flammable methane in North-Eastern Pennsylvania. Although the presented study was confined to the contamination of water, the impact in other areas that would be dug out for shale extraction purposes was not discussed.

Over the past few years, various research works have been published suggesting, in the context of optimal production policies, a selection area, supply chain network, and socio-economic balancing during shale gas extraction at the commercial level. Lutz et al. [12] presented a theoretical overview of shale gas development in the context of a more prominent resource-producing country such as the United States. The quantification of shale gas energy and wastewater generation throughout Pennsylvania was revealed with consolidated data obtained from 2189 wells. The concluding remarks were contrary to the current perception regarding the shale gas extraction-to-wastewater evulsion ratio, transportation, disposal facilities, treatment strategies, and the associated factors in the shale gas extraction processes. Yang et al. [13] presented the optimum usage of the water life cycle for drilled well-peds through a discontinuous-time bi-stage stochastic mixed-integer linear programming (SMILP) optimization framework under uncertainty. The model was optimized with a set of long-term historical data. The discussed approach was applied to two Marcellus shale gas uses, which showed the effectiveness of the addressed study. Yang et al. [14] discussed the optimal usage of water in the fracking and drilling mechanism during shale gas extraction processes at commercial scale, and also formulated a new mixed-integer linear programming (MILP) problem that inherently optimizes the capital investment for an optimal shale gas yielding scheme. A case study was implemented in the proposed optimization scenario. Li and Peng [15] investigated a new solution scheme based on interval-valued hesitant fuzzy information for the selection of promising shale gas areas, and discussed the applicability of the proposed approach by selecting the shale gas areas using multi-criteria decision making (MCDM). Although shale gas extraction has been done for over 100 years in two different prominent basins of the United States (i.e., the Appalachian Basin and the Illinois Basin), the wells seldom result in profitable production. The current shale gas extraction process, consisting of horizontal drilling and hydraulic fracturing, has made shale gas synthesis more advantageous.

In April 2012 [16], the cost of extraction incurred over shale gas in different coastal parts of the UK was approximated to be much higher than $200 per barrel, which was compared to oil prices of about $120 per barrel in the UK North Sea. North America has emerged as one of the potential leaders in developing and producing shale energy. In the US and Canada, after the successful economic accomplishment of the Barnett Shale use in Texas, the exploration of promising new sources of shale gas is being made. Gao and You [17] designed an active water cycle configuration for the shale gas extraction process and re-formulated it as a mixed-integer linear fractional programming (MILFP) optimization model under different objectives and sets of constraints. The models were globally optimized using various approaches such as the parametric method, a reformulation linearization approach, the branch and bound method, and the Charnes–Cooper transformation technique. The addressed mathematical models were applied to two case studies based on Marcellus shale play, in which on-site treatment techniques of wastewater gained importance in generating freshwater storage. Sang et al. [18] discussed a numerical optimization model of desorption and adsorption processes for hydraulic fracturing stimulations that was optimized by assuming polar co-ordinate and balance space, respectively. To estimate the receptacle volume of drilled horizontal shale well reservoirs, Gao and You [19] addressed a practical framework for the optimal flow of shale energy networks. The designed configuration comprises various coherent components such as freshwater, shale energy, wastewater management, transportation, and disposal facility with treatment plant options. The formulated models were built in the form of a mixed-integer nonlinear programming (MINLP) problem. The obtained results revealed the trade-off between economic and environmental objectives. Furthermore, Guerra et al. [20] also discussed the mathematical formulation and implementation of a comprehensive shale gas production framework with the integration of the water supply chain management system. The proposed optimization framework was illustrated with two case studies with different leading components of the shale gas production systems. Bartholomew and Mauter [21] also developed a multiobjective mixed-integer linear programming framework to highlight the trade-off between financial cost and human health & environment (HHE) costs in the overall shale gas water management system. The system's objectives were defined effectively, inherently representing different financial aspects of the shale gas production planning problem. Zhang et al. [22] presented a specific study on shale gas wastewater management systems under uncertainty. The presented optimization framework for shale gas wastewater management system corresponds to the disposal and treatment facilities under the expansion of treatment capacity. The proposed model has been designed by considering fuzzy and stochastic parameters with feasibility degree and probability distribution function at the different significance level. The concluding remarks revealed the optimal wastewater management in cases where underground disposal capacity is scarce. The uncertainty involved in the parameter reduced the reliability risk factor in shale gas production. In the present competitive epoch, different shale-gas-producing countries have motivated the wholesome and challenging study of shale gas production policies and the optimal supply chain network configuration. Lira-Barragán et al. [23] investigated a mathematical programming formulation for integrating water networks consumed for hydraulic fracturing processes in shale gas extraction. The proposed uncertainty pertained to the use of water for a different purpose and highlighted probabilistic aspects. Moreover, the developed models also cover the scheduling problem associated with the whole modeling framework for shale gas extraction. The different expected objective functions were incorporated, which led to the existence of uncertainty in the modeling approach. Interested researchers can find recent publications on shale gas development and future research scope in Chen et al. [24], Knee and Masker [25], Lan et al. [26], Ren et al. [27], Zhang et al. [28], Denham et al. [29], Al-Aboosi and El-Halwagi [30], Jin et al. [31], Ren and Zhang [32], and Wang et al. [33].

Research Gaps and Contribution

Shale gas extraction planning models and optimal strategic implementation inherently depend on various parametric factors that are actively indulged in the decision-making process. The requirement of a tremendous amount of freshwater for hydraulic fracturing (i.e., between 7000 and 40,000 m^3 per well) becomes challenging. The assessment of different freshwater sources is somewhat uneconomic, but other extractions can fulfill freshwater demand. The produced wastewater management system is also an indispensable issue and is very important in shale gas production planning models.

Many recent publications, such as Guo et al. [34], Gao et al. [35], Chebeir et al. [36], Chen et al. [37], Drouven and Grossmann [38], He et al. [39], and Wang et al. [40] have discussed different optimal modeling approaches for shale gas water management systems with socio-economic and environmental concerns. All of the above studies are confined to only uncertain modeling approaches, and have not discussed uncertainty among parameters' values; however, Zhang et al. [22] incorporated vagueness among parameters and represented it by fuzzy and stochastic quantification methodology. However, the study proposed by Zhang et al. [22] is lagging in two more practical aspects. First, it may not always be possible to have historical data for to the stochastic technique can be applied; additionally, due to some hesitation regarding imprecise parameters, the fuzzy number may not be an appropriate representative of uncertain parameters. Hence, better representation of the degree of hesitation under vagueness or imprecision can be made by using the intuitionistic fuzzy number, which considers the degree of belongingness as well as non-belongingness of the element in the possible set. Second, Zhang et al. [22] only designed the optimization framework for the optimal management of wastewater throughout the shale gas extraction processes, and did not consider the management of freshwater, which is also an integrated part of the whole shale gas extraction over time horizons. Thus, in this study we propose the unification of the two aspects discussed above. The proposed multiobjective shale gas water management system optimization model was designed after considering the most critical aspects of overall water management planning and optimization epoch. Furthermore, the concept of a neutrosophic goal programming approach is new and has not yet been applied in the field of such an emerging source of energy. The proposed model also ensures the trade-off between the socio-economic and environmental effects of shale gas production policies more realistically. The proposed shale gas optimization model also provides an opportunity to adopt the available on-site treatment technology along with the option of expanding the treatment plant, which would be beneficial for Pennsylvania because underground disposal facilities are scarce and most often wastewater is supplied to nearby cities in Ohio. The rest of the paper is summarized as follows:

In Section 2, the methodologies and technical definitions regarding intuitionistic fuzzy parameters and the neutrosophic goal programming approach (NGPA) are discussed, while Section 3 represents the multiobjective shale gas water management optimization model and implementation of the NGPA algorithm. A hypothetical case study is examined in Section 4 that shows the applicability and validity of the proposed approach. Finally, concluding remarks and findings are highlighted based on the present work in Section 5.

2. Methodology

The shale gas optimization and modeling framework discussed in this paper enviably involve significant work-flow procedures. The involvement of various critical terminological aspects in the proposed modeling and computational approach makes the shale gas optimization problem more pervasive. In order to represent these aspects, we have used some technical terminology which is able to define each and every aspect of the proposed model effectively and efficiently. The mathematical technical terminologies used in this study are intuitionistic fuzzy parameters [41–43] and those from multiobjective optimization problems [44–46] and the neutrosophic goal programming approach, which is based on the neutrosophic decision set (see [47–49]). On the basis of these mathematical technical terminologies, we developed an effective modeling and optimization framework for a shale gas water management system that dynamically characterizes the freshwater requirement and the dispensation of the generated wastewater from shale gas wells. The proposed model for shale gas water management systems contemplates different kinds of cost parameters (e.g., acquisition cost, transportation cost, treatment cost, disposal cost, and capital investment) involved in the accumulation process of freshwater and the dispensation of the generated wastewater from the shale gas extraction process. Apart from the cost, different parameters such as the freshwater storage capacity, underground injection disposal capacity of wastewater, wastewater treatment capacity, and the capacity for the expansion of wastewater treatment plants were considered in this study. Moreover, these parameters are not always in deterministic/crisp form, despite containing some kind of ambiguity and vagueness. This ambiguousness and vagueness can be represented by different uncertain parameters, such as fuzzy Zhang et al. [22], intuitionistic fuzzy, stochastic Zhang et al. [22], and other uncertain forms. The fuzzy parameters are only concerned with the maximization of membership degree (belongingness), whereas an intuitionistic fuzzy set is based on more intuition than a fuzzy set, as it deals with the maximization of membership (belongingness) and the minimization of non-membership degree (non-belongingness) of the element in the set. A stochastic parameter involves a probability distribution function with known mean and variances based on the randomly occurring events.

Furthermore, the proposed modeling approach was designed and incorporated with socio-economic and environmental facts. The potential production and distribution of shale gas energy at the commercial level is not a boon unless and until the proper pertinent initiatives are undertaken in order to overcome the by-products released by the shale gas extraction processes. Therefore, the proposed modeling and optimization approach inherently involves more than one objective (known as a multiobjective optimization problem), which is sufficient to justify the trade-off among different critical socio-economic and environmental aspects of shale gas energy. The mathematical formulation of multiple objectives ensures the economic and environmental aspects of shale gas extraction procedures. To deal with the proposed multiobjective shale gas water management optimization model, a neutrosophic goal programming approach was developed that reveals the actual situation more appropriately. The proposed NGPA considers the independent indeterminacy/neutral degree, which is the area of incognizance of a proposition's value. The selection of the proposed NGPA technique is quite effective, explanatory, and a good representative of real-life situations.

2.1. Intuitionistic Fuzzy Set

Definition 1 ([50]). *(Intuitionistic fuzzy set (IFS)) Let there be a universal set Y; then, an IFS $\tilde{W}$ in Y is given by the ordered triplets as follows:*

$$\tilde{W} = \{y,\ \mu_{\tilde{W}}(y),\ \nu_{\tilde{W}}(y) |\ y \in\ Y\},$$

where

$$\mu_{\tilde{W}}(y) : Y\ \to\ [0,\ 1];\ \ \nu_{\tilde{W}}(y) : Y\ \to\ [0,\ 1],$$

with conditions

$$0\ \leq\ \mu_{\tilde{W}}(y) + \nu_{\tilde{W}}(y)\ \leq 1,$$

where $\mu_{\tilde{W}}(y)$ and $\nu_{\tilde{W}}(y)$ denote the membership and non-membership functions of the element $y \in Y$ into the set $\tilde{W}$.

Definition 2 ([51] (Intuitionistic fuzzy number)). *An intuitionistic fuzzy set $\tilde{W} = \{y,\ \mu_{\tilde{W}}(y),\ \nu_{\tilde{W}}(y) |\ y\ \in\ Y\}$ of the real number R is said to be an intuitionistic fuzzy number if the following condition holds:*

(i) $\tilde{W}$ should be intuitionistic fuzzy normal and convex.
(ii) $\mu_{\tilde{W}}(y)$ and $\nu_{\tilde{W}}(y)$ should be upper and lower semi-continuous functions.
(iii) $Supp\ \tilde{W} = \{y \in R;\ \nu_W(y) < 1\}$ should be bounded.

Definition 3 ([43]). *(Triangular intuitionistic fuzzy number) An intuitionistic fuzzy number $\tilde{W}$ is said to be a triangular intuitionistic fuzzy number if the membership function $\mu_{\tilde{W}}(y)$ and non-membership function $\nu_{\tilde{W}}(y)$ are given by:*

$$\mu_{\tilde{W}}(y) = \begin{cases} \dfrac{y - a_1}{b - a_1}, & if\ a_1 \leq y \leq b, \\ 1, & if\ y = b, \\ \dfrac{a_2 - y}{a_2 - b}, & if\ b \leq y \leq a_2 \end{cases}$$

and

$$\nu_{\tilde{W}}(y) = \begin{cases} \dfrac{b-y}{b-a_3}, & if\ a_3 \le y \le b, \\ 0, & if\ y = b, \\ \dfrac{y-b}{a_4-b}, & if\ b \le y \le a_4, \end{cases}$$

where $a_3 \le a_1 \le b \le a_2 \le a_4$ *and is denoted by* $\tilde{W} = \{(a_1, b, a_2; \mu_{\tilde{W}}), (a_3, b, a_4; \nu_{\tilde{W}})\}$.

Definition 4 (**[42]**). *(Expected interval for intuitionistic fuzzy number) Let us consider that there exists an intuitionistic fuzzy number* $\tilde{W}$ *which belongs to the set of real numbers R with* $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) \in \mathbb{R}$ *such that* $a_1 \le a_2 \le a_3 \le a_4 \le b_1 \le b_2 \le b_3 \le b_4$. *The four functions* $f_{\tilde{W}}(y), g_{\tilde{W}}(y), h_{\tilde{W}}(y), k_{\tilde{W}}(y) : \mathbb{R} \to [0,1]$ *such that* $f_{\tilde{W}}(y)$ *and* $g_{\tilde{W}}(y)$ *are non-decreasing and* $h_{\tilde{W}}(y)$ *and* $k_{\tilde{A}}(y)$ *are non-increasing functions, then the intuitionistic fuzzy number* $\tilde{W} = \{y,\ \mu_{\tilde{W}}(y),\ \nu_{\tilde{W}}(y) :\ y \in\ Y\}$ *can be represented by membership and non-membership functions stated as follows:*

$$\mu_{\tilde{W}}(y) = \begin{cases} 0, & if\ y \le a_1\ or\ y \ge a_4, \\ f_{\tilde{W}}(y), & if\ a_1 \le y \le a_2, \\ g_{\tilde{W}}(y), & if\ a_3 \le y \le a_4, \\ 1, & if\ a_2 \le y \le a_3 \end{cases}$$

and

$$\nu_{\tilde{W}}(y) = \begin{cases} 1, & if\ y \le b_1\ or\ y \ge b_4, \\ h_{\tilde{W}}(y), & if\ b_1 \le y \le b_2, \\ k_{\tilde{W}}(y), & if\ b_3 \le y \le b_4, \\ 0, & if\ b_2 \le y \le b_3. \end{cases}$$

Furthermore, Grzegrorzewski [52] discussed the expected interval for the intuitionistic fuzzy number $\tilde{W} = \{a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4\}$ *as a crisp interval and presented it as follows:*

$$EI(\tilde{W}) = [E_1(\tilde{W}), E_2(\tilde{W})]. \quad (1)$$

The lower and upper values of the expected interval for the intuitionistic fuzzy number $\tilde{W}$ *is defined as given below:*

$$E_1(\tilde{W}) = \frac{b_1 + a_2}{2} + \int_{b_1}^{b_2} h_{\tilde{W}}(y) - \int_{a_1}^{a_2} f_{\tilde{W}}(y),$$

$$E_2(\tilde{W}) = \frac{a_3 + b_4}{2} + \int_{a_3}^{a_4} g_{\tilde{W}}(y) - \int_{b_3}^{b_4} k_{\tilde{W}}(y),$$

where

$$h_{\tilde{W}}(y) = \frac{y - b_1}{b_2 - b_1}, \quad f_{\tilde{W}}(y) = \frac{y - a_1}{a_2 - a_1},$$

$$k_{\tilde{W}}(y) = \frac{y - b_4}{b_3 - b_4}, \quad g_{\tilde{W}}(y) = \frac{y - a_4}{a_3 - a_4}.$$

Definition 5 (**[42]**). *(Expected interval and value for triangular intuitionistic fuzzy number) Suppose that* $\tilde{W} = \{(a_1, b, a_2; \mu_{\tilde{W}}), (a_3, b, a_4; \nu_{\tilde{W}})\}$ *is a triangular intuitionistic fuzzy number with membership and non-membership functions* $\mu_{\tilde{W}}(y)$ *and* $\nu_{\tilde{W}}(y)$*; then, the expected interval of the triangular intuitionistic fuzzy number by using the above definition can be obtained as follows:*

$$E_1(\tilde{W}) = \frac{b_1 + a_2}{2} + \int_{b_1}^{b_2} h_{\tilde{W}}(y) - \int_{a_1}^{a_2} f_{\tilde{W}}(y) = \frac{3a + b_1 + (a - b_1)\nu_{\tilde{W}} - (a - a_1)\mu_{\tilde{W}}}{4} \quad (2)$$

and

$$E_2(\tilde{W}) = \frac{a_3 + b_4}{2} + \int_{a_3}^{a_4} g_{\tilde{W}}(y) - \int_{b_3}^{b_4} k_{\tilde{W}}(y) = \frac{3a + b_2 + (a_2 - a)\mu_{\tilde{W}} + (a - b_2)\nu_{\tilde{W}}}{2}. \quad (3)$$

Grzegrorzewski [52] also suggested the expected value for the intuitionistic fuzzy number with the help of lower and upper values of the expected interval. Therefore, the expected value for the triangular intuitionistic fuzzy number is obtained as follows:

$$EV(\tilde{W}) = \left[\frac{E_1(\tilde{W}) + E_2(\tilde{W})}{2}\right]. \quad (4)$$

Definition 6. *The general mathematical programming formulation of a multiobjective optimization problem with k objectives, j constraints, and q variables can be stated as follows:*

$$\begin{aligned} Optimize\ & (Z_1, Z_2, \cdots, Z_k) \;\; k = 1, 2, \cdots, K \\ s.t.\ & g_j(x) \le d_j, \;\; j = 1, 2, \cdots, j_1; \\ & g_j(x) \ge d_j, \;\; j = j_1 + 1, j_1 + 2, \cdots, j_2; \\ & g_j(x) = d_j, \;\; j = j_2 + 1, j_2 + 2, \cdots, J; \\ & x_q \ge 0, \qquad q = 1, 2, 3, \cdots, Q; \;\; x_q \in X, \end{aligned} \quad (5)$$

where Z_k are a set of k different conflicting objectives, g_j are real-valued functions, and d_j are real numbers. x_q is a q-dimensional decision variable vector and X is a feasible solution set.

2.2. Neutrosophic Goal Programming Approach (NGPA)

In the past few decades, the extended version of the fuzzy set (FS) and intuitionistic fuzzy set (IFS) have been introduced. In order to reflect the insightful concept of indeterminacy or neutral thoughts in decision making, a new set called the *neutrosophic set* was introduced by Smarandache [47]. The technical erm *neutrosophic* holds two different words, which are *neutre* derived from French and meaning "neutral", and *sophia*, adopted from Greek and meaning "skill" or "wisdom". Therefore, the word "neutrosophic" concretely means "knowledge of neutral thoughts". The FS is mainly concerned with the maximization of the degree of belongingness (membership function) of an element in the set, whereas the IFS deals with two aspects, namely, the degree of belongingness (membership function) and the degree of non-belongingness (non-membership function) of the element in the set. The incorporation of the independent neutral/indeterminacy concept in the neutrosophic set differentiates itself from FS and IFS, providing more strength to decision-making processes.

Moreover, many real-life circumstances may not be easy to tackle with only the degree of belongingness and non-belongingness of the element in the set. However, the degree up to some level of belongingness and non-belongingness would be a significant touchstone in the decision-making process. For example, if we take the opinion about the victory of team X in a cricket match, and supposing they have the possible chance of winning equalling 0.8, the chance team X has of losing would be 0.4 and the chance that match would be a tie would be 0.5 (see [53]). All the possibilities are independent of each other and can take any value between 0 and 1. Therefore, this sort of decision-making problem is outside of the domain of FS and IFS, and consequently beyond the periphery of fuzzy programming and intuitionistic fuzzy programming approaches, respectively. Hence, independent indeterminacy conditions under the uncertainty domain are a more technical perspective in real-life optimization problems (see [48,49,53]).

An efficient approach called the neutrosophic goal programming approach (NGPA) based on the neutrosophic decision set [47] was designed in order to reach the best compromise solution of multiobjective optimization problems. The NGPA inherently comprises three membership functions, namely, the maximization of truth and indeterminacy degrees and the minimization of the falsity degree present in any optimization problem. It permits policymakers to manifest independent neutral inferences about decision-making processes and provides an opportunity to effectively reach goals using the NGPA technique.

Definition 7 ([47] (Neutrosophic Set)). *Let there be a universal discourse Y such that $y \in Y$, then a neutrosophic set W in Y is defined by three membership functions, namely, truth $T_W(y)$, indeterminacy $I_W(y)$, and falsity $F_W(y)$, denoted by the following form:*

$$W = \{< y, T_W(y), I_W(y), F_W(y) > | y \in Y\},$$

where $T_W(y), I_W(y)$, and $F_W(y)$ are real standard or non-standard subsets belonging to $]0^-, 1^+[$, also given as, $T_W(y) : y \rightarrow]0^-, 1^+[$, $I_W(y) : Y \rightarrow]0^-, 1^+[$, and $F_W(y) : Y \rightarrow]0^-, 1^+[$. There is no restriction on the sum of $T_W(y), I_W(y)$ and $F_W(y)$, so we have

$$0^- \leq \ sup \ T_W(y) + I_W(y) + \ sup \ F_W(y) \leq 3^+.$$

Definition 8 ([47]). *Let there be two single-valued neutrosophic sets A and B, then $C = (A \cup B)$ with truth $T_C(y)$, indeterminacy $I_C(y)$, and falsity $F_C(y)$ membership functions are given by:*

$T_C(y) = max\ (T_A(y), T_B(y))$,
$I_C(y) = min\ (I_A(y), I_B(y))$,
$F_C(y) = min\ (F_A(y), F_B(y))$ *for each* $y \in Y$.

Definition 9 ([47]). *Let there be two single-valued neutrosophic sets A and B, then $C = (A \cap B)$ with truth $T_C(y)$, indeterminacy $I_C(y)$, and falsity $F_C(y)$ membership functions are given by*

$T_C(y) = min\ (T_A(y), T_B(y))$,
$I_C(y) = max\ (I_A(y), I_B(y))$,
$F_C(y) = max\ (F_A(y), F_B(y))$ *for each* $y \in Y$.

The concept of fuzzy decision (D), fuzzy goal (G), and fuzzy constraint (C) was first discussed by Bellman and Zadeh [44] and extensively used in many real-life decision-making problems under fuzziness. Therefore, a fuzzy decision set can be defined as follows:

$$D = G \cap C.$$

Equivalently, the neutrosophic decision set D_N, with the set of neutrosophic goals and constraints, can be defined as:

$$D_N = (\cap_{k=1}^{K} G_k)(\cap_{j=1}^{J} C_j) = (y,\ T_D(y),\ I_D(y),\ F_D(y)\ ),$$

where

$$T_D(y) = min \left\{ \begin{array}{c} T_{G_1}(y), T_{G_2}(y), ..., T_{G_K}(y) \\ T_{C_1}(y), T_{C_2}(y), ..., T_{C_J}(y) \end{array} \right\} \forall\ y \in Y,$$

$$I_D(y) = max \left\{ \begin{array}{c} I_{G_1}(y), I_{G_2}(y), ..., I_{G_K}(y) \\ I_{C_1}(y), I_{C_2}(y), ..., I_{C_J}(y) \end{array} \right\} \forall\ y \in Y,$$

$$F_D(y) = max \left\{ \begin{array}{c} F_{G_1}(y), F_{G_2}(y), ..., F_{G_K}(y) \\ F_{C_1}(y), F_{C_2}(y), ..., F_{C_J}(y) \end{array} \right\} \forall\ y \in Y,$$

where $T_D(y), I_D(y)$, and $F_D(y)$ are the truth, indeterminacy, and falsity membership functions of neutrosophic decision set D_N, respectively.

In order to formulate the different membership functions for multiobjective optimization problems (MOOPs), we defined the bounds for each objective function. The lower and upper bounds for each objective function are represented by L_k and U_k which can be obtained as follows:

First, we solved each objective function as a single objective under the given constraints of the problem. After solving k objectives individually, we have the k solutions set, $X^1, X^2, ..., X^k$. After that, the obtained solutions are substituted for each objective function to provide the lower and upper bounds for each objective, as given below:

$$U_k = \max\,[Z_k(X^k)] \text{ and } L_k = \min\,[Z_k(X^k)] \quad \forall\, k = 1, 2, 3, ..., K. \tag{6}$$

The bounds for k objective functions under the neutrosophic environment can be obtained as follows:

$$U_k^T = U_k, \quad L_k^T = L_k \quad \text{for truth membership,}$$
$$U_k^I = L_k^T + s_k, \quad L_k^I = L_k^T \quad \text{for indeterminacy membership,}$$
$$U_k^F = U_k^T, \quad L_k^F = L_k^T + t_k \quad \text{for falsity membership,}$$

where s_k and $t_k \in (0, 1)$ are predetermined real numbers assigned by decision maker(s) (DM(s)). By using the above lower and upper bounds, we defined the linear membership functions under a neutrosophic environment:

$$T_k(Z_k(x)) = \begin{cases} 1 & if\;\; Z_k(x) < L_k^T \\ \frac{U_k^T - Z_k(x)}{U_k^T - L_k^T} & if\;\; L_k^T \le Z_k(x) \le U_k^T \\ 0 & if\;\; Z_k(x) > U_k^T, \end{cases} \tag{7}$$

$$I_k(Z_k(x)) = \begin{cases} 1 & if\;\; Z_k(x) < L_k^I \\ \frac{U_k^I - Z_k(x)}{U_k^I - L_k^I} & if\;\; L_k^I \le Z_k(x) \le U_k^I \\ 0 & if\;\; Z_k(x) > U_k^I, \end{cases} \tag{8}$$

$$F_k(Z_k(x)) = \begin{cases} 1 & if\;\; Z_k(x) > U_k^F \\ \frac{Z_k(x) - L_k^F}{U_k^F - L_k^F} & if\;\; L_k^F \le Z_k(x) \le U_k^F \\ 0 & if\;\; Z_k(x) < L_k^F. \end{cases} \tag{9}$$

In the above case, $L_k^{(.)} \neq U_k^{(.)}$ for all k objective functions. If for any membership $L_k^{(.)} = U_k^{(.)}$, then the value of these memberships will be equal to 1. The diagrammatic representation of the objective function with different components of membership functions under a neutrosophic environment is shown in Figure 1.

Moreover all the above three discussed membership degrees can be transformed into membership goals according to their respective degrees of attainment. The highest degree of truth membership function that can be achieved is unity (1), the indeterminacy membership function is neutral and independent with the highest attainment degree half (0.5), and the falsity membership function can be achieved with the highest attainment degree zero (0). Now the transformed membership goals under a neutrosophic environment can be expressed as follows:

$$T_k(Z_k(x)) + d_{kT}^- - d_{kT}^+ = 1, \tag{10}$$

$$I_k(Z_k(x)) + d_{kI}^- - d_{kI}^+ = 0.5, \tag{11}$$

$$F_k(Z_k(x)) + d_{kF}^- - d_{kF}^+ = 0, \tag{12}$$

where $d_{kT}^-, d_{kT}^+, d_{kI}^-, d_{kI}^+, d_{kF}^-$, and d_{kF}^+ are the over and under deviations such that $d_{kT}^-.d_{kT}^+ = 0$, $d_{kF}^-.d_{kF}^+ = 0$, and $d_{kF}^-.d_{kF}^+ = 0$ for truth membership, indeterminacy membership, and falsity membership goals under a neutrosophic environment.

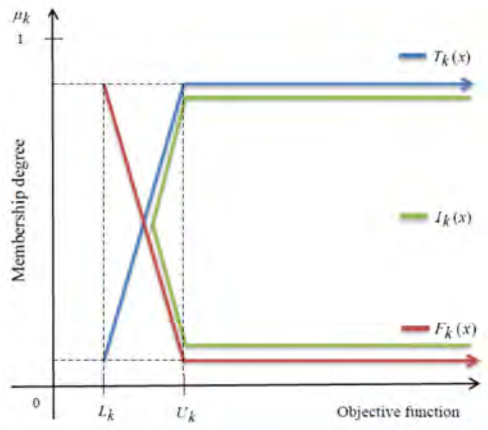

Figure 1. Diagrammatic representation of truth, indeterminacy, and falsity membership degree for the objective function.

Intuitionally, the aims are to maximize the truth and indeterminacy membership degrees of neutrosophic objectives and constraints, and minimize the falsity membership degree of neutrosophic objectives and constraints. The general formulation of the neutrosophic goal programming (NGP) model for multiobjective optimization problem (5) is represented as follows:

$$
\begin{aligned}
&\text{Minimize } Z = \sum_{k=1}^{K} w_{kT}.d_{kT}^- + \sum_{k=1}^{K} w_{kI}.d_{kI}^- + \sum_{k=1}^{K} w_{kF}.d_{kF}^+,\\
&\text{subject to}\\
&T_k(Z_k(x)) + d_{kT}^- - d_{kT}^+ \geq 1;\\
&I_k(Z_k(x)) + d_{kI}^- - d_{kI}^+ \geq 0.5;\\
&F_k(Z_k(x)) + d_{kF}^- - d_{kF}^+ \leq 0;\\
&T_k(Z_k(x)) \geq I_k(Z_k(x));\\
&T_k(Z_k(x)) \geq F_k(Z_k(x));\\
&F_k(Z_k(x)) \geq 0, \;\; d_{kT}^-.d_{kT}^+ = 0;\\
&d_{kI}^-.d_{kI}^+ = 0, \;\; d_{kF}^-.d_{kF}^+ = 0;\\
&g_j(x) \leq d_j, \;\; j = 1, 2, \cdots, m_1;\\
&g_j(x) \geq d_j, \;\; j = m_1 + 1, m_1 + 2, \cdots, m_2;\\
&g_j(x) = d_j, \;\; j = m_2 + 1, m_2 + 2, \cdots, m;\\
&x_i \geq 0, \qquad i = 1, 2, 3, \cdots, q; \;\; x_i \in X;\\
&d_{kT}^-, \; d_{kT}^+, \; d_{kI}^-, \; d_{kI}^+ \; d_{kF}^-, \; d_{kF}^+ \geq 0 \; \forall \, k,
\end{aligned}
\tag{13}
$$

where w_{kT}, w_{kI}, and w_{kF} are the weights assigned to deviations of the truth, indeterminacy, and falsity membership goals of each objective function, respectively. Now the assignment of corresponding weighting schemes of different weights can be obtained as follows:
$w_{kT} = \dfrac{1}{U_k^T - L_k^T}$, $w_{kI} = \dfrac{1}{U_k^I - L_k^I}$, and $w_{kF} = \dfrac{1}{U_k^F - L_k^F}$.

Hence, the optimum evaluation of multiobjective optimization problems by using the NGP approach is a very useful technique as it involves the degree of indeterminacy, which is independent and certainly ensures the achievement of marginal evaluation of each membership goal by reducing the deviational values under a neutrosophic environment.

3. Shale Gas Water Management System: Modeling and Optimization under Uncertainty

Shale gas is a rapidly emerging and unconventional source of energy found trapped in shale rocks. The extraction process at the wholesale level is very complicated. Since shales usually possess low permeability to permit significant fluid inflow to a well-bore, most shale wells are not adequate sources of natural gas for commercial production. Other sources of natural gases include coal bed methane, methane hydrates, and tight sandstones. Most commonly, the area in which shale gases are trapped are known as resource plays. Shale has comparatively low matrix permeability, which affects gas production at the commercial level and requires the fracturing process to supply permeability. In the past few decades, shale gas has been produced from shale rocks with natural fractures. Shale gas production seems to be booming in recent years due to the latest potential technology in hydraulic fracturing (fracking), which has led in the direction of pervasive artificial fractures around good bores. Horizontal drilling is often used in the shale gas extraction process. Lateral lengths up to 10,000 feet (3000 m) within shale wells are dug out to create maximum borehole surface area in contact with the shale. While injecting water with high pressure into shale rocks, chemicals are added to facilitate the underground fracturing process, which releases natural gas. The fracturing fluid is primarily water and contains approximately 0.5% chemical additives that are fully dissolved into the water. Depending on the size of the area, millions of liters of water are used for fracking, which signifies that thousands of liters of chemicals are injected into the subsurface.

The massive amount of contaminated surface water and groundwater with fracking fluids has emerged as a problematic issue. Generally, accrued shale gas is usually trapped several thousand feet below ground. Different challenging environmental concerns are often observed. For example, methane migration, improper treatment of produced wastewater, and lack of an underground injection disposal site. About 50% to 70% of the injected volume of contaminated water is generated after fracking, and sufficient storage capacity for wastewater management is required. The remaining volume of water remains in the subsurface. The hydraulic fracturing process leads to the perception that it can lead to the contamination of groundwater aquifers. However, foul odors and very toxic local water supply above-ground are also unavoidable truths about shale gas. Acid mine wastewater can be released into groundwater, but it might cause significant contamination of underground freshwater. Usually, the harmful impact and water pollution associated with wastewater and coal production can be reduced to a certain extent in shale gas production. Apart from using water and industrial chemicals, it may also be feasible to frack shale gas with only liquefied propane gas. This extraction option simultaneously reduces water and environmental degradation. It can be implemented in regions like Pennsylvania that have experienced a marginal increment in the freshwater requirement for energy production. More explicitly, shale gas development in the United States represents less than half a percent of total domestic freshwater consumption, although this quantity can reach as high as 25% in particularly arid regions.

Therefore, the proposed shale gas water management strategy has been designed to optimize the allocation of water requirement for different purposes. The designed water supply chain network configuration contains various components, such as the acquisition of freshwater and its transportation, on-site treatment with different technology, underground injection disposal sites, and treatment plants for wastewater with an option for expansion with a to and fro transportation network. Different potential objectives addressing the project planning strategy were also considered in this present study. A well-defined set of dynamic constraints were imposed to represent the modeling approach more realistically. The integrated water flow supply chain network within shale gas planning periods is shown in Figure 2. In Figure 2, the different echelons are presented to highlight the proposed shale gas water management design. The flow of freshwater initiates from different freshwater sources S and is then shipped to various shale sites I. After fracking processes, a possible part of the generated toxic water would be treated by on-site treatment technology O, and the remaining wastewater would be used to dispatch for further management to different treatment plants or disposal sites J, respectively. To enhance the treatment capacity of sewage treatment plants, an opportunity to adopt the different expansion options M was incorporated along with the associated capital investment. Hence, the treated wastewater can be reused for household purposes and in turn can yield significant revenues from the reuse of water. The whole integrated water cycle continues to flow over different time horizons T. Therefore, to assure the optimal flow of water among different echelons, the proposed water flow network captures the actual behavior of flow-back and produced water during shale gas extraction processes. The shale gas project planning model explicitly includes different indices' set, decision variables, and values of parameters shown in Table 1, which presents the significant characteristic features during the shale gas synthesis process.

The proposed shale gas water flow network configuration is based on the following assumptions:

1. There is no scope for the transportation of water using pipelines throughout the planning horizons.
2. The expansion options of underground injection disposal sites have not been considered due to the financial crisis or uneconomic aspects throughout the planning horizons.
3. The expansion of the treatment plant has been considered in order to avoid excess wastewater at the subsurface level of underground water during all the planning periods.
4. An absolute option of on-site treatment technology has been included that enables the reuse of wastewater within the shale sites throughout the planning horizons.
5. The restrictive margin was designed for the minimum and maximum capacity of wastewater treatment by using different on-site treatment technologies throughout the planning horizons.
6. The overall produced wastewater volume was successfully managed by the proposed system during all the planning horizons.

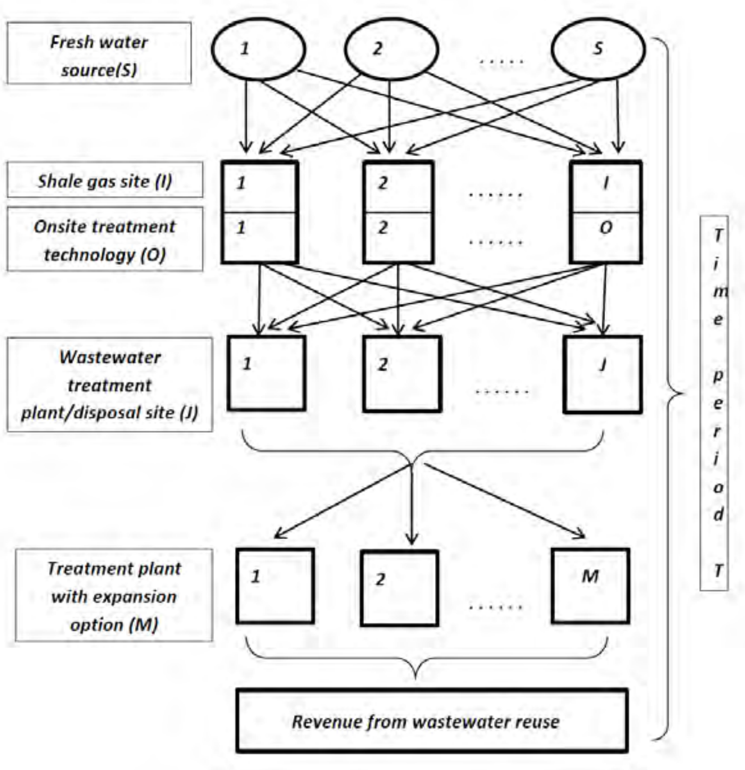

Figure 2. Representation of shale gas integrated water flow optimization network over time.

Table 1. Notions and descriptions.

Indices	Descriptions
i	Denotes the number of shale sites
j	Represents the number of disposal sites and treatment plants
m	Denotes the available options for the expansion capacity of the treatment plant
o	Denotes the on-site treatment technologies
t	Represents the time period
s	Denotes the source of freshwater
Decision variables	
$FW_{s,i,t}$	Amount of freshwater acquired from source s at shale site i in time period t
$WTO_{i,o,t}$	Amount of wastewater treated by on-site treatment technology o at shale site i in time period t
$WW_{i,j,t}$	Total amount of wastewater generated at shale site i and received by disposal site and treatment plant j in time period t
$WWD_{i,j,t}$	Amount of wastewater generated at shale site i and received by disposal site j in time period t
$WWT_{i,j,t}$	Amount of wastewater generated at shale site i and received by treatment plant j in time period t
$Y_{j,m,t}$	Binary variable representing the expansion capacity of the disposal site and treatment plant j by expansion option m in time period t
$YO_{i,o}$	Binary variable representing that on-site technology o is applied at shale site i
Parameters	
lo_o	Recovery factor for treating wastewater with on-site treatment technology o
$fdw_{i,t}$	Freshwater demand at shale site i in time period t
$fca_{s,t}$	Freshwater supply capacity at source s in time period t
rf_o	Ratio of freshwater to wastewater required for blending after treatment with on-site treatment technology o
$wwds_{j,t}$	Capacity for wastewater at disposal site j in time period t
$wwtp_{j,t}$	Capacity for wastewater at treatment plant j in time period t
$wdw_{j,t}$	Total wastewater capacity at disposal site and treatment plant j in time period t
$eo_{j,m,t}$	Represents increased treatment capacity of wastewater treatment plant j by using available expansion option m in time period t
$caq_{s,t}$	Denotes the unit acquisition cost of freshwater at source s in time period t
$ctf_{s,i,t}$	Denotes the unit transportation cost of freshwater from source s to shale site i in time period t
$ctw_{i,j,t}$	Denotes the unit transportation cost of wastewater from shale site i to disposal site and treatment plant j in time period t
$ctr_{j,t}$	Denotes the unit treatment cost of wastewater at treatment plant j in time period t
$cd_{j,t}$	Denotes the unit disposal cost of wastewater at disposal site j in time period t
$re_{j,t}$	Denotes the revenues from wastewater reuse from treatment plant j in time period t
$rr_{j,t}$	Denotes the reuse rate from wastewater treatment plant j in time period t
$cex_{j,m,t}$	Represents the investment cost of expanding the disposal site and treatment plant j by expansion option m in time period t
ocl_o	Denotes the minimum capacity for the on-site treatment of wastewater
ocu_o	Denotes the maximum capacity for the on-site treatment of wastewater

3.1. Objective Function

The first objective function is concerned with a different kind of cost incurred over the freshwater. It is quite a challenging task to collect the optimal amount of freshwater directly from natural freshwater sources; however, the option exists to acquire the freshwater from nearby the shale gas plant, which results in a lower acquisition cost. The transportation of freshwater is also required, which again appears as a transportation cost from source s to shale site i over period t, and both are of minimization type. Therefore, the cost function (14) related to freshwater can be furnished as follows:

$$\text{Minimize } Z_1 = \sum_{s=1}^{S}\sum_{i=1}^{I}\sum_{t=1}^{T}(caq_{s,t} + ctf_{s,i,t})FW_{s,i,t}. \tag{14}$$

The second objective mainly focuses on a different kinds of cost levied over the wastewater. It is crucial to manage the huge amount of contaminated or toxic wastewater released during the shale gas energy generation process. The produced amount of wastewater can be handled by either sending it to the treatment plants or by dumping into underground wastewater disposal sites. Both techniques are associated with some cost known as treatment and disposal facility costs, respectively. The transportation of wastewater from shale sites to different treatment plants and disposal sites results in additional transportation costs associated with the wastewater. The total revenues from the reuse of wastewater with some reuse rate are also associated with wastewater from shale site i to disposal and treatment plant j over period t. Therefore, the cost function (15) related to wastewater can be presented as follows:

$$\text{Minimize } Z_2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}(ctr_{j,t} + cd_{j,t} + ctw_{i,j,t} - rr_{j,t}.re_{j,t})WW_{i,j,t}. \tag{15}$$

The third objective function provides the facility of proliferation at treatment plants and underground disposal sites with some predetermined expansion option. The different expansion options require capital investment, which is to be minimized with binary variable taking value 1 if the expansion option m is adopted at treatment plant j over time period t; otherwise 0. Therefore the total capital investment (16) for the expansion of wastewater treatment plant capacity can be summarized as follows:

$$\text{Minimize } Z_3 = \sum_{j=1}^{J}\sum_{m=1}^{M}\sum_{t=1}^{T}(cex_{j,m,t}).Y_{j,m,t}. \tag{16}$$

3.2. Constraints

The constraint given by (17) is related to freshwater demand at shale sites:

At each shale site, a certain quantity of freshwater is required for the hydraulic fracturing process. The total amount of freshwater obtained from different sources is not sufficient to meet the demand at shale sites, but it is indispensable to build up the other sources or by developing other techniques to obtain the freshwater. Therefore on-site treatment technology with a recovery factor for treating wastewater is also an important option to fulfill the demand of such a tremendous amount of freshwater. Hence, the sum of the total amount freshwater acquired from different freshwater sources s and freshwater obtained from various on-site treatment technologies o with the recovery factor for treating wastewater must be greater than or equal to its total requirement at each shale site i over period t:

$$\sum_{s=1}^{S} FW_{s,i,t} + \sum_{o=1}^{O} lo_o * WTO_{i,o,t} \geq fdw_{i,t} \quad \forall\ i,t. \tag{17}$$

The constraint has given in (18) is related to the freshwater capacity at each source:

The total amount of freshwater obtained from different sources has some limitations in terms of storage capacity at different sources. The optimal stock of freshwater at different sources may differ marginally. It is necessary to ensure that the total amount of freshwater can be obtained without substantially affecting the storage capacity of each freshwater source. Therefore, the total amount of freshwater acquired from different sources s with the consumption at each shale site i must be less than or equal to its storage capacity at source s over period t:

$$\sum_{i=1}^{I} FW_{s,i,t} \leq fca_{s,t} \quad \forall\ s,t. \tag{18}$$

The constraint given in (19) is related to wastewater capacity at underground disposal sites:

The amount of wastewater generated after the fracturing process contains various toxic chemicals dissolved in it. A proper disposal system with its associated available capacity must be built to overcome fatal environmental effects. Therefore, it must be assured that the amount of wastewater received at different disposal sites can be fully tackled. Thus, the amount of wastewater released from shale site *i* and received at each disposal site *j* must be less than or equal to the presumable capacity of each disposal site *j* over period *t*:

$$\sum_{i=1}^{I} WWD_{i,j,t} \leq wwds_{j,t} \quad \forall \ j,t. \tag{19}$$

The constraint given in (20) is related to the wastewater capacity at each treatment plant with its prevalence:

The wastewater treatment facility leads to the option of reusing wastewater. The amount of wastewater liberated from different shale sites restrains the tremendous amount of harmful chemicals that must be treated at the water treatment plant to ensure its reuse for different household purposes. Thus, the amount of wastewater released from different shale sites *i* and dispatched to different treatment plants *j* must be less than or equal to the sum of the total capacity of each treatment plant with its several expansion options *m* over period *t*:

$$\sum_{i=1}^{I} WWT_{i,j,t} \leq wwtp_{j,t} + \sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} \quad \forall \ j,t. \tag{20}$$

The constraint given in (21) is related to the overall wastewater capacity at the treatment plant and disposal site:

The total amount of wastewater generated during the shale gas extraction process must be confronted with proper cautionary measures. The option of the treatment plant and disposal site for dealing with wastewater must be sufficient to conquer its harmful effects. Therefore it must be ensured that the total amount of wastewater generated from the hydraulic fracturing process at shale site *i* is less than or equal to its total capacity at the treatment plant and disposal site *j* over period *t*:

$$\sum_{i=1}^{I} WW_{i,j,t} \leq wdw_{j,t} \quad \forall \ j,t. \tag{21}$$

The constraint given in (22) is related to different wastewater capacities at the treatment plant and disposal site:

This constraint ensures that regardless of what the excess amount of wastewater released from shale sites is, it must be fully managed by expanding the treatment plant capacity. Therefore, the different treatment plants have a potential storage capacity increment option within the investment costs. Thus, the sum of total wastewater capacity enhanced by expanding treatment plant *j* with expansion option *m*, the total capacity of underground disposal and treatment plant *j* must be less than or equal to the assorted capacity of disposal site and treatment plant *j* over time period *t*:

$$\sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} + wwds_{j,t} + wwtp_{j,t} \leq wdw_{j,t} \quad \forall \ i,j,t. \tag{22}$$

The constraint given in (23) is related to the different wastewater capacities at the treatment plant and disposal site:

The necessity and utilization of a huge amount of freshwater in the whole process of shale gas extraction requires thought regarding its acquisition. Various techniques are used to recycle freshwater. Therefore, one of the most trending techniques is on-site treatment with different technologies. Thus, the reuse specification for hydraulic fracturing with the blending ratio of freshwater to wastewater after the treatment of on-site treatment technology o must be less than or equal to the total amount of freshwater acquired at source s transported to shale site i over period t:

$$\sum_{o=1}^{O} rf_o.lo_o.WTO_{i,o,t} \leq FW_{s,i,t} \quad \forall \ s,i,t. \tag{23}$$

The constraint given in (24) is related to the minimum capacity of the ton-site treatment of wastewater:

This restriction was imposed with the fact that a minimal amount of freshwater must be obtained by using on-site treatment technology. The capital investment towards the setup of on-site treatment plant steers the utilization of on-site treatment technology. Thus, the minimum capacity of on-site wastewater treatment with technology o along with the binary variable taking value one if the certain technology is used (and otherwise 0) at shale site i must be less than or equal to the amount of wastewater treated by on-site treatment technology o over period t:

$$\sum_{o=1}^{O} ocl_o.YO_{i,o} \leq WTO_{i,o,t} \quad \forall \ i,t. \tag{24}$$

The constraint given in (25) is related to the maximum capacity of on-site wastewater treatment:

This restriction ensures that the maximal amount of freshwater is acquired by using on-site treatment technology. The upper limit for the on-site treatment of wastewater restricts the excessive holding of wastewater at the on-site treatment plant. Thus, this constraint provides the surety that the minimum capacity of on-site treatment of wastewater with technology o along with the binary variable taking value 1 if the certain technology is used (otherwise 0) at shale site i is greater than or equal to the amount of wastewater treated by on-site treatment technology o over time period t:

$$\sum_{o=1}^{O} ocu_o.YO_{i,o} \geq WTO_{i,o,t} \quad \forall \ i,t. \tag{25}$$

The constraint given in (26) is related to the total wastewater produced during the shale gas extraction process:

It must be ensured that the total amount of wastewater generated during the fracking procedures is strictly equal to the sum of different amounts of wastewater distributed to the treatment plant and disposal site. Therefore, the sum of the total amount of wastewater at the treatment plant and disposal site j dispatched from shale site i must be equal to the assorted wastewater capacity at disposal site and treatment plant j over time period t:

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWD_{i,j,t} + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWT_{i,j,t} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WW_{i,j,t} \quad \forall \ i,j,t. \tag{26}$$

The proposed multiobjective shale gas optimization model under uncertainty is presented in model $\mathbf{M}_1$ with the fact that parameter values inherently contain vagueness and ambiguousness in the real-life decision-making process. The decision maker(s) or policy maker(s) is(are) not very sure about the exact parameter values due to a lack of proper information, relatively little experience, environmental issues, and other humanitarian logical perception. To overcome these issues, the settings are taken as the triangular intuitionistic fuzzy number and are more elaborately discussed in Section 3.3.

$$
\begin{aligned}
\mathbf{M}_1 : \textit{Minimize } Z_1 &= \sum_{s=1}^{S}\sum_{i=1}^{I}\sum_{t=1}^{T}\{c\tilde{a}q_{s,t} + c\tilde{t}f_{s,i,t}\}FW_{s,i,t} \\
\textit{Minimize } Z_2 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\{c\tilde{t}r_{j,t} + c\tilde{d}_{j,t} + c\tilde{t}w_{i,j,t} - rr_{j,t}.re_{j,t}\}WW_{i,j,t} \\
\textit{Minimize } Z_3 &= \sum_{j=1}^{J}\sum_{m=1}^{M}\sum_{t=1}^{T}\{c\tilde{e}x_{j,m,t}\}.Y_{j,m,t}
\end{aligned}
$$

subject to:

$$
\begin{aligned}
&\sum_{s=1}^{S} FW_{s,i,t} + \sum_{o=1}^{O} lo_o.WTO_{i,o,t} \geq f\tilde{d}w_{i,t} \quad \forall\ i,t \\
&\sum_{i=1}^{I} FW_{s,i,t} \leq f\tilde{c}a_{s,t} \quad \forall\ s,t \\
&\sum_{i=1}^{I} WWD_{i,j,t} \leq w\tilde{w}ds_{j,t} \quad \forall\ j,t \\
&\sum_{i=1}^{I} WWT_{i,j,t} \leq w\tilde{w}tp_{j,t} + \sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} \quad \forall\ j,t \\
&\sum_{i=1}^{I} WW_{i,j,t} \leq w\tilde{d}w_{j,t} \quad \forall\ j,t \\
&\sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} + w\tilde{w}ds_{j,t} + w\tilde{w}tp_{j,t} \leq w\tilde{d}w_{j,t} \quad \forall\ i,j,t \\
&\sum_{o=1}^{O} rf_o.lo_o.WTO_{i,o,t} \leq FW_{s,i,t} \quad \forall\ s,i,t \\
&\sum_{o=1}^{O} ocl_o.YO_{i,o} \leq WTO_{i,o,t} \quad \forall\ i,t \\
&\sum_{o=1}^{O} ocu_o.YO_{i,o} \geq WTO_{i,o,t} \quad \forall\ i,t \\
&\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWD_{i,j,t} + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWT_{i,j,t} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WW_{i,j,t} \quad \forall\ i,j,t \\
&FW_{s,i,t} \geq 0,\ \ WW_{i,j,t} \geq 0 \quad \forall\ s,i,j,t \\
&WTO_{i,o,t} \geq 0,\ \ WWD_{i,j,t} \geq 0,\ \ WWT_{i,j,t} \geq 0 \quad \forall\ i,o,j \textit{ and } t \\
&0 \leq Y_{j,m,t}\ \ , YO_{i,o} \leq 1,\quad Y_{j,m,t} \textit{ and } YO_{i,o} = \textit{integer},\quad \forall\ i,o,j,m \textit{ and } t,
\end{aligned}
$$

where the notation ($\tilde{\ }$) over different parameters represents the triangular intuitionistic fuzzy number for the indices' whole set.

3.3. Intuitionistic Fuzzy Parameters

The proposed multiobjective shale gas water management optimization model discussed in Section 3 inherently involves uncertainty or impreciseness. The existence of ambiguity among parameters makes it uncertain. It is not always feasible for decision maker(s) or project manager(s) to assign crisp/exact parameter values. Actual perceptions behind the uncertainty involve a lack of proper information, environmental conditions, the condition of roads, natural calamities, abrupt changes in the prices of fuel, different routes of transportation, shortages of freshwater on sunny days, etc. In such cases, only some vague and inconsistent pieces of information are available regarding the parameter values. Therefore, uncertainty can take different forms, such as fuzzy numbers, stochastic random variables, and other forms of change. Based on this confluent information, one may assume imprecise parameters and easily overcome uncertainty by applying the different techniques to obtain the best estimates of the parameters. In brief, we may distinguish between stochastic and fuzzy methods while dealing with the uncertain dataset. The uncertainty involved in the data due to randomness can be handled with a stochastic programming approach while it can be dealt with using fuzzy techniques due to vagueness or ambiguousness. In the present study, all the parameters were assumed to be triangular intuitionistic fuzzy numbers, which is more realistic as compared to fuzzy numbers as it simultaneously reveals both the degree of belongingness and the degree of non-belongingness. The defuzzification/ranking method of triangular intuitionistic fuzzy parameters is based on the expected interval and expected values of a lower and upper member of the set. Imprecise parameters involved in the different objective functions were converted to their crisp forms by using expected values, whereas uncertain parameters present in constraints were transformed into their deterministic forms using expected intervals. All the pieces of information regarding triangular intuitionistic fuzzy settings used in THE shale gas optimization model are summarized in Table 2.

Table 2. Information regarding the triangular intuitionistic fuzzy parameters of the shale gas model.

Intuitionistic Fuzzy Parameters	Triangular Intuitionistic Fuzzy Number	$EI(.) = [E_1(.), E_2(.)]$	$EV(.)$
$c\tilde{a}q_{s,t}$	$\{(caq_{s,t}^{(1)}, caq_{s,t}, caq_{s,t}^{(2)}); (caq_{s,t}^{(3)}, caq_{s,t}, caq_{s,t}^{(4)})\}$	$\{\frac{3.caq + caq^{(3)} + (caq - caq^{(3)})\nu_{c\tilde{a}q} - (caq - caq^{(1)})\mu_{c\tilde{a}q}}{4}, \frac{3.caq + caq^{(4)} + (caq^{(2)} - caq)\mu_{c\tilde{a}q} + (caq - caq^{(4)})\nu_{c\tilde{a}q}}{2}\}$	$\frac{E_1(c\tilde{a}q) + E_2(c\tilde{a}q)}{2}$
$c\tilde{t}f_{s,i,t}$	$\{(ctf_{s,i,t}^{(1)}, ctf_{s,i,t}, ctf_{s,i,t}^{(2)}); (ctf_{s,i,t}^{(3)}, ctf_{s,i,t}, ctf_{s,i,t}^{(4)})\}$	$\{\frac{3.ctf + ctf^{(3)} + (ctf - ctf^{(3)})\nu_{c\tilde{t}f} - (ctf - ctf^{(1)})\mu_{c\tilde{t}f}}{4}, \frac{3.ctf + ctf^{(4)} + (ctf^{(2)} - ctf)\mu_{c\tilde{t}f} + (ctf - ctf^{(4)})\nu_{c\tilde{t}f}}{2}\}$	$\frac{E_1(c\tilde{t}f) + E_2(c\tilde{t}f)}{2}$
$c\tilde{t}r_{j,t}$	$\{(ctr_{j,t}^{(1)}, ctr_{j,t}, ctr_{j,t}^{(2)}); (ctr_{j,t}^{(3)}, ctr_{j,t}, ctr_{j,t}^{(4)})\}$	$\{\frac{3.ctr + ctr^{(3)} + (ctr - ctr^{(3)})\nu_{c\tilde{t}r} - (ctr - ctr^{(1)})\mu_{c\tilde{t}r}}{4}, \frac{3.ctr + ctr^{(4)} + (ctr^{(2)} - ctr)\mu_{c\tilde{t}r} + (ctr - ctr^{(4)})\nu_{c\tilde{t}r}}{2}\}$	$\frac{E_1(c\tilde{t}r) + E_2(c\tilde{t}r)}{2}$
$c\tilde{t}w_{i,j,t}$	$\{(ctw_{i,j,t}^{(1)}, ctw_{i,j,t}, ctw_{i,j,t}^{(2)}); (ctw_{i,j,t}^{(3)}, ctw_{i,j,t}, ctw_{i,j,t}^{(4)})\}$	$\{\frac{3.ctw + ctw^{(3)} + (ctw - ctw^{(3)})\nu_{c\tilde{t}w} - (ctw - ctw^{(1)})\mu_{c\tilde{t}w}}{4}, \frac{3.ctw + ctw^{(4)} + (ctw^{(2)} - ctw)\mu_{c\tilde{t}w} + (ctw - ctw^{(4)})\nu_{c\tilde{t}w}}{2}\}$	$\frac{E_1(c\tilde{t}w) + E_2(c\tilde{t}w)}{2}$
$\tilde{cd}_{j,t}$	$\{(cd_{j,t}^{(1)}, cd_{j,t}, cd_{j,t}^{(2)}); (cd_{j,t}^{(3)}, cd_{j,t}, cd_{j,t}^{(4)})\}$	$\{\frac{3.cd + cd^{(3)} + (cd - cd^{(3)})\nu_{\tilde{cd}} - (cd - cd^{(1)})\mu_{\tilde{cd}}}{4}, \frac{3.cd + cd^{(4)} + (cd^{(2)} - cd)\mu_{\tilde{cd}} + (cd - cd^{(4)})\nu_{\tilde{cd}}}{2}\}$	$\frac{E_1(\tilde{cd}) + E_2(\tilde{cd})}{2}$
$c\tilde{e}x_{j,m,t}$	$\{(cex_{j,m,t}^{(1)}, cex_{j,m,t}, cex_{j,m,t}^{(2)}); (cex_{j,m,t}^{(3)}, cex_{j,m,t}, cex_{j,m,t}^{(4)})\}$	$\{\frac{3.cex + cex^{(3)} + (cex - cex^{(3)})\nu_{c\tilde{e}x} - (cex - cex^{(1)})\mu_{c\tilde{e}x}}{4}, \frac{3.cex + cex^{(4)} + (cex^{(2)} - cex)\mu_{c\tilde{e}x} + (cex - cex^{(4)})\nu_{c\tilde{e}x}}{2}\}$	$\frac{E_1(c\tilde{e}x) + E_2(c\tilde{e}x)}{2}$
$f\tilde{d}w_{i,t}$	$\{(fdw_{i,t}^{(1)}, fdw_{i,t}, fdw_{i,t}^{(2)}); (fdw_{i,t}^{(3)}, fdw_{i,t}, fdw_{i,t}^{(4)})\}$	$\{\frac{3.fdw + fdw^{(3)} + (fdw - fdw^{(3)})\nu_{f\tilde{d}w} - (fdw - fdw^{(1)})\mu_{f\tilde{d}w}}{4}, \frac{3.fdw + fdw^{(4)} + (fdw^{(2)} - fdw)\mu_{f\tilde{d}w} + (fdw - fdw^{(4)})\nu_{f\tilde{d}w}}{2}\}$	$\frac{E_1(f\tilde{d}w) + E_2(f\tilde{d}w)}{2}$
$f\tilde{c}a_{s,t}$	$\{(fca_{s,t}^{(1)}, fca_{s,t}, fca_{s,t}^{(2)}); (fca_{s,t}^{(3)}, fca_{s,t}, fca_{s,t}^{(4)})\}$	$\{\frac{3.fca + fca^{(3)} + (fca - fca^{(3)})\nu_{f\tilde{c}a} - (fca - fca^{(1)})\mu_{f\tilde{c}a}}{4}, \frac{3.fca + fca^{(4)} + (fca^{(2)} - fca)\mu_{f\tilde{c}a} + (fca - fca^{(4)})\nu_{f\tilde{c}a}}{2}\}$	$\frac{E_1(f\tilde{c}a) + E_2(f\tilde{c}a)}{2}$
$w\tilde{w}ds_{j,t}$	$\{(wwds_{j,t}^{(1)}, wwds_{j,t}, wwds_{j,t}^{(2)}); (wwds_{j,t}^{(3)}, wwds_{j,t}, wwds_{j,t}^{(4)})\}$	$\{\frac{3.wwds + wwds^{(3)} + (wwds - wwds^{(3)})\nu_{w\tilde{w}ds} - (wwds - wwds^{(1)})\mu_{w\tilde{w}ds}}{4}, \frac{3.wwds + wwds^{(4)} + (wwds^{(2)} - wwds)\mu_{w\tilde{w}ds} + (wwds - wwds^{(4)})\nu_{w\tilde{w}ds}}{2}\}$	$\frac{E_1(w\tilde{w}ds) + E_2(w\tilde{w}ds)}{2}$
$w\tilde{w}tp_{j,t}$	$\{(wwtp_{j,t}^{(1)}, wwtp_{j,t}, wwtp_{j,t}^{(2)}); (wwtp_{j,t}^{(3)}, wwtp_{j,t}, wwtp_{j,t}^{(4)})\}$	$\{\frac{3.wwtp + wwtp^{(3)} + (wwtp - wwtp^{(3)})\nu_{w\tilde{w}tp} - (wwtp - wwtp^{(1)})\mu_{w\tilde{w}tp}}{4}, \frac{3.wwtp + wwtp^{(4)} + (wwtp^{(2)} - wwtp)\mu_{w\tilde{w}tp} + (wwtp - wwtp^{(4)})\nu_{w\tilde{w}tp}}{2}\}$	$\frac{E_1(w\tilde{w}tp) + E_2(w\tilde{w}tp)}{2}$
$w\tilde{d}w_{j,t}$	$\{(wdw_{j,t}^{(1)}, wdw_{j,t}, wdw_{j,t}^{(2)}); (wdw_{j,t}^{(3)}, wdw_{j,t}, wdw_{j,t}^{(4)})\}$	$\{\frac{3.wdw + wdw^{(3)} + (wdw - wdw^{(3)})\nu_{w\tilde{d}w} - (wdw - wdw^{(1)})\mu_{w\tilde{d}w}}{4}, \frac{3.wdw + wdw^{(4)} + (wdw^{(2)} - wdw)\mu_{w\tilde{d}w} + (wdw - wdw^{(4)})\nu_{w\tilde{d}w}}{2}\}$	$\frac{E_1(w\tilde{d}w) + E_2(w\tilde{d}w)}{2}$

Therefore, the crisp/deterministic version of the proposed multiobjective shale gas water management system optimization model $\mathbf{M}_1$ based on the different crisp values of the parameters can be represented in the model $\mathbf{M}_2$ as follows:

$$\mathbf{M}_2 : Minimize\ Z_1 = \sum_{s=1}^{S}\sum_{i=1}^{I}\sum_{t=1}^{T}\{EV(c\tilde{aq}_{s,t}) + EV(c\tilde{tf}_{s,i,t})\}FW_{s,i,t},$$

$$Minimize\ Z_2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T}\{EV(c\tilde{tr}_{j,t}) + EV(c\tilde{d}_{j,t}) + EV(c\tilde{fw}_{i,j,t}) - rr_{j,t}.re_{j,t}\}WW_{i,j,t},$$

$$Minimize\ Z_3 = \sum_{j=1}^{J}\sum_{m=1}^{M}\sum_{t=1}^{T}\{EV(c\tilde{ex}_{j,m,t})\}Y_{j,m,t},$$

subject to:

$$\sum_{s=1}^{S} FW_{s,i,t} + \sum_{o=1}^{O} lo_o.WTO_{i,o,t} \geq E_1^{fdw_{i,t}} \quad \forall\ i,t$$

$$\sum_{i=1}^{I} FW_{s,i,t} \leq E_2^{fca_{s,t}} \quad \forall\ s,t$$

$$\sum_{i=1}^{I} WWD_{i,j,t} \leq E_2^{wwds_{j,t}} \quad \forall\ j,t$$

$$\sum_{i=1}^{I} WWT_{i,j,t} \leq E_2^{wwtp_{j,t}} + \sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} \quad \forall\ j,t$$

$$\sum_{i=1}^{I} WW_{i,j,t} \leq E_2^{wdw_{j,t}} \quad \forall\ j,t$$

$$\sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} + E_1^{wwds_{j,t}} + E_1^{wwtp_{j,t}} \leq E_2^{wdw_{j,t}} \quad \forall\ j,t$$

$$\sum_{o=1}^{O} rf_o.lo_o.WTO_{i,o,t} \leq FW_{s,i,t} \quad \forall\ s,i,t$$

$$\sum_{o=1}^{O} ocl_o.YO_{i,o} \leq WTO_{i,o,t} \quad \forall\ i,t$$

$$\sum_{o=1}^{O} ocu_o.YO_{i,o} \geq WTO_{i,o,t} \quad \forall\ i,t$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWD_{i,j,t} + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWT_{i,j,t} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WW_{i,j,t} \quad \forall\ i,j,t$$

$$FW_{s,i,t} \geq 0,\ \ WW_{i,j,t} \geq 0 \quad \forall\ s,i,j,t$$

$$WTO_{i,o,t} \geq 0,\ \ WWD_{i,j,t} \geq 0,\ \ WWT_{i,j,t} \geq 0 \quad \forall\ i,o,j\ and\ t$$

$$0 \leq Y_{j,m,t}\ ,YO_{i,o} \leq 1,\ \ Y_{j,m,t}\ and\ YO_{i,o} = integer, \quad \forall\ i,o,j,m\ and\ t,$$

where $EV(.)$, $E_1^{(.)}$, and $E_2^{(.)}$ are the expected value and lower and upper intervals of triangular intuitionistic fuzzy numbers for the entire indices' set, respectively.

The discussed solution technique (i.e., the neutrosophic goal programming approach (NGPA)) is based on the neutrosophic decision set, which ensures the efficient implementation of the independent neutral thoughts of the decision maker(s). The obtained crisp model $\mathbf{M}_2$ can be transformed into $\mathbf{M}_3$ to achieve the globally optimal solution of the proposed multiobjective shale gas water management system optimization model.

$$\begin{aligned}
\mathbf{M}_3: &\text{Minimize } Z = (w_{1T}.d_{1T}^- + w_{2T}.d_{2T}^- + w_{3T}.d_{3T}^-) + (w_{1I}.d_{1I}^- + w_{2I}.d_{2I}^- + w_{3I}.d_{3I}^-) \\
&+ (w_{1F}.d_{1F}^+ + w_{2F}.d_{2F}^+ + w_{3F}.d_{3F}^+) \\
&\text{subject to} \\
&\frac{U_1^T - \sum_{s=1}^S \sum_{i=1}^I \sum_{t=1}^T \{EV(c\tilde{a}q_{s,t}) + EV(c\tilde{t}f_{s,i,t})\}FW_{s,i,t}}{U_1^T - L_1^T} + d_{1T}^- - d_{1T}^+ = 1 \\
&\frac{U_1^I - \sum_{s=1}^S \sum_{i=1}^I \sum_{t=1}^T \{EV(c\tilde{a}q_{s,t}) + EV(c\tilde{t}f_{s,i,t})\}FW_{s,i,t}}{U_1^I - L_1^I} + d_{1I}^- - d_{1I}^+ = 0.5 \\
&\frac{\sum_{s=1}^S \sum_{i=1}^I \sum_{t=1}^T \{EV(c\tilde{a}q_{s,t}) + EV(c\tilde{t}f_{s,i,t})\}FW_{s,i,t} - L_1^F}{U_1^F - L_1^F} + d_{1F}^- - d_{1F}^+ = 0 \\
&\frac{U_2^T - \sum_{i=1}^I \sum_{j=1}^J \sum_{t=1}^T \{EV(c\tilde{t}r_{j,t}) + EV(c\tilde{d}_{j,t}) + EV(c\tilde{t}w_{i,j,t}) - rr_{j,t}.re_{j,t}\}WW_{i,j,t}}{U_2^T - L_2^T} + d_{2T}^- - d_{2T}^+ = 1 \\
&\frac{U_2^I - \sum_{i=1}^I \sum_{j=1}^J \sum_{t=1}^T \{EV(c\tilde{t}r_{j,t}) + EV(c\tilde{d}_{j,t}) + EV(c\tilde{t}w_{i,j,t}) - rr_{j,t}.re_{j,t}\}WW_{i,j,t}}{U_2^I - L_2^I} + d_{2I}^- - d_{2I}^+ = 0.5 \\
&\frac{\sum_{i=1}^I \sum_{j=1}^J \sum_{t=1}^T \{EV(c\tilde{t}r_{j,t}) + EV(c\tilde{d}_{j,t}) + EV(c\tilde{t}w_{i,j,t}) - rr_{j,t}.re_{j,t}\}WW_{i,j,t} - L_2^F}{U_2^F - L_2^F} + d_{2F}^- - d_{2F}^+ = 0 \\
&\frac{U_3^T - \sum_{j=1}^J \sum_{m=1}^M \sum_{t=1}^T \{EV(c\tilde{e}x_{j,m,t})\}Y_{j,m,t}}{U_3^T - L_3^T} + d_{3T}^- - d_{3T}^+ = 1 \\
&\frac{U_3^I - \sum_{j=1}^J \sum_{m=1}^M \sum_{t=1}^T \{EV(c\tilde{e}x_{j,m,t})\}Y_{j,m,t}}{U_3^I - L_3^I} + d_{3I}^- - d_{3I}^+ = 0.5 \\
&\frac{\sum_{j=1}^J \sum_{m=1}^M \sum_{t=1}^T \{EV(c\tilde{e}x_{j,m,t})\}Y_{j,m,t} - L_3^F}{U_3^F - L_3^F} + d_{3F}^- - d_{3F}^+ = 0 \\
&\sum_{s=1}^{S} FW_{s,i,t} + \sum_{o=1}^{O} lo_o.WTO_{i,o,t} \geq E_1^{fdw_{i,t}} \quad \forall\ i,t \\
&\sum_{i=1}^{I} FW_{s,i,t} \leq E_2^{fca_{s,t}} \quad \forall\ s,t \\
&\sum_{i=1}^{I} WWD_{i,j,t} \leq E_2^{wwds_{j,t}} \quad \forall\ j,t \\
&\sum_{i=1}^{I} WWT_{i,j,t} \leq E_2^{wwtp_{j,t}} + \sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} \quad \forall\ j,t \\
&\sum_{i=1}^{I} WW_{i,j,t} \leq E_2^{wdw_{j,t}} \quad \forall\ j,t \\
&\sum_{m=1}^{M} eo_{j,m,t}.Y_{j,m,t} + E_1^{wwds_{j,t}} + E_1^{wwtp_{j,t}} \leq E_2^{wdw_{j,t}} \quad \forall\ j,t \\
&\sum_{o=1}^{O} rf_o.lo_o.WTO_{i,o,t} \leq FW_{s,i,t} \quad \forall\ s,i,t \\
&\sum_{o=1}^{O} ocl_o.YO_{i,o} \leq WTO_{i,o,t} \quad \forall\ i,t \\
&\sum_{o=1}^{O} ocu_o.YO_{i,o} \geq WTO_{i,o,t} \quad \forall\ i,t \\
&\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWD_{i,j,t} + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WWT_{i,j,t} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{t=1}^{T} WW_{i,j,t} \quad \forall\ i,j,t
\end{aligned}$$

$$FW_{s,i,t} \geq 0, \;\; WW_{i,j,t} \geq 0 \quad \forall \; s, i, j, t$$
$$WTO_{i,o,t} \geq 0, \;\; WWD_{i,j,t} \geq 0, \;\; WWT_{i,j,t} \geq 0 \quad \forall \; i, o, j \; and \; t$$
$$0 \leq Y_{j,m,t} \;\; , YO_{i,o} \leq 1, \quad Y_{j,m,t} \; and \; YO_{i,o} = integer, \quad \forall \; i, o, j, m \; and \; t$$
$$d_{kT}^{-}.d_{kT}^{+} = 0, \quad d_{kI}^{-}.d_{kI}^{+} = 0, \quad d_{kF}^{-}.d_{kF}^{+} = 0,$$

where $w_{1T}, w_{1I}, w_{1F}, w_{2T}, w_{2I}, w_{2F}, w_{3T}, w_{3I}$, and w_{3F} are the parameter weights assigned to different deviational variables of the neutrosophic membership goals.

3.4. Solution Algorithm

To reformulate the shale gas water management optimization model into the neutrosophic goal programming model, one needs to solve each objective function individually and has to determine the maximum and minimum values of each objective. With the help of these values, the upper and lower bounds for each membership function under a neutrosophic environment were obtained. Then, the truth, indeterminacy, and falsity membership functions for each objective were constructed. The transformation of membership functions into membership goals can be done by using the different deviational variables. The weighting scheme of each aim was designed based on the difference between the best and worst values of the respective objective function. The developed framework for the optimal shale gas water management computational model was transmuted under a neutrosophic environment. The stepwise solution procedures for the proposed neutrosophic goal programming approach can be summarized as follows:

Step 1. Design the proposed multiobjective shale gas water management optimization model as given in $\mathbf{M}_1$.

Step 2. Convert each intuitionistic fuzzy parameter involved in model $\mathbf{M}_1$ into its crisp form by using the expected interval and values method as given in Equations (2)–(4) or presented in Table 2.

Step 3. Modify model $\mathbf{M}_1$ into $\mathbf{M}_2$ and solve model $\mathbf{M}_2$ for each objective function individually in order to obtain the best and worst solutions.

Step 4. Determine the upper and lower bounds for each objective function by using Equation (6). Using U_k and L_k, define the upper and lower bounds for truth, indeterminacy, and falsity membership as given in Equations (7)–(9).

Step 5. Transform the truth, indeterminacy, and falsity membership degrees into their respective membership goals and deviational variables as defined in Equations (10)–(12).

Step 6. Formulate the neutrosophic goal programming model defined in $\mathbf{M}_3$ and solve the multiobjective shale gas water management optimization model in order to obtain the compromise solution using suitable techniques or some optimization software packages.

4. A Computational Study

The integrated framework representative of the multiobjective shale gas water management optimization model is presented based on a real-life scenario, hypothetical proposition, data, information, and a quick review of the published research (Lutz et al. [12], Rahm and Riha [54], Rahm et al. [55], Zhang et al. [22], Alawattegama [56]). The unified optimal shale gas water planning model was structured to manifest the real-life scenario in the current and future characteristic features of shale gas extraction processes. The proposed model includes the optimal acquisition of freshwater, on-site treatment of wastewater, expansion of treatment plant facility, underground injection disposal site, treatment plant facility, and primary socio-economic concerns and environmental issues with the technical and potential aspects in major shale gas plays in the United States. The acquisition of freshwater from different sources and the inventory holding of freshwater to a certain level for the smooth operation of the shale gas extraction processes is quite a challenging task. Therefore, the acquisition of freshwater is allowed some predetermined budget allocation at the different freshwater sources. The flow-back-produced water from shale play is a matter of grave concern.

The privilege of an on-site wastewater treatment facility for reuse purposes at a moderate scale is also feasible and laid down as a base of future technologies. Various burning socio-environmental issues are being raised against the contaminated wastewater generated from shale wells after fracturing processes at the national and international political levels. To overcome these issues, the orientation of wastewater underground disposal sites and treatment facilities with their expansion options have been taken under consideration. There is no scope for pipelines to any extent throughout the shale gas extraction process. All sorts of to and fro flow of freshwater and wastewater have been depicted with roadways. The planning periods are designed in such a way that shale gas production turnover results in an economically profitable scenario.

In this study, the shale gas water management system optimization model comprises one freshwater source, five shale sites with one drilled well at each shale site, and three on-site wastewater treatment facilities. The toxic wastewater management system includes one wastewater underground injection disposal site, two wastewater treatment plants with three expansion options for each treatment plant facility over three planning periods of 5 years each which are capable of representing the whole shale gas production process more realisticallyy. All the summarized parameters were assumed to be a triangular intuitionistic fuzzy number, and their defuzzified version can be obtained from Table 2. The acquisition costs (in $/bbl) of freshwater at source and transportation cost (in $/bbl) of freshwater by road over three planning periods are presented in Table 3. The various costs incurred on account of wastewater, such as transportation cost (in $/bbl) from different shale sites to disposal site and treatment plants, underground injection disposal cost (in $/bbl), and wastewater operational cost at different treatment plants (in $/bbl) over three planning periods are summarized in Table 4. The capital investment costs(in $/bbl) for alternative options for the expansion of treatment plant capacity with the respective enhanced potential volume (in bbl/day) over three planning periods are presented in Table 4. The crisp parameters which include revenues/profits from the reuse of wastewater (in $), reuse rate (in bbl/day), recovery factor for treating wastewater with different treatment technology, and the required ratio of freshwater to sewer for blending after on-site treatment technology, along with the minimum and maximum capacities for on-site treatment with conflicting technology over three time periods are summarized in Table 5. The different restrictive intuitionistic fuzzy parameters (for freshwater and wastewater) were introduced for the optimal allocation of freshwater and wastewater according to their speculated destination. The freshwater acquisition capacity at the source, the requirement of freshwater at different shale sites, the underground wastewater disposal capacity, the wastewater treatment plant capacity, and the overall generated wastewater permitted for managerial purposes are summarized in Table 5. Throughout the project planning scheme, the decision maker(s) or project manager(s) intend to adopt the certainly feasible strategy that ensures the optimal allocation of freshwater and wastewater to their predetermined consumption points. However, during the whole planning periods, the decision maker(s) are confronted with the different multiple conflicting objectives which are to be optimized in order to achieve the global benefits from the production of shale gas energy as well as its commercial distribution. Hence, the proposed multiobjective shale gas water management optimization model experiments with these hypothetical datasets and was applied to tackle the project planning scheme.

Table 3. Acquisition and transportation costs of freshwater ($/bbl).

Freshwater Acquisition Cost at Source $(\tilde{caq})$	**Time Period** $t = 1$	$t = 2$	$t = 3$
Source	**(1.9,2.1,2.3;1.8,2.1,2.4)**	**(1.6,1.8,2;1.5,1.8,2.1)**	**(0.9,1.2,1.5;0.8,1.2,1.6)**
Transportation costs of freshwater from source to shale site $(\tilde{ctf})$			
Source to shale site 1	(1.2,1.4,1.6;1.1,1.4,1.7)	(4.2,4.4,4.6;4.1,4.4,4.7)	(4.1,4.3,4.5;4.0,4.3,4.6)
Source to shale site 2	(2.1,2.3,2.5;1.9,2.3,2.7)	(3.2,3.4,3.6;3.1,3.4,3.7)	(3.2,3.4,3.6;3.0,3.4,3.8)
Source to shale site 3	(3.4,3.6,3.8;3.2,3.6,4.0)	(2.2,2.4.2.6;2.1,2.4,2.7)	(2.2,2.4,2.6;2.0,2.4,2.8)
Source to shale site 4	(2.2,2.4,2.6;2.1,2.4,2.7)	(1.5,1.8,1.9;1.4,1.8,2.1)	(1.5,1.7,1.9;1.4,1.7,2.0)
Source to shale site 5	(1.4,1.6,1.8;1.2,1.6,2)	(1.8,2,2.2;1.8,2,2.2)	(2.6,2.8,3.0;2.5,2.8,3.1)

Table 4. Different costs related to wastewater and capital investment for treatment plant expansions ($/bbl).

Transportation Cost From Shale Site to Facility ($\tilde{ctw}$)		**Time Period**		
Source	**Treatment and Disposal Facility**	**$t = 1$**	**$t = 2$**	**$t = 3$**
Shale site 1	Disposal site	(1.4,2.4,3.4)	(2,3,4)	(3.4,3.6,3.8)
Shale site 1	Treatment plant 1	(3.0,3.2,3.4)	(3.4,3.6,3.8)	(3.8,4.0,4.2)
Shale site 1	Treatment plant 2	(5.2,5.6,6.0)	(6.0,6.3,6.6)	(6.6,6.7,6.8)
Shale site 2	Disposal site	(6.0,6.5,7.0)	(6.6,6.9,7.2)	(7.1,7.4,7.7)
Shale site 2	Treatment plant 1	(2.8,2.9,3.0)	(3.5,3.7,3.9)	(4.2,4.4,4.6)
Shale site 2	Treatment plant 2	(3.2,3.4,3.6)	(3.5,3.9,4.3)	(4.1,4.2,4.3)
Shale site 3	Disposal site	(4.0,4.2,4.4)	(4.5,4.8,5.1)	(5.0,5.5,6.0)
Shale site 3	Treatment plant 1	(4.4,4.8,5.2)	(5.0,5.3,5.6)	(5.5,5.9,6.3)
Shale site 3	Treatment plant 2	(5.0,5.1,5.2)	(5.0,5.5,6.0)	(6.0,6.3,6.6)
Shale site 4	Disposal site	(2.5,2.7,2.9)	(3.0,3.2,3.4)	(3.5,3.9,7.3)
Shale site 4	Treatment plant 1	(5.5,6.0,6.5)	(6.5,6.7,6.9)	(7.1,7.3,7.5)
Shale site 4	Treatment plant 2	(3.3,3.6,3.9)	(4.0,4.3,4.6)	(4.4,4.9,5.4)
Shale site 5	Disposal site	(6.8,7.1,7.4)	(7.3,7.5,7.7)	(7.8,7.9,8.0)
Shale site 5	Treatment plant 1	(3.0,3.2,3.4)	(3.4,3.6,3.8)	(3.8,3.9,4.0)
Shale site 5	Treatment plant 2	(2.8,3.1,3.4)	(3.6,3.8,4.0)	(4.0,4.3,4.6)
Operational costs of treatment facility ($\tilde{ctr}$) and disposal facility ($\tilde{cd}$)	Disposal site	(0.5,0.7,0.9;0.4,0.7,1.0)	(0.4,0.6,0.8;0.3,0.6,0.9)	(2.1,2.3,2.6;2.1,2.3,2.6)
	Treatment plant 1	(3.6,3.8,4.0;3.5,3.8,4.1)	(0.5,0.7,0.9;0.4,0.7,1.0)	(1.4,1.6,1.8;1.2,1.6,2.0)
	Treatment plant 2	(2.5,2.7,2.9;2.4,2.7,3.0)	(1.5,1.7,1.9;1.4,1.7,2.0)	(1.5,1.7,1.9;1.4,1.7,2.0)
		Time period		
Capital cost of expanding treatment plant ($\tilde{cex}$)	Expansion option m	$t = 1$	$t = 2$	$t = 3$
Treatment plant 1	1	(15.6,15.8,16.0;15.4,15.8,16.2)	(17.2,17.4,17.6;17.1,17.4,17.7)	(14.3,14.6,14.9;14.2,14.6,15.0)
Treatment plant 1	2	(09.6,09.8,10.0;09.5,09.8,10.1)	(16.2,16.4,16.6;16.1,16.4,16.7)	(12.2,12.4,12.6;12.1,12.4,12.7)
Treatment plant 1	3	(12.2,12.4,12.6;12.0,12.4,12.8)	(13.3,13.5,13.7;13.2,13.5,13.8)	(11.2,11.4,11.6;11.1,11.4,11.7)
Treatment plant 2	1	(14.2,14.4,14.6;14.0,14.4,14.8)	(12.1,12.3,12.5;12.0,12.3,12.6)	(13.1,13.3,13.5;13.0,13.3,13.6)
Treatment plant 2	2	(13.2,13.4,13.6;13.0,13.4,13.8)	(11.2,11.4,11.6;11.1,11.4,11.7)	(16.2,16.4,16.6;16.1,16.4,16.7)
Treatment plant 2	3	(12.2,12.5,12.8;12.1,12.5,12.9)	(11.3,11.5,11.7;11.2,11.5,11.8)	(17.2,17.4,17.6;17.1,17.4,17.7)
Increased treatment capacity (eo)				
Treatment plant 1	1	600	600	600
Treatment plant 1	2	750	750	750
Treatment plant 1	3	850	850	850
Treatment plant 2	1	550	550	550
Treatment plant 2	2	650	650	650
Treatment plant 2	3	800	800	800

Table 5. Capacity restrictions on freshwater and wastewater (bbl/day).

Freshwater Acquisition Capacity at Source ($\tilde{fca}$)	**Time Period**		
	$t = 1$ **(200, 300, 400; 100, 300, 500)**	**$t = 2$** **(300, 500, 700; 200, 500, 800)**	**$t = 3$** **(500, 600, 700; 400, 600, 800)**
Freshwater demand at shale site ($\tilde{fdw}$)			
Shale site 1	(300,000, 500,000, 700,000, 900,000)	(500,000, 700,000, 900,000, 1,100,000)	(1,300,000, 1,400,000, 1,500,000, 1,600,000)
Shale site 2	(500,000, 600,000, 700,000, 800,000)	(600,000, 700,000, 800,000, 900,000)	(1,000,000, 1,100,000, 1,200 000, 1,300,000)
Shale site 3	(700,000, 900,000, 1,100,000, 1,300,000)	(300,000, 400,000, 500,000, 600,000)	(600,000, 800,000, 1,000,000, 1,200,000)
Shale site 4	(800,000, 900,000, 1,000,000, 1,100,000)	(1,000,000, 1,100,000, 1,200,000, 1,300,000)	(1,000,000, 1,200,000, 1,400,000, 1,600,000)
Shale site 5	(600,000, 800,000, 1,000,000, 1,200,000)	(1,000,000, 1,200,000, 1,400,000, 1,600,000)	(1,000,000, 1,500,000, 2,000,000, 2,500,000)
Wastewater capacity at disposal site ($\tilde{wwds}$)			
Disposal site	(200, 300, 400; 100, 300, 500)	(600, 800, 1000; 500, 800, 1100)	(400, 600, 800; 300, 600, 900)
Wastewater capacity at treatment plant ($\tilde{wwtp}$)			
Treatment plant 1	(100,000, 200,000, 300,000, 400,000)	(200,000, 300,000, 400,000, 500,000)	(1,000,000, 1,200,000, 1,400,000, 1,600,000)
Treatment plant 2	(200,000, 400,000, 600,000, 800,000)	(1,300,000, 1,600,000, 1,800,000, 2,200,000)	(3,000,000, 3,200,000, 3,400,000, 3,600,000)
Overall wastewater capacity ($\tilde{wdw}$)			
Disposal site	(620,000, 630,000, 640,000, 650,000)	(2,473,000, 2,474,000, 2,475,000, 2,476,000)	(4,460,000, 4,460,000, 4,470,000, 4,480,000)
Treatment plant 1	(600,000, 700,000, 800,000, 900,000)	(2,000,000, 3,000,000, 4,000,000, 5,000,000)	(4,070,000, 4,080,000, 4,090,000, 4,500,000)
Treatment plant 2	(3,002,000, 3,004,000, 3,006,000, 3,008,000)	(4,010,000, 4,020,000, 4,030,000, 4,040,000)	(5,100,000, 5,200 000, 5,300,000, 5,400,000)
Revenues from wastewater reuse (re)			
Treatment plant 1	1.20	1.30	1.50
Treatment plant 2	1.00	1.20	1.40
Reuse rate (rr)			
Treatment plant 1	0.75	0.85	0.95
Treatment plant 2	0.70	0.80	0.90
		Onsite treatment technology o	
	1	2	3
Recovery factor (lo)	0.15	0.45	0.65
Ratio of freshwater to wastewater for blending (rf)	0.43	0.40	0.38
Minimum capacity for on-site treatment (ocl)	150	200	300
Maximum capacity for on-site treatment (ocu)	5000	8000	9000

Results Analyses

The multiobjective shale gas water management optimization model was written in the AMPL language and solved using the BARON solver through NEOS server version 5.0 in the on-line facility provided by Wisconsin Institutes for Discovery at the University of Wisconsin in Madison for solving optimization problems, see Dolan [57], Drud [58], Server [59], and Gropp, W. Moré [60]. The technical description of the problem is presented as follows: The final multiobjective shale gas water management optimization model along with a set of well-defined multiple objectives comprised 219 variables including 42 binary variables, 27 non-linear variables, 150 linear variables, and 336 constraints, including 15 non-linear constraints and 321 linear constraints, 66 equality, and 270 inequality constraints. The total computational time for obtaining the final solution was 0.095 s (CPU time). The proposed multiobjective shale gas water management optimization model was solved with three weight parameters assigned to deviational variables of each membership goal with respect to their marginal membership degree. The first weight parameter w_{kT} was assigned to the truth deviational variable of each membership goal. The second weight parameter w_{kI} was assigned to the indeterminacy deviational variable of each membership goal, and the third weight parameter w_{kF} was assigned to the falsity deviational variable of each membership goal included in all three objective functions. The obtained optimal results were categorized into five main parts: (i) the optimal acquisition of freshwater from various sources to different shale sites in order to ensure smooth operation of the shale gas energy generation system; (ii) prominent emerging technologies for the on-site treatment of wastewater; (iii) the optimal wastewater management system strategy, which is challenging from the environmental point of view; (iv) the optimal expansion plan to enhance the treatment plant capacity; and (v) the optimal values of different conflicting objectives with their corresponding assigned weights. The optimal amounts of freshwater from source to different shale sites are summarized in Table 6. In planning period 1, the amount of freshwater requirements from source to five shale sites were 700.000, 186.765, 700.000, 300.480, and 74.100 bbl/day, respectively. In planning period 2, the requirements of freshwater at each shale site were obtained as 1125.000, 1125.000, 654.419, 131.542 and 212.553 bbl/day, respectively. In planning period 3, the consumption of freshwater at each shale site was 1275.000, 187.613, 528.153, 131.542, and 212.553 bbl/day in order to ensure smooth operation of the shale gas extraction processes. However, with the exception of shale sites 1 and 5, the requirements for freshwater increased for each planning horizon. The maximum requirement of freshwater was in shale site 5 with a volume 1275.000 bbl/day, whereas the minimum freshwater requirement was observed at shale site 1 during planning period 3, with 74.100 bbl/day due to the low and high cost of acquisition and transportation incurred over the amount of freshwater, respectively.

The most promising characteristic features of on-site wastewater treatment are the different technologies which are being used to reutilize the wastewater within candidate shale sites. The optimal allocation of wastewater for on-site treatment is summarized in Table 7. The on-site treatment of wastewater by different technologies are emerging options for generating freshwater, which was included in the proposed modeling and optimization framework. At shale site 1, the amount of freshwater after treatment by technology 1 was 150 bbl/day in all three planning horizons; the generation of freshwater after treatment by using technology 2 was 200 bbl/day in each planning period; and by applying technology 3 the values were 1551.71, 2401.65, and 2701.62 bbl/day, which was consistently increasing and ensuring the reuse of wastewater in these three planning horizons. At shale site 2, the amount of freshwater generated by the on-site treatment facility using technology 1 was 127.352 bbl/day in each planning period; the generation of freshwater after treatment using technology 2 was 200, 6250, and 200 bbl/day each; and by applying on-site treatment technology 3 they were 525.313, 4554.66, and 527.01 bbl/day in each planning horizon, respectively. At shale site 3, the amount of freshwater after treatment by technology 1 was 150 bbl/day in all three planning horizons; the generation of freshwater after treatment by using technology 2 was 200 bbl/day in the first and second planning periods, which were the same as shale site 1; whereas it was 2865.32 bbl/day in the third planning slot, and unlike by applying technology 3 the obtained amounts were 1551.71, 2060.51, and 2208.04 in all planning horizons, resulting in a significant increase in the freshwater generation pattern by on-site treatment. At shale site 4, the generation of freshwater using on-site treatment technology 1 was 147.779, 2023.26, and 147.779 bbl/day; by implementing on-site treatment technology 2 they were 200, 272.266, and 730.788 bbl/day, revealing the significant increment in the regenerated wastewater volumes in three planning slots. The amount of freshwater by using on-site treatment technology 3 was 751.275, 300.000, and 300.000 bbl/day in each planning horizon respectively. At shale site 5, the generation of freshwater using on-site treatment technology 1 was 150 bbl/day in each planning slot; by applying on-site treatment technology 2 it was 342.801, 861.73, and 861.73 bbl/day; and after implementing on-site treatment technology 3 it was 300, 860.539, and 860.539 bbl/day in each planning horizon, respectively. Therefore, the optimal regeneration of freshwater at each shale sites was effectively designed by implementing the on-site treatment technology component in the proposed shale gas water management study and could be potentially achieved using these technologies in an efficient manner under many adverse circumstances, especially where wastewater managerial issues are often encountered at the political level.

The presented wastewater managerial study includes one underground injection disposal site and two treatment plants with its three expansion options which are capable of representing the wastewater management system for the shale plays. Optimal distribution of total wastewater for underground injection disposal and treatment facility is summarized in Table 7. The toxic wastewater produced at shale site 1 was 6.75, 17.25, and 13.25 bbl/day, which is directly transported to the underground injection disposal site; whereas the total volume shipped to the treatment plant was 645, 842.50, and 0 bbl/day in all three time horizons. At shale site 2, the whole volume of wastewater was directly sent to the underground injection disposal site and it was not feasible to facilitate the usage of a treatment plant facility. At shale site 3, a certain volume of wastewater was delivered to an underground injection disposal site and treatment plant 2 without allocating any volume to treatment plant 1. The amount of wastewater shipped to the underground injection disposal site was 6.75, 17.25, and 13.25 bbl/day, and the optimal allocations to treatment plant 1 were 137.71, 675, and 850 bbl/day in the three planning periods. At shale sites 4 and 5, the overall volume of produced wastewater that would be delivered from both shale sites were the same and found to be 6.75, 17.25, and 13.25 bbl/day for underground injection disposal purposes: 645, 842.50, and 937.50 bbl/day towards treatment plant 1 whereas the optimal shipment volumes of wastewater from both shale sites to treatment plant 2 were 137.71, 675, and 850 bbl/day in all three planning horizons, respectively. The optimal allocation strategy for the total wastewater volumes was described in such a fashion that the optimal contribution of each wastewater management system components had equal significance. At all five shale sites, the generated amount of wastewater sent from each shale site to the underground injection disposal site were 6.75, 17.25, and 13.25 bbl/day over the three planning horizons, respectively, revealing the maximum permitted amount at the underground injection disposal site and restraining the subsurface water for a certain period. More elaborately, it could be concluded that during the various time horizons it was not found optimal and feasible to flow the wastewater towards the underground injection disposal site due to the significant cost of transportation and the underground injection disposal facility. At shale site 1, the amount of wastewater that would be shipped to treatment plant 1 was 645 and 842.50 bbl/day for planning horizons 1 and 2, respectively. The shipment of wastewater from shale site 1 to treatment plant 2 was not found to be feasible due to the significant increase in the transportation cost incurred over wastewater. At shale site 2, the allocation of any wastewater amount to treatment plants 1 and 2 was not found to be justified in all three planning periods. At shale site 3, it was not feasible to deliver any amount of wastewater to treatment plant 1 during all three planning periods, although the amount of wastewater that would be shipped to treatment plant 2 was 137.71, 675, and 850 bbl/day in the three planning horizons, respectively. At shale sites 4 and 5, the volume of wastewater that would be delivered from both shale sites were the same and found to be 645, 842.50, and 937.5 bbl/day towards treatment plant 1, whereas the optimal shipment volume of wastewater from both shale sites to treatment plant 2 were 137.71, 675, and 850 bbl/day in all three planning horizons respectively.

During all three planning horizons, treatment plant expansion options played a significant role in dealing with the excess volume of wastewater produced at different shale sites. The vital dominant characteristic of treatment plant expansion was mainly due to limited and rare existence of underground injection disposal site facilities in some places. The limitations imposed on underground injection disposal sites enabled the expanded scope of treatment plant expansions. The optimal strategy for the expansion of treatment plants is presented in Table 7. The optimal expansion results of treatment plant 1 during planning periods 1 and 3 by using expansion option 1 were 600 bbl/day each. By using expansion option 2 in planning periods 1 and 3, the optimal capacity was 750 bbl/day each; and by using expansion option 3 in planning periods 1 and 3, the optimal capacity was 850 bbl/day each. There was no need to expand the treatment capacity of treatment plant 1 in planning period 2. Moreover, the optimal expansion strategy for treatment plant 2 by using all three expansion options during planning horizon 1 were obtained as 550, 650, and 850 bbl/day, whereas in planning period 2, only expansion option 2 was suggested to enhance the treatment capacity. There was no more optimal strategy indicated for the rest of the expansion options. The compromise solution results obtained by solving the proposed multiobjective shale gas water management model are summarized in Table 6. The minimum total cost of acquisition and transportation of freshwater at the source and from different sources to shale sites was USD $525126.00, whereas the net cost incurred over the entire amount of wastewater management during the three planning periods was obtained as USD $4025940.00. The optimal strategy to expand the treatment plant capacity with the predetermined expansion option was presented efficiently and the total capital investment levied on the expansion of the wastewater treatment plant was USD $5548.97, which reveals that there is still adequate opportunity to expand the capacity of the treatment plant. Shale gas water management systems play an important role in the whole process of generating shale gas energy. The acquisition of a huge amount of freshwater for the fracturing process is a challenging task. The wastewater released from shale sites is toxic in nature and contains various harmful dissolved elements. Therefore, a well-organized wastewater management system includes disposal sites (underground injections) and the establishment of different treatment plants with expansion options.

The overall shale gas water modeling approach was presented, inevitably revealing more practical aspects of decision-making scenarios. Uncertainty among parameters due to vagueness and hesitation were addressed with the triangular intuitionistic fuzzy number, which complies over the degree of acceptance and non-acceptance simultaneously. For example, if the decision maker intends to quantify the value of freshwater requirement with some estimated value, such as each shale site requires approximately 54,800 bbl/day for fracking and horizontal drilling purposes, then the most likely estimated interval would be 54,750–54,850 bbl/day, along with some hesitation degree that may be given as 54,700–54900 bbl/day, which ensures less violation of risks with degree of acceptance and non-acceptance. The representation of different constraints imposed over various parameters also reflects the real scenario of Pennsylvania. In Pennsylvania, underground disposal facilities are very rare and most often wastewater is shipped to nearby cities in Ohio. The solution results have shown a similar situation, and less sewage has been allocated to a different underground disposal facility. Furthermore, the scope for on-site treatment technology and expansion capacity options of treatment plants have been optimally utilized. The resulting optimal allocation of wastewater for on-site treatment at different shale sites shows another advantage by reducing the transportation cost incurred over the treatment and disposal facilities. The opportunity for the expansion capacity option of the treatment plant—if needed—was propounded, and results show that some expansion option was adopted due to the lesser capital investment. The determination of the wastewater reuse rate at the treatment plant also yielded a significant amount of freshwater generation and ensured a lesser burden on the underground disposal facility, which again exhibits substantial characteristic features of the shale gas modeling approach of Pennsylvania. Thus, the proposed shale gas water management model can be easily applied to shale gas energy project planning problems that inherently involve uncertain parameters. The decision maker(s) or project manager(s) can conclusively determine the optimal allocation of each water component with a set of multiple conflicting objectives along with a profitable and economic strategy.

Table 6. Optimal amount of freshwater and value of objective functions.

	Amount of Freshwater $FW_{s,i,t}$
1 1 1	700.000
1 1 2	1125.000
1 1 3	1275.000
1 2 1	186.765
1 2 2	1125.000
1 2 3	187.613
1 3 1	700.000
1 3 2	654.419
1 3 3	528.153
1 4 1	300.480
1 4 2	131.542
1 4 3	131.542
1 5 1	74.100
1 5 2	212.553
1 5 3	212.553
Optimal objective values	
Minimum Z_1	525,126.00
Minimum Z_2	4,025,940.00
Minimum Z_3	5548.97

Table 7. Optimal amount of wastewater allocation and treatment plant expansion strategy.

	Total Amount of wastewater $WW_{i,j,t}$	Amount of Wastewater at Disposal Site $WWD_{i,j,t}$	Amount of Wastewater at Treatment Plant $WWT_{i,j,t}$	Amount of Wastewater for on-Site Treatment $WTO_{i,o,t}$
1 1 1	6.75	6.75	0	150
1 1 2	17.25	17.25	0	150
1 1 3	13.25	13.25	0	150
1 2 1	645	0	645	200
1 2 2	842.5	0	842.5	200
1 2 3	0	0	0	200
1 3 1	0	0	0	1551.71
1 3 2	0	0	0	2401.65
1 3 3	0	0	0	2701.62
2 1 1	6.75	6.75	0	127.352
2 1 2	17.25	17.25	0	127.352
2 1 3	13.25	13.25	0	127.352
2 2 1	0	0	0	200
2 2 2	0	0	0	6250
2 2 3	0	0	0	200
2 3 1	0	0	0	525.313
2 3 2	0	0	0	4554.66
2 3 3	0	0	0	527.01
3 1 1	6.75	6.75	0	150
3 1 2	17.25	17.25	0	150
3 1 3	13.25	13.25	0	150
3 2 1	0	0	0	200
3 2 2	0	0	0	200
3 2 3	0	0	0	2865.32
3 3 1	137.71	0	137.71	1551.32
3 3 2	675	0	675	2060.51
3 3 3	850	0	850	2208.04
4 1 1	6.75	6.75	0	147.779
4 1 2	17.25	17.25	0	2023.26
4 1 3	13.25	13.25	0	147.779
4 2 1	645	0	645	200
4 2 2	842.5	0	842.5	272.266
4 2 3	937.5	0	937.5	730.788
4 3 1	137.71	0	137.71	751.275
4 3 2	675	0	675	300
4 3 3	850	0	850	300
5 1 1	6.75	6.75	0	150
5 1 2	17.25	17.25	0	150
5 1 3	13.25	13.25	0	150
5 2 1	645	0	645	342.801
5 2 2	842.5	0	842.5	861.73
5 2 3	937.5	0	937.5	861.73
5 3 1	137.71	0	137.71	300
5 3 2	675	0	675	360.539
5 3 3	850	0	850	360.539
Increased treatment plant capacity (eo)	Expansion option (m)	$t = 1$	Time period $t = 2$	$t = 3$
Treatment plant 1	1	600	-	600
Treatment plant 1	2	750	-	750
Treatment plant 1	3	850	-	850
Treatment plant 2	1	550	550	-
Treatment plant 2	2	650	-	-
Treatment plant 2	3	800	-	-

5. Conclusions

The multiobjective shale gas water management optimization model addressed within synthesizes the optimum allocation of water resources for shale gas extraction processes. It assures the optimal distribution of freshwater and wastewater, which are the complementary components of shale gas energy production problems. The proposed shale gas modeling outlook is reliable and provides a helpful tool to investigate and analyze the trade-off between socio-economic and environmental concerns globally. The different costs incurred over freshwater, charges levied on wastewater, and capital investment of expanding treatment plant capacity along with the set of shale gas water management system constraints were optimized simultaneously. Uncertainty measures were incorporated among different parameters to demonstrate the actual situations encountered in real-life shale gas optimization frameworks. The accumulation of freshwater from various sources is a crucial task to fulfill commercial needs. However, alternate options were suggested for the generation of freshwater by using on-site treatment technology, which simultaneously reduced the transportation costs for freshwater. Underground injection disposal sites and treatment plant facilities are two major consumption points of generated wastewater from shale sites. A critical factor in the reuse of water in shale gas is the detailed coordination of activities. For greater convenience, auxiliary options have also been introduced to tackle the excess amount of wastewater in the form of on-site treatment technology and different potential expansions of treatment plant capacity at each shale site during each planning horizon. Unlike the various existing conventional solution techniques, the neutrosophic goal programming approach was suggested, which also considers the independent neutral thoughts of decision makers in the decision-making process. Since the proposed approach was applied to a small-scale shale gas extraction process (see Figure 2), it resulted in the globally optimal solution for all objectives simultaneously. However, it may not always be possible to have a globally optimal solution when dealing with large-scale dataset problems. The discussed approach cannot capture the stochastic nature of parameters, which consequently cannot be applied to stochastic optimization problems.

The significant contributions of the proposed multiobjective shale gas water management system are summarized as follows:

- The proposed study considers the overall shale gas water management system which consists of freshwater acquisition at sources, on-site wastewater treatment facilities at each shale site, underground injection disposal sewage facilities, different treatment plant options for the reuse of wastewater and the total wastewater capacity which are feasible to handle without affecting the environmental issues. The decision maker(s) or project manager(s) may adopt the presented shale gas modeling framework, which has a magnetic orientation concerning the overall water management system. However, pipeline facilities have not been included throughout the shale gas energy extraction due to their uneconomic aspect.
- Uncertainty among the parameter values is commonly known in the decision-making process. In this shale gas optimization model, the different parameters (e.g., acquisition cost, transportation cost, treatment cost, disposal cost, and capital investment) are taken as the triangular intuitionistic fuzzy number, which is based on more intuition and leads to more realistic uncertainty modeling texture. It also ensures that the system costs the reliability of each component (costs related to freshwater and wastewater) more realistically. The crisp versions of uncertain parameters were determined in terms of expected interval and expected values.
- A neutrosophic-based computational decision-making algorithm for such a complex and dynamic multiobjective shale gas water management optimization model provides benefits while obtaining globally optimal solutions. The indeterminacy/neutral thought is the region of the propositions' value uncertainly and originates from the independent and impartial thoughts. Therefore, the proposed NGPA is a dominating and suitable conventional optimization technique that is preferred over others due to the existence of its independent indeterminacy degree.

- The multiobjective shale gas project planning model was implemented with a possible dataset and the obtained optimal results were analyzed for each component of the shale gas system in a well-organized and efficient manner. Hence, it was concluded that the proposed optimal strategy for shale gas production could be adopted for more sophisticated and quite typical Marcellus shale plays in large-scale long-term scenarios.

Due to manuscript drafting constraints and space limitations, some important aspects remain untouched and may be explored as a future research scope. The presented shale gas water management modeling approach could be extended by considering different essential aspects such as the to and fro movement of water through the pipeline which was not considered in this paper. The presented computational study was demonstrated for small-scale shale-plays, which could be further explored for large-scale and long-term time horizons by enhancing the number of shale sites and different sources of freshwater and various destinations for wastewater.

Flow-back water does not exit instantaneously, but follows a decline curve. Most of the water exits in the first 3–4 weeks, but there is small and a continuous flow of produced water during all the shale-sites life. Therefore the presented modeling approach may be extended by capturing the above-discussed behavior of flow-back produced water. On-site treatment technology exerts less pressure on the underground disposal of wastewater and provides an opportunity to reuse the treated wastewater for fracking purposes within the shale sites itself. If there are no NORMs (normally occurring radioactive materials), the most costly part of water treatment is desalination. Therefore, the sort of on-site treatment technologies may be specified along with their actual cost, and the possibility of being used for on-site treatment purposes may be explored as a future study. Most of the water management system (e.g., the water treated in municipal wastewater treatment facilities that are usually not prepared to deal with hypersaline water) are presently forbidden and may be implemented and executed by including them under a good practices scheme in future work. From the decision-making point of view, hierarchical decision-making processes could be adopted, ensuring a decentralized decision-making scenario and providing more flexibility compared to multiobjective optimization techniques with a single decision maker. Apart from conventional solution techniques, some metaheuristic algorithms could be applied to solve such shale gas water management planning problems. Furthermore, the propounded neutrosophic modeling approach could be applied to real-life dataset problems such as supplier selection problems, inventory control problems, supply chain management, humanitarian logistic problems, etc. The proposed approach could be further extended by incorporating the multi-choice and stochastic parameters along with bi-level and multi-level decision-making scenarios.

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Combination of the Single-Valued Neutrosophic Fuzzy Set and the Soft Set with Applications in Decision-Making

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Abstract: In this article, we propose a novel concept of the single-valued neutrosophic fuzzy soft set by combining the single-valued neutrosophic fuzzy set and the soft set. For possible applications, five kinds of operations (e.g., subset, equal, union, intersection, and complement) on single-valued neutrosophic fuzzy soft sets are presented. Then, several theoretical operations of single-valued neutrosophic fuzzy soft sets are given. In addition, the first type for the fuzzy decision-making based on single-valued neutrosophic fuzzy soft set matrix is constructed. Finally, we present the second type by using the AND operation of the single-valued neutrosophic fuzzy soft set for fuzzy decision-making and clarify its applicability with a numerical example.

1. Introduction

Many areas (e.g., physics, social sciences, computer sciences, and medicine) work with vague data that require fuzzy sets [1], intuitionistic fuzzy sets [2], picture fuzzy sets [3], and other mathematical tools. Molodtsov [4] presented a novel approach termed “soft set theory”, which plays a very significant role in different fields. Therefore, several researchers have developed some methods and operations of soft set theory. For instance, Maji et al. [5] introduced some notions of and operations on soft sets. In addition, Maji et al. [6] gave an application of soft sets to solve fuzzy decision-making. Maji et al. [7] proposed the notion of fuzzy soft sets, followed by studies on inverse fuzzy soft sets [8], belief interval-valued soft sets [9], interval-valued intuitionistic fuzzy soft sets [10], interval-valued picture fuzzy soft sets [11], interval-valued neutrosophic soft sets [12], and generalized picture fuzzy soft sets [13]. Furthermore, several expansion models of soft sets have been developed very quickly, such as possibility Pythagorean fuzzy soft sets [14], possibility m-polar fuzzy soft sets [15], possibility neutrosophic soft sets [16], and possibility multi-fuzzy soft sets [17]. Karaaslan and Hunu [18] defined the notion of type-2 single-valued neutrosophic sets and gave several distance measure methods: Hausdorff, Hamming, and Euclidean distances for Type-2 single-valued neutrosophic sets. Al-Quran et al. [19] presented the notion of fuzzy parameterized complex neutrosophic soft expert sets and gave a novel approach by transforming from the complex case to the real case for decision-making. Qamar and Hassan [20] proposed a novel approach to Q-neutrosophic soft sets and studied several operations of Q-neutrosophic soft sets. Further, they generalized Q-neutrosophic soft expert sets based on uncertainty for decision-making [21]. On the other hand, Uluçay et al. [22] presented the concept of generalized neutrosophic soft expert sets and applied a novel algorithm for multiple-criteria decision-making. Zhang et al. [23] gave novel algebraic operations of totally dependent neutrosophic sets and totally dependent neutrosophic soft sets. In 2018, Smarandache [24] generalized the soft set to the hypersoft set by transforming the function F into a multi-argument function.

Fuzzy sets are used to tackle uncertainty using the membership grade, whereas neutrosophic sets are used to tackle uncertainty using the truth, indeterminacy, and falsity membership grades, which are considered as independent. As the motivation of this article, we present a novel notion of the single-valued neutrosophic fuzzy soft set, which can be seen as a novel single-valued neutrosophic fuzzy soft set model, which gives rise to some new concepts. Since neutrosophic fuzzy soft sets have some difficulties in dealing with some real-life problems due to the nonstandard interval of neutrosophic components, we introduce the single-valued neutrosophic fuzzy soft set (i.e., the single-valued neutrosophic set has a symmetric form, since the membership (T) and nonmembership (F) are symmetric with each other, while indeterminacy (I) is in the middle), which is considered as an instance of neutrosophic fuzzy soft sets. The structural operations (e.g., subset, equal, union, intersection, and complement) on single-valued neutrosophic fuzzy soft sets, and several fundamental properties of the five operations above are introduced. Lastly, two novel approaches (i.e., Algorithms 1 and 2) to fuzzy decision-making depending on single-valued neutrosophic fuzzy soft sets are discussed, in addition to a numerical example to show the two approaches we have developed.

The rest of this article is arranged as follows. Section 2 briefly introduces several notions related to fuzzy sets, neutrosophic sets, single-valued neutrosophic sets, neutrosophic fuzzy sets, single-valued neutrosophic fuzzy sets, soft sets, fuzzy soft sets, and neutrosophic soft sets. Section 3 discusses single-valued neutrosophic fuzzy soft sets (along with their basic operations and structural properties). Section 4 gives two algorithms for single-valued neutrosophic fuzzy soft sets for decision-making. Lastly, the conclusions are given in Section 5.

2. Preliminaries

In the following, we present a short survey of seven definitions which are necessary to this paper.

2.1. Fuzzy Set

Definition 1 (cf. [1])**.** *Assume that X (i.e., $X = \{x_1, x_2, ..., x_p\}$) is a set of elements and $\mu(x_p)$ is a membership function of element $x_p \in X$. Then*

(1) The following mapping (called fuzzy set), is given by

$$\mu : X \longrightarrow [0,1]$$

and $[0,1]^X$ is a set of whole fuzzy subset over X.

(2) Let

$$\mu = \left\{ \frac{\mu(x_1)}{x_1}, \frac{\mu(x_2)}{x_2}, \cdots, \frac{\mu(x_p)}{x_p} \middle| \, x_p \in X \right\} \in [0,1]^X$$

and

$$\nu = \left\{ \frac{\nu(x_1)}{x_1}, \frac{\nu(x_2)}{x_2}, \cdots, \frac{\nu(x_p)}{x_p} \middle| \, x_p \in X \right\} \in [0,1]^X.$$

Then

(1) *The union* $\mu \cup \nu$, *is defined as*

$$\mu \cup \nu = \left\{ \frac{\mu(x_1) \vee \nu(x_1)}{x_1}, \frac{\mu(x_2) \vee \nu(x_2)}{x_2}, \cdots, \frac{\mu(x_p) \vee \nu(x_p)}{x_p} \middle| x_p \in X \right\}.$$

(2) *The intersection* $\mu \cap \nu$, *is defined as*

$$\mu \cap \nu = \left\{ \frac{\mu(x_1) \wedge \nu(x_1)}{x_1}, \frac{\mu(x_2) \wedge \nu(x_2)}{x_2}, \cdots, \frac{\mu(x_p) \wedge \nu(x_p)}{x_p} \middle| x_p \in X \right\}.$$

2.2. Neutrosophic Set and Single-Valued Neutrosophic Set

Definition 2 (cf. [25,26]). *Assume that* X *(i.e.,* $X = \{x_1, x_2, ..., x_p\}$*) is a set of elements and*

$$\Phi = \left\{ \frac{(T_\Phi(x_p), I_\Phi(x_p), F_\Phi(x_p))}{x_p} \middle| x_p \in X, \ 0 \leq T_{\hat{\Phi}}(x_p) + I_{\hat{\Phi}}(x_p) + F_{\hat{\Phi}}(x_p) \leq 3 \right\}.$$

(1) *If* $T_\Phi(x_p) \in]0^-, 1^+[$ *(i.e., the degree of truth membership),* $I_\Phi(x_p) \in]0^-, 1^+[$ *(i.e., the degree of indeterminacy membership), and* $F_\Phi(x_p)$ *(i.e., the degree of falsity membership), then* Φ *is called a neutrosophic set on* X*, denoted by* $(\mathbb{NS})^X$.

(2) *If* $T_\Phi(x_p) \in [0,1]$ *(i.e., the degree of truth membership),* $I_\Phi(x_p) \in [0,1]$ *(i.e., the degree of indeterminacy membership), and* $F_\Phi(x_p) \in [0,1]$ *(i.e., the degree of falsity membership), then* Φ *is called a single-valued neutrosophic set on* X*, denoted by* $(\mathbb{SVNS})^X$.

2.3. Neutrosophic Fuzzy Set and Single-Valued Neutrosophic Fuzzy Set

Definition 3 (cf. [27]). *Assume that* X *(i.e.,* $X = \{x_1, x_2, ..., x_p\}$*) is a set of elements and*

$$\hat{\Phi} = \left\{ \frac{(T_{\hat{\Phi}}(x_p), I_{\hat{\Phi}}(x_p), F_{\hat{\Phi}}(x_p), \mu(x_p))}{x_p} \middle| x_p \in X, \ 0 \leq T_{\hat{\Phi}}(x_p) + I_{\hat{\Phi}}(x_p) + F_{\hat{\Phi}}(x_p) \leq 3 \right\}.$$

(1) *If* $T_{\hat{\Phi}}(x_p) \in]0^-, 1^+[$ *(i.e., the degree of truth membership),* $I_{\hat{\Phi}}(x_p) \in]0^-, 1^+[$ *(i.e., the degree of indeterminacy membership), and* $F_{\hat{\Phi}}(x_p)$ *(i.e., the degree of falsity membership), then* $\hat{\Phi}$ *is called a neutrosophic fuzzy set on* X*, denoted by* $(\mathbb{NFS})^X$.

(2) *If* $T_{\hat{\Phi}}(x_p) \in [0,1]$ *(i.e., the degree of truth membership),* $I_{\hat{\Phi}}(x_p) \in [0,1]$ *(i.e., the degree of indeterminacy membership), and* $F_{\hat{\Phi}}(x_p) \in [0,1]$ *(i.e., the degree of falsity membership), then* $\hat{\Phi}$ *is called a single-valued neutrosophic fuzzy set on* X*, denoted by* $(\mathbb{SVNFS})^X$.

Definition 4 (cf. [27]). *Let* $\hat{\Phi}, \hat{\Psi} \in (\mathbb{SVNFS})^X$*, where*

$$\hat{\Phi} = \left\{ \frac{(T_{\hat{\Phi}}(x_p), I_{\hat{\Phi}}(x_p), F_{\hat{\Phi}}(x_p), \mu(x_p))}{x_p} \middle| x_p \in X, \ 0 \leq T_{\hat{\Phi}}(x_p) + I_{\hat{\Phi}}(x_p) + F_{\hat{\Phi}}(x_p) \leq 3 \right\}$$

and

$$\hat{\Psi} = \left\{ \frac{(T'_{\hat{\Psi}}(x_p), I'_{\hat{\Psi}}(x_p), F'_{\hat{\Psi}}(x_p), \mu'(x_p))}{x_p} \middle| x_p \in X, \ 0 \leq T_{\hat{\Psi}}(x_p) + I_{\hat{\Psi}}(x_p) + F_{\hat{\Psi}}(x_p) \leq 3 \right\}.$$

The following operations (i.e., complement, inclusion, equal, union, and intersection) are defined by

(1) $$\hat{\Phi}^c = \left\{ \frac{(F_{\hat{\Phi}}(x_p), 1 - I_{\hat{\Phi}}(x_p), T_{\hat{\Phi}}(x_p), 1 - \mu(x_p))}{x_p} \middle| x_p \in X \right\}.$$

(2) $\hat{\Phi} \subseteq \hat{\Psi} \iff T_{\hat{\Phi}}(x_p) \leq T'_{\hat{\Psi}}(x_p), I_{\hat{\Phi}}(x_p) \geq I'_{\hat{\Psi}}(x_p), F_{\hat{\Phi}}(x_p) \geq F'_{\hat{\Psi}}(x_p)$ *and* $\mu(x_p) \leq \mu'(x_p)\ (\forall x_p \in X)$.

(3) $\hat{\Phi} = \hat{\Psi} \iff \hat{\Phi} \subseteq \hat{\Psi}$ *and* $\hat{\Psi} \subseteq \hat{\Phi}$.

(4) $\hat{\Phi} \cup \hat{\Psi} = \left\{ \frac{(F_{\hat{\Phi}}(x_p) \vee F'_{\hat{\Psi}}(x_p), I_{\hat{\Phi}}(x_p) \wedge I'_{\hat{\Psi}}(x_p), T_{\hat{\Phi}}(x_p) \wedge T'_{\hat{\Psi}}(x_p), \mu(x_p) \vee \mu'(x_p))}{x_p} \,\middle|\, x_p \in X \right\}$.

(5) $\hat{\Phi} \cap \hat{\Psi} = \left\{ \frac{(F_{\hat{\Phi}}(x_p) \wedge F'_{\hat{\Psi}}(x_p), I_{\hat{\Phi}}(x_p) \vee I'_{\hat{\Psi}}(x_p), T_{\hat{\Phi}}(x_p) \vee T'_{\hat{\Psi}}(x_p), \mu(x_p) \wedge \mu'(x_p))}{x_p} \,\middle|\, x_p \in X \right\}$.

2.4. Soft Set, Fuzzy Soft Set, and Neutrosophic Soft Set

Definition 5 (cf. [4,7,28]). *Assume that X (i.e., $X = \{x_1, x_2, ..., x_p\}$) is a set of elements and I (i.e., $I = \{i_1, i_2, ..., i_q\}$) is a set of parameters, where $(p, q \in N, N$ are natural numbers). Then*

(1) The following mapping (called a soft set), is given by

$$S : I \to P(X),$$

where $P(X)$ is a set of all subsets over X.

(2) The following mapping (called a fuzzy soft set), is given by

$$\widetilde{S} : I \to [0,1]^X,$$

where $[0,1]^X$ is a set of whole fuzzy subset over X.

(3) The following mapping (called a neutrosophic soft set), is given by

$$\widetilde{\hat{S}} : I \to (\mathbb{NS})^X,$$

where $(\mathbb{NS})^X$ is a set of whole neutrosophic subset over X.

Example 1. *Assume that the two brothers Mr. Z and Mr. M plan to go the car dealership office to purchase a new car. Suppose that the car dealership office contains types of new cars $X = \{x_1, x_2, x_3, x_4\}$ and $I = \{i_1, i_2, i_3\}$ characterize three parameters, where i_1 is "cheap", i_2 is "expensive", and i_3 is "beautiful". Then*

(1) *By Definition 5(1) we can describe the soft sets as $S_{(i_1)} = \{x_1, x_3\}, S_{(i_2)} = \{x_3, x_4\}$, and $S_{(i_3)} = \{x_2\}$. Therefore,*

$$\mathcal{S} = \left\{ \frac{\{x_1, x_3\}}{i_1}, \frac{\{x_3, x_4\}}{i_2}, \frac{\{x_2\}}{i_3} \right\}.$$

(2) *It is obvious to replace the crisp number 0 or 1 by a membership of fuzzy information. Therefore, by Definition 5(2) we can describe the fuzzy soft sets by $\widetilde{S}_{(i_1)} = \left\{ \frac{0.3}{x_1}, \frac{0.4}{x_2}, \frac{0.6}{x_3}, \frac{0.5}{x_4} \right\}$, $\widetilde{S}_{(i_2)} = \left\{ \frac{0.6}{x_1}, \frac{0.9}{x_2}, \frac{0.1}{x_3}, \frac{0.2}{x_4} \right\}$, $\widetilde{S}_{(i_3)} = \left\{ \frac{0.7}{x_1}, \frac{0.5}{x_2}, \frac{0.2}{x_3}, \frac{0.9}{x_4} \right\}$. Then,*

$$\widetilde{S} = \left\{ \frac{\left\{ \frac{0.3}{x_1}, \frac{0.4}{x_2}, \frac{0.6}{x_3}, \frac{0.5}{x_4} \right\}}{i_1}, \frac{\left\{ \frac{0.6}{x_1}, \frac{0.9}{x_2}, \frac{0.1}{x_3}, \frac{0.2}{x_4} \right\}}{i_2}, \frac{\left\{ \frac{0.7}{x_1}, \frac{0.5}{x_2}, \frac{0.2}{x_3}, \frac{0.9}{x_4} \right\}}{i_3} \right\}.$$

(3) *By Definition 5(3) we can describe the neutrosophic soft sets as*

$$\widetilde{\hat{S}}_{(i_1)} = \left\{ \frac{(0.3, 0.7, 0.5)}{x_1}, \frac{(0.1, 0.8, 0.5)}{x_2}, \frac{(0.2, 0.6, 0.8)}{x_3}, \frac{(0.4, 0.7, 0.6)}{x_4} \right\},$$

$$\widetilde{\hat{S}}_{(i_2)} = \left\{ \frac{(0.3, 0.7, 0.5)}{x_1}, \frac{(0.1, 0.8, 0.5)}{x_2}, \frac{(0.2, 0.6, 0.8)}{x_3}, \frac{(0.5, 0.8, 0.3)}{x_4} \right\},$$

and

$$\widetilde{\hat{S}}_{(i_3)} = \left\{ \frac{(0.3, 0.7, 0.5)}{x_1}, \frac{(0.1, 0.8, 0.5)}{x_2}, \frac{(0.2, 0.6, 0.8)}{x_3}, \frac{(0.8, 0.9, 0.2)}{x_4} \right\}.$$

3. Single-Valued Neutrosophic Fuzzy Soft Set

In the following, we propose the concept of a single-valued neutrosophic fuzzy soft set and study some definitions, propositions, and examples.

Definition 6. *Assume that X (i.e., $X = \{x_1, x_2, ..., x_p\}$) is a set of elements, I (i.e., $I = \{i_1, i_2, ..., i_q\}$) is a set of parameters, and $\mathbb{S}^{XI}$ is called a soft universe. A single-valued neutrosophic fuzzy soft set $\hat{\Phi}_{(i_q)}$ over X, denoted by $(\mathbb{SVNFS})^{XI}$, is defined by*

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\},$$

where $p, q \in N$ (N are natural numbers) and $\mu(x_p) \in [0,1]$. For each parameter $i_q \in I$ and for each $x_p \in X$, $T_{\hat{\Phi}_{(i_q)}}(x_p) \in [0,1]$ (i.e., the degree of truth membership), $I_{\hat{\Phi}_{(i_q)}}(x_p) \in [0,1]$ (i.e., the degree of indeterminacy membership), and $F_{\hat{\Phi}_{(i_q)}}(x_p) \in [0,1]$ (i.e., the degree of falsity membership).

Example 2. *Assume that $X = \{x_1, x_2, x_3\}$ are three kinds of novel cars and $I = \{i_1, i_2, i_3\}$ are three parameters, where i_1 is "cheap", i_2 is "expensive", and i_3 is "beautiful". Let $\mu \in [0,1]^X$ and $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ are defined as follows ($q = 1, 2, 3$):*

$$\hat{\Phi}_{(i_1)} = \left\{ \frac{(0.3, 0.7, 0.5, 0.2)}{x_1}, \frac{(0.1, 0.8, 0.5, 0.5)}{x_2}, \frac{(0.2, 0.6, 0.8, 0.7)}{x_3} \right\},$$

$$\hat{\Phi}_{(i_2)} = \left\{ \frac{(0.9, 0.4, 0.5, 0.7)}{x_1}, \frac{(0.3, 0.7, 0.5, 0.4)}{x_2}, \frac{(0.8, 0.2, 0.6, 0.8)}{x_3} \right\},$$

$$\hat{\Phi}_{(i_3)} = \left\{ \frac{(0.6, 0.3, 0.5, 0.6)}{x_1}, \frac{(0.3, 0.5, 0.6, 0.4)}{x_2}, \frac{(0.7, 0.1, 0.6, 0.3)}{x_3} \right\}.$$

Additionally, we can write by matrix form as

$$\hat{\Phi} = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (0.3, 0.7, 0.5, 0.2) & (0.1, 0.8, 0.5, 0.5) & (0.2, 0.6, 0.8, 0.7) \\ i_2 & (0.9, 0.4, 0.5, 0.7) & (0.3, 0.7, 0.5, 0.4) & (0.8, 0.2, 0.6, 0.8) \\ i_3 & (0.6, 0.3, 0.5, 0.6) & (0.3, 0.5, 0.6, 0.4) & (0.7, 0.1, 0.6, 0.3) \end{array} \right).$$

Definition 7. *Let $\hat{\Phi}_{(i_q)}, \hat{\Psi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$ and $\mu, \mu' \in [0,1]^X$, where*

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\}$$

and

$$\hat{\Psi}_{(i_q)} = \left\{ \frac{(T'_{\hat{\Psi}_{(i_q)}}(x_p), I'_{\hat{\Psi}_{(i_q)}}(x_p), F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu'(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T'_{\hat{\Psi}_{(i_q)}}(x_p) + I'_{\hat{\Psi}_{(i_q)}}(x_p) + F'_{\hat{\Psi}_{(i_q)}}(x_p) \le 3 \right\}.$$

Then, $\hat{\Phi}_{(i_q)} \Subset \hat{\Psi}_{(i_q)}$ (i.e., $\hat{\Phi}_{(i_q)}$ is a single-valued neutrosophic fuzzy soft subset of $\hat{\Psi}_{(i_q)}$) if

(1) $\mu(x_p) \le \mu'(x_p)\ \forall x_p \in X$;

(2) *For all $i_q \in I, x_p \in X$, $T_{\hat{\Phi}_{(i_q)}}(x_p) \le T'_{\hat{\Psi}_{(i_q)}}(x_p)$, $I_{\hat{\Phi}_{(i_q)}}(x_p) \ge I'_{\hat{\Psi}_{(i_q)}}(x_p)$, $F_{\hat{\Phi}_{(i_q)}}(x_p) \ge F'_{\hat{\Psi}_{(i_q)}}(x_p)$.*

Example 3. *(Continued from Example 2). Let* $\hat{\Psi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ *be defined as follows* $(q = 1, 2, 3)$:

$$\hat{\Psi} = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (0.4, 0.6, 0.4, 0.4) & (0.2, 0.7, 0.3, 0.5) & (0.3, 0.4, 0.7, 1) \\ i_2 & (1, 0.3, 0.5, 0.8) & (0.4, 0.6, 0.4, 0.6) & (0.9, 0.2, 0.4, 0.9) \\ i_3 & (0.7, 0.2, 0.4, 0.7) & (0.4, 0.5, 0.6, 0.6) & (0.8, 0.1, 0.5, 0.5) \end{array}\right).$$

Thus, $\hat{\Phi}_{(i_q)} \Subset \hat{\Psi}_{(i_q)}$ $(\forall i_q \in I)$.

Definition 8. *Let* $\hat{\Phi}_{(i_q)}, \hat{\Psi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ *over* $\mathbb{S}^{XI}$ *and* $\mu, \mu' \in [0, 1]^X$, *where*

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \middle| i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\}$$

and

$$\hat{\Psi}_{(i_q)} = \left\{ \frac{(T'_{\hat{\Psi}_{(i_q)}}(x_p), I'_{\hat{\Psi}_{(i_q)}}(x_p), F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu'(x_p))}{x_p} \middle| i_q \in I,\ x_p \in X,\ 0 \le T'_{\hat{\Psi}_{(i_q)}}(x_p) + I'_{\hat{\Psi}_{(i_q)}}(x_p) + F'_{\hat{\Psi}_{(i_q)}}(x_p) \le 3 \right\}.$$

Then, $\hat{\Phi}_{(i_q)} = \hat{\Psi}_{(i_q)}$ *(i.e.,* $\hat{\Phi}_{(i_q)}$ *is a single-valued neutrosophic fuzzy soft equal to* $\hat{\Psi}_{(i_q)}$*) if* $\hat{\Phi}_{(i_q)} \Subset \hat{\Psi}_{(i_q)}$ *and* $\hat{\Phi}_{(i_q)} \Supset \hat{\Psi}_{(i_q)}$.

Definition 9. *Let* $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ *over* $\mathbb{S}^{XI}$ *and* $\mu \in [0, 1]^X$, *where*

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \middle| i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\}$$

over $\mathbb{S}^{XI}$. *Then,*

(1) $\hat{\Phi}_{(i_q)}$ *is called a single-valued neutrosophic fuzzy soft null set (denoted by* $\hat{\varnothing}_{(i_q)}$*), defined as*

$$\hat{\varnothing}_{(i_q)} = \left\{ \frac{(0, 1, 1, 0)}{x_p} | i_q \in I, x_p \in X \right\}.$$

(2) $\hat{\Phi}_{(i_q)}$ *is called a single-valued neutrosophic fuzzy soft universal set (denoted by* $\hat{X}_{(i_q)}$*), defined as*

$$\hat{X}_{(i_q)} = \left\{ \frac{(1, 0, 0, 1)}{x_p} | i_q \in I, x_p \in X \right\}.$$

Example 4. *(Continued from Example 2). Then,* $\hat{\varnothing}_{(i_q)}, \hat{X}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ *are defined as follows:*

$$\hat{\varnothing} = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (0, 1, 1, 0) & (0, 1, 1, 0) & (0, 1, 1, 0) \\ i_2 & (0, 1, 1, 0) & (0, 1, 1, 0) & (0, 1, 1, 0) \\ i_3 & (0, 1, 1, 0) & (0, 1, 1, 0) & (0, 1, 1, 0) \end{array}\right)$$

and

$$\hat{X} = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (1, 0, 0, 1) & (1, 0, 0, 1) & (1, 0, 0, 1) \\ i_2 & (1, 0, 0, 1) & (1, 0, 0, 1) & (1, 0, 0, 1) \\ i_3 & (1, 0, 0, 1) & (1, 0, 0, 1) & (1, 0, 0, 1) \end{array}\right).$$

Definition 10. *Let $\hat{\Phi}_{(i_q)}, \hat{\Psi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$ and $\mu, \mu' \in [0,1]^X$, where*

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\}$$

and

$$\hat{\Psi}_{(i_q)} = \left\{ \frac{(T'_{\hat{\Psi}_{(i_q)}}(x_p), I'_{\hat{\Psi}_{(i_q)}}(x_p), F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu'(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T'_{\hat{\Psi}_{(i_q)}}(x_p) + I'_{\hat{\Psi}_{(i_q)}}(x_p) + F'_{\hat{\Psi}_{(i_q)}}(x_p) \le 3 \right\}.$$

Then,

(1) The union $\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)}$ is defined as

$$\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p) \circ T'_{\hat{\Psi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p) * I'_{\hat{\Psi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p) * F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu(x_p) \circ \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}.$$

(2) The intersection $\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)}$ is defined as

$$\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p) * T'_{\hat{\Psi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p) \circ I'_{\hat{\Psi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p) \circ F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu(x_p) * \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}.$$

Example 5. *(Continued from Examples 2 and 3). For $\alpha, \beta \in [0,1]$, let the t-norm (i.e., given as $\alpha * \beta = \alpha \wedge \beta$) and the t-conorm (i.e., given as $\alpha \circ \beta = \alpha \vee \beta$). Then,*

$$\hat{\Phi} \Cup \hat{\Psi} = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (0.4, 0.6, 0.4, 0.4) & (0.2, 0.7, 0.3, 0.5) & (0.3, 0.4, 0.7, 1) \\ i_2 & (1, 0.3, 0.5, 0.8) & (0.4, 0.6, 0.4, 0.6) & (0.9, 0.2, 0.4, 0.9) \\ i_3 & (0.7, 0.2, 0.4, 0.7) & (0.4, 0.5, 0.6, 0.6) & (0.8, 0.1, 0.5, 0.5) \end{array} \right)$$

and

$$\hat{\Phi} \Cap \hat{\Psi} = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (0.3, 0.7, 0.5, 0.2) & (0.1, 0.8, 0.5, 0.5) & (0.2, 0.6, 0.8, 0.7) \\ i_2 & (0.9, 0.4, 0.5, 0.7) & (0.3, 0.7, 0.5, 0.4) & (0.8, 0.2, 0.6, 0.8) \\ i_3 & (0.6, 0.3, 0.5, 0.6) & (0.3, 0.5, 0.6, 0.4) & (0.7, 0.1, 0.6, 0.3) \end{array} \right).$$

Proposition 1. *Let $\hat{\emptyset}_{(i_q)}, \hat{X}_{(i_q)}, \hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$ and $\mu \in [0,1]^X$. Then the following hold:*

(1) $\hat{\Phi}_{(i_q)} \Cup \hat{\Phi}_{(i_q)} = \hat{\Phi}_{(i_q)}$;
(2) $\hat{\Phi}_{(i_q)} \Cap \hat{\Phi}_{(i_q)} = \hat{\Phi}_{(i_q)}$;
(3) $\hat{\Phi}_{(i_q)} \Cup \hat{\emptyset}_{(i_q)} = \hat{\Phi}_{(i_q)}$;
(4) $\hat{\Phi}_{(i_q)} \Cap \hat{\emptyset}_{(i_q)} = \hat{\emptyset}_{(i_q)}$;
(5) $\hat{\Phi}_{(i_q)} \Cup \hat{X}_{(i_q)} = \hat{X}_{(i_q)}$;
(6) $\hat{\Phi}_{(i_q)} \Cap \hat{X}_{(i_q)} = \hat{\Phi}_{(i_q)}$.

Proof. Follows from Definitions 9 and 10. □

Proposition 2. *Let $\hat{\Phi}_{(i_q)}, \hat{\Psi}_{(i_q)}, \hat{\Gamma}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$ and $\mu, \mu', \mu'' \in [0,1]^X$. Then the following hold:*

(1) $\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)} = \hat{\Psi}_{(i_q)} \Cup \hat{\Phi}_{(i_q)}$;
(2) $\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)} = \hat{\Psi}_{(i_q)} \Cap \hat{\Phi}_{(i_q)}$;
(3) $\hat{\Phi}_{(i_q)} \Cup (\hat{\Psi}_{(i_q)} \Cup \hat{\Gamma}_{(i_q)}) = (\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)}) \Cup \hat{\Gamma}_{(i_q)}$;

(4) $\hat{\Phi}_{(i_q)} \Cap (\hat{\Psi}_{(i_q)} \Cap \hat{\Gamma}_{(i_q)}) = (\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)}) \Cap \hat{\Gamma}_{(i_q)}$;

(5) $\hat{\Phi}_{(i_q)} \Cap (\hat{\Psi}_{(i_q)} \Cup \hat{\Gamma}_{(i_q)}) = (\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)}) \Cup (\hat{\Phi}_{(i_q)} \Cap \hat{\Gamma}_{(i_q)})$;

(6) $\hat{\Phi}_{(i_q)} \Cup (\hat{\Psi}_{(i_q)} \Cap \hat{\Gamma}_{(i_q)}) = (\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)}) \Cap (\hat{\Phi}_{(i_q)} \Cup \hat{\Gamma}_{(i_q)})$.

Proof. Follows from Definition 10. □

Proposition 3. *Let $\hat{\Phi}_{(i_q)}, \hat{\Psi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}, \mu, \mu' \in [0,1]^X$, and $\hat{\Psi}_{(i_q)} \Subset \hat{\Phi}_{(i_q)}$. Then the following hold:*

(1) $\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)} = \hat{\Phi}_{(i_q)}$;

(2) $\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)} = \hat{\Psi}_{(i_q)}$.

Proof. Follows from Definitions 7 and 10. □

Next, we propose a definition, example, remark, and two propositions on the complement of $(\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$.

Definition 11. *Let $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$ and $\mu \in [0,1]^X$, where*

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\}.$$

Then, the complement $\hat{\Phi}^c_{(i_q)}$ of $\hat{\Phi}_{(i_q)}$ is defined as

$$\hat{\Phi}^c_{(i_q)} = \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p), 1 - \mu(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X \right\}.$$

Example 6. *(Continued from Example 2). The complement $\hat{\Phi}^c_{(i_q)}$ of $\hat{\Phi}_{(i_q)}$ is calculated by*

$\hat{\Phi}^c =$

I	x_1	x_2	x_3
i_1	(0.5, 0.3, 0.3, 0.8)	(0.5, 0.2, 0.1, 0.5)	(0.8, 0.4, 0.2, 0.3)
i_2	(0.5, 0.6, 0.9, 0.3)	(0.5, 0.3, 0.3, 0.6)	(0.6, 0.8, 0.8, 0.2)
i_3	(0.5, 0.7, 0.6, 0.4)	(0.6, 0.5, 0.3, 0.6)	(0.6, 0.9, 0.7, 0.7)

Proposition 4. *Let $\hat{\emptyset}_{(i_q)}, \hat{X}_{(i_q)}, \hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ over $\mathbb{S}^{XI}$, and $\mu \in [0,1]^X$. Then, the following hold:*

(1) $\hat{\emptyset}^c_{(i_q)} = \hat{X}_{(i_q)}$;

(2) $\hat{X}^c_{(i_q)} = \hat{\emptyset}_{(i_q)}$;

(3) $(\hat{\Phi}^c_{(i_q)})^c = \hat{\Phi}^c_{(i_q)}$.

Proof. Follows from Definitions 9 and 11. □

Remark 1. *The equality of $\hat{\Phi}_{(i_q)} \Cup \hat{\Phi}^c_{(i_q)} = \hat{X}_{(i_q)}$ and $\hat{\Phi}_{(i_q)} \Cap \hat{\Phi}^c_{(i_q)} = \hat{\emptyset}_{(i_q)}$ does not hold by the following example.*

Example 7. *(Continued from Examples 2 and 6). Then, $\hat{\Phi}^c_{(i_q)}$ of $\hat{\Phi}_{(i_q)}$ is calculated by*

$\hat{\Phi} \Cup \hat{\Phi}^c =$

I	x_1	x_2	x_3
i_1	(0.5, 0.3, 0.3, 0.8)	(0.5, 0.2, 0.1, 0.5)	(0.8, 0.4, 0.2, 0.3)
i_1	(0.5, 0.6, 0.9, 0.3)	(0.5, 0.3, 0.3, 0.6)	(0.6, 0.8, 0.8, 0.2)
i_1	(0.5, 0.7, 0.6, 0.4)	(0.6, 0.5, 0.3, 0.6)	(0.6, 0.9, 0.7, 0.7)

and

$$\hat{\Phi} \Cap \hat{\Phi}^c = \left(\begin{array}{c|ccc} I & x_1 & x_2 & x_3 \\ \hline i_1 & (0.3, 0.7, 0.5, 0.2) & (0.1, 0.8, 0.5, 0.5) & (0.2, 0.6, 0.8, 0.7) \\ i_2 & (0.9, 0.4, 0.5, 0.7) & (0.3, 0.7, 0.5, 0.4) & (0.8, 0.2, 0.6, 0.8) \\ i_3 & (0.6, 0.3, 0.5, 0.6) & (0.3, 0.5, 0.6, 0.4) & (0.7, 0.1, 0.6, 0.3) \end{array} \right).$$

This shows that $\hat{\Phi}_{(i_q)} \Cup \hat{\Phi}^c_{(i_q)} \neq \hat{X}_{(i_q)}$ *and* $\hat{\Phi}_{(i_q)} \Cap \hat{\Phi}^c_{(i_q)} \neq \hat{\emptyset}_{(i_q)}$.

Proposition 5. *Let* $\hat{\Phi}_{(i_q)}, \hat{\Psi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ *over* $\mathbb{S}^{XI}$ *and* $\mu, \mu' \in [0,1]^X$. *Then, the following hold:*

(1) $(\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)})^c = \hat{\Phi}^c_{(i_q)} \Cap \hat{\Psi}^c_{(i_q)}$;

(2) $(\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)})^c = \hat{\Phi}^c_{(i_q)} \Cup \hat{\Psi}^c_{(i_q)}$.

Proof. Consider $a * b = a \wedge b$ (t-norm) and $\alpha \circ \beta = \alpha \vee \beta$ (t-conorm) ($\forall \alpha, \beta \in [0,1]$). We have

(1) $(\hat{\Phi}_{(i_q)} \Cup \hat{\Psi}_{(i_q)})^c(x_p)$

$$= \left(\left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p) \circ T'_{\hat{\Psi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p) * I'_{\hat{\Psi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p) * F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu(x_p) \circ \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\} \right)^c$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) * F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (I_{\hat{\Phi}_{(i_q)}}(x_p) * I'_{\hat{\Psi}_{(i_q)}}(x_p)), T_{\hat{\Phi}_{(i_q)}}(x_p) \circ T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (\mu(x_p) \circ \mu'(x_p)))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \wedge F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (I_{\hat{\Phi}_{(i_q)}}(x_p) \wedge I'_{\hat{\Psi}_{(i_q)}}(x_p)), T_{\hat{\Phi}_{(i_q)}}(x_p) \vee T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (\mu(x_p) \vee \mu'(x_p)))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \wedge F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p) \vee 1 - I'_{\hat{\Psi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p) \vee T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - \mu(x_p) \wedge 1 - \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) * F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p) \circ 1 - I'_{\hat{\Psi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p) \circ T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - \mu(x_p) * 1 - \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p), 1 - \mu(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\} \Cap \left\{ \frac{(F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - I'_{\hat{\Psi}_{(i_q)}}(x_p), T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \hat{\Phi}^c_{(i_q)}(x_p) \Cap \hat{\Psi}^c_{(i_q)}(x_p).$$

(2) $(\hat{\Phi}_{(i_q)} \Cap \hat{\Psi}_{(i_q)})^c(x_p)$

$$= \left(\left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p) * T'_{\hat{\Psi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p) \circ I'_{\hat{\Psi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p) \circ F'_{\hat{\Psi}_{(i_q)}}(x_p), \mu(x_p) * \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\} \right)^c$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \circ F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (I_{\hat{\Phi}_{(i_q)}}(x_p) \circ I'_{\hat{\Psi}_{(i_q)}}(x_p)), T_{\hat{\Phi}_{(i_q)}}(x_p) * T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (\mu(x_p) * \mu'(x_p)))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \vee F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (I_{\hat{\Phi}_{(i_q)}}(x_p) \vee I'_{\hat{\Psi}_{(i_q)}}(x_p)), T_{\hat{\Phi}_{(i_q)}}(x_p) \wedge T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - (\mu(x_p) \wedge \mu'(x_p)))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \vee F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p) \wedge 1 - I'_{\hat{\Psi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p) \wedge T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - \mu(x_p) \vee 1 - \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \circ F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p) * 1 - I'_{\hat{\Psi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p) * T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - \mu(x_p) \circ 1 - \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p), 1 - I_{\hat{\Phi}_{(i_q)}}(x_p), T_{\hat{\Phi}_{(i_q)}}(x_p), 1 - \mu(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\} \Cup \left\{ \frac{(F'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - I'_{\hat{\Psi}_{(i_q)}}(x_p), T'_{\hat{\Psi}_{(i_q)}}(x_p), 1 - \mu'(x_p))}{x_p} \middle| i_q \in I, x_p \in X \right\}$$

$$= \hat{\Phi}^c_{(i_q)}(x_p) \Cup \hat{\Psi}^c_{(i_q)}(x_p).$$

□

4. Two Algorithms of Single-Valued Neutrosophic Fuzzy Soft Sets for Decision-Making

Depending on single-valued neutrosophic fuzzy soft sets, in the following, we introduce two new approaches for fuzzy decision-making problems.

Next, we construct Algorithm 1 as the first type for decision-making (i.e., the first application of a single-valued neutrosophic fuzzy soft set).

Algorithm 1: Determine the optimal decision based on a single-valued neutrosophic fuzzy soft set matrix.

First step: Input the single-valued neutrosophic fuzzy soft set $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ as follows:

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \,\middle|\, i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\},$$

to be evaluated by a group of experts n to element x on parameter i, where $T_{\hat{\Phi}_{(i_q)}}(x_p) \in [0,1]$ (i.e., the degree of truth membership), $I_{\hat{\Phi}_{(i_q)}}(x_p)$ (i.e., the degree of indeterminacy membership), $F_{\hat{\Phi}_{(i_q)}}(x_p)$ (i.e., the degree of falsity membership), and $\mu(x_p) \in [0,1]$.

Second step: Input the single-valued neutrosophic fuzzy soft set in matrix form (written as $\mathcal{M}_{q\times p}$, $p, q \in N$):

$$\mathcal{M}_{q\times p} = \begin{pmatrix} (T_{\hat{\Phi}_{(i_1)}}(x_1), I_{\hat{\Phi}_{(i_1)}}(x_1), F_{\hat{\Phi}_{(i_1)}}(x_1), \mu(x_1)) & (T_{\hat{\Phi}_{(i_1)}}(x_2), I_{\hat{\Phi}_{(i_1)}}(x_2), F_{\hat{\Phi}_{(i_1)}}(x_2), \mu(x_2)) & \cdots & (T_{\hat{\Phi}_{(i_2)}}(x_p), I_{\hat{\Phi}_{(i_2)}}(x_p), F_{\hat{\Phi}_{(i_2)}}(x_p), \mu(x_p)) \\ (T_{\hat{\Phi}_{(i_2)}}(x_1), I_{\hat{\Phi}_{(i_2)}}(x_1), F_{\hat{\Phi}_{(i_2)}}(x_1), \mu(x_1)) & (T_{\hat{\Phi}_{(i_2)}}(x_2), I_{\hat{\Phi}_{(i_2)}}(x_2), F_{\hat{\Phi}_{(i_2)}}(x_2), \mu(x_2)) & \cdots & (T_{\hat{\Phi}_{(i_2)}}(x_p), I_{\hat{\Phi}_{(i_2)}}(x_p), F_{\hat{\Phi}_{(i_2)}}(x_p), \mu(x_p)) \\ (T_{\hat{\Phi}_{(i_3)}}(x_1), I_{\hat{\Phi}_{(i_3)}}(x_1), F_{\hat{\Phi}_{(i_3)}}(x_1), \mu(x_1)) & (T_{\hat{\Phi}_{(i_3)}}(x_2), I_{\hat{\Phi}_{(i_3)}}(x_2), F_{\hat{\Phi}_{(i_3)}}(x_2), \mu(x_2)) & \cdots & (T_{\hat{\Phi}_{(i_3)}}(x_p), I_{\hat{\Phi}_{(i_3)}}(x_p), F_{\hat{\Phi}_{(i_3)}}(x_p), \mu(x_p)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{\hat{\Phi}_{(i_q)}}(x_1), I_{\hat{\Phi}_{(i_q)}}(x_1), F_{\hat{\Phi}_{(i_q)}}(x_1), \mu(x_1)) & (T_{\hat{\Phi}_{(i_q)}}(x_2), I_{\hat{\Phi}_{(i_q)}}(x_2), F_{\hat{\Phi}_{(i_q)}}(x_2), \mu(x_2)) & \cdots & (T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p)) \end{pmatrix}.$$

Third step: Calculate the center matrix (i.e., $\delta_{\hat{\Phi}_{(i_q)}}(x_p) = (T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p)) - \mu(x_p)$):

$$C_{q\times p} = \begin{pmatrix} \delta_{\hat{\Phi}_{(i_1)}}(x_1) & \delta_{\hat{\Phi}_{(i_1)}}(x_2) & \cdots, & \delta_{\hat{\Phi}_{(i_1)}}(x_p) \\ \delta_{\hat{\Phi}_{(i_2)}}(x_1) & \delta_{\hat{\Phi}_{(i_2)}}(x_2) & \cdots, & \delta_{\hat{\Phi}_{(i_2)}}(x_p) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\hat{\Phi}_{(i_q)}}(x_1) & \delta_{\hat{\Phi}_{(i_q)}}(x_2) & \cdots, & \delta_{\hat{\Phi}_{(i_q)}}(x_p) \end{pmatrix}.$$

Fourth step: Calculate the $d^{max}(x_j)$ (maximum decision), $d^{min}(x_j)$ (minimum decision), and $S(x_j)$ (score) of elements x_j ($j = 1, 2, \cdots, p$):

$$d^{max}(x_j) = \sum_{i=1}^{q} \left(1 - \delta_{\hat{\Phi}_{(i_q)}}(x_j)\right)^2, \quad d^{min}(x_j) = \sum_{i=1}^{q} (\delta_{\hat{\Phi}_{(i_q)}}(x_j))^2$$
$$S(x_j) = d^{max}(x_j) + d^{min}(x_j).$$

(to understand the motivation behind this method, let ρ be the Euclidean metric on R^q, $\mathbf{0} = (0, \cdots, 0)^T \in R^q$, $\mathbf{1} = (1, \cdots, 1)^T \in R^q$, and $\boldsymbol{\theta}_j = (\theta_{1,x_j}, \theta_{2,x_j}, \cdots, \theta_{q,x_j})^T \in R^q$. Thus $S(x_j) = [\rho(\boldsymbol{\theta}_j, \mathbf{1})]^2 + [\rho(\boldsymbol{\theta}_j, \mathbf{0})]^2$ $(j = 1, 2, \cdots, p)$).

Fifth step: Obtain the decision p satisfying

$$x_p = \max\left\{S(x_1), S(x_2), \cdots, S(x_j)\right\}.$$

Now, we show the principle and steps of the above Algorithm 1 by using the following example.

Example 8. *An investment company wants to choose some investment projects to make full use of idle funds. There are five alternatives $X = \{z_1, z_2, z_3, z_4, z_5\}$ that can be selected: two internet education projects (denoted as z_1 and z_2) and three film studio investments (represented as z_3, z_4, z_5). According to the project investment books, the decision-makers evaluate the five alternatives from the following three parameters $I = \{i_1, i_2, i_3\}$, where i_1 is "human resources", i_2 is "social benefits", and i_3 is "expected benefits". The data of the single-valued neutrosophic fuzzy soft set $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ is given by*

$$\hat{\Phi} = \left(\begin{array}{c|ccccc} I & z_1 & z_2 & z_3 & z_4 & z_5 \\ \hline i_1 & (0.3,0.7,0.5,0.2) & (0.1,0.8,0.5,0.5) & (0.2,0.6,0.8,0.7) & (0.5,0.6,0.5,0.2) & (0.4,0.7,0.9,0.1) \\ i_2 & (0.9,0.4,0.5,0.7) & (0.3,0.7,0.5,0.4) & (0.8,0.2,0.6,0.8) & (0.3,0.7,0.2,0.5) & (0.7,0.8,0.8,0.3) \\ i_3 & (0.6,0.3,0.5,0.6) & (0.3,0.5,0.6,0.4) & (0.7,0.1,0.6,0.3) & (0.8,0.9,0.6,0.4) & (0.7,0.8,0.9,0.6) \end{array} \right).$$

Now, we will explain the practical meaning of alternatives X by taking the alternative z_1 as an example: the single-valued neutrosophic fuzzy soft set $\hat{\Phi}_{(i_1)}(z_1) = (0.3, 0.7, 0.5, 0.2)$ is the evaluation by four expert groups; the single-valued neutrosophic fuzzy soft value 0.3 (meaning that 30% say yes in the first expert group) in $\hat{\Phi}_{(i_1)}(z_1)$, the single-valued neutrosophic fuzzy soft value 0.7 (meaning 70% say no in the second expert group) in $\hat{\Phi}_{(i_1)}(z_1)$, the single-valued neutrosophic fuzzy soft value 0.5 (meaning 50% say yes in the third expert group) in $\hat{\Phi}_{(i_1)}(z_1)$, and fuzzy value 0.2 (meaning 20% say no in the fourth expert group) in $\hat{\Phi}_{(i_1)}(z_1)$. Then, the single-valued neutrosophic fuzzy soft set in matrix form $\mathcal{M}_{3\times5}$ in the second step of Algorithm 1 is given by

$$\mathcal{M}_{3\times5} = \begin{pmatrix} (0.3,0.7,0.5,0.2) & (0.9,0.4,0.5,0.7) & (0.6,0.3,0.5,0.6) \\ (0.1,0.8,0.5,0.5) & (0.3,0.7,0.5,0.4) & (0.3,0.5,0.6,0.4) \\ (0.2,0.6,0.8,0.7) & (0.8,0.2,0.6,0.8) & (0.7,0.1,0.6,0.3) \\ (0.5,0.6,0.5,0.2) & (0.3,0.7,0.2,0.5) & (0.8,0.9,0.6,0.4) \\ (0.4,0.7,0.9,0.1) & (0.7,0.8,0.8,0.3) & (0.7,0.8,0.9,0.6) \end{pmatrix}.$$

Thus, we obtain the following center matrix $C_{3\times5}$ of $\mathcal{M}_{3\times5}$ in the third step of Algorithm 1:

$$C_{3\times5} = \begin{pmatrix} 1.3 & 1.1 & 0.8 \\ 0.9 & 1.1 & 1 \\ 0.9 & 0.8 & 1.1 \\ 1.4 & 0.7 & 1.9 \\ 1.9 & 2 & 1.8 \end{pmatrix}.$$

By calculating, we get $d^{max}(z_j)$, $d^{min}(z_j)$, and $S(z_j)$ of elements z_j $(j = 1, 2, 3, 4, 5)$:

$$d^{max}(z_1) = 0.14, d^{max}(z_2) = 0.02, d^{max}(z_3) = 0.06, d^1(z_4) = 1.06, d^{max}(z_5) = 2.45;$$

$$d^{min}(z_1) = 3.54, d^{min}(z_2) = 3.02, d^{min}(z_3) = 2.66, d^{min}(z_4) = 6.06, d^{min}(z_5) = 10.85;$$

$$S(z_1) = 3.68, S(z_2) = 3.04, S(z_3) = 2.72, S(z_4) = 7.12, S(z_5) = 13.3.$$

Finally, we can see from the fifth step that z_5 is the best decision.

Now, we present Algorithm 2 as a second type for a decision-making problem (i.e., a second application of the single-valued neutrosophic fuzzy soft set) as follows:

Algorithm 2: Determine the optimal decision based on AND operation of two single-valued neutrosophic fuzzy soft sets.

First step: Input the single-valued neutrosophic fuzzy soft sets $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ and $\hat{\Psi}_{(j_q)} \in (\mathbb{SVNFS})^{XJ}$, defined, respectively, as follows:

$$\hat{\Phi}_{(i_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p), \mu(x_p))}{x_p} \middle| i_q \in I,\ x_p \in X,\ 0 \le T_{\hat{\Phi}_{(i_q)}}(x_p) + I_{\hat{\Phi}_{(i_q)}}(x_p) + F_{\hat{\Phi}_{(i_q)}}(x_p) \le 3 \right\},$$

to be evaluated by a group of experts n to element x on parameter i, where $T_{\hat{\Phi}_{(i_q)}}(x_p) \in [0,1]$ (i.e., the degree of truth membership), $I_{\hat{\Phi}_{(i_q)}}(x_p)$ (i.e., the degree of indeterminacy membership), $F_{\hat{\Phi}_{(i_q)}}(x_p)$ (i.e., the degree of falsity membership), and $\mu(x_p) \in [0,1]$,

$$\hat{\Psi}_{(j_q)} = \left\{ \frac{(T'_{\hat{\Psi}_{(j_q)}}(x_p), I'_{\hat{\Psi}_{(j_q)}}(x_p), F'_{\hat{\Psi}_{(j_q)}}(x_p), \mu'(x_p))}{x_p} \middle| j_q \in J,\ x_p \in X,\ 0 \le T'_{\hat{\Psi}_{(j_q)}}(x_p) + I'_{\hat{\Psi}_{(j_q)}}(x_p) + F'_{\hat{\Psi}_{(j_q)}}(x_p) \le 3 \right\}$$

to be evaluated by a group of experts n to element x on parameter j, where $T_{\hat{\Psi}'_{(j_q)}}(x_p) \in [0,1]$ (i.e., the degree of truth membership), $I_{\hat{\Psi}'_{(j_q)}}(x_p)$ (i.e., the degree of indeterminacy membership), $F_{\hat{\Psi}'_{(j_q)}}(x_p)$ (i.e., the degree of falsity membership), and $\mu(x_p) \in [0,1]$.

Second step: Define and calculate the AND operation of two single-valued neutrosophic fuzzy soft sets $\hat{\Phi}_{(i_q)} \in (\mathbb{SVNFS})^{XI}$ and $\hat{\Psi}_{(j_q)} \in (\mathbb{SVNFS})^{XJ}$, denoted by $(\hat{\Phi}\overline{\wedge}\hat{\Psi})_{(i_q,j_q)}$ $(\forall i \in I, j \in J)$, defined as

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})_{(i_q,j_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p) \wedge T'_{\hat{\Psi}_{(j_q)}}(x_p), I_{\hat{\Phi}_{(i_q)}}(x_p) \vee I'_{\hat{\Psi}_{(j_q)}}(x_p), F_{\hat{\Phi}_{(i_q)}}(x_p) \vee F'_{\hat{\Psi}_{(j_q)}}(x_p), \mu(x_p) \wedge \mu'(x_p))}{x_p} \middle| i_q \in I, j_q \in J, x_p \in X \right\}.$$

Third step: Define and write the truth membership $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}$, the indeterminacy membership $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}$, and the falsity membership $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}$, respectively, as follows:

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)} = \left\{ \frac{(T_{\hat{\Phi}_{(i_q)}}(x_p) \wedge T'_{\hat{\Psi}_{(j_q)}}(x_p), \mu(x_p) \wedge \mu'(x_p))}{x_p} \middle| i_q \in I, j_q \in J, x_p \in X \right\},$$

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)} = \left\{ \frac{(I_{\hat{\Phi}_{(i_q)}}(x_p) \vee I'_{\hat{\Psi}_{(j_q)}}(x_p), \mu(x_p) \wedge \mu'(x_p))}{x_p} \middle| i_q \in I, j_q \in J, x_p \in X \right\},$$

and

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)} = \left\{ \frac{(F_{\hat{\Phi}_{(i_q)}}(x_p) \vee F'_{\hat{\Psi}_{(j_q)}}(x_p), \mu(x_p) \wedge \mu'(x_p))}{x_p} \middle| i_q \in I, j_q \in J, x_p \in X \right\}.$$

Fourth step: Define and compute the max-matrices of $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}$, $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}$, and $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}$, respectively, for every $x_p \in X$ as follows $(p = 1, 2, \cdots, N)$:

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}(x_p) = \frac{1}{2}\Big((T_{\hat{\Phi}_{(i_q)}}(x_p) \wedge T'_{\hat{\Psi}_{(j_q)}}(x_p)) + (\mu(x_p) \wedge \mu'(x_p)) \Big),$$

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}(x_p) = \Big((I_{\hat{\Phi}_{(i_q)}}(x_p) \vee I'_{\hat{\Psi}_{(j_q)}}(x_p)) \times (\mu(x_p) \wedge \mu'(x_p)) \Big),$$

and

$$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}(x_p) = \Big((F_{\hat{\Phi}_{(i_q)}}(x_p) \vee F'_{\hat{\Psi}_{(j_q)}}(x_p)) - (\mu(x_p) \wedge \mu'(x_p) \Big)^2.$$

Algorithm 2: *Cont.*

Fifth step: Calculate and write the max-decision τ_T (i.e., $\tau_T : X \to R$), τ_I (i.e., $\tau_I : X \to R$), and τ_F (i.e., $\tau_F : X \to R$) of $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}$, $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}$, and $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}$, respectively, for every $x_p \in X$ as follows ($p = 1, 2, \cdots, N$):

$$\tau_T(x_p) = \sum_{(i,j)\in I\times J} \delta_T(x_p)(i,j), \quad \tau_I(x_p) = \sum_{(i,j)\in I\times J} \delta_F(x_p)(i,j), \quad \text{and} \sum_{(i,j)\in I\times J} \delta_F(x_p)(i,j),$$

where

$$\delta_T(x_p)(i,j) = \begin{cases} (\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}(x_p), & (\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}(x_p) = \max\{(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(u_q,v_q)}(x_p) : (u,v) \in I \times J\} \\ 0, & \text{otherwise} \end{cases},$$

$$\delta_I(x_p)(i.j) = \begin{cases} (\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}(x_p), & (\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}(x_p) = \max\{(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(u_q,v_q)}(x_p) : (u,v) \in I \times J\} \\ 0, & \text{otherwise} \end{cases},$$

$$\delta_F(x_p)(i.j) = \begin{cases} (\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}(x_p), & (\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}(x_p) = \max\{(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(u_q,v_q)}(x_p) : (u,v) \in I \times J\} \\ 0, & \text{otherwise} \end{cases}.$$

Sixth step: Calculate the score $S(x_p)$ of element x_p as follows ($p = 1, 2, \cdots, N$):

$$S(x_p) = \tau_T(x_p) + \tau_I(x_p) + \tau_F(x_p).$$

Seventh step: Obtain the decision p satisfying

$$x_p = \max\{S(x_1), S(x_2), \cdots, S(x_j)\}.$$

Now, we show the principle and steps of the above Algorithm 2 using the following example.

Example 9. *(Continued from Example 11). Suppose that an investment company also adds three different parameters $J = \{j_1, j_2, j_3\}$, where j_1 is "marketing management", j_2 is "productivity of capital", and j_3 is "interest rates". The data of the single-valued neutrosophic fuzzy soft set $\hat{\Psi}_{(j_q)} \in (\mathbb{SVNFS})^{XJ}$ is given by*

$\hat{\Psi} =$

J	z_1	z_2	z_3	z_4	z_5
j_1	(0.5, 0.6, 0.7, 0.4)	(0.3, 0.2, 0.7, 0.8)	(0.6, 0.9, 0.4, 0.3)	(0.8, 0.8, 0.2, 0.1)	(0.9, 0.5, 0.4, 0.2)
j_2	(0.8, 0.4, 0.5, 0.2)	(0.7, 0.9, 0.2, 0.1)	(0.3, 0.3, 0.9, 0.4)	(0.9, 0.4, 0.5, 0.5)	(0.7, 0.8, 0.7, 0.2)
j_3	(0.9, 0.9, 0.5, 0.3)	(0.5, 0.9, 0.2, 0.1)	(0.6, 0.6, 0.1, 0.5)	(0.5, 0.7, 0.8, 0.8)	(0.6, 0.2, 0.4, 0.7)

Now, we explain the practical meaning of alternatives X by taking the alternative z_1 as an example: the single-valued neutrosophic fuzzy soft set $\hat{\Psi}_{(j_1)}(z_1) = (0.5, 0.6, 0.7, 0.4)$ is the evaluation by four expert groups; the single-valued neutrosophic fuzzy soft value 0.5 (meaning 50% say yes in the first expert group) in $\hat{\Psi}_{(j_1)}(z_1)$, the single-valued neutrosophic fuzzy soft value 0.6 (meaning 60% say no in the second expert group) in $\hat{\Psi}_{(j_1)}(z_1)$, the single-valued neutrosophic fuzzy soft value 0.7 (meaning 70% say yes in the third expert group) in $\hat{\Psi}_{(j_1)}(z_1)$, and fuzzy value 0.4 (meaning 40% say no in the fourth expert group) in $\hat{\Psi}_{(j_1)}(z_1)$. Then, by computing $(\hat{\Phi}\overline{\wedge}\hat{\Psi})_{(i_q,j_q)}$ ($q = 1, 2, 3$) in the second step of Algorithm 2, we obtain the following:

$\hat{\Phi}\overline{\wedge}\hat{\Psi}$	z_1	z_2	z_3	z_4	z_5
(i_1, j_1)	(0.3, 0.7, 0.7, 0.2)	(0.1, 0.8, 0.7, 0.5)	(0.2, 0.9, 0.8, 0.3)	(0.5, 0.8, 0.5, 0.1)	(0.4, 0.7, 0.9, 0.1)
(i_1, j_2)	(0.3, 0.7, 0.5, 0.2)	(0.1, 0.9, 0.5, 0.1)	(0.2, 0.6, 0.9, 0.4)	(0.5, 0.6, 0.5, 0.2)	(0.4, 0.8, 0.9, 0.1)
(i_1, j_3)	(0.3, 0.9, 0.5, 0.2)	(0.1, 0.9, 0.5, 0.1)	(0.2, 0.6, 0.8, 0.5)	(0.5, 0.7, 0.8, 0.2)	(0.4, 0.7, 0.9, 0.1)
(i_2, j_1)	(0.5, 0.6, 0.7, 0.4)	(0.3, 0.7, 0.7, 0.4)	(0.6, 0.9, 0.6, 0.3)	(0.3, 0.8, 0.2, 0.1)	(0.7, 0.8, 0.8, 0.2)
(i_2, j_2)	(0.8, 0.4, 0.5, 0.2)	(0.3, 0.9, 0.5, 0.1)	(0.3, 0.3, 0.9, 0.4)	(0.3, 0.7, 0.5, 0.5)	(0.7, 0.8, 0.8, 0.2)
(i_2, j_3)	(0.9, 0.9, 0.5, 0.3)	(0.3, 0.9, 0.5, 0.1)	(0.6, 0.6, 0.6, 0.5)	(0.3, 0.7, 0.8, 0.5)	(0.6, 0.8, 0.8, 0.3)
(i_3, j_1)	(0.5, 0.6, 0.7, 0.4)	(0.3, 0.5, 0.6, 0.4)	(0.7, 0.8, 0.6, 0.1)	(0.8, 0.9, 0.6, 0.1)	(0.7, 0.8, 0.9, 0.2)
(i_3, j_2)	(0.6, 0.4, 0.5, 0.2)	(0.3, 0.9, 0.6, 0.1)	(0.3, 0.3, 0.9, 0.3)	(0.8, 0.9, 0.6, 0.4)	(0.7, 0.8, 0.9, 0.2)
(i_3, j_3)	(0.6, 0.9, 0.5, 0.3)	(0.3, 0.9, 0.6, 0.1)	(0.6, 0.6, 0.6, 0.3)	(0.5, 0.9, 0.8, 0.4)	(0.6, 0.8, 0.9, 0.6)

By calculating in the third step of Algorithm 2, we get the truth membership $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}$*, the indeterminacy membership* $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}$*, and the falsity membership* $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}$*, respectively, as follows:* $(q = 1, 2, 3)$*:*

$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T$	z_1	z_2	z_3	z_4	z_5
(i_1, j_1)	(0.3, 0.2)	(0.1, 0.5)	(0.2, 0.3)	(0.5, 0.1)	(0.4, 0.1)
(i_1, j_2)	(0.3, 0.2)	(0.1, 0.1)	(0.2, 0.4)	(0.5, 0.2)	(0.4, 0.1)
(i_1, j_3)	(0.3, 0.2)	(0.1, 0.1)	(0.2, 0.5)	(0.5, 0.2)	(0.4, 0.1)
(i_2, j_1)	(0.5, 0.4)	(0.3, 0.4)	(0.6, 0.3)	(0.3, 0.1)	(0.7, 0.2)
(i_2, j_2)	(0.8, 0.2)	(0.3, 0.1)	(0.3, 0.4)	(0.3, 0.5)	(0.7, 0.2)
(i_2, j_3)	(0.9, 0.3)	(0.3, 0.1)	(0.6, 0.5)	(0.3, 0.5)	(0.6, 0.3)
(i_3, j_1)	(0.5, 0.4)	(0.3, 0.4)	(0.7, 0.1)	(0.8, 0.1)	(0.7, 0.2)
(i_3, j_2)	(0.6, 0.2)	(0.3, 0.1)	(0.3, 0.3)	(0.8, 0.4)	(0.7, 0.2)
(i_3, j_3)	(0.6, 0.3)	(0.3, 0.1)	(0.6, 0.3)	(0.5, 0.4)	(0.6, 0.6)

$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I$	z_1	z_2	z_3	z_4	z_5
(i_1, j_1)	(0.7, 0.2)	(0.8, 0.5)	(0.9, 0.3)	(0.8, 0.1)	(0.7, 0.1)
(i_1, j_2)	(0.7, 0.2)	(0.9, 0.1)	(0.6, 0.4)	(0.6, 0.2)	(0.8, 0.1)
(i_1, j_3)	(0.9, 0.2)	(0.9, 0.1)	(0.6, 0.5)	(0.7, 0.2)	(0.7, 0.1)
(i_2, j_1)	(0.6, 0.4)	(0.7, 0.4)	(0.9, 0.3)	(0.8, 0.1)	(0.8, 0.2)
(i_2, j_2)	(0.4, 0.2)	(0.9, 0.1)	(0.3, 0.4)	(0.7, 0.5)	(0.8, 0.2)
(i_2, j_3)	(0.9, 0.3)	(0.9, 0.1)	(0.6, 0.5)	(0.7, 0.5)	(0.8, 0.3)
(i_3, j_1)	(0.6, 0.4)	(0.5, 0.4)	(0.8, 0.1)	(0.9, 0.1)	(0.8, 0.2)
(i_3, j_2)	(0.4, 0.2)	(0.9, 0.1)	(0.3, 0.3)	(0.9, 0.4)	(0.8, 0.2)
(i_3, j_3)	(0.9, 0.3)	(0.9, 0.1)	(0.6, 0.3)	(0.9, 0.4)	(0.8, 0.6)

$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F$	z_1	z_2	z_3	z_4	z_5
(i_1, j_1)	(0.7, 0.2)	(0.7, 0.5)	(0.8, 0.3)	(0.5, 0.1)	(0.9, 0.1)
(i_1, j_2)	(0.5, 0.2)	(0.5, 0.1)	(0.9, 0.4)	(0.5, 0.2)	(0.9, 0.1)
(i_1, j_3)	(0.5, 0.2)	(0.5, 0.1)	(0.8, 0.5)	(0.8, 0.2)	(0.9, 0.1)
(i_2, j_1)	(0.7, 0.4)	(0.7, 0.4)	(0.6, 0.3)	(0.2, 0.1)	(0.8, 0.2)
(i_2, j_2)	(0.5, 0.2)	(0.5, 0.1)	(0.9, 0.4)	(0.5, 0.5)	(0.8, 0.2)
(i_2, j_3)	(0.5, 0.3)	(0.5, 0.1)	(0.6, 0.5)	(0.8, 0.5)	(0.8, 0.3)
(i_3, j_1)	(0.7, 0.4)	(0.6, 0.4)	(0.6, 0.1)	(0.6, 0.1)	(0.9, 0.2)
(i_3, j_2)	(0.5, 0.2)	(0.6, 0.1)	(0.9, 0.3)	(0.6, 0.4)	(0.9, 0.2)
(i_3, j_3)	(0.5, 0.3)	(0.6, 0.1)	(0.6, 0.3)	(0.8, 0.4)	(0.9, 0.6)

By calculating in the fourth step of Algorithm 2, we obtain the max-matrices of $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T_{(i_q,j_q)}$, $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I_{(i_q,j_q)}$, *and* $(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F_{(i_q,j_q)}$ $(p = 1,2,3,4,5;\ q = 1,2,3)$*, respectively, for every* $z_p \in X$ *as follows:*

$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^T$	z_1	z_2	z_3	z_4	z_5
(i_1,j_1)	0.25	$\underline{0.3}$	0.25	$\underline{0.3}$	0.25
(i_1,j_2)	0.25	0.1	0.3	$\underline{0.35}$	0.25
(i_1,j_3)	0.25	0.1	$\underline{0.35}$	$\underline{0.35}$	0.25
(i_2,j_1)	$\underline{0.45}$	0.35	$\underline{0.45}$	0.2	$\underline{0.45}$
(i_2,j_2)	$\underline{0.5}$	0.2	0.35	0.4	0.45
(i_2,j_3)	$\underline{0.6}$	0.2	0.55	0.4	0.45
(i_3,j_1)	$\underline{0.45}$	0.35	0.4	$\underline{0.45}$	$\underline{0.45}$
(i_3,j_2)	0.4	0.2	0.3	$\underline{0.6}$	0.45
(i_3,j_3)	0.45	0.2	0.45	0.45	$\underline{0.6}$

,

$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^I$	z_1	z_2	z_3	z_4	z_5
(i_1,j_1)	0.14	$\underline{0.4}$	0.27	0.08	0.07
(i_1,j_2)	0.14	0.09	$\underline{0.24}$	0.12	0.08
(i_1,j_3)	0.18	0.09	$\underline{0.3}$	0.14	0.07
(i_2,j_1)	0.24	$\underline{0.28}$	0.27	0.08	0.16
(i_2,j_2)	0.08	0.08	0.12	$\underline{0.35}$	0.16
(i_2,j_3)	0.27	0.09	0.3	$\underline{0.35}$	0.24
(i_3,j_1)	$\underline{0.24}$	0.2	0.08	0.09	0.16
(i_3,j_2)	0.08	0.09	0.09	$\underline{0.36}$	0.16
(i_3,j_3)	0.27	0.09	0.18	0.36	$\underline{0.48}$

,

$(\hat{\Phi}\overline{\wedge}\hat{\Psi})^F$	x_1	z_2	z_3	z_4	z_5
(i_1,j_1)	0.25	0.04	0.25	0.16	$\underline{0.64}$
(i_1,j_2)	0.09	0.16	0.25	0.09	$\underline{0.64}$
(i_1,j_3)	0.09	0.16	0.09	0.36	$\underline{0.64}$
(i_2,j_1)	0.09	0.09	0.09	0.01	$\underline{0.36}$
(i_2,j_2)	0.09	0.16	0.25	0	$\underline{0.36}$
(i_2,j_3)	0.04	0.16	0.01	0.09	$\underline{0.25}$
(i_3,j_1)	0.09	0.04	0.25	0.25	$\underline{0.49}$
(i_3,j_2)	0.09	0.25	0.36	0.04	$\underline{0.49}$
(i_3,j_3)	0.04	$\underline{0.25}$	0.09	0.16	0.09

.

By calculating in the fifth step of Algorithm 2, we obtain the max-decision τ_T, τ_I*, and* τ_F *of elements* z_p*, respectively, as follows* $(p = 1,2,3,4,5)$*:*

$$\tau_T(z_1) = 2,\ \tau_T(z_2) = 0.3,\ \tau_T(z_3) = 0.8,\ \tau_T(z_4) = 2.05,\ \tau_T(z_5) = 1.5;$$

$$\tau_I(z_1) = 0.24,\ \tau_I(z_2) = 0.68,\ \tau_I(z_3) = 0.54,\ \tau_I(z_4) = 1.06,\ \tau_I(z_5) = 0.48;$$

$$\tau_F(z_1) = 0,\ \tau_F(z_2) = 0.25,\ \tau_F(z_3) = 0,\ \tau_F(z_4) = 0,\ \tau_F(z_5) = 3.87.$$

By calculating in the sixth step of Algorithm 2, the scores $S(z_p)$ *of elements* $z_p (p = 1,2,3,4,5)$*, respectively, are as follows:*

$$S(z_1) = 2.24,\ S(z_2) = 1.23,\ S(z_3) = 1.34,\ S(z_4) = 3.11,\ S(z_5) = 5.85.$$

Finally, we know from the seventh step that z_5 has a high value. Therefore, the experts should select z_5 as the best choice.

Remark 2.

(1) By means of Algorithms 1 and 2, we can see that the final results are in agreement. Thus, x_5 is the most accurate and refinable.

(2) By comparing the steps in Algorithms 1 and 2, we can see that step 4 and step 5 in Algorithm 2 are complicated in their process compared to step 2 and step 3 in Algorithm 1, respectively. So, if we take the complexity of these steps into consideration, Algorithm 2 gives its decision concisely.

(3) Algorithms 1 and 2 that we have elaborated here arrive at their decisions by combining the concept of single-valued neutrosophic fuzzy set theory and soft set theory. As result, we can apply Algorithm 1 to picture fuzzy soft sets [29], generalized picture fuzzy soft sets [13], and interval-valued neutrosophic soft sets [12]. Further, Algorithm 2 can be applied to possibility m-polar fuzzy soft sets [15] and possibility multi-fuzzy soft sets [17].

5. Conclusions

We introduced the notion of the single-valued neutrosophic fuzzy soft set as a novel neutrosophic soft set model. We discussed the five operations of the single-valued neutrosophic fuzzy soft set, such as subset, equal, union, intersection, and complement. The structure properties of the single-valued neutrosophic fuzzy soft set are explained. Then, a novel approach (i.e., Algorithm 1) is presented as a single-valued neutrosophic fuzzy soft set decision method. Lastly, an application (i.e., Algorithm 2) of a single-valued neutrosophic fuzzy soft set for fuzzy decision-making is constructed, and the two approaches (i.e., Algorithms 1 and 2) introduce an important contribution to further research and relevant applications. Therefore, in the future, we will provide a real application with a real dataset or we will apply the two approaches (i.e., Algorithms 1 and 2) to lung cancer disease [30] and coronary artery disease [31]. In addition, we will describe in more detail in order to clarify if the methods (i.e., Algorithms 1 and 2) converge or diverge from standard approaches such as fuzzy sets [1], intuitionistic fuzzy sets [2], picture fuzzy sets [3].

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Generalized Neutrosophic TOPSIS to Solve Multi-Criteria Decision-Making Problems

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Abstract: Multi-criteria decision making (MCDM) is the technique of selecting the best alternative from multiple alternatives and multiple conditions. The technique for order preference by similarity to an ideal solution (TOPSIS) is a crucial practical technique for ranking and selecting different options by using a distance measure. In this article, we protract the fuzzy TOPSIS technique to neutrosophic fuzzy TOPSIS, and prove the accuracy of the method by explaining the MCDM problem with single-value neutrosophic information, and use the method for supplier selection in the production industry. We hope that this article will promote future scientific research on numerous existence issues based on multi-criteria decision making.

Keywords: Neutrosophic set, Single valued Neutrosophic set, TOPSIS, MCDM

1. Introduction

We faced a lot of complications in different areas of life which contains vagueness such as engineering, economics, modeling, and medical diagnoses, etc. However, a general question is raised that in mathematical modeling how we can express and use the uncertainty. A lot of researchers in the world proposed and recommended different approaches to solve those problems that contain uncertainty. In decision-making problems, multiple attribute decision making (MADM) is the most essential part which provides us to find the most appropriate and extraordinary alternative. However, to choose the appropriate alternative is very difficult because of vague information in some cases. To overcome such situations, Zadeh developed the notion of fuzzy sets (FSs) [1] to solve those problems which contain uncertainty and vagueness. It is observed that in some cases circumstances cannot be handled by fuzzy sets, to overcome such types of situations Turksen [2] gave the idea of interval-valued fuzzy sets (IVFSs). In some cases, we must deliberate membership unbiassed as the non- membership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by FSs nor IVFSs. To overcome these difficulties Atanassov offered the concept of Intuitionistic fuzzy sets (IFSs) [3]. The theory which was presented by Atanassov only deals the insufficient data considering both the membership and non-membership values, but the intuitionistic fuzzy set theory cannot handle the incompatible and imprecise information. To deal with such incompatible and imprecise data Smarandache [4] extended the work of Atanassov IFSs and proposed a powerful tool comparative to FSs and IFSs to deal with indeterminate, incomplete, and inconsistent information's which faced in real-life problems. Since the direct use of Neutrosophic sets (NSs) for TOPSIS is somewhat difficult. To apply the NSs, Wang et al. introduced a subclass of NSs known as single-valued Neutrosophic sets (SVNSs) in [5]. In [6] the author proposed a geometric interpretation by using NSs. Gulfam et al. [7] introduced a new distance formula for SVNSs and developed some new techniques under the Neutrosophic environment. The concept of a single-valued Neutrosophic soft expert set proposed in [8] by combining the SVNSs and soft expert sets.

To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [9], they constructed the concept of cut sets of SVNNs. On the base of the correlation of IFSs, the term correlation coefficient of SVNSs [10] introduced and proposed a decision-making method by using a weighted correlation coefficient or the weighted cosine similarity measure of SVNSs. In [11] the idea of simplified Neutrosophic sets introduced with some operational laws and aggregation operators such as real-life Neutrosophic weighted arithmetic average operator and weighted geometric average operator. They constructed an MCDM method on the base of proposed aggregation operators and cosine similarity measure for simplified neutrosophic sets. Sahin and Yiğider [12] extended the TOPSIS method to MCDM with a single-valued neutrosophic technique.

The TOPSIS method is presented in [13] to solve multi-criteria decision problems with different choices. In [14], Chen & Hwang extended the idea of the TOPSIS method and proposed a new TOPSIS model. The author uses the newly proposed decision-making method to solve uncertain data [15]. In [16], the authors applied this method to the prediction of diabetic patients in medical diagnosis. In [17–19] the authors studied the soft set TOPSIS, fuzzy TOPSIS, and Intuitionistic Fuzzy TOPSIS respectively and used for decision making. In [20], for the solution of single-valued neutrosophic soft set expert based multi-attribute decision-making problems, the authors proposed the TOPSIS technique. Generalized fuzzy TOPSIS was given in [21,22] with accuracy function. Maji [23] proposed the concept of neutrosophic soft sets (NSSs) with some properties and operations. Authors studied NSSs and gave some new definitions on NSSs [24], they also gave the idea of neutrosophic soft matrices with some operations and proposed a decision-making method. Many researchers developed the decision-making models by using the NSSs reported in the literature [25–27]. Elhassouny and Smarandache [28] extended the work on a simplified TOPSIS method and by using single-valued Neutrosophic information they proposed Neutrosophic simplified TOPSIS method. Saqlain et.al [21] presented generalized neutrosophic TOPSIS using accuracy function for the neutrosophic hypersoft set environment. The concept of single-valued neutrosophic cross-entropy measure introduced by Jun [29], he also constructed an MCDM method and claimed that this proposed method is more appropriate than previous methods for decision making.

Saha and Broumi [31], studied the interval-valued neutrosophic sets (IVNSs) and developed some new set-theoretic operations on IVNSs with their properties. The idea of an Interval-valued generalized single valued neutrosophic trapezoidal number (IVGSVTrN) was presented by Deli [32] with some operations and discussed their properties based on neutrosophic numbers. Hashim et al [33], studied the vague set and interval neutrosophic set and established a new theory known as interval neutrosophic vague set (INVS), they also presented some operations for INVS with their properties and derived the properties by using numerical examples. In [34], Abdel basset et al. applied TODIM and TOPSIS methods based on the best-worst method to increase the accuracy of evaluation under uncertainty according to the NSs. They also used the plithogenic set theory to resolve the indeterminate information and evaluate the economic performance of manufacturing industries, they used the AHP method to find the weight vector of the financial ratios to achieve this goal after that they used the VIKOR and TOPSIS methods to utilize the companies ranking [35, 36].

In the following paragraph, we explain some positive impacts of this research. The concentration of this study is to evaluate the best supplier for the production industry. This research is a very suitable illustration of Neutrosophic TOPSIS. A group of decision-makers chooses the best supplier for the production industry. The Neutrosophic TOPSIS method increases alternative performances based on the best and worst solutions.

1.1 Motivation and Contribution

Classical TOPSIS uses clear techniques for language assessment, but due to the imprecision and ambiguity of language assessment, we propose neutrosophic TOPSIS. In this paper, we discuss the

NSs and SVNSs with some operations. We presented the generalization of TOPSIS for the SVNSs and use the proposed method for supplier selection.

1.2 Structure of Article

In Section 2, some basic definitions have been added, which will help the rest of this article. Section 3 consists of the main work of the article, which defines the neutrosophic TOPSIS algorithm. The application of the proposed method and calculations are presented in section 4 and finally, the conclusion draws in Section 5.

2. Preliminaries

In this section, we remind some basic definitions such as NSs and SVNSs with some operations that will be used in the following sequel.

Neutrosophic Set (NS) [30]: Let X be a space of points and x be an arbitrary element of X. A neutrosophic set A in X is defined by a Truth-membership function $T_A(x)$, an Indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ i.e.; $T_A(x), I_A(x), F_A(x): X \to]0^-, 1^+[$, and $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Single Valued Neutrosophic Sets [5]**:** Let E be a universe. An SVNS over E is an NS over E, but truthiness, indeterminacy, and falsity membership functions are defined

$T_A(x): X \to [0, 1]$, $I_A(x): X \to [0, 1]$, $F_A(x): X \to [0, 1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Multiplication of SVNS [11]: Let A = $\{\alpha_1, \alpha_2, \alpha_3\}$ and B = $\{\beta_1, \beta_2, \beta_3\}$ are two SVN numbers, then their multiplication is defined as follows A ⊗ B = $(\alpha_1\beta_1,\ \alpha_2 + \beta_2 - \alpha_2\beta_2,\ \alpha_3 + \beta_3 - \alpha_3\beta_3)$.

3. Neutrosophic TOPSIS [11]

3. 1. Algorithm for Neutrosophic TOPSIS using SVNNs

To explain the procedure of Neutrosophic TOPSIS using SVNNs the following steps are followed. Let A = $\{A_1, A_2, A_3, \ldots, A_m\}$ be a set of alternatives and C = $\{C_1, C_2, C_3, \ldots, C_n\}$ be a set of evaluation criteria and *DM* be a set of "l" decision-makers as follows *DM* = $\{DM_1, DM_2, DM_3, \ldots, DM_l\}$. In the form of linguistic variables, the importance of the evaluation criteria, DMs, and alternative ratings are given in Table 1.

Step 1: Computation of weights of the DMs

Let the SVN number for rating the k^{th} DM is denoted by

$D_k = (T_k^{dm},\ I_k^{dm},\ F_k^{dm})$

Weight of the k^{th} DM can be found by the following formula

$$\lambda_k = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)}\ ;\text{ where } \lambda_k \geq 0 \text{ and } \sum_{k=1}^{l}\lambda_k = 1$$

Step 2: Computation of the Aggregated Neutrosophic Decision Matrix (ANDM)

The ANDM is given as follows

$$D = \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} = [r_{ij}]_{m\times n}$$

where r_{ij} can be defined as

$r_{ij} = (T_{ij},\ I_{ij},\ F_{ij}) = (T_{A_i}(x_j),\ I_{A_i}(x_j),\ F_{A_i}(x_j))$, where $i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n$

Therefore, ANDM written as follows

$$D = \begin{bmatrix} (T_{A_1}(x_1), I_{A_1}(x_1), F_{A_1}(x_1)) & (T_{A_1}(x_2), I_{A_1}(x_2), F_{A_1}(x_2)) & \cdots & (T_{A_1}(x_n), I_{A_1}(x_n), F_{A_1}(x_n)) \\ (T_{A_2}(x_1), I_{A_2}(x_1), F_{A_2}(x_1)) & (T_{A_2}(x_2), I_{A_2}(x_2), F_{A_2}(x_2)) & \cdots & (T_{A_2}(x_n), I_{A_2}(x_n), F_{A_2}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m}(x_1), I_{A_m}(x_1), F_{A_m}(x_1)) & (T_{A_m}(x_2), I_{A_m}(x_2), F_{A_m}(x_2)) & \cdots & (T_{A_m}(x_n), I_{A_m}(x_n), F_{A_m}(x_n)) \end{bmatrix}$$

rating for the i^{th} alternative w.r.t. the j^{th} criterion by the k^{th} *DM*

$$r_{ij}^{(k)} = (T_{ij}^{(k)},\ I_{ij}^{(k)},\ F_{ij}^{(k)})$$

For *DM* weights and alternative ratings r_{ij} can be calculated by using a single-valued neutrosophic weighted averaging operator (SVNWAO)

$$r_{ij} = \left[1 - \prod_{k=1}^{l}(1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_{ij}^{(k)})^{\lambda_k}\right]$$

Step 3: Computation of the weights for the criteria

Let an SVNN allocated to the criterion by X_j the k^{th} *DM* is denoted as

$$w_j^{(k)} = \left(T_j^{(k)}, I_j^{(k)}, F_j^{(k)}\right)$$

SVNWAO to compute the weights of the criteria is given as follows

$$w_j = \left[1 - \prod_{k=1}^{l}(1 - T_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_j^{(k)})^{\lambda_k}\right]$$

The aggregated weight for the criterion X_j is represented as

$$w_j = (T_j,\ I_j,\ F_j) \qquad j = 1, 2, 3, \ldots, n$$

$$W = [w_1, w_2, w_3, \ldots, w_n]^{Transpose}$$

Step 4: Computation of Aggregated Weighted Neutrosophic Decision Matrix (AWNDM)

The AWNDM is calculated as follows

$$R' = \begin{bmatrix} r'_{11} & r'_{12} & \cdots & r'_{1n} \\ r'_{21} & r'_{22} & \cdots & r'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r'_{m1} & r'_{m2} & \cdots & r'_{mn} \end{bmatrix} = \left[r'_{ij}\right]_{m\times n}$$

where $r'_{ij} = \left(T_{A_i.W}(x_j), I_{A_i.W}(x_j), F_{A_i.W}(x_j)\right)$ where $i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n.$

Therefore, R' can be written as

$$R' = \begin{bmatrix} (T_{A_1.W}(x_1), I_{A_1.W}(x_1), F_{A_1.W}(x_1)) & (T_{A_1.W}(x_2), I_{A_1.W}(x_2), F_{A_1.W}(x_2)) & \cdots & (T_{A_1.W}(x_n), I_{A_1.W}(x_n), F_{A_1.W}(x_n)) \\ (T_{A_2.W}(x_1), I_{A_2.W}(x_1), F_{A_2.W}(x_1)) & (T_{A_2.W}(x_2), I_{A_2.W}(x_2), F_{A_2.W}(x_2)) & \cdots & (T_{A_2.W}(x_n), I_{A_2.W}(x_n), F_{A_2.W}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m.W}(x_1), I_{A_m.W}(x_1), F_{A_m.W}(x_1)) & (T_{A_m.W}(x_2), I_{A_m.W}(x_2), F_{A_m.W}(x_2)) & \cdots & (T_{A_m.W}(x_n), I_{A_m.W}(x_n), F_{A_m.W}(x_n)) \end{bmatrix}$$

To find $T_{A_i.W}(x_j)$, $I_{A_i.W}(x_j)$ and $F_{A_i.W}(x_j)$ we used

$$R \otimes W = \left\{\langle x, T_{A_i.W}(x)\rangle, \langle x, I_{A_i.W}(x)\rangle, \langle x, F_{A_i.W}(x)\rangle \,\middle|\, x \in X\right\}$$

The components of the product given as

$T_{A_i.W}(x) = T_{A_i}(x).\ T_j$

$I_{A_i.W}(x) = I_{A_i}(x) + I_j(x) - I_{A_i}(x) \times I_j(x)$

$F_{A_i.W}\ (x)\ =\ F_{A_i}\ (x)\ +\ F_j\ (x)\ -\ \ F_{A_i}\ (x)\times F_j\ (x)$

Step 5: Computation of Single Valued Neutrosophic Positive Ideal Solution (SVN-PIS) and Single Valued Neutrosophic Positive Ideal Solution (SVN-NIS)

Let J_1 be the benefit criteria and J_2 be the cost criteria. A^* be an SVN-PIS and A' be an SVN-NIS as follows

$A^*\ = (T_{A^*W}\ (x_j),\ I_{A^*W}\ (x_j),\ F_{A^*W}\ (x_j))$ and

$A'\ = (T_{A'W}\ (x_j),\ I_{A'W}\ (x_j),\ F_{A'W}\ (x_j))$

The components of SVN-PIS and SVN-NIS are following

$$T_{A^*W}\ (x_j)\ =\ \left(\left(\max_i T_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_1\right),\left(\min_i T_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_2\right)\right)$$

$$I_{A^*W}\ (x_j)\ =\ \left(\left(\min_i I_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_1\right),\left(\max_i I_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_2\right)\right)$$

$$F_{A^*W}\ (x_j)\ =\ \left(\left(\min_i F_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_1\right),\left(\max_i F_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_2\right)\right)$$

$$T_{A'W}\ (x_j)\ =\ \left(\left(\min_i T_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_1\right),\left(\max_i T_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_2\right)\right)$$

$$I_{A'W}\ (x_j)\ =\ \left(\left(\max_i I_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_1\right),\left(\min_i I_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_2\right)\right)$$

$$F_{A'W}\ (x_j)\ =\ \left(\left(\max_i F_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_1\right),\left(\min_i F_{A_i.W}(x_j)\ \middle|\ j\ \in\ j_2\right)\right)$$

Step 6: Computation of Separation Measures

For the separation measures $d^*$and d', Normalized Euclidean Distance is used as given as

$$d_i^*= \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j) - T_{A^*W}\ (x_j)\right)^2 +\ \left(I_{A_i.W}(x_j) - I_{A^*W}\ (x_j)\right)^2 + \left(F_{A_i.W}(x_j) - F_{A^*W}\ (x_j)\right)^2\right]\right)^{0.5}$$

$$d_i'= \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j) - T_{A'W}\ (x_j)\right)^2 +\ \left(I_{A_i.W}(x_j) - I_{A'W}\ (x_j)\right)^2 + \left(F_{A_i.W}(x_j) - F_{A'W}\ (x_j)\right)^2\right]\right)^{0.5}$$

Step 7: Computation of Relative Closeness Coefficient (*RCC*)

The *RCC* of an alternative *Ai* w.r.t. the SVN-PIS A^* is computed as

$$RCCi = \frac{d_i'}{d_i'+d_i^*} \qquad \text{where } 0 \leq RCCi \leq 1$$

Step 8: Ranking alternatives

After computation of *RCCi* for each alternative A_i, the rank of the alternatives presented in descending orders of *RCCi*.

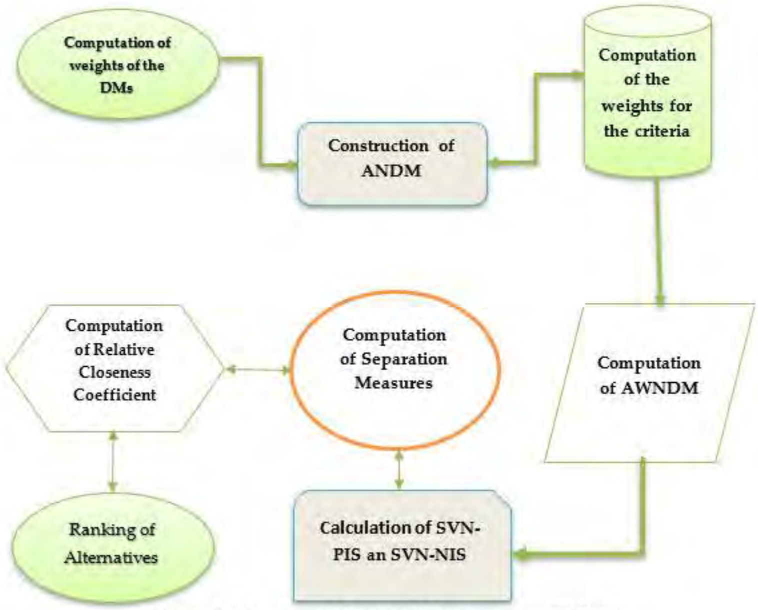

Figure 1 *Algorithm of Proposed Neutrosophic TOPSIS*

4. Application of Neutrosophic TOPSIS in decision making

A production industry wants to hire a supplier, for the selection of supplier managing director of the industry decides the criteria for supplier selection. The industry hires a team of decision-makers for the selection of the best supplier. Consider A = {A_i: i = 1, 2, 3, 4, 5} be a set of supplier and $DM = \{DM_1, DM_2, DM_3, DM_4\}$ be a team of decision-makers (l = 4). The evaluation criteria (n = 5) for the selection of supplier given as follows,

$$C = \begin{cases} Benifit\ Criteria \\ Cost\ Criteria \end{cases} \quad j_1 = \begin{cases} X_1: & Delivery \\ X_2: & Quality \\ X_3: & Flexibility \\ X_4: & Service \end{cases} \quad j_2 = \{X_5 : Price$$

Calculations of the problem using the proposed SVN-TOPSIS for the importance of criteria and DMs SVN rating scale is given in the following Table

Table 1. Linguistic variables LV's for rating the importance of criteria and decision-makers

LVs	SVNNs
VI	(.90, .10, .10)
I	(.75, .25, .20)
M	(.50, .50, .50)
UI	(.35, .75, .80)
VUI	(.10, .90, .90)

Where VI, I, M, UI, VUI stand for very important, important, medium, unimportant, very unimportant respectively. The alternative ratings are given in the following table

Table 2. Alternative Ratings for Linguistic Variables

LVs	SVNNs
EG	(1.0, 0.0,0.0)

VVG	(.90, .10, .10)
VG	(.80, .15, .20)
G	(.70, .25, .30)
MG	(.60, .35, .40)
M	(.50, .50, .50)
MB	(.40, .65, .60)
B	(.30, .75, .70)
VB	(.20, .85, .80)
VVB	(.10, .90, .90)
EB	(0.0,1.0,1.0)

Where EG, VVG, VG, G, MG, M, MB, B, VB, VVB, EB are representing extremely good, very very good, very good, good, medium good, medium, medium bad, bad, very bad, very very bad, extremely bad respectively.

Step 1: Determine the weights of the DMs

Weights for the *DMs* are calculated as follows

$$\lambda_k = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)} \ ; \ \lambda_k \geq 0 \text{ and } \sum_{k=1}^{l}\lambda_k = 1$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)}$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}}{\begin{array}{c}1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{\left(1-T_2^{dm}(x)\right)^2+\left(I_2^{dm}(x)\right)^2+\left(F_2^{dm}(x)\right)^2\right\}\right]^{0.5}+\\ 1-\left[\frac{1}{3}\left\{\left(1-T_3^{dm}(x)\right)^2+\left(I_3^{dm}(x)\right)^2+\left(F_3^{dm}(x)\right)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{\left(1-T_4^{dm}(x)\right)^2+\left(I_4^{dm}(x)\right)^2+\left(F_4^{dm}(x)\right)^2\right\}\right]^{0.5}\end{array}}$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{(1-0.9)^2+(0.10)^2+(0.10)^2\right\}\right]^{0.5}}{\begin{array}{c}1-\left[\frac{1}{3}\left\{(1-0.9)^2+(0.10)^2+(0.10)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{(1-0.75)^2+(0.25)^2+(0.20)^2\right\}\right]^{0.5}+\\ 1-\left[\frac{1}{3}\left\{(1-0.50)^2+(0.50)^2+(0.50)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{(1-0.35)^2+(0.75)^2+(0.80)^2\right\}\right]^{0.5}\end{array}}$$

$$\lambda_1 = \frac{0.9}{0.9+0.76548+0.5+0.26402}$$

$$\lambda_1 = \frac{0.9}{2.42950} = 0.37045$$

$\lambda_1 = 0.37045$

Similarly, we get the weights for the other decision-makers as follows

$$\lambda_2 = \frac{0.76548}{2.42950} = 0.31508$$

$\lambda_2 = 0.31508$

$$\lambda_3 = \frac{0.5}{2.42950} = 0.20580$$

$\lambda_3 = 0.20580$

$$\lambda_4 = \frac{0.26402}{2.42950} = 0.10867$$

$\lambda_4 = 0.10867$

The weights for *DMs* are given in the following Table

Table 3. Weights of Decision Makers

Criteria	**Alternatives**	**Decision Makers**			
		DM1	**DM2**	**DM3**	**DM4**
X1	*A1*	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{11}^{(1)} = (T_{11}^{(1)}, I_{11}^{(1)}, F_{11}^{(1)})$	$r_{11}^{(2)} = (T_{11}^{(2)}, I_{11}^{(2)}, F_{11}^{(2)})$	$r_{11}^{(3)} = (T_{11}^{(3)}, I_{11}^{(3)}, F_{11}^{(3)})$	$r_{11}^{(4)} = (T_{11}^{(4)}, I_{11}^{(4)}, F_{11}^{(4)}$
	A2	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{21}^{(1)} = (T_{21}^{(1)}, I_{21}^{(1)}, F_{21}^{(1)})$	$r_{21}^{(2)} = (T_{21}^{(2)}, I_{21}^{(2)}, F_{21}^{(2)})$	$r_{21}^{(3)} = (T_{21}^{(3)}, I_{21}^{(3)}, F_{21}^{(3)})$	$r_{21}^{(4)} = (T_{21}^{(4)}, I_{21}^{(4)}, F_{21}^{(4)})$
	A3	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{31}^{(1)} = (T_{31}^{(1)}, I_{31}^{(1)}, F_{31}^{(1)})$	$r_{31}^{(2)} = (T_{31}^{(2)}, I_{31}^{(2)}, F_{31}^{(2)})$	$r_{31}^{(3)} = (T_{31}^{(3)}, I_{31}^{(3)}, F_{31}^{(3)})$	$r_{31}^{(4)} = (T_{31}^{(4)}, I_{31}^{(4)}, F_{31}^{(4)})$
	A4	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{41}^{(1)} = (T_{41}^{(1)}, I_{41}^{(1)}, F_{41}^{(1)})$	$r_{41}^{(2)} = (T_{41}^{(2)}, I_{41}^{(2)}, F_{41}^{(2)})$	$r_{41}^{(3)} = (T_{41}^{(3)}, I_{41}^{(3)}, F_{41}^{(3)})$	$r_{41}^{(4)} = (T_{41}^{(4)}, I_{41}^{(4)}, F_{41}^{(4)})$
	A5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{51}^{(1)} = (T_{51}^{(1)}, I_{51}^{(1)}, F_{51}^{(1)})$	$r_{51}^{(2)} = (T_{51}^{(2)}, I_{51}^{(2)}, F_{51}^{(2)})$	$r_{51}^{(3)} = (T_{51}^{(3)}, I_{51}^{(3)}, F_{51}^{(3)})$	$r_{51}^{(4)} = (T_{51}^{(4)}, I_{51}^{(4)}, F_{51}^{(4)})$
X2	*A1*	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{12}^{(1)} = (T_{12}^{(1)}, I_{12}^{(1)}, F_{12}^{(1)})$	$r_{12}^{(2)} = (T_{12}^{(2)}, I_{12}^{(2)}, F_{12}^{(2)})$	$r_{12}^{(3)} = (T_{12}^{(3)}, I_{12}^{(3)}, F_{12}^{(3)})$	$r_{12}^{(4)} = (T_{12}^{(4)}, I_{12}^{(4)}, F_{12}^{(4)})$
	A2	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{22}^{(1)} = (T_{22}^{(1)}, I_{22}^{(1)}, F_{22}^{(1)})$	$r_{22}^{(2)} = (T_{22}^{(2)}, I_{22}^{(2)}, F_{22}^{(2)})$	$r_{22}^{(3)} = (T_{22}^{(3)}, I_{22}^{(3)}, F_{22}^{(3)})$	$r_{22}^{(4)} = (T_{22}^{(4)}, I_{22}^{(4)}, F_{22}^{(4)})$
	A3	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{32}^{(1)} = (T_{32}^{(1)}, I_{32}^{(1)}, F_{32}^{(1)})$	$r_{32}^{(2)} = (T_{32}^{(2)}, I_{32}^{(2)}, F_{32}^{(2)})$	$r_{32}^{(3)} = (T_{32}^{(3)}, I_{32}^{(3)}, F_{32}^{(3)})$	$r_{32}^{(4)} = (T_{32}^{(4)}, I_{32}^{(4)}, F_{32}^{(4)})$
	A4	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{42}^{(1)} = (T_{42}^{(1)}, I_{42}^{(1)}, F_{42}^{(1)})$	$r_{42}^{(2)} = (T_{42}^{(2)}, I_{42}^{(2)}, F_{42}^{(2)})$	$r_{42}^{(3)} = (T_{42}^{(3)}, I_{42}^{(3)}, F_{42}^{(3)})$	$r_{42}^{(4)} = (T_{42}^{(4)}, I_{42}^{(4)}, F_{42}^{(4)})$
	A5	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{52}^{(1)} = (T_{52}^{(1)}, I_{52}^{(1)}, F_{52}^{(1)})$	$r_{52}^{(2)} = (T_{52}^{(2)}, I_{52}^{(2)}, F_{52}^{(2)})$	$r_{52}^{(3)} = (T_{52}^{(3)}, I_{52}^{(3)}, F_{52}^{(3)})$	$r_{52}^{(4)} = (T_{52}^{(4)}, I_{52}^{(4)}, F_{52}^{(4)})$
X3	*A1*	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)
		$r_{13}^{(1)} = (T_{13}^{(1)}, I_{13}^{(1)}, F_{13}^{(1)})$	$r_{13}^{(2)} = (T_{13}^{(2)}, I_{13}^{(2)}, F_{13}^{(2)})$	$r_{13}^{(3)} = (T_{13}^{(3)}, I_{13}^{(3)}, F_{13}^{(3)})$	$r_{13}^{(4)} = (T_{13}^{(4)}, I_{13}^{(4)}, F_{13}^{(4)})$
	A2	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{23}^{(1)} = (T_{23}^{(1)}, I_{23}^{(1)}, F_{23}^{(1)})$	$r_{23}^{(2)} = (T_{23}^{(2)}, I_{23}^{(2)}, F_{23}^{(2)})$	$r_{23}^{(3)} = (T_{23}^{(3)}, I_{23}^{(3)}, F_{23}^{(3)})$	$r_{23}^{(4)} = (T_{23}^{(4)}, I_{23}^{(4)}, F_{23}^{(4)})$
	A3	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{33}^{(1)} = (T_{33}^{(1)}, I_{33}^{(1)}, F_{33}^{(1)})$	$r_{33}^{(2)} = (T_{33}^{(2)}, I_{33}^{(2)}, F_{33}^{(2)})$	$r_{33}^{(3)} = (T_{33}^{(3)}, I_{33}^{(3)}, F_{33}^{(3)})$	$r_{33}^{(4)} = (T_{33}^{(4)}, I_{33}^{(4)}, F_{33}^{(4)})$
	A4	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{43}^{(1)} = (T_{43}^{(1)}, I_{43}^{(1)}, F_{43}^{(1)})$	$r_{43}^{(2)} = (T_{43}^{(2)}, I_{43}^{(2)}, F_{43}^{(2)})$	$r_{43}^{(3)} = (T_{43}^{(3)}, I_{43}^{(3)}, F_{43}^{(3)})$	$r_{43}^{(4)} = (T_{43}^{(4)}, I_{43}^{(4)}, F_{43}^{(4)})$
	A5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{53}^{(1)} = (T_{53}^{(1)}, I_{53}^{(1)}, F_{53}^{(1)})$	$r_{53}^{(2)} = (T_{53}^{(2)}, I_{53}^{(2)}, F_{53}^{(2)})$	$r_{53}^{(3)} = (T_{53}^{(3)}, I_{53}^{(3)}, F_{53}^{(3)})$	$r_{53}^{(4)} = (T_{53}^{(4)}, I_{53}^{(4)}, F_{53}^{(4)})$
X4	*A1*	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{14}^{(1)} = (T_{14}^{(1)}, I_{14}^{(1)}, F_{14}^{(1)})$	$r_{14}^{(2)} = (T_{14}^{(2)}, I_{14}^{(2)}, F_{14}^{(2)})$	$r_{14}^{(3)} = (T_{14}^{(3)}, I_{14}^{(3)}, F_{14}^{(3)})$	$r_{14}^{(4)} = (T_{14}^{(4)}, I_{14}^{(4)}, F_{14}^{(4)})$
	A2	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)
		$r_{24}^{(1)} = (T_{24}^{(1)}, I_{24}^{(1)}, F_{24}^{(1)})$	$r_{24}^{(2)} = (T_{24}^{(2)}, I_{24}^{(2)}, F_{24}^{(2)})$	$r_{24}^{(3)} = (T_{24}^{(3)}, I_{24}^{(3)}, F_{24}^{(3)})$	$r_{24}^{(4)} = (T_{24}^{(4)}, I_{24}^{(4)}, F_{24}^{(4)})$

	A_3	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{34}^{(1)} = (T_{34}^{(1)}, I_{34}^{(1)}, F_{34}^{(1)})$	$r_{34}^{(2)} = (T_{34}^{(2)}, I_{34}^{(2)}, F_{34}^{(2)})$	$r_{34}^{(3)} = (T_{34}^{(3)}, I_{34}^{(3)}, F_{34}^{(3)})$	$r_{34}^{(4)} = (T_{34}^{(4)}, I_{34}^{(4)}, F_{34}^{(4)})$
	A_4	M (0.50,0.50,0.50)	MB (0.40,0.65,0.60)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{44}^{(1)} = (T_{44}^{(1)}, I_{44}^{(1)}, F_{44}^{(1)})$	$r_{44}^{(2)} = (T_{44}^{(2)}, I_{44}^{(2)}, F_{44}^{(2)})$	$r_{44}^{(3)} = (T_{44}^{(3)}, I_{44}^{(3)}, F_{44}^{(3)})$	$r_{44}^{(4)} = (T_{44}^{(4)}, I_{44}^{(4)}, F_{44}^{(4)})$
	A_5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{54}^{(1)} = (T_{54}^{(1)}, I_{54}^{(1)}, F_{54}^{(1)})$	$r_{54}^{(2)} = (T_{54}^{(2)}, I_{54}^{(2)}, F_{54}^{(2)})$	$r_{54}^{(3)} = (T_{54}^{(3)}, I_{54}^{(3)}, F_{54}^{(3)})$	$r_{54}^{(4)} = (T_{54}^{(4)}, I_{54}^{(4)}, F_{54}^{(4)})$
X_5	A_1	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{15}^{(1)} = (T_{15}^{(1)}, I_{15}^{(1)}, F_{15}^{(1)})$	$r_{15}^{(2)} = (T_{15}^{(2)}, I_{15}^{(2)}, F_{15}^{(2)})$	$r_{15}^{(3)} = (T_{15}^{(3)}, I_{15}^{(3)}, F_{15}^{(3)})$	$r_{15}^{(4)} = (T_{15}^{(4)}, I_{15}^{(4)}, F_{15}^{(4)})$
	A_2	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{25}^{(1)} = (T_{25}^{(1)}, I_{25}^{(1)}, F_{25}^{(1)})$	$r_{25}^{(2)} = (T_{25}^{(2)}, I_{25}^{(2)}, F_{25}^{(2)})$	$r_{25}^{(3)} = (T_{25}^{(3)}, I_{25}^{(3)}, F_{25}^{(3)})$	$r_{25}^{(4)} = (T_{25}^{(4)}, I_{25}^{(4)}, F_{25}^{(4)})$
	A_3	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{35}^{(1)} = (T_{35}^{(1)}, I_{35}^{(1)}, F_{35}^{(1)})$	$r_{35}^{(2)} = (T_{35}^{(2)}, I_{35}^{(2)}, F_{35}^{(2)})$	$r_{35}^{(3)} = (T_{35}^{(3)}, I_{35}^{(3)}, F_{35}^{(3)})$	$r_{35}^{(4)} = (T_{35}^{(4)}, I_{35}^{(4)}, F_{35}^{(4)})$
	A_4	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{45}^{(1)} = (T_{45}^{(1)}, I_{45}^{(1)}, F_{45}^{(1)})$	$r_{45}^{(2)} = (T_{45}^{(2)}, I_{45}^{(2)}, F_{45}^{(2)})$	$r_{45}^{(3)} = (T_{45}^{(3)}, I_{45}^{(3)}, F_{45}^{(3)})$	$r_{45}^{(4)} = (T_{45}^{(4)}, I_{45}^{(4)}, F_{45}^{(4)})$
	A_5	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{55}^{(1)} = (T_{55}^{(1)}, I_{55}^{(1)}, F_{55}^{(1)})$	$r_{55}^{(2)} = (T_{55}^{(2)}, I_{55}^{(2)}, F_{55}^{(2)})$	$r_{55}^{(3)} = (T_{55}^{(3)}, I_{55}^{(3)}, F_{55}^{(3)})$	$r_{55}^{(4)} = (T_{55}^{(4)}, I_{55}^{(4)}, F_{55}^{(4)})$

Table 4. Importance and Weights of Decision-Makers

	DM_1	*DM_2*	*DM_3*	*DM_4*
Linguistic	VI(0.90,0.10,0.10)	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	UI (0.35,0.75,0.80)
Variables	$(T_1^{dm}, I_1^{dm}, F_1^{dm})$	$(T_2^{dm}, I_2^{dm}, F_2^{dm})$	$(T_3^{dm}, I_3^{dm}, F_3^{dm})$	$(T_4^{dm}, I_4^{dm}, F_4^{dm})$
Weights	λ_{DM_1}= 0.37045	λ_{DM_2}= 0.31508	λ_{DM_3}= 0.20580	λ_{DM_4}= 0.10867

Step 2: Computation of Aggregated Single Valued Neutrosophic Decision Matrix (ASVNDM)

To find the ASVNDM not only the weights of the DMs, but the alternative ratings are also required. The alternative ratings, according to the *DMs* given in the following table.

Now by using the alternative ratings $r_{ij}^{(k)}$ and the DM weights λ_k we get

$r_{ij}= \lambda_1 r_{ij}^{(1)} \oplus \lambda_2 r_{ij}^{(2)} \oplus \lambda_3 r_{ij}^{(3)} \oplus \cdots \oplus \lambda_l r_{ij}^{(l)}$

$r_{ij} = \left(1 - \prod_{k=1}^{l}(1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_{ij}^{(k)})^{\lambda_k}\right)$

where *i* = 1, 2, 3, 4, 5; *j* = 1, 2, 3, 4, 5 and (*l* = 4).

For *i* = *j* = 1 and *l* = 4

$r_{11}= \lambda_1 r_{11}^{(1)} \oplus \lambda_2 r_{11}^{(2)} \oplus \lambda_3 r_{11}^{(3)} \oplus \cdots \oplus \lambda_l r_{11}^{(l)}$

$r_{11} = \left(1 - \prod_{k=1}^{4}(1 - T_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(I_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(F_{11}^{(k)})^{\lambda_k}\right)$

r_{11} = (1- $(1 - T_{11}^{(1)})^{\lambda_1}(1 - T_{11}^{(2)})^{\lambda_2}(1 - T_{11}^{(3)})^{\lambda_3}(1 - T_{11}^{(4)})^{\lambda_4}$, $(I_{11}^{(1)})^{\lambda_1}(I_{11}^{(2)})^{\lambda_2}(I_{11}^{(3)})^{\lambda_3}(I_{11}^{(4)})^{\lambda_4}$, $(F_{11}^{(1)})^{\lambda_1}(F_{11}^{(2)})^{\lambda_2}(F_{11}^{(3)})^{\lambda_3}(F_{11}^{(4)})^{\lambda_4}$)

r_{11} = (1-($(1 - 0.8)^{0.37045}(1 - 0.6)^{0.31508}(1 - 0.8)^{0.20580}(1 - 0.7)^{0.10867}$),

($(0.15)^{0.37045}(0.35)^{0.31508}(0.15)^{0.20580}(0.25)^{0.10867}$) ,

($(0.20)^{0.37045}(0.40)^{0.31508}(0.20)^{0.20580}(0.30)^{0.10867}$))

r_{11} = (0.740, 0.207, 0.260)

Similarly, we can find other values

r_{21} = (0.711, 0.237, 0.289)

r_{31} = (0.593, 0.373, 0.407)
r_{41} = (0.661, 0.288, 0.339)
r_{51} = (0.706, 0.241, 0.294)
r_{12} = (0.682, 0.268, 0.318)
r_{22} = (0.676, 0.275, 0.324)
r_{32} = (0.681, 0.275, 0.324)
r_{42} = (0.619, 0.342, 0.381)
r_{52} = (0.695, 0.253, 0.305)
r_{13} = (0.505, 0.392, 0.429)
r_{23} = (0.773, 0.176, 0.227)
r_{33} = (0.603, 0.359, 0.397)
r_{43} = (0.661, 0.288, 0.339)
r_{53} = (0.693, 0.255, 0.307)
r_{14} = (0.605, 0.359, 0.395)
r_{24} = (0.748, 0.203, 0.252)
r_{34} = (0.600, 0.350, 0.400)
r_{44} = (0.542, 0.443, 0.458)
r_{54} = (0.693, 0.339, 0.307)
r_{15} = (0.614, 0.349, 0.386)
r_{25} = (0.697, 0.257, 0.303)
r_{35} = (0.656, 0.299, 0.344)
r_{45} = (0.548, 0.431, 0.452)
r_{55} = (0.768, 0.181, 0.232)

Table 5. Aggregated Single Valued Neutrosophic Decision Matrix D = $[r_{ij}]_{5\times4}$

	X1	X2	X3	X4	X5
*A*1	r_{11} = (0.740, 0.207, 0.260)	r_{12} = (0.682, 0.268, 0.318)	r_{13} = (0.505, 0.392, 0.429)	r_{14} = (0.605, 0.359, 0.395)	r_{15} = (0.614, 0.349, 0.386)
*A*2	r_{21} = (0.711, 0.237, 0.289)	r_{22} = (0.676, 0.275, 0.324)	r_{23} = (0.773, 0.176, 0.227)	r_{24} = (0.748, 0.203, 0.252)	r_{25} = (0.697, 0.257, 0.303)
*A*3	r_{31} = (0.593, 0.373, 0.407)	r_{32} = (0.681, 0.275, 0.324)	r_{33} = (0.603, 0.359, 0.397)	r_{34} = (0.600, 0.350, 0.400)	r_{35} = (0.656, 0.299, 0.344)
*A*4	r_{41} = (0.661, 0.288, 0.339)	r_{42} = (0.619, 0.342, 0.381)	r_{43} = (0.661, 0.288, 0.339)	r_{43} = (0.661, 0.288, 0.339)	r_{45} = (0.548, 0.431, 0.452)
*A*5	r_{51} = (0.706, 0.241, 0.294)	r_{52} = (0.695, 0.253, 0.305)	r_{53} = (0.693, 0.255, 0.307)	r_{54}= (0.693, 0.339, 0.307)	r_{55} = (0.768, 0.181, 0.232)

Step 3: Computation of the weights of the criteria

The individual weights given by each DM is given in Table 6.

Table 6. Weights of alternatives determined by the DMs $w_j^{(k)}= (T_j^{(k)}, I_j^{(k)}, F_j^{(k)})$

Criteria	***DM*1**	***DM*2**	***DM*3**	***DM*4**
X1	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)
(DELIVERY)	$w_1^{(1)} =(T_1^{(1)}, I_1^{(1)}, F_1^{(1)})$	$w_1^{(2)} = (T_1^{(2)}, I_1^{(2)}, F_1^{(2)})$	$w_1^{(3)} = (T_1^{(3)}, I_1^{(3)}, F_1^{(3)})$	$w_1^{(4)} = (T_1^{(4)}, I_1^{(4)}, F_1^{(4)})$

X_2 (QUALITY)	I (0.75,0.25,0.20) $w_2^{(1)} = (T_2^{(1)}, I_2^{(1)}, F_2^{(1)})$	M (0.50,0.50,0.50) $w_2^{(2)} = (T_2^{(2)}, I_2^{(2)}, F_2^{(2)})$	M (0.50,0.50,0.50) $w_2^{(3)} = (T_2^{(3)}, I_2^{(3)}, F_2^{(3)})$	I (0.75,0.25,0.20) $w_2^{(4)} = (T_2^{(4)}, I_2^{(4)}, F_2^{(4)})$
X_3 (FLEXIBILITY)	VI (0.90,0.10,0.10) $w_3^{(1)} = (T_3^{(1)}, I_3^{(1)}, F_3^{(1)})$	VI (0.90,0.10,0.10) $w_3^{(2)} = (T_3^{(2)}, I_3^{(2)}, F_3^{(2)})$	I (0.75,0.25,0.20) $w_3^{(3)} = (T_3^{(3)}, I_3^{(3)}, F_3^{(3)})$	VI (0.90,0.10,0.10) $w_3^{(4)} = (T_3^{(4)}, I_3^{(4)}, F_3^{(4)})$
X_4 (SERVICE)	I (0.75,0.25,0.20) $w_4^{(1)} = (T_4^{(1)}, I_4^{(1)}, F_4^{(1)})$	I (0.75,0.25,0.20) $w_4^{(2)} = (T_4^{(2)}, I_4^{(2)}, F_4^{(2)})$	M (0.50,0.50,0.50) $w_4^{(3)} = (T_4^{(3)}, I_4^{(3)}, F_4^{(3)})$	UI (0.35,0.75,0.80) $w_4^{(4)} = (T_4^{(4)}, I_4^{(4)}, F_4^{(4)})$
X_5 (PRICE)	M (0.50,0.50,0.50) $w_5^{(1)} = (T_5^{(1)}, I_5^{(1)}, F_5^{(1)})$	M (0.50,0.50,0.50) $w_5^{(2)} = (T_5^{(2)}, I_5^{(2)}, F_5^{(2)})$	VI (0.90,0.10,0.10) $w_5^{(3)} = (T_5^{(3)}, I_5^{(3)}, F_5^{(3)})$	VI (0.90,0.10,0.10) $w_5^{(4)} = (T_5^{(4)}, I_5^{(4)}, F_5^{(4)})$

By using the values from Table 6, the aggregated criteria weights are calculated as follows

$w_j = (T_j, I_j, F_j) = \lambda_1 w_j^{(1)} \oplus \lambda_2 w_j^{(2)} \oplus \lambda_3 w_j^{(3)} \oplus \cdots \oplus \lambda_l w_j^{(l)}$

$w_j = (1-\prod_{k=1}^{l}(1-T_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_j^{(k)})^{\lambda_k})$ where j = 1, 2, 3, 4, 5 and (l = 4).

For j = 1 and l = 4

$w_1 = \lambda_1 w_1^{(1)} \oplus \lambda_2 w_1^{(2)} \oplus \lambda_3 w_1^{(3)} \oplus \lambda_4 w_1^{(4)}$

$w_1 = (1-\prod_{k=1}^{4}(1-T_1^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(I_1^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(F_1^{(k)})^{\lambda_k})$

$w_1 = (1- (1-T_1^{(1)})^{\lambda_1}(1-T_1^{(2)})^{\lambda_2}(1-T_1^{(3)})^{\lambda_3}(1-T_1^{(4)})^{\lambda_4}, (I_1^{(1)})^{\lambda_1}(I_1^{(2)})^{\lambda_2}(I_1^{(3)})^{\lambda_3}(I_1^{(4)})^{\lambda_4},$
$(F_1^{(1)})^{\lambda_1}(F_1^{(2)})^{\lambda_2}(F_1^{(3)})^{\lambda_3}(F_1^{(4)})^{\lambda_4})$

$w_1 = (1 - ((1-0.9)^{0.37045}(1-0.9)^{0.31508}(1-0.9)^{0.20580}(1-0.75)^{0.10867}),$
$((0.10)^{0.37045}(0.10)^{0.31508}(0.10)^{0.20580}(0.25)^{0.10867})$,
$((0.10)^{0.37045}(0.10)^{0.31508}(0.10)^{0.20580}(0.20)^{0.10867}))$

r_{11} = (0.740, 0.207, 0.260)

$w_1 = (T_1, I_1, F_1)$ = (0.890, 0.110, 0.108)

Similarly, we can get other values

Therefore

$$W_{\{X_1,X_2,X_3,X_4\}} = \begin{bmatrix} (0.890, 0.110, 0.108) \\ (0.641, 0.359, 0.322) \\ (0.879, 0.121, 0.115) \\ (0.680, 0.325, 0.281) \\ (0.699, 0.301, 0.301) \end{bmatrix}^T$$

Step 4: Construction of Aggregated Weighted Single Valued Neutrosophic Decision Matrix (AWSVNDM)

After finding the weights of the criteria and the alternative ratings, the aggregated weighted single-valued neutrosophic ratings are calculated as follows

$r'_{ij} = (T'_{ij}, I'_{ij}, rF'_{ij}) = (T_{A_i}(x).T_j, I_{A_i}(x) + I_j - I_{A_i}(x).I_j, F_{A_i}(x) + F_j - F_{A_i}(x).F_j)$

By using the above equation, we can get an aggregated weighted single-valued neutrosophic decision matrix.

Table 7. Aggregated Weighted Single Valued Neutrosophic Decision Matrix $R' = [r'_{ij}]_{5\times5}$

	X_1	X_2	X_3	X_4	X_5
A1	r'_{11} = (0.659,0.294,0.340)	r'_{12} = (0.437,0.531,0.538)	r'_{13} = (0.444,0.466,0.495)	r'_{14} = (0.411,0.567,0.565)	r'_{15} = (0.429,0.545,0.571)

A2	$r'_{21} =$ (0.633,0.321,0.366)	$r'_{22} =$ (0.433,0.535,0.542)	$r'_{23} =$ (0.679,0.276,0.316)	$r'_{24} =$ (0.509,0.462,0.462)	$r'_{25} =$ (0.487,0.481,0.513)
A3	$r'_{31} =$ (0.528,0.442,0.471)	$r'_{32} =$ (0.437,0.535,0.542)	$r'_{33} =$ (0.530,0.437,0.466)	$r'_{34} =$ (0.408,0.561,0.569)	$r'_{35} =$ (0.459,0.510,0.541)
A4	$r'_{41} =$ (0.588,0.366,0.410)	$r'_{42} =$ (0.397,0.578,0.580)	$r'_{43} =$ (0.581,0.374,0.415)	$r'_{44} =$ (0.037,0.624,0.610)	$r'_{45} =$ (0.383,0.602,0.617)
A5	$r'_{51} =$ (0.628,0.324,0.3700	$r'_{52} =$ (0.445,0.521,0.529)	$r'_{53} =$ (0.609,0.345,0.387)	$r'_{54} =$ (0.471,0.554,0.502)	$r'_{55} =$ (0.537,0.428,0.463)

Step 5: Computation of SVN-PIS and SVN-NIS

Since Delivery, Quality, Flexibility, and Services are benefit criteria that is why they are in the set

$$J_1 = \{X_1, X_2, X_3, X_4\}$$

whereas Price being the cost criteria, so it is in the set $J_2 = \{X_2\}$ SVN-PIS and SVN-NIS are calculated as,

Table 8. SVN-PIS and SVN-NIS

SVN-PIS	**SVN-NIS**
$\boldsymbol{T_1^+}$ = max {0.659,0.633,0.528,0.588,0.628} = 0.659	T_1^- = min {0.659,0.633,0.528,0.588,0.628} = 0.528
$\boldsymbol{I_1^+}$ = min {0.294,0.321,0.442,0.366,0.324} = 0.294	I_1^- = max {0.294,0.321,0.442,0.366,0.324} = 0.442
$\boldsymbol{F_1^+}$ = min {0.340,0.366,0.471,0.410,0.370} = 0.340	F_1^- = max {0.340,0.366,0.471,0.410,0.370} = 0.471
$\boldsymbol{T_2^+}$ = max {0.437,0.433,0.437,0.397,0.445} = 0.445	T_2^- = min {0.437,0.433,0.437,0.397,0.445} = 0.397
$\boldsymbol{I_2^+}$ = min {0.531,0.535,0.535,0.578,0.521} = 0.521	I_2^- = max {0.531,0.535,0.535,0.578,0.521} = 0.578
$\boldsymbol{F_2^+}$ = min {0.538,0.542,0.542,0.580,0.529} = 0.529	F_2^- = max {0.538,0.542,0.542,0.580,0.529} = 0.580
$\boldsymbol{T_3^+}$= max {0.444,0.679,0.530,0.581,0.609} = 0.679	T_3^- = min {0.444,0.679,0.530,0.581,0.609} = 0.444
$\boldsymbol{I_3^+}$ = min {0.466,0.276,0.437,0.374,0.345} = 0.276	I_3^- = max {0.466,0.276,0.437,0.374,0.345} = 0.466
$\boldsymbol{F_3^+}$ = min {0.495,0.316,0.466,0.415,0.387} = 0.316	F_3^- = max {0.495,0.316,0.466,0.415,0.387} = 0.495
$\boldsymbol{T_4^+}$ = max {0.411,0.509,0.408,0.037,0.471} = 0.509	T_4^- = min {0.411,0.509,0.408,0.037,0.471} = 0.037
$\boldsymbol{I_4^+}$ = min {0.567,0.462,0.561,0.624,0.554} = 0.462	I_4^- = max {0.567,0.462,0.561,0.624,0.554} = 0.624
$\boldsymbol{F_4^+}$ = min {0.565,0.462,0.569,0.610,0.502} = 0.462	F_4^- = max {0.565,0.462,0.569,0.610,0.502} = 0.610
$\boldsymbol{T_5^+}$ = min {0.429,0.487,0.459,0.383,0.537} = 0.383	T_5^- = max {0.429,0.487,0.459,0.383,0.537} = 0.537
$\boldsymbol{I_5^+}$ = max {0.545,0.481,0.510,0.602,0.428} = 0.602	I_5^- = min {0.545,0.481,0.510,0.602,0.428} = 0.428
$\boldsymbol{F_5^+}$ = max {0.571,0.513,0.541,0.617,0.463} = 0.617	F_5^- = min {0.571,0.513,0.541,0.617,0.463} = 0.463

$$A^+ = \begin{Bmatrix} (0.659, 0.294, 0.340), \\ (0.445, 0.521, 0.529), \\ (0.679, 0.276, 0.316), \\ (0.509, 0.462, 0.462), \\ (0.383, 0.602, 0.617) \end{Bmatrix} \qquad A^- = \begin{Bmatrix} (0.528, 0.442, 0.471), \\ (0.397, 0.578, 0.580), \\ (0.444, 0.466, 0.495), \\ (0.037, 0.624, 0.610), \\ (0.537, 0.428, 0.463) \end{Bmatrix}$$

Step 6: Computation of Separation Measures

Normalized Euclidean Distance Measure is used to find the negative and positive separation measures $\boldsymbol{d}^{+}$ and $\boldsymbol{d}^{-}$respectively. Now for the SVN-PIS, we use

$$d_i^{+}= \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j)-T_{A^*W}\,(x_j)\right)^2+\left(I_{A_i.W}(x_j)-I_{A^*W}\,(x_j)\right)^2+\left(F_{A_i.W}(x_j)-F_{A^*W}\,(x_j)\right)^2\right]\right)^{0.5}$$

For i = 1and n = 5

$$d_1^{+}= \left(\frac{1}{3(5)}\sum_{j=1}^{5}\left[\left(T_{A_1.W}(x_j)-T_{A^*W}\,(x_j)\right)^2+\left(I_{A_1.W}(x_j)-I_{A^*W}\,(x_j)\right)^2+\left(F_{A_1.W}(x_j)-F_{A^*W}\,(x_j)\right)^2\right]\right)^{0.5}$$

$$d_1^{+}= \left(\frac{1}{15}\begin{bmatrix}\left(T_{A_1.W}(X_1)-T_{A^*W}\,(X_1)\right)^2+\left(I_{A_1.W}(X_1)-I_{A^*W}\,(X_1)\right)^2+\left(F_{A_1.W}(X_1)-F_{A^*W}\,(X_1)\right)^2+\\ \left(T_{A_1.W}(X_2)-T_{A^*W}\,(X_2)\right)^2+\left(I_{A_1.W}(X_2)-I_{A^*W}\,(X_2)\right)^2+\left(F_{A_1.W}(X_2)-F_{A^*W}\,(X_2)\right)^2+\\ \left(T_{A_1.W}(X_3)-T_{A^*W}\,(X_3)\right)^2+\left(I_{A_1.W}(X_3)-I_{A^*W}\,(X_3)\right)^2+\left(F_{A_1.W}(X_3)-F_{A^*W}\,(X_3)\right)^2+\\ \left(T_{A_1.W}(X_4)-T_{A^*W}\,(X_4)\right)^2+\left(I_{A_1.W}(X_4)-I_{A^*W}\,(X_4)\right)^2+\left(F_{A_1.W}(X_4)-F_{A^*W}\,(X_4)\right)^2+\\ \left(T_{A_1.W}(X_5)-T_{A^*W}\,(X_5)\right)^2+\left(I_{A_1.W}(X_5)-I_{A^*W}\,(X_5)\right)^2+\left(F_{A_1.W}(X_5)-F_{A^*W}\,(X_5)\right)^2\end{bmatrix}\right)^{0.5}$$

$$d_1^{+}= \left(\frac{1}{15}\begin{bmatrix}(0659-0.659)^2+(0.294-0.294)^2+(0.340-0.340)^2+\\ (0.437-0.445)^2+(0.531-0.521)^2+(0.538-0.529)^2+\\ (0.444-0.679)^2+(0.466-0.276)^2+(0.495-0.316)^2+\\ (0.411-0.509)^2+(0.567-0.462)^2+(0.565-0.462)^2+\\ (0.429-0.383)^2+(0.545-0.602)^2+(0.571-0.617)^2\end{bmatrix}\right)^{0.5}$$

$$d_1^{+}= \left[\frac{1}{15}\,(0.000245+0.123366+0.031238+0.007481)\right]^{0.5}$$

d_1^{+}= 0.1040

Similarly, we can find other separation measures.

Step 7: Computation of Relative Closeness Coefficient (RCC)

The RCC is calculated by using

$$\text{RCCi} = \frac{d_i'}{d_i'+d_i^*}\ ;\ \text{i} = 1, 2, 3, 4, 5$$

$$\text{RCC}_1 = \frac{d_1'}{d_1'+d_1^*} = \frac{0.127532}{0.127532+0.104029} = 0.551$$

RCC_2 = 0.896

RCC_3 = 0.505

RCC_4 = 0.363

RCC5 = 0.757

The separation measure and the value of relative closeness coefficient (RCC) expressed in the following figure.

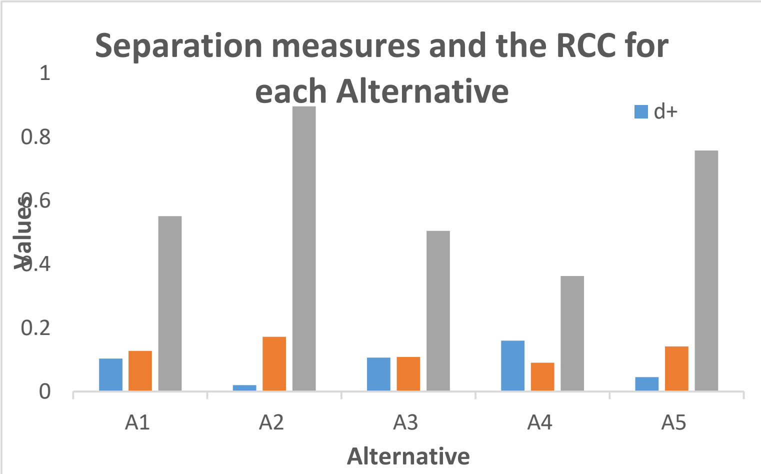

Figure 1. Separation measure and the RCC for each Alternative

Step 8: Ranking alternatives

From the above figure, we can see the RCC are ranked as follows

$RCC_2 > RCC_5 > RCC_1 > RCC_3 > RCC_4 \Rightarrow A_2 > A_5 > A_1 > A_3 > A_4$

By using the presented technique, we choose the best supplier for the production industry and observe that A_2 is the best alternative.

5. Conclusion

In this paper, we studied neutrosophic set and SVNSs with some basic operations and developed the generalized neutrosophic TOPSIS by using single-valued neutrosophic numbers. By using crisp data, it is more difficult to solve decision-making problems under uncertain environments, to overcome such uncertainties single-valued neutrosophic sets are more appropriate. We also developed the graphical model for generalized neutrosophic TOPSIS. Finally, to show the validity of the proposed technique an illustrated example of the best supplier in the production industry is presented and observed that A_2 is the best supplier for the production industry. We consider this technique will be helpful in problem-solving and will expand the area of investigations for more accuracy in real-life issues.

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Intelligent Algorithm for Trapezoidal Interval Valued Neutrosophic Network Analysis

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Abstract: The shortest path problem has been one of the most fundamental practical problems in network analysis. One of the good algorithms is Bellman-Ford, which has been applied in network, for the last some years. Due to complexity in the decision-making process, the decision makers face complications to express their view and judgment with an exact number for single valued membership degrees under neutrosophic environment. Though the interval number is a special situation of the neutrosophic, it did not solve the shortest path problems in an absolute manner. Hence, in this work, the authors have introduced the score function and accuracy function of trapezoidal interval valued neutrosophic numbers with their illustrative properties. These properties provide important theoretical base of the trapezoidal interval valued neutrosophic number. Also, they proposed an intelligent algorithm called trapezoidal interval valued neutrosophic version of Bellman's algorithm to solve neutrosophic shortest path problem in network analysis. Further, comparative analysis has been made with the existing algorithm.

1 Introduction

In wireless communication and digital electronics, the short distance is communicated using multifunctional sensors. This type of sensors consists of sensing and processing the data. Embedded systems, wireless communication, distributed processing, micro-electro-mechanical systems and applications using wireless sensors are the developments in the technology of sensors and these developments have contributed to large transformation in wireless sensor networks. Sensors help and boost work performed in the field of both industry and our daily life. Sensor network system is away from the actual phenomenon and could collect and process a huge number of data. These sensors are using sense perception. The sensor network and algorithm must possess self-organising capabilities.

Neighbourhood nodes are close to each other and the nodes are used for constant sensing. Multichip sensor networks are used to consume low power than other sensors. The topological information would be provided by every each node of the sensor network. Interconnection network can be used for parallel computing. Shortest path algorithms are used to message from any source to any destination. Bellman-Ford is mostly applied for a large network with a stable node. A set whose elements have degrees of membership called fuzzy set (FS) in 1965 [1] and mainly deals with numerous real-world situations, where the data possesses some sort of uncertainty.

The concept of FS deals with only the membership value of the elements, not the non-membership value. This issue was sorted out by intuitionistic FS (IFS) introduced by Atannasov in 1975 [2] which allows both the membership function (MF) and non-membership function. Since the real-world situations may contain indeterminacy in the data, FS and IFS could not deal with indeterminacy of the data. This problem was rectified by neutrosophic set (NS), which is the generalisation of FS and IFS, introduced by Smarandache [3].

NS is a set in which, all the elements have degree of membership, indeterminacy and non-membership and the sum of these MFs should be less than or equal to 3. All three MFs are independent of each other. Since the NSs are difficult to apply in real-world problems, Wang *et al.* introduced single valued NSs [4]. Uncertainty of the elements can be captured using fuzzy numbers and intuitionistic fuzzy numbers. In the same way, neutrosophic numbers are very useful in capturing uncertainty and indeterminacy of the elements. Hence, it is a special case of the NS which enhances the domain of real numbers to neutrosophic numbers. Fuzzy shortest path problems (FSPP) can be solved by considering the edge weights of the network as fuzzy or uncertain using Bellman dynamic programming approach and multi-objective linear programming technique [5].

Shortest path problem (SPP) has been solved by many researchers under fuzzy and intuitionistic fuzzy environments [6–8]. The concept of Bellman's algorithm has been applied in a fuzzy network [9] for solving SPP and it is not applied in neutrosophic network so far. Distance measure can be obtained using single and interval valued trapezoidal neutrosophic numbers in a multi-attribute decision-making problem [10]. Dijkstra algorithm is a very useful and optimised one to solve the SPP but incapable to handle negative weights, whereas Bellman can deal with negative weights. SPP also can be solved by using single valued neutrosophic graph. The information that a sender does convey in communication with a receiver is called the linguistic information involved with the nature of language and communication. This information is a correlation between the knowledge of the people about the old and new information and deviation in grammatical structure, reacting to this knowledge.

Finding a shortest path between two or more vertices is called SPP in which the sum of edge weights should be minimum. The practicability of a path is resolved its length under familiar measures such as distance and its corresponding nature. In the transportation system, with mode options, for a passenger to reach the destination from a source point, the selection of the mode and destination guide of the journey for an optimised route would be specified by linguistic terms. Hence, the linguistic information and SPPs are connected.

To deal with inconsistency, uncertainty, ambiguity, impreciseness and indeterminate, many methods have been recommended by the researchers under different environments, namely FS, IFS, interval valued IFS, triangular IFS, trapezoidal IFS and NS where the information can be represented in the form of triangles and trapezoid under all these environments. Also, the membership values lie in the real unit interval [0, 1]. Hence, trapezoidal interval valued neutrosophic number (TrIVNN) helps in real-world problems where the information is uncertain and indeterminate between some ranges of acceptable behaviour. Therefore, TrIVNN is the key to extract the MFs of truth, indeterminacy and falsity whose values depend on both trapezoidal neutrosophic numbers and the intervals [11–15].

In [16, 17], Broumi *et al.* made an overview of the SPP under various environments and solved SPP using single valued and triangular and trapezoidal interval valued neutrosophic environments. Aggregation operators for interval valued generalised single valued neutrosophic trapezoidal number have been derived and applied in decision-making problem [18]. An extension of FSs called type-2 FSs and its special cases called interval type-2 FSs have growing applications in control systems, edge detection in image processing and other medical fields. SPPs can be solved using triangular and trapezoidal interval neutrosophic environments as an extension of NSs. From the overview of solving SPP under various sets environments, one can understand the difference and capacity of handling uncertainty with various levels [19–25].

In [26], the authors considered the concept of (extended) derivable single-valued neutrosophic graph as the energy clustering of wireless sensor networks and applied this concept as a tool in wireless sensor (hyper) networks. In [27], Broumi *et al.* applied single-valued neutrosophic techniques for analysis of WIFI connection. Jan *et al.* [28] developed the concept of constant single valued neutrosophic graphs and applied it to a real-world problem of Wi-Fi system. For more information on the application of neutrosophic theory, we refer the readers to [29–36]. NSs are usually applied to model linguistic information. In our previous work, we solved SPP for a network with triangular and trapezoidal interval valued neutrosophic edge weights using an improved algorithm with the operational laws and new score function of interval valued neutrosophic numbers. Also, comparative analysis has been done with the existing methods.

In this paper, we are motivated to present neutrosophic version of Bellman's algorithm for solving neutrosophic SPP (NSPP). Therefore for the first time, we proposed trapezoidal interval valued neutrosophic version of Bellman's algorithm to solve NSPP in network analysis, where the edge weight is characterised by TrIVNN.

The rest of this paper is organised as follows. In Section 2, literature review is given with the existing work and application side. In Section 3, some concepts and theories are reviewed. Section 4 introduces the score function and accuracy function of TrIVNNs with their illustrative properties. In Section 5, an intelligent algorithm called trapezoidal interval valued neutrosophic version of Bellman-Ford algorithm is proposed with a numerical example as an application of our proposed algorithm. Section 6 gives the significance of the proposed algorithm. Section 7 gives the comparative analysis of the proposed algorithm with the existing algorithm to solve NSPP in network analysis. The last but not least, in Section 8 the conclusion is drawn with the advantages and limitations of the proposed work and some hints for further research is given.

2 Literature review

In this section, literature review on existing work and application side is given for solving SPP under fuzzy, intuitionistic fuzzy and neutrosophic environments.

2.1 Existing work

Zadeh [1] proposed FSs. Atanassov [2] introduced IFSs. Smarandache [3] proposed neutrosophic logic, set and probability. Wang *et al.* [4] invented single valued NSs. Bellman [13] proposed routing problem with functional equation approach. Bellman-Ford algorithm is explained in [14]. Lathamaheswari *et al.* [22] analysed the different applications of type-2 fuzzy in the field of bio-medicine. Lathamaheswari *et al.* [24] re-examined the usage of type-2 fuzzy controller in the area of control system.

2.2 Application side

De and Bhincher [5] described two different methods to solve SPP namely Bellman dynamic programming and multi-objective linear programming. Kumar *et al.* [6] introduced a new algorithm to solve SPP under interval valued intuitionistic trapezoidal fuzzy environment. Meenakshi and Kaliraja [7] determined shortest path for interval valued fuzzy network. Elizabeth and Sujatha [8] solved FSPP using interval valued fuzzy number matrices. Das and De [9] figured out SPP under intuitionistic fuzzy setting. Biswas *et al.* [10] introduced a new strategy for multi-attribute decision-making problem under interval trapezoidal neutrosophic environment. Broumi *et al.* [11] estimated a shortest path using single valued trapezoidal neutrosophic number as the edge weights. Broumi *et al.* [12] solved NSPP using Dijkstra algorithm. Broumi *et al.* [15] dealt with SPP using single valued neutrosophic graphs. Broumi *et al.* [16] made an analysis on SPP under various environments. Broumi *et al.* [17] solved SPP under interval valued trapezoidal and triangular neutrosophic setting. Deli [18] introduced interval valued generalised single valued neutrosophic trapezoidal number and its aggregation operators, also applied the proposed concept in decision-making problem. Giri *et al.* [19] solved a decision-making problem using TOPSIS method under interval trapezoidal neutrosophic environment. Deli *et al.* [20] determined a decision-making problem using single and interval valued trapezoidal and triangular neutrosophic numbers. Nagarajan *et al.* [21] introduced a new technique for edge detection on DICOM image under type-2 fuzzy environment. Nagarajan *et al.* [23] proposed a technique for image extraction on DICOM image under type-2 fuzzy environment. Sellappan *et al.* [25] evaluated risk priority number in design failure modes and used factor analysis for effects analysis. Mohammad and Arsham Borumand [26] introduced achievable single valued neutrosophic graphs and applied in wireless sensor networks. Broumi *et al.* [27] estimated information processing using mobile ad-hoc network with an example under neutrosophic environment. Jan *et al.* [28] introduced and studied the characteristics of constant single valued neutrosophic graph and applied in Wi-Fi network system. Harish [29] solved multi-attribute group decision-making problem using novel neutrality aggregation operators under single valued neutrosophic setting. Dimple and Harish [30] introduced some modified results of the subtraction and division operations on interval NSs. Harish and Nancy [31] solved multi-criteria decision-making (MCDM) problem using Frank Choquet Heronian mean operator under single valued neutrosophic setting. Nancy and Harish [32] introduced a novel divergence measure and used in TOPSIS method for MCDM problem under single-valued neutrosophic environment. Harish and Nancy [33] proposed some hybrid weighted aggregation operators and applied in MCDM problem under NS environment. Harish and Nancy [34] introduced new logarithmic operational laws for single-valued neutrosophic numbers and applied in multi-attribute decision-making problem. Harish and Nancy [35] proposed non-linear programming method under interval NS setting and applied in MCDM problem. Nancy and Harish [36] introduced an improved score function for the ranking proposed of NSs and applied in decision-making process.

3 Overview of trapezoidal interval valued neutrosophic number

In this section, we review some basic concepts regarding NSs, single valued NSs, trapezoidal NSs and some existing ranking functions for trapezoidal neutrosophic numbers which are the background of this study and will help us to further research.

3.1 Neutrosophic set [3]

Let ξ be points (objects) set and its generic elements denoted by x; we define the neutrosophic set A (NS A) as the form $\ddot{A} = \{\langle x{:}T_{\ddot{A}}(x), I_{\ddot{A}}(x), F_{\ddot{A}}(x)\rangle, x \in \xi\}$, where the functions T, I, F: $\xi \to]{-0},1+[$ are called the truth-MF, an indeterminacy-membership function, and a falsity-membership function, respectively, and they satisfy the following condition:

$$-0 \le T_{\ddot{A}}(x) + I_{\ddot{A}}(x) + F_{\ddot{A}}(x) \le 3+. \tag{1}$$

The values of these three MFs $T_{\ddot{A}}(x)$, $I_{\ddot{A}}(x)$ and $F_{\ddot{A}}(x)$ are real standard or non-standard subsets of $]{-0},1+[$. As we have difficulty in applying NSs to practical problems. Wang *et al.* [4] proposed the concept of a SVNS that represents the simplification of a NS and can be applied to real scientific and technical applications.

3.2 Single valued NS [4]

A single valued NS $\ddot{A}$ (SVNS $\ddot{A}$) in the universe set ξ is defined by the set

$$\ddot{A} = \{ < x{:}T_{\ddot{A}}(x), I_{\ddot{A}}(x), F_{\ddot{A}}(x) > , x \in \xi\} \tag{2}$$

where $T_{\ddot{A}}(x), I_{\ddot{A}}(x), F_{\ddot{A}}(x) \in [0, 1]$ satisfying the condition

$$0 \le T_{\ddot{A}}(x) + I_{\ddot{A}}(x) + F_{\ddot{A}}(x) \le 3 \tag{3}$$

3.3 Trapezoidal interval valued neutrosophic set [10]

Let x be TrIVNN. Then its truth, indeterminacy and falsity MFs are given by

$$T_x(z) = \begin{cases} \dfrac{(z-a)t_x}{(b-a)}, & a \le z < b, \\ t_x, & b \le z \le c \\ \dfrac{(d-z)t_x}{(d-c)}, & c \le z \le d \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Its indeterminacy MF is

$$I_x(z) = \begin{cases} \dfrac{(b-z)+(z-a)i_x}{(b-a)}, & a \le z < b \\ i_x, & b \le z \le c \\ \dfrac{z-c+(d-z)i_x}{d-c} & c < z \le d \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Its falsity MF is

$$F_x(z) = \begin{cases} \dfrac{b-z+(z-a)f_x}{b-a}, & a \le z < b \\ f_x, & b \le z \le c \\ \dfrac{z-c+(d-z)f_x}{d-c}, & c < z \le d \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where $0 \le T_x(z) \le 1$, $0 \le I_x(z) \le 1$ and $0 \le F_x(z) \le 1$, also t_x, i_x, f_x are subset of $[0,1]$ and $0 \le a \le b \le c \le d \le 1$, $0 \le \sup(t_x) + \sup(i_x) + \sup(f_x) \le 3$; Then x is called an interval trapezoidal neutrosophic number $x = ([a, b, c, d]; t_x, i_x, f_x)$. We take $t_x = [\underline{t}, \overline{t}]$, $i_x = [\underline{i}, \overline{i}]$ and $f_x = [\underline{f}, \overline{f}]$.

3.4 Ranking technique [10]

Let $\tilde{a}$ and $\tilde{r}$ be two TrIVNNs, the ranking of $\tilde{a}$ and $\tilde{r}$ by score function and accuracy function is described as follows:

(i) if $s(\hat{r}^N) \prec s(\hat{s}^N)$ then $\hat{r}^N \prec \hat{s}^N$
(ii) if $s(\hat{r}^N) \simeq s(\hat{s}^N)$ and if

(a) $a(\hat{r}^N) \prec a(\hat{s}^N)$ then $\hat{r}^N \prec \hat{s}^N$
(b) $a(\hat{r}^N) \succ a(\hat{s}^N)$ then $\hat{r}^N \succ \hat{s}^N$
(c) $a(\hat{r}^N) \simeq a(\hat{s}^N)$ then $\hat{r}^N \simeq \hat{s}^N$

3.5 Bellman dynamic programming [13]

Given an acyclic directed connected graph $G=(V, E)$ with 'n' vertices where node '1' is the source node and 'n' is the destination node. The nodes of the given network are organised with the topological ordering (Eij: $i<j$). Now for the given network the shortest path can be obtained based on the formulation of Bellman dynamic programming by forward pass computation method.

The formulation of Bellman dynamic programming is described as follows:

$f(1)=0$

$$f(i) = \min_{i<j}\left\{f(i) + d_{ij}\right\} \tag{7}$$

where d_{ij} is the weight of the directed edge E_{ij}, $f(i)$ is the length of the shortest path of ith node from the source node 1.

3.6 Advantages of trapezoidal interval valued neutrosophic number [17]

There are some advantages of using TrIVNN as follows:

(i) Interval trapezoidal neutrosophic number is a generalised form of single valued trapezoidal neutrosophic number.
(ii) In this number, the trapezoidal number is characterised by three independent membership degrees, which are in interval form.
(iii) The number can flexibly express neutrosophic information than the single valued neutrosophic trapezoidal number.

Therefore, the number can be employed to solve neutrosophic multiple attribute decision-making problem, where the preference values cannot be expressed in terms of single valued trapezoidal neutrosophic number.

4 Proposed concepts

Score function and accuracy function are measurement functions to rank fuzzy, intuitionistic and neutrosophic numbers. While solving NSPP, score function finds the aggregate value of each path and measures their accuracy and provides the relevant score that measures how well the path satisfies the requirement. Accuracy function gives the most intuitive performance measure. Here, the score and accuracy functions are introduced with their illustrative properties for trapezoidal interval neutrosophic numbers.

4.1 Score function of trapezoidal interval valued neutrosophic number

Let $x = ([a, b, c, d]; [\underline{t}, \overline{t}], [\underline{i}, \overline{i}], [\underline{f}, \overline{f}])$ be a TrIVNN then its score function is defined by

$$S(x) = \frac{1}{16}(a + b + c + d)(2 + \underline{t} + \overline{t} - \underline{i} - \overline{i} - \underline{f} - \overline{f}) \quad (8)$$

and $S(x) \in [0, 1]$. Here we take $0 \leq a \leq b \leq c \leq d \leq 1$, t_x, i_x, f_x are subset of [0, 1] where $t_x = [\underline{t}, \overline{t}]$, $i_x = [\underline{i}, \overline{i}]$ and $f_x = [\underline{f}, \overline{f}]$.

4.1.1 Property: Score function is bounded on [0, 1].

Proof: Since, $0 \leq a \leq b \leq c \leq d \leq 1$, we have

$$0 \leq a + b + c + d \leq 4 \quad (9)$$

Now

$$\begin{aligned} &-4 \leq \underline{t} + \overline{t} - \underline{i} - \overline{i} - \underline{f} - \overline{f} \leq 2 \\ &=> 2 - 4 \leq 2 + \underline{t} + \overline{t} - \underline{i} - \overline{i} - \underline{f} - \overline{f} \leq 4 \\ &=> -2 \leq 2 + \underline{t} + \overline{t} - \underline{i} - \overline{i} - \underline{f} - \overline{f} \leq 4 \end{aligned} \quad (10)$$

Multiplying (9) and (10), we get

$$\begin{aligned} &0 \leq (a + b + c + d)(2 + \underline{t} + \overline{t} - \underline{i} - \overline{i} - \underline{f} - \overline{f}) \leq 16 \\ &=> 0 \leq \frac{1}{16}(a + b + c + d)(2 + \underline{t} + \overline{t} - \underline{i} - \overline{i} - \underline{f} - \overline{f}) \leq 1 \end{aligned}$$

Therefore, score function is bounded.
Example: Let a = ([0.1,0.2,0.3,0.4]; [0.1,0.2], [0.2,0.3], [0.4,0.5]) be a TrIVNN then its score value is $Sc(a) = \frac{1}{16}(0.1 + 0.2 + 0.3 + 0.4)$ $(2 + .1 + .2 - .2 - .3 - .4 - .5) = 0.07875 \in [0, 1]$, hence the result. □

4.2 Accuracy function of trapezoidal interval valued neutrosophic number

Let $x = ([a, b, c, d]; [\underline{t}, \overline{t}], [\underline{i}, \overline{i}], [\underline{f}, \overline{f}])$ be a TrIVNN then its accuracy function is defined by

$$Ac(x) = \frac{1}{8}(c + d - a - b)(2 + \underline{t} + \overline{t} - \underline{f} - \overline{f}) \quad (11)$$

and $Ac(x) \in [0, 1]$. Here we take $0 \leq a \leq b \leq c \leq d \leq 1$ and t_x, i_x, f_x are the subset of [0, 1] where $t_x = [\underline{t}, \overline{t}]$, $i_x = [\underline{i}, \overline{i}]$ and $f_x = [\underline{f}, \overline{f}]$.

4.2.1 Property: Accuracy function is bounded on [0, 1].

Proof: Since $0 \leq a \leq b \leq c \leq d \leq 1$, we have

$$-2 \leq c + d - a - b \leq 2 \quad (12)$$

$$\begin{aligned} &\Rightarrow -2 \leq \underline{t} + \overline{t} - \underline{f} - \overline{f} \leq 2 \\ &=> 0 \leq 2 + \underline{t} + \overline{t} - \underline{f} - \overline{f} \leq 4 \end{aligned} \quad (13)$$

Multiplying (12) and (13), we get

$$\begin{aligned} &0 \leq (c + d - a - b)(2 + \underline{t} + \overline{t} - \underline{f} - \overline{f}) \leq 8 \\ &=> 0 \leq \frac{1}{8}(a + b + c + d)(2 + \underline{t} + \overline{t} - \underline{f} - \overline{f}) \leq 1 \end{aligned}$$

Therefore, accuracy function is bounded and is proved by a numerical illustration in Section 4.2.2. □

4.2.2 Numerical illustration: Let x = ([0.1,0.2,0.3,0.4]; [0.1,0.2],[0.2,0.3],[0.4,0.5]) be the TrIVNN then its accuracy value is

$$\begin{aligned} Ac(x) &= \frac{1}{8}(0.1 + 0.2 + 0.3 + 0.4)(2 + .1 + .2 - .4 - .5) \\ &= 0.175 \in [0, 1] \end{aligned}$$

Hence the result.

5 Computation of shortest path based on TrIVNN

This section introduces an algorithmic approach to solve NSPP. Consider a network with 'n' nodes where the node '1' is the source node and the node 'n' is the destination node under trapezoidal interval valued neutrosophic environment. The neutrosophic distance between the nodes is denoted by d_{ij} (node 'i' to node 'j'). Here $M_{N(i)}$ denotes the set of all nodes having a relation with the node 'i'.

5.1 Revised version of trapezoidal interval valued neutrosophic Bellman-Ford algorithm

Applying the concept of Bellman's algorithm in neutrosophic environment, we get trapezoidal interval valued neutrosophic version of Bellman-Ford algorithm (Algorithm 1, see Fig. 1).

5.2 Illustrative example

The revised version of Bellman-Ford algorithm under trapezoidal interval valued neutrosophic environment is demonstrated by an illustrative example as follows for a better understanding.

For the illustrative purpose, a numerical problem from [11] is considered, to prove the inherent application of the proposed algorithm. It shows the clear procedure of the proposed algorithm.

Consider a network (Fig. 2) with six nodes and eight edges and the edge weights are characterised by TpIVNNs, where the first node is the source node and the sixth node is the destination node. Trapezoidal interval valued neutrosophic distance is given in Table 1.

In this situation, we need to evaluate the shortest distance from source node, i.e. node 1 to destination node, i.e. node 6. Table 1 represents the edges and their trapezoidal interval valued neutrosophic distance.

For all the edges, trapezoidal interval valued neutrosophic distance is reduced into crisp numbers using score function as a deneutrosophication process and is represented by Table 2.

According to the proposed neutrosophic Bellman-Ford algorithm in Section 5.1, the shortest path from node one to node six can be computed by Algorithm 2 (see Fig. 3).

Therefore, the path P: 1→2→5→6 is identified as the trapezoidal interval valued neutrosophic shortest path, and the crisp shortest path is 0, 85.

The neutrosophic shortest path can be obtained for the network with a large number of vertices and edges also.

6 Significance of the proposed work

The proposed trapezoidal interval valued neutrosophic version of Bellman-Ford algorithm has a potential significance as it has the following qualities:

(i) It deals with the network in which the edge weights are TrIVNNs and so that it characterises membership, indeterminacy and falsity of each edge.

```
nrank[s] ← 0
ndist[s] ← Empty neutrosophic number
Add s into Q
For each node i (except the s) in the neutrosophic graph G
rank[i] ← ∞
        Add i into Q
End For
u←s
While(Q is not empty)
remove the vertex u from Q
For each adjacent vertex v of vertex u
        relaxed←False
temp_ndist[v] ← ndist[u] ⊕ edge_weight(u,v)
//⊕ represents the addition of neutrosophic//
temp_nrank[v]
←rank_of_neutrosophic(temp_ndist[v])
Iftemp_nrank[v]<nrank[v]  then
ndist[v] ←temp_ndist[v]
nrank[v] ← temp_nrank[v]
prev[v] ←u
        End If
End For
If relaxed equals False then
exit the loop
        End If
u ←Node in Q with minimum rank value
End While
  For each arc(u,v) in neutrosophic graph G do
If nrank[v]> rank_of_neutrosophic(ndist[u] ⊕
edge_weight(u,v))
return false
End If
End For
The neutrosophic number ndist[u] is a neutrosophic
number and it represents the shortest path between
source node s and end node u.
```

Fig. 1 *Algorithm 1: Trapezoidal interval valued neutrosophic Bellman-Ford algorithm for shortest path analysis of the network*

(ii) It proceeds with the concept of relaxation, whither approximations to the exact distance are replaced by better ones until they finally reach the solution.

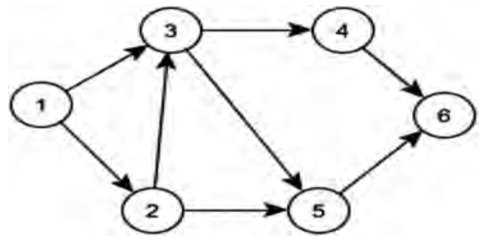

Fig. 2 *Network with six vertices and eight edges [Broumi et al. [11]]*

Table 1 Details of edge information in terms of TrIVNNs

Edges	Trapezoidal interval valued neutrosophic distance
1-2(e_1)	⟨(0.1, 0.2, 0.3, 0.4); [0.1, 0.2], [0.2, 0.3], [0.4, 0.5]⟩
1-3(e_2)	⟨(0.2, 0.5, 0.7, 0.8); [0.2, 0.4], [0.3, 0.5], [0.1, 0.2]⟩
2-3(e_3)	⟨(0.3, 0.7, 0.8, 0.9); [0.3, 0.4], [0.1, 0.2], [0.3, 0.5]⟩
2-5(e_4)	⟨(0.1, 0.5, 0.7, 0.9); [0.1, 0.3], [0.3, 0.4], [0.2, 0.3]⟩
3-4(e_5)	⟨(0.2, 0.4, 0.8, 0.9); [0.2, 0.3], [0.2, 0.5], [0.4, 0.5]⟩
3-5(e_6)	⟨(0.3, 0.4, 0.5, 1); [0.3, 0.6], [0.1, 0.2], [0.1, 0.4]⟩
4-6(e_7)	⟨(0.7, 0.8, 0.9, 1); [0.4, 0.6], [0.2, 0.4], [0.1, 0.3]⟩
5-6(e_8)	⟨(0.2, 0.4, 0.5, 0.7); [0.2, 0.3], [0.3, 0.4], [0.1, 0.5]⟩

Table 2 Details of deneutrosophication value of edge (*i, j*)

Edges	Score function	Edges	Score function
e_{12}	0,05625	e_{34}	0,416875
e_{13}	0,48125	e_{35}	0,56375
e_{23}	0,6075	e_{46}	0,85
e_{25}	0,44	e_{56}	0,36

f(1)=0

f(2)=$\min_{i<2}\{f(1)+c_{12}\}=c^*_{12}$=0,05625

f(3)=$\min_{i<3}\{f(i)+c_{i3}\}=\min\{f(1)+c_{13}; f(2)+c_{23}\}$

={0+0,48125, 0,05625+0,6075}={0,48125; 0,66375}=0,48125

f(4)=$\min_{i<4}\{f(i)+c_{i4}\}=\min\{f(3)+c_{34}\}$={0,48125+0,416875}=0,89

f(5)=$\min_{i<5}\{f(i)+c_{i5}\}=\min\{f(2)+c_{25}; f(3)+c_{35}\}$

={0,05625+0,44; 0,48125+0,56375}={0,49, 1,045}=0,49

f(6)=$\min_{i<6}\{f(i)+c_{i6}\}=\min\{f(4)+c_{46}; f(5)+c_{56}\}$

={0,89+0,85;0,49+0,36}={1.74, 0,85}=0,85

Thus, f(6)=$f(5)+c_{56}=f(2)+c_{25}+c_{56}=f(1)+c_{12}+c_{25}+c_{56}$

$=c_{12}+c_{25}+c_{56}$.

Fig. 3 *Algorithm 2: Steps involved in finding trapezoidal interval valued neutrosophic shortest path*

(iii) This revised version of Bellman-Ford algorithm simply relaxes all the edges for $|V|-1$ times. In all these repetitions, the number of vertices with properly calculate distances become larger, from which it follows that, finally all vertices will get their exact distances. Here $|V|$ is the number of vertices in the trapezoidal interval valued neutrosophic network.

Hence, this proposed trapezoidal interval valued neutrosophic revised version of Bellman-Ford algorithm can be applied to a large number of inputs.

Table 3 Comparison of neutrosophic shortest path using the proposed method and existing method

Method	Neutrosophic shortest path length
in [18], Broumi *et al.* solved NSPP with triangular interval valued neutrosophic numbers and TrIVNNs as the edge weights of the network with six edges and eight edges	0.485 (using improved algorithm)
in this present work, we solved NSPP for the network with TrIVNNs as the edge weights	0.85 (using an intelligent algorithm called revised version of Bellman-Ford algorithm)

7 Comparative analysis

In [18], the authors solved NSPP under triangular and trapezoidal neutrosophic environment using an improved algorithm. However, NSPP is not solved using Bellman algorithm under trapezoidal interval neutrosophic environment to date. Hence, the comparative analysis is made in Table 3.

8 Conclusions

Hence in this work, the new definitions of score function and accuracy functions of trapezoidal interval neutrosophic numbers and their properties with numerical example are proposed. Also, the neutrosophic version of Bellman's algorithm based on the TrIVNN called an intelligent algorithm, which expresses the flexibility of the neutrosophic information absolutely under trapezoidal interval valued neutrosophic environment with a numerical example is proposed. In the future, the bipolar neutrosophic version of Bellman algorithm can be introduced.

8.1 Advantages of the proposed work

The proposed algorithm under trapezoidal interval valued neutrosophic environment has the following advantages:

(i) indeterminacy of the information can be dealt with efficiency.
(ii) cost of the neutrosophic shortest path can be minimised
(iii) the performance of the network can be maximised through the data have indeterminacy
(iv) Indeterminacy can be captured and shortest path can be obtained by splitting the various paths and hence performance of the system can be increased.

8.2 Limitations of the proposed work

The proposed revised version of Bellman-Ford algorithm has the following limitations:

(i) It runs only $O(|V|.|E|)$ times, where $|E|$ is the number of edges in the network.
(ii) It is unable to deal with the degree of contradiction of the edges.

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Neutrosophic Optimization of Industry 4.0 Production Inventory Model

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Abstract: In this age of information, the industrial sectors are embedding its functioning principles with the components of Industry 4.0. This article proposes a production inventory model discussing the paradigm shift towards smart production process involving many new cost parameters in addition to the conventional inventory costs. The proposed Industry 4.0 production inventory model is discoursed and compared in both deterministic and neutrosophic environments. The trapezoidal neutrosophic number representation of the parameters enhances the efficiency of the model in determining the optimal order time that minimizes the total costs. The model is highly comprehensive in nature and it is validated with a numerical example.

Keywords: Neutrosophic sets, Industry 4.0, production inventory model, optimization, decision making.

1. Introduction

Presently the industrial sectors are incorporating the techniques of digitalization to meet the requirements of the customer's demands at all ends. The production sectors practice new production methods to ease the process of production that comprises of several sequential steps and new cost parameters. The optimizing principle of manufacturing companies is costs minimization and profit maximization and the inventory models are utilized to make optimal decisions on order time and quantity. The Economic Production Quantity (EPQ) model proposed by Taft [1], a basic production inventory model to manage the levels of inventory by the production sectors. This model is the underlying model and it was developed and extended based on decision-making situations. The fundamental EPQ model was further modified with the integration of the cost parameters of shortages, trade discount, imperfect items, supply chain, deteriorating items, remanufacturing, waste disposal and so on. The production inventory models are extended to cater the requirements of the production sectors. Presently, the fourth industrial revolution is gaining significance amidst the developed and developing nations. Industry 4.0 will certainly bring a paradigm transition at all the levels of organization and control over the different stages of the product's life. The entire process of product production beginning from product conception, product design, product development, initialization of product production, manufacturing of the product, product delivery and ending with product rework, recycle and disposal will get into the digitalized mode based on customer- centric approach.

The elements of Industry 4.0 are stepping into the production sectors of large, medium and small-sized and at all phases of production processes. Christian Decker et al [2] introduced a cost-benefit model for smart items in the supply chain which is an initial initiative in calculating the advantages of introducing smart items into the network of the supply chain. Andrew Kusiak [3] presented the benefits of smart manufacturing; its core components and the production pattern in future. Fei et al [4] developed IT -driven assistance arranged shrewd assembling with its structure and attributes. Sameer et al [5] introduced a basic audit on keen assembling and Industry 4.0 development models and the suggestions for the entrance of medium and little enterprises. Xiulong et al [6] developed CPS-based smart production system for Industry 4.0 based on the review of the existing literature on smart production systems. Pietro et al [7] built up a digital flexibly chain through the powerful stock and smart agreements. Marc Wins[8] introduced a wide depiction of the highlights of a smart stock administration framework. Souvik et al [9] investigated the savvy stock administration framework dependent on the web of things (Iot). Poti et al [10] introduced the prerequisite examination for shrewd flexibly chain the board for SMEs. Ghadge et al [11] tended to the effect of Industry 4.0 execution on flexibly chains; introduced the benefits and confinements of industry 4.0 in supply chain arrange alongside its cutting-edge headings; clarified the core Industry 4.0 innovations and their business applications and investigated the ramifications of Industry 4.0 with regards to operational and gainful proficiency. Iqra Asghar et al [12] presented a digitalized smart EPQ-based stock model for innovation subordinate items under stochastic failure and fix rate. The above examined stock models are deterministic in nature and the costs boundaries are traditional in nature and they don't mirror the real costs boundaries identified with industry 4.0 components.

In this paper, manufacturing inventory model incorporating a new range of smart costs is formulated, also in this industry 4.0 model, the cost parameters are characterized as neutrosophic sets. This is the novelty of this research work and as for as the literature is concerned, industry 4.0 neutrosophic production inventory models have not been discussed so far and related literature does not exist. Smarandache [13] introduced neutrosophic sets that deal with truth, indeterminacy and falsity membership functions. Neutrosophic sets are widely applied to handle the situations of indeterminacy and it has extensive applications in diverse fields. Sahidul et al [14] developed neutrosophic goal programming for choosing the optimal green supplier, Abdel Nasser [15] used an integrated neutrosophic approach for supplier selection, Lyzbeth [16] constructed neutrosophic decision- making model to determine the operational risks in financial management, Ranjan Kumar et al [17,18] developed neutrosophic multi-objective programming for finding the solution to shortest path problem, Vakkas et al [19] proposed MADM method with bipolar neutrosophic sets.

Abdel-Basst, Mohamed et al [20] has developed neutrosophic decision-making models for effective identification of COVID-19; constructed bipolar neutrosophic MCDM for professional selection [21]; formulated a model to solve supply chain problem using best-worst method [22]and to measure the financial performance of the manufacturing industries [23]. Also, Abdel-Basset proposed presented a new framework for evaluating the innovativeness of the smart product – service systems [24]. As neutrosophic sets are highly viable, neutrosophic inventory models are formulated by many researchers. Chaitali Kar et al[25] developed inventory model with neutrosophic geometric programming approach. Mullai and Broumi[26] discussed neutrosophic inventory model without shortages, Mullai [27] developed neutrosophic model with price breaks. Mullai et al [28] constructed neutrosophic inventory model dealing with single-valued neutrosophic representation.

In all these neutrosophic inventory models, the cost parameters of the conventional inventory models are represented as neutrosophic sets or numbers, but these models did not discuss any new kind of cost parameters reflecting the transitions in the production processes. But the proposed model reflects the paradigm shift towards smart production process and incorporates new kinds of costs to cater the requirements of smart production inventory model. The industry 4.0 neutrosophic production inventory model with the inclusion of the respective costs to the core elements of smart production systems is highly essential as the existing production sectors are adapting to the environment of smart production set up, but to the best of our knowledge such models are still uncovered. This model primarily focuses on increase productivity and high quality of the product within low investment of finance. The composition of several components of industry 4.0 production inventory model result in diverse costs parameters such as smart ordering cost , internet connectivity initialization cost, holding costs ,smart product design cost, data management cost, customer data analysis cost, supplier data analysis cost, smart technology cost, production monitoring cost, reworking cost, smart training work personnel cost, smart tools purchase cost , smart disposal costs , smart environmental costs, holding cost. The term smart refers to the costs incurred with the integration of digital gadgets to the respective production departments.

The article is structured into the following sections: section 2 consists of the preliminary definitions of neutrosophic sets and its arithmetic operation; section 3 presents the industry 4.0 production inventory model; section 4 validates the proposed model with neutrosophic parameters; section 5 discusses the results and the last section concludes the paper.

2. Basics of Neutrosophic sets and operations

This section presents the fundamentals of neutrosophic sets, arithmetic operations and defuzzification

2.1 Neutrosophic set [13]

A neutrosophic set is characterized independently by a truth-membership function $\alpha(x)$, an indeterminacy-membership function $\beta(x)$, and a falsity-membership function$\gamma(x)$ and each of the function is defined from X → [0,1]

2.2**Single valued Trapezoidal Neutrosophic Number**

A single valued trapezoidal neutrosophic number $\tilde{A} = \langle (a,b,c,d): \rho_{\overline{A}}, \sigma_{\overline{A}}, \tau_{\overline{A}} \rangle$ is a special neutrosophic set on the real number set R, whose truth –membership, indeterminacy-membership, and a falsity –membership is given as follows.

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-a)\rho_{\tilde{A}}/(b-a) & (a \le x < b) \\ \rho_{\tilde{A}} & (b \le x \le c) \\ (d-x)\rho_{\tilde{A}}/(d-c) & (c < x \le d) \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_{\tilde{A}}(x) = \begin{cases} (b - x + \sigma_{\tilde{A}}(x-a)))/(b-a) & (a \le x < b) \\ \sigma_{\tilde{A}} & (b \le x \le c) \\ (x-c+\sigma_{\tilde{A}}(d-x))/(d-c) & (c < x \le d) \\ 1 & \text{otherwise} \end{cases}$$

$$\varphi_{\tilde{A}}(x) = \begin{cases} (b - x + \tau_{\tilde{A}}(x-a)))/(b-a) & (a \le x < b) \\ \tau_{\tilde{A}} & (b \le x \le c) \\ (x-c+\tau_{\tilde{A}}(d-x))/(d-c) & (c < x \le d) \\ 1 & \text{otherwise} \end{cases}$$

2.3. Operations on Single valued Trapezoidal Neutrosophic Numbers

Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1): \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle$ and $\tilde{B} = \langle (a_2, b_2, c_2, d_2): \rho_{\tilde{B}}, \sigma_{\tilde{B}}, \tau_{\tilde{B}} \rangle$ be two single valued trapezoidal neutrosophic numbers and $\mu \neq 0$, then

1. $\tilde{A} + \tilde{B} = \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle$
2. $\tilde{A} - \tilde{B} = \langle (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle$
3. $\tilde{A}\,\tilde{B} = \begin{cases} \langle (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 > 0, d_2 > 0) \\ \langle (a_1 d_2, b_1 c_2, c_1 b_2, d_1 a_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 > 0) \\ \langle (d_1 d_2, c_1 c_2, b_1 b_2, a_1 a_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 < 0) \end{cases}$
4. $\tilde{A}/\tilde{B} =$

   $\begin{cases} \langle (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 > 0, d_2 > 0) \\ \langle (d_1/d_2, c_1/c_2, b_1/b_2, a_1/a_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 > 0) \\ \langle (d_1/a_2, c_1/b_2, b_1/c_2, a_1/d_2): \rho_{\tilde{A}} \wedge \rho_{\tilde{B}}, \sigma_{\tilde{A}} \vee \sigma_{\tilde{B}}, \tau_{\tilde{A}} \vee \tau_{\tilde{B}} \rangle & (d_1 < 0, d_2 < 0) \end{cases}$
5. $\mu\tilde{A} = \begin{cases} \langle (\mu a_1, \mu b_1, \mu c_1, \mu d_1): \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle & (\mu > 0) \\ \langle (\mu d_1, \mu c_1, \mu b_1, \mu a_1): \rho_{\tilde{A}}, \sigma_{\tilde{A}}, \tau_{\tilde{A}} \rangle & (\mu < 0) \end{cases}$
6. $\tilde{A}^{-1} = \langle (1/d_1, 1/c_1, 1/b_1, 1/a_1): \rho_{\tilde{B}}, \sigma_{\tilde{B}}, \tau_{\tilde{B}} \rangle$ $(\tilde{A} \neq 0)$.

2.4 Defuzzification of Neutrosophic set

A single valued trapezoidal neutrosophic numbers of the form $\tilde{A} = \langle (a, b, c, d); \rho, \sigma, \tau \rangle$ can be defuzzified by finding its respective score value $K(\tilde{A})$

$K(\tilde{A}) = \frac{1}{16}[a + b + c + d] \times (2 + \mu_{\tilde{A}} - \pi_{\tilde{A}} - \varphi_{\tilde{A}})$.

3. Model Development

3.1 Assumptions

Shortages are not allowed.

Demand is not deterministic in nature.

The products are not of deteriorating type.

Planning horizon is infinite.

3.2 Notations

The below notations are used throughout this paper.

P – Smart production rate per cycle

D → Uniform demand rate per cycle

General Costs

O_s – Smart Ordering cost

I_c–Internet Connectivity Initialization Cost

Costs for time period $0 \leq t \leq t_1$

PD_s – Smart Product design cost

DM - Data management Cost

CD – Customer Data Analysis cost

SD – Supplier Data Analysis cost

T_s- Smart Technology Cost

M – Production Monitoring cost

r - defective rate

R – Reworking Cost

TR_s – Smart training work personnel cost

TO_s – Smart tools purchase cost

Costs for time period $t_1 \leq t \leq T$

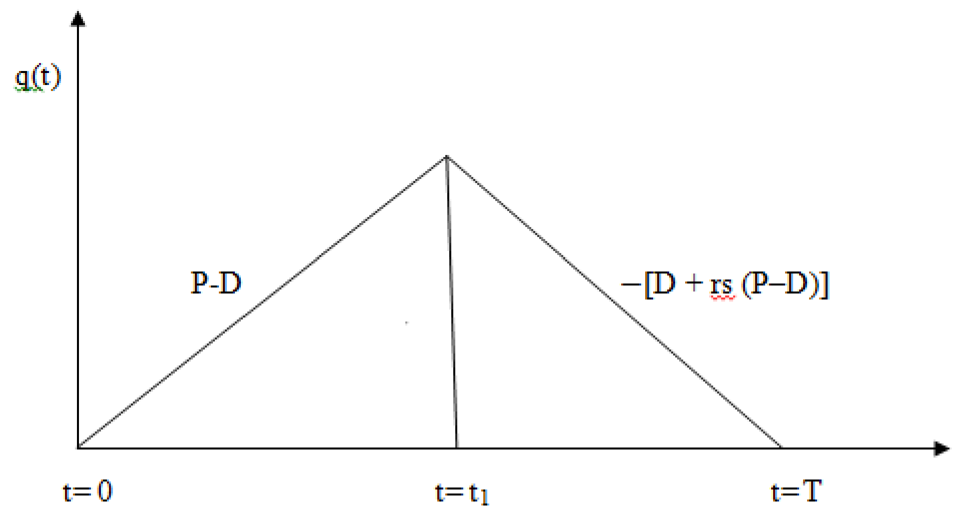
s – disposal rate

D_s– Smart disposal costs

E_s- Smart Environmental costs

Costs common for both the time periods

H - Holding costs



If q(t) represents the inventory level at time t∈ [0, T], so the differential equation for the instantaneous inventory q(t) at any time t over [0, T] is

$\frac{dq(t)}{dt}$ = P – D $0 \le t \le t_1$ → (1)

= −[D + rs (P–D)] $t_1 \le t \le T$ (2)

With initial condition q(0) = 0 and

Boundary condition q(T) = 0

$\frac{dq(t)}{dt}$ = P – D

dq(t) = (P – D) dt

q(t) = (P – D) t + c

with initial condition q(0) = 0

q(0) =(P – D) 0 + c

0 = c

q(t) =(P – D) t $0 \le t \le t_1$ → (3)

solving equation (3)

$\frac{dq(t)}{dt}$ = − [D + rs (P–D)] $t_1 \le t \le T$

dq(t) = − [D + rs (P–D)] dt

q(t) = −[D + rs (P–D)] t + c

With boundary condition q(T) = 0

q(T) = − [D + rs(P–D)] T + c

0 = − [D + rs(P–D)] T + c

c =[D + rs(P–D)] T

q(t) = −[D + rs (P–D)] t + [D +rs (P–D)] T   →(4)

using equation (3),(4), we get

I_{max} = (P–D) t_1

I_{max} =[D + rs (P–D)] (T− t_1)

$t_1 = \frac{I_{max}}{P-D}$

$T - t_1 = \frac{I_{max}}{D+rs\,(P-D)}$

We adding ,we get

$t_1 + T - t_1 = I_{max}\left[\frac{1}{(P-D)} + \frac{1}{D+rs(P-D)} \right]$

$T = I_{max}\left[\frac{1}{(P-D)} + \frac{1}{D+rs(P-D)} \right]$

$T = I_{max}\left[\left(\frac{P+(P-D)rs}{P-D[D+rs(P-D)]}\right)\right]$

I_{max}=[$\frac{P-D[D+rs(P-D)}{P+(P-D)rs}$] T

Smart product design cost = $PD_s\int_0^{t_1} q(t)\,dt$

= $PD_s\int_0^{t_1}(P-D)tdt$

= $\frac{PD_S}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Data management cost = DM $\int_0^{t_1} q(t)\,dt$

= DM $\int_0^{t_1}(P-D)tdt$

= $\frac{DM}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Customer data analysis cost = CD $\int_0^{t_1} q(t)dt$

= CD $\int_0^{t_1}(P-D)tdt$

= $\frac{CD}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Supplier data analysis cost = SD $\int_0^{t_1} q(t)dt$

= SD $\int_0^{t_1}(P-D)tdt$

= $\frac{SD}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Smart Technology cost = $T_s\int_0^{t_1} q(t)dt$

= $T_s\int_0^{t_1}(P-D)tdt$

= $\frac{T_S}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Production Monitoring cost = M $\int_0^{t_1} q(t)dt$

= M $\int_0^{t_1}(P-D)tdt$

= $\frac{M}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Reworking cost = R $\int_0^{t_1} q(t)dt$

= R $\int_0^{t_1}(P-D)tdt$

= $\frac{R}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Smart training work personal cost = $TR_s\int_0^{t_1} q(t)dt$

= $TR_s\int_0^{t_1}(P-D)tdt$

= $\frac{TR_S}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Smart tools purchase cost = $TO_s\int_0^{t_1} q(t)dt$

= $TO_s\int_0^{t_1}(P-D)tdt$

= $\frac{TO_S}{2}$[P-D ($\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$]

Smart disposal cost = $D_s\int_{t_1}^{T} q(t)dt$

= $D_s\int_{t_1}^{T} D+rs(P-D)tdt$

$$= \frac{D_S}{2} \quad [D + rs(P-D) \quad (\frac{(P-D)}{P+(P-D)rs}T)^2]$$

Smart Environmental cost = $E_s\int_{t_1}^{T} q(t)dt$

$$= E_s\int_{t_1}^{T} D + rs(P-D)tdt$$

$$= \frac{E_S}{2} \quad [D + rs(P-D) \quad (\frac{(P-D)}{P+(P-D)rs}T)^2]$$

∴Holding cost =C_1 $[\int_0^{t_1} q(t)dt + \int_{t_1}^{T} q(t)dt]$

$$= C_1 \; [(P–D)\frac{t_1^2}{2} + [D + rs(P-D)]\frac{(T-t_1)^2}{2}]$$

$$= \frac{C_1}{2} [\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}] T^2$$

∴**Total Cost** = Smart Ordering cost + Internet Connectivity Initialization Cost+ Holding Costs +Smart Product design cost+ Data management Cost+ Customer Data Analysis cost+ Supplier Data Analysis cost+ Smart Technology Cost+ Production Monitoring cost+ Reworking Cost+ Smart training work personnel cost+ Smart tools purchase cost + Smart disposal costs + Smart Environmental costs

$= O_s + I_c + \frac{C_1}{2} [\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}] T^2 + \frac{PD_s}{2}[P\text{-}D (\frac{D+rs(P-D)}{P+(P-D)rs}T)^2] +$

$\frac{DM}{2}[P\text{-}D (\frac{D+rs(P-D)}{P+(P-D)rs}T)^2] + \frac{CD}{2} [P\text{-}D (\frac{D+rs(P-D)}{P+(P-D)rs}T)^2] +$

$\frac{SD}{2}[P\text{-}D (\frac{D+rs(P-D)}{P+(P-D)rs}T)^2] + \frac{T_S}{2} [P\text{-}D (\frac{D+rs(P-D)}{P+(P-D)rs}T)^2] +$

$\frac{M}{2}$[P-D ([illegible])^2] + [illegible] [P-D ([illegible])^2] +

[illegible] [P-D ([illegible])^2] + [illegible] [P-D ([illegible])^2] +

[illegible] [illegible] + [illegible]

= $O_s + I_c +$ [illegible] T^2 + [P-D ([illegible])^2][illegible]

[illegible]]+ [illegible]

= $O_s + I_c +$ [illegible] $T^2 + \frac{1}{2}$ [P-D ([illegible])^2 (PD_s+ [illegible] + T_s+ [illegible] +TR_s+TO_s)+

[illegible]$(\frac{(P-D)}{P+(P-D)rs}T)^2]$ (D_s+ E_s)]

Total Average cost= $\frac{1}{T}$ [O_s + I_c+$\frac{C_1}{2}$ $[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}]$ T^2 + $\frac{1}{2}$ [P-D $(\frac{D+rs(P-D)}{P+(P-D)rs}T)^2$($PD_s$+ DM+ CD+ SD+ T_s+ M+ R+TR_s+TO_s)+ $[D + rs(P-D) \quad (\frac{(P-D)}{P+(P-D)rs}T)^2]$ (D_s + E_s)]

$=\frac{O_S}{T}+\frac{I_C}{T}+\frac{C_1}{2}\left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}\right]T+\frac{1}{2}\left[P-D\left(\frac{D+rs(P-D)}{P+(P-D)rs}\right)^2 T(PD_s+DM+CD+SD+T_s+M+R+TR_s+TO_s)+\right.$

$\left.[D+rs(P-D)\quad\left(\frac{(P-D)}{P+(P-D)rs}\right)^2 T](D_s+E_s)\right]$

So the Classical EPQ model is

$$\text{Min}\quad TAC(T)=\frac{O_S}{T}+\frac{I_C}{T}+\frac{C_1}{2}\left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}\right]T+\frac{1}{2}\left[P-D\left(\frac{D+rs(P-D)}{P+(P-D)rs}\right)^2 T(PD_s+DM+CD+SD+T_s+M+R+TR_s+TO_s)+[D+rs(P-D)\quad\left(\frac{(P-D)}{P+(P-D)rs}\right)^2 T](D_s+E_s)\right]$$

Such that T > 0

We can show that TAC(T) will be minimum for

$$T^*=\sqrt{\frac{2(O_S+I_C)}{C_1\left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}\right]+[P-D\left(\frac{D+rs(P-D)}{P+(P-D)rs}\right)^2(PD_s+DM+CD+SD+T_s+M+R+TR_s+TO_s)+[D+rs(P-D)\left(\frac{(P-D)}{P+(P-D)rs}\right)^2](D_s+E_s)]}}$$

$$TAC^*(T^*)=\sqrt{\begin{array}{l}2(O_s+I_c)+C_1\left[\frac{P-D[D+rs(P-D)]}{P+(P-D)rs}\right]+[P-D\left(\frac{D+rs(P-D)}{P+(P-D)rs}\right)^2(PD_s+DM+CD+SD+\\ \quad T_s+M+R+TR_s+TO_s)+[D+rs(P-D)\left(\frac{(P-D)}{P+(P-D)rs}\right)^2](D_s+E_s)]\end{array}}$$

4.Illustration

To validate the developed model, an inventory system with the below characteristics is taken into consideration

Smart production rate per cycle = Rs.500unit/per month , Uniform demand rate per cycle = Rs.250/month, Smart Ordering cost =Rs.310/run, Internet Connectivity Initialization Cost = Rs.370/year, Smart Product design cost = Rs.25/unit, Data management Cost = Rs.50/unit, Customer Data Analysis cost = Rs.45/unit, Supplier Data Analysis cost = Rs.25/unit, Smart Technology Cost = Rs.15/unit, Production Monitoring cost = Rs.45/unit, defective rate = Rs.1, Reworking Cost = Rs.22/run, Smart training work personnel cost = Rs.30/unit, Smart tools purchase cost= Rs.10/unit, disposal rate = Rs. 3/unit, Smart disposal costs = Rs. 5/unit, Smart Environmental costs = Rs.7/unit, Holding costs = Rs.1/unit/year. Find the time interval and find the total average cost.

The value of T* and TAC(T*) is 0.179 and Rs.208.39 respectively

This model can be validated with the single valued neutrosophic trapezoidal fuzzy value representations as follows,

D = ⟨(250,350,450,550):0.7,0.2,0.1⟩

Os= ⟨(350,450,550,650):0.9,0.3,0.1⟩

Ic= ⟨(550,650,750,850):0.8,0.3,0.4⟩

PDs = ⟨(25,35,45,55):0.7,0.3,0.2⟩

DM = ⟨(65,75,85,95):0.9,0.3,0.4⟩

CD = ⟨(55,65,75,85):0.8,0.1,0.2⟩

SD = ⟨(20,30,40,50):0.8,0.3,0.2⟩

Ts = ⟨(15,18,22,24):0.7,0.1,0.2⟩

M = ⟨(60,70,80,90):0.7,0.2,0.4⟩
r= ⟨(1,1.5,2.5,3):0.9,0.1,0.2⟩
R = ⟨(20,25,35,40):0.8,0.2,0.1⟩
TRs =⟨(35,45,55,65):0.7,0.1,0.3⟩
TOs= ⟨(8,12,16,20):0.7,0.1,0.4⟩
S = ⟨(3,4,6,8):0.8,0.1,0.4⟩
Ds= ⟨(5,7,9,11):0.7,0.2,0.3⟩
Es= ⟨(6,9,12,15):0.8,0.2,0.3⟩
C_1= ⟨(1,1.5,2.5,3):0.9,0.3,0.2⟩
The value of T* = 0.178 and TAC*(T*) = 210.29

5. Discussion

A neutrosophic production inventory model incorporating the costs parameters of industry 4.0 is developed together with the presentation of its conceptual framework. Several key benefits of neutrosophic production inventory model have been emphasized in this paper, together with the additional cost parameters. Another point of discussion is the usage of the production inventory model to find the feasible time to place orders that confines the total expenses. The representation of these costs parameters as single valued trapezoidal neutrosophic number tackles the conditions of uncertainty.

The constructed manufacturing inventory model is validated with deterministic parameters and neutrosophic parameters. The optimal time that yields minimum costs is nearly equal in both the cases of deterministic and neutrosophic validation. The neutrosophic representation makes this model more comprehensive. In this paper shortages are not allowed, the products are not of deteriorating type, planning horizon is infinite. The developed model can be extended to neutrosophic production inventory model with shortages and deteriorating items. This model primarily focuses on increases productivity of high-quality products within low investment of finance. The discussion is summarized as follows, a novel neutrosophic production inventory model is developed with the cost parameters pertaining to the fourth industrial revolution. This proposed model will certainly assist the production sectors to incorporate new types of costs. A deeper investigation on the effects of our decision making is clearly an obligation for upcoming work.

6. Conclusion

The proposed industry 4.0 inventory model is a novel approach integrating the concept of smart production principles, and neutrosophic representations of cost parameters. This model is an underlying smart production model and this model can be further developed based on the needs of the production sectors. The proposed model is pragmatic in nature and it can be extended by including the concepts of customer acquisition and product propagation with additional cost parameters. These models will certainly unveil the new requirements of production scenario to meet the demands of the customers of this information age. The model constructed in this paper presents the present need of the production environment and it will certainly assist the decision makers to optimize profit. The cost parameters of this model can be scaled to the requirements of small and medium sized enterprises which could be the future work.

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New type of neutrosophic supra connected space

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Abstract Neutrosophic α supra-connected space is defined and its properties are studied in this paper. The purpose of this theory is to investigate the common relationship between two objects after dropping an axiom in neutrosophic topological spaces. Also, defined herein is a new compactness in neutrosophic supra topological spaces and some of its properties are investigated.

Key words Neutrosophic supra topology, neutrosophic α supra open set, neutrosophic α supra closed set, neutrosophic α supra connected space, neutrosophic α supra compact sapce.

1 Introduction

F. Smarandache [16, 17] developed neutrosophic theory as a generalization of Zadeh's [18] fuzzy set (FS) theory and Atanassov's [2] (IFS) theory. Neutrosophic sets gained attention in many fields such as topology [8, 10–12], image processing, algebra, graph theory, medicine, etc. Fuzzy topological space introduced by D. Coker [3] then Salama and Alblowi [15] defined neutrosophic topology. Later on researchers developed his theory and investigated the various types of open and closed sets, continuous function, homeomorphism in neutrosophic topological spaces.
Kuratowski [4] first used the notion of connected space in general topology. Parimala et al. [7, 9, 13] developed various open and closed sets in nano topological space. Parimala et al. [6, 14] also introduced neutrosophic $\alpha\psi$-closed sets, neutrosophic $\alpha\psi$-connected space. Karthika et al. [5] redefined the notion of neutrosophic topology in the extended range, i.e., the range of neutrosophic components from the unit interval to the complex plane and then studied the relationship between neutrosophic complex $\alpha\psi$ connectedness and neutrosophic complex connected space and the properties of neutrosophic complex $\alpha\psi$ connected space.

1.1 Motivation and objective

The notion of neutrosophic sets and connected space motivates us to generate this novel neutrosophic α supra connected space. Our objective in this paper is to define the neutrosophic α supra connected space and the notion of the neutrosophic α supra compactness and to study their properties. The purpose of these connected spaces in real life situations is to investigate the common relationship between two objects, such as, two different branded cars, the common symptoms between two diseases, etc. The paper is constructed as follows: in section 2, the basic definitions such as neutrosophic set, neutrosophic supra topological space, neutrosophic α open set, arithmetic operations are discussed. The definition of q-coincident, the interior and the closure of a neutrosophic alpha open set, the neutrosophic α supra connected space and its properties are presented in section 3. The neutrosophic α supra compactness is defined and its properties are investigated in section 4. The conclusions and future directions of work of these novel concepts are presented in section 5.

2 Preliminaries

The definitions which are relevant to our work are presented in this section.

Definition 2.1. [16,17] A neutrosophic set (NS) D on $\mathfrak{W} \neq \emptyset$ is defined by

$$D = \{\langle \xi, \mu_D(\xi), \sigma_D(\xi), \nu_D(\xi)\rangle : \xi \in \mathfrak{W}\}$$

where MF μ_D, INDF σ_D, NMF ν_D maps from $\mathfrak{W}$ to [0,1] for each $\xi \in \mathfrak{W}$ to D and $0 \leq \mu_D(\xi)+\sigma_D(\xi)+\nu_D(\xi) \leq 3$ for each $\xi \in \mathfrak{W}$.

Definition 2.2. [15] Let $D_1 = \{\langle \xi, \mu_{D_1}(\xi), \sigma_{D_1}(\xi), \nu_{D_1}(\xi)\rangle : \xi \in \mathfrak{W}\}$ and $D_2 = \{\langle \xi, \mu_{D_2}(\xi), \sigma_{D_2}(\xi), \nu_{D_2}(\xi)\rangle : \xi \in \mathfrak{W}\}$ be NSs. Then

(i) $D_1 \subseteq D_2$ if and only if $\mu_{D_1}(\xi) \leq \mu_{D_2}((\xi))$, $\sigma_{D_1}(\xi) \geq \sigma_{D_2}(\xi)$ and $\nu_{D_1}(\xi) \geq \nu_{D_2}(\xi)$;

(ii) ${D_1}^C = \{\langle \xi, \nu_{D_1}(\xi), 1-\sigma_{D_1}(\xi), \mu_{D_1}(\xi)\rangle : \xi \in \mathfrak{W}\}$;

(iii) $D_1 \cap D_2 = \{\langle \xi, \mu_{D_1}(\xi) \wedge \mu_{D_2}(\xi), \sigma_{D_1}(\xi) \vee \sigma_{D_2}(\xi), \nu_{D_1}(\xi) \vee \nu_{D_2}(\xi)\rangle : \xi \in \mathfrak{W}\}$;

(iv) $D_1 \cup D_2 = \{\langle \xi, \mu_{D_1}(\xi) \vee \mu_{D_2}(\xi), \sigma_{D_1}(\xi) \wedge \sigma_{D_2}(\xi), \nu_{D_1}(\xi) \wedge \nu_{D_2}(\xi)\rangle : \xi \in \mathfrak{W}\}$.

The symbols $\vee, \wedge$ denotes the maximum and minimum operator. The NS $(D_1)^C$ denotes the complement of NS D_1.

Definition 2.3. [15] Let $\mathfrak{D}$ be a family of NSs on $\mathfrak{W}$. The pair $(\mathfrak{W}, \mathfrak{D})$ is called a neutrosophic supra topology, if the following conditions are satisfied:

(T1) $0_{\mathfrak{D}}, 1_{\mathfrak{D}} \in \mathfrak{W}$,

(T2) An arbitrary union of NSs D_i is in $\mathfrak{D}$.

Definition 2.4. [1] A subset D of a NTS $(\mathfrak{W}, \mathfrak{D})$ is called

1. a NPOS, if $D \subseteq (\overline{D})^o$ and a NPCS if $\overline{(D^o)} \subseteq D$,
2. a NSOS, if $D \subseteq \overline{(D^o)}$ and a NSCS if $(\overline{D})^o \subseteq D$,
3. a NαSOS, if $D \subseteq (\overline{(D^o)})^o$ and NαSCS if $\overline{(\overline{D})^o} \subseteq D$.

3 On the neutrosophic α supra connected space

Definition 3.1. The interior and closure of NS D in NTS $\mathfrak{W}$ are denoted by D^o, $\overline{D}$ and defined by $D^o = \cup\{C : C$ is an NαSOS in $\mathfrak{W}$ and C $\subseteq D\}$, $\overline{D} = \cap\{C : C$ is an NαSCS in $\mathfrak{W}$ and C $\supseteq D\}$.

Definition 3.2. Two NαSOSs C and D of $\mathfrak{W}$ are said to be q-coincident if and only if there exists an element $\zeta \in \xi$ such that $C(\zeta) + D(\zeta) > 1$ or, $\mu_C(\xi) > \nu_D(\xi), \sigma_C(\xi) < 1 - \sigma_D(\xi), \nu_C(\xi) < \mu_D(\xi)$.

Lemma 3.3. *For any two NSs D and F of $\mathfrak{W}$, $\neg(DqF)$ if and only if $D \subset F^c$.*

Example 3.4. Let $\mathfrak{W} = \{x, y\}$ and $C = \{\frac{x}{(0.5,0.2,0.2)}, \frac{y}{(0.4,0.4,0.6)}\}$, $D = \{\frac{x}{(0.5,0.3,0.3)}, \frac{y}{(0.2,0.5,0.6)}\}$ be two neutrosophic open sets. Let $\mathfrak{D} = \{0_{\mathfrak{D}}, 1_{\mathfrak{D}}, C, D\}$. We know that each neutrosophic open set is a neutrosophic α open set. C and D are q-coincident since the intersection of these two sets is a non-empty set and also C is not a subset of the complement of D.

Definition 3.5. An NTS is said to be neutrosophic α supra connected space, if the intersection of two NαSOSs C and D is non-empty or, if there does not exist an NαSOS NαSCS F in $\mathfrak{W}$ such that $C \subset F \subset D^c$. An NTS is said to be separation of $\mathfrak{W}$, if it is not a neutrosophic α supra connected space.

Theorem 3.6. *If C, D are NαSOSs in $\mathfrak{W}$ which forms a separation of $\mathfrak{W}$ and U is a neutrosophic α-connected subspace of $\mathfrak{W}$, then U is either in C or in D.*

Proof. Let C, D be an NαSOSs in $\mathfrak{W}$. The intersection of an NαSOS C and a neutrosophic α-connected subspace U is an NαSOS in U and the intersection of an NαSOS D and neutrosophic α-connected subspace is an NαSOS in U. These two NαSOSs are disjoint and their union is a neutrosophic α-connected subspace U. These two NαSOSs constitute a separation of U if these two NαSOSs are nonempty. Thus one of the NαSOSs is empty. Hence the neutrosophic α-connected subspace U is definitely either in C or in D. □

Theorem 3.7. *If an NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space. Then it is a neutrosophic connected space.*

Proof. Let C and D be two neutrosophic open sets in $\mathfrak{W}$. If $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic connected space, then there exists a neutrosophic closed open set F in $\mathfrak{W}$ such that $D \subset F$. Then we know that every neutrosophic open(resp. closed) set is an NαSOS(resp. NαSCS), therefore, F is an NαSOS NαSCS in $\mathfrak{W}$ such that $D \subset F$ and $\neg(FqB)$. Hence $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. This is in contradiction to our hypothesis. Therefore, NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic connected space. □

Theorem 3.8. *An NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space if and only if there is no NαSOS NαSCS E in $\mathfrak{W}$ such that C is a subset of E and E is a subset of the complement of D.*

Proof. Let NTS $(\mathfrak{W}, \mathfrak{D})$ be a neutrosophic α supra connected space. Suppose E is NαSOS NαSCS in $\mathfrak{W}$ such that C is a subset of E and E is a subset of complement of D. This implies that $\neg(EqD)$. Therefore, C is NαSOS NαSCS in $\mathfrak{W}$ such that $C \subset D^c$ and $\neg(EqD)$. Hence $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space, which is a contradiction. Hence there is no NαSOS NαSCS E in $\mathfrak{W}$ such that C is a subset of E and E is a subset of complement of D.
Suppose NTS $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. Then there is an NαSOS NαSCS E in $\mathfrak{W}$ such that C is a subset of E and E is a subset of complement of D. Now, $\neg(EqD)$ which implies that E is a subset of D^c.
$\therefore$ E is an NαSOS NαSCS in $\mathfrak{W}$ such that C is a subset of E and E is a subset of the complement of D, which contradicts our assumption. □

Theorem 3.9. *If an NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space, then NαSOSs $C \neq \emptyset$ and $D \neq \emptyset$.*

Proof. If an NαSOS C is empty, then C is a neutrosophic α supra open neutrosophic α supra closed set in $\mathfrak{W}$. Now to prove $\neg(CqD)$. If NαSOSs C and D are q-coincident, then there is a $\xi \in \mathfrak{W}$ such that $\mu_C(\xi) > \nu_D(\xi)$ or, $\nu_C(\xi) < \mu_D(\xi)$. But $\mu_C(\xi) = 0_{\mathfrak{D}}$ and $\nu_C(\xi) = 1_{\mathfrak{D}}$ for all $\xi \in \mathfrak{W}$. Therefore, there exists no point $\xi \in \mathfrak{W}$ for which $\mu_C(\xi) > \nu_D(\xi)$ or, $\sigma_C(\xi) < 1 - \sigma_D(\xi)$ or, $\nu_C(\xi) < \mu_D(\xi)$. Hence $\neg(CqD)$ and an NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space. This contradicts the hypothesis.
$\therefore$ both NαSOSs $C \neq \emptyset$ and $D \neq \emptyset$. □

Theorem 3.10. *If an NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space and $C \subset C_1$ and $D \subset D_1$, then NαSOSs $C_1 \neq \emptyset$ and $D_1 \neq \emptyset$ are not disjoint sets.*

Proof. Suppose an NTS $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. Then there is a NαSOS NαSCS E in $\mathfrak{W}$ such that $C_1 \subset E$ and $\neg(EqD_1)$. Clearly, $C \subset E$. Now we claim that $\neg(EqB)$. If FqB, then there exists a point $\xi \in \mathfrak{W}$ such that $\mu_E(\xi) > \nu_D(\xi)$ or, $\nu_E(\xi) < \mu_D(\xi)$. Suppose $\xi \in \mathfrak{W}$ such that $\mu_E(\xi) > \nu_B(\xi)$. Now $D \subset D_1$, $\mu_D(\xi) \leq \mu_{D_1}(\xi)$, $\sigma_D(\xi) \geq \sigma_{D_1}(\xi)$ $\nu_D(\xi) \geq \nu_{D_1}(\xi)$. So $\mu_E(\xi) > \nu_{D_1}(\xi)$, $\nu_E(\xi) < \mu_{D_1}(\xi)$ and EqD_1, a contradiction. Consequently, NTS $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. □

Example 3.11. Let $\mathfrak{W} = \{x, y\}$ and $C = \{\frac{x}{(0.5,0.2,0.2)}, \frac{y}{(0.4,0.4,0.6)}\}$, $D = \{\frac{x}{(0.5,0.3,0.3)}, \frac{y}{(0.2,0.5,0.6)}\}$, $D_1 = \{\frac{x}{(0.6,0.1,0.2)}, \frac{y}{(0.7,0.2,0.3)}\}$, $C_1 = \{\frac{x}{(0.5,0.2,0.2)}, \frac{y}{(0.4,0.4,0.4)}\}$ be neutrosophic α open sets on $\mathfrak{W}$. Let $\mathfrak{D} = \{0_{\mathfrak{D}}, 1_{\mathfrak{D}}, C, D, C_1, D_1\}$ be neutrosophic topology on $\mathfrak{W}$ and $C \subseteq C_1, D \subseteq D_1$. Then $(\mathfrak{W}, \mathfrak{D})$ is neutrosophic α connected between C_1 and D_1.

Theorem 3.12. *An NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space iff NαSOSs $\overline{C} \neq \emptyset$ and $\overline{D} \neq \emptyset$ are not disjoint sets.*

Proof. The proof of the necessary part follows from Theorem 3.5, since we know that $C \subset \overline{C}$ and $D \subset \overline{D}$. For the sufficient part we assume that NTS $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. Then NαSOS NαSCS E of $\mathfrak{W}$ is such that $C \subset E$ and $\neg(EqD)$. Since E is an NαSCS and $C \subset E$, $\overline{C} \subset \overline{E}$. Now, $\neg(EqD)$, which implies that E is a subset of the complement of D. Therefore, the interior of E is a subset of the interior of the complement of D. Hence $\neg(Eq\overline{D})$ and $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space between $\overline{C}$ and $\overline{D}$. □

Theorem 3.13. *Let C and D be two NαSOSs in $(\mathfrak{W}, \mathfrak{D})$. If C and D are q-coincident, then $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space.*

Theorem 3.14. *An NTS $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space iff every pair of NαSOSs forms a neutrosophic α supra connected space.*

Proof. Necessity: Let C, D be any pair of neutrosophic α open subsets of $\mathfrak{W}$. Suppose $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. Then there is an NαSOS NαSCS E of $\mathfrak{W}$ such that A is a subset of E and $\neg(AqD)$. E is a nonempty NαSOS NαSCS in $(\mathfrak{W}, \mathfrak{D})$ since, NαSOS C and D are non-empty. Hence $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space.
Sufficiency: Suppose NTS $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. Then there is a proper NαSOS NαSCS $E \neq \emptyset$ of $\mathfrak{W}$. Consequently, $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space between E and E^c, which is a contradiction. □

Theorem 3.15. *Let $(\mathfrak{W}_1, \mathfrak{D}_1)$ be a neutrosophic α subspace of an NTS $(\mathfrak{W}, \mathfrak{D})$ and C, D be neutrosophic α open subsets of $\mathfrak{W}_1$. If $(\mathfrak{W}_1, \mathfrak{D}_1)$ be a neutrosophic α supra connected space then $(\mathfrak{W}, \mathfrak{D})$ is also a neutrosophic α supra connected space.*

Proof. Suppose $(\mathfrak{W}, \mathfrak{D})$ is not a neutrosophic α supra connected space. Then there is an NαSOS NαSCS E of $\mathfrak{W}$ such that C is a subset of E and $\neg(EqD)$. But $U = E \cap \mathfrak{D}_1$. Then U is an NαSOS NαSCS in $\mathfrak{D}_1$ such that C is a subset of U and $\neg(UqD)$. Hence $(\mathfrak{W}_1, \mathfrak{D}_1)$ is not a neutrosophic α supra connected space, a contradiction, thus, $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space. □

Theorem 3.16. *If $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic semi-connected space then $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space.*

Proof. Let C, D be two neutrosophic semi-open sets in $\mathfrak{D}$. Suppose $(\mathfrak{W}, \mathfrak{D})$ be not a neutrosophic α supra connected space. Then there is an NαSOS NαSCS E in $\mathfrak{D}$ such that $C \subset E$ and $\neg(EqD)$. We know that every NαSOS is a neutrosophic semi-open set. Therefore E is a neutrosophic semi-open neutrosophic semi-closed set such that $C \subset E$, which is a contradiction. Hence $(\mathfrak{W}, \mathfrak{D})$ is a neutrosophic α supra connected space. □

4 Neutrosophic α supra compactness

The concepts of a neutrosophic α supra open cover and the neutrosophic α supra compactness in a neutrosophic supra topological space are defined and their properties are studied in this section.

Definition 4.1. 1. Let $\mathcal{N} = \{\langle a, \mu_{D_i}, \sigma_{D_i}, \nu_{D_i}\rangle : i \in J\}$ be a collection of neutrosophic α supra open sets. The collection $\mathcal{N}$ of subsets of neutrosophic supra topological space $(\mathfrak{W}, \mathfrak{D})$ is said to be a neutrosophic α supra cover of $\mathfrak{W}$ if the union of $\{\langle a, \mu_{D_i}, \sigma_{D_i}\nu_{D_i}\rangle : i \in J\}$ is $1_{\mathfrak{D}}$. It is called a neutrosophic α supra open covering of $\mathfrak{W}$ if its NSs are neutrosophic α open subsets of $\mathfrak{W}$.

2. Let $\mathcal{N}_i$ be a subfamily of neutrosophic α supra open covers on $\mathfrak{W}$. A neutrosophic supra topological space is said to be neutrosophic α supra compact if every neutrosophic α supra open covering $\mathcal{N}$ of $\mathfrak{W}$ contains a finite sub collection that also covers $\mathfrak{W}$.

Proposition 4.2. *A neutrosophic supra topological space* $(\mathfrak{W}, \mathfrak{D})$ *is called neutrosophic* α *supra compact if and only if every neutrosophic* α *supra open cover of* $\mathfrak{W}$ *has a finite neutrosophic* α *supra subcover.*

Lemma 4.3. *The subspace* $\mathfrak{V}$ *of* $\mathfrak{W}$ *is neutrosophic* α *supra compact if and only if every covering of* $\mathfrak{V}$ *by sets which are neutrosophic* α *supra open in* $\mathfrak{W}$ *contains a finite subcollection covering* $\mathfrak{V}$.

Proof. Suppose $\mathfrak{V}$ is compact, so the collection of neutrosophic α supra open sets $\mathcal{N} = \{\mathcal{N}_i : i \in J\}$ covers $\mathfrak{V}$. Then $\{\mathcal{N}_i \bigcap \mathfrak{V} : i \in J\}$ covers $\mathfrak{V}$. These sets are neutrosophic α supra open sets in $\mathfrak{V}$. The finite subcollection $\{\mathcal{N}_{i_1} \bigcap \mathfrak{V}, \ldots, \mathcal{N}_{i_n} \bigcap \mathfrak{V}\}$ also covers $\mathfrak{V}$. This implies that the finite subcollection $\{\mathcal{N}_{i_1}, \ldots, \mathcal{N}_{i_n}\}$ of $\mathfrak{N}$ covers $\mathfrak{V}$.
Conversely, assume that every neutrosophic α supra open set covering of $\mathfrak{V}$ contains a finite subcollection which also covers $\mathfrak{V}$. Let $\mathcal{B}$ be the family of neutrosophic α supra open sets which covers $\mathfrak{V}$. These neutrosophic α supra open sets are in $\mathfrak{V}$. For each i, choose a neutrosophic α supra open set $\mathcal{N}_i$ in $\mathfrak{W}$ such that $\mathcal{B} = \mathcal{N}_i \bigcap \mathfrak{V}$. The collection of neutrosophic α supra open set $\mathcal{N} = \{\mathcal{N}_i\}$ covers $\mathfrak{V}$, by our assumption, some finite subcollection $\{\mathcal{N}_{i_1}, \ldots, \mathcal{N}_{i_n}\}$ covers $\mathfrak{V}$. Then the finite subcollection $\mathcal{B}_1, \ldots, \mathcal{B}_n$ covers $\mathfrak{V}$. □

Theorem 4.4. *Every neutrosophic* α *closed subspace of a neutrosophic* α *supra compact space is neutrosophic* α *supra compact.*

Theorem 4.5. *If the map* $f : \mathfrak{W} \to \mathfrak{V}$ *is neutrosophic* α *supra continuous and* $\mathfrak{W}$ *is a neutrosophic* α *supra compact space, then* $\mathfrak{V}$ *is a neutrosophic* α *supra compact space.*

Proof. Given that the map $f : \mathcal{W} \longrightarrow \mathfrak{V}$ is a neutrosophic α supra continuous map and $\mathfrak{W}$ is a neutrosophic α supra compact space. Let the collection $\mathcal{C}$ be a covering of neutrosophic α supra open sets $f(\mathfrak{W})$ in $\mathfrak{V}$. Since the map f is continuous, the collection $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a neutrosophic α supra open set covering of X. Hence $f^{-1}(C_1), \ldots, f^{-1}(C_n)$ cover $\mathfrak{W}$. Then the neutrosophic α supra open sets $C_1, \ldots, C_n$ cover $f(\mathfrak{W})$. □

Definition 4.6. A collection of neutrosophic α supra open sets $\mathcal{N}$ of subsets of $\mathfrak{W}$ is said to have finite intersection property if for every finite subcollection $\{\mathcal{N}_1, \ldots, \mathcal{N}_n\}$ of $\mathcal{N}$, the intersection of $\mathcal{N}_1, \ldots, \mathcal{N}_n$ is nonempty.

Theorem 4.7. *Let* $\mathfrak{W}$ *be a neutrosophic supra topological space. An NSTS* $\mathfrak{W}$ *is neutrosophic* α *supra compact if and only if for every collection* $\mathcal{N}$ *of neutrosophic* α *supra closed sets in* $\mathfrak{W}$ *having finite intersection property,* $\cap_{A \in \mathcal{N}} A$ *of all elements of* $\mathcal{N}$ *is non-empty.*

Proof. Let $\mathcal{C}$ be a collection of neutrosophic α supra open sets and $\mathcal{D}$ be a neutrosophic α supra closed set, i.e., $\mathcal{D} = \{\mathfrak{W} - C : C \in \mathcal{C}\}$. We know that C covers $\mathfrak{W}$ if and only if $\cap_{D \in \mathcal{D}} D$ of all elements of $\mathcal{D}$ is empty.
The finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$ covers $\mathfrak{W}$ if and only if the intersection of the corresponding subcollection $\{\mathfrak{W} - C_1, \ldots, \mathfrak{W} - C_n\}$ is empty.
Contrarily, if there is no finite subcollection of given neutrosophic α supra open sets $\mathcal{C}$ covering $\mathfrak{W}$, then the given collection $\mathcal{C}$ does not cover $\mathfrak{W}$. This implies that $\cap_{D \in \mathcal{D}}$ is non-empty. This is $\Longrightarrow \Longleftarrow$. □

Theorem 4.8. *A finite union of neutrosophic* α *supra compact subspaces of* $\mathfrak{W}$ *is neutrosophic* α *supra compact.*

5 Conclusions

The neutrosophic α supra connected space is defined in this paper and we studied the relationship between two neutrosophic sets and the properties of neutrosophic α supra connected space. Further the neutrosophic α supra compact space is introduced and its properties were also discussed. The local α connectedness and local α compactness on neutrosophic supra topological space will be our future work.

Abbreviations:

FS	- Fuzzy set
IFS	- Intuitionistic Fuzzy set
MF	- membership function
INDF	- indeterminacy
NMF	- non-membership function
NTS	- Neutrosophic Topological space
NS	- Neutrosophic set
NαSOS	- Neutrosophic alpha supra open set
NαSCS	- Neutrosophic alpha supra closed set

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On Product of Smooth Neutrosophic Topological Spaces

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Abstract: In this paper, we develop the notion of the basis for a smooth neutrosophic topology in a more natural way. As a sequel, we define the notion of symmetric neutrosophic quasi-coincident neighborhood systems and prove some interesting results that fit with the classical ones, to establish the consistency of theory developed. Finally, we define and discuss the concept of product topology, in this context, using the definition of basis.

Keywords: neutrosophic sets; smooth neutrosophic topology; basis; subbasis; smooth neutrosophic product topology

1. Introduction

The idea of neutrosophy was initiated and developed by Smarandache [1] in 1999. In recent decades the theory was used at various junctions of mathematics. More precisely, the theory made an outstanding advancement in the field of topological spaces. Salama et al. and Hur et al. [2–6] are some who posted their works of neutrosophic topological spaces, following the approach of Chang [7] in the context of fuzzy topological spaces. One can easily observe that the fuzzy topology introduced by Chang is a crisp collection of fuzzy subsets.

Šostak [8] observed that Chang's approach is crisp in nature and so he redefined the notion of fuzzy topology, often referred as smooth fuzzy topology, as a function from the collection of all fuzzy subsets of X to $[0,1]$; Fang Jin-ming et al. and Vembu et al. [9,10] are some who discussed the concept of basis as a function from a suitable collection of fuzzy subsets of X to $[0,1]$. Yan, Wang, Nanjing, Liang and Yan [11,12] developed a parallel theory in the context of intuitionistic I-fuzzy topological spaces.

The notion of a single-valued neutrosophic set was proposed by Wang [13] in 2010. In 2016, Gayyar [14] introduced the concept of smooth neutrosophic topological spaces. The notion of the basis for an ordinary single-valued neutrosophic topology was defined and discussed by Kim [15]. Salama, Alblowi, Shumrani, Muhammed Gulisten, Smarandache, Saber, Alsharari, Zhang and Sunderraman [4,16,17] are some others who posted their work in the context of single-valued neutrosophic topological spaces.

In Section 2, we give all basic definitions and results, which are important prerequisites that are needed to go through the theory developed in this paper. In Section 3, we define the notion of the basis and subbasis for a smooth neutrosophic topology; further, we develop the theory using the concept of neutrosophic quasi-coincident neighborhood systems. In addition, we prove some results which are similar to the classical ones, to establish the consistency of theory developed. Finally, in Section 4, we define and discuss the product of smooth neutrosophic spaces using our definition of basis.

2. Preliminaries

In this section, we give all basic definitions and results which we need to go through our work. As usual $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of all real numbers and rationals respectively. First we give the definition of a neutrosophic set [1,4].

Definition 1. *Let X be a non-empty set. A neutrosophic set in X is an object having the form*

$$\mathtt{N} = \{\langle x, T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}}\rangle : x \in X\}$$

where

$$T_{\mathtt{N}} : X \to \rfloor^{-}0, 1^{+}\lfloor\,,\ I_{\mathtt{N}} : X \to \rfloor^{-}0, 1^{+}\lfloor\,,\ F_{\mathtt{N}} : X \to \rfloor^{-}0, 1^{+}\lfloor$$

and

$$^{-}0 \leq T_{\mathtt{N}}(x) + I_{\mathtt{N}}(x) + F_{\mathtt{N}}(x) \leq 3^{+},$$

represent the degree of membership (namely, $T_{\mathtt{N}}(x)$), the degree of indeterminacy (namely, $I_{\mathtt{N}}(x)$) and the degree of non-membership (namely, $F_{\mathtt{N}}(x)$), for all $x \in X$ to the set to the set $\mathtt{N}$.

Here $^{-}0 = 1 - \epsilon$ and $1^{+} = 1 + \epsilon$ where ϵ is infinitesimal number and $\epsilon > 0$; further, 1 and ϵ denote standard part and non-standard part of $1 + \epsilon$; 0 and ϵ denote the standard part and non-standard part of $0 - \epsilon$. While dealing with scientific and engineering problems in real life applications, it is difficult to use a neutrosophic set with values from $\rfloor^{-}0, 1^{+}\lfloor$. In order to overcome this draw back, Wang et al. [13] defined the single-valued neutrosophic set, which is a particular case of the neutrosophic set.

Definition 2. *[13] Let X be a space of points (objects) with a generic element in X denoted by x. Then $\mathtt{N}$ is called a single-valued neutrosophic set in X if $\mathtt{N}$ has of the form $\mathtt{N} = \langle T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}}\rangle$, where $T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}} : X \to [0,1]$. In this case, $T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}}$ are called the truth membership function, indeterminancy membership function and falsity membership function, respectively.*

For conventional reasons and as there is no ambiguity, we refer a single-valued neutrosophic set simply as a neutrosophic set throughout this paper; we also restate the definition, in order to view it explicitly as a function from a non-empty set X to $\zeta = [0,1]^3$, in the following way:

Let X be a nonempty set and $I = [0,1]$. A neutrosophic set $\mathtt{N}$ on X is a mapping defined as $\mathtt{N} = \langle T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}}\rangle : X \to \zeta$, where $\zeta = I^3$ and $T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}} : X \to I$ such that $0 \leq T_{\mathtt{N}} + I_{\mathtt{N}} + F_{\mathtt{N}} \leq 3$.

We denote the set of all neutrosophic sets of X by ζ^X and the neutrosophic sets $\langle 0,1,1\rangle$ and $\langle 1,0,0\rangle$ by 0_X and 1_X respectively. Let $(r,s,t), (l,m,n) \in \zeta$; then

- $(r,s,t) \sqcup (l,m,n) = (r \vee l, s \wedge m, t \wedge n)$;
- $(r,s,t) \sqcap (l,m,n) = (r \wedge l, s \vee m, t \vee n)$;
- $(r,s,t) \sqsubseteq (l,m,n) = (r \leq l, s \geq m, t \geq n)$;
- $(r,s,t) \sqsupseteq (l,m,n) = (r \geq l, s \leq m, t \leq n)$.

Definition 3. *[1,4] Let X be a non-empty set and let $\mathtt{N}, \mathtt{M} \in \zeta^X$ be given by $\mathtt{N} = \langle T_{\mathtt{N}}, I_{\mathtt{N}}, F_{\mathtt{N}}\rangle$ and $\mathtt{M} = \langle T_{\mathtt{M}}, I_{\mathtt{M}}, F_{\mathtt{M}}\rangle$. Then*

- *The complement of $\mathtt{N}$ denoted by $\mathtt{N}^c$ is given by*

$$\mathtt{N}^c = \langle 1 - T_{\mathtt{N}}, 1 - I_{\mathtt{N}}, 1 - F_{\mathtt{N}}\rangle\,.$$

- *The union of* $\mathtt{N}$ *and* $\mathtt{M}$ *denoted by* $\mathtt{N} \sqcup \mathtt{M}$ *is an neutrosophic set in* X *given by*

$$\mathtt{N} \sqcup \mathtt{M} = \langle T_{\mathtt{N}} \vee T_{\mathtt{M}}, I_{\mathtt{N}} \wedge I_{\mathtt{M}}, F_{\mathtt{N}} \wedge F_{\mathtt{M}} \rangle .$$

- *The intersection of* $\mathtt{N}$ *and* $\mathtt{M}$ *denoted by* $\mathtt{N} \sqcap \mathtt{M}$ *is an neutrosophic set in* X *given by*

$$\mathtt{N} \sqcap \mathtt{M} = \langle T_{\mathtt{N}} \wedge T_{\mathtt{M}}, I_{\mathtt{N}} \vee I_{\mathtt{M}}, F_{\mathtt{N}} \vee F_{\mathtt{M}} \rangle .$$

- *The product of* $\mathtt{N}$ *and* $\mathtt{M}$ *denoted by* $\mathtt{N} \times \mathtt{M}$ *is given by*

$$(\mathtt{N} \times \mathtt{M})(x, y) = \mathtt{N}(x) \sqcap \mathtt{M}(y), \forall\, (x, y) \in X \times Y.$$

- *We say that* $\mathtt{N} \sqsubseteq \mathtt{M}$ *if* $T_{\mathtt{N}} \leq T_{\mathtt{M}}, I_{\mathtt{N}} \geq I_{\mathtt{M}}, F_{\mathtt{N}} \geq F_{\mathtt{M}}$.

For an any arbitrary collection $\{\mathtt{N}_i\}_{i \in J} \subseteq \zeta^X$ of neutrosophic sets the union and intersection are given by

- $\bigsqcup_{i \in J} \mathtt{N}_i = \left\langle \bigvee_{i \in J} T_{\mathtt{N}_i}, \bigwedge_{i \in J} I_{\mathtt{N}_i}, \bigwedge_{i \in J} F_{\mathtt{N}_i} \right\rangle$
- $\bigsqcap_{i \in J} \mathtt{N}_i = \left\langle \bigwedge_{i \in J} T_{\mathtt{N}_i}, \bigvee_{i \in J} I_{\mathtt{N}_i}, \bigvee_{i \in J} F_{\mathtt{N}_i} \right\rangle$.

Definition 4. *Let* X *be a nonempty set and* $x \in X$. *If* $r \in (0, 1]$, $s \in [0, 1)$ *and* $t \in [0, 1)$, *then a neutrosophic point* $x_{r,s,t}$ *in* X *given by*

$$x_{r,s,t}(z) = \begin{cases} (r, s, t), & \text{if } z = x, \\ (0, 1, 1), & \text{otherwise.} \end{cases}$$

We say $x_{r,s,t} \in \mathtt{N}$ *if* $x_{r,s,t} \sqsubseteq \mathtt{N}$. *To avoid the ambiguity, we denote the set of all neutrosophic points by* $pt(\zeta^X)$.

Definition 5. *A neutrosophic set* $\mathtt{N}$ *is said to be quasi-coincident with another neutrosophic set* $\mathtt{M}$, *denoted by* $\mathtt{N}[q]\mathtt{M}$, *if there exists an element* $x \in X$ *such that*

$$T_{\mathtt{N}}(x) + T_{\mathtt{M}}(x) > 1 \text{ or } I_{\mathtt{N}}(x) + I_{\mathtt{M}}(x) < 1 \text{ or } F_{\mathtt{N}}(x) + F_{\mathtt{M}}(x) < 1.$$

If $\mathtt{M}$ *is not quasi-coincident with* $\mathtt{N}$, *then we write* $\mathtt{M}[\bar{q}]\mathtt{N}$.

Definition 6. *[14] Let* X *be a nonempty set. Then a neutrosophic set* $\mathfrak{T} = \langle T_{\mathfrak{T}}, I_{\mathfrak{T}}, F_{\mathfrak{T}} \rangle : \zeta^X \to \zeta$ *is said to be a smooth neutrosophic topology on* X *if it satisfies the following conditions:*

C1 $\mathfrak{T}(0_X) = \mathfrak{T}(1_X) = (1, 0, 0)$.
C2 $\mathfrak{T}(\mathtt{N} \sqcap \mathtt{M}) \sqsupseteq \mathfrak{T}(\mathtt{N}) \sqcap \mathfrak{T}(\mathtt{M}), \forall\, \mathtt{N}, \mathtt{M} \in \zeta^X$.
C3 $\mathfrak{T}(\sqcup_{i \in J} \mathtt{N}_i) \sqsupseteq \sqcap_{i \in J} \mathfrak{T}(\mathtt{N}_i),\ \forall\, \mathtt{N}_i \in \zeta^X, i \in J$.

The pair $(X, \mathfrak{T})$ *is called a smooth neutrosophic topological space.*

3. The Basis for a Smooth Neutrosophic Topology

The main objective of this section is to define and discuss the concept of basis for a neutrosophic topology. Many fundamental classical statements and theories describe ways to obtain a topology from a basis; every topology is a basis for itself; characterizations of a set to form a basis; comparison of two topologies is a way to get a basis from a subbasis; quasi-neighborhood systems are discussed. Though the structural development of the theory is same as the ones followed in the context of classical and fuzzy topological spaces, the strategies following the proofs of the statements are entirely different. We start with the definition of a basis for a smooth neutrosophic topology.

Definition 7. *Let $\mathfrak{B} : \zeta^X \to \zeta$ be a function that satisfies:*

B1 *If $x \in X$ and $\epsilon, \delta > 0$, then there exists $\mathtt{M} \in \zeta^X$ such that*

$$\mathtt{M}(x) \sqsupseteq 1_X(x) - (\delta, 0, 0) \text{ and } \mathfrak{B}(\mathtt{M}) \sqsupseteq (1,0,0) - (\epsilon, 0, 0).$$

B2 *If $x \in X$, $\mathtt{M}, \mathtt{N} \in \zeta^X$ and $\epsilon, \delta > 0$, then there exists $\mathtt{L} \in \zeta^X$ such that $\mathtt{L} \sqsubseteq \mathtt{M} \sqcap \mathtt{N}$,*

$$\mathtt{L}(x) \sqsupseteq (\mathtt{M}(x) \sqcap \mathtt{N}(x)) - (\delta, 0, 0) \text{ and } \mathfrak{B}(\mathtt{L}) \sqsupseteq (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon, 0, 0).$$

Then $\mathfrak{B}$ is called a basis for a smooth neutrosophic topology on X.

Any function $\mathfrak{S} : \zeta^X \to \zeta$ satisfying **B1** is called a subbasis of a smooth neutrosophic topology on X. A collection $\{\mathtt{M}_\lambda\}_{\lambda \in \Lambda}$ of neutrosophic sets is said to be an inner cover for a neutrosophic set $\mathtt{M}$ if $\mathtt{M} = \sqcup \mathtt{M}_\lambda$.

Definition 8. *Let $\mathfrak{B}$ be a basis for a smooth neutrosophic topology on X. Then the smooth neutrosophic topology $\mathfrak{T} : \zeta^X \to \zeta$ generated by $\mathfrak{B}$ is defined as follows:*

$$\mathfrak{T}(\mathtt{M}) = \begin{cases} (1,0,0) & \text{if } \mathtt{M} = 0_X \\ \bigsqcup_{\Lambda \in \Gamma} \{ \bigsqcap_{\mathtt{M}_\lambda \in \mathcal{L}_\Lambda} \{\mathfrak{B}(\mathtt{M}_\lambda)\}\} & \text{if } \mathtt{M} \neq 0_X \end{cases}$$

where $\{\mathcal{L}_\Lambda\}_{\Lambda \in \Gamma}$ is the collection of all inner covers $\mathcal{L}_\Lambda = \{\mathtt{M}_\lambda\}_{\lambda \in \Lambda}$ of $\mathtt{M}$.

It is clear to see that $\mathfrak{T}(\mathtt{M}) \sqsupseteq \mathfrak{B}(\mathtt{M})$; the strict inequality may hold; in fact, it may happen that $\mathfrak{B}(\mathtt{M}) = (0,1,1)$ and $\mathfrak{T}(\mathtt{M}) = (1,0,0)$; however, this is not unnatural as even in the crisp theory a subset that is not an element of a basis may be an element of the topology generated by it. However, we have a question: "If $\mathfrak{B}(\mathtt{M}) \sqsupset (0,1,1)$, can $\mathfrak{T}(\mathtt{M}) \sqsupset \mathfrak{B}(\mathtt{M})$?" Of course this may happen, as seen in the following example.

Example 1. *Let $X = [0,1]$. For any subset $A \subseteq [0,1]$, let ξ_A denote the neutrosophic set in X defined by*

$$\xi_A(x) = \begin{cases} (1,0,0) & \text{if } x \in A \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Define $\mathfrak{B} : \zeta^X \to \zeta$ by

$$\mathfrak{B}(\mathtt{M}) = \begin{cases} (1,0,0) & \text{if } \mathtt{M} = 1_X \\ (1,0,0) & \text{if } \mathtt{M} = \xi_{\{(q,1)\}}, \text{ where } q \text{ is rational} \\ (\frac{1}{2}, 0, 0) & \text{if } \mathtt{M} = \xi_{\{1\}} \\ (\frac{n}{n+1}, 0, 0) & \text{if } \mathtt{M} = \xi_{\{\frac{n}{n+1}\}}, \text{ where } n \in \mathbb{N} \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Then $\mathfrak{B}$ is a basis for a smooth neutrosophic topology $\mathfrak{T}$ on X. We note that $\mathfrak{B}(\xi_{\{(a,1)\}}) = (0,1,1)$, whereas $\mathfrak{T}(\xi_{\{(a,1)\}}) = (1,0,0)\ \forall\ a \in [0,1] \cap \mathbb{Q}^c$ and $\mathfrak{B}\left(\xi_{\{1\}}\right) = (\frac{1}{2}, 0, 0)$, whereas $\mathfrak{T}(\xi_{\{1\}}) = (1,0,0) \sqsupset \mathfrak{B}\left(\xi_{\{1\}}\right)$.

Theorem 1. *Let $\mathfrak{B}$ be a basis and $\mathfrak{T}$ be as defined in Definition 8; then $\mathfrak{T}$ is a smooth neutrosophic topology on X.*

Proof. From the definition of $\mathfrak{T}$ it directly follows that $\mathfrak{T}(0_X) = (1,0,0)$. Next we wish to show that $\mathfrak{T}(1_X) = (1,0,0)$. Indeed, let $x \in X$ and $\delta, \epsilon > 0$; then by the definition of a basis for a smooth neutrosophic topology, there exists $\mathtt{M}_{x,\delta,\epsilon} \in \zeta^X$ such that $\mathtt{M}_{x,\delta,\epsilon}(x) \sqsupseteq 1_X(x) - (\delta,0,0)$ and $\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon}) \sqsupseteq (1,0,0) - (\epsilon,0,0)$, which in turn implies that $T_{\mathtt{M}_{x,\delta,\epsilon}}(x) \geq 1-\delta$, $I_{\mathtt{M}_{x,\delta,\epsilon}}(x) \leq 0$ and $F_{\mathtt{M}_{x,\delta,\epsilon}}(x) \leq 0$. Thus it follows that

$$\begin{aligned} \bigsqcup_{x,\delta} \mathtt{M}_{x,\delta,\epsilon} &= \left\langle \bigvee_{x,\delta} T_{\mathtt{M}_{x,\delta,\epsilon}}, \bigwedge_{x,\delta} I_{\mathtt{M}_{x,\delta,\epsilon}}, \bigwedge_{x,\delta} F_{\mathtt{M}_{x,\delta,\epsilon}} \right\rangle \\ &= (1,0,0) \\ &= 1_X. \end{aligned}$$

If we let $\mathcal{L}_\epsilon = \{\mathtt{M}_{x,\delta,\epsilon}\}_{x,\delta}$, then it is easy to see that $\mathcal{L}_\epsilon$ is an inner cover for 1_X. However, since $\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon}) \sqsupseteq (1,0,0) - (\epsilon,0,0)$, we have

$$T_{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})} \geq 1-\epsilon, I_{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})} \leq 0 \text{ and } F_{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})} \leq 0.$$

Therefore

$$\begin{aligned} \bigsqcap_{x\in X,\delta>0} \{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})\} &= \left\langle \bigwedge_{x,\delta} T_{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})}, \bigvee_{x,\delta} I_{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})}, \bigvee_{x,\delta} F_{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})} \right\rangle \\ &\sqsupseteq (1-\epsilon,0,0). \end{aligned}$$

Thus for every $\epsilon > 0$, there exists an inner cover $\mathcal{L}_\epsilon = \{\mathtt{M}_{x,\delta,\epsilon}\}_{x,\delta}$ of 1_X such that

$$\bigsqcap_{x\in X,\delta>0} \{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})\} \sqsupseteq (1-\epsilon,0,0).$$

Therefore

$$\mathfrak{T}(1_X) \sqsupseteq \bigsqcup_{\mathcal{L}_\epsilon} \{ \bigsqcap_{x\in X,\delta>0} \{\mathfrak{B}(\mathtt{M}_{x,\delta,\epsilon})\}\} \sqsupseteq (1,0,0)$$

and hence $\mathfrak{T}(1_X) = (1,0,0)$.

Next we claim that $\mathfrak{T}(\mathtt{M} \sqcap \mathtt{N}) \sqsupseteq \mathfrak{T}(\mathtt{M}) \wedge \mathfrak{T}(\mathtt{N})$ for any two neutrosophic sets $\mathtt{M}$, $\mathtt{N}$ in ζ^X. Suppose $\mathtt{M} \sqcap \mathtt{N} = 0_X$; then there is nothing to prove. Let $\mathtt{M} \sqcap \mathtt{N} \neq 0_X$ and let $\epsilon > 0$. Then there exist inner covers $\{\mathtt{M}_\lambda\}_{\lambda\in\Lambda_1}$ and $\{\mathtt{N}_\gamma\}_{\gamma\in\Lambda_2}$, such that $\bigsqcap_{\lambda\in\Lambda_1} \{\mathfrak{B}(\mathtt{M}_\lambda)\} = \mathfrak{T}(\mathtt{M}) - (\frac{\epsilon}{2},0,0)$ and

$$\bigsqcap_{\gamma\in\Lambda_2} \{\mathfrak{B}(\mathtt{N}_\gamma)\} \sqsupseteq \mathfrak{T}(\mathtt{N}) - (\frac{\epsilon}{2},0,0).$$

Let $\mathtt{L}_{\lambda,\gamma} = \mathtt{M}_\lambda \sqcap \mathtt{N}_\gamma$ for $\lambda \in \Lambda_1$ and $\gamma \in \Lambda_2$ and let Λ denote the set of all pairs (λ,γ) for which $\mathtt{L}_{\lambda,\gamma} \neq 0_X$. Now since $\mathtt{M} \sqcap \mathtt{N} \neq 0_X$ there exists an $x \in X$ such that $\mathtt{M}(x) \sqcap \mathtt{N}(x) \neq (0,1,1)$, which implies $\mathtt{M}(x) \neq (0,1,1)$ and $\mathtt{N}(x) \neq (0,1,1)$; then by the definition of an inner cover there exist $\mathtt{M}_{\lambda_0}$ and $\mathtt{N}_{\gamma_0}$ in the corresponding inner covers, such that $\mathtt{M}_{\lambda_0}(x) \sqcap \mathtt{N}_{\gamma_0}(x) \neq (0,1,1)$ and hence $(\lambda_0,\gamma_0) \in \Lambda$. Thus we have $\Lambda \neq \emptyset$. Now for any $(\lambda,\gamma) \in \Lambda$, $x \in X$ and $\delta > 0$, let $\mathtt{D}_{\lambda,\gamma,x,\delta} \in \zeta^X$ be such that

$$\begin{aligned} \mathtt{D}_{\lambda,\gamma,x,\delta}(x) &\sqsupseteq \mathtt{M}_\lambda(x) \sqcap \mathtt{N}_\gamma(x) - (\delta,0,0) \\ &= \mathtt{L}_{\lambda,\gamma}(x) - (\delta,0,0), \end{aligned}$$

$\mathtt{D}_{\lambda,\gamma,x,\delta} \sqsubseteq \mathtt{L}_{\lambda,\gamma}$ and

$$\mathfrak{B}(\mathtt{D}_{\lambda,\gamma,x,\delta}) \sqsupseteq (\mathfrak{B}(\mathtt{M}_\lambda) \sqcap \mathfrak{B}(\mathtt{N}_\gamma)) - (\frac{\epsilon}{2},0,0).$$

Then the collection $\{\mathtt{D}_{\lambda,\gamma,x,\delta}\}_{x,\delta}$ is an inner cover for $\mathtt{L}_{\lambda,\gamma}$ and hence the collection $\{\mathtt{D}_{\lambda,\gamma,x,\delta}\}_{\lambda,\gamma,x,\delta}$ is an inner cover for $\mathtt{M} \sqcap \mathtt{N}$.

Additionally, we have,

$$\begin{aligned}
\underset{\substack{x\in X,\delta>0\\(\lambda,\gamma)\in\Lambda}}{\sqcap}\left\{\mathfrak{B}(\mathtt{D}_{\lambda,\gamma,x,\delta})\right\}P &\sqsupseteq \underset{(\lambda,\gamma)\in\Lambda}{\sqcap}\left\{\mathfrak{B}(\mathtt{M}_\lambda)\sqcap\mathfrak{B}(\mathtt{N}_\gamma)-(\frac{\epsilon}{2},0,0)\right\}\\
&= \underset{(\lambda,\gamma)\in\Lambda}{\sqcap}\{\mathfrak{B}(\mathtt{M}_\lambda)\sqcap\mathfrak{B}(\mathtt{N}_\gamma)\}-(\frac{\epsilon}{2},0,0)\\
&\sqsupseteq \left\{\underset{(\lambda,\gamma)\in\Lambda}{\sqcap}\{\mathfrak{B}(\mathtt{M}_\lambda)\}\sqcap\underset{(\lambda,\gamma)\in\Lambda}{\sqcap}\{\mathfrak{B}(\mathtt{N}_\gamma)\}\right\}-(\frac{\epsilon}{2},0,0)\\
&\sqsupseteq \left\{\underset{\lambda\in\Lambda_1}{\sqcap}\{\mathfrak{B}(\mathtt{M}_\lambda)\}\sqcap\underset{\gamma\in\Lambda_2}{\sqcap}\{\mathfrak{B}(\mathtt{N}_\gamma)\}\right\}-(\frac{\epsilon}{2},0,0)\\
&\sqsupseteq (\mathfrak{T}(\mathtt{M})\sqcap\mathfrak{T}(\mathtt{N}))-(\frac{\epsilon}{2},0,0)-(\frac{\epsilon}{2},0,0)\\
&= (\mathfrak{T}(\mathtt{M})\sqcap\mathfrak{T}(\mathtt{N}))-(\epsilon,0,0).
\end{aligned}$$

Since this is true for every $\epsilon>0$ and

$$\mathfrak{T}(\mathtt{M}\sqcap\mathtt{N})\sqsupseteq\underset{\substack{x\in X,\delta>0\\(\lambda,\gamma)\in\Lambda}}{\sqcap}\left\{\mathfrak{B}(\mathtt{D}_{\lambda,\gamma,x,\delta})\right\}$$

we have $\mathfrak{T}(\mathtt{M}\sqcap\mathtt{N})\sqsupseteq(\mathfrak{T}(\mathtt{M})\sqcap\mathfrak{T}(\mathtt{N}))$ for any $\mathtt{M},\mathtt{N}$ in ζ^X.

Finally we prove that $\mathfrak{T}(\underset{\lambda\in\Lambda}{\sqcup}\mathtt{M}_\lambda)\sqsupseteq\underset{\lambda\in\Lambda}{\sqcap}\mathfrak{T}(\mathtt{M}_\lambda)$ for any collection $\{\mathtt{M}_\lambda\}_{\lambda\in\Lambda}\subseteq\zeta^X$. For each $\epsilon>0$ and for each $\mathtt{M}_\lambda$, let $\{\mathtt{M}_{\lambda,\gamma}\}_{\gamma\in\Gamma_\lambda}$ be an inner cover for $\mathtt{M}_\lambda$ such that $\underset{\gamma\in\Gamma_\lambda}{\sqcap}\{\mathfrak{B}(\mathtt{M}_{\lambda,\gamma})\}\sqsupseteq\mathfrak{T}(\mathtt{M}_\lambda)-(\epsilon,0,0)$. Since $\{\mathtt{M}_{\lambda,\gamma}\}_{\gamma\in\Gamma_\lambda}$ is an inner cover for $\mathtt{M}_\lambda$, we have $\{\mathtt{M}_{\lambda,\gamma}\}_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda}$ is an inner cover for $\underset{\lambda\in\Lambda}{\sqcup}\mathtt{M}_\lambda$. Thus it follows that

$$\begin{aligned}
\mathfrak{T}(\underset{\lambda\in\Lambda}{\sqcup}\mathtt{M}_\lambda) &\sqsupseteq \underset{\lambda\in\Lambda,\gamma\in\Gamma_\lambda}{\sqcap}\{\mathfrak{B}(\mathtt{M}_{\lambda,\gamma})\}\\
&= \underset{\lambda\in\Lambda}{\sqcap}\left\{\underset{\gamma\in\Gamma_\lambda}{\sqcap}\{\mathfrak{B}(\mathtt{M}_{\lambda,\gamma})\}\right\}\\
&\sqsupseteq \underset{\lambda\in\Lambda}{\sqcap}\{\mathfrak{T}(\mathtt{M}_\lambda)-(\epsilon,0,0)\}\\
&= \underset{\lambda\in\Lambda}{\sqcap}\{\mathfrak{T}(\mathtt{M}_\lambda)\}-(\epsilon,0,0),
\end{aligned}$$

which implies $\mathfrak{T}(\underset{\lambda\in\Lambda}{\sqcup}\mathtt{M}_\lambda)\sqsupseteq\underset{\lambda\in\Lambda}{\sqcap}\mathfrak{T}(\mathtt{M}_\lambda)$ for any collection $\{\mathtt{M}_\lambda\}_{\lambda\in\Lambda}\subseteq\zeta^X$ as desired. □

Definition 9. *Let $(X,\mathfrak{T})$ be smooth neutrosophic topological space. For all $x_{r,s,t}\in pt(\zeta^X)$ and $\mathtt{N}\in\zeta^X$, the mapping $Q^{\mathfrak{T}}_{x_{r,s,t}}:\zeta^X\to\zeta$ is defined as follows:*

$$Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N}) = \begin{cases}\underset{x_{r,s,t}[q]\mathtt{M}\sqsubseteq\mathtt{N}}{\sqcup}\mathfrak{T}(\mathtt{M}); & \text{if } x_{r,s,t}[q]\mathtt{N}\\ (0,1,1) & \text{otherwise.}\end{cases}$$

The set $Q^{\mathfrak{T}}=\{Q^{\mathfrak{T}}_{x_{r,s,t}}:x_{r,s,t}\in pt(\zeta^X)\}$ is called a neutrosophic quasi-coincident neighborhood system. Further, a neutrosophic quasi-coincident neighborhood system $Q^{\mathfrak{T}}$ is said to be symmetric if for any $x_{r,s,t},y_{l,m,n}\in pt(\zeta^X)$, $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M})\sqsupset(0,1,1)$, $Q^{\mathfrak{T}}_{y_{l,m,n}}(\mathtt{N})\sqsupset(0,1,1)$, $x_{r,s,t}[q]\mathtt{N}$ implies $y_{l,m,n}[q]\mathtt{M}$.

Theorem 2. *Let $(X,\mathfrak{T})$ be neutrosophic topological space. Then for all $\mathtt{M},\mathtt{N}\in\zeta^X$,*

(i) $Q^{\mathfrak{T}}_{x_{r,s,t}}(0_X)=(0,1,1)$;

(ii) $Q^{\mathfrak{T}}_{x_{r,s,t}}(1_X)=(1,0,0)$;

(iii) $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) \sqsupseteq (0,1,1)$ *implies* $x_{r,s,t}[q]\mathtt{M}$*;*

(iv) $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M} \sqcap \mathtt{N}) = Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) \sqcap Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N})$*;*

(v) $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) = \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \bigsqcap_{y_{l,m,n}[q]\mathtt{N}} Q^{\mathfrak{T}}_{y_{l,m,n}}(\mathtt{N})$.

Proof. As (i), (ii) and (iii) follow directly from the definition of $Q^{\mathfrak{T}}_{x_{r,s,t}}$, we skip their proof. To prove (iv), first we observe that

$$\begin{aligned} Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M} \sqcap \mathtt{N}) &= \bigsqcup_{x_{r,s,t}[q]\mathtt{L}\sqsubseteq\mathtt{M}\sqcap\mathtt{N}} \mathfrak{T}(\mathtt{L}) \\ &\sqsubseteq \bigsqcup_{x_{r,s,t}[q])\mathtt{L}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{L}) \\ &= Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}). \end{aligned}$$

Similarly, it follows that $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M} \sqcap \mathtt{N}) \sqsubseteq Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N})$, which implies

$$Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M} \sqcap \mathtt{N}) \sqsubseteq Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) \sqcap Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N}).$$

To prove the reverse inequality, consider

$$\begin{aligned} Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) \sqcap Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N}) &= \bigsqcup_{x_{r,s,t}[q]\mathtt{A}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{A}) \sqcap \bigsqcup_{x_{r,s,t}[q]B\sqsubseteq\mathtt{N}} \mathfrak{T}(\mathtt{B}) \\ &= \bigsqcup_{x_{r,s,t}[q](\mathtt{A}\sqcap\mathtt{B})\sqsubseteq(\mathtt{M}\sqcap\mathtt{N})} (\mathfrak{T}(\mathtt{A}) \sqcap \mathfrak{T}(\mathtt{B})) \\ &\sqsubseteq \bigsqcup_{x_{r,s,t}[q](\mathtt{A}\sqcap\mathtt{B})\sqsubseteq(\mathtt{M}\sqcap\mathtt{N})} (\mathfrak{T}(\mathtt{A} \sqcap \mathtt{B})) \\ &\sqsubseteq \bigsqcup_{x_{r,s,t}[q]\mathtt{L}\sqsubseteq(\mathtt{M}\sqcap\mathtt{N})} (\mathfrak{T}(\mathtt{L})) \\ &= Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M} \sqcap \mathtt{N}). \end{aligned}$$

To prove (v), for any $\mathtt{N} \in \zeta^X$ with $x_{r,s,t}[q]\mathtt{N} \sqsubseteq \mathtt{M}$, we have $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N}) \sqsupseteq \mathfrak{T}(\mathtt{N})$, and therefore,

$$\mathfrak{T}(\mathtt{N}) \sqsubseteq \sqcap_{y_{l,m,n}[q]\mathtt{N}} Q^{\mathfrak{T}}_{y_{l,m,n}}(\mathtt{N}) \sqsubseteq Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N}) \sqsubseteq Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}).$$

Hence, we have

$$\begin{aligned} Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) &= \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{N}) \\ &\sqsubseteq \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \bigsqcap_{y_{l,m,n}[q]\mathtt{N}} Q^{\mathfrak{T}}_{y_{l,m,n}}(\mathtt{N}) \\ &\sqsubseteq Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) \end{aligned}$$

as desired. □

Theorem 3. *Let* $\mathfrak{B} : \zeta^X \to \zeta$ *be a mapping. Then* $\mathfrak{B}$ *is a basis of a smooth neutrosophic topology* $\mathfrak{T}$ *if and only if* $\mathfrak{B} \sqsubseteq \mathfrak{T}$ *and for all* $\mathtt{M} \in \zeta^X$, $x_{r,s,t} \in pt(\zeta^X)$, $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) \sqsubseteq \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{B}(\mathtt{N})$.

Proof. Let $\mathfrak{B}$ be a basis for given smooth neutrosophic topology; then clearly $\mathfrak{B} \sqsubseteq \mathfrak{T}$. Let $\mathtt{M} \in \zeta^X$ and $x_{r,s,t} \in pt(\zeta^X)$; then $Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{M}) = \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{N})$. Let $\mathcal{N} = \{\mathtt{N} : x_{r,s,t}[q]\mathtt{N} \sqsubseteq \mathtt{M}\}$; then for every $\mathtt{N} \in \mathcal{N}$, we have

$$\mathfrak{T}(\mathtt{N}) = \bigsqcup_{\bigsqcup_{\lambda\in\Lambda} \mathtt{N}_\lambda = \mathtt{N}} \bigsqcap_{\lambda\in\Lambda} \mathfrak{B}(\mathtt{N}_\lambda).$$

Let $\epsilon > 0$; then there exists $\{\mathtt{N}_\lambda : \lambda \in \Lambda\}$ with $\bigsqcup_{\lambda\in\Lambda} \mathtt{N}_\lambda = \mathtt{N}$ such that

$$\bigsqcap_{\lambda\in\Lambda} \mathfrak{B}(\mathtt{N}_\lambda) \sqsupseteq \mathfrak{T}(\mathtt{N}) - (\epsilon, 0, 0).$$

Thus there exists an $\mathtt{N}_{\lambda_0}$ such that $x_{r,s,t}[q]\mathtt{N}_{\lambda_0}$ and $\mathfrak{B}(\mathtt{N}_{\lambda_0}) \sqsupseteq \mathfrak{T}(\mathtt{N}) - (\epsilon, 0, 0)$. Hence for every $\mathtt{N} \in \mathcal{N}$, there exists an $\mathtt{N}_{\lambda_0}$ such that $\mathfrak{B}(\mathtt{N}_{\lambda_0}) \sqsupseteq \mathfrak{T}(\mathtt{N}) - (\epsilon, 0, 0)$, which in turn implies that

$$\bigsqcup_{\mathtt{N}\in\mathcal{N}} \mathfrak{B}(\mathtt{N}_{\lambda_0}) \sqsupseteq \bigsqcup_{\mathtt{N}\in\mathcal{N}} \mathfrak{T}(\mathtt{N}) - (\epsilon, 0, 0).$$

Thus it follows that,

$$\begin{aligned}
\bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{B}(\mathtt{N}) &\sqsupseteq \bigsqcup_{\mathtt{N}\in\mathcal{N}} \mathfrak{B}(\mathtt{N}_{\lambda_0}) \\
&\sqsupseteq \bigsqcup_{\mathtt{N}\in\mathcal{N}} \mathfrak{T}(\mathtt{N}) - (\epsilon, 0, 0) \\
&= \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{N}) - (\epsilon, 0, 0).
\end{aligned}$$

This implies that

$$\bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{B}(\mathtt{N}) \sqsupseteq \bigsqcup_{x_{r,s,t}[q]\mathtt{N}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{N}) = Q^{\mathfrak{T}}_{x_{r,s,t}}(\mathtt{N})$$

as desired.

Conversely, let $x \in X$ and $\epsilon, \delta > 0$; then clearly $x_{\delta,0,0} \in pt(\zeta^X)$. However, since

$$Q^{\mathfrak{T}}_{x_{\delta,0,0}}(1_X) \sqsubseteq \bigsqcup_{x_{\delta,0,0}[q]\mathtt{N}\sqsubseteq 1_X} \mathfrak{B}(\mathtt{N})$$

and $(1, 0, 0) = Q^{\mathfrak{T}}_{x_{\delta,0,0}}(1_X)$, it is possible to find an $\mathtt{N} \in \zeta^X$ such that $\mathtt{N}(x) \sqsupseteq 1_X(x) - (\delta, 0, 0)$, such that $\mathfrak{B}(\mathtt{N}) \sqsupseteq (1, 0, 0) - (\epsilon, 0, 0)$. Thus, **B1** of Definition 8 follows.

Let $x \in X$, $\mathtt{M}, \mathtt{N} \in \zeta^X$ and $\epsilon, \delta > 0$. First we claim that, $Q^{\mathfrak{T}}_{x_{\delta,0,0}}(\mathtt{M} \sqcap \mathtt{N}) \sqsupseteq [\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})]$; consider

$$\begin{aligned}
Q^{\mathfrak{T}}_{x_{\delta,0,0}}(\mathtt{M} \sqcap \mathtt{N}) &= [Q^{\mathfrak{T}}_{x_{\delta,0,0}}(\mathtt{M}) \sqcap Q^{\mathfrak{T}}_{x_{\delta,0,0}}(\mathtt{N})] \\
&= [\bigsqcup_{x_{\delta,0,0}[q]\mathtt{L}\sqsubseteq\mathtt{M}} \mathfrak{T}(\mathtt{L}) \sqcap \bigsqcup_{x_{\delta,0,0}[q]\mathtt{L}\sqsubseteq\mathtt{N}} \mathfrak{T}(\mathtt{L})] \\
&\sqsupseteq [\mathfrak{T}(\mathtt{M}) \sqcap \mathfrak{T}(\mathtt{N})] \\
&\sqsupseteq [\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})].
\end{aligned}$$

If $x_{\delta,0,0}[q]\mathtt{M} \sqcap \mathtt{N}$, then for every $\mathtt{L} \in \zeta^X$ with $\mathtt{L} \sqsubseteq \mathtt{M} \sqcap \mathtt{N}$ and $x_{\delta,0,0}[q]\mathtt{L}$, we have

$$\mathtt{L}(x) \sqsupset 1_X(x) - (\delta, 0, 0) \sqsupseteq (\mathtt{M} \sqcap \mathtt{N})(x) - (\delta, 0, 0).$$

Let $\epsilon > 0$; then there exists $\mathtt{L}$ such that

$$\begin{aligned}
\mathfrak{B}(\mathtt{L}) &\sqsupseteq \bigsqcup_{x_{\delta,0,0}[q]\mathtt{L}\sqsubseteq(\mathtt{M}\sqcap\mathtt{N})} \mathfrak{B}(\mathtt{L}) - (\epsilon, 0, 0) \\
&\sqsupseteq Q^{\mathfrak{T}}_{x_{\delta,0,0}}(\mathtt{M} \sqcap \mathtt{N}) - (\epsilon, 0, 0) \\
&\sqsupseteq [\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})] - (\epsilon, 0, 0).
\end{aligned}$$

Suppose $x_{\delta,0,0}[\bar{q}]\mathtt{M} \sqcap \mathtt{N}$. Let

$$\mathcal{L}_x = \{\mathtt{L} \in \zeta^X : \mathtt{L}(x) \sqsupseteq (\mathtt{M} \sqcap \mathtt{N})(x) - (\delta,0,0)\}.$$

Then there exists $\mathtt{L} \in \mathcal{L}_x$ such that

$$\begin{aligned} \mathfrak{B}(\mathtt{L}) &\sqsupseteq \bigsqcup_{\mathtt{L} \in \mathcal{L}_x} \mathfrak{B}(\mathtt{L}) - (\epsilon,0,0) \\ &\sqsupseteq Q^{\mathfrak{T}}_{x_{\delta,0,0}}(\mathtt{M} \sqcap \mathtt{N}) - (\epsilon,0,0) \\ &\sqsupseteq [\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})] - (\epsilon,0,0) \end{aligned}$$

Thus, **B2** of Definition 8 follows in both cases. □

Here we note that, "If $(X,\mathfrak{T})$ is a smooth neutrosophic topological space, then $\mathfrak{T}$ is a basis for a smooth fuzzy topology on X and the smooth fuzzy topology generated by $\mathfrak{T}$ is itself." In the following, we give certain theorems which can be proved in a similar fashion to Theorems 3.8, 3.9 and 3.10 in [10].

Theorem 4. *Let $\mathfrak{T}$ be a smooth neutrosophic topology on X. Let $\mathfrak{B} : \zeta^X \to \zeta$ be a function satisfying*

i. $\mathfrak{T}(\mathtt{M}) \sqsupseteq \mathfrak{B}(\mathtt{M})$ *for all* $\mathtt{M} \in \zeta^X$;

ii. *If* $\mathtt{M} \in \zeta^X$, $x \in X$, $\delta > 0$ *and* $\epsilon > 0$, *then there exists* $\mathtt{N} \in \zeta^X$ *such that* $\mathtt{N}(x) \sqsupseteq \mathtt{M}(x) - (\delta,0,0)$, $\mathtt{N} \sqsubseteq \mathtt{M}$ *and* $\mathfrak{B}(\mathtt{N}) \sqsupseteq \mathfrak{T}(\mathtt{M}) - (\epsilon,0,0)$.

Then $\mathfrak{B}$ is a basis for the smooth neutrosophic topology $\mathfrak{T}$ on X.

Theorem 5. *If $\mathfrak{B}$ is a basis for the smooth fuzzy topological space $(X,\mathfrak{T})$, then*

i. $\mathfrak{T}(\mathtt{M}) \sqsupseteq \mathfrak{B}(\mathtt{M})$ *for all* $\mathtt{M} \in \zeta^X$.

ii. *If* $x \in X$, $\mathtt{M} \in \zeta^X$, $\delta > 0$ *and* $\epsilon > 0$, *then there exists* $\mathtt{N} \in \zeta^X$ *such that* $\mathtt{N}(x) \sqsupseteq \mathtt{M}(x) - (\delta,0,0)$, $\mathtt{N} \sqsubseteq \mathtt{M}$ *and* $\mathfrak{B}(\mathtt{N}) \sqsupseteq \mathfrak{T}(\mathtt{M}) - (\epsilon,0,0)$.

Theorem 6. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be bases for the smooth neutrosophic topologies $\mathfrak{T}$ and $\mathfrak{T}'$, respectively, on X. Then the following conditions are equivalent.*

i. $\mathfrak{T}'$ *is finer than* $\mathfrak{T}$.

ii. *If* $\mathtt{M} \in \zeta^X$, $x \in X$, $\delta > 0$ *and* $\epsilon > 0$, *there exists* $\mathtt{N} \in \zeta^X$ *such that* $\mathtt{N}(x) \sqsupseteq \mathtt{M}(x) - (\delta,0,0)$, $\mathtt{N} \sqsubseteq \mathtt{M}$ *and* $\mathfrak{B}'(\mathtt{N}) \sqsupseteq \mathfrak{B}(\mathtt{M}) - (\epsilon,0,0)$.

To end this section, we present a theorem which gives a way to get a basis from a subbasis, from which a smooth neutrosophic topology can be generated.

Theorem 7. *Let $\mathfrak{S} : \zeta^X \to \zeta$ be a subbasis for a smooth neutrosophic topology on X. Define $\mathfrak{B} : \zeta^X \to \zeta$ as*

$$\mathfrak{B}(\mathtt{M}) = \bigsqcup_{D \in \mathcal{D}} \{\bigsqcap_{i \in I_D} \{\mathfrak{S}(\mathtt{M}_i)\}\},$$

where $\mathcal{D}$ is the family of all finite collections $D = \{\mathtt{M}_i\}_{i \in I_D}$ of members of ζ^X such that $\mathtt{M} = \bigsqcap_{i \in I_D} \mathtt{M}_i$. Then the $\mathfrak{B}$ is a basis for a smooth neutrosophic topology on X.

Proof. Since $\mathcal{D} \neq \emptyset$, every $\mathtt{M} \in \zeta^X$, and by the definition of $\mathfrak{S}$, $\mathfrak{B}$ is well defined. As $\mathfrak{B}$ clearly satisfies **B1** of Definition 7, it is enough to prove **B2**. Let $x \in X$, $\mathtt{M}$, $\mathtt{N}$ in ζ^X and $\delta, \epsilon > 0$. Then by the definition of $\mathfrak{B}$ there exist collections $\{\mathtt{M}_i\}_{i=1,2,\ldots,n}$ and $\{\mathtt{N}_j\}_{j=1,2,\ldots,m}$ such that

$$\mathtt{M} = \bigsqcap_{i=1}^{n} \mathtt{M}_i, \quad \sqcap_i\{\mathfrak{B}(\mathtt{M}_i)\} \sqsupseteq \mathfrak{B}(\mathtt{M}) - (\epsilon,0,0)$$

and

$$\mathtt{N} = \bigsqcap_{j=1}^{m} \mathtt{N}_j, \ \ \sqcap_j\{\mathfrak{B}(\mathtt{N}_j)\} \sqsupseteq \mathfrak{B}(\mathtt{N}) - (\epsilon, 0, 0).$$

Now let us define a collection of neutrosophic sets $\mathtt{L}_k \sqsubseteq 1_X$, for $k = 1, 2, \ldots, n+m$, as

$$\mathtt{L}_k = \begin{cases} \mathtt{M}_k & \text{if } k \leq n \\ \mathtt{N}_{k-n} & \text{if } k > n. \end{cases}$$

If we let $\mathtt{L} = \bigsqcap_{k=1}^{n+m} \mathtt{L}_k$, then $\mathtt{L} = \mathtt{M} \sqcap \mathtt{N}$ and therefore

$$\mathtt{L}(x) \sqsupseteq (\mathtt{M} \sqcap \mathtt{N})(x) - (\delta, 0, 0).$$

Now by definition of $\mathfrak{B}$, we have

$$\mathfrak{B}(\mathtt{L}) = \bigsqcup_{D \in \mathcal{D}} \{ \bigsqcap_{i \in I_D} \{\mathfrak{S}(\mathtt{L}_i)\}\},$$

where $\mathcal{D}$ is the family of all finite collections $D = \{\mathtt{L}_i\}_{i \in I_D}$ of members of ζ^X such that $\mathtt{L} = \bigsqcap_{i \in I_D} \mathtt{L}_i$. Thus it follows that

$$\begin{aligned} \mathfrak{B}(\mathtt{L}) &= \bigsqcup_{D \in \mathcal{D}} \{ \bigsqcap_{i \in I_D} \{\mathfrak{S}(\mathtt{L}_i)\}\} \\ &\sqsupseteq \sqcap_k\{\mathfrak{S}(\mathtt{L}_k)\} \\ &= \bigsqcap_i \{\mathfrak{S}(\mathtt{M}_i)\} \sqcap \bigsqcap_j \{\mathfrak{S}(\mathtt{N}_j)\} \\ &\sqsupseteq (\mathfrak{B}(\mathtt{M}) - (\epsilon, 0, 0)) \sqcap (\mathfrak{B}(B) - (\epsilon, 0, 0)) \\ &= (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon, 0, 0) \end{aligned}$$

as desired. □

4. Product of Neutrosophic Topologies

In this section, we first define the concept of a finite product of smooth neutrosophic topologies, using the notion of basis defined in the previous section. We present a way to obtain the product topology from the given bases; in the following we present a subbasis for a product topology. Later, we generalize the discussed contents in the context of an arbitrary product of smooth neutrosophic topologies.

Definition 10. *Let $(X, \mathfrak{T}_1)$ and $(Y, \mathfrak{T}_2)$ be smooth neutrosophic topological spaces. Let $\mathfrak{B} : \zeta^{X \times Y} \to \zeta$ be defined as follows:*

> *Let $\mathtt{A} \in \zeta^{X \times Y}$. If $\mathtt{A} \neq \mathtt{M} \times \mathtt{N}$ for any $\mathtt{M} \in \zeta^X$ and $\mathtt{N} \in \zeta^Y$, then define $\mathfrak{B}(\mathtt{A}) = (0, 1, 1)$. Otherwise, define*
>
> $$\mathfrak{B}(\mathtt{A}) = \bigsqcup_{\lambda \in \Lambda} \{\mathfrak{T}_1(\mathtt{M}_\lambda) \sqcap \mathfrak{T}_2(\mathtt{N}_\lambda)\},$$
>
> *where $\{\mathtt{M}_\lambda \times \mathtt{N}_\lambda\}_{\lambda \in \Lambda}$ is the collection of all possible ways of writing $\mathtt{A}$ as $\mathtt{A} = \mathtt{M}_\lambda \times \mathtt{N}_\lambda$, where $\mathtt{M}_\lambda \in \zeta^X$, $\mathtt{N}_\lambda \in \zeta^Y$.*

Then $\mathfrak{B}$ is a basis for the smooth neutrosophic topology called the smooth neutrosophic product topology on $X \times Y$.

Example 2. *Let $X_1 = X_2 = \mathbb{R}$ and let $\mathtt{M}_1$ and $\mathtt{N}_1$ be defined by*

$$\mathtt{M}_1(x) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \textit{ for all } x \in X_1$$

and

$$\mathtt{N}_1(x) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \text{ for all } x \in X_2.$$

Let $\mathfrak{T}_1 : \zeta^{X_1} \to \zeta$ and $\mathfrak{T}_2 : \zeta^{X_2} \to \zeta$ be the functions defined by

$$\mathfrak{T}_1(\mathtt{M}) = \begin{cases} (1,0,0) & \text{if } \mathtt{M} = 1_{X_1} \text{ or } \mathtt{M} = 0_{X_1} \\ (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{M} = \mathtt{M}_1 \\ (0,1,1) & \text{otherwise} \end{cases}$$

and

$$\mathfrak{T}_2(\mathtt{N}) = \begin{cases} (1,0,0) & \text{if } \mathtt{N} = 1_{X_2} \text{ or } \mathtt{N} = 0_{X_2} \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{N} = \mathtt{N}_1 \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Then clearly $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are smooth neutrosophic topologies on X_1 and X_2. From the above definition, we get $\mathfrak{B} : \zeta^{X_1 \times X_2} \to \zeta$ given by

$$\mathfrak{B}(\mathtt{E}) = \begin{cases} (1,0,0) & \text{if } \mathtt{E} = 1_{X_1 \times X_2} \text{ or } \mathtt{E} = 0_{X_1 \times X_2} \\ (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{E} = \mathtt{M}_1 \times 1_{X_2} \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{E} = 1_{X_1} \times \mathtt{N}_1 \\ (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{E} = \mathtt{M}_1 \times \mathtt{N}_1 \\ (0,1,1) & \text{otherwise} \end{cases}$$

which is a basis for a smooth neutrosophic topology $\mathfrak{T}$ on $X_1 \times X_2$ and the smooth neutrosophic topology (product topology) generated by $\mathfrak{B}$ is given by

$$\mathfrak{T}(\mathtt{E}) = \begin{cases} (1,0,0) & \text{if } \mathtt{E} = 1_{X_1 \times X_2} \text{ or } \mathtt{E} = 0_{X_1 \times X_2} \\ (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{E} = \mathtt{M}_1 \times 1_{X_2} \\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{E} = 1_{X_1} \times \mathtt{N}_1 \\ (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}) & \text{if } \mathtt{E} = \mathtt{M}_1 \times \mathtt{N}_1 \\ (0,1,1) & \text{otherwise.} \end{cases}$$

Theorem 8. *Let $\mathfrak{B} : \zeta^{X \times Y} \to \zeta$ be the function defined in Definition 10. Then $\mathfrak{B}$ is a basis for a smooth neutrosophic topology on $X \times Y$.*

Proof. If we let $\mathtt{M} = 1_{X \times Y}$, then clearly **B1** of Definition 7 follows.

Let $(x,y) \in X \times Y$, $\mathtt{M}, \mathtt{N}$ in $\zeta^{X \times Y}$ and $\delta, \epsilon > 0$. We wish to show that there exists $\mathtt{L} \in \zeta^{X \times Y}$ such that $\mathtt{L} \sqsubseteq \mathtt{M} \sqcap \mathtt{N}$,

$$\mathtt{L}(x,y) \sqsupseteq (\mathtt{M}(x,y) \sqcap \mathtt{N}(x,y)) - (\delta, 0, 0)$$

and

$$\mathfrak{B}(\mathtt{L}) \sqsupseteq (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon, 0, 0).$$

Suppose any one of $\mathtt{M}$ and $\mathtt{N}$, say $\mathtt{M}$ cannot be written as $\mathtt{M}_1 \times \mathtt{M}_2$ for any $\mathtt{M}_1 \in \zeta^X$ and $\mathtt{M}_2 \in \zeta^Y$; then by letting $\mathtt{L} = \mathtt{M} \sqcap \mathtt{N}$, we have $\mathtt{L}(x,y) \sqsupseteq (\mathtt{M}(x,y) \sqcap \mathtt{N}(x,y)) - (\delta, 0, 0)$. However, by the definition of $\mathfrak{B}$, it follows that $\mathfrak{B}(\mathtt{M}) = (0,1,1)$ and therefore $\mathfrak{B}(\mathtt{L}) \sqsupseteq (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon, 0, 0)$ as desired. If both $\mathtt{M}$ and $\mathtt{N}$ can be written as $\mathtt{A} \times \mathtt{A}'$ and $\mathtt{B} \times \mathtt{B}'$ for some $\mathtt{A}, \mathtt{B} \in \zeta^X$ and $\mathtt{A}', \mathtt{B}' \in \zeta^Y$, then by the definition of $\mathfrak{B}$, there exist $\mathtt{M}_1, \mathtt{N}_1 \in \zeta^X$ and $\mathtt{M}_2, \mathtt{N}_2 \in \zeta^Y$ such that $\mathtt{M} = \mathtt{M}_1 \times \mathtt{M}_2$, $\mathtt{N} = \mathtt{N}_1 \times \mathtt{N}_2$,

$$\mathfrak{T}_1(\mathtt{M}_1) \sqcap \mathfrak{T}_2(\mathtt{M}_2) \sqsupseteq \mathfrak{B}(\mathtt{M}) - (\epsilon, 0, 0)$$

and

$$\mathfrak{T}_1(\mathtt{N}_1) \sqcap \mathfrak{T}_2(\mathtt{N}_2) \sqsupseteq \mathfrak{B}(\mathtt{N}) - (\epsilon, 0, 0).$$

Now if we let $\mathtt{L} = \mathtt{M} \sqcap \mathtt{N}$, then $\mathtt{L}(x,y) = (\mathtt{M}(x,y) \sqcap \mathtt{N}(x,y)) - (\delta, 0, 0)$ and

$$\begin{aligned}
(\mathtt{M} \sqcap \mathtt{N})(x,y) &= ((\mathtt{M}_1 \times \mathtt{M}_2) \sqcap (\mathtt{N}_1 \times \mathtt{N}_2))(x,y) \\
&= (\mathtt{M}_1(x) \sqcap \mathtt{M}_2(y)) \sqcap (\mathtt{N}_1(x) \sqcap \mathtt{N}_2(y)) \\
&= (\mathtt{M}_1(x) \sqcap \mathtt{N}_1(x)) \sqcap (\mathtt{M}_2(y) \sqcap \mathtt{N}_2(y)) \\
&= (\mathtt{M}_1(x) \sqcap \mathtt{N}_1(x)) \times (\mathtt{M}_2(y) \sqcap \mathtt{N}_2(y)) \\
&= ((\mathtt{M}_1 \sqcap \mathtt{N}_1) \times (\mathtt{M}_2 \sqcap \mathtt{N}_2))(x,y).
\end{aligned}$$

Now consider,

$$\begin{aligned}
\mathfrak{B}(\mathtt{L}) &= \mathfrak{B}(\mathtt{M} \sqcap \mathtt{N}) \\
&= \mathfrak{B}((\mathtt{M}_1 \sqcap \mathtt{N}_1) \times (\mathtt{M}_2 \sqcap \mathtt{N}_2)) \\
&\sqsupseteq \mathfrak{T}_1(\mathtt{M}_1 \sqcap \mathtt{N}_1) \sqcap \mathfrak{T}_2(\mathtt{M}_2 \sqcap \mathtt{N}_2) \\
&\sqsupseteq (\mathfrak{T}_1(\mathtt{M}_1) \sqcap \mathfrak{T}_1(\mathtt{N}_1)) \sqcap (\mathfrak{T}_2(\mathtt{M}_2) \sqcap \mathfrak{T}_2(\mathtt{N}_2)) \\
&= \{\mathfrak{T}_1(\mathtt{M}_1) \sqcap \mathfrak{T}_2(\mathtt{M}_2)\} \sqcap \{\mathfrak{T}_1(\mathtt{N}_1) \sqcap \mathfrak{T}_2(\mathtt{N}_2)\} \\
&\sqsupseteq (\mathfrak{B}(\mathtt{M}) - (\epsilon, 0, 0)) \sqcap (\mathfrak{B}(\mathtt{N}) - (\epsilon, 0, 0)) \\
&= (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon, 0, 0)
\end{aligned}$$

and hence **B2** of Definition 7 follows in this case also. □

Theorem 9. *Let $\mathfrak{B}_1, \mathfrak{B}_2$ be bases for the smooth neutrosophic topologies $\mathfrak{T}_1, \mathfrak{T}_2$ respectively. Define $\mathfrak{B} : \zeta^{X \times Y} \to \zeta$ as follows:*

If $\mathtt{A} \in \zeta^{X \times Y}$ cannot be written as $\mathtt{M} \times \mathtt{N}$ for any $\mathtt{M} \in \zeta^X$ and $\mathtt{N} \in \zeta^Y$, then define $\mathfrak{B}(\mathtt{A}) = (0,1,1)$. Otherwise define

$$\mathfrak{B}(\mathtt{A}) = \bigsqcup_{\lambda \in \Lambda} \{\mathfrak{B}_1(\mathtt{M}_\lambda) \sqcap \mathfrak{B}_1(\mathtt{N}_\lambda)\}$$

where $\{\mathtt{M}_\lambda \times \mathtt{N}_\lambda\}_{\lambda \in \Lambda}$ is the collection of all possible ways of writing $\mathtt{A}$ as $\mathtt{A} = \mathtt{M}_\lambda \times \mathtt{N}_\lambda$, where $\mathtt{M}_\lambda \in \zeta^X$, $\mathtt{N}_\lambda \in \zeta^Y$.

Then $\mathfrak{B}$ is a basis for the product topology on $X \times Y$.

Proof. First we claim that $\mathfrak{B}$ is a basis for a smooth neutrosophic topology on $X \times Y$. Let $(x,y) \in X \times Y$, $\delta > 0$ and $\epsilon > 0$. Now since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are bases for the smooth neutrosophic topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$, there exist $\mathtt{M} \in \zeta^X$ and $\mathtt{N} \in \zeta^Y$ such that

$$\mathtt{M}(x) \sqsupseteq 1_X(x) - (\delta, 0, 0),\ \mathfrak{B}_1(\mathtt{M}) \sqsupseteq (1,0,0) - (\epsilon, 0, 0)$$

and

$$\mathtt{N}(y) \sqsupseteq 1_Y(y) - (\delta, 0, 0),\ \mathfrak{B}_2(\mathtt{N}) \sqsupseteq (1,0,0) - (\epsilon, 0, 0).$$

Let $\mathtt{A} = \mathtt{M} \times \mathtt{N}$; then we have

$$\begin{aligned}
\mathtt{A}(x,y) &= (\mathtt{M} \times \mathtt{N})(x,y) \\
&= \mathtt{M}(x) \sqcap \mathtt{N}(y) \\
&\sqsupseteq (1_X(x) - (\delta, 0, 0)) \sqcap (1_Y(y) - (\delta, 0, 0)) \\
&= (1_X(x) \sqcap 1_Y(y)) - (\delta, 0, 0) \\
&\sqsupseteq 1_{X \times Y}(x,y) - (\delta, 0, 0)
\end{aligned}$$

and
$$\mathfrak{B}(\mathtt{A}) \sqsupseteq \mathfrak{B}_1(\mathtt{M}) \sqcap \mathfrak{B}_1(\mathtt{N}) \sqsupseteq (1,0,0) - (\epsilon,0,0).$$
Thus **B1** of Definition 7 follows.

To prove **B2**, let $(x,y) \in X \times Y$, $\mathtt{M},\mathtt{N} \in \zeta^{X\times Y}$ and $\delta,\epsilon > 0$. If any one of $\mathtt{M}$ and $\mathtt{N}$, say $\mathtt{M}$, cannot be written as $\mathtt{M}_1 \times \mathtt{M}_2$ for any $\mathtt{M}_1 \in \zeta^X$ and $\mathtt{M}_2 \in \zeta^Y$, then by letting $\mathtt{L} = \mathtt{M} \sqcap \mathtt{N}$, as in the above theorem, **B2** of Definition 7 follows. On the other hand, suppose both $\mathtt{M}$ and $\mathtt{N}$ can be written as $\mathtt{A} \times \mathtt{A}'$ and $\mathtt{B} \times \mathtt{B}'$ for some $\mathtt{A},\mathtt{B} \in \zeta^X$ and $\mathtt{A}',\mathtt{B}' \in \zeta^Y$; then by definition of $\mathfrak{B}$, there exist $\mathtt{M}_1,\mathtt{N}_1 \in \zeta^X$, and $\mathtt{M}_2,\mathtt{N}_2 \in \zeta^Y$ such that $\mathtt{M} = \mathtt{M}_1 \times \mathtt{M}_2$, $\mathtt{N} = \mathtt{N}_1 \times \mathtt{N}_2$,
$$\mathfrak{B}_1(\mathtt{M}_1) \sqcap \mathfrak{B}_2(\mathtt{M}_2) \sqsupseteq \mathfrak{B}(\mathtt{M}) - (\frac{\epsilon}{2},0,0)$$
and
$$\mathfrak{B}_1(\mathtt{N}_1) \sqcap \mathfrak{B}_2(\mathtt{N}_2) \sqsupseteq \mathfrak{B}(\mathtt{N}) - (\frac{\epsilon}{2},0,0).$$
Here it is easy to see that there exists $\mathtt{L}_1 \in \zeta^X$ such that $\mathtt{L}_1 \sqsubseteq \mathtt{M}_1 \sqcap \mathtt{N}_1$, $\mathtt{L}_1(x) \sqsupseteq (\mathtt{M}_1 \sqcap \mathtt{N}_1)(x) - (\delta,0,0)$ and $\mathfrak{B}_1(\mathtt{L}_1) \sqsupseteq (\mathfrak{B}_1(\mathtt{M}_1) \sqcap \mathfrak{B}_1(\mathtt{N}_1)) - (\frac{\epsilon}{2},0,0)$, as $x \in X$ and $\mathtt{M}_1,\mathtt{N}_1$ are in ζ^X.

Analogously, since $y \in Y$ and $\mathtt{M}_2$, $\mathtt{N}_2$ are in ζ^Y, there exists $\mathtt{L}_2$ in ζ^Y such that $\mathtt{L}_2 \sqsubseteq \mathtt{M}_2 \sqcap \mathtt{N}_2$, $\mathtt{L}_2(y) \sqsupseteq (\mathtt{M}_2 \sqcap \mathtt{N}_2)(y) - (\delta,0,0)$ and $\mathfrak{B}_2(\mathtt{L}_2) \sqsupseteq (\mathfrak{B}_2(\mathtt{M}_2) \sqcap \mathfrak{B}_2(\mathtt{N}_2)) - (\frac{\epsilon}{2},0,0)$.

Let $\mathtt{L} = \mathtt{L}_1 \times \mathtt{L}_2$; then we have
$$\begin{aligned}
\mathtt{L}(x,y) &= (\mathtt{L}_1 \times \mathtt{L}_2)(x,y)\\
&= \mathtt{L}_1(x) \sqcap \mathtt{L}_2(y)\\
&\sqsupseteq \{(\mathtt{M}_1(x) \sqcap \mathtt{N}_1(x)) \sqcap (\mathtt{M}_2(y) \sqcap \mathtt{N}_2(y))\} - (\delta,0,0)\\
&= \{(\mathtt{M}_1(x) \times \mathtt{M}_2(y)) \sqcap (\mathtt{N}_1(x) \times \mathtt{N}_2(y))\} - (\delta,0,0)\\
&= (\mathtt{M}(x,y) \sqcap \mathtt{N}(x,y)) - (\delta,0,0)
\end{aligned}$$
and
$$\begin{aligned}
\mathfrak{B}(\mathtt{L}) &= \mathfrak{B}(\mathtt{L}_1 \times \mathtt{L}_2)\\
&\sqsupseteq \mathfrak{B}_1(\mathtt{L}_1) \sqcap \mathfrak{B}_2(\mathtt{L}_2)\\
&\sqsupseteq \Big\{(\mathfrak{B}_1(\mathtt{M}_1) \sqcap \mathfrak{B}_1(\mathtt{N}_1)) - (\frac{\epsilon}{2},0,0)\Big\}\\
&\qquad m \sqcap \Big\{(\mathfrak{B}_2(\mathtt{M}_2) \sqcap \mathfrak{B}_2(\mathtt{N}_2)) - (\frac{\epsilon}{2},0,0)\Big\}\\
&= \{(\mathfrak{B}_1(\mathtt{M}_1) \sqcap \mathfrak{B}_1(\mathtt{N}_1)) \sqcap (\mathfrak{B}_2(\mathtt{M}_2) \sqcap \mathfrak{B}_2(\mathtt{N}_2))\} - (\frac{\epsilon}{2},0,0)\\
&= \{(\mathfrak{B}_1(\mathtt{M}_1) \sqcap \mathfrak{B}_2(\mathtt{M}_2)) \sqcap (\mathfrak{B}_1(\mathtt{N}_1) \sqcap \mathfrak{B}_2(\mathtt{N}_2))\} - (\frac{\epsilon}{2},0,0)\\
&\sqsupseteq \Big(\mathfrak{B}(\mathtt{M}) - (\frac{\epsilon}{2},0,0)\Big) \sqcap \Big(\mathfrak{B}(\mathtt{N}) - (\frac{\epsilon}{2},0,0)\Big) - (\frac{\epsilon}{2},0,0)\\
&= (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\frac{\epsilon}{2},0,0) - (\frac{\epsilon}{2},0,0)\\
&= (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon,0,0).
\end{aligned}$$
Thus **B2** of Definition 7 follows in this case also. Hence $\mathfrak{B}$ is a basis for a smooth neutrosophic topology on $X \times Y$. Thus, proving that the smooth neutrosophic topology generated by this basis coincides with the smooth neutrosophic product topology remains.

Let $\mathfrak{T}$ be the smooth fuzzy topology generated by $\mathfrak{B}$. Let $\mathfrak{T}_p$ be the product topology on $X \times Y$ and $\mathfrak{B}_p$ be the basis for $\mathfrak{T}_p$ as described in Definition 10. Now we prove that $\mathfrak{T}_p = \mathfrak{T}$. Let $\mathtt{A} \in \zeta^{X\times Y}$; then
$$\mathfrak{T}_p(\mathtt{A}) = \bigsqcup_{\Lambda\in\Gamma} \left\{ \bigsqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda} \{\mathfrak{B}_p(\mathtt{A}_\lambda)\} \right\},$$

where $\{\mathcal{L}_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all inner covers $\mathcal{L}_\Lambda = \{\mathtt{A}_\lambda\}_{\lambda\in\Lambda}$ of $\mathtt{A}$. Now we divide the collection $\{\mathcal{L}_\Lambda\}_{\Lambda\in\Gamma}$, say $\mathcal{L}$, into two subcollections $\mathcal{L}'$ and $\mathcal{L}''$ where $\mathcal{L}'$ is the collection all possible inner covers $\{\mathtt{A}_\lambda\}_{\lambda\in\Lambda}$ of $\mathtt{A}$ so that for all $\lambda\in\Lambda$, $\mathtt{A}_\lambda$ is of the form $\mathtt{M}_\lambda\times\mathtt{N}_\lambda$ for at least one $\mathtt{M}_\lambda\in\zeta^X$ and one $\mathtt{N}_\lambda\in\zeta^Y$, and $\mathcal{L}''$ is the complement of $\mathcal{L}'$ in $\mathcal{L}$.

If an inner cover $\mathcal{L}_\Lambda = \{\mathtt{A}_\lambda\}_{\lambda\in\Lambda}$ of $\mathtt{A}$ is in $\mathcal{L}''$, then for at least one $\lambda_0\in\Lambda$, $\mathtt{A}_{\lambda_0}$ is not of the form $\mathtt{M}\times\mathtt{N}$ for any $\mathtt{M}\in\zeta^X$ and $\mathtt{N}\in\zeta^Y$; hence $\mathfrak{B}_p(\mathtt{A}_{\lambda_0}) = (0,1,1)$ and therefore

$$\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\mathfrak{B}_p(\mathtt{A}_\lambda)\} = (0,1,1)$$

and

$$\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\mathfrak{B}(\mathtt{A}_\lambda)\} = (0,1,1).$$

If $\mathcal{L}' = \emptyset$, then $\mathfrak{T}_p(\mathtt{A}) = \mathfrak{T}(\mathtt{A}) = (0,1,1)$ and hence it is enough to consider the case $\mathcal{L}'\neq\emptyset$. Now consider

$$\begin{aligned}
\mathfrak{T}_p(\mathtt{A}) &= \sqcup_{\mathcal{L}}\{\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\mathfrak{B}_p(\mathtt{A}_\lambda)\}\} \\
&= \sqcup_{\mathcal{L}'}\{\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\mathfrak{B}_p(\mathtt{A}_\lambda)\}\} \\
&= \sqcup_{\mathcal{L}'}\{\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\sqcup_{\mathtt{A}_\lambda=\mathtt{M}_\lambda\times\mathtt{N}_\lambda}\{\mathfrak{T}_1(\mathtt{M}_\lambda)\sqcap\mathfrak{T}_2(\mathtt{N}_\lambda)\}\}\} \\
&\sqsupseteq \sqcup_{\mathcal{L}'}\{\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\sqcup_{\mathtt{A}_\lambda=\mathtt{M}_\lambda\times\mathtt{N}_\lambda}\{\mathfrak{B}_1(\mathtt{M}_\lambda)\sqcap\mathfrak{B}_2(\mathtt{N}_\lambda)\}\}\} \\
&= \sqcup_{\mathcal{L}'}\{\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\mathfrak{B}(\mathtt{A}_\lambda)\}\} \\
&= \sqcup_{\mathcal{L}}\{\sqcap_{\mathtt{A}_\lambda\in\mathcal{L}_\Lambda}\{\mathfrak{B}(\mathtt{A}_\lambda)\}\} \\
&= \mathfrak{T}(\mathtt{A}).
\end{aligned}$$

This implies that, $\mathfrak{T}_p\sqsupseteq\mathfrak{T}$.

To prove the reverse inequality, let $\mathtt{A}\in\zeta^{X\times Y}$, $\epsilon>0$ and $\mathcal{L},\mathcal{L}',\mathcal{L}''$ be as above. Let $\mathcal{L}_\Lambda = \{\mathtt{A}_\lambda\}_{\lambda\in\Lambda}$ be an inner cover for $\mathtt{A}$. As above it is enough to consider the case $\mathcal{L}'\neq\emptyset$. Now let $\mathcal{L}_\Lambda\in\mathcal{L}'$. Then for all $\lambda\in\Lambda$, we have $\mathtt{A}_\lambda = \mathtt{M}\times\mathtt{N}$ for at least one $\mathtt{M}\in\zeta^X$ and one $\mathtt{N}\in\zeta^Y$. Fix a $\lambda\in\Lambda$. Let $\mathcal{B}_\lambda$ denote the set of all pairs $(\mathtt{M},\mathtt{N})$ such that $\mathtt{A}_\lambda = \mathtt{M}\times\mathtt{N}$. Let $(\mathtt{M},\mathtt{N})\in\mathcal{B}_\lambda$. Since $\mathfrak{B}_1$, $\mathfrak{B}_2$ are bases for $\mathfrak{T}_1$, $\mathfrak{T}_2$, by Theorem 5, for any $x\in X$, $y\in Y$ and $\delta>0$ there exist $\mathtt{M}_{x,\delta}\in\zeta^X$ and $\mathtt{N}_{y,\delta}\in\zeta^Y$ such that

$$\mathtt{M}_{x,\delta}(x)\sqsupseteq\mathtt{M}(x)-(\delta,0,0),\ \mathtt{M}_{x,\delta}\sqsubseteq\mathtt{M}$$

and

$$\mathtt{N}_{y,\delta}(y)\sqsupseteq\mathtt{N}(y)-(\delta,0,0),\ \mathtt{N}_{y,\delta}\sqsubseteq\mathtt{N}$$

with

$$\mathfrak{B}_1(\mathtt{M}_{x,\delta})+(\epsilon,0,0)\sqsupseteq\mathfrak{T}_1(\mathtt{M})$$

and

$$\mathfrak{B}_2(\mathtt{N}_{y,\delta})+(\epsilon,0,0)\sqsupseteq\mathfrak{T}_2(\mathtt{N}).$$

Clearly the collection $\{\mathtt{M}_{x,\delta}\}_{x\in X,\delta>0}$ is an inner cover for $\mathtt{M}$ and the collection $\{\mathtt{N}_{y,\delta}\}_y\in\zeta^Y,\delta>0$ is an inner cover for $\mathtt{N}$. Therefore, the collection $\{\mathtt{M}_{x,\delta}\times\mathtt{N}_{y,\delta}\}_{x\in X,y\in Y,\delta>0}$ is an inner cover for $\mathtt{M}\times\mathtt{N}$ which is equal to $\mathtt{A}_\lambda$. Thus for any pair $(\mathtt{M},\mathtt{N})\in\mathcal{B}_\lambda$ with $\mathtt{M}\times\mathtt{N} = \mathtt{A}_\lambda$, we have an inner cover $\{\mathtt{M}_{x,\delta}\times\mathtt{N}_{y,\delta}\}_{x\in X,\,y\,\in Y,\,\delta>0}$ of $\mathtt{A}_\lambda$ such that

$$\mathfrak{B}_1(\mathtt{M}_{x,\delta})+(\epsilon,0,0)\sqsupseteq\mathfrak{T}_1(\mathtt{M}) \tag{1}$$

and

$$\mathfrak{B}_2(\mathtt{N}_{y,\delta}) + (\epsilon, 0, 0) \sqsupseteq \mathfrak{T}_2(\mathtt{N}) \tag{2}$$

for all $x \in X$, $y \in X$ and $\delta > 0$.
Now since

$$\begin{aligned} \mathfrak{T}_p(\mathtt{A}) &= \bigsqcup_{\mathcal{L}}\{\bigsqcap_{\mathtt{A}_\lambda \in \mathcal{L}_\Lambda}\{\mathfrak{B}_p(\mathtt{A}_\lambda)\}\} \\ &= \bigsqcup_{\mathcal{L}'}\{\bigsqcap_{\mathtt{A}_\lambda \in \mathcal{L}_\Lambda}\{\bigsqcup_{(\mathtt{M},\mathtt{N}) \in \mathfrak{B}_\lambda}\{\mathfrak{T}_1(\mathtt{M}) \sqcap \mathfrak{T}_2(\mathtt{N})\}\}\}, \end{aligned}$$

using (1) and (2), we have

$$\begin{aligned} \mathfrak{T}_p(\mathtt{A}) &\sqsubseteq \bigsqcup_{\mathcal{L}'}\{\bigsqcap_{\mathtt{A}_\lambda \in \mathcal{L}_\Lambda}\{\mathfrak{T}(\mathtt{A}_\lambda)\}\} + (\epsilon, 0, 0) \\ &= \bigsqcup_{\mathcal{L}}\{\bigsqcap_{\mathtt{A}_\lambda \in \mathcal{L}_\Lambda}\{\mathfrak{T}(\mathtt{A}_\lambda)\}\} + (\epsilon, 0, 0) \\ &= \mathfrak{T}(\mathtt{A}) + (\epsilon, 0, 0). \end{aligned}$$

Since this is true for every $\epsilon > 0$, it follows that $\mathfrak{T}_p(\mathtt{A}) \sqsubseteq \mathfrak{T}(\mathtt{A})$ and hence we get $\mathfrak{T}_p \sqsubseteq \mathfrak{T}$ as desired. □

Theorem 10. *Let $(X, \mathfrak{T}_1)$ and $(Y, \mathfrak{T}_2)$ be smooth neutrosophic topological spaces. Let*

$$\mathcal{A}_1 = \{\mathtt{M}/\ \mathtt{A} = \mathtt{M} \times 1_Y, \mathtt{M} \in \zeta^X\}$$

and

$$\mathcal{A}_2 = \{\mathtt{N}/\ \mathtt{A} = 1_X \times \mathtt{N}, \mathtt{N} \in \zeta^Y\}.$$

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Define $\mathfrak{S} : \zeta^{X \times Y} \to \zeta$ as

$$\mathfrak{S}(\mathtt{A}) = \begin{cases} \bigsqcup_{\mathcal{A}}\{\mathfrak{T}_1(\mathtt{M}), \mathfrak{T}_2(\mathtt{N})\} & \text{if } \mathcal{A} \neq \emptyset \\ (0, 1, 1) & \text{otherwise.} \end{cases}$$

Then $\mathfrak{S}$ is a subbasis for the smooth neutrosophic product topology on $X \times Y$.

Proof. Since $\mathfrak{S}(1_{X \times Y}) = (1, 0, 0)$, by letting $\mathtt{M} = 1_{X \times Y}$, it clearly follows that $\mathfrak{S}$ is a subbasis for a smooth neutrosophic topology on $X \times Y$. Thus all that remains is to show the smooth neutrosophic topology induced by this subbasis is the same as the product topology on $X \times Y$. We do this by proving that the basis induced by this subbasis is the same as the basis defined in Definition 10.

Let $\mathfrak{B}'$ be the basis generated by $\mathfrak{S}$. Then for any $\mathtt{A}$ in $\zeta^{X \times Y}$, we have

$$\mathfrak{B}'(\mathtt{A}) = \bigsqcup_{D \in \mathcal{D}}\left\{\bigsqcap_{i \in I_D}\{\mathfrak{S}(\mathtt{A}_i)\}\right\},$$

where $\mathcal{D}$ is the family of all finite collections $D = \{\mathtt{A}_i\}_{i \in I_D}$ of neutrosophic sets in $\zeta^{X \times Y}$ for some finite indexing set I_D such that $\mathtt{A} = \bigsqcap_{i \in I_D} \mathtt{A}_i$, where each $\mathtt{A}_i \in \zeta^{X \times Y}$. Let $\mathfrak{B}$ be the basis for the smooth neutrosophic product topology on $X \times Y$ as in Definition 10. Let $\mathtt{A} \in \zeta^{X \times Y}$; then we claim that $\mathfrak{B}(\mathtt{A}) = \mathfrak{B}'(\mathtt{A})$. Suppose $\mathtt{A}$ is not of form $\mathtt{M} \times \mathtt{N}$ for any $\mathtt{M} \sqsubseteq 1_X$ and $\mathtt{N} \sqsubseteq 1_Y$. Then by Definition 10 we have, $\mathfrak{B}(\mathtt{A}) = (0, 1, 1)$. Now let us compute $\mathfrak{B}'(\mathtt{A})$. Let $\mathtt{A} = \mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n$ be any representation of $\mathtt{A}$ as a finite intersection of neutrosophic sets of $X \times Y$. First we claim that $\mathtt{A}_i$ is neither of the form $(\mathtt{M}_i \times 1_Y)$ nor of the form $(1_X \times \mathtt{N}_i)$ for at least one i. If $\mathtt{A}_i = (\mathtt{M}_i \times 1_Y)$ or $\mathtt{A}_i = (1_X \times \mathtt{N}_i)$ for all i.

Without loss of generality, let us assume that $\mathtt{A}_i = (\mathtt{M}_i \times 1_Y)$ for $i = 1, 2, \ldots, m$ and $\mathtt{A}_i = (1_X \times \mathtt{N}_i)$ for $i = m+1, m+2, \ldots, n$, then we have

$$\begin{aligned} \mathtt{A} &= \mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n \\ &= \{(\mathtt{M}_1 \times 1_Y) \sqcap \cdots \sqcap (\mathtt{M}_m \times 1_Y)\} \\ &\qquad \sqcap \{(1_X \times \mathtt{N}_{m+1}) \sqcap \cdots \sqcap (1_X \times \mathtt{N}_n)\} \\ &= \{(\mathtt{M}_1 \sqcap \mathtt{M}_2 \sqcap \cdots \sqcap \mathtt{M}_m) \times 1_Y\} \\ &\qquad \sqcap \{1_X \times (\mathtt{N}_{m+1} \sqcap \mathtt{N}_{m+2} \sqcap \cdots \sqcap \mathtt{N}_n)\}. \end{aligned}$$

Now if we let $\mathtt{M} = \mathtt{M}_1 \sqcap \mathtt{M}_2 \sqcap \cdots \sqcap \mathtt{M}_m$ and $\mathtt{N} = \mathtt{N}_{m+1} \sqcap \mathtt{N}_{m+2} \sqcap \cdots \sqcap \mathtt{N}_n$, then it follows that $\mathtt{A} = (\mathtt{M} \times 1_Y) \sqcap (1_X \times \mathtt{N}) = \mathtt{M} \times \mathtt{N}$, which is a contradiction to our assumption that $\mathtt{A}$ is not of the form $\mathtt{M} \times \mathtt{N}$. This proves the claim and hence $\sqcap\{\mathfrak{S}(\mathtt{A}_i)\} = (0,1,1)$. Since this is true for any representation of $\mathtt{A}$ as a finite intersection, by the definition of $\mathfrak{B}'$ we have $\mathfrak{B}'(\mathtt{A}) = (0,1,1)$. Thus $\mathfrak{B} = \mathfrak{B}'$ in this case.

If $\mathtt{A}$ is of the form $\mathtt{M} \times \mathtt{N}$ for some $\mathtt{M} \sqsubseteq 1_X$, $\mathtt{N} \sqsubseteq 1_Y$. First we claim that $\mathfrak{B}(\mathtt{A}) \sqsupseteq \mathfrak{B}'(\mathtt{A})$. For, let $\mathtt{A} = \mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n$ be a representation of $\mathtt{A}$ as a finite intersection of neutrosophic sets in $\zeta^{X \times Y}$. If $\mathtt{A}_i$ is neither of the form $(\mathtt{M}_i \times 1_Y)$ nor of the form $(1_X \times \mathtt{N}_i)$, for at least one j, then it follows that $\sqcap\{\mathfrak{S}(\mathtt{A}_i)\} = (0,1,1)$ and hence $\mathfrak{B}(\mathtt{A}) \sqsupseteq \sqcap\{\mathfrak{S}(\mathtt{A}_i)\}$. Suppose all $\mathtt{A}_i$'s are either of the form $(\mathtt{M}_i \times 1_Y)$ or of the form $(1_X \times \mathtt{N}_i)$ for some $\mathtt{M}_i \in \zeta^X$ and $\mathtt{N}_i \in \zeta^Y$; then we have $\mathfrak{S}(\mathtt{A}_i) \sqsupseteq (0,1,1)$ for all i. Let $\epsilon > 0$; then there exist $\mathtt{M}_i \in \zeta^X$ and $\mathtt{N}_i \in \zeta^Y$ such that

$$\mathtt{A}_i = (\mathtt{M}_i \times 1_Y),\ \mathfrak{T}_1(\mathtt{M}_i) \sqsupseteq \mathfrak{S}(\mathtt{A}_i) - (\epsilon, 0, 0)$$

for $i = 1, 2, \ldots, m$ and

$$\mathtt{A}_i = (1_X \times \mathtt{N}_i),\ \mathfrak{T}_1(\mathtt{N}_i) \sqsupseteq \mathfrak{S}(\mathtt{A}_i) - (\epsilon, 0, 0).$$

for $i = m+1, m+2, \ldots, n$. Then,

$$\begin{aligned} \mathtt{A} &= \mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n \\ &= \{(\mathtt{M}_1 \sqcap \mathtt{M}_2 \sqcap \cdots \sqcap \mathtt{M}_m) \times 1_Y\} \\ &\qquad \sqcap \{1_X \times (\mathtt{N}_{m+1} \sqcap \mathtt{N}_{m+2} \sqcap \cdots \sqcap \mathtt{N}_n)\}. \end{aligned}$$

Let $\mathtt{M}' = \mathtt{M}_1 \sqcap \cdots \sqcap \mathtt{M}_m$ and $\mathtt{N}' = \mathtt{N}_{m+1} \sqcap \cdots \sqcap \mathtt{N}_n$. Then we have $\mathtt{A} = (\mathtt{M}' \times 1_Y) \sqcap (1_X \times \mathtt{N}') = \mathtt{M}' \times \mathtt{N}'$.

Now consider

$$\begin{aligned} \mathfrak{B}(\mathtt{A}) &\sqsupseteq \sqcap\{\mathfrak{T}_1(\mathtt{M}'), \mathfrak{T}_2(\mathtt{N}')\} \\ &\sqsupseteq \sqcap\{\bigsqcap_{j=1}^{m} \mathfrak{T}_1(\mathtt{M}_j), \bigsqcap_{j=m+1}^{n} \mathfrak{T}_2(\mathtt{N}_j)\} \\ &= \sqcap\{\mathfrak{T}_1(\mathtt{M}_1), \ldots, \mathfrak{T}_1(\mathtt{M}_m), \mathfrak{T}_2(\mathtt{N}_{m+1}), \ldots, \mathfrak{T}_2(\mathtt{N}_n)\} \\ &\sqsupseteq \sqcap\{\mathfrak{S}(\mathtt{A}_1) - (\epsilon,0,0), \ldots, \mathfrak{S}(\mathtt{A}_n) - (\epsilon,0,0)\} \\ &= \sqcap\{\mathfrak{S}(\mathtt{A}_i)\} - (\epsilon,0,0). \end{aligned}$$

Since this is true for any representation of $\mathtt{A}$ as a finite intersection of neutrosophic sets in $\zeta^{X \times Y}$, we have

$$\mathfrak{B}(\mathtt{A}) \sqsupseteq \mathfrak{B}'(\mathtt{A}).$$

To prove the reverse inequality, let $\epsilon > 0$; then by Definition 10, there exist $\mathtt{M} \in \zeta^X$ and $\mathtt{N} \in \zeta^Y$ such that $\mathtt{A} = \mathtt{M} \times \mathtt{N}$ and

$$\mathfrak{T}_1(\mathtt{M}) \sqcap \mathfrak{T}_2(\mathtt{N}) \sqsupseteq \mathfrak{B}(\mathtt{A}) - (\epsilon, 0, 0).$$

However, $\mathtt{M} \times \mathtt{N} = (\mathtt{M} \times 1_Y) \sqcap (1_X \times \mathtt{N})$; thus, we have,

$$\begin{aligned} \mathfrak{B}'(\mathtt{A}) &= \mathfrak{B}'(\mathtt{M} \times \mathtt{N}) \\ &\sqsupseteq \sqcap\{\mathfrak{S}(\mathtt{M} \times 1_Y), \mathfrak{S}(1_X \times \mathtt{N})\} \\ &\sqsupseteq \sqcap\{\mathfrak{T}_1(\mathtt{M}), \mathfrak{T}_2(\mathtt{N})\} \\ &\sqsupseteq \mathfrak{B}(\mathtt{A}) - (\epsilon, 0, 0) \end{aligned}$$

which implies $\mathfrak{B}'(\mathtt{A}) \sqsupseteq \mathfrak{B}(\mathtt{A})$ as desired. □

Definition 11. *Let $\{(X_i, \mathfrak{T}_i)\}_{i\in J}$ be a collection of smooth neutrosophic topological spaces, for some indexing set J. Now define a function $\mathfrak{B} : \zeta^{\prod_{i\in J} X_i} \to \zeta$ as follows:*

Let $\mathtt{A} \in \zeta^{\prod_{i\in J} X_i}$. If $\mathtt{A} \neq \prod_{i\in J} \mathtt{A}_i$ where $\mathtt{A}_i \in \zeta^{X_i}$ and $\mathtt{A}_i = 1_{X_i}$ except for finitely many $i \in J$, then define $\mathfrak{B}(\mathtt{A}) = (0, 1, 1)$. Otherwise define

$$\mathfrak{B}(\mathtt{A}) = \bigsqcup_{\mathcal{A}_\mathtt{A}} \{ \bigsqcap_{i\in J} \{\mathfrak{T}_i(\mathtt{A}_i)\}\},$$

where $\mathcal{A}_\mathtt{A}$ is the collection of all $\{\prod_{i\in J} \mathtt{A}_i\}$ such that $\mathtt{A} = \prod_{i\in J} \mathtt{A}_i$, $\mathtt{A}_i \in \zeta^{X_i}$ and $\mathtt{A}_i = 1_{X_i}$ except for finitely many $i \in J$.

Then $\mathfrak{B}$ is a basis for a smooth neutrosophic topology called the smooth product topology on $\prod_{i\in J} X_i$.

Theorem 11. *Let $\{(X_i, \mathfrak{T}_i)\}_{i\in J}$ be a collection of smooth neutrosophic topological spaces, for some indexing set J. Let $\mathfrak{B}$ be as defined in Definition 11; then $\mathfrak{B}$ is a basis for a smooth neutrosophic topology on $\prod_{i\in J} X_i$.*

Proof. Since $\mathfrak{B}(1_{\prod_{i\in J} X_i}) = (1, 0, 0)$, **B1** of Definition 7 follows trivially.

To prove **B2**, let $\mathtt{M}, \mathtt{N} \in \zeta^{\prod_{i\in J} X_i}$, $\mathbf{x} \in \prod_{i\in J} X_i$ and $\epsilon, \delta > 0$. Let $\mathcal{A}_\mathtt{M}$ be the collection of all $\{\prod_{i\in J} \mathtt{M}_i\}$ such that $\mathtt{M} = \prod_{i\in J} \mathtt{M}_i$, $\mathtt{M}_i \in \zeta^{X_i}$ and $\mathtt{M}_i = 1_{X_i}$ except for finitely many $i \in J$ and let $\mathcal{A}_\mathtt{N}$ be the collection of all $\{\prod_{i\in J} \mathtt{N}_i\}$ such that $\mathtt{N} = \prod_{i\in J} \mathtt{N}_i$, $\mathtt{N}_i \in \zeta^{X_i}$ and $\mathtt{N}_i = 1_{X_i}$ except for finitely many $i \in J$.

Suppose any one of the collections $\mathcal{A}_\mathtt{M}$ and $\mathcal{A}_\mathtt{N}$, say $\mathcal{A}_\mathtt{M}$, is empty. Then by the definition of $\mathfrak{B}$, we get that $\mathfrak{B}(\mathtt{M}) = (0, 1, 1)$. Thus **B2** of Definition 7 follows in this case. If both collections $\mathcal{A}_\mathtt{M}$ and $\mathcal{A}_\mathtt{N}$ are nonempty, then there exist $\mathtt{A}_\mathtt{M} = \Pi \mathtt{M}_i$ in $\mathcal{A}_\mathtt{M}$ and $\mathtt{A}_\mathtt{N} = \Pi \mathtt{N}_i$ in $\mathcal{A}_\mathtt{N}$ such that

$$\sqcap\{\mathfrak{T}_i(\mathtt{M}_i)\} \sqsupseteq \mathfrak{B}(\mathtt{M}) - (\epsilon, 0, 0)$$

and

$$\sqcap\{\mathfrak{T}_i(\mathtt{N}_i)\} \sqsupseteq \mathfrak{B}(\mathtt{N}) - (\epsilon, 0, 0).$$

Let $\mathtt{L} = \mathtt{M} \sqcap \mathtt{N}$; then clearly

$$\mathtt{L}(\mathbf{x}) \sqsupseteq (\mathtt{M}(\mathbf{x}) \sqcap \mathtt{N}(\mathbf{x})) - (\delta, 0, 0), \ \forall\, \mathbf{x} \in \Pi X_i$$

and

$$\begin{aligned}
\mathfrak{B}(\mathtt{L}) &= \mathfrak{B}(\mathtt{M} \sqcap \mathtt{N}) \\
&= \mathfrak{B}(\Pi \mathtt{M}_i \sqcap \Pi \mathtt{N}_i) \\
&= \mathfrak{B}(\Pi(\mathtt{M}_i \sqcap \mathtt{N}_i)) \\
&\sqsupseteq \sqcap\{\mathfrak{T}_i(\mathtt{M}_i \sqcap \mathtt{N}_i)\} \\
&\sqsupseteq \sqcap\{\mathfrak{T}_i(\mathtt{M}_i) \sqcap \mathfrak{T}_i(\mathtt{N}_i)\} \\
&\sqsupseteq \sqcap\{\mathfrak{T}_i(\mathtt{M}_i)\} \sqcap \sqcap\{\mathfrak{T}_i(\mathtt{N}_i)\} \\
&\sqsupseteq (\mathfrak{B}(\mathtt{M}) - (\epsilon, 0, 0)) \sqcap (\mathfrak{B}(\mathtt{N}) - (\epsilon, 0, 0)) \\
&\sqsupseteq (\mathfrak{B}(\mathtt{M}) \sqcap \mathfrak{B}(\mathtt{N})) - (\epsilon, 0, 0).
\end{aligned}$$

Thus, **B2** of Definition 7 follows in this case also, and hence $\mathfrak{B}$ is a basis for a smooth neutrosophic topology on ΠX_i. □

Theorem 12. *Let $\{(X_i, \mathfrak{T}_i)\}_{i\in J}$ be a collection of smooth neutrosophic topological spaces. For any $\mathtt{A} \in \zeta^{\prod\limits_{i\in J} X_i}$, let $\mathcal{A}_{\mathtt{A}}$ be the collection of all $\{\prod \mathtt{A}_i\}$ such that $\mathtt{A} = \prod \mathtt{A}_i$, $\mathtt{A}_i \in \zeta^{X_i}$ and $\mathtt{A}_i = 1_{X_i}$ except for finitely many $i \in J$. Let $\mathfrak{S} : \zeta^{\prod\limits_{i\in J} X_i} \to \zeta$ be defined as follows:*

$$\mathfrak{S}(\mathtt{A}) = \begin{cases} (1, 0, 0) & \text{if } \mathtt{A} = 1_{\Pi X_i} \\ \bigsqcup\limits_{\mathcal{A}_{\mathtt{A}}} \{ \bigsqcup\limits_{\mathtt{A}_i \neq 1_{X_i}} \{\mathfrak{T}_i(\mathtt{A}_i)\}\} & \text{if } \mathtt{A} \neq 1_{\Pi X_i}, \mathcal{A}_{\mathtt{A}} \neq \emptyset \\ (0, 1, 1) & \text{if } \mathcal{A}_{\mathtt{A}} = \emptyset. \end{cases}$$

Then $\mathfrak{S}$ is a subbasis for a smooth neutrosophic product topology on ΠX_i.

Proof. Since $\mathfrak{S}(1_{\Pi X_i}) = (1, 0, 0)$, **B1** of Definition 7 follows. Thus $\mathfrak{S}$ is a subbasis for a smooth neutrosophic topology on $\prod X_i$. Thus, proving that the smooth neutrosophic topology generated from $\mathfrak{S}$ is the smooth neutrosophic product topology on $\prod X_i$ needs proving.

Now let $\mathfrak{B}'$ be the basis generated by $\mathfrak{S}$ and let $\mathfrak{B}$ be the basis for the smooth neutrosophic product topology defined in Definition 11. To prove the topologies generated by $\mathfrak{B}$ and $\mathfrak{B}'$ are same, we prove the stronger result that $\mathfrak{B} = \mathfrak{B}'$.

As $\mathfrak{B}(1_{\Pi X_i}) = \mathfrak{B}'(1_{\Pi X_i}) = (1, 0, 0)$ follows trivially, we prove the other cases. Let $\mathtt{A} \in \zeta^{\Pi X_i}$ and let $\mathcal{A}_{\mathtt{A}}$ be the collection of all $\{\Pi \mathtt{A}_i\}$ such that $\mathtt{A} = \Pi \mathtt{A}_i$, $\mathtt{A}_i \in \zeta^{X_i}$ and $\mathtt{A}_i = 1_{X_i}$ except for finitely many $i \in J$. If $\mathcal{A}_{\mathtt{A}} = \emptyset$, then by the definition of $\mathfrak{B}$, we have $\mathfrak{B}(\mathtt{A}) = (0, 1, 1)$. Now to compute $\mathfrak{B}'(\mathtt{A})$, let $\mathtt{A} = \mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n$; we claim that there must exist at least one $\mathtt{A}_k$ which is not of the form $\prod \mathtt{A}_{ki}$ where $\mathtt{A}_{ki} \in \zeta^{X_i}$ and $\mathtt{A}_{ki} = 1_{X_i}$ except for finitely many $i \in J$. Suppose not; instead, let $\mathtt{A}_j = \Pi \mathtt{A}_{ji}$ where $\mathtt{A}_{ji} \in \zeta^{X_i}$ and $\mathtt{A}_{ji} = 1_{X_i}$ except for finitely many $i \in J$, for all $j = 1, 2, \ldots, n$. Then using these finitely many $\mathtt{A}_{ji}$'s, $\mathtt{A}$ can be written in the form $\Pi \mathtt{A}_i$ where $\mathtt{A}_i \in \zeta^{X_i}$ and $\mathtt{A}_i = 1_{X_i}$ except for finitely many $i \in J$, which is a contradiction to our assumption that $\mathcal{A}_{\mathtt{A}} = \emptyset$. Thus there exists at least one $\mathtt{A}_k$ which is not of the form $\prod \mathtt{A}_{ki}$ where $\mathtt{A}_{ki} \in \zeta^{X_i}$ and $\mathtt{A}_{ki} = 1_{X_i}$ except for finitely many $i \in J$ and hence $\mathfrak{S}(\mathtt{A}_k) = (0, 1, 1)$. Thus we have

$$\sqcap\{\mathfrak{S}(\mathtt{A}_j)/j = 1, 2, \ldots, n\} = (0, 1, 1).$$

Since this is true for any possible finite representation $\mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n$ of $\mathtt{A}$, we have $\mathfrak{B}'(\mathtt{A}) = (0, 1, 1)$ and hence $\mathfrak{B}'(\mathtt{A}) = \mathfrak{B}(\mathtt{A})$ in this case.

If $\mathcal{A}_{\mathtt{A}} \neq \emptyset$, then there must exist a representation $\mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n$ of $\mathtt{A}$ such that $\mathcal{A}_{\mathtt{A}_j} \neq \emptyset$ for all $j = 1, 2, \ldots, n$, where $\mathcal{A}_{\mathtt{A}_j}$ is the collection of all $\{\Pi \mathtt{A}_{ji}\}$ such that $\mathtt{A}_j = \Pi \mathtt{A}_{ji}$, $\mathtt{A}_{ji} \in \zeta^{X_i}$ and $\mathtt{A}_{ji} = 1_{X_i}$

except for finitely many $i \in J$. Let $\epsilon > 0$. Then for each $\mathtt{A}_j$ we can find a collection $\{\mathtt{A}_{ji}\}_{i\in J}$ such that $\mathtt{A}_j = \Pi\mathtt{A}_{ji}$ where $\mathtt{A}_{ji} = 1_{X_i}$ except for finitely many $i \in J$ and $\mathfrak{T}_i(\mathtt{A}_{ji}) \sqsupseteq \mathfrak{S}(\mathtt{A}_j) - (\epsilon, 0, 0)$. Now since

$$\begin{aligned} \mathtt{A} &= \mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n \\ &= \Pi\mathtt{A}_{1i} \sqcap \Pi\mathtt{A}_{2i} \sqcap \cdots \sqcap \Pi\mathtt{A}_{ni} \\ &= \Pi(\mathtt{A}_{1i} \sqcap \mathtt{A}_{2i} \sqcap \cdots \sqcap \mathtt{A}_{ni}) \end{aligned}$$

we have

$$\begin{aligned} \mathfrak{B}(\mathtt{A}) &= \mathfrak{B}(\mathtt{A}_1 \sqcap \mathtt{A}_2 \sqcap \cdots \sqcap \mathtt{A}_n) \\ &= \mathfrak{B}(\Pi\mathtt{A}_{1i} \sqcap \Pi\mathtt{A}_{2i} \sqcap \cdots \sqcap \Pi\mathtt{A}_{ni}) \\ &= \mathfrak{B}(\Pi(\mathtt{A}_{1i} \sqcap \mathtt{A}_{2i} \sqcap \cdots \sqcap \mathtt{A}_{ni})) \\ &\sqsupseteq \sqcap\{\mathfrak{T}_i(\mathtt{A}_{1i} \sqcap \mathtt{A}_{2i} \sqcap \cdots \sqcap \mathtt{A}_{ni})\} \\ &\sqsupseteq \sqcap\{\mathfrak{T}_i(\mathtt{A}_{1i}) \sqcap \mathfrak{T}_i(\mathtt{A}_{2i}) \sqcap \cdots \sqcap \mathfrak{T}_i(\mathtt{A}_{ni})\} \\ &\sqsupseteq \sqcap\{\mathfrak{S}(\mathtt{A}_1) - (\epsilon, 0, 0) \sqcap \mathfrak{S}(\mathtt{A}_2) - (\epsilon, 0, 0) \\ &\qquad\qquad \sqcap \cdots \sqcap \mathfrak{S}(\mathtt{A}_n) - (\epsilon, 0, 0)\} \\ &= \sqcap\{\mathfrak{S}(\mathtt{A}_1) \sqcap \mathfrak{S}(\mathtt{A}_2) \sqcap \cdots \sqcap \mathfrak{S}(\mathtt{A}_n)\} - (\epsilon, 0, 0) \\ &= \sqcap\{\mathfrak{S}(\mathtt{A}_j)\} - (\epsilon, 0, 0). \end{aligned}$$

Since this is true for any representation of $\mathtt{A}$ as a finite intersection of neutrosophic sets in $\zeta^{\Pi X_i}$, we have

$$\mathfrak{B}(\mathtt{A}) \sqsupseteq \mathfrak{B}'(\mathtt{A}).$$

To prove the reverse inequality, let $\epsilon > 0$. Since $\mathcal{A}_{\mathtt{A}} \neq \emptyset$, we can find a collection $\{\mathtt{A}_i\}_{i\in J}$ such that $\Pi\mathtt{A}_i \in \mathcal{A}_{\mathtt{A}}$ and

$$\sqcap\{\mathfrak{T}_i(\mathtt{A}_i)\} \sqsupseteq \mathfrak{B}(\mathtt{A}) - (\epsilon, 0, 0).$$

Thus it follows that

$$\mathfrak{B}'(\mathtt{A}) = \mathfrak{B}'(\Pi\mathtt{A}_i) \sqsupseteq \sqcap\{\mathfrak{T}_i(\mathtt{A}_i)\} \sqsupseteq \mathfrak{B}(\mathtt{A}) - (\epsilon, 0, 0)$$

and hence $\mathfrak{B}'(\mathtt{A}) \sqsupseteq \mathfrak{B}(\mathtt{A})$. Thus $\mathfrak{B}'(\mathtt{A}) = \mathfrak{B}(\mathtt{A})$ in this case also and hence in all the cases. □

5. Conclusions

In this paper, we have defined the notion of a basis and subbasis for a neutrosophic topology as a neutrosophic set from a suitable collection of neutrosophic sets of X to $[0,1]^3$. Using this idea of considering a basis as a neutrosophic set, we developed a theory of smooth neutrosophic topological spaces that fits exactly with the theory of classical and fuzzy topological spaces. Next, we introduced and investigated the concept of quasi-coincident neighborhood systems in this context. Finally, we defined and discussed the notion of both finite and infinite products of smooth neutrosophic topologies.

6. A Discussion for Future Works

The theory can extended in the following natural ways. One may

- Study the properties of neutrosophic metric topological spaces using the concept of basis defined in this paper;
- Investigate the products of Hausdorff, regular, compact and connected spaces in the context of neutrosophic topological spaces.

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On Single-Valued Neutrosophic Ideals in Šostak Sense

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Abstract: Neutrosophy is a recent section of philosophy. It was initiated in 1980 by Smarandache. It was presented as the study of origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. In this paper, we introduce the notion of single-valued neutrosophic ideals sets in Šostak's sense, which is considered as a generalization of fuzzy ideals in Šostak's sense and intuitionistic fuzzy ideals. The concept of single-valued neutrosophic ideal open local function is also introduced for a single-valued neutrosophic topological space. The basic structure, especially a basis for such generated single-valued neutrosophic topologies and several relations between different single-valued neutrosophic ideals and single-valued neutrosophic topologies, are also studied here. Finally, for the purpose of symmetry, we also define the so-called single-valued neutrosophic relations.

Keywords: single-valued neutrosophic closure; single-valued neutrosophic ideal; single-valued neutrosophic ideal open local function; single-valued neutrosophic ideal closure; single-valued neutrosophic ideal interior; single-valued neutrosophic ideal open compatible

1. Introduction

The notion of fuzzy sets, employed as an ordinary set generalization, was introduced in 1965 by Zadeh [1]. Later on, using fuzzy sets through the fuzzy topology concept was initially introduced in 1968 by Chang [2]. Afterwards, many properties in fuzzy topological spaces have been explored by various researchers [3–13]

Paradoxically, it is to be emphasized that being fuzzy or what is termed as fuzzy topology in fuzzy openness concept is not highlighted and well-studied. Meanwhile, Samanta et al. [14,15] introduced what is called the graduation of openness of fuzzy sets. Later on, Ramadan [16] introduced smooth continuity, a number of their properties, and smooth topology. Demirci [17] investigated properties and systems of smooth Q-neighborhood and smooth neighborhood alike. It is worth mentioning that Chattopadhyay and Samanta [18] have initiated smooth connectedness and smooth compactness. On the other hand, Peters [19] tackled the notion of primary fuzzy smooth characteristics and structures together with smooth topology in Lowen sense. He [20] further evidenced that smooth topologies collection constitutes a complete lattice. Furthermore, Onassanya and Hošková-Mayerová [21] inspected certain features of subsets of α-level as an integral part of a fuzzy subset topology. Likewise, more specialists in the field like Çoker and Demirci [22], in addition to Samanta and Mondal [23,24], have provided definitions to the concept of graduation intuitionistic openness of fuzzy sets based on Šostak's sense [25] according to Atanassov's [26] intuitionistic fuzzy sets. Essentially, they focused on intuitionistic gradation of openness in light of Chang. On the other hand, Lim et al. [27] examined

Lowen's framework smooth intuitionistic topological spaces. In recent times, Kim et al. [28] considered systems of neighborhood and continuities within smooth intuitionistic topological spaces. Moreover, Choi et al. [29] scrutinized smooth interval-valued topology through graduation of the concept of interval-valued openness of fuzzy sets, as suggested by Gorzalczany [30] and Zadeh [31], respectively. Ying [32] put forward a topology notion termed as fuzzifying topology, taking into consideration the extent of ordinary subset of a set openness. General properties in ordinary smooth topological spaces were elaborated in 2012 by Lim et al. [33]. In addition, they [34–36] inspected compactness, interiors, and closures within normal smooth topological spaces. In 2014, Saber et al. [37] shaped the notion of fuzzy ideal and r-fuzzy open local function in fuzzy topological spaces in view of the definition of Šostak. In addition, they [38,39] inspected intuitionistic fuzzy ideals, fuzzy ideals and fuzzy open local function in fuzzy topological spaces in view of the definition of Chang.

Smarandache [40] determined the notion of a neutrosophic set as intuitionistic fuzzy set generalization. Meanwhile, Salama et al. [41,42] familiarized the concepts of neutrosophic crisp set and neutrosophic crisp relation neutrosophic set theory. Correspondingly, Hur et al. [43,44] initiated classifications NSet(H) and NCSet including neutrosophic crisp and neutrosophic sets, where they examined them in a universe topological position. Furthermore, Salama and Alblowi [45] presented neutrosophic topology as they claimed a number of its characteristics. Salama et al. [46] defined a neutrosophic crisp topology and studied some of its properties. Others, such as Wang et al. [47], defined the single-valued neutrosophic set concept. Currently, Kim et al. [48] has come to grips with a neutrosophic partition single-value, neutrosophic equivalence relation single-value, and neutrosophic relation single-value.

Preliminaries of single-value neutrosophic sets and single-valued neutrosophic topology are reviewed in Section 2. Section 3 is devoted to the concepts of single-valued neutrosophic closure space and single-valued neutrosophic ideal. Some of their characteristic properties are considered. Finally, the concepts of single-valued neutrosophic ideal open local function has been introduced and studied. Several preservation properties and some characterizations concerning single-valued neutrosophic ideal open compatible have been obtained.

2. Preliminaries

In this section, we attempt to cover enough of the fundamental concepts and definitions.

Definition 1 ([49])**.** *A neutrosophic set $\mathcal{H}$ (**NS**, for short) on a nonempty set $\mathcal{S}$ is defined as*

$$\mathcal{H} = \langle \kappa, T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} : \kappa \in \mathcal{S} \rangle,$$

where

$$T_{\mathcal{H}} : \mathcal{S} \to \rfloor^{-}0, 1^{+}\lfloor, \quad I_{\mathcal{H}} : \mathcal{S} \to \rfloor^{-}0, 1^{+}\lfloor, \quad F_{\mathcal{H}} : \mathcal{S} \to \rfloor^{-}0, 1^{+}\lfloor$$

and

$$^{-}0 \leq T_{\mathcal{H}}(\kappa) + I_{\mathcal{H}}(\kappa) + F_{\mathcal{H}}(\kappa) \leq 3^{+},$$

representing the degree of membership (namely, $T_{\mathcal{H}}(\kappa)$), the degree of indeterminacy (namely, $I_{\mathcal{H}}(\kappa)$), and the degree of nonmembership (namely, $F_{\mathcal{H}}(\kappa)$); for all $\kappa \in \mathcal{S}$ to the set $\mathcal{H}$.

Definition 2 ([49])**.** *Let $\mathcal{H}$ and $\mathcal{R}$ be fuzzy neutrosophic sets in $\mathcal{S}$. Then, $\mathcal{H}$ is a subset of $\mathcal{R}$ if, for each $\kappa \in \mathcal{S}$,*

$$\inf T_{\mathcal{H}}(x) \leq \inf T_{\mathcal{R}}(\kappa), \quad \inf I_{\mathcal{H}}(x) \geq \inf I_{\mathcal{R}}(\kappa), \quad \inf F_{\mathcal{H}}(x) \geq \inf F_{\mathcal{R}}(\kappa)$$

and

$$\sup T_{\mathcal{H}}(\kappa) \leq \sup T_{\mathcal{R}}(\kappa), \quad \sup I_{\mathcal{H}}(\kappa) \geq \sup I_{\mathcal{R}}(\kappa), \quad \sup F_{\mathcal{H}}(\kappa) \geq \sup F_{\mathcal{R}}(\kappa).$$

Definition 3 ([47]). *Let $\mathcal{H}$ be a space of points (objects) with a generic element in $\mathcal{S}$ denoted by κ. Then, $\mathcal{H}$ is called a single-valued neutrosophic set (in short,* **SVNS***) in $\mathcal{S}$ if $\mathcal{H}$ has the form $\mathcal{H} = \langle T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} \rangle$, where $T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} : \mathcal{S} \to [0,1]$.*

In this case, $T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}}$ are called truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively, and we will denote the set of all **SVNS's** *in $\mathcal{S}$ as* **SVNS**$(\mathcal{S})$.

Moreover, we will refer to the Null (empty) **SVNS** *(or the absolute (universe)* **SVNS***) in $\mathcal{S}$ as 0_N (or 1_N) and define by $0_N = (0,1,1)$ (or $1_N = (1,0,0)$) for each $\kappa \in \mathcal{S}$.*

Definition 4 ([47]). *Let $\mathcal{H} = \langle T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}} \rangle$ be an* **SVNS** *on $\mathcal{S}$. The complement of the set $\mathcal{H}$ ($\mathcal{H}^c$, for short) and is defined as follows: for every $\kappa \in \mathcal{S}$,*

$$T_{\mathcal{H}^c}(\kappa) = F_{\mathcal{H}}(\kappa), \quad I_{\mathcal{H}^c}(\kappa) = 1 - I_{\mathcal{H}}(\kappa), \quad F_{\mathcal{H}^c}(\kappa) = T_{\mathcal{H}}(\kappa).$$

Definition 5 ([50]). *Suppose that $\mathcal{H} \in$* **SVNS**$(\mathcal{S})$. *Then,*

(i) $\mathcal{H}$ is said to be contained in $\mathcal{R}$, denoted by $\mathcal{H} \subseteq \mathcal{R}$, if, for every $\kappa \in \mathcal{S}$,

$$T_{\mathcal{H}}(\kappa) \leq T_{\mathcal{R}}(\kappa), \quad I_{\mathcal{H}}(\kappa) \geq I_{\mathcal{R}}(\kappa), \quad F_{\mathcal{H}}(\kappa) \geq F_{\mathcal{R}}(\kappa);$$

(ii) $\mathcal{H}$ is said to be equal to $\mathcal{R}$, denoted by $\mathcal{H} = \mathcal{H}$, if $\mathcal{R} \subseteq \mathcal{R}$ and $\mathcal{H} \supseteq \mathcal{R}$.

Definition 6 ([51]). *Suppose that $\mathcal{H}, \mathcal{R} \in$* **SVNS**$(\mathcal{S})$. *Then,*

(i) the union of $\mathcal{H}$ and $\mathcal{R}$ ($\mathcal{H} \cup \mathcal{R}$, for short) is an **SVNS** *in $\mathcal{S}$ defined as*

$$\mathcal{H} \cup \mathcal{R} = (T_{\mathcal{H}} \cup T_{\mathcal{R}}, I_{\mathcal{H}} \cap I_{\mathcal{R}}, F_{\mathcal{H}} \cap F_{\mathcal{R}}),$$

where $(T_{\mathcal{H}} \cup T_{\mathcal{R}})(\kappa) = T_{\mathcal{H}}(\kappa) \cup T_{\mathcal{R}}(\kappa)$ and $(F_{\mathcal{H}} \cap F_{\mathcal{R}})(\kappa) = F_{\mathcal{H}}(\kappa) \cap F_{\mathcal{R}}(\kappa)$, for each $\kappa \in \mathcal{S}$;

(ii) the intersection of $\mathcal{H}$ and $\mathcal{R}$, ($\mathcal{H} \cap \mathcal{R}$, for short), is an **SVNS** *in $\mathcal{S}$ defined as*

$$\mathcal{H} \cap \mathcal{R} = (T_{\mathcal{H}} \cap T_{\mathcal{R}}, I_{\mathcal{H}} \cup I_{\mathcal{R}}, F_{\mathcal{H}} \cup F_{\mathcal{R}}).$$

Definition 7 ([45]). *Let $\mathcal{H} \in$* **SVNS**$(\mathcal{S})$. *Then,*

(i) the union of $\{\mathcal{H}_i\}_{i \in J}$ ($\bigcup_{i \in J} \mathcal{H}_i$, for short) is an **SVNS** *in $\mathcal{S}$ defined as follows: for every $\kappa \in \mathcal{S}$,*

$$(\bigcup_{i \in J} \mathcal{H}_i)(\kappa) = (\bigcup_{i \in J} T_{\mathcal{H}_i}(\kappa), \bigcap_{i \in J} I_{\mathcal{H}_i}(\kappa), \bigcap_{i \in J} F_{\mathcal{H}_i}(\kappa);$$

(ii) the intersection of $\{\mathcal{H}_i\}_{i \in J}$ ($\bigcap_{i \in J} \mathcal{H}_i$, for short) is an **SVNS** *in $\mathcal{S}$ defined as follows: for every $\kappa \in \mathcal{S}$,*

$$(\bigcap_{i \in J} \mathcal{H}_i)(\kappa) = (\bigcap_{i \in J} T_{\mathcal{H}_i}(\kappa), \bigcup_{i \in J} I_{\mathcal{H}_i}(\kappa), \bigcup_{i \in J} F_{\mathcal{H}_i}(\kappa).$$

Definition 8 ([52]). *A single-valued neutrosophic topology on $\mathcal{S}$ is a map $(\tau^T, \tau^I, \tau^F) : I^{\mathcal{S}} \to I$ satisfying the following three conditions:*

(SVNT1) $\tau^T(\underline{0}) = \tau^T(\underline{1}) = 1$ *and* $\tau^I(\underline{0}) = \tau^I(\underline{1}) = \tau^F(\underline{0}) = \tau^F(\underline{1}) = 0$,

(SVNT2) $\tau^T(\mathcal{H} \cap \mathcal{R}) \geq \tau^T(\mathcal{H}) \cap \tau^T(\mathcal{R})$, $\quad \tau^I(\mathcal{H} \cap \mathcal{R}) \leq \tau^I(\mathcal{H}) \cup \tau^I(\mathcal{R})$, $\tau^F(\mathcal{H} \cap \mathcal{R}) \leq \tau^F(\mathcal{H}) \cup \tau^F(\mathcal{R})$, *for any* $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$,

(SVNT3) $\tau^T(\cup_{i\in j}\mathcal{H}_i) \geq \cap_{i\in j}\tau^T(\mathcal{H}_i)$, $\tau^I(\cup_{i\in j}\mathcal{H}_i) \leq \cup_{i\in j}\tau^I(\mathcal{H}_i)$, $\tau^F(\cup_{i\in j}\mathcal{H}_i) \leq \cup_{i\in j}\tau^F(\mathcal{H}_i)$, *for any* $\{\mathcal{H}_i\}_{i\in J} \in I^{\mathcal{S}}$.

The pair $(X, \tau^T, \tau^I, \tau^F)$ is called single-valued neutrosophic topological spaces (**SVNTS**, *for short*). We will occasionally write τ^{TIF} for (τ^T, τ^I, τ^F) and it will cause no ambiguity.

3. Single-Valued Neutrosophic Closure Space and Single-Valued Neutrosophic Ideal in Šostak Sense

This section deals with the definition of single-valued neutrosophic closure space. The researchers examine the connection between single-valued neutrosophic closure space and **SVNTS** based in Šostak sense. Moreover, the researchers focused on the single-valued neutrosophic ideal notion where they obtained fundamental properties. Based on Šostak's sense, where a single-valued neutrosophic ideal takes the form $(\mathcal{S}, \mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$ and the mappings $\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F : I^{\mathcal{S}} \to I$, where $(\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$ are the degree of openness, the degree of indeterminacy, and the degree of non-openness, respectively.

In this paper, $\mathcal{S}$ is used to refer to nonempty sets, whereas I is used to refer to closed interval $[0, 1]$ and I_o is used to refer to the interval $(0, 1]$. Concepts and notations that are not described in this paper are standard, instead, $\mathcal{S}$ is usually used.

Definition 9. *A mapping* $\mathbb{C} : I^{\mathcal{S}} \times I_0 \to I^{\mathcal{S}}$ *is called a single-valued neutrosophic closure operator on* $\mathcal{S}$ *if, for every* $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$ *and* $r, s \in I_0$*, the following axioms are satisfied:*

($\mathbb{C}_1$) $\mathbb{C}((0.1.1), s) = (0.1.1)$,
($\mathbb{C}_2$) $\mathcal{H} \leq \mathbb{C}(\mathcal{H}, s)$,
($\mathbb{C}_3$) $\mathbb{C}(\mathcal{H}, s) \vee \mathbb{C}(\mathcal{R}, s) = \mathbb{C}(\mathcal{H} \vee \mathcal{R}, s)$,
($\mathbb{C}_4$) $\mathbb{C}(\mathcal{H}, s) \leq \mathbb{C}(\mathcal{H}, r)$ *if* $s \leq r$,
($\mathbb{C}_5$) $\mathbb{C}(\mathbb{C}(\mathcal{H}, s), s) = \mathbb{C}(\mathcal{H}, s)$.

The pair $(X, \mathbb{C})$ is a single-valued neutrosophic closure space ($\mathcal{SVNCS}$, *for short*).

Suppose that $\mathbb{C}_1$ and $\mathbb{C}_2$ are single-valued neutrosophic closure operators on $\mathcal{S}$. Then, $\mathbb{C}_1$ is finer than $\mathbb{C}_2$, denoted by $\mathbb{C}_2 \leq \mathbb{C}_1$ iff $\mathbb{C}_1(\mathcal{H}, s) \leq \mathbb{C}_2(\mathcal{H}, s)$, for every $\mathcal{H} \in I^{\mathcal{S}}$ and $s \in I_0$.

Theorem 1. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS**. *Then, for any* $\mathcal{H} \in I^{\mathcal{S}}$ *and* $s \in I_0$*, we define an operator* $\mathbb{C}_{\tau^{TIF}} : I^{\mathcal{S}} \times I_0 \to I^{\mathcal{S}}$ *as follows:*

$$\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) = \bigwedge\{\mathcal{R} \in I^X : \mathcal{H} \leq \mathcal{R}, \quad \tau^T(\underline{1} - \mathcal{R}) \geq s, \quad \tau^I(\underline{1} - \mathcal{R}) \leq 1 - s, \quad \tau^F(\underline{1} - \mathcal{R}) \leq 1 - s\}.$$

Then, $(\mathcal{S}, \mathbb{C}_{\tau^{TIF}})$ *is an* $\mathcal{SVNCS}$.

Proof. Suppose that $(\mathcal{S}, \tau^{TIF})$ is an **SVNTS**. Then, $\mathbb{C}_1$, ($\mathbb{C}_2$) and ($\mathbb{C}_4$) follows directly from the definition of $\mathbb{C}_{\tau^{TIF}}$.

($\mathbb{C}_3$) Since $\mathcal{R}, \mathcal{H} \leq \mathcal{H} \cup \mathcal{R}$, $\mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s)$ and $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s)$, therefore,

$$\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s).$$

Let (X, τ^{TIF}) be an **SVNTS**. From ($\mathbb{C}_2$), we have

$$\mathcal{H} \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), \quad \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \geq s, \ \tau^I(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \leq 1 - s \ \text{and} \ \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \leq 1 - s,$$

$$\mathcal{R} \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s), \quad \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \geq s, \ \tau^I(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s$$
$$\textit{and } \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s.$$

It implies that $\mathcal{H} \cup \mathcal{R} \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)$,

$$\begin{aligned}\tau^T(\underline{1} - (\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) &= \tau^T((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap (\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) \\ &\geq \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap \tau^T(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \geq s,\end{aligned}$$

$$\begin{aligned}\tau^I(\underline{1} - (\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) &= \tau^I((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap (\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) \\ &\leq \tau^I((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cup \tau^I(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s,\end{aligned}$$

$$\begin{aligned}\tau^F(\underline{1} - (\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) &= \tau^F((\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cap (\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s))) \\ &\leq \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)) \cup \tau^F(\underline{1} - \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, s)) \leq 1 - s.\end{aligned}$$

Hence, $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s) \geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s)$. Therefore,

$$\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \cup \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s) = \mathbb{C}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, s).$$

($\mathbb{C}_5$) Suppose that there exists $s \in I_0$, $\mathcal{H} \in I^{\mathcal{S}}$, and $\kappa \in \mathcal{S}$ such that

$$\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s)(\kappa) > \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)(\kappa).$$

By the definition of $\mathbb{C}_{\tau^{TIF}}$, there exists $\mathcal{D} \in I^{\mathcal{S}}$ with $\mathcal{D} \geq \mathcal{H}$, and $\tau^T(\underline{1} - \mathcal{D}) \geq s$, $\tau^I(\underline{1} - \mathcal{D}) \leq 1 - s$ and $\tau^F(\underline{1} - \mathcal{D}) \leq 1 - s$ such that

$$\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s)(\kappa) > \mathcal{D}(\kappa) \geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)(\kappa).$$

Since $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s) \leq \mathcal{D}$ and $\tau^T(\underline{1} - \mathcal{D}) \geq s$, $\tau^I(\underline{1} - \mathcal{D}) \leq 1 - s$, and $\tau^F(\underline{1} - \mathcal{D}) \leq 1 - s$, by the definition of $\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}})$, we have

$$\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s) \leq \mathcal{D}.$$

It is a contradiction. Thus, $\mathbb{C}_{\tau^{TIF}}(\mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s), s) = \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, s)$. Hence, $\mathbb{C}_{\tau^{TIF}}$ is a single-valued neutrosophic closure operator on $\mathcal{S}$. □

Theorem 2. *Let $(\mathcal{S}, \mathbb{C})$ be an $\mathcal{SVNCS}$ and $\mathcal{H} \in \mathcal{S}$. Define the mapping $\tau_C^{TIF} : I^{\mathcal{S}} \to I$ on $\mathcal{S}$ by*

$$\tau_{\mathbb{C}}^T(\mathcal{H}) = \bigcup\{s \in I_0 \mid \mathbb{C}(\overline{1} - \mathcal{H}, s) = \overline{1} - \mathcal{H}\},$$

$$\tau_{\mathbb{C}}^I(\mathcal{H}) = \bigcap\{1 - s \in I_0 \mid \mathbb{C}(\overline{1} - \mathcal{H}, s) = \overline{1} - \mathcal{H}\},$$

$$\tau_{\mathbb{C}}^F(\mathcal{H}) = \bigcap\{1 - s \in I_0 \mid \mathbb{C}(\overline{1} - \mathcal{H}, s) = \overline{1} - \mathcal{H}\},$$

Then,

(1) $\tau_{\mathbb{C}}^{TIF}$ *is an* **SVNTS** *on* $\mathcal{S}$*;*
(2) $\mathbb{C}_{\tau_{\mathbb{C}}^{TIF}}$ *is finer than* $\mathbb{C}$*.*

Proof. (SVNT1) Let $(\mathcal{S}, \mathbb{C})$ be an $\mathcal{SVNCS}$. Since $\mathbb{C}((0.1.1), r) = (0.1.1)$ and $\mathbb{C}(1,0,0), r) = (1,0,0)$ for every $s \in I_0$, (SVNT1).

(SVNT2) Let $(\mathcal{S}, \mathbb{C})$ be an $\mathcal{SVNCS}$. Suppose that there exists $\mathcal{H}_1, \mathcal{H}_2 \in I^{\mathcal{S}}$ such that

$$\tau_{\mathbb{C}}^T(\mathcal{H}_1 \cap \mathcal{H}_2) < \tau_{\mathbb{C}}^T(\mathcal{H}_1) \cap \tau_{\mathbb{C}}^T(\mathcal{H}_2), \quad \tau_{\mathbb{C}}^I(\mathcal{H}_1 \cap \mathcal{H}_2) > \tau_{\mathbb{C}}^I(\mathcal{H}_1) \cup \tau_{\mathbb{C}}^I(\mathcal{H}_2),$$

$$\tau_{\mathbb{C}}^F(\mathcal{H}_1 \cap \mathcal{H}_2) > \tau_{\mathbb{C}}^F(\mathcal{H}_1) \cup \tau_{\mathbb{C}}^F(\mathcal{H}_2).$$

There exists $s \in I_0$ such that

$$\tau_{\mathbb{C}}^T(\mathcal{H}_1 \cap \mathcal{H}_2) < s < \tau_{\mathbb{C}}^T(\mathcal{H}_1) \cap \tau_{\mathbb{C}}^T(\mathcal{H}_2), \qquad \tau_{\mathbb{C}}^I(\mathcal{H}_1 \cap \mathcal{H}_2) > 1 - s > \tau_{\mathbb{C}}^I(\mathcal{H}_1) \cup \tau_{\mathbb{C}}^I(\mathcal{H}_2),$$

$$\tau_{\mathbb{C}}^F(\mathcal{H}_1 \cap \mathcal{H}_2) > 1 - s > \tau_{\mathbb{C}}^F(\mathcal{H}_1) \cup \tau_{\mathbb{C}}^F(\mathcal{H}_2).$$

For each $i \in \{1,2\}$, there exists $s \in I_0$ with $\mathbb{C}(\mathcal{H}_i, s_i) = \overline{1} - \mathcal{H}_i$ such that

$$s < s_i \leq \tau_{\mathbb{C}}^T(\mathcal{H}_i), \quad \tau_{\mathbb{C}}^I(\mathcal{H}_i) \leq 1 - s_i < 1 - s, \quad \tau_{\mathbb{C}}^F(\mathcal{H}_i) \leq 1 - s_i < 1 - s.$$

In addition, since $(\overline{1} - \mathcal{H}_i, r) = \overline{1} - \mathcal{H}_i$ by $\mathbb{C}_2$ and $\mathbb{C}_4$ of Definition 9, for any $i \in \{1,2\}$,

$$\mathbb{C}((\overline{1} - \mathcal{H}_1) \cup (\overline{1} - \mathcal{H}_2), s) = (\overline{1} - \mathcal{H}_1) \cup (\overline{1} - \mathcal{H}_2).$$

It follows that $\tau_{\mathbb{C}}^T(\mathcal{H}_1 \cap \mathcal{H}_2) \geq s$, $\tau_{\mathbb{C}}^I(\mathcal{H}_1 \cap \mathcal{H}_2) \leq 1 - s$, and $\tau_{\mathbb{C}}^F(\mathcal{H}_1 \cap \mathcal{H}_2) \leq 1 - s$. It is a contradiction. Thus, for every $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$, $\tau_{\mathbb{C}}^T(\mathcal{H} \cap \mathcal{R}) \geq \tau_{\mathbb{C}}^T(\mathcal{H}) \cap \tau_{\mathbb{C}}^T(\mathcal{B})$, $\tau_{\mathbb{C}}^I(\mathcal{H} \cap \mathcal{R}) \leq \tau_{\mathbb{C}}^I(\mathcal{H}) \cup \tau_{\mathbb{C}}^I(\mathcal{R})$, and $\tau_{\mathbb{C}}^F(\mathcal{H} \cap \mathcal{R}) \leq \tau_{\mathbb{C}}^F(\mathcal{H}) \cup \tau_{\mathbb{C}}^F(\mathcal{R})$.

(SVNT3) Suppose that there exists $\mathcal{H} = \bigcup_{i \in I} \mathcal{H}_i \in I^{\mathcal{S}}$ such that

$$\tau_{\mathbb{C}}^T(\mathcal{H}) < \bigcup_{i \in I} \tau_{\mathbb{C}}^T(\mathcal{H}_i), \quad \tau_{\mathbb{C}}^I(\mathcal{H}) > \bigcup_{i \in I} \tau_{\mathbb{C}}^I(\mathcal{H}_i), \quad \tau_{\mathbb{C}}^F(\mathcal{H}) > \bigcup_{i \in I} \tau_{\mathbb{C}}^F(\mathcal{H}_i).$$

There exists $s_0 \in I_0$ such that

$$\tau_C^T(\mathcal{H}) < s_0 < \bigcup_{i \in I} \tau_{\mathbb{C}}^T(\mathcal{H}_i), \quad \tau_{\mathbb{C}}^I(\mathcal{H}) > 1 - s_0 > \bigcup_{i \in I} \tau_{\mathbb{C}}^I(\mathcal{H}_i), \quad \tau_{\mathbb{C}}^F(\mathcal{H}) > 1 - s_0 > \bigcup_{i \in I} \tau_{\mathbb{C}}^F(\mathcal{H}_i).$$

For every $i \in I$, there exists $\mathbb{C}(\mathcal{H}_i, s_i) = \overline{1} - \mathcal{H}_i$ and $s_i \in I_0$ such that

$$s_0 < s_i \leq \tau_{\mathbb{C}}^T(\mathcal{H}_i), \quad 1 - s_0 > 1 - s_i \geq \tau_{\mathbb{C}}^I(\mathcal{H}_i), \quad 1 - s_i > 1 - s_0 \geq \tau_{\mathbb{C}}^F(\mathcal{H}_i).$$

In addition, since $\mathbb{C}(\overline{1} - \mathcal{H}_i, r_0) \leq \mathbb{C}(\overline{1} - \mathcal{H}_i, s_i) = \overline{1} - \mathcal{H}_i$, by $\mathbb{C}_2$ of Definition 9,

$$\mathbb{C}(\overline{1} - \mathcal{H}_i, s_0) = \overline{1} - \mathcal{H}_i.$$

It implies, for all $i \in I$,

$$\mathbb{C}(\overline{1} - \mathcal{H}, s_0) \leq \mathbb{C}(\overline{1} - \mathcal{H}_i, s_0) = \overline{1} - \mathcal{H}_i.$$

It follows that

$$\mathbb{C}(\overline{1} - \mathcal{H}, r_0) \leq \bigcap_{i \in J} (\overline{1} - \mathcal{H}_i) = \overline{1} - \mathcal{H}.$$

Thus, $\mathbb{CI}(\overline{1}-\mathcal{H},s_0)=\overline{1}-\mathcal{H}$, that is, $\tau^T_{\mathbb{C}}(\mathcal{H})\geq s_0$, $\tau^I_{\mathbb{C}}(\mathcal{H})\leq 1-s_0$, and $\tau^F_{\mathbb{C}}(\mathcal{H})\leq 1-s_0$. It is a contradiction. Hence, $\tau^{TIF}_{\mathbb{C}}$ is an **SVNTS** on $\mathcal{S}$.

(2) Since $\mathcal{H}\leq\mathbb{C}(\mathcal{H},r)$,

$$\tau^T_{\mathbb{C}}(\overline{1}-\mathbb{C}(\mathcal{H},s))\geq s,\ \tau^I_{\mathbb{C}}(\overline{1}-\mathbb{C}(\mathcal{H},s))\leq 1-s,\ \tau^F_{\mathbb{C}}(\overline{1}-\mathbb{C}(\mathcal{H},s))\leq 1-s.$$

From $\mathbb{C}_5$ of Definition 9, we have $\mathbb{C}_{\tau^{TIF}_{\mathbb{C}}}(\mathcal{H},s)\leq\mathbb{C}(\mathcal{H},s)$. Thus, $\mathbb{C}_{\tau^{TIF}_{\mathbb{C}}}$ is finer than $\mathbb{C}$. □

Example 1. *Let $\mathcal{S}=\{a,b\}$. Define $\mathcal{B},\mathcal{H},\mathcal{A}\in I^{\mathcal{S}}$ as follows:*

$$\mathcal{B}=\langle(0.2,0.2),(0.3,0.3),(0.3,0.3)\rangle;\mathcal{H}=\langle(0.5,0.5),(0.1,0.1),(0.1,0.1)\rangle.$$

We define the mapping $\mathbb{C}:I^{\mathcal{S}}\times I_0\to I^{\mathcal{S}}$ as follows:

$$\mathbb{C}(\mathcal{A},s)=\begin{cases}(0.1.1), & \text{if}\quad \mathcal{A}=(0.1.1),\quad s\in I_0,\\ \mathcal{B}\cap\mathcal{H}, & \text{if}\quad 0\neq\mathcal{A}\leq\mathcal{B}\cap\mathcal{H},\ 0<r<\frac{1}{2},\\ \mathcal{B}, & \text{if}\quad \mathcal{A}\leq\mathcal{B},\mathcal{A}\not\leq\mathcal{H},\quad 0<r<\frac{1}{2},\\ & \text{or}\ 0\neq\mathcal{A}\leq\mathcal{B}\quad \frac{1}{2}<r<\frac{2}{3},\\ \mathcal{H}, & \text{if}\quad \mathcal{A}\leq\mathcal{H},\mathcal{A}\not\leq\mathcal{B},\ 0<r<\frac{1}{2},\\ \mathcal{B}\cup\mathcal{H}, & \text{if}\quad 0\neq\mathcal{A}\leq\mathcal{B}\cup\mathcal{H},\ 0<r<\frac{1}{2},\\ \overline{1}, & \text{otherwise}.\end{cases}$$

Then, $\mathbb{C}$ is a single-valued neutrosophic closure operator.
From Theorem 2, we have a single-valued neutrosophic topology $(\tau^T_{\mathbb{C}},\tau^I_{\mathbb{C}},\tau^F_{\mathbb{C}})$ on $\mathcal{S}$ as follows:

$$\tau^T_{\mathbb{C}}(\mathcal{A})=\begin{cases}1, & \text{if}\ \mathcal{A}=(1,0,0)\ \text{or}\ (0,1,1),\\ \frac{2}{3}, & \text{if}\ \mathcal{A}=\mathcal{B}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{H}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{B}^c\cup\mathcal{H}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{B}^c\cap\mathcal{H}^c,\\ 0, & \text{otherwise}.\end{cases}$$

$$\tau^I_{\mathbb{C}}(\mathcal{A})=\begin{cases}0, & \text{if}\ \mathcal{A}=(1,0,0)\ \text{or}\ (0,1,1),\\ \frac{1}{3}, & \text{if}\ \mathcal{A}=\mathcal{B}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{H}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{B}^c\cup\mathcal{H}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{B}^c\cap\mathcal{H}^c,\\ 1, & \text{otherwise}.\end{cases}$$

$$\tau^F_{\mathbb{C}}(\mathcal{A})=\begin{cases}0, & \text{if}\ \mathcal{A}=(1,0,0)\ \text{or}\ (0,1,1),\\ \frac{1}{3}, & \text{if}\ \mathcal{A}=\mathcal{B}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{H}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{B}^c\cup\mathcal{H}^c,\\ \frac{1}{2}, & \text{if}\ \mathcal{A}=\mathcal{B}^c\cap\mathcal{H}^c,\\ 1, & \text{otherwise}.\end{cases}$$

Thus, the $\tau^{TIF}_{\mathbb{C}}$ is a single-valued neutrosophic topology on $\mathcal{S}$.

Definition 10. *A single-valued neutrosophic ideal (**SVNI**) on $\mathcal{S}$ in Šostak's sense on a nonempty set $\mathcal{S}$ is a family $\mathcal{L}^T,\mathcal{L}^I,\mathcal{L}^F$ of single-valued neutrosophic sets in $\mathcal{S}$ satisfying the following axioms:*

(L_1) *$\mathcal{L}^T(\underline{0}) = 1$ and $\mathcal{L}^I(\underline{0}) = \mathcal{L}^F(\underline{0}) = 0$.*

(L_2) *If $\mathcal{H} \leq \mathcal{B}$, then $\mathcal{L}^T(\mathcal{R}) \leq \mathcal{L}^T(\mathcal{H})$, $\mathcal{L}^I(\mathcal{R}) \geq \mathcal{L}^I(\mathcal{H})$, and $\mathcal{L}^F(\mathcal{R}) \geq \mathcal{L}^F(\mathcal{H})$, for each single-valued neutrosophic set $\mathcal{R}, \mathcal{H}$ in $I^{\mathcal{S}}$.*

(L_3) *$\mathcal{L}^T(\mathcal{R} \cup \mathcal{H}) \geq \mathcal{L}^T(\mathcal{R}) \cap \mathcal{L}^T(\mathcal{H})$, $\mathcal{L}^I(\mathcal{R} \cup \mathcal{H}) \leq \mathcal{L}^I(\mathcal{R}) \cup \mathcal{L}^I(\mathcal{H})$, and $\mathcal{L}^F(\mathcal{R} \cup \mathcal{H}) \leq \mathcal{L}^F(\mathcal{R}) \cup \mathcal{L}^F(\mathcal{H})$, for each single-valued neutrosophic set $\mathcal{R}, \mathcal{H}$ in $I^{\mathcal{S}}$.*

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are **SVNI** *on $\mathcal{S}$, we say that $\mathcal{L}_1$ is finer than $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \leq \mathcal{L}_2$, iff $\mathcal{L}_1^T(\mathcal{H}) \leq \mathcal{L}_2^T(\mathcal{H})$, $\mathcal{L}_1^I(\mathcal{H}) \geq \mathcal{L}_2^I(\mathcal{A})$, and $\mathcal{L}_1^F(\mathcal{H}) \geq \mathcal{L}_2^F(\mathcal{H})$, for $\mathcal{H} \in I^{\mathcal{S}}$.*

*The triable $(X, (\tau^T, \tau^I, \tau^F), (\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$ is called a single-valued neutrosophic ideal topological space in Šostak sense (***SVNITS***, for short).*

We will occasionally write $\mathcal{L}^{TIF}$, $\mathcal{L}_i^{TIF}$, and $\mathcal{L}^{TIF} : I^X \to I$ for $(\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F)$, $(\mathcal{L}_i^T, \mathcal{L}_i^I, \mathcal{L}_i^F)$, and $\mathcal{L}^T, \mathcal{L}^I, \mathcal{L}^F : I^{\mathcal{S}} \to I$, respectively.

Remark 1. *The conditions (L_2) and (L_3), which are given in Definition 10, are equivalent to the following axioms: $\mathcal{L}^T(\mathcal{H} \cup \mathcal{R}) = \mathcal{L}^T(\mathcal{H}) \cap \mathcal{L}^T(\mathcal{R})$, $\mathcal{L}^I(\mathcal{H} \cup \mathcal{R}) \neq \mathcal{L}^I(\mathcal{H}) \cup \mathcal{L}^I(\mathcal{R})$, and $\mathcal{L}^F(\mathcal{H} \cup \mathcal{R}) \neq \mathcal{L}^F(\mathcal{H}) \cup \mathcal{L}^F(\mathcal{R})$, for every $\mathcal{R}, \mathcal{H} \in I^{\mathcal{S}}$.*

Example 2. *Let $\mathcal{S} = \{a, b\}$. Define the single-valued neutrosophic sets $\mathcal{R}, \mathcal{C}, \mathcal{H}, \mathcal{A}$ and $(\mathcal{L}^T, \mathcal{L}^T, \mathcal{L}^T) : I^{\mathcal{S}} \to I$ as follows:*

$$\mathcal{R} = \langle (0.3, 0.5), (0.4, 0.5), (0.5, 0.5) \rangle; \quad \mathcal{C} = \langle (0.3, 0.4), (0.5, 0.5), (0.3, 0.4) \rangle,$$

$$\mathcal{H} = \langle (0.1, 0.2), (0.5, 0.5), (0.5, 0.5) \rangle.$$

$$\mathcal{L}^T(\mathcal{A}) = \begin{cases} 1, & \text{if } \mathcal{B} = (0.1.1), \\ \frac{1}{2}, & \text{if } \mathcal{A} = \mathcal{R}, \\ \frac{2}{3}, & \text{if } (0.1.1) < \mathcal{A} < \mathcal{R}, \\ 0, & \text{otherwise}. \end{cases}$$

$$\mathcal{L}^I(\mathcal{A}) = \begin{cases} 0, & \text{if } \mathcal{A} = (0.1.1), \\ \frac{1}{2}, & \text{if } \mathcal{A} = \mathcal{C}, \\ \frac{1}{4}, & \text{if } (0.1.1) < \mathcal{A} < \mathcal{C}, \\ 1, & \text{otherwise}. \end{cases}$$

$$\mathcal{L}^T(\mathcal{B}) = \begin{cases} 0, & \text{if } \mathcal{A} = (0, 1, 1), \\ \frac{1}{2}, & \text{if } \mathcal{A} = \mathcal{H}, \\ \frac{1}{4}, & \text{if } (0.1.1) < \mathcal{A} < \mathcal{H}, \\ 1, & \text{otherwise}. \end{cases}$$

Then, $\mathcal{L}^{TIF}$ is an **SVNI** *on $\mathcal{S}$.*

Remark 2.

(i) If $\mathcal{L}^T(\underline{1}) = 1$, $\mathcal{L}^I(\underline{1}) = 0$, and $\mathcal{L}^F(\underline{1}) = 0$, then $\mathcal{L}^{TIF}$ is called a single-valued neutrosophic proper ideal.

(ii) If $\mathcal{L}^T(\underline{1}) = 0$, $\mathcal{L}^I(\underline{1}) = 1$, and $\mathcal{L}^F(\underline{1}) = 1$, then $\mathcal{L}^{TIF}$ is called a single-valued neutrosophic improper ideal.

Proposition 1. *Let $\{\mathcal{L}_i^{TIF}\}_{i \in J}$ be a family of* **SVNI** *on $\mathcal{S}$. Then, their intersection $\bigcap_{i \in J} \mathcal{L}_i^{TIF}$ is also* **SVNI***.*

Proof. Directly from Definition 7. □

Proposition 2. *Let $\{\mathcal{L}_i^{TIF}\}_{i\in J}$ be a family of* **SVNI** *on $\mathcal{S}$. Then, their union $\bigcup_{i\in J}\mathcal{L}_i^{TIF}$ is also an* **SVNI**.

Proof. Directly from Definition 7. □

4. Single-Valued Neutrosophic Ideal Open Local Function in Šostak Sense

In this section, we study the single-valued neutrosophic ideal open local function in Šostak's sense and present some of their properties. Additionally, properties preserved by single-valued neutrosophic ideal open compatible are examined.

Definition 11. *Let $s,t,p \in I_0$ and $s+t+p \leq 3$. A single-valued neutrosophic point $x_{s,t,r}$ of $\mathcal{S}$ is the single-valued neutrosophic set in $I^{\mathcal{S}}$ for each $\kappa \in \mathcal{H}$, defined by*

$$x_{s,t,p}(\kappa) = \begin{cases} (s,t,p), & \text{if } x = \kappa, \\ (0,1,1), & \text{if } x \neq \kappa. \end{cases}$$

A single-valued neutrosophic point $x_{s,t,p}$ is said to belong to a single-valued neutrosophic set $\mathcal{H} = \langle T_{\mathcal{H}}, I_{\mathcal{H}}, F_{\mathcal{H}}\rangle \in I^{\mathcal{S}}$, denoted by $x_{s,t,p} \in \mathcal{H}$ iff $s < T_{\mathcal{H}}$, $t \geq I_{\mathcal{H}}$ and $p \geq F_{\mathcal{H}}$. 1. We indicate the set of all single-valued neutrosophic points in $\mathcal{S}$ as **SVNP**$(\mathcal{S})$.

For every $x_{s,t,p} \in \mathbf{SVNP}(\mathcal{S})$ and $\mathcal{H} \in I^{\mathcal{S}}$ we shall write $x_{s,t,p}$ quasi-coincident with $\mathcal{H}$, denoted by $x_{s,t,p}q\mathcal{H}$, if

$$s + T_{\mathcal{H}}(\kappa) > 1, \qquad t + I_{\mathcal{H}}(\kappa) \leq 1, \qquad p + F_{\mathcal{H}}(\kappa) \leq 1.$$

For every $\mathcal{R},\mathcal{H} \in \mathcal{S}$ we shall write $\mathcal{H}\bar{q}\mathcal{R}$ to mean that $\mathcal{H}$ is quasi-coincident with $\mathcal{R}$ if there exists $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}}(\kappa) + T_{\mathcal{R}}(\kappa) > 1, \quad I_{\mathcal{H}}(\kappa) + I_{\mathcal{R}}(\kappa) \leq 1, \quad F_{\mathcal{H}}(\kappa) + F_{\mathcal{R}}(\kappa) \leq 1.$$

Definition 12. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS**. *For each $r \in I_0$, $\mathcal{H} \in I^{\mathcal{S}}$, $x_{s,t,p} \in \mathbf{SVNP}(\mathcal{S})$, a single-valued neutrosophic open $Q_{\tau^{TIF}}$-neighborhood of $x_{s,t,p}$ is defined as follows:*

$$Q_{\tau^{TIF}}(x_{s,t,p}, r) = \{\mathcal{H}|(x_{s,t,p})q\mathcal{H}, \qquad \tau^{T}(\mathcal{H}) \geq r, \qquad \tau^{I}(\mathcal{H}) \leq 1-r, \qquad \tau^{F}(\mathcal{H}) \leq 1-r\}.$$

Lemma 1. *A single-valued neutrosophic point $x_{s,t,p} \in \mathbb{C}_{\tau^{TIF}}(\mathcal{R}, r)$ iff every single-valued neutrosophic open $Q_{\tau^{TIF}}$-neighborhood of $x_{s,t,p}$ is quasi-coincident with $\mathcal{H}$.*

Definition 13. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS** *for each $\mathcal{H} \in I^{\mathcal{S}}$. Then, the single-valued neutrosophic ideal open local function $\mathcal{H}_r^{\star}(\tau^{TIF}, \mathcal{L}^{TIF})$ of $\mathcal{H}$ is the union of all single-valued neutrosophic points $x_{s,t,p}$ such that if $\mathcal{R} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and $\mathcal{L}^{T}(\mathcal{C}) \geq r$, $\mathcal{L}^{I}(\mathcal{C}) \leq 1-r$, $\mathcal{L}^{F}(\mathcal{C}) \leq 1-r$, then there is at least one $\kappa \in \mathcal{S}$ for which $T_{\mathcal{R}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{R}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{R}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$.*

Occasionally, we will write $\mathcal{H}_r^{\star}$ for $\mathcal{H}_r^{\star}(\tau^{TIF}, \mathcal{L}^{TIF})$ and it will have no ambiguity.

Example 3. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS**. *The simplest single-valued neutrosophic ideal on $\mathcal{S}$ is $\mathcal{L}_0^{TIF} : I^{\mathcal{S}} \to I$, where*

$$\mathcal{L}_0^{TIF}(\mathcal{R}) = \begin{cases} 1, & \text{if } \mathcal{R} = (1,0,0), \\ 0, & \text{otherwise}. \end{cases}$$

If we take $\mathcal{L}^{TIF} = \mathcal{L}_0^{TIF}$, for each $\mathcal{H} \in I^{\mathcal{S}}$ we have $\mathcal{H}_r^{\star} = \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$.

Theorem 3. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS** *and $\mathcal{L}_1^{TIF}, \mathcal{L}_2^{TIF} \in$* **SVNI**$(\mathcal{S})$*. Then, for any $\mathcal{H}, \mathcal{R} \in I^{\mathcal{S}}$ and $r \in I_0$, we have*

(1) If $\mathcal{H} \leq \mathcal{R}$, then $\mathcal{H}_r^\star \leq \mathcal{R}_r^\star$;
(2) If $\mathcal{L}_1^T \leq \mathcal{L}_2^T$, $\mathcal{L}_1^I \geq \mathcal{L}_2^I$ and $\mathcal{L}_1^F \geq \mathcal{L}_2^F$, then $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \geq \mathcal{H}_r^\star((\mathcal{L}_2^{TIF}, \tau^{TIF})$;
(3) $\mathcal{H}_r^\star = C_{\tau^{TIF}}(\mathcal{A}_r^\star, r) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$;
(4) $(\mathcal{H}_r^\star)_r^\star \leq \mathcal{H}_r^\star$;
(5) $(\mathcal{H}_r^\star \vee \mathcal{R}_r^\star) = (\mathcal{H} \vee \mathcal{R})_r^\star$;
(6) If $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{R}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{R}) \leq 1-r$ then $(\mathcal{H} \vee \mathcal{R})_r^\star = \mathcal{A}_r^\star \vee \mathcal{R}_r^\star = \mathcal{H}_r^\star$;
(7) If $\tau^T(\mathcal{R}) \geq r$, $\tau^I(\mathcal{R}) \leq 1-r$, and $\tau^F(\mathcal{R}) \leq 1-r$, then $(\mathcal{R} \wedge \mathcal{H}_r^\star) \leq (\mathcal{R} \wedge \mathcal{H})_r^\star$;
(8) $(\mathcal{H}_r^\star \wedge \mathcal{R}_r^\star) \geq (\mathcal{H} \wedge \mathcal{R})_r^\star$.

Proof. (1) Suppose that $\mathcal{H} \in I^{\mathcal{S}}$ and $\mathcal{H}_r^\star \not\leq \mathcal{R}_r^\star$. Then, there exists $\kappa \in \mathcal{S}$ and $s, t, p \in I_0$ such that

$$T_{\mathcal{H}_r^\star}(\kappa) \geq s > T_{\mathcal{R}_r^\star}(\kappa), \quad I_{\mathcal{H}_r^\star}(\kappa) < t \leq I_{\mathcal{R}_r^\star}(\kappa), \quad F_{\mathcal{H}_r^\star}(\kappa) < p \leq F_{\mathcal{R}_r^\star}(\kappa). \tag{1}$$

Since $T_{\mathcal{R}_r^\star}(\kappa) < s$, $I_{\mathcal{R}_r^\star}(\kappa) \geq t$, and $F_{\mathcal{R}_r^\star}(\kappa) \geq p$. Then, there exists $\mathcal{D} \in Q_{(\tau^{TIF})}(x_{s,t,p}, r)$, $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$ such that for any $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{R}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{R}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{R}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Since $\mathcal{H} \leq \mathcal{R}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

So, $T_{\mathcal{H}_r^\star}(\kappa) < s$, $I_{\mathcal{H}_r^\star}(\kappa) \geq t$, and $F_{\mathcal{H}_r^\star}(\kappa) \geq p$ and we arrive at a contradiction for Equation (1). Hence, $\mathcal{H}_r^\star \leq \mathcal{R}_r^\star$.

(2) Suppose $\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF}) \not\geq \mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) < s \leq T_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa),$$

$$I_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq t > I_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa), \tag{2}$$

$$F_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq p > F_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa).$$

Since $T_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq t$, and $F_{\mathcal{H}_r^\star(\mathcal{L}_1^{TIF}, \tau^{TIF})}(\kappa) \geq p$, $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ with $\mathcal{L}_1^T(\mathcal{C}) \geq r$, $\mathcal{L}_1^I(\mathcal{C}) \leq 1-r$ and $\mathcal{L}_1^F(\mathcal{C}) \leq 1-r$. Thus, for every $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Since $\mathcal{L}_2^T(\mathcal{C}) \geq \mathcal{L}_1^T(\mathcal{C})) \geq r$, $\mathcal{L}_2^I(\mathcal{C}) \leq \mathcal{L}_1^I(\mathcal{C})) \leq 1-r$, and $\mathcal{L}_2^F(\mathcal{C}) \leq \mathcal{L}_1^F(\mathcal{C})) \leq 1-r$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}(\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Thus, $T_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa) < s$, $I_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa) \geq t$, and $F_{\mathcal{H}_r^\star(\mathcal{L}_2^{TIF}, \tau^{TIF})}(\kappa) \geq p$. This is a contradiction for Equation (2). Hence, $\mathcal{H}_r^\star((\mathcal{L}_1^{TIF}, \tau^{TIF})) \geq \mathcal{H}_r^\star((\mathcal{L}_2^{TIF}, \tau^{TIF}))$.

(3)($\Rightarrow$) Suppose $\mathcal{H}_r^\star \not\leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^\star}(\kappa) \geq s > T_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H},r)}(\kappa), \quad I_{\mathcal{H}_r^\star}(\kappa) < t \leq I_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H},r)}(\kappa), \quad F_{\mathcal{H}_r^\star}(\kappa) < p \leq F_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H},r)}(\kappa). \tag{3}$$

Since $T_{\mathcal{H}_r^\star}(\kappa) \geq s$, $I_{\mathcal{H}_r^\star}(\kappa) < t$ and $F_{\mathcal{H}_r^\star}(\kappa) < p$, $x_{s,t,p} \in \mathcal{H}_r^\star$. So there is at least one $\kappa_1 \in \mathcal{S}$ for every $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ with $\mathcal{L}_1^T(\mathcal{C}) \geq r$, $\mathcal{L}_1^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}_1^F(\mathcal{C}) \leq 1-r$ such that

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) > T_{\mathcal{C}}(\kappa_1) + 1, \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) \leq I_{\mathcal{C}}(\kappa_1) + 1, \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) \leq F_{\mathcal{C}}(\kappa_1) + 1.$$

Therefore, by Lemma 1, $x_{s,t,p} \in \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$ which is a contradiction for Equation (3). Hence, $\mathcal{H}_r^\star \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$.

($\Leftarrow$) Suppose $\mathcal{H}_r^\star \not\geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}_r^\star}(\kappa) < s \leq T_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa), \quad I_{\mathcal{H}_r^\star}(\kappa) \geq t > I_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa), \quad F_{\mathcal{H}_r^\star}(\kappa) \geq p > F_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa). \tag{4}$$

Since $T_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa) \geq t$, $I_{\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)}(\kappa) < s$, $\mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)(\kappa) < p$ we have $x_{s,t,p} \in \mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)$. So, there is at least one $\kappa_1 \in \mathcal{S}$ with $\mathcal{R} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that

$$T_{\mathcal{R}}(\kappa_1) + T_{\mathcal{H}_r^\star}(\kappa_1) > 1, \quad I_{\mathcal{R}}(\kappa_1) + I_{\mathcal{H}_r^\star}(\kappa_1) \leq 1, \quad F_{\mathcal{R}}(\kappa_1) + F_{\mathcal{H}_r^\star}(\kappa_1) \leq 1.$$

Therefore, $\mathcal{H}_r^\star(\kappa_1) \neq 0$. Let $s_1 = T_{\mathcal{H}_r^\star}(\kappa_1)$, $t_1 = I_{\mathcal{H}_r^\star}(\kappa_1)$, and $p_1 = F_{\mathcal{H}_r^\star}(\kappa_1)$. Then, $(\kappa_1)_{s_1,t_1,p_1} \in \mathcal{H}_r^*$ and $s_1 + T_{\mathcal{R}}(\kappa_1) > 1$, $t_1 + I_{\mathcal{R}}(\kappa_1) \leq 1$, and $p_1 + F_{\mathcal{R}}(\kappa_1) \leq 1$ so that $\mathcal{R} \in Q_{\tau^{TIF}}((\kappa_1)_{s_1,t_1,p_1}, r)$. Now, $(\kappa_1)_{s_1,t_1,p_1} \in \mathcal{H}_r^\star$ implies there is at least one $\kappa' \in \mathcal{S}$ such that $T_{\mathcal{D}}(\kappa') + T_{\mathcal{H}}(\kappa') - 1 > T_{\mathcal{C}}(\kappa')$, $I_{\mathcal{D}}(\kappa') + I_{\mathcal{H}}(\kappa') - 1 \leq I_{\mathcal{C}}(\kappa')$, and $F_{\mathcal{D}}(\kappa') + F_{\mathcal{H}}(\kappa') - 1 \leq F_{\mathcal{C}}(\kappa')$, for all $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}) \leq 1-r$, and $\mathcal{D} \in Q_{\tau^{TIF}}((\kappa_1)_{s_1,t_1,p_1}, r)$. That is also true for $\mathcal{R}$. So there is at least one $\kappa'' \in \mathcal{S}$ such that $T_{\mathcal{R}}(\kappa'') + T_{\mathcal{H}}(\kappa'') - 1 > T_{\mathcal{C}}(\kappa'')$, $I_{\mathcal{R}}(\kappa'') + I_{\mathcal{H}}(\kappa'') - 1 \leq I_{\mathcal{C}}(\kappa'')$, and $F_{\mathcal{R}}(x'') + F_{\mathcal{H}}(\kappa'') - 1 \leq F_{\mathcal{C}}(\kappa'')$. Since $\mathcal{R} \in Q_{\tau^{TIF}}(\kappa_{s,t,p}, r)$ and $\mathcal{R}$ is arbitrary; then $T_{\mathcal{H}_r^\star}(\kappa) > s$, $I_{\mathcal{H}_r^\star}(\kappa) \leq t$ and $T_{\mathcal{H}_r^\star}(\kappa) \leq p$. It is a contradiction for (4). Thus, $\mathcal{H}_r^\star \geq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}_r^\star, r)$.

(4) ($\Rightarrow$) Can be easily established using standard technique.

(5) ($\Rightarrow$) Since $\mathcal{H}, \mathcal{R} \leq \mathcal{H} \cup \mathcal{R}$. By (1), $\mathcal{H}_r^\star \leq (\mathcal{H} \cup \mathcal{R})_r^\star$ and $\mathcal{R}_r^\star \leq (\mathcal{H} \cup \mathcal{R})_r^\star$. Hence, $\mathcal{H}_r^\star \cup \mathcal{B}_r^\star \leq (\mathcal{H} \cup \mathcal{R})_r^\star$.

($\Leftarrow$) Suppose $(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star) \not\geq (\mathcal{H} \cup \mathcal{R})_r^\star$. Then, there exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$T_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) < s \leq T_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa), \ I_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq t > I_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa), \ F_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq p > F_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa). \tag{5}$$

Since $T_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) < s$, $I_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq t$, and $F_{(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star)}(\kappa) \geq p$, we have $T_{\mathcal{H}_r^\star}(\kappa) < s$, $I_{\mathcal{H}_r^\star}(\kappa) \geq t$, $F_{\mathcal{H}_r^\star}(\kappa) \geq p$ or $T_{\mathcal{R}_r^\star}(\kappa) < t$, $I_{\mathcal{R}_r^\star}(\kappa) \geq t$, $F_{\mathcal{R}_r^\star}(\kappa) \geq t$. So, there exists $\mathcal{D}_1 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa_1 \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}_1) \geq r$, $\mathcal{L}^I(\mathcal{C}_1) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}_1) \leq 1-r$, we have

$$T_{\mathcal{D}_1}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_1}(\kappa_1), \quad I_{\mathcal{D}_1}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}_1}(\kappa_1), \quad F_{\mathcal{D}_1}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}_1}(\kappa_1).$$

Similarly, there exists $\mathcal{D}_2 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa_1 \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}_2) \geq r$, $\mathcal{L}^I(\mathcal{C}_2) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}_2) \leq 1-r$, we have

$$T_{\mathcal{D}_2}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_2}(\kappa_1), \quad I_{\mathcal{D}_2}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}_2}(\kappa_1), \quad F_{\mathcal{D}_2}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}_2}(\kappa_1).$$

Since $\mathcal{D} = \mathcal{D}_1 \wedge \mathcal{D}_2 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and by (L_3), $\mathcal{L}^T(\mathcal{C}_1 \cup \mathcal{C}_2) \geq \mathcal{L}^T(\mathcal{C}_1) \cap \mathcal{L}^T(\mathcal{C}_2) \geq r$, $\mathcal{L}^I(\mathcal{C}_1 \cup \mathcal{C}_2) \leq \mathcal{L}^I(\mathcal{C}_1) \cup \mathcal{L}^I(\mathcal{C}_2) \leq 1-r$, and $\mathcal{L}^F(\mathcal{C}_1 \cup \mathcal{C}_2) \leq \mathcal{L}^T(\mathcal{C}_1) \cup \mathcal{L}^T(\mathcal{C}_2) \leq 1-r$. Thus, for every $\kappa_1 \in \mathcal{S}$,

$$\begin{aligned} T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{R} \cup \mathcal{H}}(\kappa_1) - 1 &\leq T_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1), \\ I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{R} \cup \mathcal{H}}(\kappa_1) - 1 &\geq I_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1), \\ F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{R} \cup \mathcal{H}}(\kappa_1) &\geq F_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1). \end{aligned}$$

Therefore, $T_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa) < s$, $I_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa) \geq t$, and $F_{(\mathcal{H} \cup \mathcal{R})_r^\star}(\kappa) \geq p$. So, we arrive at a contradiction for (5). Hence, $(\mathcal{H}_r^\star \cup \mathcal{R}_r^\star) \geq (\mathcal{H} \cup \mathcal{R})_r^\star$.

(6), (7), and (8) can be easily established using the standard technique. □

Example 4. *Let $\mathcal{S} = \{a, b\}$. Define $\mathcal{R}, \mathcal{C}, \mathcal{H} \in \mathcal{S}$ as follows:*

$\mathcal{R}_1 = \langle (0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle$; $\mathcal{R}_2 = \langle (0.4, 0.4, 0.4), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1) \rangle$;

$\mathcal{R}_3 = \langle (0.3, 0.3, 0.3), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1)$; $\mathcal{C}_1 = \langle (0.3, 0.3, 0.3), (0.3, 0.3, 0.3), (0.1, 0.1, 0.1) \rangle$;

$\mathcal{C}_2 = \langle (0.2, 0.2, 0.2), (0.2, 0.2, 0.2), (0.1, 0.1, 0.1)$; $\mathcal{C}_3 = \langle (0.1, 0.1, 0.1), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1) \rangle$.

Define $\tau^{TIF}, \mathcal{L}^{TIF} : I^X \to I$ as follows:

$$\tau^T(\mathcal{H}) = \begin{cases} 1, & \text{if } \mathcal{H} = (0,1,1), \\ 1, & \text{if } \mathcal{H} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{R}_1; \end{cases} \qquad \mathcal{L}^T(\mathcal{H}) = \begin{cases} 1, & \text{if } \mathcal{H} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{C}_1, \\ \frac{2}{3}, & \text{if } \underline{0} < \mathcal{H} < \mathcal{C}_1; \end{cases}$$

$$\tau^I(\mathcal{H}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ 0, & \text{if } \mathcal{H} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{R}_2; \end{cases} \qquad \mathcal{L}^I(\mathcal{R}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{C}_2, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{H} < \mathcal{C}_2; \end{cases}$$

$$\tau^F(\mathcal{H}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ 0, & \text{if } \mathcal{H} = (1,0,0), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{R}_3; \end{cases} \qquad \mathcal{L}^F(\mathcal{H}) = \begin{cases} 0, & \text{if } \mathcal{H} = (0,1,1), \\ \frac{1}{2}, & \text{if } \mathcal{H} = \mathcal{C}_3, \\ \frac{1}{4}, & \text{if } \underline{0} < \mathcal{H} < \mathcal{C}_3. \end{cases}$$

Let $\mathcal{G} = \langle (0.4, 0.4, 0.4), (0.4, 0.4, 0.4), (0.4, 0.4, 0.4) \rangle$. Then, $\mathcal{G}^\star_{\frac{1}{2}} = \mathcal{R}_1$.

Theorem 4. *Let $\{\mathcal{H}_i\}_{i \in J} \subset I^{\mathcal{S}}$ be a family of single-valued neutrosophic sets on $\mathcal{S}$ and $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS***. Then,*

(1) $(\bigcup (\mathcal{H}_i)^\star_r : i \in J) \le (\bigcup \mathcal{H}_i : i \in J)^\star_r$;
(2) $(\bigcap (\mathcal{H}_i)^\star_r : i \in J) \ge (\bigcap \mathcal{H}_i : i \in J)^\star_r$.

Proof. (1) Since $\mathcal{H}_i \le \bigcup \mathcal{H}_i$ for all $i \in J$, and by Theorem 3 (1), we obtain $(\bigcup (\mathcal{H}_i)^\star_r, i \in J) \le (\bigcup \mathcal{H}_i, i \in J)^\star_r$. Then, (1) holds.

(2) Easy, so omitted. □

Remark 3. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS** *and $\mathcal{H} \in I^{\mathcal{S}}$, we can define*

$$\mathbb{C}^\star_{\tau^{TIF}}(\mathcal{H}, r) = \mathcal{H} \cup \mathcal{H}^\star_r, \qquad int^\star_{\tau^{TIF}}(\mathcal{H}, r) = \mathcal{H} \wedge [\underline{1} - (\underline{1} - \mathcal{H})^\star_r].$$

It is clear, $\mathbb{C}^\star_{\tau^{TIF}}$ is a single-valued neutrosophic closure operator and $(\tau^{T\star}(\mathcal{L}^T), \tau^{I\star}(\mathcal{L}^I), \tau^{F\star}(\mathcal{L}^F)$ is the single-valued neutrosophic topology generated by $\mathbb{C}^\star_{\tau^{TIF}}$, i.e.,

$$\tau^\star(\mathcal{I})(\mathcal{H}) = \bigcup \{r | \ \mathbb{C}^\star_{\tau^{TIF}}(\underline{1} - \mathcal{H}, r) = \underline{1} - \mathcal{H}\}.$$

Now, if $\mathcal{L}^{TIF} = \mathcal{L}^{TIF}_0$, then, $\mathbb{C}^\star_{\tau^{TIF}}(\mathcal{H}, r) = \mathcal{H}^*_r \cup \mathcal{H} = \mathbb{C}^\star_{\tau^{TIF}}(\mathcal{H}, r) \cup \mathcal{H} = \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$, for $\mathcal{H} \in I^{\mathcal{S}}$. So, $\tau^{TIF\star}(\mathcal{L}^{TIF}_0) = \tau^{TIF}$.

Proposition 3. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS***, $r \in I_0$, and $\mathcal{H} \in I^{\mathcal{S}}$. Then,*

(1) $\mathbb{C}^\star_{\tau^{TIF}}(\underline{1}, r) = \underline{1}$;

(2) $\mathbb{C}^{\star}_{\tau^{TIF}}(\underline{0}, r) = \underline{0}$;
(3) $int^{\star}_{\tau^{TIF}}(\mathcal{H} \cup \mathcal{R}, r) \leq int^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \cup int^{\star}_{\tau^{TIF}}(\mathcal{R}, r)$;
(4) $int^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \leq \mathcal{H} \leq \mathbb{C}^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \leq \mathbb{C}_{\tau^{TIF}}(\mathcal{H}, r)$;
(5) $\mathbb{C}^{\star}_{\tau^{TIF}}(\underline{1} - \mathcal{H}, r) = \underline{1} - int^{\star}_{\tau^{TIF}}(\mathcal{H}, r)$ and $\underline{1} - \mathbb{C}^{\star}_{\tau^{TIF}}\star(\mathcal{H}, r) = int^{\star}_{\tau^{TIF}}(\underline{1} - \mathcal{H}, r)$;
(6) $int^{\star}_{\tau^{TIF}}(\mathcal{H} \cap \mathcal{R}, r) = int^{\star}_{\tau^{TIF}}(\mathcal{H}, r) \cap int^{\star}_{\tau^{TIF}}(\mathcal{R}, r)$.

Proof. Follows directly from definitions of $\mathbb{C}^{\star}_{\tau^{TIF}}$, $int^{\star}_{\tau^{TIF}}$, $\mathbb{C}_{\tau^{TIF}}$, and Theorem 3 (5). □

Theorem 5. *Let $(\mathcal{S}, \tau_1^{TIF}, \mathcal{L}^{TIF})$ and $(\mathcal{S}, \tau_2^{TIF}, \mathcal{L}^{TIF})$ be* **SVNTS′s** *and $\tau_1^{TIF} \leq \tau_2^{TIF}$. Then, $\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF}) \leq \mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})$.*

Proof. Suppose $\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF}) \not\leq \mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})$. Then, there exists $s, t, p \in I_0$, $\kappa \in \mathcal{S}$ such that

$$T_{\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq s > T_{\mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa),$$

$$I_{\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) < t \leq I_{\mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa), \tag{6}$$

$$F_{\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) < t \leq F_{\mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa).$$

Since $T_{\mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa) < s$, $I_{\mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq t$, $F_{\mathcal{H}^{\star}_r(\tau_1^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq p$, there exists $\mathcal{D} \in Q_{\tau_1^{TIF}}(x_{s,t,p}, r)$ with $\mathcal{L}^T(\mathcal{C}_1) \geq r$, $\mathcal{L}^I(\mathcal{C}_1) \leq 1 - r$ and $\mathcal{L}^F(\mathcal{C}_1) \leq 1 - r$, such that for any $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}}\kappa_1), \quad I_{\mathcal{D}}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 > I_{\mathcal{C}}(\kappa_1), \quad F_{\mathcal{D}}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 > F_{\mathcal{C}}(\kappa_1).$$

Since $\tau_1^{TIF} \leq \tau_2^{TIF}$, $\mathcal{D} \in Q_{\tau_2^{TIF}}(x_{s,t,p}, r)$. Thus, $T_{\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) < s$, $I_{\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF}))}(\kappa) \geq t$, $F_{\mathcal{H}^{\star}_r(\tau_2^{TIF}, \mathcal{L}^{TIF})}(\kappa) \geq p$. It is a contradiction for Equation (6). □

Theorem 6. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}_1^{TIF})$ and $(\mathcal{S}, \tau^{TIF}, \mathcal{L}_2^{TIF})$ be* **SVNTS′s** *and $\mathcal{L}_1^{TIF} \leq \mathcal{L}_2^{TIF}$. Then, $\mathcal{H}^{\star}_r(\mathcal{L}_1^{TIF}, \tau^{TIF}) \geq \mathcal{H}^{\star}_r(\mathcal{L}_2^{TIF}, \tau^{TIF})$.*

Proof. Clear. □

Definition 14. *Let Θ be a subset of $I^{\mathcal{S}}$, and $\underline{0} \notin \Theta$. A mapping $\beta^T, \beta^I, \beta^F : \Theta \to I$ is called a single-valued neutrosophic base on $\mathcal{S}$ if it satisfies the following conditions:*

(1) $\beta^T(\underline{1}) = 1$ *and* $\beta^I(\underline{1}) = \beta^F(\underline{1}) = 0$;
(2) *For all* $\mathcal{H}, \mathcal{R} \in \Theta$,

$$\beta^T(\mathcal{H} \cap \mathcal{R}) \geq \beta^T(\mathcal{H}) \cap \beta^T(\mathcal{R}), \quad \beta^I(\mathcal{H} \cap \mathcal{R}) \leq \beta^I(\mathcal{H}) \cup \beta^I(\mathcal{R}), \quad \beta^F(\mathcal{H} \cap \mathcal{R}) \leq \beta^F(\mathcal{H}) \cup \beta^F(\mathcal{R}).$$

Theorem 7. *Define a mapping $\beta : \Theta \to I$ on $\mathcal{S}$ by*

$$\beta^I(\mathcal{H}) = \bigcup \{\tau^T(\mathcal{R}) \cap \mathcal{I}^T(\mathcal{C}) |\ \mathcal{H} = \mathcal{R} \cap (\underline{1} - \mathcal{C})\},$$

$$\beta^I(\mathcal{H}) = \bigcap \{\tau^I(\mathcal{R}) \cup \mathcal{I}^I(\mathcal{C}) |\ \mathcal{H} = \mathcal{R} \cap (\underline{1} - \mathcal{C})\},$$

$$\beta^F(\mathcal{H}) = \bigcap \{\tau^F(\mathcal{R}) \cup \mathcal{I}^F(\mathcal{C}) |\ \mathcal{H} = \mathcal{R} \cap (\underline{1} - \mathcal{C})\}.$$

Then, β^{TIF} is a base for the single-valued neutrosophic topology $\tau^{TIF\star}$.

Proof.

(1) Since $\mathcal{L}^T(\underline{0}) = 1$ and $\mathcal{L}^I(\underline{0}) = \mathcal{L}^F(\underline{0}) = 0$, we have $\beta^T(\underline{1}) = 1$ and $\beta^I(\underline{1}) = \beta^F(\underline{1}) = 0$;

(2) Suppose that there exists $\mathcal{H}_1, \mathcal{H}_2 \in \Theta$ such that

$$\begin{aligned}\beta^T(\mathcal{H}_1 \cap \mathcal{H}_2) &\ngeq \beta^T(\mathcal{H}_1) \cap \beta^T(\mathcal{H}_2),\\ \beta^I(\mathcal{H}_1 \cap \mathcal{H}_2) &\nleq \beta^I(\mathcal{H}_1) \cup \beta^I(\mathcal{H}_2),\\ \beta^F(\mathcal{H}_1 \cap \mathcal{H}_2) &\nleq \beta^F(\mathcal{H}_1) \cup \beta^F(\mathcal{H}_2).\end{aligned}$$

There exists $s, t, p \in I_0$ and $\kappa \in \mathcal{S}$ such that

$$\beta^T(\mathcal{H}_1 \cap \mathcal{H}_2)(\kappa) < s \leq \beta^T(\mathcal{H}_1)(x) \cap \beta^T(\mathcal{H}_2)(\kappa),$$

$$\beta^I(\mathcal{H}_1 \cap \mathcal{H}_2)(\kappa) \geq t > \beta^I(\mathcal{H}_1)(\kappa) \cap \beta^I(\mathcal{H}_2)(\kappa), \tag{7}$$

$$\beta^F(\mathcal{H}_1 \cap \mathcal{H}_2)(\kappa) \geq p > \beta^F(\mathcal{H}_1)(\kappa) \cup \beta^F(\mathcal{H}_2)(\kappa).$$

Since $\beta^T(\mathcal{H}_1)(\kappa) \geq s$, $\beta^I(\mathcal{H}_1)(\kappa) < t$, $\beta^F(\mathcal{H}_1)(\kappa) < p$, and $\beta^T(\mathcal{H}_2)(\kappa) \geq s$, $\beta^I(\mathcal{H}_2)(\kappa) < t$, $\beta^F(\mathcal{H}_2)(\kappa) < p$, then there exists $\mathcal{R}_1, \mathcal{R}_1, \mathcal{C}_1, \mathcal{C}_2 \in \Theta$ with $\mathcal{H}_1 = \mathcal{R}_1 \cap (\underline{1} - \mathcal{C}_1)$ and $\mathcal{H}_2 = \mathcal{R}_2 \cap (\underline{1} - \mathcal{C}_2)$, such that $\beta^T(\mathcal{H}_1) \geq \tau^T(\mathcal{R}_1) \cap \mathcal{L}^T(\mathcal{C}_1) \geq s$, $\beta^I(\mathcal{H}_1) \leq \tau^I(\mathcal{R}_1) \cup \mathcal{L}^I(\mathcal{C}_1) < t$, $\beta^F(\mathcal{H}_1) \leq \tau^F(\mathcal{R}_1) \cup \mathcal{L}^F(\mathcal{C}_1) < p$, and $\beta^T(\mathcal{H}_2) \geq \tau^T(\mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_2) \geq s$, $\beta^I(\mathcal{H}_2) \leq \tau^I(\mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_2) < t$, $\beta^F(\mathcal{H}_2) \leq \tau^F(\mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_2) < p$. Therefore,

$$\begin{aligned}\mathcal{H}_1 \cap \mathcal{H}_2 &= (\mathcal{R}_1 \cap (\underline{1} - \mathcal{C}_1)) \cap (\mathcal{R}_2 \cap (\underline{1} - \mathcal{C}_2))\\ &= (\mathcal{R}_1 \cap \mathcal{R}_2) \cap ((\underline{1} - \mathcal{C}_1) \cap (\underline{1} - \mathcal{C}_2))\\ &= (\mathcal{R}_1 \cap \mathcal{R}_2) \cap (\underline{1} - (\mathcal{C}_1 \cup \mathcal{C}_2)).\end{aligned}$$

Hence, from Definition 14, we have

$$\begin{aligned}\beta^T(\mathcal{H}_1 \cap \mathcal{H}_2) &\geq \tau^T(\mathcal{R}_1 \cap \mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_1 \cup \mathcal{C}_2)\\ &\geq \tau^T(\mathcal{R}_1) \cap \tau^T(\mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_1) \cap \mathcal{L}^T(\mathcal{C}_2)\\ &= (\tau^T(\mathcal{R}_1) \cap \mathcal{L}^T(\mathcal{C}_1)) \cap (\tau^T(\mathcal{R}_2) \cap \mathcal{L}^T(\mathcal{C}_2)) \geq s,\end{aligned}$$

$$\begin{aligned}\beta^I(\mathcal{H}_1 \cap \mathcal{H}_2) &\leq \tau^I(\mathcal{R}_1 \cap \mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_1 \cup \mathcal{C}_2)\\ &\leq \tau^I(\mathcal{R}_1) \cup \tau^I(\mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_1) \cup \mathcal{L}^I(\mathcal{C}_2)\\ &= (\tau^I(\mathcal{R}_1) \cup \mathcal{L}^F(\mathcal{C}_1)) \cup (\tau^I(\mathcal{R}_2) \cup \mathcal{L}^I(\mathcal{C}_2)) < t,\end{aligned}$$

$$\begin{aligned}\beta^F(\mathcal{H}_1 \cap \mathcal{H}_2) &\leq \tau^F(\mathcal{R}_1 \cap \mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_1 \cup \mathcal{C}_2)\\ &\leq \tau^F(\mathcal{R}_1) \cup \tau^F(\mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_1) \cup \mathcal{L}^F(\mathcal{C}_2)\\ &= (\tau^F(\mathcal{R}_1) \cup \mathcal{L}^F(\mathcal{C}_1)) \cup (\tau^F(\mathcal{R}_2) \cup \mathcal{L}^F(\mathcal{C}_2)) < p.\end{aligned}$$

It is a contradiction for Equation (7). Thus,

$$\beta^T(\mathcal{H}_1 \cap \mathcal{H}_2) \geq \beta^T(\mathcal{H}_1) \cap \beta^T(\mathcal{H}_2), \beta^I(\mathcal{H}_1 \cap \mathcal{H}_2) \leq \beta^I(\mathcal{H}_1) \cup \beta^I(\mathcal{H}_2), \beta^F(\mathcal{H}_1 \cap \mathcal{H}_2) \leq \beta^F(\mathcal{H}_1) \cup \beta^F(\mathcal{H}_2).$$

□

Theorem 8. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS**, *and* $\mathcal{L}_1^{TIF}$ *and* $\mathcal{L}_1^{TIF}$ *be two single-valued neutrosophic ideals on* $\mathcal{S}$. *Then, for every* $r \in I_0$ *and* $\mathcal{H} \in I^{\mathcal{S}}$,

(1) $\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) = \mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^{\star}(\mathcal{L}_2^{TIF}, \tau^{TIF})$,
(2) $\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF} \cup \mathcal{L}_2^{TIF}, \tau) = \mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{T\star}(\mathcal{L}_2^{TIF})) \cap \mathcal{H}^{\star}(\mathcal{L}_2^{TIF}, \tau^{T\star}(\mathcal{L}_1^{TIF}))$.

Proof. (1) Suppose that $\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) \not\leq \mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^{\star}(\mathcal{L}_2^{TIF}, \tau^{TIF})$, there exists $\kappa \in \mathcal{S}$ and $s, t, p \in I_0$ such that

$$T_{\mathcal{H}_r^{\star}(\mathcal{L}_1^T \cap \mathcal{L}_2^T, \tau^T)}(\kappa) \geq s > T_{\mathcal{H}_r^{\star}(\mathcal{L}_1^T, \tau^T)}(\kappa) \cup T_{\mathcal{H}_r^{\star}(\mathcal{L}_2^T, \tau^T)}(\kappa), \quad (8)$$
$$I_{\mathcal{H}_r^{\star}(\mathcal{L}_1^I \cap \mathcal{L}_2^I, \tau^I)}(\kappa) < t \leq I_{\mathcal{H}_r^{\star}(\mathcal{L}_1^I, \tau^I)}(\kappa) \cup I_{\mathcal{H}_r^{\star}(\mathcal{L}_2^I, \tau^I)}(\kappa),$$

$$F_{\mathcal{H}_r^{\star}(\mathcal{L}_1^F \cap \mathcal{L}_2^F, \tau^F)}(\kappa) < p \leq F_{\mathcal{H}_r^{\star}(\mathcal{L}_1^F, \tau^F)}(\kappa) \cap F_{\mathcal{H}_r^{\star}(\mathcal{L}_2^F, \tau^F)}(\kappa).$$

Since $T_{\mathcal{H}_r^{\star}(\mathcal{L}_1^T, \tau^T)}(\kappa) \cup T_{\mathcal{H}_r^{\star}(\mathcal{L}_2^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\mathcal{L}_1^I, \tau^I)}(\kappa) \cap I_{\mathcal{H}_r^{\star}(\mathcal{L}_2^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\mathcal{L}_1^F, \tau^F)}(\kappa) \cap F_{\mathcal{H}_r^{\star}(\mathcal{L}_2^F, \tau^F)}(\kappa) \geq p$, we have, $T_{\mathcal{H}_r^{\star}(\mathcal{L}_1^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\mathcal{L}_1^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\mathcal{L}_1^F, \tau^F)}(\kappa) \geq p$, and $I_{\mathcal{H}_r^{\star}(\mathcal{L}_2^I, \tau^I)}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\mathcal{L}_2^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\mathcal{L}_2^F, \tau^F)}(\kappa) \geq p$.

Now, $T_{\mathcal{H}_r^{\star}(\mathcal{L}_1^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\mathcal{L}_1^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\mathcal{L}_1^F, \tau^F)}(\kappa) \geq p$ implies that there exists $\mathcal{D}_1 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and for some $\mathcal{L}_1^T(\mathcal{C}_1) \geq r$, $\mathcal{L}_1^I(\mathcal{C}_1) \leq 1 - r$ and $\mathcal{L}_1^F(\mathcal{C}_1) \leq 1 - r$ such that for every $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}_1}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_1}(\kappa_1), \quad I_{\mathcal{D}_1}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 \geq I_{\mathcal{C}_1}(\kappa_1), \quad F_{\mathcal{D}_1}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 \geq F_{\mathcal{C}_1}(\kappa_1).$$

Once again, $T_{\mathcal{H}_r^{\star}(\mathcal{L}_2^T, \tau^T)}(\kappa) < s$, $I_{\mathcal{H}_r^{\star}(\mathcal{L}_2^I, \tau^I)}(\kappa) \geq t$, $F_{\mathcal{H}_r^{\star}(\mathcal{L}_2^F, \tau^F)}(\kappa) \geq p$, implies there exists $\mathcal{D}_2 \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and for some $\mathcal{L}_2^T(\mathcal{C}_2) \geq r$, $\mathcal{L}_2^I(\mathcal{C}_2) \leq 1 - r$ and $\mathcal{L}_2^F(\mathcal{C}_2) \leq 1 - r$, such that for $\kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{D}_2}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_2}(\kappa_1), \quad I_{\mathcal{D}_2}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 \geq I_{\mathcal{C}_2}(\kappa), \quad F_{\mathcal{D}_2}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 \geq F_{\mathcal{C}_2}(\kappa_1),$$

Therefore, for every $\kappa_1 \in \mathcal{S}$, we have

$$T_{\mathcal{D}_1 \cap \mathcal{D}_2}(\kappa_1) + T_{\mathcal{H}}(\kappa_1) - 1 \leq T_{\mathcal{C}_1 \cap \mathcal{C}_2}(\kappa_1), \quad I_{\mathcal{D}_1 \cup \mathcal{D}_2}(\kappa_1) + I_{\mathcal{H}}(\kappa_1) - 1 \geq I_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1),$$

$$F_{\mathcal{D}_1 \cup \mathcal{D}_2}(\kappa_1) + F_{\mathcal{H}}(\kappa_1) - 1 \geq F_{\mathcal{C}_1 \cup \mathcal{C}_2}(\kappa_1).$$

Since $(\mathcal{D}_1 \wedge \mathcal{D}_2) \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and $(\mathcal{L}_1^T \cap \mathcal{L}_2^T)(\mathcal{C}_1 \cap \mathcal{C}_2) \geq r$, $(\mathcal{L}_1^I \cap \mathcal{L}_2^I)(\mathcal{C}_1 \cup \mathcal{C}_2) \leq 1 - r$, and $(\mathcal{L}_1^F \cap \mathcal{L}_2^F)(\mathcal{C}_1 \cup \mathcal{C}_2) \geq 1 - r$ we have $T_{\mathcal{H}_r^{\star}(\mathcal{L}_1^T \cap \mathcal{L}_2^T, \tau^T)}(\kappa) \leq s$, $I_{\mathcal{H}_r^{\star}(\mathcal{L}_1^I \cap \mathcal{L}_2^I, \tau^I)}(\kappa) > t$, and $F_{\mathcal{H}_r^{\star}(\mathcal{L}_1^F \cap \mathcal{L}_2^F, \tau^F)}(\kappa) > t$ and this is a contradiction for Equation (8). So that

$$\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) \leq \mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^{\star}(\mathcal{L}_2^{TIF}, \tau^{TIF}).$$

On the opposite direction, $\mathcal{L}_1^{TIF} \geq \mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}$ and $\mathcal{L}_2^{TIF} \geq \mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}$, so by Theorem 3 (2),

$$\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^T) \geq \mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^{\star}(\mathcal{L}_2^{TIF}, \tau^{TIF}).$$

Then,

$$\mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF} \cap \mathcal{L}_2^{TIF}, \tau^{TIF}) = \mathcal{H}_r^{\star}(\mathcal{L}_1^{TIF}, \tau^{TIF}) \cup \mathcal{H}_r^{\star}(\mathcal{L}_2^{TIF}, \tau^{TIF}).$$

(2) Straightforward. □

The above theorem results in an important consequence. $\tau^{TIF\star}(\mathcal{L}^{TIF})$ and $[\tau^{TIF\star}(\mathcal{L}^{TIF})]^\star(\mathcal{L}^{TIF})$ (in short $\tau^{\star\star}$) are equal for any single-valued neutrosophic ideal on $\mathcal{S}$.

Corollary 1. *Let $(\mathcal{S}, \tau^{TIF}, \mathcal{L}^{TIF})$ be an* **SVNITS**. *For every $r \in I_0$ and $\mathcal{H} \in I^X$, $\mathcal{H}^\star_r(\mathcal{L}^{TIF}) = \mathcal{H}^\star_r(\mathcal{L}^{TIF}, \tau^{TIF\star})$ and $\tau^{TIF\star}(\mathcal{L}^{TIF}) = \tau^{TIF\star\star}$.*

Proof. Putting $\mathcal{L}^{TIF}_1 = \mathcal{L}^{TIF}_2$ in Theorem 8 (2), we have the required result. □

Corollary 2. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS**, *and $\mathcal{L}^{TIF}_1$ and $\mathcal{L}^{TIF}_1$ be two single-valued neutrosophic ideals on $\mathcal{S}$. Then, for any $\mathcal{H} \in I^{\mathcal{S}}$ and $r \in I_0$,*

(1) $\tau^{T\star}(\mathcal{L}^{TIF}_1 \cup \mathcal{I}^{TIF}_2) = (\tau^{TIF\star}(\mathcal{L}^{TIF}_2))^\star(\mathcal{L}^T_1) = (\tau^{TIF\star}(\mathcal{L}^{TIF}_1))^\star(\mathcal{L}^T_2)$,
(2) $\tau^{T\star}(\mathcal{L}^{TIF}_1 \cap \mathcal{L}^{TIF}_2) = \tau^{TIF\star}(\mathcal{L}^{TIF}_1) \cap \tau^{T\star}(\mathcal{L}^{TIF}_2)$.

Proof. Straightforward. □

Definition 15. *For an* **SVNTS** *$(\mathcal{S}, \tau^{TIF})$ with a single-valued neutrosophic ideal $\mathcal{I}^{TIF}$, τ^{TIF} is said to be single-valued neutrosophic ideal open compatible with $\mathcal{I}^{TIF}$, denoted by $\tau^{TIF} \sim \mathcal{L}^{TIF}$, if for each $\mathcal{H}, \mathcal{C} \in I^{\mathcal{S}}$ and $x_{s,t,p} \in \mathcal{H}$ with $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$, there exists $\mathcal{D} \in Q_{\tau^{TIF}}(x_t, r)$ such that $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ holds for any $\kappa \in \mathcal{S}$, then $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$ and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$.*

Definition 16. *Let $\{\mathcal{R}_j\}_{j\in J}$ be an indexed family of a single-valued neutrosophic set of $\mathcal{S}$ such that $\mathcal{R}_j q \mathcal{H}$ for each $j \in J$, where $\mathcal{H} \in I^{\mathcal{S}}$. Then, $\{\mathcal{R}_j\}_{j\in J}$ is said to be a single-valued neutrosophic quasi-cover of $\mathcal{H}$ iff $T_{\mathcal{H}}(\kappa) + T_{\bigvee_{j\in J}(\mathcal{R}_j)}(\kappa) \geq 1$, $I_{\mathcal{H}}(\kappa) + I_{\bigvee_{j\in J}(\mathcal{R}_j)}(\kappa) < 1$, and $F_{\mathcal{H}}(\kappa) + F_{\bigvee_{j\in J}(\mathcal{R}_j)}(\kappa) < 1$, for every $\kappa \in \mathcal{S}$.*

Further, let $(\mathcal{S}, \tau^{TIF})$ be an **SVNTS**, for each $\tau^T(\mathcal{R}_j) \geq r$, $\tau^I(\mathcal{R}_j) \leq 1-r$, and $\tau^F(\mathcal{R}_j) \leq 1-r$. Then, any single-valued neutrosophic quasi-cover will be called single-valued neutrosophic quasi open-cover of $\mathcal{H}$.

Theorem 9. *Let $(\mathcal{S}, \tau^{TIF})$ be an* **SVNTS** *with single-valued neutrosophic ideal $\mathcal{L}^{TIF}$ on $\mathcal{S}$. Then, the following conditions are equivalent:*

(1) $\tau \sim \mathcal{L}$.
(2) *If for every $\mathcal{H} \in I^{\mathcal{S}}$ has a single-valued neutrosophic quasi open-cover of $\{\mathcal{R}_j\}_{j\in J}$ such that for each j, $T_{\mathcal{H}}(\kappa) + T_{R_j}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{H}}(\kappa) + I_{R_j}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{H}}(\kappa) + F_{R_j}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ for every $\kappa \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$, then $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$,*
(3) *For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \wedge \mathcal{H}^\star_r = (0,1,1)$ implies $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$,*
(4) *For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1-r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1-r$, where $\widetilde{\mathcal{H}} = \bigvee x_{s,t,p}$ such that $x_{s,t,p} \in \mathcal{H}$ but $x_{s,t,p} \notin \mathcal{H}^*_r$,*
(5) *For every $\tau^{T\star}(\underline{1} - \mathcal{H}) \geq r$, $\tau^{I\star}(\underline{1} - \mathcal{H}) \leq 1-r$, and $\tau^{F\star}(\underline{1} - \mathcal{H}) \leq 1-r$ we have $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1-r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1-r$,*
(6) *For every $\mathcal{H} \in I^{\mathcal{S}}$, if $\mathcal{A}$ contains no $\mathcal{R} \neq (0,1,1)$ with $\mathcal{R} \leq \mathcal{R}^\star_r$, then $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$.*

Proof. It is proved that most of the equivalent conditions ultimately prove the all the equivalence.

(1)⇒(2): Let $\{\mathcal{R}_j\}_{j\in J}$ be a single-valued neutrosophic quasi open-cover of $\mathcal{H} \in I^{\mathcal{S}}$ such that for $j \in J$, $T_{\mathcal{H}}(\kappa) + T_{R_j}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{H}}(\kappa) + I_{R_j}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{H}}(\kappa) + F_{R_j}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ for every $\kappa \in \mathcal{R}$ and for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$. Therefore, as $\{\mathcal{R}_j\}_{j\in J}$ is a single-valued neutrosophic quasi open-cover of $\mathcal{R}$, for each $x_{s,t,p} \in \mathcal{H}$, there exists at least one $\mathcal{R}_{j\circ}$ such that $x_{s,t,p} q \mathcal{R}_{j\circ}$ and for every $\kappa \in \mathcal{S}$, $T_{\mathcal{H}}(\kappa) + T_{R_{j\circ}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{H}}(\kappa) + I_{R_{j\circ}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{H}}(\kappa) + F_{R_{j\circ}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ for every $\kappa \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$ and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$. Obviously, $\mathcal{R}_{j\circ} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$. By (1), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$.

(2)⇒(1): Clear from the fact that a collection of $\{\mathcal{R}_j\}_{j\in J}$, which contains at least one $\mathcal{R}_{j\circ} \in Q_{\tau^T IF}(x_{s,t,p}, r)$ of each single-valued neutrosophic point of $\mathcal{H}$, constitutes a single-valued neutrosophic quasi-open cover of $\mathcal{H}$.

(1)⇒(3): Let $\mathcal{H} \cap \mathcal{H}_r^\star = (0,1,1)$, for every $\kappa \in \mathcal{S}$, $x_t \in \mathcal{H}$ implies $x_{s,t,p} \notin \mathcal{H}_r^\star$. Then, there exists $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ and $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}) \leq 1-r$ such that for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$. Since $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$, By (1), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$.

(3)⇒(1): For every $x_{s,t,p} \in \mathcal{H}$, there exists $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$, for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}) \leq 1-r$. This implies $x_{s,t,p} \notin \mathcal{H}_r^\star$. Now, there are two cases: either $\mathcal{H}_r^\star = (0,1,1)$ or $\mathcal{H}_r^\star \neq (0,1,1)$ but $s > T_{\mathcal{H}_r^\star}(\kappa) \neq 0$, $t \leq I_{\mathcal{H}_r^\star}(\kappa) \neq 1$, and $p \leq F_{\mathcal{H}_r^\star}(\kappa) \neq 1$. Let, if possible, $x_{s,t,p} \in \mathcal{H}$ such that $t > T_{\mathcal{H}_r^\star}(\kappa) \neq 0$, $t \leq I_{\mathcal{H}_r^\star}(\kappa) \neq 1$, and $t \leq F_{\mathcal{H}_r^\star}(\kappa) \neq 1$. Let $s' = T_{\mathcal{H}_r^\star}(\kappa) \neq 0$, $t' = I_{\mathcal{H}_r^\star}(\kappa) \neq 1$, and $p' = F_{\mathcal{H}_r^\star}(\kappa) \neq 1$. Then, $x_{s',t',p'} \in \mathcal{H}_r^\star(\kappa)$. In addition, $x_{s',t',p'} \in \mathcal{H}$. Thus, for every $\mathcal{V} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$, for every $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$, there is at least one $\kappa \in \mathcal{S}$ such that $T_{\mathcal{V}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{V}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{V}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$. Since $x_{s,t,p} \in \mathcal{H}$, this contradicts the assumption for every single-valued neutrosophic point of $\mathcal{H}$. So, $\mathcal{H}_r^\star = (0,1,1)$. That means $x_{s,t,p} \in \mathcal{H}$ implies $x_{s,t,p} \notin \mathcal{H}_r^\star$. Now this is true for every $\mathcal{H} \in I^{\mathcal{S}}$. So, for any $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \cap \mathcal{H}_r^\star = (0,1,1)$. Hence, by (3), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, $\mathcal{L}^F(\mathcal{H}) \leq 1-r$, which implies $\tau^{TIF} \sim \mathcal{L}^{TIF}$.

(3)⇒(4): Let $x_{s,t,p} in \widetilde{\mathcal{H}}$. Then, $x_{s,t,p} \in \mathcal{H}$ but $x_{s,t,p} \notin \mathcal{H}_r^\star$. So, there exists a $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$, for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}) \leq 1-r$. Since $\widetilde{\mathcal{H}} \leq \mathcal{H}$, for every $\kappa \in \mathcal{S}$, $T_{\mathcal{D}}(\kappa) + T_{\widetilde{\mathcal{H}}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\widetilde{\mathcal{H}}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\widetilde{\mathcal{H}}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$, for some $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$ and $\mathcal{L}^F(\mathcal{C}) \leq 1-r$. Therefore, $x_{s,t,p} \notin \widetilde{\mathcal{H}}_r^\star$ implies that $\widetilde{\mathcal{H}}_r^\star = (0,1,1)$ or $\widetilde{\mathcal{H}}_r^\star \neq (0,1,1)$ but $s > T_{\widetilde{\mathcal{H}}_r^\star}$, $t \leq I_{\widetilde{\mathcal{H}}_r^\star}$, and $p \leq F_{\widetilde{\mathcal{H}}_r^\star}$. Let $x_{s',t',p'}$ in $SVNP(\mathcal{S})$ such that $s' \leq T_{\widetilde{\mathcal{H}}_r^\star}(\kappa) < s$, $t \leq I_{\widetilde{\mathcal{A}}_r^\star}(\kappa) < t'$, and $p \leq F_{\widetilde{\mathcal{H}}_r^\star}(\kappa) < p'$, i.e., $x_{s',t',p'} \in \widetilde{\mathcal{H}}_r^\star$. Then, for each $\mathcal{V} \in Q_{\tau^{TIF}}(x_{s',t',p'}, r)$ and for each $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}) \leq 1-r$, there is at least one $\kappa \in \mathcal{S}$ such that $T_{\mathcal{V}}(\kappa) + T_{\widetilde{\mathcal{H}}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{V}}(\kappa) + I_{\widetilde{\mathcal{H}}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{V}}(\kappa) + F_{\widetilde{\mathcal{H}}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$. Since $\widetilde{\mathcal{H}} \leq \mathcal{H}$, then for each $\mathcal{V} \in Q_{\tau^{TIF}}(x_{s',t',p'}, r)$ and for each $\mathcal{L}^T(\mathcal{C}) \geq r$, $\mathcal{L}^I(\mathcal{C}) \leq 1-r$, $\mathcal{L}^F(\mathcal{C}) \leq 1-r$, there is at least one $\kappa \in \mathcal{S}$ such that $T_{\mathcal{V}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 > T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{V}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 \leq I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{V}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 \leq F_{\mathcal{C}}(\kappa)$. This implies $x_{s',t',p'} \in \mathcal{H}_r^\star$. But as $s' < s$, $t' < t$, and $p' < p$, $x_{s,t,p} \in \widetilde{\mathcal{H}}$ implies $x_{s',t',p'} \in \widetilde{\mathcal{H}}$, and therefore, $x_{s',t',p'} \notin \mathcal{H}_r^\star$. This is a contradiction. Hence, $\mathcal{H}_r^\star = (0,1,1)$, so that $x_{s,t,p} \in \widetilde{\mathcal{H}}$ implies $x_{s,t,p} \notin \widetilde{\mathcal{H}}_r^\star$ with $\widetilde{\mathcal{H}}_r^\star = (0,1,1)$. Thus, $\widetilde{\mathcal{H}} \cap \widetilde{\mathcal{H}}_r^* = \underline{0}$, for every $\mathcal{H} \in I^X$. Hence, by (3), $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1-r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1-r$.

(4)⇒(5): Straightforward.

(4)⇒(6): Let $\mathcal{H} \in I^{\mathcal{S}}$ and $\mathcal{H} \leq R \neq (0,1,1)$ with $\mathcal{R} \leq \mathcal{R}_r^\star$. Then, for any $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} = \widetilde{\mathcal{H}} \cup (\mathcal{H} \cap \mathcal{H}_r^\star)$. Therefore, $\mathcal{H}_r^\star = (\widetilde{\mathcal{A}} \cup (\mathcal{H} \cap \mathcal{H}_r^\star))_r^\star = \widetilde{\mathcal{H}}_r^\star \cup (\mathcal{H} \cap \mathcal{H}_r^\star)_r^\star$. by Theorem 3 (5).

Now, by (4), we have $\mathcal{L}^T(\widetilde{\mathcal{H}}) \geq r$, $\mathcal{L}^I(\widetilde{\mathcal{H}}) \leq 1-r$, and $\mathcal{L}^F(\widetilde{\mathcal{H}}) \leq 1-r$, then $\widetilde{\mathcal{H}}_r^\star = (0,1,1)$. Hence, $(\mathcal{H} \cap \mathcal{H}_r^\star)_r^\star = \mathcal{H}_r^\star$ but $\mathcal{H} \cap \mathcal{H}_r^\star \leq \mathcal{H}_r^\star$, then $\mathcal{H} \cap \mathcal{A}_r^\star \leq (\mathcal{H} \cap \mathcal{H}_r^\star)_r^\star$. This contradicts the hypothesis about every single-valued neutrosophic set $\mathcal{H} \in I^{\mathcal{S}}$, if $(0,1,1) \neq \mathcal{R} \leq \mathcal{H}$ with $\mathcal{R} \leq \mathcal{R}_r^\star$. Therefore, $\mathcal{H} \cap \mathcal{H}_r^\star = (0,1,1)$, so that $\mathcal{H} = \widetilde{\mathcal{H}}$ by (4), we have $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$.

(6)⇒(4): Since, for every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \cap \mathcal{H}_r^\star = (0,1,1)$. Therefore, by (6), as $\mathcal{H}$ contains no non-empty single-valued neutrosophic subset $\mathcal{R}$ with $\mathcal{R} \leq \mathcal{R}_r^\star$, $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1-r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1-r$.

(5)⇒(1): For every $\mathcal{H} \in I^{\mathcal{S}}$, $x_{s,t,p} \in \mathcal{H}$, there exists an $\mathcal{D} \in Q_{\tau^{TIF}}(x_{s,t,p}, r)$ such that $T_{\mathcal{D}}(\kappa) + T_{\mathcal{H}}(\kappa) - 1 \leq T_{\mathcal{C}}(\kappa)$, $I_{\mathcal{D}}(\kappa) + I_{\mathcal{H}}(\kappa) - 1 > I_{\mathcal{C}}(\kappa)$, and $F_{\mathcal{D}}(\kappa) + F_{\mathcal{H}}(\kappa) - 1 > F_{\mathcal{C}}(\kappa)$ holds for every $\kappa \in \mathcal{S}$ and for some $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$. This implies $x_{s,t,p} \notin \mathcal{H}_r^\star$. Let $\mathcal{R} = \mathcal{H} \cup \mathcal{H}_r^*$. Then, $\mathcal{R}_r^* = (\mathcal{H} \cup \mathcal{H}_r^*)_r^\star = \mathcal{H}_r^\star \cup (\mathcal{H}_r^\star)_r^\star = \mathcal{H}_r^\star$ by Theorem 3(4). So, $C^\star_{\tau^{TIF}}(\mathcal{R}, r) = \mathcal{R} \cup \mathcal{R}_r^\star = \mathcal{R}$. That means $\tau^{T\star}(\underline{1} - \mathcal{R}) \geq r$, $\tau^{I\star}(\underline{1} - \mathcal{R}) \leq 1 - r$, and $\tau^{F\star}(\underline{1} - \mathcal{R}) \leq 1 - r$. Therefore, by (5), we have $\mathcal{L}^T(\mathcal{R}) \geq r$, $\mathcal{L}^I(\mathcal{R}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{R}) \leq 1 - r$.

Once again, for any $x_{s,t,p}$ in $SVNP(X)$, $x_{s,t,p} \notin \widetilde{\mathcal{R}}_r^\star$ implies $x_{s,t,p} \in \mathcal{R}$ but $x_{s,t,p} \notin \mathcal{R}_r^\star = \mathcal{H}_r^\star$ So, as $\mathcal{B} = \mathcal{H} \vee \mathcal{H}_r^\star$, $x_{s,t,p} \in \mathcal{H}$. Now, by hypothesis about $\mathcal{H}$. Then, for any $x_{s,t,p} \in \mathcal{H}_r^\star$. So, $\widetilde{\mathcal{R}} = \mathcal{H}$. Hence, $\mathcal{L}^T(\mathcal{H}) \geq r$, $\mathcal{L}^I(\mathcal{H}) \leq 1 - r$, and $\mathcal{L}^F(\mathcal{H}) \leq 1 - r$, i.e., $\tau^{TIF} \sim \mathcal{L}^{TIF}$. □

Theorem 10. *Let* $(\mathcal{S}, \tau^{TIF})$ *be an* **SVNTS** *with single-valued neutrosophic ideal* $\mathcal{L}^{TIF}$ *on* $\mathcal{S}$. *Then, the following are equivalent and implied by* $\tau \sim \mathcal{L}$.

(1) For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \wedge \mathcal{H}_r^* = (0, 1, 1)$ *implies* $\mathcal{H}_r^* = (0, 1, 1)$;
(2) For any $\mathcal{H} \in I^{\mathcal{S}}$, $\widetilde{\mathcal{H}}_r^\star = (0, 1, 1)$;
(3) For every $\mathcal{H} \in I^{\mathcal{S}}$, $\mathcal{H} \wedge \mathcal{H}_r^\star = \mathcal{H}_r^\star$.

Proof. Clear from Theorem 9. □

The following corollary is an important consequence of Theorem 10.

Corollary 3. *Let* $\tau^{TIF} \sim \mathcal{L}^{TIF}$. *Then,* $\beta(\tau^{TIF}, \mathcal{L}^{TIF})$ *is a base for* $\tau^{TIF\star}$ *and also* $\beta(\tau^{TIF}, \mathcal{L}^{TIF}) = \tau^{TIF\star}$.

Definition 17. *Let* $\mathcal{H}, \mathcal{R} \in$ **SVNS** *on* $\mathcal{S}$. *If* $\mathcal{H}$ *is a single-valued neutrosophic relation on a set* $\mathcal{S}$, *then* $\mathcal{H}$ *is called a single-valued neutrosophic relation on* $\mathcal{B}$ *if, for every* $\kappa, \kappa_1 \in \mathcal{S}$,

$T_{\mathcal{R}}(\kappa, \kappa_1) \leq \min(T_{\mathcal{H}}(\kappa), T_{\mathcal{H}}(\kappa_1))$,
$I_{\mathcal{R}}(\kappa, \kappa_1) \geq \max(I_{\mathcal{H}}(\kappa), I_{\mathcal{H}}(\kappa_1))$, *and*
$F_{\mathcal{R}}(\kappa, \kappa_1) \geq \max(F_{\mathcal{H}}(\kappa), F_{\mathcal{H}}(\kappa_1))$.

A single-valued neutrosophic relation $\mathcal{H}$ *on* $\mathcal{S}$ *is called symmetric if, for every* $\kappa, \kappa_1 \in \mathcal{S}$,

$$T_{\mathcal{H}}(\kappa, \kappa_1) = T_{\mathcal{H}}(\kappa_1, \kappa), \quad I_{\mathcal{H}}(\kappa, \kappa_1) = I_{\mathcal{H}}(\kappa_1, \kappa), \quad F_{\mathcal{H}}(\kappa, \kappa_1) = F_{\mathcal{H}}(\kappa_1, \kappa); \text{ and}$$

$$T_{\mathcal{R}}(\kappa, \kappa_1) = T_{\mathcal{R}}(\kappa_1, \kappa) \quad I_{\mathcal{R}}(\kappa, \kappa_1) = I_{\mathcal{R}}(\kappa_1, \kappa), \quad F_{\mathcal{R}}(\kappa, \kappa_1) = F_{\mathcal{R}}(\kappa_1, \kappa).$$

In the purpose of symmetry, we can replace Definition 3 with Definition 17.

5. Conclusions

In this paper, we defined a single-valued neutrosophic closure space and single-valued neutrosophic ideal to study some characteristics of neutrosophic sets and obtained some of their basic properties. Next, the single-valued neutrosophic ideal open local function, single-valued neutrosophic ideal closure, single-valued neutrosophic ideal interior, single-valued neutrosophic ideal open compatible, and ordinary single-valued neutrosophic base were introduced and studied.

Discussion for further works:
We can apply the following ideas to the notion of single-valued ideal topological spaces.

(a) The collection of bounded single-valued sets [53];
(b) The concept of fuzzy bornology [54];
(c) The notion of boundedness in topological spaces. [54].

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Sentiment Analysis of Tweets Using Refined Neutrosophic Sets

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A B S T R A C T

In the last decade, opinion mining and sentiment analysis have been the subject of fascinating interdisciplinary research. Alongside the evolution of social media networks, the sheer volume of social media text available for sentiment analysis has increased multi-fold, leading to a formidable corpus. Sentiment analysis of tweets have been carried out to gauge public opinion on breaking news, various policies, legislations, personalities and social movements. Fuzzy logic has been used in the sentiment analysis of twitter data, whereas neutrosophy which factors in the concept of indeterminacy has not been used to analyse tweets. In this paper, the concept of multi refined neutrosophic set (MRNS) with two positive, three indeterminate and two negative memberships is proposed. Single valued neutrosophic set (SVNS), triple refined indeterminate neutrosophic set (TRINS) and MRNS have been used in the sentiment analysis of tweets on ten different topics. Eight of these topics chosen for sentiment analysis are related to Indian scenario and two topics to international scenario. A comparative analysis of the methods show that the approach with MRNS provides better refinement to the indeterminacy present in the data.

1. Introduction

The evolution of public opinion as an influential force in the political sphere can be dated back to French revolution in 17th century [1]. The analysis of public opinion began in the 20th century. Opinion mining is a research domain that has seen speedy evolution in the preceding decade. In a contemporary trend mostly sentiment analysis is carried out on social media texts from Twitter and Facebook. A recent review paper on sentiment analysis states that more than 7000 research papers have been published on this topic and that modern sentiment analysis has established a 50-fold growth in over ten years (2005 to 2016) [2]. Conventional sentiment analysis does not deal with a neutral or an indeterminate opinion, it merely gives an overall opinion as positive or negative.

Fuzzy theory has been helpful in improving sentiment analysis techniques on twitter data [3]. Fuzzy set theory [4] that permits soft partition of sets, is stretched to Intuitionistic Fuzzy Set (A-IFS), in which a membership and a non-membership degree is allotted to every single constituent element [5], whereas in neutrosophic set, an indeterminacy membership is represented independently, together with truth membership and falsity membership to separately represent indeterminate, unpredictable, vague and uncertain information from the real world [6]. It simplifies from a philosophical point of view the idea of several sets, and its functions; $T_A(x)$, $I_A(x)$, and $F_A(x)$ and the functions are real standard or non-standard subsets for any object x in the universal space of points or objects.

Wang et al. [7] presented a single valued neutrosophic set (SVNS), to achieve an improved solution to the problem of applying neutrosophy in real world scientific and engineering problems. Neutrosophy and neutrosophic logic have found manifold applications in real world practical problems like image processing [8–10], decision-making [11–18], social network analysis [19] and social issues [20,21] etc.

In double valued neutrosophic set (DVNS) [22,23], an indeterminacy membership of the neutrosophic set has been characterised into two memberships to enable more accuracy in the indeterminacy present. Distance measure, cross entropy measure, dice measure, and clustering algorithm of DVNS was introduced and studied in [22,24]. The indeterminacy notion was separated into three, as indeterminacy inclined towards truth, indeterminacy and indeterminacy inclined towards false memberships in triple refined indeterminate neutrosophic set (TRINS), to improve the accuracy and precision of the uncertain data and to adapt it to the Likert's scale which is a habitually used psychometric scale. It was utilized for personality testing and classification [25]. TRINS was refined recently with positive, positive indeterminate, indeterminate, negative indeterminate and negative memberships, to give the finest conceivable mapping of the Likert scaling. This was defined as indeterminate likert scaling [26,27].

To capture the indeterminacy present in sentiment analysis of tweets, neutrosophy is used. Multi refined neutrosophic set (MRNS) with 2 positive, 3 indeterminate and 2 negative memberships is introduced and utilized for sentiment analysis. These seven memberships aid in capturing the polarity with better accuracy.

Section one is preliminary in nature. The rest of the paper is planned as follows: Section two presents some elementary concepts about sentiment analysis and different neutrosophic sets like SVNS, DVNS, TRINS and refined neutrosophic sets. In section three MRNS with 3 indeterminate memberships is defined and its properties are discussed. Section four discusses the limitation and problems with normal sentiment analysis and provides justification for using indeterminacy in sentiment analysis. In section five, sentiment analysis using neutrosophy is proposed. In the next section sentiment analysis of tweets of eight domestic and two international issues using three neutrosophic sets namely SVNS, TRINS and MRNS is carried out. Evaluations and discussions of these different models are analyzed in section seven. A sample topic from SemEval 2017 is taken for comparative analysis of our models. Results and further probable studies in this direction are provided in the last section.

2. Basic concepts

2.1. Sentiment Analysis

News articles, blogs, film reviews and social media information have been investigated extensively to comprehend public opinion. Typically, tweets are scrutinized and classified as positive, neutral or negative; this methodology is carried out to discover how society is feeling about a specific trending topic. Usually keyword-based tools are used to classify data (mostly social media posts, news, reviews, etc.) as positive or neutral or negative.

With the increase in data available online from early 2000, modern sentiment analysis started to take shape in mid-2000s. It has resulted in various concepts like web product reviews [28], prediction of financial markets [29], reactions to terrorist attacks [30] and multi-lingual support [31].

Sentiment analysis overlaps or relies heavily on information and knowledge management, data mining, text mining, web mining, natural language processing (NLP) and computational linguistics. Recently work is being carried out in evolving from humble polarity detection to complex gradations of emotions and distinguishing negative emotions such as anger from grief [32] and figurative language [33].

Irony detection [34–36] has a huge impact on sentiment analysis, since they write the opposite of what they feel. Recent deep learning based approaches like transfer learning have been applied in irony detection [37], aspect-based sentiment analysis using attentive long short-term memory (LSTM) [38], word vectors representations for sentiment analysis [39] and capsule networks for sentiment classification [40]. Other deep learning techniques used in sentiment analysis are reviewed in [41,42].

Fuzzy theory has been used for sentiment analysis in [3,43]. Fuzzy theory captures the positive and negative of the topic but fails to address the indeterminacy present. To address the indeterminacy present, the concept of neutrosophy in the form of SVNS, TRINS and MRNS are used to analyse the twitter data. Recently in [44] sentiment analysis of tweets about #MeToo movement, each tweet was analysed and represented as SVNS.

The discussion about the tools and methodology used for sentiment analysis in this paper are carried out in later sections. We briefly describe the notion of neutrosophy, SVNS and TRINS in the following subsection.

2.2. Neutrosophy, SVNS and TRINS

Neutrosophy, studies an opinion or sentiment, "A" and its relation to its opposite sentiment, "anti-A" and not A, "non-A", and as neither "A" nor "anti-A".

Definition 1. Let X be a space of points (objects) with generic elements x in X. The set A in X is characterized by three functions $T_A(x)$ truth membership, $I_A(x)$ indeterminacy membership, and $F_A(x)$ falsity membership. For each $x \in X$, $T_A(x), I_A(x), F_A(x) \in [0, 1]$ and $0 \le T_A(x)+I_A(x)+F_A(x) \le 3$. Single valued neutrosophic set (SVNS) A is represented by $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$.

The refined neutrosophic set [45] is defined as follows:

Definition 2. The truth T is divided into several types of truths: $T_1, T_2, \ldots, T_p$, and I into various indeterminacies: $I_1, I_2, \ldots, I_r$, and F into various falsities: $F_1, F_2, \ldots, F_s$, where all $p, r, s \ge 1$ are integers, and $p + r + s = n$.

Definition 3. A triple refined indeterminate neutrosophic set (TRINS) A in X, as given above is characterized by positive $P_A(x)$, positive indeterminate $I_{PA}(x)$, indeterminate $I_A(x)$, negative indeterminate $I_{NA}(x)$ and negative $N_A(x)$ membership functions. Each has a weight $w_m \in [0, 5]$ associated with it. For each $x \in X$, there are

$P_A(x), I_{PA}(x), I_A(x), I_{NA}(x), N_A(x) \in [0, 1]$, with weights $w_P^m(P_A(x))$, $w_{I_P}^m(I_{PA}(x))$, $w_I^m(I_A(x))$, $w_{I_N}^m(I_{NA}(x))$, $w_N^m(N_A(x)) \in [0, 5]$,
and $0 \le P_A(x)+I_{PA}(x)+I_A(x)+I_{NA}(x)+N_A(x) \le 5$.
Therefore, TRINS A can be represented by
$A = \{\langle x, P_A(x), I_{PA}(x), I_A(x), I_{NA}(x), N_A(x)\rangle \mid x \in X\}$.

The different properties and set theoretic operators like commutativity, idempotency, distributivity, associativity, absorption and the DeMorgan's Laws have been defined over TRINS [25]. As future research it is proposed to map the middle 3 terms of TRINS to neutrosophic triplets [46] and then they can be automatically mapped to neutrosophic duplets [47,48] in case of the indeterminacy leaning towards false is zero.

Neutrosophy has been applied to several different fields ranging from medical diagnosis [49,50] image processing [51], decision making [52,53], personnel selection [54], supply chain management [55,56], internet of things [57], psychology [25,58] and social science [21,59], but has not been used in sentiment analysis, until recently in [44].

3. Multi refined neutrosophic set (MRNS)

In the newly proposed Multi refined neutrosophic set (MRNS), the concept of positive (truth) is divided into two memberships, as strong positive and positive, similarly the concept of negative (false) is divided into two memberships as strong negative and negative. Also the indeterminate membership is divided into three memberships as positive indeterminate, indeterminate and negative indeterminate. This division helps in increasing the accuracy and precision in data analysis and fits the multipoint likert scale kind of structure where there are different degrees of acceptance. This refined neutrosophic set is defined as MRNS.

Definition 4. A multi refined neutrosophic set (MRNS) A in X is characterized by strong positive $SP_A(x)$, positive $P_A(x)$, positive indeterminate $PI_A(x)$, indeterminate $I_A(x)$, negative indeterminate $NI_A(x)$, negative $N_A(x)$ and strong negative $SN_A(x)$ membership functions. Each has a weight $w_m \in [0, 7]$ associated with it. For each $x \in X$, there are

$SP_A(x), P_A(x), PI_A(x), I_A(x), NI_A(x), N_A(x), SN_A(x) \in [0, 1]$,

and $0 \leq SP_A(x) + P_A(x) + PI_A(x) + I_A(x) + NI_A(x) + N_A(x) + SN_A(x) \leq 7$.

Therefore, a MRNS A can be represented by

$A = \{\langle x, SP_A(x), P_A(x), PI_A(x), I_A(x), NI_A(x), N_A(x), SN_A(x)\rangle \mid x \in X\}$.

To illustrate the applications of MRNS in a real world problem, consider parameters like work satisfaction, occupational stress and role of technology that are commonly used to measure work-life balance. The evaluation of work-life balance is used to illustrate set-theoretic operations on MRNSs.

Example 1. Let $WL = [w_1, w_2, w_3]$ where w_1 is work satisfaction, w_2 is occupational stress and w_3 is role of technology. The values of w_1, w_2 and w_3 are in [0, 1]. They are obtained using a questionnaire answered by an anonymous working women. A is a MRNS of WL defined by

$A = \langle 0.3, 0.2, 0.1, 0.2, 0.1, 0.2, 0.5\rangle/w_1 + \langle 0.4, 0.1, 0.1, 0.1, 0.2, 0.2, 0.3\rangle/w_2 + \langle 0.5, 0.2, 0.2, 0.1, 0, 0.1, 0.1\rangle/w_3$.

B is a MRNS of WL defined by

$B = \langle 0.4, 0.2, 0.1, 0.2, 0.2, 0.1, 0.2\rangle/w_1 + \langle 0.2, 0.2, 0, 0.1, 0.1, 0.2, 0.4\rangle/w_2 + \langle 0.4, 0.2, 0.1, 0.1, 0.1, 0.2, 0.3\rangle/w_3$

Definition 5. The complement $c(A)$ of A is defined as

1 $SP_{c(A)}(x) = SN_A(x)$
2 $P_{c(A)}(x) = N_A(x)$
3 $PI_{c(A)}(x) = 1 - PI_A(x)$
4 $I_{c(A)}(x) = 1 - I_A(x)$
5 $NI_{c(A)}(x) = 1 - NI_A(x)$
6 $N_{c(A)}(x) = P_A(x)$
7 $SN_{c(A)}(x) = SP_A(x)$ for all x in X.

Definition 6. A is contained in B, that is $A \subseteq B$, if and only if

1 $SP_A(x) \leq SP_B(x)$
2 $P_A(x) \leq P_B(x)$
3 $PI_A(x) \leq PI_B(x)$
4 $I_A(x) \leq I_B(x)$
5 $NI_A(x) \leq NI_B(x)$
6 $N_A(x) \geq N_B(x)$
7 $SN_A(x) \geq SN_B(x)$ for all x in X.

Note that by the definition of containment relation, X is a partially ordered set and not a totally ordered set.

For example, consider the MRNS A and B as mentioned in Example 1, then A is not contained in B and vice versa.

Definition 7. Two MRNS A and B are equal, that is $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 8. The union of A and B is a MRNS G, denoted as $G = A \cup B$, whose seven membership functions are related to those of A and B by the following

1 $SP_G(x) = max(SP_A(x), SP_B(x))$
2 $P_G(x) = max(P_A(x), P_B(x))$
3 $PI_G(x) = max(PI_A(x), PI_B(x))$
4 $I_G(x) = max(I_A(x), I_B(x))$
5 $NI_G(x) = max(NI_A(x), NI_B(x))$
6 $N_G(x) = min(N_A(x), N_B(x))$
7 $SN_G(x) = min(SN_A(x), SN_B(x))$ for all x in X.

Definition 9. The intersection of A and B is a MRNS F, denoted as $F = A \cap B$, whose seven membership functions are related to those of A and B by the following

1 $SP_F(x) = min(SP_A(x), SP_B(x))$
2 $P_F(x) = min(P_A(x), P_B(x))$
3 $PI_F(x) = min(PI_A(x), PI_B(x))$
4 $I_F(x) = min(I_A(x), I_B(x))$
5 $NI_F(x) = min(NI_A(x), NI_B(x))$
6 $N_F(x) = max(N_A(x), N_B(x))$
7 $SN_F(x) = max(SN_A(x), SN_B(x))$ for all $x \in X$.

Theorem 1. *$A \cap B$ is the largest MRNS contained in both A and B.*

Proof. It is straightforward from the definition of intersection operator. □

Definition 10. The difference D, written as $D = A\ B$, whose seven membership functions are related to those of A and B by

1 $SP_D(x) = min(SP_A(x), SN_B(x))$
2 $P_D(x) = min(P_A(x), N_B(x))$
3 $PI_D(x) = min(PI_A(x), 1 - PI_B(x))$
4 $I_D(x) = min(I_A(x), 1 - I_B(x))$
5 $NI_D(x) = min(NI_A(x), 1 - NI_B(x))$
6 $N_D(x) = max(N_A(x), P_B(x))$
7 $SN_D(x) = max(SN_A(x), SP_B(x))$ for all x in X.

Two operators positive favourite ($\triangle$) and negative favourite ($\triangledown$) are defined to remove the indeterminacy in the MRNSs and transform it into intuitionistic fuzzy sets or paraconsistent sets. Similarly a MRNS can be transformed into a SVNS by applying the indeterminacy neutral (∇) operator that combines the indeterminacy values of the MRNS.

Definition 11. The positive favourite of A represented as $B = \triangle A$, whose membership functions are related to those of A by

1 $T_B(x) = min(SP_A(x) + P_A(x) + PI_A(x), 1)$
2 $F_B(x) = N_A(x)$

Definition 12. The negative favourite of A, represented as $B = \triangledown A$, whose membership functions are related to those of A by

1 $T_B(x) = P_A(x)$
2 $F_B(x) = min(SN_A(x) + N_A(x) + NI_A(x), 1)$

Definition 13. The indeterminacy neutral of a MRNS A, written as $B = \nabla A$, whose membership functions are related to those of A by

1 $T_B(x) = min(SP_A(x) + P_A(x), 1)$
2 $I_{TB}(x) = min(PI_A(x) + I_A(x) + NI_A(x), 1)$
3 $F_B(x) = min(SN_A(x) + N_A(x), 1)$

The set theoretic operators like commutativity, associativity, distributivity, idempotency, absorption, involution and De Morgan's Laws are similar to the ones defined on SVNS. MRNS satisfies most of the properties of classical set, fuzzy set, intuitionistic fuzzy set and SVNS. Similar to fuzzy set, IFS set, SVNS, DVNS and TRINS, MRNS does not satisfy the principle of middle exclude.

4. Justification for applying indeterminacy

In opinion mining and sentiment analysis, the major division of opinion is done in terms of positive, neutral and negative opinion. Mostly Likert scaling based questionnaires are used for opinion mining. Even if Likert scaling is used in gauging the opinion of the user, the user is forced to select the most dominant choice. Generally, a person has feelings which actually vary from "strongly agree" to "strongly disagree", and which are indefinite in nature, mostly a mixture of feelings. A little disagreement might force the opinion from "strongly agree" to "agree"; whereas a different person might still choose to go ahead with the dominant opinion of "strongly agree" ignoring the little disagreement. A different respondent might mark the option "neither agree nor disagree" due to a little disagreement.

Evidently people respond differently to experiences and issues (political, economic or social in nature) while answering the questions. The questionnaire based on Likert scale will fail to capture the feelings accurately. The respondent generally goes with the dominant choice or the choice which he feels at that time or the choice which may be only a shade dominant than the other choice, thereby the degree of the memberships with other choices is completely lost. Only a measure of coarse ordinal scale with closed format is used by Likert method [60].

Similarly, when opinion mining is carried out on a specific topic to gauge the public reaction; only positive, neutral or negative categorization is done. Every person will have opinion that has various degrees of different memberships and the analysis needs to go with the dominant choice. The innumerable degrees and choices has to be captured accurately with greater precision; in fact, in a sensitive, accurate and realistic way and not in an approximate way. This will eventually aid in better understanding of people opinion or public or customers.

There is actually a lot of difference between someone who is undecided and someone who is taking a neutral stance, in a MRNS, there can be a separate option for undecided, since equal amount of agreement and disagreement can be represented in degree of weak agreement and degree of weak disagreement, individually.

Consider the seven point Likert scale, the various options given will be strongly disagree, disagree, weakly disagree, neither agree or disagree, weakly agree, agree and strongly agree. They will get mapped in MRNS; independently and appropriately under the seven heads.

Take a typical situation were the researchers must ascertain the public opinion about a political party. Usually they will project as positive or neutral or negative opinion. If the same is ascertained by making use of MRNS, the results obtained will be very accurate and clearly show the different degrees of strong negative, negative, indeterminate negative, indeterminate, indeterminate positive, positive and strong positive memberships.

5. Sentiment analysis using neutrosophy

Commonly sentiment analysis is done on tweets to classify the tweet as positive or neutral or negative. The typical scenario in which sentiment analysis of tweets is carried out is to discover how people are feeling about a specific trending topic. It is well known that there are many shades of agreement, disagreement and neutrality and there is indeterminacy involved in that neutrality. Two tweets which are classified as positive need not exactly have the same amount of positivity in them. One of them might be very strongly positive, whereas the other might be a little positive and have a lot of uncertainty or indeterminacy in it. This newly proposed method of analyzing using neutrosophic sets will give greater complexity but the accuracy in prediction of the tweet's polarity is better.

Table 1
Case study and data collection time line

Case study	Date of data collection
Farm loan wavier (*FL*)	28-12-2018
Onion price (*OP*)	29-12-2018
Foreign trips of PM (*FT*)	29-12-2018
Women reservation bill (*WB*)	29-12-2018
Triple talaq bill (*TT*)	29-12-2018
POCSO act (*PA*)	29-12-2018
UP mob violence (*UP*)	30-12-2018
Trump wall (*TW*)	30-12-2018
Yellow vests protest (*YV*)	30-12-2018
#Metoo movement (*MM*)	30-12-2018

5.1. Tools and methodology

A Twitter application programming interface (API) was created and consumer key, consumer secret, access token and access token secret were generated for obtaining tweets. Python language was used for data analysis of the collected tweets. Tweets were fetched from Twitter using tweepy python client. TextBlob is a Python library that helps in processing textual data. It provides a simple API for NLP. In our study, TextBlob was utilized for sentiment analysis. The sentiment property of TextBlob returns a tuple named Sentiment. It is of the form Sentiment(polarity, subjectivity), it returns two properties, polarity, and subjectivity. The polarity score is a float value in the range $[-1.0, 1.0]$, where 1 means positive statement and -1 means a negative statement. Let $p(x)$ denote the polarity score of the tweet. A part of the tweets extracted and used for analysis is available at [61].

5.2. Pre-processing of the Twitter data

The presence of mention, numbers, special characters, stop-words, hashtags, links and other jargon decreases the efficiency of the model and hence the tweets were cleaned. Several python libraries like pandas, numpy, matplotlib, BeautifulSoup and WordPunctTokenizer were used for cleaning. The cleaned tweet was saved as.CSV file and later used for analysis, using TextBlob.

The methodology of using different neutrosophic sets are discussed below:

5.3. Using SVNS

The normal classification of tweets as positive or negative or neutral is generally carried out. This classification is represented as SVNS. If the polarity calculated is greater than 0, i.e., $p(x) \in (0, 1]$, it is mapped to positive membership, if polarity is less than 0; $p(x) \in [-1, 0)$, it is mapped as negative membership, and if polarity is 0 it is mapped to indeterminate membership.

In the considered scenario of farm loan (denoted by *FL*), from the analysis it was obtained that, 41.90% was positive, 43.8% was indeterminate and 14.30% was negative as given in Table 2. Tweets are classified as neutral, when the classifier is not able to decide on the polarity of the tweet, that is when it is indeterminate. The aggregated result is normalized before it is converted to a neutrosophic set representation. The result of the analysis carried out is represented as SVNS is $FL_{SVNS} = \langle 0.419, 0.438, 0.143 \rangle$. It is clearly seen that positive membership and indeterminate membership have a very small difference.

Table 2
Case 1: Farm loan waiver (*FL*) and Case 2: Onion prices (*OP*)

	SVNS	TRINS	MRNS		SVNS	TRINS	MRNS
SP		0.048	0.026	SP		0.025	0.012
P	0.419	0.311	0.164	P	0.253	0.228	0.036
PI			0.229	PI			0.205
I	0.438	0.438	0.438	I	0.664	0.664	0.664
NI			0.111	NI			0.059
N	0.143	0.136	0.025	N	0.082	0.076	0.017
SN		0.007	0.007	SN		0.006	0.006

5.4. Using TRINS

The classification can be made more precise by dividing the positive polarity tweets into two different classifications and the negative polarity into two different classifications. The classification is strong positive, positive, indeterminate, negative and strong negative. If polarity of the tweet is from 1 to 0.5, i.e., $p(x) \in (0.5, 1]$, it is classified as strong positive, if $p(x) \in (0, 0.5]$ is from 0.5 to greater than 0, it is classified as positive, if $p(x) \in (0)$ is mapped to indeterminate, less than 0 to −0.5 i.e., $p(x) \in [-0.5, 0)$, it is classified as negative, greater than −0.5 to −1 i.e., $p(x) \in [-1, -0.5)$ is classified as strong negative. The data given is normalised and it is represented as TRINS is $FL_{TRINS} = \langle 0.048, 0.371, 0.438, 0.136, 0.007 \rangle$, in the case study of farm loan.

5.5. Using MRNS

The classification is made even more precise by dividing the positive polarity tweets and the negative polarity tweets into three different classification. The classification scheme that is introduced is strong positive, positive, positive indeterminate, indeterminate, negative indeterminate, negative, strong negative. If polarity of the tweet is from +1 to greater than 0.6; $p(x) \in (0.6, 1]$, it is classified as strong positive, from 0.6 to greater than 0.3; $p(x) \in (0.3, 0.6]$ it is classified as positive, from 0.3 to greater than 0; $p(x) \in (0, 0.3]$ it is mapped as positive indeterminate, if $p(x) \in (0)$; it is mapped to indeterminate, less than 0 to −0.3, $p(x) \in [-0.3, 0)$ it is classified as negative indeterminate, lesser than −0.3 to −0.6; $p(x) \in [-0.6, -0.3)$ it is mapped as negative, less than −0.6 to −1; $p(x) \in [-1, -0.6)$ it is classified as strong negative. According to this classification the results obtained is normalized and represented as MRNS is $FL_{MRNS} = \langle 0.026, 0.164, 0.229, 0.438, 0.111, 0.025, 0.007 \rangle$.

The analysis of each individual case scenario is carried out in next section.

6. Sample case scenarios

Utilizing the twitter API created for this purpose, 1000 tweets were obtained for each case under consideration. Preprocessing of the tweets were carried out to remove links and emojis, after which sentiment analysis was done. For each case study a background is provided and then analysis of tweets is discussed. The topic with abbreviation and the period of data collection is tabulated in Table 1. All tweets collected were in English language and from across the world, despite some topics being related only to some geographic location. The 10 topics selected were trending topics at the time of data collection, since we needed at least 1000 tweets about the topic.

The analysis result of each case study is given in the form of tables were the following abbreviations are used. In the first column, SP refers to strong positive membership value, P is positive membership value, PI is positive indeterminate membership value, I is indeterminate membership value, NI is negative indeterminate membership value, N is negative membership value and SN is strong negative membership value. Along the header row, SVNS stands for single valued neutrosophic set, TRINS is triple refined indeterminate neutrosophic set and MRNS is used for multi refined neutrosophic set.

6.1. Case 1: Farm loan wavier by government

Introduction: Agriculture remains to be the primary source of livelihood for nearly more than half of India's population. The country is dependent on farmers, the systematic failures of the state and center government to address their issues, pushes farmers to protest regularly and in recent years many farmers have committed suicide. Farmers have marched in Delhi and Mumbai cities to highlight the reality of their deprivation and anger. This is due to lack of compensation from drought and natural disasters like cyclone etc., crop insurance scheme failures, and the deficit created due to prices decreasing below the minimum support prices and so on. These losses are estimated to be around thousands of crores every year.

Most leaders of major Indian political parties have pledged their support to the farmers issue. In late 2018, the first step taken by three newly formed state governments (Rajasthan, Madhya Pradesh and Chhattisgarh) was a farm loan waiver. This has understandably started a debate about the usefulness of loan waiver since it is only an element of immediate relief. Farm loans and wavier have been a topic that has invoked mixed responses from people. Here tweets were collected about farm loan for analysis using the search term "farm loan wavier".

Analysis: While applying SVNS for analysis, the result obtained is $FL_{SVNS} = \langle 0.419, 0.438, 0.143 \rangle$. The indeterminate and positive membership values have little difference; hence the opinion is indeterminate and positive opinion. Next TRINS was used for analysing the same set of tweets, the result obtained is $FL_{TRINS} = \langle 0.048, 0.311, 0.438, 0.136, 0.007 \rangle$, which also implies an indeterminate and positive opinion. When MRNS was applied a change in the scenario is seen. The resultant obtained is $FL_{MRNS} = \langle 0.026, 0.164, 0.229, 0.438, 0.111, 0.025, 0.007 \rangle$; most of the positive is indeterminate positive, even the negative opinion is mostly indeterminate negative. It is seen that the public are undecided about farm loan wavier. The resultant of each neutrosophic representation is given in Table 2.

6.2. Case 2: Decrease in onion price

Introduction: The last weeks of December 2018 saw steep drops in the prices of onions and potatoes in India, it crashed as much as 86 percent. Both are staple food for the India's huge population, such a steep decrease has badly hit the rural economy in large states. Onion price hit a low of Re.1 per kilogram, while it cost nearly Rs.8 to produce one kilogram. These kind of unsteady market prices cause more distress to farmers. This also does not benefit the urban population because there are too many middlemen between the farmer and customer. The search term "onion price" was used to collect 1000 tweets for analysis.

Analysis: In SVNS representation the result obtained is $OP_{SVNS} = \langle 0.253, 0.664, 0.082 \rangle$. It is indeterminate, even though the steep decrease in onion price, has affected the farmers adversely. While using TRINS it is observed that $OP_{TRINS} = \langle 0.025, 0.228, 0.664, 0.076, 0.006 \rangle$; it indicates that people neither have a strong negative or strong positive opinion about the price decrease, but it is in general more positive than negative. MRNS was used to analyse the same dataset of tweets. The resultant is $OP_{MRNS} = \langle 0.012, 0.036, 0.205, 0.664, 0.059, 0.017, 0.006 \rangle$ as given in Table 2. It shows that even most of the positive opinion was tending towards indeterminacy. Hence, the indeterminate positive has the second highest value. People who are twitter users are unaffected by the steep drop in price which affects farmers unfavorably. More so the affected

Table 3
Case 3: Women reservation bill and Case 4: Triple talaq bill

	SVNS	TRINS	MRNS		SVNS	TRINS	MRNS
SP		0.005	0.004	SP		0.152	0.149
P	0.084	0.079	0.038	P	0.315	0.163	0.109
PI			0.042	PI			0.057
I	0.165	0.165	0.165	I	0.538	0.538	0.538
NI			0.019	NI			0.110
N	0.751	0.750	0.731	N	0.146	0.145	0.035
SN		0.001	0.001	SN		0.001	0.001

Table 4
Case 5: POCSO act death penalty and Case 6: MeToo movement

	SVNS	TRINS	MRNS		SVNS	TRINS	MRNS
SP		0.001	0.001	SP		0.04	0.014
P	0.909	0.899	0.863	P	0.562	0.522	0.375
PI			0.036	PI			0.173
I	0.038	0.038	0.038	I	0.291	0.291	0.291
NI			0.052	NI			0.086
N	0.053	0.053	0.001	N	0.147	0.132	0.046
SN		0	0	SN		0.015	0.015

farmers are not the tweeters; so only this opinion of tweeter clearly reflects the situation is indeterminable.

6.3. *Case 3: Women reservation bill*

Introduction: The women reservation bill (108th amendment to the constitution of India) is a lapsed bill in the parliament of India that was proposed in 2008. It proposed to amend the constitution of India to reserve for women 33% of all seats in the lok sabha (lower house of parliament of India), and in all state legislative assemblies, in rotational basis. With the 2019 general elections in a few months' time, the demand for the bill in the parliament has been gathering support. The bill has been around for nearly ten years and the people have mixed opinion. The term "women reservation bill" was used to query and collect 1000 tweets for analysis.

Analysis: Sentiment analysis was carried out on women reservation bill using SVNS. The resultant is $WB_{SVNS} = \langle 0.084, 0.165, 0.751\rangle$; it clearly shows that the general public (that are on twitter) are against the bill. Even when TRINS is applied, the resultant $WB_{TRINS} = \langle 0.005, 0.079, 0.165, 0.750, 0.001\rangle$, shows the same sentiment with the value of 0.750 for negative and 0.001 for strong negative. When MRNS is applied for analysis, the resultant obtained is $WB_{MRNS} = \langle 0.004, 0.038, 0.042, 0.165, 0.019, 0.731, 0.001\rangle$, it shows a meagre amount of negative indeterminate (0.019), decreasing the value of negative membership. It is clearly seen that most people are openly against the bill. The results are tabulated in Table 3.

6.4. *Case 4: Triple talaq bill*

Introduction: From 2011 census, it is known that 2.37 million women across India have identified themselves as "separated", though it is not known if these women voluntarily separated from their husbands or were abandoned or worse sent away. The vast majority (1.9 million) are hindu women, and nearly 0.28 million were "separated" muslim women. It is known that India's family laws permit for divorce, but they also allow husbands to leave a marriage without the divorce formalities. The salient features of the triple talaq bill states that any declaration of talaq by a muslim man upon his wife shall be void and illegal, shall be punished with an imprisonment term and are liable to fine. The custom of triple talaq is both harsh and unjust, muslim women have crusaded long to get free of it. Despite the law, men can choose to walk out of the marriage without saying talaq and they go free without punishment. The bill does not address the issue of non-muslim women who are abandoned by their husbands and provide punishment for those people who are equally guilty of abandonment. The keyword "triple talaq bill" was used for collecting 1000 tweets for analysis.

Analysis: Using SVNS, the resultant obtained is $TT_{SVNS} = \langle 0.315, 0.538, 0.146\rangle$, it clearly shows that the general public have not made up their mind, they are undecided, and the second leading opinion was positive. Even when TRINS is applied, the resultant $TT_{TRINS} = \langle 0.152, 0.163, 0.538, 0.145, 0.001\rangle$, shows the absence of a strong negative, whereas the positive sentiment is divided with the value of 0.152 for positive and 0.163 for strong positive. When the newly constructed MRNS is applied, the resultant is $TT_{MRNS} = \langle 0.149, 0.109, 0.057, 0.538, 0.110, 0.035, 0.001\rangle$, a meagre amount of positive indeterminate (0.019) and negative indeterminate (0.110) comes into picture, increasing the indeterminacy of the opinion. It is clearly seen that mostly people are undecided about the bill. The resultants are given in Table 3.

6.5. *Case 5: Protection of children from sexual offences (POCSO) act death penalty amendment*

Introduction: The Indian cabinet has approved amendments to the protection of children from sexual offences (POCSO) act in December 2018, to give more stringent punishment for committing sexual crimes against children. To discourage the current trend of child sexual abuse to act as a warning, it has been amended to provide death penalty in case of aggravated penetrative sexual assault on a child as option of stringent punishment. The search term "POCSO Act" was used for collecting the tweets for analysis.

Analysis: When sentimental analysis was carried out using SVNS, we obtained $PA_{SVNS} = \langle 0.909, 0.038, 0.053\rangle$, where a 90% majority felt it be a good move as given in Table 4. When the analysis was carried out with TRINS, we obtained $PA_{TRINS} = \langle 0.001, 0.899, 0.038, 0.053, 0\rangle$, it was clearly seen that no one felt strongly negative about it. People have a positive opinion about the law. Only a meagre amount of people had a negative opinion. Lastly, MRNS was used for analysing the same set of tweets. The result obtained was $PA_{MRNS} = \langle 0.001, 0.863, 0.036, 0.038, 0.052, 0.001, 0\rangle$. We are clearly able to capture that even the meagre negative is negative indeterminate; implying that people who are not sure about the amendments in the act, but they have some indeterminate negative opinion. Hence, we conclude that the negative opinion is also an indeterminate negative making up the overall opinion to be a positive opinion.

6.6. *Case 6: #Metoo movement*

Introduction: In past year the "#MeToo" movement was started against sexual harassment and assault and has gather a lot of attention and created several controversies. The term "MeToo" was used to extract related tweets for analysis, the term is ambiguous and can also lead to tweets unconnected to the movement.

Analysis: The result of the analysis is given in Table 4. Using SVNS, we obtained $MM_{SVNS} = \langle 0.559, 0.291, 0.144\rangle$, where a majority have a positive opinion. While using TRINS, we obtained $MM_{TRINS} = \langle 0.04, 0.519, 0.291, 0.13, 0.014\rangle$, it was clearly seen that the opinion is positive. Only a meagre amount of people had a negative opinion. Lastly, MRNS was used for analysis. The result obtained was $MM_{MRNS} = \langle 0.013, 0.374, 0.172, 0.291, 0.085, 0.045, 0.014\rangle$. We could clearly capture that even the negative is mostly negative indeterminate opinion. The overall opinion happens to be a positive opinion.

6.7. *Case 7: Foreign trips of prime minister*

Introduction: Over Rs. 2,021 crores (from June 2014) was spent on chartered flights, aircraft maintenance and hot-line facilities for the Indian Prime Minister Narendra Modi's visits to foreign coun-

Table 5
Case 7: Foreign trip and Case 8: UP mob violence

	SVNS	TRINS	MRNS		SVNS	TRINS	MRNS
SP		0.001	0.001	SP		0	0
P	0.474	0.473	0.003	P	0.104	0.104	0.013
PI			0.470	PI			0.091
I	0.123	0.123	0.123	I	0.223	0.223	0.223
NI			0.180	NI			0.434
N	0.403	0.399	0.219	N	0.673	0.670	0.236
SN		0.004	0.004	SN		0.003	0.003

Table 6
Case 9: Trump wall shutdown and Case 10: Yellow vests protest

	SVNS	TRINS	MRNS		SVNS	TRINS	MRNS
SP		0.030	0.020	SP		0	0
P	0.188	0.158	0.046	P	0.018	0.018	0
PI			0.122	PI			0.018
I	0.581	0.581	0.581	I	0.319	0319	0.319
NI			0.199	NI			0.654
N	0.231	0.224	0.025	N	0.663	0.662	0.008
SN		0.007	0.007	SN		0.001	0.001

tries. The numerous visits of the PM have been a topic of debate. The search term used for extracting tweets is “Foreign trip”.

Analysis: Here 1000 tweets were analysed using SVNS, the resultant obtained is $FT_{SVNS} = \langle 0.474, 0.123, 0.403\rangle$. It shows an overall positive opinion about the amount spent on foreign visits by the PM, the negative opinion is also prevalent among the public as the positive membership and negative membership values have very little difference. When TRINS is used for analysis, $FT_{TRINS} = \langle 0.001, 0.473, 0.123, 0.399, 0.004\rangle$ is the resultant. Despite an overall positive opinion; it is seen that very few had a strong positive opinion, but more people had a strong negative opinion about the trips. While using MRNS we arrive at a clearer picture where the resultant is $FT_{MRNS} = \langle 0.001, 0.003, 0.470, 0.180, 0.219, 0.004\rangle$. Most of the positive opinion is indeterminate positive and not actually positive; whereas most of the negative is negative. People are undecided about PM’s foreign visits, and more people have a decisive negative opinion than a positive opinion about the visit. It can be clearly seen that MRNS provides a better realistic picture of the actual sentiment of public opinion.

6.8. Case 8: Uttar Pradesh mob violence

Introduction: In Uttar Pradesh (India), a police constable was stoned to death in Ghazipur district, the head constable’s death is the second such incident in a month. A police inspector was killed in Bulandshahr when he tried to stop a mob from keeping cattle carcasses to block traffic. While reacting to such horrific incidents, BJP MP Udit Raj called it an isolated incident and refused to admit law and order lapse saying that such incidents can happen in a huge state like UP. Nearly 1000 tweets were collected using the term “UP mob violence”, they were used for analysis.

Analysis: While using SVNS, we obtained $UP_{SVNS} = \langle 0.104, 0.223, 0.673\rangle$; it shows that the public opinion was a negative one, whereas while using TRINS; $UP_{TRINS} = \langle 0, 0.104, 0.223, 0.67, 0.003\rangle$; it was found that no one had a strong positive opinion and very little population had a positive opinion, the majority was negative. When analysis was further carried out using MRNS, the result was $UP_{MRNS} = \langle 0, 0.013, 0.091, 0.223, 0.434, 0.236, 0.003\rangle$; most of the positive also turns out to be indeterminate positive. It is clearly seen that much of the negative opinion is negative indeterminate. The overall major opinion was indeterminate in nature. MRNS highlights the indeterminacy involved in this case accurately, whereas SVNS failed to capture the data with this amount of accuracy as shown in Table 5.

6.9. Case 9: Trump wall shutdown

Introduction: A critical feature of the standoff that has led US President Trump in 2018 to partially shut down the government is for funding of billions of dollars to build wall on the US-Mexico border.

Analysis: The result of the analysis is given in Table 6. While using SVNS for analysis it is seen that majority have an indeterminate opinion about the wall and related forced shut down of government. The resultant obtained is $TW_{SVNS} = \langle 0.186, 0.581, 0.229\rangle$; it is also noted that the difference between positive and negative is minimal. While TRINS was used for analysis, the resultant is $TW_{TRINS} = \langle 0.03, 0.156, 0.581, 0.223, 0.006\rangle$; it is seen from the membership values that not many people felt strongly negative about it, but people felt strongly positive about it. When MRNS was used to analyze the same set of tweets, the resultant obtained is $TW_{MRNS} = \langle 0.02, 0.045, 0.121, 0.581, 0.199, 0.024, 0.006\rangle$; it is clearly seen that majority fell into indeterminate, positive indeterminate and negative indeterminate membership intervals. Meagre value is associated with positive or negative membership. This clearly proves that people neither hold a positive nor negative opinion that is they are not able to make up their mind.

6.10. Case 10: Yellow vests protest

Introduction: The yellow vests movement is a populist political movement from grass roots for economic justice which started in France in 2018. Regular mass demonstrations against the French government began on 17 November 2018. The movement is motivated by high fuel prices and high cost of living and together with the burden on the working and middle classes due to government’s tax reforms. The protests had turned violent and tear gas was used on the protesters. The tweets taken for analysis are right after this news broke out.

Analysis: The majority opinion about yellow vests protest and the police action was negative in nature when analysis with SVNS was carried out, the resultant SVNS was $YV_{SVNS} = \langle 0.018, 0.319, 0.663\rangle$. When TRINS is used, the resultant TRINS is $YV_{TRINS} = \langle 0, 0.018, 0.319, 0.662, 0.001\rangle$; it is seen that there is a meagre strong negative and no strong positive. When MRNS is used the resultant obtained is $YV_{MRNS} = \langle 0, 0, 0.018, 0.319, 0.654, 0.008, 0.001\rangle$; it is clearly seen that most people had a negative indeterminate opinion. MRNS provides a clear result about the indeterminacy involved and is tabulated in Table 6.

7. Comparison and discussions

Till date neutrosophy has not been used for sentiment analysis of twitter data. A brief comparison of conventional, fuzzy and neutrosophic sentiment analysis is illustrated by Table 7 to highlight the results of our study.

If fuzzy set theory based sentiment analysis given in [3] is carried on the same set of tweets, only positive and negative memberships will get mapped, indeterminacy concept will not be dealt with, hence there will be information loss, and results will not be accurate. Henceforth it is clear that when splitting of these three memberships is done, better accuracy is achieved. There is an increased complexity in handling seven memberships when MRNS is taken, but better accuracy is achieved.

A comprehensive tabulation of all the ten topics is given in Table 8 with respect to fuzzy sentiment analysis, analysis using SVNS, TRINS, and MRNS. It is evident from Table 8 each of the case studies given above, that MRNS gives more accurate results than TRINS or SVNS or fuzzy.

Table 7
Comparison of different sentiment analysis approaches

Normal sentiment analysis	Fuzzy sentiment analysis	Neutrosophic sentiment analysis
Major overall sentiment	Percentage of positive sentiment Percentage of negative sentiment	Proportion of positive sentiment Proportion of indeterminate sentiment Proportion of negative sentiment

Table 8
Neutrosophic representation of each case study

Type of NS	Positive			Neutral	Negative		
Case 1: Farm loan Wavier							
Fuzzy		0.857				0.143	
SVNS		0.419		0.438		0.143	
TRINS	0.048	0.371		0.438		0.136	0.007
MRNS	0.026	0.164	0.229	0.438	0.111	0.255	0.007
Case 2: Onion Price							
Fuzzy		0.917				0.083	
SVNS		0.253		0.664		0.083	
TRINS	0.026	0.228		0.664		0.076	0.006
MRNS	0.012	0.037	0.205	0.664	0.059	0.017	0.006
Case 3: Women Reservation Bill							
Fuzzy		0.249				0.751	
SVNS		0.084		0.165		0.751	
TRINS	0.005	0.079		0.165		0.750	0.001
MRNS	0.004	0.038	0.042	0.165	0.019	0.731	0.001
Case 4: Triple Talaq Bill							
Fuzzy		0.854				0.146	
SVNS		0.316		0.538		0.146	
TRINS	0.153	0.163		0.538		0.145	0.001
MRNS	0.149	0.11	0.057	0.538	0.11	0.035	0.001
Case 5: POCSO Act Death Penalty							
Fuzzy		0.947				0.053	
SVNS		0.909		0.038		0.053	
TRINS	0.01	0.899		0.038		0.053	0
MRNS	0.01	0.863	0.036	0.038	0.052	0.001	0
Case 6: #Me too Movement							
Fuzzy		0.853				0.147	
SVNS		0.562		0.291		0.147	
TRINS	0.04	0.522		0.291		0.132	0.015
MRNS	0.014	0.375	0.173	0.291	0.086	0.046	0.015
Case 7: Foreign Trip PM							
Fuzzy		0.597				0.403	
SVNS		0.474		0.123		0.403	
TRINS	0.001	0.473		0.123		0.399	0.004
MRNS	0.001	0.003	0.470	0.123	0.180	0.219	0.004
Case 8: UP Mob Violence							
Fuzzy		0.327				0.673	
SVNS		0.104		0.223		0.673	
TRINS	0	0.104		0.223		0.670	0.003
MRNS	0	0.013	0.091	0.223	0.434	0.236	0.003
Case 9: Trump Wall Shutdown							
Fuzzy		0.769				0.231	
SVNS		0.188		0.581		0.231	
TRINS	0.03	0.158		0.581		0.224	0.007
MRNS	0.02	0.046	0.122	0.581	0.199	0.025	0.007
Case 10: Yellow Vest Protest							
Fuzzy		0.337				0.663	
SVNS		0.018		0.319		0.663	
TRINS	0	0.018		0.319		0.662	0.001
MRNS	0	0	0.018	0.319	0.654	0.008	0.001

To further enhance the results and enable comparison, indeterminacy neutral operator (Definition 13) was used on MRNS to convert it to SVNS kind of representation. The comparison of fuzzy, SVNS and indeterminate neutral of MRNS is given in Table 9, it is seen in several cases that a positive opinion or a negative opinion is actually indeterminate in nature. Also in some cases a positive opinion turns out to indeterminate positive and a negative opinion has an edge over positive opinion.

It is clearly seen in case of "foreign trips of PM" and "Me too", the overall positive opinion turns out to be indeterminate in nature when analysed with MRNS. Similarly the overall negative opinion of "yellow vest protest" and "UP mob violence" turns out to be indeterminate opinion with analysed with MRNS. In case of "farm loan wavier" the opinion changes from positive to negative, and in case of "Trump wall", the opinion changes from more negative to more positive when analysed with MRNS. In the other cases, like "Women reservation bill", "triple talaq bill", "POSCO act" and "onion price", the opinion obtained with SVNS and TRINS is confirmed using MRNS for analysis. The trendy that is predicted by fuzzy is sometimes not exactly accurate, mostly when indeterminacy happens to be the actual trend.

Table 9
Comparison between Fuzzy, SVNS and Indeterminate neutral of MRNS

Case no	Fuzzy	SVNS	Indeterminate neutral MRNS
1	Positive	Indeterminate; More positive than negative	⟨0.19, 0.778, 0.263⟩ Indeterminate more negative than positive
2	Positive	Indeterminate more positive than negative	⟨0.048, 0.928, 0.023⟩ Indeterminate; meagre negative and positive
3	Negative	Negative	⟨0.042, 0.226, 0.732⟩ Negative; negligible positive
4	Positive	Indeterminate more positive than negative	⟨0.258, 0.705, 0.03⟩ Indeterminate; more positive than negative
5	Positive	Positive	⟨0.873, 0.126, 0.001⟩ Positive; no negative
6	Positive	Positive	⟨0.387, 0.548, 0.059⟩ Indeterminate more positive than negative
7	Positive	Positive	⟨0.004, 0.773, 0.223⟩ Indeterminate more negative than positive
8	Negative	Negative	⟨0.014, 0.743, 0.239⟩ Indeterminate more negative than positive
9	Positive	Indeterminate more negative than positive	⟨0.065, 0.802, 0.03⟩ Indeterminate; more positive than negative
10	Negative	Negative	⟨0, 0.991, 0.009⟩ Indeterminate; negligible negative and no positive

7.1. Working on SemEval2017 Task 4 dataset

Recently, SemEval-2017 Task 4 [62], the sentiment analysis in Twitter task was conducted for the fifth year. Several tasks are given to the participating teams. It includes identifying the overall sentiment of the tweet, sentiment towards a topic with classification on a two point and on a five-point ordinal scale. The five-point scale used was strongly positive, weakly positive, neutral, weakly negative, and strongly negative. This five-point scale is similar to the TRINS concept in neutrosophy.

SemEval-2017 Task 4: Sentiment analysis in twitter is an important benchmark in sentiment analysis of twitter data. The various subtasks associated with it are subtasks A, B, C, D and E. Subtask A is about message polarity classification, subtasks B-C are about topic based message polarity classification into two-point and five-point scale and subtasks D-E are tweet quantification into two-point and five-point scale.

Our model using neutrosophy is closely related with polarity classification and tweet quantification (Subtask E), it works on three-point scale (SVNS), five-point scale (TRINS) and seven point scale (MRNS). An in-depth analysis of using neutrosophy on SemEval 2017 dataset will be done later for all the 5 subtasks of task 4, since almost all participating teams had made use of several machine learning algorithms. Our existing model is not making use of any machine learning algorithm and hence is at disadvantage for complete comparison with the baselines and benchmarks set by them. The possible comparative analysis can be carried out on five-point scale (TRINS) as subtask E of tweet quantification is on a five-point scale. For illustrative purpose we have worked with just one topic from SemEval-2017 dataset. We have dealt with 100 tweets related to the topic "Amazon" from the SemEval-2017 training dataset. The original values are taken from the polarity value given in the dataset, it is mapped to the percentage of $-2, -1, 0, 1, 2$ present in the polarity column. The values obtained for subtask E in case of topic "Amazon" using TRINS is tabulated below in Table 10.

Table 10
Case study of Amazon with SemEval 2017

Values	Positive			Neutral	Negative		
Original Values	0.05	0.52		0.27		0.15	0.01
TRINS	0.06	0.47		0.28		0.16	0.03
MRNS	0.06	0.03	0.44	0.28	0.13	0.06	0

The TRINS values are obtained by polarity classification and tweet quantification of the tweets in the dataset. The values obtained are ⟨0.06, 0.47, 0.28, 0.16, 0.03⟩. There is a slight variation when compared to the expected values given by the experts. The same data set was used for analysis using MRNS and the following was obtained ⟨0.06, 0.03, 0.44, 0.28, 0.13, 0.06, 0⟩ on a 7-point scale. This cannot be compared with the expected values of the SemEval dataset, since they are in different point scale, but it is pertinent to mention that the 7-point scale gives more detailed accuracy to the analysis. For example, it can be seen that none had a strongly negative opinion about Amazon, this is not captured with TRINS.

Since SemEval-2017 dataset is an interesting, important and bench-marked dataset to work on, a detailed analysis of the SemEval 2017 dataset using neutrosophy will be carried out soon. It shall be a separate analysis on its own along with creation of appropriate machine learning and natural language processing algorithms which make use of neutrosophy. This exciting and innovative study will be taken up in future. For further study, we are planning to work on datasets from [33–35], combined together with appropriate NLP and machine learning algorithms that are based on neutrosophy.

8. Results and future study

Both conventional and fuzzy sentiment analysis fail to capture the indeterminacy and neutrality that is present in the content. To handle the indeterminacy, neutrosophy is used for analysis of tweets. In this paper, a new concept called MRNS with two positive memberships, three indeterminate memberships and two negative memberships is defined. Its properties and various operators are discussed. For the purposes of this research work, ten subjects (data sets) which are political or social in nature were taken for sentiment analysis and for each case study, 1000 tweets were collected and used for analysis. The collection was carried out through a specifically created twitter API, through which tweets were extracted, programming was carried out using Python and necessary libraries for NLP. It is the first time that refined neutrosophic sets have been used for sentiment analysis of tweets. Three neutrosophic sets namely SVNS, TRINS and MRNS were used to analyse the tweets. In each case, it was clearly seen that MRNS gives a better and accurate result when compared to SVNS or TRINS. An illustrative comparison with one topic from SemEval-2017 was also done with TRINS and MRNS. From the discussions and comparisons carried out, it is seen that when MRNS is used the best accuracy was obtained in sentiment analysis.

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Singular Neutrosophic Extended Triplet Groups and Generalized Groups

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Abstract

Neutrosophic extended triplet group (NETG) is an interesting extension of the concept of classical group, which can be used to express general symmetry. This paper further studies the structural characterizations of NETG. First, some examples are given to show that some results in literature are false. Second, the differences between generalized groups and neutrosophic extended triplet groups are investigated in detail. Third, the notion of singular neutrosophic extended triplet group (SNETG) is introduced, and some homomorphism properties are discussed and a Lagrange-like theorem for finite SNETG is proved. Finally, the following important result is proved: a semigroup is a singular neutrosophic extended triplet group (SNETG) if and only if it is a generalized group.

Keywords: Neutrosophic extended triplet group; Generalized group; Semigroup; Singular neutrosophic extended triplet group; Kernel of homomorphism

1. Introduction and basic concepts

The theory of neutrosophic set was introduced by Smarandache, and it is applied to many fields (see Smarandache, 2005; Ye, 2014; Liu, Khan, Ye, & Mahmood, 2018; Zhang, Bo, Smarandache, & Dai, 2018; Zhang, Bo, Smarandache, & Park, 2018). In recent years, the ideology of neutrosophic set has been applicable in related algebraic structures. In particular, Smarandache and Ali (2018) introduced the notion of neutrosophic triplet group, which is a new extension of the concept of classical group. Now, this new algebraic structure has aroused scholars' interest, and some new research papers have been published one after another (see Smarandache, 2017; Zhang, Smarandache, & Liang, 2017; Jaiyeola and Smarandache, 2018; Smarandache, Şahin, & Kargin, 2018; Ali, Smarandache, & Khan, 2018; Zhang, Hu, Smarandache, & An, 2018). In fact, neutrosophic triplet structures are closely connected with related non-classical logic algebras (see Zhang, Wu, Smarandache, & Hu, 2018; Zhang, 2017; Zhang, Park, & Wu, 2018). In (Smarandache, 2017), the notion of neutrosophic extended triplet group (NETG) was introduced as a generalization of neutrosophic triplet group.

On the other hand, Molaei (Molaei, 1999) introduced the notion of generalized group, as a class of algebras of interest in physics. After that, some scholars studied the properties of generalized groups (see (Araujo and Konieczny, 2002; Akinmoyewa, 2009; Adeniran, Akinmoyewa, Solarin, & Jaiyeola, 2011). According to Araujo & Konieczny, 2002), generalized group is equivalent to the notion of completely simple semigroup.

Intuitively, as two generalizations of classical group, the notion of neutrosophic extended triplet group is very close to generalized group. However, the comparative analysis of the two kinds of algebraic structures is far from perfect. This paper will further analyze their connections and differences. First, we recall some basic concepts.

Definition 1 Smarandache, 2017. Let N be a set together with a binary operation $*$. Then, N is called a neutrosophic extended triplet set if for any $a \in N$, there exist a neutral of "a" denoted by $neut(a)$, and an opposite of "a" denoted by $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$a * neut(a) = neut(a) * a = a$;

$a * anti(a) = anti(a) * a = neut(a)$.

The triple $a, neut(a)$ and $anti(a)$ is referred to as a neutrosophic triplet, and denoted by $(a, neut(a), anti(a))$.

Remark 1. The above definition is a generalization of original definition of a neutrosophic triplet set. For a neutrosophic extended triplet, the neutral of x is allowed to also be equal to the classical identity element as a special case.

Note that, for a neutrosophic triplet set $(N, *)$, $a \in N, anti(a)$ may not be unique. In order not to cause ambiguity, we use the following notations to distinguish:

$anti(a)$: denote any certain one of opposite of a;
$\{anti(a)\}$: denote the set of all opposite of a.

Definition 2 (*Smarandache and Ali*, 2018; *Smarandache*, 2017). Let $(N, *)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group, if the following conditions are satisfied:

(1) If $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$.
(2) If $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if for all $a, b \in N$, $a * b = b * a$.

Remark 2. The most prominent character of neutrosophic extended triplet group (NETG), which is different from other algebraic structure features, is "triplet". According to the definition above, just as $a * b = b * a = a$, the b cannot be called a neutral element of a. It is only when there exists c, at the same time, such that $a * c = c * a = b$, the b can be called neutral element of a. Therefore, "neutral element" and "identity element" are two different concepts.

Here are some other related notions: let G be a non-empty set, define a binary operation $*$ on G. If $x * y \in G$, $\forall x, y \in G$,$(G, *)$ is called a groupoid. If the equations $a * x = b$ and $y * a = b$ have unique solutions relative to x and y respectively, then $(G, *)$ is called a quasigroup.

Definition 3 (*Molaei*, 1999; *Akinmoyewa*, 2009). A generalized group $(G, *)$ is a non-empty set admitting a binary operation $*$ called multiplication subject to the set of rules given below:

(i) $(x * y) * z = x * (y * z)$ for all $x, y, z \in G$.
(ii) For each $x \in G$, there exists a unique $e(x) \in G$ such that $x * e(x) = e(x) * x = x$.
(iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e(x)$.

Definition 4 (*Akinmoyewa*, 2009; *Adeniran et al.*, 2011). Let $(G, *)$ be a generalized group. If $e(x * y) = e(x) * e(y)$ for all $x, y \in G$, then G is called normal generalized group.

Theorem 1 (*Araujo and Konieczny*, 2002; *Akinmoyewa*, 2009; *Adeniran et al.*, 2011). *For each element x in a generalized group $(G, *)$, there exists a unique $x^{-1} \in G$.*

Theorem 2 (*Araujo and Konieczny*, 2002; *Akinmoyewa*, 2009; *Adeniran et al.*, 2011). *Let $(G, *)$ be a generalized group. If $x * y = y * x$ for all $x, y \in G$, then G is a group.*

2. Some counterexamples on neutrosophic extended triplet groups

For a neutrosophic extended triplet group, the Ref. Jaiyeola and Smarandache (2018) gives some important research directions, but there were some errors. In this section, some counterexamples will be constructed.

Example 1. Let $X = \{a, b, c, d\}$. The operation $*$ on X is defined as Table 1. Then, $(X, *)$ is a non-commutative neutrosophic extended triplet group, and

$neut(a) = a, anti(a) = a; neut(b) = b, anti(b) = b;$

$neut(c) = c, anti(c) = c; neut(d) = d, \{anti(d)\} = \{c, d\}.$

(1) Obviously, each element x in X has a unique $neut(x)$, but $(X, *)$ is not a generalized group, since $d * d = d$, $c * d = d * c = d$. Thus, the condition in Definition 3 (ii) is not satisfied for $(X, *)$. It follows

Table 1
Neutrosophic extended triplet group $(X, *)$.

*	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	d	d	c	d
d	d	d	d	d

that Lemma 1 (2) in (Jaiyeola & Smarandache, 2018) is not true.

(2) Since $\{anti(anti(d))\} = \{c, d\}$, so putting $anti(anti(d)) = c \in \{anti(anti(d))\}$, then $anti(anti(d)) \neq d$. This means that Theorem 1 in (Jaiyeola & Smarandache, 2018) is not true.

(3) Let $H = \{a, b, d\}$, then $(H, *)$ is a neutrosophic extended triplet group, that is,$(H, *)$ is a neutrosophic extended triplet subgroup of $(X, *)$. Consider $d \in H, c \in \{anti(d)\}$ but $c \notin H$. It follows that Lemma 2 (iii) in (Jaiyeola & Smarandache, 2018) is not true.

(4) Let $Y = \{1, 2, 3\}$. The operation $*$ on Y is defined as Table 2. Then, $(Y, *)$ is a non-commutative neutrosophic extended triplet group, and

$$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2;$$
$$neut(3) = 3, \{anti(3)\} = \{2, 3\}.$$

Denote $f : X \to Y; a \mapsto 1, b \mapsto 1, c \mapsto 2, d \mapsto 3$. Then f is a homomorphism.

Putting

$$anti(d) = c \in \{anti(d)\},\ anti(3) = 3 \in \{anti(3)\},$$

Then

$$f(anti(d)) = f(c) = 2 \neq 3 = anti(3) = anti(f(d))$$

This means that Theorem 5 (2) in (Jaiyeola & Smarandache, 2018) is not true. Moreover, according to Definition 6 in (Jaiyeola & Smarandache, 2018) (about X_a, $\ker f_a$, $\ker f$), we have

$$X_a = \{a\},\ \ker f_a = \{a, b\};\ X_b = \{b\},\ \ker f_b = \{a, b\};$$
$$X_c = \{c\},\ \ker f_c = \{c\};\ X_d = \{d\},\ \ker f_d = \{d\};$$
$$\ker f = \{a, b, c, d\}.$$

Thus,

$$|X_a| = 1 \neq 2 = [X_a : \ker f_a] \times |\ker f_a|,$$
$$|X_b| = 1 \neq 2 = [X_b : \ker f_b] \times |\ker f_b|;$$
$$\sum_{a \in X} [Xa : \ker f_a] \cdot |\ker f_a|$$
$$= 1 \times 2 + 1 \times 2 + 1 \times 1 + 1 \times 1 = 6.$$

Therefore,

$$|X| < \sum_{a \in X} [X_a : \ker f_a] \cdot |\ker f_a|$$

It follows that Thereom 6 (6) and (8) in (Jaiyeola & Smarandache, 2018) are not true.

Table 2
Neutrosophic extended triplet group $(Y, *)$.

*	1	2	3
1	1	1	1
2	3	2	3
3	3	3	3

Moreover, in the proof of Thereom 6 (6) in (Jaiyeola & Smarandache, 2018), an assertion is used: if X is finite, $|\ker f_a| = |c * \ker f_a|$ for all c in X_a. In fact, it is not true, since in this example, $|\ker f_a| = 2 \neq 1 = |a * \ker f_a|$.

Example 2. Let $X = \{a, b, c, d, e, f\}$. The operation $*$ on X is defined as Table 3. Then, $(X, *)$ is a non-commutative neutrosophic extended triplet group, and

$neut(a) = a, \{anti(a)\} = \{a, c, d, e, f\}; neut(b) = b,$
$anti(b) = b; neut(c) = c, anti(c) = c; neut(d) = c,$
$anti(d) = d; neut(e) = e, anti(e) = e; neut(f) = e,$
$anti(f) = f.$

(1) Obviously, each element x in X has a unique $neut(x)$, but $(X, *)$ is not a generalized group, since $a * a = a, c * a = a * c = a, d * a = a * d = a, e * a = a * e = a, f * a = a * f = a$. So, the condition in Definition 3 (ii) is not satisfied for $(X, *)$. It follows that Lemma 1 (2) in (Jaiyeola & Smarandache, 2018) is not true.

(2) Since $\{anti(anti(a))\} = \{a, c, d, e, f\}$, then putting $anti(anti(a)) = c \in \{anti(anti(a))\}$, we get $anti(anti(a)) \neq a$. This means that Theorem 1 in (Jaiyeola & Smarandache, 2018) is not true.

(3) Let $H = \{a, b, c, d\}$, then $(H, *)$ is a neutrosophic extended triplet group, that is, $(H, *)$ is a neutrosophic extended triplet subgroup of $(X, *)$. Consider $a \in H, e \in \{anti(a)\}$ but $e \notin H$. It follows that Lemma 2 (iii) in (Jaiyeola & Smarandache, 2018) is not true.

(4) According to Definition 6 in (Jaiyeola & Smarandache, 2018) (about X_a), we have $X_a = \{a\}$. Consider $d \in \{anti(a)\}$, we have $anti(a * a) = anti(a) = d \neq c = d * d = anti(a) * anti(a)$. It follows that Theorem 6 (4) in (Jaiyeola & Smarandache, 2018) is not true.

(5) Let $Y = \{1, 2, 3, 4\}$. The operation $*$ on Y is defined as Table 4. Then, $(Y, *)$ is a non-commutative neutrosophic extended triplet group, and

$$neut(1) = 1, \{anti(1)\} = \{1, 3, 4\}; neut(2) = 2, anti(2) = 2;$$
$$neut(3) = 3, anti(3) = 3; neut(4) = 4, anti(4) = 4.$$

Denote $f : X \to Y; a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 3, e \mapsto 4, f \mapsto 4$. Then f is a homomorphism. Putting $anti(a) = a \in \{anti(a)\}$, and $anti(1) = 3 \in \{anti(1)\}$, then $f(anti(a)) = f(a) = 1 \neq 3 = anti(1) = anti(f(a))$. This means that Theorem 5 (2) in (Jaiyeola & Smarandache, 2018) is not true. Moreover, according to Definition 6 in (Jaiyeola & Smarandache, 2018) (about X_a, $\ker f_a$, $\ker f$), we have

Table 3
Neutrosophic extended triplet group $(X, *)$.

*	a	b	c	d	e	f
a	a	b	a	a	a	a
b	a	b	a	a	a	a
c	a	b	c	d	a	a
d	a	a	d	c	a	a
e	a	b	a	a	e	f
f	a	b	a	a	f	e

Table 4
Neutrosophic extended triplet group $(Y, *)$.

*	1	2	3	4
1	1	2	1	1
2	1	2	1	1
3	1	2	3	1
4	1	2	1	4

$X_a = \{a\},\ \ker f_a = \{a\};\ X_b = \{b\},\ \ker f_b = \{b\};$
$X_c = \{c, d\},\ \ker f_c = \{c, d\};$
$X_d = \{d, c\},\ \ker f_d = \{d, c\};\ X_e = \{e, f\},\ \ker f_e = \{e, f\};$
$X_f = \{f, e\},\ \ker f_f = \{f, e\};\ \ker f = \{a, b, c, d, e, f\}.$

Thus,

$$\sum_{a \in X} [X_a : \ker f_a] \cdot |\ker f_a| = $$
$$= 1 \times 1 + 1 \times 1 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2$$
$$= 10.$$

Therefore, $|X| < \sum_{a \in X} [X_a : \ker f_a] \cdot |\ker f_a|$. It follows that Theorem 6 (6) and (8) in (Jaiyeola & Smarandache, 2018) are not true.

3. The differences between neutrosophic extended triplet groups and generalized groups

In this section, the differences between neutrosophic extended triplet group (NETG) and generalized group (GG) are summarized.

Note 1. For a neutrosophic extended triplet group $(X, *)$, the opposite element of an element x in X may not be unique (see Example 1 and Example 2). But, for each element x in a generalized group $(G, *)$, there exists a unique $x^{-1} \in G$ (see Theorem 1).
Note 2. For a generalized group $(G, *)$, if x, a in G such that $a * x = x * a = x$, then $a = x^{-1}$. But, for a neutrosophic extended triplet group $(X, *)$, if x, a in X such that $a * x = x * a = x$, we cannot get that $a = neut(x)$ in general (see Remark 2, and in Example 2, $c * a = a * c = a$, but $c \neq neut(a) = a$).
Note 3. For a generalized group $(G, *)$, if a in G, then $e(a^{-1}) = e(a)$, see (Araujo and Konieczny, 2002; Akinmoyewa, 2009; Adeniran et al., 2011). But, for a neutrosophic extended triplet group $(X, *)$, if a in X, we cannot get that $neut(anti(a)) = neut(a)$ in general (in Example 1, putting $anti(d) = c$, we have $neut(anti(d)) = neut(c) = c \neq neut(d) = d$).
Note 4. There exists some commutative neutrosophic extended triplet groups which are not classical groups. But, every commutative generalized group is a classical group (see Theorem 2).
Note 5. For a generalized group $(G, *)$, if a in G, then $(a^{-1})^{-1} = a$, see Theorem 3.1 in (Adeniran, Akinmoyewa, Solarin, & Jaiyeola, 2011). But, for a neutrosophic extended triplet group $(X, *)$, if a in X, we cannot get that $anti(anti(a)) = a$ in general (see Example 1 (2) and Example 2 (2)).
Note 6. Let $f : G \to H$ be a homomorphism where G and H are two distinct generalized groups, then $f(a^{-1}) = (f(a))^{-1}$ for all a in G. But, for neutrosophic extended triplet groups, $f(anti(a)) \neq anti(f(a))$ in general (see Example 1 (4) and Example 2 (5)).

Definition 5 Araujo & Konieczny, 2002. A semigroup $(S, *)$ is said to be completely simple if it satisfies the following conditions:

(C1) $S * a * S = S$ for every $a \in S$.
$(C2)$ if $e, f \in G$ are idempotents such that $e * f = f * e = e$, then $e = f$.

Theorem 3 Araujo & Konieczny, 2002. *Let* $(S, *)$ *be a semigroup. Then the following are equivalent*:

(1) $(S, *)$ is completely simple;
(2) $(S, *)$ is a generalized group.

The following example shows that there exists neutrosophic extended triplet group in which the conditions (C1) and (C2) are not satisfied.

Example 3. Let $X = \{1, 2, 3, 4\}$. The operation $*$ on X is defined as Table 5. Then, $(X, *)$ is a non-commutative neutrosophic extended triplet group, and

$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2;$
$neut(3) = 3, anti(3) = 3; neut(4) = 4, \{anti(4)\} = \{1, 2, 4\}.$
$X * 4 * X = \{3, 4\} \neq X;$
$4 * 4 = 4, 2 * 2 = 2, 4 * 2 = 2 * 4 = 4,$ but $4 \neq 2.$

Note 7. Every generalized group is a completely simple semigroup, but there exists some neutrosophic extended triplet groups which are not completely simple semigroups. In fact, a generalized group is a special type of neutrosophic extended triplet group. Thus, neutrosophic extended triplet group is a generalization of generalized group.

Table 5
Neutrosophic extended triplet group $(X, *)$.

*	1	2	3	4
1	1	1	4	4
2	2	2	4	4
3	3	3	3	3
4	4	4	4	4

4. Singular neutrosophic extended tripet groups

Theorem 4 Zhang, Hu, Smarandache, & An, 2018. *Let* $(N, *)$ *be a neutrosophic extended triplet group. Then*

(1) $neut(a)$ is unique for any a in N.
(2) $neut(a) * neut(a) = neut(a)$ for any a in N.

Theorem 5 Zhang, Hu et al., 2018. *Let* $(N, *)$ *be a neutrosophic extended triplet group. Then* $\forall a \in N, \forall anti(a) \in \{anti(a)\}$,

(1) $neut(a) * p = q * neut(a)$, for any $p, q \in \{anti(a)\}$;
(2) $neut(neut(a)) = neut(a)$;
(3) $anti(neut(a)) * anti(a) \in \{anti(a)\}$;
(4) $neut(a * a) * a = a * neut(a * a) = a$;
$neut(a * a) * neut(a) = neut(a) * neut(a * a) = neut(a)$;
(5) $neut(anti(a)) * a = a * neut(anti(a)) = a$;
$neut(anti(a)) * neut(a) = neut(a) * neut(anti(a)) = neut(a)$;
(6) $anti(neut(a)) * a = a * anti(neut(a)) = a$, for any $anti(neut(a)) \in \{anti(neut(a))\}$.
(7) $a \in \{anti(neut(a) * anti(a))\}$;
(8) $neut(a) * anti(a) \in \{anti(a)\}$;
$anti(a) * neut(a) \in \{anti(a)\}$;
(9) $a \in \{anti(anti(a))\}$, that is, there exists $p \in \{anti(a)\}$ such that $a \in \{anti(p)\}$;
(10) $neut(a) * anti(anti(a)) = a$.

Definition 6. A neutrosophic extended triplet group $(X, *)$ is said to be singular, if $anti(a)$ is unique for any $a \in X$.

Applying Theorem 5 we can get the following results.

Theorem 6. *Let* $(X, *)$ *be a singular neutrosophic extended triplet group. Then* $\forall a \in X$,

(1) $neut(a) * anti(a) = anti(a) * neut(a) = anti(a)$;
(2) $anti(neut(a)) = neut(a)$;
(3) $a = anti(anti(a))$;
(4) $neut(anti(a)) = neut(a)$.

Proof.

(1) Since $(X, *)$ is a singular neutrosophic extended triplet group, using Definition 6 and Theorem 5 (1) and (8), we have $neut(a) * anti(a) = anti(a) * neut(a) = anti(a)$ for all a in X.
(2) Applying (1),

$$neut(neut(a)) * anti(neut(a)) = anti(neut(a)), \forall a \in X.$$

By Theorem 5 (2), $neut(neut(a)) = neut(a)$. It follows that

$$neut(a) * anti(neut(a)) = anti(neut(a)), \forall a \in X.$$

On the other hand, using Definition 1, $neut(a) * anti(neut(a)) = neut(neut(a))$. Applying Theorem 5 (2) again, $neut(a) * anti(neut(a)) = neut(neut(a)) = neut(a)$. Therefore,

$$anti(neut(a)) = neut(a) * anti(neut(a)) = neut(a), \forall a \in X.$$

(3) Since $(X, *)$ is a singular neutrosophic extended triplet group, using Definition 6 and Theorem 5 (9), we get $a = anti(anti(a))$ for all a in X.
(4) Since $(X, *)$ is a singular neutrosophic extended triplet group, applying Definition 1,

$$anti(a) * anti(anti(a)) = neut(anti(a)), \forall a \in X.$$

Using (3), $anti(anti(a)) = a$, thus

$$anti(a) * a = anti(a) * anti(anti(a)) = neut(anti(a)), \forall a \in X.$$

On the other hand, $anti(a) * a = neut(a)$ (from Definition 1). Therefore,

$$neut(a) = anti(a) * a = anti(a) * anti(anti(a)) = neut(anti(a)), \forall a \in X.$$

Definition 7 Zhang, Hu et al., 2018. Let $(X, *)$ be a neutrosophic extended triplet group and H be a non-empty subset of X. Then H is called a NT-subgroup of X if

(1) $a * b \in H$ for all $a, b \in H$;
(2) there exists $anti(a) \in \{anti(a)\}$ such that $anti(a) \in H$ for all $a \in H$, where $\{anti(a)\}$ is the set of opposite element of a in $(X, *)$.

Proposition 1. *If* H *is a NT- subgroup of a neutrosophic extended triplet group* $(X, *)$, *then* $neut(a) \in H$ *for all* $a \in H$, *where* $neut(a)$ *is the neutral element of* a *in* $(X, *)$.

By Definition 6, Definition 7 and Proposition 1 we have

Proposition 2.. *If* H *is a non-empty subset of a singular neutrosophic extended triplet group* $(X, *)$, *then* H *is a NT-subgroup of* $(X, *)$ *if and only if it satisfies*

(1) $a * b \in H$ *for all* $a, b \in H$;
(2) $anti(a) \in H$ *for all* $a \in H$.

Definition 8 Jaiyeola & Smarandache, 2018. Let $(X, *)$ be a neutrosophic extended triplet group. Whenever $neut(a * b) = neut(a) * neut(b)$ for all $a, b \in X$, then X is referred to as a normal neutrosophic extended triplet group. Let $H \subseteq X$, if H is a NT-subgroup of X, then the relation of H and X can be denoted by $H \lhd X$. Whence, for any fixed $a \in X$, H is called a-normal NT-subgroup of X, written by $H \lhd_a X$, if $a * y * anti(a) \in H$ for all $y \in H$.

Definition 9 Jaiyeola & Smarandache, 2018. Let $f : X \to Y$ be a mapping such that X and Y are two neutrosophic extended triplet groups. Then f is referred to as a neutrosophic extended triplet group homomorphism if $f(c * d) = f(c) * f(d)$ for all $c, d \in X$. The kernel of f at $a \in X$ is defined by

$\ker f_a = \{x \in X : f(x) = neut(f(a))\}$

The kernel of f is defined by

$\ker f = \underset{a \in X}{\cup} \ker f_a$

where $X_a = \{x \in X : neut(x) = neut(a)\}$.

Theorem 7. *Let $f : X \to Y$ be a homomorphism, where X and Y are two singular neutrosophic extended triplet groups.*

(1) For all $a \in X$, $f(neut(a)) = neut(f(a))$ and $f(anti(a)) = anti(f(a))$.
(2) If H is a NT-subgroup of X, then $f(H)$ is a NT-subgroup of Y.
(3) If K is a NT-subgroup of Y and $f^{-1}(K) \neq \emptyset$, then $f^{-1}(K)$ is a NT-subgroup of X.
(4) If X is a normal neutrosophic extended triplet group and the set $X_f = \{(neut(a), f(a)) : a \in X\}$ with the product $(neut(a), f(a)) * (neut(b), f(b)) := (neut(a * b), f(a * b))$, then X_f is a neutrosophic extended triplet group.

Proof.

(1) For all a in X,
$f(neut(a)) * f(a) = f(neut(a) * a) = f(a)$,
$f(a) * f(neut(a)) = f(a * neut(a)) = f(a)$.

On the other hand,

$f(anti(a)) * f(a) = f(anti(a) * a) = f(neut(a))$,

$f(a) * f(anti(a)) = f(a * anti(a)) = f(neut(a))$.

Combining above facts, we get that $f(neut(a)) = neut(f(a))$. Thus,

$f(anti(a)) * f(a) = f(anti(a) * a) = f(neut(a)) = neut(f(a))$,

$f(a) * f(anti(a)) = f(a * anti(a)) = f(neut(a)) = neut(f(a))$.

Since X is singular, so $anti(f(a))$ is unique. It follows that $f(anti(a)) = anti(f(a))$.

(2) For any $f(h_1), f(h_2) \in f(H) = \{f(h) : h \in H\}$, where $h_1, h_2 \in H$. Since H is a NT-subgroup of X, then $h_1 * h_2 \in H$. Thus, $f(h_1) * f(h_2) = f(h_1 * h_2) \in f(H)$.

Moreover, for all $f(h) \in f(H)$, where $h \in H$. Since H is a NT-subgroup of X, then $anti(h) \in H$, by Proposition 2. Thus, applying (1) we have

$anti(f(h)) = f(anti(h)) \in f(H)$.

Therefore, using Proposition 2, we know that $f(H)$ is a NT-subgroup of Y.

(3) Suppose that K is a NT-subgroup of Y and $f^{-1}(K) \neq \emptyset$. For any $a, b \in f^{-1}(K)$, there exists $k_1, k_2 \in K$ such that $f(a) = k_1, f(b) = k_2$. Since K is a NT-subgroup of Y, then (by Proposition 2) $k_1, k_2 \in K$, $anti(k_1) \in K$, and (applying Definition 9 and (1))

$f(a * b) = f(a) * f(b) = k_1 * k_2 \in K$;
$f(anti(a)) = anti(f(a)) = anti(k_1) \in K$.

Thus, $a, b \in f^{-1}(K)$, $anti(a) \in f^{-1}(K)$. Therefore, by Proposition 2, we get that $f^{-1}(K)$ is a NT-subgroup of X.

(4) Applying Theorem 6 and (1), we can verify that (4) holds. □

Theorem 8. *Let $f : X \to Y$ be a neutrosophic extended triplet group homomorphism, where X and Y are two singular neutrosophic extended triplet groups. Then*

(1) $a \in X_a$ for all a in X and $X = \underset{a \in X}{\cup} X_a$.
(2) for all a and b belong to X, $X_a \cap X_b = \emptyset$ or $X_a = X_b$.
(3) for all a in X, $neut(a) \in X_a$ and $anti(a) \in X_a$.
(4) if $x \in X_a$, then $neut(x) \in X_a$ and $anti(x) \in X_a$.
(5) if $x, y \in X_a$, then $x * y \in X_a$ and $anti(x * y) = anti(y) * anti(x)$.
(6) for all a in X, $(X_a, *)$ is a NT-subgroup of X and it is a classical group.
(7) $X_a \lhd_a X$.
(8) X_a is a normal neutrosophic extended triplet group.
(9) $\ker f_a \lhd_a X$.
(10) the binary relation $\approx$ is an equivalence relation on X_a, where $\approx$ is defined as follows: for $x, y \in X_a$, $x \approx y$ if and only if $anti(x) * y \in \ker f_a$.
(11) $X_a = \underset{c \in X_a}{\cup} (c * \ker f_a), \forall a \in X$.
(12) If X is finite, $|X_a| = \sum_{c*\ker\ f_a,\ c \in X_a} |c * \ker f_a| = [X_a : \ker f_a] \cdot |c * \ker f_a|$, for all $a \in X$, where $[X_a : \ker f_a]$ is the number of distinct $c * \ker f_a$ in X_a, and the summation is done for all the different $c * \ker f_a$.
(13) If X is finite, $|X| = \sum_{X_a,\ a \in X,\ c \in X_a} [X_a : \ker f_a] \cdot |c * \ker f_a|$, the summation is done for all the different X_a and corresponding $\ker f_a$. That is, only one calculation coincides with X_a and X_b.

Proof.

(1) For any a in X, by the definition of X_a, we know that $a \in X_a$. Then, $X \subseteq \underset{a \in X}{\cup} X_a$. Moreover, obviously, $\underset{a \in X}{\cup} X_a \subseteq X$. Thus, $X = \underset{a \in X}{\cup} X_a$.
(2) For any a, b in X, if $neut(a) = neut(b)$, then $X_a = X_b$.

If $neut(a) \neq neut(b)$, then $X_a \cap X_b = \emptyset$. In fact, assuming that there exists $x \in X_a \cap X_b$, then we have $neut(x) = neut(a)$ and $neut(x) = neut(b)$. From this and using Theorem 4 (1), $neut(a) = neut(b) = neut(x)$, this is a contradiction with $neut(a) \neq neut(b)$.

(3) By Theorem 5 (2), $neut(neut(a)) = neut(a)$, then $neut(a) \in X_a$. Applying Theorem 6 (4), $neut(anti(a)) = neut(a)$, it follows that $anti(a) \in X_a$.

(4) Assume $x \in X_a$, then $neut(x) = neut(a)$. Using Theorem 5 (2) and Theorem 6 (4) we have

$$neut(neut(x)) = neut(x) = neut(a),$$
$$neut(anti(x)) = neut(x) = neut(a).$$

Therefore, $neut(x) \in X_a$, $anti(x) \in X_a$.

(5) Suppose $x, y \in X_a$, then $neut(x) = neut(y) = neut(a)$. Thus,

$$neut(a) * (x * y) = neut(x) * (x * y) = (neut(x) * x) * y = x * y,$$

$$(x * y) * neut(a) = (x * y) * neut(y) = x * (y * neut(y)) = x * y.$$

On the other hand,

$$(x * y) * (anti(y) * anti(x)) = x * (y * anti(y)) * anti(x) = x * neut(y) * anti(x)$$
$$= x * neut(x) * anti(x) = x * anti(x) = neut(x) = neut(a).$$
$$(anti(y) * anti(x)) * (x * y) = anti(y) * (anti(x) * x) * y$$
$$= anti(y) * neut(x) * y = anti(y) * neut(y) * y = anti(y) * y = neut(y) = neut(a).$$

Therefore, $neut(x * y) = neut(a)$, and $anti(x * y) = anti(y) * anti(x)$. It follows that $x * y \in X_a$.

(6) By (4) and (5), applying Proposition 2 we know that $(X_a, *)$ is a NT-subgroup of X. Since X is singular, and for all $x \in X_a, neut(x) = neut(a)$, so $(X_a, *)$ is a classical group.

(7) From (6), $(X_a, *)$ is a NT-subgroup of X. If $x \in X_a$, then $neut(x) = neut(a)$, and $a * x * anti(a) \in X_a$, by (1), (3) and (5). Thus, by Theorem 6(1),

$$(a * x * anti(a)) * neut(a) = (a * x) * (anti(a) * neut(a)) = a * x * anti(a),$$
$$neut(a) * (a * x * anti(a)) = (neut(a) * a) * (x * anti(a)) = a * x * anti(a).$$

It follows that $neut(a) = neut(a * x * anti(a))$, since $(X_a, *)$ is a classical group. Therefore, $a * x * anti(a) \in X_a$, by Definition 8, we get that $X_a \lhd_a X$.

(8) For all a in X, by (6), $(X_a, *)$ is a neutrosophic extended triplet group. If $x, y \in X_a$, then $neut(x) = neut(y) = neut(a)$. From the proof of (5), we have $neut(x * y) = neut(a)$. Applying Theorem 4 (2), $neut(a) * neut(a) = neut(a)$. Thus,

$$neut(x*y) = neut(a) = neut(a) * neut(a) = neut(x) * neut(y).$$

According to Definition 8, we get that X_a is a normal neutrosophic extended triplet group.

(9) Suppose $x, y \in \ker f_a$, then $f(x) = f(y) = neut(f(a))$. Thus, by Theorem 4 (2), Theorem 7 (1), and Theorem 6 (2),

$$f(x * y) = f(x) * f(y) = neut(f(a)) * neut(f(a)) = neut(f(a)).$$
$$f(anti(x)) = anti(f(x)) = anti(neut(f(a))) = neut(f(a)).$$

It follows that $x * y \in \ker f_a$, $anti(x) \in \ker f_a$. Applying Proposition 2 we get that X_a is a NT-subgroup.

If $x \in \ker f_a$, then $f(x) = neut(f(a))$. Considering $a * x * anti(a)$, we have

$$\begin{aligned} f(a * x * anti(a)) &= f(a) * f(x) * f(anti(a)) \\ &= f(a) * neut(f(a)) * f(anti(a)) \\ &= f(a) * f(anti(a)) \\ &= f(a * anti(a)) \\ &= f(neut(a)) \\ &= neut(f(a)). \end{aligned}$$

Thus, $a * x * anti(a) \in \ker f_a$, according to Definition 8, we get that $\ker f_a \lhd_a X$.

(10) Define the binary relation $\approx$ on X_a as follows:

for $x, y \in X_a$, $x \approx y$ if and only if $anti(x) * y \in \ker f_a$. Then,

(i) for any x in X_a, $x \approx x$. In fact, $f(anti(x) * x) = f(neut(x)) = f(neut(a)) = neut(f(a))$, that is, $anti(x) * x \in \ker f_a$.

(ii) if $x \approx y$ then $y \approx x$. In fact, if $x \approx y$, then $anti(x) * y \in \ker f_a$. Using (9), we have $anti(anti(x) * y) \in \ker f_a$. On the other hand,

$$(anti(y) * x) * (anti(x) * y) = anti(y) * (x * anti(x)) * y$$
$$= anti(y) * neut(x) * y = anti(y) * neut(y) * y = neut(y) = neut(a).$$
$$(anti(x) * y) * (anti(y) * x) = anti(x) * (y * anti(y)) * x$$
$$= anti(x) * neut(y) * x = anti(x) * neut(x) * x = neut(x) = neut(a).$$

From this, applying (6),

$$anti(y) * x = anti(anti(x) * y).$$

Thus, $anti(y) * x = anti(anti(x) * y) \in \ker f_a$. That is, $y \approx x$.

(iii) If $x \approx y$ and $y \approx z$, then $x \approx z$. In fact, from $x \approx y$ and $y \approx z$, we have $anti(x) * y \in \ker f_a$, $anti(y) * z \in \ker f_a$. Using (6) and (9), we have

$$anti(x) * z = anti(x) * neut(z) * z = anti(x) * neut(y) * z$$
$$= anti(x) * (y * anti(y)) * z = (anti(x) * y) * (anti(y) * z) \in \ker f_a.$$

That is, $x \approx z$.

(11) Now, we prove that the equivalence class $[c]_{\approx}$ of c is equal to $c * kerf_a$, for all c in X_a.

(i) for c in X_a, $c \in c * \ker f_a$. In fact, by the definition of $\ker f_a$, $neut(a) \in \ker f_a$. From this, applying (6), we have

$$c = c * neut(c) = c * neut(a) \in c * \ker f_a.$$

(ii) for any x in $\ker f_a$, $c * x \approx c$, that is, $c * \ker f_a \subseteq [c]_{\approx}$. Since,

$$anti(c) * (c * x) = (anti(c) * c) * x = neut(c) * x = neut(a) * x \in \ker f_a.$$

That is, $c * x \approx c$.

(iii) for any a in $[c]_{\approx}$, there exists $x \in \ker f_a$ such that $a = c * x$, that is, $[c]_{\approx} \subseteq c * \ker f_a$. Since $a \approx c$, then $anti(c) * a \in \ker f_a$. Denote $x = anti(c) * a$, then $x \in \ker f_a$ and

$$a = neut(a) * a = neut(c) * a = c * anti(c) * a = c * x \in c * \ker f_a.$$

Therefore, $[c]_{\approx} = c * \ker f_a$, and $X_a = \bigcup_{c \in X_a} (c * \ker\ f_a)$.

(12) It follows from (10) and (11).
(13) It follows from (1), (2) and (12).

Theorem 9. *Let X and Y be two singular neutrosophic extended triplet groups, $a \in X$ and $f : X \to Y$ be a neutrosophic extended triplet group homomorphism. Then f is a monomorphism if and only if $\ker f_a = \{neut(a)\}$ for all $a \in X$.*

Proof.

(1) Assuming that f is a monomorphism, then for any x in $\ker f_a$ we have $f(x) = neut(f(a)) = f(neut(a))$, thus $x = neut(a)$. That is, $\ker f_a = \{neut(a)\}$.

Conversely, assuming $\ker f_a = \{neut(a)\}$ for all a in X, if $f(x) = f(y), x, y \in X$, then

$$f(anti(x) * y) = f(anti(x)) * f(y) = f(anti(x)) * f(x) = f(anti(x) * x) = f(neut(x)) = neut(f(x)).$$

$$f(y * anti(x)) = f(y) * f(anti(x)) = f(x) * f(anti(x)) = f(x * anti(x)) = f(neut(x)) = neut(f(x)).$$

Thus, $anti(x) * y \in \ker f_x$ and $y * anti(x) \in \ker f_x$. Since $\ker f_x = \{neut(x)\}$, so $anti(x) * y = neut(x)$, $y * anti(x) = neut(x)$. Applying Theorem 6 (4), we get

$$anti(x) * y = neut(x) = neut(anti(x)),\ y * anti(x) = neut(x) = neut(anti(x)).$$

This means that is the opposite element of $anti(x)$. Since X is singular, the opposite element is unique, thus

$$y = anti(anti(x)).$$

Using Theorem 6 (3), we have $y = anti(anti(x)) = x$. That is, f is a monomorphism. □

Theorem 10. *$(X, *)$ is a singular neutrosophic extended triplet group if and only if $(X, *)$ is a generalized group.*

Proof. By Definition 6, we know that every generalized group is a singular neutrosophic extended triplet group.

Conversely, assuming that $(X, *)$ is a singular neutrosophic extended triplet group, then, we only need to prove that

for any $x, a \in X$, $a * x = x * a = x$ implies $a = neut(x)$.

In fact, from $a * x = x$, and applying Theorem 5 (2), we have

$$a * neut(x) = a * (x * anti(x)) = (a * x) * anti(x) = x * anti(x) = neut(x) = neut(neut(x)).$$

Similarly, we can get $neut(x) * a = neut(neut(x))$. That is,

$$a * neut(x) = neut(x) * a = neut(neut(x)).$$

This means that a is a opposite element of $neut(x)$. Since $(X, *)$ is a singular, it follows that $a = anti(neut(x))$. Using Theorem 6(2), we obtain $anti(neut(x)) = neut(x)$.

Therefore, $a = anti(neut(x)) = neut(x)$. From this, we know that $(X, *)$ is a generalized group. □

5. Conclusions

In this paper, neutrosophic extended triplet group (NETG) is discussed in depth, thereby some erroneous conclusions in the literature are corrected, and the differences between NETG and generalized group highlighted. From the results of this paper, the following algebraic structures: generalized group (GG), singular neutrosophic extended triplet group (SNETG) and completely simple semigroup are equivalent to each other. Therefore, it is discovered that a generalized group is a special type of neutrosophic extended triplet group, and NETG is a more extensive algebraic system than group and generalized group. On one hand, NETG preserves some properties of group (such as the Lagrange-like theorem obtained in this article). On the other hand, NETG has many characteristics different from group and generalized group. In the future, it is needful to do a more detailed and in-depth study to reveal its structural characteristics, and we will expand our research on some new developments in algebras and neutrosophic sets (see Zhang, Borzooei, & Jun, 2018; Liu, Zhang, Liu & Wang, 2016; Liu and Tang, 2016; Liu and Shi, 2017; Liu and Zhang, 2018; Ye, 2018).

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A Re-Introduction of Pancasila from Neutrosophic Logic Perspective: In search of the root cause of deep problems of modern societies

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Victor Christianto, Robby I. Chandra, Florentin Smarandache (2021). A Re-Introduction of Pancasila from Neutrosophic Logic Perspective: In search of the root cause of deep problems of modern societies. *New Perspective in Theology and Religious Studies*, 2(2), 21-36

Abstract

The present economic crises induced by covid pandemic have called our attention to reconsider where we are heading as a global community; because as we know with the emergence of ubiquitous Internet, then the world has become a global village in real sense.. Shall we lend ourselves to directive and -at times- insistence to move to new economy called the industrial revolution 4.0? Or is there another way, even if it seems like a less traveled path for now? In this article, we also re-introduce *Pancasila* from Indonesian *weltanschauung* (fundamental tenets) to become one of these less travelled path available at our table. The essence of the Indonesian Five Principles (Pancasila) is to return to spirit of communal values, but in a peaceful way, not via revolution. That is a path that in Indonesia, is called as "*gotong royong*" (or to put it in a more scientific term: *cooperative collective dynamics*).

Keywords: *Pancasila, Indonesia studies, state capitalism, welfare state, free market, cooperative collective dynamics*.

Introduction

The present economic crises induced by covid pandemic have called for our attention to reconsider where we are heading at a global community. Shall we lend ourselves to directive and at time rather insistence to move to new economy called the Industrial Revolution version 4.0? Or is there another way, even if it seems like a less traveled path for now? We tend to think that the insistence to move in a global level towards Industrial Revolution 4.0 as advocated by Klaus Schwab and other Davos Club proponents will only benefit those 1% globalist elites (see also Vandana Shiva's book "*The Oneness versus 1%*"[8-9]). No wonder that many people begin to protest several Davos meetings in the past, because they realize that those elites often decide to maximize their own interests, while they act in the name of global society as a whole.

In this article, we also try to re-introduce *Pancasila* or Five Principles from Indonesian *weltanschauung* to become one of these less travelled path available at our table.

Problem statement

This article is a continuation to two papers by us, one paper is a contribution chapter to Nova Science book [10], and one is a recent paper for an upcoming paper. The problem discussed here can be summarized as follows:

"*What is the root cause of problems in modern societies, be it in psychological term, theological term, or spiritual term?*"

Scope and Limitations of this article

The scope of this article is around psychological and theological meaning among the present tensions among worldviews (or *weltanschauung*).

Limitations of this article are that we do not offer in depth economics or political analysis in each country. Instead, we focus on deep societal problems as a wholeness.

Methodology

The methodology used in this article is literature survey along with analysis of recent issues, especially in psychological and theological analysis.

What is really going on in USA?

While we admit that none of these authors are political analysts by profession, let us put our discussion in present days context. The following is a short conversation by one of us (VC) and a professor of mathematics and logics in Canada. He asked on what is VC's opinion on what presently happens in USA. The following is a quote of VC's response to address that question:

> "First of all, there are satanists who are working out to turn USA into communist socialism society. There are lots of effort to implement cultural marxism and also cloward-piven strategy[1] into USA (that is also obama plan and then it is continued by the present administration).

Actually, I do not really like to comment on who is right, socialist countries or USA? To me both are only playing extreme sides: capitalism versus socialism.[2] On the root cause: it is dialectical philosophy of Hegel all along. Thesis meets antithesis and then conflicts and more conflicts, and the modern version is a book by Samuel Huntington. And the essence is people especially scientists rely too much on rationalism.

Thanks God, there are Coptic church leaders like Milad Hanna (already deceased) who did not agree with Huntington's clashism. He offers a much more humane term: Acceptance of the Others.

On a deeper level, too much reliance on rationalism will wipe off the entire humanity. But there is an old mathematician, he is also a Christian, the name is Dr Dennis P. Allen, Jr.. He wrote a memoir on possibility to work out a "realism" part of mathematics. That is a hope to us, as I see it.

And plus, from history we learn that relying too much on rationalism can be traced back to Pythagoreans. They worshipped rationalism until they forced a pupil who invented irrational number to drowning in the sea."

Tracking the problems in psychological term

One of these problems can be stated as follows: that human being has been reduced into *homo economicus;* in other words philosophers and economists alike

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[1] If some readers don't know what is cultural marxism or what is cloward-piven strategy, you are advised to googling, for instance check on writings by Prof. Jordan Peterson etc. See also for instance Ref. [12-14].

[2] Or to be more precise in Beginda Pakpahan's term: between *progressive capitalism* and *state (driven) capitalism.*

have put economics terms as our sole goal. Such a deep flaw can be traced back to materialist view predominating in sciences, see Mario Beauregard [7]. Therefore, most non-materialist view is ignored.

As VC wrote in a chapter in Nova Science book [10]:

> "And also we can recall from Genesis 3 that the first fall of our ancestors came from greediness. How far we have fallen in this modern society, where greed has been hailed as highest virtue. To emphasize this root problem in our modern society, allow me to quote Grekko's remark: Greed is good. Quote from Wall Street movie:

"The point is, ladies and gentleman, that greed, for lack of a better word, is good. Greed is right, greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit. Greed, in all of its forms; greed for life, for money, for love, knowledge has marked the upward surge of mankind."1

We consider this is one basis of modern reality: Most of us have been consumed by greed and are drowning in an ocean of greed. The real irony is that greed has eaten us alive, from our childhood until we die."

So, what are socio-economic implications? Again, as one of us (VC) wrote in Nova science book [10]:

> Yes, normally you read numerous political-economics jargons, e.g. leftist, right wing, centrist left or centrist right and so on. But it is not our intention to submit yet another ideological parlance. In fact, these authors are scientists and mathematicians, so we are not inclined to any such parlance.[3]

In our opinion, our tendency to cooperate or compete is partly influenced by the culture that we inherit from our ancestors. One of us (VC) once lived for a while in Russia, and he found that many people there are rather cold and distant (of course not all of them, some are friendly). He learned that such a trait is quite common in many countries in Europe. They tend to be individual and keep a distant to each other. In physics term, they are like fermions.7

[3] See also for instance Ref. [12-14].

There is a developmental psychology hypothesis that suggests that perhaps such a trait correlates to the fact that many children in Europe lack nurturing and human touch from their parents, which makes them rather cold and individualistic. Of course, whether this is true correlation or not, it should be verified.

On the other hand, most people in Asia are gregariously “groupie” in behaviors (except perhaps in big metropolitan areas). They tend to spend much time with family and friends, just like many Italians. They attend religious rituals regularly, and so on. In physics term, they are bosons. Of course, this sweeping generalisation may be oversimplifying.8

That is why we choose to work out Mancur Olson’s theorem in more detail,because he is able to condense complicated game theory reasoning (whether one should cooperate or not) into a matter of collective actions. So, which is better: to be like fermions or bosons? Our opinion is: both fermions and bosons are required. In the same way, fermion behaviour and boson behaviour are both needed to advance the quality of life. Fermion people tend to strive toward human progress, while boson people are those who make us alive.”

That is why we tend to argue in favor of theo-antropological view, called: *indivi-group*, i.e. human being is both individuals and also part of their societies/communities. Or in particle term, we may call that human beings are like *fersons* (composed of mixed fermion and bosons). This is our hypothesis in this article.

Tracking the problems in spiritual term: How can we connect those fundamental problems in society to the divided brain? One of most interesting insight came from Iain McGilchrist. In his book, *The master and his emissary* [6], he suggests us to look at our divided brain: the deep polarization caused by two hemispheres of human brain have led mankind astray. In essence, his arguments can be summarized as follows: learning from church fathers until St Augustine, we can read an integrative perspective and harmony between left and right brain. But since the work of scholastic theologians, including Thomas Aquinas, our theological thoughts have gone down the road where the left brain predominated the entire brain function.

If we borrow Iain McGilchrist’s term, then the emissary has become the master; or in other words; left brain function seems to take over the entire human societies.[6] Therefore, people should better learn how to think more intuitively, more holistic and *more with heart not just logic and reason* [3].

In the same way, we can capture the essence of many problems in scientific development is caused by our too much reliance on the left-side of human brain. Or in terms of Yin-Yang (Asian philosophy) we need more Yin touch, who are more adept to intuition, holistic thinking and respect to life and care.

In essence we can say that what McGilchrist wishes to say is that too much logic and left-brain functions will make the emissary rule out the entire humanity and the result is doom, that is the meaning of the sixth extinction which is already in our door (see again the book by Dr. Vandana Shiva, she is a physicist turns ecologist and activist [8-9]).

And if we put McGilchrist's term of *divided brain* into a more spiritual realm, then it is a call to all of us, as the entire humanity, that if we don't want to succumb to darkness. That we shall not eat too much fruit of knowledge, and we ought to learn to eat fruit of life (Referring to Genesis chapter 3).

More on fruit of life: Re-introducing Pancasila, the Five Principles of Indonesia

Enough is enough for fruit of knowledge. Now what shall we do in order to eat more on fruit of life? Let us consider again Iain McGilchrist. As a psychiatrist, his argument on left and right function of human brain can be captured in essence as follows: the left hemisphere which usually processes in detailed manner any problem (logically) should not predominate the right brain, which capture holistic and spiritual process. In the words of Blaise Pascal: ”*The heart has its own logic, which reason cannot understand*.”[4]

In that sense, both heart as spiritual brain function should not be governed by the left brain function. In other words, in spirituality realm especially in worshiping God, we should not let the emissary (Logical process) to lead the master. It should be the other way around.

[4] https://headhearthand.org/blog/2015/05/07/21256/

This problem of choosing between Logic or going beyond Logic, or from rationality to go beyond rational thinking can be traced back even to classical history of mathematics. It is known that Pythagoreans worshiped rationality and Logic in mathematics so much, up to the point when they were shocked when one of their disciples found an irrational number, those Pythagoreans left that disciple to drowning in the sea.

So, we know that what McGilchrist described is a real issue, and not just a joke. Therefore, we need to shift our emphasis intentionally from knowledge-seeker toward more wisdom-insight-intuition seeker.[5] We need to learn to care for each other, to be more compassionate for those in needs around us. We guess that we can connect those compassionate and caring to Asian traditional values, which seem to us returning us to the main point of this article: it is now the time to not being captured in dialectical logic *between [A]* free market capitalism of the West which then has evolved into progressive capitalism, and [B] state driven capitalism or socialism.

That there is no middle ground between [A] and [B} entities is the one of basic premise of Aristotelian logic. We ought to move forward to non-Aristotelian Logic. For instance in Neutrosophic Logic, we can consider that there is dynamics of neutrality between [A] and [B} entities, in other words, we shall consider included middle theorem.

In that way, perhaps we can consider what Beginda Pakpahan wrote:

> "I would argue that the concept of a *Pancasila-based economy is positioned between progressive capitalism and new socialism* and can be seen as a middle way for Indonesia to respond to the global economic crisis and to secure its national interests. It is a mixed model, demonstrating the role of the state in institutional reform, policy design and socio-economic development, while simultaneously promoting the spirit of social justice through effective partnerships between the public and private sectors and other relevant stakeholders."[1]

[5] In the Old Testament thoughts, it is supposed that knowledge and intelligence are often connected to wisdom from heart, not just brain/intellectual capacity, that is why it was thought that wisdom comes from God Almighty. See for instance the Book of Proverbs.

That is the essence of our message in this article, we don't have to catch up endlessly with Industrial Revolution 4.0 which tends to disrupt the entire global economy without many benefit of majority, but instead only will make the top 1% even more greedy to capture the entire global resources. (In the next section, we will discuss two ways we can do that to go beyond disruptive changes, which also tend to be destructive to the entire economy.)

Instead, we shall begin to learn to develop national and regional economies based on caring and empowering, or in scientific term: *cooperative collective dynamics*.[6] For instance, the dynamics of Subak as community based irrigation system in Bali can be taught in engineering or sociology schools. That is a good way to return to nature, live in *slow living (in Danish term: hygge, or Swedish term: lagom)*, to love our neighbors and develop based on communities [5].[7]

Two examples on how we can implement the relational economics concept into more practical way

In the preceding sections, we discussed the root cause of deep problems in modern societies including our economics approach and technological approach, and in essence our dichotomic approach toward separating human and nature. In this section we discuss how to put the aforementioned ideas into actions.

A. Koinomics : doing economics in Trintarian perichoresis way [15]

Deeply embedded in Christian theology, God, the Father, is understood as the Creator, the Giver of Life, and the universe. Further, by God's grace, human beings receive the potential to become the stewards of God's creation, grow, multiply, and glorify the Almighty. However, it is the basic tenet of Christian faith that God has the intention that human beings enjoy a relationship with their Creator and with themselves but not being forced to do so. Therefore, God endows human beings a capability to make a choice.

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[6] H. Guo et al. *New Journal of Physics*. Url: https://iopscience.iop.org/article/10.1088/1367-2630/ab9e89

[7] See also our recent draft article [5], and also next section, where we outlined how we can implement the relational economics concept into more practical way in micro and ultramicro economics setting.

As human beings make a wrong choice by focusing on their wish and centeredness, they live with total depravity, broken relationship with God and their own, thus bearing the dire consequences. Self-centeredness, competition, domination, self-protection, and the likes become the game rules in social, political, and economic life. As the creation gradually evolves toward extinction, once again, God gives a special grace through the life of Jesus Christ. God dwells among humans and redeems them. God also offered reconciliation between the creature and the Creator freely. Further, Christian theologians then, point out that Christian ethics should root on gratefulness for such a reconciliation act of God and to foster a communal relation based on such thankfulness.

The process is incomplete. Churches have long neglected in their theology that God has invited the forgiven human beings to enter a gradual transformation process. In theology, people understand that God has entered their lives as the Holy Spirit. They who have been living in God's grace should develop their capacity to make choices to live primarily for themselves or live in relation with others and with God. If they choose the latter, they should live by following God's internal relationality. It is God's transformative grace. Thus, grace and relationship are two central tenets in Christian theology.

As in the relational dimension of God, known as a communion, fellowship, or in Greek, koinonia, each person or community learns to view themselves as an inseparable part of humanity. Participating or being in connection does not mean only taking part in a program or embracing doctrines. It means to enter other peoples' lives and allow others to join our own life. It also means to have a life rotates or centres in grace.

Another word in Greek might also express the concept sharply. The term is koinoikos. Its meaning is social, friendly, apt to form and maintain communion or fellowship. It also means the inclination to make others share in one's possessions and impart or be free in giving.

This dimension of koinonia or koinonikos will be incomplete without being tied up with the term perichoresis. Slobodan Stamatović state that "… *perichoresis as a theological terminus technicus originally appeared in the late Patristics (7th and*

8th century) and that it irretrievably entered the theological endeavour through the influential work of John Damascene (†750 AD).[8]

In this context, the introduction of a new term, *koinomics* is in order. The name derives from two terms "koinonia" and "economics." The word *"koinonia"* or *"koinon"* – comes from the New Testament. Koinonia itself in the New Testament does not have a single meaning as *koinonia* appears nineteen times.

The words related to and the root-word koinon occur 46 times, mostly in Paul's letters and some in John's letters, Peter's letters, letters Hebrews, and Acts. In the gospels, the word *koinonia* does not appear. However, some terms have roots in *Koino*.

From various sources, it is evident that the word koinonia's meaning comes from the word *koinos*, which means *joint* or *communal*. *Koinon or koinonia* has a broad definition of fellowship, friendship, and close relationships (Fuchs, 2008). A nun and activist of the ecumenical movement, Lorelei Fuchs, also explained that *koinonia* has many meanings. The meanings are communion, acting together, friendship, reciprocity, participating, helping, sharing, solidarity, togetherness, cohesion, unity, and wholeness.[9]

In the context of the Trinity, the word koinonia interconnects with *perichoresis. Perichoresis* means The Triune God moves to one another in a cosmic dance, complementary to each other.[10]

Case study:

Twelve pastors and an agricultural engineer from Indonesia studied the national food production system, supply, and demand in Indonesia. At that time, many farmers burned their harvests as no middlemen appear as usual to buy anything from them. COVID-19 disrupted the food source and the supply line. The study of those pastors triggered A *Food Terminal* program with an objective: to bridge the farmers who live in remote areas as producers with the customers who live in big cities. The

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[8] Stamatović, Slobodan. (2016). The Meaning of Perichoresis. Open Theology. 2. 10.1515/opt-2016-0026. p. 303.

[9] Lorelei Fuchs. Koinonia and the quest for ecumenical ecclesiology. Wm. B. Eerdmans.

[10] *See also Tihomir Lazic. Koinonia. MA Thesis submitted to Newbold College, April 2008. http://n10308uk.eos-intl.eu/eosuksql01_N10308UK_Documents/Dissertations/Lazic.pdf*

farmers will receive higher income for what they produce while the customers will mostly have fresh and organic products.

The idea was then, supported by the Kayu Putih Church, a church in the capital city that allows the Terminal Pangan to use the church's space as storage for the products. A couple of donors from other churches supported them with the initial capital, about USD 1500.00 to buy the food products, refrigerators, and operational expenditure.

Given that COVID-19 was rampant, the Food Terminal management asks purchase orders from their customers through digital channels and afterwards sends requests to several farming communities. Three days later, the food products will arrive at the church complex.

Later, besides the fruits and vegetables, fresh seafood from a nearby fishing community started to enter the Food Terminal. The Food Terminal opens once a week. The customers will receive their orders as the terminal hires church members who lost their job or need additional income as the delivery team members. Thus, if this is a drama, the actors are the Food Terminal workers who are mostly voluntary, the fishermen and farmers, the customers, and the delivery team.

In a month, four responses emerged from the congregation members. The first was a rejection that a church got involved in the business world. The second, a harsh critique came concerning the quality of products that were not at the level that the customers wanted. The third response was that the Food Terminal management is not professional enough as frequently they made mistakes as they sent few products that the customers did not order. In many cases, even, they forgot to send bills for the food.

Illustration 1. Staffs prepared the vegetables before delivering to customer (Food Terminal Jakarta)

B. Smooth changes: how to develop non-disruptive creations instead of disruptive technologies [16]

In recent years, there is an alternative scheme in corporate strategy discourse, called Blue Ocean (shift) Strategy by W. Chan Kim and R. Mauborgne. In this paper we offer a new insight based on Neutrosophic Logic perspective, which combines red ocean and blue ocean, while a company moves forward and shift to blue ocean space.

In their Blue Ocean Shift, W. Chan Kim and R. Mauborgne offer some clear and good examples of organizations who have made such a transition to blue ocean [17]. There is the case of an inn network that applied the demonstrated advances plot in the book to break out of the exceptionally serious inn industry – which is 'redder than red'– to make the new market of moderate lavish inns offering five-star comfort at three-star costs. Today it has 90% inhabitance rates, visitor appraisals called it 'magnificent' and 'spectacular' on booking destinations, and portrayed it as the least expenses in the most stylish areas. It is turning out to significant urban communities over the world. The book likewise clarifies how a worldwide, little machine organization with over 100 years of history turned an industry, whose worth was declining by 10% every year, into a high-development one. The organization did that by reclassifying its contribution so much that it permits we all today to make scrumptious French fries with no browning and practically no oil. The aftereffect of its work day: Not just requested develop by 40%, its stock cost lifted by 5 percent.

Problems of transition

While the Blue Ocean Shift book has offered some practical tools to help organizations mapping their position and going toward blue ocean, such a transition or shift to become blue ocean organization is not so easy. In physics term, this process can be called as *transition phase.*

In this context, Tantau and Mateesescu offer a bit more realistic pathway, that they call: green ocean, where a mixture of red ocean space and blue ocean space is allowed while an organizations move gradually toward blue ocean.[18]

Such a transition can be seen from Neutrosophic Logic Perspective, albeit with a bit rather different lingo, i.e. in Neutrosophic Logic it is known (T,I,F) means: degree of truth, indeterminacy, and falsehood. Meanwhile, in green ocean scheme, there are R,I,B: x percent of (R) red ocean, indeterminacy, and y percent of (B) blue ocean.

In the meantime, instead of neutrosophic logic we can use Neutrosophy, since in neutrosophy we have in general <A> and <antiA>, the opposites, and the neutral <neutA>. In this case we take Red = <A> and Blue = <antiA>, while green (or other color in between) as part of <neutA>.

To summarize such an approach, we offer the following table:

Description	Red Ocean	Indeterminacy	Blue Ocean
Analogy with Neutrosophic Logic	Truth	Indeterminacy	falsehood

Green Ocean	X percentage of red		Y percentage of blue
In neutrosophy framework	Red <A>	Green <neutA>	Blue <antiA>
Main strategy	Competitive	A mixture	Non-competitive
Porter scheme	Value or low budget trade off	A mixture	Value leap while keeping low budget
Disruptive/non-disruptive pattern	Disruptive innovations		Non-disruptive creations (value leap)

Table 1. Neutrosophic Logic perspective to red-blue ocean mixture

To simplify the above notions, perhaps we should not call it “green ocean strategy” which only makes it more complicated, but perhaps “*brue*” from a mixture of blue ocean-red ocean strategy. (perhaps we can call it : Brue strategy: from “red in mixture with blue.”)

We hope a simple scheme as outlined above can be developed further in the near future. Allow us to remark here that despite some innovation books by those management luminaries emphasize disruptive changes (perhaps they follow Schumpeter train of thought: changes must be creative destruction), we submit more in tune with Kim & Mauborgne, that there is always possibility to introduce non-disruptive creations instead of disruptive changes. See for instance our new draft paper [21].

Concluding remarks

The present economic crises induced by covid pandemic have called for our attention to reconsider where we are heading at a global community. Shall we lend ourselves to directive and at time rather insistence to move to new economy called the Industrial Revolution version 4.0? Or is there another way, even if it seems like a less traveled path for now? We argue here that the insistence to move in a global level towards Industrial Revolution 4.0 as advocated by Klaus Schwab and other Davos Club proponents will only benefit those 1% globalist elites (see also Vandana Shiva's book "*The Oneness versus 1%*"[8-9]). No wonder that many people begin to protest several Davos meetings in the past, because they realize that those elites often decide to maximize their own interests, while they act in the name of global society as a whole.

In this article, we also try to re-introduce Pancasila on the basis of traditional Asian values (which may be linked to the fruit of life in Genesis chapter 3) to become one of these less travelled path available at our table.

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Developing a Novel Approach for Determining the Reliability of Bipolar Neutrosophic Sets and its Application in Multi-Criteria Decision-Making

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This article aims to propose an approach for determining reliability of information collected using questionnaires and bipolar neutrosophic sets. Bipolar neutrosophic sets use six membership functions that express the truth membership, the falsity membership, as well as the indeterminacy membership to the set and complementary set. Therefore, bipolar neutrosophic numbers may be suitable for applying in multi-criteria evaluation when a smaller number of complex evaluation criteria are used. Unfortunately, a significant number of membership functions make them somewhat complex for collecting data by surveying respondents. Using reliability of data decision-makers can identify respondents who did not want to participate in the survey, or did not understand the application of BNNs, and take appropriate action. The usability of the proposed approach is presented through two illustrative examples and conclusions were drawn based on obtained results.

Keywords: Bipolar fuzzy set, neutrosophistic, single-valued bipolar fuzzy number, data reliability, MCDM, decision making

1 INTRODUCTION

Decision-making is a process that is continuously taking place everywhere and by everyone. Therefore, the challenge that arises is to make the optimal decision for a given problem or situation. Decision-making is an essential part of operational research. Due to its popularity and rapid growth, so far it has been used for solving numerous complex decision-making problems [1-5].

MCDM enables the choice of the suitable alternative from the finite set of alternatives, respecting the values of the criterion attributes, i.e. enables the decision-making process in the presence of multiple, generally conflicting criteria [6-7]. Multi-criteria decision-making (MCDM) allows tolerating ambiguities that arise when solving many problems in the decision-making system [8]. Depending on the domain of alternatives, MCDM is divided into Multi-Objective Decision Making (MODM) and Multi-Attribute Decision Making (MADM). The fundamental problem of multi-criteria decision-making is how to reconcile criteria, different preferences, and conflicting interests. Therefore, the primary task of multi-criteria decision-making is to find the best solution in terms of evaluated criteria [9-10].

In due course of time, due to extremely rapid development, many prominent and widely-used MCDM methods are developed, such as: Simple additive weighting – SAW [11]; Analytic hierarchy process – AHP [12]; Elimination et choix traduisant la realit – ELECTRE [13]; Preference ranking organization method for enrichment evaluation – PROMETHEE [14]; Technique for order performance by similarity to ideal solution – TOPSIS [15]; Višekriterijumska optimizacija i kompromisno rešenje – VIKOR [16]; and so forth. Besides, it is worth mentioning a new generation of the MCDM methods and MCDM approaches, such as: A new additive ratio assessment method – ARAS [17]; Multiobjective optimization by ratio analysis plus full multiplicative form – MULTIMOORA [18]; Pivot pair-wise relative criteria importance assessment method – PIPRECIA [19]; Full consistency method – FUCOM [20]; Measurement of alternatives and ranking according to COmpromise solution – MARCOS [21]; Multi-attributive ideal-real comparative analysis method – MAIRCA [22]; A new combinative distance-based assessment – CODAS [23]; and so forth.

To provide a reliable methodology for solving complex decision-making problems, Zadeh [24] introduced a fuzzy set theory. Fuzzy set theory has been used successfully for solving a number of decision-making problems, which is why Bellman and Zadeh proposed fuzzy MCDM [25]. A concise overview of the application of fuzzy logic can be found in [26-29]. However, the membership function to the set, presented in this theory, has not been sufficient for solving some classes of complex decision-making problems, or its determination was difficult. Therefore, some extensions of the fuzzy set theory have been proposed. For example, Attanasov [30] proposed intuitionistic fuzzy sets by introducing non-membership function to the set, while Zhang [31] introduced the concept of bipolar fuzzy sets and suggested the usage of the two membership functions that represent membership to a set and membership to a complementary set.

Smarandache [32] introduced the neutrosophic sets (NS) theory, as the generalization of fuzzy sets and intuitionistic fuzzy sets, introducing truth-membership function, indeterminacy membership function and falsity-membership function, while Wang, Smarandache, Zhang, and Sunderraman [33] further introduced the single-valued neutrosophic sets (SVNS) that are more suitable for solving many real-world decision-making problems. Deli, Ali and Smarandache [34] introduced Bipolar Neutrosophic Sets (BNS) by generalizing the concept of bipolar fuzzy sets.

Neutrosophic set theory has enabled forming extensions of a number of MCDM methods, such as the AHP [35] and EDAS [36] methods, and has also been used for solving a number of decision-making problems, such as the diagnosis of bipolar disorder diseases [37]. A comprehensive overview of the application of neutrosophic sets to solve decision-making problems can be found in [40-41].

One of the areas where NS and BNS can be effectively applied is multiple decision-making criteria based on the use of complex evaluation criteria. However, the use of several membership functions can make evaluation somewhat complex, especially when the evaluation is based on data collected by the survey. Therefore, Smarandache, Stanujkic and Karabasevic [42] proposed an approach to determine the reliability of information collected using questionnaires and single-valued neutrosophic numbers. This research was continued in [43].

In this article, the mentioned approach is extended to the use of bipolar neutrosophic numbers. Therefore, the rest of the manuscript is organized as follows: in Section 2, the basic concepts of the bipolar neutrosophic sets are presented and in Section 3, a procedure for determining the reliability of the information contained in bipolar neutrosophic numbers is proposed. In

Section 4, the usability of the proposed approach is shown in two numerical illustrations. Finally, the conclusions are given.

2 THE BASIC CONCEPTS OF A BIPOLAR NEUTROSOPHIC SET

As is previously mentioned, Zadeh [24] proposed a fuzzy set theory and introduced the membership function.

Definition 1. ***A fuzzy set***. *Let X be a nonempty set. Then, a fuzzy set A in X is a set of ordered pairs [24]:*

$$A = \{ \langle x, \mu_A(x) \rangle | \, x \in X \} \tag{1}$$

where the membership function $\mu_{A(x)}$ denotes the degree of the membership of an element x to the set A, and $\mu_A(x) \in [0, 1]$.

Definition 2. ***A bipolar fuzzy set***. *Let X be a nonempty set. Then, a bipolar fuzzy set is defined as follows [44]:*

$$A = \left\{ \left\langle x, \mu_A^+(x), \upsilon_A^+(x) \right\rangle \middle| \, x \in X \right\} \tag{2}$$

where: the positive membership function $\mu_A^+(x)$denotes the satisfaction degree of the element x to the property corresponding to a bipolar-valued fuzzy set, and the negative membership function $\upsilon_A^+(x)$, denotes the degree of the satisfaction degree of the element x to a corresponding complementary bipolar-valued fuzzy set, respectively; $\mu_A^+ : X \to [0, 1]$ and $\upsilon_A^+ : X \to [0, 1]$.

Definition 3. ***A neutrosophic set***. *Let X be a nonempty set. Then, NS A in X is defined as [32]:*

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle | \, x \in X \} \tag{3}$$

where: $T_A(x)$, $I_A(x)$ and $F_A(x)$, denote the truth-membership $T_A(x)$, the indeterminacy-membership $I_A(x)$ and the falsity-membership functions $F_A(x)$, and T_A, I_A, $F_A : X \to \left]0^-, 1^+\right[$.

In contrast to intuitionistic sets, the restriction regarding the sum of the membership functions is eliminated, so that $0^- \leq T_A(x) + I_A(x) + U_A(x) \leq 3^+$.

Definition 4. ***A single-valued neutrosophic set***. *Let X be a nonempty set. Then, an SVNS A over X is an object having the form [32, 33]:*

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle | \, x \in X \} \tag{4}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively, T_A, I_A, $F_A : X \to [0,1]$and $0 \leq T_A(x) + I_A(x) + U_A(x) \leq 3$.

Smarandache [32] also introduced the Single Valued Neutrosophic Number (SVNN), which can be designated as follows $x = \langle t_x, i_x, f_x \rangle$ for convenience.

Definition 5. ***Bipolar neutrosophic sets***. *Let X be a nonempty set. Then, a BNS A in X is as follows [34]:*

$$A = \left\{ \left\langle x, T_A^+(x), I_A^+(x), F_A^+(x), T_A^-(x), I_A^-(x), F_A^-(x) \right\rangle \middle| x \in X \right\} \quad (5)$$

where: T_A^+, I_A^+, F_A^+ denote the truth-membership, the indeterminacy-membership and the falsity-membership of x to the BNS A, and T_A^-, I_A^-, F_A^- denote the truth membership, the indeterminacy-membership, and the falsity-membership of x to a complementary BNS; $T_A^+, I_A^+, F_A^+ : X \to [0,1]$ and $T_A^-, I_A^-, F_A^- : X \to [-1, 0]$.

Deli, Ali and Smarandache [34] also introduced the Bipolar Neutrosophic Number (BNN), which can be denoted as follows $x = \left\langle t_x^+, i_x^+, f_x^+, t_x^-, i_x^-, f_x^- \right\rangle$ for convenience.

Definition 6. ***Score function of BNN***. *Let be a $x = \left\langle t_x^+, i_x^+, f_x^+, t_x^-, i_x^-, f_x^- \right\rangle$ BNN. The score function $s_{(x)}$ of a BNN is as follows [34]:*

$$s_{(x)} = \left(t_x^+ + 1 - i_x^+ + 1 - f_x^+ + 1 + t_x^- - i_x^- - f_x^- \right)/6 \quad (6)$$

Definition 7. ***A bipolar neutrosophic weighted average operator of BNNs***. *Let $a_j = \left\langle t_j^+, i_j^+, f_j^+, t_j^-, i_j^-, f_j^- \right\rangle$be a collection of BNNs. The bipolar neutrosophic weighted average operator (A_w) of the n dimensions is mapping $A_w : Q_n \to Q$ as follows [34]:*

$$\begin{aligned} &A_w(a_1, a_2, \ldots, a_n) \\ &= \sum_{j=1}^{n} w_j a_j = \left(1 - \prod_{j=1}^{n} \left(1 - t_j^+\right)^{w_j}, \prod_{j=1}^{n} \left(i_j^+\right)^{w_j}, \prod_{j=1}^{n} \left(f_j^+\right)^{w_j}, \right. \\ &\left. - \prod_{j=1}^{n} \left(-t_j^-\right)^{w_j}, - \left(1 - \prod_{j=1}^{n} \left(1 - \left(-i_j^-\right)\right)^{w_j} \right), - \left(1 - \prod_{j=1}^{n} \left(1 - \left(-f_j^-\right)\right)^{w_j} \right) \right) \end{aligned} \quad (7)$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3 RELIABILITY OF THE INFORMATION CONTAINED IN BNNS

NS theory has numerous applications. One of them is in decision making where different types of neutrosophic numbers are used for gathering attitudes from DMs, experts, and/or respondents. Therefore, Smarandache, Stanujkic and Karabasevic [42] proposed an approach to assess the reliability of the information $r_{(x)}$ contained in an SVNN, as follows:

$$r_{(x)} = \frac{t - f}{1 + i^{1/n}} \tag{8}$$

where: t, i, f denote the truth, the indeterminacy and the falsity of information contained in SVNN $x = \langle t,\ \mathrm{i},\ \mathrm{f}\ \rangle$, n denotes a parameter and $n \in [0, 1]$.

For simplicity, the value of the variable n can be set to 0, in which case the previous Eq. (9) has the following form:

$$r_{(x)} = \frac{t - f}{1 + i} \tag{9}$$

Due to their complexity, the reliability of the information contained in the BNN should also include the reliability of the information contained in the complementary NS, as follows:

$$r_{(x)} = r^{+}_{(x)} + r^{-}_{(x)} = 0.5\left(\frac{\left|t^{+} - f^{+}\right|}{1 + i^{-}} + \frac{\left|t^{-} - f^{-}\right|}{1 - i^{-}}\right) \tag{10}$$

where: $r_{(x)}$ denote reliability of the information contained in a BNN, $r_{(x)} \in [0, 1]$ and higher value of $r_{(x)}$ indicates the higher reliability, $r^{+}_{(x)}$ and $r^{-}_{(x)}$ denote reliability contained in the complementary NS and complementary NS.

And finally, each decision matrix in the MCDM contains more rows and more columns, which is why the following equations can be used to determine the average reliability of the information contained in their rows, columns or them, respectively:

$$r_r = \frac{1}{m}\sum_{i=1}^{m} r_{(x_{\mathrm{ij}})} \tag{11}$$

$$r_c = \frac{1}{n}\sum_{j=1}^{n} r_{(x_{\mathrm{ij}})} \tag{12}$$

$$r_m = \frac{1}{m \times n}\sum_{i=1}^{m}\sum_{j=1}^{n} r_{(x_{\mathrm{ij}})} \tag{13}$$

where: m denotes the number of alternatives, n denotes the number of criteria, and x_{ij} denotes rating of alternative i in relation to the criterion j in the form of BNN.

4 NUMERICAL ILLUSTRATIONS

In order to demonstrate the usability of the proposed approach, two numerical illustrations are presented and discussed in this section. The first example is borrowed from Ulucay, Deli and Sahin [45] and slightly modified to accurately demonstrate the usability of the proposed approach. The second example shows the application of the proposed approach in the case of a cloud service provider selection.

4.1 The First Numerical Illustration

Ulucay, Deli and Sahin [45] evaluated four green suppliers based on three criteria, namely: product quality, technology capability, and pollution control.

The decision matrix, whose elements are BNN, is shown in Table 1. The reliability of the information contained in the elements of the decision matrix, determined using Eq. (10), and the average reliability of data contained in their columns and rows, determined by Eqs. (11) and (12) are shown in Table 2. Table 2 also shows the overall reliability, which is 0.39.

It has already been stated that the higher value of the reliability of the information contained in the BNN is more preferable. The average reliability of all information contained in the decision matrix is 0.39. Such a decision

	C_1	C_2	C_3
A_1	< 0.8, 0.0, 0.2, −0.6, −0.1, −0.1 >	< 0.6, 0.1, 0.2, −0.4, 0.0, −0.2 >	< 0.8, 0.1, 0.5, −0.3, 0.0, −0.1 >
A_2	< 0.7, 0.0, 0.2, −0.4, 0.0, −0.2 >	< 0.6, 0.2, 0.3, −0.6, −0.1, −0.3 >	< 0.7, 0.2, 0.5, −0.1, 0.0, −0.8 >
A_3	< 0.7, 0.1, 0.2, −0.2, −0.1, −0.7 >	< 0.9, 0.1, 0.6, 0.1, 0.0, −0.5 >	< 0.6, 0.1, 0.5, −0.2, −0.1, −0.6 >
A_4	< 0.8, 0.0, 0.1, −0.2, 0.0, −0.8 >	< 0.6, 0.0, 0.3, −0.1, −1.0, −0.4 >	< 0.9, 0.1, 0.4, 0.0, 0.0, −0.8 >

TABLE 1
The initial decision matrix.

Source: Ulucay et al. (2018)

	C_1	C_2	C_3	***Average***
A_1	0.53	0.28	0.24	0.35
A_2	0.35	0.26	0.43	0.35
A_3	0.45	0.44	0.23	0.37
A_4	0.65	0.23	0.63	0.50
Average	0.50	0.30	0.38	0.39

TABLE 2
The reliability of the information.

Source: Authors' calculation

matrix can be accepted for further evaluation, but it is noticeable that there is a possibility to increase the reliability of the information.

Using values calculated by Eqs. (10), (11) and (12) decision-makers could analyze the reliability of collected information, make some modifications, and accept the decision matrix for further evaluation or rejected it.

4.2 The Second Numerical Illustration

The second illustration shows the partial results of an evaluation of four cloud service providers based on four criteria. For the sake of simplicity and more precise presentation, only the characteristic results obtained from three DMs and four evaluation criteria are presented below. The criteria used for evaluation are: Security – (C_1), Service levels - (C_2), Support - (C_3), and Costs - (C_4).

During the completion of the questionnaire, the respondents expressed their attitudes using values from the interval [0, 1], for positive membership functions, and values from the interval [0, 1], for negative membership functions. The use of linguistic variables in this research would significantly facilitate the collection of respondents' attitudes. However, the use of linguistic variables, i.e. predefined values of membership functions, also can limit expressing real respondents' attitudes. Therefore, linguistic variables are omitted in this case in order to perform the most realistic collection of respondents' attitudes and realistically consider the use of bipolar neutrosophic numbers for that purpose. The results of the evaluation obtained from the first decision-maker are shown in Table 3.

As can be seen from Table 4, the reliability of all responses obtained from the first respondent is 0.76, where the highest value of reliability is achieved based on criterion C_1 and the lowest value of reliability is achieved based on criterion C_2. In relation to the considered alternatives, the highest reliability of the collected responses is achieved in relation to alternative A_2, i.e. 0.84, and the lowest in relation to alternative A_4, and it is 0.62. The difference in the reliability of these alternatives is 0.22, which indicates that the evaluation

	C_1	C_2	C_3	C_4
A_1	< 1.0, 0.0, 0.0, 0.0, 0.0, −1.0 >	< 0.9, 1.0, 0.1, 0.0, −0.1, −0.7 >	< 0.9, 0.0, 0.1, −0.1, 0.0, −0.9 >	< 0.9, 0.0, 0.0, 0.0, 0.0, −0.8 >
A_2	< 1.0, 0.0, 0.0, 0.0, 0.0, −1.0 >	< 0.8, 0.0, 0.0, −0.2, 0.0, −0.8 >	< 0.9, 0.0, 0.0, 0.0, −0.1, −0.9 >	< 0.8, 0.0, 0.0, 0.0, 0.0, −0.8 >
A_3	< 1.0, 0.1, 0.1, 0.0, 0.0, −1.0 >	< 1.0, 0.1, 0.1, −0.4, 0.0, 0.0 >	< 0.9, 0.0, 0.0, 0.0, 0.0, −0.8 >	< 0.7, 0.0, 0.0, 0.0, 0.0, −0.9 >
A_4	< 0.9, 0.1, 0.1, −0.1, −0.1, −0.9 >	< 0.9, 0.1, 0.2, 0.0, 0.0, −0.8 >	< 0.7, 2.0, 0.2, −0.1, −0.1, −0.5 >	< 0.8, 0.0, 0.0, 0.0, 0.0, −0.7 >

TABLE 3
The responses obtained from the first respondent.

Source: Authors' calculation

	C_1	C_2	C_3	C_4	***Average***
A_1	1.00	0.52	0.80	0.85	0.79
A_2	1.00	0.70	0.86	0.80	0.84
A_3	0.91	0.61	0.85	0.80	0.79
A_4	0.73	0.72	0.27	0.75	0.62
Average	0.91	0.64	0.69	0.80	0.76

TABLE 4
The reliability of the information obtained from the first DM.

Source: Authors' calculation

	C_1	C_2	C_3	C_4
A_1	< 1.0, 0.0, 0.0, 0.0, 0.0, −1.0 >	< 0.4, 0.1, 0.5, −0.5, −0.1, −0.4 >	< 0.7, 0.0, 0.5, −0.8, 0.0, −0.6 >	< 0.1, 0.0, 0.7, −0.1, 0.0, −0.8 >
A_2	< 0.9, 0.0, 0.0, 0.0, 0.0, −1.0 >	< 0.7, 0.0, 0.8, −0.2, 0.0, −0.4 >	< 0.9, 0.0, 0.6, −0.1, 0.0, −0.5 >	< 0.5, 0.1, 0.7, −0.3, 0.0, −0.9 >
A_3	< 0.9, 0.2, 0.1, 0.0, 0.0, −1.0 >	< 0.3, 0.0, 0.2, −0.4, −1.0, −0.8 >	< 0.9, 0.0, 0.5, −0.6, 0.0, −0.2 >	< 0.7, 0.0, 0.3, −0.2, 0.0, −0.2 >
A_4	< 0.9, 0.1, 0.1, −0.1, −0.1, −0.9 >	< 0.6, 0.0, 0.2, −0.5, 0.0, 0.1 >	< 0.7, 0.0, 0.1, −0.1, 0.0, −0.9 >	< 0.4, 0.0, 0.2, −0.6, 0.0, −0.3 >

TABLE 5
The responses obtained from the second respondent.

Source: Authors' calculation

was performed correctly, carefully, and with an understanding of the use of BNNs.

The responses obtained from the second and the third respondents are shown in Tables 5 and 6, while the calculated reliability values are shown in Tables 7 and 8.

	C_1	C_2	C_3	C_4
A_1	$< 0.8, 0.0, 0.5, 0.0, 0.0, -0.3 >$	$< 0.4, 0.0, 0.5, -0.4, -0.8, -0.4 >$	$< 0.7, 0.0, 0.5, -0.8, 0.0, -0.6 >$	$< 0.8, 0.0, 0.0, 0.0, 0.0, 0.0 >$
A_2	$< 0.8, 0.2, 0.3, 0.0, -0.3, -0.2 >$	$< 0.7, 0.1, 0.8, -0.3, -0.5, -0.1 >$	$< 0.9, 0.0, 0.6, -0.1, 0.0, -0.5 >$	$< 0.9, 0.0, 0.0, 0.0, 0.0, 0.0 >$
A_3	$< 0.7, 0.0, 0.3, 0.0, -0.2, -0.2 >$	$< 0.2, 0.1, 0.2, -0.4, -0.7, -0.4 >$	$< 0.9, 0.0, 0.5, -0.6, 0.0, -0.2 >$	$< 0.9, 0.0, 0.0, 0.0, 0.0, 0.0 >$
A_4	$< 0.8, 0.0, 0.2, -0.1, -0.1, -0.1 >$	$< 0.3, 0.1, 0.2, -0.5, -0.5, -0.1 >$	$< 0.5, 0.0, 0.5, -0.1, 0.0, -0.2 >$	$< 0.7, 0.1, 0.1, -0.1, 0.0, 0.0 >$

TABLE 6
The responses obtained from the third respondent.

Source: Authors' calculation

	C_1	C_2	C_3	C_4	***Average***
A_1	1.00	0.09	0.20	0.65	0.49
A_2	0.95	0.15	0.35	0.39	0.46
A_3	0.83	0.15	0.40	0.20	0.40
A_4	0.73	0.50	0.70	0.25	0.54
Average	0.88	0.22	0.41	0.37	0.47

TABLE 7
The reliability of the information obtained from the second respondent.

Source: Authors' calculation

	C_1	C_2	C_3	C_4	***Average***
A_1	0.30	0.05	0.20	0.40	0.24
A_2	0.29	0.11	0.35	0.45	0.30
A_3	0.28	0.00	0.40	0.45	0.28
A_4	0.30	0.18	0.05	0.32	0.21
Average	0.29	0.09	0.25	0.41	0.26

TABLE 8
The reliability of the information obtained from the third respondent.

Source: Authors' calculation

As can be seen from Tables 7 and 8, the mentioned decision-makers achieved significantly lower values of reliability, where the reliability of responses obtained from the third decision-maker is evidently low, which is why these responses have to be rechecked or omitted from further evaluation.

In this case, responses obtained from the third decision-maker are omitted from further evaluation, and a group decision matrix was constructed using Eq. (7), as shown in Table 9.

	C_1	C_2	C_3	C_4
A_1	$< 1.0, 0.0, 0.0, 0.0, 1.0, 0.0 >$	$< 0.8, 0.3, 0.2, 0.0, 0.9, 0.4 >$	$< 0.8, 0.0, 0.2, -0.3, 1.0, 0.2 >$	$< 0.7, 0.0, 0.0, 0.0, 1.0, 0.2 >$
A_2	$< 1.0, 0.0, 0.0, 0.0, 1.0, 0.0 >$	$< 0.8, 0.0, 0.0, -0.2, 1.0, 0.3 >$	$< 0.9, 0.0, 0.0, 0.0, 0.9, 0.2 >$	$< 0.7, 0.0, 0.0, 0.0, 1.0, 0.1 >$
A_3	$< 1.0, 0.1, 0.1, 0.0, 1.0, 0.0 >$	$< 1.0, 0.0, 0.1, -0.4, 0.0, 0.4 >$	$< 0.9, 0.0, 0.0, 0.0, 1.0, 0.4 >$	$< 0.7, 0.0, 0.0, 0.0, 1.0, 0.3 >$
A_4	$< 0.9, 0.1, 0.1, -0.1, 0.9, 0.1 >$	$< 0.8, 0.0, 0.2, 0.0, 1.0, 0.5 >$	$< 0.7, 0.0, 0.1, -0.1, 0.9, 0.2 >$	$< 0.7, 0.0, 0.0, 0.0, 1.0, 0.5 >$

TABLE 9
A group decision matrix.

Source: Authors' calculation

	C_1	Score	Rank
A_1	$< 1.00, 0.00, 0.00, 0.00, 0.98, 0.18 >$	0.47	3
A_2	$< 1.00, 0.00, 0.00, 0.00, 0.99, 0.16 >$	0.48	2
A_3	$< 1.00, 0.00, 0.00, 0.00, 0.70, 0.25 >$	0.51	1
A_4	$< 0.79, 0.00, 0.00, 0.00, 0.96, 0.30 >$	0.42	4

TABLE 10
Overall utilities, values of score function and ranking orders of alternatives.

Source: Authors' calculation

	C_1	Score	Rank
A_1	$< 1.00, 0.00, 0.00, 0.00, 0.88, 0.16 >$	0.493	2
A_2	$< 1.00, 0.00, 0.00, 0.00, 0.91, 0.15 >$	0.491	3
A_3	$< 1.00, 0.00, 0.00, 0.00, 0.68, 0.23 >$	0.515	1
A_4	$< 0.75, 0.00, 0.00, 0.00, 0.91, 0.29 >$	0.425	4

TABLE 11
Overall utilities, values of score function and ranking orders of alternatives based on responses of three DMs

Source: Authors' calculation

The overall utility of each alternative, calculated by using Eq. (6), are shown in Table 10, where the following vector weight $w_j = (0.29, 0.24, 0.21, 0.26)$ was used. The values of the score function and ranking orders for the considered alternatives are also shown in Table 10.

As can be seen from Table 11, the alternative A_3 is the most acceptable. In order to verify the proposed approach, an additional calculation is performed with ratings of all three DMs, as it is shown in Table 11.

As can be seen from Table 11, the scores obtained from the third respondent did not affect the best-ranked alternative but reflected on the rank of the second and third-ranked alternatives. However, in some other cases, the ratings of respondents with low data reliability may affect the ranking order of alternatives. Using reliability of data decision-makers can identify respondents who did not want to participate in the survey, or did not understand the application of BNNs, and take appropriate action.

5 AN ANALYSIS OF THE PROPOSED APPROACH

In order to verify the proposed approach, some characteristic cases, i.e. characteristic BNNs, and their reliability of information are analyzed below. Table 12 shows two “ideal” cases of BNNs, as well as their reliability.

As can be seen from Table 12, both BNNs have the reliability of the information equal to one. Table 13 shows three characteristic, but less desirable, cases in which the respondent fails to consistently express his preferences.

It can also be seen from Table 13 that inconsistency in assessment is reflected in reliability.

The influence of the value of the indeterminacy-membership function on the reliability of the information contained in the BNNs is shown in Table 14.

From Table 14 it can be concluded that the increase in the value of the indeterminacy-membership function affects the decrease in the reliability of the information contained in the BNN.

Based on the cases discussed above, it can be concluded that the proposed approach can be used with high reliability for accessing the reliability of the information contained in BNNs.

BNN	$r^{+}_{(x)}$	$r^{-}_{(x)}$	$r_{(x)}$
$< 1.0, 0.0, 0.0, 0.0, 0.0, -1.0 >$	1.00	1.00	1.00
$< 0.0, 0.0, 1.0, -1.0, 0.0, 0.0 >$	1.00	1.00	1.00

TABLE 12
“Ideal” BNNs and their reliability.

Source: Authors’ calculation

	BNN	$r^{+}_{(x)}$	$r^{-}_{(x)}$	$r_{(x)}$
I	$< 1.0, 0.0, 1.0, 0.0, 0.0, -1.0 >$	0.00	1.00	0.50
II	$< 0.0, 0.0, 1.0, -1.0, 0.0, -1.0 >$	1.00	0.00	0.50
II	$< 1.0, 0.0, 1.0, -1.0, 0.0, -1.0 >$	0.00	0.00	0.00

TABLE 13
Some characteristic BNNs and their reliability.

Source: Authors’ calculation

BNN	$r^{+}_{(x)}$	$r^{-}_{(x)}$	$r_{(x)}$
$< 1.0, 0.5, 1.0, 0.0, -0.5, -1.0 >$	0.00	0.67	0.33
$< 0.0, 0.5, 1.0, -1.0, -0.5, -1.0 >$	0.67	0.00	0.33
$< 1.0, 0.5, 1.0, -1.0, -0.5, -1.0 >$	0.00	0.00	0.00
$< 1.0, 1.0, 1.0, 0.0, -1.0, -1.0 >$	0.00	0.50	0.25
$< 0.0, 1.0, 1.0, -1.0, -1.0, -1.0 >$	0.50	0.00	0.25
$< 1.0, 1.0, 1.0, -1.0, -1.0, -1.0 >$	0.00	0.00	0.00

TABLE 14
Influence of indeterminacy-membership function on the reliability of the information contained in BNNs.

Source: Authors' calculation

BNN	$r^{+}_{(x)}$	$r^{-}_{(x)}$	$r_{(x)}$
$< 1.0, 0.0, 0.0, -1.0, 0.0, 0.0 >$	1.00	1.00	1.00
$< 0.0, 0.0, 1.0, 0.0, 0.0, -1.0 >$	1.00	1.00	1.00

TABLE 15
Some characteristic BNNs and their reliability.

Source: Authors' calculation

Finally, Table 15 also shows two possible cases of BNNs, as well as their reliability.

From Table 15 it can be seen that the proposed approach, i.e. Eq. (10), does not include the interrelationship between positive and negative truth-membership, the indeterminacy-membership, and the falsity-membership functions. The proposed approach is an extension of the approach proposed in [42] and [43] to BNNs, proposed with the aim of simply assessing the reliability of the information contained in BNNs and identifying inconsistently in completed questionnaires. However, the proposed approach can be further extended to include the aforementioned relationships between membership functions.

6 CONCLUSIONS

Bipolar neutrosophic sets use six membership functions that express the truth membership, the falsity membership, as well as the indeterminacy membership to the set and complementary set. Therefore, bipolar neutrosophic numbers may be suitable for applying in multiple criteria evaluation when a smaller number of complex evaluation criteria are used. Unfortunately, a significant number of membership functions make them more complex for collecting data by surveying respondents.

Numerous studies are dedicated to the investigation of the use of fuzzy and neutrosophic numbers for solving complex decision-making problems. However, an evident lack of researches deals with the reliability of data collected by using surveys based on the use of fuzzy or neutrosophic numbers, as well as bipolar neutrosophic numbers. Using reliability of data decision-makers can identify respondents who did not want to participate in the survey, or did not understand the application of bipolar neutrosophic numbers, and take appropriate action. With the main aim of increasing the reliability of the decision-making process based on the bipolar neutrosophic sets, the novel approach for assessing the consistency of the respondents' responses is proposed in this article.

The research conducted in this study indicates that the proposed approach can be used to assess the reliability of the data collected using bipolar fuzzy numbers. As a shortcoming of the proposed approach can be mentioned the fact that the use of bipolar neutrosophic numbers can be complicated and even confusing for some respondents. However, the proposed approach actually allows the identification of such respondents, i.e. respondents who failed to complete the questionnaire correctly. As a weakness of the proposed approach, it can be stated that some characteristic values of reliability of the information contained in a bipolar neutrosophic number on which basis surveys could be classified into certain groups are not been defined in this article. This can be mentioned as a possible direction of further research if the proposed approach is accepted by other researchers.

The proposed approach does not include the interrelationship between positive and negative truth-membership, the indeterminacy-membership, and the falsity-membership functions. However, the proposed approach can be further extended to include the aforementioned relationships, and this can also be mentioned as one of the potential directions of future research.

Finally, the extension of the proposed approach to other types of fuzzy and neutrosophic sets, such as spherical neutrosophic sets, can be mentioned as directions for the future development of the proposed approach.

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Leading From Powerlessness: A Third-way Neutrosophic Leadership Model For Developing Countries

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Victor Christianto, Florentin Smarandache (2021). Leading From Powerlessness: A Third-way Neutrosophic Leadership Model For Developing Countries. *International Journal of Neutrosophic Science*, 13(2), 95-103

Abstract

We argue that there are essentially two chief leadership models: the hard-style and soft-style leadership. From Neutrosophic point of view, there can be a third way, between hard-style leadership and soft-style leadership model, which may be more relevant to many of people in developing countries as well as in developed countries, who feel "powerless" and "hopeless" especially in this pandemic situation. We prefer to call this new approach: leading from powerlessness. The third-way Neutrosophic leadership model may also mean partially hard-style and partially soft-style leadership.

Keywords: Leadership development, leading from powerlessness, leading at zero, community leaders, developing countries, neutrosophic logic, Leipzig Leadership Model.

1. Introduction

Leadership has always been challenging. This holds particularly true in times of fundamental change, which, driven by globalization and digitalization, we are experiencing nowadays.[3]

Timo Meynhardt wrote in his article on public value creation [2]:

> "…leaders are highly instrumental not only in making markets, but in doing so also building societies. In modern times, such value creation for society has had an indispensable impact, improving the quality of life on our planet in many respects. … For many leaders, thinking in public value terms comes naturally; for others, seeing themselves as creating or destroying public value requires considerably more effort."

Most of us may think that to lead well, one needs power. Not infrequently, prospective leaders who tend to be charismatic think that "I have to be successful and rich first, then people will listen to what I have to say. Because if I can't prove the success of God's words, how can people believe? " At first glance, maybe many think that this argument makes sense, but if we think about it, this mindset is actually a worldly mindset, that a leader must be someone who is strong, powerful, authoritative, and if possible super-rich and so on. We might call this pattern a hard-style leadership.

However, there are other patterns, such as Jesus, Gautama Buddha, Mahatma Gandhi or Martin Luther King, Jr. more aptly called, leading with softness (soft-style leadership).

From Neutrosophic point of view, there can be a third way, between hard-style leadership and soft-style leadership model, which may be more relevant to many of people in developing countries as well as in developed countries, who feel “powerless” and “hopeless” especially in this pandemic situation. So what can we do?

This article addresses the topic of leadership from the slightly different perspective we are familiar with, with an emphasis on "leading from powerlessness."

2. Accept our weaknesses

One of the basic premises of various (Western) leadership theories is that a leader must take all the initiatives, and also must be a demigod figure. Yet this is clearly impossible to sustain in the long term. There are many failures of modern leaders today due to the impossibility of demands to be superhuman, to work long hours a day, and still have to lead this and that events, counsel people and so on. And when he failed, their people became disappointed and then frustrated.

Even though every human being has their own strengths and weaknesses, there is also a leader who has a talent in teaching, wisdom, execution skills and so on.

The author is inspired by the example of the book by Furtick, (un)-qualified, and Joe Vitale's book on the ancient Hawaiian method (Zero Limit).

The point is that being a leader today, you need to be an authentic, learn to accept your weaknesses and go from there. Like a SWOT analysis, a prospective leader must identify the strengths, weaknesses and talents that the Universe has given, and learn to develop these strengths, while surrounding himself with reliable people who can complement his weaknesses.

So it's not by creating a superman image, but instead developing other people with a dialogical leadership pattern. That's a good way to develop authentic leadership patterns in today's digital era: be yourself, focus on your strengths, keep your weaknesses at check, and stay humble.

3. Implications of leadership at zero

Maybe someone here asks: why shall we propose a new leadership concept? Isn't there a natural leadership pattern that is widely applied in industry, seminaries and other organizations?

In this article, we submit to a new term: “leading from powerlessness,” where people without real power at hands, still can do many initiatives for public good [2]. For instance, local farmers in Bali Island, Indonesia, used to coordinate by themselves on how to share water resources for their farms, without much influence from authority (it is called Subak system). There is need for local leaders who sometimes are referred to as informal leaders. And Alvin Toffler has predicted that informal economy become increasingly important nowadays. For clarity, we don’t think that our model of leading from powerlessness is similar to servant leadership, because servant leadership still assumes that a leader to be almost perfect superhuman. The third-way Neutrosophic leadership model may also mean partially hard-style and partially soft-style leadership. However, it should be clear that we don't say that formal leaders are not required, but there should be coherent and constant communications in order to achieve public good [2].

Indeed, servant leadership has been known for a long time, especially by Greenleaf. The concept of Servant Leadership from Robert Greenleaf, a leader at the American Telephone and Telegraph company in the 1970s was initially considered an expression of an anti-establishment attitude popular at that time. It turns out that the concept was welcomed to India. From 2015 to 2019 alone, there were more than 100 articles and two meta-analyzes published on Servant Leadership.

The essence of the concept of Servant Leadership is leadership that involves followers in various dimensions both relational, ethically, emotionally, and spiritually so that they grow into complete personalities according to their potential. Greenleaf, the originator of this concept, states that the leader is able to do this because, he lives his main role as a servant, then as a leader. Also he displays Servant leadership by empowering and developing others through humility, authenticity, acceptance, and stewardship and giving direction to himself as a leader. So, the Servant leader is someone who strives to recognize the uniqueness of each of his followers, gives them space to independently learn with his guidance, and is given warm support. Thus, followers are treated not only as objects of the program, planning, or development process of the institution in which they work but as subjects.

So, servant leadership is a leadership model that rests on service in the sense of providing service to others by synergizing with those being led, and building togetherness so that together can share when making an organizational decision (Spears, 2010). Northouse (2013) states that Servant Leadership focuses on making leaders more sensitive and attentive to the problems that the people they lead have, a sense of empathy and can develop them towards a better direction.

However, there are some criticisms of the servant leadership model in practical application in the real world, for example that servant leadership may not be suitable in the military or in prisons.

That is why, in the opinion of these authors, "leading from powerlessness" model may be more suitable for the real situation in developing countries, when many informal leaders do not hold positions of authority in government.

4. Comparison with Leipzig Leadership Model

There is not much similar concept available at now that we can learn toward developing this idea of leading from powerlessness, except a short article by Vaclav Havel, from which he wrote it in a book: The power of powerless.

Of one particular development in leadership theory that we can mention here is : the Leipzig Leadership model. Leadership is about more than simply wielding power. The Leipzig Leadership Model places the importance of consistently contributing to a greater good at the centre of the concept of leadership. The critical factor is what leaders use their power for and what they use as orientation in the process.

As Tessen von Heydebreck wrote, a leader is required to find a balance between corporate/organization values/goals and public values/goals:

> "All individuals, from simple laborers to the executive board, are constantly confronted with a number of leadership tasks in their field of activity but remain dependent on somebody else's leadership in many other areas within their position in society as a whole. Good leadership is, in this respect, a substantial link amongst humans living together successfully. … Entrepreneurial optimism and responsible action are central theoretical as well as practical guiding principles which determine the successful realization of forward-looking prospects of our present time both on an individual level as well as for society as a whole. The Leipzig Leadership Model presented in this publication is a trendsetting step in that direction."[3]

HHL's Leipzig Leadership model is developed from such a premise.

See the following illustration.

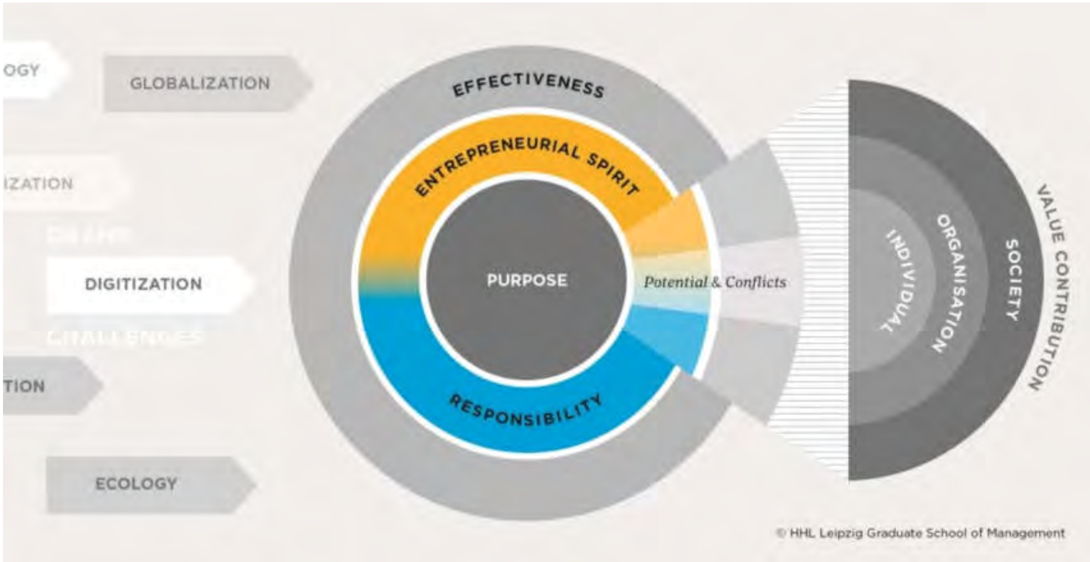

Illustration 1. Leipzig Leaderhip Model

From the above illustration, it shall be clear that a good leader should bring a balance between internal values such as effectiveness, entrepreneurial spirit and responsibility of their actions, in tune with external factors such as globalization, digitization and ecology.

5. Comment to Leipzig Leadership Model

While LLM/HHL is a welcome development of leadership model for business and modern organizations, nonetheless it is quite lacking in giving some role to informal leaders, who are typically considered outside the decision making structure of the corporations. Yes, that is one problem in this highly industrial society that decisions are often made from the top-to-bottom, while people on the streets are typically considered as outside of the equations.

Such a problem of technocratic policy making method has been predicted in the last chapter of Alvin Toffler's book: Future Shock. Writing during the late 1960s Toffler summarized this thesis thus [5]: "[I]n three short decades between now and the turn of the next millennium, millions of psychologically normal people will experience an abrupt collision with the future. Affluent, educated citizens of the world's richest and most technically advanced nations, they will fall victim to tomorrow's most menacing malady: the disease of change. Unable to keep up with the supercharged pace of change, brought to the edge of breakdown by incessant demands to adapt to novelty, many will plunge into future shock. For them the future will have arrived too soon" (Cross 1974).

In the last chapter of his best-selling futuristic book, Toffler suggested that it would be highly imperative to get out from the failure of technocratic decision making processes.

In other words, we need to go to post-technocratic decision making toward inclusion of informal leaders and also other participants in the society instead pursuing elite-only camps, be it WHO or WEF.

In that sense, we think that our proposed model of leading from powerlessness can be considered as necessity to be included for community leaders. This approach can be combined with coach-leadership style [8].

6. A story on how leading from powerlessness was put into practice: Art as cultural resistance in Romania

As one of us (FS) experienced around 70s in his native country back then, art can be used as cultural resistance; and it can be seen as a way of leading from powerlessness. During the Ceausescu's era he got in conflict with authorities. In 1986 he did the hunger strike for being refused to attend the International Congress of Mathematicians at the University of Berkeley, then published a letter in the Notices of the American Mathematical Society for the freedom of circulating of scientists, and became a dissident. As a consequence, he remained unemployed for almost two years, living from private tutoring done to students. The Swedish Royal Academy Foreign Secretary Dr. Olof G. Tandberg contacted him by telephone from Bucharest. Not being allowed to publish, he tried to get his manuscripts out of the country through the French School of Bucharest and tourists, but for many of them he lost track. Escaped from Romania in September 1988 and waited almost two years in the political refugee camps of Turkey, where he did unskilled works in construction in order to survive: cleaner, house painter, whetstoner. Here he kept in touch with the French Cultural Institutes that facilitated him the access to books and rencontres with personalities. Before leaving the country he buried some of his manuscripts in a metal box in his parents vineyard, near a peach tree, that he retrieved four years later, after the 1989 Revolution, when he returned for the first time to his native country. Other manuscripts, that he tried to mail to a translator in France, Chantal Signoret from the Université de Provence, were confiscated by the secret police and never returned. He wrote hundreds of pages of diary about his life in the Romanian dictatorship (unpublished), as a cooperative teacher in Morocco ("Professor in Africa", 1999), in the Turkish refugee camp ("Escaped... / Diary From the Refugee Camp", Vol. I, II, 1994, 1998), and in the American exile - diary which is still going on. But he's internationally known as the literary school leader for the "paradoxism" movement which has many advocates in the world, that he set up in 1980, based on an excessive use of antitheses, antinomies, paradoxes in creation paradoxes - both at the small level and the entire level of the work - making an interesting connection between mathematics, philosophy, and literature [http://fs.unm.edu/a/paradoxism.htm]. He introduced the 'paradoxist distich', 'tautologic distich', and 'dualistic distich', inspired from the mathematical logic [http://fs.unm.edu/a/literature.htm].

Literary experiments he realized in his dramas: Country of the Animals, where there is no dialogue!, and An Upside-Down World, where the scenes are permuted to give birth to one billion of billions of distinct dramas!
http://fs.unm.edu/a/theatre.htm].
He stated: "Paradoxism started as an anti-totalitarian protest against a closed society, where the whole culture was manipulated by a small group. Only their ideas and publications counted. They couldn't publish almost anything. Then, I said: Let's do literature... without doing literature! Let's write... without actually writing anything. How? Simply: literature-object! 'The flight of a bird', for example, represents a "natural poem", that is not necessary to write down, being more palpable and perceptible in any language that some signs laid on the paper, which, in fact, represent an "artificial poem": deformed, resulted from a translation by the observant of the observed, and by translation one falsifies. Therefore, a mute protest we did!
And so on, until he migrated to USA and gradually became appointed as a full professor of mathematics at The University of New Mexico.

7. Concluding remarks

From Neutrosophic point of view, there can be a third way, between hard-style leadership and soft-style leadership model, which may be more relevant to many of people in developing countries as well as in developed countries, who feel “powerless” and “hopeless” especially in this pandemic situation.
This article addresses the topic of leadership from a slightly different perspective than what we are familiar with, emphasizing on "leading from powerlessness."
We also discuss two stories of our own, on how this new concept can be put into practice.

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Introduction to Neutrosophic Genetics

Florentin Smarandache

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Abstract

Neutrosophic Genetics is the study of genetics using neutrosophic logic, set, probability, statistics, measure and other neutrosophic tools and procedures.

In this paper, based on the Neutrosophic Theory of Evolution (that includes degrees of Evolution, Neutrality (or Indeterminacy), and Involution) – as extension of Darwin's Theory of Evolution, we show the applicability of neutrosophy in genetics, and we present within the frame of neutrosophic genetics the following concepts: neutrosophic mutation, neutrosophic speciation, and neutrosophic coevolution.

Keywords: Genetics, Mutation, Speciation, Coevolution, Neutrosophic Genetics, Neutrosophic Theory of Evolution, Degrees of Evolution / Neutrality or Indeterminacy / Involution, neutrosophic mutation, neutrosophic speciation, neutrosophic coevolution

1. Introduction to Mutation

The common definition is that a Mutation is a change (a permanent alteration) in the genetic (DNA) sequence. During the cell division, if a mistake is made in DNA copying, we have a mutation.

The mutation can result from random mistake of DNA copying, or due to environmental factors (such as exposure to chemicals that are called mutagens, to ionizing radiation, or infection by viruses) [1].

If the mutation occurs in the body cells, it is called Somatic Mutation, and it is not passed on to the offspring. But if the mutation occurs in the female eggs and male sperm, it is called Germ-Line Mutation, and it is passed on [1].

The mutation is part of Darwin's Evolution and thanks to mutation we have much biodiversity of species on Earth.

2. Neutrosophic Theory of Evolution

As an extension of Darwin's Evolution, Neutrosophic Theory of Evolution [9] comprises three types of degrees:

a) Degree of Evolution (as Darwin's).

b) Degree of Neutrality (neither evolution, nor involution) or Indeterminacy (not sure if the change is towards evolution or involution).

c) Degree of Involution.

Mutations alterate genes, or create new genes.

3. Neutrosophic Mutation

As in neutrosophy, we meet three types of mutations:

a) Positive Mutations, or mutation that produces benefic (positive) effect in the sense of evolution (adaptation) of the individual to the environment.

b) Neutral Mutations, or mutation that have no effect on evolution or on involution (adaptation or inadaptation) of the individual to the environment.

The overwhelming number of mutations are neutral.

This is also due to the mechanisms that many organisms have for repairing the DNA initial changes and for removing somatic cells that were mutated.

c) Negative Mutations, or mutation that produces malefic (acrimonious, negative) effect in the sense of involution (inadaptation) of the individual to the environment.

Since mutation may weaken the immune system and produces genetic disorder, negative mosaicism, birth defects, infections, cancer, abnormal biological processes, etc.

4. Species

In the frame of a species, with respect to its individuals all together, there occur:

degrees of positive mutation, neutral mutation, and negative mutation – denoted by T (truth), I (neutral or indeterminate), and F (falsehood) respectively, where $T, I, F \in [0,1]$.

Let $\alpha_T, \beta_T, \alpha_I, \beta_I, \alpha_F, \beta_F \in [0,1]$, with:

$\alpha_T > \beta_T, \alpha_I > \beta_I, \alpha_F > \beta_F$,

where α_T is the upper treshold of T,

and β_T is the lower treshold of T,

and similarly for α_I, β_I, and respectively α_F, β_F.

Of course, the thresholds depend on each species and on its environment.

5. Neutrosophic Speciation

Each Species has a degree of speciation (T), a degree of continuation (I), and degree of extinction (F), where $T, I, F \in [0,1]$. We use the neutrosophic notation: Species(T, I, F).

Each Species(T, I, F) neutrosophically tends towards:

a) Speciation, or formation of a new species, if $T \geq \alpha_T$, and $I \leq \beta_I$, $F \leq \beta_F$;

b) Continuation, as the same species, if $I \geq \alpha_I$, and $T \leq \beta_T$, $F \leq \beta_F$;

c) Extinction, if $F \geq \alpha_F$, and $T \leq \beta_F$, $I \leq \beta_I$.

6. Neutrosophic Coevolution

Two species in the same environment may be in some:

a) Degree of cooperation (T);

b) Degree of neutrality (I);

c) Degree of conflict (F).

Of course these degrees $T, I, F \in [0, 1]$ are dynamic, and continuously change according to the environment and the species that live and interact with each other.

Conclusion

The Neutrosophy [1998], as a new branch of philosophy [9], is based on triads of the form (<A>, <neutA>, <antiA>), where <A> and <antiA> are opposites of each other, while <neutA> is the neutral (or indeterminate) between them. In general <A> may be an item (concept, idea, notion, theory, etc.).

We have introduced for the first time the Neutrosophic Genetics, which is the study of genetics using neutrosophic logic, set, probability, statistics, measure and other neutrosophic tools and procedures.

Within the frame of neutrosophic genetics, we have extended the classical concepts of mutation, speciation, and coevolution, to respectively neutrosophic mutation, neutrosophic speciation, and neutrosophic coevolution in order to better describe our real world.

Acknowledgement

The author brings deep thanks to Dr. Said Broumi for many discutions related to the applications of neutrosophy in genetics.

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Addendum (Definitions)

Since this paper is intended for the general public, in order for the paper to be self-contained, we provide below dictionary definitions of principal genetic terms.

Allele: any of the alternative forms of a gene that may occur at a given locus; “allele,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/allele. Accessed 12/7/2020.

Chromosome: any of the rod-shaped or threadlike DNA-containing structures of cellular organisms that are located in the nucleus of eukaryotes, are usually ring-shaped in prokaryotes (such as bacteria), and contain all or most of the genes of the organism; “chromosome,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/chromosome. Accessed 12/7/2020.

DNA: any of various nucleic acids that are usually the molecular basis of heredity, are constructed of a double helix held together by hydrogen bonds between purine and pyrimidine bases which project inward from two chains containing alternate links of deoxyribose and phosphate, and that in eukaryotes are localized chiefly in cell nuclei; “DNA,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/DNA. Accessed 12/7/2020.

Gene: a specific sequence of nucleotides in DNA or RNA that is located usually on a chromosome and that is the functional unit of inheritance controlling the transmission and expression of one or more traits by specifying the structure of a particular polypeptide and especially a protein or controlling the function of other genetic material; “gene,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/gene. Accessed 12/7/2020.

Genome: one haploid set of chromosomes with the genes they contain; “genome,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/genome. Accessed 12/7/2020.

Germ : a small mass of living substance capable of developing into an organism or one of its parts; “germ,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/germ. Accessed 12/7/2020.

Phenotype: the observable characteristics or traits of an organism that are produced by the interaction of the genotype and the environment; “phenotype,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/phenotype. Accessed 12/7/2020.

Somatic: of, relating to, or affecting the body especially as distinguished from the germplasm (germ cells and their precursors serving as the bearers of heredity); “somatic,” Merriam-Webster.com Dictionary, https://www.merriam-webster.com/dictionary/somatic. Accessed 12/7/2020.

Using Sieve of Eratosthenes for the Factor Analysis of Neutrosophic Form of the Five Facet Mindfulness Questionnaire as an Alternative Confirmatory Factor Analysis

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DOI: 10.32604/cmes.2021.016696

ABSTRACT

In this study, the Five Facet Mindfulness Questionnaire which was adapted from the short form of the Five Facet Mindfulness Questionnaire was evaluated and this scale into neutrosophic form was converted and the results of the scale were compared for proposing new type confirmatory analysis procedure as well as developing neutrosophic scales. The exploratory factor analysis was used in the analysis of the data. Besides, test results were analyzed for Kaiser–Meyer–Olkin and Bartlett values, common factor variance values, scree plot graphs, and the principal component analysis results. The sample of the study consists of 194 students in mathematics departments at Bitlis Eren University and Iğdır University in Turkey by convenience sampling method. A convenience sampling is a kind of non-probability sampling procedure in which the sample is obtained from a group of individuals easily accessible or reachable. The convenience sampling method was chosen in this study because the study aims to examine the structure of the measurement tool rather than the psychological characteristics of a particular population. First of all, it is observed that if any classical scale can be converted into a neutrosophic one. It is observed that the sub-dimensions of a neutrosophic scale as agree, disagree, and undecided might not have a similar factor structure to the classical one. Interestingly, in the factor analysis of the neutrosophic scale, both classical and the agreement part of the neutrosophic scales have the same factors, implying that the one-dimensional classical scale measures the agreement degree of the participants. When the factor analysis was conducted to disagreement and vagueness dimensions, it seemed that some factors were eliminated and even some new factors emerged, indicating that in human cognition those three dimensions can be taken as independent of each other, just as assumed by neutrosophic logic. The another important implication of the factor analysis is that the neutrosophic forms of any questionnaire can be used for the validity of the classical ones. Loads of items or their accumulation into factors are compared to the classical scale and the three-dimensional neutrosophic scale in the factor, so that the corresponding ones in the same factors and the items or factors that do not correspond to each other are eliminated. It is very similar to the Sieve of Eratosthenes, which is an ancient algorithm for finding prime numbers up to any given limit where each prime is taken as an independent base or dimension and multiples of the selected primes in a given interval are eliminated until there are only prime numbers left. Finally, the reliability of three independent dimensions of the neutrosophic forms of any questionnaire can also be used to check whether the measurement tool is reliable. Low-reliability results in any dimensions may imply that the scale has some problems in terms of meaning, language, or other factors.

KEYWORDS

Neutrosophic scales; factor analysis; scale development; explanatory analysis; reliability analysis

1 Introduction

Neutrosopy is all about looking at the world with fresh eyes, and then tailoring the perspective to account for uncertainty. Neutrosophy offers a third logic alternative to the binary model of true or false, which goes by the name of neutrals. In summary, Neutrosophy replaces the binary method in logics by offering indeterminancy, which may also be interpreted as ambiguous, uncertain, or inconsistent. Neutrosophy was conceptualized by Smarandache et al. [1] in 1988, and development since then has rapidly grown with the use of logical extensions, such as measure, sets, graphs, and even practical applications in various areas. The field of neutrosophy has shown its power and effectiveness in a variety of contexts. This created a big backlog of contributions which were theoretical in nature and confirmed only using mathematical examples or restricted data sets. Neutrosophic logic could be used in both natural science and social science, and recently publications have been emphasized the use of the neutrosophic logic in social sciences. Neutrosophic sets are better than fuzzy sets for surveys because they provide a wider range of answers. Through its membership indeterminacy function, the former allows respondents to more clearly articulate their actual thoughts and feelings. Neutrosophy is beneficial to those who want to express themselves since it better captures their thoughts and emotions due to its embrace of indeterminacy and independent membership function of falsehood. Therefore the study's primary goal is to use the principles of neutrosophy in social sciences, particularly in education and assessment and evaluation techniques of scale development [2–4].

The main purpose of the survey or scale development is to gather accurate and relevant data. In social sciences, the reliability and validity of scale and questionnaire formats are, therefore, used to enable to gather accurate and relevant data [5]. The data space and data range in this respect are essential parameters for developing scales since they often alter the data type, logical analytical space, methods, validity and reliability of the findings (Fig. 1).

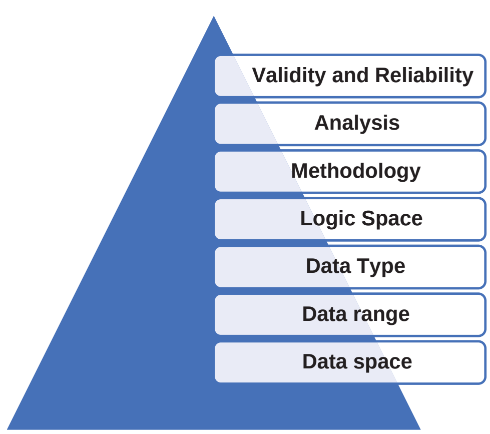

Figure 1: Data space and data range determines the validity and reliability of any scale

For measurement instruments such as scales, data space refers to an independent collection of choices for a particular measurement item. For example, there is only one choice in every Likert-type scale that the individual may express his/her ideas or feelings and its data space is 1d, but there are three different dimensions about each aspect in the neutrosophic scale as undecided, agreeable, and disagreeable. The data space is 1d for any form of Likert type scale a whereas 3d for neutrosophic space. Such a extension can be done for more dimensions. For instance, the more qualitative-oriented measurement tools like providing items that require more free opinions in a paragraph like preferences are supposed to have more dimensions as well. Though n-dimensional space is more suitable for clearer and more accurate outcomes, the representation of the data in less dimensional spaces can easily be statistically analyzed. Besides, the measurement tool's objectivity in terms of estimation of common features decreases as the dimension of space rises. The benefit of the 3-dimensional neutrosophic scale is that the participants are both involved in the degree of agreement, disagreement, and uncertainty. The difference among classical logic, fuzzy logic and neurosophic logic can be describes as in Fig. 2. In the classical logic the space is in 0 dimension where there is only discrete points as 0 and 1. In the fuzzy logic the spaces can be represented as 1 dimensional continuum or segment where there is a continuous possible admixture of the states of 0 and 1. Finally, in the Fuzzy logic there are three independent states constituting 3 dimensional logic space.

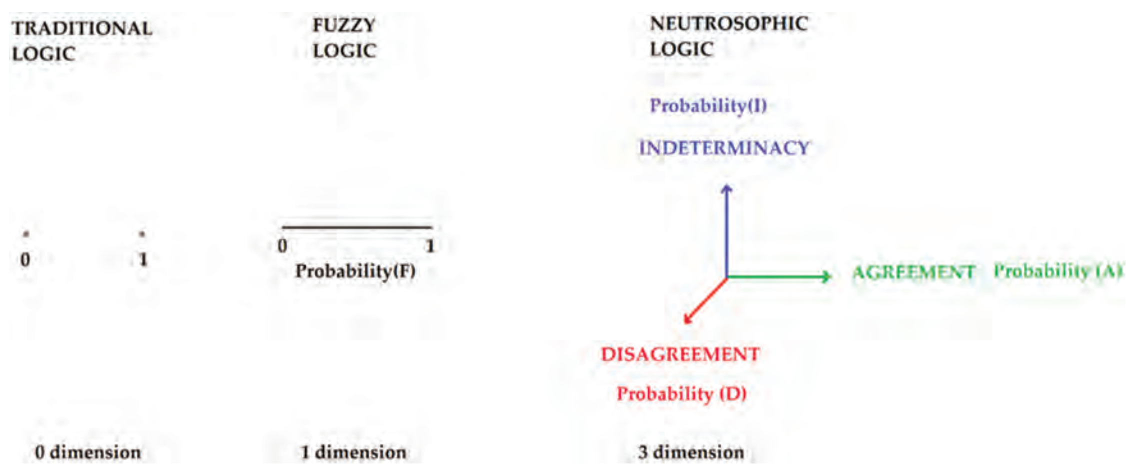

Figure 2: The difference difference among classical logic, fuzzy logic and neurosophic logic

It should be noted that there is no study focusing on 2d data space in the literature because the possible combinations of the agreement, disagreement and intederminacy in the forms of two independent states such as (a, d), (a, i), (d, i). Such a 2d data space is very limited because it disregards indeterminany, agreement, disagreement dimesions. For instance if 2d scale having agreement, disagreement dimesions firstly ignore the indeterminancy dimension. Secondly, sometimes agreement, disagreement dimesions are complement to each other as in the case od classical logic or fuzzy logic but the indeterminancy is important for the analysis. Such an example can be extended into the all possible combinations of (a, d), (a, i), (d, i). The degree of freedom of 2d space may dismiss the other two parameters that cannot be ignored in the actual case. These hidden variables can lead to huge differences especially in the case of the analysis of the options of a huge number of participants and even this cannot be realized (Fig. 3).

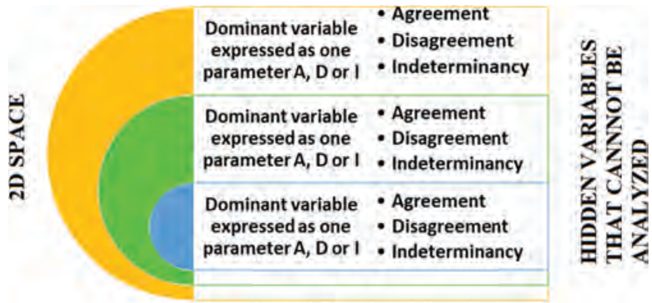

Figure 3: The degree of freedom of 2d space may dismiss the other two parameters that cannot be ignored in the actual case

However, in neutrosophic logic, it is impossible to dismiss three parameters since the researchers must give their opinions on them (Fig. 4).

Figure 4: In neutrosophic logic, it is impossible to dismiss three parameters since the researchers must give their opinions on them

In everyday life, humans are not confined within one dimension space in terms of the expressions of the agreement, disagreement and interdeterminancy dimensions. Neutrosophic logic is more compatible with this fact since the participants express in the three-dimensional neutrosophic space both their agreement and their contradictions and the ambiguity of the items or scale parameters. We often believe that a sentence is understood, but one term in the statement leaves us unsure if it is the 'right message' the source intends. We often approve of such proposals, but we sometimes disagree with the item only because of the source of the message itself. The neutrosophic scale is therefore distinct in terms of data space from the classical Likert scale (Fig. 5).

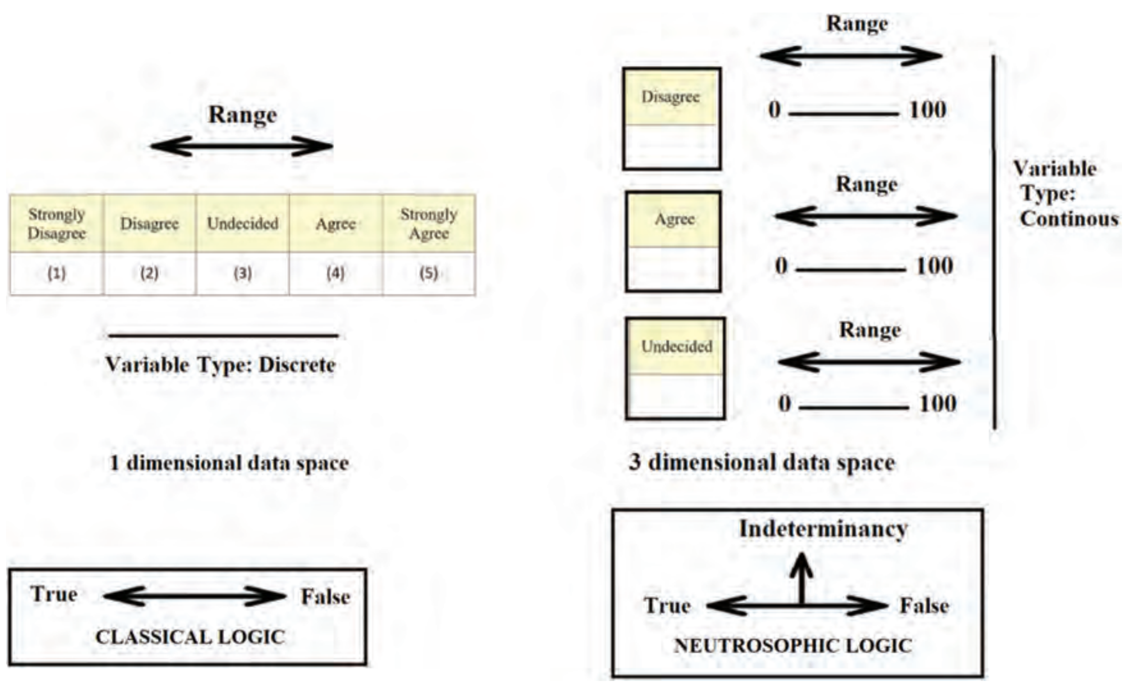

Figure 5: Data space of classical Likert type scale, neutrosophic scale

The second main point that differentiates between measuring tools is the range of the data that every scale is dependent on. The range of the data set is the difference from the highest value to the lowest value in any setting. Data may well be organized from 3 points in Likert form to 10 points on the Likert-type scale. The neutrosophic scale is, however, broader than the scales of such a Likert kind measurement tool. It contains all numbers ranging from 0 to 100. There are therefore continuous variable forms of neutrosophic scales, while Likert scales have discrete values in terms of rational numbers such that the data processing can differ. In this sense, this will help increase the sensitivity of the measuring instrument. This is actually what is called as neutrosophic Data in some recent researches is the piece of information that contains some indeterminacy. Similar to the classical statistics, it can be classified as [4]:

– Discrete neutrosophic data, if the values are isolated points.
– Continuous neutrosophic data, if the values form one or more intervals.
– Quantitative (numerical) neutrosophic data; for example: a number in the interval [6,7] (we do not know exactly), 47, 52, 67 or 69 (we do not know exactly);
– Qualitative (categorical) neutrosophic data; for example: blue or red (we do not know exactly), white, black or green or yellow (not knowing exactly).
– The univariate neutrosophic data is a neutrosophic data that consists of observations on a neutrosophic single attribute.

The logic space of a measuring instrument is the third essential point. Logic space is important because "in any field of knowledge, each structure is composed from two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy of the form (t, i, f) $\neq$ (1, 0, 0), that structure is a (t, i, f)-Neutrosophic Structure" [6]. The logic we are focused on, Neutrosophic Logic [7], is an emergent field where the percentage (percentage) of truth in a T subset, the percentage of indeterminacy in an I subset, and the proportion of falsity in an F subset are listed. We here consider a subset of truth (or falsity or indeterminacy), rather than just a number, since in many situations we can not precisely determine the proportions of truth and falsity but we can only approach them. For example, suppose that a statement (or proposition) is between 22% and 43% true and 51% to 82% false; indeterminancy (undecided): 32% to 39% or 40% to 52% true (according to various observers) and 57% or 62% to 71% false. The subsets are not simple intervals but are arranged in line with the proposition (open or closed or semi-open/semi-closed intervals, isolated, constant, or intersected or united by previous sets, etc.). On the other hand, there are many ways to evaluate and interpret data. Some recent studies reveal important developments based on the interpretation and effective use of data [8,9].

Although in Likert-type scales, there are mostly three options as agreement, disagreement, and vagueness, classical logic is located one valued option located on the opposite sides of true and false values. The neutrosophic set has three independent components, giving more freedom for analysis so that it brings different logical operations as well. Therefore, the methodology of the analysis of the data should be changed based on the logical structure of the scale. For instance, while factor analysis is used for classical Likert-type scales, as shown in this paper, we can not directly assume that all the sub-dimensions of any neutrosophic scale directly correspond to the factor structure of the classical one. Nevertheless, it should be noted that classical analysis and methods can indeed be used for neutrosophic scales based on different analysis procedures. Hence, we can conclude that the validity and reliability of the measurement tools can change based on the logical structure of the scale. Therefore, in this study, the Five Facet Mindfulness Questionnaire which was adapted by [10] from the short form of the Five Facet Mindfulness Questionnaire (FFMQ) was evaluated and this scale into neutrosophic form was converted and the results of the scale were compared for proposing new type confirmatory analysis procedure as well as developing neutrosophic scales.

2 Methodology

2.1 Procedure for the Analysis of the Neutrosophic Form of the Five Facet Mindfulness Questionnaire

Firstly, it is thought that a valid and reliable scale should be chosen that has appropriate psychometric properties such as its adequacy, relevance, and usefulness since we see the characteristics of the neutrosophic scale in the reliable and valid foundations. Otherwise, we must do the reliability and validity analysis for the neutrosophic scale, but we want to check our method based on a more solid context since there is not so much research on this subject. The exploratory factor analysis includes the determination and clustering of objects by researchers to measure the same characteristic and offers insights into the reliability of objects and the test [11].

2.1.1 Kaiser–Meyer–Olkin (KMO) and Bartlett Tests

This method determines the proportion of the total variation in given variables that is most likely to be caused by latent factors. If values are very close to 1.0, then one may benefit from doing a factor analysis on the data. The findings of the factor analysis are unlikely to be particularly relevant if the value is less than 0.50. To test the hypothesis that the correlation matrix is an identity matrix, Bartlett's test can be used. Component component analysis is usually quite effective when one has small values at the significance level (less than 0.05) [12]. Therefore the test results were analyzed for KMO and Bartlett values, common factor variance values, scree plot graphs, and the principal component analysis results. KMO and Bartlett tests also examined the adequacy of the scale for factor analysis. The KMO measure of sample adequacy is a test of how much variation can be explained by factors inside the data. A KMO value of 0.5 is poor as a measure of its factorability; 0.6 is acceptable; a value nearer to 1 is better [13]. The fact that the value of the ratio exceeds 0.80 suggests that the results are positive for factor analysis [13,14]. Bartlett's test reveals that the data is likely to be factorable if $p < 0.05$, but it is called a sensitive test, but it is best to use it the other way round: if $p > 0.05$, do not continue; however, if $p > 0.05$, review other factorability metrics before proceeding [13]. The higher correlation between the factors suggests that the model was developed correctly and that the model's hypotheses may also be evaluated. This illustrates the explained variance rate observed by the factor scale study. The variance explanation rate should be at least 50% [15].

2.1.2 Scree-Plot

In the Factor Analysis [13]: extraction dialog box, the graph generated by the Scree-Plot option may be used to determine which components should be removed as an alternative to eigenvalues >1.0. In factor analysis, the number of factors was decided by Eigenvalues statistics and Scree test (Line chart). Expressions greater than 1 in the eigenvalue statistics are accepted. The scree diagram shows the point at which the curve slope declines and flattens and the corresponding amount of the factor is determined [16]. The direct oblique rotation technique was used in the factor study. The Cronbach alpha reliability coefficient has been examined for the reliability of any scale. Cronbach alpha is a metric used to predict the stability of the inner consistency of the scale. When the internal coefficient of Cronbach alpha consistency is 0.70 and beyond, it can be said that the scale has adequate internal consistency [11,17,18].

2.2 Population

The sample of the study consists of 194 students in various departments at Bitlis Eren University and Iğdır University in Turkey by convenience sampling method. A convenience sampling is a kind of non-probability sampling procedure in which the sample is obtained from a group of individuals easily accessible or reachable. The convenience sampling method was chosen in this study because the study aims to examine the structure of the measurement tool rather than the psychological characteristics of a particular population (Tab. 1).

Table 1: The characteristics of the sample in terms of age and gender

	Age												Total
	17,00	18,00	19,00	20,00	21,00	22,00	23,00	24,00	25,00	26,00	27,00	41,00	
Gender													
Female	1	24	42	45	15	7	6	0	3	1	1	0	145
Male	0	3	10	21	7	2	1	3	0	1	0	1	49
Total	1	27	52	66	22	9	7	3	3	2	1	1	194

3 Findings

3.1 Factor Analysis for Agreement Dimension

Before doing to assess the suitability of the data for the factor analysis, two methodological measures are used. KMO and Bartlett's test are used for this [11]. Both are spherical tests. The KMO coefficient determines if the sample size for factor analysis is appropriate. The KMO value should be at least 0.60 and above if the sample size is sufficient; the Barlett test should also be important ($p < 0.05$) [18]. KMO and Bartlett's test for agree dimension shows that data is suitable for the data to the factor analysis (KMO = 0.816, $p < 0.05$) (Tab. 2).

Table 2: KMO and Bartlett's test for agree dimension

KMO and Bartlett's test		
Kaiser–Meyer–Olkin measure of sampling adequacy		0.816
Bartlett's test of sphericity	Approx. Chi-square	1223.922
	Df	190
	Sig.	0.000

After assessing if the data was appropriate for factor analysis, the data are evaluated for an exploratory factor to evaluate the factor structure in the scale. The first analysis showed that five factors had an eigenvalue of 1 and higher, which explains the total variance of 46,283 points as given in Tab. 3.

Table 3: Total variance for agreement dimension

Total variance explained							
Factor	Initial eigenvalues			Extraction sums of squared loadings			Rotation sums of squared loadings[a]
	Total	% of variance	Cumulative %	Total	% of variance	Cumulative %	Total
1	5.011	25.054	25.054	4.476	22.379	22.379	3.167
2	2.592	12.961	38.016	1.851	9.257	31.636	2.319
3	1.573	7.866	45.881	1.181	5.903	37.538	2.879
4	1.431	7.157	53.038	.899	4.497	42.035	2.949
5	1.217	6.087	59.126	.849	4.247	46.283	1.375
6	.966	4.829	63.954				
7	.877	4.384	68.339				
8	.813	4.066	72.405				
9	.664	3.321	75.726				
10	.652	3.259	78.985				
11	.618	3.089	82.074				
12	.552	2.762	84.836				
13	.496	2.481	87.318				
14	.453	2.263	89.581				
15	.394	1.971	91.552				
16	.390	1.952	93.504				
17	.365	1.826	95.329				
18	.346	1.729	97.059				
19	.313	1.563	98.622				
20	.276	1.378	100.000				

Notes: Extraction method: maximum likelihood. [a]When factors are correlated, sums of squared loadings cannot be added to obtain a total variance.

The Scree plot was also examined to determine how many factors the scale consists of. The Scree plot is given in Fig. 6. The scree plot in Fig. 6 indicates that after the fifth point, the breaks are diminished and the chart is continued horizontally.

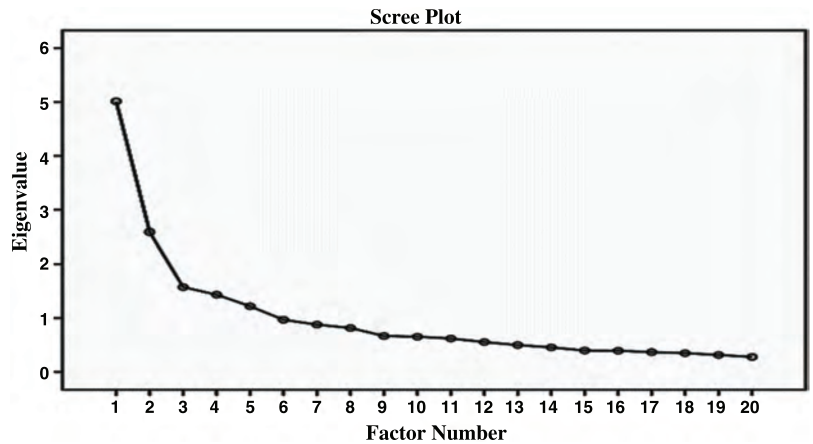

Figure 6: Scree plot for agreement dimension

The research was then carried out using the direct oblique rotation method. The data then proceeded. Researchers use the oblique rotation technique since the relations between factors exist [11,19]. In this analysis, the direct oblique rotation approach was chosen because the variables may be related. After rotation, no items were deleted from the scale with a factor load value of under 0.30 and overlaps of more than one factor. Then items used in the measuring scale with the same feature with a high factor load were held inside the scale. It is observed that no item with a load with a low factor was excluded. As a consequence of the analysis, five factors were taken into account for the remaining 20 items on the scale (Tab. 4).

Table 4: Pattern matrix for agreement dimension

Pattern matrix[a]					
	Factor				
	1	2	3	4	5
v15aIagree	0.778				
v14aIagree	0.759				
v13aIagree	0.756				
v16aIagree	0.496				
v19aIagree		0.722			
v18aIagree		0.706			
v17aIagree		0.629			
v20aIagree		0.399			
v5aIagree			−0.902		
v7aIagree			−0.606		
v6aIagree			−0.406		
v8aIagree			−0.402		

(Continued)

Table 4 (continued)

Pattern matrix[a]					
	Factor				
	1	2	3	4	5
v9aIAgree				0.773	
v12aIagree				0.665	
v10aagree				0.530	
v11aIagree				0.493	
v1aIagree					0.594
v2aIagree					0.480
v3aIagree					0.450
v4aIagree					0.448

Notes: Extraction method: maximum likelihood. Rotation method: Oblimin with Kaiser normalization.
[a]Rotation converged in 9 iterations.

When the factor structure for agreement dimension is compared to the original classic Five Facet Mindfulness Questionnaire (FFMQ), it is observed that all items are directly correlated with the same dimensions of the original classic FFMQ (Tab. 5).

Table 5: Comparison of the items in agreement dimension in the neutrosophic Five Facet Mindfulness Questionnaire with the items in the dimensions original classic FFMQ

Act with awareness (Factor 4)	Nonjudge items (Factor 1)	Nonreact items (Factor 2)	Observe (Factor 5)	Describe (Factor 3)
9*	13*	17	1	5*
10*	14*	18	2	6
11*	15*	19	3	7*
12*	16*	20	4	8

Reliability statistics show that the structure and assessment are highly reliable since reliability refers not only to the instrument itself but also to assessments obtained with a measurement tool [20–22] (Tab. 6).

Table 6: Reliability statistics for agreement dimension

Reliability statistics	
Cronbach's alpha	N of items
0.829	18

3.2 Factor Analysis for Disagreement Dimension

KMO and Bartlett's test for disagreement dimension shows that data is suitable for data factor analysis (KMO = 740, $p < 0.05$). After assessing if the data was appropriate for factor analysis, the data are evaluated for an exploratory factor to evaluate the factor structure in the scale. The first analysis showed that five factors had an eigenvalue of 1 and higher, which explains the total variance of 45,221. The research was then carried out using the direct oblique rotation method. After rotation, one item (Item 1) was deleted from the scale since it overlaps with more than one factor because it overlaps with Factor 1 and factor four having similar factor loads as given 0.327 and 0.356. Then items used in the measuring scale with the same feature with a high factor load were held inside the scale and those with a load with a low factor were excluded. KMO and Bartlett's test for disagreement dimension shows that data is suitable for data factor analysis (KMO = 0.731, $p < 0.05$) (Tab. 7).

Table 7: KMO and Bartlett's test results for disagreement dimension

KMO and Bartlett's test		
Kaiser–Meyer–Olkin measure of sampling adequacy		0.731
Bartlett's test of sphericity	Approx. Chi-square	947.066
	Df	153
	Sig.	0.000

After assessing if the data was appropriate for factor analysis, the data are evaluated for an exploratory factor to evaluate the factor structure in the scale. The first analysis showed that four factors had an eigenvalue of 1 and higher, which explains the total variance of 41.035 points as given in Tab. 8.

Table 8: Total variance for disagreement dimension

Factor	Initial eigenvalues			Extraction sums of squared loadings			Rotation sums of squared loadings[a]
	Total	% of variance	Cumulative %	Total	% of variance	Cumulative %	Total
1	3.699	20.548	20.548	3.115	17.305	17.305	2.461
2	2.928	16.265	36.814	2.424	13.468	30.773	2.138
3	1.543	8.571	45.384	1.033	5.740	36.514	2.025
4	1.381	7.670	53.054	0.814	4.522	41.035	2.413
5	0.999	5.552	58.606				
6	0.979	5.440	64.046				
7	0.872	4.846	68.893				
8	0.737	4.094	72.987				
9	0.716	3.978	76.965				
10	0.684	3.801	80.767				
11	0.646	3.587	84.353				
12	0.563	3.130	87.484				
13	0.512	2.842	90.326				
14	0.417	2.318	92.644				
15	0.405	2.250	94.893				
16	0.368	2.046	96.940				
17	0.305	1.693	98.633				
18	0.246	1.367	100.000				

Notes: Extraction method: maximum likelihood. [a]When factors are correlated, sums of squared loadings cannot be added to obtain a total variance.

Scree plot was also examined in order to determine how many factors the scale consists of. Scree plot is given in Fig. 7. The scree pilot in Fig. 7 indicates that after the fourth point, the breaks are diminished and the chart is continued horizontally.

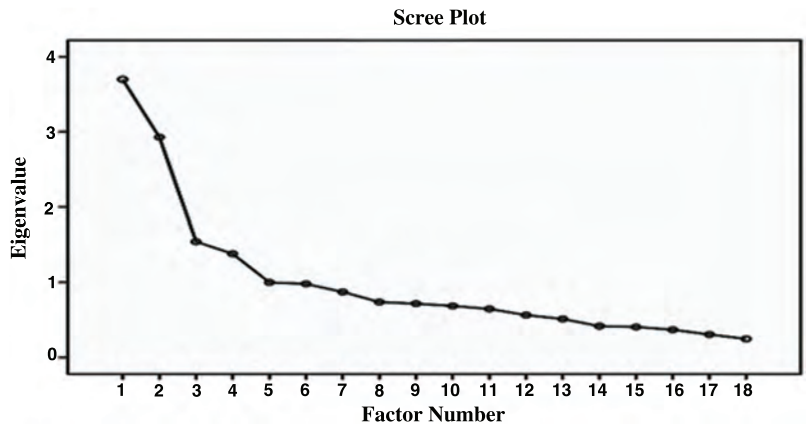

Figure 7: Scree plot for disagreement dimension

While one out of 20 items was removed on the draft scale and the analysis for the other 19 items was repeated. As a consequence of the study, four factors were taken into account for the remaining 19 items on the scale (Tab. 9).

Table 9: Pattern matrix for disagree dimension

Pattern matrix[a]				
	Factor			
	1	2	3	4
v7cIDİİSagree	0.702			
v5cIDİİSagree	0.682			
v8cIdisagree	0.498			
v9cIDİİSagree	0.490			
v12cIDİİSaagree	0.323			
v11cIDİİSaagree	0.309			
v18cIdisagree		0.833		
v19cIdisagree		0.638		
v17cIdisagree		0.581		
v20cdisagree		0.442		
v3bIdisagree			0.808	
v2cIdisagree			0.674	
v6cIdisagree			0.457	
v4cIdisagree			0.436	
v15cIDİİSaagree				0.788
v14cIDİİSaagree				0.759
v13cIDİİSaagree				0.667
v16cIDİİSaagree				0.439

Notes: Extraction method: maximum likelihood. Rotation method: Oblimin with Kaiser normalization.
[a]Rotation converged in 10 iterations.

When the factor structure for disagreement dimension is compared to the original classic FFMQ, it is observed that all Item 6 is removed from the Describe dimension and it moved to the Observe dimension. It is observed that Items 9, 11, 12 were removed from the Act with Awareness they moved into the Describe. Additionally, the factor is eliminated because no items are accumulated there. It seems that as the dimension of the classical scale has changed, the general structure of the scale has also changed (Tab. 10).

Table 10: Comparison of the items in disagreement dimension in the neutrosophic FFMQ with the items in the dimensions original classic FFMQ

Act with awareness (Factor 4) eliminated factor	Nonjudge items (Factor 4)	Nonreact items (Factor 2)	Observe (Factor 3)	Describe (Factor 1)
9*	13*	17	6 (moved there)	5*
10*	14*	18	2	6
11*	15*	19	3	7*
12*	16*	20	4	8
				9 (moved there)
				11 (moved there)
				12 (moved there)

Reliability statistics show that the structure and assessment are regarded as reliable (Tab. 11).

Table 11: Reliability statistics for disagreement dimension

Reliability statistics	
Cronbach's alpha	N of items
0.722	18

3.3 Factor Analysis for the Uncertainty Dimension

KMO and Bartlett's test for uncertainty dimension shows that data is suitable for data factor analysis (KMO = 0.891, $p < 0.05$). After assessing if the data was appropriate for factor analysis, the data are evaluated for an exploratory factor to evaluate the factor structure in the scale. The first analysis showed that five factors had an eigenvalue of 1 and higher, which explains the total variance of 52.890 points. After rotation, three elements were deleted from the scale because they overlapped more than one-factor having similar loads (Item 17 having factor loads as 0.306 and −0.303 in Factor 1 and Factor 3, Item 20 having factor loads as 0.388 and −0.333 in Factor 2 and Factor 3, Item 6 having factor loads as −0.618 and 0.348 in Factor 3 and Factor 5). While 3 out of 20 items were removed on the draft scale and the analysis for the other 17 items was repeated. KMO and Bartlett's test for uncertainty dimension shows that data is suitable for data factor analysis (KMO = 0.892, $p < 0.05$). After assessing if the data was appropriate for factor analysis, the data are evaluated for an exploratory factor to evaluate the factor structure in the scale. The first analysis showed that four factors had an eigenvalue of 1 and higher, which explains the total variance of 48.643 points. After rotation, three elements were deleted from the scale because they overlapped more than one-factor having similar loads (Item 16 having factor loads as, 418 and −0.314 in Factor 1 and Factor 2, Item 14 having factor loads as −0.738 and −0.316 in Factor 2 and Factor 3, Item 19 having factor loads as 0.386, −0.397 in Factor 1 and Factor 2). (Tab. 12).

As for the last analysis, KMO and Bartlett's test for uncertainty dimension shows that data is suitable for the data for the factor analysis (KMO = 0.879, $p < 0.05$).

Table 12: KMO and Bartlett's test for the uncertainty dimension

KMO and Bartlett's test		
Kaiser–Meyer–Olkin measure of sampling adequacy		0.879
Bartlett's test of sphericity	Approx. Chi-square	926,646
	Df	91
	Sig.	0.000

After assessing if the data was appropriate for factor analysis, the data are evaluated for an exploratory factor to evaluate the factor structure in the scale. The first analysis showed that three factors had an eigenvalue of 1 and higher, which explains the total variance of 44,498 points (Tab. 13).

Table 13: Total variance for uncertainty dimension

Total variance explained							
Factor	Initial eigenvalues			Extraction sums of squared loadings			Rotation sums of squared loadings[a]
	Total	% of variance	Cumulative %	Total	% of variance	Cumulative %	Total
1	5.446	38.901	38.901	4.881	34.864	34.864	3.811
2	1.198	8.557	47.458	0.727	5.190	40.054	3.747
3	1.102	7.873	55.331	0.622	4.444	44.498	3.322
4	0.924	6.602	61.933				
5	0.888	6.343	68.276				
6	0.725	5.181	73.457				
7	0.639	4.562	78.019				
8	0.619	4.425	82.443				
9	0.532	3.799	86.243				
10	0.499	3.563	89.805				
11	0.400	2.860	92.666				
12	0.372	2.654	95.320				
13	0.351	2.506	97.826				
14	0.304	2.174	100.000				

Notes: Extraction method: maximum likelihood. [a]When factors are correlated, sums of squared loadings cannot be added to obtain a total variance.

The scree plot was also examined to determine how many factors the scale consists of. The Scree plot is given in Fig. 8. The scree plot in Fig. 8 indicates that after the third point, the breaks are diminished and the chart is continued horizontally.

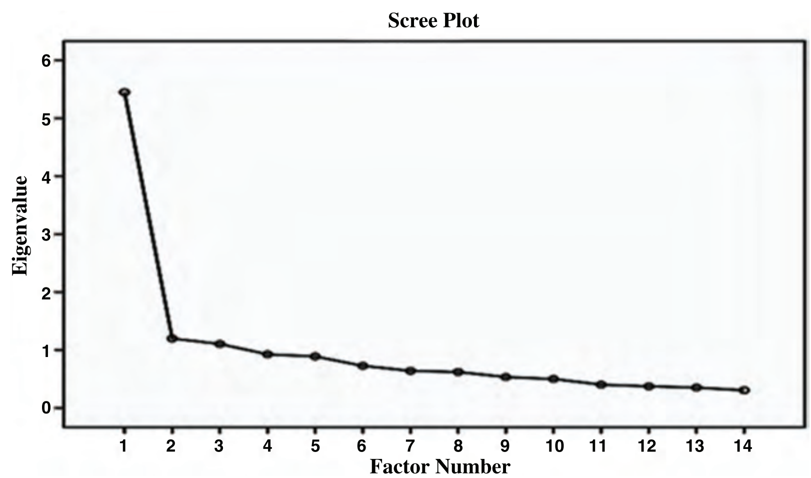

Figure 8: Scree plot for uncertainty dimension

After rotation, no items were deleted from the scale with a factor load value of under 0.30 and overlaps of more than one factor. As a consequence of the study, three factors were taken into account for the remaining 13 items on the scale (Tab. 14).

Table 14: Pattern matrix for the uncertainty dimension

Pattern matrix[a]			
	Factor		
	1	2	3
v1bluncertain	0.741		
v4bIuncertain	0.598		
v15bIuncertain	0.530		
v13buncertain	0.447		
v3bIuncertain	0.423		
v2bIuncertain	0.420		
v18bIuncertain			
v12bIuncertain		−0.805	
v11bIuncertain		−0.746	
v9bIuncertain		−0.555	
v10bIuncertain		−0.494	
v5bIuncertain			−0.782
v8bIuncertain			−0.641
v7bIuncertain			−0.510

Notes: Extraction method: maximum likelihood. Rotation method: Oblimin with Kaiser Normalization.
[a] Rotation converged in 9 iterations.

When the factor structure for uncertainty dimension is compared to the original classic Five Facet Mindfulness Questionnaire, it is observed that it seems that Observe is partially merged with Nonjudge items. Therefore, Nonjudge items are eliminated. Act with Awareness is originally correlated with Factor 2 so that it does not change its position. Nonreact factor items are also eliminated because they have no items corresponding to the original classical scale. Factor 3 corresponds to Describe except Item 6 because it was removed from there (Tab. 15).

Table 15: Comparison of the items in uncertainty dimension in the neutrosophic five facet mindfulness questionnaire with the items in the dimensions original classic five facet mindfulness questionnaire

Act with awareness (Factor 2)	Nonjudge items eliminated factor	Nonreact items eliminated factor	Observe (Factor 1)	Describe (Factor 3)
9*	13*	17	1	5*
10*	14*	18	2	6
11*	15*	19	3	7*
12*	16*	20	4	8
			13* (moved there)	
			15* (moved there)	

Reliability statistics show that the structure and assessment are regarded as reliable (Tab. 16).

Table 16: Reliability statistics for uncertainty dimension

Reliability statistics	
Cronbach's alpha	N of Items
0.875	13

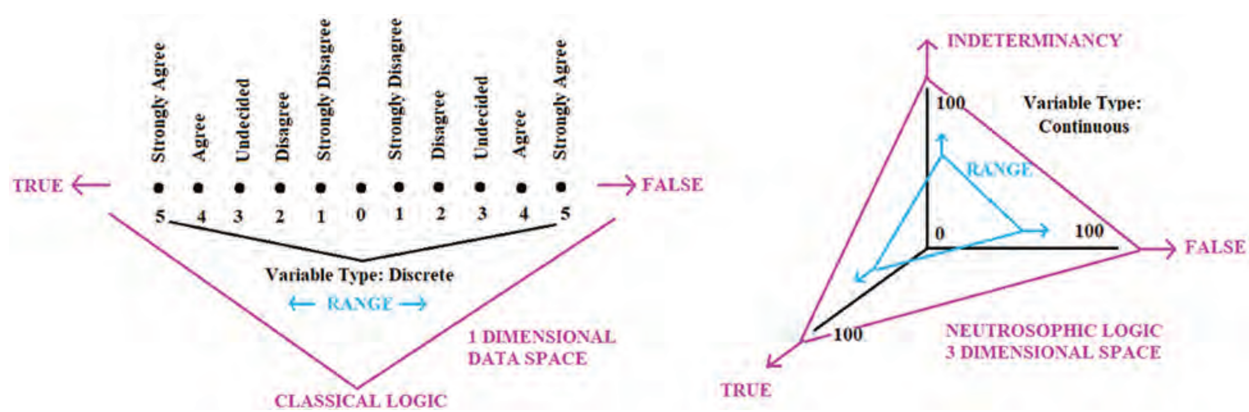

Figure 9: Factor structure of neutrosophic Questionnaire may not be same as the factor structure of the classical Questionnaire

Table 17: Comparison of the items in the agreement, disagreement, and uncertainty dimensions in the neutrosophic Questionnaire with the items in the dimensions original classic Questionnaire

Agreement dimension	Act with awareness (Factor 4)	Nonjudge items (Factor 1)	Nonreact items (Factor 2)	Observe (Factor 5)	Describe (Factor 3)
	9* 10* 11* 12*	13* 14* 15* 16*	17 18 19 20	1 2 3 4	5* 6 7* 8
Disaggrement Dimension	Act with awareness (Factor 4) eliminated factor	Nonjudge items (Factor 4)	Nonreact items (Factor 2)	Observe (Factor 3)	Describe (Factor 1)
	9* 10* 11* 12*	13* 14* 15* 16*	17 18 19 20	6 (moved there) 2 3 4	5* 6 7* 8 9 (moved there) 11 (moved there) 12 (moved there)
Uncertainty	Act with Awareness (Factor 2)	Nonjudge items eliminated factor (Factor 1)	Nonreact items eliminated factor	Observe	Describe (Factor 3)
9* 10* 11* 12*	13* 14* 15* 16*	17 18 19 20	1 2 3 4 13* (moved there) 15* (moved there)	5* 6 7* 8	

4 Discussion and Conclusion

The comparison of the items in the agreement, disagreement, and vagueness dimensions in the neutrosophic and classic Five Facet Mindfulness Questionnaire gives us many clues about how the structure of any questionnaire, survey, or scale can change as the dimensions or more generally, their space change. Therefore, if we convert any classical scale into a neutrosophic one, we shouldn't directly assume that all of the sub-dimensions of a neutrosophic scale as agree, disagree, and undecided have a similar factor structure to the classical one. This is an important point, because, for further analysis of the data, such a wrong assumption may lead to wrong conclusions since neutrosophic logic requires three independent truth values while classical one takes two dependent truth values (Fig. 9).

Interestingly, in the factor analysis of the neutrosophic scale, both classical and neutrosophic scales have the same factors, implying that the one-dimensional classical scale measures the agreement degree of the participants. When the factor analysis was conducted to disagreement and vagueness dimensions, it seemed that some factors were eliminated and even some new factors emerged, indicating that in human cognition those three dimensions can be taken as independent of each other, just as assumed by neutrosophic logic (Tab. 17).

The second important implication of the factor analysis is that the neutrosophic forms of any questionnaire can be used for the validity of the classical ones. Although it is not required that the dimensions of the neutrosophic forms of any questionnaire have the same or similar factors, since these different structures should be evaluated within their realms in terms of their structure, the classical forms of questionnaires can be checked based on neutrosophic forms. When Tab. 17 is examined, it is observed that some factors are eliminated on the neutrosophic scale while some of them stay in the same state. Since we used a valid and reliable scale having smaller items to check whether it is neutrosophic form can be used to evaluate it rather than a draft of a questionnaire having more items like 100 or 120 items, it seems that this scale is invalid, but for draft scales the similar procedure can be applied and more coherent scales having same factor structure in three dimensions with same items can be achieved so that items and dimensions can more sensitively measure the intended meaning of the items and factors. It is very similar to the Sieve of Eratosthenes, which is an ancient algorithm for finding the prime numbers up to any given limit where each prime is taken as an independent base or dimension and multiples of the selected prime in a given interval are eliminated until there are only prime numbers left.

Additionally, although it was said that this structure is deemed to be invalid for the general procedure, actually it is still used as a valid one because both factors, at least in two dimensions, were not eliminated. For instance, Act with Awareness (Factor 4) was eliminated in the disagreement dimension but it is still the same in two other dimensions as well, indicating that it has an approximately valid structure. Similar arguments can be made for items individually. For example, although Item 6 corresponds to the same structure in the classical one, indicating that it belongs to this factor, it changes its position in the other dimensions, possibly because of its dependence on other items in the realms of these two dimensions in the context of classical interdependent logic. Finally, the reliability of three independent dimensions of the neutrosophic forms of any questionnaire can also be used to check whether the measurement tool is reliable. Low-reliability results in any dimensions may imply that the scale has some problems in terms of meaning, language, or other factors.

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