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Introduction to the MultiNeutrosophic Set

Florentin Smarandache[1]

[1] University of New Mexico; Mathematics, Physics and Natural Sciences Division; 705 Gurley Ave., Gallup, NM 87301, United States of America; smarand@unm.edu

Abstract: In the real word, in most cases, everything (an attribute, event, proposition, theory, idea, person, object, action, etc.) is evaluated in general by many *sources* (called *experts*), not only one. The more sources evaluate a subject, the better accurate result (after fusioning all evaluations). That's why, in this paper, we straightforwardly extend the Refined Neutrosophic Set to the MultiNeutrosophic Set, and we show that the last two are isomorphic. A MultiNeutrosophic Set is a Neutrosophic Set whose all elements' degrees of truth/indeterminacy/falsehood are evaluated by many (Multi) sources.

Afterwards, we introduce a total order on the set of n-plets of the form (p, r, s), we build the operators on the (p, r, s)-plets, and show several applications of the MultiNeutrosophic Sets.

Several particular cases of the MultiNeutrosophic Sets are presented: such as MultiFuzzy Set, MultiIntuitionistic Fuzzy Set, MultiPicture Fuzzy Set, and other Multi(Fuzzy Extension) Set.

1. General Definition of the Neutrosophic Set (or Subset Neutrosophic Set - *SNS*)

Let $\mathcal{U}$ be a universe of discourse and a subset N of it.

Then:

$N = \{x, (T, I, F), x \ \epsilon \ \ U\}$

is called a *Neutrosophic Set*, where *T, I, F* are subsets of [0, 1], and they are called respectively degrees of Truth (*T*), Indeterminacy (*I*), and Falsehood (*F*) of the element x with respect to the set *A*. No other restrictions on *T, I,* and *F*. Of course, it implies that:

$0 \le \inf T + \inf I + \inf F \quad \le \sup T + \sup I + \sup F \quad \le 3.$

The most used (particular cases are):

i) If *T, I, F* are all single-values (numbers) from [0, 1], then one has a *Singe-Valued Neutrosophic Set* (SVNS);
ii) If *T, I, F* are intervals included in [0, 1], then one has an *Interval-Valued Neutrosophic Set* (IVNS).

2. The MultiNeutrosophic Set

In the real word, in most cases, everything: an attribute, event, proposition, theory, idea, person, object, action, etc., is evaluated in general by many *sources* (called *experts*), let's denote them by S_1, S_2, …, S_n, where the number of sources $n \quad \ge 2$ (to ensure the MultiSource). The more sources evaluate a subject, the better accurate result (after fusioning all evaluations).

Therefore, let's assume the degree of truth (or membership) of the generic element x with respect to the set *N* is evaluated by p sources of information, that give the following results,

respectively $T_1, \ T_2, \dots, T_p$;

and the degree of indeterminacy (neither truth/membership, nor falsehood/nonmembership) of the element x with respect to the set *N* is evaluated by *r* sources of information, that give the following results, respectively: $I_1, \ I_2, \dots, I_r$;

while the degree of falsehood (or nonmembership) of the element x with respect to the set *N* is also evaluated by *s* sources of information that give the following results, respectively: $F_1, \ F_2, \dots, F_s$;

where all $T_1,\ T_2, \dots, T_p,\ I_1,\ I_2, \dots, I_r,\ F_1,\ F_2, \dots, F_s$ are subsets of [0,1], with p, r, s integers ≥ 0, and at least one of p, r, s is ≥ 2 (in order to ensure the multiplicity of at least one of: truth, indeterminacy, or falsehood), with $p + r + s = n \quad \geq 2$.

All n sources may be either totally independent two by two, or partially independent and partially dependent, or totally dependent - according to the need of each specific application.

In the situation where there is some dependence between sources, we understand that either they communicate with each other and share information (influencing each other), or the same source may evaluate two or three of the components: truth, indeterminacy, falsehood of the same element.

3. General Definition of MultiNeutrosophic Set (or Subset MultiNeutrosophic Set - *SMNS*)

Let $\mathcal{U}$ be a universe of discourse and M a subset of it. Then, a *MultiNeutrosophic Set* is:

$M = \{x, x(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s), x \in \mathcal{U},$

where p, r, s are integers $\geq 0,\ p + r + s = n \geq 2,$

and at least one of p, r, s is ≥ 2, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth/membership, indeterminacy, or falsehood/nonmembership;

all subsets $T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s \subseteq [0, 1];$

$0 \leq \sum_{j=1}^{p} infT_j + \sum_{k=1}^{r} infI_k + \sum_{l=1}^{s} infF_l \leq \sum_{j=1}^{p} supT_j + \sum_{k=1}^{r} supI_k + \sum_{l=1}^{s} supF_l \leq n\}.$

No other restrictions apply on these neutrosophic multicomponents.

$T_1, T_2, \dots, T_p$ are multiplicities of the truth, each one provided by a different source of information (expert).

Similarly, $I_1, I_2, \dots, I_r$ are multiplicities of the indeterminacy, each one provided by a different source.

And $F_1, F_2, \dots, F_s$ are multiplicities of the falsehood, each one provided by a different source.

The **Degree of MultiTruth** (MultiMembership), also called *MultiDegree of Truth*, of the element x with respect to the set M are $T_1, T_2, \dots, T_p$;

the **Degree of MultiIndeterminacy** (MultiNeutrality), also called *MultiDegree of Indeterminacy*, of the element x with respect to the set M are $I_1, I_2, \dots, I_r$;

and the **Degree of MultiFalsehood** (MultiNonmembership), also called *MultiDegree of Falsehood*, of element x with respect to the set M are $F_1, F_2, \dots, F_s$.

All these $p + r + s = n \geq 2$ are assigned by n sources (experts) that may be:

— either totally independent;

— or partially independent and partially dependent;

— or totally dependent;

according or as needed to each specific application.

A generic element x with regard to the MultiNeutrosophic Set A has the form:

$$x(\underbrace{T_1, T_2, \dots, T_p}_{\text{multi-truth}};\qquad \underbrace{I_1, I_2, \dots, I_r}_{\text{multi-indeterminacy}};\qquad \underbrace{F_1, F_2, \dots, F_s}_{\text{multi-falsehood}})$$

In many particular cases $p = r = s$, and a source (expert) assigns all three degrees of truth, indetermincay, and falsehood (T_j, I_j, F_j) for the same element.

4. Particular Cases of MultiNeutrosophic Set (MNS)

Upon the types of sets that the neutrosophic components are, one has:

i. *Single-Valued MultiNeutrosophic Set (SVMNS)*,

when all neutrosophic components

$T_j,\ 1 \leq j \leq p;\ T_j,\ I_k, 1 \leq k \leq r,$ and $F_l, 1 \leq l \leq s,$

are single-values (numbers), such that all $T_j, I_k, F_l \in [0, 1]$.

ii. *Interval-Valued MultiNeutrosophic Set (IVMNS)*,

when all neutrosophic components

$T_j,\ 1 \le j \le p;\ T_j,\ I_k, 1 \le k \le r,$ and $F_l, 1 \le l \le s,$
are interval-values, such that all $T_j, I_k, F_l \subseteq [0, 1]$.

5. Particular Cases of Single-Valued MultiNeutrosophic Set (*SVMNS*)

a. *MultiFuzzy Set*, by setting $p \ge 2$, and $r = s = 0$, into the above SVMNS Definition.
b. *MultiIntuitionistic Set*, by setting $r = 0,\ p$ and $s \ge 1$, with $p + s \ge 3$, and $0 \le T_j + F_l \le 1$, for all $j \in \{1, 2, \dots p\}, l \in \{1, 2, \dots s\}$, into the SVMNS Definition.
c. *MultiPythagorean Fuzzy Set*, by letting $r = 0$, and $p, s \ge 3$, with $p + s \ge 3$, and $0 \le T_j^2 + F_l^2 \le 1$, for $j \in \{1, 2, \dots, p\},\ l \in \{1, 2, \dots, s\}$, into the SVMNS Definition.
d. *MultiFermatean Fuzzy Set*, by letting $r = 0$, and $p, s \ge 1$, with $p + s \ge 3$, and $0 \le T_j^3 + F_l^3 \le 1$, for $j \in \{1, 2, \dots, p\},\ l \in \{1, 2, \dots, s\}$, into SVMNS Definition.
e. *Multi q-Rung Orthopair Fuzzy Set*, by letting $r = 0$, and $p, s \ge 1$, with $p + s \ge 3$, and $0 \le T_j^q + F_l^q \le 1$, with $q \ge 1$, for $j \in \{1, 2, \dots, p\},\ l \in \{1, 2, \dots, s\}$, into SVMNS Definition.
f. *MultiPicture Fuzzy Set*, by letting $p, r, s \ge 1$, with $p + r + s \ge 4$, and $0 \le T_j + N_k + F_e \le 1$, for $j \in \{1, 2, \dots, p\},\ k \in \{1, 2, \dots, r\},\ l \in \{1, 2, \dots, s\}$, where N_k is considered neutral (as in neutrosophic set is ideterminacy) into SVMNS Definition.
g. *MultiSpherical Set*, by setting $p, r, s \ge 1$, with $p + r + s \ge 4$, and $0 \le T_j^2 + I_k^2 + F_e^2 \le 1$, and $R_{jkl} = \sqrt{1 - T_j^2 - I_k^2 - F_e^2}$, for all $j \in \{1, 2, \dots, p\},\ k \in \{1, 2, \dots, r\},\ l \in \{1, 2, \dots, s\}$, into the SVMNS Definition.

6. Particular Cases of Interval-Valued MultiNeutrosophic Set (*IVMNS*)

In an identical way we get the *Particular Cases of Interval-Valued MultiNeutrosophic Set*, as being Interval-Valued (fuzzy and fuzzy-extension) sets, replacing the single-valued components by interval components and using the operations of intervals:
For any $[a, b],\ [c, d] \subseteq [0, 1]$, where $a \le b$ and $c \le d$, one has:
$[a, b] + [c + d] = [min\{a + c, 1\}, min\{b + d, 1\}]$
$[a, b]^n = [min\{a^n, 1\}, min\{b^n, 1\}]$
$1 - [a, b] = [1 - b, 1 - a]$
$[a, b] - [c, d] = [max\{a - d, 0\}, max\{b - c, 0\}]$.

7. Application of Single-Valued MultiNeutrosophic Set

Let $M = \{A, B, C, D\}$ be a group of students.
Their performance in science is evaluated by several professors (= sources of information, experts).
Let's assume that three professors $P_1,\ P_2,\ P_3$ evaluate the degrees of positive knowledge (truth) of the students, and:
– Professor P_1 assigns the value T_1 respectively to all the students;
– Professor P_2 assigns the value T_2 respectively to all the students;
– Professor P_3 assigns the value T_3 respectively to all the students, as follows:

A($T_1 = 0.8,\ T_2 = 0.6,\ T_3 = 0.7$),
B($T_1 = 0.6,\ T_2 = 0.9,\ T_3 = 0.5$),
C($T_1 = 0.4,\ T_2 = 0.4,\ T_3 = 0.6$),
D($T_1 = 0.7,\ T_2 = 0.0,\ T_3 = 0.4$).

But two professors Q_1 and Q_2 are not very sure of the students' performances and assign indeterminate degrees (I_1 and I_2 respectively) to the students:
A ($I_1 = 0.2, I_2 = 0.3$),
B ($I_1 = 0.5, I_2 = 0.4$),
C ($I_1 = 0.1, I_2 = 0.0$),
D ($I_1 = 0.3, I_2 = 0.1$).

Further on, four professor $R_1, R_2, R_3,$ and R_4, dissatisfied with the students' performance, assign negative evaluations (falsehood degrees), $F_1, F_2, F_3,$ and F_4 respectively:

$A \ (F_1 \ = 0.7, F_2 = 0.4, F_3 = 0.5, F_4 = 0.4)$,
$B \ (F_1 \ = 0.6, F_2 = 0.3, F_3 = 0.5, F_4 = 0.1)$,
$C \ (F_1 \ = 0.2, F_2 = 0.1, F_3 = 0.2, F_4 = 0.3)$,
$D \ (F_1 \ = 0.5, F_2 = 0.2, F_3 = 0.1, F_4 = 0.2)$.

The students have been evaluated by $3 + 2 + 4$ sources of information. In the case that all sources were independent two by two, one has 9 sources. But, if there was some dependence (i.e. the same professor assigning, for example, not only the truth, but also the indeterminacy and/or the falsehood, the number of independent sources is < 9).

The more sources evaluate a subject, the better accurate result.

We got the following single-valued MultiNeutrosophic Set, where each element has the form:
$x(\{T_1, T_2, T_3\}, \{I_1, I_2\}, \{F_1, F_2, F_3, F_4\})$.

$M = \{A(\{0.8, 0.6, 0.7\}, \{0.2, 0.3\}, \{0.7, 0.4, 0.5, 0.4\}),$

$B(\{0.6, 0.9, 0.3\}, \{0.5, 0.4\}, \{0.6, 0.3, 0.5, 0.1\}),$

$C(\{0.4, 0.4, 0.6\}, \{0.1, 0.0\}, \{0.2, 0.1, 0.2, 0.3\}),$

$D(\{0.7, 0.0, 0.4\}, \{0.3, 0.1\}, \{0.5, 0.2, 0.1, 0.2\}).$

7.1. Remark on previous Application

The Single-Valued MultiNeutrosophic Set (SVMNS) coincides in form with the particular case of the Subset Neutrosophic Set (SNS) by taking the neutrosophic components as discrete subsets of the form $\{a_1, a_2, \ldots, a_m\} \subset [0,1], m \geq 1$.

For example, considering the student A, his degree of truth (membership) is $T(A) = \{0..8, 0.6, 0.7\}$, his degree of indeterminacy-membership is $I(A) = \{0.2, 0.3\}$, and his degree of falsehood (nonmembership) is $F(A) = \{0.7, 0.4, 0.5, 0.4\}$, from the point of view of Subset Neutrosophic Set.

i. The first distinction is that in the case of Subset Neutrosophic Set, *only one source (expert)* provides information about let's say the student degree $T(A) = \{0.8, 0.6, 0.7\}$, while in the case of Single-Value Multi Neutrosophic Set, three sources provide information on $T(A)$, i.e. one source evaluates the student A degree of truth as 0.8, the second one as 0.6, and the third one as 0.7. The more experts evaluating, the better accuracy, whence the SVMNS better evaluates than the SNS. Similarly for the degree of indeterminacy $I(A)$, and the degree of falsehood $F(A)$.
ii. The second distinction is in applying the neutrosophic operators, since in general the operators for the Subset Neutrosophic Sets are different from the operators for the Single-Valued MultiNeutrosophic Set (we'll see it below on Section 13).

8. Ranking of n-valued MultiNeutrosophic types of the same (*p, r, s*)-form

$(T_1, T_2, \ldots, T_p; \quad I_1, I_2, \ldots, I_r; \quad F_1, F_2, \ldots, F_s)$,

where p, r, s are integers ≥ 0, and $p + r + s = n \geq 2$, and at least one of $p, r, s \geq 2$ to be sure that we have multiplicity for at least one neutrosophic component (either truth, or indeterminacy, or falsehood).

The first research in n-ranking neutrosophic triplets was done in 2023 by V. Lakshmana Gomathi Nayagam, and Bharanidharan R. [3], using the dictionary ranking.

We propose an easier n-ranking, but this is rather an approximation. Let's compute the following.

1. Average Positivity:

$$\frac{\sum_{j=1}^{p} T_j + \sum_{k=1}^{r}(1 - I_k) + \sum_{e=1}^{s}(1 - F_e)}{p + r + s}$$

2. Average (Truth-Falsehood):

$$\frac{\sum_{j=1}^{p} T_j - \sum_{e=1}^{s} F_e}{p+s}$$

3. Average Truth

$$\frac{\sum_{j=1}^{p} T_j}{p}$$

Let's compare $(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s) \equiv N$
with $(T_1', T_2', \dots, T_p';\ \ I_1', I_2', \dots, I_r';\ \ F_1', F_2', \dots, F_s') \equiv N'$.

If their *Average Positivity* is the same, one gets (1):

$$\sum_{j=1}^{p} T_j - \sum_{k=1}^{r} I_k - \sum_{e=1}^{s} F_e = \sum_{j=1}^{p} T_j' - \sum_{k=1}^{r} I_k' - \sum_{e=1}^{s} F_e'$$

If their *Average (Truth-Falsehood)* is the same, one gets (2):

$$\sum_{j=1}^{p} T_j - \sum_{e=1}^{s} F_e = \sum_{j=1}^{p} T_j' - \sum_{e=1}^{s} F_e'$$

whence, by combining (1) and (2), one gets (3):

$$\sum_{k=1}^{r} I_k = \sum_{k=1}^{r} I_k'$$

If their *Average Truth* is the same, one gets (4):

$$\sum_{j=1}^{p} T_j = \sum_{j=1}^{p} T_j'$$

Then, from (2) and (4), one gets:

$$\sum_{e=1}^{s} F_e = \sum_{e=1}^{s} F_e'$$

Therefore $N = N'$ means that their corresponding averages of truths, indeterminacies, and falsehoods respectively are equal:

$$\begin{cases} \frac{1}{p}\sum_{j=1}^{p} T_j = \frac{1}{p}\sum_{j=1}^{p} T_j' \\ \frac{1}{r}\sum_{k=1}^{r} I_k = \frac{1}{r}\sum_{k=1}^{r} I_k' \\ \frac{1}{s}\sum_{e=1}^{s} F_e = \frac{1}{s}\sum_{e=1}^{s} F_e' \end{cases}$$

9. Ranking *n*-valued MultiNeutrosophic tuples of different (*p, r, s*)–forms

Let's consider two n-valued multi neutrosophic tuples of the forms (p_1, r_1, s_1) and respectively (p_2, r_2, s_2), where $p_1, r_1, s_1, p_2, r_2, s_2$ are integers ≥ 0, and $p_1 + r_1 + s_1 = n_1 \geq 2$, and at least one of p_1, r_1, s_1 $is \geq 2$ to be sure that we have multiplicity for at least one neutrusophic component (either truth, or indeterminacy, or falsehood); similarly $p_2 + r_2 + s_2 \geq 2$, and at least one of $p_2, r_2, s_2 \geq 2$. Let's take the following Single-Valued Multi Neutrosophic Tulpes (SVMNT):

$SVMNT = (T_1, T_2, \dots, T_{p_1};\ \ I_1, I_2, \dots, I_{r_1};\ \ F_1, F_2, \dots, F_{s_1})$ of (p_1, r_1, s_1)-form, and

$SVMNT' = (T_1', T_2', \dots, T_{p_2}'; I_1', I_2', \dots, I_{r_2}';\ \ F_1', F_2', \dots, F_{s_2}'\ )$ of (p_2, r_2, s_2)-form.

We make the classical averages of truth (T_a), indeterminancies (I_a) and falsehood (F_a), respectively for:

$SVMNT = (T_a, I_a, F_a)$

and the averages of truths (T_a'), indeterminancies (I_a'), and falsehood (F_a') respectively for:

$SVMNT' = (T'_a, I'_a, F'_a)$.

And then we apply the Score (S), Accuracy (A), and Certainty (C) Functions, as for the single valued neutrosophic set:

1. *Compute the Score Function* (average of positiveness)

$$S(T_a, I_a, F_a) = \frac{(T_a + (1 - I_a) + (1 - F_a)}{3}$$

$$S(T'_a, I'_a, F'_a) = \frac{T'_a + 1 - I'_a + \quad F'_a)}{3}$$

(i) if $S(T_a, I_a, F_a) > S(T'_a, I'_a, F'_a)$, then $SVMNT > SVMNT'$;

(ii) if $S(T_a, I_a, F_a) < S(T'_a, I'_a, F'_a)$, then $SVMNT < SVMNT'$;

(iii) and if $S(T_a, I_a, F_a) = S(T'_a, I'_a, F'_a)$,
then go to the second step.

2. *Compute the Accuracy Function* (difference between the truth and falsehood)

$$A(T_a, I_a, F_a) = T_a - \quad F_a$$

$$A(T'_a, I'_a, F'_a) = T'_a - F'_a$$

(i) if $A(T_a, I_a, F_a) > A(T'_a, I'_a, F'_a)$, then $SVMNT > SVMNT'$;

(ii) if $A(T_a, I_a, F_a) < A(T'_a, I'_a, F'_a)$, then $SVMNT < SVMNT'$;

(iii) and if $A(T_a, I_a, F_a) = A(T'_a, I'_a, F'_a)$,
then go to the third step.

3. *Compute the Certainty Function* (truth)

$$C(T_a, I_a, F_a) = T_a$$

$$C(T'_a, I'_a, F'_a) = T'_a$$

(i) if $C(T_a, I_a, F_a) > C(T'_a, I'_a, F'_a)$, then $SVMNT > SVMNT'$;

(ii) if $C(T_a, I_a, F_a) < C(T'_a, I'_a, F'_a)$, then $SVMNT < SVMNT'$;

(iii) if $C(T_a, I_a, F_a) = C(T'_a, I'_a, F'_a)$, then $SVMNT$ and $SVMNT'$ are multi-neutrosophically equal, i.e. $T_a = T'_a$, $I_a = I'_a$, $F_a = F'_a$, or their corresponding truth, indeterminancy, and falsehood averages are equal.

10. Example 1

Example where all sources providing information have equal weights.

Assume the student George is evaluated by several professors from his university with respect to his skills in science:

George$(\{0.8, 0.9, 0.3\}, \{0.2\}, \{0.6, 0.7\})$

While the student John is evaluated with respect to the same scientific skills by some of the previous professors and by other professors from the same university:

John$(\{0.7, 1.0, 0.6, 0.5\}, \{01., 0.4\}, \{0.2, 0.8, 0.7\})$

Which student does better than the others?

Let's compute the averages.

John$\left(\frac{0.7+1.0+0.6+0.5}{4}, \frac{0.1+0.4}{2}, \frac{0.2+0.8+0.7}{3}\right) \simeq$ John(0.70, 0.25, 0.57).

George$\left(\frac{0.8+0.9+0.3}{3}, \frac{0.2}{1}, \frac{0.6+0.7}{2}\right) \simeq$ George(0.67, 0.20, 0.65).

The Score Function:

S(George) = $\frac{0.67+(1-0.20)\_(1-0.65)}{3} \simeq 0.61$.

S(John) = $\frac{0.70+(1-0.25)\_(1-0.57)}{3} \simeq 0.63$.

John has better scientific skills than George, since S(John) $\simeq 0.63 > 0.61 \approx$S(George).

This may be explained from the fact that if more or less sources evaluate the same element x of a given set, we make the average of evaluations.

In cases some sources have a greater weight in evaluation than others, one uses the *weighted averages*, indexed as T_{wa}, I_{ua}, F_{va} and $T'_{w'a'}$, $I'_{u'a'}$, $F'_{v'a'}$ respectively.

Because the sources may be independent or partially independent, the sum of weights should not necessarily be equal to 1. As such, one has:

$$T_{wa} = \frac{w_1T_1 + w_2T_2 + \cdots + w_{p_1}T_{p_1}}{w_1 + w_2 + \cdots + w_{p_1}}$$

where $w_1, w_2, \ldots, w_{p_1} \in [0,1]$, while the sum $w_1 + w_2 + \cdots + w_{p_1}$ may be < 1, or $= 1$, or > 1;

$$I_{ua} = \frac{u_1I_1 + u_2I_2 + \cdots + u_{r_1}I_{r_1}}{u_1 + u_2 + \cdots + u_{r_1}}$$

where $u_1, u_2, \ldots, u_{r_1} \in [0,1]$, while the sum $u_1 + u_2 + \cdots + u_{r_1}$ may be < 1, or $= 1$, or > 1;

$$F_{va} = \frac{v_1F_1 + v_2F_2 + \cdots + v_{s_1}F_{s_1}}{v_1 + v_2 + \cdots + v_{s_1}}$$

where $v_1, v_2, \ldots, v_{s_1} \in [0,1]$, while the sum $v_1 + v_2 + \cdots + v_s$ may be < 1, or $= 1$, or > 1.

Similarly,

$$T'_{w'a} = \frac{w'_1T'_1 + w'_2T'_2 + \cdots + w'_{p_2}T'_{p_2}}{w'_1 + w'_2 + \cdots + w'_{p_2}}$$

where $w'_1, w'_2, \ldots, w'_{p_2} \in [0,1]$, while the sum $w'_1 + w'_2 + \cdots + w'_{p_1}$ may be < 1, or $= 1$, or > 1;

$$I'_{u'a} = \frac{u'_1I'_1 + u'_2I'_2 + \cdots + u'_{r_2}I'_{r_2}}{u_1 + u_2 + \cdots + u_{r_1}}$$

where $u'_1, u'_2, \ldots, u'_{r_2} \in [0,1]$, while the sum $u'_1 + u'_2 + \cdots + u'_{r_1}$ may be < 1, or $= 1$, or > 1;

$$F'_{v'a} = \frac{v'_1F'_1 + v'_2F'_2 + \cdots + v'_{s_2}F'_{s_2}}{v'_1 + v'_2 + \cdots + v'_{s_2}}$$

where $v'_1, v'_2, \ldots, v'_{s_2} \in [0,1]$, while the sum $v'_1 + v'_2 + \cdots + v'_s$ may be < 1, or $= 1$, or > 1.

And, similarly, one applies the Score, Accuracy, and Certainty Functions on these weighted averages to rank them.

$S(T_{wa}, I_{ua}, F_{va}) = \frac{T_{wa}+(1-I_{ua})+(1-F_{va})}{3}$

$A(T_{wa}, I_{ua}, F_{va}) = T_{wa} - F_{va}$

$C(T_{wa}, I_{ua}, F_{va}) = T_{wa}$

$S(T'_{w'a}, I'_{u'a}, F'_{v'a}) = \frac{T'_{w'a}+(1-I'_{u'a})+(1-F'_{v'a})}{3}$

$A(T'_{w'a}, I'_{u'a}, F'_{v'a}) = T'_{w'a} - F'_{v'a}$

$C(T'_{w'a}, I'_{u'a}, F'_{v'a}) = T'_{w'a}$

11. Example 2

Let's retake the *Example 1*:

George($\{0.8, 0.9, 0.3\}, \{0.2\}, \{0.6, 0.7\}$),

and John($\{0.7, 1.0, 0.6, 0.5\}, \{0.1, 0.4\}, \{0.2, 0.8, 0.7\}$),

and assume the six evaluators of George have the following corresponding weights respectively:

0.6, 0.7, 0.4; 0.3; 0.8, 0.7;

while the nine evaluators of John have the following corresponding weights respectively:

0.7, 0.2, 0.5, 0.1; 0.8, 0.3; 0.9, 0.4, 0.6.

Let's compute the weighted averages.

For George:

$$T_{wa} = \frac{0.8\cdot(0.6) + 0.9\cdot(0.7) + 0.3\cdot(0.4)}{0.6 + 0.7 + 0.4} \simeq 0.72$$

$$I_{ua} = \frac{0.2\cdot(0.3)}{0.3} = 0.20$$

$$F_{va} = \frac{0.6\cdot(0.8) + 0.7\cdot(0.7)}{0.8 + 0.7} \simeq 0.65.$$

We got George (0.72, 0.20, 0.65).

For John:

$$T'_{w'a} = \frac{0.7\cdot(0.7) + 1.0\cdot(0.2) + 0.6\cdot(0.5) + 0.5\cdot(0.1)}{0.7 + 0.2 + 0.5 + 0.1} \simeq 0.69$$

$$I'_{u'a} = \frac{0.1\cdot(0.8) + 0.4\cdot(0.3)}{0.8 + 0.3} \simeq 0.18$$

$$F'_{v'a} = \frac{0.2\cdot(0.9) + 0.8\cdot(0.4) + 0.7\cdot(0.6)}{0.9 + 0.4 + 0.6} \simeq 0.48$$

We got John (0.69, 0.18, 0.48).

Compute the score functions in order to rank them.

S(George) = S(0.72, 0.20, 0.65) = $\frac{0.72+(1-0.20)+(1-0.65}{3} \simeq 0.62$.

S(John) = S(0.69, 0.20, 0.65) = $\frac{0.69+(1-0.20)+(1-0.65)}{3} \simeq 0.61$.

Therefore, now George is better, because S(George) = 0.62 $>$ 0.61 = S(John).

12. Isomorphism between Subset Refined Neutrosophic Set (*SRNS*) and Subset MultiNeutrosophic Set (*SMNS*)

The Subset Refined Neutrosophic Set was first introduced by Smarandache [4} in 2013.

12.1. Definition of Subset Refined Neutrosophic Set (SRNS)

Let $\mathcal{U}$ be a universe of discourse, and a set $R \subset \mathcal{U}$.

Then a Subset Refined Neutrosophic R is defined as follows:

$R = \{x, x(T, I, F), x \in \mathcal{U}\}$,

where T is refined/split into p sub-truths,

I is refined/split into r sub-indeterminacies,

$I = \langle I_1, I_2, \dots, I_r\rangle, I_k \subseteq [0, 1],\ 1 \le k \le r,$

and F is refined/split into s sub-falsehoods,

$F = \langle F_1, F_2, \dots, F_s\rangle, F_l \subseteq [0, 1],\ 1 \le l \le s,$

where $p, r, s \ge 0$ are integers, and $p + r + s = n \ge 2$, and at least one of p, r, s is ≥ 2 in order to ensure the existence of refinement (splitting).

Similarly, in particular cases, $p = r = s$, meaning that each component *T, I, F* is refined/split into the same member of sub-components.

The isomorphism is obvious:

$$\varphi: SMNS \to SRNS$$
$$\varphi(T_j) = T_j,\ 1 \le j \le p,$$
$$\varphi(I_k) = I_k,\ 1 \le k \le r,$$
$$\varphi(F_l) = F_l,\ 1 \le l \le s.$$

But while $T_j,\ I_k,\ F_l$ from SMNS are duplicates (or multi-truth, multi-indeterminacy, multi-falsehood respectively), the corresponding $T_j,\ I_k,\ F_l$ from SRNS are parts (or sub-truth, sub-indeterminacy, sub-falsehood respectively).

13. Operators on Multi (and Refined) Neutrosophic Sets/Logic

i. The case when the neutrosophic tuples have the same (p, r, s)-format.

Let $\vee_N,\ \wedge_N,\ \neg_N,\ \rightarrow_N,\ \leftrightarrow_N$ be the neutrosophic union, intersection, complement (negation), inclusion (implication), equality (equivalence) respectively.

While $\vee_F,\ \wedge_F,\ \neg_F,\ \rightarrow_F,\ \leftrightarrow_F$ the fuzzy operators respectively, where $\vee_F$ and $\wedge_F$ are t-conorm and t-norm respectively, afterwards fuzzy negation, fuzzy implication, and fuzzy equivalence respectively.

Also, by notation, one considers:

$(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s) \equiv (T_j, 1 \le j \le p;\ \ I_k, 1 \le k \le r; F_l, 1 \le l \le s\ \ )$.

The operations will be a straightforward extension from the (1, 1, 1)-format (*T, I, F*) to the (p, r, s)-format.

Multi/Refined Neutrosophic Union

$$(T_1, T_2, \dots, T_{p_1};\ \ I_1, I_2, \dots, I_{r_1};\ \ F_1, F_2, \dots, F_s) \vee_N (T_1', T_2', \dots, T_p'; I_1', I_2', \dots, I_r';\ \ F_1', F_2', \dots, F_s')$$
$$= (T_1 \vee_F T_1', T_2 \vee_F T_2'\ \ , \dots, T_p \vee_F T_p';\ \ I_1 \wedge_F I_2', \dots, I_r \wedge_F I_r'; F_1 \wedge_F F_1', F_2 \wedge_F F_2', \dots, F_s \wedge_F F_s')$$

Shortly, we may write:

$$(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s) \vee_N (T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s)$$
$$= (T_j \vee_F T_j', 1 \le j \le p; I_k \wedge_F I'_k, 1 \le k \le r; F_l \wedge_F F_l', 1 \le l \le s)$$

Multi/Refined Neutrosophic Intersection

$$(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s) \wedge_N (T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s)$$
$$= (T_j \wedge_F T'_j, 1 \le j \le p; I_k \vee_F I'_k, 1 \le k \le r; F_l \vee_F F'_l, 1 \le l \le s)$$

Multi/Refined Neutrosophic Negation

$$\neg_N(T_j, 1 \le j \le p; 1 \le k \le r; F_l, 1 \le l \le s) = (F_l, 1 \le l \le s; 1 - I_k, 1 \le k \le r; T_j, 1 \le j \le p)$$

Multi/Refined Neutrosophic Implication and Equivalence

Let $A = (T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s)$

and $A' = (T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s)$.

Then $A \rightarrow_N A'$ means $(\neg_N A) \vee_N A'$

and $A \leftrightarrow_N A'$ means $[[A \rightarrow_N A']\text{and}[A' \rightarrow_N A]]$.

ii. The case when the neutrosophic tuples have different (p, r, s)-formats.

Let $B_1 = (T_j, 1 \le j \le p_1;\ \ I_k, 1 \le k \le r_1;\ \ F_l, 1 \le l \le s_1)$ of (p_1, r_1, s_1)-format,

and $B_2 = (T'_j, 1 \le j \le p_2;\ \ I'_k, 1 \le k \le r_2;\ \ F'_l, 1 \le l \le s_2)$ of (p_2, r_2, s_2)-format.

We compute the weight average of each neutrosophic component of both tuples, and we get:

$B_1 = (T_{wa}, I_{ua}, F_{va})$,

and $B_2 = (T'_{w\prime a}, I'_{u\prime a}, F'_{v\prime a})$, which have the (1, 1, 1)-form, let's simplify their notation under the form:

$B_1 = (T, I, F)$

and $B_2 = (T', I', F')$.

and one applies the well-known and most used neutrosophic operators:

$$(T, I, F) \vee_N (T', I', F') = (T \vee_F T', I \wedge_F I', F \wedge_F F')$$
$$(T, I, F) \wedge_N (T', I', F') = (T \wedge_F T', I \vee_F I', F \vee_F F')$$
$$\neg(T, I, F) \vee_N (F, 1-I, T)$$

$(T, I, F) \to_N (T', I', F')$ is the same as $(F, 1-I, T) \vee_N (T', I', F')$, or $(F \vee_F T', (1-I) \wedge_F I', T \wedge_F F')$

$(T, I, F) \leftrightarrow_N (T', I', F')$ is the same as $(T, I, F) \to_N (T', I', F')$ and $(T', I', F') \to_N (T, I, F)$

or $(F \vee_F T', (1-I) \wedge_F I', T \wedge_F F')$ neutrosophic and $(F' \vee_F T, (1-I_1') \wedge_F I, T' \wedge_F F)$,

or $([(F \vee_F T') \wedge_F (F' \vee_F T)], [(1-I) \wedge_F I'] \vee_F [(1-I_1') \wedge_F I], [(T \wedge_F F') \wedge_F (T' \wedge_F F)])$.

13.1. Weight Averaging and Neutrosophic Operators

The (weight) averaging and the neutrosophic operators for (*p, r, s*)-tuples, in general, do not commute.

13.2. Counter-Example

Let's consider the (2,3,2)-tuples:

A = $(\{0.2, 0.3\}, \{0.1, 0.4, 0.5\}, \{0.6, 0.9\})$

and B = $(\{0.8, 0.4\}, \{0.6, 0.0, 0.3\}, \{0.5, 0.6\})$

i. Union, then Averaging Union:

$A \vee_N B =$

$= \left(\left\{max\{0.2, 0.8\}, max\{0.3, 0.4\}, \{min\{0.1, 0.6\}\}, min\{0.4, 0.0\}, min\{0.5, 0.3\}, \{min\{0.6, 0.5\}\}, min\{0.9, 0.6\}\right\}\right)$

$= (\{0.8, 0.4\}, \{0.1, 0.0, 0.3\}, \{0.5, 0.6\})$

Averaging:

$$A \vee_N B = \left(\frac{0.8+0.4}{2}, \frac{0.1+0.0+0.3}{3}, \frac{0.5+0.6}{2}\right) \simeq (0.60, 0.13, 0.55)$$

ii. Reversely: Averaging, then Union.

Averaging:

$$A = \left(\frac{0.2+0.3}{2}, \frac{0.1+0.4+0.5}{3}, \frac{0.6+0.9}{2}\right) \simeq (0.25, 0.33, 0.75)$$
$$B = \left(\frac{0.8+0.4}{2}, \frac{0.6+0.0+0.3}{3}, \frac{0.5+0.6}{2}\right) \simeq (0.60, 0.30, 0.55)$$

Union:

$$A \vee_N B = (max\{0.25, 0.60\}, min\{0.33, 0.30\}, min\{0.75, 0.53\})$$
$$= (0.60, 0.30, 0.55) \neq$$

$(0.60, 0.13, 0.55)$.

Conclusion: The MultiNeutrosophic Set was introduced now for the first time. It is a neutrosophic set whose elements' degrees of truth / indeterminacy / falsehood are evaluated by many sources to get a better accurate result. The ranking of the n-tuples of the form (p, r, s) and their operators were also built on.

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