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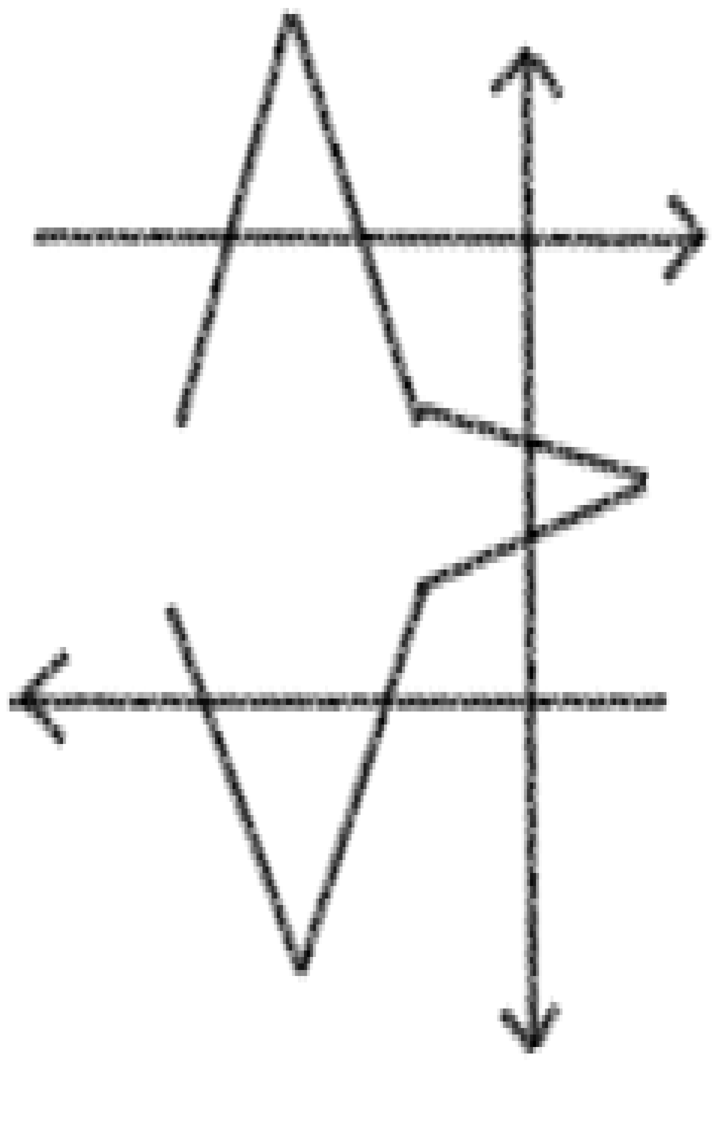
Neutrosophic Algebraic Structures and Their Applications



Editors:

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University of New Mexico, USA

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Gaziantep University, Turkey

Assoc. Prof. Dr. Derya Bakbak
Ankara, Turkey

Assoc. Prof. Dr. Vakkas Uluçay
Kilis 7 Aralık University, Turkey

Dr. Abdullah Kargın
Gaziantep University, Turkey

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Aims and Scope

Neutrosophic theory and its applications have been expanding in all directions at an astonishing rate especially after of the introduction the journal entitled "Neutrosophic Sets and Systems". New theories, techniques, algorithms have been rapidly developed. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, hesitant fuzzy set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic hesitant fuzzy set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been an important tool in the application of various areas such as data mining, decision making, e-learning, engineering, medicine, social science, and some more.

Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay & Abdullah Kargın

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Preface

Neutrosophic set has been derived from a new branch of philosophy, namely Neutrosophy. Neutrosophic set is capable of dealing with uncertainty, indeterminacy and inconsistent information. Neutrosophic set approaches are suitable to modeling problems with uncertainty, indeterminacy and inconsistent information in which human knowledge is necessary, and human evaluation is needed.

Neutrosophic set theory firstly proposed in 1998 by Florentin Smarandache, who also developed the concept of single valued neutrosophic set, oriented towards real world scientific and engineering applications. Since then, the single valued neutrosophic set theory has been extensively studied in books and monographs introducing neutrosophic sets and its applications, by many authors around the world. Also, an international journal - Neutrosophic Sets and Systems started its journey in 2013.

http://fs.unm.edu/neutrosophy.htm.

This first volume collects original research and applications from different perspectives covering different areas of neutrosophic studies, such as decision-making, neutroalgebra, neutro metric, and some theoretical papers.

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Florentin Smarandache
University of New Mexico, USA

Memet Şahin
Gaziantep University, Turkey

Derya Bakbak
Ankara, Turkey

Vakkas Uluçay
Kilis 7 Aralık University, Turkey

Abdullah Kargın
Gaziantep University, Turkey

Chapter One

History of SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra (revisited again)

Florentin Smarandache
The University of New Mexico
Mathematics, Physics, and Natural Science Division,
705 Gurley Ave., Gallup, NM 87301, USA
E-mail: smarand@unm.edu

ABSTRACT

We recall the topic of the nth-Powerset of a Set, and the concepts built on it such as SuperHyperOperation, Super-HyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation, Neutrosophic Super-HyperAxiom and Neutrosophic SuperHyperAlgebra are re-called and then prolonged to the Neutrosophic SuperHyperStructures {or more accurately Neutrosophic (m,n)-SuperHyperStructures}.

Keywords: nth-Powerset of a Set, HyperAxiom; HyperOperation; Hy-perAlgebra; SuperHyperAxiom, SuperHyperOperation; Su-perHyperAlgebra, Neutrosophic SuperHyperAlgebra; Su-perHyperStructure; Neutrosophic SuperHyperStructure.

1. History of HyperAlgebra and SuperHyperAlgebra

We revisit the SuperHyperAgebra and Neutrosophic SuperHyperAlgebra introduced and developed by Smarandache [2, 3, 4] between 2016 – 2022.

We recall that F. Marty [1] has introduced in 1934 the

HyperAlgebra that is based on HyperOperations and consequently on HyperAxioms. More information and the evolution from HyperAlgebra to SuperHyperAlgebra & Neutrosophic SuperHyperAlgebra are presented below.

2. **Definition of classical HyperOperations**

Let U be a universe of discourse and H a non-empty set, $H \subset U$.

A **classical Binary HyperOperation** $\circ_2^*$ is defined as follows:

$\circ_2^* : H^2 \rightarrow P_*(H)$

where H is a discrete or continuous set, and $P_*(H)$ is the powerset of H without the empty-set ϕ or $P_*(H) = P(H) - \{\phi\}$.

A **classical m-ary HyperOperation** $\circ_m^*$ is defined as:

$\circ_m^* : H^m \rightarrow P_*(H)$

for integer $m \geq 1$. For $m = 1$ one gets a **Unary HyperOperation**.

The **classical HyperStructures** are structures endowed with classical HyperOperations and classical HyperAxioms.

3. Definition of the n^{th}-Powerset of a Set [2]

The n^{th}-Powerset of a Set was introduced in [2, 3, 4] in the following way:

$P^n(H)$, as the n^{th}-Powerset of the Set H, for integer $n \geq 1$, is recursively defined as:

$P^2(H) = P\big(P(H)\big)$, $P^3(H) = P\big(P^2(H)\big) = P\Big(P\big(P(H)\big)\Big)$, … ,

$P^n(H) = P\big(P^{n-1}(H)\big)$,

where $P^\circ(H) \overset{\text{def}}{=} H$, and $P^1(H) \overset{\text{def}}{=} P(H)$.

The nth-Powerset of a Set better reflects our complex reality, since a set H (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general any items) is organized onto subsets *P(H)*, and these subsets are again organized onto subsets of subsets *P(P(H))*, and so on [Smarandache, 2016].

That's our world.

4. Neutrosophic HyperOperation and Neutrosophic HyperStructures [1, 2]

In the classical HyperOperation and classical HyperStructures, the empty-set ϕ does not belong to the power set, or $P_*(H) = P(H) - \{\phi\}$.

However, in the real world we encounter many situations when a HyperOperation $\circ$ is:

- *indeterminate*, for example $a \circ b = \phi$ (unknown, or undefined),
- or *partially indeterminate*, for example: $c \circ d = \{[0.2, 0.3], \phi\}$.

In our everyday life, there are many more operations and laws that have some degrees of indeterminacy (vagueness, unclearness, unknowingness, contradiction, etc.), than those that are totally determinate.

That's why in 2016 we have extended the classical HyperOperation to the Neutrosophic HyperOperation, by taking the whole power *P(H)* (that includes the empty-set ϕ as well), instead of $P_*(H)$ (that does not include the empty-set ϕ), as follow.

3.1 Definition of Neutrosophic HyperOperation

Let U be a universe of discourse and H a non-empty set, $H \subset U$.

A **Neutrosophic Binary HyperOperation** $\circ_2$ is defined as follows:

$\circ_2 : H^2 \rightarrow P(H)$

where H is a discrete or continuous set; *P(H)* is the powerset of H that includes the empty-set ϕ.

A **Neutrosophic m-ary HyperOperation** $\circ_m$ is defined as:

$\circ_m : H^m \to P(H)$

for integer *m* ≥ *1*. Similarly, for *m = 1* one gets a **Neutrosophic Unary HyperOperation**.

3.2 Neutrosophic HyperStructures

A Neutrosophic HyperStructure is a structured endowed with Neutrosophic HyperOperations.

5. Definition of SuperHyperOperations

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [2].

Let $P_*^n(H)$ be the n[th]-powerset of the set H such that none of $P(H), P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

Also, let $P^n(H)$ be the n[th]-powerset of the set H such that at least one of the $P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has **classical-type SuperHyperOperations**, or $P^n(H)$ and in this case one has **Neutrosophic SuperHyperOperations**, for integer $n \geq 2$.

5.1 Classical-type Binary SuperHyperOperation

A **classical-type Binary SuperHyperOperation** $\circ_{(2,n)}^*$ is defined as follows:

$\circ_{(2,n)}^* : H^2 \to P_*^n(H)$

where $P_*^n(H)$ is the n^{th}-powerset of the set H, with no empty-set.

5.2 Examples of classical-type Binary SuperHyperOperation

1) Let $H = \{a, b\}$ be a finite discrete set; then its power set, without the empty-set ϕ, is:

$P(H) = \{a, b, \{a, b\}\}$, and:

$P^2(H) = P(P(H)) = P(\{a, b, \{a, b\}\}) =$
$\{a, b, \{a, b\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}, \{a, b, \{a, b\}\}\}$.

$\circ_{(2,2)}^* : H^2 \to P_*^2(H)$

Table 1. Example 1 of classical-type Binary SuperHyperOperation.

$\circ_{(2,2)}^*$	a	b
a	$\{a, \{a, b\}\}$	$\{b, \{a, b\}\}$
b	a	$\{a, b, \{a, b\}\}$

2) Let $H = [0, 2]$ be a continuous set.
$P(H) = P([0,2]) = \{A|A \subseteq [0, 2], A = subset\}$,
$P^2(H) = P(P([0, 2]))$.
Let $c, d \in H$.
$\circ^*_{(2,2)} : H^2 \to P^2_*(H)$

Table 2. Example 2 of classical-type Binary SuperHyperOperation.

$\circ^*_{(2,2)}$	c	d
c	$\{[0, 0.5], [1,2]\}$	$\{0.7, 0.9, 1.8\}$
d	$\{2.5\}$	$\{(0.3, 0.6), \{0.4, 1.9\}, 2\}$

4.3 Classical-type m-ary SuperHyperOperation {or more accurate denomination (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:
$\circ^*_{(m,n)} : H^m \to P^n_*(H)$
where the integers $m, n \geq 1$,
$H^m = \underbrace{H \times H \times \ldots \times H}_{m\ times}$,
and $P^n_*(H)$ is the n^{th}-powerset of the set H that includes the empty-set.

This SuperHyperOperation is a m-ary operation defined from the set H to the n^{th}-powerset of the set H.

4.4 Neutrosophic m-ary SuperHyperOperation {or more accurate denomination Neutrosophic (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:
$\circ_{(m,n)} : H^m \to P^n(H)$
where the integers $m, n \geq 1$; $P^n(H)$ - the n-th powerset of the set H that includes the empty-set.

6. SuperHyperAxiom

A **classical-type SuperHyperAxiom** or more accurately a ***(m, n)*-SuperHyperAxiom** is an axiom based on classical-type SuperHyperOperations.

Similarly, a **Neutrosophic SuperHyperAxiom** {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- **Strong SuperHyperAxioms**, when the left-hand side is equal to the right-hand side as in non-hyper axioms,
- and **Week SuperHyperAxioms**, when the intersection between the left-hand side and the right-hand side is non-empty.

For examples, one has:

- Strong SuperHyperAssociativity, when $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H^m$, where the law $\circ^*_{(m,n)} : H^m \to P_*^n(H)$;
- and Week SuperHyperAssociativity, when $[(x \circ y) \circ z] \cap [x \circ (y \circ z)] \neq \phi$, for all $x, y, z \in H^m$.

7. SuperHyperAlgebra and SuperHyperStructure

A **SuperHyperAlgebra** or more accurately ***(m-n)*-SuperHyperAlgebra** is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a **Neutrosophic SuperHyperAlgebra** {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have **SuperHyperStructures** {or *(m-n)*-SuperHyperStructures}, and corresponding **Neutrosophic SuperHyperStructures**.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

8. Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra

i. If none of the power sets $P^k(H)$, $1 \leq k \leq n$, do not include the empty set ϕ, then one has a classical-type SuperHyperAlgebra;

ii. If at least one power set, $P^k(H)$, $1 \leq k \leq n$, includes the empty set ϕ, then one has a Neutrosophic SuperHyperAlgebra.

9. Conclusion

A set *H* (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general any items) is organized onto subsets *P(H)*, and these subsets in their turn are again organized onto subsets of subsets *P(P(H))*, and so on, the n^{th}-PowerSet of a Set [2] was introduced to better reflect our world.

The most general form of algebras, which is based on the n^{th}-Powerset of a Set, called SuperHyperAlgebra {or more accurate denomination (m, n)-SuperHyperAlgebra} and the Neutrosososphic SuperHyperAlgebra, and their extensions to SuperHyperStructures and respectively Neutrosophic SuperHyperAlgebra in any field of knowledge are recalled.

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Chapter Two

A Study on the Properties of AntiTopological Space

Bhimraj Basumatary[1] and Jeevan Krishna Khaklary[2]
[1]Department of Mathematical Sciences, Bodoland University, Kokrajhar, 783370, India
[2]Central Institute of Technology, Kokrajhar, 783370, India
E-mail: brbasumatary14@gmail.com, jk.khaklary@cit.ac.in

ABSTRACT

In the current study, the properties of Interior, Closure and Boundary points of the antiTopological studies have been observed and studied by introducing the ideas of AntiInterior, AntiClosure, and AntiBoundary. It has been found that some of the properties that are valid in general topological spaces are also valid in anti-topological spaces while some of the properties are found to be not valid, as in the case that the AntiInterior of a set is not the smallest closed set that contains the set as in the general topological spaces.

Keywords: AntiInterior, AntiClosure, AntiBoundary, AntiTopological space.

INTRODUCTION

General topology is the branch where most of the studies had been done by all the founders of topology and the various properties that the subsets of the topology have, like continuity, connectedness, compactness, etc. But, most of the properties that have been accepted to be of the topological spaces are, as put forward by the ones who defined them, without any actual testing on whether they apply to the real-world situations, whether they are true for all cases or whether there may exist some cases where those cases are not applicable in general. That is, where the proposal of a fuzzy set came in 1965 by Lofti A. Zadeh [39], and it is where elements of a set are assigned degree of membership and degree of non-membership. And, in due course of time, the case of neutrosophy had to be ushered in by Florentine Smarandache in 1998. The neutrosophic set encompasses three components, namely the truth (T), the indeterminacy (I), and the falsity (F) of a statement or a property. Many authors (Sahin *et al.* [40, 41, 56, 57], Hassan *et al.* [42], Ulucay *et al.* [43-45, 48-50], Broumi *et al.* [46]) applied the concepts of the neutrosophic set to various field [58-84]. The present study deals with the falsity component of the neutrosophic set. Anti-topological space was defined along with neutro-topological space by Sahin *et al.* [25].

In recent years, there has been a surge in academic interest in neutrosophic set theory. The concept of neutro-structures and anti-structures was first defined by Florentin Smarandache [30, 31]. Also, a lot of researchers studied neutroalgebra [51-55]. Şahin *et al.* [25] discussed the idea of neutro-topological space and anti-topological space. Smarandache [33] studied NeutroAlgebra as a generalization of partial algebra. Agboola [1-3] investigated the idea of NeutroRings, NeutroGroups, and finite NeutroGroups of type-NG.

Smarandache [34] proposed the generalizations and alternatives of Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures. Al-Tahan *et al.* [6] studied the NeutroOrderedAlgebra, NeutroHyper structures, and their properties.

Smarandache [30-31] founded and studied the concept of neutro-structures and anti-structures. From the concepts of NeutroAlgebra, he showed that if a statement (theorem, lemma, consequence, property, etc.) is totally true in a classical Algebra, it does not mean that it is also totally true in a NeutroAlgebra or in an AntiAlgebra. It depends on the operations and axioms (if they are totally true, partially true, totally false, or partially or totally indeterminate) it is based upon.

For examples:

(1) Let $(A,*)$ be a NeutroAlgebra (it has NeutroOperations or NeutroAxioms while the others are classical Operations and classical Axioms, and no AntiOperation and no AntiAxiom).

Statement: If x, y in A, then $x * y$ in A.
This statement is true for classical Algebra.
But for a NeutroAlgebra we have:

(a) The Statement is true if the operation $*$ is a classical Operation (totally true).
(b) The Statement is true if the operation $*$ is a NeutroAxiom,
but x, y both belong to the partially true subset;
(c) The Statement is false if the operation $*$ is a NeutroAxiom,
and at least one of x or y belongs to the partially false subset.

(2) Similarly, for the NeutroGroup.
Let A be a NeutroGroup, and x in A. Then its inverse x^{-1} is also in A. This is true for the classical Group.
For the NeutroGroup:
(a) This is true if the inverse element axiom is totally true;
(b) This is true if the NeutroInverse element axiom is partially true, and x belongs to the true subset;
(c) This is false otherwise.

By observing the above concepts, the properties of Interior, Closure and Boundary points of the AntiTopological studies have been observed.

BACKGROUND

Definition 2.1: [34] The NeutroSophication of the Law

(i) Let X be a non-empty set and $*$ be binary operation. For some elements $(a, b) \in (X, X)$, $(a * b) \in X$ (degree of well defined (T)) and for other elements $(x, y), (p, q) \in (X, X)$; [$x * y$ is indeterminate (degree of indeterminacy (I)), or $p * q \notin X$ (degree of outer-defined (F)],where (T, I, F) is different from (1,0,0) that represents the Classical Law, and from (0,0,1) that represents the AntiLaw.

(ii) In NeutroAlgebra, the classical well-defined for $*$ binary operation is divided into three regions: degree of well-defined (T), degree of indeterminacy (I) and degree of outer-defined (F) similar to neutrosophic set and neutrosophic logic.

Definition 2.2: [25] Let X be the non-empty set and τ be a collection of subsets of X. Then τ is said to be a NeutroTopology on X and the pair (X, τ) is said to be a NeutroTopological space, if at least one of the following conditions hold good:

(i) $[(\emptyset_N \in \tau, X_N \notin \tau)\ or\ (X_N \in \tau, \emptyset_N \notin \tau)]$ or $[\emptyset_N, X_N \in_{\sim} \tau]$.

(ii) For some n elements $a_1, a_2, \dots, a_n \in \tau, \bigcap_{i=1}^{n} a_i \in \tau$ [degree of truth T] and for other n elements $b_1, b_2, \dots, b_n \in \tau, p_1, p_2, \dots, p_n \in \tau$; $[(\bigcap_{i=1}^{n} b_i \notin \tau)$ [degree of falsehood F] or $(\bigcap_{i=1}^{n} p_i$ is indeterminate (degree of indeterminacy I)], where n is finite; where (T, I, F) is different from (1,0,0) that represents the Classical Axiom, and from (0,0,1) that represents the AntiAxiom.)].

(iii) For some n elements $a_1, a_2, \dots, a_n \in \tau, \bigcup_{i=1} a_i \in \tau$ [degree of truth T] and for other n elements $b_1, b_2, \dots, b_n \in \tau, p_1, p_2, \dots, p_n \in \tau$; $[(\bigcup_{i=I} b_i \notin \tau)$ [degree of falsehood F] or $(\bigcup_{i=I} p_i$ is indeterminate (degree of indeterminacy I)], where n is finite; where (T, I, F) is different from (1,0,0) that represents the Classical Axiom, and from (0,0,1) that represents the AntiAxiom.)].

Remark 2.1: [25] The symbol "$\in_{\sim}$" will be used for situations where it is an unclear appurtenance (not sure if an element belongs or not to a set). For example, if it is not certain whether "a" is a member of the set P, then it is denoted by a $\in_{\sim} P$.

Theorem 2.1: [25] Let (X, τ) be a classical topological space. Then $(X, \tau - \emptyset)$ is a NeutroTopological space.

Theorem 2.2: [25] Let (X, τ) be a classical topological space. Then $(X, \tau - X)$ is a NeutroTopological space.

Definition 2.3: [34]**: The Anti-sophication of the Law** (totally outer-defined)

Let X be a non-empty set and * be a binary operation. For all double elements $(x, y) \in (X, X), x * y \notin X$ (totally outer-defined).

Definition 2.4: [25]: AntiTopological space: Let X be a non-empty set, τbe a collection of subsets of X. If the following conditions {i, ii, iii} are satisfied then, $\tau$is called an anti-topology and (X, τ) is called an anti-topological space.

i) $\emptyset, X \notin \tau$
ii) For all $q_1, q_2, q_3, \dots, q_n \in \tau,\ \bigcap_{i=1}^{n} q_i \notin \tau$, where n is finite.
iii) For all $q_1, q_2, q_3, \dots, q_n \in \tau,\ \bigcup_{i \in I} q_i \notin \tau$, where I is an index set.

MAIN FOCUS OF THE CHAPTER

Proposition 3.1: In an AntiTopological space. The following conditions (i), (ii), and (iii) are satisfied.

(i) Empty set and X is not AntiOpen.
(ii) Union of the AntiOpen sets is not AntiOpen.
(iii) Intersection of the AntiOpen sets is not AntiOpen.

Examples 3.1: Let $X = \{a, b, c, d\}$ and $\tau = \{\{a, b\}, \{c, d\}, \{b, c\}\}$. Then (X, τ) is antiTopological space.

(i) Here $\emptyset\ and\ X$ are not AntiOpen.
(ii) $\{a, b\} \cup \{c, d\} = \{a, b, c, d\}; \{a, b\} \cup \{b, c\} = \{a, b, c\}; \{c, d\} \cup \{b, c\} = \{b, c, d\}$ which are all not AntiOpen in (X, τ).
(iii) Also,$\{a, b\} \cap \{c, d\} = \emptyset; \{a, b\} \cap \{b, c\} = \{b\}; \{c, d\} \cap \{b, c\} = \{c\}$, which are all not AntiOpen in (X, τ).

Definition 3.1: Let (X, τ) be anAntiTopological space over X and A is subset on X. Then, the AntiInterior of A is the union of all AntiOpen subsets of A. Clearly, AntiInterior of A is the biggest AntiOpen set over X which is contained A.

That is, $AntiInt(A) = \cup \{B, where\ B\ is\ open\ and\ B \subseteq A\}$

Proposition 3.2: Let (X, τ) be an AntiTopological space over X and A is subset on X. If A is AntiOpen, then $AntiInt(A) = A$.

Proof: By definition, $AntiInt(A) = \cup \{B, where\ B\ is\ open\ and\ B \subseteq A\}$.

If A is AntiOpen, and $B \subset A$ and B is AntiOpen then $A \cap B = B$ and it will violate the condition (iii) of the definition of the AntiTopological Spaces. Hence, $B \not\subset A$. So $B = A$. Hence, $AntiInt(A) = A$.

Proposition 3.3: In an AntiTopological space (X, τ), $AntiInt(A) \notin \tau$ if A is not AntiOpen.

Proof: By definition, $AntiInt(A) = \cup \{B, where\ B\ is\ AntiOpen\ and\ B \subseteq A\}$.

By **Proposition 3.2**, if A is AntiOpen, then $AntiInt(A) = A$. If A is not AntiOpen, then either $AntiInt(A) = \emptyset$ or, $AntiInt(A) = B \cup C$, where B and C are AntiOpen. And $B \cup C$ cannot be contained in τ otherwise it will violate condition (ii) of the Proposition 3.1.

Example 3.2: Let $X = \{1,2,3,4\}$ and $\tau = \{\{1,2\}, \{2,3\}, \{3,4\}\}$.
Let $A = \{1,2,3\}$, then $AntiInt(A) = \{1,2\} \cup \{2,3\} = \{1,2,3\} \notin \tau$.
And, $A = \{2,4\}$, then $AntiInt(A) = \emptyset \notin \tau$.
Observation: From Example 3.2, it is observed that *AntiInt(A)* is equal to A even if A is not AntiOpen.

Proposition 3.4: Let (X, τ) be AntiTopological space. Then

(i) $A \subseteq B \Rightarrow AntiInt(A) \subseteq AntiInt(B)$
(ii) $AntiInt(A \cap B) \subseteq AntiInt(A) \cap AntiInt(B)$
(iii) $AntiInt(A) \cup AntiInt(B) \subseteq AntiInt(A \cup B)$
(iv) $AntiInt(AntiInt(A)) = AntiInt(A)$ if A is AntiOpen.

Proof:

(i) Both A and B cannot be AntiOpen at the same time because in that case A cannot be a subset of B. Suppose that A is AntiOpen, and B is not. Then, $AntiInt(A) = A$ and $AntiInt(B) = \{A \cup C_i\}$ since $A \subseteq B$ and A is AntiOpen, where C_i are AntiOpen. Hence, $AntiInt(A) \subseteq AntiInt(B)$ in this case. Next, suppose that B is AntiOpen while A is not, then $AntiInt(B) = B$ and $AntiInt(A) = \cup \{C, C$ is AntiOpen$\}$ and $C \neq B$. By Proposition 3.3, $AntiInt(A) \notin \tau$ and $A \subseteq B$. The only possibility for this is that $AntiInt(A) = \emptyset$.

(ii) For any A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

So, we have: $AntiInt(A \cap B) \subseteq AntiInt(A)$ and $AntiInt(A \cap B) \subseteq AntiInt(B)$

Hence, $AntiInt(A \cap B) \subseteq AntiInt(A) \cap AntiInt(B)$

(iii) For any A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

So, we have: $AntiInt(A) \subseteq AntiInt(A \cup B)$ and $AntiInt(B) \subseteq AntiInt(A \cup B)$

Hence, $AntiInt(A) \cup AntiInt(B) \subseteq AntiInt(A \cup B)$

(iv) The proof is direct by Proposition 3.2.

Definition 3.2: Let (X, τ) be an AntiTopological space and a subset A of X is said to be τ-AntiClosed set if and only if its complement A^C is an AntiOpen set.

Proposition 3.5: In an AntiTopological space. The conditions (i) and (ii) are satisfied.

(i) The intersection of AntiClosed sets isnot AntiClosed.

(ii) Union of AntiClosed sets is notAntiClosed.

Definition 3.3: Let (X, τ) be anAntiTopological space over X and A is subset on X. Then, the AntiClosure of A is the intersection of all AntiClosed super sets of A. Clearly, AntiClosure of A is *not* the smallest AntiClosed set over X containing A, which is shown in the Proposition 3.6 (ii) below.

That is, $AntiCl(A) = \cap \{G: G \supseteq A \text{ and } G \text{ is AntiClosed}\}$

Example 3.3: Let $X = \{1,2,3,4,5\}$ and $\tau = \{\{1\},\{2\},\{3\},\{5\}\}$. Then, the AntiClosed sets are: $\{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}$ and $\{1,2,3,4\}$. Let $A = \{1,2\}$, then $AntiCl(A) = \{1,2,4,5\} \cap \{1,2,3,4\} = \{1,2,4\}$.

Proposition 3.6: Let (X, τ) be an AntiTopological space. Then

(i) $AntiCl(A)$ is not the smallest AntiClosed set containing A.

(ii) If A is AntiClosed, then $A = AntiCl(A)$.

Proof:

(i) We prove it by a counter example. Let $X = \{1,2,3,4,5\}$ and $\tau = \{\{1\},\{2\},\{3\},\{5\}\}$. Then, the AntiClosed sets are: $\{2,3,4,5\}, \{1,3,4,5\}, \{1,2,4,5\}$ and $\{1,2,3,4\}$. Let $A = \{1,2\}$, then $AntiCl(A) = \{1,2,4,5\} \cap \{1,2,3,4\} = \{1,2,4\}$ which is not AntiClosed.

We may consider another example by considering A as an AntiOpen set, say $A = \{1\}$, then $AntiCl(A) = \{1,3,4,5\} \cap \{1,2,4,5\} \cap \{1,2,3,4\} = \{1,4\}$ which is also not AntiClosed.

Thus, AntiClosure of A is *not* the smallest AntiClosed set over X containing A.

(ii) Proof is obvious from the definition of anti-topology.

Proposition 3.7: Let (X, τ) be AntiTopological space and let $A, B \subseteq X$. If B is AntiClosed, then

(i) $A \subseteq AntiCl(A)$

(ii) $A \subseteq B \implies AntiCl(A) \subseteq AntiCl(B)$

(iii) $AntiCl(A) \cup AntiCl(B) \subseteq AntiCl(A \cup B)$

(iv) $AntiCl(A \cap B) \subseteq AntiCl(A) \cap AntiCl(B)$

(v) $AntiCl\big(AntiCl(B)\big) = AntiCl(B)$

Proof:

(i) By definition, we have $AntiCl(A)$ is a set containing A. So, $A \subseteq AntiCl(A)$

(ii) If B is closed then, $AntiCl(B) = B$. Thus, $A \subseteq B \Rightarrow \mathrm{A} \subseteq AntiCl(B)$ which give, $AntiCl(A) \subseteq AntiCl(B)$.

(iii) $A \subseteq A \cup B \Rightarrow AntiCl(A) \subseteq AntiCl(A \cup B)$ by (i) above. Also, $B \subseteq A \cup B \Rightarrow AntiCl(B) \subseteq AntiCl(A \cup B)$ by (i) above. Hence, $AntiCl(A) \cup AntiCl(B) \subseteq AntiCl(A \cup B)$

(iv) $A \cap B \subseteq A \Rightarrow \mathrm{AntiCl(A \cap B)} \subseteq AntiCl(A)$ by (i) above. Also, $A \cap B \subseteq B \Rightarrow \mathrm{AntiCl(A \cap B)} \subseteq AntiCl(B)$ by (i) above. Hence, $AntiCl(A \cap B) \subseteq AntiCl(A) \cap AntiCl(B)$.

(v) Since B is AntiClosed, we have $AntiCl(B) = B$. So, $AntiCl\big(AntiCl(B)\big) = AntiCl(B)$

Remark 3.1: In Proposition 3.7, if B is not AntiClosed then the results are not generally true. It is because the AntiClosure of every subset of X will not always exist because X is not AntiClosed.

Proposition 3.8: Let (X,τ) be AntiTopological space and let $A \subseteq X$. Then

(i) $AntiInt(A) = \left(AntiCl(A^C)\right)^C$, if A is AntiOpen.

(ii) $AntiCl(A^C) = \left(AntiInt(A)\right)^C$, if A is AntiOpen.

(iii) $AntiCl(A) = \left(AntiInt(A^C)\right)^C$, if A is AntiClosed.

(iv) $AntiCl\big(AntiInt(A)\big) = AntiCl(A)$

Proof:

(i) Let $x \in AntiInt(A) \Rightarrow x \in A \Rightarrow x \notin A^C \Rightarrow x \notin AntiCl(A^C) \Rightarrow x \in \left(AntiCl(A^C)\right)^C$.

Hence, $AntiInt(A) \subseteq \left(AntiCl(A^C)\right)^C$.

Conversely, let $x \in \left(AntiCl(A^C)\right)^C \Rightarrow x \notin AntiCl(A^C) \Rightarrow x \notin A^C \Rightarrow x \in A$.

Hence, $x \in AntiInt(A)$. So, $\left(AntiCl(A^C)\right)^C \subseteq AntiInt(A)$.

(ii) Let $x \in AntiCl(A^C) \Leftrightarrow x \in A^C \Leftrightarrow x \notin A \Leftrightarrow x \notin AntiInt(A) \Leftrightarrow x \in \left(AntiInt(A)\right)^C$.

Hence, $AntiCl(A^C) = \left(AntiInt(A)\right)^C$.

(iii) Let $x \in AntiCl(A) \Leftrightarrow x \in A \Leftrightarrow x \notin A^C \Leftrightarrow x \notin AntiInt(A^C) \Leftrightarrow x \in \left(AntiInt(A^C)\right)^C$.

Hence, $AntiCl(A) = \left(AntiInt(A^C)\right)^C$.

(iv) Let $x \in AntiCl\big(AntiInt(A)\big) \Leftrightarrow x \in AntiInt(A) \Leftrightarrow x \in A \Leftrightarrow x \in AntiCl(A)$.

Hence, $AntiCl\big(AntiInt(A)\big) = AntiCl(A)$.

Definition 3.4: Let (X,τ) be an AntiTopological space over X and A is subset on X. Then AntiBoundary of A is defined as $AntiBd(A) = AntiCl(A) \cap AntiCl(A^C)$.

Example 3.4: Let $X = \{1,2,3,4\}$ and $\tau = \big\{\{2\},\{3\}\big\}$. Then, the AntiClosed sets are: $\{1,3,4\}$ and $\{1,2,4\}$.

Let $A = \{3\}$, then $A^c = \{1,2,4\}$. Now, $AntiCl(A) = \{1,3,4\}$ and $AntiCl(A^C) = \{1,2,4\}$. So, the $AntiBd(A) = AntiCl(A) \cap AntiCl(A^C) = \{1,3,4\} \cap \{1,2,4\} = \{1,4\}$.

Proposition 3.9: Let (X,τ) be Anti-Topological space and let $A, B \subseteq X$. Then

(i) $AntiCl(A) - AntiInt(A) = AntiBd(A)$

(ii) $AntiInt(A) = A - AntiBd(A)$

(iii) $AntiInt(A) \cup AntiInt(A^C) = [AntiBd(A)]^C$

(iv) $AntiBd\big(Int(A)\big) = AntiBd(A)$

(v) $AntiBd\big(AntiCl(A)\big) \subseteq AntiBd(A)$

(vi) $AntiBd(A \cup B) \subseteq AntiBd(A) \cup AntiBd(B)$

(vii) $Bd(A \cap B) \subseteq Bd(A) \cup Bd(B)$.

Proof:

(i) Let $x \in AntiCl(A) - AntiInt(A)$
Now,

$x \in AntiCl(A) - AntiInt(A)$
$\Leftrightarrow x \in AntiCl(A)\ and\ x \notin AntiInt(A)$
$\Leftrightarrow x \in AntiCl(A)\ and\ x \notin A$
$\Leftrightarrow x \in AntiCl(A)\ and\ x \in A^C$
$\Leftrightarrow x \in AntiCl(A)\ and\ x \in AntiCl(A^C)$
$\Leftrightarrow x \in AntiCl(A) \cap AntiCl(A^C)$
$\Leftrightarrow x \in AntiBd(A)$.
Hence, $AntiCl(A) - AntiInt(A) = AntiBd(A)$.

(ii) Let $x \in AntiInt(A)$
Now,
$x \in AntiInt(A)$
$\Leftrightarrow x \in A\ and\ x \notin A^C$
$\Leftrightarrow x \in A\ and\ x \in AntiCl(A)\ and\ \ x \notin AntiCl(A^C)$
$\Leftrightarrow x \in A\ and\ x \notin AntiBd(A)$
$\Leftrightarrow x \in A - Bd(A)$
Hence, $AntiInt(A) = A - AntiBd(A)$.

(iii) From definition, we have
$AntiBd(A) = AntiCl(A) \cap AntiCl(A^C)$
$\Leftrightarrow [AntiBd(A)]^C = [AntiCl(A) \cap AntiCl(A^C)]^C$
$\Leftrightarrow [AntiBd(A)]^C = [AntiCl(A)]^C \cup [AntiCl(A^C)]^C$
$\Leftrightarrow [AntiBd(A)]^C = AntiInt(A^C) \cup AntiInt(A)$, by Proposition 3.8.
Hence, $AntiInt(A) \cup AntiInt(A^C) = [AntiBd(A)]^C$

(iv) $AntiBd\big(AntiInt(A)\big) = AntiCl\big(AntiInt(A)\big) \cap AntiCl\left[\big(AntiInt(A)\big)^C\right]$ [by Proposition 3.8 (i)]
$= AntiCl\big(AntiInt(A)\big) \cap AntiCl\left[\left\{\big(AntiCl(A^C)\big)^C\right\}^C\right]$ $\left[as\ \big(AntiCl(A^C)\big)^C = AntiInt(A)\right]$
$= AntiCl\big(AntiInt(A)\big) \cap AntiCl(A^C)[as\ (P^C)^C$
$= P\ and\ AntiCl\big(AntiCl(P)\big) = AntiCl(P)$, for any set P
$= AntiCl(A) \cap AntiCl(A^C)$ [by Proposition 3.8 (iv)]
$= AntiBd(A)$ [by definition]
Hence, $AntiBd\big(AntiInt(A)\big) = AntiBd(A)$.

(v) $AntiBd(AntiCl(A)) = AntiCl\big(AntiCl(A)\big) \cap AntiCl\left[\big(AntiCl(A)\big)^C\right]$
Now, $A \subseteq AntiCl(A) \Rightarrow \big(AntiCl(A)\big)^C \subseteq A^C$
$\Rightarrow AntiCl\left[\big(AntiCl(A)\big)^C\right] \subseteq AntiCl(A^C)$ $[A \subseteq B \Rightarrow AntiCl(A) \subseteq AntiCl(B)]$
Hence, $AntiBd\big(AntiCl(A)\big) \subseteq AntiCl(A) \cap AntiCl(A^C) = AntiBd(A)$
i.e., $AntiBd\big(AntiCl(A)\big) \subseteq AntiBd(A)$.

(vi) $AntiBd(A \cup B) = AntiCl(A \cup B) \cap AntiCl(A \cup B)^C$
$\subseteq [AntiCl(A) \cup AntiCl(B)] \cap [AntiCl(A^C) \cap AntiCl(B^C)]$
$= [AntiCl(A) \cap \{AntiCl(A^C) \cap AntiCl(B^C)\}] \cup [AntiCl(B) \cap \{AntiCl(A^C) \cap AntiCl(B^C)\}]$
$= [\{AntiCl(A) \cap AntiCl(A^C)\} \cap AntiCl(B^C)] \cup [\{AntiCl(B) \cap AntiCl(B^C)\} \cap AntiCl(A^C)]$
$= [AntiBd(A) \cap AntiCl(B^C)] \cup [AntiBd(B) \cap AntiCl(A^C)]$
$\subseteq AntiBd(A) \cup AntiBd(B)$
Hence, $AntiBd(A \cup B) \subseteq AntiBd(A) \cup AntiBd(B)$.

(vii) $AntiBd(A \cap B) = AntiCl(A \cap B) \cap AntiCl[(A \cap B)^C]$

$\subseteq [AntiCl(A) \cap AntiCl(B)] \cap AntiCl(A^C \cup B^C)$

$= [AntiCl(A) \cap AntiCl(B)] \cap [AntiCl(A^C) \cup AntiCl(B^C)]$

$= [\{AntiCl(A) \cap AntiCl(B)\} \cap AntiCl(A^C)] \cup [\{AntiCl(A) \cap AntiCl(B)\} \cap AntiCl(B^C)]$

$= [\{AntiCl(A) \cap AntiCl(A^C)\} \cap AntiCl(B)] \cup [AntiCl(A) \cap \{AntiCl(B) \cap AntiCl(B^C)\}]$

$= [AntiBd(A) \cap AntiCl(B)] \cup [AntiCl(A) \cap AntiBd(B)]$

$\subseteq AntiBd(A) \cup AntiBd(B)$

Hence, $AntiBd(A \cap B) \subseteq AntiBd(A) \cup AntiBd(B)$.

Proposition 3.10: Let (X, τ) be AntiTopological space and let $A \subseteq X$. If A is AntiOpen, then $AntiCl(A) - A = AntiBd(A)$

Proof: Since A is AntiOpen, therefore $AntiInt(A) = A$ [from Proposition 3.2]

and $AntiInt(A) = \left(AntiCl(A^C)\right)^C$ [from Proposition 3.8 (i)]

$AntiCl(A) - A = AntiCl(A) - Int(A)$

$= AntiCl(A) - \left(AntiCl(A^C)\right)^C$

$= AntiCl(A) \cap \left\{\left(AntiCl(A^C)\right)^C\right\}^C$

$= AntiCl(A) \cap AntiCl(A^C)$

$= AntiBd(A)$.

Hence $AntiCl(A) - A = AntiBd(A)$.

Remark 3.2: If the subset A of X is not AntiOpen, then the equality in Proposition 3.10 may not hold. We will show it by an example:

Let $X = \{1,2,3,4,5\}$ and $\tau = \{\{3\}, \{1,2\}, \{1,4\}, \{4,5\}\}$.

The AntiClosed sets are: $\{1,2,4,5\}, \{3,4,5\}, \{2,3,5\}, \{1,2,3\}$.

Let A= $\{1,3\}$, then $A^c = \{2,4,5\}$.

Now, $AntiCl(A) = \{1,2,3\}$ and $AntiCl(A^c) = \{1,2,4,5\}$.

So, $AntiBd(A) = AntiCl(A) \cap AntiCl(A^c) = \{1,2\}$.

Now, $AntiCl(A) - A = \{1,2,3\} - \{1,3\} = \{2\} \neq AntiBd(A)$.

Conclusion

In this study, it is observed that many properties of AntiTopological space are not the same as general topological space and NeutroTopological Spaces. Then we have investigated the properties of the interior, closure, and boundary of AntiTopological spaces. Hope our work will help in further study of AntiTopological space. This may lead to a new beginning for further research on the study of Topological space.

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Chapter Three

Some Kinds of γ-Irresolute Functions in N-Neutrosophic Crisp Topological Spaces

A. Vadivel[1] and C. John Sundar[2]

[1]Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002;
PG and Research Department of Mathematics, Government Arts College (A), Karur - 639 005, India
E-mail: avmaths@gmail.com
[2]Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India
E-mail: johnphdau@hotmail.com

ABSTRACT

The aim of this article is to introduce a N-neutrosophic crisp γ-irresolute, completely N-neutrosophic crisp γ-irresolute and completely weakly N-neutrosophic crisp γ-irresolute functions in a N-neutrosophic crisp topological space and also discuss a relation between them in N-neutrosophic crisp topological spaces. We also investigate some of their properties using these N-neutrosophic crisp γ-irresolute functions via N-neutrosophic crisp γ-continuous function in N-neutrosophic crisp topological spaces. Also, we interact with separation axioms and open mapping functions using these N-neutrosophic crisp γ-irresolute functions.

Keywords: N-neutrosophic crisp γ-open set, N-neutrosophic crisp γ-irresolute, completely N-neutrosophic crisp γ-irresolute, completely weakly N-neutrosophic crisp γ-irresolute.

INTRODUCTION

In our daily routine, we have used the crisp sets in most of our life. The concepts of neutrosophy and neutrosophic set are the recent tools in a topological space. It was first introduced by Smarandache [14, 15] in the beginning of 20^{th} century. In 2014, Salama, Smarandache and Kroumov [12] has provided the basic concept of neutrosophic crisp set in a topological space. After that Al-Omeri [1] also investigated some fundamental properties of neutrosophic crisp topological Spaces. Al-Hamido [9] explore the possibility of expanding the concept of neutrosophic crisp topological spaces into N-topology and investigate some of their basic properties in N-terms. By using N-terms of topological spaces, we can define $1_{nc}ts$, $2_{nc}ts$,…, $N_{nc}ts$.

In 1996, Andrijevic [2] introduced b-open sets and develop some of their works in general topology. The notion of γ-open set (originally called γ-sets) in topological spaces was introduced by Min [7] and worked in the field of general topology. Vadivel and John Sundar [24] presented γ-open sets in neutrosophic crisp topological spaces via N-terms of topology.

The strong and weak forms of continuous functions are introduced by Levine in 1960 [6] and also introduced in strong continuity in topological spaces. In 1967, Naimpally [8] also discussed strongly continuous functions in a topology. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory [3-5,10,11,17-23,31-59] and Vadivel et al. [26, 27, 28, 30] introduced some strongly continuous functions in N_{nc} topological spaces.

The chapter is organized as follows: In section 2, introduces some concepts and basic operations are reviewed. In section 3, we extend the continuous functions into a irresolute functions such as N-neutrosophic crisp γ-irresolute function in $N_{nc}ts$ and investigates their properties. In section 4, presents a completely N-neutrosophic crisp γ-irresolute function in $N_{nc}ts$. In section 5, study a completely weakly N-neutrosophic crisp γ-irresolute functions in $N_{nc}ts$. Finally, Conclusions and further research are contained.

BACKGROUND

Definition 1 [13] For any non-empty fixed set X, a neutrosophic crisp set (briefly, ncs) K, is an object having the form $K = \langle K_1, K_2, K_3 \rangle$ where K_1, K_2 & K_3 are subsets of X satisfying any one of the types

(T1) $K_\eta \cap K_\xi = \phi, \eta \neq \xi$ & $\cup_{\eta=1}^{3} K_\eta \subset X, \forall \eta, \xi = 1,2,3$.

(T2) $K_\eta \cap K_\xi = \phi, \eta \neq \xi$ & $\cup_{\eta=1}^{3} K_\eta = X, \forall \eta, \xi = 1,2,3$.

(T3) $\cap_{\eta=1}^{3} K_\eta = \phi$ & $\cup_{\eta=1}^{3} K_\eta = X, \forall \eta = 1,2,3$.

Definition 2 [13] Types of ncs's $\emptyset_N$ and X_N in X are as

(i) $\emptyset_N = \langle \emptyset, \emptyset, X \rangle$ or $\langle \emptyset, X, X \rangle$ or $\langle \emptyset, X, \emptyset \rangle$ or $\langle \emptyset, \emptyset, \emptyset \rangle$.

(ii) $X_N = \langle X, \emptyset, \emptyset \rangle$ or $\langle X, X, \emptyset \rangle$ or $\langle X, \emptyset, X \rangle$ or $\langle X, X, X \rangle$.

Definition 3 [13] Let X be a non-empty set & the ncs's K & M in the form $K = \langle K_{11}, K_{22}, K_{33} \rangle$, $M = \langle M_{11}, M_{22}, M_{33} \rangle$, then

(i) $K \subseteq M \Leftrightarrow K_{11} \subseteq M_{11}$, $K_{22} \subseteq M_{22}$ & $K_{33} \supseteq M\_33$ or $K_{11} \subseteq M_{11}$, $K_{22} \supseteq M_{22}$ & $K_{33} \supseteq M_{33}$.

(ii) $K \cap M = \langle K_{11} \cap M_{11}, K_{22} \cap M_{22}, K_{33} \cup M_{33} \rangle$ or $\langle K_{11} \cap M_{11}, K_{22} \cup M_{22}, K_{33} \cup M_{33} \rangle$

(iii) $K \cup M = \langle K_{11} \cup M_{11}, K_{22} \cup M_{22}, K_{33} \cap M_{33} \rangle$ or $\langle K_{11} \cup M_{11}, K_{22} \cap M_{22}, K_{33} \cap M_{33} \rangle$.

Definition 4 [13] Let $K = \langle K_1, K_2, K_3 \rangle$ a ncs on X, then the complement of K (briefly, K^c) may be defined in three different ways:

(C1) $K^c = \langle K_1^c, K_2^c, K_3^c \rangle$, or

(C2) $K^c = \langle K_3, K_2, K_1 \rangle$, or

(C3) $K^c = \langle K_3, K_2^c, K_1 \rangle$.

Definition 5 [9] Let X be a non-empty set. Then ${}_{nc}\Gamma_1$, ${}_{nc}\Gamma_2$, …, ${}_{nc}\Gamma_N$ are N-arbitrary crisp topologies defined on X and the collection $N_{nc}\Gamma$ is called N_{nc}-topology on X is

$$N_{nc}\Gamma = \{A \subseteq X : A = (\cup_{\eta j=1}^{N} E_{\eta j}) \cup (\cap_{\eta j=1}^{N} F_{\eta j}), E_{\eta j}, F_{\eta j} \in \ _{nc}\Gamma_j\}$$

and it satisfies the following axioms:

(i) $\emptyset_N$, $X_N \in N_{nc}\Gamma$.

(ii) $\cup_{j=1}^{\infty} K_\eta \in N_{nc}\Gamma \ \forall \ \{K_\eta\}_{\eta=1}^{\infty} \in N_{nc}\Gamma$.

(iii) $\cap_{j=1}^{n} K_\eta \in N_{nc}\Gamma \ \forall \ \{K_\eta\}_{\eta=1}^{n} \in N_{nc}\Gamma$.

Then $(X, N_{nc}\Gamma)$ is called a N_{nc}-topological space (briefly, $N_{nc}ts$) on X. The N_{nc}-open sets ($N_{nc}os$) are the elements of $N_{nc}\Gamma$ in X and the complement of $N_{nc}os$ is called N_{nc}-closed sets ($N_{nc}cs$) in X. The elements of X are known as N_{nc}-sets ($N_{nc}s$) on X.

Definition 6 [9] Let $(X, N_{nc}\ \Gamma)$ be $N_{nc}ts$ on X and K be a $N_{nc}s$ on X, then the N_{nc} interior of K (briefly, $N_{nc}int(K)$) and N_{nc} closure of K (briefly, $N_{nc}cl(K)$) are defined as

$$N_{nc}int(K) = \cup \{A : A \subseteq K \ \& \ G \text{ is a } N_{nc}os\}$$

$$N_{nc}cl(K) = \cap \{D : K \subseteq D \ \& \ D \text{ is a } N_{nc}cs\}.$$

Definition 7 [9] Let $(X, N_{nc}\Gamma)$ be any $N_{nc}ts$. Let K be a $N_{nc}s$ in $(X, N_{nc}\Gamma)$. Then K is said to be a N-neutrosophic crisp

(i) regular open [24] set (briefly, $N_{nc}ros$) if $K = N_{nc}int(N_{nc}cl(K))$.

(ii) pre open set (briefly, $N_{nc}\mathcal{P}os$) if $K \subseteq N_{nc}int(N_{nc}cl(K))$.

(iii) semi open set (briefly, $N_{nc}\mathcal{S}os$) if $K \subseteq N_{nc}cl(N_{nc}int(K))$.

(iv) α-open set (briefly, $N_{nc}\alpha os$) if $K \subseteq N_{nc}int(N_{nc}cl(N_{nc}int(K)))$.

(v) γ-open [24] set (briefly, $N_{nc}\gamma os$) set if $K \subseteq N_{nc}cl(N_{nc}int(K)) \cup N_{nc}int(N_{nc}cl(K))$.

The complement of a $N_{nc}\mathcal{P}os$ (resp. $N_{nc}\mathcal{S}os$, $N_{nc}\alpha os$, $N_{nc}ros$ & $N_{nc}\gamma os$) is called a N_{nc} pre (resp. N_{nc} semi, $N_{nc}\alpha$, N_{nc}-regular & $N_{nc}\gamma$) closed set (briefly, $N_{nc}\mathcal{P}cs$ (resp. $N_{nc}\mathcal{S}cs$, $N_{nc}\alpha cs$, $N_{nc}rcs$ & $N_{nc}\gamma cs$)) in X.

The family of all $N_{nc}\mathcal{P}os$ (resp. $N_{nc}\mathcal{P}cs$, $N_{nc}\mathcal{S}os$, $N_{nc}\mathcal{S}cs$, $N_{nc}\alpha os$, $N_{nc}\alpha cs$, $N_{nc}\gamma os$ & $N_{nc}\gamma cs$) of X is denoted by $N_{nc}\mathcal{P}OS(X)$ (resp. $N_{nc}\mathcal{P}CS(X)$, $N_{nc}\mathcal{S}OS(X)$, $N_{nc}\mathcal{S}CS(X)$, $N_{nc}\alpha OS(X)$, $N_{nc}\alpha CS(X)$, $N_{nc}\gamma OS(X)$ & $N_{nc}\gamma CS(X)$).

Definition 8 [24] Let $(X, N_{nc}\Gamma)$ be a $N_{nc}ts$ on X and K be a $N_{nc}s$ on X then

(i) $N_{nc}\gamma int(K) = \cup \ \{D : D \subseteq K$ and D is a $N_{nc}\gamma os\}$.

(ii) $N_{nc}\gamma cl(K) = \cap \ \{D : K \subseteq D$ and D is a $N_{nc}\gamma cs\}$.

Definition 9 Let $(X_1, N_{nc}\Gamma)$ and $(X_2, N_{nc}\Psi)$ be any two $N_{nc}ts$'s. A map $\rho : (X_1, N_{nc}\Gamma) \rightarrow (X_2, N_{nc}\Psi)$ is said to be

(i) N_{nc} (resp. $N_{nc}\gamma$)-continuous (briefly, $N_{nc}Cts$ [16] (resp. $N_{nc}\gamma Cts$ [30])) if the inverse image of every $N_{nc}os$ in $(X_2, N_{nc}\Psi)$ is a $N_{nc}os$ (resp. $N_{nc}\gamma os$) in $(X_1, N_{nc}\Gamma)$.

(ii) strongly N_{nc} continuous (briefly, $StN_{nc}Cts$ [28]) function if the inverse image of every subset in $(X_2, N_{nc}\Psi)$ is N-neutrosophic crisp clopen (i.e both $N_{nc}o$ and $N_{nc}c$) (briefly, $N_{nc}clo$) in $(X_1, N_{nc}\Gamma)$.

(iii) completely N_{nc} continuous (briefly, $CN_{nc}Cts$ [28]) function if the inverse image of every $N_{nc}os$ in $(X_2, N_{nc}\Psi)$ is $N_{nc}ros$ in $(X_1, N_{nc}\Gamma)$.

Definition 10 [28] A $N_{nc}ts$ $(X, N_{nc}\Gamma)$ is said to be $N_{nc}r$-T_1 if for each pair of distinct points x and y of X, there exist $N_{nc}ro$ sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$, $x \notin U_2$ and $y \notin U_1$.

Definition 11 [28] A $N_{nc}ts$ $(X, N_{nc}\Gamma)$ is said to be $N_{nc}r$-T_2 for each pair of distinct points x and y of X, there exist $N_{nc}ro$ sets L and M in X such that $x \in L$ and $y \in M$.

N-Neutrosophic Crisp γ-Irresolute Functions

Definition 12 Let $(X_1, N_{nc}\Gamma)$ and $(X_2, N_{nc}\Psi)$ be any two $N_{nc}ts$'s. A map $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is said to be N_{nc} γ-irresolute function (briefly, $N_{nc}\gamma Irr$), if for the inverse image of every $N_{nc}\gamma cs$ in $(X_2, N_{nc}\Psi)$ is a $N_{nc}\gamma cs$ in $(X_1, N_{nc}\Gamma)$.

Theorem 13 Let $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ be a mapping, if $N_{nc}\gamma Irr$, then ρ is $N_{nc}\gamma Cts$.

Proof: Let C be $N_{nc}cs$ in X_2, then C is $N_{nc}\gamma cs$ in X_2, since every $N_{nc}cs$ is $N_{nc}\gamma cs$. By hypothesis, $\rho^{-1}(C)$ is $N_{nc}\gamma cs$. Therefore ρ is $N_{nc}\gamma Cts$.

Remark 14 The converse of the above theorem need not be true as shown in the following example.

Example 15 Let $X = \{l_1, m_1, n_1, o_1, p_1\} = Y$, ${}_{nc}\Gamma_1 = \{\phi_N, X_N, L, M, N\}$, ${}_{nc}\Gamma_2 = \{\phi_N, X_N \backslash\}$. $L = \langle\{n_1\}, \{\phi\}, \{l_1, m_1, o_1, p_1\}\rangle$, $M = \langle\{l_1, m_1\}, \{\phi\}, \{n_1, o_1, p_1\}\rangle$, $N = \langle\{l_1, m_1, n_1\}, \{\phi\}, \{o_1, p_1\}\rangle$, then we have $2_{nc}\Gamma = \{\phi_N, X_N, L, M, N\}$. $\_{nc}\Psi_1 = \{\phi_N, Y_N, O, P, Q\}$, $\_{nc}\Psi_2 = \{\phi_N, Y_N\}$. $O = \langle\{l_1, m_1\}, \{\phi\}, \{n_1, o_1, p_1\}\rangle$, $P = \langle\{n_1, o_1\}, \{\phi\}, \{l_1, m_1, p_1\}\rangle$, $Q = \langle\{l_1, m_1, n_1, o_1\}, \{\phi\}, \{p_1\}\rangle$, then we have $2_{nc}\Psi = \{\phi_N, Y_N, O, P, Q\}$.

Define $\rho : (X, 2_{nc}\Gamma) \to (Y, 2_{nc}\Psi)$ as $\rho(l_1) = l_1$, $\rho(m_1) = m_1$, $\rho(n_1) = n_1$, $\rho(o_1) = p_1$ & $\rho(p_1) = p_1$, then $2_{nc}\gamma Cts$ mapping but not $2_{nc}\gamma Irr$ mapping, the set $\rho^{-1}(\langle\{m_1, o_1, p_1\}, \{\phi\}, \{l_1, n_1\}\rangle) = \langle\{m_1, o_1, p_1\}, \{\phi\}, \{l_1, n_1\}\rangle$ is a $2_{nc}\gamma os$ in Y but not $2_{nc}\gamma os$ in X.

Theorem 16 A function $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is $N_{nc}\gamma Irr$ if and only if for every $N_{nc}\gamma os$ K in X_2, $\rho^{-1}(K)$ is $N_{nc}\gamma os$ in X_1.

Proof: Follows from the fact that the complement of $N_{nc}\gamma os$ is $N_{nc}\gamma cs$ and vice versa.

Theorem 17 If $\rho_1 : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ and $\rho_2 : (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ are both $N_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1 : (X_1, N_{nc}\Gamma) \to (X_3, N_{nc}\Phi)$ is $N_{nc}\gamma Irr$.

Proof: Let K be $N_{nc}\gamma os$ in X_3 . Then $\rho_2^{-1}(K)$ is $N_{nc}\gamma os$ in X_2 , since ρ_2 is $N_{nc}\gamma Irr$ and $\rho_1^{-1}(\rho_2^{-1}(K)) = (\rho_2 \circ \rho_1)^{-1}(K)$ is $N_{nc}\gamma os$ in X_1, since ρ_1 is $N_{nc}\gamma Irr$. Hence $\rho_2 \circ \rho_1$ is $N_{nc}\gamma Irr$.

Theorem 18 (i) If $\rho_1 : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is $N_{nc}\gamma Irr$ and $\rho_2 : (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ is $N_{nc}\gamma Cts$, then $\rho_2 \circ \rho_1 : (X_1, N_{nc}\Gamma) \to (X_3, N_{nc}\Phi)$ is $N_{nc}\gamma Cts$.

(ii) If $\rho_1 : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is $N_{nc}\gamma Cts$ and $\rho_2 : (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ is $N_{nc}Cts$, then $\rho_2 \circ \rho_1 : (X_1, N_{nc}\Gamma) \to (X_3, N_{nc}\Phi)$ is $N_{nc}\gamma Cts$.

Proof: (i) Let K be $N_{nc}os$ in X_3. Then, $\rho_2^{-1}(K)$ is $N_{nc}\gamma os$ in X_2, since ρ_2 is $N_{nc}\gamma Cts$ & $\rho_1^{-1}(\rho_2^{-1}(K)) = (\rho_2 \circ \rho_1)^{-1}(K)$ is $N_{nc}\gamma os$ in X_1, since ρ_1 is $N_{nc}\gamma Irr$. Hence $\rho_2 \circ \rho_1$ is $N_{nc}\gamma Cts$.

(ii) Let K be $N_{nc}os$ in X_3. Then, $\rho_2^{-1}(K)$ is $N_{nc}os$ in X_2, since ρ_2 is $N_{nc}Cts$ & $\rho_1^{-1}(\rho_2^{-1}(K)) = (\rho_2 \circ \rho_1)^{-1}(K)$ is $N_{nc}\gamma os$ in X_1, since ρ_1 is $N_{nc}\gamma Cts$. Hence $\rho_2 \circ \rho_1$ is $N_{nc}\gamma Cts$.

Completely N-neutrosophic crisp γ irresolute functions

Definition 19 Let $(X_1, N_{nc}\Gamma)$ and $(X_2, N_{nc}\Psi)$ be any two $N_{nc}ts$'s. A map $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is said to be Completely N-neutrosophic crisp γ irresolute (briefly, $CN_{nc}\gamma Irr$) if the inverse image of every $N_{nc}\gamma os$ of X_2 is $N_{nc}ros$ in X_1.

Remark 20 The following implications are true.

$$\boxed{StN_{nc}Cts} \longrightarrow \boxed{CN_{nc}\gamma Irr} \longrightarrow \boxed{N_{nc}\gamma Irr}$$

But not converse as shown by the following examples.

Example 21 Let $X = \{u_1, v_1, w_1\}$, ${}_{nc}\Gamma_1 = \{\Phi_n, X_n, L\}$, ${}_{nc}\Gamma_2 = \{\Phi_n, X_n\}$. $L = \langle\{u_1\}, \{\Phi\}, \{v_1, w_1\}\rangle$, then we have $2_{nc}\Gamma = \{\Phi_n, X_n, L\}$. Define $\rho : (X, 2_{nc}\Gamma) \to (X, 2_{nc}\Gamma)$ as $\rho(u_1) = u_1$, $\rho(v_1) = v_1$ & $\rho(w_1) = w_1$, then it is $C2_{nc}\gamma Irr$ but not $St2_{nc}Cts$, the set $\rho^{-1}(\langle\{u_1\}, \{\Phi\}, \{v_1, w_1\}\rangle) = \langle\{u_1\}, \{\Phi\}, \{v_1, w_1\}\rangle$ is a $2_{nc}os$ but not $2_{nc}cs$ in X.

Example 22 Let $X = \{u_1, v_1, w_1, x_1\}$, ${}_{nc}\Gamma_1 = \{\Phi_n, X_n, L, M, N\}$, ${}_{nc}\Gamma_2 = \{\Phi_n, X_n\}$. $L = \langle\{w_1\}, \{\Phi\}, \{u_1, v_1, x_1\}\rangle$, $M = \langle\{u_1, v_1\}, \{\Phi\}, \{w_1, x_1\}\rangle$, $N = \langle\{u_1, v_1, w_1\}, \{\Phi\}, \{x_1\}\rangle$, then we have $2_{nc}\Gamma = \{\Phi_n, X_n, L, M, N\}$. Define $\rho : (X, 2_{nc}\Gamma) \to (X, 2_{nc}\Gamma)$ as $\rho(u_1) = u_1$, $\rho(v_1) = v_1$, $\rho(w_1) = w_1$ & $\rho(x_1) = x_1$, then it is $2_{nc}\gamma Irr$ but not $C2_{nc}\gamma Irr$, the set $\rho^{-1}(\langle\{u_1, v_1\}, \{\Phi\}, \{w_1, x_1\}\rangle) = \langle\{u_1, v_1\}, \{\Phi\}, \{w_1, x_1\}\rangle$ is a $2_{nc}\gamma os$ but not $2_{nc}ros$ in X.

Theorem 23 Let $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ be a function, then statements

(i) ρ is $CN_{nc}\gamma Irr$,

(ii) $\rho^{-1}(N_{nc}\gamma int(C)) \subseteq N_{nc}int\,(\rho^{-1}(C))$ for every $N_{nc}s$ C of X_2,

(iii) $\rho(N_{nc}cl(D)) \subseteq N_{nc}\gamma cl(\rho(D))$ for every $N_{nc}s$ D of X_1,

(iv) $N_{nc}cl(\rho^{-1}(C)) \subseteq \rho^{-1}(N_{nc}\gamma cl(C))$ for every $N_{nc}s$ C of X_2,

(v) $\rho^{-1}(E)$ is $N_{nc}rc$ in X_1 for each $N_{nc}\gamma cs$ E in X_2,

(vi) $\rho^{-1}(E)$ is $N_{nc}ro$ in X_1 for each $N_{nc}\gamma os$ E in X_2

are equivalent.

Proof: (i) ⇒ (ii): Let $C \subseteq X_2$ and $x \in \rho^{-1}(N_{nc}\gamma int(C))$.

$$x \in \rho^{-1}(N_{nc}\gamma int(C)) \Rightarrow N_{nc}\gamma int(C) \in N_{nc}\gamma O(X_2, \rho(x))$$

$\Rightarrow (\exists\ U \in N_{nc}RO(X_1, x))(\rho(U) \subseteq N_{nc}\gamma int(C) \subseteq C)$

$\Rightarrow (\exists\ U \in N_{nc}RO(X_1, x))(U \subseteq \rho^{-1}(C))$

$\Rightarrow x \in N_{nc}int\,(\rho^{-1}(C))$.

(ii) ⇒ (iii): Let $D \subseteq X_1$.

$D \subseteq X_1 \Rightarrow \rho(D) \subseteq X_2 \Rightarrow X_2 \setminus \rho(D) \subseteq X\_2$

$\Rightarrow \rho^{-1}\left(N_{nc}\gamma int(X_2 \setminus \rho(D))\right) \subseteq N_{nc}int\,(\rho^{-1}(X_2 \setminus \rho(D)))$

$\Rightarrow X_1 \setminus \rho^{-1}\left(N_{nc}\gamma cl(\rho(D))\right) \subseteq X_1 \setminus N_{nc}cl(\rho^{-1}(\rho(D)))$

$\Rightarrow N_{nc}cl(D) \subseteq N_{nc}cl\left(\rho^{-1}(\rho(D))\right) \subseteq \rho^{-1}(N_{nc}\gamma cl(\rho(D)))$

$\Rightarrow \rho(N_{nc}cl(D)) \subseteq N_{nc}\gamma cl(\rho(D))$.

(iii) ⇒ (iv): Let $C \subseteq X_2$.

$C \subseteq X_2 \Rightarrow \rho^{-1}(C) \subseteq X_1$

$\Rightarrow \rho\left(N_{nc}cl(\rho^{-1}(C))\right) \subseteq N_{nc}\gamma cl\left(\rho(\rho^{-1}(C))\right) \subseteq N_{nc}\gamma cl(C)$

$\Rightarrow N_{nc}cl(\rho^{-1}(C)) \subseteq \rho^{-1}(N_{nc}\gamma cl(C))$.

(iv) ⇒ (v): Let $E \in N_{nc}\gamma C(X_2)$.

$E \in N_{nc}\gamma C(X_2) \Rightarrow E = N_{nc}\gamma cl(E)$

$\Rightarrow N_{nc}cl(\rho^{-1}(E)) \subseteq \rho^{-1}(N_{nc}\gamma cl(E)) = \rho^{-1}(E)$

$\Rightarrow \rho^{-1}(E) = N_{nc}cl(\rho^{-1}(E))$

$\Rightarrow \rho^{-1}(E) \in N_{nc}C(X_1)$.

(v) ⇒ (vi): Obvious.

(vi) ⇒ (i): Let $E \in N_{nc}\gamma O(X_2)$ and $x \in \rho^{-1}(E)$.

$(E \in N_{nc}\gamma O(X_2))(x \in \rho^{-1}(E)) \Rightarrow E \in N_{nc}\gamma O(X_2, \rho(x))$

$\Rightarrow (U = \rho^{-1}(E) \in N_{nc}RO(X_1, x))(\rho(U) \subseteq E)$.

Theorem 24 Let $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ be a bijective function, then statements

(i) ρ is $CN_{nc}\gamma Irr$,

(ii) $N_{nc}\gamma int(\rho(O)) \subseteq \rho(N_{nc}int(O))$ for every $N_{nc}s$ of X_1

are equivalent.

Proof: (i) ⇒ (ii): Let $O \subseteq X_1$.

$O \subseteq X_1 \Rightarrow X_1 \setminus O \subseteq X_1$

$\Rightarrow \rho(X_1 \setminus N_{nc}int(O)) = \rho(N_{nc}cl(X_1 \setminus O)) \subseteq N_{nc}\gamma cl(\rho(X_1 \setminus O)), \rho$ is bijection

$\Rightarrow X_2 \setminus \rho(N_{nc}int(O)) \subseteq X_2 \setminus N_{nc}\gamma int(\rho(O))$

$\Rightarrow N_{nc}\gamma int(\rho(O)) \subseteq \rho(N_{nc}int(O))$.

(ii) $\Rightarrow$ (i): Let $O \subseteq X_1$.

$O \subseteq X_1 \Rightarrow X_1 \setminus O \subseteq X_1$

$\Rightarrow N_{nc}\gamma int(\rho(X_1 \setminus O)) \subseteq \rho(N_{nc}int(X_1 \setminus O)), \rho$ is bijection

$\Rightarrow X_2 \setminus N_{nc}\gamma cl(\rho(O)) \subseteq X_2 \setminus \rho(N_{nc}cl(O))$

$\Rightarrow \rho(N_{nc}cl(O)) \subseteq N_{nc}\gamma cl(\rho(O))$.

Lemma 25 Let K be a $N_{nc}os$ of a $N_{nc}ts$ $(X, N_{nc}\Gamma)$. Then the following hold:

(i) If C is $N_{nc}ro$ in X, then so is $C \cap K$ in the subspace $(K, N_{nc}\Gamma_K)$,

(ii) If D is $N_{nc}ro$ in $(K, N_{nc}\Gamma_K)$, then there exists a $N_{nc}ros$ S in X such that $D = S \cap K$.

Theorem 26 If $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is a $CN_{nc}\gamma Irr$ function and C is any $N_{nc}os$ of X_1, then the restriction $\rho_C : (C, N_{nc}\Gamma_C) \to (X_2, N_{nc}\Psi)$ is $CN_{nc}\gamma Irr$.

Proof: Let $F \in N_{nc}\gamma O(X\_2)$.

$F \in N_{nc}\gamma O(X_2) \Rightarrow \rho^{-1}(F) \in N_{nc}RO(X_1)$. Since $C \in X_1$, by Lemma 25 $(\rho_C)^{-1}(F) = \rho^{-1}(F) \cap C \in N_{nc}RO(C)$.

Lemma 27 Let $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ be a function and K be a $N_{nc}\mathcal{P}os$ of X_1. Then $K \cap C$ is $N_{nc}ro$ in X_2 for each $N_{nc}ros$ C of X_1.

Theorem 28 If $\rho : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is a $CN_{nc}\gamma Irr$ function and C is $N_{nc}\mathcal{P}os$ of X_1, then $\rho_C : (C, N_{nc}\Gamma_C) \to (X_2, N_{nc}\Psi)$ is $CN_{nc}\gamma Irr$.

Proof: It is similar to the proof of Theorem 26.

Theorem 29 Let $\rho_1 : (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ and $\rho_2 : (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ be two functions. Then the following hold. If

(i) ρ_1 is $CN_{nc}\gamma Irr$ and ρ_2 is $N_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CN_{nc}\gamma Irr$,

(ii) ρ_1 is $CN_{nc}Cts$ and ρ_2 is $CN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CN_{nc}\gamma Irr$,

(iii) ρ_1 is $CN_{nc}\gamma Irr$ and ρ_2 is $N_{nc}\gamma Cts$, then $\rho_2 \circ \rho_1$ is $CN_{nc}Cts$.

Proof: Straightforward.

Definition 30 A $N_{nc}ts$ $(X, N_{nc}\Gamma)$ is said to be $N_{nc}\gamma$ -T_1 if for each pair of distinct points l and m of X, there exist $N_{nc}\gamma os$'s U_1 and U_2 such that $l \in U_1$ and $m \in U_2$, $l \notin U_2$ and $m \notin U_1$.

Theorem 31 If $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is $CN_{nc}\gamma Irr$ injection and X_2 is $N_{nc}\gamma$ -T_1, then X_1 is $N_{nc}r$-T_1.

Proof: Let $l, m \in X_1$ and $l \neq m$.

$(l, m \in X_1)(l \neq m) \Rightarrow \rho(l) \neq \rho(m)$, ($X_2$ is $N_{nc}\gamma$ -T_1)

$$\Rightarrow (\exists F_1 \in N_{nc}\gamma O(X_2, \rho(l)))\ (\exists F_2 \in N_{nc}\gamma O(X_2, \rho(m)))\ (\rho(l) \notin F_2)\ (\rho(m) \notin F_1)$$

$$\Rightarrow (\rho^{-1}(F_1) \in N_{nc}RO(X_1, l))\ (\rho^{-1}(F_2) \in N_{nc}RO(X_1, m))\ (l \notin \rho^{-1}(F_2))\ (m \notin \rho^{-1}(F_1)).$$

Since, ρ is injective and $CN_{nc}\gamma Irr$, then we have X_1 is $N_{nc}r$-T_1.

Definition 32 A $N_{nc}ts\ (X, N_{nc}\Gamma)$ is said to be $N_{nc}\gamma$-T_2 for each pair of distinct points l and m in X, there exist disjoint $N_{nc}\gamma o$ sets L and M in X such that $l \in L$ and $m \in M$.

Theorem 33 If $\rho: (X_1, N_{nc}\Gamma) \rightarrow (X_2, N_{nc}\Psi)$ is $CN_{nc}\gamma Irr$ injection and X_2 is $N_{nc}\gamma$-T_2, then X_1 is $N_{nc}r$-T_2.

Proof: Let $l, m \in X_1$ and $l \neq m$.

$$(l, m \in X_1)(l \neq m) \Rightarrow \rho(l) \neq \rho(m),\ (X_2 \text{ is } N_{nc}\gamma\text{-}T_2)$$

$$\Rightarrow (\exists L \in N_{nc}\gamma O(X_2, \rho(l)))\ (\exists M \in N_{nc}\gamma O(X_2, \rho(m)))\ (L \cap M = \Phi)$$

$$\Rightarrow (\rho^{-1}(L) \in N_{nc}RO(X_1, l))(\rho^{-1}(M) \in N_{nc}RO(X_1, m))\ ((\rho^{-1}(L) \cap \rho^{-1}(M)) = \Phi).$$

Since, ρ is injective and $CN_{nc}\gamma Irr$, then we have X_1 is $N_{nc}r$-T_2.

Theorem 34 Let X_2 be a $N_{nc}\gamma$-T_2 space. If ρ_1 & $\rho_2: (X_1, N_{nc}\Gamma) \rightarrow (X_2, N_{nc}\Psi)$ are $CN_{nc}\gamma Irr$, then the set $L = \{l \mid \rho_1(l) = \rho_2(l)\} \in N_{nc}C(X_1)$.

Proof: Let $l \notin L$.

$$l \notin L \Rightarrow \rho_1(l) \neq \rho_2(l),\ (X_2 \text{ is } N_{nc}\gamma\text{-}T_2)$$

$$\Rightarrow \left(\exists O_1 \in N_{nc}\gamma O\big(X_2, \rho_1(l)\big)\right)\left(\exists O_2 \in N_{nc}\gamma O\big(X_2, \rho_2(l)\big)\right), (O_1 \cap O_2 = \Phi)$$

$$\Rightarrow (\rho_1^{-1}(O_1) \in N_{nc}RO(X_1, l))\ (\rho_2^{-1}(O_2) \in N_{nc}RO(X_1, l))(\rho_1^{-1}(O_1 \cap O_2) = \Phi)\ (\rho_2^{-1}(O_1 \cap O_2) = \Phi)$$

ρ_1 and ρ_2 are $CN_{nc}\gamma Irr$

$$\Rightarrow \left(U = (\rho_1^{-1}(O_1) \cap \rho_2^{-1}(O_2) \in N_{nc}RO(X_1, l)\right)(U \cap L \neq \Phi)$$

$$\Rightarrow l \notin N_{nc}cl(L).$$

Then L is $N_{nc}\ c$ in X_1.

Theorem 35 Let X_2 be a $N_{nc}\gamma$-T_2 space. If $\rho: (X_1, N_{nc}\Gamma) \rightarrow (X_2, N_{nc}\Psi)$ is $CN_{nc}\gamma Irr$, then the set $M = \{(l, m) | \rho(l) = \rho(m) \in N_{nc}C(X_1 \times X_1)\}$.

Proof: $(l, m) \notin M$.

$$(l, m) \notin M \Rightarrow \rho(l) \neq \rho(m),\ (X_2 \text{ is } N_{nc}\gamma\text{-}T_2)$$

$$\Rightarrow (\exists O_1 \in N_{nc}\gamma O(X_2, \rho(l)))\ (\exists O_2 \in N_{nc}\gamma O(X_2, \rho(m))), (O_1 \cap O_2 = \Phi)$$

$$\Rightarrow (\rho^{-1}(O_1) \in N_{nc}RO(X_1, l))\ (\rho^{-1}(O_2) \in N_{nc}RO(X_1, m))\ (\rho^{-1}(O_1) \cap \rho^{-1}(O_2) = \Phi)\ \rho \text{ is } CN_{nc}\gamma Irr$$

$$\Rightarrow (U = (\rho^{-1}(O_1) \times \rho^{-1}(O_2) \in N_{nc}RO(X_1 \times X_1, (l, m))))\ (U \cap M \neq \Phi)$$

$$\Rightarrow (l, m) \notin N_{nc}cl(M).$$

Then M is $N_{nc}c$ in $X_1 \times X_1$.

Completely Weakly N-neutrosophic crisp γ irresolute function

Definition 36 A function $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is said to be completely weakly $N_{nc}\gamma$-irresolute (briefly, $CWN_{nc}\gamma Irr$) if for each $l \in X_1$ and for any $N_{nc}\gamma os$ C containing $\rho(l)$, there exists a $N_{nc}os$ D containing l such that $\rho(D) \subseteq C$.

Remark 37 The following implications are true.

$$\boxed{CN_{nc}\gamma Irr} \longrightarrow \boxed{CWN_{nc}\gamma Irr} \longrightarrow \boxed{N_{nc}\gamma Irr}$$

But not converse as shown by the following examples.

Example 38 In Example 22, then it is

(i) $CW2_{nc}\gamma Irr$ but not $C2_{nc}\gamma Irr$, the set $\rho^{-1}(\langle\{u_1, v_1\}, \{\Phi\}, \{w_1, x_1\}\rangle) = \langle\{u_1, v_1\}, \{\Phi\}, \{w_1, x_1\}\rangle$ is a $2_{nc}\gamma os$ but not $2_{nc}ros$ in X.

(ii) $2_{nc}\gamma Irr$ but not $CW2_{nc}\gamma Irr$, the set $\rho(\langle\{v_1, w_1\}, \{\Phi\}, \{u_1, x_1\}\rangle) \subsetneq \langle\{v_1\}, \{\Phi\}, \{u_1, w_1, x_1\}\rangle$.

$\langle\backslash\{v_1\}, \{\Phi\}, \{u_1, w_1, x_1\}\rangle$ is a $2_{nc}\gamma os$ and $\langle\{v_1, w_1\}, \{\Phi\}, \{u_1, x_1\}\rangle$ is a $2_{nc}os$.

Theorem 39 Let $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ be a function, then statements

(i) ρ is $CWN_{nc}\gamma Irr$,

(ii) $\rho^{-1}(N_{nc}\gamma int(D)) \subseteq N_{nc}int(\rho^{-1}(D))$ for every $N_{nc}s$ D of X_2,

(iii) $\rho(N_{nc}cl(C)) \subseteq N_{nc}\gamma cl(\rho(C))$ for every $N_{nc}s$ C of X_1,

(iv) $N_{nc}cl(\rho^{-1}(D) \subseteq \rho^{-1}(N_{nc}\gamma cl(D))$ for every $N_{nc}s$ D of X_2,

(v) $\rho^{-1}(E)$ is $N_{nc}c$ in X_1 for each $N_{nc}\gamma c$s E in X_2,

(vi) $\rho^{-1}(E)$ is $N_{nc}o$ in X_1 for each $N_{nc}\gamma os$ E in X_2

are equivalent.

Proof: (i) ⇒ (ii): Let $D \subseteq X_2$ and $x \in \rho^{-1}(N_{nc}\gamma int(D))$.

$x \in \rho^{-1}(N_{nc}\gamma int(D)) \Rightarrow N_{nc}\gamma int(D) \in N_{nc}\gamma O(X_2, \rho(x))$

$\Rightarrow (\exists\, U \in N_{nc}O(x))\, (\rho(U) \subseteq N_{nc}\gamma int(D) \subseteq D)$

$\Rightarrow (\exists\, U \in N_{nc}O(x))\, (U \subseteq \rho^{-1}(D))$

$\Rightarrow x \in N_{nc}int(\rho^{-1}(D))$.

(ii) ⇒ (iii): Let $C \subseteq X_1$.

$C \subseteq X_1 \Rightarrow \rho(C) \subseteq X\_2$

$\Rightarrow X_2 \backslash \rho(C) \subseteq X_2$

$$\Rightarrow \rho^{-1}\left(N_{nc}\gamma int(X_2 \setminus \rho(C))\right) \subseteq N_{nc}int(\rho^{-1}(X_2 \setminus \rho(C)))$$

$$\Rightarrow X_1 \setminus \rho^{-1}\left(N_{nc}\gamma cl(\rho(C))\right) \subseteq X_1 \setminus N_{nc}cl(\rho^{-1}(\rho(C)))$$

$$\Rightarrow N_{nc}cl(C) \subseteq N_{nc}cl\left(\rho^{-1}(\rho(C))\right) \subseteq \rho^{-1}(N_{nc}\gamma cl(\rho(C)))$$

$$\Rightarrow \rho(N_{nc}cl(C)) \subseteq N_{nc}\gamma cl(\rho(C)).$$

(iii) ⇒ (iv): Let $D \subseteq X_2$.

$$D \subseteq X_2 \Rightarrow \rho^{-1}(D) \subseteq X_1$$

$$\Rightarrow \rho\left(N_{nc}cl(\rho^{-1}(D))\right) \subseteq N_{nc}\gamma cl\left(\rho(\rho^{-1}(D))\right) \subseteq N_{nc}\gamma cl(D)$$

$$\Rightarrow N_{nc}cl(\rho^{-1}(D)) \subseteq \rho^{-1}(N_{nc}\gamma cl(D)).$$

(iv) ⇒ (v): Let $O \in N_{nc}\gamma C(X_2)$.

$$O \in N_{nc}\gamma C(X_2) \Rightarrow O = N_{nc}\gamma cl(E)$$

$$\Rightarrow N_{nc}cl(\rho^{-1}(E)) \subseteq \rho^{-1}(N_{nc}\gamma cl(E)) = \rho^{-1}(E)$$

$$\Rightarrow \rho^{-1}(E) = N_{nc}cl(\rho^{-1}(E))$$

$$\Rightarrow \rho^{-1}(E) \in N_{nc}C(X_1).$$

(v) ⇒ (vi): Obvious.

(vi) ⇒ (i): Let $O \in N_{nc}\gamma O(X_2)$ and $x \in \rho^{-1}(E)$.

$$(O \in N_{nc}\gamma O(X_2))(x \in \rho^{-1}(E)) \Rightarrow O \in N_{nc}\gamma O(X_2, \rho(x))$$

$$\Rightarrow (U = \rho^{-1}(E) \in N_{nc}O(X_1, x))(\rho(U) \subseteq E).$$

Theorem 40 Let $\rho: (X_1, N_{nc}\Gamma) \rightarrow (X_2, N_{nc}\Psi)$ be a bijective function, Then statements

(i) ρ is $CWN_{nc}\gamma Irr$,

(ii) $N_{nc}\gamma int(\rho(O)) \subseteq \rho(N_{nc}int(O))$ for every $N_{nc}s$ of X_1

are equivalent.

Proof: (i) ⇒ (ii): Let $O \subseteq X_1$.

$$O \subseteq X_1 \Rightarrow X_1 \setminus O \subseteq X_1$$

$$\Rightarrow \rho(X_1 \setminus N_{nc}int(O)) = \rho(N_{nc}cl(X_1 \setminus O)) \subseteq N_{nc}\gamma cl(\rho(X_1 \setminus O)),\ \rho \setminus \text{is bijection}$$

$$\Rightarrow X_2 \setminus \rho(N_{nc}int(O)) \subseteq X_2 \setminus N_{nc}int(\rho(O))$$

$$\Rightarrow N_{nc}\gamma int(\rho(O)) \subseteq \rho(N_{nc}int(O)).$$

(ii) ⇒ (i): Let $O \subseteq X_1$.

$$O \subseteq X_1 \Rightarrow X_1 \setminus O \subseteq X_1$$

$$\Rightarrow N_{nc}\gamma int(\rho(X_1 \backslash O)) \subseteq \rho(N_{nc}int(X_1 \backslash O)) \text{ } (\rho \text{ is bijection})$$

$$\Rightarrow X_2 \backslash N_{nc}\gamma cl(\rho(O)) \subseteq X_2 \backslash \rho(N_{nc}cl(O))$$

$$\Rightarrow \rho(N_{nc}cl(O)) \subseteq N_{nc}\gamma cl(\rho(O)).$$

Theorem 41 Let $\rho_1: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ and $\rho_2: (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ be any two functions such that $\rho_2 \circ \rho_1: (X_1, N_{nc}\Gamma) \to (X_3, N_{nc}\Phi)$. Then the following statements hold:

(i) If ρ_1 is $CWN_{nc}\gamma Irr$ and ρ_2 is $N_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CWN_{nc}\gamma Irr$,

(ii) If ρ_1 is $CN_{nc}Cts$ and ρ_2 is $CWN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CN_{nc}\gamma Irr$,

(iii) If ρ_1 is $StN_{nc}Cts$ and ρ_2 is $CN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CN_{nc}\gamma Irr$,

(iv) If ρ_1 and ρ_2 are $CN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CN_{nc}\gamma Irr$,

(v) If ρ_1 is $CN_{nc}\gamma Irr$ and ρ_2 is $CWN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CN_{nc}\gamma Irr$,

(vi) If ρ_1 is $CWN_{nc}\gamma Irr$ and ρ_2 is $N_{nc}\gamma Cts$, then $\rho_2 \circ \rho_1$ is $N_{nc}Cts$,

(vii) If ρ_1 is $N_{nc}\gamma Cts$ and ρ_2 is $CWN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $N_{nc}\gamma Irr$,

(viii) If ρ_1 is $N_{nc}Cts$ and ρ_2 is $CWN_{nc}\gamma Irr$, then $\rho_2 \circ \rho_1$ is $CWN_{nc}\gamma Irr$.

Proof: Straightforward.

Definition 42 A mapping $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is N_{nc}-open (briefly, $N_{nc}O$) if the image of every $N_{nc}os$ in $(X_1, N_{nc}\Gamma)$ is a $N_{nc}os$ in $(X_2, N_{nc}\Psi)$.

Definition 43 A function $\rho: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is said to be almost N_{nc}-open (briefly, $\mathcal{A}N_{nc}O$) if the image of every $N_{nc}ros$ in $(X_1, N_{nc}\Gamma)$ is a $N_{nc}os$ in $(X_2, N_{nc}\Psi)$.

Theorem 44 If $\rho_1: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is $\mathcal{A}N_{nc}O$ surjection and $\rho_2: (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ is any function such that $\rho_2 \circ \rho_1: (X_1, N_{nc}\Gamma) \to (X_3, N_{nc}\Phi)$ is $CN_{nc}\gamma Irr$, then ρ_2 is $CWN_{nc}\gamma Irr$.

Proof: Let $O \in N_{nc}\gamma O(X_3)$.

$$O \in N_{nc}\gamma O(X_3) \Rightarrow (\rho_2 \circ \rho_1)^{-1}(O) = \rho_1^{-1}(\rho_2^{-1}(O)) \in N_{nc}RO(P), \rho_1 \text{ is a } \mathcal{A}N_{nc}O$$

$$\Rightarrow \rho_1\left(\rho_1^{-1}(\rho_2^{-1}(O))\right) = \rho_2^{-1}(O) \in (X_2, N_{nc}\Psi).$$

Theorem 45 If $\rho_1: (X_1, N_{nc}\Gamma) \to (X_2, N_{nc}\Psi)$ is $N_{nc}O$ surjection and $\rho_2: (X_2, N_{nc}\Psi) \to (X_3, N_{nc}\Phi)$ is any function such that $\rho_2 \circ \rho_1: (X_1, N_{nc}\Gamma) \to (X_3, N_{nc}\Phi)$ is $CN_{nc}\gamma Irr$, then ρ_2 is $CWN_{nc}\gamma Irr$.

Proof: Let $O \in N_{nc}\gamma O(X_3)$.

$$O \in N_{nc}\gamma O(X_3) \Rightarrow (\rho_2 \circ \rho_1)^{-1}(O) = \rho_1^{-1}(\rho_2^{-1}(O)) \in (X_1, N_{nc}\Gamma), \rho_1 \text{ is a } N_{nc}\, O \text{ surjection}$$

$$\Rightarrow \rho_1\left(\rho_1^{-1}(\rho_2^{-1}(O))\right) = \rho_2^{-1}(O) \in (X_2, N_{nc}\Psi).$$

Conclusions

In this chapter, a new type of $N_{nc}\gamma$-irresolute map, completely $N_{nc}\gamma$-irresolute function and completely weakly $N_{nc}\gamma$-irresolute function in $N_{nc}ts$ are presented and analyzed the difference between these maps. This can be improved to $N_{nc}\gamma$-open mapping function, $N_{nc}\gamma$-closed mapping function, $N_{nc}\gamma$-homeomorphism functions of $N_{nc}ts$ are the further research areas can be covered in future tasks. In addition, authors hope that to investigate further on some fundamental properties between these new notions with separation and covering of $N_{nc}\gamma$-open set in a N-neutrosophic crisp topological space.

Future Research Directions

Using these $N_{nc}\gamma$-irresolute map, completely $N_{nc}\gamma$-irresolute function and completely weakly $N_{nc}\gamma$-irresolute function of $N_{nc}\gamma$-open set in $N_{nc}ts$, try to relate and properties to $N_{nc}\gamma$-open mapping and $N_{nc}\gamma$-closed mapping functions of $N_{nc}\gamma$-open set.

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Chapter Four

On dense, rare and nowhere dense sets in anti-topological spaces

Tomasz Witczak

[1] Institute of Mathematics, University of Silesia, Bankowa 14, Katowice, Poland

E-mail: tm.witczak@gmail.com

ABSTRACT

Anti-topological spaces have been defined by Şahin, Kargın and Yücel in 2021. They investigated some relationships and connections between these structures and so-called neutro topological spaces which they introduced in the same paper. Recently, we have extended their research by analyzing the notions of interior, closure, continuity, doorness, density and nowhere density in anti-topological setting. In the present chapter we attempt to compile previous information on density and to add some statements on rare sets. Moreover, we give some new examples of anti-topological spaces. Some other general theorems are presented too.

Keywords: Anti-topological spaces, Neutro-topological spaces, Generalized weak structures, Generalized topology.

INTRODUCTION

Anti-topological spaces have been introduced by Şahin, Kargın and Yücel in [12]. The most striking difference between topologies (or supra and infra topologies, minimal structures or weak structures|) and anti-topologies is that the former are based on the idea of *closure* of a family of sets under some operations (like finite intersections and arbitrary unions) and on the assumption that some special sets (like empty set and the whole universe) must be considered as *open*, while the latter concentrate on *exclusion*. This is clear from their definition: our family of subsets can be called anti-topology if we have a guarantee that any finite intersection and any union of its elements is *beyond* this family. Of course we do not mean trivial intersections and unions, e.g. those in which a single set is intersected or joined with itself. Moreover, both the empty set and the whole universe are excluded.

We may investigate many classical topological notions in this new context. We have already examined some basic properties of (anti-) closure, interior, continuity, doorness, density and nowhere density in [23]. In general, it is always an interesting question: which of the standard features and properties remain the same when we replace topology with some other structure (maybe more general or just changed in some way)?

In this chapter we would like to gather some earlier results on density and nowhere density in anti-topological spaces. Moreover, we want to define and investigate the notion of rarity in this new setting. Finally, there are also some additional facts on interior and closure stated.

One should emphasize the fact that Şahin et al. examined anti-topologies together with neutro-topological spaces. Besides, they are something different than, say, neutrosophic crisp topological spaces defined by Salama et al. in [14].

Neutrosophy, founded by Smarandache, is a branch of science which extends the idea of fuzzy sets and gives some new concepts of uncertainty and ambiguity. Clearly, this line of research (namely, the connection between anti-topologies and neutrosophic sets) should also be studied.

Note that nowadays there are many new trends in neutrosophic studies. Some of them are largely concentrated on practical applications in multi-criteria decision making (see [4], [5], [13], [15], [17 - 19], [21],[25-52]). Other papers deal with pattern recognition (see [6]) and even medical diagnosis (see [16]). However, there are also purely mathematical and theoretical considerations (see [20]).

On the same principle, one can imagine the potential use of anti-topological spaces in decision making or data clustering. The idea of exclusion outside the family of distinguished sets should be studied in these contexts. Moreover, even if anti-topologies in their basic form are crisp, then it is still possible to reintroduce them in fuzzy, vague, neutrosophic or soft environment. Furthermore, one can think about measuring the grade of anti-openness and exclusion by applying the idea of smooth topological space (as it was presented in, say, [22] and other articles, starting from the initial research of Badard from 1986).

Finally, we may point out that the general idea of "anti-structures" (that is, various algebraic structures obtained by the assumption that some traditionally accepted conditions have been *rejected*) is now studied extensively. The reader may check papers [7 - 11].

BACKGROUND AND SOME INITIAL NOTIONS

First, let us define anti-topological spaces. The definition below is taken from our paper [23]. In general, it is based on the definition from [12] but with some small adjustments.

Definition 1. [23] Let X be a non-empty universe and T be a collection of subsets of X . We say that (X,T) is an anti-topological space if the following conditions are satisfied:

1. $\emptyset, X \notin T$.
2. For any $n \in \mathbb{N}$, if $A_1, A_2, \dots, A_n \in T$, then $\bigcap_{i=1}^{n} A_i \notin T$. We assume that the sets in question are not all identical (later we will call such families *non-trivial*).
3. For any collection $\{A_i\}_{i \in J \neq \emptyset}$ such that $A_i \in T$ for each $i \in J$, $\bigcup_{i \in J} A_i \notin T$. We assume that the sets in question are not all identical.

The elements of T are called *anti-open* sets, while their complements are *anti-closed* sets. The set of all anti-closed sets (with respect to some particular T) will be denoted by T_{Cl} .

One can check in [23] (Lemma 2.5, Lemma 2.6, Lemma 2.7) that anti-topology excludes not only finite but, in fact, arbitrary intersections. Moreover, conditions (2) and (3) from Definition 1 are equivalent. Finally, both these conditions hold for anti-closed sets too (see Lemma 2.8 and Lemma 2.9 in [23]).

Let us show some examples of anti-topological spaces. The first one is taken from [12], the rest is our own invention. The vast majority of them has been already presented in [23] (with some additional reflections which have been omitted here).

Example 2. (Compare [12] and [23]). The following structures are anti-topological spaces:

1. $X = \{1, 2, 3, 4\}$ and $T = \{\{1,2\}, \{2,3\}, \{3,4\}\}$.
2. $X = \{1,2\}$ and $T = \{\{1\}, \{2\}\}$.
3. $X = \{a, b, c\}$ and $T = \{\{a, b\}, \{c\}\}$. Note that adding X and $\emptyset$ to this family transforms it into topology.
4. $X = \{a, b, c, d, e, f\}$ and $T = \{\{a, b\}, \{c, d\}, \{e\}\}$.
5. $X = \mathbb{N}^+$ and T_k consists only of these finite subsets of X which have cardinality k, where k is a fixed positive natural number.
6. $X = \mathbb{N}^+$ and $T = \{\{1\}, \{2\}, \{3\}, \ldots\}$. This is a special case of the anti-topology introduced in the previous point. Clearly, this is just T_1 (not to confuse with T_1 separation axiom).
7. $X = \mathbb{R}$ and T_y contains only these closed intervals which have length y, where y is a fixed positive real number.
8. X is arbitrary and $T = \{A, X - A\}$ for some distinguished set $A \neq \emptyset$.
9. $X = \mathbb{R}$ and $T = \{\mathbb{R}^-, \mathbb{R}^+\}$.
10. $X = \mathbb{N}$ and $T = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \ldots\}$.
11. Let $X = \mathbb{R}^2$ with usual Euclidean metric. We may define T_r as the set of all these closed balls which have radius $r > 0$.
12. Let $X = \mathbb{R}^+$ and $T = \{(0,1), \{1\}, (1,2), \{2\}, (2,3), \{3\}, \ldots\}$.
13. Let $X = \mathbb{R}$ and T be the family of all those subsets of $\mathbb{R}$ which are of the form $A = \mathbb{R} - \{x\}$ for some real x. Now assume that $A, B \in T$ and they are different. That is $A = \mathbb{R} - \{x\}$, $B = \mathbb{R} - \{y\}$ for some $x, y \in \mathbb{R}$ such that $x \neq y$. Then:

$A \cap B = (\mathbb{R} - \{x\}) \cap (\mathbb{R} - \{y\}) = (\mathbb{R} \cap \mathbb{R}) - (\{x\} \cup \{y\}) = \mathbb{R} - \{x, y\} \notin T$. We used the following identity here: $(E - F) \cap (G - H) = (E \cap G) - (F \cup H)$.

Now assume that $\{A_i\}_{i \in J \neq \emptyset}$ is a non-trivial family of members of T. Consider $\bigcup_{i \in J} A_i$. It is enough to discuss any two members of our family, say $A = \mathbb{R} - \{x\}$, $B = \mathbb{R} - \{y\}$, $x \neq y$. Now $A \cup B \subseteq \bigcup_{i \in J} A_i$ and (according to the basic properties of set difference) $A \cup B = (\mathbb{R} - \{x\}) \cup (\mathbb{R} - \{y\}) = \mathbb{R} - (\{x\} \cap \{y\}) = \mathbb{R} - \emptyset = \mathbb{R}$. Here we used this identity: $(E - F) \cap (E - G) = E - (F \cap G)$.

Hence, $A \cup B = \mathbb{R} \subseteq \bigcup_{i \in J} A_i = \mathbb{R} \notin T$. Clearly, $\emptyset \notin T$. This structure may be called *co-singleton* anti-topology. Note that adding X and $\emptyset$ to this family transforms it into strong generalized topology in the sense of Császár.

One can prove that:

Lemma 3. The intersection of two anti-topological spaces on the same universe is an anti-topological space too.

Proof. Suppose that T, S are two anti-topologies on the same universe X. Consider $W = T \cap S$. Let $A_1, A_2, \ldots, A_n \in W$. In particular, it means that each of these sets belongs to T (without loss of generality).

Clearly, $\bigcap_{i=1}^{n} A_i \notin T$, hence $\bigcap_{i=1}^{n} A_i \notin T \cap S = W$. Now consider $\{A_i\}_{i \in J \neq \emptyset}$such that $A_i \in W$for any $i \in J$. Hence $A_i \in T \cap S$for any $i \in J$ and (without loss of generality) $A_i \in T$. But then$\bigcup_{i \in J} A_i \notin T$. All the more it does not belong to $T \cap S = W$. Clearly, $\emptyset, X \notin W$ because they do not belong neither to T . Note that according to our previous mention it was enough to check that W excludes intersections (or unions).

Lemma 4. The union of two anti-topological spaces on the same universe need not to be an anti-topological space.

Proof. Consider the following counter-example: $X = \{a, b, c, d, e\}$, $T = \{\{a\},\{b\},\{c\}\}$, $S = \{\{b, c\}, \{a, e\}\}$. Then let us think about $W = T \cup S$. This is not anti-topology because $\{a\} \cap \{a, e\} = \{a\} \in W$. Analogously, $\{b\} \cup \{c\} = \{b, c\} \in W$.

We would like to recall one lemma from [23] which is simple but, in some sense, important.

Lemma 5. [23] Assume that (X, T) is an anti-topological space, $B \in T$ and $A \subseteq B$. Suppose that $A \neq B$. Then $A \notin T$.

Now we may introduce anti-interior and anti-closure.

Definition 6. [23] Assume that (X, T) is an anti-topological space and $A \subseteq X$. Then we define *anti-interior* of A (that is, $aInt(A)$) and its anti-closure (namely, $aCl(A)$) as follows:

1. $aInt(A) = \bigcup \quad \{U;\ U \subseteq A,\ U \in T\}$
2. $aCl(A) = \bigcap \quad \{F;\ A \subseteq F,\ F \in T_{Cl}\}$

The reader can find some examples of anti-interiors and anti-closures in [18]. Besides, some other examples will be later presented in the present paper. Clearly, it is possible that $aInt(A) \notin T$. In [23] we proposed to call these sets which have anti-open interior, *anti-genuine* sets. The idea is taken from [24] where it was applied to infra topological spaces. Each anti-open set is identical with its anti-interior and thus is anti-genuine too. In Example 2 (4) we have $\{a, b, c\}$which is not anti-open but is anti-genuine.

As for the properties of anti-interior and anti-closure, we may recall the following theorem from [23]:

Theorem 7. [23]

Let (X, T) be an anti-topological space. Let $A \subseteq X$. Then the following statements are true:

1. $aInt(A) \subseteq A$.
2. If $A \in T$, then $aInt(A) = T$. The converse may not be true.
3. If $A \subseteq B$, then $aInt(A) \subseteq aInt(B)$ and $aCl(A) \subseteq aCl(B)$.
4. $aInt(aInt(A)) = aInt(A)$ and $aCl(aCl(A)) = aCl(A)$.
5. $A \subseteq aCl(A)$.
6. If $A \in T_{Cl}$, then $aCl(A) = A$. The converse may not be true.
7. $-aInt(A) = aCl(-A)$.
8. $aInt(-A) = -aCl(A)$.
9. $x \in aInt(A)$ if and only if there is $U \in T$ such that $x \in U \subseteq A$.
10. $x \in aCl(A)$ if and only if for any $U \in T$ such that $x \in U$ we have that $U \cap A \neq \emptyset$.

Then we have the following two lemmas:

Lemma 8. Assume that (X,T) is an anti-topological space and $\{A_i\}_{i\in J\neq\emptyset}$ is a family of sets. Then $aInt\left(\bigcap_{i\in J} A_i\right) \subseteq \bigcap_{i\in J} aInt(A_i)$.

Proof. The proof is rather typical and, in fact, is true for generalized weak structures too. Generalized weak structures, introduced by Avila and Molina in [1], are just arbitrary families of subsets. In this general setting the authors reconstruct some basic topological notions.

Clearly, $\bigcap_{i\in J} A_i \subseteq A_k$ for any $k \in J$. Now we use monotonicity of interior to get that $aInt\left(\bigcap_{i\in J} A_i\right) \subseteq aInt(A_k)$. But this is true for any $k \in J$ and thus we obtain our expected conclusion.

Remark 9. Note that the converse is not true even for binary intersections. The reader may find an appropriate counter-example in [23]. The converse for binary intersections is true in these spaces which are *closed* under finite intersections, while in case of anti-topologies we can say that they are *anti-closed* under this operation (and the same with arbitrary unions).

Lemma 10. Assume that (X,T) is an anti-topological space and $\{A_i\}_{i\in J\neq\emptyset}$ is a family of sets. Then $\bigcup_{i\in J} aCl(A_i) \subseteq aCl\left(\bigcup_{i\in J} A_i\right)$.

Proof. The proof is analogous to the proof of the previous lemma. Again, the converse need not to be true even for finite unions.

Now let us recall Theorem 7 (2). The fact that the converse may not be true allows us to define pseudo-anti-open sets:

Definition 11. [23] Let (X,T) be an anti-topological space. Assume that $A \subseteq X$. If $aInt(A) = A$ then we say that A is *pseudo-anti-open*.

One can prove that the family of all pseudo-anti-open sets (with respect to a given anti-topology T) is closed under arbitrary unions. The same holds for pseudo-open sets in minimal structures. The reader can check [3] where Bhattacharya calls them *open* m_X while the elements of minimal structure are named m_X *open*.

RARITY, DENSITY AND NOWHERE DENSITY

In this section we study some properties of (anti-) rare, dense and nowhere dense sets. First, let us define all these classes.

Definition 12. Let (X,T) be an anti-topological space and $A \subseteq X$. We say that A is:

1. *anti-dense* if and only if $aCl(A) = X$.
2. *anti-nowhere dense* if and only if $aInt\big(aCl(A)\big) = \emptyset$.
3. *strongly anti-nowhere dense* if and only if it has empty intersection with any anti-open set.
4. *anti-rare* if and only if $aInt(A) = \emptyset$.

Remark 13. It seems that many topologists identify rare sets with nowhere dense sets. This is not our case. Our approach is based on the one presented e.g. in [3] (but also in some other papers). Hence, our rare (or rather anti-rare) sets are analogous to boundary sets.

Example 14. Consider $X = \{1,2,3,4\}, T = \big\{\{1,2\},\{2,3\},\{3,4\}\big\}$. Then $\{1,2,3\},\{2,3,4\}$ are anti-dense. Moreover, their intersection, namely $\{2,3\}$, is anti-dense too. However, this is not always true. Think about $Y = \{a,b,c,d,e\}, S = \big\{\{a,b\},\{c,d\},\{e\}\big\}$. Then $\{a,c,e\},\{b,d,e\}$are anti-dense but their intersection $\{e\}$

is not anti-dense. Clearly, $S_{Cl} = \{\{c,d,e\},\{a,b,e\},\{a,b,c,d\}\}$ and $aCl(\{e\}) = \{c,d,e\} \cap \{a,b,e\} = \{e\}$. Note that we may consider $\{e\}$ as pseudo-anti-closed, *per analogiam* with pseudo-anti-open sets. It means that $aCl(A) = A$.

Example 15. Consider $X = \{1,2,3,4\}, T = \{\{1,2\},\{2,3\},\{3,4\}\}$. Then $A = \{1,4\}$ is anti-nowhere dense. One can check that $aCl(A) = \{1,4\} = A$ and $aInt(A) = \emptyset$ because there are no anti-open sets contained in A (but *not* because empty set is contained in A : as we already know, empty set is never anti-open). Besides, note that the fact that A is anti-nowhere dense does not mean that it is strongly anti-nowhere dense. Just take $B = \{1,2\}$. Clearly, $A \cap B = \{1\} \neq \emptyset$.

Now take $Y = \{a,b,c,d,e,f\}$, $T = \{\{a,b\},\{c,d\},\{e\}\}$. Clearly, $\{f\}$ is strongly anti-nowhere dense, having empty intersection with any anti-open set from T .

Remark 16. One could ask why the definition of strong anti-nowhere density is so strict. For example, in topological spaces we would say that A is nowhere dense if for any $B \in T$ we may find $C \in T$ (not necessarily B itself!) such that $C \subseteq B$, $A \cap C = \emptyset$. Of course we could use this approach but it would be irrelevant. Note that anti-open sets *do not have* proper anti-open subsets (recall Lemma 5).

Example 17. Let $X = \{1,2,3,4\}, T = \{\{1,2\},\{2,3\},\{3,4\}\}$. Consider $\{1,3\},\{1,4\},\{2,4\}$. These sets are anti-rare. Now take $Y = \{1,2\}$ and $T = \{\{1\},\{2\}\}$. Here there are no non-empty anti-rare sets. Now consider $Z = \mathbb{N}^+$ with anti-topology T_k. Now every set which has cardinality $< k$ is anti-rare. Analogously, if $Z = \mathbb{R}$ and we have anti-topology T_y then any closed interval of the length $< y$ is anti-rare.

Now we may prove some lemmas and theorems about these classes.

Theorem 18. Every strongly anti-nowhere dense set in an anti-topological space is anti-nowhere dense too.

Proof. Assume that A is strongly anti-nowhere dense but $aInt(aCl(A)) \neq \emptyset$. Then there is $x \in aInt(aCl(A))$. Hence, $x \in aCl(A)$. But then for any $V \in T$ such that $x \in V$, $V \cap A \neq \emptyset$. This is contradiction because we assumed that A has empty intersection with any anti-open set.

Lemma 19. If A is anti-rare, then for any $B \in T$, B is not contained in A .

Proof. Assume the contrary: that there is some anti-open B contained in A . Then B is contained in the union of all anti-open sets contained in A , that is in $aInt(A)$. Moreover, B is non-empty as a member of T . Thus $aInt(A) \neq \emptyset$ and this is contradiction.

Lemma 20. Any non-empty and proper subset of anti-open set is anti-rare.

Proof. Recall the already mentioned fact that anti-open sets do not have proper anti-open subsets.

Theorem 21. Every anti-nowhere dense set is anti-rare.

Proof. Assume that (X,T) is an anti-topological space and A is anti-nowhere dense. Assume that it is not anti-rare. Hence, $aInt(A) \neq \emptyset$. Then there is some $x \in aInt(A)$ and some $B \in T$ such that $B \subseteq A$, $x \in A$. But $B \subseteq aCl(A)$, hence $x \in aInt(Cl(A))$. Contradiction.

Remark 22. Anti-rare sets need not to be anti-nowhere dense. Consider $X = \{a,b,c,d,e\}, T = \{\{a,b\},\{c,d\},\{e\}\}$. Then $T_{Cl} = \{\{c,d,e\},\{a,b,e\},\{a,b,c,d\}\}$. Take $A = \{b,c\}$. On the one hand,

$aInt(A) = \emptyset$, so A is anti-rare. On the other hand, $aCl(A) = \{a,b,c,d\}$and $aInt\big(aCl(A)\big) = \{a,b\} \cup \{c,d\} = \{a,b,c,d\} \neq \emptyset$.

Remark 23. Anti-rare sets can be anti-dense. Consider $X = \{a,b,c,d\}, T = \big\{\{a,b\},\{c,d\}\big\}$and $A = \{b,c\}$. On the one hand, $aInt(A) = \emptyset$. On the other hand, $T_{Cl} = T$ and $aCl(A) = \cap \quad \emptyset = X$. Note that A from Remark 22 *was not* anti-dense.

Theorem 24. Assume that $(X,T)$is an anti-topological space and $A \subseteq X$. Then A is anti-rare if and only if $-A$ is anti-dense.

Proof. From left to right: assume that $aInt(A) = \emptyset$. Assume that $-A$ is not anti-dense. Hence, $aCl(-A) \neq X$. But $aCl(-A) = -aInt(A) = -\emptyset = X$.

From right to left: let $aCl(-A) = X$. But $aCl(-A) = -aInt(A)$. Hence $-aInt(A) = X$ and thus $aInt(A) = -X = \emptyset$.

Theorem 25. Any intersection of anti-rare sets is anti-rare too.

Proof. Assume that (X,T) is an anti-topological space. Let $A = \bigcap_{i \in J \neq \emptyset} A_i$, where for any $i \in J$, $A_i$is anti-rare. Thus, $aInt(A_i) = \emptyset$ for any $i \in J$. Now suppose that $aInt(A) \neq \emptyset$. Hence there is some $B \in T$ such that $B \subseteq A$. But then $B \subseteq A_i$ for any $i \in J$. This is contradiction.

Remark 26. Finite unions of anti-rare sets may not be anti-rare. Again, consider $X = \{1,2,3,4\}, T = \big\{\{1,2\},\{2,3\},\{3,4\}\big\}$. Now $\{1,3\},\{2,4\}$are anti-rare but their union, namely $\{1,2,3,4\}$, is not anti-rare: its anti-interior is $\{1,2,3\} \neq \emptyset$.

Theorem 24. Any union of anti-dense sets is anti-dense too.

Proof. Assume that $(X,T)$is an anti-topological space. Let$A = \bigcup_{i \in J \neq \emptyset} A_i$ where for any $i \in J$, A_i is anti-dense, i. e. $aCl(A_i) = X$. Hence for any $i \in J$, the set $Z_i = \{B \in T_{Cl}; A_i \subseteq B\}$is empty. Assume the contrary: if it is not empty then $X \in T_{Cl}$ and thus $-X = \emptyset \in T$. But this is not possible by the very definition of anti-topology. Now assume that $aCl(A) \neq X$. This means that $Z = \{B \in T_{Cl}; A \subseteq B\} \neq \emptyset$. Hence there is some $C \in T_{Cl}$ such that $A \subseteq C$ and $C \neq X$. But then $A_i \subseteq C \in T_{Cl}$ for any $j \in J$ and this is contradiction.

Remark 27. We could use different reasoning in this proof. For example: as we know from Lemma 10, $\bigcup_{i \in J} aCl(A_i) \subseteq aCl\big(\bigcup_{i \in J} A_i\big)$. Now, if $aCl(A_i) = X$ for any $j \in J$, then the union on the left side is just X . Thus the set on the right side must be X too.

Theorem 28. [23] Let (X,T) be an anti-topological space and $A \subseteq X$. Then A is anti-dense if and only if it has non-empty intersection with each anti-open set from T .

Remark 29. Recall the proof of Theorem 24. Alternatively, we could think in the following way (using Theorem 28 and Theorem 7 (10)). Assume that $aCl(A) \neq X$ where A is some union of anti-dense sets. Then there is some $x \in X - aCl(A)$. Hence there exists certain $V \in T$ such that $x \in V$ and $V \cap A = \emptyset$. This means that $\cap \bigcup_{i \in J} A_i = \emptyset$. Hence, V has empty intersection with each of the sets A_i in A . But this is not possible because they are all anti-dense.

Finally, we get:

Theorem 30. Let (X,T)be an anti-topological space and $A \subseteq X$ is anti-dense. Then for any pseudo-anti-open set G such that $A \subseteq G$ we have that $G \subseteq aCl(A)$.

Proof. Assume that G is some pseudo-anti-open set such that $A \subseteq G$. Then $G \subseteq X$ but $X = aCl(A)$. Note that the assumption about pseudo-anti-openness of is superfluous. We left it just to compare the whole thing with Theorem 4.3. in [3] (check Remark 31 below).

Remark 31. What about the converse of Theorem 30? In general, it is not true. Hence the situation is different than in case of minimal structures studied by Bhattacharya. Take $X = \{a, b, c, d, e\}, T = \{\{a, b\}, \{c\}, \{d\}\}$and think about $A = \{e\}$. This singleton is beyond any pseudo-anti-open set. Hence, the implication "if G is pseudo-anti-open and $A \subseteq G$, then $G \subseteq aCl(A)$" is trivially satisfied. On the other hand, $\{e\}$is not anti-dense: in fact, it has empty intersection with any anti-open set. Thus, it is strongly anti-nowhere dense.

However, we may prove the following statement:

Theorem 32. Let (X,T) be an anti-topological space and $A \subseteq X$. Assume that for any pseudo-anti-open set G such that $A \subseteq G$, $G \subseteq aCl(A)$. Suppose that the class of such pseudo-anti-open sets is non-empty. Then A is anti-dense.

Proof. First, it is clear that $\bigcup T$, namely the union of all anti-open sets, is pseudo-anti-open. Let us take some pseudo-anti-open G such that $A \subseteq G \subseteq aCl(A)$. If $x \in G$, then $x \in aInt(G)$which means that x belongs to some anti-open set contained in G . This implies that $G \subseteq \bigcup T$. Hence $A \subseteq \bigcup T$ and $\bigcup T \subseteq aCl(A)$. Assume now that $aCl(A) \neq X$. Hence, if $A \subseteq B$ such that $B \in T_{Cl}$, then $\bigcup T \subseteq B$. Then $-B \subseteq - \bigcup T$. But $-B$ is anti-open (because B is anti-closed) so it cannot be contained in the complement of the union of *all* anti-open sets. Contradiction.

CONCLUSION AND FUTURE WORK

In this chapter we have analyzed some properties of anti-topological spaces. One can gather them together with those lemmas and theorems which have been already proved in [12] and [23] to obtain some kind of general framework for anti-topological spaces. Clearly, some of these results are more general than it seems at first glance: they are true even for generalized weak structures. However, some of them require specific properties of anti-topology. The reader is encouraged to continue this line of research. We should study separation axioms, connectedness and compactness in the context of anti-topological spaces. Moreover, one can imagine that anti-topologies could serve as models of some very specific non-classical modal logics. In [23] we gave a short discussion of this issue.

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Chapter Fıve

Neutrosophic h-ideal in INK-Algebra

Murugan Kaviayarsu [1] and Arunachalam Jayapreethi [2]
[1]Assistant Professor, Department of Mathematics, Sri Vidya Mandir Arts and Science College, Katteri- 636902, Tamilnadu, India.
[2]PG Research Scholar, Department of Mathematics, Sri Vidya Mandir Arts and Science College, Katteri- 636902, Tamilnadu, India.
E-mail: kavitamilm@gmail.com, jppreethijaya2392@gmail.com

ABSTRACT

In this paper, first we define the notion neutrosophic h-ideal in INK-Algebra, neutrosophic union and intersection of neutrosophic h-ideals in INK-algebras. We prove some theorems which show that there is some relation between these notions. Finally, we define the INK-sub algebra, neutrosophic T-ideals and neutrosophic p-ideals of INK-algebra and then we give related theorem about complements of neutrosophic h-ideals.

Keywords: INK-Algebra- Neutrosophic ideal, Neutrosophic h-ideals, T-ideal, p-ideal, Union, Intersection.

İ. INTRODUCTION

In 1986, Atanassov Introduced the Intuitionistic fuzzy set and later intuitionistic fuzzy set was applied in BCI/BCK-algebra, Introduced by Imai and Iseki in the 1980s. Following this, various researchers published articles using the intuitionistic fuzzy set concept. In 2005, Smarandache invented the new notion of the neutrosophic set in 1998 and it is a common code from the intuitionistic fuzzy set [1-8] and [15-55]. This has been followed by a lot of researchers publishing various articles over the last few years. In 2018 [44] Establish the intuitionistic fuzzification of the concept of P-Ideals and H-Ideals In BCI-Algebras and investigate some of their properties In [9], [10], [11], [13], [14] and [12] Kaviyarasu et. al published an article using the fuzzy concept set in INK-algebra and later in solve they neutrosophic set in INKalgebra. In this paper we have introduced a neutrosophic h-ideal of INK-algebra. We are also examining the relationship between neutrosophic INK- sub algebra and neutrosophic h-ideal, T-ideal, p-ideal and its conditions.

The chapter is organized as follows: In section 2 is devoted to basic definition of BCI.BCK, INK-Algebras, fuzzy h-ideal, fuzzy p-ideal and intuitionistic fuzzy h-ideal of INK-Algebra. 3, presents a neutrosophic h-ideal, T-ideal and p-ideal of INK-algebra and investigates their properties. Finally, conclusions are contained.

II. PRELIMINARY

Definition 1[9] An algebra (X,*, 0) is called a INK-algebra if you meet the ensuing conditions for every x, y, z ∈X.

1. ((x*y)*(x*z))*(z*y) = 0

2. ((x*z)*(y*z))*(x*y) = 0
3. x * 0 = x
4. x * y = 0 and y*x = 0 imply x = y.

Definition 2[9] Let (X,*, 0) be a INK-algebra. A nonempty subset I of X is called an ideal of X if it satisfies

1. 0 ∈ I
2. x * y ∈I and y∈I imply x∈I for all x, y∈ X.

Any ideal I has the property that y∈I and x ≤ y imply x ∈ I.

Definition 3[44] Let (X,*, 0) be a BCI-algebra. A nonempty subset I of X is called an h-ideal of X if it satisfies

1. 0 ∈ I
2. x * (y*z) ∈I and y∈I imply x∈I for all x, y, z∈ X.

Definition 4 [45] Let (X,*, 0) be a TM-algebra. A nonempty subset I of X is called an T-ideal of X if it satisfies

1. 0 ∈ I
2. (x * y)*z ∈I and y∈I imply x∈I for all x, y, z∈ X.

Definition 5 [9] Let (X,*, 0) be a INK-algebra. A nonempty subset I of X is called an p-ideal of X if it satisfies

1. 0 ∈ I
2. (x * z)*(y*z) ∈I and y∈I imply x∈I for all x, y, z∈ X.

Definition 6 [9] Let I be a non-empty subset S algebra of a INK-algebra X.

1. 0 ∈ I
2. x*y ∈I and y∈I imply x∈I for all x, y, z ∈ X.

Definition 7 [7] Let X be a non-empty set. A Fuzzy set can be defined as an object of the form μ = {(X, μ (x)) : x∈ X }, where the function ϑ: X → [0, 1] is the degree of membership.

Definition 8[7] A fuzzy set μ in a BCK-algebra X is called fuzzy sub algebra of X if

μ (x*y) ≥ min{μ(x), μ(y)}, for all x, y ∈ X.

Definition 9 [22] Let X be a BCK-algebra. A fuzzy subset μ in X is called a fuzzy ideal of X if it satisfies the following conditions:

1. μ(0) ≥ μ(x)
2. μ(x) ≥ min{μ(x*y)}, for all x, y ∈ X.

Definition 10 [22] Let μ be a fuzzy set in BCI-algebra X. μ is called a fuzzy h-ideal if it satisfies:

1. 0 ∈ I
2. μ(x) ≥ min {μ((x ∗ (y ∗ z)), μ(y)}, ∀ x, y, z ∈ X.

Definition 11 [25] Let μ be a fuzzy set in BCI-algebra X. μ is called a fuzzy T-ideal if it satisfies:

1. 0 ∈ I
2. μ(x) ≥ min {μ((x ∗ y) ∗ z)), μ(y)}, ∀ x, y, z ∈ X.

Definition 12 [9] Let μ be a fuzzy set in BCI-algebra X. μ is called a fuzzy p-ideal if it satisfies:

1. 0 ∈ I
2. μ(x) ≥ min {μ((x ∗ z) *(y∗ z)), μ(y)}, ∀ x, y, z ∈ X.

Definition 13 [1] An intuitionistic fuzzy set A in a non-empty set X is an object having a form A = {X, μ_A (x), ν_A (x): x ∈ X}, where the function μ_A: X → [0, 1] and ν_A: X → [0, 1] denote the degree of membership and the degree of non-membership of each element x ∈ X to set A respectively, and 0 ≤ μ_A (x)+ ν_A (x) ≤ 1, for all x ∈ X. For the sake of simplicity, symbol A = (X, μ_A, ν_A) is used for the IFs A = {X, μ_A (x), ν_A (x): x ∈ X}.

Definition 14 [1] An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy sub X if

1. $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$
2. $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in X$.

Definition 15 [1] An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy ideal of X if it satisfies

for all $x, y \in X$,

1. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
2. $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$
3. $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\}$

Definition 16 [44] An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy h- ideal of X if it satisfies for all $x, y, z \in X$,

1. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
2. $\mu_A(x) \geq \min\{\mu_A(x * (y * z), \mu_A(y)\}$
3. $\nu_A(x) \leq \max\{\nu_A(x * (y * z), \nu_A(y)\}$.

Definition 17 [25] An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy T- ideal of X if it satisfies for all $x, y, z \in X$,

1. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
2. $\mu_A(x) \geq \min\{\mu_A(x * y) * z), \mu_A(y)\}$
3. $\nu_A(x) \leq \max\{\nu_A(x * y) * z), \nu_A(y)\}$.

Definition 18[44] An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy p- ideal of X if it satisfies for all $x, y, z \in X$,

1. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
2. $\mu_A(x) \geq \min\{\mu_A(x * z) * (y * z), \mu_A(y)\}$
3. $\nu_A(x) \leq \max\{\nu_A(x * z) * (y * z), \nu_A(y)\}$.

III. NEUTROSOPHIC h-IDEAL

Definition 19 A neutrosophic set μ in a nonempty set X is a structure of the form $\mu = \{X, \mu_T(x), \mu_I(x), \mu_F(x) | x \in X\}$, Where μ_T: X→ [0,1] is a truth membership function, μ_I: X→ [0,1] is a indeterminacy membership function and μ_F: X→ [0,1] is a false membership function.

Definition 20 A neutrosophic set μ in X is called a neutrosophic INK-subalgebra of X if it satisfies the following condition, for all x, y, z ∈ X.

1. $\mu_T(x * y) \geq min\{\mu_T(x), \mu_T(y)\}$
2. $\mu_I(x * y) \geq min\{\mu_I(x), \mu_I(y)\}$
3. $\mu_F(x * y) \leq max\{\mu_F(x), \mu_F(y)\}$.

Definition 21 A neutrosophic set μ in X is called a neutrosophic ideal of X if it satisfies the following condition, for all $x, y \in X$

1. $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0)\} \geq \mu_I(x)$, and $\mu_F(0) \leq \mu_F(x)$
2. $\mu_T(x) \geq min\{\mu_T(x * y), \mu_T(y)\}$
3. $\mu_I(x) \geq min\{\mu_I(x * y), \mu_I(y)\}$
4. $\mu_F(x) \leq max\{\mu_F(x * y), \mu_F(y)\}$

Definition 22 A neutrosophic set μ in X is called a neutrosophic INK-algebra of p-ideal X if it satisfies the following condition, for all $x, y, z \in X$

1. $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0)\} \geq \mu_I(x)$, and $\mu_F(0) \leq \mu_F(x)$
2. $\mu_T(x) \geq \min\{\ \mu_T(x * z) * (y * z)),\ \mu_T(y)\}$
3. $\mu_I(x) \geq \min\ \{\mu_I((x * z) * (y * z)),\ \mu_I(y)\}$
4. $\mu_F(x) \leq \max\{\ \mu_F(x * z) * (y * z)),\ \mu_F(y)\}$

Definition 23 A neutrosophic set μ in X is called a neutrosophic INK-algebra of T-ideal X if it satisfies the following condition, for all x, y,z $\in X$

1. $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0)\} \geq \mu_I(x)$, and $\mu_F(0) \leq \mu_F(x)$
2. $\mu_T(x) \geq \min\{ \mu_T(x * y) * z)), \mu_T(y)\}$
3. $\mu_I(x) \geq \min \{\mu_I((x* y) * z), \mu_I(y)\}$
4. $\mu_F(x) \leq \max\{ \mu_F(x* y) * z)), \mu_F(y)\}$

Definition 24 A neutrosophic set μ *in* X is called a neutrosophic INK-algebra of h-ideal X if it satisfies the following condition, for all x, y,z $\in X$

1. $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0)\} \geq \mu_I(x)$, and $\mu_F(0) \leq \mu_F(x)$
2. $\mu_T(x) \geq \min\{ \mu_T(x * (y * z)), \mu_T(y)\}$
3. $\mu_I(x) \geq \min \{\mu_I((x* (y * z), \mu_I(y)\}$
4. $\mu_F(x) \leq \max\{ \mu_F(x* (y * z)), \mu_F(y)\}$

Example 25 Consider a set X= {0, 1, a, b}with the binary operation * which is given in Table 1. Then

*	0	1	a	b
0	0	0	a	a
1	1	0	a	a
a	a	a	0	0
b	b	a	1	0

(X,*, 0) is a INK-algebra. Let μ be a neutrosophic set in X defined by Table 2. It is routine to verify that M is a neutrosophic h-ideal of X.

X	0	1	a	b
$\mu_T(x)$	0.8	0.5	0.4	0.4
$\mu_I(x)$	0.3	0.5	0.6	0.8
$\mu_F(x)$	0.2	0.3	0.4	0.7

Theorem 26 Every neutrosophic h-ideal is a neutrosophic ideal in INK-Algebra.

Proof. For all x, y, z ϵ X

We have $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0) \geq \mu_I(x)$ and $\mu_F(0) \leq \mu_F(x)$

$$\mu_T(x) \geq \min\{\mu_T(x * (y * z)), \mu_T(y)\}$$

Put z = 0

$$\geq \min \{\mu_T(x * (y * 0)), \mu_T(y)\}$$

$$\mu_T(x) \geq \min\{ \mu_T(x * y), \mu_T(y)\},$$

$$\mu_I(x) \geq \min \{\mu_T(x * (y * z)), \mu_T(y)\}$$

Put z = 0

$$\mu_I(x) \geq \min \{ \mu_I(x * (y * 0)), \mu_I(y)\}$$

$$\geq \min \{ \mu_I(x * y), \mu_I(y)\}$$

and

$$\mu_F(x) \leq \max\{ \mu_F(x * (y * z)), \mu_F(y)\}$$

Put z = 0

$$\mu_F(x) \leq \max\{ \mu_F(x * (y * 0)), \mu_F(y)\}$$

$$\leq \max\{ \mu_F(x * y), \mu_F(y)\}$$

Theorem 27 Every neutrosophic h-ideal is a neutrosophic T-ideal in INK-Algebra.

Proof. We have $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0) \geq \mu_I(x)$ and $\mu_F(0) \leq \mu_F(x)$

$$\mu_T(x) \geq \min\{ \mu_T(x*(y*z)), \mu_T(y)\}$$

$$\geq \min\{ \mu_T((x*0)*(y*z)), \mu_T(y)\}$$
$$\mu_T(x*0) \geq \min\{ \mu_T((x*(z*z))*(y*z), \mu_T(y)\}$$

Put 0 = z

$$\mu_T(x*z) \geq \min\{ \mu_T(x*(z*0)*(y*0)), \mu_T(y)\}$$
$$\geq \min\{ \mu_T(x*z)*y)), \mu_T(y)\}$$
$$\geq \min\{ \mu_T(x*(y*z)), \mu_T(y)\}$$
$$\mu_T(x*z) \geq \min\{\mu_T(x*y)*z), \mu_T(y)\},$$

$$\mu_I(x) \geq \min\{\mu_I(x*0)*(y*z)), \mu_I(y)\}$$
$$\mu_I(x*0) \geq \min\{\mu_I(x*(z*z)*(y*z)), \mu_I(y)\}$$

Put 0 = z

$$\mu_I(x*z) \geq \min\{\mu_I(x*(z*0)*(y*0)), \mu_I(y)\}$$
$$\geq \min\{\mu_I(x*z)*y)), \mu_I(y)\}$$
$$\geq \min\{\mu_I(x*(y*z)), \mu_I(y)\}$$
$$\mu_I(x*z) \geq \min\{\mu_I((x*y)*z)), \mu_I(y)\}$$

and

$$\mu_F(x) \leq \max\{ \mu_F(x*(y*z)), \mu_F(y)\}$$
$$\mu_F(x*0) \leq \max\{ \mu_F(x*(z*z)*(y*z)), \mu_F(y)\}$$

Put 0 = z

$$\mu_F(x*z) \leq \max\{ \mu_F(x*(z*0)*(y*0)), \mu_F(y)\}$$
$$\leq \max\{ \mu_F((x*z)*y), \mu_F(y)\}$$
$$\leq \max\{ \mu_F(x*(y*z)), \mu_F(y)\}$$
$$\mu_F(x*z) \leq \max\{ \mu_F((x*y)*z)), \mu_F(y)\}$$

Theorem 28 Every neutrosophic h-ideal is a neutrosophic p-ideal in INK-Algebra.

Proof. We have $\mu_T(0) \geq \mu_T(x)$, $\mu_I(0) \geq \mu_I(x)$ and $\mu_F(0) \leq \mu_F(x)$

$$\mu_T(x) \geq \min\{ \mu_T(x*(y*z)), \mu_T(y)\}$$
$$\geq \min\{ \mu_T((x*z)*y)), \mu_T(y)\}$$
$$\geq \min\{ \mu_T((x*z)*(y*0)), \mu_T(y)\}$$
$$\mu_T(x) \geq \min\{ \mu_T((x*z)*(y*z)), \mu_T(y)\}$$
$$\mu_I(x) \geq \min\{\mu_I(x*(y*z)), \mu_I(y)\}$$
$$\mu_I(x) \geq \min\{\mu_I((x*z)*y), \mu_I(y)\}$$
$$\mu_I(x) \geq \min\{\mu_I((x*z)*(y*0)), \mu_I(y)\}$$
$$\geq \min\{\mu_I((x*z)*(y*z)), \mu_I(y)\}$$
$$\mu_F(x) \leq \max\{ \mu_F((x*(y*z)), \mu_F(y)\}$$
$$\leq \max\{ \mu_F((x*z)*(y*0)), \mu_F(y)\}$$
$$\mu_F(x) \leq \max\{ \mu_F((x*z)*(y*z)), \mu_F(y)\}$$

Theorem 29 If μ is a neutrosophic h-ideal of INK-algebra X, Then μ^m is a neutrosophic h-ideal of INK-Algebra of X.

Proof. We have $\mu_T(0) \geq \mu_T(x)$

$$\{ \mu_T(0)\}^m \geq \{\mu_T(x)\}^m$$
$$\mu_T(0)^m \geq \mu_T(x)^m$$
$$\mu_T{}^m(0) \geq \mu_T{}^m(x)$$
$$\mu_I(0) \geq \mu_I(x)$$
$$\{\mu_I(0)\}^m \geq \{\mu_I(x)\}^m$$
$$\mu_I(0)^m \geq \mu_I(x)^m$$
$$\mu_I{}^m(0) \geq \mu_I{}^m(x)$$
$$\mu_F(0) \leq \mu_F(x)$$
$$\{\mu_F(0)\}^m \leq \{ \mu_F(x)\}^m$$
$$\mu_F(0)^m \leq \mu_F(x)^m$$
$$\mu_F{}^m(0) \leq \mu_F{}^m(x)$$

and

$$\mu_T(x) \geq \min\{ \mu_T(x*(y*z)), \mu_T(y)\}$$

$$\{\mu_T(x)\}^m \geq \min\{\mu_T(x*(y*z)),\ \mu_T(y)\}^m$$
$$\mu_T(x)^m \geq \min\{\mu_T(x*(y*z))^m,\ \mu_T(y)^m\}$$
$$\mu_T{}^m(x) \geq \min\{\mu_T{}^m(x*(y*z)),\ \mu_T{}^m(y)\}$$
$$\mu_I(x) \geq \min\{\mu_I((x*(y*z)),\ \mu_I(y)\}$$
$$\{\mu_I(x)\}^m \geq \min\{\mu_I((x*(y*z),\ \mu_I(y)\}^m$$
$$\mu_I(x)^m \geq \min\{\mu_I((x*(y*z))^m,\ \mu_I(y)^m\}$$
$$\mu_I{}^m(x) \geq \min\{\mu_I{}^m((x*((y*z)),\ \mu_I{}^m(y)\}$$
$$\mu_F(x) \leq \max\{\mu_F(x*(y*z)),\ \mu_F(y)\}$$
$$\{\mu_F(x)\}^m \leq \max\{\mu_F(x*(y*z)),\ \mu_F(y)\}^m$$
$$\mu_F(x)^m \leq \max\{\mu_F(x*(y*z))^m,\ \mu_F(y)^m\}$$
$$\mu_F{}^m(x) \leq \max\{\mu_F{}^m(x*(y*z)),\ \mu_F{}^m(y)\}.$$

Theorem 30 Let μ_1 and μ_2 are two neutrosophic h-ideal of INK-Algebra, Then $\mu_1 \cap \mu_2$ is a neutrosophic h-ideal of INK-Algebra.

Proof. Since $\mu_{T1}(0) \geq \mu_{T1}(x)$ and $\mu_{T2}(0) \geq \mu_{T2}(x)$, for all x in X.
We get,

$$\min\{\mu_{T1}(0),\mu_{T2}(0)\} \geq \min\{\mu_{T1}(x),\mu_{T2}(x)\}$$
$$\mu_{T1}\cap_{T2}(0) \geq \mu_{T1}\cap_{T2}(x)$$
$$\mu_{T1}(x) \geq \min\{\mu_{T1}(x*(y*z)),\mu_{T1}(x)\}$$
$$\mu_{T2}(x) \geq \min\{\mu_{T2}(x*(y*z)),\mu_{T2}(x)\}$$
$$\min\{\mu_{T1}(x),\mu_{T2}(x)\} \geq \{\min\{\mu_{T1}(x*(y*z)),\mu_{T1}(y)\},\min\{\mu_{T2}(x*(y*z)),\mu_{T2}(y)\}\}$$
$$\mu_{T1}\cap_{T2}(x) \geq \{\min\{\mu_{T1}(x*(y*z)),\mu_{T2}(x*(y*z))\},\min\{\mu_{T1}(y),\mu_{T2}(y)\}\}$$
$$\mu_{T1}\cap_{T2}(x) \geq \min\{\mu_{T1}\cap_{T2}(x*(y*z)),\mu_{T1}\cap_{T2}(y)\},$$

Since $\mu_{I1}(0) \geq \mu_{I1}(x)$ and $\mu_{I2}(0) \geq \mu_{I2}(x)$,
We get,

$$\min\{\mu_{I1}(0),\mu_{I2}(0)\} \geq \min\{\mu_{I1}(x),\mu_{I2}(x)\}$$
$$\mu_{I1}\cap_{I2}(0) \geq \mu_{I1}\cap_{I2}(x)$$
$$\mu_{I1}(x) \geq \min\{\mu_{I1}(x*(y*z)),\mu_{I1}(x)\}$$
$$\mu_{I2}(x) \geq \min\{\mu_{I2}(x*(y*z)),\mu_{I2}(x)\}$$
$$\min\{\mu_{I1}(x),\mu_{I2}(x)\} \geq \{\min\{\mu_{I1}(x*(y*z)),\mu_{I1}(y)\},\min\{\mu_{I2}(x*(y*z)),\mu_{I2}(y)\}\}$$
$$\mu_{I1}\cap_{I2}(x) \geq \{\min\{\mu_{I1}(x*(y*z)),\mu_{I2}(x*(y*z))\},\min\{\mu_{I1}(y),\mu_{I2}(y)\}\}$$
$$\mu_{I1}\cap_{I2}(x) \geq \min\{\mu_{I1}\cap_{I2}(x*(y*z)),\mu_{I1}\cap_{I2}(y)\}.$$

Since $\mu_{F1}(0) \geq \mu_{F1}(x)$ and $\mu_{F2}(0) \geq \mu_{F2}(x)$,
We get,

$$\max\{\mu_{F1}(0),\mu_{F2}(0)\} \leq \max\{\max\{\mu_{F1}(x),\mu_{F2}(x)\}\}$$
$$\mu_{F1}\cap_{F2}(0) \leq \max\{\mu_{F1}\cap_{F2}(x)\}$$
$$\mu_{F1}(x) \leq \max\{\mu_{F1}(x*(y*z)),\mu_{F1}(x)\}$$
$$\mu_{F2}(x) \leq \max\{\mu_{F2}(x*(y*z)),\mu_{F2}(x)\}$$
$$\max\{\mu_{F1}(x),\mu_{F2}(x)\} \leq \{\max\{\mu_{F1}(x*(y*z)),\mu_{F1}(y)\},\max\{\mu_{F1}(x*(y*z)),\mu_{F2}(y)\}\}$$
$$\mu_{F1}\cap_{F2}(x) \leq \{\max\{\mu_{F1}(x*(y*z)),\mu_{F2}(x*(y*z))\},\max\{\mu_{F1}(y),\mu_{F2}(y)\}\}$$
$$\mu_{F1}\cap_{F2}(x) \leq \max\{\mu_{F1}\cap_{F2}(x*(y*z)),\mu_{F1}\cap_{F2}(Y)\}$$

Theorem 20 Let μ_1 and μ_2 are two neutrosophic h-ideal of INK-Algebra, Then $\mu_1 \cup \mu_2$ is a neutrosophic h-ideal of INK-Algebra.

Proof. Since $\mu_{T1}(0) \geq \mu_{T1}(x)$ and $\mu_{T2}(0) \geq \mu_{T2}(x)$,
We get,

$$\min\{\mu_{T1}(0),\mu_{T2}(0)\} \geq \min\{\mu_{T1}(x),\mu_{T2}(x)\}$$
$$\mu_{T1}\cup_{T2}(0) \geq \mu_{T1}\cup_{T2}(x)$$
$$\mu_{T1}(x) \geq \min\{\mu_{T1}(x*(y*z)),\mu_{T1}(y)\}$$
$$\mu_{T2}(x) \geq \min\{\mu_{T2}(x*(y*z)),\mu_{T2}(y)\}$$
$$\min\{\mu_{T1}(x),\mu_{T2}(x)\} \geq \max\{\min\{\mu_{T1}(x*(y*z)),\mu_{T1}(y)\},\min\{\mu_{T2}(x*(y*z)),\mu_{T2}(y)\}\}$$
$$\mu_{T1}\cup_{T2}(x) \geq \min\{\min\{\mu_{T1}(x*(y*z)),\mu_{T2}(x*(y*z))\},\min\{\mu_{T1}(y),\mu_{T2}(y)\}\}$$
$$\mu_{T1}\cup_{T2}(x) \geq \min\{\mu_{T1}\cup_{T2}(x*(y*z)),\mu_{T1}\cup_{T2}(y)\}$$

since $\mu_{I1}(0) \geq \mu_{I1}(x)$ and $\mu_{I2}(0) \geq \mu_{I2}(x)$, We get,

$$\min\{\mu_{I1}(0),\mu_{I2}(0)\} \geq \min\{\mu_{I1}(x),\mu_{I2}(x)\}$$

$$\mu_{I1}\cup_{I2}(0) \geq \mu_{I1}\cup_{I2}(x)$$

$$\mu_{I1}(x) \geq \min\{\mu_{I1}(x * (y * z)),\mu_{I1}(y)\}$$

$$\mu_{I2}(x) \geq \min\{\mu_{I2}(x * (y * z)),\mu_{I2}(y)\}$$

$$\min\{\mu_{I1}(x),\mu_{I2}(x)\} \geq \min\{ \min\{\mu_{I1}(x * (y * z)),\mu_{I1}(y)\}, \min\{\mu_{I2}(x * (y * z)),\mu_{I2}(y)\}\}$$

$$\mu_{I1}\cup_{I2}(x) \geq \min\{ \min\{\mu_{I1}(x * (y * z)), \mu_{I2}(x * (y * z))\}, \min\{\mu_{I1}(y),\mu_{I2}(y)\}\}$$

$$\mu_{I1}\cup_{I2}(x) \geq \min\{ \mu_{I1}\cup_{I2}(x * (y * z)), \mu_{I1}\cup_{I2}(y)\}$$

Since $\mu_{F1}(0) \leq \mu_{F1}(x)$ and $\mu_{F2}(0) \leq \mu_{F2}(x)$,

We get,

$$\max\{\mu_{F1}(0),\mu_{F2}(0)\} \leq \max\{\mu_{T1}(x),\mu_{T2}(x)\}$$

$$\mu_{F1}\cup_{F2}(0) \leq \mu_{F1}\cup_{F2}(x)$$

$$\mu_{F1}(x) \leq \max\{\mu_{F1}(x * (y * z)),\mu_{F1}(y)\}$$

$$\mu_{F2}(x) \geq \max\{\mu_{F2}(x * (y * z)),\mu_{F2}(y)\}$$

$$\max\{\mu_{F1}(x),\mu_{F2}(x)\} \leq \max\{ \max\{\mu_{F1}(x * (y * z)),\mu_{F1}(y)\}, \max\{\mu_{F2}(x * (y * z)),\mu_{F2}(y)\}\}$$

$$\mu_{F1}\cup_{F2}(x) \leq \max\{ \max\{\mu_{F1}(x * (y * z)), \mu_{F2}(x * (y * z))\}, \max\{ \mu_{F1}(y),\mu_{F2}(y)\}\}.$$

$$\mu_{F1}\cup_{F2}(x) \leq \max\{ \mu_{F1}\cup_{F2}(x * (y * z)), \mu_{F1}\cup_{F2}(y)\}$$

SUMMARY

The chapter is organized as follows: In section 2, introduces some concepts and basic operations are reviewed. In section 3, presents a neutrosophic INK-sub algebra, neutrosophic h-ideal, neutrosophic T-ideal and neutrosophic p-ideal of INK-algebra and investigates their properties. Finally, conclusions are contained.

CONCLUSION

In this paper, we have introduced the notion of a neutrosophic h-ideal in INK-algebras, and investigated several properties. We have considered relations between a neutrosophic h-ideal and T-ideal, p-ideal and neutrosophic sub algebra. We have discussed characterizations of a neutrosophic h-ideal. Finaly we discussed some characterization of neutrosophic set in INK-algebra and union and intersection of neutrosophic h-ideal. These concepts are illustrated through example.

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Chapter Six

Neutro Ordered $R - module$

Necati Olgun, Ahmed Hatip
Department of Mathematics, Gaziantep University, Gaziantep27310-Turkey
E-mail: olgun@gantep.edu.tr, kollnaar5@gmail.com

ABSTRACT

Modules are one of fundamental and rich algebraic structure with respect to some binary operation in the study of algebra. The objective of this paper is to introduce the concept of Neutro $R - module$ and NeutroOrdered $R - module$. Several interesting results and examples on Neutro $R - module$, NeutroOrdered $R - module$,NeutroOrdered Sub $R - module$, Neutro $R - module$ Homomorphisms, the kernel, the image of Neutro $R - Module$ Homomorphism and NeutroOrdered $R - module$ Homomorphisms are presented.

Keywords: Neutro-Group, Neutro-Ring, Neutro $R - module$, NeutroOrdered $R - module$, NeutroOrdered Sub $R - module$, Neutro $R - module$ Homomorphism and NeutroOrdered $R - module$ Homomorphisms.

INTRODUCTION

Vagueness or uncertainty is a critical issue in the representation of incomplete knowledge in the fields of Computer Science and artificial intelligence. To deal with the uncertainty, the fuzzy set introduced by Zadeh [20] allows the uncertainty of a set with a membership degree between 0 and 1. Then, Atanassov [1] introduced an intuitionistic Fuzzy set (IFS) as a generalization of the Fuzzy set. The IFS represents the uncertainty with respect to both membership and non-membership. However, it can only handle incomplete information but not the indeterminate and inconsistent information which exists commonly in real situations. Therefore, Smarandache [12] proposed a neutrosophic set. It can independently express truth-membership degree T, indeterminacy-membership degree I, and false membership degree F and deal with incomplete, indeterminate, and inconsistent information. The indeterminate element I is such that ordinary multiplication $.I = I^2 = I$, I^{-1} the inverse of I is not defined and hence does not exist. Moreover $I + I + \cdots I = nI{:}\, n \in N$.Also, several generalizations of the set theories made such as fuzzy multi-set theory [15, 16], intuitionistic fuzzy multi-set theory [10, 11] and refined neutrosophic set theory [3, 4, 6, 8, 13, 18, 27, 28, 39- 41]. Many research treating imprecision and uncertainty have been developed and studied. Since then, it is applied to various areas, such as decision-making problems [2, 5, 7, 9, 14, 17, 19, 26, 29, 55-90] machine learning [30, 31], intelligent disease diagnosis [32, 33] communication services [34] pattern recognition [35] social network analysis and e-learning systems [36] physics [37, 38], … etc.

Smarandache [22] recently introduced new fields of research in neutrosophy called Neutro-Structures and Anti-Structures respectively. In,[23] Smarandache introduced the concepts of Neutro-Algebras and Anti-Algebras and in,[21] he revisited the concept of Neutro-Algebras and Anti-Algebras where he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that Neutro-Algebras are generalization of Partial Algebras. Şahin M et al. studied neutro-R module [44-54]; Agboola [21] introduced the concept of Neutro-Group. Inspired by NeutroAlgebra and ordered Algebra [43] Introduced NeutroOrdered Algebra and some related terms such as NeutroOrdered Sub Algebra and NeutroOrdered Homomorphism.

In continuation of this work the present research is devoted to the presentation of the concept of Neutro $R-module$ and NeutroOrdered $R-module$.Several interesting results and examples on Neutro $R-modules$, Neutro-Sub $R-modules$, NeutroOrdered $R-module$ and NeutroOrdered $R-module$ Homomorphisms are presented.

BACKGROUND

In this section, we will give some definitions, examples and results that will be useful in other sections of the research.

2.1. Neutrosophic Sets [12]

Let $\mathcal{U}$ be a universe. $\mathcal{A}$ neutrosophic sets $\mathcal{A}$ over $\mathcal{U}$ is defined by

$$\mathcal{A} = \{\prec u, (T_{\mathcal{A}}(u), I_{\mathcal{A}}(u), F_{\mathcal{A}}(u)) \succ : u \in \mathcal{U}\}$$

where, $T_{\mathcal{A}}(u)$, $I_{\mathcal{A}}(u)$ and $F_{\mathcal{A}}(u)$ are called truth-membership function, indeterminacy-membership function and falsity- membership function, respectively. They are respectively defined by

$$T_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[\,, \qquad I_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[\,, \qquad F_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

such that $0^{-} \le T_{\mathcal{A}}(u)+I_{\mathcal{A}}(u)+F_{\mathcal{A}}(u) \le 3^{+}$.

2.2. Single Valued Neutrosophic Set [18]

Let $\mathcal{U}$ be a universe. A single valued neutrosophic set (SVN-set) over $\mathcal{U}$ is a neutrosophic set over $\mathcal{U}$, but the truth-membership function T, indeterminacy-membership function I and falsity- membership function F are respectively defined by

$$T_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[\,, \qquad I_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[\,, \qquad F_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

Such that $0 \le T_{\mathcal{A}}(u)+I_{\mathcal{A}}(u)+F_{\mathcal{A}}(u) \le 3$.

2.3. Neutro-Axiom, Anti-Axiom [21]

i- A classical axiom defined on a nonempty set is an axiom that is totally true (i.e., true for all set's elements).

ii- A Neutro-Axiom (or Neutrosophic Axiom) defined on a nonempty set is an axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements. [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the classical axiom, and $(T, I, F) \neq (0,0,1)$ that represents the Anti-Axiom.

iii- An Anti-Axiom defined on a nonempty set is an axiom that is false for all set's elements. Therefore, we have the neutrosophic triplet:

$$< Axiom, Neutro - Axiom, Anti - Axiom >.$$

2.4. Neutro-Group [22]

Let G be a nonempty set and let $*: G \times G \to G$ be a binary operation on G. The couple $(G,*)$ is called a Neutro-Group if the following conditions are satisfied:

I- $*$ is Neutro-Associative that is there exists at least one triplet $(a, b, c) \in G$ such that

$a * (b * c) = (a * b) * c$ [degree of truth (T)], one triplet $(d, e, f) \in G$ such that

$d * (e * f)$ or $(d * e) * f$ are indeterminate[degree of indeterminacy (I)] and there exists at least one triplet $(x, y, z) \in G$ such that $x * (y * z) \neq (x * y) * z$ [degree of falsehood (F)]. with $(T, I, F) \neq (1,0,0)$ that represents the classical axiom, and $(T, I, F) \neq (0,0,1)$ that represents the Anti-Axiom.

II- There exists a Neutro-Neutral element in G that is there exists at least an element $a \in G$ that has a single neutral element that is we have $e \in G$ such that $a * e = e * a = a$[degree of truth (T)], for $c \in G$ that has a single neutral element that is we have $e \in G$ such that $c * e$ or $c * a$ are indeterminate[degree of indeterminacy (I)] and for $b \in G$ there does not exist $e \in G$ such that $b * e = e * b = b$ or there exist $e_1, e_2 \in G$ such that $b * e_1 = e_1, * b = b$ or $b * e_2 = e_2 * b = b$ with $e_1 \neq e_2$ [degree of falsehood (F)].

III- There exists a Neutro-Inverse element that is there exists at least one element $a \in G$ that has an inverse $b \in G$ with respect to a unit element $e \in G$ that is $a * b = b * a = e$ [degree of truth (T)], there exists at least one element $c \in G$ that is $a * c$ or $b * a$ are indeterminate[degree of indeterminacy (I)] and that has two or more inverses $c, d \in G$ with respect to some unit element $u \in G$ that is $b * c = c * b = u$, $b * d = d * b = u$ [degree of falsehood (F)].

In addition, if $*$ is Neutro-Commutative that is there exists at least a duplet $(a, b) \in G$ such that $a * b = b * a$, there exists at least a duplet $(x, y) \in G$ such that $x * y$ or $y * x$ are indeterminate and there exists at least a duplet $(c, d) \in G$ such that $c * d \neq d * c$, then $(G,*)$ is called a Neutro-Commutative Group or a Neutro-Abelian Group.

If only condition I is satisfied, then $(G,*)$ is called a Neutro-Semi Group and if only conditions I and II are satisfied, then $(G,*)$ is called a Neutro-Monoid.

2.5. Neutro-Ring [25]

(a) A Neutro-Ring $(R, +, .)$ is a ring structure that has at least one Neutro-Operation among

" $+$ " and or " $\cdot$ " at least one Neutro-Axiom.

(b) Let R be a nonempty set and let $+, .: R \times R \to R$ be binary operations of ordinary addition and multiplication on R. The triple $(R, +, .)$ is called a Neutro-Ring if the following conditions are satisfied:

I- $(R,+)$ is a Neutro-Abelian Group.

II- $(R,.)$ is a Neutro-Semi Group.

III- " $\cdot$ " is both left and right Neutro-Distributive over " $+$ " that is there exists at least a triplet $(a,b,c) \in R$ and at least a triplet $(d,e,f) \in R$ such that

$a.(b + c) = a.b + a.c$

$(b + c).a = b.a + c.a$

$d.(e + f) \neq d.e + d.f$ $(e + f).d \neq e.d + f.d.$

If" $\cdot$ " is Neutro-Commutative, then $(R,+,.)$ is called a Neutro-Commutative Ring.

2.6. Neutro-R module [45]

Let $(G,\#)$ be an abelian neutro-group, $(R,+_1,._1)$ a commutative neutro-ring and let $*: RxG \to R$ be a binary operation. If at least one of the following conditions $\{i, ii, iii, iv, v\}$ is satisfied, then $(G,\#)$ is called a neutro-R module.

i) There exists a double $(b,n) \in (R,G)$ such that $b * n \in G$ (degree of truth T) and there exist two doubles

(u,v) and $(p,q) \in (R,G)$ such that [$p * q \notin R$ (degree of falsehood F) or $u * v \in^U V$ (indeterminacy (I))]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

ii) There exists a triplet $(b,n,m) \in (R,G,G)$ such that

$b * (n \# m) =^U (b * n)\#(b * m)$ (degree of truth T) and there exist two triplets (p,q,r) and $(u,v,w) \in (R,G,G)$ such that [$p * (q \# r) =^U (p * q)\#(p * r)$ (degree of indeterminacy I) or [$u * (v \# w) \neq (u * v) \# (u * w)$ (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iii) There exists a triplet $(b,n,m) \in (R,G,G)$ such that $(b +_1 n) * m = (b * m) +_1 (n * m)$ (degree of truth T) and there exist two triplets (p,q,r) and $(u,v,w) \in (R,R,G)$ such that

[$(p +_1 q) * r =^U (p * r) +_1 (q * r)$ (degree of indeterminacy I) or [$(u +_1 v) * w \neq (u * w) +_1 (v * w)$ (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

iv) There exists a triplet $(b,n,m) \in (R,G,G)$ such that

$*(n ._1 m) = (b * n) ._1$ m (degree of truth T) and there exist two triplets (p,q,r) and $(u,v,w) \in (R,R,G)$ such that [$p * (q ._1 r) =^U (p * q) ._1 r$ (degree of indeterminacy I) or $u * (v ._1 w) \neq (u * v) ._1 w$ (degree of falsehood F)]; where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

v) For a double (a,e) (R,G), there exists an $e \in G$ such that $a * e = a$ (degree of truth T) and (for two doubles $(b,e),(c,e) \in (R,G)$, there exists $e \in G$ such that $b * e \neq b$ (degree of falsehood F) or $c * e =^U c$ (degree of indeterminacy I)); where (T, I, F) is different from (1, 0, 0) and (0, 0, 1).

2.7. Ordered Algebra [42]

Let A be an Algebra with n operations "$*_i$" and "$\leq$" be a partial order relation (reflexive, anti-symmetric, and transitive) on A. Then $(A, *_1, *_2, \dots, *_n, \leq)$ is an Ordered Algebra if the following conditions hold.

If $x \le y \in A$ then $z *_i x \le z *_i y$ *and* $x *_i z \le y *_i z$ for all $i = 1, \ldots, n$ and $z \in A$.

2.8. Neutro Ordered Algebra [43]

Let A be a Neutro Algebra with n (Neutro) operations "i" and " $\le$ " be a partial order (reflexive, anti-symmetric, and transitive) on A. Then $(A, *_1, *_2, \ldots, *_n, \le)$ is a NeutroOrdered Algebra if the following conditions hold.

(1) There exist $x \le y \in A$ with $x \ne y$ such that $z *_i x \le z *_i y$ *and* $x *_i z \le y *_i z$ for all $i = 1, \ldots, n$ and $z \in A$ (This condition is called degree of truth, "T").

(2) There exist $x \le y \in A$ and $z \in A$ such that $z *_i x \not\lesssim z *_i y$ *and* $x *_i z \not\lesssim y *_i z$ for some $i = 1, \ldots, n$. (This condition is called degree of falsity, "F".)

(3) There exist $x \le y \in A$ and $z \in A$ such that $z *_i x$ *or* $z *_i y$ *or* $x *_i z$ *or* $y *_i z$ are indeterminate, or the relation between that $z *_i x$ and $z *_i y$, or the relation between $x *_i z$ and $y *_i z$ are indeterminate for some $i = 1, \ldots, n$. (This condition is called degree of indeterminacy, "I".) Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered Algebra as well from $(0, 0, 1)$ that represents the AntiOrderedAlgebra.

2.9. NeutroTotalOrdered Algebra [43]

Let $(A, *_1, *_2, \ldots, *_n, \le)$ be a NeutroOrdered Algebra. If " $\le$ " is a total order on A then A is called NeutroTotalOrdered Algebra.

NeutroOrdered $R - Module$ and their properties

In this section, we use the defined notion of NeutroOrdered Algebra and apply it to NeutroOrdered $R - Module$. As a result, we define NeutroOrdered $R - Module$ and other related concepts. Moreover, we study some properties of NeutroOrdered $R - Module$ and, NeutroOrdered $R - Module$ homomorphism.

3.1. NeutroOrdered $R - Module$

Let R be a Neutro-Ring and let $({}_R M, +)$ be a Neutro abelian group and " $\cdot$ " be a binary operation such that

$\cdot : R \times M \to M$. Then $({}_R M, +, \cdot)$ is called a Neutro left $R - Module$ on Neutro-Ring $(R, +, .)$ if the following conditions are satisfied:

1) " + " is left Neutro-Distributive over " $\cdot$ " that is there exists at least some $r \in R$ *and* $m, n \in {}_R M$ such that $r \cdot (m + n) = r \cdot m + r \cdot n$,there exists at least $q \in R$ *and* $t, v \in {}_R M$ such that $q \cdot (t + v)$ *or* $q \cdot t + q \cdot v$ are indeterminate and there exists at least $s \in R, x, y \in {}_R M$ such that $s \cdot (x + y) \ne s \cdot x + s \cdot y$.
2) " + " is right Neutro-Distributive over " $\cdot$ " that is there exists at least some $r, s \in R$ *and* $m \in {}_R M$ such that $(r + s) \cdot m = r \cdot m + s \cdot m$,there exists at least $x, y \in R$ *and* $z \in {}_R M$ such

that $(x+y)\cdot z$ or $x\cdot z+y\cdot z$ are indeterminate and there exists at least some $t,q\in R$, $n\in {}_RM$ such that
$(t+q)\cdot n\neq t\cdot n+q\cdot n$.

3) " $\cdot$ " is Neutro-Associative that is there exists at least some $r,s\in R$ and $m\in M$ such that $(rs)\cdot m=r\cdot(s\cdot m)$, there exists at least some $x,y\in R$, $z\in {}_RM$ such that $(x\cdot y)\cdot z$ or $x\cdot(y\cdot z)$ are indeterminate and there exists at least some $t,q\in R$, $n\in {}_RM$ such that $(tq)\cdot n\neq t\cdot(q\cdot n)$.
4) There is an element e (Neutro-Neutral element in R) that is there exists at least some $m\epsilon M$ such that
$e\cdot m=m$ there exists at least some $x\in {}_RM$ such that $e\cdot x$ is indeterminate and there exists at least some $n\in {}_RM$ such that $e\cdot n\neq n$.

Similarly, the form $(M_R,+,\cdot)$ is known as Neutro right $R-Module$ over a Neutro-Ring .

Notes:

1- If we have R as a commutative Neutro-Ring, then every Neutro left $R-Module$ is a Neutro right $R-Module$.
2- M is called a finite Neutro $R-Module$ of order n if the number of elements in M is n that is $o(M)=n$. If no such n exists, then M is called an infinite Neutro $R-Module$ and we write $o(M)=\infty$.
3- An element $x\in M$ is called a NeutroIdempotent element if $x^2=x$.
4- An element $x\in M$ is called a NeutroINilpotent element if for the least positive integer n, we have
$x^n=e$ where e is Neutro-Neutral element in M.

3.2. Example: Let R be a commutative Neutro-Ring. A very important example of an Neutro $R-Module$ is R Neutro-Ring itself:

3.3. Example: Let $X=\{m,n,p,q,t\}$ be a universe of discourse and let $M=\{m,n,p\}$ be a subset of . let ■ and $*$ be binary operation defined on M as shown in the Cayely tables below:

■	m	n	p
m	m	n	n or p
n	p or n	m or n	p
p	n	p	n

$*$	m	n	p
m	m	m	m
n	m or n	p	m
p	m	p	n

It is clear from the table that it $(R, \blacksquare, *)$ is a Neutro-Commutative Ring with Neutro-Unity and:

1- $m * (n \blacksquare p) = m * p = m$

$(m * n) \blacksquare (m * p) = m \blacksquare m = m$ [degree of truth (T)],

$p * (n \blacksquare m) = p \text{ or } n$

$(p * n) \blacksquare (p * m) = n \text{ or } m \text{ are indeterminacy}$[degree of indeterminacy (I)]

and $n * (p \blacksquare m) = n * n = p$

$(n * p) \blacksquare (n * m) = m \blacksquare m = m$[degree of falsehood (F)].

This shows that "$\blacksquare$"is both left Neutro-Distributive over "$*$".

2- $(m \blacksquare n) * p = n * p = m$

$(m * p) \blacksquare (n * p) = m \blacksquare m = m$ [degree of truth (T)],

$(n \blacksquare m) * p = n \text{ or } m$

$(n * p) \blacksquare (m * p) = m$[degree of indeterminacy (I)]

and $(p \blacksquare m) * n = n * n = p$

$(p * n) \blacksquare (m * n) = p \blacksquare m = n$ [degree of falsehood (F)].

This shows that "$\blacksquare$"is both right Neutro-Distributive over "$*$".

3- $m * (n * p) = m * m = m$

$(m * n) * p = m * p = m$

$(n * m) * p = m$[degree of truth (T)],

$n * (m * p) = n \text{ or } m$[degree of indeterminacy (I)]

and $p * (n * n) = p * p = n$

$(p * n) * n = p * n = p$ [degree of falsehood (F)].

This shows that "$*$" is a Neutro-Associative.

4- $p * n = p$, $m * n = m$ [degree of truth (T)],

$m * n = m$, $n * m = m \text{ or } n$ [degree of indeterminacy (I)]

$n * n = p \neq n$ [degree of falsehood (F)].

It follows that $(M, \blacksquare, *)$ Neutro $R - Module$ over Neutro-Ring $(R, \blacksquare, *)$.

3.4. Neutro-Sub $R - Module$

Let M be a Neutro $R - Module$. A nonempty subset N of M is called a Neutro-Sub $R - Module$ of M if N is also a Neutro $R - Module$.

3.5. Example: Let M be a Neutro $R-Module$. M is a Neutro-Sub $R-Module$ called a trivial Neutro-Sub $R-Module$.

3.6. Theorem: Let M be a Neutro $R-Module$ over a Neutro-Ring R and let N be a nonempty subset of M.

N is a Neutro-Sub $R-Module$ of M if the following conditions hold:

(1) That is there exists at least some $m, n \in N$ such that $m + n \in N$.
(2) That is there exists at least some $m \in N$, $r \in R$ such that $rm \in N$.

3.7. Corollary: Let M be a Neutro $R-Module$ over a Neutro-Ring R and let N be a nonempty subset of M.

N is a Neutro-Sub $R-Module$ of M if the following conditions hold:

That is there exists at least some $m, n \in N$, $r, s \in R$ such that $rm + sn \in N$ **.**

3.8 Example: Let $(M, \blacksquare, *)$ be a the Neutro $R-Module$ of **3.3. Example** and let $N = \{p, n\}$:

1- $p, n \in N$, $p \blacksquare n = p \in N$ but $n \blacksquare n = m$
2- $p, n \in N$, $p \in R$, $p * n = p \in N$ but $n * p = m$

It follows that N is Neutro-Sub $R-Module$ of M.

3.9. Theorem: Let M be a Neutro $R-Module$ over a Neutro-Ring R and let $\{N_n\}_{n \in \lambda}$ be a family of Neutro-Sub $R-Module$ of M .Then $\cap N_n$ is a Neutro-Sub $R-Module$.

3.10. Neutro $R-Module$ Homomorphism

Let $(M, +, \cdot)$ and $(N, \blacksquare, *)$ be any two Neutro $R-Modules$. The mapping $\varphi : M \to N$ is called a Neutro $R-Module$ Homomorphism if the following conditions hold:

for at least a pair $(x, y) \in M$, we have:

$$\varphi(x + y) = \varphi(x) \blacksquare \varphi(y)$$

$$\varphi(x \cdot y) = \varphi(x) * \varphi(y)$$

If in addition φ is a Neutro-Bijection, then φ is called a Neutro $R-Module$ Isomorphism and we write $M \cong N$. Neutro $R-Module$ Epimorphism, Neutro $R-Module$ Monomorphism, Neutro $R-Module$ Endomorphism and Neutro $R-Module$ Automorphism are defined similarly.

3.11. The kernel and the image of Neutro $R-Module$ Homomorphism

The kernel of φ denoted by $Ker\varphi$ is defined as

$Ker\varphi = \{x : \varphi(x) = e_N\}$ where $e_N \in N$ is Neutro-Neutral element in N.

The image of φ denoted by $Im\varphi$ is defined as

$Im\varphi = \{y \in N : y = \varphi(x)\, for\ at\ least\ one\ y \in N\}$.

3.12. Example: Let $(M, \blacksquare, *)$ be a the Neutro $R - Module$ of **3.3. Example** and let $\varphi: (M, \blacksquare, *) \to (M, \blacksquare, *)$ be

a mapping defined by:

$$\varphi(m) = m * m$$

It can be shown that φ is a Neutro $R - Module$ Homomorphism such that

for $m, n, p \in M$, we have:

1- $\varphi(m \blacksquare m) = \varphi(m) = m * m = m$

$\varphi(m) \blacksquare \varphi(m) = (m * m) \blacksquare (m * m) = m \blacksquare m = m$ but

$\varphi(m \blacksquare n) = \varphi(n) = n * n = p$

$\varphi(m) \blacksquare \varphi(n) = (m * m) \blacksquare (n * n) = m \blacksquare p = n$

2- $\varphi(m * n) = \varphi(m) = m * m = m$
$\varphi(m) * \varphi(n) = m * p = m$ but

$\varphi(p * n) = \varphi(p) = p * p = n$

$\varphi(p) * \varphi(n) = n * p = m$

The kernel of φ is $Ker\varphi = \{x : \varphi(x) = e_M\} = \{m, p\}$ where $e_M \in M$ is Neutro-Neutral element in M.

The image of φ is $Im\varphi = \{y \in N : y = \varphi(x) for\ at\ least\ one\ y \in N\} = \{m, n, p\}$

3.13. Theorem: Let $(M, \cdot, +)$ and $(N, \blacksquare, *)$ be any two Neutro $R - Modules$. Suppose that $\varphi : M \to N$ is a Neutro $R - Module$ Homomorphism. Then:

I- $\varphi(e_M)$ is not necessarily equals e_N.
II- $Ker\varphi$ is a Neutro-Sub $R - Module$ of M.
III- $Im\varphi$ is not necessarily a Neutro-Sub $R - Module$ of N.
IV- φ is NeutroInjective if and only if $Ker\varphi = \{e_M\}$ for at least one $e_M \in M$.

3.14. The composition of Neutro $R - Module$ Homomorphism:

Let K, M and N be Neutro $R - Modules$ over a Neutro-Ring R and let

$$\phi : K \to M, \psi : M \to N$$

be Neutro $R - Module$ homomorphisms. The composition $\psi\phi : K \to N$ is defined by

$$\psi\phi(k) = \psi(\phi(k))\ for\ all\ k \in K.$$

3.15. Theorem: Let K, M and N be Neutro $R - Modules$ over a Neutro-Ring R and let

$$\phi : K \to M, \psi : M \to N$$

be Neutro $R - Module$ homomorphisms. Then the composition $\psi\phi : K \to N$ is a Neutro $R - Module$ homomorphisms.

3.16. Theorem: Let K, M and N be Neutro $R - Modules$ over a Neutro-Ring R and let

$$\phi : K \to M, \psi : M \to N$$

be Neutro $R - Module$ homomorphisms. Then

1- If $\psi\phi$ is Monomorphism Neutro $R - Module$, then ϕ Monomorphism Neutro $R - Module$.
2- If $\psi\phi$ is Neutro $R - Module$ Epimorphism, then ψ is Neutro $R - Module$ Epimorphism.
3- If ψ and ϕ are Monomorphism Neutro $R - Module$, then $\psi\phi$ is Monomorphism Neutro $R - Module$.

3.17. Neutro Ordered $R - Module$

Let M be a Neutro $R - Module$ with n (Neutro) operations "i" and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on M. Then $(M, *_1, *_2, \leq)$ is a NeutroOrdered $R - Module$ if the following conditions hold.

(1) There exist $x \leq y \in M$ with $x \neq y$ such that $z *_i x \leq z *_i y$ *and* $x *_i z \leq y *_i z$ for all $i = 1, 2$ and $z \in M$ (This condition is called degree of truth, "T".)

(2) There exist $x \leq y \in M$ and $z \in A$ such that $z *_i x \nleq z *_i y$ *and* $x *_i z \nleq y *_i z$ for some $i = 1,2$. (This condition is called degree of falsity, "F".)

(3) There exist $x \leq y \in M$ and $z \in A$ such that $z *_i x$ *or* $z *_i y$ *or* $x *_i z$ *or* $y *_i z$ are indeterminate, or the relation between that $z *_i x$ and $z *_i y$, or the relation between $x *_i z$ and $y *_i z$ are indeterminate for some $i = 1, 2$. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered $R - Module$ as well from $(0, 0, 1)$ that represents the AntiOrdered $R - Module$.

3.18. Neutro Total Ordered $R - Module$

Let $(M, *_1, *_2, \leq)$ be a NeutroOrdered $R - Module$. If "$\leq$" is a total order on A then M is called NeutroTotalOrdered $R - Module$.

3.19. Neutro Ordered Sub $R - Module$

Let $(M, *_1, *_2, \leq)$ be a Neutro Ordered $R - Module$ and $\emptyset \neq S \subseteq M$. Then S is a Neutro Ordered Sub $R - Module$ of S if $(S, *_1, *_2, \leq)$ is a Neutro Ordered $R - Module$ and there exist.

a. **Example:** Let $M = \{m, n, p\}$ and $(M, \blacksquare, *,)$ be defined by the following table.

■	m	n	p
m	m	n	n
n	p or n	m or n	p
p	n	p	n

$*$	m	n	p
m	m	m	m
n	m or n	p	m
p	m	p	n

As showed $(M, ■, *,)$ in **3.3. Example** is a Neutro $R-Module$.

By defining the total order

$$\leq = \{(m,m),(n,n),(p,p),(m,n),(m,p),(n,p)\}$$

on M, we get that $(M, ■, *, \leq)$ is a NeutroTotalOrdered $R-Module$. This is easily seen as:

1- $m \leq p$ implies that $m * x \leq p * x$ and $x * m \leq x * p$ for all $x \in M$.

And having $n \leq p$ but $p■n \not\leq p■p$.

2- $m \leq n$ implies that $m * x \leq n * x$ and $x * m \leq x * n$ for all $x \in M$.

And having $n \leq p$ but $p * n \not\leq p * p$.

3.21. Example: Let $(M, ■, *, \leq)$ be a the Neutro $R-Module$ of **3.3. Example** and let $N = \{p, n\}$:

1- $p, n \in N$, $p■n = p \in N$ but $n■n = m$
2- $p, n \in N$, $p \in R$, $p * n = p \in N$ but $n * p = m$

By defining the total order

$$\leq = \{(m,m),(n,n),(p,p),(m,n),(m,p),(n,p)\}$$

It follows that $(N, ■, *, \leq)$ is Neutro-Sub $R-Module$ of M.

3.22. Neutro Ordered $R-Module$ Homomorphism

Let $(M, *_1, *_2, \leq_1)$ and $(N, ■_1, ■_2, \leq_2)$ be any two Neutro Ordered $R-Modules$. The mapping $\varphi : M \to N$ is called a Neutro Ordered $R-Module$ Homomorphism if the following conditions hold:

for some $(x, y) \in M$, we have:

- $\varphi(x *_1 y) = \varphi(x)■_1 \varphi(y)$
- $\varphi(x *_2 y) = \varphi(x)■_2\varphi(y)$
- and there exist $a \leq_1 b$, $a \neq b$, $\varphi(a) \leq_2 \varphi(b)$

φ is called Neutro Ordered $R - Module$ Isomorphism if φ is a bijective NeutroOrdered $R - Module$ Homomorphism.

i)There exists a double $(p, q) \in M$ such that $\varphi(p *_1 q) = \varphi(p) \blacksquare_1 \varphi(q)$ (degree of truth T) and there exist two doubles $(s, t), (k, m)$ (F, V) such that $[\varphi(s *_1 t) \neq \varphi(s) \blacksquare_1 \varphi(t)$ (degree of falsehood F) or $\varphi(k *_1 m) =_{\text{indeterminacy}} \varphi(k) \blacksquare_1 \varphi(m)$ (degree of indeterminacy I)]; where (T, I, F) is different from $(1, 0, 0)$ and $(0, 0, 1)$.

ii)There exists a double $(p, q) \in M$ such that $\varphi(p *_2 q) = \varphi(p) \blacksquare_2 \varphi(q)$ (degree of truth T) and there exist two doubles $(s, t), (k, m)$ (F, V) such that $[\varphi(s *_2 t) \neq \varphi(s) \blacksquare_2 \varphi(t)$ (degree of falsehood F) or $\varphi(k *_2 m) =_{\text{indeterminacy}} \varphi(k) \blacksquare_2 \varphi(m)$ (degree of indeterminacy I)]; where (T, I, F) is different from $(1, 0, 0)$ and $(0, 0, 1)$.

3.23. Example: Let $\varphi: (M, \blacksquare, *, \leq) \rightarrow (M, \blacksquare, *, \leq)$ be a mapping defined by:

$$\varphi(m) = m * m$$

It can be shown that φ is a Neutro Ordered $R - Module$ Homomorphism such that

for $m, n, p \in M$, we have:

1- and 2- it proved in **3.12. Example**

3- there exist $m \leq n$ sush that $\varphi(m) \leq \varphi(n)$

Conclusions

This paper contributed to the study of Neutro Algebra by introducing, for the first time, Neutro $R - module$ and Neutro Ordered$R - module$. Many interesting properties were proved as well illustrative examples were given on Neutro $R - module$ and Neutro Ordered$R - module$.

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Chapter Seven

On Neutrosophic Double Controlled Metric-like Type Spaces

Khalil Javed[1], Umar Ishtiaq[2], Fahim Uddin[3] and Khalil Ahmed[4]

[1,2,3,4]Department of Math & Stats, International Islamic University Islamabad, Pakistan

E-mail: khalil.msma551@iiu.edu.pk, umarishtiaq000@gmail.com, fahim.uddin@iiu.edu.pk, khalil7066616@gmail.com

ABSTRACT

In this manuscript, we introduce the new concept of Neutrosophic double controlled metric-like spaces that generalize the concept of Neutrosophic metric spaces. We prove and generalize the concept of Banach contraction principle and fuzzy contractive mappings in the sense of neutrosophic double controlled metric-like spaces. These results and illustrative examples generalize several comparable results from the current literature.

Keywords: Fixed point, Controlled metric like space; Double controlled metric like space; Neutrosophic double controlled metric-like spaces;

INTRODUCTION

In the field of fixed point theory, the notion of metric spaces and the Banach contraction principle play crucial roles. Many researches are drawn to metric spaces because of its axiomatic clarity. There have been a lot of generalizations to the metric spaces so far. This demonstrates the allurement and scope of the definition of the metric spaces.

The notion given by Zadeh [3] is known as fuzzy sets (FSs) acquire an ultra-attraction for researchers. This concept succeeded in shifting a lot of mathematical structures within itself. In this continuation, Kramosil and Michalek [9] originate the notion of fuzzy metric spaces and Garbiec [10] gave the fuzzy interpretation of Banach contraction principle in fuzzy metric spaces. Harandi [22] is credited with coining the term metric like spaces (MLS) which elegantly generalises the idea of metric spaces. [25] N. Mlaiki introduced the concept of controlled metric type spaces and controlled metric-like spaces [24]. Shukla and Abbas [23] reformulated definition (MLS) in this context, resulting in fuzzy metric like spaces (FMLS). Fuzzy metric spaces discuss only for memberships functions, so for dealing with membership and non-membership functions intuitionistic fuzzy metric spaces introduced by J. H. Park [11]. The neutrosophic set and theory are given by Smarandache. A lot of studies are given based on neutrosophic sets [28-32,40-76]. Also, a lot of studies are given based on some type of neutrosophic triplet metric space [33-39]. M. Kirişci, N. Simsek [20] tossed the approach of neutrosophic metric spaces (NMSs) that deals with membership, non-membership and naturalness functions. N. Simsek, M. Kirişci [19] and S. Sowndrarajan, M. Jeyarama, F. Smarandache [21] prove some fixed point (FP) results in the setting of NMSs. Recently N. Saleem [7] introduce the notion of fuzzy double controlled metric spaces (FDCMSs) and generalized the Banach contraction principle. For related articles see [1, 2, 4-6, 8, 12-18, 26,27].

In this manuscript, our aim is to generalize the concept of NMSs by using the approach in [7] and toss the concept of neutrosophic double controlled metric-like spaces (NDCMLSs). Fixed point (FP) results and non-trivial examples are imparted in this work

BACKGROUND

Definition 1.1 [1] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangle norm (briefly CTN) if:

1. $\pi * \mu = \mu * \pi, (\forall)\ \pi, \mu \in [0, 1]$;
2. $*$ is continuous;
3. $\pi * 1 = \pi, (\forall)\ \pi \in [0, 1]$;
4. $(\pi * \mu) * \rho = \pi * (\mu * \rho), (\forall)\ \pi, \mu, \rho \in [0, 1]$;
5. If $\pi \leq \rho$ and $\mu \leq \sigma$, with $\pi, \mu, \rho, \sigma \in [0, 1]$, then $\pi * \mu \leq \rho * \sigma$.

Example 1.1 [1, 2] Some fundamental examples of CTNs are: $\pi * \mu = \pi \cdot \mu, \pi * \mu = \min\{\pi, \mu\}$ and $\pi * \mu = \max\{\pi + \mu - 1, 0\}$.

Definition 1.2 [1] A binary operation $\circ: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangle conorm (briefly CTCN) if it meets the below assertions:

1. $\pi \circ \mu = \mu \circ \pi$, for all $\pi, \mu \in [0, 1]$;
2. $\circ$ is continuous;
3. $\pi \circ 0 = 0$, for all $\pi \in [0, 1]$;
4. $(\pi \circ \mu) \circ \rho = \pi \circ (\mu \circ \rho)$, for all $\pi, \mu, \rho \in [0, 1]$;
5. If $\pi \leq \rho$ and $\mu \leq \sigma$, with $\pi, \mu, \rho, \sigma \in [0, 1]$, then $\pi \circ \mu \leq \rho \circ \sigma$.

Example 1.2 [1] $\pi \circ \mu = \max\{\pi, \mu\}$ and $\pi \circ \mu = \min\{\pi + \mu, 1\}$ are examples of CTCNs.

Definition 1.3 [3] Let a set $\mathfrak{D} \neq \emptyset$, then a pair $(\mathfrak{D}, P)$ is named to be fuzzy set, here P is a function from $\mathfrak{D}$ to $[0,1]$ i.e. $P: \mathfrak{D} \to [0,1]$ for each $\partial \in \mathfrak{D}$, $P(\partial)$ is called the grade of membership of $\mathfrak{D}$, in $(\mathfrak{D}, P)$ and P is called a membership function of $(\mathfrak{D}, P)$.

Definition 1.4 [27] Let a set $\mathfrak{D} \neq \emptyset$ and $\partial \in \mathfrak{D}$. A neutrosophic set G in $\mathfrak{D}$ is categorized by a truth-membership function, $T_G(\partial)$, an indeterminacy-membership function $U_G(\partial)$ and a falsity –membership function $V_G(\partial)$. The functions $T_G(\partial)$, $U_G(\partial)$ and $V_G(\partial)$ are real standard or non-standard subsets of $]0^-, 1^+[$, that is $T_G(\partial): X \to]0^-, 1^+[$, $U_G(\partial): X \to]0^-, 1^+[$ and $V_G(\partial): X \to]0^-, 1^+[$. So,

$0^- \leq \sup T_G(\partial) + \sup U_G(\partial) + \sup V_G(\partial) \leq 3^+$.

Definition 1.5 [22] A mapping $P: \mathfrak{D} \times \mathfrak{D} \to [1, \infty)$, where $\mathfrak{D} \neq \emptyset$, fulfill the below circumstances:

a. $P(\partial, \beta) = 0$ implies $\partial = \beta$;
b. $P(\partial, \beta) = P(\beta, \partial)$;
c. $P(\partial, \beta) \leq P(\partial, \delta) + P(\delta, \beta)$;

for all $\partial, \beta, \delta \in \mathfrak{D}$. Then P is called a metric-like and $(\mathfrak{D}, P)$ is named metric-like space.

Definition 1.6 [24] Let a function $\psi: \mathfrak{D} \times \mathfrak{D} \to [1, \infty)$ and a mapping $P: \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}^+$, where $\mathfrak{D} \neq \emptyset$, fulfill the below circumstances:

a. $P(\partial, \beta) = 0$ implies $\partial = \beta$;
b. $P(\partial, \beta) = P(\beta, \partial)$;
c. $P(\partial, \beta) \leq \psi((\partial, \delta)P(\partial, \delta) + \psi(\delta, \partial)P(\delta, \beta)$;

for all $\partial, \beta, \delta \in \mathfrak{D}$. Then P is called a controlled metric-like and $(\mathfrak{D}, P)$ is named controlled metric-like space.

Definition 1.7 [8] Given functions $\phi, \eta: \Sigma \times \Sigma \to [1, \infty)$ are non-comparable. If $\partial: \Sigma \times \Sigma \to [0, \infty)$ fulfil:

d. $\partial(\alpha, \beta) = 0$ iff $\alpha = \beta$;
e. $\partial(\alpha, \beta) = \partial(\beta, \alpha)$;
f. $\partial(\alpha, \beta) \leq \phi(\alpha, \lambda)\partial(\alpha, \lambda) + \eta(\lambda, \beta)\partial(\lambda, \beta)$;

for all $\alpha, \beta, \lambda \in \Sigma$. Then ∂ is called a double controlled metric and (Σ, ∂) is named double controlled metric space.

Definition 1.8 [7] Let $\Sigma \neq \emptyset$ and $\phi, \eta: \Sigma \times \Sigma \to [1, \infty)$ given non-comparable functions, and $*$ is a CTN. P be a FS on $\Sigma \times \Sigma \times (0, \infty)$ is named fuzzy double controlled metric on Σ, if for all $\alpha, \beta, \lambda \in \Sigma$, the below circumstances fulfil:

i. $P(\alpha, \beta, 0) = 0$;
ii. $P(\alpha, \beta, \mathcal{T}) = 1$ for all $\mathcal{T} > 0$, if and only if $\alpha = \beta$;
iii. $P(\alpha, \beta, \mathcal{T}) = P(\beta, \alpha, \mathcal{T})$;
iv. $P(\alpha, \lambda, \mathcal{T} + \mathcal{S}) \geq P\left(\alpha, \beta, \frac{\mathcal{T}}{\phi(\alpha, \beta)}\right) * P\left(\beta, \lambda, \frac{\mathcal{S}}{\eta(\beta, \lambda)}\right)$;
v. $P(\alpha, \beta, \cdot): (0, \infty) \to [0,1]$ is left continuous.

Then $(\Sigma, P, Q, *)$ be named a FDCMS.

Definition 1.9 [4] Take $\Sigma \neq \emptyset$. Let $*$ be a CTN, $\circ$ be a CTCN, $b \geq 1$ and P, Q be FSs on $\Sigma \times \Sigma \times (0, \infty)$. If $(\Sigma, P, Q, *, \circ)$ verifies the following for all $\alpha, \beta \in \Sigma$ and $\mathcal{S}, \mathcal{T} > 0$:

I. $P(\alpha, \beta, \mathcal{T}) + Q(\alpha, \beta, \mathcal{T}) \leq 1$;
II. $P(\alpha, \beta, \mathcal{T}) > 0$;
III. $P(\alpha, \beta, \mathcal{T}) = 1 \Leftrightarrow \alpha = \beta$;
IV. $P(\alpha, \beta, \mathcal{T}) = P(\beta, \alpha, \mathcal{T})$;
V. $P(\alpha, \lambda, b(\mathcal{T} + \mathcal{S})) \geq P(\alpha, \beta, \mathcal{T}) * P(\beta, \lambda, \mathcal{T})$;
VI. $P(\alpha, \beta, \cdot)$ is a non-decreasing function of $\mathbb{R}^+$ and $\lim_{\mathcal{T} \to \infty} P(\alpha, \beta, \mathcal{T}) = 1$;

VII. $Q(\alpha,\beta,\mathcal{T}) > 0$;
VIII. $Q(\alpha,\beta,\mathcal{T}) = 0 \Leftrightarrow \alpha = \beta$;
IX. $Q(\alpha,\beta,\mathcal{T}) = Q(\beta,\alpha,\mathcal{T})$;
X. $Q\big(\alpha,\lambda,b(\mathcal{T}+\mathcal{S})\big) \leq Q(\alpha,\beta,\mathcal{T})\circ Q(\beta,\lambda,\mathcal{T})$;
XI. $Q(\alpha,\beta,\cdot)$ is a non-increasing function of $\mathbb{R}^+$ and $\lim_{\mathcal{T}\to\infty} Q(\alpha,\beta,\mathcal{T}) = 0$,

then $(\Sigma,P,Q,*,\circ)$ is an Intuitionistic fuzzy b-metric space.

Definition 1.10 [20] Let $\Sigma \neq \emptyset$ and $*$ is a CTN and $\circ$ be a CTCN. P,Q,S are Neutrosophic sets on $\Sigma\times\Sigma\times(0,\infty)$ is named Neutrosophic metric on Σ, if for all $\alpha,\beta,\lambda\in\Sigma$, the below circumstances fulfil:

1) $P(\alpha,\beta,\mathcal{T}) + Q(\alpha,\beta,\mathcal{T}) + S(\alpha,\beta,\mathcal{T}) \leq 3$ for all $\mathcal{T}\in\mathbb{R}^+$;
2) $P(\alpha,\beta,\mathcal{T}) > 0$ *for all* $\mathcal{T} > 0$;
3) $P(\alpha,\beta,\mathcal{T}) = 1$ for all $\mathcal{T} > 0$, if and only if $\alpha = \beta$;
4) $P(\alpha,\beta,\mathcal{T}) = P(\beta,\alpha,\mathcal{T})$ for all $\mathcal{T} > 0$;
5) $P(\alpha,\lambda,\mathcal{T}+\mathcal{S}) \geq P(\alpha,\beta,\mathcal{T}) * P(\beta,\lambda,\mathcal{S})$ for all $\mathcal{T},\mathcal{S} > 0$;
6) $P(\alpha,\beta,\ \cdot): (0,\infty)\to[0,1]$ is continuous and $\lim_{\mathcal{T}\to\infty} P(\alpha,\beta,\mathcal{T}) = 1$ for all $\mathcal{T} > 0$;
7) $Q(\alpha,\beta,\mathcal{T}) < 1$ for all $\mathcal{T} > 0$;
8) $Q(\alpha,\beta,\mathcal{T}) = 0$ for all $\mathcal{T} > 0$, if and only if $\alpha = \beta$;
9) $Q(\alpha,\beta,\mathcal{T}) = Q(\beta,\alpha,\mathcal{T})$ for all $\mathcal{T} > 0$;
10) $Q(\alpha,\lambda,\mathcal{T}+\mathcal{S}) \leq Q(\alpha,\beta,\mathcal{T}) \circ Q(\beta,\lambda,\mathcal{S})$ for all $\mathcal{T},\mathcal{S} > 0$;
11) $Q(\alpha,\beta,\ \cdot): (0,\infty)\to[0,1]$ is continuous and $\lim_{\mathcal{T}\to\infty} Q(\alpha,\beta,\mathcal{T}) = 0$ for all $\mathcal{T} > 0$;
12) $S(\alpha,\beta,\mathcal{T}) < 1$ for all $\mathcal{T} > 0$;
13) $S(\alpha,\beta,\mathcal{T}) = 0$ for all $\mathcal{T} > 0$, if and only if $\alpha = \beta$;
14) $S(\alpha,\beta,\mathcal{T}) = S(\beta,\alpha,\mathcal{T})$ for all $\mathcal{T} > 0$;
15) $S(\alpha,\lambda,\mathcal{T}+\mathcal{S}) \leq S(\alpha,\beta,\mathcal{T}) \circ S(\beta,\lambda,\mathcal{S})$ for all $\mathcal{T},\mathcal{S} > 0$;
16) $S(\alpha,\beta,\ \cdot): (0,\infty)\to[0,1]$ is continuous and $\lim_{\mathcal{T}\to\infty} S(\alpha,\beta,\mathcal{T}) = 0$ for all $\mathcal{T} > 0$;
17) If $\mathcal{T} \leq 0$, then $P(\alpha,\beta,\mathcal{T}) = 0$, $Q(\alpha,\beta,\mathcal{T}) = 1$ and $S(\alpha,\beta,\mathcal{T}) = 1$.

Then $(\Sigma,P,Q,S,*,\circ)$ be called an NMS.

NEUTROSOPHIC DOUBLE CONTROLLED METRIC-LIKE SPACE

In this section, we introduce the concept of NDCMLSs and prove some FP results.

Definition 2.1 Let $\Sigma \neq \emptyset$ and $\phi,\eta:\Sigma\times\Sigma\to[1,\infty)$ given non-comparable functions, and $*$ is a CTN and $\circ$ be a CTCN. P,Q,R are Neutrosophic sets on $\Sigma\times\Sigma\times(0,\infty)$ is named Neutrosophic double controlled metric-like on Σ, if for all $\alpha,\beta,\lambda\in\Sigma$, the below circumstances fulfil:

(i) $P(\alpha,\beta,\mathcal{T}) + Q(\alpha,\beta,\mathcal{T}) + R(\alpha,\beta,\mathcal{T}) \leq 3$ for all $\mathcal{T} > 0$;
(ii) $P(\alpha,\beta,\mathcal{T}) > 0$ for all $\mathcal{T} > 0$;
(iii) $P(\alpha,\beta,\mathcal{T}) = 1$ for all $\mathcal{T} > 0$, implies $\alpha = \beta$;
(iv) $P(\alpha,\beta,\mathcal{T}) = P(\beta,\alpha,\mathcal{T})$ for all $\mathcal{T} > 0$;
(v) $P(\alpha,\lambda,\mathcal{T}+\mathcal{S}) \geq P\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) * P\left(\beta,\lambda,\frac{\mathcal{S}}{\eta(\beta,\lambda)}\right)$ for all $\mathcal{T},\mathcal{S} > 0$;

(vi) $P(\alpha,\beta,\cdot):(0,\infty)\to[0,1]$ is continuous and and $\lim_{\mathcal{T}\to\infty} P(\alpha,\beta,\mathcal{T})=1$ for all $\mathcal{T}>0$;

(vii) $Q(\alpha,\beta,\mathcal{T})<1$ for all $\mathcal{T}>0$;

(viii) $Q(\alpha,\beta,\mathcal{T})=0$ for all $\mathcal{T}>0$, implies $\alpha=\beta$;

(ix) $Q(\alpha,\beta,\mathcal{T})=Q(\beta,\alpha,\mathcal{T})$ for all $\mathcal{T}>0$;

(x) $Q(\alpha,\lambda,\mathcal{T}+\mathcal{S})\leq Q\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right)\circ Q\left(\beta,\lambda,\frac{\mathcal{S}}{\eta(\beta,\lambda)}\right)$ for all $\mathcal{T},\mathcal{S}>0$;

(xi) $Q(\alpha,\beta,\cdot):(0,\infty)\to[0,1]$ is continuous and $\lim_{\mathcal{T}\to\infty} Q(\alpha,\beta,\mathcal{T})=0$ for all $\mathcal{T}>0$;

(xii) $R(\alpha,\beta,\mathcal{T})<1$ for all $\mathcal{T}>0$;

(xiii) $R(\alpha,\beta,\mathcal{T})=0$ for all $\mathcal{T}>0$, implies $\alpha=\beta$;

(xiv) $R(\alpha,\beta,\mathcal{T})=R(\beta,\alpha,\mathcal{T})$ for all $\mathcal{T}>0$;

(xv) $R(\alpha,\lambda,\mathcal{T}+\mathcal{S})\leq R\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right)\circ R\left(\beta,\lambda,\frac{\mathcal{S}}{\eta(\beta,\lambda)}\right)$ for all $\mathcal{T},\mathcal{S}>0$;

(xvi) $R(\alpha,\beta,\cdot):(0,\infty)\to[0,1]$ is continuous and and $\lim_{\mathcal{T}\to\infty} R(\alpha,\beta,\mathcal{T})=0$ for all $\mathcal{T}>0$;

Then $(\Sigma,P,Q,R,*,\circ)$ be called a NDCMLS.

Example 2.1 Let $\Sigma=\{1,2,3\}$ and $\phi,\eta:\Sigma\times\Sigma\to[1,\infty)$ be two non-comparable functions given by $\phi(\alpha,\beta)=\alpha+\beta+1$ and $\eta(\alpha,\beta)=\alpha^2+\beta^2-1$. Define $P,Q,R:\Sigma\times\Sigma\times(0,\infty)\to[0,1]$ as

$$P(\alpha,\beta,\mathcal{T})=\frac{\mathcal{T}}{\mathcal{T}+\max\{\alpha,\beta\}}$$

$$Q(\alpha,\beta,\mathcal{T})=\frac{\max\{\alpha,\beta\}}{\mathcal{T}+\max\{\alpha,\beta\}},$$

and

$$R(\alpha,\beta,\mathcal{T})=\frac{\max\{\alpha,\beta\}}{\mathcal{T}}.$$

Then $(\Sigma,P,Q,R,*,\circ)$ is an NDCMLS with CTN $\pi*\mu=\pi\mu$ and CTCN $\pi\circ\mu=\max\{\pi,\mu\}$.

Proof: Conditions (i)-(iv), (vi)-(ix), (xi)-(xiv) and (xvi) are easy to examine, here we prove (v), (x) and (xv).

Let $\alpha=1,\beta=2$ and $\lambda=3$. Then

$$P(1,3,\mathcal{T}+\mathcal{S})=\frac{\mathcal{T}+\mathcal{S}}{\mathcal{T}+\mathcal{S}+\max\{1,3\}}$$

$$=\frac{\mathcal{T}+\mathcal{S}}{\mathcal{T}+\mathcal{S}+3}.$$

On the other hand,

$$P\left(1,2,\frac{\mathcal{T}}{\phi(1,2)}\right)=\frac{\frac{\mathcal{T}}{\phi(1,2)}}{\frac{\mathcal{T}}{\phi(1,2)}+\max\{1,2\}}$$

$$=\frac{\frac{\mathcal{T}}{4}}{\frac{\mathcal{T}}{4}+2}=\frac{\mathcal{T}}{\mathcal{T}+8}$$

and

$$P\left(2,3,\frac{\mathcal{S}}{\eta(2,3)}\right) = \frac{\frac{\mathcal{S}}{\eta(2,3)}}{\frac{\mathcal{S}}{\eta(2,3)} + \max\{2,3\}}$$

$$= \frac{\frac{\mathcal{S}}{12}}{\frac{\mathcal{S}}{12} + 3} = \frac{\mathcal{S}}{\mathcal{S} + 36}.$$

That is,

$$\frac{\mathcal{T} + \mathcal{S}}{\mathcal{T} + \mathcal{S} + 3} \geq \frac{\mathcal{T}}{\mathcal{T} + 8} \cdot \frac{\mathcal{S}}{\mathcal{S} + 36}.$$

Then it satisfies for all $\mathcal{T}, \mathcal{S} > 0$. Hence,

$$P(\alpha, \lambda, \mathcal{T} + \mathcal{S}) \geq P\left(\alpha, \beta, \frac{\mathcal{T}}{\phi(\alpha, \beta)}\right) * P\left(\beta, \lambda, \frac{\mathcal{S}}{\eta(\beta, \lambda)}\right).$$

Now,

$$Q(1,3, \mathcal{T} + \mathcal{S}) = \frac{\max\{1,3\}}{\mathcal{T} + \mathcal{S} + \max\{1,3\}}$$

$$= \frac{3}{\mathcal{T} + \mathcal{S} + 3}.$$

On the other hand,

$$Q\left(1,2,\frac{\mathcal{T}}{\eta(1,2)}\right) = \frac{\max\{1,2\}}{\frac{\mathcal{T}}{\eta(1,2)} + \max\{1,2\}}$$

$$= \frac{2}{\frac{\mathcal{T}}{4} + 2} = \frac{8}{\mathcal{T} + 8}$$

and

$$Q\left(2,3,\frac{\mathcal{S}}{\eta(2,3)}\right) = \frac{\max\{2,3\}}{\frac{\mathcal{S}}{\eta(2,3)} + \max\{2,3\}}$$

$$= \frac{3}{\frac{\mathcal{S}}{12} + 3} = \frac{36}{\mathcal{S} + 36}.$$

That is,

$$\frac{3}{\mathcal{T} + \mathcal{S} + 3} \leq \max\left\{\frac{8}{\mathcal{T} + 8}, \frac{36}{\mathcal{S} + 36}\right\}.$$

Then it satisfies for all $\mathcal{T}, \mathcal{S} > 0$. Hence,

$$Q(\alpha, \lambda, \mathcal{T} + \mathcal{S}) \leq Q\left(\alpha, \beta, \frac{\mathcal{T}}{\phi(\alpha, \beta)}\right) \circ Q\left(\beta, \lambda, \frac{\mathcal{S}}{\eta(\beta, \lambda)}\right).$$

Now,

$$R(1,3,\mathcal{T}+\mathcal{S}) = \frac{\max\{1,3\}}{\mathcal{T}+\mathcal{S}}$$

$$= \frac{3}{\mathcal{T}+\mathcal{S}}.$$

On the other hand,

$$R\left(1,2,\frac{\mathcal{T}}{\eta(1,2)}\right) = \frac{\max\{1,2\}}{\frac{\mathcal{T}}{\eta(1,2)}}$$

$$= \frac{2}{\frac{\mathcal{T}}{4}} = \frac{8}{\mathcal{T}}$$

and

$$R\left(2,3,\frac{\mathcal{S}}{\eta(2,3)}\right) = \frac{\max\{2,3\}}{\frac{\mathcal{S}}{\eta(2,3)}}$$

$$= \frac{3}{\frac{\mathcal{S}}{12}} = \frac{36}{\mathcal{S}}.$$

That is,

$$\frac{3}{\mathcal{T}+\mathcal{S}} \leq \max\left\{\frac{8}{\mathcal{T}}, \frac{36}{\mathcal{S}}\right\}.$$

Then it satisfies for all $\mathcal{T},\mathcal{S} > 0$. Hence,

$$R(\alpha,\lambda,\mathcal{T}+\mathcal{S}) \leq R\left(\alpha,\beta,\frac{\mathcal{T}}{\phi(\alpha,\beta)}\right) \circ R\left(\beta,\lambda,\frac{\mathcal{S}}{\eta(\beta,\lambda)}\right).$$

On the same lines, one can examine all other cases. Hence, $(\Sigma,P,Q,R,*,\circ)$ is a NDCMLS.

Remark 2.1 Above example also satisfied for CTN $\pi * \mu = \min\{\pi,\mu\}$ and CTCN $\pi \circ \mu = \max\{\pi,\mu\}$.

Example 2.2 Let $\Sigma = (0,\infty)$ and $\phi,\eta: \Sigma\times\Sigma \to [1,\infty)$ be two non-comparable functions given by $\phi(\alpha,\beta) = \alpha+\beta+1$ and $\eta(\alpha,\beta) = \alpha^2+\beta^2-1$.

Define $P,Q,R: \Sigma\times\Sigma\times(0,\infty) \to [0,1]$ as

$$P(\alpha,\beta,\mathcal{T}) = \frac{\mathcal{T}}{\mathcal{T}+\max\{\alpha,\beta\}^2}, \qquad Q(\alpha,\beta,\mathcal{T}) = \frac{\max\{\alpha,\beta\}^2}{\mathcal{T}+\max\{\alpha,\beta\}^2}, \qquad R(\alpha,\beta,\mathcal{T}) = \frac{\max\{\alpha,\beta\}^2}{\mathcal{T}}$$

Then $(\Sigma,P,Q,R,*,\circ)$ is a NDCMLS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \circ \mu = \max\{\pi,\mu\}$.

Remark 2.2 Above example also holds for

$$\phi(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha=\beta, \\ \dfrac{1+\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}} & \text{if } \alpha\neq\beta \end{cases}$$

and

$$\eta(\alpha,\beta) = \begin{cases} 1 & \text{if} \quad \alpha=\beta, \\ \dfrac{1+\max\{\alpha^2,\beta^2\}}{\min\{\alpha^2,\beta^2\}} & \text{if } \alpha\neq\beta \end{cases}.$$

Remark 2.3 Above example also satisfied for CTN $\pi * \mu = \min\{\pi,\mu\}$ and CTCN $\pi \circ \mu = \max\{\pi,\mu\}$.

Definition 2.2 Let $(\Sigma, P, Q, R, *, \circ)$ is a NDCMLS, then we define an open ball $B(\alpha, r, \mathcal{T})$ with centre α, radius r, $0<r<1$ and $\mathcal{T}>0$ as follows:

$B(\alpha, r, \mathcal{T}) = \{\beta \in \Sigma : P(\alpha,\beta,\mathcal{T}) > 1-r, Q(\alpha,\beta,\mathcal{T}) < r, R(\alpha,\beta,\mathcal{T}) < r\}$.

Definition 2.3 Let $(\Sigma, P, Q, R, *, \circ)$ is an NDCMLS and $\{\alpha_n\}$ be a sequence in Σ.then $\{\alpha_n\}$ is named to be:

(i) a convergent, if there exists $\alpha \in \Sigma$ such that $\lim_{n\to\infty} P(\alpha_n,\alpha,\mathcal{T}) = P(\alpha,\alpha,\mathcal{T})$, $\lim_{n\to\infty} Q(\alpha_n,\alpha,\mathcal{T}) = Q(\alpha,\alpha,\mathcal{T})$, and $\lim_{n\to\infty} R(\alpha_n,\alpha,\mathcal{T}) = R(\alpha,\alpha,\mathcal{T})$, for all $\mathcal{T}>0$

(ii) a Cauchy sequence (CS), if and only if for each $\mathcal{T}>0$, there exists $n_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} P(\alpha_n,\alpha_{n+q},\mathcal{T})$, $\lim_{n\to\infty} Q(\alpha_n,\alpha_{n+q},\mathcal{T})$ and $\lim_{n\to\infty} Q(\alpha_n,\alpha_{n+q},\mathcal{T})$ exists and finite.

(iii) If every Cauchy sequence convergent in Σ, then $(\Sigma, P, Q, *, \circ)$ is called complete NDCMLS.
$\lim_{n\to\infty} P(\alpha_n,\alpha_{n+q},\mathcal{T}) = \lim_{n\to\infty} P(\alpha_n,\alpha,\mathcal{T}) = P(\alpha,\alpha,\mathcal{T})$,
$\lim_{n\to\infty} Q(\alpha_n,\alpha_{n+q},\mathcal{T}) = \lim_{n\to\infty} Q(\alpha_n,\alpha,\mathcal{T}) = Q(\alpha,\alpha,\mathcal{T})$,
$\lim_{n\to\infty} R(\alpha_n,\alpha_{n+q},\mathcal{T}) = \lim_{n\to\infty} R(\alpha_n,\alpha,\mathcal{T}) = R(\alpha,\alpha,\mathcal{T})$.

Theorem 2.1 Suppose $(\Sigma, P, Q, R, *, \circ)$ be a complete NDCMLS in the company of $\phi, \eta : \Sigma \times \Sigma \to \left[1, \frac{1}{\theta}\right)$ with $0<\theta<1$ and suppose that

$$\lim_{\mathcal{T}\to\infty} P(\alpha,\beta,\mathcal{T}) = 1, \quad \lim_{\mathcal{T}\to\infty} Q(\alpha,\beta,\mathcal{T}) = 0 \text{ and } \lim_{\mathcal{T}\to\infty} R(\alpha,\beta,\mathcal{T}) = 0 \qquad (1)$$

for all $\alpha,\beta \in \Sigma$ and $\mathcal{T}>0$. Let $\Psi : \Sigma \to \Sigma$ be a mapping satisfying

$$P(\Psi\alpha,\Psi\beta,\theta\mathcal{T}) \geq P(\alpha,\beta,\mathcal{T}), \quad Q(\Psi\alpha,\Psi\beta,\theta\mathcal{T}) \leq Q(\alpha,\beta,\mathcal{T}) \text{ and } R(\Psi\alpha,\Psi\beta,\theta\mathcal{T}) \leq R(\alpha,\beta,\mathcal{T}) \qquad (2)$$

for all $\alpha,\beta \in \Sigma$ and $\mathcal{T}>0$. Then Ψ has a unique FP.

Proof: Let α_0 be a random integer of Σ and describe a sequence α_n by $\alpha_n = \Psi^n\alpha_0 = \Psi\alpha_{n-1}$, $n \in \mathbb{N}$. By using (1) for all $\mathcal{T}>0$, we have

$$P(\alpha_n,\alpha_{n+1},\theta\mathcal{T}) = P(\Psi\alpha_{n-1},\Psi\alpha_n,\theta\mathcal{T}) \geq P(\alpha_{n-1},\alpha_n,\mathcal{T}) \geq P\left(\alpha_{n-2},\alpha_{n-1},\frac{\mathcal{T}}{\theta}\right)$$

$$\geq P\left(\alpha_{n-3},\alpha_{n-2},\frac{\mathcal{T}}{\theta^2}\right) \geq \cdots \geq P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{\theta^{n-1}}\right),$$

$$Q(\alpha_n,\alpha_{n+1},\theta\mathcal{T}) = Q(\Psi\alpha_{n-1},\Psi\alpha_n,\theta\mathcal{T}) \leq Q(\alpha_{n-1},\alpha_n,\mathcal{T}) \leq Q\left(\alpha_{n-2},\alpha_{n-1},\frac{\mathcal{T}}{\theta}\right)$$

$$\leq Q\left(\alpha_{n-3},\alpha_{n-2},\frac{\mathcal{T}}{\theta^2}\right) \leq \cdots \leq Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{\theta^{n-1}}\right)$$

and

$$R(\alpha_n,\alpha_{n+1},\theta\mathcal{T}) = R(\Psi\alpha_{n-1},\Psi\alpha_n,\theta\mathcal{T}) \le R(\alpha_{n-1},\alpha_n,\mathcal{T}) \le R\left(\alpha_{n-2},\alpha_{n-1},\frac{\mathcal{T}}{\theta}\right)$$

$$\le R\left(\alpha_{n-3},\alpha_{n-2},\frac{\mathcal{T}}{\theta^2}\right) \le \cdots \le R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{\theta^{n-1}}\right).$$

We obtain

$$P(\alpha_n,\alpha_{n+1},\theta\mathcal{T}) \ge P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{\theta^{n-1}}\right),$$

$$Q(\alpha_n,\alpha_{n+1},\theta\mathcal{T}) \le Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{\theta^{n-1}}\right) \text{ and } R(\alpha_n,\alpha_{n+1},\theta\mathcal{T}) \le R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{\theta^{n-1}}\right) \quad (3)$$

for any $q \in \mathbb{N}$, using (v), (x) and (xv), we deduce

$$P(\alpha_n,\alpha_{n+q},\mathcal{T}) \ge P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\eta(\alpha_{n+1},\alpha_{n+q})\right)}\right)$$

$$\ge P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_{n+2},\alpha_{n+q},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\right)}\right)$$

$$\ge P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$* P\left(\alpha_{n+3},\alpha_{n+q},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\right)}\right)$$

$$\ge P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$* P\left(\alpha_{n+3},\alpha_{n+4},\frac{\mathcal{T}}{(2)^4\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\phi(\alpha_{n+3},\alpha_{n+4})\right)}\right) * \cdots *$$

$$P\left(\alpha_{n+q-2},\alpha_{n+q-1},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$*P\left(\alpha_{n+q-1},\alpha_{n+q},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right),$$

$$Q(\alpha_n,\alpha_{n+q},\mathcal{T})\leq Q\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ Q\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\eta(\alpha_{n+1},\alpha_{n+q})\right)}\right)$$

$$\leq Q\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ Q\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_{n+2},\alpha_{n+q},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\right)}\right)$$

$$\leq Q\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ Q\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$\circ Q\left(\alpha_{n+3},\alpha_{n+q},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\right)}\right)$$

$$\leq Q\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ Q\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$\circ Q\left(\alpha_{n+3},\alpha_{n+4},\frac{\mathcal{T}}{(2)^4\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\phi(\alpha_{n+3},\alpha_{n+4})\right)}\right)\circ\cdots\circ$$

$$Q\left(\alpha_{n+q-2},\alpha_{n+q-1},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ Q\left(\alpha_{n+q-1},\alpha_{n+q},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)$$

and

$$R(\alpha_n,\alpha_{n+q},\mathcal{T})\leq R\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ R\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\eta(\alpha_{n+1},\alpha_{n+q})\right)}\right)$$

$$\leq R\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ R\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$\circ R\left(\alpha_{n+2}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\right)}\right)$$

$$\leq R\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ R\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$\circ R\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\phi(\alpha_{n+2}, \alpha_{n+3})\right)}\right)$$

$$\circ R\left(\alpha_{n+3}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\right)}\right)$$

$$\leq R\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ R\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$\circ R\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\phi(\alpha_{n+2}, \alpha_{n+3})\right)}\right)$$

$$\circ R\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\phi(\alpha_{n+3}, \alpha_{n+4})\right)}\right) \circ \cdots \circ$$

$$R\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q}) \cdots \eta(\alpha_{n+q-2}, \alpha_{n+q})\phi(\alpha_{n+q-2}, \alpha_{n+q-1})\right)}\right)$$

$$\circ R\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q}) \cdots \eta(\alpha_{n+q-1}, \alpha_{n+q})\right)}\right)$$

Using (3) in the above inequalities, we deduce

$$\geq P\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{2(\theta)^{n-1}(\phi(\alpha_n, \alpha_{n+1}))}\right) * P\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^2(\theta)^n\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^3(\theta)^{n+1}\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\phi(\alpha_{n+2}, \alpha_{n+3})\right)}\right)$$

$$* P\left(\alpha_0, \alpha_1, \frac{\mathcal{T}}{(2)^4(\theta)^{n+2}\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\phi(\alpha_{n+3}, \alpha_{n+4})\right)}\right) * \cdots *$$

$$P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\theta)^{n+q-2}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$*P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\theta)^{n+q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right),$$

$$\leq Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2(\theta)^{n-1}\left(\phi(\alpha_n,\alpha_{n+1})\right)}\right)\circ Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2(\theta)^n\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3(\theta)^{n+1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$\circ Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^4(\theta)^{n+2}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\phi(\alpha_{n+3},\alpha_{n+4})\right)}\right)$$

$$\circ\cdots\circ$$

$$Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\theta)^{n+q-2}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\theta)^{n+q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)$$

and

$$\leq R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2(\theta)^{n-1}\left(\phi(\alpha_n,\alpha_{n+1})\right)}\right)\circ R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2(\theta)^n\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3(\theta)^{n+1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$\circ R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^4(\theta)^{n+2}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\phi(\alpha_{n+3},\alpha_{n+4})\right)}\right)$$

$$\circ\cdots\circ$$

$$R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\theta)^{n+q-2}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}(\theta)^{n+q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)$$

Using (1), for $n \to \infty$, we deduce

$$\lim_{n\to\infty} P(\alpha_n, \alpha_{n+q}, \mathcal{T}) = 1 * 1 * \cdots * 1 = 1,$$

$$\lim_{n\to\infty} Q(\alpha_n, \alpha_{n+q}, \mathcal{T}) = 0 \circ 0 \circ \cdots \circ 0 = 0$$

and

$$\lim_{n\to\infty} R(\alpha_n, \alpha_{n+q}, \mathcal{T}) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

i.e., $\{\alpha_n\}$ is a CS. Since $(\Sigma, P, Q, R, *, \circ)$ be a complete NDCMLS, there exists

$$\lim_{n\to\infty} P(\alpha_n, \alpha_{n+q}, \mathcal{T}) = \lim_{n\to\infty} P(\alpha_n, \alpha, \mathcal{T}) = P(\alpha, \alpha, \mathcal{T}),$$
$$\lim_{n\to\infty} Q(\alpha_n, \alpha_{n+q}, \mathcal{T}) = \lim_{n\to\infty} Q(\alpha_n, \alpha, \mathcal{T}) = Q(\alpha, \alpha, \mathcal{T}),$$
$$\lim_{n\to\infty} R(\alpha_n, \alpha_{n+q}, \mathcal{T}) = \lim_{n\to\infty} R(\alpha_n, \alpha, \mathcal{T}) = R(\alpha, \alpha, \mathcal{T}).$$

Now investigate that α is a FP of Ψ, using (v), (x), (xv) and (1), we obtain

$$P(\alpha, \Psi\alpha, \mathcal{T}) \geq P\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) * P\left(\alpha_{n+1}, \Psi\alpha, \frac{\mathcal{T}}{2(\eta(\alpha_{n+1}, \Psi\alpha))}\right)$$

$$P(\alpha, \Psi\alpha, \mathcal{T}) \geq P\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) * P\left(\Psi\alpha_n, \Psi\alpha, \frac{\mathcal{T}}{2(\eta(\alpha_{n+1}, \Psi\alpha))}\right)$$

$$P(\alpha, \Psi\alpha, \mathcal{T}) \geq P\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) * P\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\theta(\eta(\alpha_{n+1}, \Psi\alpha))}\right) \to 1 * 1 = 1$$

as $n \to \infty$,

$$Q(\alpha, \Psi\alpha, \mathcal{T}) \leq Q\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) \circ Q\left(\alpha_{n+1}, \Psi\alpha, \frac{\mathcal{T}}{2(\eta(\alpha_{n+1}, \Psi\alpha))}\right)$$

$$Q(\alpha, \Psi\alpha, \mathcal{T}) \leq Q\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) \circ Q\left(\Psi\alpha_n, \Psi\alpha, \frac{\mathcal{T}}{2(\eta(\alpha_{n+1}, \Psi\alpha))}\right)$$

$$Q(\alpha, \Psi\alpha, \mathcal{T}) \leq Q\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) \circ Q\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\theta(\eta(\alpha_{n+1}, \Psi\alpha))}\right) \to 0 \circ 0 = 0$$

as $n \to \infty$, and

$$R(\alpha, \Psi\alpha, \mathcal{T}) \leq R\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) \circ R\left(\alpha_{n+1}, \Psi\alpha, \frac{\mathcal{T}}{2(\eta(\alpha_{n+1}, \Psi\alpha))}\right)$$

$$R(\alpha, \Psi\alpha, \mathcal{T}) \leq R\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) \circ R\left(\Psi\alpha_n, \Psi\alpha, \frac{\mathcal{T}}{2(\eta(\alpha_{n+1}, \Psi\alpha))}\right)$$

$$R(\alpha, \Psi\alpha, \mathcal{T}) \leq R\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha, \alpha_{n+1}))}\right) \circ R\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\theta(\eta(\alpha_{n+1}, \Psi\alpha))}\right) \to 0 \circ 0 = 0$$

as $n \to \infty$. This implies that $\Psi\alpha = \alpha$, a FP. Now we show the uniqueness, suppose $\Psi\rho = \rho$ for some $\rho \in \Sigma$, then

$$1 \geq P(\rho, \alpha, \mathcal{T}) = P(\Psi\rho, \Psi\alpha, \mathcal{T}) \geq P\left(\rho, \alpha, \frac{\mathcal{T}}{\theta}\right) = P\left(\Psi\rho, \Psi\alpha, \frac{\mathcal{T}}{\theta}\right)$$

$$\geq P\left(\rho, \alpha, \frac{\mathcal{T}}{\theta^2}\right) \geq \cdots \geq P\left(\rho, \alpha, \frac{\mathcal{T}}{\theta^n}\right) \to 1 \text{ as } n \to \infty,$$

$$0 \leq Q(\rho, \alpha, \mathcal{T}) = Q(\Psi\rho, \Psi\alpha, \mathcal{T}) \leq Q\left(\rho, \alpha, \frac{\mathcal{T}}{\theta}\right) = Q\left(\Psi\rho, \Psi\alpha, \frac{\mathcal{T}}{\theta}\right)$$

$$\leq Q\left(\rho, \alpha, \frac{\mathcal{T}}{\theta^2}\right) \leq \cdots \leq Q\left(\rho, \alpha, \frac{\mathcal{T}}{\theta^n}\right) \to 0 \text{ as } n \to \infty,$$

and

$$0 \leq R(\rho, \alpha, \mathcal{T}) = R(\Psi\rho, \Psi\alpha, \mathcal{T}) \leq R\left(\rho, \alpha, \frac{\mathcal{T}}{\theta}\right) = R\left(\Psi\rho, \Psi\alpha, \frac{\mathcal{T}}{\theta}\right)$$

$$\leq R\left(\rho, \alpha, \frac{\mathcal{T}}{\theta^2}\right) \leq \cdots \leq R\left(\rho, \alpha, \frac{\mathcal{T}}{\theta^n}\right) \to 0 \text{ as } n \to \infty,$$

by using (iii), (viii) and (xiii), $\alpha = \rho$.

Definition 2.4 Let $(\Sigma, P, Q, R, *, \circ)$ be a NDCMLS. A map $\Psi: \Sigma \to \Sigma$ is ND-controlled like contraction if there exists $0 < \theta < 1$, such that

$$\frac{1}{P(\Psi\alpha, \Psi\beta, \mathcal{T})} - 1 \leq \theta\left[\frac{1}{P(\alpha, \beta, \mathcal{T})} - 1\right] \qquad (4)$$

$$Q(\Psi\alpha, \Psi\beta, \mathcal{T}) \leq \theta Q(\alpha, \beta, \mathcal{T}), \qquad (5)$$

and

$$R(\Psi\alpha, \Psi\beta, \mathcal{T}) \leq \theta R(\alpha, \beta, \mathcal{T}), \qquad (6)$$

for all $\alpha, \beta \in \Sigma$ and $\mathcal{T} > 0$.

Now we prove the theorem for ND-controlled like contraction.

Theorem 2.2 Let $(\Sigma, P, Q, R, *, \circ)$ be a complete NDCMLS with $\phi, \eta: \Sigma \times \Sigma \to [1, \infty)$ and suppose that

$$\lim_{\mathcal{T} \to \infty} P(\alpha, \beta, \mathcal{T}) = 1, \lim_{\mathcal{T} \to \infty} Q(\alpha, \beta, \mathcal{T}) = 0 \text{ and } \lim_{\mathcal{T} \to \infty} R(\alpha, \beta, \mathcal{T}) = 0 \qquad (7)$$

for all $\alpha, \beta \in \Sigma$ and $\mathcal{T} > 0$. Let $\Psi: \Sigma \to \Sigma$ be a ND-controlled like contraction. Further, suppose that for an arbitrary $\alpha_0 \in \Sigma$, and $n, q \in \mathbb{N}$, where $\alpha_n = \Psi^n \alpha_0 = \Psi\alpha_{n-1}$. Then Ψ has a unique FP.

Proof: Let α_0 be a random integer of Σ and describe a sequence α_n by $\alpha_n = \Psi^n \alpha_0 = \Psi\alpha_{n-1}$, $n \in \mathbb{N}$. By using (4), (5) and (6) for all $\mathcal{T} > 0$, $n > q$, we have

$$\frac{1}{P(\alpha_n, \alpha_{n+1}, \mathcal{T})} - 1 = \frac{1}{P(\Psi\alpha_{n-1}, \alpha_n, \mathcal{T})} - 1$$

$$\leq \theta\left[\frac{1}{P(\alpha_{n-1}, \alpha_n, \mathcal{T})} - 1\right] = \frac{\theta}{P(\alpha_{n-1}, \alpha_n, \mathcal{T})} - \theta$$

$$\Rightarrow \frac{1}{P(\alpha_n, \alpha_{n+1}, \mathcal{T})} \leq \frac{\theta}{P(\alpha_{n-1}, \alpha_n, \mathcal{T})} + (1 - \theta)$$

$$\leq \frac{\theta^2}{P(\alpha_{n-2},\alpha_{n-1},\mathcal{T})} + \theta(1-\theta) + (1-\theta)$$

Continuing in this way, we get

$$\frac{1}{P(\alpha_n,\alpha_{n+1},\mathcal{T})} \leq \frac{\theta^n}{P(\alpha_0,\alpha_1,\mathcal{T})} + \theta^{n-1}(1-\theta) + \theta^{n-2}(1-\theta) + \cdots + \theta(1-\theta) + (1-\theta)$$

$$\leq \frac{\theta^n}{P(\alpha_0,\alpha_1,\mathcal{T})} + (\theta^{n-1} + \theta^{n-2} + \cdots + 1)(1-\theta) \leq \frac{\theta^n}{P(\alpha_0,\alpha_1,\mathcal{T})} + (1-\theta^n)$$

We obtain

$$\frac{1}{\frac{\theta^n}{P(\alpha_0,\alpha_1,\mathcal{T})} + (1-\theta^n)} \leq P(\alpha_n,\alpha_{n+1},\mathcal{T}) \qquad (8)$$

$$Q(\alpha_n,\alpha_{n+1},\mathcal{T}) = Q(\Psi\alpha_{n-1},\alpha_n,\mathcal{T}) \leq \theta Q(\alpha_{n-1},\alpha_n,\mathcal{T}) = Q(\Psi\alpha_{n-2},\alpha_{n-1},\mathcal{T})$$

$$\leq \theta^2 Q(\alpha_{n-2},\alpha_{n-1},\mathcal{T}) \leq \cdots \leq \theta^n Q(\alpha_0,\alpha_1,\mathcal{T}) \qquad (9)$$

and

$$R(\alpha_n,\alpha_{n+1},\mathcal{T}) = R(\Psi\alpha_{n-1},\alpha_n,\mathcal{T}) \leq \theta R(\alpha_{n-1},\alpha_n,\mathcal{T}) = R(\Psi\alpha_{n-2},\alpha_{n-1},\mathcal{T})$$

$$\leq \theta^2 R(\alpha_{n-2},\alpha_{n-1},\mathcal{T}) \leq \cdots \leq \theta^n R(\alpha_0,\alpha_1,\mathcal{T}) \qquad (10)$$

for any $q \in \mathbb{N}$, using (v), (x) and (xv) , we deduce

$$P(\alpha_n,\alpha_{n+q},\mathcal{T}) \geq P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\eta(\alpha_{n+1},\alpha_{n+q})\right)}\right)$$

$$\geq P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_{n+2},\alpha_{n+q},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\right)}\right)$$

$$\geq P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$* P\left(\alpha_{n+3},\alpha_{n+q},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\right)}\right)$$

$$\geq P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right) * P\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$* P\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\phi(\alpha_{n+2}, \alpha_{n+3})\right)}\right)$$

$$* P\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\phi(\alpha_{n+3}, \alpha_{n+4})\right)}\right) * \cdots *$$

$$P\left(\alpha_{n+q-2}, \alpha_{n+q-1}, \frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\cdots\eta(\alpha_{n+q-2}, \alpha_{n+q})\phi(\alpha_{n+q-2}, \alpha_{n+q-1})\right)}\right)$$

$$* P\left(\alpha_{n+q-1}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\cdots\eta(\alpha_{n+q-1}, \alpha_{n+q})\right)}\right),$$

$$Q(\alpha_n, \alpha_{n+q}, \mathcal{T}) \leq Q\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ Q\left(\alpha_{n+1}, \alpha_{n+q}, \frac{\mathcal{T}}{2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\right)}\right)$$

$$\leq Q\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ Q\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_{n+2}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\right)}\right)$$

$$\leq Q\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ Q\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\phi(\alpha_{n+2}, \alpha_{n+3})\right)}\right)$$

$$\circ Q\left(\alpha_{n+3}, \alpha_{n+q}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\right)}\right)$$

$$\leq Q\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2(\phi(\alpha_n, \alpha_{n+1}))}\right) \circ Q\left(\alpha_{n+1}, \alpha_{n+2}, \frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1}, \alpha_{n+q})\phi(\alpha_{n+1}, \alpha_{n+2})\right)}\right)$$

$$\circ Q\left(\alpha_{n+2}, \alpha_{n+3}, \frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\phi(\alpha_{n+2}, \alpha_{n+3})\right)}\right)$$

$$\circ Q\left(\alpha_{n+3}, \alpha_{n+4}, \frac{\mathcal{T}}{(2)^4\left(\eta(\alpha_{n+1}, \alpha_{n+q})\eta(\alpha_{n+2}, \alpha_{n+q})\eta(\alpha_{n+3}, \alpha_{n+q})\phi(\alpha_{n+3}, \alpha_{n+4})\right)}\right) \circ \cdots \circ$$

$$Q\left(\alpha_{n+q-2},\alpha_{n+q-1},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ\, Q\left(\alpha_{n+q-1},\alpha_{n+q},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)$$

and

$$R(\alpha_n,\alpha_{n+q},\mathcal{T})\le R\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ R\left(\alpha_{n+1},\alpha_{n+q},\frac{\mathcal{T}}{2\left(\eta(\alpha_{n+1},\alpha_{n+q})\right)}\right)$$

$$\le R\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ R\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ\, R\left(\alpha_{n+2},\alpha_{n+q},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\right)}\right)$$

$$\le R\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ R\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ\, R\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$\circ\, R\left(\alpha_{n+3},\alpha_{n+q},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\right)}\right)$$

$$\le R\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ R\left(\alpha_{n+1},\alpha_{n+2},\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ\, R\left(\alpha_{n+2},\alpha_{n+3},\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)$$

$$\circ\, R\left(\alpha_{n+3},\alpha_{n+4},\frac{\mathcal{T}}{(2)^4\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\phi(\alpha_{n+3},\alpha_{n+4})\right)}\right)\circ\cdots\circ$$

$$R\left(\alpha_{n+q-2},\alpha_{n+q-1},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ\, R\left(\alpha_{n+q-1},\alpha_{n+q},\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right).$$

$$P(\alpha_n,\alpha_{n+q},\mathcal{T}) \geq \frac{1}{\frac{\theta^n}{P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)}+(1-\theta^n)}$$

$$*\frac{1}{\frac{\theta^{n+1}}{P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)}+(1-\theta^{n+1})}$$

$$*\frac{1}{\frac{\theta^{n+2}}{P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)}+(1-\theta^{n+2})}*\cdots*$$

$$\frac{1}{\frac{\theta^{n+q-2}}{P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)}+(1-\theta^{n+q-2})}$$

$$*\frac{1}{\frac{\theta^{n+q-1}}{P\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)}+(1-\theta^{n+q-1})},$$

$$Q(\alpha_n,\alpha_{n+q},\mathcal{T}) \leq \theta^n Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ\theta^{n+1}Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ\,\theta^{n+2}Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)\circ\cdots\circ$$

$$\theta^{n+q-2}Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ\,\theta^{n+q-1}Q\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)$$

and

$$R(\alpha_n,\alpha_{n+q},\mathcal{T}) \leq \theta^n R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{2(\phi(\alpha_n,\alpha_{n+1}))}\right)\circ\theta^{n+1}R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^2\left(\eta(\alpha_{n+1},\alpha_{n+q})\phi(\alpha_{n+1},\alpha_{n+2})\right)}\right)$$

$$\circ\,\theta^{n+2}R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^3\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\phi(\alpha_{n+2},\alpha_{n+3})\right)}\right)\circ\cdots\circ$$

$$\theta^{n+q-2}R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\cdots\eta(\alpha_{n+q-2},\alpha_{n+q})\phi(\alpha_{n+q-2},\alpha_{n+q-1})\right)}\right)$$

$$\circ\,\theta^{n+q-1}R\left(\alpha_0,\alpha_1,\frac{\mathcal{T}}{(2)^{q-1}\left(\eta(\alpha_{n+1},\alpha_{n+q})\eta(\alpha_{n+2},\alpha_{n+q})\eta(\alpha_{n+3},\alpha_{n+q})\cdots\eta(\alpha_{n+q-1},\alpha_{n+q})\right)}\right)$$

Therefore,

$$\lim_{n\to\infty}P(\alpha_n,\alpha_{n+q},\mathcal{T})=1*1*\cdots*=1,$$

$$\lim_{n\to\infty}Q(\alpha_n,\alpha_{n+q},\mathcal{T})=0\circ 0\circ\cdots\circ 0=0$$

and

$$\lim_{n\to\infty}R(\alpha_n,\alpha_{n+q},\mathcal{T})=0\circ 0\circ\cdots\circ 0=0$$

i.e., $\{\alpha_n\}$ is a CS. Since $(\Sigma,P,Q,R,*,\circ)$ be a complete NDCMLS, there exists

$$\lim_{n\to\infty}P(\alpha_n,\alpha_{n+q},\mathcal{T})=\lim_{n\to\infty}P(\alpha_n,\alpha,\mathcal{T})=P(\alpha,\alpha,\mathcal{T}),$$
$$\lim_{n\to\infty}Q(\alpha_n,\alpha_{n+q},\mathcal{T})=\lim_{n\to\infty}Q(\alpha_n,\alpha,\mathcal{T})=Q(\alpha,\alpha,\mathcal{T}),$$
$$\lim_{n\to\infty}R(\alpha_n,\alpha_{n+q},\mathcal{T})=\lim_{n\to\infty}R(\alpha_n,\alpha,\mathcal{T})=R(\alpha,\alpha,\mathcal{T}).$$

Now investigate that α is a FP of Ψ, using (v), (x) and (xv), we obtain

$$\frac{1}{P(\Psi\alpha_n,\Psi\alpha,\mathcal{T})}-1\le\theta\left[\frac{1}{P(\alpha_n,\alpha,\mathcal{T})}-1\right]=\frac{\theta}{P(\alpha_n,\alpha,\mathcal{T})}-\theta$$

$$\Rightarrow\frac{1}{\frac{\theta}{P(\alpha_n,\alpha,\mathcal{T})}+(1-\theta)}\le P(\Psi\alpha_n,\Psi\alpha,\mathcal{T})$$

Using above inequality, we obtain

$$P(\alpha,\Psi\alpha,\mathcal{T})\ge P\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi(\alpha,\alpha_{n+1})}\right)*P\left(\alpha_{n+1},\Psi\alpha,\frac{\mathcal{T}}{2\eta(\alpha_{n+1},\Psi\alpha)}\right)$$

$$\ge P\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi(\alpha,\alpha_{n+1})}\right)*P\left(\Psi\alpha_n,\Psi\alpha,\frac{\mathcal{T}}{2\eta(\alpha_{n+1},\Psi\alpha)}\right)$$

$$\ge P\left(\alpha_n,\alpha_{n+1},\frac{\mathcal{T}}{(2\phi(\alpha,\alpha_{n+1}))}\right)*\frac{1}{\frac{\theta}{P\left(\alpha_n,\alpha,\frac{\mathcal{T}}{2\eta(\alpha_{n+1},\Psi\alpha)}\right)}+(1-\theta)}\to 1*1=1$$

as $n\to\infty$,

$$Q(\alpha,\Psi\alpha,\mathcal{T})\le Q\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi(\alpha,\alpha_{n+1})}\right)\circ Q\left(\alpha_{n+1},\Psi\alpha,\frac{\mathcal{T}}{2\eta(\alpha_{n+1},\Psi\alpha)}\right)$$

$$\le Q\left(\alpha,\alpha_{n+1},\frac{\mathcal{T}}{2\phi(\alpha,\alpha_{n+1})}\right)\circ Q\left(\Psi\alpha_n,\Psi\alpha,\frac{\mathcal{T}}{2\eta(\alpha_{n+1},\Psi\alpha)}\right)$$

$$\leq Q\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\phi(\alpha, \alpha_{n+1})}\right) \circ \theta Q\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\eta(\alpha_{n+1}, \Psi\alpha)}\right) \to 0 \circ 0 = 0 \text{ as } n \to \infty$$

and

$$R(\alpha, \Psi\alpha, \mathcal{T}) \leq R\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\phi(\alpha, \alpha_{n+1})}\right) \circ R\left(\alpha_{n+1}, \Psi\alpha, \frac{\mathcal{T}}{2\eta(\alpha_{n+1}, \Psi\alpha)}\right)$$

$$\leq R\left(\alpha, \alpha_{n+1}, \frac{\mathcal{T}}{2\phi(\alpha, \alpha_{n+1})}\right) \circ R\left(\Psi\alpha_n, \Psi\alpha, \frac{\mathcal{T}}{2\eta(\alpha_{n+1}, \Psi\alpha)}\right)$$

$$\leq R\left(\alpha_n, \alpha_{n+1}, \frac{\mathcal{T}}{2\phi(\alpha, \alpha_{n+1})}\right) \circ \theta R\left(\alpha_n, \alpha, \frac{\mathcal{T}}{2\eta(\alpha_{n+1}, \Psi\alpha)}\right) \to 0 \circ 0 = 0 \text{ as } n \to \infty.$$

This implies that $\Psi\alpha = \alpha$, a FP. Now we show the uniqueness, suppose $\Psi\rho = \rho$ for some $\rho \in \Sigma$, then

$$\frac{1}{P(\alpha, \rho, \mathcal{T})} - 1 = \frac{1}{P(\Psi\alpha, \Psi\rho, \mathcal{T})} - 1$$

$$\leq \theta\left[\frac{1}{P(\alpha, \rho, \mathcal{T})} - 1\right] < \frac{1}{P(\alpha, \rho, \mathcal{T})} - 1$$

a contradiction,

$$Q(\alpha, \rho, \mathcal{T}) = Q(\Psi\alpha, \Psi\rho, \mathcal{T}) \leq \theta Q(\alpha, \rho, \mathcal{T}) < Q(\alpha, \rho, \mathcal{T})$$

a contradiction, and

$$R(\alpha, \rho, \mathcal{T}) = R(\Psi\alpha, \Psi\rho, \mathcal{T}) \leq \theta R(\alpha, \rho, \mathcal{T}) < R(\alpha, \rho, \mathcal{T})$$

a contradiction. Therefore, we must have $P(\alpha, \rho, \mathcal{T}) = 1, Q(\alpha, \rho, \mathcal{T}) = 0$ and $R(\alpha, \rho, \mathcal{T}) = 0$, hence $\alpha = \rho$.

Example 2.3 Let $\Sigma = [0,1]$ and $\phi, \eta: \Sigma \times \Sigma \to [1, \infty)$ be two non-comparable functions given by

$$\phi(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha = \beta, \\ \dfrac{1 + \max\{\alpha, \beta\}}{\min\{\alpha, \beta\}} & \text{if } \alpha \neq \beta \end{cases}$$

and

$$\eta(\alpha, \beta) = \begin{cases} 1 & \text{if} \quad \alpha = \beta, \\ \dfrac{1 + \max\{\alpha^2, \beta^2\}}{\min\{\alpha^2, \beta^2\}} & \text{if } \alpha \neq \beta \end{cases}.$$

Define $P, Q, R: \Sigma \times \Sigma \times (0, \infty) \to [0,1]$ as

$$P(\alpha, \beta, \mathcal{T}) = \frac{\mathcal{T}}{\mathcal{T} + \max\{\alpha, \beta\}^2}, \qquad Q(\alpha, \beta, \mathcal{T}) = \frac{\max\{\alpha, \beta\}^2}{\mathcal{T} + \max\{\alpha, \beta\}^2}, \qquad Q(\alpha, \beta, \mathcal{T}) = \frac{\max\{\alpha, \beta\}^2}{\mathcal{T}}$$

Then $(\Sigma, P, Q, R, *, \circ)$ is a complete NDCMS with CTN $\pi * \mu = \pi\mu$ and CTCN $\pi \circ \mu = \max\{\pi, \mu\}$.

Define $\Psi: \Sigma \to \Sigma$ by $\Psi(\alpha) = \frac{1-2^{-\alpha}}{3}$ and take $\theta \in \left[\frac{1}{2}, 1\right)$, then

$$P(\Psi\alpha, \Psi\beta, \theta\mathcal{T}) = P\left(\frac{1 - 2^{-\alpha}}{3}, \frac{1 - 2^{-\beta}}{3}, \theta\mathcal{T}\right)$$

$$= \frac{\theta\mathcal{T}}{\theta\mathcal{T} + \max\left\{\frac{1-2^{-\alpha}}{3}, \frac{1-2^{-\beta}}{3}\right\}^2} \geq \frac{\mathcal{T}}{\mathcal{T} + \max\{\alpha, \beta\}^2} = P(\alpha, \beta, \mathcal{T}),$$

$$Q(\Psi\alpha, \Psi\beta, \theta\mathcal{T}) = Q\left(\frac{1-2^{-\alpha}}{3}, \frac{1-2^{-\beta}}{3}, \theta\mathcal{T}\right)$$

$$= \frac{\max\left\{\frac{1-2^{-\alpha}}{3}, \frac{1-2^{-\beta}}{3}\right\}^2}{\theta\mathcal{T} + \max\left\{\frac{1-2^{-\alpha}}{3}, \frac{1-2^{-\beta}}{3}\right\}^2} \leq \frac{\max\{\alpha, \beta\}^2}{\mathcal{T} + \max\{\alpha, \beta\}^2} = Q(\alpha, \beta, \mathcal{T})$$

and

$$R(\Psi\alpha, \Psi\beta, \theta\mathcal{T}) = R\left(\frac{1-2^{-\alpha}}{3}, \frac{1-2^{-\beta}}{3}, \theta\mathcal{T}\right)$$

$$= \frac{\max\left\{\frac{1-2^{-\alpha}}{3}, \frac{1-2^{-\beta}}{3}\right\}^2}{\theta\mathcal{T}} \leq \frac{\max\{\alpha, \beta\}^2}{\mathcal{T}} = R(\alpha, \beta, \mathcal{T}).$$

Hence, all circumstances of theorem 2.1 are fulfilled and 0 is a unique fixed point for Ψ.

Conclusions

Fixed point technique is used to solve many mathematical problems as it gets involved with differential and integral equations, integro-differential equation, game theory, economics and more disciplines. The intent of this manuscript is to present a new space neutrosophic double controlled metric like space. Ultimately, to illustrate the practical side of the theoretical results. Moreover, we provided a non-trivial example to demonstrate the viability of the proposed methods. We have supplemented this work with an application that demonstrates how the built method outperforms those found in the literature. Since our structure is more general than the class of fuzzy and double controlled metric like spaces, our results and notions expand and generalize a number of previously published results.

Abbreviations

FS: Fuzzy Set

MLS: Metric-like space

FMLS: Fuzzy metric-like space

NMS: Neutrosophic metric space

NDCMLS: Neutrosophic double controlled metric-like space

FP: Fixed point

CTN: Continuous triangle norm

CTCN: Continuous triangle conorm

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Chapter Eight

Bipolar Generalized Set Valued Neutrosophic Quadruple Sets and Numbers

Abdullah Kargın[1, *] and Memet Şahin[2]

[1, 2]Department of Mathematics, Gaziantep University, Gaziantep27310-Turkey
[*]Corresponding Author: abdullahkargin27@gmail.com
E-mail: abdullahkargin27@gmail.com, mesahin@gantep.edu.tr

ABSTRACT

Generalized set valued neutrosophic quadruple sets have an important role in neutrosophic quadruple theory and single valued neutrosophic theory. Thanks to generalized set valued neutrosophic quadruple sets, the solutions of decision-making problems in which single valued neutrosophic numbers are used can be obtained more objectively. Also, bipolar single valued neutrosophic sets are more useful in neutrosophic theory, especially at decision making problems. In this chapter, we obtain bipolar generalized set valued neutrosophic quadruple sets and numbers. We give some basic properties for bipolar generalized set valued neutrosophic quadruple sets and numbers. Also, we define some new operations for bipolar generalized set valued neutrosophic quadruple sets and numbers. Thus, we obtain a new structure based on generalized set valued neutrosophic quadruple sets and bipolar single valued neutrosophic numbers. In this way, we obtain new results for generalized set valued neutrosophic quadruple set and bipolar single valued neutrosophic set. Furthermore, thanks to this new structure; the solutions of decision-making problems in which bipolar neutrosophic numbers are used will be obtained more objectively.

Keywords: bipolar single valued neutrosophic set, neutrosophic quadruple number, set valued neutrosophic quadruple set, bipolar generalized set valued neutrosophic quadruple set

INTRODUCTION

Neutrosophic logic and the concept of neutrosophic set are defined in 1998 by Florentin Smarandache [1]. In the concept of neutrosophic logic and neutrosophic sets, there is the degree of membership T, degree of uncertainty I and degree of non-membership F. These degrees are defined independently from each other. It has the form of a neutrosophic value (T, I, F). In other words, a situation is handled in neutrosophy according to its accuracy, its falsehood, and its uncertainty. In addition, many researchers have conducted studies on neutrosophic set theory [2-8]. Recently, Şahin and Kargın obtained neutrosophic triplet normed ring space [9]; Zeng et al. studied a novel similarity measure of single-valued neutrosophic sets based on modified Manhattan distance [10]; Şahin et al. introduced neutrosophic triplet partial g – metric space [11]; Şahin and Kargın obtained a new similarity measure based on single valued neutrosophic sets and decision-making applications in professional proficiencies [12]; Alhasan et al. studied neutrosophic reliability theory [13]; Bordbar et al. introduced positive implicative ideals of BCK-algebras based on neutrosophic sets and falling shadows [14].

Deli et al. studied bipolar neutrosophic sets and logic in 2015 [15] and Broumi et al. obtained bipolar single valued neutrosophic set in 2016 [16]. The bipolar single valued neutrosophic sets have an important role in neutrosophic theory and decision making problems. The use of negative and positive integers (from [-1, 0] and [0, 1] intervals) as values in bipolar single valued neutrosophic sets makes this set superior to other sets in many problem situations. Because while it is often difficult to reach a definite judgment in a decision-making situation, a decision given as an negative and positive integers will be more useful. Hence, many researchers studied based on bipolar neutrosophic sets and logic [17-19]. Recently, Sugapriya et al. obtained two-warehouse system for trapezoidal bipolar neutrosophic disparate expeditious worsen items with power demand pattern [20]; Jamil et al. studied multicriteria decision-making methods using bipolar neutrosophic hamacher geometric aggregation operators [21]; Arulpandy and Pricilla introduced bipolar neutrosophic graded soft sets and their topological spaces [22].

Smarandache obtained neutrosophic quadruple set and numbers in 2015 [23]. While neutrosophic quadruple set have T, I and F components as in neutrosophic sets; unlike neutrosophic sets, there is a known part and an unknown part. Therefore, neutrosophic quadruple sets are a generalization of neutrosophic sets. For this reason, neutrosophic quadruple sets are widely used in the algebraic and application areas [24-29]. Recently, Li et al. introduced neutrosophic extended triplet group based on neutrosophic quadruple numbers [30]; Şahin et al. obtained neutrosophic triplet field and neutrosophic triplet vector space based on set valued neutrosophic quadruple number [31]; Borzooei et al. studied positive implicative neutrosophic quadruple BCK-algebras and ideals [32]; Şahin and Kargın introduced neutrosophic triplet metric space based on set valued neutrosophic quadruple number [33]; Smarandache et al. obtained neutrosophic quadruple groups [34]; Şahin et al. studied multi-criteri decision-making applications based on set valued generalized neutrosophic quadruple sets for law [35]. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory [36-70].

In this chapter, we obtain bipolar generalized set valued neutrosophic quadruple sets (BgsvNqs) and numbers (BgsvNqsn) using generalized set valued neutrosophic quadruple sets and bipolar single valued neutrosophic sets. Thanks to BgsvNqs and BgsvNqsn, generalized set valued neutrosophic quadruple sets and bipolar single valued neutrosophic sets will more useful together. Also, we obtain some basic properties and some operations ($\cup_A$, $\cup_O$, $\cup_P$, $\cap_A$, $\cap_O$, $\cap_P$, $/_A$, $/_O$, $/_P$). In fact, we generalize the some operations in [24] for BgsvNqs. In Section 2, we introduced some basic definitions for bipolar single valued neutrosophic set [15], neutrosophic quadruple sets [24], [27].

BACKGROUND

Definition 1. [15] Let A be a universal set. Bipolar set valued neutrosophic set N; is identified as

$$N = \{(a, T_N^+(a), I_N^+(a), F_N^+(a), T_N^-(a), I_N^-(a), F_N^-(a)): a \in A\}.$$

Where the functions

$T_N^+,\ I_N^+,\ F_N^+ : X \to [0,1]$ are the positive degrees of truth functions, uncertainly functions and falsity functions; respectively.

$T_N^-,\ I_N^-,\ F_N^- : X \to [-1, 0]$ are the negative degrees of truth functions, uncertainly functions and falsity functions; respectively.

Definition 2. [15] Let A be a universal set and

$$N_1 = \{(a, T_{N_1}^+(a), I_{N_1}^+(a), F_{N_1}^+(a), T_{N_1}^-(a), I_{N_1}^-(a), F_{N_1}^-(a)): a \in A\}$$

and

$$N_2 = \{(a, T_{N_2}^+(a), I_{N_2}^+(a), F_{N_2}^+(a), T_{N_2}^-(a), I_{N_2}^-(a), F_{N_2}^-(a)): a \in A\}$$

be two bipolar single valued neutrosophic sets.

i) N_2 is subset of N_1 if and only if

$T^+{}_{N_1(a)} \geq T^+{}_{N_2(a)}, T^-{}_{N_1(a)} \geq T^-{}_{N_2(a)}$

$I^+{}_{N_1(a)} \leq I^+{}_{N_2(a)}, I^-{}_{N_1(a)} \leq I^-{}_{N_2(a)}$

$F^+{}_{N_1(a)} \leq F^+{}_{N_2(a)}, F^-{}_{N_1(a)} \leq F^-{}_{N_2(a)}$

ii) N_2 is equal to N_1 if and only if

${T^+}_{N_1(\mathrm{a})} = {T^+}_{N_2(\mathrm{a})}, {T^-}_{N_1(\mathrm{a})} = {T^-}_{N_2(\mathrm{a})}$;

${I^+}_{N_1(\mathrm{a})} = {I^+}_{N_2(\mathrm{a})}, {I^-}_{N_1(\mathrm{a})} = {I^-}_{N_2(\mathrm{a})}$;

${F^+}_{N_1(\mathrm{a})} = +_{N_2(\mathrm{a})}, {F^-}_{N_1(\mathrm{a})} = {F^-}_{N_2(\mathrm{a})}$.

iii) $N_1 \cup N_2$ = {(a: $\max\{{T^+}_{N_1(\mathrm{a})}, {T^+}_{N_2(\mathrm{a})}\}, \min\{{I^+}_{N_1(\mathrm{a})}, {I^+}_{N_2(\mathrm{a})}\}$, $\min\{{F^+}_{N_1(\mathrm{a})}, {F^+}_{N_2(\mathrm{a})}\}, \max\{{T^-}_{N_1(\mathrm{a})}, {T^-}_{N_2(\mathrm{a})}\}$, $\min\{{I^-}_{N_1(\mathrm{a})}, {I^-}_{N_2(\mathrm{a})}\}$, , $\min\{{F^-}_{N_1(\mathrm{a})}, {F^-}_{N_2(\mathrm{a})}\}$), a∈A}.

iv) $N_1 \cap N_2$ = {(a: $\min\{{T^+}_{N_1(\mathrm{a})}, {T^+}_{N_2(\mathrm{a})}\}, \max\{{I^+}_{N_1(\mathrm{a})}, {I^+}_{N_2(\mathrm{a})}\}$, $\max\{{F^+}_{N_1(\mathrm{a})}, {F^+}_{N_2(\mathrm{a})}\}, \min\{{T^-}_{N_1(\mathrm{a})}, {T^-}_{N_2(\mathrm{a})}\}$, $\max\{{I^-}_{N_1(\mathrm{a})}, {I^-}_{N_2(\mathrm{a})}\}$, , $\max\{{F^-}_{N_1(\mathrm{a})}, {F^-}_{N_2(\mathrm{a})}\}$), a∈A}.

Definition 3: [27] Let N be a set and P(N) be power set of N. A set valued neutrosophic quadruple set is shown by the form

$$(A_1, A_2\mathrm{T}, A_3\mathrm{I}, A_4\mathrm{F}).$$

Where, T, I and F are degree of membership, degree of undeterminacy, degree of non-membership in neutrosophic theory, respectively. Also, A_1, A_2, A_3, A_4 ∈ P(N); A_1 is called the known part and (A_1, A_2T, A_3I, A_4F) is called the unknown part.

Definition 4: [27] Let A = (A_1, A_2T, A_3I, A_4F) and B = (B_1, B_2T, B_3I, B_4F) be set valued neutrosophic quadruple set s. We define the following operations, well known operators in set theory, such that

A ∪ B = ($A_1 \cup B_1$, $(A_2 \cup B_2)$T, $(A_3 \cup B_3)$I, $(A_4 \cup B_4)$F)

A ∩ B = ($A_1 \cap B_1$, $(A_2 \cap B_2)$T, $(A_3 \cap B_3)$I, $(A_4 \cap B_4)$F)

A \ B = ($A_1 \setminus B_1$, $(A_2 \setminus B_2)$T, $(A_3 \setminus B_3)$I, $(A_4 \setminus B_4)$F)

A' = (${A'}_1$, ${A'}_2$T, ${A'}_3$I, ${A'}_4$F)

Definition 5: [27] Let A = (A_1, A_2T, A_3I, A_4F), B = (B_1, B_2T, B_3I, B_4F) be set valued neutrosophic quadruple sets. If

$$A_1 \subset B_1, A_2 \subset B_2 \text{ and } A_3 \subset B_3, A_4 \subset B_4,$$

then it is called that A is subset of B. It is shown by

$$A \subset B.$$

Definition 6: [27] Let A = (A_1, A_2T, A_3I, A_4F), B = (B_1, B_2T, B_3I, B_4F) be set valued neutrosophic quadruple sets. If

$$A \subset B \text{ and } B \subset A,$$

then it is called that A is equal to B. It is shown by

$$A = B.$$

Definition 7: [24] Let A be a universal set and P(A) be power set of A. A generalized set valued neutrosophic quadruple set N; is identified as

$$N = \{<{K^1}_{N_1}, T_{N_1(a)}{L^1}_{N_1}, I_{N_1(a)}\, {M^1}_{N_1}, F_{N_1(a)}\, {P^1}_{N_1};$$

$${K^2}_{N_2}, T_{N_2(a)}{L^2}_{N_2}, I_{N_2(a)}\, {M^2}_{N_2}, F_{N_2(a)}\, {P^2}_{N_2};$$

$${K^i}_{N_i}, T_{N_i(a)}{L^i}_{N_i}, I_{N_i(a)}\, {M^i}_{N_i}, F_{N_i(a)}\, {P^i}_{N_i} >,$$

$${K^n}_{N_n},\ {L^n}_{N_n},\ {M^n}_{N_n},\ {P^n}_{N_n} \in P(A);\ n = 1, 2, 3, \dots, i\}.$$

Where,

$$T_{N_n(a)}, I_{N_n(a)}, F_{N_n(a)}\ (n = 1, 2, 3, \dots, i)$$

have their usual single valued neutrosophic logic means and a generalized set valued neutrosophic quadruple number ${N^n}_1$; is identified as

$${N^N}_1 = \{<{K^1}_{N_1}, T_{N_1(a)}{L^1}_{N_1}, I_{N_1(a)}\, {M^1}_{N_1}, F_{N_1(a)}\, {P^1}_{N_1} >\}.$$

As in neutrosophic quadruple number, for a generalized set valued neutrosophic quadruple number

$${K^1}_{N_1}$$

is called known part and

$$T_{N_1(a)}{L^1}_{N_1}, I_{N_1(a)}\, {M^1}_{N_1}, F_{N_1(a)}\, {P^1}_{N_1}$$

is called the unknown part.

Also, we can show that

$$N = \{N^N{}_n: n = 1, 2, 3, \ldots, i\}.$$

BIPOLAR GENERALIZED SET VALUED NEUTROSOPHIC QUADRUPLE SETS AND NUMBERS

Definition 8: Let A be a universal set and P(A) be power set of A. Bipolar generalized set valued neutrosophic quadruple set (BgsvNqs) N; is identified as

$$N = \{<K^1{}_{N_1}, (T^-{}_{N_1(a)}, T^+{}_{N_1(a)})L^1{}_{N_1}, (I^-{}_{N_1(a)}, I^+{}_{N_1(a)})\, M^1{}_{N_1}, (F^-{}_{N_1(a)}, F^+{}_{N_1(a)})\, P^1{}_{N_1};$$

$$K^2{}_{N_2}, (T^-{}_{N_2(a)}, T^+{}_{N_2(a)})L^2{}_{N_2}, (I^-{}_{N_2(a)}, I^+{}_{N_2(a)})\, M^2{}_{N_2}, (F^-{}_{N_2(a)}, F^+{}_{N_2(a)})\, P^2{}_{N_2};$$

$$K^i{}_{N_i}, (T^-{}_{N_i(a)}, T^+{}_{N_i(a)})L^i{}_{N_i}, (I^-{}_{N_i(a)}, I^+{}_{N_i(a)})\, M^i{}_{N_i}, (F^-{}_{N_i(a)}, F^+{}_{N_i(a)})\, P^i{}_{N_i} >,$$

$$K^n{}_{N_n},\ L^n{}_{N_n},\ M^n{}_{N_n},\ P^n{}_{N_n} \in P(A); n = 1, 2, 3, \ldots, i\}.$$

Where,

$$T^-{}_{N_n(a)}, I^-{}_{N_n(a)}, F^-{}_{N_n(a)}, T^+{}_{N_n(a)}, I^+{}_{N_n(a)} \text{ and } F^+{}_{N_n(a)}\ (n = 1, 2, 3, \ldots, i)$$

have their usual bipolar single valued neutrosophic logic means.

Also, an bipolar generalized neutrosophic quadruple number (BgsvNqn) $N^n{}_1$; is identified as

$$N^N{}_1 = \{<K^1{}_{N_1}, (T^-{}_{N_1(a)}, T^+{}_{N_1(a)})L^1{}_{N_1}, (I^-{}_{N_1(a)}, I^+{}_{N_1(a)})\, M^1{}_{N_1}, (F^-{}_{N_1(a)}, F^+{}_{N_1(a)})\, P^1{}_{N_1} >\}$$

as in neutrosophic quadruple number, for a BgsvNqn,

$$K^1{}_{N_1}$$

is called known part and

$$(T^-{}_{N_1(a)}, T^+{}_{N_1(a)})L^1{}_{N_1}, (I^-{}_{N_1(a)}, I^+{}_{N_1(a)})\, M^1{}_{N_1}, (F^-{}_{N_1(a)}, F^+{}_{N_1(a)})\, P^1{}_{N_1}$$

is called the unknown part.

Also, we can show that

$$N = \{N^N{}_n: n = 1, 2, 3, \ldots, i\}.$$

Example 1: Let A = {k, l, m, n, p, r} be a set. Then;

$$N = \{<\{k, l, m, n\}, (0, 0.7)\{k, l\}, (0.5, 0.6)\, \{m\}, (-0.4, 0.5)\, \{n\};$$

$$\{k, l, p, r\}, (-0.1, 0.9)\{k, p\}, (-0.2, 0.3)\, \{l\}, (-0.2, 0.7)\, \{r\} >\}$$

and

$$R = \{<\{l, p, m, n, k\}, (-0.4, 0.8)\{l, p\}, -0, 0.3)\, \{p, m\}, (-0.2, 0.6)\, \{n\};$$

$\{m, l, p, r\}$, $(-0.3, 0.7)\{p\}$, $(-0.2, 0.5)$ $\{m, l\}$, $(-0.1, 0.5)$ $\{r\} >\}$

are two BgsvNqs.

Also,

$N^N{}_1$=$\{\{k, l, m, n\}$, $(0, 0.7)\{k, l\}$, $(0.5, 0.6)$ $\{m\}$, $(-0.4, 0.5)$ $\{n\}\}$

and

$N^N{}_2 = \{\{k, l, p, r\}$, $(-0.1, 0.9)\{k, p\}$, $(-0.2, 0.3)$ $\{l\}$, $(-0.2, 0.7)$ $\{r\}\}$

are two BgsvNqn of N such that

N = $\{N^N{}_1, N^N{}_2\}$.

Similarly,

$R^N{}_1$ =$\{\{l, p, m, n, k\}$, $(-0.4, 0.8)\{l, p\}$, $-0, 0.3)$ $\{p, m\}$, $(-0.2, 0.6)$ $\{n\}\}$;

and

$R^N{}_2 =$ $\{\{m, l, p, r\}$, $(-0.3, 0.7)\{p\}$, $(-0.2, 0.5)$ $\{m, l\}$, $(-0.1, 0.5)$ $\{r\}\}$

are two BgsvNqn of R such that

R = $\{R^N{}_1, R^N{}_2\}$.

Definition 9: Let

N = {<$K^1{}_{N_1}$, $(T^-{}_{N_1(\mathrm{a})}, T^+{}_{N_1(\mathrm{a})})L^1{}_{N_1}$, $(I^-{}_{N_1(\mathrm{a})}, I^+{}_{N_1(\mathrm{a})})$ $M^1{}_{N_1}$, $(F^-{}_{N_1(\mathrm{a})}, F^+{}_{N_1(\mathrm{a})})$ $P^1{}_{N_1}$;

$K^2{}_{N_2}$, $(T^-{}_{N_2(\mathrm{a})}, T^+{}_{N_2(\mathrm{a})})L^2{}_{N_2}$, $(I^-{}_{N_2(\mathrm{a})}, I^+{}_{N_2(\mathrm{a})})$ $M^2{}_{N_2}$, $(F^-{}_{N_2(\mathrm{a})}, F^+{}_{N_2(\mathrm{a})})$ $P^2{}_{N_2}$;

$K^i{}_{N_i}$, $(T^-{}_{N_i(\mathrm{a})}, T^+{}_{N_i(\mathrm{a})})L^i{}_{N_i}$, $(I^-{}_{N_i(\mathrm{a})}, I^+{}_{N_i(\mathrm{a})})$ $M^i{}_{N_i}$, $(F^-{}_{N_i(\mathrm{a})}, F^+{}_{N_i(\mathrm{a})})$ $P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {<$K^1{}_{R_1}$, $(T^-{}_{R_1(\mathrm{a})}, T^+{}_{R_1(\mathrm{a})})L^1{}_{R_1}$, $(I^-{}_{R_1(\mathrm{a})}, I^+{}_{R_1(\mathrm{a})})$ $M^1{}_{R_1}$, $(F^-{}_{R_1(\mathrm{a})}, F^+{}_{R_1(\mathrm{a})})$ $P^1{}_{R_1}$;

$K^2{}_{R_2}$, $(T^-{}_{R_2(\mathrm{a})}, T^+{}_{R_2(\mathrm{a})})L^2{}_{R_2}$, $(I^-{}_{R_2(\mathrm{a})}, I^+{}_{R_2(\mathrm{a})})$ $M^2{}_{R_2}$, $(F^-{}_{R_2(\mathrm{a})}, F^+{}_{R_2(\mathrm{a})})$ $P^2{}_{R_2}$;

$K^i{}_{R_i}$, $(T^-{}_{R_i(\mathrm{a})}, T^+{}_{R_i(\mathrm{a})})L^i{}_{R_i}$, $(I^-{}_{R_i(\mathrm{a})}, I^+{}_{R_i(\mathrm{a})})$ $M^i{}_{R_i}$, $(F^-{}_{R_i(\mathrm{a})}, F^+{}_{R_i(\mathrm{a})})$ $P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss.

i) N is subset of R (N⊂R) if and only if

$K^n{}_{N_n} \subset K^n{}_{R_n}$, $L^n{}_{N_n} \subset L^n{}_{R_n}$, $M^n{}_{N_n} \subset M^n{}_{R_n}$, $P^n{}_{N_n} \subset P^n{}_{R_n}$;

$T^{-}{}_{N_n(\mathrm{a})} \leq T^{-}{}_{R_n(\mathrm{a})}, T^{+}{}_{N_n(\mathrm{a})} \leq T^{+}{}_{R_n(\mathrm{a})};$

$I^{-}{}_{N_n(\mathrm{a})} \geq I^{-}{}_{R_n(\mathrm{a})}\}, I^{+}{}_{N_n(\mathrm{a})} \geq I^{+}{}_{R_n(\mathrm{a})};$

$F^{-}{}_{N_n(\mathrm{a})} \geq F^{-}{}_{R_n(\mathrm{a})}\}, F^{+}{}_{N_n(\mathrm{a})} \geq F^{+}{}_{R_n(\mathrm{a})}.$

ii) N is equal to R if and only if

$K^{n}{}_{N_n} = K^{n}{}_{R_n}, L^{n}{}_{N_n} = L^{n}{}_{R_n}, M^{n}{}_{N_n} = M^{n}{}_{R_n}, P^{n}{}_{N_n} = P^{n}{}_{R_n};$

$T^{-}{}_{N_n(\mathrm{a})} = T^{-}{}_{R_n(\mathrm{a})}, T^{+}{}_{N_n(\mathrm{a})} = T^{+}{}_{R_n(\mathrm{a})};$

$I^{-}{}_{N_n(\mathrm{a})} = I^{-}{}_{R_n(\mathrm{a})}\}, I^{+}{}_{N_n(\mathrm{a})} = I^{+}{}_{R_n(\mathrm{a})};$

$F^{-}{}_{N_n(\mathrm{a})} = F^{-}{}_{R_n(\mathrm{a})}\}, F^{+}{}_{N_n(\mathrm{a})} = F^{+}{}_{R_n(\mathrm{a})}.$

Example 2: From Example 1,

$$\mathrm{N} = \{<\{k, l, m, n\}, (0, 0.7)\{k, l\}, (0.5, 0.6)\{m\}, (-0.4, 0.5)\{n\};$$

$$\{k, l, p, r\}, (-0.1, 0.9)\{k, p\}, (-0.2, 0.3)\{l\}, (-0.2, 0.7)\{r\} >\}$$

is a IgsvNqss. Also, it is clear that

$$\mathrm{Y} = \{<\{k, m, n\}, (0, 0.5)\{k\}, (-0.6, 0.7)\{m\}, (-0.6, 0.8)\{n\};$$

$$\{, l, p, r\}, (-0.3, 0.9)\{p\}, (-0.4, 0.5)\{l\}, (-0.3, 0.8)\{r\} >\}$$

is a BgsvNqss. Thus,

$$\mathrm{Y} \subset \mathrm{N}.$$

Definition 10: Let

$\mathrm{N} = \{<K^{1}{}_{N_1}, (T^{-}{}_{N_1(\mathrm{a})}, T^{+}{}_{N_1(\mathrm{a})})L^{1}{}_{N_1}, (I^{-}{}_{N_1(\mathrm{a})}, I^{+}{}_{N_1(\mathrm{a})})\, M^{1}{}_{N_1}, (F^{-}{}_{N_1(\mathrm{a})}, F^{+}{}_{N_1(\mathrm{a})})\, P^{1}{}_{N_1};$

$K^{2}{}_{N_2}, (T^{-}{}_{N_2(\mathrm{a})}, T^{+}{}_{N_2(\mathrm{a})})L^{2}{}_{N_2}, (I^{-}{}_{N_2(\mathrm{a})}, I^{+}{}_{N_2(\mathrm{a})})\, M^{2}{}_{N_2}, (F^{-}{}_{N_2(\mathrm{a})}, F^{+}{}_{N_2(\mathrm{a})})\, P^{2}{}_{N_2};$

$K^{i}{}_{N_i}, (T^{-}{}_{N_i(\mathrm{a})}, T^{+}{}_{N_i(\mathrm{a})})L^{i}{}_{N_i}, (I^{-}{}_{N_i(\mathrm{a})}, I^{+}{}_{N_i(\mathrm{a})})\, M^{i}{}_{N_i}, (F^{-}{}_{N_i(\mathrm{a})}, F^{+}{}_{N_i(\mathrm{a})})\, P^{i}{}_{N_i} >,$

$K^{n}{}_{N_n}, L^{n}{}_{N_n}, M^{n}{}_{N_n}, P^{n}{}_{N_n} \in \mathrm{P(A)}; \mathrm{n} = 1, 2, 3, \dots, \mathrm{i}\}.$

and

$\mathrm{R} = \{<K^{1}{}_{R_1}, (T^{-}{}_{R_1(\mathrm{a})}, T^{+}{}_{R_1(\mathrm{a})})L^{1}{}_{R_1}, (I^{-}{}_{R_1(\mathrm{a})}, I^{+}{}_{R_1(\mathrm{a})})\, M^{1}{}_{R_1}, (F^{-}{}_{R_1(\mathrm{a})}, F^{+}{}_{R_1(\mathrm{a})})\, P^{1}{}_{R_1};$

$K^{2}{}_{R_2}, (T^{-}{}_{R_2(\mathrm{a})}, T^{+}{}_{R_2(\mathrm{a})})L^{2}{}_{R_2}, (I^{-}{}_{R_2(\mathrm{a})}, I^{+}{}_{R_2(\mathrm{a})})\, M^{2}{}_{R_2}, (F^{-}{}_{R_2(\mathrm{a})}, F^{+}{}_{R_2(\mathrm{a})})\, P^{2}{}_{R_2};$

$K^{i}{}_{R_i}$, $(T^{-}{}_{R_i(a)}, T^{+}{}_{R_i(a)}) L^{i}{}_{R_i}$, $(I^{-}{}_{R_i(a)}, I^{+}{}_{R_i(a)})\, M^{i}{}_{R_i}$, $(F^{-}{}_{R_i(a)}, F^{+}{}_{R_i(a)})\, P^{i}{}_{R_i}$ >,

$K^{n}{}_{R_n}$, $L^{n}{}_{R_n}$, $M^{n}{}_{R_n}$, $P^{n}{}_{R_n} \in$ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss.

i) We define the "average ∪" operations for N and R such that

N $\cup_A$ R = {< $K^{1}{}_{N_1R_1}$, $(T^{-}{}_{N_1R_1(a)}, T^{+}{}_{N_1R_1(a)})$ $L^{1}{}_{N_1R_1}$, $(I^{-}{}_{N_1R_1(a)}, I^{+}{}_{N_1R_1(a)})$ $M^{1}{}_{N_1R_1}$, $(F^{-}{}_{N_1R_1(a)}, F^{u}{}_{N_1R_1(a)})\, P^{1}{}_{N_1R_1}$;

$K^{2}{}_{N_2R_2}$, $(T^{-}{}_{N_2R_2(a)}, T^{+}{}_{N_2R_2(a)}) L^{2}{}_{N_2R_2}$, $(I^{-}{}_{N_2R_2(a)}, I^{+}{}_{N_2R_2(a)})\, M^{2}{}_{N_2R_2}$, $(F^{-}{}_{N_2R_2(a)}, F^{u}{}_{N_2R_2(a)})\, P^{2}{}_{N_2R_2}$;

$K^{i}{}_{N_iR_i}$, $(T^{-}{}_{N_iR_i(a)}, T^{+}{}_{N_iR_i(a)}) L^{i}{}_{N_iR_i}$, $(I^{-}{}_{N_iR_i(a)}, I^{+}{}_{N_iR_i(a)})\, M^{i}{}_{N_iR_i}$, $(F^{-}{}_{N_iR_i(a)}, F^{u}{}_{N_iR_i(a)})\, P^{i}{}_{N_iR_i}$ >,

$K^{n}{}_{N_nR_n}$, $L^{n}{}_{N_nR_n}$, $M^{n}{}_{N_nR_n}$, $P^{n}{}_{N_nR_n} \in$ P(A); n = 1, 2, 3, … , i}.

Where,

$K^{n}{}_{N_nR_n} = K^{n}{}_{N_n} \cup K^{n}{}_{R_n}$, $L^{n}{}_{N_nR_n} = L^{n}{}_{N_n} \cup L^{n}{}_{R_n}$, $M^{n}{}_{N_nR_n} = M^{n}{}_{N_n} \cup M^{n}{}_{R_n}$, $P^{n}{}_{N_nR_n} = P^{n}{}_{N_n} \cup P^{n}{}_{R_n}$;

$T^{-}{}_{N_nR_n(a)} = \frac{T^{-}{}_{N_n(a)} + T^{-}{}_{R_n(a)}}{2}$, $T^{+}{}_{N_nR_n(a)} = \frac{T^{+}{}_{N_n(a)} + T^{+}{}_{R_n(a)}}{2}$;

$I^{-}{}_{N_nR_n(a)} = \frac{I^{-}{}_{N_n(a)} + I^{-}{}_{R_n(a)}}{2}$, $I^{+}{}_{N_nR_n(a)} = \frac{I^{+}{}_{N_n(a)} + I^{+}{}_{R_n(a)}}{2}$;

$F^{-}{}_{N_nR_n(a)} = \frac{F^{-}{}_{N_n(a)} + F^{-}{}_{R_n(a)}}{2}$, $F^{+}{}_{N_nR_n(a)} = \frac{F^{+}{}_{N_n(a)} + F^{+}{}_{R_n(a)}}{2}$; (n = 1, 2, 3, … , i).

ii) We define the "average ∩ " operations for N and R such that

N $\cap_A$ R = {< $K^{1}{}_{N_1R_1}$, $(T^{-}{}_{N_1R_1(a)}, T^{+}{}_{N_1R_1(a)})$ $L^{1}{}_{N_1R_1}$, $(I^{-}{}_{N_1R_1(a)}, I^{+}{}_{N_1R_1(a)})$ $M^{1}{}_{N_1R_1}$, $(F^{-}{}_{N_1R_1(a)}, F^{u}{}_{N_1R_1(a)})\, P^{1}{}_{N_1R_1}$;

$K^{2}{}_{N_2R_2}$, $(T^{-}{}_{N_2R_2(a)}, T^{+}{}_{N_2R_2(a)}) L^{2}{}_{N_2R_2}$, $(I^{-}{}_{N_2R_2(a)}, I^{+}{}_{N_2R_2(a)})\, M^{2}{}_{N_2R_2}$, $(F^{-}{}_{N_2R_2(a)}, F^{u}{}_{N_2R_2(a)})\, P^{2}{}_{N_2R_2}$;

$K^{i}{}_{N_iR_i}$, $(T^{-}{}_{N_iR_i(a)}, T^{+}{}_{N_iR_i(a)}) L^{i}{}_{N_iR_i}$, $(I^{-}{}_{N_iR_i(a)}, I^{+}{}_{N_iR_i(a)})\, M^{i}{}_{N_iR_i}$, $(F^{-}{}_{N_iR_i(a)}, F^{u}{}_{N_iR_i(a)})\, P^{i}{}_{N_iR_i}$ >,

$K^{n}{}_{N_nR_n}$, $L^{n}{}_{N_nR_n}$, $M^{n}{}_{N_nR_n}$, $P^{n}{}_{N_nR_n} \in$ P(A); n = 1, 2, 3, … , i}.

Where,

$K^{n}{}_{N_nR_n} = K^{n}{}_{N_n} \cap K^{n}{}_{R_n}$, $L^{n}{}_{N_nR_n} = L^{n}{}_{N_n} \cap L^{n}{}_{R_n}$, $M^{n}{}_{N_nR_n} = M^{n}{}_{N_n} \cap M^{n}{}_{R_n}$, $P^{n}{}_{N_nR_n} = P^{n}{}_{N_n} \cap P^{n}{}_{R_n}$;

$T^{-}{}_{N_nR_n(a)} = \frac{T^{-}{}_{N_n(a)} + T^{-}{}_{R_n(a)}}{2}$, $T^{+}{}_{N_nR_n(a)} = \frac{T^{+}{}_{N_n(a)} + T^{+}{}_{R_n(a)}}{2}$;

$I^{-}{}_{N_nR_n(a)} = \frac{I^{-}{}_{N_n(a)} + I^{-}{}_{R_n(a)}}{2}$, $I^{+}{}_{N_nR_n(a)} = \frac{I^{+}{}_{N_n(a)} + I^{+}{}_{R_n(a)}}{2}$;

$F^{-}{}_{N_nR_n(a)} = \frac{F^{-}{}_{N_n(a)} + F^{-}{}_{R_n(a)}}{2}$, $F^{+}{}_{N_nR_n(a)} = \frac{F^{+}{}_{N_n(a)} + F^{+}{}_{R_n(a)}}{2}$; (n = 1, 2, 3, … , i).

Example 3: From Example 1,

N = {<$\{k, l, m, n\}$, (0, 0.7)$\{k, l\}$, (0.5, 0.6) $\{m\}$, (−0.4, 0.5) $\{n\}$;

$\{k, l, p, r\}$, (−0.1, 0.9)$\{k, p\}$, (−0.2, 0.3) $\{l\}$, (−0.2, 0.7) $\{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, (−0.4, 0.8)$\{l, p\}$, −0, 0.3) $\{p, m\}$, (−0.2, 0.6) $\{n\}$;

$\{m, l, p, r\}$, (−0.3, 0.7)$\{p\}$, (−0.2, 0.5) $\{m, l\}$, (−0.1, 0.5) $\{r\}$ >}

are two BgsvNqs. Thus,

i) N $\cup_A$ R = {<$\{k, l, m, n, p\}$, (−0.2, 0.75)$\{k, l, p\}$, (−0.25, 0.45) $\{p, m\}$, (−0.3, 0.6) $\{n\}$;

$\{m, k, l, p, r\}$, (−0.2, 0.8)$\{k, p\}$, (−0.2, 0.4) $\{m, l\}$, (−0.15, 0.6) $\{r\}$ >}

ii) N $\cap_A$ R = {<$\{k, l, m, n\}$, (−0.2, 0.75)$\{l\}$, (−0.25, 0.45)$\{m\}$, (−0.3, 0.6) $\{n\}$;

$\{l, p, r\}$, (−0.2, 0.8)$\{p\}$, (−0.2, 0.4) $\{l\}$, (−0.15, 0.6) $\{r\}$ >}

Definition 11: Let

N = {<$K^1{}_{N_1}$, $(T^-{}_{N_1(a)}, T^+{}_{N_1(a)})L^1{}_{N_1}$, $(I^-{}_{N_1(a)}, I^+{}_{N_1(a)})\, M^1{}_{N_1}$, $(F^-{}_{N_1(a)}, F^+{}_{N_1(a)})\, P^1{}_{N_1}$;

$K^2{}_{N_2}$, $(T^-{}_{N_2(a)}, T^+{}_{N_2(a)})L^2{}_{N_2}$, $(I^-{}_{N_2(a)}, I^+{}_{N_2(a)})\, M^2{}_{N_2}$, $(F^-{}_{N_2(a)}, F^+{}_{N_2(a)})\, P^2{}_{N_2}$;

$K^i{}_{N_i}$, $(T^-{}_{N_i(a)}, T^+{}_{N_i(a)})L^i{}_{N_i}$, $(I^-{}_{N_i(a)}, I^+{}_{N_i(a)})\, M^i{}_{N_i}$, $(F^-{}_{N_i(a)}, F^+{}_{N_i(a)})\, P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {<$K^1{}_{R_1}$, $(T^-{}_{R_1(a)}, T^+{}_{R_1(a)})L^1{}_{R_1}$, $(I^-{}_{R_1(a)}, I^+{}_{R_1(a)})\, M^1{}_{R_1}$, $(F^-{}_{R_1(a)}, F^+{}_{R_1(a)})\, P^1{}_{R_1}$;

$K^2{}_{R_2}$, $(T^-{}_{R_2(a)}, T^+{}_{R_2(a)})L^2{}_{R_2}$, $(I^-{}_{R_2(a)}, I^+{}_{R_2(a)})\, M^2{}_{R_2}$, $(F^-{}_{R_2(a)}, F^+{}_{R_2(a)})\, P^2{}_{R_2}$;

$K^i{}_{R_i}$, $(T^-{}_{R_i(a)}, T^+{}_{R_i(a)})L^i{}_{R_i}$, $(I^-{}_{R_i(a)}, I^+{}_{R_i(a)})\, M^i{}_{R_i}$, $(F^-{}_{R_i(a)}, F^+{}_{R_i(a)})\, P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss.

i) We define the "optimistic ∪" operations for N and R such that

N $\cup_O$ R = {< $K^1{}_{N_1R_1}$, ($T^-{}_{N_1R_1(a)}, T^+{}_{N_1R_1(a)}$) $L^1{}_{N_1R_1}$, ($I^-{}_{N_1R_1(a)}, I^+{}_{N_1R_1(a)}$) $M^1{}_{N_1R_1}$, $(F^-{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)})\, P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $(T^-{}_{N_2R_2(a)}, T^+{}_{N_2R_2(a)})L^2{}_{N_2R_2}$, $(I^-{}_{N_2R_2(a)}, I^+{}_{N_2R_2(a)})\, M^2{}_{N_2R_2}$, $(F^-{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)})\, P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $(T^-{}_{N_iR_i(a)}, T^+{}_{N_iR_i(a)}) L^i{}_{N_iR_i}$, $(I^-{}_{N_iR_i(a)}, I^+{}_{N_iR_i(a)})$ $M^i{}_{N_iR_i}$, $(F^-{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)})$ $P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 10's (i).

$T^-{}_{N_nR_n(a)}$= max{$T^-{}_{N_n(a)}$, $T^-{}_{R_n(a)}$}, $T^+{}_{N_nR_n(a)}$= max{$T^+{}_{N_n(a)}$, $T^+{}_{R_n(a)}$};

$I^-{}_{N_nR_n(a)}$= min{$I^-{}_{N_n(a)}$, $I^-{}_{R_n(a)}$}, $I^u{}_{N_nR_n(a)}$= min{$I^+{}_{N_n(a)}$, $I^+{}_{R_n(a)}$};

$F^-{}_{N_nR_n(a)}$= min{$F^-{}_{N_n(a)}$, $F^-{}_{R_n(a)}$}, $F^u{}_{N_nR_n(a)}$= min{$F^+{}_{N_n(a)}$, $F^+{}_{R_n(a)}$}; (n = 1, 2, 3, … , i).

ii) We define the "optimistic ∩ " operations for N and R such that

N $\cap_O$ R = {< $K^1{}_{N_1R_1}$, ($T^-{}_{N_1R_1(a)}, T^+{}_{N_1R_1(a)}$) $L^1{}_{N_1R_1}$, ($I^-{}_{N_1R_1(a)}, I^+{}_{N_1R_1(a)}$) $M^1{}_{N_1R_1}$, $(F^-{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)})$ $P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $(T^-{}_{N_2R_2(a)}, T^+{}_{N_2R_2(a)}) L^2{}_{N_2R_2}$, $(I^-{}_{N_2R_2(a)}, I^+{}_{N_2R_2(a)})$ $M^2{}_{N_2R_2}$, $(F^-{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)})$ $P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $(T^-{}_{N_iR_i(a)}, T^+{}_{N_iR_i(a)}) L^i{}_{N_iR_i}$, $(I^-{}_{N_iR_i(a)}, I^+{}_{N_iR_i(a)})$ $M^i{}_{N_iR_i}$, $(F^-{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)})$ $P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 11's (ii).

$T^-{}_{N_nR_n(a)}$= max{$T^-{}_{N_n(a)}$, $T^-{}_{R_n(a)}$}, $T^+{}_{N_nR_n(a)}$= max{$T^+{}_{N_n(a)}$, $T^+{}_{R_n(a)}$};

$I^-{}_{N_nR_n(a)}$= min{$I^-{}_{N_n(a)}$, $I^-{}_{R_n(a)}$}, $I^u{}_{N_nR_n(a)}$= min{$I^+{}_{N_n(a)}$, $I^+{}_{R_n(a)}$};

$F^-{}_{N_nR_n(a)}$= min{$F^-{}_{N_n(a)}$, $F^-{}_{R_n(a)}$}, $F^u{}_{N_nR_n(a)}$= min{$F^+{}_{N_n(a)}$, $F^+{}_{R_n(a)}$}; (n = 1, 2, 3, … , i).

Example 4 From Example 1,

N = {<$\{k, l, m, n\}$, $(0, 0.7)\{k, l\}$, $(0.5, 0.6)$ $\{m\}$, $(-0.4, 0.5)$ $\{n\}$;

$\{k, l, p, r\}$, $(-0.1, 0.9)\{k, p\}$, $(-0.2, 0.3)$ $\{l\}$, $(-0.2, 0.7)$ $\{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, $(-0.4, 0.8)\{l, p\}$, $-0, 0.3)$ $\{p, m\}$, $(-0.2, 0.6)$ $\{n\}$;

$\{m, l, p, r\}$, $(-0.3, 0.7)\{p\}$, $(-0.2, 0.5)$ $\{m, l\}$, $(-0.1, 0.5)$ $\{r\}$ >}

are two BgsvNqs. Thus,

i) N $\cup_O$ R = {<$\{k, l, m, n, p\}$, $(-0.4, 0.8)\{k, l, p\}$, $(0, 0.3)$ $\{p, m\}$, $(-0.2, 0.5)$ $\{n\}$;

$\{m, k, l, p, r\}$, $(-0.3, 0.9)\{k, p\}$, $(-0.2, 0.3)\{m, l\}$, $(-0.1, 0.5)\{r\}$ >}

ii) N $\cap_O$ R = {<$\{k, l, m, n\}$, $(-0.4, 0.8)\{l\}$, $(0, 0.3)\{m\}$, $(-0.2, 0.5)\{n\}$;

$\{l, p, r\}$, $(-0.3, 0.9)\{p\}$, $(-0.2, 0.3)\{l\}$, $(-0.1, 0.5)\{r\}$ >}

Definition 12: Let

N = {<$K^1{}_{N_1}$, $(T^-{}_{N_1(a)}, T^+{}_{N_1(a)})L^1{}_{N_1}$, $(I^-{}_{N_1(a)}, I^+{}_{N_1(a)})\,M^1{}_{N_1}$, $(F^-{}_{N_1(a)}, F^+{}_{N_1(a)})\,P^1{}_{N_1}$;

$K^2{}_{N_2}$, $(T^-{}_{N_2(a)}, T^+{}_{N_2(a)})L^2{}_{N_2}$, $(I^-{}_{N_2(a)}, I^+{}_{N_2(a)})\,M^2{}_{N_2}$, $(F^-{}_{N_2(a)}, F^+{}_{N_2(a)})\,P^2{}_{N_2}$;

$K^i{}_{N_i}$, $(T^-{}_{N_i(a)}, T^+{}_{N_i(a)})L^i{}_{N_i}$, $(I^-{}_{N_i(a)}, I^+{}_{N_i(a)})\,M^i{}_{N_i}$, $(F^-{}_{N_i(a)}, F^+{}_{N_i(a)})\,P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {<$K^1{}_{R_1}$, $(T^-{}_{R_1(a)}, T^+{}_{R_1(a)})L^1{}_{R_1}$, $(I^-{}_{R_1(a)}, I^+{}_{R_1(a)})\,M^1{}_{R_1}$, $(F^-{}_{R_1(a)}, F^+{}_{R_1(a)})\,P^1{}_{R_1}$;

$K^2{}_{R_2}$, $(T^-{}_{R_2(a)}, T^+{}_{R_2(a)})L^2{}_{R_2}$, $(I^-{}_{R_2(a)}, I^+{}_{R_2(a)})\,M^2{}_{R_2}$, $(F^-{}_{R_2(a)}, F^+{}_{R_2(a)})\,P^2{}_{R_2}$;

$K^i{}_{R_i}$, $(T^-{}_{R_i(a)}, T^+{}_{R_i(a)})L^i{}_{R_i}$, $(I^-{}_{R_i(a)}, I^+{}_{R_i(a)})\,M^i{}_{R_i}$, $(F^-{}_{R_i(a)}, F^+{}_{R_i(a)})\,P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss.

i) We define the "pessimistic ∪" operations for N and R such that

N $\cup_p$ R = {< $K^1{}_{N_1R_1}$, ($T^-{}_{N_1R_1(a)}, T^+{}_{N_1R_1(a)}$) $L^1{}_{N_1R_1}$, ($I^-{}_{N_1R_1(a)}, I^+{}_{N_1R_1(a)}$) $M^1{}_{N_1R_1}$, $(F^-{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)})\,P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $(T^-{}_{N_2R_2(a)}, T^+{}_{N_2R_2(a)})L^2{}_{N_2R_2}$, $(I^-{}_{N_2R_2(a)}, I^+{}_{N_2R_2(a)})\,M^2{}_{N_2R_2}$, $(F^-{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)})\,P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $(T^-{}_{N_iR_i(a)}, T^+{}_{N_iR_i(a)})L^i{}_{N_iR_i}$, $(I^-{}_{N_iR_i(a)}, I^+{}_{N_iR_i(a)})\,M^i{}_{N_iR_i}$, $(F^-{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)})\,P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 10's (i).

$T^-{}_{N_nR_n(a)}$= min{$T^-{}_{N_n(a)}, T^-{}_{R_n(a)}$}, $T^+{}_{N_nR_n(a)}$= min{$T^+{}_{N_n(a)}, T^+{}_{R_n(a)}$};

$I^-{}_{N_nR_n(a)}$= max{$I^-{}_{N_n(a)}, I^-{}_{R_n(a)}$}, $I^u{}_{N_nR_n(a)}$= max{$I^+{}_{N_n(a)}, I^+{}_{R_n(a)}$};

$F^-{}_{N_nR_n(a)}$= max{$F^-{}_{N_n(a)}, F^-{}_{R_n(a)}$}, $F^u{}_{N_nR_n(a)}$= max{$F^+{}_{N_n(a)}, F^+{}_{R_n(a)}$}; (n = 1, 2, 3, … , i).

ii) We define the "pessimistic ∩ " operations for N and R such that

N $\cap_p$ R = {< ${K^1}_{N_1R_1}$, (${T^-}_{N_1R_1(a)}, {T^+}_{N_1R_1(a)}$) ${L^1}_{N_1R_1}$, (${I^-}_{N_1R_1(a)}, {I^+}_{N_1R_1(a)}$) ${M^1}_{N_1R_1}$, $({F^-}_{N_1R_1(a)}, {F^u}_{N_1R_1(a)})\ {P^1}_{N_1R_1}$;

$${K^2}_{N_2R_2}, ({T^-}_{N_2R_2(a)}, {T^+}_{N_2R_2(a)}){L^2}_{N_2R_2}, ({I^-}_{N_2R_2(a)}, {I^+}_{N_2R_2(a)})\ {M^2}_{N_2R_2}, ({F^-}_{N_2R_2(a)}, {F^u}_{N_2R_2(a)})\ {P^2}_{N_2R_2};$$

$${K^i}_{N_iR_i}, ({T^-}_{N_iR_i(a)}, {T^+}_{N_iR_i(a)}){L^i}_{N_iR_i}, ({I^-}_{N_iR_i(a)}, {I^+}_{N_iR_i(a)})\ {M^i}_{N_iR_i}, ({F^-}_{N_iR_i(a)}, {F^u}_{N_iR_i(a)})\ {P^i}_{N_iR_i} >,$$

${K^n}_{N_nR_n},\ {L^n}_{N_nR_n},\ {M^n}_{N_nR_n},\ {P^n}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

${K^n}_{N_nR_n}, {L^n}_{N_nR_n}, {M^n}_{N_nR_n}$ and ${P^n}_{N_nR_n}$ are same as in Definition 10's (ii).

${T^-}_{N_nR_n(a)} = \min\{{T^-}_{N_n(a)}, {T^-}_{R_n(a)}\}$, ${T^+}_{N_nR_n(a)} = \min\{{T^+}_{N_n(a)}, {T^+}_{R_n(a)}\}$;

${I^-}_{N_nR_n(a)} = \max\{{I^-}_{N_n(a)}, {I^-}_{R_n(a)}\}$, ${I^u}_{N_nR_n(a)} = \max\{{I^+}_{N_n(a)}, {I^+}_{R_n(a)}\}$;

${F^-}_{N_nR_n(a)} = \max\{{F^-}_{N_n(a)}, {F^-}_{R_n(a)}\}$, ${F^u}_{N_nR_n(a)} = \max\{{F^+}_{N_n(a)}, {F^+}_{R_n(a)}\}$; (n = 1, 2, 3, … , i).

Example 5: From Example 1,

N = {<$\{k, l, m, n\}$, $(0, 0.7)\{k, l\}$, $(-0.5, 0.6)\ \{m\}$, $(-0.4, 0.5)\ \{n\}$;

$\{k, l, p, r\}$, $(-0.1, 0.9)\{k, p\}$, $(-0.2, 0.3)\ \{l\}$, $(-0.2, 0.7)\ \{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, $(-0.4, 0.8)\{l, p\}$, $-0, 0.3)\ \{p, m\}$, $(-0.2, 0.6)\ \{n\}$;

$\{m, l, p, r\}$, $(-0.3, 0.7)\{p\}$, $(-0.2, 0.5)\ \{m, l\}$, $(-0.1, 0.5)\ \{r\}$ >}

are two BgsvNqs. Thus,

i) N $\cup_P$ R = {<$\{k, l, m, n, p\}$, $(0, 0.7)\{k, l, p\}$, $(-0.5, 0.6)\ \{p, m\}$, $(-0.4, 0.6)\ \{n\}$;

$\{m, k, l, p, r\}$, $(-0.1, 0.7)\{k, p\}$, $(-0.2, 0.5)\ \{m, l\}$, $(-0.2, 0.7)\ \{r\}$ >}

ii) N $\cap_P$ R = {<$\{k, l, m, n\}$, $(0, 0.7)\{l\}$, $(-0.5, 0.6)\ \{m\}$, $(-0.4, 0.6)\ \{n\}$;

$\{l, p, r\}$, $(-0.1, 0.7)\{p\}$, $(-0.2, 0.5)\ \{l\}$, $(-0.2, 0.7)\ \{r\}$ >}

Definition 13: Let

N = {<${K^1}_{N_1}$, $({T^-}_{N_1(a)}, {T^+}_{N_1(a)}){L^1}_{N_1}$, $({I^-}_{N_1(a)}, {I^+}_{N_1(a)})\ {M^1}_{N_1}$, $({F^-}_{N_1(a)}, {F^+}_{N_1(a)})\ {P^1}_{N_1}$;

${K^2}_{N_2}$, $({T^-}_{N_2(a)}, {T^+}_{N_2(a)}){L^2}_{N_2}$, $({I^-}_{N_2(a)}, {I^+}_{N_2(a)})\ {M^2}_{N_2}$, $({F^-}_{N_2(a)}, {F^+}_{N_2(a)})\ {P^2}_{N_2}$;

$K^{i}{}_{N_i}$, $(T^{-}{}_{N_i(a)}, T^{+}{}_{N_i(a)})L^{i}{}_{N_i}$, $(I^{-}{}_{N_i(a)}, I^{+}{}_{N_i(a)})\, M^{i}{}_{N_i}$, $(F^{-}{}_{N_i(a)}, F^{+}{}_{N_i(a)})\, P^{i}{}_{N_i}$ >,

$K^{n}{}_{N_n}$, $L^{n}{}_{N_n}$, $M^{n}{}_{N_n}$, $P^{n}{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {<$K^{1}{}_{R_1}$, $(T^{-}{}_{R_1(a)}, T^{+}{}_{R_1(a)})L^{1}{}_{R_1}$, $(I^{-}{}_{R_1(a)}, I^{+}{}_{R_1(a)})\, M^{1}{}_{R_1}$, $(F^{-}{}_{R_1(a)}, F^{+}{}_{R_1(a)})\, P^{1}{}_{R_1}$;

$K^{2}{}_{R_2}$, $(T^{-}{}_{R_2(a)}, T^{+}{}_{R_2(a)})L^{2}{}_{R_2}$, $(I^{-}{}_{R_2(a)}, I^{+}{}_{R_2(a)})\, M^{2}{}_{R_2}$, $(F^{-}{}_{R_2(a)}, F^{+}{}_{R_2(a)})\, P^{2}{}_{R_2}$;

$K^{i}{}_{R_i}$, $(T^{-}{}_{R_i(a)}, T^{+}{}_{R_i(a)})L^{i}{}_{R_i}$, $(I^{-}{}_{R_i(a)}, I^{+}{}_{R_i(a)})\, M^{i}{}_{R_i}$, $(F^{-}{}_{R_i(a)}, F^{+}{}_{R_i(a)})\, P^{i}{}_{R_i}$ >,

$K^{n}{}_{R_n}$, $L^{n}{}_{R_n}$, $M^{n}{}_{R_n}$, $P^{n}{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss.

i) We define the "average /" operations for N and R such that

N $/_A$ R = {< $K^{1}{}_{N_1R_1}$, $(T^{-}{}_{N_1R_1(a)}, T^{+}{}_{N_1R_1(a)})\, L^{1}{}_{N_1R_1}$, $(I^{-}{}_{N_1R_1(a)}, I^{+}{}_{N_1R_1(a)})\, M^{1}{}_{N_1R_1}$, $(F^{-}{}_{N_1R_1(a)}, F^{u}{}_{N_1R_1(a)})\, P^{1}{}_{N_1R_1}$;

$K^{2}{}_{N_2R_2}$, $(T^{-}{}_{N_2R_2(a)}, T^{+}{}_{N_2R_2(a)})L^{2}{}_{N_2R_2}$, $(I^{-}{}_{N_2R_2(a)}, I^{+}{}_{N_2R_2(a)})\, M^{2}{}_{N_2R_2}$, $(F^{-}{}_{N_2R_2(a)}, F^{u}{}_{N_2R_2(a)})\, P^{2}{}_{N_2R_2}$;

$K^{i}{}_{N_iR_i}$, $(T^{-}{}_{N_iR_i(a)}, T^{+}{}_{N_iR_i(a)})L^{i}{}_{N_iR_i}$, $(I^{-}{}_{N_iR_i(a)}, I^{+}{}_{N_iR_i(a)})\, M^{i}{}_{N_iR_i}$, $(F^{-}{}_{N_iR_i(a)}, F^{u}{}_{N_iR_i(a)})\, P^{i}{}_{N_iR_i}$ >,

$K^{n}{}_{N_nR_n}$, $L^{n}{}_{N_nR_n}$, $M^{n}{}_{N_nR_n}$, $P^{n}{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^{n}{}_{N_nR_n} = K^{n}{}_{N_n} / K^{n}{}_{R_n}$, $L^{n}{}_{N_nR_n} = L^{n}{}_{N_n} / L^{n}{}_{R_n}$, $M^{n}{}_{N_nR_n} = M^{n}{}_{N_n} / M^{n}{}_{R_n}$, $P^{n}{}_{N_nR_n} = P^{n}{}_{N_n} / P^{n}{}_{R_n}$;

$T^{l}{}_{N_nR_n(a)}$, $T^{u}{}_{N_nR_n(a)}$, $I^{l}{}_{N_nR_n(a)}$, $I^{u}{}_{N_nR_n(a)}$, $F^{l}{}_{N_nR_n(a)}$ and $F^{u}{}_{N_nR_n(a)}$ are same as in Definition 10' (i).

ii) We define the "optimistic / " operations for N and R such that

N $/_O$ R = {< $K^{1}{}_{N_1R_1}$, $(T^{-}{}_{N_1R_1(a)}, T^{+}{}_{N_1R_1(a)})\, L^{1}{}_{N_1R_1}$, $(I^{-}{}_{N_1R_1(a)}, I^{+}{}_{N_1R_1(a)})\, M^{1}{}_{N_1R_1}$, $(F^{-}{}_{N_1R_1(a)}, F^{u}{}_{N_1R_1(a)})\, P^{1}{}_{N_1R_1}$;

$K^{2}{}_{N_2R_2}$, $(T^{-}{}_{N_2R_2(a)}, T^{+}{}_{N_2R_2(a)})L^{2}{}_{N_2R_2}$, $(I^{-}{}_{N_2R_2(a)}, I^{+}{}_{N_2R_2(a)})\, M^{2}{}_{N_2R_2}$, $(F^{-}{}_{N_2R_2(a)}, F^{u}{}_{N_2R_2(a)})\, P^{2}{}_{N_2R_2}$;

$K^{i}{}_{N_iR_i}$, $(T^{-}{}_{N_iR_i(a)}, T^{+}{}_{N_iR_i(a)})L^{i}{}_{N_iR_i}$, $(I^{-}{}_{N_iR_i(a)}, I^{+}{}_{N_iR_i(a)})\, M^{i}{}_{N_iR_i}$, $(F^{-}{}_{N_iR_i(a)}, F^{u}{}_{N_iR_i(a)})\, P^{i}{}_{N_iR_i}$ >,

$K^{n}{}_{N_nR_n}$, $L^{n}{}_{N_nR_n}$, $M^{n}{}_{N_nR_n}$, $P^{n}{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^{n}{}_{N_nR_n} = K^{n}{}_{N_n} / K^{n}{}_{R_n}$, $L^{n}{}_{N_nR_n} = L^{n}{}_{N_n} / L^{n}{}_{R_n}$, $M^{n}{}_{N_nR_n} = M^{n}{}_{N_n} / M^{n}{}_{R_n}$, $P^{n}{}_{N_nR_n} = P^{n}{}_{N_n} / P^{n}{}_{R_n}$;

$T^{l}{}_{N_nR_n(a)}$, $T^{u}{}_{N_nR_n(a)}$, $I^{l}{}_{N_nR_n(a)}$, $I^{u}{}_{N_nR_n(a)}$, $F^{l}{}_{N_nR_n(a)}$ and $F^{u}{}_{N_nR_n(a)}$ are same as in Definition 11' (i).

iii) We define the "pessimistic / " operations for N and R such that

N $/_P$ R = {< $K^1{}_{N_1R_1}$, ($T^-{}_{N_1R_1(\text{a})}, T^+{}_{N_1R_1(\text{a})}$) $L^1{}_{N_1R_1}$, ($I^-{}_{N_1R_1(\text{a})}, I^+{}_{N_1R_1(\text{a})}$) $M^1{}_{N_1R_1}$, $(F^-{}_{N_1R_1(\text{a})}, F^u{}_{N_1R_1(\text{a})})$ $P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $(T^-{}_{N_2R_2(\text{a})}, T^+{}_{N_2R_2(\text{a})})L^2{}_{N_2R_2}$, $(I^-{}_{N_2R_2(\text{a})}, I^+{}_{N_2R_2(\text{a})})$ $M^2{}_{N_2R_2}$, $(F^-{}_{N_2R_2(\text{a})}, F^u{}_{N_2R_2(\text{a})})$ $P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $(T^-{}_{N_iR_i(\text{a})}, T^+{}_{N_iR_i(\text{a})})L^i{}_{N_iR_i}$, $(I^-{}_{N_iR_i(\text{a})}, I^+{}_{N_iR_i(\text{a})})$ $M^i{}_{N_iR_i}$, $(F^-{}_{N_iR_i(\text{a})}, F^u{}_{N_iR_i(\text{a})})$ $P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n} = K^n{}_{N_n} / K^n{}_{R_n}$, $L^n{}_{N_nR_n} = L^n{}_{N_n} / L^n{}_{R_n}$, $M^n{}_{N_nR_n} = M^n{}_{N_n} / M^n{}_{R_n}$, $P^n{}_{N_nR_n} = P^n{}_{N_n} / P^n{}_{R_n}$;

$T^l{}_{N_nR_n(\text{a})}$, $T^u{}_{N_nR_n(\text{a})}$, $I^l{}_{N_nR_n(\text{a})}$, $I^u{}_{N_nR_n(\text{a})}$, $F^l{}_{N_nR_n(\text{a})}$ and $F^u{}_{N_nR_n(\text{a})}$ are same as in Definition 12' (i).

Example 6: From Example 1,

N = {<$\{k, l, m, n\}$, (0, 0.7)$\{k, l\}$, (−0.5, 0.6) $\{m\}$, (−0.4, 0.5) $\{n\}$;

$\{k, l, p, r\}$, (−0.1, 0.9)$\{k, p\}$, (−0.2, 0.3) $\{l\}$, (−0.2, 0.7) $\{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, (−0.4, 0.8)$\{l, p\}$, −0, 0.3) $\{p, m\}$, (−0.2, 0.6) $\{n\}$;

$\{m, l, p, r\}$, (−0.3, 0.7)$\{p\}$, (−0.2, 0.5) $\{m, l\}$, (−0.1, 0.5) $\{r\}$ >}

are two BgsvNqs. Thus,

i) N $/_A$ R = {<∅, (−0.2, 0.75)$\{l\}$, (−0.25, 0.45) ∅, (−0.3, 0.6) ∅;

$\{k\}$, (−0.2, 0.8)$\{k\}$, (−0.2, 0.4) ∅, (−0.15, 0.6) ∅ >}.

ii) N $/_O$ R = {<∅, (−0.4, 0.8)$\{l\}$, (0, 0.3) ∅, (−0.2, 0.5) ∅;

$\{k\}$, (−0.3, 0.9)$\{k\}$, (−0.2, 0.3)∅, (−0.1, 0.5)∅ >}.

iii) N $/_P$ R = {<∅, (−0.2, 0.75)$\{l\}$, (−0.25, 0.45) ∅, (−0.3, 0.6) ∅;

$\{k\}$, (−0.2, 0.8)$\{k\}$, (−0.2, 0.4) ∅, (−0.15, 0.6) ∅ >}.

Properties 1: Let

N = {<$K^1{}_{N_1}$, $(T^-{}_{N_1(\text{a})}, T^+{}_{N_1(\text{a})})L^1{}_{N_1}$, $(I^-{}_{N_1(\text{a})}, I^+{}_{N_1(\text{a})})$ $M^1{}_{N_1}$, $(F^-{}_{N_1(\text{a})}, F^+{}_{N_1(\text{a})})$ $P^1{}_{N_1}$;

$K^2{}_{N_2}$, $(T^-{}_{N_2(\text{a})}, T^+{}_{N_2(\text{a})})L^2{}_{N_2}$, $(I^-{}_{N_2(\text{a})}, I^+{}_{N_2(\text{a})})$ $M^2{}_{N_2}$, $(F^-{}_{N_2(\text{a})}, F^+{}_{N_2(\text{a})})$ $P^2{}_{N_2}$;

$K^i{}_{N_i}$, $(T^-{}_{N_i(\text{a})}, T^+{}_{N_i(\text{a})})L^i{}_{N_i}$, $(I^-{}_{N_i(\text{a})}, I^+{}_{N_i(\text{a})})$ $M^i{}_{N_i}$, $(F^-{}_{N_i(\text{a})}, F^+{}_{N_i(\text{a})})$ $P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {<$K^1{}_{R_1}$, $(T^-{}_{R_1(a)}, T^+{}_{R_1(a)})L^1{}_{R_1}$, $(I^-{}_{R_1(a)}, I^+{}_{R_1(a)})\, M^1{}_{R_1}$, $(F^-{}_{R_1(a)}, F^+{}_{R_1(a)})\, P^1{}_{R_1}$;

$K^2{}_{R_2}$, $(T^-{}_{R_2(a)}, T^+{}_{R_2(a)})L^2{}_{R_2}$, $(I^-{}_{R_2(a)}, I^+{}_{R_2(a)})\, M^2{}_{R_2}$, $(F^-{}_{R_2(a)}, F^+{}_{R_2(a)})\, P^2{}_{R_2}$;

$K^i{}_{R_i}$, $(T^-{}_{R_i(a)}, T^+{}_{R_i(a)})L^i{}_{R_i}$, $(I^-{}_{R_i(a)}, I^+{}_{R_i(a)})\, M^i{}_{R_i}$, $(F^-{}_{R_i(a)}, F^+{}_{R_i(a)})\, P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

Y = {<$K^1{}_{Y_1}$, $(T^-{}_{Y_1(a)}, T^+{}_{Y_1(a)})L^1{}_{Y_1}$, $(I^-{}_{Y_1(a)}, I^+{}_{Y_1(a)})\, M^1{}_{Y_1}$, $(F^-{}_{Y_1(a)}, F^+{}_{Y_1(a)})\, P^1{}_{Y_1}$;

$K^2{}_{Y_2}$, $(T^-{}_{Y_1(a)}, T^+{}_{Y_2(a)})L^2{}_{Y_2}$, $(I^-{}_{Y_2(a)}, I^+{}_{Y_2(a)})\, M^2{}_{Y_2}$, $(F^-{}_{Y_2(a)}, F^+{}_{Y_2(a)})\, P^2{}_{Y_2}$;

$K^i{}_{R_i}$, $(T^-{}_{Y_i(a)}, T^+{}_{Y_i(a)})L^i{}_{Y_i}$, $(I^-{}_{Y_i(a)}, I^+{}_{Y_i(a)})\, M^i{}_{Y_i}$, $(F^-{}_{Y_i(a)}, F^+{}_{Y_i(a)})\, P^i{}_{Y_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be three BgsvNqss.

From Definition 8, Definition 9, Definition 10, Definition 11, Definition 12 and Definition 13; it is clear that

i) N $\cup_A$ R = R $\cup_A$ N; N $\cup_O$ R = R $\cup_O$ N; N $\cup_P$ R = R $\cup_P$ N.

ii) N $\cap_A$ R = R $\cap_A$ N; N $\cap_O$ R = R $\cap_O$ N; N $\cap_P$ R = R $\cap_P$ N.

iii) N $\cup_A$ **(**R $\cup_A$ Y) = (N $\cup_A$ R) $\cup_A$ Y,

N $\cup_O$ **(**R $\cup_O$ Y) = (N $\cup_O$ R) $\cup_O$ Y,

N $\cup_P$ **(**R $\cup_P$ Y) = (N $\cup_P$ R) $\cup_P$ Y.

iv) N $\cap_A$ **(**R $\cap_A$ Y) = (N $\cap_A$ R) $\cap_A$ Y,

N $\cap_O$ **(**R $\cap_O$ Y) = (N $\cap_O$ R) $\cap_O$ Y,

N $\cap_P$ **(**R $\cap_P$ Y) = (N $\cap_P$ R) $\cap_P$ Y.

v) N $\cap_A$ **(**R $\cup_A$ Y) = (N $\cap_A$ R) $\cup_A$ (N $\cap_A$ Y),

N $\cap_O$ **(**R $\cup_O$ Y) = (N $\cap_O$ R) $\cup_O$ (N $\cap_O$ Y),

N $\cap_P$ **(**R $\cup_P$ Y) = (N $\cap_P$ R) $\cup_P$ (N $\cap_P$ Y).

vi) N $\cup_A$ **(**R $\cap_A$ Y) = (N $\cup_A$ R) $\cap_A$ (N $\cup_A$ Y),

N $\cup_O$ **(**R $\cap_O$ Y) = (N $\cup_O$ R) $\cap_O$ (N $\cup_O$ Y),

N $\cup_P$ **(**R $\cap_P$ Y) = (N $\cup_P$ R) $\cap_P$ (N $\cup_P$ Y).

v) If N = R, then

$$N \cup_A R = N \cup_O R = N \cup_P R = R$$

and

$$N \cap_A R = N \cap_O R = N \cap_P R = R.$$

Theorem 1: Let

N = {< $K^1{}_{N_1}$, $(T^-{}_{N_1(a)}, T^+{}_{N_1(a)})L^1{}_{N_1}$, $(I^-{}_{N_1(a)}, I^+{}_{N_1(a)})\, M^1{}_{N_1}$, $(F^-{}_{N_1(a)}, F^+{}_{N_1(a)})\, P^1{}_{N_1}$;

$K^2{}_{N_2}$, $(T^-{}_{N_2(a)}, T^+{}_{N_2(a)})L^2{}_{N_2}$, $(I^-{}_{N_2(a)}, I^+{}_{N_2(a)})\, M^2{}_{N_2}$, $(F^-{}_{N_2(a)}, F^+{}_{N_2(a)})\, P^2{}_{N_2}$;

$K^i{}_{N_i}$, $(T^-{}_{N_i(a)}, T^+{}_{N_i(a)})L^i{}_{N_i}$, $(I^-{}_{N_i(a)}, I^+{}_{N_i(a)})\, M^i{}_{N_i}$, $(F^-{}_{N_i(a)}, F^+{}_{N_i(a)})\, P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {< $K^1{}_{R_1}$, $(T^-{}_{R_1(a)}, T^+{}_{R_1(a)})L^1{}_{R_1}$, $(I^-{}_{R_1(a)}, I^+{}_{R_1(a)})\, M^1{}_{R_1}$, $(F^-{}_{R_1(a)}, F^+{}_{R_1(a)})\, P^1{}_{R_1}$;

$K^2{}_{R_2}$, $(T^-{}_{R_2(a)}, T^+{}_{R_2(a)})L^2{}_{R_2}$, $(I^-{}_{R_2(a)}, I^+{}_{R_2(a)})\, M^2{}_{R_2}$, $(F^-{}_{R_2(a)}, F^+{}_{R_2(a)})\, P^2{}_{R_2}$;

$K^i{}_{R_i}$, $(T^-{}_{R_i(a)}, T^+{}_{R_i(a)})L^i{}_{R_i}$, $(I^-{}_{R_i(a)}, I^+{}_{R_i(a)})\, M^i{}_{R_i}$, $(F^-{}_{R_i(a)}, F^+{}_{R_i(a)})\, P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss. Then,

i) $(N \cap_P R) \subset (N \cap_A R) \subset (N \cap_O R)$

ii) $(N \cup_P R) \subset (N \cup_A R) \subset (N \cup_O R)$

iii) $(N \cap_A R) \subset (N \cup_A R)$, $(N \cap_O R) \subset (N \cup_O R)$ and $(N \cap_P R) \subset (N \cup_P R)$.

Proof:

i) From Definition 10, Definition 11 and Definition 12; we obtain that

$$\min\{T^-{}_{N_n(a)}, T^-{}_{R_n(a)}\} \leq \frac{T^-{}_{N_n(a)}+T^-{}_{R_n(a)}}{2} \leq \max\{T^-{}_{N_n(a)}, T^-{}_{R_n(a)}\} \quad (1)$$

$$\min\{T^+{}_{N_n(a)}, T^+{}_{R_n(a)}\} \leq \frac{T^+{}_{N_n(a)}+T^+{}_{R_n(a)}}{2} \leq \max\{T^+{}_{N_n(a)}, T^+{}_{R_n(a)}\} \quad (2)$$

$$\max\{I^-{}_{N_n(a)}, I^-{}_{R_n(a)}\} \geq \frac{I^-{}_{N_n(a)}+I^-{}_{R_n(a)}}{2} \geq \min\{I^-{}_{N_n(a)}, I^-{}_{R_n(a)}\} \quad (3)$$

$$\max\{I^+{}_{N_n(a)}, I^+{}_{R_n(a)}\} \geq \frac{I^+{}_{N_n(a)}+I^+{}_{R_n(a)}}{2} \geq \min\{I^+{}_{N_n(a)}, I^+{}_{R_n(a)}\} \quad (4)$$

$$\max\{F^-{}_{N_n(a)}, F^-{}_{R_n(a)}\} \geq \frac{F^-{}_{N_n(a)}+F^-{}_{R_n(a)}}{2} \geq \min\{F^-{}_{N_n(a)}, F^-{}_{R_n(a)}\} \quad (5)$$

$$\max\{F^{+}{}_{N_n(\mathrm{a})}, F^{+}{}_{R_n(\mathrm{a})}\} \geq \frac{F^{+}{}_{N_n(\mathrm{a})}+F^{+}{}_{R_n(\mathrm{a})}}{2} \geq \min\{F^{+}{}_{N_n(\mathrm{a})}, F^{+}{}_{R_n(\mathrm{a})}\} \quad (6)$$

Also,

for $\cap_P$, $\cap_O$ and $\cap_A$,

$$K^{n}{}_{N_nR_n} = K^{n}{}_{N_n} \cap K^{n}{}_{R_n}, L^{n}{}_{N_nR_n} = L^{n}{}_{N_n} \cap L^{n}{}_{R_n}, M^{n}{}_{N_nR_n} = M^{n}{}_{N_n} \cap M^{n}{}_{R_n}, P^{n}{}_{N_nR_n} = P^{n}{}_{N_n} \cap P^{n}{}_{R_n} \quad (7)$$

is hold. Thus, from 1-7 and Definition 9; we obtain that

$$(\mathrm{N} \cap_P \mathrm{R}) \subset (\mathrm{N} \cap_A \mathrm{R}) \subset (\mathrm{N} \cap_O \mathrm{R}).$$

Proofs of {ii, iii} can be given similarly to proof of i.

Theorem 2: Let

N = {<$K^{1}{}_{N_1}$, $(T^{-}{}_{N_1(\mathrm{a})}, T^{+}{}_{N_1(\mathrm{a})}) L^{1}{}_{N_1}$, $(I^{-}{}_{N_1(\mathrm{a})}, I^{+}{}_{N_1(\mathrm{a})})\ M^{1}{}_{N_1}$, $(F^{-}{}_{N_1(\mathrm{a})}, F^{+}{}_{N_1(\mathrm{a})})\ P^{1}{}_{N_1}$;

$K^{2}{}_{N_2}$, $(T^{-}{}_{N_2(\mathrm{a})}, T^{+}{}_{N_2(\mathrm{a})}) L^{2}{}_{N_2}$, $(I^{-}{}_{N_2(\mathrm{a})}, I^{+}{}_{N_2(\mathrm{a})})\ M^{2}{}_{N_2}$, $(F^{-}{}_{N_2(\mathrm{a})}, F^{+}{}_{N_2(\mathrm{a})})\ P^{2}{}_{N_2}$;

$K^{i}{}_{N_i}$, $(T^{-}{}_{N_i(\mathrm{a})}, T^{+}{}_{N_i(\mathrm{a})}) L^{i}{}_{N_i}$, $(I^{-}{}_{N_i(\mathrm{a})}, I^{+}{}_{N_i(\mathrm{a})})\ M^{i}{}_{N_i}$, $(F^{-}{}_{N_i(\mathrm{a})}, F^{+}{}_{N_i(\mathrm{a})})\ P^{i}{}_{N_i}$ >,

$K^{n}{}_{N_n}$, $L^{n}{}_{N_n}$, $M^{n}{}_{N_n}$, $P^{n}{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

and

R = {<$K^{1}{}_{R_1}$, $(T^{-}{}_{R_1(\mathrm{a})}, T^{+}{}_{R_1(\mathrm{a})}) L^{1}{}_{R_1}$, $(I^{-}{}_{R_1(\mathrm{a})}, I^{+}{}_{R_1(\mathrm{a})})\ M^{1}{}_{R_1}$, $(F^{-}{}_{R_1(\mathrm{a})}, F^{+}{}_{R_1(\mathrm{a})})\ P^{1}{}_{R_1}$;

$K^{2}{}_{R_2}$, $(T^{-}{}_{R_2(\mathrm{a})}, T^{+}{}_{R_2(\mathrm{a})}) L^{2}{}_{R_2}$, $(I^{-}{}_{R_2(\mathrm{a})}, I^{+}{}_{R_2(\mathrm{a})})\ M^{2}{}_{R_2}$, $(F^{-}{}_{R_2(\mathrm{a})}, F^{+}{}_{R_2(\mathrm{a})})\ P^{2}{}_{R_2}$;

$K^{i}{}_{R_i}$, $(T^{-}{}_{R_i(\mathrm{a})}, T^{+}{}_{R_i(\mathrm{a})}) L^{i}{}_{R_i}$, $(I^{-}{}_{R_i(\mathrm{a})}, I^{+}{}_{R_i(\mathrm{a})})\ M^{i}{}_{R_i}$, $(F^{-}{}_{R_i(\mathrm{a})}, F^{+}{}_{R_i(\mathrm{a})})\ P^{i}{}_{R_i}$ >,

$K^{n}{}_{R_n}$, $L^{n}{}_{R_n}$, $M^{n}{}_{R_n}$, $P^{n}{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

be two BgsvNqss. We assume that N ⊂ R. Then,

i) N ⊂ (N $\cap_A$ R) ⊂ R, N ⊂ (N $\cap_O$ R) ⊂ R and N = (N $\cap_P$ R) ⊂ R.

ii) N ⊂ (N $\cup_A$ R) ⊂ R, N ⊂ (N $\cup_O$ R) = R and N ⊂ (N $\cup_P$ R) ⊂ R.

iii) (N $/_A$ R) ⊂ R, (R $/_A$ N) ⊂ R, (N $/_O$ R) ⊂ R, (R $/_O$ N) ⊂ R, (N $/_P$ R) ⊂ R and (R $/_P$ N) ⊂ R.

Proof:

i) From Definition 9; we obtain that

$$K^{n}{}_{N_n} \subset K^{n}{}_{R_n}, L^{n}{}_{N_n} \subset L^{n}{}_{R_n}, M^{n}{}_{N_n} \subset M^{n}{}_{R_n}, P^{n}{}_{N_n} \subset P^{n}{}_{R_n};$$

$$T^{-}{}_{N_n(\mathrm{a})} \leq T^{-}{}_{R_n(\mathrm{a})}, T^{+}{}_{N_n(\mathrm{a})} \leq T^{+}{}_{R_n(\mathrm{a})};$$

$I^{-}_{N_n(a)} \geq I^{-}_{R_n(a)}\}, I^{+}_{N_n(a)} \geq I^{+}_{R_n(a)};$

$$F^{-}_{N_n(a)} \geq F^{-}_{R_n(a)}\}, F^{+}_{N_n(a)} \geq F^{+}_{R_n(a)}. \quad (8)$$

Thus, we obtain that

$$K^{n}_{N_nR_n} = K^{n}_{N_n} \cap K^{n}_{R_n} = K^{n}_{R_n}, L^{n}_{N_nR_n} = L^{n}_{N_n} \cap L^{n}_{R_n} = L^{n}_{R_n},$$

$$M^{n}_{N_nR_n} = M^{n}_{N_n} \cap M^{n}_{R_n} = M^{n}_{R_n}, P^{n}_{N_nR_n} = P^{n}_{N_n} \cap P^{n}_{R_n} = P^{n}_{R_n}. \quad (9)$$

Also, from Proof of (i) of Theorem 1; conditions 1-7 are hold. Hence, thanks to Definition 10, Definition 11, Definition 12, 1-7 and 9; we obtain that

$$N \subset (N \cap_A R) \subset R, N \subset (N \cap_O R) \subset R \text{ and } N = (N \cap_P R) \subset R.$$

Proofs of {ii, iii} can be given similarly to proof of i.

Conclusions

In this chapter, we obtain BgsvNqs and BgsvNqsn using generalized set valued neutrosophic quadruple sets with bipolar single valued neutrosophic sets. Thanks to BgsvNqs and BgsvNqsn, generalized set valued neutrosophic quadruple sets and bipolar single valued neutrosophic sets will more useful together. Also, we obtain some basic properties and some operations ($\cup_A$, $\cup_O$, $\cup_P$,$\cap_A$, $\cap_O$, $\cap_P$,$/_A$, $/_O$, $/_P$). Especially, for decision making problems; these operations will more useful. Furthermore, thanks to definitions of BgsvNqs, BgsvNqsn and operations ($\cup_A$, $\cup_O$, $\cup_P$,$\cap_A$, $\cap_O$, $\cap_P$,$/_A$, $/_O$, $/_P$); researchers can define similarity measures, some specific decision making methods (TOPSIS, VIKOR, DEMATEL, AHP, …), arithmetic operations, aggregation operations based on BgsvNqs and BgsvNqsn for decision making problems.

Abbreviations

BgsvNqs: Bipolar generalized set valued neutrosophic quadruple set

BgsvNqsn: Bipolar generalized set valued neutrosophic quadruple number

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Chapter Nine

Interval Generalized Set Valued Neutrosophic Quadruple Sets and Numbers

Abdullah Kargın[1, *] and Memet Şahin[2]

[1, 2]Department of Mathematics, Gaziantep University, Gaziantep27310-Turkey
[*]Corresponding Author: abdullahkargin27@gmail.com
E-mail: abdullahkargin27@gmail.com, mesahin@gantep.edu.tr

ABSTRACT

Interval neutrosophic sets are more useful in neutrosophic theory, especially at decision making problems. Also, generalized set valued neutrosophic quadruple sets have an important role in neutrosophic quadruple theory and single valued neutrosophic theory. Thanks to generalized set valued neutrosophic quadruple sets, the solutions of decision-making problems in which single-valued neutrosophic numbers are used can be obtained more objectively. In this chapter, we obtain interval generalized set valued neutrosophic quadruple sets and numbers. We give some basic properties for interval generalized set valued neutrosophic quadruple sets and numbers. Also, we define some new operations for interval generalized set valued neutrosophic quadruple sets and numbers. Thus, we obtain a new structure based on generalized set valued neutrosophic quadruple sets and interval neutrosophic numbers. In this way, we obtain new results for generalized set valued neutrosophic quadruple set and interval neutrosophic set. Furthermore, thanks to this new structure; the solutions of decision-making problems in which interval neutrosophic numbers are used will be obtained more objectively.

Keywords: interval neutrosophic set, neutrosophic quadruple number, set valued neutrosophic quadruple set, interval generalized set valued neutrosophic quadruple set

INTRODUCTION

Neutrosophic logic and the concept of neutrosophic set are defined in 1998 by Florentin Smarandache [1]. In the concept of neutrosophic logic and neutrosophic sets, there is the degree of membership T, degree of uncertainty I and degree of non-membership F. These degrees are defined independently from each other. It has the form of a neutrosophic value (T, I, F). In other words, a situation is handled in neutrosophy according to its accuracy, its falsehood, and its uncertainty. In addition, many researchers have conducted studies on neutrosophic set theory [2-4]. Recently, Hamidi and Smarandache studied single-valued neutro Hyper BCK-subalgebras [5]; Mohana and Mohanasundari introduced on some similarity measures of single valued neutrosophic rough sets [6]; Ali et al. obtained neutrosophic triplet ring and neutrosophic triplet field [7]; Şahin et al. studied generalized Euclid measures on generalized neutrosophic quadruple numbers [8]; Aslan et al. obtained neutrosophic modeling of Talcott Parsons's action [9]; Kargın et al. introduced Hamming similarity measure on generalized neutrosophic quadruple numbers [10]; Şahin and Dayan studied generalized neutrosophic quadruple numbers based on Hamming measure for law [11]; Alhasan et al. obtained neutrosophic reliability theory [12]; Şahin and Uz introduced multi-criteria decision-making applications based neutrosophic quadruple sets for law [13].

Wang et al. studied interval neutrosophic sets and logic in 2005 [14]. The interval neutrosophic sets have an important role in neutrosophic theory and decision making problems. The use of intervals as values in interval neutrosophic sets makes this set superior to other sets in many problem situations. Because while it is often difficult to reach a definite judgment in a decision-making situation, a decision given as an interval will be more useful. Hence, many researchers studied based on interval neutrosophic sets and logic [15-17]. Recently, Chi and Liu studied TOPSIS method based on interval neutrosophic set [18]; Liu and Tang obtained some power generalized aggregation operators based on the interval neutrosophic sets [19]; Hashim et al. introduced entropy measures for interval neutrosophic vague sets [20]; Pai and Gaonkar studied the safety assessment in dynamic conditions using interval neutrosophic sets [21]; Karthikeyan and Karuppaiya obtained reverse subsystems of interval neutrosophic automata [22].

Smarandache obtained neutrosophic quadruple set and numbers in 2015 [23]. While neutrosophic quadruple set have T, I and F components as in neutrosophic sets; unlike neutrosophic sets, there is a known part and an unknown part. Therefore, neutrosophic quadruple sets are a generalization of neutrosophic sets. For this reason, neutrosophic quadruple sets are widely used in the algebraic and application areas [24-27]. Recently, Muhiuddin et al. studied implicative neutrosophic quadruple BCK-algebras and ideals [28]; Şahin et al. obtained generalized set valued neutrosophic quadruple sets and numbers [29]; Li et al. introduced neutrosophic extended triplet group based on neutrosophic quadruple numbers [30]; Şahin and Kargın obtained neutrosophic triplet groups based on set valued neutrosophic quadruple numbers [31]; Borzooei et al. studied positive implicative neutrosophic quadruple BCK-algebras and ideals [32]; Şahin and Kargın introduced single valued neutrosophic quadruple graphs [33]; Smarandache et al. obtained neutrosophic quadruple groups [33];

Şahin et al. studied generalized set valued neutrosophic quadruple numbers and decision making applications [34].

In this chapter, we obtain interval generalized set valued neutrosophic quadruple sets (IgsvNqs) and numbers (IgsvNqsn) using generalized set valued neutrosophic quadruple sets and interval neutrosophic sets. Thanks to IgsvNqs and IgsvNqsn, generalized set valued neutrosophic quadruple sets and interval neutrosophic sets will more useful together. Also, we obtain some basic properties and some operations ($\cup_A$, $\cup_O$, $\cup_P$,$\cap_A$, $\cap_O$, $\cap_P$,$/_A$, $/_O$, $/_P$). In fact, we generalize the some operations in [29] for IgsvNqs. In Section 2, we introduced some basic definitions for interval neutrosophic set [14], neutrosophic quadruple sets [31], [29]. In recent years, the academic community has witnessed growing research interests in neutrosophic set theory [36-72].

BACKGROUND

Definition 1. [14] Let A be a universal set. Interval neutrosophic set N; is identified as

$$N = \{<a: [T^l{}_{N(a)}, T^u{}_{N(a)}], [I^l{}_{N(a)}, I^u{}_{N(a)}], [F^l{}_{N(a)}, F^u{}_{N(a)}], >, a \in A\}.$$

Where the functions

$T^l{}_N$:A → [0,1], $T^u{}_N$:A → [0,1] is truth functions;

$I^l{}_N$:A → [0,1], $I^u{}_N$:A → [0,1] is uncertainly functions;

and $F^l{}_N$:A → [0,1] and $F^u{}_N$:A → [0,1] is falsity functions.

Definition 2. [14] Let

$$N_1 = \{<a: [T^l{}_{N_1(a)}, T^u{}_{N_1(a)}], [I^l{}_{N_1(a)}, I^u{}_{N_1(a)}], [F^l{}_{N_1(a)}, F^u{}_{N_1(a)}], >, a \in A\}$$

and

$$N_2 = \{<a: [T^l{}_{N_2(a)}, T^u{}_{N_2(a)}], [I^l{}_{N_2(a)}, I^u{}_{N_2(a)}], [F^l{}_{N_2(a)}, F^u{}_{N_2(a)}], >, a \in A\}$$

be two interval neutrosophic sets.

i) N_2 is subset of N_1 if and only if

$T^l{}_{N_1(a)} \geq T^l{}_{N_2(a)}$, $T^u{}_{N_1(a)} \geq T^u{}_{N_2(a)}$

$I^l{}_{N_1(a)} \leq I^l{}_{N_2(a)}$, $I^u{}_{N_1(a)} \leq I^u{}_{N_2(a)}$

$F^l{}_{N_1(\mathrm{a})} \leq F^l{}_{N_2(\mathrm{a})}$, $F^u{}_{N_1(\mathrm{a})} \leq F^u{}_{N_2(\mathrm{a})}$

ii) N_2 is equal to N_1 if and only if

$T^l{}_{N_1(\mathrm{a})} = T^l{}_{N_2(\mathrm{a})}$, $T^u{}_{N_1(\mathrm{a})} = T^u{}_{N_2(\mathrm{a})}$;

$I^l{}_{N_1(\mathrm{a})} = I^l{}_{N_2(\mathrm{a})}$, $I^u{}_{N_1(\mathrm{a})} = I^u{}_{N_2(\mathrm{a})}$;

$F^l{}_{N_1(\mathrm{a})} = F^l{}_{N_2(\mathrm{a})}$, $F^u{}_{N_1(\mathrm{a})} = F^u{}_{N_2(\mathrm{a})}$.

iii) $N_1 \cup N_2$ = {<a: [$\max\{T^l{}_{N_1(\mathrm{a})}, T^l{}_{N_2(\mathrm{a})}\}, \max\{T^u{}_{N_1(\mathrm{a})}, T^u{}_{N_2(\mathrm{a})}\}$], [$\min\{I^l{}_{N_1(\mathrm{a})}, I^l{}_{N_2(\mathrm{a})}\}, \min\{I^u{}_{N_1(\mathrm{a})}, I^u{}_{N_2(\mathrm{a})}\}$], [$\min\{F^l{}_{N_1(\mathrm{a})}, F^l{}_{N_2(\mathrm{a})}\}, \min\{F^u{}_{N_1(\mathrm{a})}, F^u{}_{N_2(\mathrm{a})}\}$] >, a∈A}.

iv) $N_1 \cap N_2$ = {<a: [$\min\{T^l{}_{N_1(\mathrm{a})}, T^l{}_{N_2(\mathrm{a})}\}, \min\{T^u{}_{N_1(\mathrm{a})}, T^u{}_{N_2(\mathrm{a})}\}$], [$\max\{I^l{}_{N_1(\mathrm{a})}, I^l{}_{N_2(\mathrm{a})}\}, \max\{I^u{}_{N_1(\mathrm{a})}, I^u{}_{N_2(\mathrm{a})}\}$], [$\max\{F^l{}_{N_1(\mathrm{a})}, F^l{}_{N_2(\mathrm{a})}\}, \max\{F^u{}_{N_1(\mathrm{a})}, F^u{}_{N_2(\mathrm{a})}\}$] >, a∈A}.

Definition 3: [31] Let N be a set and P(N) be power set of N. A set valued neutrosophic quadruple set is shown by the form

$$(A_1, A_2\mathrm{T}, A_3\mathrm{I}, A_4\mathrm{F}).$$

Where, T, I and F are degree of membership, degree of undeterminacy, degree of non-membership in neutrosophic theory, respectively. Also, A_1, A_2, A_3, A_4 ∈ P(N); A_1 is called the known part and (A_1, A_2T, A_3I, A_4F) is called the unknown part.

Definition 4: [31] Let A = (A_1, A_2T, A_3I, A_4F) and B = (B_1, B_2T, B_3I, B_4F) be set valued neutrosophic quadruple set s. We define the following operations, well known operators in set theory, such that

A ∪ B = ($A_1 \cup B_1$, $(A_2 \cup B_2)$T, $(A_3 \cup B_3)$I, $(A_4 \cup B_4)$F)

A ∩ B = ($A_1 \cap B_1$, $(A_2 \cap B_2)$T, $(A_3 \cap B_3)$I, $(A_4 \cap B_4)$F)

A \ B = ($A_1 \setminus B_1$, $(A_2 \setminus B_2)$T, $(A_3 \setminus B_3)$I, $(A_4 \setminus B_4)$F)

A' = (A'_1, A'_2T, A'_3I, A'_4F)

Definition 5: [31] Let A = (A_1, A_2T, A_3I, A_4F), B = (B_1, B_2T, B_3I, B_4F) be set valued neutrosophic quadruple sets. If

$A_1 \subset B_1$, $A_2 \subset B_2$ and $A_3 \subset B_3$, $A_4 \subset B_4$,

then it is called that A is subset of B. It is shown by

$$A \subset B.$$

Definition 6: [31] Let A = (A_1, A_2T, A_3I, A_4F), B = (B_1, B_2T, B_3I, B_4F) be set valued neutrosophic quadruple sets. If

$$A \subset B \text{ and } B \subset A,$$

then it is called that A is equal to B. It is shown by

$$A = B.$$

Definition 7: [29] Let A be a universal set and P(A) be power set of A. A generalized set valued neutrosophic quadruple set N; is identified as

$$N = \{<{K^1}_{N_1}, T_{N_1(a)}{L^1}_{N_1}, I_{N_1(a)}\ {M^1}_{N_1}, F_{N_1(a)}\ {P^1}_{N_1};$$

$${K^2}_{N_2}, T_{N_2(a)}{L^2}_{N_2}, I_{N_2(a)}\ {M^2}_{N_2}, F_{N_2(a)}\ {P^2}_{N_2};$$

$${K^i}_{N_i}, T_{N_i(a)}{L^i}_{N_i}, I_{N_i(a)}\ {M^i}_{N_i}, F_{N_i(a)}\ {P^i}_{N_i} >,$$

$${K^n}_{N_n},\ {L^n}_{N_n},\ {M^n}_{N_n},\ {P^n}_{N_n} \in P(A);\ n = 1, 2, 3, \dots, i\}.$$

Where,

$$T_{N_n(a)}, I_{N_n(a)}, F_{N_n(a)}\ (n = 1, 2, 3, \dots, i)$$

have their usual single valued neutrosophic logic means and a generalized set valued neutrosophic quadruple number ${N^n}_1$; is identified as

$${N^N}_1 = \{<{K^1}_{N_1}, T_{N_1(a)}{L^1}_{N_1}, I_{N_1(a)}\ {M^1}_{N_1}, F_{N_1(a)}\ {P^1}_{N_1} >\}.$$

As in neutrosophic quadruple number, for a generalized set valued neutrosophic quadruple number

$${K^1}_{N_1}$$

is called known part and

$$T_{N_1(\mathrm{a})}{L^1}_{N_1}, I_{N_1(\mathrm{a})}\,{M^1}_{N_1}, F_{N_1(\mathrm{a})}\,{P^1}_{N_1}$$

is called the unknown part.

Also, we can show that

$$\mathrm{N} = \{{N^N}_n\text{: n} = 1, 2, 3, \ldots, \mathrm{i}\}.$$

INTERVAL GENERALIZED SET VALUED NEUTROSOPHIC QUADRUPLE SETS AND NUMBERS

Definition 8: Let A be a universal set and P(A) be power set of A. Interval generalized set valued neutrosophic quadruple set (IgsvNqs) N; is identified as

$$\mathrm{N} = \{<{K^1}_{N_1}, [{T^l}_{N_1(\mathrm{a})}, {T^u}_{N_1(\mathrm{a})}]{L^1}_{N_1}, [{I^l}_{N_1(\mathrm{a})}, {I^u}_{N_1(\mathrm{a})}]\,{M^1}_{N_1}, [{F^l}_{N_1(\mathrm{a})}, {F^u}_{N_1(\mathrm{a})}]\,{P^1}_{N_1};$$

$${K^2}_{N_2}, [{T^l}_{N_2(\mathrm{a})}, {T^u}_{N_2(\mathrm{a})}]{L^2}_{N_2}, [{I^l}_{N_2(\mathrm{a})}, {I^u}_{N_2(\mathrm{a})}]\,{M^2}_{N_2}, [{F^l}_{N_2(\mathrm{a})}, {F^u}_{N_2(\mathrm{a})}]\,{P^2}_{N_2};$$

$${K^i}_{N_i}, [{T^l}_{N_i(\mathrm{a})}, {T^u}_{N_i(\mathrm{a})}]{L^i}_{N_i}, [{I^l}_{N_i(\mathrm{a})}, {I^u}_{N_i(\mathrm{a})}]\,{M^i}_{N_i}, [{F^l}_{N_i(\mathrm{a})}, {F^u}_{N_i(\mathrm{a})}]\,{P^i}_{N_i} >,$$

$${K^n}_{N_n},\ {L^n}_{N_n},\ {M^n}_{N_n},\ {P^n}_{N_n} \in \mathrm{P(A)}; \mathrm{n} = 1, 2, 3, \ldots, \mathrm{i}\}.$$

Where,

$${T^l}_{N_n(\mathrm{a})}, {I^l}_{N_n(\mathrm{a})}, {F^l}_{N_n(\mathrm{a})}, {T^u}_{N_n(\mathrm{a})}, {I^u}_{N_n(\mathrm{a})} \text{ and } {F^u}_{N_n(\mathrm{a})} \ (\mathrm{n} = 1, 2, 3, \ldots, \mathrm{i})$$

have their usual interval neutrosophic logic means and an interval generalized neutrosophic quadruple number (IgsvNqn) ${N^n}_1$; is identified as

$${N^N}_1 = \{<{K^1}_{N_1}, [{T^l}_{N_1(\mathrm{a})}, {T^u}_{N_1(\mathrm{a})}]{L^1}_{N_1}, [{I^l}_{N_1(\mathrm{a})}, {I^u}_{N_1(\mathrm{a})}]\,{M^1}_{N_1}, [{F^l}_{N_1(\mathrm{a})}, {F^u}_{N_1(\mathrm{a})}]\,{P^1}_{N_1} >\}$$

As in neutrosophic quadruple number, for a IgsvNqn,

$${K^1}_{N_1}$$

is called known part and

$$[{T^l}_{N_1(\mathrm{a})}, {T^u}_{N_1(\mathrm{a})}]{L^1}_{N_1}, [{I^l}_{N_1(\mathrm{a})}, {I^u}_{N_1(\mathrm{a})}]\,{M^1}_{N_1}, [{F^l}_{N_1(\mathrm{a})}, {F^u}_{N_1(\mathrm{a})}]\,{P^1}_{N_1}$$

is called the unknown part.

Also, we can show that

$$\mathrm{N} = \{{N^N}_n\text{: n} = 1, 2, 3, \ldots, \mathrm{i}\}.$$

Example 1: Let A = {k, l, m, n, p, r} be a set. Then;

$$\mathrm{N} = \{<\{k, l, m, n\}, [0, 0.7]\{k, l\}, [0.5, 0.6]\,\{m\}, [0.4, 0.5]\,\{n\};$$

$\{k, l, p, r\}$, [0.1, 0.9]$\{k, p\}$, [0.2, 0.3] $\{l\}$, [0.2, 0.7] $\{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, [0.4, 0.8]$\{l, p\}$, [0, 0.3] $\{p, m\}$, [0.2, 0.6] $\{n\}$;

$\{m, l, p, r\}$, [0.3, 0.7]$\{p\}$, [0.2, 0.5] $\{m, l\}$, [0.1, 0.5] $\{r\}$ >}

are two IgsvNqs.

Also,

$N^N{}_1$={$\{k, l, m, n\}$, [0, 0.7]$\{k, l\}$, [0.5, 0.6] $\{m\}$, [0.4, 0.5] $\{n\}$}

and

$N^N{}_2$ = {$\{k, l, p, r\}$, [0.3, 0.7]$\{k, p\}$, [0.2, 0.5] $\{l\}$, [0.1, 0.5] $\{r\}$}

are two IgsvNqn of N such that

N = $\{N^N{}_1, N^N{}_2\}$.

Similarly,

$R^N{}_1$ ={$\{l, p, m, n, k\}$, [0.4, 0.8]$\{l, p\}$, [0, 0.3] $\{p, m\}$, [0.2, 0.6] $\{n\}$}

and

$R^N{}_2$ = {$\{m, l, p, r\}$, [0.1, 0.9]$\{p\}$, [0.2, 0.3] $\{m, l\}$, [0.2, 0.7] $\{r\}$}

are two IgsvNqn of R such that

R = $\{R^N{}_1, R^N{}_2\}$.

Definition 9: Let

N = {<$K^1{}_{N_1}$, $[T^l{}_{N_1(a)}, T^u{}_{N_1(a)}]L^1{}_{N_1}$, $[I^l{}_{N_1(a)}, I^u{}_{N_1(a)}]\, M^1{}_{N_1}$, $[F^l{}_{N_1(a)}, F^u{}_{N_1(a)}]\, P^1{}_{N_1}$;

$K^2{}_{N_2}$, $[T^l{}_{N_2(a)}, T^u{}_{N_2(a)}]L^2{}_{N_2}$, $[I^l{}_{N_2(a)}, I^u{}_{N_2(a)}]\, M^2{}_{N_2}$, $[F^l{}_{N_2(a)}, F^u{}_{N_2(a)}]\, P^2{}_{N_2}$;

$K^i{}_{N_i}$, $[T^l{}_{N_i(a)}, T^u{}_{N_i(a)}]L^i{}_{N_i}$, $[I^l{}_{N_i(a)}, I^u{}_{N_i(a)}]\, M^i{}_{N_i}$, $[F^l{}_{N_i(a)}, F^u{}_{N_i(a)}]\, P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}

and

R = {<$K^1{}_{R_1}$, $[T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}$, $[I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]\, M^1{}_{R_1}$, $[F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]\, P^1{}_{R_1}$;

$K^2{}_{R_2}$, $[T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}$, $[I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]\, M^2{}_{R_2}$, $[F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]\, P^2{}_{R_2}$; …

$K^i{}_{R_i}$, $[T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}$, $[I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]\, M^i{}_{R_i}$, $[F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]\, P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i} be two IgsvNqss.

i) N is subset of R (N⊂R) if and only if

${K^n}_{N_n} \subset {K^n}_{R_n}$, ${L^n}_{N_n} \subset {L^n}_{R_n}$, ${M^n}_{N_n} \subset {M^n}_{R_n}$, ${P^n}_{N_n} \subset {P^n}_{R_n}$;

${T^l}_{N_n(a)} \leq {T^l}_{R_n(a)}$, ${T^u}_{N_n(a)} \leq {T^u}_{R_n(a)}$;

${I^l}_{N_n(a)} \geq {I^l}_{R_n(a)}\}$, ${I^u}_{N_n(a)} \geq {I^u}_{R_n(a)}$;

${F^l}_{N_n(a)} \geq {F^l}_{R_n(a)}\}$, ${F^u}_{N_n(a)} \geq {F^u}_{R_n(a)}$.

ii) N is equal to R if and only if

${K^n}_{N_n} = {K^n}_{R_n}$, ${L^n}_{N_n} = {L^n}_{R_n}$, ${M^n}_{N_n} = {M^n}_{R_n}$, ${P^n}_{N_n} = {P^n}_{R_n}$;

${T^l}_{N_n(a)} = {T^l}_{R_n(a)}$, ${T^u}_{N_n(a)} = {T^u}_{R_n(a)}$;

${I^l}_{N_n(a)} = {I^l}_{R_n(a)}\}$, ${I^u}_{N_n(a)} = {I^u}_{R_n(a)}$;

${F^l}_{N_n(a)} = {F^l}_{R_n(a)}\}$, ${F^u}_{N_n(a)} = {F^u}_{R_n(a)}$.

Example 2: From Example 1,

$$N = \{<\{k, l, m, n\}, [0, 0.7]\{k, l\}, [0.5, 0.6]\ \{m\}, [0.4, 0.5]\ \{n\};$$

$$\{k, l, p, r\}, [0.1, 0.9]\{k, p\}, [0.2, 0.3]\ \{l\}, [0.2, 0.7]\ \{r\} >\}$$

is a IgsvNqss. Also, it is clear that

$$Y = \{<\{k, m, n\}, [0, 0.5]\{k\}, [0.6, 0.7]\ \{m\}, [0.6, 0.8]\ \{n\};$$

$$\{, l, p, r\}, [0.3, 0.9]\{p\}, [0.4, 0.5]\ \{l\}, [0.3, 0.8]\ \{r\} >\}$$

is a IgsvNqss. Thus,

$$Y \subset N.$$

Definition 10: Let

$N = \{<{K^1}_{N_1}, [{T^l}_{N_1(a)}, {T^u}_{N_1(a)}]{L^1}_{N_1}, [{I^l}_{N_1(a)}, {I^u}_{N_1(a)}]\ {M^1}_{N_1}, [{F^l}_{N_1(a)}, {F^u}_{N_1(a)}]\ {P^1}_{N_1};$

${K^2}_{N_2}, [{T^l}_{N_2(a)}, {T^u}_{N_2(a)}]{L^2}_{N_2}, [{I^l}_{N_2(a)}, {I^u}_{N_2(a)}]\ {M^2}_{N_2}, [{F^l}_{N_2(a)}, {F^u}_{N_2(a)}]\ {P^2}_{N_2};$

${K^i}_{N_i}, [{T^l}_{N_i(a)}, {T^u}_{N_i(a)}]{L^i}_{N_i}, [{I^l}_{N_i(a)}, {I^u}_{N_i(a)}]\ {M^i}_{N_i}, [{F^l}_{N_i(a)}, {F^u}_{N_i(a)}]\ {P^i}_{N_i} >,$

${K^n}_{N_n}, {L^n}_{N_n}, {M^n}_{N_n}, {P^n}_{N_n} \in P(A); n = 1, 2, 3, \dots, i\}$

and

R = $\{<K^1{}_{R_1}, [T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}, [I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]\, M^1{}_{R_1}, [F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]\, P^1{}_{R_1};$

$K^2{}_{R_2}, [T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}, [I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]\, M^2{}_{R_2}, [F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]\, P^2{}_{R_2};$

$K^i{}_{R_i}, [T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}, [I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]\, M^i{}_{R_i}, [F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]\, P^i{}_{R_i}>,$

$K^n{}_{R_n},\ L^n{}_{R_n},\ M^n{}_{R_n},\ P^n{}_{R_n} \in$ P(A); n = 1, 2, 3, … , i}

be two IgsvNqss.

i) We define the "average ∪" operations for N and R such that

N $\cup_A$ R = $\{<K^1{}_{N_1R_1}, [T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}, [I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]\, M^1{}_{N_1R_1}, [F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]\, P^1{}_{N_1R_1};$

$K^2{}_{N_2R_2}, [T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}, [I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]\, M^2{}_{N_2R_2}, [F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]\, P^2{}_{N_2R_2};$

$K^i{}_{N_iR_i}, [T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}, [I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\, M^i{}_{N_iR_i}, [F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\, P^i{}_{N_iR_i}>,$

$K^n{}_{N_nR_n},\ L^n{}_{N_nR_n},\ M^n{}_{N_nR_n},\ P^n{}_{N_nR_n} \in$ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n} = K^n{}_{N_n} \cup K^n{}_{R_n}, L^n{}_{N_nR_n} = L^n{}_{N_n} \cup L^n{}_{R_n}, M^n{}_{N_nR_n} = M^n{}_{N_n} \cup M^n{}_{R_n}, P^n{}_{N_nR_n} = P^n{}_{N_n} \cup P^n{}_{R_n};$

$T^l{}_{N_nR_n(a)} = \frac{T^l{}_{N_n(a)} + T^l{}_{R_n(a)}}{2}, T^u{}_{N_nR_n(a)} = \frac{T^u{}_{N_n(a)} + T^u{}_{R_n(a)}}{2};$

$I^l{}_{N_nR_n(a)} = \frac{I^l{}_{N_n(a)} + I^l{}_{R_n(a)}}{2}, I^u{}_{N_nR_n(a)} = \frac{I^u{}_{N_n(a)} + I^u{}_{R_n(a)}}{2};$

$F^l{}_{N_nR_n(a)} = \frac{F^l{}_{N_n(a)} + F^l{}_{R_n(a)}}{2}, F^u{}_{N_nR_n(a)} = \frac{F^u{}_{N_n(a)} + F^u{}_{R_n(a)}}{2};$ (n = 1, 2, 3, … , i).

ii) We define the "average ∩ " operations for N and R such that

N $\cap_A$ R = $\{<K^1{}_{N_1R_1}, [T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}, [I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]\, M^1{}_{N_1R_1}, [F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]\, P^1{}_{N_1R_1};$

$K^2{}_{N_2R_2}, [T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}, [I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]\, M^2{}_{N_2R_2}, [F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]\, P^2{}_{N_2R_2};$

$K^i{}_{N_iR_i}, [T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}, [I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\, M^i{}_{N_iR_i}, [F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\, P^i{}_{N_iR_i}>,$

$K^n{}_{N_nR_n},\ L^n{}_{N_nR_n},\ M^n{}_{N_nR_n},\ P^n{}_{N_nR_n} \in$ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n} = K^n{}_{N_n} \cap K^n{}_{R_n}, L^n{}_{N_nR_n} = L^n{}_{N_n} \cap L^n{}_{R_n}, M^n{}_{N_nR_n} = M^n{}_{N_n} \cap M^n{}_{R_n}, P^n{}_{N_nR_n} = P^n{}_{N_n} \cap P^n{}_{R_n};$

$T^l{}_{N_nR_n(a)} = \frac{T^l{}_{N_n(a)} + T^l{}_{R_n(a)}}{2}, T^u{}_{N_nR_n(a)} = \frac{T^u{}_{N_n(a)} + T^u{}_{R_n(a)}}{2};$

$I^l{}_{N_nR_n(a)} = \frac{I^l{}_{N_n(a)} + I^l{}_{R_n(a)}}{2}, I^u{}_{N_nR_n(a)} = \frac{I^u{}_{N_n(a)} + I^u{}_{R_n(a)}}{2};$

$F^l{}_{N_nR_n(a)} = \frac{F^l{}_{N_n(a)} + F^l{}_{R_n(a)}}{2}, F^u{}_{N_nR_n(a)} = \frac{F^u{}_{N_n(a)} + F^u{}_{R_n(a)}}{2};$ (n = 1, 2, 3, … , i).

Example 3: From Example 1,

N = {<$\{k, l, m, n\}$, [0, 0.7]$\{k, l\}$, [0.5, 0.6] $\{m\}$, [0.4, 0.5] $\{n\}$;

$\{k, l, p, r\}$, [0.1, 0.9]$\{k, p\}$, [0.2, 0.3] $\{l\}$, [0.2, 0.7] $\{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, [0.4, 0.8]$\{l, p\}$, [0, 0.3] $\{p, m\}$, [0.2, 0.6] $\{n\}$;

$\{m, l, p, r\}$, [0.3, 0.7]$\{p\}$, [0.2, 0.5] $\{m, l\}$, [0.1, 0.5] $\{r\}$ >}

are two IgsvNqs. Thus,

i) N $\cup_A$ R = {<$\{k, l, m, n, p\}$, [0.2, 0.75]$\{k, l, p\}$, [0.25, 0.45] $\{p, m\}$, [0.3, 0.6] $\{n\}$;

$\{m, k, l, p, r\}$, [0.2, 0.8]$\{k, p\}$, [0.2, 0.4] $\{m, l\}$, [0.15, 0.6] $\{r\}$ >}

ii) N $\cap_A$ R = {<$\{k, l, m, n\}$, [0.2, 0.75]$\{l\}$, [0.25, 0.45] $\{m\}$, [0.3, 0.6] $\{n\}$;

$\{l, p, r\}$, [0.2, 0.8]$\{p\}$, [0.2, 0.4] $\{l\}$, [0.15, 0.6] $\{r\}$ >}

Definition 11: Let

N = {<${K^1}_{N_1}$, $[{T^l}_{N_1(a)}, {T^u}_{N_1(a)}]{L^1}_{N_1}$, $[{I^l}_{N_1(a)}, {I^u}_{N_1(a)}]\, {M^1}_{N_1}$, $[{F^l}_{N_1(a)}, {F^u}_{N_1(a)}]\, {P^1}_{N_1}$;

${K^2}_{N_2}$, $[{T^l}_{N_2(a)}, {T^u}_{N_2(a)}]{L^2}_{N_2}$, $[{I^l}_{N_2(a)}, {I^u}_{N_2(a)}]\, {M^2}_{N_2}$, $[{F^l}_{N_2(a)}, {F^u}_{N_2(a)}]\, {P^2}_{N_2}$;

${K^i}_{N_i}$, $[{T^l}_{N_i(a)}, {T^u}_{N_i(a)}]{L^i}_{N_i}$, $[{I^l}_{N_i(a)}, {I^u}_{N_i(a)}]\, {M^i}_{N_i}$, $[{F^l}_{N_i(a)}, {F^u}_{N_i(a)}]\, {P^i}_{N_i}$ >,

${K^n}_{N_n}$, ${L^n}_{N_n}$, ${M^n}_{N_n}$, ${P^n}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}

and

R = {<${K^1}_{R_1}$, $[{T^l}_{R_1(a)}, {T^u}_{R_1(a)}]{L^1}_{R_1}$, $[{I^l}_{R_1(a)}, {I^u}_{R_1(a)}]\, {M^1}_{R_1}$, $[{F^l}_{R_1(a)}, {F^u}_{R_1(a)}]\, {P^1}_{R_1}$;

${K^2}_{R_2}$, $[{T^l}_{R_2(a)}, {T^u}_{R_2(a)}]{L^2}_{R_2}$, $[{I^l}_{R_2(a)}, {I^u}_{R_2(a)}]\, {M^2}_{R_2}$, $[{F^l}_{R_2(a)}, {F^u}_{R_2(a)}]\, {P^2}_{R_2}$;

${K^i}_{R_i}$, $[{T^l}_{R_i(a)}, {T^u}_{R_i(a)}]{L^i}_{R_i}$, $[{I^l}_{R_i(a)}, {I^u}_{R_i(a)}]\, {M^i}_{R_i}$, $[{F^l}_{R_i(a)}, {F^u}_{R_i(a)}]\, {P^i}_{R_i}$ >,

${K^n}_{R_n}$, ${L^n}_{R_n}$, ${M^n}_{R_n}$, ${P^n}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}

be two IgsvNqss.

i) We define the "optimistic ∪" operations for N and R such that

N $\cup_O$ R = {<${K^1}_{N_1R_1}$, $[{T^l}_{N_1R_1(a)}, {T^u}_{N_1R_1(a)}]{L^1}_{N_1R_1}$, $[{I^l}_{N_1R_1(a)}, {I^u}_{N_1R_1(a)}]\, {M^1}_{N_1R_1}$, $[{F^l}_{N_1R_1(a)}, {F^u}_{N_1R_1(a)}]\, {P^1}_{N_1R_1}$;

${K^2}_{N_2R_2}$, $[{T^l}_{N_2R_2(a)}, {T^u}_{N_2R_2(a)}]{L^2}_{N_2R_2}$, $[{I^l}_{N_2R_2(a)}, {I^u}_{N_2R_2(a)}]\, {M^2}_{N_2R_2}$, $[{F^l}_{N_2R_2(a)}, {F^u}_{N_2R_2(a)}]\, {P^2}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\, M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\, P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 10's (i).

$T^l{}_{N_nR_n(a)}$= max{$T^l{}_{N_n(a)}$, $T^l{}_{R_n(a)}$}, $T^u{}_{N_nR_n(a)}$= max{$T^u{}_{N_n(a)}$, $T^u{}_{R_n(a)}$};

$I^l{}_{N_nR_n(a)}$= min{$I^l{}_{N_n(a)}$, $I^l{}_{R_n(a)}$}, $I^u{}_{N_nR_n(a)}$= min{$I^u{}_{N_n(a)}$, $I^u{}_{R_n(a)}$};

$F^l{}_{N_nR_n(a)}$= min{$F^l{}_{N_n(a)}$, $F^l{}_{R_n(a)}$}, $F^u{}_{N_nR_n(a)}$= min{$F^u{}_{N_n(a)}$, $F^u{}_{R_n(a)}$}; (n = 1, 2, 3, … , i).

ii) We define the "optimistic ∩ " operations for N and R such that

N $\cap_O$ R = {<$K^1{}_{N_1R_1}$, $[T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}$, $[I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]\, M^1{}_{N_1R_1}$, $[F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]\, P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $[T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}$, $[I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]\, M^2{}_{N_2R_2}$, $[F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]\, P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\, M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\, P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 11's (ii).

$T^l{}_{N_nR_n(a)}$= max{$T^l{}_{N_n(a)}$, $T^l{}_{R_n(a)}$}, $T^u{}_{N_nR_n(a)}$= max{$T^u{}_{N_n(a)}$, $T^u{}_{R_n(a)}$};

$I^l{}_{N_nR_n(a)}$= min{$I^l{}_{N_n(a)}$, $I^l{}_{R_n(a)}$}, $I^u{}_{N_nR_n(a)}$= min{$I^u{}_{N_n(a)}$, $I^u{}_{R_n(a)}$};

$F^l{}_{N_nR_n(a)}$= min{$F^l{}_{N_n(a)}$, $F^l{}_{R_n(a)}$}, $F^u{}_{N_nR_n(a)}$= min{$F^u{}_{N_n(a)}$, $F^u{}_{R_n(a)}$}; (n = 1, 2, 3, … , i).

Example 4: From Example 1,

N = {<$\{k, l, m, n\}$, $[0, 0.7]\{k, l\}$, $[0.5, 0.6]\,\{m\}$, $[0.4, 0.5]\,\{n\}$;

$\{k, l, p, r\}$, $[0.1, 0.9]\{k, p\}$, $[0.2, 0.3]\,\{l\}$, $[0.2, 0.7]\,\{r\}$ >}

and

R = {<$\{l, p, m, n, k\}$, $[0.4, 0.8]\{l, p\}$, $[0, 0.3]\,\{p, m\}$, $[0.2, 0.6]\,\{n\}$;

$\{m, l, p, r\}$, $[0.3, 0.7]\{p\}$, $[0.2, 0.5]\,\{m, l\}$, $[0.1, 0.5]\,\{r\}$ >}

are two IgsvNqs. Thus,

i) N $\cup_O$ R = {<$\{k, l, m, n, p\}$, $[0.4, 0.8]\{k, l, p\}$, $[0, 0.3]\,\{p, m\}$, $[0.2, 0.5]\,\{n\}$;

$\{m, k, l, p, r\}$, $[0.3, 0.9]\{k, p\}$, $[0.2, 0.3]\,\{m, l\}$, $[0.1, 0.5]\,\{r\}$ >}

ii) N $\cap_O$ R = {<$\{k, l, m, n\}$, $[0.4, 0.8]\{l\}$, $[0, 0.3]$ $\{m\}$, $[0.2, 0.5]$ $\{n\}$;

$\{l, p, r\}$, $[0.3, 0.9]\{p\}$, $[0.2, 0.3]$ $\{l\}$, $[0.1, 0.5]$ $\{r\}$ >}

Definition 12: Let

N = {<$K^1{}_{N_1}$, $[T^l{}_{N_1(a)}, T^u{}_{N_1(a)}]L^1{}_{N_1}$, $[I^l{}_{N_1(a)}, I^u{}_{N_1(a)}]$ $M^1{}_{N_1}$, $[F^l{}_{N_1(a)}, F^u{}_{N_1(a)}]$ $P^1{}_{N_1}$;

$K^2{}_{N_2}$, $[T^l{}_{N_2(a)}, T^u{}_{N_2(a)}]L^2{}_{N_2}$, $[I^l{}_{N_2(a)}, I^u{}_{N_2(a)}]$ $M^2{}_{N_2}$, $[F^l{}_{N_2(a)}, F^u{}_{N_2(a)}]$ $P^2{}_{N_2}$;

$K^i{}_{N_i}$, $[T^l{}_{N_i(a)}, T^u{}_{N_i(a)}]L^i{}_{N_i}$, $[I^l{}_{N_i(a)}, I^u{}_{N_i(a)}]$ $M^i{}_{N_i}$, $[F^l{}_{N_i(a)}, F^u{}_{N_i(a)}]$ $P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}

and

R = {<$K^1{}_{R_1}$, $[T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}$, $[I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]$ $M^1{}_{R_1}$, $[F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]$ $P^1{}_{R_1}$;

$K^2{}_{R_2}$, $[T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}$, $[I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]$ $M^2{}_{R_2}$, $[F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]$ $P^2{}_{R_2}$;

$K^i{}_{R_i}$, $[T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}$, $[I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]$ $M^i{}_{R_i}$, $[F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]$ $P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}

be two IgsvNqss.

i) We define the "pessimistic ∪" operations for N and R such that

N $\cup_p$ R = {<$K^1{}_{N_1R_1}$, $[T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}$, $[I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]$ $M^1{}_{N_1R_1}$, $[F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]$ $P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $[T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}$, $[I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]$ $M^2{}_{N_2R_2}$, $[F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]$ $P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]$ $M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]$ $P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 10's (i).

$T^l{}_{N_nR_n(a)}$= min{$T^l{}_{N_n(a)}$, $T^l{}_{R_n(a)}$}, $T^u{}_{N_nR_n(a)}$= min{$T^u{}_{N_n(a)}$, $T^u{}_{R_n(a)}$};

$I^l{}_{N_nR_n(a)}$= max{$I^l{}_{N_n(a)}$, $I^l{}_{R_n(a)}$}, $I^u{}_{N_nR_n(a)}$= max{$I^u{}_{N_n(a)}$, $I^u{}_{R_n(a)}$};

$F^l{}_{N_nR_n(a)}$= max{$F^l{}_{N_n(a)}$, $F^l{}_{R_n(a)}$}, $F^u{}_{N_nR_n(a)}$= max{$F^u{}_{N_n(a)}$, $F^u{}_{R_n(a)}$}; (n = 1, 2, 3, … , i).

ii) We define the "pessimistic ∩ " operations for N and R such that

N $\cap_p$ R = {<$K^1{}_{N_1R_1}$, $[T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}$, $[I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]$ $M^1{}_{N_1R_1}$, $[F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]$ $P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $[T^l{}_{N_2R_2(\text{a})}, T^u{}_{N_2R_2(\text{a})}]L^2{}_{N_2R_2}$, $[I^l{}_{N_2R_2(\text{a})}, I^u{}_{N_2R_2(\text{a})}]\, M^2{}_{N_2R_2}$, $[F^l{}_{N_2R_2(\text{a})}, F^u{}_{N_2R_2(\text{a})}]\, P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(\text{a})}, T^u{}_{N_iR_i(\text{a})}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(\text{a})}, I^u{}_{N_iR_i(\text{a})}]\, M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(\text{a})}, F^u{}_{N_iR_i(\text{a})}]\, P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

where,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$ and $P^n{}_{N_nR_n}$ are same as in Definition 10's (ii).

$T^l{}_{N_nR_n(\text{a})}$= min{$T^l{}_{N_n(\text{a})}, T^l{}_{R_n(\text{a})}$}, $T^u{}_{N_nR_n(\text{a})}$= min{$T^u{}_{N_n(\text{a})}, T^u{}_{R_n(\text{a})}$};

$I^l{}_{N_nR_n(\text{a})}$= max{$I^l{}_{N_n(\text{a})}, I^l{}_{R_n(\text{a})}$}, $I^u{}_{N_nR_n(\text{a})}$= max{$I^u{}_{N_n(\text{a})}, I^u{}_{R_n(\text{a})}$};

$F^l{}_{N_nR_n(\text{a})}$= max{$F^l{}_{N_n(\text{a})}, F^l{}_{R_n(\text{a})}$}, $F^u{}_{N_nR_n(\text{a})}$= max{$F^u{}_{N_n(\text{a})}, F^u{}_{R_n(\text{a})}$}; (n = 1, 2, 3, … , i).

Example 5: From Example 1,

N = {<{k, l, m, n}, [0, 0.7]{k, l}, [0.5, 0.6] {m}, [0.4, 0.5] {n};

{k, l, p, r}, [0.1, 0.9]{k, p}, [0.2, 0.3] {l}, [0.2, 0.7] {r} >}

and

R = {<{l, p, m, n, k}, [0.4, 0.8]{l, p}, [0, 0.3] {p, m}, [0.2, 0.6] {n};

{m, l, p, r}, [0.3, 0.7]{p}, [0.2, 0.5] {m, l}, [0.1, 0.5] {r} >}

are two IgsvNqs. Thus,

i) N $\cup_P$ R = {<{k, l, m, n, p}, [0, 0.7]{k, l, p}, [0.5, 0.6] {p, m}, [0.4, 0.6] {n};

{m, k, l, p, r}, [0.1, 0.7]{k, p}, [0.2, 0.5] {m, l}, [0.2, 0.7] {r} >}

ii) N $\cap_P$ R = {<{k, l, m, n}, [0, 0.7]{l}, [0.5, 0.6] {m}, [0.4, 0.6] {n};

{l, p, r}, [0.1, 0.7]{p}, [0.2, 0.5] {l}, [0.2, 0.7] {r} >}

Definition 13: Let

N = {<$K^1{}_{N_1}$, $[T^l{}_{N_1(\text{a})}, T^u{}_{N_1(\text{a})}]L^1{}_{N_1}$, $[I^l{}_{N_1(\text{a})}, I^u{}_{N_1(\text{a})}]\, M^1{}_{N_1}$, $[F^l{}_{N_1(\text{a})}, F^u{}_{N_1(\text{a})}]\, P^1{}_{N_1}$;

$K^2{}_{N_2}$, $[T^l{}_{N_2(\text{a})}, T^u{}_{N_2(\text{a})}]L^2{}_{N_2}$, $[I^l{}_{N_2(\text{a})}, I^u{}_{N_2(\text{a})}]\, M^2{}_{N_2}$, $[F^l{}_{N_2(\text{a})}, F^u{}_{N_2(\text{a})}]\, P^2{}_{N_2}$;

$K^i{}_{N_i}$, $[T^l{}_{N_i(\text{a})}, T^u{}_{N_i(\text{a})}]L^i{}_{N_i}$, $[I^l{}_{N_i(\text{a})}, I^u{}_{N_i(\text{a})}]\, M^i{}_{N_i}$, $[F^l{}_{N_i(\text{a})}, F^u{}_{N_i(\text{a})}]\, P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}

and

R = {< $K^1{}_{R_1}$, $[T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}$, $[I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]\ M^1{}_{R_1}$, $[F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]\ P^1{}_{R_1}$;

$K^2{}_{R_2}$, $[T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}$, $[I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]\ M^2{}_{R_2}$, $[F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]\ P^2{}_{R_2}$; …

$K^i{}_{R_i}$, $[T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}$, $[I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]\ M^i{}_{R_i}$, $[F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]\ P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}

be two IgsvNqss.

i) We define the "average /" operations for N and R such that

N $/_A$ R = {< $K^1{}_{N_1R_1}$, $[T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}$, $[I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]\ M^1{}_{N_1R_1}$, $[F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]\ P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $[T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}$, $[I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]\ M^2{}_{N_2R_2}$, $[F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]\ P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\ M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\ P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n} = K^n{}_{N_n} / K^n{}_{R_n}$, $L^n{}_{N_nR_n} = L^n{}_{N_n} / L^n{}_{R_n}$, $M^n{}_{N_nR_n} = M^n{}_{N_n} / M^n{}_{R_n}$, $P^n{}_{N_nR_n} = P^n{}_{N_n} / P^n{}_{R_n}$;

$T^l{}_{N_nR_n(a)}$, $T^u{}_{N_nR_n(a)}$, $I^l{}_{N_nR_n(a)}$, $I^u{}_{N_nR_n(a)}$, $F^l{}_{N_nR_n(a)}$ and $F^u{}_{N_nR_n(a)}$ are same as in Definition 10' (i).

ii) We define the "optimistic / " operations for N and R such that

N $/_O$ R = {< $K^1{}_{N_1R_1}$, $[T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}$, $[I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]\ M^1{}_{N_1R_1}$, $[F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]\ P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $[T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}$, $[I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]\ M^2{}_{N_2R_2}$, $[F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]\ P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\ M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\ P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n}$, $L^n{}_{N_nR_n}$, $M^n{}_{N_nR_n}$, $P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n} = K^n{}_{N_n} / K^n{}_{R_n}$, $L^n{}_{N_nR_n} = L^n{}_{N_n} / L^n{}_{R_n}$, $M^n{}_{N_nR_n} = M^n{}_{N_n} / M^n{}_{R_n}$, $P^n{}_{N_nR_n} = P^n{}_{N_n} / P^n{}_{R_n}$;

$T^l{}_{N_nR_n(a)}$, $T^u{}_{N_nR_n(a)}$, $I^l{}_{N_nR_n(a)}$, $I^u{}_{N_nR_n(a)}$, $F^l{}_{N_nR_n(a)}$ and $F^u{}_{N_nR_n(a)}$ are same as in Definition 11' (i).

iii) We define the "pessimistic / " operations for N and R such that

N $/_P$ R = {< $K^1{}_{N_1R_1}$, $[T^l{}_{N_1R_1(a)}, T^u{}_{N_1R_1(a)}]L^1{}_{N_1R_1}$, $[I^l{}_{N_1R_1(a)}, I^u{}_{N_1R_1(a)}]\ M^1{}_{N_1R_1}$, $[F^l{}_{N_1R_1(a)}, F^u{}_{N_1R_1(a)}]\ P^1{}_{N_1R_1}$;

$K^2{}_{N_2R_2}$, $[T^l{}_{N_2R_2(a)}, T^u{}_{N_2R_2(a)}]L^2{}_{N_2R_2}$, $[I^l{}_{N_2R_2(a)}, I^u{}_{N_2R_2(a)}]\ M^2{}_{N_2R_2}$, $[F^l{}_{N_2R_2(a)}, F^u{}_{N_2R_2(a)}]\ P^2{}_{N_2R_2}$;

$K^i{}_{N_iR_i}$, $[T^l{}_{N_iR_i(a)}, T^u{}_{N_iR_i(a)}]L^i{}_{N_iR_i}$, $[I^l{}_{N_iR_i(a)}, I^u{}_{N_iR_i(a)}]\ M^i{}_{N_iR_i}$, $[F^l{}_{N_iR_i(a)}, F^u{}_{N_iR_i(a)}]\ P^i{}_{N_iR_i}$ >,

$K^n{}_{N_nR_n},\ L^n{}_{N_nR_n},\ M^n{}_{N_nR_n},\ P^n{}_{N_nR_n}$ ∈ P(A); n = 1, 2, 3, … , i}.

Where,

$K^n{}_{N_nR_n} = K^n{}_{N_n} / K^n{}_{R_n}$, $L^n{}_{N_nR_n} = L^n{}_{N_n}/L^n{}_{R_n}$, $M^n{}_{N_nR_n} = M^n{}_{N_n}/M^n{}_{R_n}$, $P^n{}_{N_nR_n} = P^n{}_{N_n} / P^n{}_{R_n}$;

$T^l{}_{N_nR_n(a)}$, $T^u{}_{N_nR_n(a)}$, $I^l{}_{N_nR_n(a)}$, $I^u{}_{N_nR_n(a)}$, $F^l{}_{N_nR_n(a)}$ and $F^u{}_{N_nR_n(a)}$ are same as in Definition 12' (i).

Example 6: From Example 1,

$$N = \{<\{k,l,m,n\}, [0, 0.7]\{k,l\}, [0.5, 0.6]\ \{m\}, [0.4, 0.5]\ \{n\};$$

$$\{k,l,p,r\}, [0.1, 0.9]\{k,p\}, [0.2, 0.3]\ \{l\}, [0.2, 0.7]\ \{r\} >\}$$

and

$$R = \{<\{l,p,m,n,k\}, [0.4, 0.8]\{l,p\}, [0, 0.3]\ \{p,m\}, [0.2, 0.6]\ \{n\};$$

$$\{m,l,p,r\}, [0.3, 0.7]\{p\}, [0.2, 0.5]\ \{m,l\}, [0.1, 0.5]\ \{r\} >\}$$

are two IgsvNqs. Thus,

i) N $/_A$ R = {<∅, [0.2, 0.75]{l}, [0.25, 0.45] ∅, [0.3, 0.6] ∅;

{k}, [0.2, 0.8]{k}, [0.2, 0.4] ∅, [0.15, 0.6] ∅ >}.

ii) N $/_O$ R = {<∅, [0.4, 0.8]{l}, [0, 0.3] ∅, [0.2, 0.5] ∅;

{k}, [0.3, 0.9]{k}, [0.2, 0.3]∅, [0.1, 0.5]∅ >}.

iii) N $/_P$ R = {<∅, [0.2, 0.75]{l}, [0.25, 0.45] ∅, [0.3, 0.6] ∅;

{k}, [0.2, 0.8]{k}, [0.2, 0.4] ∅, [0.15, 0.6] ∅ >}.

Properties 1: Let

N = {<$K^1{}_{N_1}$, $[T^l{}_{N_1(a)}, T^u{}_{N_1(a)}]L^1{}_{N_1}$, $[I^l{}_{N_1(a)}, I^u{}_{N_1(a)}]\ M^1{}_{N_1}$, $[F^l{}_{N_1(a)}, F^u{}_{N_1(a)}]\ P^1{}_{N_1}$;

$K^2{}_{N_2}$, $[T^l{}_{N_2(a)}, T^u{}_{N_2(a)}]L^2{}_{N_2}$, $[I^l{}_{N_2(a)}, I^u{}_{N_2(a)}]\ M^2{}_{N_2}$, $[F^l{}_{N_2(a)}, F^u{}_{N_2(a)}]\ P^2{}_{N_2}$;

$K^i{}_{N_i}$, $[T^l{}_{N_i(a)}, T^u{}_{N_i(a)}]L^i{}_{N_i}$, $[I^l{}_{N_i(a)}, I^u{}_{N_i(a)}]\ M^i{}_{N_i}$, $[F^l{}_{N_i(a)}, F^u{}_{N_i(a)}]\ P^i{}_{N_i}$ >,

$K^n{}_{N_n},\ L^n{}_{N_n},\ M^n{}_{N_n},\ P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i},

R = {<$K^1{}_{R_1}$, $[T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}$, $[I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]\ M^1{}_{R_1}$, $[F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]\ P^1{}_{R_1}$;

$K^2{}_{R_2}$, $[T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}$, $[I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]\ M^2{}_{R_2}$, $[F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]\ P^2{}_{R_2}$; …

$K^i{}_{R_i}$, $[T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}$, $[I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]\ M^i{}_{R_i}$, $[F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]\ P^i{}_{R_i}$ >,

$K^n{}_{R_n},\ L^n{}_{R_n},\ M^n{}_{R_n},\ P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}

and

Y = {< $K^1{}_{Y_1}$, [$T^l{}_{Y_1(a)}$, $T^u{}_{Y_1(a)}$]$L^1{}_{Y_1}$, [$I^l{}_{Y_1(a)}$, $I^u{}_{Y_1(a)}$] $M^1{}_{Y_1}$, [$F^l{}_{Y_1(a)}$, $F^u{}_{Y_1(a)}$] $P^1{}_{Y_1}$;

$K^2{}_{Y_2}$, [$T^l{}_{Y_2(a)}$, $T^u{}_{Y_2(a)}$]$L^2{}_{Y_2}$, [$I^l{}_{Y_2(a)}$, $I^u{}_{Y_2(a)}$] $M^2{}_{Y_2}$, [$F^l{}_{Y_2(a)}$, $F^u{}_{Y_2(a)}$] $P^2{}_{Y_2}$; …

$K^i{}_{Y_i}$, [$T^l{}_{Y_i(a)}$, $T^u{}_{Y_i(a)}$]$L^i{}_{Y_i}$, [$I^l{}_{Y_i(a)}$, $I^u{}_{Y_i(a)}$] $M^i{}_{Y_i}$, [$F^l{}_{Y_i(a)}$, $F^u{}_{Y_i(a)}$] $P^i{}_{Y_i}$ >,

$K^n{}_{Y_n}$, $L^n{}_{Y_n}$, $M^n{}_{Y_n}$, $P^n{}_{Y_n}$ ∈ P(A); n = 1, 2, 3, … , i}

be three IgsvNqss. From Definition 8, Definition 9, Definition 10, Definition 11, Definition 12 and Definition 13; it is clear that

i) N $\cup_A$ R = R $\cup_A$ N; N $\cup_O$ R = R $\cup_O$ N; N $\cup_P$ R = R $\cup_P$ N.

ii) N $\cap_A$ R = R $\cap_A$ N; N $\cap_O$ R = R $\cap_O$ N; N $\cap_P$ R = R $\cap_P$ N.

iii) N $\cup_A$ **(**R $\cup_A$ Y) = (N $\cup_A$ R) $\cup_A$ Y,

N $\cup_O$ **(**R $\cup_O$ Y) = (N $\cup_O$ R) $\cup_O$ Y,

N $\cup_P$ **(**R $\cup_P$ Y) = (N $\cup_P$ R) $\cup_P$ Y.

iv) N $\cap_A$ **(**R $\cap_A$ Y) = (N $\cap_A$ R) $\cap_A$ Y,

N $\cap_O$ **(**R $\cap_O$ Y) = (N $\cap_O$ R) $\cap_O$ Y,

N $\cap_P$ **(**R $\cap_P$ Y) = (N $\cap_P$ R) $\cap_P$ Y.

v) N $\cap_A$ **(**R $\cup_A$ Y) = (N $\cap_A$ R) $\cup_A$ (N $\cap_A$ Y),

N $\cap_O$ **(**R $\cup_O$ Y) = (N $\cap_O$ R) $\cup_O$ (N $\cap_O$ Y),

N $\cap_P$ **(**R $\cup_P$ Y) = (N $\cap_P$ R) $\cup_P$ (N $\cap_P$ Y).

vi) N $\cup_A$ **(**R $\cap_A$ Y) = (N $\cup_A$ R) $\cap_A$ (N $\cup_A$ Y),

N $\cup_O$ **(**R $\cap_O$ Y) = (N $\cup_O$ R) $\cap_O$ (N $\cup_O$ Y),

N $\cup_P$ **(**R $\cap_P$ Y) = (N $\cup_P$ R) $\cap_P$ (N $\cup_P$ Y).

v) If N = R, then

$$N \cup_A R = N \cup_O R = N \cup_P R = R$$

and

$$N \cap_A R = N \cap_O R = N \cap_P R = R.$$

Theorem 1: Let

N = {< $K^1{}_{N_1}$, [$T^l{}_{N_1(a)}$, $T^u{}_{N_1(a)}$]$L^1{}_{N_1}$, [$I^l{}_{N_1(a)}$, $I^u{}_{N_1(a)}$] $M^1{}_{N_1}$, [$F^l{}_{N_1(a)}$, $F^u{}_{N_1(a)}$] $P^1{}_{N_1}$;

$K^2{}_{N_2}$, $[T^l{}_{N_2(a)}, T^u{}_{N_2(a)}]L^2{}_{N_2}$, $[I^l{}_{N_2(a)}, I^u{}_{N_2(a)}]\ M^2{}_{N_2}$, $[F^l{}_{N_2(a)}, F^u{}_{N_2(a)}]\ P^2{}_{N_2}$;

$K^i{}_{N_i}$, $[T^l{}_{N_i(a)}, T^u{}_{N_i(a)}]L^i{}_{N_i}$, $[I^l{}_{N_i(a)}, I^u{}_{N_i(a)}]\ M^i{}_{N_i}$, $[F^l{}_{N_i(a)}, F^u{}_{N_i(a)}]\ P^i{}_{N_i}$ >,

$K^n{}_{N_n}$, $L^n{}_{N_n}$, $M^n{}_{N_n}$, $P^n{}_{N_n}$ ∈ P(A); n = 1, 2, 3, … , i}

and

R = {<$K^1{}_{R_1}$, $[T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}$, $[I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]\ M^1{}_{R_1}$, $[F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]\ P^1{}_{R_1}$;

$K^2{}_{R_2}$, $[T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}$, $[I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]\ M^2{}_{R_2}$, $[F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]\ P^2{}_{R_2}$; …

$K^i{}_{R_i}$, $[T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}$, $[I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]\ M^i{}_{R_i}$, $[F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]\ P^i{}_{R_i}$ >,

$K^n{}_{R_n}$, $L^n{}_{R_n}$, $M^n{}_{R_n}$, $P^n{}_{R_n}$ ∈ P(A); n = 1, 2, 3, … , i}

be two IgsvNqss. Then,

i) (N $\cap_P$ R) ⊂ (N $\cap_A$ R) ⊂ (N $\cap_O$ R)

ii) (N $\cup_P$ R) ⊂ (N $\cup_A$ R) ⊂ (N $\cup_O$ R)

iii) (N $\cap_A$ R) ⊂ (N $\cup_A$ R), (N $\cap_O$ R) ⊂ (N $\cup_O$ R) and (N $\cap_P$ R) ⊂ (N $\cup_P$R).

Proof:

i) From Definition 10, Definition 11 and Definition 12; we obtain that

$$\min\{T^l{}_{N_n(a)}, T^l{}_{R_n(a)}\} \le \frac{T^l{}_{N_n(a)} + T^l{}_{R_n(a)}}{2} \le \max\{T^l{}_{N_n(a)}, T^l{}_{R_n(a)}\} \qquad (1)$$

$$\min\{T^u{}_{N_n(a)}, T^u{}_{R_n(a)}\} \le \frac{T^u{}_{N_n(a)} + T^u{}_{R_n(a)}}{2} \le \max\{T^u{}_{N_n(a)}, T^u{}_{R_n(a)}\} \qquad (2)$$

$$\max\{I^l{}_{N_n(a)}, I^l{}_{R_n(a)}\} \ge \frac{I^l{}_{N_n(a)} + I^l{}_{R_n(a)}}{2} \ge \min\{I^l{}_{N_n(a)}, I^l{}_{R_n(a)}\} \qquad (3)$$

$$\max\{I^u{}_{N_n(a)}, I^u{}_{R_n(a)}\} \ge \frac{I^u{}_{N_n(a)} + I^u{}_{R_n(a)}}{2} \ge \min\{I^u{}_{N_n(a)}, I^u{}_{R_n(a)}\} \qquad (4)$$

$$\max\{F^l{}_{N_n(a)}, F^l{}_{R_n(a)}\} \ge \frac{F^l{}_{N_n(a)} + F_{R_n(a)}}{2} \ge \min\{F^l{}_{N_n(a)}, F^l{}_{R_n(a)}\} \qquad (5)$$

$$\max\{F^u{}_{N_n(a)}, F^u{}_{R_n(a)}\} \ge \frac{F^u{}_{N_n(a)} + F^u{}_{R_n(a)}}{2} \ge \min\{F^u{}_{N_n(a)}, F^u{}_{R_n(a)}\} \qquad (6)$$

Also,

for $\cap_P$, $\cap_O$ and $\cap_A$,

$$K^n{}_{N_nR_n} = K^n{}_{N_n} \cap K^n{}_{R_n},\ L^n{}_{N_nR_n} = L^n{}_{N_n} \cap L^n{}_{R_n},\ M^n{}_{N_nR_n} = M^n{}_{N_n} \cap M^n{}_{R_n},\ P^n{}_{N_nR_n} = P^n{}_{N_n} \cap P^n{}_{R_n} \qquad (7)$$

is hold. Thus, from 1-7 and Definition 9; we obtain that

$$(N \cap_P R) \subset (N \cap_A R) \subset (N \cap_O R).$$

Proofs of {ii, iii} can be given similarly to proof of i.

Theorem 2: Let

$N = \{<K^1{}_{N_1}, [T^l{}_{N_1(a)}, T^u{}_{N_1(a)}]L^1{}_{N_1}, [I^l{}_{N_1(a)}, I^u{}_{N_1(a)}]\, M^1{}_{N_1}, [F^l{}_{N_1(a)}, F^u{}_{N_1(a)}]\, P^1{}_{N_1};$

$K^2{}_{N_2}, [T^l{}_{N_2(a)}, T^u{}_{N_2(a)}]L^2{}_{N_2}, [I^l{}_{N_2(a)}, I^u{}_{N_2(a)}]\, M^2{}_{N_2}, [F^l{}_{N_2(a)}, F^u{}_{N_2(a)}]\, P^2{}_{N_2};$

$K^i{}_{N_i}, [T^l{}_{N_i(a)}, T^u{}_{N_i(a)}]L^i{}_{N_i}, [I^l{}_{N_i(a)}, I^u{}_{N_i(a)}]\, M^i{}_{N_i}, [F^l{}_{N_i(a)}, F^u{}_{N_i(a)}]\, P^i{}_{N_i} >,$

$K^n{}_{N_n}, L^n{}_{N_n}, M^n{}_{N_n}, P^n{}_{N_n} \in P(A); n = 1, 2, 3, \dots, i\}$

and

$R = \{<K^1{}_{R_1}, [T^l{}_{R_1(a)}, T^u{}_{R_1(a)}]L^1{}_{R_1}, [I^l{}_{R_1(a)}, I^u{}_{R_1(a)}]\, M^1{}_{R_1}, [F^l{}_{R_1(a)}, F^u{}_{R_1(a)}]\, P^1{}_{R_1};$

$K^2{}_{R_2}, [T^l{}_{R_2(a)}, T^u{}_{R_2(a)}]L^2{}_{R_2}, [I^l{}_{R_2(a)}, I^u{}_{R_2(a)}]\, M^2{}_{R_2}, [F^l{}_{R_2(a)}, F^u{}_{R_2(a)}]\, P^2{}_{R_2}; \dots$

$K^i{}_{R_i}, [T^l{}_{R_i(a)}, T^u{}_{R_i(a)}]L^i{}_{R_i}, [I^l{}_{R_i(a)}, I^u{}_{R_i(a)}]\, M^i{}_{R_i}, [F^l{}_{R_i(a)}, F^u{}_{R_i(a)}]\, P^i{}_{R_i} >,$

$K^n{}_{R_n}, L^n{}_{R_n}, M^n{}_{R_n}, P^n{}_{R_n} \in P(A); n = 1, 2, 3, \dots, i\}$

be two IgsvNqss. We assume that N ⊂ R. Then,

i) $N \subset (N \cap_A R) \subset R$, $N \subset (N \cap_O R) \subset R$ and $N = (N \cap_P R) \subset R$.

ii) $N \subset (N \cup_A R) \subset R$, $N \subset (N \cup_O R) = R$ and $N \subset (N \cup_P R) \subset R$.

iii) $(N /_A R) \subset R$, $(R /_A N) \subset R$, $(N /_O R) \subset R$, $(R /_O N) \subset R$, $(N /_P R) \subset R$ and $(R /_P N) \subset R$.

Proof:

i) From Definition 9; we obtain that

$$K^n{}_{N_n} \subset K^n{}_{R_n}, L^n{}_{N_n} \subset L^n{}_{R_n}, M^n{}_{N_n} \subset M^n{}_{R_n}, P^n{}_{N_n} \subset P^n{}_{R_n};$$

$$T^l{}_{N_n(a)} \leq T^l{}_{R_n(a)}, T^u{}_{N_n(a)} \leq T^u{}_{R_n(a)};$$

$$I^l{}_{N_n(a)} \geq I^l{}_{R_n(a)}\}, I^u{}_{N_n(a)} \geq I^u{}_{R_n(a)};$$

$$F^l{}_{N_n(a)} \geq F^l{}_{R_n(a)}\}, F^u{}_{N_n(a)} \geq F^u{}_{R_n(a)}. \quad (8)$$

Thus, we obtain that

$$K^n{}_{N_nR_n} = K^n{}_{N_n} \cap K^n{}_{R_n} = K^n{}_{R_n}, L^n{}_{N_nR_n} = L^n{}_{N_n} \cap L^n{}_{R_n} = L^n{}_{R_n},$$

$$M^n{}_{N_nR_n} = M^n{}_{N_n} \cap M^n{}_{R_n} = M^n{}_{R_n}, P^n{}_{N_nR_n} = P^n{}_{N_n} \cap P^n{}_{R_n} = P^n{}_{R_n}. \quad (9)$$

Also, from Proof of (i) of Theorem 1; conditions 1-7 are hold. Hence, thanks to Definition 10, Definition 11, Definition 12, 1-7 and 9; we obtain that

$$N \subset (N \cap_A R) \subset R,\ N \subset (N \cap_O R) \subset R \text{ and } N = (N \cap_P R) \subset R.$$

Proofs of {ii, iii} can be given similarly to proof of i.

Conclusions

In this chapter, we define IgsvNqs, IgsvNqsn using generalized set valued neutrosophic quadruple sets and interval neutrosophic sets. Thanks to IgsvNqs and IgsvNqsn, generalized set valued neutrosophic quadruple sets and interval neutrosophic sets will more useful together. Also, we obtain some basic properties and some operations ($\cup_A$, $\cup_O$, $\cup_P$,$\cap_A$, $\cap_O$, $\cap_P$,$/_A$, $/_O$, $/_P$). Especially, for decision making problems; these operations will more useful. Furthermore, thanks to definitions of IgsvNqs, IgsvNqsn and operations ($\cup_A$, $\cup_O$, $\cup_P$,$\cap_A$, $\cap_O$, $\cap_P$,$/_A$, $/_O$, $/_P$); researchers can define similarity measures, some specific decision making methods (TOPSIS, VIKOR, DEMATEL, AHP, …), arithmetic operations, aggregation operations based on IgsvNqs and IgsvNqsn for decision making problems.

Abbreviations

IgsvNqs: Interval generalized set valued neutrosophic quadruple set

IgsvNqsn: Interval generalized set valued neutrosophic quadruple number

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Chapter Ten

National Human Rights in the Protection and Promotion of Human Rights Influence of Institutions: Fuzzy Method

Enver Bozkurt[1], Necmiye Merve Şahin[2] and Abdullah Kargın[3]

[1, 2]Faculty of Law, Hasan Kalyoncu University, Gaziantep 27410, Turkey,
[3]Department of Mathematics, Gaziantep University, Gaziantep27310-Turkey,

E-mail: enver.bozkurt@hku.edu.tr, necmiyemerve.sahin@gmail.com, abdullahkargin27@gmail.com

ABSTRACT

In this study, we first tried to study the impact of the concept of human rights from a historical perspective totally to explain the creation of national institutions and organizations. We then tried to explain in detail the human rights documents and human rights systems found at universal and regional level to complete this integrity. Finally, in order to explain how important the existence of human rights institutions and organizations to achieve human rights and freedoms, we analyzed their benefits for states and individuals. In addition, we created an artificial intelligence application to determine the impact of national human rights in the protection and promotion of human rights. Thus, we obtained a fuzzy application method in which more objective results can be obtained compared to previous methods in determining this effect.

Keywords: Human Rights and freedom, National Human Rights, Fuzzy logic, Artificial Intelligence Application

INTRODUCTION

Human rights has tried to take part in international law from the first era to the present. In this process, many studies have been done at the universal and national level. The issue of human rights was formally established by the United Nations on 24 October 1945. The UN requirement and human rights that have been officially introduced to the international field have been discussed in detail with the Universal Declaration of Human Rights. IHEB has been a guide in internal law practices and has contributed to the embodiment of abstract rules. Although human rights have been subject to studies at a universal level, they are also discussed in different forms of practice at a regional level. The American Human Rights System, the African Human Rights System, the European Human Rights System are examples of this. And so, as a result of all these studies, the topic of human rights has developed extendedly.

International documents have been issued so that individuals can reach their rights and freedoms fairly; national human rights institutions or equality institutions have been established to ensure that the articles in these documents are best implemented by the states. These institutions and organizations are very important for the effective implementation of human rights. We tried to reach certain conclusions by trying to study the beneficial consequences of human rights institutions and equality institutions in artificial intelligence with our own interests.

There are many uncertainties in our daily life. Many times, classical logic is insufficient to describe these uncertainties. Because in classical logic, an element is either an element of a set or it is not. That is, the membership value of an element belongs to the set {0, 1}. For example, according to classical logic, the color of an apple is either red or not. But it cannot explain the tones of red in classical logic. Due to such situations, classical logic is insufficient to explain the uncertainties. Fuzzy set and logic are defined by Zadeh in 1965 to explain uncertainties more precisely mathematically [1]. In fuzzy logic, the membership degree of each element of a set takes a value in the range of [0, 1]. Thus, unlike classical logic, the membership of each element is graded. For example, the speed of a vehicle is too fast, too fast, too slow, too slow, etc. It can be specified with expressions such as and with different membership degrees. Thus, a more sensitive type of logic including classical logic has been obtained in explaining uncertainties. Fuzzy logic is one of the most used logic types in almost every field of science, especially in artificial intelligence applications and decision-making applications, from the date it was defined to the present.

In this study, we obtained eight criteria determine the impact of national human rights in the protection and promotion of human rights. We have gathered these eight items under three headings as International Human Rights Influence, Government Influence and Legislative-Judicial Influence. With these three criteria, we obtained a fuzzy matlab algorithm to calculate the rate of this effect for countries. Thus, we have obtained a decision-making algorithm using artificial intelligence, which can be applied and objective results can be obtained. For researchers who want to use or improve this algorithm, we have created an example of decision making with imaginary data.

BACKGROUND

Although human rights do not change and have no common definition, it encompasses all human rights, in a broad sense resulting from being an individual. These rights are universal libertarian, peaceful, responsible and based on ethical foundations aiming material and spiritual development of human being. they are fundamental rights. In the historical process, human rights were considered as human rights in an abstract foundation before the establishment of states, and accepted as the rights and freedoms that man have had from the birth. And then because of the individual' living in society, it is based on positive law and bounding certain legal assurances it became materialized [2]. From here, it would be beneficial to briefly mention the development of human rights from an abstract concept to concrete one and what stages have passed in the historical process to understand the effects of human rights on international and national levels.

In the first era, the idea of the rights and freedoms of individual were firstly mentioned in Greek and Roman civilization The first examples of human rights implementation were seen in these civilizations. The "citizens" ,which are actually a minority group, participated in state government and made laws and decided to war and peace. The rest of the site were created by slaves, who had no rights, who were considered property or an animal legally , who manufactured tools. The important point here in terms of human rights is that the minority group of the people was involved in the management of the site. In the ancient Roman law system, they could only bring government art and management further than the old Greek [3]. Unlike the ancient Greek, the concept of citizenship was expanded. The people in the Roman Empire were given citizenship right except women, children, and slaves [4].

In Indian civilization, individual with his birth lived according to the rules of certain caste . The persons were subjected to a standard system, depending on the status of the class within the caste. In this civilization, no one had ever embraced the notion that they had certain rights because they were just human. The word right corresponded to the person's status within the class in this system. The individual must have done his/her duty in order to claim and use a right and. For example, in the case of old or disability, individual had the right to get support, if only he fulfilled his duty [3]. Chinese civilization contributed to the development of human rights through the teaching of limiting political power. All these minor developments in the early ages were too inadequate to fill the legal concept of human rights, but they were still the first steps toward embodying human rights.

In Middle Ages the struggle between the ruled and the rulers, had provided important step toward the development of human rights. In this age, Magna Carta Libertatum which was the legal development of human rights in England was signed in 1215 by preventing the king's keyship and indicating people’s having the security of their property and life. The limitation of the king's keyness and the extension of the rights and freedoms of the people made it the first and most important document in the field of human rights [5].

Thanks to the natural law which was reshaped in the New Age in the 15st century although human rights were not discussed as an independent topic, the issues that human rights could not be transferred and were inborn right were moved to the political area. Human rights began to take part in positive law in this era with the weakening of the understanding of absolute sovereignty. In the 16 st century, philosophers such as John Locke, Jean-Jacques Rousseau and Montesquieu, who argued that the individual should be protected from the pressures of the ruling class, were the pioneers of this argument. Locke, a human rights activist, argued that people who based the basis of political thought on a human being equipped with natural rights had equal and same rights because they were born as human beings. Montesquie argued that the only freedom of the state is the freedom of individuals [6].

During World War I. the idea of the League of Nations was suggested by US President Woodrow Wilson in 1917, within the framework of Wilson principles. The idea was brought to life with the support of the Allied States at the Paris Peace Conference that ended World War I. But among the aims of the society, which was established for world peace and security, protecting human rights and the human value associated with it were not included , not even the word human rights had expressed. But some articles have been linked to human rights rules. The fair and humane operating conditions set out in Article 22. article 5. this is an example of the idea of prohibiting slave trade, guaranteeing the freedom of conscience and religion of people under the administration of buffalo, including the people of Central Africa [7].

The League of Nations did not provide the peace and trust and did not prevent World War II. After World War II, Germany and Japan were defeated and the victorious states organized conferences to establish an organization in order to ensure their own security, international stability against other states. One of them is the San Francisco Conference, which had the signatures of 50 states. The United Nations clout was signed on October 24, 1945, with the signatures of the states participating in this conference. Under the UN requirement, human rights have gained an international identity and has become a matter that must be protected internationally [8].

In general, there is a significant link between the protection of international peace and security and trust which is the main purpose of the UN and human rights. In order to prevent any disrespect or violation of human rights, no matter where it is in the world, and to ensure World peace, the UN has encouraged and supported the states to develop and respect human rights. Although the provisions of human rights were not systematically regulated under the circumstances, the initial chapter emphasized fundamental human rights, the honor and value of human personality, the belief in men and women and equality the rights of big and small nations, and stated creation of conditions for respecting the obligations arising from the agreement. It also included six specific points of human rights. These are; (md. 1/3), (md. 13/1), (md. 55c), (md. 62/2), (md. 68) and (md. 76c) [3].

Although there were many regulations on human rights under the UN requirement, there were no regulations related to the content of these rights. Therefore, a human rights catalog had to be created to protect human rights. The Economic and Social Council had been appointed to work in this area; Human Rights Commission formed by this council were tasked with preparing a human rights declaration. In response to this, the Universal Declaration of Human Rights (IHEB) was prepared and the vote submitted to the General Assembly was adopted. This declaration provided the recognition of the rights within the IHEB of member states in the world and It was the first step in promoting and raising respect for the fundamental freedoms and human rights of the UN. Although there had been no binding document since it was published, it had been a fundamental document in the development and spread of human rights thought and had received the approval of the international community. The IHEB is the abstract principle of it has been a guide in transforming concrete rules and transferring human rights to internal law [9].

In the post-Magna Carta UK, documents such as 1628 Petiton of Rights, 1679 Habeas corpus Act, 1689 Bill of Rights and 1701 Act of settlement hav tried to expand individual rights and freedoms. These documents have affected human rights developments in the United States and later France. On June 12, 1776, in the process of independence for the United States, in the beginning of Virginia Constitution, The Amendment of Rights (Bill of Rights) stated that the people had equal, irreversible and indispensable rights due to creation, these fundamental rights were determined as happiness, security rights, life and freedom, property rights. Later, classical political rights and freedoms, such as freedom of speech, conscience right, freedom of press, freedom of assembly, individual security, were included in the American Declaration of independence and other state constitutional [4].

The regional human rights agreements in American States Human Rights Systems were the terms of the United States organizations (1951), the Declaration on American Human Rights and duties (1948), the Announcement of American Human Rights (1969/1978). Although the American Declaration of Human Rights and duties is considered a similar arrangement to the UN Universal Declaration, it has covered different details in terms of its assignments, including statements, definitions. For example, 26. it contains both the execution of the sentence and the sentence given in its article, out of cruel, insulting or unusual. In 1969, the American Convention on Human Rights was accepted and put into force in 1978. The Convention includes a broad range of rights including legal personality, legislative, human conundrum, nonslave, freedom of thought and conscience, equality in front of the law, right to sue; these rights have been imposed assignments to the States in order to take economic, cultural, social measures and are supported by the ban on discrimination [10].

At this point, the French Declaration of Human and citizen Rights, affected by the American Declaration of independence, will be appropriate to be stated Based on the American Declaration of Rights and the modern legal concept expressed by 18thcentury philosophers, The French Declaration of Human and citizen Rights which was written. For a self-sufficient and self-confident person, this document, created by the destruction of the old regime, had made up of seventeen substances and had been written in French language which makes human rights and freedoms known by large audiences. It was emphasized that human rights should first be based on clear and simple principles. According to the introduction text, this declaration would remind them of the rights that all members of society have. As the title of the declaration states, man won citizenship status because he had natural rights, preserved them, and behaved with the rights that existed in his nature. The role of the citizen was to protect the rights making man's development and existence. Although many more documents were published after this declaration, none were as effective as FIYHB. All the laws adopted in France have been referred to the FIYHB and have formally accepted all the rights and freedoms listed in the declaration and declared that they will be bound to them. Universal statements in the declaration have been effective in legal documents of other countries [11].

The right to live in daily life in Africa during the pre-colonial period, self-defense, the man of sacrifice, freedom of expression, freedom of religion, rights such as freedom of organization, freedom of travel were recognized and used by orf and customs. But in the colonial period, the African people were subjected to discrimination, intense pressure, human rights violations, and slave trade. These events showed their results in 20.th century and accelerated the independence process. After 1960 focusing on the achievement of political independence, African states established the Organization of the African Union in 1963 for the purpose of establishing regional unity. One of the objectives of the establishment of this union was to improve international co-operation in accordance with the UN Convention and the Universal Declaration of Human Rights. The African states submitted human rights to the UN Universal Declaration on Human Rights and the UN requirement. However, although they were a party to various regional or international documents and were stated a comprehensive part for list of rights in their constitution, they were divided by ethnic, religious, racial reasons, the failure of national integrity, the military's frequent involvement in politics, economic incompetence in the hands of the minority class, Human rights violations could not be terminated in practice because they moved on with a Western imitation system instead of a traditional system [3].

Four years after the UN in 1949. The Council of Europe was established in London to prevent the World War II ruins from being repeated. One of the main areas of this council's work had been human rights. In the First article1 of the status it was stated that The article also stipulates that contracts would be made and joint action would be taken to protect and carried human rights and their main freedoms to a further level. Again in article 8 It has been stated that in case of human rights and fundamental freedoms not being observed ,membership would be suspended and the right to representation of the member country might be terminated. The European Council's most important regulation on human rights has been the European Court of Human Rights and the European Court of Human Rights [12].

The European Convention on Human Rights was signed in Rome on November 4, 1950, inspired by the EHIB but kept in a narrower scope. The AIHS, which envisions a powerful mechanism for legal protection of human rights, had established a highly developed legal basis with its terms and expressions. With the AIHS, each member state and a signatory state had the obligation to comply with the human rights and fundamental freedoms listed in the contract. The obligation to grant rights and freedoms to all individuals in the entity states to the contract is governed by the first article. These obligations have been met in their own method, but they have been granted the freedom to comply with the agreement [11].

In the historical process, on the definition of national human rights institutions with the framework of Principles on the Status of National Human Rights institutions(Principles of Paris), at the end of the 1991 seminar realized in UN , minimal definition for national human rights institutions was made [13]. At this point, it is appropriate to address the points that the Principles of Paris address about human rights institutions. The Parisian Principle lists various responsibilities for national human rights institutions. Firstly, these institutions will review every situation in the face of human rights violations and will have enough personnel to monitor developments anywhere in the country. Secondly, human rights institutions will be able to advise governments, parliament and other authorized bodies on the implementation of and compliance with international human rights documents in the event of human rights violations. Therefore, some communication channels officially and informally between the state bodies concerned with the institution will be formed. Third, these institutions will be in contact with regional and international organizations, and contribute to reports submitted by the states to regional or international institutions. Fourth, it will support human rights research and educational human rights programs and will be involved in the implementation of them in universities, schools and professionally.

Finally, some human rights institutions will be granted semi-judicial powers. If it doesn't fulfill even one of them, Principles of Paris are consired not being implemented. In addition, it may be authorized to listen to individual complaints and evaluate petitions outside of its authority [14].

The UN handbook, published in 1995, it is stated that these institutions are structures established by states ,by constitution, law or regulatory procedures, and its mandate is to improve and protect human rights in particular. Also, a book belonging to the UNHCR stated that it is a state body, having constitutional or legal basis, and is part of a state-funded device, and is established to improve and protect human rights rights for human rights institutions [13].

National human rights institutions act as a bridge between local practices and international norms. In order to carry out its organizational functions in terms of preventing rıghts violations and the development of the human rights and to make international human right rules more functional in local level , international human rights rules must be placed in public institutions [15].

The national human rights institutions are defined as responsible structures for the development of human rights, which are autonomous and independent, created by public authorities. The sole purpose of these institutions is to gather information/data about the human rights practices of states and to report them to the public. At this point, human rights activists have embarked on a search for the standards of institutional structures that can be considered as human rights institutions, which is reflected up to the Paris Principles. The Parisian Principles have stated that the authority of national human rights institutions for the protection and development of the rights of the human should be broad, but have not given any insight into the number of these institutions. However, the Global Elevation of the National Human Rights institutions has indicated that only one national human rights institution should be available from each state. EU equal treatment Directives have established the framework of the minimum standards of all equity institutions, and two approaches have been taken accordingly. Some EU member states have specialized equity institutions in line with obligations arising from the EU equality directive and are focused solely on discrimination and equality. Other member states have human rights institutions based on the Parisian Principles, focusing on equality. In some other EU member states, the Equality institutions and national human rights institutions have established a single institutional model to fulfill their functions. For example, England, Belgium, France, the Netherlands [15].

Definition 1: [1] Let $\mathcal{B}$ be the universal set. A fuzzy set $\mathcal{A}$ on $\mathcal{B}$ is defined by

$$\mathcal{A} = \{\langle a, \mu_{\mathcal{A}}(a)\rangle : a \in \mathcal{B}\}.$$

Here, $\mu_A(a)$ is membership function such that $\mu_{\mathcal{A}}: \mathcal{B} \to [0,1]$.

Definition 2: [16] A triangular fuzzy number $\tilde{n}$ = $[k_1,\ l_1,\ m_1]$ is a special fuzzy set on the real number set $\mathbb{R}$, whose membership function is defined as follows

$$\mu_{\tilde{n}}(a) = \begin{cases} (a-k_1)/(l_1-k_1), & if\ (k_1 \le x < l_1) \\ 1, & if\ (a = l_1) \\ (m_1-a)/(m_1-l_1), & if\ (l_1 < a \le m_1) \\ 0, & if\ otherwise \end{cases}$$

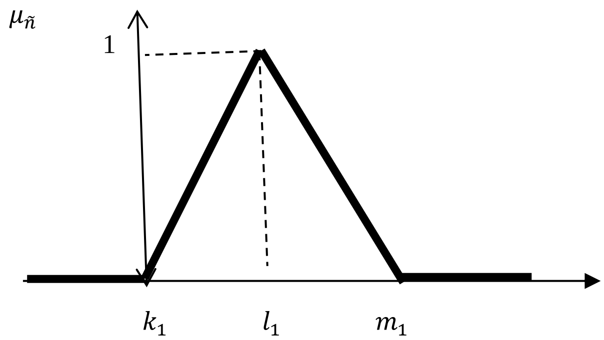

Figure 1. $\tilde{n} = [k_1,\ l_1,\ m_1]$ triangular fuzzy membership function

NATIONAL HUMAN RIGHTS IN THE PROTECTION AND PROMOTION OF HUMAN RIGHTS INFLUENCE OF INSTITUTIONS: FUZZY METHOD

Human rights have a place in the field of international law from the First Age to the present. tried to do. In this process, many studies have been carried out at the global and national level. This As a result of the studies, the subject of human rights has developed considerably. rights of individuals and international documents have been drawn up in order for them to reach their freedoms in a just way; In order for the articles in these documents to be implemented by the states in the best way, national human rights institutions or equality bodies have been established. We listed these useful results as follows:

1) Exhibiting a holistic collection of human rights
2) Ensure compliance with international human rights standards
3) Ensuring access and improvement to education, health and housing
4) Making a positive contribution to the regulations on personal integrity Rights
5) Contributing to an inclusive democratic and political regime
6) Facilitating changes in human rights regulations at the local level
7) Collaboration with judicial authorities in the prevention of human rights Violations
8) Establishing principles regarding process of Impersonal general assemblies and general functioning of intergovernmental human rights
(i)

Fuzzy Method

In this section, we will collect the eight criteria in (i) under three headings. These headings: International Human Rights Influence (1, 2, 3 and 4 in (i)), Government Influence (5 and 8 in (i)) and Legislative-Judicial Influence (6 and 7 in (i)).

Now we give a fuzzy matlab application for determine the impact of national human rights in the protection and promotion of human rights. In the fuzzy matlab application, the process is given at Figure 1.

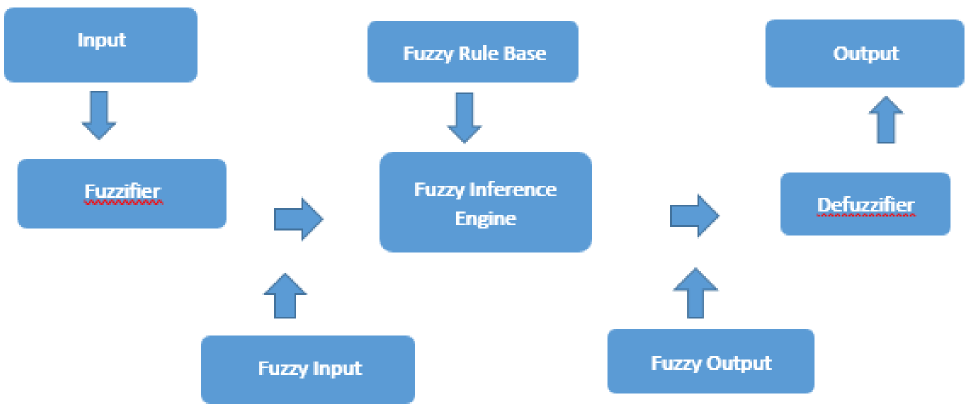

Figure 2. Fuzzy Matlab Algorithm

We give the inputs for this fuzzy matlab application in Table 1 and output for this fuzzy matlab application in Table 2.

Table 1. Inputs for this fuzzy matlab application

Input	Abbreviation
International Human Rights Influence	IHRI
Government Influence	GI
Legislative-Judicial Influence	LJI

Table 2. Output for this fuzzy matlab application

Output	Abbreviation
National Institutions Influence	NII

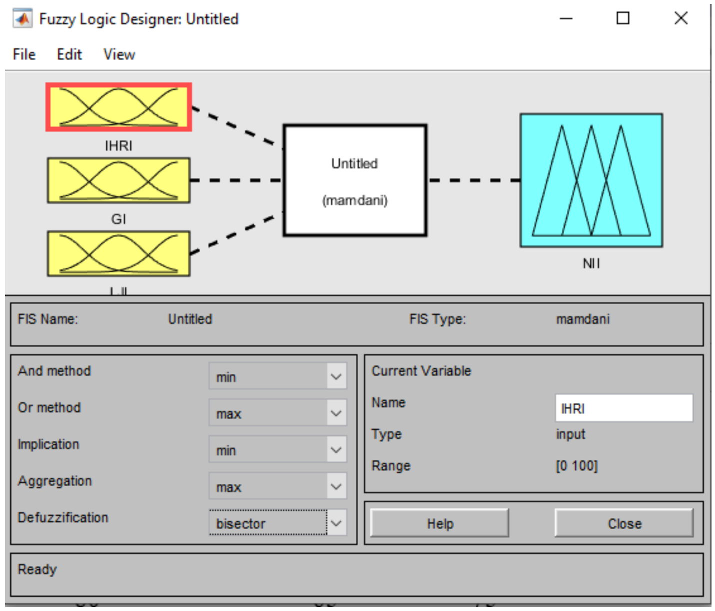

Figure 3. Fuzzy Matlab's input (IHRI, GI, LJI) and output (NII)

We give the fuzzy membership functions of these inputs and the representation of these functions as triangular fuzzy numbers in Table 3.

Table 3. *Fuzzy Membership Functions of IHR, GI and LJI.*

Fuzzy Membership Functions	Abbreviation	Fuzzy Number
Little	L	[0, 25, 50]
Medium	M	[30, 50, 70]
High	H	[60, 75, 90]
Very High	V.H	[80, 100, 100]

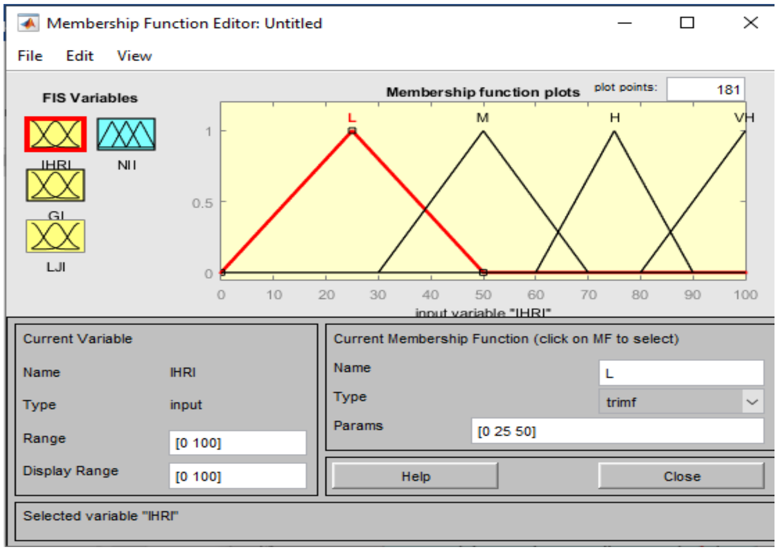

Figure 4. Fuzzy Membership Functions of Fuzzy Matlab for IHR, GI and LJI

We give the fuzzy membership functions of output and the representation of these functions as triangular fuzzy numbers in Table 4.

Table 4: *Fuzzy Membership Functions of NII*

Triangular Fuzzy Membership Functions	Abbreviation	Triangular Fuzzy Number
Very Little	VL	[0, 0, 25]
Little	L	[20, 40, 60]
Medium	M	[40, 60, 80]
High	H	[60, 80, 90]
Very High	VH	[85, 100, 100]

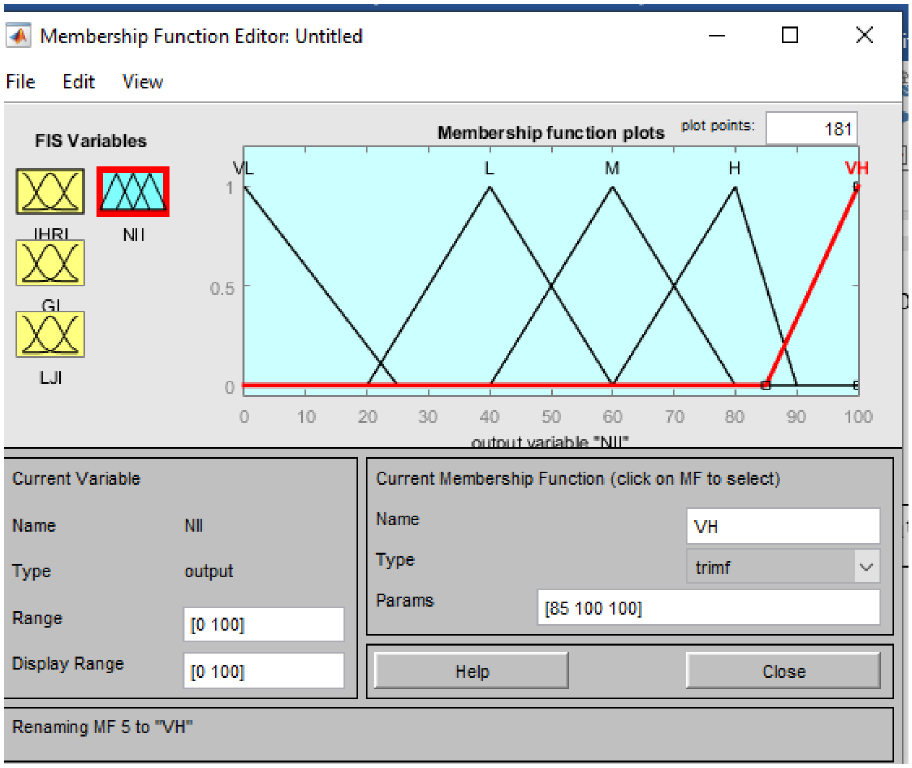

Figure 5. Fuzzy Membership Functions of Fuzzy Matlab for output

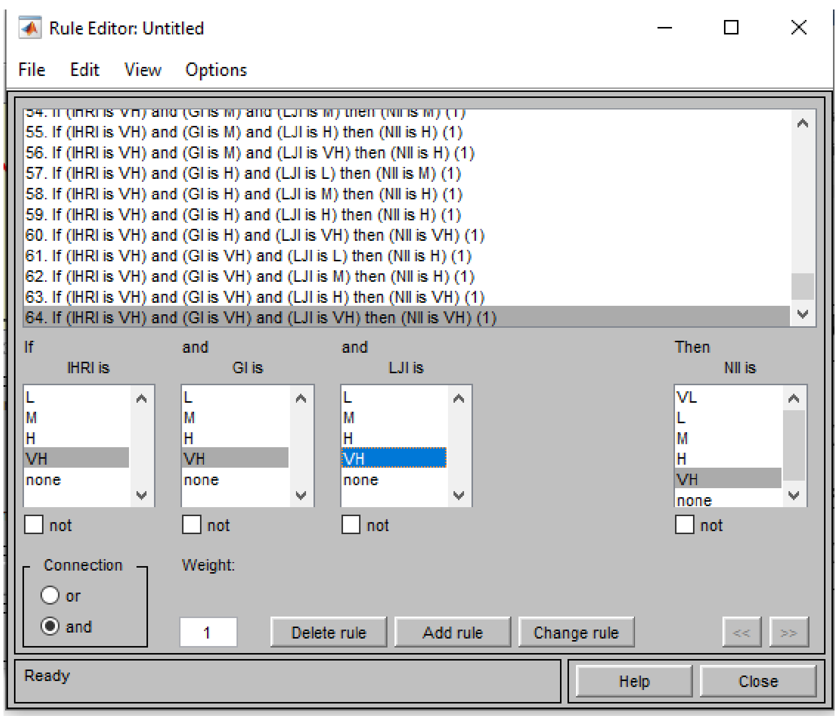

Figure 6. Representation of Fuzzy Rules in Fuzzy Matlab

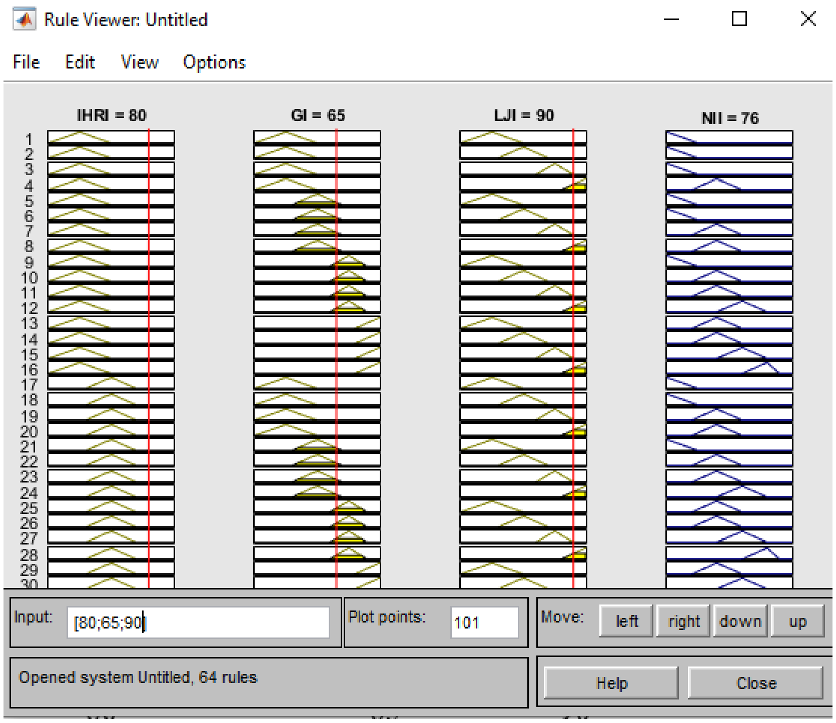

Figure 7. Getting Results with Fuzzy Matlab Rules

Now, let's give the imaginary three inputs (IHRI, GI and LJI) values for imaginary five countries (S_1, S_2, S_3, S_4 and S_5) in Table 5.

Table 5: *Input values for countries* (S_1, S_2, S_3, S_4 and S_5)

States	IHRI	GI	LJI
S_1	75	65	55
S_2	85	70	45
S_3	50	65	90
S_4	70	70	75
S_5	80	65	90

If the data in Table 5 is calculated in the fuzzy matlab algorithm we obtained for each country, Table 6 is obtained.

Table 6: *Output values for countries* (S_1, S_2, S_3, S_4 and S_5)

States	NII
S_1	53
S_2	58
S_3	68
S_4	77
S_5	76

Therefore, according to Table 6, the countries with the highest impact of national human rights in the protection and promotion of human rights are S_4, S_5, S_3, S_2 and S_1, respectively.

Conclusions

Human rights have grown to an important position in international law since first ages. The progress of human rights in the historical process has come to this point today due to the efforts of individuals, communities, states. In this study, as detailed in this process, the human rights steps in the first Age, Middle Ages and New Age have gradually started to sound, and this voice has become the basis for international law. We have covered international studies, human rights has become defined phenomenon that should be accepted by all, and this led to a number of studies to reach individuals. These studies, which are the basis of our work, have focused on the benefits of human rights institutions and equality institutions established by states for better application of human rights. From this framework the effectiveness of these institution was determined by using artificial intelligent method. Thanks to this method, we determine the impact of national human rights in the protection and promotion of human rights. Also, using (or improving) the data and decision-making method in this study, researchers can conduct new studies on international human rights and law.

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Chapter Eleven

VIKOR Method-based on Entropy Measure for Decision-Making Method with N-valued Neutrosophic Trapezoidal Numbers: Application of Architucture

Derya BAKBAK[1], Vakkas ULUÇAY[2,*]

[1]TBMM Public relations building 2nd Floor, B206 room Ministries, Ankara06543- TURKEY;

[2]Kilis 7 Aralık University, Faculty of Arts and Sciences, Department of Mathematics, Kilis/ TURKEY

E-mail: derya.bakbak@tbmm.gov.tr,

***** Correspondence: vulucay27@gmail.com

ABSTRACT

VIKOR is a popular strategy for multi-criteria decision making. As an extension of the neutrosophic trapezoidal numbers, the N-valued neutrosophic trapezoidal numbers, which are special neutrosophic multi-sets on real number, are used to effectively solve the repetitive uncertainty of decision-makers in multi-criteria decision-making problems. The aim of the this chapter the VIKOR strategy for MCDM problems in N-valued neutrosophic trapezoidal numbers. In decision making situation, N-valued neutrosophic trapezoidal numbers are employed to express the criteria values. Then we develop an extended VIKOR strategy to deal with MCDM in N-valued neutrosophic trapezoidal numbers environment. To show the advantages of our proposed VIKOR strategy, a decision-making problem of architecture to illustrate the effectiveness of the developed method is solved in N-valued neutrosophic trapezoidal numbers environment.

Keywords: Neutrosophic sets, neutrosophic multi-sets, N-valued neutrosophic trapezoidal number, generalized distance measure, entropy measure, VIKOR method, multi-criteria decision-making.

1. Introduction

Production materials; metal, ceramic and organic materials are divided into three main groups. Each of these materials has their own advantages and disadvantages. The new material obtained as a result of the process of combining the superior properties of two or more of them in one material is composite.

The reason why the composite is preferred is; This is because it is resistant to heat and moisture, is lighter than metal, and has high strength. In addition, it is an economical alternative for every sector with its low cost. Wear resistance, thermal expansion feature, stylish appearance are also advantages. For such reasons, composite material has become more preferred in recent years. Composite materials have found the opportunity to be used in a wide area. We aim to solve the uncertainty arising from these possibilities by using the decision-making method. Since decision making problems which contain uncertain are difficult to model and solve, and it is a need for us to develop some mathematical theories. Recent years, fuzzy set theory by using only one degree of membership proposed by Zadeh [64] and intuitionistic fuzzy set theory by using two degrees of membership introduced by Atanassov [2] have been received great attention in solving various decision-making problems. These theories can better solve the fuzziness of the uncertain decision making therefore the theories are all very successfully studied in Hu et al. [18], Liu et al. [19], Narayanamoorthy et al. [20] and [32-44].

By using truth-membership function, indeterminacy-membership function and falsity-membership functions, in 1998, Smarandache [51] proposed the concept of neutrosophic sets (N-sets). In 2013, Smarandache [52] generalized the classical neutrosophic logic to neutrosophic refined logic which have more than one with the possibility of the same or the different membership functions. Moreover, Ye and Ye [62], Chatterjee et al. [11] and Ye and Smarandache [63] introduced the concept single valued neutrosophic multi sets as a further generalization of that of neutrosophic sets based on both the neutrosophic refined logic and multi sets of Yager [61]. The multisets and single valued neutrosophic multisets has received more and more attention since its appearance in [1,3-10,12,14-17,21-31,45-50,53-59,61,63-72]

In order to use the concept of single valued neutrosophic multi sets to define an uncertain quantity or a quantity difficult to quantify, in Deli et al. [13] the authors put forward the concept of continuous N-valued neutrosophic trapezoidal numbers (NVNT-numbers). They developed a TOPSIS method by giving some operational laws of NVNT-numbers and some aggregation operators of NVNT-numbers.

Distance measure is an important information measure in the study of single valued neutrosophic multi sets but there are few distance formulas for NVNT-numbers proposed in studies. There, this paper will first propose some new generalized distance measures for NVNT-numbers then use it to develop a decision-making method based on a entropy measure which find weight of criterias.

The remainder of this paper is arranged as follows. The “Preliminaries” section gives a brief introduction to single valued neutrosophic sets, single valued neutrosophic multi sets, N-valued neutrosophic trapezoidal number. In the “NVNT-numbers VIKOR method” section, a NVNT-numbers -based decision-making approach is proposed, and in the “Illustrative example” section, an illustrative example is provided to demonstrate the effectiveness of the above method. Later, we compare the proposed example with different distance measures and existing methods.

2. Preliminary

This section firstly introduces several the known definitions and propositions that would be helpful for better study of this paper.

Definition 2.1 [60] Assume that E is the universe. Then, a single valued neutrosophic set (N-set) A in E defined as

$$A = \{< x, T_A(x)\,,\ I_A(x)\,,\ F_A(x) >: x \in E\} \tag{1}$$

where $T_A(x)\,,\ I_A(x)\,,\ F_A(x) \in [0,1]$ for each point *x* in such that $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.2 [52] Let E be a universe. A_1 neutrosophic multi-set set A_1 on E can be defined as follows:

$$A_1=\{<x,(T^1_{A_1}(x), T^2_{A_1}(x), \dots, T^P_{A_1}(x)), (I^1_{A_1}(x), I^2_{A_1}(x), \dots, I^P_{A_1}(x)), (F^1_{A_1}(x), F^2_{A_1}(x), \dots, F^P_{A_1}(x)) >: x \in E\},$$

where

$$T^1_{A_1}(x), T^2_{A_1}(x), \dots, T^P_{A_1}(x), I^1_{A_1}(x), I^2_{A_1}(x), \dots, I^p_{A_1}(x), F^1_{A_1}(x), F^2_{A_1}(x), \dots, F^P_{A_1}(x): E \rightarrow [0,1]$$

such that $0 \leq \sup T_{A_1}^{i}(x) + \sup I_{A_1}^{i}(x) + \sup F_{A_1}^{i}(x) \leq 3$ (i=1,2,…,P) for any $x \in E$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively.

Definition 2.3 [13] Let $\eta_{A_1}^{i}, \vartheta_{A_1}^{i}, \theta_{A_1}^{i} \in [0,1]$ $(i \in \{1,2,\ldots,p\})$ and $a,b,c,d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. Then, an N-valued neutrosophic trapezoidal number (NVNT-number)

$\tilde{a} = \langle [a,b,c,d]; (\eta_{A_1}^{1}, \eta_{A_1}^{2}, \ldots, \eta_{A_1}^{P}), (\vartheta_{A_1}^{1}, \vartheta_{A_1}^{2}, \ldots, \vartheta_{A_1}^{P}), (\theta_{A_1}^{1}, \theta_{A_1}^{2}, \ldots, \theta_{A_1}^{P}) \rangle$ is a neutrosophic multi-set on the real number set $\mathbb{R}$, whose truth-membership functions, indeterminacy-membership functions and falsity-membership functions are defined as, respectively.

$$T_{\tilde{\alpha}}^{i}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\eta_{\tilde{\alpha}}^{i}, & a \leq x < b \\ \eta_{\tilde{\alpha}}^{i}, & b \leq x \leq c \\ \frac{(d-x)}{(d-c)}\eta_{\tilde{\alpha}}^{i}, & c < x \leq d \\ 0, & \text{otherwise}, \end{cases} \quad , \quad I_{\tilde{\alpha}}^{i}(x) = \begin{cases} \frac{(b-x)+\vartheta_{\tilde{\alpha}}^{i}(x-a)}{(b-a)}, & a \leq x < b \\ \vartheta_{\tilde{\alpha}}^{i}, & b \leq x \leq c \\ \frac{(x-c)+\vartheta_{\tilde{\alpha}}^{i}(d-x)}{(d-c)}, & c < x \leq d \\ 1, & \text{otherwise}, \end{cases}$$

and

$$F_{\tilde{\alpha}}^{i}(x) = \begin{cases} \frac{(b-x)+\theta_{A}^{i}(x-a)}{(b-a)}, & a \leq x < b \\ \theta_{\tilde{\alpha}}^{i}, & b \leq x \leq c \\ \frac{(x-c)+\theta_{A}^{i}(d-x)}{(d-c)}, & c < x \leq d \\ 1, & \text{otherwise}, \end{cases}$$

Note that the set of all NVNT-numbers on $\mathbb{R}$ will be denoted by Λ.

Definition 2.4 [13] Let $A_1 = \langle (a_1,b_1,c_1,d_1); (\eta_{A_1}^{1}, \eta_{A_1}^{2}, \ldots, \eta_{A_1}^{P}), (\vartheta_{A_1}^{1}, \vartheta_{A_1}^{2}, \ldots, \vartheta_{A_1}^{P}), (\theta_{A_1}^{1}, \theta_{A_1}^{2}, \ldots, \theta_{A_1}^{P}) \rangle \in \Lambda$. If A_1 is not normalized NVTN-number($a_1,b_1,c_1,d_1 \notin [0,1]$), the normalized NVTN-number of A_1, denoted by $\overline{A}_1$ is given by;

$$\overline{A}_1 = \langle \left[\frac{a_1}{a_1+b_1+c_1+d_1}, \frac{b_1}{a_1+b_1+c_1+d_1}, \frac{c_1}{a_1+b_1+c_1+d_1}, \frac{d_1}{a_1+b_1+c_1+d_1}\right]; \left(\eta_{\overline{A_1}}^{1}, \eta_{\overline{A_1}}^{2}, \ldots, \eta_{\overline{A_1}}^{P}\right), \left(\vartheta_{\overline{A_1}}^{1}, \vartheta_{\overline{A_1}}^{2}, \ldots, \vartheta_{\overline{A_1}}^{P}\right), (\theta_{\overline{A_1}}^{1}, \theta_{\overline{A_1}}^{2}, \ldots, \theta_{\overline{A_1}}^{P}) \rangle. \quad (5)$$

Definition 2.5 [65] Let $\overline{\mathcal{A}} = \langle [a_1,b_1,c_1,d_1]; (\eta_{\overline{\mathcal{A}}}^{1}, \eta_{\overline{\mathcal{A}}}^{2}, \ldots, \eta_{\overline{\mathcal{A}}}^{P}), (\vartheta_{\overline{\mathcal{A}}}^{1}, \vartheta_{\overline{\mathcal{A}}}^{2}, \ldots, \vartheta_{\overline{\mathcal{A}}}^{P}), (\theta_{\overline{\mathcal{A}}}^{1}, \theta_{\overline{\mathcal{A}}}^{2}, \ldots, \theta_{\overline{\mathcal{A}}}^{P}) \rangle$ and $\overline{\mathcal{B}} = \langle [a_2,b_2,c_2,d_2]; (\eta_{\overline{\mathcal{B}}}^{1}, \eta_{\overline{\mathcal{B}}}^{2}, \ldots, \eta_{\overline{\mathcal{B}}}^{P}), (\vartheta_{\overline{\mathcal{B}}}^{1}, \vartheta_{\overline{\mathcal{B}}}^{2}, \ldots, \vartheta_{\overline{\mathcal{B}}}^{P}), (\theta_{\overline{\mathcal{B}}}^{1}, \theta_{\overline{\mathcal{B}}}^{2}, \ldots, \theta_{\overline{\mathcal{B}}}^{P}) \rangle$ be two normalized NVNT-numbers then, respectively, the weighted Hamming and Euclidean distance measures between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are given below;

$$d_r^{w}(\overline{\mathcal{A}}, \overline{\mathcal{B}}) = \frac{1}{16p} \cdot (\sum_{i=1}^{p} \left[\left(|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i} - \vartheta_{\overline{\mathcal{A}}}^{i} - \theta_{\overline{\mathcal{A}}}^{i})a_1 - w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i} - \vartheta_{\overline{\mathcal{B}}}^{i} - \theta_{\overline{\mathcal{B}}}^{i})a_2|\right)^{r} + \right.$$
$$\left(|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i} - \vartheta_{\overline{\mathcal{A}}}^{i} - \theta_{\overline{\mathcal{A}}}^{i})b_1 - w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i} - \vartheta_{\overline{\mathcal{B}}}^{i} - \theta_{\overline{\mathcal{B}}}^{i})b_2|\right)^{r} +$$

$$\left(\left|w_{\overline{\mathcal{A}}}\left(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}}\right)c_1-w_{\overline{\mathcal{B}}}\left(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}}\right)c_2\right|\right)^r+$$
$$\left(\left|w_{\overline{\mathcal{A}}}\left(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}}\right)d_1-w_{\overline{\mathcal{B}}}\left(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}}\right)d_2\right|\right)^r\Big]\Big)^{\frac{1}{r}} \tag{3}$$

For r=1, the equation 3 is given as;

$$d^{w}_{1}(\overline{\mathcal{A}},\overline{\mathcal{B}}) = \frac{1}{16p}.\sum_{i=1}^{p}\Big[\left|w_{\overline{\mathcal{A}}}\left(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}}\right)a_1-w_{\overline{\mathcal{B}}}\left(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}}\right)a_2\right|+$$
$$\left|w_{\overline{\mathcal{A}}}\left(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}}\right)b_1-w_{\overline{\mathcal{B}}}\left(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}}\right)b_2\right|+$$
$$\left|w_{\overline{\mathcal{A}}}\left(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}}\right)c_1-w_{\overline{\mathcal{B}}}\left(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}}\right)c_2\right|+$$
$$\left|w_{\overline{\mathcal{A}}}\left(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}}\right)d_1-w_{\overline{\mathcal{B}}}\left(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}}\right)d_2\right|\Big] \tag{4}$$

For r=2, the equation 3 is given as;

$$d^{w}_{2}(\overline{\mathcal{A}},\overline{\mathcal{B}}) = \frac{1}{16p}.\sum_{i=1}^{p}\Big[\left(w_{\overline{\mathcal{A}}}(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}})a_1-w_{\overline{\mathcal{B}}}(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}})a_2\right)^2+$$
$$\left(w_{\overline{\mathcal{A}}}(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}})b_1-w_{\overline{\mathcal{B}}}(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}})b_2\right)^2+$$
$$\left(w_{\overline{\mathcal{A}}}(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}})c_1-w_{\overline{\mathcal{B}}}(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}})c_2\right)^2+$$
$$\left(w_{\overline{\mathcal{A}}}(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}})d_1-w_{\overline{\mathcal{B}}}(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}})d_2\right)^2\Big]^{\frac{1}{2}} \tag{5}$$

Theorem 2.6 [65] Let $\overline{\mathcal{A}} = \langle(a_1,b_1,c_1,d_1);\left(\eta^{1}_{\overline{\mathcal{A}}},\eta^{2}_{\overline{\mathcal{A}}},\dots,\eta^{P}_{\overline{\mathcal{A}}}\right),\left(\vartheta^{1}_{\overline{\mathcal{A}}},\vartheta^{2}_{\overline{\mathcal{A}}},\dots,\vartheta^{P}_{\overline{\mathcal{A}}}\right),\left(\theta^{1}_{\overline{\mathcal{A}}},\theta^{2}_{\overline{\mathcal{A}}},\dots,\theta^{P}_{\overline{\mathcal{A}}}\right)\rangle$,
$\overline{\mathcal{B}} = \langle(a_2,b_2,c_2,d_2);\left(\eta^{1}_{\overline{\mathcal{B}}},\eta^{2}_{\overline{\mathcal{B}}},\dots,\eta^{P}_{\overline{\mathcal{B}}}\right),\left(\vartheta^{1}_{\overline{\mathcal{B}}},\vartheta^{2}_{\overline{\mathcal{B}}},\dots,\vartheta^{P}_{\overline{\mathcal{B}}}\right),\left(\theta^{1}_{\overline{\mathcal{B}}},\theta^{2}_{\overline{\mathcal{B}}},\dots,\theta^{P}_{\overline{\mathcal{B}}}\right)\rangle$ and
$\overline{\mathcal{C}} = \langle(a_3,b_3,c_3,d_3);\left(\eta^{1}_{\overline{\mathcal{C}}},\eta^{2}_{\overline{\mathcal{C}}},\dots,\eta^{P}_{\overline{\mathcal{C}}}\right),\left(\vartheta^{1}_{\overline{\mathcal{C}}},\vartheta^{2}_{\overline{\mathcal{C}}},\dots,\vartheta^{P}_{\overline{\mathcal{C}}}\right),\left(\theta^{1}_{\overline{\mathcal{C}}},\theta^{2}_{\overline{\mathcal{C}}},\dots,\theta^{P}_{\overline{\mathcal{C}}}\right)\rangle$ be three normalized NVNT- numbers.
Then, $d^{w}_{r}(\overline{\mathcal{A}},\overline{\mathcal{B}})$ satisfies the following properties:

i. $0 \le d^{w}_{r}(\overline{\mathcal{A}},\overline{\mathcal{B}}) \le 1$,
ii. $\overline{\mathcal{A}} = \overline{\mathcal{B}} \Rightarrow d^{w}_{r}(\overline{\mathcal{A}},\overline{\mathcal{B}}) = 0$,
iii. $d^{w}_{r}(\overline{\mathcal{A}},\overline{\mathcal{B}}) = d^{w}_{r}\left(\overline{\mathcal{B}},\overline{\mathcal{A}}\right)$,
iv. $d^{w}_{r}\left(\overline{\mathcal{A}},\overline{\mathcal{B}}\right) \le d^{w}_{r}\left(\overline{\mathcal{A}},\overline{\mathcal{C}}\right) + d^{w}_{r}\left(\overline{\mathcal{C}},\overline{\mathcal{B}}\right)$.

Definition 2.7 [65] A real-valued function $\boldsymbol{\mathcal{E}_r}: \mu \to [0,1]$ is called an entropy on NVNT-numbers if it satisfies the following properties:

$\boldsymbol{\mathcal{EP}}\mathbf{1}$. $\mathcal{A} = \begin{cases}\langle[a,b,c,d];(1,1,\dots,1),(0,0,\dots,0),(0,0,\dots,0)\rangle\\ \langle[a,b,c,d];(0,0,\dots,0),(0,0,\dots,0),(1,1,\dots,1)\rangle\end{cases} \Rightarrow \mathcal{E}_r(\mathcal{A}) = 0$;

$\boldsymbol{\mathcal{EP}}\mathbf{2}$. $\mathcal{E}_r(\mathcal{A}) = \mathcal{E}_r(\mathcal{A}^c)$ for all $\mathcal{A} \in$ NVNT-numbers, where

$$\mathcal{A}^c = \langle[a,b,c,d];\left(\theta^{1}_{\mathcal{A}},\theta^{2}_{\mathcal{A}},\dots,\theta^{P}_{\mathcal{A}}\right),\left(1-\vartheta^{1}_{\mathcal{A}},1-\vartheta^{2}_{\mathcal{A}},\dots,1-\vartheta^{P}_{\mathcal{A}}\right),\left(\eta^{1}_{\mathcal{A}},\eta^{2}_{\mathcal{A}},\dots,\eta^{P}_{\mathcal{A}}\right)\rangle.$$

$\boldsymbol{\mathcal{EP}}$**3.** $d_r(\mathcal{A},\mathcal{A}^-) = d_r(\mathcal{A},\mathcal{A}^+) \Leftrightarrow \mathcal{E}_r(\mathcal{A}) = 1$ for all $\mathcal{A} \in$ NVNT-numbers, where $d_r(\mathcal{A},\mathcal{A}^+)$ is a distance from $\mathcal{A}$ to $\mathcal{A}^+$ and $d_r(\mathcal{A},\mathcal{A}^-)$ is a distance from $\mathcal{A}$ to $\mathcal{A}^-$;

$\boldsymbol{\mathcal{EP}}$**4.** For all $\mathcal{A},\mathcal{B} \in$ NVNT-numbers, if

$$\left|\frac{d_r(\mathcal{A},\mathcal{A}^-)}{d_r(\mathcal{A},\mathcal{A}^+) + d_r(\mathcal{A},\mathcal{A}^-)} - \frac{1}{2}\right| \geq \left|\frac{d_r(\mathcal{B},\mathcal{B}^-)}{d_r(\mathcal{B},\mathcal{B}^+) + d_r(\mathcal{B},\mathcal{B}^-)} - \frac{1}{2}\right| \qquad (6)$$

then $\mathcal{E}(\mathcal{A}) \leq \mathcal{E}(\mathcal{B})$, where $d_r(\mathcal{B},\mathcal{B}^+)$ is a distance from $\mathcal{B}$ to $\mathcal{B}^+$ and $d_r(\mathcal{B},\mathcal{B}^-)$ is a distance from $\mathcal{B}$ to $\mathcal{B}^-$, where

$$\mathcal{A}^+ = \langle [a,b,c,d]; (1,1,\ldots,1), (0,0,\ldots,0), (0,0,\ldots,0)\rangle$$

and

$$\mathcal{A}^- = \langle [a,b,c,d]; (0,0,\ldots,0), (1,1,\ldots,1), (1,1,\ldots,1)\rangle.$$

Theorem 2.8 [65] Assume that d_r is an distance measure for NVNT-numbers. Then, for any $\mathcal{A} \in$ NVNT-numbers,

$$\mathcal{E}_r(\mathcal{A}) = 1 - 2\left|\frac{d_r(\mathcal{A},\mathcal{A}^-)}{d_r(\mathcal{A},\mathcal{A}^+) + d_r(\mathcal{A},\mathcal{A}^-)} - \frac{1}{2}\right| \qquad (7)$$

is entropy of NVNT-numbers based on TOPSIS.

3. NVNT-numbers VIKOR method

In this section, we proposed a normalized NVNT-numbers VIKOR method with entropy-based weights for solving multi-criteria decision-making problems.

Definition 3.1 Assume that $F = \{F_1, F_2, \ldots, F_m\}$ be the set of altenatives and $Z = \{z_1, z_2, \ldots, z_n\}$ be the set of criterias. In Deli et al. [13], the normalized NVNT-numbers decision matrix is given as;

$$(F_{kj})_{mxn} = \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1n} \\ F_{21} & F_{22} & \cdots & F_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ F_{m1} & F_{m2} & \cdots & F_{mn} \end{pmatrix}$$

such that

$F_{kj} = \langle [a_{kj}, b_{kj}, c_{kj}, d_{kj}], \left(\eta^1_{kj}, \eta^2_{kj}, \eta^3_{kj}, \ldots, \eta^p_{kj}\right), \left(\vartheta^1_{kj}, \vartheta^2_{kj}, \vartheta^3_{kj}, \ldots, \vartheta^p_{kj}\right), \left(\theta^1_{kj}, \theta^2_{kj}, \theta^3_{kj}, \ldots, \theta^p_{kj}\right)\rangle$, (k=1,2,…,m) and (j=1,2,…,n).

It is carried out the following algorithm to get best choice:

Algorithm:

Step 1: Create an evaluation matrix $(F_{kj})_{mxn}$, (k=1,2,…,m; j=1,2,…,n)

Step 2: Find of the weights of the criteria vector $w = \{w_1, w_2, \dots, w_n\}$ by using equation in Theorem 2.6 as;

$$w_j = \frac{m - \sum_{k=1}^{m} \mathcal{E}_{kj}}{m.n - \sum_{k=1}^{m} \sum_{j=1}^{n} \mathcal{E}_{kj}}, \quad (j = 1,2, \dots, n).$$

where the entropy matrix $(\mathcal{E}_{kj})_{mxn}$ (k=1,2,…,m; j=1,2,…,n) of the decision matrix $(F_{kj})_{mxn}$ and where

$$\mathcal{E}_{kj} = 1 - 2\left|\frac{d_r(F_{kj}, F_{kj}{}^{-})}{d_r(F_{kj}, K_{kj}{}^{+}) + d_r(F_{kj}, F_{kj}{}^{-})} - \frac{1}{2}\right|$$

$(k = 1,2, \dots, m;\ j = 1,2, \dots, n)$.

Note that if the entropy matrix $(\mathcal{E}_{kj})_{mxn}$ (k=1,2,…,m; j=1,2,…,n) is not normalized then, the entropy matrix must be normalized as;

$$\bar{\mathcal{E}}_{kj} = \frac{\mathcal{E}_{kj}}{\max\{\mathcal{E}_{kj} : \mathcal{E}_{kj} \in (\mathcal{E}_{kj})_{mxn}, k = 1,2, \dots, m \text{ and } j = 1,2, \dots, n\}}.$$

Step 3: Determine the positive ideal solution r^{+} and negative ideal solution r^{-}, respectively as;

$$r^{+} = \langle [a_{kj}^{+}, b_{kj}^{+}, c_{kj}^{+}, d_{kj}^{+}]; (\eta_{kj}^{1\,+}, \eta_{kj}^{2\,+}, \eta_{kj}^{3\,+}, \dots, \eta_{kj}^{p\,+}), (\vartheta_{kj}^{1\,+}, \vartheta_{kj}^{2\,+}, \vartheta_{kj}^{3\,+}, \dots, \vartheta_{kj}^{p\,+}), (\theta_{kj}^{1\,+}, \theta_{kj}^{2\,+}, \theta_{kj}^{3\,+}, \dots, \theta_{kj}^{p\,+}) \rangle$$

$$= \Big\langle \Big[\max_k\{a_{kj}\}, \max_k\{b_{kj}\}, \max_k\{c_{kj}\}, \max_k\{d_{kj}\}\Big]; \Big(\max_k\{\eta_{kj}^{1}\}, \max_k\{\eta_{kj}^{2}\}, \max_k\{\eta_{kj}^{3}\}, \dots, \max_k\{\eta_{kj}^{p}\}\Big)$$
$$\Big(\min_k\{\vartheta_{kj}^{1}\}, \min_k\{\vartheta_{kj}^{2}\}, \min_k\{\vartheta_{kj}^{3}\}, \dots, \min_k\{\vartheta_{kj}^{p}\}\Big), \Big(\min_k\{\theta_{kj}^{1}\}, \min_k\{\theta_{kj}^{2}\}, \min_k\{\theta_{kj}^{3}\}, \dots, \min_k\{\theta_{kj}^{p}\}\Big)\Big\rangle.$$

and

$$r^{-} = \langle [a_{kj}^{-}, b_{kj}^{-}, c_{kj}^{-}, d_{kj}^{-}]; (\eta_{kj}^{1\,-}, \eta_{kj}^{2\,-}, \eta_{kj}^{3\,-}, \dots, \eta_{kj}^{p\,-}), (\vartheta_{kj}^{1\,-}, \vartheta_{kj}^{2\,-}, \vartheta_{kj}^{3\,-}, \dots, \vartheta_{kj}^{p\,-}), (\theta_{kj}^{1\,-}, \theta_{kj}^{2\,-}, \theta_{kj}^{3\,-}, \dots, \theta_{kj}^{p\,-}) \rangle$$

$$= \Big\langle \Big[\min_k\{a_{kj}\}, \min_k\{b_{kj}\}, \min_k\{c_{kj}\}, \min_k\{d_{kj}\}\Big]; \Big(\min_k\{\eta_{kj}^{1}\}, \min_k\{\eta_{kj}^{2}\}, \min_k\{\eta_{kj}^{3}\}, \dots, \min_k\{\eta_{kj}^{p}\}\Big),$$
$$\Big(\max_k\{\vartheta_{kj}^{1}\}, \max_k\{\vartheta_{kj}^{2}\}, \max_k\{\vartheta_{kj}^{3}\}, \dots, \max_k\{\vartheta_{kj}^{p}\}\Big), \Big(\max_k\{\theta_{kj}^{1}\}, \max_k\{\theta_{kj}^{2}\}, \max_k\{\theta_{kj}^{3}\}, \dots, \max_k\{\theta_{kj}^{p}\}\Big)\Big\rangle$$

for all (k=1,2,…,m) and (j=1,2,…,n).

Step 4: According to Equation (9) positive value of $\mathrm{V}^{+}(\mathrm{F}_{\mathrm{kj}})$ based on positive ideal solution r^{+} and negative value of $\mathrm{V}^{-}(\mathrm{F}_{\mathrm{kj}})$ based on negative ideal solution r^{-} of alternative F_{k} (k=1,2,…,m) calculated as follows:

$$\mathrm{V}^{+}(\mathrm{F}_{\mathrm{kj}}) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{d}_r^{\mathrm{w}}(\mathrm{F}_{\mathrm{kj}}, r^{+})$$

$$= \frac{1}{16.n.p}.\sum_{j=1}^{n}\sum_{k=1}^{p}\Big[\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{a}_{\mathrm{kj}}-w_{r^{+}}(1+\eta_{\mathrm{kj}}^{i}{}^{+}-\vartheta_{\mathrm{kj}}^{i}{}^{+}-\theta_{\mathrm{kj}}^{i}{}^{+})\mathrm{a}_{\mathrm{kj}}\Big)^{r}$$

$$+\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{b}_{\mathrm{kj}}-w_{r^{+}}(1+\eta_{\mathrm{kj}}^{i}{}^{+}-\vartheta_{\mathrm{kj}}^{i}{}^{+}-\theta_{\mathrm{kj}}^{i}{}^{+})\mathrm{b}_{\mathrm{kj}}\Big)^{r}+$$

$$\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{c}_{\mathrm{kj}}-w_{r^{+}}(1+\eta_{\mathrm{kj}}^{i}{}^{+}-\vartheta_{\mathrm{kj}}^{i}{}^{+}-\theta_{\mathrm{kj}}^{i}{}^{+})\mathrm{c}_{\mathrm{kj}}\Big)^{r}+$$

$$\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{d}_{\mathrm{kj}}-w_{r^{+}}(1+\eta_{\mathrm{kj}}^{i}{}^{+}-\vartheta_{\mathrm{kj}}^{i}{}^{+}-\theta_{\mathrm{kj}}^{i}{}^{+})\mathrm{d}_{\mathrm{kj}}\Big)^{r}\Big]^{\frac{1}{r}}$$

and

$$\mathrm{V}^{-}(\mathrm{F}_{\mathrm{kj}}) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{d}_r^{\mathrm{w}}(\mathrm{F}_{\mathrm{kj}}, r^{-})$$

$$\frac{1}{16.n.p}.\sum_{j=1}^{n}\sum_{k=1}^{p}\Big[\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{a}_{\mathrm{kj}}-w_{r^{-}}(1+\eta_{\mathrm{kj}}^{i}{}^{-}-\vartheta_{\mathrm{kj}}^{i}{}^{-}-\theta_{\mathrm{kj}}^{i}{}^{-})\mathrm{a}_{\mathrm{kj}}\Big)^{r}$$

$$\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{b}_{\mathrm{kj}}-w_{r^{-}}(1+\eta_{\mathrm{kj}}^{i}{}^{-}-\vartheta_{\mathrm{kj}}^{i}{}^{-}-\theta_{\mathrm{kj}}^{i}{}^{-})\mathrm{b}_{\mathrm{kj}}\Big)^{r}+$$

$$\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{c}_{\mathrm{kj}}-w_{r^{-}}(1+\eta_{\mathrm{kj}}^{i}{}^{-}-\vartheta_{\mathrm{kj}}^{i}{}^{-}-\theta_{\mathrm{kj}}^{i}{}^{-})\mathrm{c}_{\mathrm{kj}}\Big)^{r}+$$

$$\Big(w_{\mathrm{F}_{\mathrm{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{d}_{\mathrm{kj}}-w_{r^{-}}(1+\eta_{\mathrm{kj}}^{i}{}^{-}-\vartheta_{\mathrm{kj}}^{i}{}^{-}-\theta_{\mathrm{kj}}^{i}{}^{-})\mathrm{d}_{\mathrm{kj}}\Big)^{r}\Big]^{\frac{1}{r}}$$

where $w_{r^{+}} = \max\{w_{\mathrm{j}}: \mathrm{j} = 1,2,\ldots,\mathrm{n}\}$ and $w_{r^{-}} = \min\{w_{\mathrm{j}}: \mathrm{j} = 1,2,\ldots,\mathrm{n}\}$.

Step 5: Compute the group utility δ_k values for the maximum and individual regret σ_k values for the opponent

$$\delta_k = \sum_{j=1}^{n}(w_{r^{+}})\frac{\mathrm{d}_r^{\mathrm{w}}(\mathrm{F}_{\mathrm{kj}}, r^{+})}{\mathrm{d}_r^{\mathrm{w}}(r^{-}, r^{+})}$$

$$\sigma_k = max\left\{\frac{\mathrm{d}_r^{\mathrm{w}}(\mathrm{F}_{\mathrm{kj}}, r^{+})}{\mathrm{d}_r^{\mathrm{w}}(r^{-}, r^{+})}\right\}$$

Step 6: Compute the index values θ_i as follows;

$$\theta_k = \rho.\left(\frac{\delta_k-\delta^{-}}{\delta^{+}+\delta^{-}}\right)+(1-\rho).\left(\frac{\sigma_k-\sigma^{-}}{\sigma^{+}+\sigma^{-}}\right), \tag{14}$$

where $\delta^{+} = \min\delta_k$, $\delta^{-} = \max\delta_k, \sigma^{+} = \min\sigma_k$ and $\sigma^{-} = \max\sigma_k$. Here ρ denotes decision-making mechanism coefficient.

a. θ_k is the minimal if $\rho < 0.5$,

b. θ_k is the maximum if $\rho > 0.5$,

c. θ_k is both minimal and maximum if $\rho = 0.5$.

Step 7: Rank the all alternatives by sorting δ, σ, θ values in decreasing order. Thus the result is a set of three ranking list denoted by $\delta_{[k]}$, $\sigma_{[k]}$, $\theta_{[k]}$.

Consider the alternative k, corresponding to $\theta_{[k]}$ (smallest among $\theta_{[k]}$ values) as a compromise solution if the following two conditions are satisfied.

(A1) Feasible benefit:

If top most two alternatives in $\theta_{[k]}$ are $[F_2]$ and $[F_1]$ then

$$\theta([F_2]) - \theta([F_1]) \geq \frac{1}{m-1}$$

where m stands for the cardinality of the set of attributes.

(A2) Acceptable stability:

The choice $[F_k]$ must be top ranked by at least one of $\delta_{[k]}$ and $\sigma_{[k]}$. If one of the condition is not satisfied then a set of compromise solution is proposed, which consist of,

a. If only (A1) is met then both alternatives $F_{[1]}$ and $F_{[2]}$ will serve as the compromise solution.
b. If (A1) is not met then there will be a series of compromise solutions, which are alternatives may be located by making use of

$$\theta([F_m]) - \theta([F_1]) \geq \frac{1}{m-1}$$

for the maximum m.

The minimal value of θ determines the best alternative.

Example 3.2. In engineering calculations, it is very important to use the material according to its purpose and according to its properties. As a composite word, it means a material consisting of two or more parts. Composite materials have found use in every field in parallel with today's technological developments and are among the indispensable materials of modern technology. Such materials are widely and effectively used in aerospace, medicine, automobile industry and sports equipment due to their high strength and lightness. Being light is a great advantage in terms of saving energy and fuel. Therefore, we want to choose the best and economical composite for the company that prefers composite. That is, the company, using which is the set of altenatives as

$K = \{k_1 = \text{particulate composite}, k_2 = \text{Discontinuous fiber composite}, k_3 = \text{Particle} - \text{reinforced metal matrix composites}, k_4 = \text{short fiber reinforcement metal matrix composites}, k_5 = \text{polymer matrix Composite}\}$ and according to three criteria determined $G = \{g_1 = \text{Combining at least two materials separated by specific interfaces with different chemical compositions}, g_2 = \text{Combining different materials in three dimensions}, g_3 = \text{having features that none of the components have on their own}$. Then, we try to choose and rank all alternatives F_k for all k=1, 2, … ,5 by using the following algorithm.

Algorithm:

Step 1: The evaluation matrix$(F_{kj})_{5x3}$ is given by an expert as;

$$(F_{kj})_{5x3} = \begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{matrix} \left(\begin{matrix} \langle [0.32,0.44,0.51,0.69];(0.3,0.5,0.7,0.8),(0.1,0.3,0.2,0.3),(0.6,0.3,0.5,0.6)\rangle \\ \langle [0.23,0.25,0.41,0.45];(0.4,0.2,0.3,0.5),(0.2,0.5,0.7,0.6),(0.7,0.5,0.6,0.8)\rangle \\ \langle [0.66,0.72,0.79,0.83];(0.7,0.6,0.4,0.8),(0.5,0.5,0.5,0.6),(0.4,0.3,0.4,0.5)\rangle \\ \langle [0.52,0.63,0.76,0.91];(0.8,0.7,0.5,0.6),(0.4,0.3,0.7,0.5),(0.1,0.5,0.7,0.7)\rangle \\ \langle [0.13,0.35,0.41,0.58];(0.7,0.8,0.9,0.9),(0.2,0.7,0.5,0.6),(0.1,0.7,0.8,0.4)\rangle \\ \langle [0.28,0.32,0.38,0.43];(0.5,0.3,0.4,0.6),(0.2,0.1,0.5,0.4),(0.4,0.6,0.5,0.7)\rangle \\ \langle [0.12,0.15,0.18,0.23];(0.3,0.7,0.9,0.9),(0.1,0.2,0.3,0.7),(0.3,0.4,0.7,0.5)\rangle \\ \langle [0.65,0.66,0.72,0.75];(0.6,0.8,0.9,0.8),(0.2,0.5,0.4,0.3),(0.2,0.3,0.6,0.6)\rangle \\ \langle [0.08,0.15,0.27,0.37];(0.3,0.9,0.8,0.4),(0.8,0.7,0.6,0.5),(0.1,0.1,0.4,0.3)\rangle \\ \langle [0.09,0.13,0.19,0.69];(0.2,0.5,0.7,0.9),(0.1,0.3,0.5,0.4),(0.6,0.7,0.8,0.8)\rangle \\ \langle [0.12,0.27,0.60,0.65];(0.2,0.7,0.8,0.9),(0.4,0.3,0.8,0.5),(0.2,0.5,0.6,0.4)\rangle \\ \langle [0.22,0.48,0.43,0.73];(0.3,0.8,0.9,0.7),(0.5,0.7,0.7,0.6),(0.1,0.4,0.8,0.6)\rangle \\ \langle [0.14,0.33,0.43,0.83];(0.1,0.6,0.9,0.5),(0.8,0.5,0.6,0.7),(0.1,0.3,0.5,0.8)\rangle \\ \langle [0.63,0.73,0.83,0.93];(0.4,0.5,0.7,0.6),(0.2,0.6,0.8,0.3),(0.3,0.6,0.7,0.2)\rangle \\ \langle [0.41,0.43,0.68,0.74];(0.5,0.7,0.8,0.3),(0.1,0.2,0.6,0.4),(0.4,0.2,0.9,0.7)\rangle \end{matrix} \right)$$

Step 2: Since the normalized entropy matrix is

$$\left(\mathcal{E}_{kj}\right)_{5\times3} = \begin{pmatrix} 0.561587 & 0.869281 & 0.848676 \\ 0.621053 & 0.506173 & 0.900990 \\ 0.739696 & 0.403579 & 0.936975 \\ 0.789368 & 0.714889 & 0.394178 \\ 0.609610 & 0.966851 & 0.760479 \end{pmatrix}_{5\times3}$$

we have calculated the weights of the criteria w= (w_1, w_2, w_3) as;

$$w_1 = \frac{5-(\mathcal{E}_{11}+\mathcal{E}_{21}+\mathcal{E}_{31}+\mathcal{E}_{41}+\mathcal{E}_{51})}{15-(\mathcal{E}_{11}+\mathcal{E}_{12}+\mathcal{E}_{13}+\mathcal{E}_{21}+\mathcal{E}_{22}+\mathcal{E}_{23}+\mathcal{E}_{31}+\mathcal{E}_{32}+\cdots+\mathcal{E}_{42}+\mathcal{E}_{43}+\mathcal{E}_{51}+\mathcal{E}_{52}+\mathcal{E}_{53})}$$

$$= \frac{5-(0.561587+0.621053+0.739696+0.789368+0.609610)}{15-(0.561587+0.869281+0.848676+0.621053+\cdots+0.60961+0.966851+0.760479)}$$

$= 0.383558117$

similarly, we have $w_2 = 0.35169353$, and $w_3 = 0.264748353$.

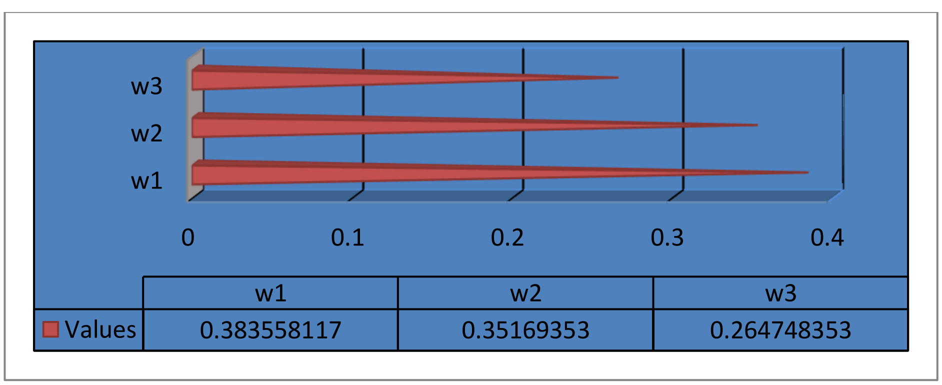

Figure 1: Weights of the criteria by normalized NVNT- numbers

Step 3: The positive ideal solution r^+ and negative ideal solution r^-, respectively calculated as;

$$\mathrm{r}^+ = \langle [0.66,0.73,0.83,0.93]; (0.8,0.9,0.9,0.9), (0.1,0.1,0.2,0.3), (0.1,0.1,0.4,0.2)\rangle$$

and

$$\mathrm{r}^- = \langle [0.08,0.13,0.18,0.23]; (0.1,0.2,0.3,0.3), (0.8,0.7,0.8,0.7), (0.7,0.7,0.9,0.8)\rangle.$$

Step 4: According to Equation (9) positive value of $\mathrm{V}^+(\mathrm{F}_{\mathrm{kj}})$ based on positive ideal solution r^+ and negative value of $\mathrm{V}^-(\mathrm{F}_{\mathrm{kj}})$ based on negative ideal solution r^- of alternative k_k (k=1, 2, …, 5) calculated as follows:

$$\mathrm{V}^+(\mathrm{F}_{11}) = 0.0118,\ \mathrm{V}^+(\mathrm{F}_{12}) = 0.0144, \mathrm{V}^+(\mathrm{F}_{13}) = 0.0131, \mathrm{V}^+(\mathrm{F}_{21}) = 0.0165, \mathrm{V}^+(\mathrm{F}_{22}) = 0.0153$$

$$\mathrm{V}^+(\mathrm{F}_{23}) = 0.0136,\ \mathrm{V}^+(\mathrm{F}_{31}) = 0.0312, \mathrm{V}^+(\mathrm{F}_{32}) = 0.0327, \mathrm{V}^+(\mathrm{F}_{33}) = 0.0441, \mathrm{V}^+(\mathrm{F}_{41}) = 0.011$$

$$\mathrm{V}^+(\mathrm{F}_{42}) = 0.0151,\ \mathrm{V}^+(\mathrm{F}_{43}) = 0.0112, \mathrm{V}^+(\mathrm{F}_{51}) = 0.0132, \mathrm{V}^+(\mathrm{F}_{52}) = 0.0153, \mathrm{V}^+(\mathrm{F}_{53}) = 0.011.$$

$$\mathrm{d}_r^{\mathrm{w}}(r^-, r^+) = 0.0745$$

Step 5: Computed the group utility δ_k (k=1, 2, … ,5) values for the maximum and individual regret σ_k values for the opponent

$$\delta_1 = \sum_{j=1}^{n} (w_{r^+}) \frac{\mathrm{V}^+(\mathrm{F}_{\mathrm{kj}})}{\mathrm{d}_r^{\mathrm{w}}(r^-, r^+)}$$

$$= (w_{r^+}) \frac{\mathrm{V}^+(\mathrm{F}_{11}) + \mathrm{V}^+(\mathrm{F}_{12}) + \mathrm{V}^+(\mathrm{F}_{13})}{\mathrm{d}_r^{\mathrm{w}}(r^-, r^+)}$$

$$= (0.383558117) \frac{0.0118 + 0.0144 + 0.0131}{0.0745}$$

$$= 0.2026$$

Similar to

$\delta_2 = 0.2334, \delta_3 = 0.5560, \delta_4 = 0.1918, \delta_5 = 0.2077.$

$$\sigma_k = max\left\{\frac{\mathrm{d}_r^{\mathrm{w}}(\mathrm{F}_{\mathrm{kj}}, r^+)}{\mathrm{d}_r^{\mathrm{w}}(r^-, r^+)}\right\}$$

$$\sigma_1 = max\{0.0118, 0.0144, 0.0131\} = 0.0144$$

$$\sigma_2 = max\{0.0165, 0.0153, 0.0136\} = 0.0165$$

$$\sigma_3 = max\{0.0312, 0.0327, 0.0441\} = 0.0441$$

$$\sigma_4 = max\{0.011, 0.0151, 0.0112\} = 0.0151$$

$$\sigma_5 = max\{0.0132, 0.0153, 0.011\} = 0.0153.$$

Step 6: Let $\rho = 0.5$, compute the index values θ_k as follows;

$$\theta_k = \rho.\left(\frac{\delta_k - \delta^-}{\delta^+ + \delta^-}\right) + (1-\rho).\left(\frac{\sigma_k - \sigma^-}{\sigma^+ + \sigma^-}\right),$$

$$\theta_1 = (0.5).\left(\frac{0.2026 - 0.1918}{0.5560 + 0.1918}\right) + (1-0.5).\left(\frac{0.0144 - 0.0144}{0.441 + 0.0144}\right)$$

$$= 0.01489$$

Similar to

$$\theta_2 = 0.0922, \theta_3 = 1, \theta_4 = 0.0122,\ \theta_5 = 0.03697$$

Step 7: Based on the index values $CC_k (k = 1,2,...,5)$ the ranking of alternatives $k_k (k = 1,2,...,5)$ are shown in Figure 2 and given as;

$$k_3 > k_2 > k_5 > k_1 > k_4.$$

Finally the best alternative is k_3.

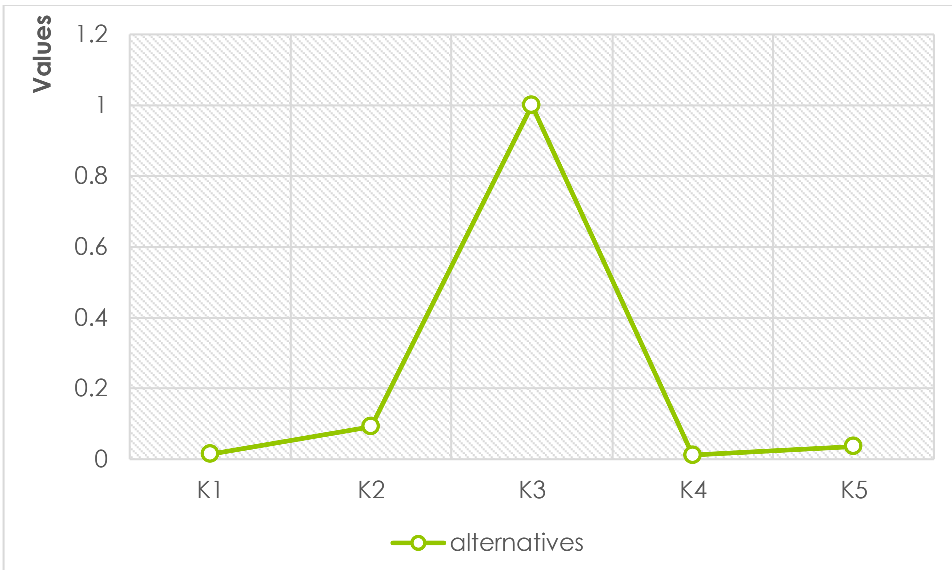

Figure 2 The ranking of alternatives K_k $(k = 1,2,...,5)$

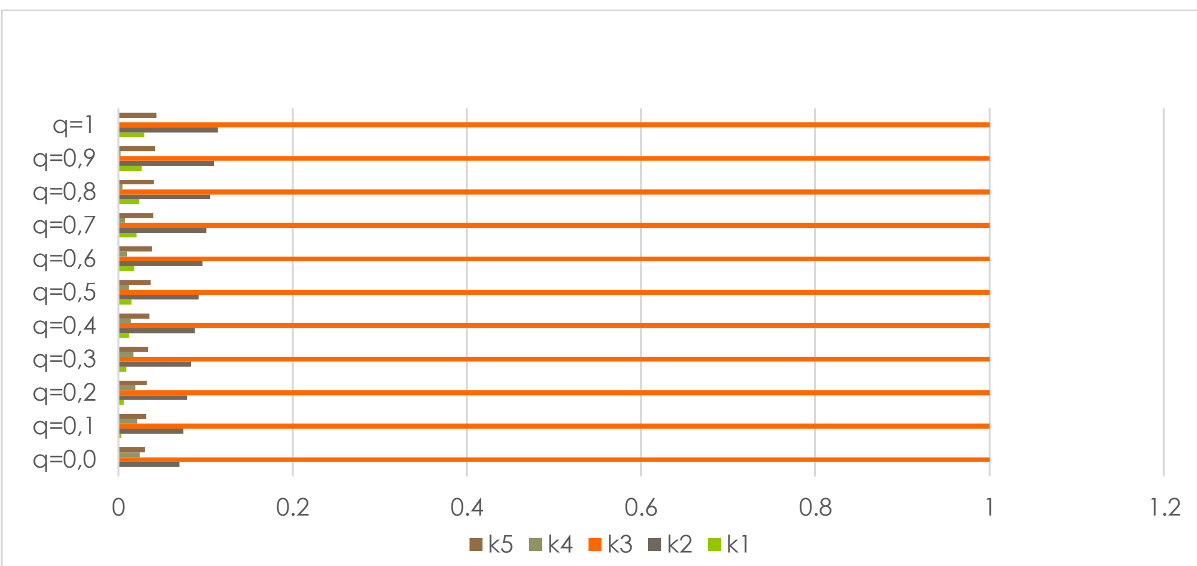

Figure 3: VIKOR index for all ρ values

The results from the different distance measures used to resolve the MCDM problem in section 4 are shown in Figure 4

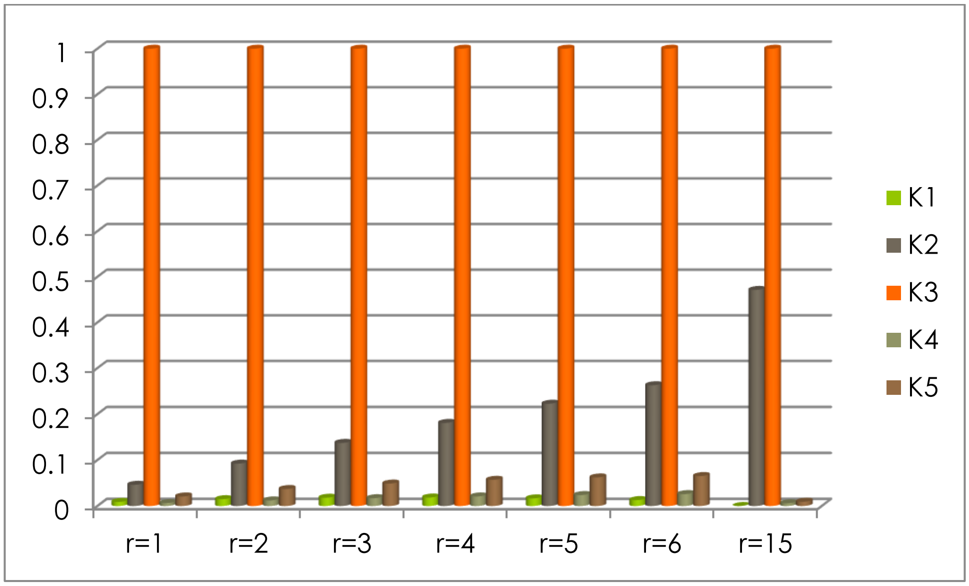

Figure 4: The results from the different distance measures

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Chapter Twelve

In Determining the Level of Teachers' Commitment to the Teaching Profession Using Classical and Fuzzy Logic

Sevilay Şahin[1], Merve Kısaoğlu[2] and Abdullah Kargın[3]

[1, 2]Department of Educational Sciences, Gaziantep University, Gaziantep 27310, Turkey;

[3]Department of Mathematics, Gaziantep University, Gaziantep 27310, Turkey;

E-mail: ssahin@gantep.edu.tr, mervekisaoglu06@gmail.com, abdullahkargin27@gmail.com,

ABSTRACT

The aim of this study is to apply a measurement tool designed with fuzzy logic in order to determine the level of teachers' commitment to the teaching profession. The working group of the study is composed of teachers from various branches working in the 2021-2022 academic year in Gaziantep province. A percentage value item related to decision making has been added to each item of the Commitment to the Teaching Profession scale. Thus, the necessary preparation has been made for the evaluation of the data in fuzzy matlab application. The data obtained in the study are evaluated in fuzzy matlab application. It has been found that the results obtained from fuzzy matlab application are more detailed and reflect the individual better in the decision-making process related to the problem. It has been concluded that the results obtained from the fuzzy questionnaire are more valid because fuzzy questionnaire and fuzzy matlab provide more accurate and precise results in decision-making processes.

Keywords: Teaching Profession, Teachers' commitment, Fuzzy Logic, Fuzzy Survey, Fuzzy Matlab

INTRODUCTION

The professions that people prefer to do have a very important place in their life. This is a result that the job people prefer affects their private, social, and work lives directly and indirectly. It is thought that both perception, thoughts, and feelings about the job chosen and the perspective of the people's live in that society on the job chosen have very important roles in terms of commitment to the profession. It is believed that the concept of commitment to the profession which is studied for a long time in very different fields and disciplines has still importance especially in educational sciences department. A commitment to a profession can cause people to be more motivated, to work productively by solving the problems more easily while they are doing their jobs. Through the commitment to the profession, people can work more efficiently and effectively by feeling devoted to the institution they are on duty.

Some descriptions about commitment to a profession in the literature are as follows: Commitment to profession is the importance people give to their work or career [1]; commitment to profession means people understand the importance of their jobs in their lives as a result of the skills and the expertise they have, not to consider leaving their jobs and to have a positive relationship with their life satisfaction [2]. Bagraim takes attention to three different components of commitment to profession: emotional commitment which means to keep up with the profession and to have a strong emotional commitment to the profession, continuance commitment which means to realize the cost related to leaving the job and normative commitment which means to have a responsibility for the job and to keep the profession [3]. Commitment to profession is related to adopt the job you have [4]. Baysal and Paksoy explain that the job has an important and a central place in people's lives as the result of the studies people do to have skills and expertise in a specific branch [5]. According to Meyer et al. professional commitment is an emotional relation between people themselves and their jobs [6]. Lee et al. also explain professional commitment as psychological relation based on emotional relation between people themselves and their job [7]. Bienkowska explains professional commitment as a term accompanying study of people's motivation [8]. According to him professional commitment is related to desire for career development and desire to specialize. A sense of identity is given to the people who are devoted to their jobs and this commitment is redirected by the need for live and work according to the values and rules which manage their profession groups they belong to. When the descriptions in the literature are examined, professional commitment is to have positive thoughts and feelings for the chosen job and to be perceived by the people at the level of consciousness, in other words, to have metacognitive awareness for these positive thoughts and feelings to the profession. Also, it can be defined as all of the efforts people plan and have to do their professions in a good way.

Studies about dimensions of professional commitment started many years ago [9]. One of the studies which can be considered a pioneer of all these studies is "A three-component conceptualization of organizational commitment" by Meyer and Allen [10]. Meyer et al. defend to examine professional commitment multidimensionally instead of one-dimensionally and they did a detailed study "Commitment to Organizations and Occupations: Extension and Test of a Three-Component Conceptualization" which is about commitment to profession [6]. According to this study, professional commitment has dimensions of emotional, continuance and normative commitment. Emotional commitment means to keep up with the profession and to have a strong emotional commitment to the profession, continuance commitment means to realize the cost related to leaving the job and normative commitment means to have a responsibility for the job and to keep the profession with ethical values.

It can be expressed that there are many factors that affect the professional commitment. These factors can be classified as individually and professionally. Each individual has different tendencies, interests, knowledge and skills which distinguish him/her from others. It causes individuals to interest a particular professional field and some professions to seem more interesting [11]. Age, gender, marital status, seniority, education from individual factors are effective in professional commitment. Individual factors such as expectation of a reward is also effective. Society's view about the profession, stress level of the job, the responsibilities in the job, economic and social return in the job, opportunities, people's wishes, desires and communication and motivation status in the job can be seen as the other factors that affect the professional commitment [9]. This complex structure of the professional commitment is also important in terms of teaching profession.

Teacher who is the subject of teaching profession is defined as "the one whose aim is to teach knowledge" according to Turkish Language Society [12]. In Basic Law of National Education 1739/43, the teaching profession is defined as a specialized profession that includes government's educational and administrative tasks [13]. Shulman mentioned 7 categories while defining teaching profession. These are general pedagogical knowledge, content knowledge, curriculum knowledge, pedagogical content knowledge, education system knowledge, students' characteristics knowledge and knowledge of educational goals, values, historical and philosophical bases [14]. According to the study of Higher Education Institute and Ministry of Education, competence of teaching profession is organized as compatible with European Union countries and "personal and professional values -professional development, get to know the student, teaching and learning process, monitoring and evaluating learning progress, school, family and community relations, program and content information" are identified as six main productivity areas [15]. Considering these explanations, teaching profession; adopting universal and social ethical values and after it has become a part of life; it can be explained as reflecting these values to other people and at the same time, in a certain area programming learning of general and special information for this area.

Commitment to the teaching profession is determined as the attitudes of an educator beyond official and normative expectations his/her commitment, enthusiasm and passion for regular educational processes and students. In this sense, from the concept of a teacher who is extremely committed to his profession, it can be understood that he is physically and spiritually ready to carry out educational activities [16]. The professional commitment of teachers has an impact on the quality of education in schools and the academic success of students, if it is in a broader context as well as their professional competence, skills, knowledge, attitudes and values of teachers [17]. In addition, “adherence to the teaching profession is very important in terms of making a direct and positive contribution to educators' teaching methodology, understanding, personality, characteristics and attitudes” [16].

From the classical point of view that its efficiency depends on external factors other than the person himself then, along with the neo-classical understanding in which other factors are bracketed and the individual is centered, a perspective that is more humane has been developed. In 20 and 21. Centuries, along with critical theory, it can be said that the idea that the individual himself, society and other factors are effective in the choice of profession, the meaning / value of the chosen profession, the way the profession is built is effective. From this point of view, it is necessary to consider the feelings, thoughts and behaviors that he has while performing the profession of an individual, the environment in which he lives, and his way of thinking as a whole. Therefore, it is considered that his commitment to the profession and the way he perceives himself in this regard are quite important. In this framework, a teacher needs a more sensitive tool rather than a limited tool to accurately reflect his/her own opinion/decision about his/her commitment to his/her profession. From the classical point of view, the boundies, framework of a person's assessments about himself or a situation are specific; the understanding of 0-1/ yes-no prevails; in fuzzy logic [27], the degree of membership of each element of a set can be the range [0, 1]. Thus, unlike classical logic, the membership of each element is graded [18]. Therefore, it is considered that it is very important to use measurement tools prepared with fuzzy logic in order to take healthier data and get more accurate results.

BACKGROUND

2.1 Teachers' Commitment to the Teaching Profession

The issue of commitment to the teaching profession has been studied by many researchers due to the fact that it is quite important. The following are some of those studies. Yıldız's study titled “Development of the Scale of Commitment to the Teaching Profession" was made in order to develop a measurement tool that can provide to measure the level of commitment of candidates to the teaching profession in a valid and reliable way. Factors are defined as professional identity, professional value, professional effort and professional dedication. The validity and reliability evidence obtained as a result of the research was found to be sufficient [19]. Kozikoglu and Senemoglu “Development of the scale of dedication to the teaching profession: The research titled “Validity and reliability study" was conducted in order to develop a scale aimed at determining the dedication of teachers to the teaching profession. According to the results of the scale, it is concluded that there is a valid and reliable scale for determining their dedication to their profession [20].

The study of Kayadelen and Koçak titled “Examining the Relationships between Leadership Capacity in Schools and Teachers' Dedication to the Profession" was conducted in order to determine the relationships between the leadership capacities of secondary education institutions and the professional dedication levels of teachers working in these schools .The study was conducted with 399 teachers who continue to work in secondary education institutions. The results of the research showed that there is a weak positive relationship between school leadership capacity and teachers' dedication to the profession. However, it has been observed that leadership capacity in schools is a significant predictor of teachers' dedication to the profession [21].

The study titled “Commitment to the Teaching Profession” by Yıldız and Çelik was conducted with the aim of shedding light on teachers who are connected to their profession and the effects of commitment on the learning and teaching process. As a result of the study, it has been concluded that teachers armed with commitment, passion and enthusiasm, will be a role model not only for students, but also for his colleagues [16].

In Dalaman's thesis titled ”Examining the attitudes of secondary school teachers towards learning and the teaching profession", it was aimed to examine the attitudes of secondary school teachers towards learning and the teaching profession. As a result of the analysis of the findings, in the attitudes of teachers about learning and the teaching profession, it has been concluded that the attitudes of male and female teachers are similar in the sub-dimensions of the nature of learning, expectations about learning, openness to learning, anxiety about learning, value giving to the teaching profession, professional burnout and disinterest in the teaching profession. According to the results of this research, it has been seen that the positive attitudes of teachers towards learning and the teaching profession should be increased and their negative attitudes should be reduced .In this sense, especially in studies such as seminars, it has been concluded that it will be useful of obtaining and evaluating teachers' opinions about the teaching profession [22].

In the study of Ataç titled “The relationship between teachers' supervision foci and their professional commitment”, it was aimed to expose whether there is a relationship between the supervision foci and the professional commitment of teachers in this study conducted on teachers. 400 teachers from various levels in public schools in İstanbul have participated in the research. As a result of the research, significant differences have been found between sub-dimensions of professional commitment, accumulated cost and limitation of alternatives and teachers’ graduation rates. It has been found that as the teachers’ graduation rate rises, the professional commitment rises. At the end of the study, it has been offered we can rise teachers’ professional commitment rising their graduation rate [23].

Some part of Sinclair’s study “Initial and changing student teacher motivation and commitment to teaching” is about how initial teacher education courses and internship affect primary school teacher trainees’ motivation and their commitment to the teaching profession. As a result of the study, when the students studying in teaching department take the initial teacher education course, they have the motivation related to the teaching content and the aim of it and have the commitment to the teaching profession [24].

2.2 Fuzzy Sets

Definition 1: [27] Let L be the universal set. A fuzzy set K on L is defined by

$$K = \{\langle l, \mu_K(l)\rangle : l \in K\}.$$

Here, μ_K is membership function such that $\mu_K : L \to [0,1]$.

Definition 2: [28] A triangular fuzzy number $\tilde{k} = [s_1,\ p_1,\ r_1]$ is a special fuzzy set on the real number set R, whose membership function is defined as follows

$$\mu_{\tilde{k}}(\beta) = \begin{cases} (\beta - s_1)/(p_1 - s_1), & if\ (s_1 \le \beta < p_1) \\ 1, & if\ (\beta = l_1) \\ (r_1 - \beta)/(r_1 - p_1), & if\ (p_1 < \beta \le r_1) \\ 0, & if\ otherwise \end{cases}$$

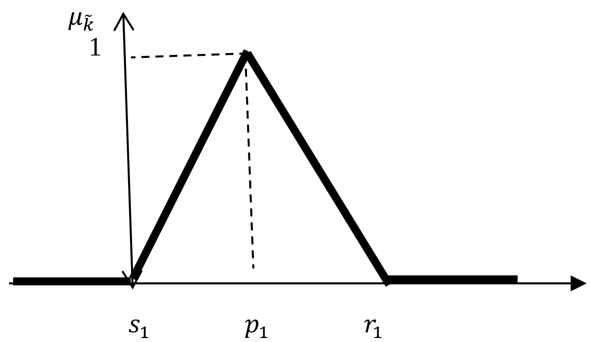

Figure 1. $\tilde{k} = [s_1,\ p_1,\ r_1]$ triangular fuzzy membership function [28]

CLASSICAL METHOD

3.1 Research Design

This study designed with survey method in order to determine the level of commitment of teachers from different branches to the teaching profession. The survey method, which is sometimes called method and sometimes technique by researchers, is the method used in scientific research to determine the thoughts of individuals [25]

3.2 Sampling

The sample of the study consists of 60 teachers working at pre-school, secondary and high school levels. The teachers participating in the study were determined by the typical case sampling method, one of the purposeful sampling methods that are not random. Typical case sampling is based on selecting the element with average values for the case to be examined [26].

3.3 Data Collection

In order to determine the level of teachers' commitment to the teaching profession, the Teacher's Occupational Commitment scale developed by Yıldız was used [19]. The scale has four factors; It is a structure consisting of 33 items: professional identification (12 items), professional value (6 items), professional effort (8 items), and professional dedication (7 items). The five-point Likert scale is graded as strongly disagree (1), disagree (2), undecided (3), agree (4), and strongly disagree (5).

3.4 Data Analysis

The data was analyzed with SPSS 21.0 program. Statistical analyzes were calculated in accordance with the purpose of the research. The appropriateness, score and arithmetic mean levels used in the interpretation of the descriptive statistics on teachers' commitment to the teaching profession are given in Table 1.

Table 1. Score Ranges and Classifications Used in İnterpretation of Teachers Commitment To The Teaching Profession

Suitability	Score	Limits (arithmetic mean)
Strongly Disagree	1	1.00-1.79
Disagree	2	1.80-2.59
Partly Agree	3	2.60-3.39
Agree	4	3.40-4.19
Strongly Agree	5	4.20-5.00

3.5 Classical Findings

Table 2. *Mean of* Teachers' Commitment to the Teaching Profession *to Teacher's Perceptions according to classical survey*

Teachers	Mean
Physical Education	3,50
Information Technologies	4,17
Biology	3,73
Geography	4,05
Religious Culture and Moral Knowledge	3,68
Philosophy	4,33
Physics	3,83
English	3,57
Chemical	4,12
Math	3,85
Accounting and Finance	4,02
Music	3,22
Pre-school	3,94

Psychological Counseling and Guidance	4,29
History	4,09
Turkish Language and Literature	4,16
Visual arts	4,46
Turkish	4,31

When Table 2 is examined, the levels of commitment to the teaching profession are shown according to the opinions of teachers in different branches. Accordingly, the average of the music branch (X=3.22) is partially agree. The averages of Physical educaiton (X=3,50), İnformation technologies (X=4,17), Biology (X=3,73), Geography (X=4,05), Religious culture and moral knowledge (X=3,68), Physics (X=3,83), English (X=3,57), Chemical (X=4,12), Math (X=3,85), Accounting and finance (X=4,02), Pre-school (X=3,94), History (X=4,09) and Turkish language and literature (X=4,16) branches are concentrated in the agree part. However, the averages of Philosophy (x=4,33), Psychological counseling and guidance (X=4,29), Visual arts (X=4,46) and Turkish (X=4,31) branches are, strongly agree.

FUZZY METHOD

In this section, we evaluate the data obtained with the survey using the fuzzy matlab application and examine teachers' commitment to the teaching profession. The difference of this method from the method in the Section 3 (Classical Method) is that the item answers in the survey are requested as %, and the fuzzy matlab application is used in the evaluation and comparison part. In both methods, the conceptual classification, sample, frequency and dimensions of teachers' commitment to the teaching profession are the same. Now, we give some properties of fuzzy matlab applications.

4.1 Fuzzy Matlab Application

Fuzzy logic controller; fuzzifier, fuzzy inference engine, defuzzifier and knowledge base consists of four main components. By using linguistic variables, the input information specific to the problem for which the fuzzy logic model will be established The process of expressing and converting into fuzzy logic information is called fuzzification. The linguistic variables formed after the fuzzification process are represented by triangular, trapezoidal, bell-shaped and many more geometric shapes specific to the structure of the problem, taking membership degrees [29]. In the fuzzy matlab application, the process is given at Figure 2.

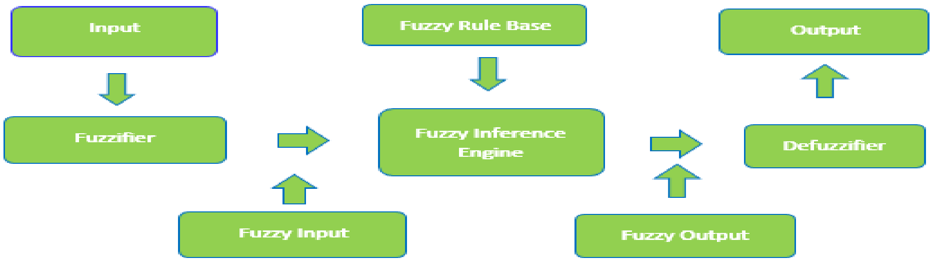

Figure 2. Process of Fuzzy Matlab Algorithm

In this chapter, inputs are "Professional Identification, Professional Value, Professional Effort, Professional Dedication" for fuzzy matlab application in Table 3.

Table 3. Inputs for this fuzzy matlab application

Input	Abbreviation
Professional Identification	PI
Professional Value	PV
Professional Effort	PE
Professional Dedication	PD

We give the triangular fuzzy membership functions of inputs in Table 3 and the representation of these functions as triangular fuzzy numbers in Table 4 (also, in Figure 3).

Table 4. *Triangular Fuzzy Membership Functions for Inputs in Table 3*

Triangular Fuzzy Membership Functions	Abbreviation	Triangular Fuzzy Number
I do not agree	N.A	[0, 25, 50]
I am indecisive	I	[25, 55, 85]
I am agree	A	[55, 100, 100]

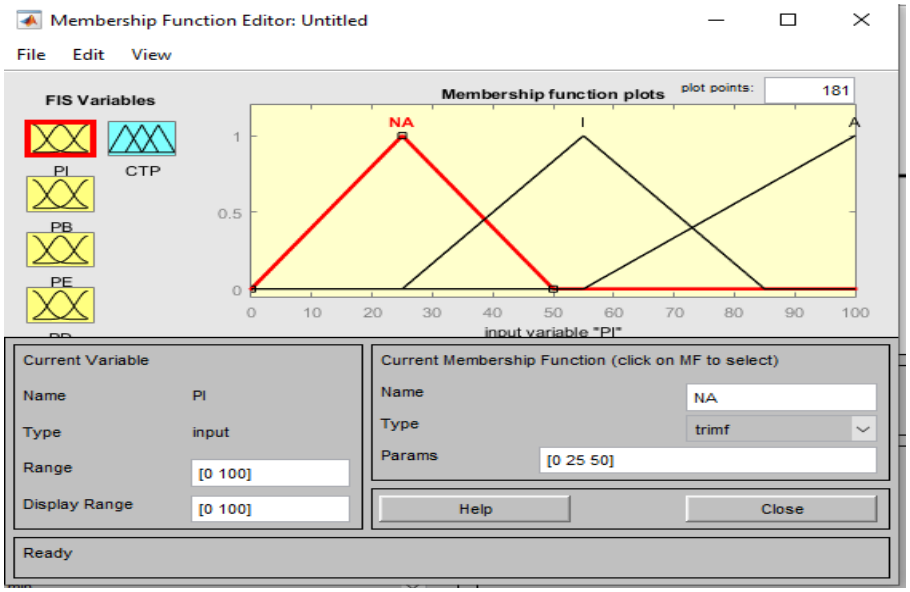

Figure 3. Triangular Fuzzy Membership Functions of Fuzzy Matlab's Inputs

In this chapter, output is "Commitment to the Teaching Profession " for fuzzy matlab application in Table 5.

Table 5. Outputs for this fuzzy matlab application

Output	**Abbreviation**
Commitment to the Teaching Profession	CTP

We give the triangular fuzzy membership functions of output in Table 5 and the representation of these functions as triangular fuzzy numbers in Table 6 (also, in Figure 4).

Table 6. *Triangular Fuzzy Membership Functions of Outputs*

Triangular Fuzzy Membership Functions	**Abbreviation**	**Triangular Fuzzy Number**
Very Little	V.L	[0, 0, 40]
Little	L	[20, 45, 65]
Medium	M	[45, 70, 90]
High	H	[75, 100, 100]

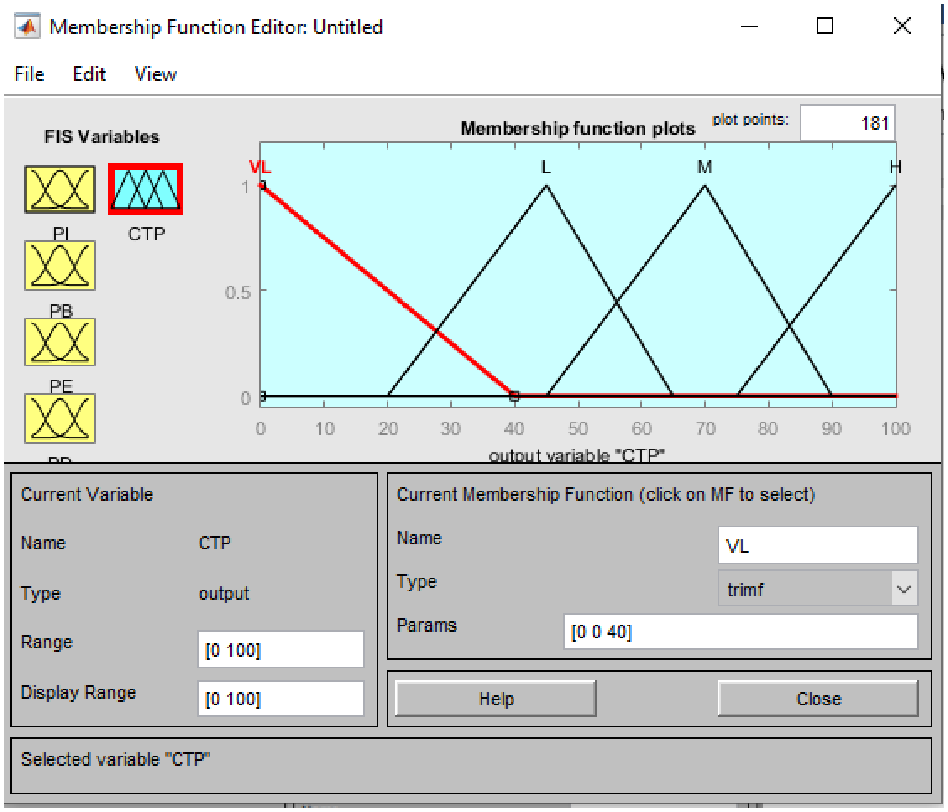

Figure 4. Triangular Fuzzy Membership Functions of Fuzzy Matlab's Output

Also, in this chapter, "Mandami fuzzy inference engine" was used. The "som method" was used for defuzzifier (in Figure 5).

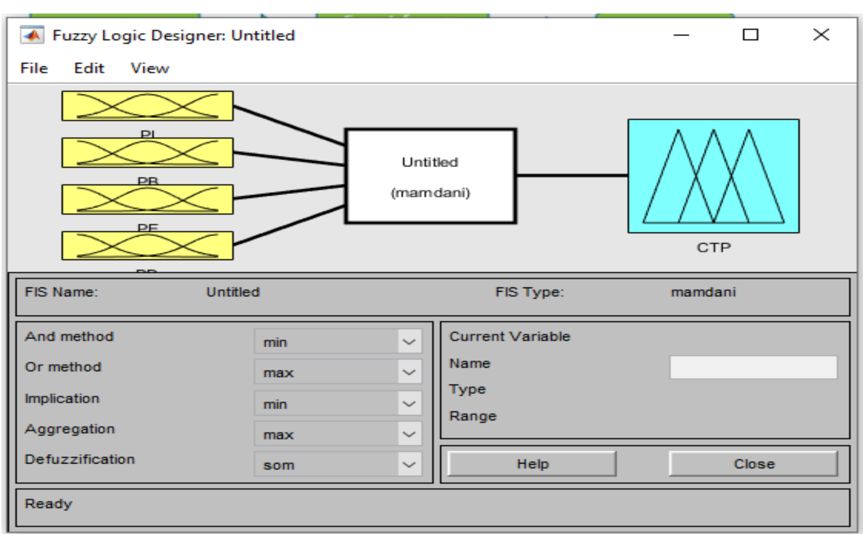

Figure 5. Fuzzy Logic Designer of Fuzzy Matlab

In Figure 6, there is the rule editor for our fuzzy matlab application.

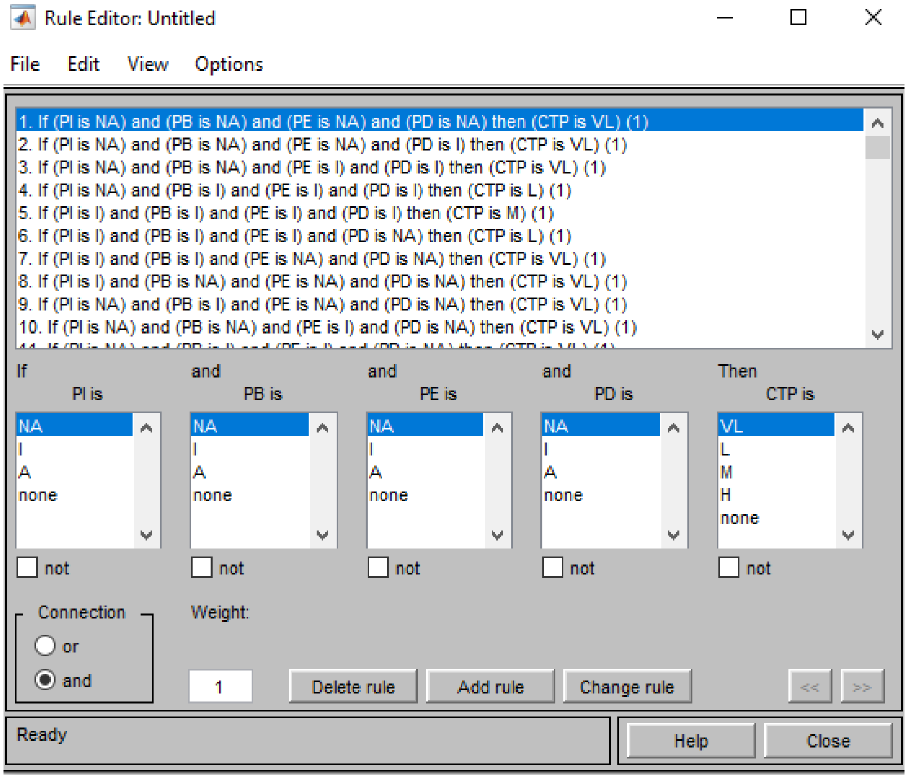

Figure 6. Representation of Fuzzy Rules in Fuzzy Matlab

Figure 6. Getting Results with Fuzzy Matlab Rules

4.3 Fuzzy Findings

In Table 7, we obtain mean of teachers' commitment to the teaching profession to teacher's perceptions according to fuzzy survey and fuzzy matlab using the average of the dimensions from the data obtained from fuzzy survey.

Table 7. *Mean of* Teachers' Commitment to the Teaching Profession *to Teacher's Perceptions according to fuzzy survey and fuzzy matlab*

	Mean (out of 5)	**Mean (out of 100)**
Physical Education	4,35	87
Information Technologies	4,55	91
Biology	4,3	86
Geography	4,3	86
Religious Culture and Moral Knowledge	4,3	86
Philosophy	4,6	92
Physics	4,4	88
English	2,95	59
Chemical	4,3	86
Math	4,3	86
Accounting and Finance	4,55	91
Music	1,6	32
Pre-school	4,5	90
Psychological Counseling and Guidance	4,85	97
History	4,3	86
Turkish Language and Literature	4,55	91
Visual arts	4,5	90
Turkish	4,55	91

When table 7 is examined, teachers in different branches were evaluated according to the opinions of teachers in the fuzzy survey. Accordingly, the average of the music branch (X=1,6) is strongly disagree. The average of the English branch (X=2,95) is, partially agree. The averages of Physical education (X=4,35), Information technologies (X=4,55), Biology (X=4,3), Geography (X=4,3), Religious culture and moral knowledge (X=4,3), Philosophy (X=4,6), Physics (X=4,4), Chemical (X=4,3), Math (X=4,3), Accounting and finance (X=4,55), Pre-school (X=4,5), Psychological counseling and guidance (X=4,85), History (X=4,3), Turkish Language and literature (X=4,55), Visual arts (X=4,5) and Turkish (X=4,55) branches are, strongly agree. That is, the majority of the branches are concentrated at the level of strongly agree.

CONCLUSIONS

If we compare the classical survey results obtained in Table 2 in Section 2 with the fuzzy survey results obtained in Table 7 in Section 3, we obtain Table 8.

Table 8. *Comparison of Classical Survey and Fuzzy Survey Results*

Teachers	Mean of Fuzzy Survey	Mean of Classical Survey
Physical Education	4,35	3,50
Information Technologies	4,55	4,17
Biology	4,3	3,73
Geography	4,3	4,05
Religious Culture and Moral Knowledge	4,3	3,68
Philosophy	4,6	4,33
Physics	4,4	3,83
English	2,95	3,57
Chemical	4,3	4,12
Math	4,3	3,85
Accounting and Finance	4,55	4,02
Music	1,6	3,22
Pre-school	4,5	3,94
Psychological Counseling and Guidance	4,85	4,29
History	4,3	4,09
Turkish Language and Literature	4,55	4,16
Visual arts	4,5	4,46
Turkish	4,55	4,31

When Table 9 is examined, it is found that the results obtained from the classical survey and the fuzzy survey are different. In the classical survey, while the meaning of the value corresponding to the average of the music branch (X=3,22) is partially agree; meaning of value (X=1,6) in fuzzy survey strongly disagree. Because of fuzzy questionnaire and fuzzy matlab provide a more exhaustive and more objective evaluation, the result of fuzzy questionnaire is more coherent than the result of classical questionnaire. Again in the classical survey, while the average of the English branch (X=3.57) I agree; I partially agree with the average of the same branch (X=2,95) in the fuzzy survey. In the classical method, with the value of X=4,46, the highest average was the Visual arts branch; In the fuzzy method, the Psychological Counseling and Guidance branch has the highest average with X=4,85. In addition, while the averages of the branches in the classical method are concentrated at the level of agree; In the fuzzy method, most of the branches are concentrated at the level of strongly agree. As can be seen, the fuzzy method and the classical method give different results and more accurate results are obtained with the fuzzy method. Because the classical survey is rated in a five-point Likert type (Strongly disagree, disagree, partly agree, agree, strongly agree). A value between 0 and 100 was requested for the fuzzy questionnaire. With another expression, while evaluating with 5 options in the classical method; In the fuzzy method, evaluation is made with 100 options. For instance, while in the classical survey interval between 2.60-3.39 is accepted as partly agree, in the fuzzy survey and fuzzy matlab there is a separate membership value for each real number between 2.60-3.39.

In this study, mean of teachers' commitment to the teaching profession to teacher's perceptions according to fuzzy survey and fuzzy matlab using the average of the dimensions was obtained. In addition, each teacher can be compared with the fuzzy matlab application in separate dimensions. In the fuzzy matlab application, we used three different triangular fuzzy membership functions for each input, and four different triangular fuzzy membership functions for the output. As the number of these triangular fuzzy membership functions is increased, more precise results can be obtained. In addition, the triangular fuzzy membership function was used for inputs and outputs in the fuzzy matlab application. Researchers can also use other membership functions (trapezoidal fuzzy membership function, Gaussian fuzzy membership function, etc.) suitable for their problems. Also, in this chapter, "Mandami fuzzy inference engine" was used and "som method" was used for defuzzifier. Furthermore, researchers can use other rinse functions (centroid, bisector, mom, lom, etc.) or the Sugeno method to suit their problem.

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Chapter Thirteen

A Comparative Analysis for Multi-Criteria Decision-Making Methods: TOPSIS and VIKOR methods using NVTN-numbers for Application of Circular Economy

Nuh OKUMUŞ[1] and Vakkas ULUÇAY [2,*]

[1]Faculty of Economics and Administrative Sciences, Hasan Kalyoncu University, Gaziantep, Turkey

okumus27@gmail.com

[*,2]Kilis 7 Aralık University, 79000 Kilis, Turkey

Corresponding author: vakkas.ulucay@kilis.edu.tr,

ABSTRACT

While changing in the new world order, it is necessary to design new business models in order to use existing resources more effectively and effectively. If the Circular Economy, which is a new concept in our country, is included in government policies in the future, an important step will be achieved in terms of Sustainable Environmental Management. Reducing waste and making it reusable will lead to the protection of our natural resources and a serious cost reduction. Therefore, this chapter, N-valued neutrosophic trapezoidal numbers are used in methods of multi-criteria decision making. Some techniques are used in each method to use N-valued neutrosophic trapezoidal numbers information. Therefore, crisp methods can be changed to use N-valued neutrosophic trapezoidal numbers information. The latter are used in TOPSIS and VIKOR based on entropy measure with N-valued neutrosophic trapezoidal numbers. We apply these methods in circular economy and we compare them to distinguish differences between used techniques.

Keywords: N-valued neutrosophic trapezoidal number, generalized distance measure, entropy measure, VIKOR method, TOPSIS mthod, multi-criteria decision-making.

3. Introduction

Ideas about the concept of circular economy began to emerge in the 1960s. In 1966 Kenneth Boulding began to argue that the economy should be transformed into a circular ecological system. In the 1970s, Walter Stahel proposed the idea of a self-regenerative economic system based on the spiral loop system. Circular economy promotes cyclical flows to reduce environmental impacts and maximize resource efficiency instead of linear flows of materials and products [15]. The circular economy [25] is an economic model in which planning, sourcing, supply, production and reprocessing are designed and managed as both processes and outputs to maximize the functioning of the ecosystem and human well-being. Since decision making problems which contain uncertain are difficult to model and solve, and it is a need for us to develop some mathematical theories. Recent years, fuzzy set theory by using only one degree of membership proposed by Zadeh [85] and intuitionistic fuzzy set theory by using two degrees of membership introduced by Atanassov [1] have been received great attention in solving various decision-making problems. These theories can better solve the fuzziness of the uncertain decision making therefore the theories are all very successfully studied in Narayanamoorthy et al. [28], Liu et al. [27] and Hu et al. [26].

By using truth-membership function, indeterminacy-membership function and falsity-membership functions, in 1998, Smarandache [66] proposed the concept of neutrosophic sets (N-sets), which is a generalization of the concept of fuzzy set Zadeh [85] and intuitionistic fuzzy sets Atanassov [1]. In 2013, Smarandache [67] generalized the classical neutrosophic logic to neutrosophic refined logic which have more than one with the possibility of the same or the different membership functions. Moreover, Ye and Ye [83], Chatterjee et al. [14] and Ye and Smarandache [84] introduced the concept single valued neutrosophic multi sets as a further generalization of that of neutrosophic sets based on both the neutrosophic refined logic and multi sets of Yager [82]. The multisets and single valued neutrosophic multisets have received more and more attention since its appearance in [2-7,9-13, 16, 18, 20-24, 29-65, 68-92].

In order to use the concept of single valued neutrosophic multi sets to define an uncertain quantity or a quantity difficult to quantify, in Deli et al. [17] the authors put forward the concept of continuous N-valued neutrosophic trapezoidal numbers (NVNT-numbers). They developed a TOPSIS method by giving some operational laws of NVNT-numbers and some aggregation operators of NVNT-numbers.

4. Preliminary

This section firstly introduces several the known definitions and propositions that would be helpful for better study of this paper.

Definition 2.1 [81] Assume that E is the universe. Then, a single valued neutrosophic set (N-set) A in E defined as

$$A = \{< x, T_A(x)\ ,\ I_A(x)\ ,\ F_A(x) >: x \in E\} \tag{1}$$

where $T_A(x)\ ,\ I_A(x)\ ,\ F_A(x) \in [0,1]$ for each point *x* in such that $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.2 [83] Let E be a universe. A_1 neutrosophic multi-set set A_1 on E can be defined as follows:

$$A_1=\{<x,(T^1_{A_1}(x), T^2_{A_1}(x), \dots, T^P_{A_1}(x)), (I^1_{A_1}(x), I^2_{A_1}(x), \dots, I^P_{A_1}(x)), (F^1_{A_1}(x), F^2_{A_1}(x), \dots, F^P_{A_1}(x)) >: x \in E\},$$

where

$$T^1_{A_1}(x), T^2_{A_1}(x), \dots, T^P_{A_1}(x), I^1_{A_1}(x), I^2_{A_1}(x), \dots, I^p_{A_1}(x), F^1_{A_1}(x), F^2_{A_1}(x), \dots, F^P_{A_1}(x): E \longrightarrow [0,1]$$

such that $0 \le \sup T^i_{A_1}(x) + \sup I^i_{A_1}(x) + \sup F^i_{A_1}(x) \le 3$ (i=1,2,…,P) for any $x \in E$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively.

Definition 2.3 [17] Let $\eta^i_{A_1}, \vartheta^i_{A_1}, \theta^i_{A_1} \in [0,1]$ $(i \in \{1,2,\dots,p\})$ and $a, b, c, d \in \mathbb{R}$ such that $a \le b \le c \le d$. Then, an N-valued neutrosophic trapezoidal number (NVNT-number)
$\tilde{a} = \langle [a, b, c, d]; (\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1}) \rangle$ is a neutrosophic multi-set on the real number set $\mathbb{R}$, whose truth-membership functions, indeterminacy-membership functions and falsity-membership functions are defined as, respectively.

$$T^i_{\tilde{\alpha}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\eta^i_{\tilde{\alpha}}, & a \le x < b \\ \eta^i_{\tilde{\alpha}}, & b \le x \le c \\ \frac{(d-x)}{(d-c)}\eta^i_{\tilde{\alpha}}, & c < x \le d \\ 0, & \text{otherwise,} \end{cases} \quad , \quad I^i_{\tilde{\alpha}}(x) = \begin{cases} \frac{(b-x)+\vartheta^i_{\tilde{\alpha}}(x-a)}{(b-a)}, & a \le x < b \\ \vartheta^i_{\tilde{\alpha}}, & b \le x \le c \\ \frac{(x-c)+\vartheta^i_{\tilde{\alpha}}(d-x)}{(d-c)}, & c < x \le d \\ 1, & \text{otherwise,} \end{cases}$$

and

$$F_{\tilde{\alpha}}^{i}(x)=\begin{cases}\dfrac{(b-x)+\theta_{A}^{i}(x-a)}{(b-a)}, & a\le x<b\\ \theta_{\tilde{\alpha}}^{i}, & b\le x\le c\\ \dfrac{(x-c)+\theta_{A}^{i}(d-x)}{(d-c)}, & c<x\le d\\ 1, & \text{otherwise,}\end{cases}$$

Note that the set of all NVNT-numbers on $\mathbb{R}$ will be denoted by Λ.

Definition 2.4 [17] Let $A_1=\langle(a_1,b_1,c_1,d_1);(\eta_{A_1}^{1},\eta_{A_1}^{2},\ldots,\eta_{A_1}^{P}),(\vartheta_{A_1}^{1},\vartheta_{A_1}^{2},\ldots,\vartheta_{A_1}^{P}),(\theta_{A_1}^{1},\theta_{A_1}^{2},\ldots,\theta_{A_1}^{P})\rangle\in\Lambda$. If A_1 is not normalized NVTN-number($a_1,b_1,c_1,d_1\notin[0,1]$), the normalized NVTN-number of A_1, denoted by $\overline{A}_1$ is given by;

$$\overline{A}_1=\langle\left[\frac{a_1}{a_1+b_1+c_1+d_1},\frac{b_1}{a_1+b_1+c_1+d_1},\frac{c_1}{a_1+b_1+c_1+d_1},\frac{d_1}{a_1+b_1+c_1+d_1}\right];\left(\eta_{A_1}^{1},\eta_{A_1}^{2},\ldots,\eta_{A_1}^{P}\right),\left(\vartheta_{A_1}^{1},\vartheta_{A_1}^{2},\ldots,\vartheta_{A_1}^{P}\right),(\theta_{A_1}^{1},\theta_{A_1}^{2},\ldots,\theta_{A_1}^{P})\rangle. \quad (2)$$

Definition 2.5 [80] Let $\overline{\mathcal{A}}=\langle[a_1,b_1,c_1,d_1];(\eta_{\overline{\mathcal{A}}}^{1},\eta_{\overline{\mathcal{A}}}^{2},\ldots,\eta_{\overline{\mathcal{A}}}^{P}),(\vartheta_{\overline{\mathcal{A}}}^{1},\vartheta_{\overline{\mathcal{A}}}^{2},\ldots,\vartheta_{\overline{\mathcal{A}}}^{P}),(\theta_{\overline{\mathcal{A}}}^{1},\theta_{\overline{\mathcal{A}}}^{2},\ldots,\theta_{\overline{\mathcal{A}}}^{P})\rangle$ and $\overline{\mathcal{B}}=\langle[a_2,b_2,c_2,d_2];(\eta_{\overline{\mathcal{B}}}^{1},\eta_{\overline{\mathcal{B}}}^{2},\ldots,\eta_{\overline{\mathcal{B}}}^{P}),(\vartheta_{\overline{\mathcal{B}}}^{1},\vartheta_{\overline{\mathcal{B}}}^{2},\ldots,\vartheta_{\overline{\mathcal{B}}}^{P}),(\theta_{\overline{\mathcal{B}}}^{1},\theta_{\overline{\mathcal{B}}}^{2},\ldots,\theta_{\overline{\mathcal{B}}}^{P})\rangle$ be two normalized NVNT-numbers then, respectively, the weighted Hamming and Euclidean distance measures between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are given below;

$$\begin{aligned}d_r^{w}(\overline{\mathcal{A}},\overline{\mathcal{B}})=\ &\frac{1}{16p}.\Big(\sum_{i=1}^{p}\Big[\left(|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})a_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})a_2|\right)^{r}+\\&\left(|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})b_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})b_2|\right)^{r}+\\&\left(|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})c_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})c_2|\right)^{r}+\\&\left(|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})d_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})d_2|\right)^{r}\Big]\Big)^{\frac{1}{r}}\end{aligned} \quad (3)$$

For r=1, the equation 3 is given as;

$$\begin{aligned}d_1^{w}(\overline{\mathcal{A}},\overline{\mathcal{B}})=\ &\frac{1}{16p}.\sum_{i=1}^{p}\Big[|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})a_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})a_2|+\\&|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})b_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})b_2|+\\&|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})c_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})c_2|+\\&|w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})d_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})d_2|\Big]\end{aligned} \quad (4)$$

For r=2, the equation 3 is given as;

$$\begin{aligned}d_2^{w}(\overline{\mathcal{A}},\overline{\mathcal{B}})=\ &\frac{1}{16p}.\sum_{i=1}^{p}\Big[\left(w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})a_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})a_2\right)^{2}+\\&\left(w_{\overline{\mathcal{A}}}(1+\eta_{\overline{\mathcal{A}}}^{i}-\vartheta_{\overline{\mathcal{A}}}^{i}-\theta_{\overline{\mathcal{A}}}^{i})b_1-w_{\overline{\mathcal{B}}}(1+\eta_{\overline{\mathcal{B}}}^{i}-\vartheta_{\overline{\mathcal{B}}}^{i}-\theta_{\overline{\mathcal{B}}}^{i})b_2\right)^{2}+\end{aligned}$$

$$\left(w_{\overline{\mathcal{A}}}(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}})c_1-w_{\overline{\mathcal{B}}}(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}})c_2\right)^2+$$

$$\left.\left(w_{\overline{\mathcal{A}}}(1+\eta^{i}_{\overline{\mathcal{A}}}-\vartheta^{i}_{\overline{\mathcal{A}}}-\theta^{i}_{\overline{\mathcal{A}}})d_1-w_{\overline{\mathcal{B}}}(1+\eta^{i}_{\overline{\mathcal{B}}}-\vartheta^{i}_{\overline{\mathcal{B}}}-\theta^{i}_{\overline{\mathcal{B}}})d_2\right)^2\right]^{\frac{1}{2}} \tag{5}$$

Theorem 2.6 [80] Let $\overline{\mathcal{A}}=$
$\langle(a_1,b_1,c_1,d_1);(\eta^1_{\overline{\mathcal{A}}},\eta^2_{\overline{\mathcal{A}}},\dots,\eta^P_{\overline{\mathcal{A}}}),(\vartheta^1_{\overline{\mathcal{A}}},\vartheta^2_{\overline{\mathcal{A}}},\dots,\vartheta^P_{\overline{\mathcal{A}}}),(\theta^1_{\overline{\mathcal{A}}},\theta^2_{\overline{\mathcal{A}}},\dots,\theta^P_{\overline{\mathcal{A}}})\rangle$,
$\overline{\mathcal{B}}=\langle(a_2,b_2,c_2,d_2);(\eta^1_{\overline{\mathcal{B}}},\eta^2_{\overline{\mathcal{B}}},\dots,\eta^P_{\overline{\mathcal{B}}}),(\vartheta^1_{\overline{\mathcal{B}}},\vartheta^2_{\overline{\mathcal{B}}},\dots,\vartheta^P_{\overline{\mathcal{B}}}),(\theta^1_{\overline{\mathcal{B}}},\theta^2_{\overline{\mathcal{B}}},\dots,\theta^P_{\overline{\mathcal{B}}})\rangle$ and
$\overline{\mathcal{C}}=\langle(a_3,b_3,c_3,d_3);(\eta^1_{\overline{\mathcal{C}}},\eta^2_{\overline{\mathcal{C}}},\dots,\eta^P_{\overline{\mathcal{C}}}),(\vartheta^1_{\overline{\mathcal{C}}},\vartheta^2_{\overline{\mathcal{C}}},\dots,\vartheta^P_{\overline{\mathcal{C}}}),(\theta^1_{\overline{\mathcal{C}}},\theta^2_{\overline{\mathcal{C}}},\dots,\theta^P_{\overline{\mathcal{C}}})\rangle$ be three normalized NVNT- numbers.
Then, $d^w_r(\overline{\mathcal{A}},\overline{\mathcal{B}})$ satisfies the following properties:

v. $0\le d^w_r(\overline{\mathcal{A}},\overline{\mathcal{B}})\le 1$,

vi. $\overline{\mathcal{A}}=\overline{\mathcal{B}}\Rightarrow d^w_r(\overline{\mathcal{A}},\overline{\mathcal{B}})=0$,

vii. $d^w_r(\overline{\mathcal{A}},\overline{\mathcal{B}})=d^w_r(\overline{\mathcal{B}},\overline{\mathcal{A}})$,

viii. $d^w_r(\overline{\mathcal{A}},\overline{\mathcal{B}})\le d^w_r(\overline{\mathcal{A}},\overline{\mathcal{C}})+d^w_r(\overline{\mathcal{C}},\overline{\mathcal{B}})$.

Definition 2.7 [80] A real-valued function $\mathcal{E}_r:\mu\to[0,1]$ is called an entropy on NVNT-numbers if it satisfies the following properties:

$\boldsymbol{\mathcal{EP}1}$. $\mathcal{A}=\begin{cases}\langle[a,b,c,d];(1,1,\dots,1),(0,0,\dots,0),(0,0,\dots,0)\rangle\\ \langle[a,b,c,d];(0,0,\dots,0),(0,0,\dots,0),(1,1,\dots,1)\rangle\end{cases}\Rightarrow\mathcal{E}_r(\mathcal{A})=0;$

$\boldsymbol{\mathcal{EP}2}$. $\mathcal{E}_r(\mathcal{A})=\mathcal{E}_r(\mathcal{A}^c)$ for all $\mathcal{A}\in$ NVNT-numbers, where

$$\mathcal{A}^c=\langle[a,b,c,d];(\theta^1_{\mathcal{A}},\theta^2_{\mathcal{A}},\dots,\theta^P_{\mathcal{A}}),(1-\vartheta^1_{\mathcal{A}},1-\vartheta^2_{\mathcal{A}},\dots,1-\vartheta^P_{\mathcal{A}}),(\eta^1_{\mathcal{A}},\eta^2_{\mathcal{A}},\dots,\eta^P_{\mathcal{A}})\rangle.$$

$\boldsymbol{\mathcal{EP}3}$. $d_r(\mathcal{A},\mathcal{A}^-)=d_r(\mathcal{A},\mathcal{A}^+)\Leftrightarrow\mathcal{E}_r(\mathcal{A})=1$ for all $\mathcal{A}\in$ NVNT-numbers, where $d_r(\mathcal{A},\mathcal{A}^+)$ is a distance from $\mathcal{A}$ to $\mathcal{A}^+$ and $d_r(\mathcal{A},\mathcal{A}^-)$ is a distance from $\mathcal{A}$ to $\mathcal{A}^-$;

$\boldsymbol{\mathcal{EP}4}$. For all $\mathcal{A},\mathcal{B}\in$ NVNT-numbers, if

$$\left|\frac{d_r(\mathcal{A},\mathcal{A}^-)}{d_r(\mathcal{A},\mathcal{A}^+)+d_r(\mathcal{A},\mathcal{A}^-)}-\frac{1}{2}\right|\ge\left|\frac{d_r(\mathcal{B},\mathcal{B}^-)}{d_r(\mathcal{B},\mathcal{B}^+)+d_r(\mathcal{B},\mathcal{B}^-)}-\frac{1}{2}\right| \tag{6}$$

then $\mathcal{E}(\mathcal{A})\le\mathcal{E}(\mathcal{B})$, where $d_r(\mathcal{B},\mathcal{B}^+)$ is a distance from $\mathcal{B}$ to $\mathcal{B}^+$ and $d_r(\mathcal{B},\mathcal{B}^-)$ is a distance from $\mathcal{B}$ to $\mathcal{B}^-$, where

$$\mathcal{A}^+=\langle[a,b,c,d];(1,1,\dots,1),(0,0,\dots,0),(0,0,\dots,0)\rangle$$

and

$$\mathcal{A}^-=\langle[a,b,c,d];(0,0,\dots,0),(1,1,\dots,1),(1,1,\dots,1)\rangle.$$

Theorem 2.8 [80] Assume that d_r is an distance measure for NVNT-numbers. Then, for any $\mathcal{A}\in$ NVNT-numbers,

$$\mathcal{E}_r(\mathcal{A})=1-2\left|\frac{d_r(\mathcal{A},\mathcal{A}^-)}{d_r(\mathcal{A},\mathcal{A}^+)+d_r(\mathcal{A},\mathcal{A}^-)}-\frac{1}{2}\right| \tag{7}$$

is entropy of NVNT-numbers based on TOPSIS.

5. Presentation of VIKOR and TOPSIS Methods in NVTN-numbers Version

Assume that $D = \{D_1, D_2, \dots, D_m\}$ be the set of altenatives and $Z = \{z_1, z_2, \dots, z_n\}$ be the set of criterias. In Deli et al. [17], the normalized NVNT-numbers decision matrix is given as;

$$(D_{kj})_{mxn} = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mn} \end{pmatrix}$$

such that

$D_{kj} = \langle [a_{kj}, b_{kj}, c_{kj}, d_{kj}], \left(\eta^1_{kj}, \eta^2_{kj}, \eta^3_{kj}, \dots, \eta^p_{kj}\right), \left(\vartheta^1_{kj}, \vartheta^2_{kj}, \vartheta^3_{kj}, \dots, \vartheta^p_{kj}\right), \left(\theta^1_{kj}, \theta^2_{kj}, \theta^3_{kj}, \dots, \theta^p_{kj}\right)\rangle$, (k=1,2,…,m) and (j=1,2,…,n).

It is carried out the following algorithm to get best choice:

3.1 NVNT-numbers VIKOR method [8]

VIKOR Algorithm:

Step 1: Create an evaluation matrix $(D_{kj})_{mxn}$, (k=1,2,…,m; j=1,2,…,n)

Step 2: Find of the weights of the criteria vector $w = \{w_1, w_2, \dots, w_n\}$ by using equation in Theorem 2.6 as;

$$w_j = \frac{m - \sum_{k=1}^{m} \mathcal{E}_{kj}}{m.n - \sum_{k=1}^{m}\sum_{j=1}^{n} \mathcal{E}_{kj}}, \quad (j = 1,2, \dots, n).$$

where the entropy matrix $(\mathcal{E}_{kj})_{mxn}$ (k=1,2,…,m; j=1,2,…,n) of the decision matrix $(D_{kj})_{mxn}$ and where

$$\mathcal{E}_{kj} = 1 - 2\left|\frac{d_r(D_{kj}, {D_{kj}}^-)}{d_r(D_{kj}, {D_{kj}}^+) + d_r(D_{kj}, {D_{kj}}^-)} - \frac{1}{2}\right|$$

$(k = 1,2, \dots, m;\ j = 1,2, \dots, n)$.

Note that if the entropy matrix $(\mathcal{E}_{kj})_{mxn}$ (k=1,2,…,m; j=1,2,…,n) is not normalized then, the entropy matrix must be normalized as;

$$\bar{\mathcal{E}}_{kj} = \frac{\mathcal{E}_{kj}}{\max\{\mathcal{E}_{kj} : \mathcal{E}_{kj} \in (\mathcal{E}_{kj})_{mxn}, k = 1,2, \dots, m \text{ and } j = 1,2, \dots, n\}}.$$

Step 3: Determine the positive ideal solution r^+ and negative ideal solution r^-, respectively as;

$$r^{+} = \langle [a_{kj}^{+}, b_{kj}^{+}, c_{kj}^{+}, d_{kj}^{+}]; \left(\eta_{kj}^{1\,+}, \eta_{kj}^{2\,+}, \eta_{kj}^{3\,+}, \ldots, \eta_{kj}^{p\,+}\right), \left(\vartheta_{kj}^{1\,+}, \vartheta_{kj}^{2\,+}, \vartheta_{kj}^{3\,+}, \ldots, \vartheta_{kj}^{p\,+}\right), \left(\theta_{kj}^{1\,+}, \theta_{kj}^{2\,+}, \theta_{kj}^{3\,+}, \ldots, \theta_{kj}^{p\,+}\right) \rangle$$

$$= \left\langle \left[\max_k\{a_{kj}\}, \max_k\{b_{kj}\}, \max_k\{c_{kj}\}, \max_k\{d_{kj}\}\right]; \left(\max_k\{\eta_{kj}^1\}, \max_k\{\eta_{kj}^2\}, \max_k\{\eta_{kj}^3\}, \ldots, \max_k\left\{\eta_{kj}^p\right\}\right) \left(\min_k\{\vartheta_{kj}^1\}, \min_k\{\vartheta_{kj}^2\}, \min_k\{\vartheta_{kj}^3\}, \ldots, \min_k\left\{\vartheta_{kj}^p\right\}\right), \left(\min_k\{\theta_{kj}^1\}, \min_k\{\theta_{kj}^2\}, \min_k\{\theta_{kj}^3\}, \ldots, \min_k\left\{\theta_{kj}^p\right\}\right)\right\rangle.$$

and

$$r^{-} = \langle [a_{kj}^{-}, b_{kj}^{-}, c_{kj}^{-}, d_{kj}^{-}]; \left(\eta_{kj}^{1\,-}, \eta_{kj}^{2\,-}, \eta_{kj}^{3\,-}, \ldots, \eta_{kj}^{p\,-}\right), \left(\vartheta_{kj}^{1\,-}, \vartheta_{kj}^{2\,-}, \vartheta_{kj}^{3\,-}, \ldots, \vartheta_{kj}^{p\,-}\right), \left(\theta_{kj}^{1\,-}, \theta_{kj}^{2\,-}, \theta_{kj}^{3\,-}, \ldots, \theta_{kj}^{p\,-}\right) \rangle$$

$$= \left\langle \left[\min_k\{a_{kj}\}, \min_k\{b_{kj}\}, \min_k\{c_{kj}\}, \min_k\{d_{kj}\}\right]; \left(\min_k\{\eta_{kj}^1\}, \min_k\{\eta_{kj}^2\}, \min_k\{\eta_{kj}^3\}, \ldots, \min_k\left\{\eta_{kj}^p\right\}\right), \left(\max_k\{\vartheta_{kj}^1\}, \max_k\{\vartheta_{kj}^2\}, \max_k\{\vartheta_{kj}^3\}, \ldots, \max_k\left\{\vartheta_{kj}^p\right\}\right), \left(\max_k\{\theta_{kj}^1\}, \max_k\{\theta_{kj}^2\}, \max_k\{\theta_{kj}^3\}, \ldots, \max_k\left\{\theta_{kj}^p\right\}\right)\right\rangle$$

for all (k=1,2,…,m) and (j=1,2,…,n).

Step 4: According to Equation (3) positive value of $V^{+}(D_{kj})$ based on positive ideal solution r^{+} and negative value of $V^{-}\left(D_{kj}\right)$ based on negative ideal solution r^{-} of alternative D_k (k=1,2,…,m) calculated as follows:

$$V^{+}(D_{kj}) = \frac{1}{n}\sum_{j=1}^{n} d_r^w(D_{kj}, r^{+})$$

$$= \frac{1}{16.n.p}.\sum_{j=1}^{n}\sum_{k=1}^{p}\Big[\left(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)a_{kj} - w_{r^+}(1+\eta_{kj}^{i\,+}-\vartheta_{kj}^{i\,+}-\theta_{kj}^{i\,+})a_{kj}\right)^r + \left(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)b_{kj} - w_{r^+}(1+\eta_{kj}^{i\,+}-\vartheta_{kj}^{i\,+}-\theta_{kj}^{i\,+})b_{kj}\right)^r + \left(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)c_{kj} - w_{r^+}(1+\eta_{kj}^{i\,+}-\vartheta_{kj}^{i\,+}-\theta_{kj}^{i\,+})c_{kj}\right)^r + \left(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)d_{kj} - w_{r^+}(1+\eta_{kj}^{i\,+}-\vartheta_{kj}^{i\,+}-\theta_{kj}^{i\,+})d_{kj}\right)^r\Big]^{\frac{1}{r}}$$

and

$$V^{-}(D_{kj}) = \frac{1}{n}\sum_{j=1}^{n} d_r^w(D_{kj}, r^{-})$$

$$\frac{1}{16.n.p}.\sum_{j=1}^{n}\sum_{k=1}^{p}\Big[\left(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)a_{kj} - w_{r^-}(1+\eta_{kj}^{i\,-}-\vartheta_{kj}^{i\,-}-\theta_{kj}^{i\,-})a_{kj}\right)^r$$

$$\left(w_{\mathrm{D_{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{b_{kj}}-w_{r^-}(1+\eta_{\mathrm{kj}}^{i\;-}-\vartheta_{\mathrm{kj}}^{i\;-}-\theta_{\mathrm{kj}}^{i\;-})\mathrm{b_{kj}}\right)^{r}+$$
$$\left(w_{\mathrm{D_{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{c_{kj}}-w_{r^-}(1+\eta_{\mathrm{kj}}^{i\;-}-\vartheta_{\mathrm{kj}}^{i\;-}-\theta_{\mathrm{kj}}^{i\;-})\mathrm{c_{kj}}\right)^{r}+$$
$$\left.\left(w_{\mathrm{D_{kj}}}(1+\eta_{\mathrm{kj}}^{i}-\vartheta_{\mathrm{kj}}^{i}-\theta_{\mathrm{kj}}^{i})\mathrm{d_{kj}}-w_{r^-}(1+\eta_{\mathrm{kj}}^{i\;-}-\vartheta_{\mathrm{kj}}^{i\;-}-\theta_{\mathrm{kj}}^{i\;-})\mathrm{d_{kj}}\right)^{r}\right]^{\frac{1}{r}}$$

where $w_{r^+}=\max\{w_{\mathrm{j}}:\mathrm{j}=1,2,\dots,\mathrm{n}\}$ and $w_{r^-}=\min\{w_{\mathrm{j}}:\mathrm{j}=1,2,\dots,\mathrm{n}\}$.

Step 5: Compute the group utility δ_k values for the maximum and individual regret σ_k values for the opponent

$$\delta_k=\sum_{j=1}^{n}(w_{r^+})\frac{\mathrm{d}_r^{\mathrm{w}}(\mathrm{D_{kj}},r^+)}{\mathrm{d}_r^{\mathrm{w}}(r^-,r^+)}$$

$$\sigma_k=max\left\{\frac{\mathrm{d}_r^{\mathrm{w}}(\mathrm{D_{kj}},r^+)}{\mathrm{d}_r^{\mathrm{w}}(r^-,r^+)}\right\}$$

Step 6: Compute the index values θ_i as follows;

$$\theta_k=\rho.\left(\frac{\delta_k-\delta^-}{\delta^++\delta^-}\right)+(1-\rho).\left(\frac{\sigma_k-\sigma^-}{\sigma^++\sigma^-}\right), \tag{8}$$

where $\delta^+=\min\delta_k$, $\delta^-=\max\delta_k, \sigma^+=\min\sigma_k$ and $\sigma^-=\max\sigma_k$. Here ρ denotes decision-making mechanism coefficient.

d. θ_k is the minimal if $\rho<0.5$,

e. θ_k is the maximum if $\rho>0.5$,

f. θ_k is both minimal and maximum if $\rho=0.5$.

Step 7: Rank the all alternatives by sorting δ,σ,θ values in decreasing order. Thus the result is a set of three ranking list denoted by $\delta_{[k]}$, $\sigma_{[k]}$, $\theta_{[k]}$.

Consider the alternative k, corresponding to $\theta_{[k]}$ (smallest among $\theta_{[k]}$ values) as a compromise solution if the following two conditions are satisfied.

(A1) Feasible benefit:

If top most two alternatives in $\theta_{[k]}$ are $[D_2]$ and $[D_1]$ then

$$\theta([D_2])-\theta([D_1])\geq\frac{1}{m-1}$$

where m stands for the cardinality of the set of attributes.

(A2) Acceptable stability:

The choice $[D_k]$ must be top ranked by at least one of $\delta_{[k]}$ and $\sigma_{[k]}$. If one of the condition is not satisfied then a set of compromise solution is proposed, which consist of,

c. If only (A1) is met then both alternatives $D_{[1]}$ and $D_{[2]}$ will serve as the compromise solution.

d. If (A1) is not met then there will be a series of compromise solutions, which are alternatives may be located by making use of

$$\theta([D_m]) - \theta([D_1]) \geq \frac{1}{m-1}$$

for the maximum m.

The minimal value of θ determines the best alternative.

3.2 **NVNT-numbers TOPSIS** [80]

In this section, we give a multi-criteria decision-making method with normalized NVNT-numbers.

Algorithm:

Step 3: Similar to above VIKOR method from Step1 to Step-3.

Step 4: According to Equation (3) positive value of $V^+(D_k)$ based on positive ideal solution r^+ and negative value of $V^-(D_k)$ based on negative ideal solution r^- of alternative D_k (k=1,2,…,m) calculated as follows:

$$\begin{aligned} V^+(D_k) = {} & \frac{1}{n}\sum_{j=1}^{n} d_r^w(D_{kj}, r^+) \\ = {} & \frac{1}{16.n.p}.\sum_{j=1}^{n}\sum_{k=1}^{p}\Big[\Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)a_{kj} - w_{r^+}(1+\eta_{kj}^{i\;+}-\vartheta_{kj}^{i\;+}-\theta_{kj}^{i\;+})a_{kj}\Big)^r \\ & +\Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)b_{kj} - w_{r^+}(1+\eta_{kj}^{i\;+}-\vartheta_{kj}^{i\;+}-\theta_{kj}^{i\;+})b_{kj}\Big)^r + \\ & \Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)c_{kj} - w_{r^+}(1+\eta_{kj}^{i\;+}-\vartheta_{kj}^{i\;+}-\theta_{kj}^{i\;+})c_{kj}\Big)^r + \\ & \Big(w_{K_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)d_{kj} - w_{r^+}(1+\eta_{kj}^{i\;+}-\vartheta_{kj}^{i\;+}-\theta_{kj}^{i\;+})d_{kj}\Big)^r\Big]^{\frac{1}{r}} \end{aligned}$$

and

$$\begin{aligned} V^-(D_k) = {} & \frac{1}{n}\sum_{j=1}^{n} d_r^w(D_{kj}, r^-) \\ & \frac{1}{16.n.p}.\sum_{j=1}^{n}\sum_{k=1}^{p}\Big[\Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)a_{kj} - w_{r^-}(1+\eta_{kj}^{i\;-}-\vartheta_{kj}^{i\;-}-\theta_{kj}^{i\;-})a_{kj}\Big)^r \\ & \Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)b_{kj} - w_{r^-}(1+\eta_{kj}^{i\;-}-\vartheta_{kj}^{i\;-}-\theta_{kj}^{i\;-})b_{kj}\Big)^r + \\ & \Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)c_{kj} - w_{r^-}(1+\eta_{kj}^{i\;-}-\vartheta_{kj}^{i\;-}-\theta_{kj}^{i\;-})c_{kj}\Big)^r + \\ & \Big(w_{D_{kj}}(1+\eta_{kj}^i-\vartheta_{kj}^i-\theta_{kj}^i)d_{kj} - w_{r^-}(1+\eta_{kj}^{i\;-}-\vartheta_{kj}^{i\;-}-\theta_{kj}^{i\;-})d_{kj}\Big)^r\Big]^{\frac{1}{r}} \end{aligned}$$

where $w_{r^+} = \max\{w_j : j = 1,2,\dots,n\}$ and $w_{r^-} = \min\{w_j : j = 1,2,\dots,n\}$.

Step 5: Calculate the relative closeness degree CC_k of each alternative D_k $(k = 1,2,\dots,m)$ as;

$$\mathrm{CC_k} = \frac{\mathrm{V}^-(D_\mathrm{k})}{\mathrm{V}^-(D_\mathrm{k}) + \mathrm{V}^+(D_\mathrm{k})}, \qquad \mathrm{k} = 1,2,\dots,\mathrm{m}. \tag{9}$$

Step 6: Rank all alternatives according to $\mathrm{CC_k}(\mathrm{k} = 1,2,\dots,\mathrm{m})$ in decreasing order and determine the best alternatives.

6. Application and Comparision

Example 4.1. The circular economy is defined as an economic approach in which the value of products, materials and resources is kept in the economy as long as possible and the amount of waste is the lowest. The circular economy concept is based on a restorative industrial economy, the transition to renewable energy, the reduction of the use of toxic chemicals and the prevention of waste. This concept aims to redefine the production and consumption processes. The circular economy is based on three key elements that focus on both system and resource problems. Depending on the Industrial Revolution, the ever-developing technology and urban population growth bring along the demand for unplanned urbanization and ready-made consumption, causing a significant increase in urban waste. This process leads to many negative environmental effects, especially the depletion of natural resources and climate change Dindar [19]. Therefore, we want to use the two methods by comparing them to choose the country that does the best among the countries that implement the circular economy. Based on this, we want to implement it in our own country. That is, using which is the set of altenatives as $\mathrm{D} = \{\mathrm{k_1} = \mathrm{TURKEY},\ \mathrm{k_2} = \mathrm{ABD}, \mathrm{k_3} = \mathrm{France},\ \mathrm{k_4} = \mathrm{Germany},\ \mathrm{k_5} = \mathrm{Italy}\}$ and according to three criteria determined $\mathrm{Z} = \{\mathrm{z_1} = \text{To protect and develop natural capital}, \mathrm{z_2} = \text{Optimizing resource efficiency}, \mathrm{z_3} = \text{Maintain system efficiency}$. Then we try to choose and rank all alternatives $\mathrm{D_k}$ for all k=1,2,…,5 by using the following algorithm.

RANKING OF EACH ALTERNATIVE USING VIKOR

Step 1: The evaluation matrix$(\mathrm{D_{kj}})_{5x3}$ is given by an expert as;

$$(\mathrm{D_{kj}})_{5x3} = \begin{matrix} \mathrm{k_1} \\ \mathrm{k_2} \\ \mathrm{k_3} \\ \mathrm{k_4} \\ \mathrm{k_5} \end{matrix} \begin{pmatrix} \langle[0.23,0.25,0.54,0.64]; (0.2,0.6,0.7,0.8), (0.3,0.4,0.5,0.6), (0.2,0.3,0.5,0.6)\rangle & \langle[0.12,0.32,0.42,062]; (0.5,0.3,0.4,0.6), (0.2,0.1,0.5,0.4), (0.4,0.6,0.5,0.7)\rangle & \langle[0.15,0.35,0.45,0.61]; (0.1,0.6,0.8,0.9), (0.2,0.3,0.8,0.5), (0.3,0.5,0.6,0.4)\rangle \\ \langle[0.62,0.72,0.82,0.90]; (0.4,0.2,0.3,0.5), (0.1,0.2,0.3,0.4), (0.7,0.5,0.6,0.8)\rangle & \langle[0.05,0.64,0.77,0.97]; (0.5,0.6,0.8,0.8), (0.1,0.3,0.4,0.7), (0.3,0.4,0.7,0.5)\rangle & \langle[0.35,0.43,0.57,0.85]; (0.3,0.6,0.8,0.9), (0.5,0.7,0.7,0.6), (0.4,0.6,0.8,0.6)\rangle \\ \langle[0.62,0.72,0.83,0.85]; (0.3,0.4,0.5,0.8), (0.2,0.3,0.5,0.6), (0.4,0.3,0.4,0.5)\rangle & \langle[0.05,0.06,0.07,0.08]; (0.7,0.6,0.8,0.9), (0.1,0.2,0.5,0.3), (0.4,0.3,0.6,0.6)\rangle & \langle[0.07,0.08,0.09,0.11]; (0.2,0.6,0.9,0.5), (0.8,0.5,0.6,0.7), (0.1,0.3,0.5,0.8)\rangle \\ \langle[0.17,0.23,0.63,0.92]; (0.8,0.7,0.5,0.6), (0.4,0.3,0.7,0.5), (0.1,0.5,0.7,0.7)\rangle & \langle[0.01,0.006,0.007,0.58]; (0.2,0.6,0.7,0.9), (0.8,0.7,0.6,0.5), (0.1,0.1,0.4,0.3)\rangle & \langle[0.63,0.73,0.83,0.93]; (0.2,0.6,0.7,0.9), (0.8,0.7,0.6,0.5), (0.3,0.6,0.7,0.2)\rangle \\ \langle[0.63,0.73,0.83,0.98]; (0.7,0.8,0.9,0.9), (0.2,0.7,0.5,0.6), (0.1,0.7,0.8,0.4)\rangle & \langle[0.02,0.03,0.04,0.06]; (0.2,0.5,0.7,0.9), (0.1,0.3,0.5,0.4), (0.6,0.7,0.8,0.8)\rangle & \langle[0.41,0.43,0.68,0.74]; (0.5,0.7,0.8,0.3), (0.1,0.2,0.6,0.4), (0.4,0.2,0.9,0.7)\rangle \end{pmatrix}$$

Step 2: Since the normalized entropy matrix is

$$(\mathcal{E}_{kj})_{5\times 3} = \begin{pmatrix} 0.692281 & 0.803063 & 0.885496 \\ 0.621053 & 0.573171 & 0.870410 \\ 0.719292 & 0.471153 & 0.968321 \\ 0.532164 & 0.716684 & 0.394200 \\ 0.609610 & 0.966851 & 0.760479 \end{pmatrix}_{5\times 3}$$

we have calculated the weights of the criteria w= (w_1, w_2, w_3) as;

$$w_2 = \frac{5 - (\mathcal{E}_{11} + \mathcal{E}_{21} + \mathcal{E}_{31} + \mathcal{E}_{41} + \mathcal{E}_{51})}{15 - (\mathcal{E}_{11} + \mathcal{E}_{12} + \mathcal{E}_{13} + \mathcal{E}_{21} + \mathcal{E}_{22} + \mathcal{E}_{23} + \mathcal{E}_{31} + \mathcal{E}_{32} + \cdots + \mathcal{E}_{42} + \mathcal{E}_{43} + \mathcal{E}_{51} + \mathcal{E}_{52} + \mathcal{E}_{53})}$$

$$= \frac{5 - (0.803063 + 0.573171 + 0.471153 + 0.716684 + 0.966851)}{15 - (0.561587 + 0.869281 + 0.848676 + 0.621053 + \cdots + 0.60961 + 0.966851 + 0.760479)}$$

$= 0.332727274$

similarly we have $w_1 = 0.413354514$, and $w_3 = 0.253918212$.

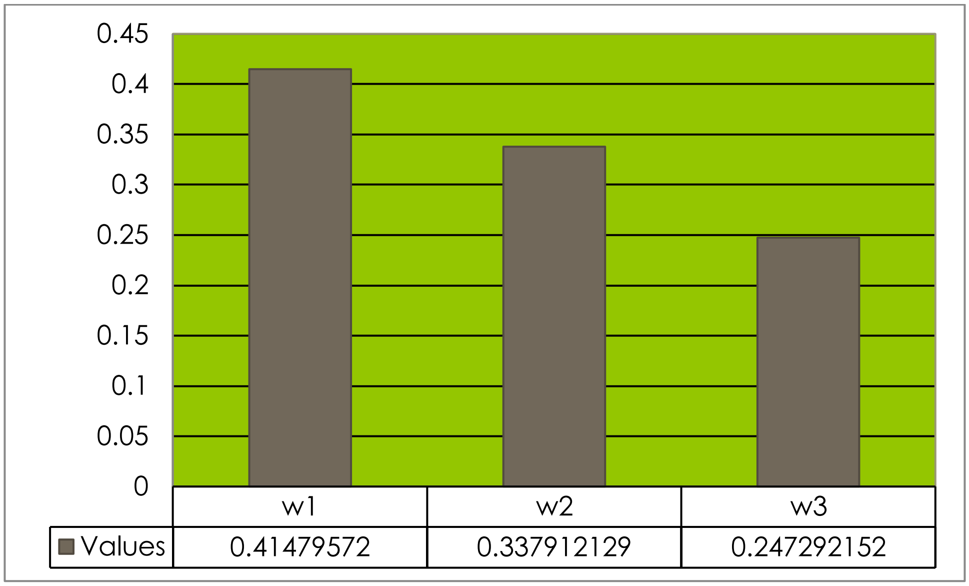

Figure 1: Weights of the criteria by normalized NVNT- numbers

Step 3: The positive ideal solution r^+ and negative ideal solution r^-, respectively calculated as;

$$r^+ = \langle [0.63,0.73,0.83,0.97]; (0.8,0.9,0.9,0.9), (0.1,0.1,0.2,0.3), (0.1,0.1,0.4,0.2) \rangle$$

and

$$r^- = \langle [0.01,0.006,0.007,0.11]; (0.1,0.2,0.3,0.3), (0.8,0.7,0.8,0.7), (0.7,0.7,0.9,0.8) \rangle.$$

Step 4: According to Equation (3) positive value of $V^+(D_{kj})$ based on positive ideal solution r^+ and negative value of $V^-(D_{kj})$ based on negative ideal solution r^- of alternative k_k (k=1,2,…,5) calculated as follows:

$$V^+(D_{11}) = 0.0148,\ V^+(D_{12}) = 0.0157, V^+(D_{13}) = 0.0152, V^+(D_{21}) = 0.0170, V^+(D_{22}) = 0.0118$$

$$V^+(D_{23}) = 0.0176,\ V^+(D_{31}) = 0.0334, V^+(D_{32}) = 0.1071, V^+(D_{33}) = 0.0495, V^+(D_{41}) = 0.0127$$

$$V^+(D_{42}) = 0.0151,\ V^+(D_{43}) = 0.0126, V^+(D_{51}) = 0.0105, V^+(D_{52}) = 0.0193, V^+(D_{53}) = 0.0134.$$

$$d_r^w(r^-, r^+) = 0.0745$$

Step 5: Computed the group utility δ_k (k=1,2,…,5) values for the maximum and individual regret σ_k values for the opponent

$$\delta_1 = \sum_{j=1}^{n} (w_{r^+}) \frac{\mathrm{V}^+(\mathrm{F_{kj}})}{\mathrm{d}_r^{\mathrm{w}}(r^-, r^+)}$$

$$= (w_{r^+}) \frac{\mathrm{V}^+(\mathrm{D}_{11}) + \mathrm{V}^+(\mathrm{D}_{12}) + \mathrm{V}^+(\mathrm{D}_{13})}{\mathrm{d}_r^{\mathrm{w}}(r^-, r^+)}$$

$$= (0.383558117) \frac{0.2137 + 0.0157 + 0.0152}{0.0711}$$

$$= 0.265606$$

Similar to

$\delta_2 = 0.252026, \delta_3 = 1.105034, \delta_4 = 0.249513, \delta_5 = 0.251712.$

$$\sigma_k = max \left\{ \frac{\mathrm{d}_r^{\mathrm{w}}(\mathrm{D_{kj}}, r^+)}{\mathrm{d}_r^{\mathrm{w}}(r^-, r^+)} \right\}$$

$$\sigma_1 = max\{0.0148, 0.0157, 0.0152\} = 0.0157$$

$$\sigma_2 = max\{0.0170, 0.0118, 0.0145\} = 0.0170$$

$$\sigma_3 = max\{0.0334, 0.1071, 0.0495\} = 0.1071$$

$$\sigma_4 = max\{0.0127, 0.0176, 0.0126\} = 0.0176$$

$$\sigma_5 = max\{0.0132, 0.0193, 0.0199\} = 0.0199.$$

Step 6: Let $\rho = 0.5$, compute the index values θ_k as follows;

$$\theta_k = \rho. \left(\frac{\delta_k - \delta^-}{\delta^+ + \delta^-} \right) + (1 - \rho). \left(\frac{\sigma_k - \sigma^-}{\sigma^+ + \sigma^-} \right),$$

$$\theta_1 = (0.5). \left(\frac{0.265606 - 0.249513}{1.105034 + 0.249513} \right) + (1 - 0.5). \left(\frac{0.0157 - 0.0157}{0.1071 + 0.0157} \right)$$

$$= 0.00941$$

Similar to

$$\theta_2 = 0.088, \theta_3 = 1, \theta_4 = 0.01072, \ \theta_5 = 0.02112$$

Step 7: Ranking the all alternatives NVNT-numbers based on VIKOR method, $\mathrm{D_k}(k = 1,2,\ldots,5)$ are shown in Figure 2 and given as;

$$\mathrm{D}_3 > \mathrm{D}_5 > \mathrm{D}_4 > \mathrm{D}_1 > \mathrm{D}_2.$$

Finally the best alternative is D_3.

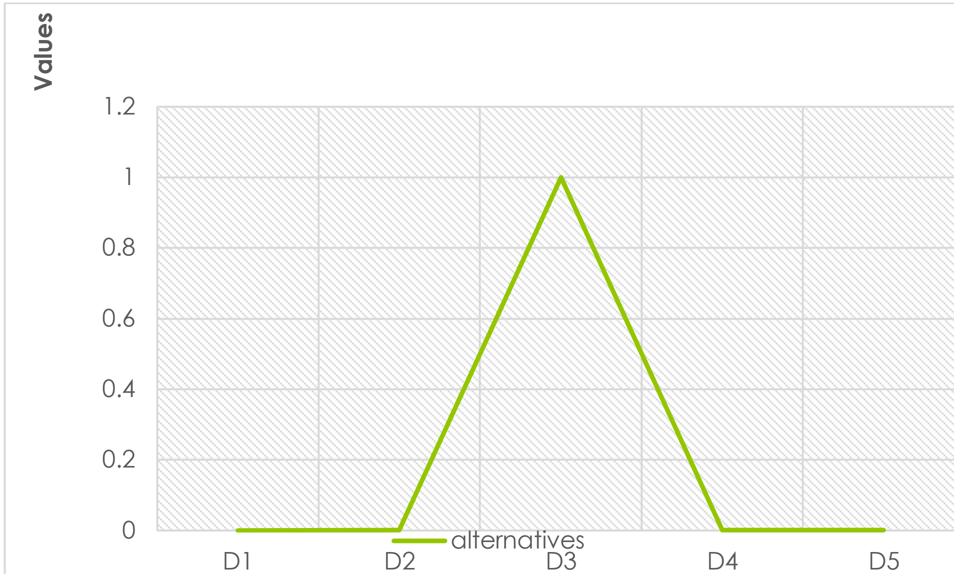

Figure 2 The ranking of alternatives D_k $(k = 1,2,\ldots,5)$

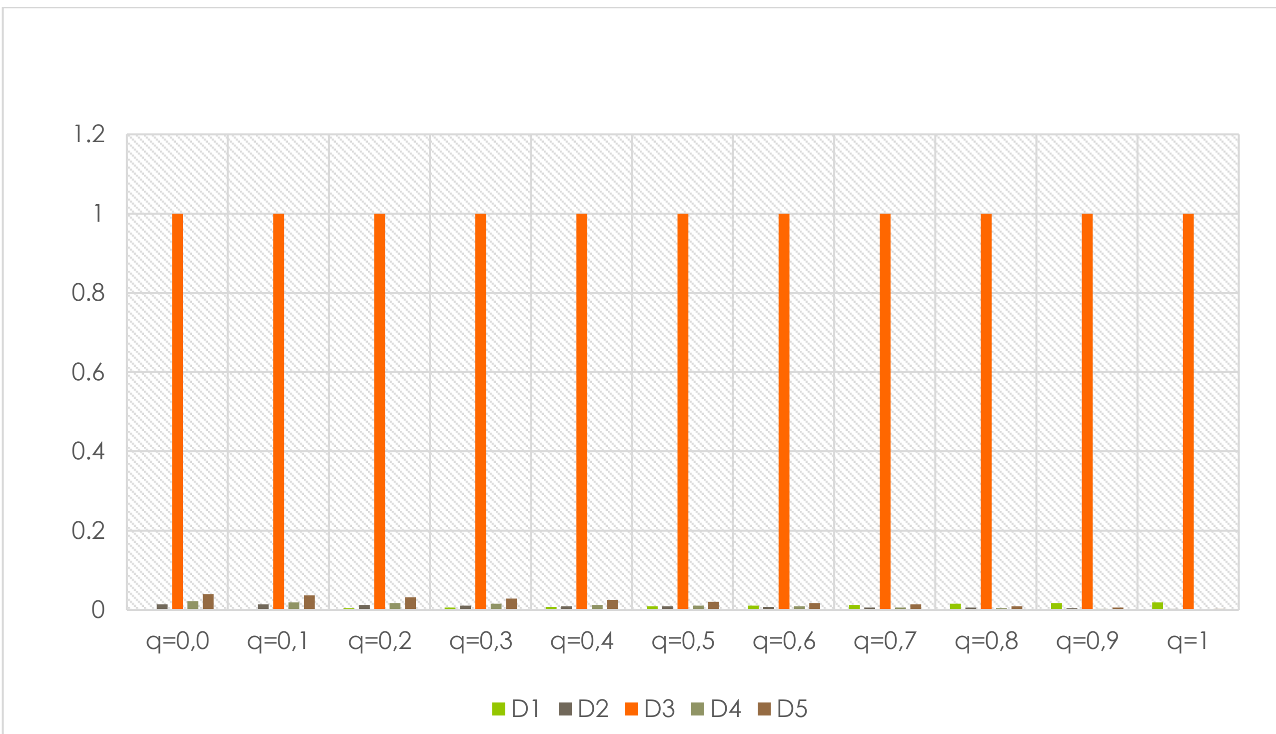

Figure 3: VIKOR index for all ρ values

The results from the different distance measures used to resolve the MCDM problem in section 4 are shown in Figure 4

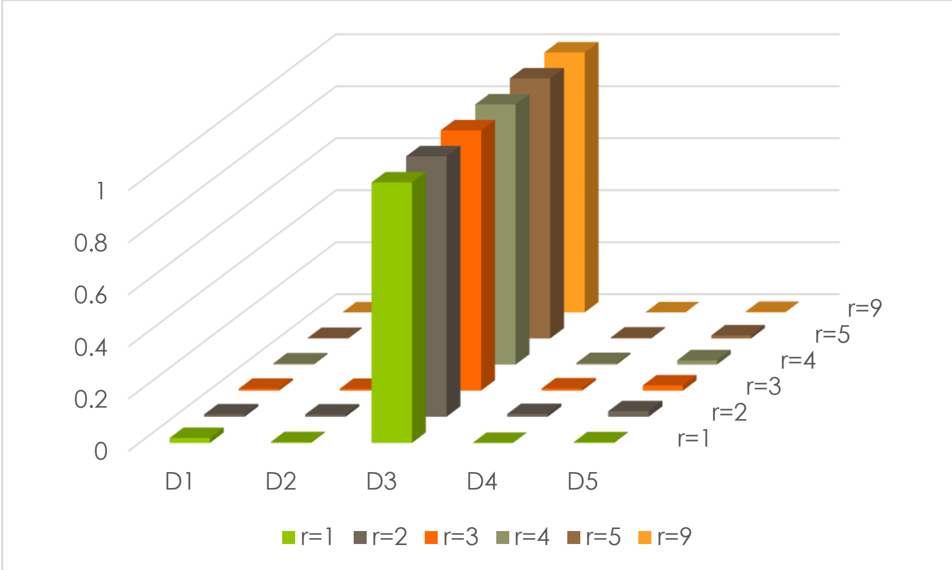

Figure 4: The results from the different distance measures

RANKING OF EACH ALTERNATIVE USING TOPSIS

Step 3: Similar to above VIKOR method from Step1 to Step-3.

Step 4: According to Equation (3) positive value of $V^+(D_k)$ based on positive ideal solution r^+ and negative value of $V^-(D_k)$ based on negative ideal solution r^- of alternative D_k (k=1,2,…,5) calculated as follows:

$$V^+(D_1) = 0.2145,\ V^+(D_2) = 0.2090, V^+(D_3) = 0.1457, V^+(D_4) = 0.2080, V^+(D_5) = 0.2089$$

$$V^-(D_1) = 0.0322, V^-(D_2) = 0.2251, V^-(D_3) = 0.0410, V^-(D_4) = 0.0877, V^-(D_5) = 0.0517$$

Step 5: We calculated the relative closeness degree CC_k of each alternative D_k $(k = 1,2,\ldots,5)$ as;

$$CC_1 = 0.1716, \quad CC_2 = 0.3650, \quad CC_3 = 0.3518, \quad CC_4 = 0.3183, \quad CC_5 = 0.1749$$

Step 6: Based on the index values $CC_k(k = 1,2,\ldots,5)$ the ranking of alternatives $D_k(k = 1,2,\ldots,5)$ are shown in Figure 5 and given as;

$$D_2 > D_4 > D_3 > D_5 > D_1.$$

Finally the best alternative is D_2.

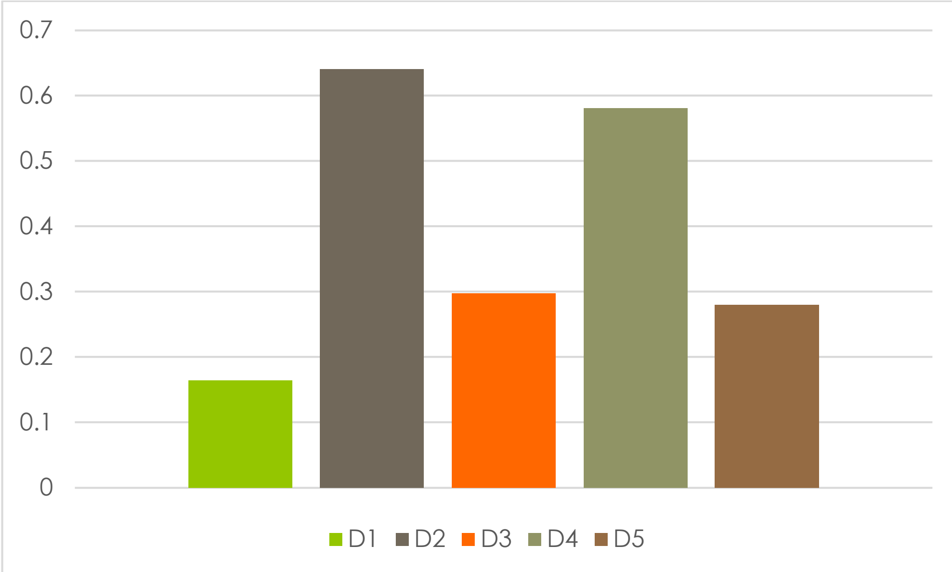

Figure 5 The ranking of alternatives D_k based on CC_k $(k = 1,2,\ldots,5)$

The results from the different distance measures used to resolve the MCDM problem in section 4 are shown in Figure 6

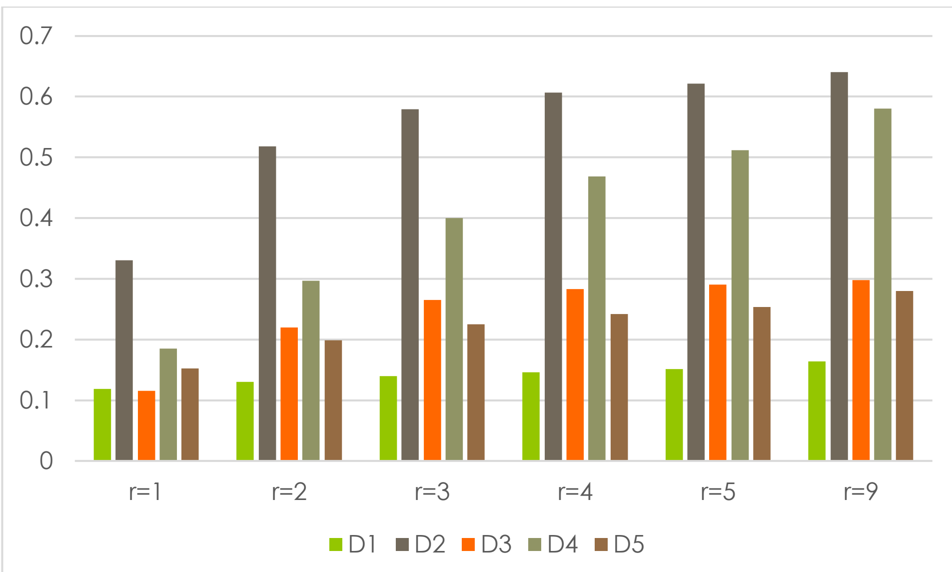

Figure 6: The results from the different distance measures

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Chapter Fourteen

Similarity Measures on N-Valued Fuzzy Numbers and Application to Multiple Attribute Decision Making Problems

İrfan DELİ [1]* and Mehmet Ali KELEŞ[2]

[1] Muallim Rıfat Faculty of Education, Kilis 7 Aralık University, 79000, Kilis, Turkey,
E-mail: irfandeli20@gmail.com,

[2] Faculty of Arts and Sciences, Kilis 7 Aralık University, 79000, Kilis, Turkey,
E-mail: kelesmehmetali36@gmail.com

Abstract: In this paper, some similarity measures of N-Valued Fuzzy Numbers **(**NVTF-numbers) by using α-cut sets/integral vector, values and ambiguities of NVTF-numbers are developed. Then some desired properties of NVTF-numbers are examined. Also, a multi attribute decision making method based on the defined similarity measures are developed. Finally, a medical diagnosis problem is given on NVTF-numbers.

Keywords: Fuzzy sets, N-valued trapezoidal fuzzy numbers, α-cut sets/integral vector, value and ambiguity of NVTF-numbers, multi attribute decision making method.

1. Introduction

A fuzzy set [28] is defined help of a function from universal set X to [0,1] to handle ambiguous and incomplete information. Fuzzy sets, especially fuzzy numbers, which are a fuzzy set on R real numbers, have study by many author in [1,3,22-29]. As a generalization of a fuzzy set, an N-valued fuzzy set (fuzzy multi-set) which an element can have more than one value in the range [0,1] was first developed by Yager [27]. After Yager, many studies have also been proposed many authors in [2,6-17].

Recently, Uluçay et al. [21] gave concept of the N-valued fuzzy numbers and Deli and Keleş [4] introduced their related concepts such as α-cut sets/integral vector, values and ambiguities. Later, various studies have also been done by many authors in [5,18-21]. Since there is not enough work in the literature on N-valued fuzzy numbers, in this study, firstly, we presented some basic definitions and operations of fuzzy sets, fuzzy numbers, N-valued fuzzy sets and N-valued trapezoidal fuzzy numbers (NVTF-numbers). Secondly,

* Corresponding author. Email address: irfandeli@kilis.edu.tr

some similarity measures of NVTF-numbers by using α-cut sets/integral vector, values and ambiguities of NVTF-numbers are introduced. Thirdly, a multi attribute decision making method based on the defined similarity measures is developed. Finally, a multi attribute decision-making problem on NVTF-numbers are solved. The present expository paper is a condensation of part of the dissertation Keleş [5].

2. Preliminaries

In the section, we give some main definitions and properties which are guide to our work.

Definition 1. [28] Let X be the universe of discourse. A fuzzy set A defined on X is an object of the form

$$A = \{\langle \mu_A(x)/x \rangle : x \in X\},$$

where $\mu_A : X \to [0,1]$.

Definition 2. . [25] Let $a_1 \leq b_1 \leq c_1 \leq d_1$ such that $a_1, b_1, c_1, d_1 \in R$. A trapezoidal fuzzy number $a =< (a_1, b_1, c_1, d_1); w_a >$ is a special fuzzy set on the real number set R, whose membership function $\mu_a : R \to [0, w_a]$can generally be defined as

$$\mu_a(x) = \begin{cases} \frac{(x-a_1)w_a}{b_1-a_1}, & a_1 \leq x < b_1, \\ w_a, & b_1 \leq x < c_1, \\ \frac{(d_1-x)w_a}{d_1-c_1}, & c_1 \leq x < d_1, \\ 0, & otherwise. \end{cases}$$

where $w_a \in [0,1]$ is a constant, If $w_{\hat{A}}$= 1, then a is a normal trapezoidal fuzzy number; otherwise, it is said to be a non-normal trapezoidal fuzzy number (or generalized trapezoidal fuzzy number). Also, if we get $b_1 = c_1$then $a = (a_1, b_1, c_1, d_1; w_a)$ reduced to triangular fuzzy number $a =< (a_1, b_1, d_1); w_a >$.

Definition 3. [29] Let $a =< (a_1, b_1, c_1) >$ ve $b =< (a_2, b_2, c_2) >$ be two triangular fuzzy numbers. Then, similarity measures between a and b are given as;

i. $S_1(a,b) = \frac{\sum_{i=1}^{n} a_i.b_i}{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 - \sum_{i=1}^{n} a_i.b_i}$

ii. $S_2(a,b) = \frac{2.\sum_{i=1}^{n} a_i.b_i}{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2}$

iii. $S_3(a,b) = \frac{\sum_{i=1}^{n} a_i.b_i}{\sqrt{\sum_{i=1}^{n} a_i^2}.\sqrt{\sum_{i=1}^{n} b_i^2}}$

Teorem 4. [29] Let $a =< (a_1, b_1, c_1) >$ and $b =< (a_2, b_2, c_2) >$ be two triangular fuzzy numbers. Then, similarity measures $S_i(a, b)$ $(i = 1,2,3)$between a and b hold following properties;

i. $0 \leq S_i(a, b) \leq 1$

ii. $S_i(a, b) = S_i(b, a)$

iii. If a =b (a_i=b_i ; i=1,2,3) then $S_i(a, b) = 1$

Definition 5. [6] Let X be the universe of discourse. A multi fuzzy set G defined on X is an object of the form

$$G = \{\langle x, \mu_G^1(x), \mu_G^2(x), \dots, \mu_G^n(x) \rangle : x \in X\}$$

where $\mu_G^i(x): X \to [0,1]$, (i=1,2,…,n).

Definition 6. [21] Let $w_{\bar{a}}^i \in [0,1]$ $(i = 1,2, \dots, n)$, $a \leq b \leq c \leq d$ such that $a, b, c, d \in R$. An N-valued trapezoidal fuzzy number (NVTF-number)

$$\bar{a} =< (a, b, c, d); w_{\bar{a}}^1, w_{\bar{a}}^2, \dots w_{\bar{a}}^n >$$

is a special fuzzy multi set on the real number set R, whose membership functions $\mu_{\bar{a}}^i : R \to [0, w_a^i]$$(i = 1,2, \dots, n)$ can generally be defined as;

$$\mu_{\bar{a}}^i(x) = \begin{cases} \dfrac{(x-a)w_{\bar{a}}^i}{b-a}, & a \leq x < b, \\ w_{\bar{a}}^i, & b \leq x \leq c, \\ \dfrac{(d-x)w_{\bar{a}}^i}{d-c}, & c < x \leq d, \\ 0, & otherwise. \end{cases}$$

where $w_{\bar{a}}^i \in [0,1]$ is a constant, If $a, b, c, d \in [0,1]$, then $\bar{a}$ is a normal NVTF-number. Also, if we get $b = c$, then $\bar{a} =< (a, b, c, d); w_{\bar{a}}^1, w_{\bar{a}}^2, \dots w_{\bar{a}}^n >$ reduced to triangular fuzzy multi number $\bar{a} =< (a, b, d); w_{\bar{a}}^1, w_{\bar{a}}^2, \dots w_{\bar{a}}^n >$.

Definition 7. [5] Let $\bar{a} =< (a, b, c, d); w_{\bar{a}}^1, w_{\bar{a}}^2, \dots, w_{\bar{a}}^n >$ be a NVTF-number and $\alpha=(\alpha_1,\alpha_2,\dots\alpha_n)$ be a vector such that $0 \leq \alpha_i \leq w_{\bar{a}}^i$ (i=1,2,…,n). Then, α-cut of $\bar{a}$, denoted by $\bar{a}_\alpha$, is defined as;

$$\begin{aligned} \bar{a}_\alpha &= \{(x_1, x_2, \dots, x_n); \mu_{\bar{a}}^1(x_1) \geq \alpha_1, \mu_{\bar{a}}^2(x_2) \geq \alpha_2, \dots, \mu_{\bar{a}}^n(x_n) \geq \alpha_n, x \in R\} \\ &= ([L_{\bar{a}}(\alpha_1), R_{\bar{a}}(\alpha_1)], [L_{\bar{a}}(\alpha_2), R_{\bar{a}}(\alpha_2)], \dots \dots, [L_{\bar{a}}(\alpha_n), R_{\bar{a}}(\alpha_n)]) \\ & \left(\left[\frac{(w_{\bar{a}}^1-\alpha_1)a+\alpha_1 b}{w_{\bar{a}}^1}, \frac{(w_{\bar{a}}^1-\alpha_1)d+\alpha_1 c}{w_{\bar{a}}^1} \right], \left[\frac{(w_{\bar{a}}^2-\alpha_2)a+\alpha_2 b}{w_{\bar{a}}^2}, \frac{(w_{\bar{a}}^2-\alpha_2)d+\alpha_2 c}{w_{\bar{a}}^2} \right], \dots, \right. \\ & \left. \left[\frac{(w_{\bar{a}}^n-\alpha_n)a+\alpha_n b}{w_{\bar{a}}^n}, \frac{(w_{\bar{a}}^n-\alpha_n)d+\alpha_n c}{w_{\bar{a}}^n} \right] \right) \end{aligned}$$

Definition 8. [5] Let $\bar{a} =< (a, b, c, d); w_{\bar{a}}^{1}, w_{\bar{a}}^{2}, \ldots, w_{\bar{a}}^{n} >$ be a NVTF-number and $\bar{a}_{\alpha} = ([L_{\bar{a}}(\alpha_1), R_{\bar{a}}(\alpha_1)], [L_{\bar{a}}(\alpha_2), R_{\bar{a}}(\alpha_2)], \ldots \ldots \ldots, [L_{\bar{a}}(\alpha_n), R_{\bar{a}}(\alpha_n)])$ be α-cut set of a such that $0 \le \alpha_i \le w_{\bar{a}}^{i}$ (i=1,2,…,n). Then,

i. the values vector of $\bar{a}$ for α-cut set $\bar{a}_{\alpha}$, denoted by $V(\bar{a})$, is defined as;

$$V(\bar{a}) = \left(\int_{0}^{w_{\bar{a}}^{1}} (L_{\bar{a}}(\alpha_1) + R_{\bar{a}}(\alpha_1)) f(\alpha_1) d\alpha_1, \int_{0}^{w_{\bar{a}}^{2}} (L_{\bar{a}}((\alpha_2)) + R_{\bar{a}}((\alpha_2))) f(\alpha_2) d(\alpha_2), \ldots, \int_{0}^{w_{\bar{a}}^{n}} (L_{\bar{a}}(\alpha_n) + R_{\bar{a}}(\alpha_n)) f(\alpha_n) d\alpha_n \right)$$

$$= \left(\frac{(a+2b+2c+d)(w_{\bar{a}}^{1})^2}{6}, \frac{(a+2b+2c+d)(w_{\bar{a}}^{2})^2}{6}, \ldots, \frac{(a+2b+2c+d)(w_{\bar{a}}^{n})^2}{6}\right)$$

ii. the ambiguities vector of the $\bar{a}$ for α-cut set $\bar{a}_{\alpha}$, denoted by $A(\bar{a})$, is defined as;

$$A(\bar{a}) = \left(\int_{0}^{w_{\bar{a}}^{1}} (R_{\bar{a}}(\alpha_1) - L_{\bar{a}}(\alpha_1)) f(\alpha_1) d\alpha_1, \int_{0}^{w_{\bar{a}}^{2}} (R_{\bar{a}}(\alpha_2) - L_{\bar{a}}(\alpha_2)) f(\alpha_2) d\alpha_2, \ldots, \int_{0}^{w_{\bar{a}}^{n}} (R_{\bar{a}}(\alpha_n) - L_{\bar{a}}(\alpha_n)) f(\alpha_n) d\alpha_n \right)$$

$$= \left(\frac{(d-a+2c-2b)(w_{\bar{a}}^{1})^2}{6}, \frac{(d-a+2c-2b)(w_{\bar{a}}^{2})^2}{6}, \ldots, \frac{(d-a+2c-2b)(w_{\bar{a}}^{n})^2}{6}\right)$$

where $f(\alpha_i) = \alpha_i$ for $0 \le \alpha_i \le w_{\bar{a}}^{i}$ (i=1, 2,…,n).
Note that the f can be any function such that monotonic and nondecreasing for $\alpha_i \in [0, w_{\bar{a}}^{i}]$.

Definition 9. [5] Let $\bar{a} =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^{1}, w_{\bar{a}_1}^{2}, \ldots w_{\bar{a}_1}^{n} >$ be a NVTF-number. Then, integral vector of $\bar{a}$, denoted by $\overline{\text{inta}}^{\alpha}$, is given as;

$$\overline{\text{inta}}^{\alpha} = ([X_1', X_1''], [X_2', X_2''], \ldots, [X_n', X_n''])$$
$$= \left(\left[\int_0^{w_{\bar{a}}^{1}} \left(\frac{(w_{\bar{a}}^{1} - \alpha)a_1 + \alpha b_1}{w_{\bar{a}}^{1}}\right) d\alpha, \int_0^{w_{\bar{a}}^{1}} \left(\frac{(w_{\bar{a}}^{1} - \alpha)d_1 + \alpha c_1}{w_{\bar{a}}^{1}}\right) d\alpha\right],\right.$$

$$\left[\int_0^{w_{\bar{a}}^2}\left(\frac{(w_{\bar{a}}^2-\alpha)a_1+\alpha b_1}{w_{\bar{a}}^2}\right)d\alpha, \int_0^{w_{\bar{a}}^2}\left(\frac{(w_{\bar{a}}^2-\alpha)d_1+\alpha c_1}{w_{\bar{a}}^2}\right)d\alpha\right],\ldots,$$
$$\left[\int_0^{w_{\bar{a}}^n}\left(\frac{(w_{\bar{a}}^n-\alpha)a_1+\alpha b_1}{w_{\bar{a}}^n}\right)d\alpha, \int_0^{w_{\bar{a}}^n}\frac{(w_{\bar{a}}^n-\alpha)a_1+\alpha b_1}{w_{\bar{a}}^n})d\alpha\right]\Big)$$
$$=\left(\left[\frac{(a_1+b_1)w_{\bar{a}}^1}{2},\frac{(c_1+d_1)w_{\bar{a}}^1}{2}\right],\left[\frac{(a_1+b_1)w_{\bar{a}}^2}{2},\frac{(c_1+d_1)w_{\bar{a}}^2}{2}\right],\ldots\right.$$
$$\left.\left[\frac{(a_1+b_1)w_{\bar{a}}^n}{2},\frac{(c_1+d_1)w_{\bar{a}}^n}{2}\right]\right)$$

3. Similarity Measures on NVTF-numbers

In this sevtion we introduce some similarity measures of NVTF-numbers by using α-cut sets/integral vector, value and ambiguity of NVTF-numbers.

Definition 10. Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \ldots w_{\bar{a}_1}^n >$ and $\bar{a}_2 =< (a_2, b_2, c_2, d_2); w_{\bar{a}_2}^1, w_{\bar{a}_2}^2, \ldots w_{\bar{a}_2}^n >$ be two NVTF-numbers having the integral vector, respectively as;

$$\overline{\imath nta}_1^{\alpha} = ([X_1', X_1''], [X_2', X_2''], \ldots, [X_n', X_n'']) =$$
$$\left(\left[\frac{(a_1+b_1)w_{\bar{a}_1}^1}{2},\frac{(c_1+d_1)w_{\bar{a}_1}^1}{2}\right],\left[\frac{(a_1+b_1)w_{\bar{a}_1}^2}{2},\frac{(c_1+d_1)w_{\bar{a}_1}^2}{2}\right],\ldots\right.$$
$$\left.\left[\frac{(a_1+b_1)w_{\bar{a}_1}^n}{2},\frac{(c_1+d_1)w_{\bar{a}_1}^n}{2}\right]\right)$$

and

$$\overline{\imath nta}_2^{\alpha} = ([Y_1', Y_1''], [Y_2', Y_2''], \ldots, [Y_n', Y_n''])$$
$$=\left(\left[\frac{(a_2+b_2)w_{\bar{a}_2}^1}{2},\frac{(c_2+d_2)w_{\bar{a}_2}^1}{2}\right],\left[\frac{(a_2+b_2)w_{\bar{a}_2}^2}{2},\frac{(c_2+d_2)w_{\bar{a}_2}^2}{2}\right],\ldots\right.$$
$$\left.\left[\frac{(a_2+b_2)w_{\bar{a}_2}^n}{2},\frac{(c_2+d_2)w_{\bar{a}_2}^n}{2}\right]\right)$$

Then

i. 1. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha -$ cut sets/integral vector, denoted by $\bar{S}_1(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_1(\bar{a}_1,\bar{a}_2)=\frac{\sum_{i=1}^{n}(X_i'.Y_i'+X_i''.Y_i'')}{\sum_{i=1}^{n}[(X_i')^2+(X_i'')^2]+\sum_{i=1}^{n}[(Y_i')^2+(Y_i'')^2]-\sum_{i=1}^{n}(X_i'.Y_i'+X_i''.Y_i'')}$$

ii. 2. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha -$ cut sets/integral vector, denoted by $\bar{S}_2(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_2(\bar{a}_1, \bar{a}_2) = \frac{2.\sum_{i=1}^{n}(X_i'.Y_i' + X_i''.Y_i'')}{\sum_{i=1}^{n}[(X_i')^2 + (X_i'')^2] + \sum_{i=1}^{n}[(Y_i')^2 + (Y_i'')^2]}$$

iii. 3. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha -$ cut sets/integral vector, denoted by $\bar{S}_3(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_3(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(X_i'.Y_i' + X_i''.Y_i'')}{\sqrt{\sum_{i=1}^{n}[(X_i')^2 + (X_i'')^2] \cdot \sqrt{\sum_{i=1}^{n}[(Y_i')^2 + (Y_i'')^2]}}}$$

Teorem 11. Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \dots w_{\bar{a}_1}^n >$ and $\bar{a}_2 =< (a_2, b_2, c_2, d_2); w_{\bar{a}_2}^1, w_{\bar{a}_2}^2, \dots w_{\bar{a}_2}^n >$ be two NVTF-numbers. Then, j. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha -$ cut sets/integral vector, denoted by $\bar{S}_j(\bar{a}_1,\bar{a}_2)$(j=1,2,3), hold following properties;

i. $0 \le \bar{S}_i(\bar{a}_1, \bar{a}_2) \le 1$

ii. $\bar{S}_i(\bar{a}_1, \bar{a}_2) = \bar{S}_i(\bar{a}_2, \bar{a}_1)$

iii. $\bar{a}_1 = \bar{a}_2 \to \bar{S}_i(\bar{a}_1, \bar{a}_2)=1$

Proof. For example; we give proof of $\bar{S}_1(\bar{a}_1, \bar{a}_2)$

i.$\bar{S}_1(\bar{a}_1, \bar{a}_2) \ge 0$ is clear. We show that $\bar{S}_1(\bar{a}_1, \bar{a}_2) \le 1$ as;

Since $(X_i' - Y_i')^2 + (X_i'' - Y_i'')^2 \ge 0, (X_i')^2 - 2X_i'Y_i' + (Y_i')^2 + (X_i'')^2 - 2X_i''Y_i'' + (Y_i'')^2 \ge 0$, $(X_i')^2 + (X_i'')^2 + (Y_i')^2 + (Y_i'')^2 \ge 2X_i'Y_i' + 2X_i''Y_i''$

and $(X_i')^2 + (X_i'')^2 + (Y_i')^2 + (Y_i'')^2 \ge 2(X_i'Y_i' + X_i''Y_i'')$.

In here we have

$$\sum_{i=1}^{n}(X_i')^2 + (X_i'')^2 + \sum_{i=1}^{n}(Y_i')^2 + (Y_i'')^2 - \sum_{i=1}^{n}(X_i'.Y_i' + X_i''.Y_i'') \ge$$

$$2X_i'Y_i' + 2X_i''Y_i'' - (X_i'Y_i' + X_i''Y_i'') \ge X_i'Y_i' + X_i''Y_i''$$

and therefore we have $\bar{S}_1(\bar{a}_1, \bar{a}_2) \le 1$.

ii. Proof of $\bar{S}_1(\bar{a}_1, \bar{a}_2)=\bar{S}_1(\bar{a}_2, \bar{a}_1)$ is clear.

iii. If $\bar{a}_1 = \bar{a}_2$ then $X_i' = Y_i'$ and $X_i'' = Y_i''$ (i=1,2,….,n) . Therefore we have

$$\bar{S}_1(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(X_i'.X_i' + X_i''.X_i'')}{\sum_{i=1}^{n}[(X_i')^2 + (X_i'')^2] + \sum_{i=1}^{n}[(X_i')^2 + (X_i'')^2] - \sum_{i=1}^{n}(X_i'.X_i' + X_i''.X_i'')}$$
$$= 1$$

Proof of $\bar{S}_2(\bar{a}_1, \bar{a}_2)$ and $\bar{S}_3(\bar{a}_1, \bar{a}_2)$ can be similarity made.

Example 12. Assume that $\bar{a}_1 =< (0.1,0.2,0.3,0.4); 0.3,0.2,0.5,0.7 >$ and $\bar{a}_2 =< (0.2,0.3,0.3,0.5); 0.5,0.7,0.4,0.1>$ be two NVTF-numbers. In here values $\overline{inta_1^{\alpha}}$ ve $\overline{inta_2^{\alpha}}$ of $\bar{a}_1$ and $\bar{a}_2$ is given, respectively, as;

$$\overline{inta_1^{\alpha}} = ([0.045,0.105],[0.030,0.070],[0.075,0.175],[0.105,0.245])$$

$$and$$

$$\overline{inta_2^{\alpha}} = ([0.125,0.200],[0.175,0.280],[0.100,0.160],[0.025,0.040])$$

Then,

i. 1. similarity measure $\bar{S}_1(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha-$cut sets/ integral vector is calculated as;

$$\bar{S}_1(\bar{a}_1,\bar{a}_2) = \frac{\sum_{i=1}^{4}(X_i'.Y_i' + X_i''.Y_i'')}{\sum_{i=1}^{4}[(X_i')^2 + (X_i'')^2] + \sum_{i=1}^{4}[(Y_i')^2 + (Y_i'')^2] - \sum_{i=1}^{4}(X_i'.Y_i' + X_i''.Y_i'')}$$

$$= 0.4336$$

ii. 2. similarity measure $\bar{S}_2(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha-$cut sets/ integral vector is calculated as;

$$\bar{S}_2(\bar{a}_1,\bar{a}_2) = \frac{2.\sum_{i=1}^{4}(X_i'.Y_i' + X_i''.Y_i'')}{\sum_{i=1}^{4}[(X_i')^2 + (X_i'')^2] + \sum_{i=1}^{4}[(Y_i')^2 + (Y_i'')^2]}$$

$$= 0.6049$$

3. 3. similarity measure $\bar{S}_3(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on $\alpha-$cut sets/ integral vector is calculated as;

$$\bar{S}_3(\bar{a}_1,\bar{a}_2) = \frac{\sum_{i=1}^{4}(X_i'.Y_i' + X_i''.Y_i'')}{\sqrt{\sum_{i=1}^{4}[(X_i')^2 + (X_i'')^2]} \cdot \sqrt{\sum_{i=1}^{4}[(Y_i')^2 + (Y_i'')^2]}}$$

$$= 0.6219$$

Definition 13. Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \dots w_{\bar{a}_1}^n >$ and $\bar{a}_2 =< (a_2, b_2, c_2, d_2); w_{\bar{a}_2}^1, w_{\bar{a}_2}^2, \dots w_{\bar{a}_2}^n >$ be two NVTF-numbers with values of $\bar{a}_1$ and $\bar{a}_2$, respectively, as;

$$V(\bar{a}_1) = (V_1', V_2', \dots, V_n')$$

$$= \left(\frac{(a_1+2b_1+2c_1+d_1)(w_{\bar{a}_1}^1)^2}{6}, \frac{(a_1+2b_1+2c_1+d_1)(w_{\bar{a}_1}^2)^2}{6}, \dots, \frac{(a_1+2b_1+2c_1+d_1)(w_{\bar{a}_1}^n)^2}{6} \right)$$

and

$$V(\bar{a}_2) = (V_1'', V_2'', \dots, V_n'')$$
$$= \left(\frac{(a_2+2b_2+2c_2+d_2)(w_{\bar{a}_2}^1)^2}{6}, \frac{(a_2+2b_2+2c_2+d_2)(w_{\bar{a}_2}^2)^2}{6}, \dots, \frac{(a_2+2b_2+2c_2+d_2)(w_{\bar{a}_2}^n)^2}{6} \right)$$

Then,

i. 4. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on values, denoted by $\bar{S}_4(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_4(\bar{a}_1,\bar{a}_2) = \frac{\sum_{i=1}^n (V_i'.V_i'')}{\sum_{i=1}^n (V_i')^2 + \sum_{i=1}^n (V_i'')^2 - \sum_{i=1}^n (V_i'.V_i'')}$$

ii. 5. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on values, denoted by $\bar{S}_5(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_5(\bar{a}_1,\bar{a}_2) = \frac{2.\sum_{i=1}^n (V_i'.V_i'')}{\sum_{i=1}^n (V_i')^2 + \sum_{i=1}^n (V_i'')^2}$$

iii. 6. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on values, denoted by $\bar{S}_6(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_6(\bar{a}_1,\bar{a}_2) = \frac{\sum_{i=1}^n (V_i'.V_i'')}{\sqrt{\sum_{i=1}^n (V_i')^2}.\sqrt{\sum_{i=1}^n (V_i'')^2}}$$

Teorem 14. Let $\bar{a}_1 =< (a_1,b_1,c_1,d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \dots w_{\bar{a}_1}^n >$ and $\bar{a}_2 =< (a_2,b_2,c_2,d_2); w_{\bar{a}_2}^1, w_{\bar{a}_2}^2, \dots w_{\bar{a}_2}^n >$ be two NVTF-numbers. Then, j. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on values, denoted by $\bar{S}_j(\bar{a}_1,\bar{a}_2)$(j=4,5,6), hold following properties;

i. $0 \le \bar{S}_i(\bar{a}_1,\bar{a}_2) \le 1$

ii. $\bar{S}_i(\bar{a}_1,\bar{a}_2) = \bar{S}_i(\bar{a}_2,\bar{a}_1)$

iii. $\bar{a}_1 = \bar{a}_2 \rightarrow \bar{S}_i(\bar{a}_1,\bar{a}_2)=1$

Proof: For example; we give proof of $\bar{S}_4(\bar{a}_1, \bar{a}_2)$.

i. $\bar{S}_4(\bar{a}_1, \bar{a}_2) \geq 0$ is clear. We show that $\bar{S}_4(\bar{a}_1, \bar{a}_2) \leq 1$ as;
Since $(V_i' - V_i'')^2 \geq 0$, $(V_i')^2 - 2V_i'V_i'' + (V_i'')^2 \geq 0$ and $(V_i')^2 + (V_i'')^2 \geq 2V_i'V_i''$ we have

$$\sum_{i=1}^{n}(V_i')^2 + \sum_{i=1}^{n}(V_i'')^2 - \sum_{i=1}^{n} V_i'.V_i'' \geq 2V_i'V_i'' - V_i'V_i'' \geq V_i'V_i''$$

and therefore we have $\bar{S}_4(\bar{a}_1, \bar{a}_2) \leq 1$.

ii. Proof of $\bar{S}_4(\bar{a}_1, \bar{a}_2) = \bar{S}_4(\bar{a}_2, \bar{a}_1)$ is clear.

iii. If $\bar{a}_1 = \bar{a}_2$ then $V_i' = V_i''$(i=1,2,….,n) .

Finally we have $\bar{S}_4(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n} V_i'V_i'}{\sum_{i=1}^{n}(V_i')^2 + \sum_{i=1}^{n}(V_i')^2 - \sum_{i=1}^{n} V_i'V_i'} = 1.$

Proof of $\bar{S}_5(\bar{a}_1, \bar{a}_2)$ and $\bar{S}_6(\bar{a}_1, \bar{a}_2)$ can be similarity made.

Example 15. Assume that $\bar{a}_1 = < (0.1,0.2,0.3,0.4); 0.3,0.2,0.5,0.7 >$ and $\bar{a}_2 = < (0.2,0.2,0.5,0.6); 0.5,0.4,0.6,0.8 >$ be two NVTF-numbers. In here value of $\bar{a}_1$ and $\bar{a}_2$ is $V(\bar{a}_1) = (0.0225,0.01, 0.0625,0.1225)$ and $V(\bar{a}_2) = (0.0916,0.0586, 0.1320,0.2346)$. Then,

i. 4. similarity measure $\bar{S}_4(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on values is computed as;

$$\bar{S}_4(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(V_i'.V_i'')}{\sum_{i=1}^{n}(V_i')^2 + \sum_{i=1}^{n}(V_i'')^2 - \sum_{i=1}^{n}(V_i'.V_i'')}$$

$$= 0.6174$$

ii. 5. similarity measure $\bar{S}_5(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on values is computed as;

$$\bar{S}_5(\bar{a}_1, \bar{a}_2) = \frac{2.\sum_{i=1}^{n}(V_i'.V_i'')}{\sum_{i=1}^{n}(V_i')^2 + \sum_{i=1}^{n}(V_i'')^2}$$

$$= 0.7634$$

iii. 6. similarity measure $\bar{S}_6(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on values is computed as;

$$\bar{S}_6(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(V_i'. V_i'')}{\sqrt{\sum_{i=1}^{n}(V_i')^2} . \sqrt{\sum_{i=1}^{n}(V_i'')^2}}$$
$$= 0.9771$$

Definition 16. Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \dots w_{\bar{a}_1}^n >$ and $\bar{a}_2 =< (a_2, b_2, c_2, d_2); w_{\bar{a}_2}^1, w_{\bar{a}_2}^2, \dots w_{\bar{a}_2}^n >$ be two NVTF-numbers with ambiguity of $\bar{a}_1$ and $\bar{a}_2$, respectively as;

$$A(\bar{a}_1) = (A_1', A_2', \dots, A_n')$$
$$= \left(\frac{(d_1 - a_1 + 2c_1 - 2b_1)(w_{\bar{a}_1}^1)^2}{6}, \frac{(d_1 - a_1 + 2c_1 - 2b_1)(w_{\bar{a}_1}^2)^2}{6}, \dots, \frac{(d_1 - a_1 + 2c_1 - 2b_1)(w_{\bar{a}_1}^n)^2}{6} \right)$$

and

$$A(\bar{a}_2) = (A_1'', A_2'', \dots, A_n'')$$
$$= \left(\frac{(d_2 - a_2 + 2c_2 - 2b_2)(w_{\bar{a}_2}^1)^2}{6}, \frac{(d_2 - a_2 + 2c_2 - 2b_2)(w_{\bar{a}_2}^2)^2}{6}, \dots, \frac{(d_2 - a_2 + 2c_2 - 2b_2)(w_{\bar{a}_2}^n)^2}{6} \right)$$

Then,

iv. 7. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity, denoted by $\bar{S}_7(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_7(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(A_i'. A_i'')}{\sum_{i=1}^{n}(A_i')^2 + \sum_{i=1}^{n}(A_i'')^2 - \sum_{i=1}^{n}(A_i'. A_i'')}$$

v. 8. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity, denoted by $\bar{S}_8(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_8(\bar{a}_1, \bar{a}_2) = \frac{2. \sum_{i=1}^{n}(A_i'. A_i'')}{\sum_{i=1}^{n}(A_i')^2 + \sum_{i=1}^{n}(A_i'')^2}$$

vi. 9. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity, denoted by $\bar{S}_9(\bar{a}_1,\bar{a}_2)$, is defined as;

$$\bar{S}_9(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(A_i'. A_i'')}{\sqrt{\sum_{i=1}^{n}(A_i')^2} . \sqrt{\sum_{i=1}^{n}(A_i'')^2}}$$

Teorem 17. Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w^1_{\bar{a}_1}, w^2_{\bar{a}_1}, \dots w^n_{\bar{a}_1} >$ and $\bar{a}_2 =< (a_2, b_2, c_2, d_2); w^1_{\bar{a}_2}, w^2_{\bar{a}_2}, \dots w^n_{\bar{a}_2} >$ be two NVTF-numbers. Then, j. similarity measure between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity, denoted by $\bar{S}_j(\bar{a}_1,\bar{a}_2)$(j=7,8,9), hold following properties;

i. $0 \leq \bar{S}_i(\bar{a}_1, \bar{a}_2) \leq 1$

ii. $\bar{S}_i(\bar{a}_1, \bar{a}_2) = \bar{S}_i(\bar{a}_2, \bar{a}_1)$

iii. $\bar{a}_1 = \bar{a}_2 \rightarrow \bar{S}_i(\bar{a}_1, \bar{a}_2)$=1

Proof: For example; we give proof of $\bar{S}_7(\bar{a}_1, \bar{a}_2)$.

i. $\bar{S}_7(\bar{a}_1, \bar{a}_2) \geq 0$ is clear. We show that $\bar{S}_7(\bar{a}_1, \bar{a}_2) \leq 1$ as;

Since $(A'_i - A''_i)^2 \geq 0$, $(A'_i)^2 - 2A'_iA''_i + (A''_i)^2 \geq 0$ and $(A'_i)^2 + (A''_i)^2 \geq 2A'_iA''_i$ we have

$$\sum_{i=1}^{n} A'_i)^2 + \sum_{i=1}^{n} (A''_i)^2 - \sum_{i=1}^{n} A'_i . A''_i \geq 2A'_iA''_i - A'_iA''_i \geq A'_iA''_i$$

and therefore we have $\bar{S}_7(\bar{a}_1, \bar{a}_2) \leq 1$.

ii. Proof of $\bar{S}_7(\bar{a}_1, \bar{a}_2) = \bar{S}_7(\bar{a}_2, \bar{a}_1)$ is clear.

iii. If $\bar{a}_1 = \bar{a}_2$ then $A'_i = A''_i$(i=1,2,….,n). Finally we have

$$\bar{S}_7(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n} A'_iA'_i}{\sum_{i=1}^{n} (A'_i)^2 + \sum_{i=1}^{n} (A'_i)^2 - \sum_{i=1}^{n} A'_iA'_i} = 1$$

Proof of $\bar{S}_8(\bar{a}_1, \bar{a}_2)$ and $\bar{S}_9(\bar{a}_1, \bar{a}_2)$ can be similarity made.

Example 18. Assume that $\bar{a}_1 =< (0.1,0.2,0.3,0.4); 0.3,0.2,0.5,0.7 >$ and $\bar{a}_2 =< (0.2,0.3,0.4,0.5); 0.5,0.4,0.6,0.8>$ be two NVTF-numbers. In here ambiguity of $\bar{a}_1$ and $\bar{a}_2$, respectively, is $A(\bar{a}_1) = (0.0075,0.0033, 0.0208,0.0408)$ and $A(\bar{a}_2) = (0.0208,0.0133, 0.0300,0.0533)$. Then,

i. 7. similarity measure $\bar{S}_7(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity is computed as;

$$\bar{S}_7(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(A'_i . A''_i)}{\sum_{i=1}^{n}(A'_i)^2 + \sum_{i=1}^{n}(A''_i)^2 - \sum_{i=1}^{n}(A'_i . A''_i)}$$

$$= 0.8528$$

ii. 8. similarity measure $\bar{S}_8(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity is computed as;

$$\bar{S}_8(\bar{a}_1, \bar{a}_2) = \frac{2. \sum_{i=1}^{n}(A'_i . A''_i)}{\sum_{i=1}^{n}(A'_i)^2 + \sum_{i=1}^{n}(A''_i)^2}$$

$$= 0.9206$$

iii. 8. similarity measure $\bar{S}_8(\bar{a}_1,\bar{a}_2)$ between $\bar{a}_1$ and $\bar{a}_2$ based on ambiguity is computed as;

$$\bar{S}_9(\bar{a}_1, \bar{a}_2) = \frac{\sum_{i=1}^{n}(A'_i . A''_i)}{\sqrt{\sum_{i=1}^{n}(A'_i)^2} . \sqrt{\sum_{i=1}^{n}(A''_i)^2}}$$

$$= 0.9771$$

4. Application

In this section, inspired by Rajarajeswari and Uma [28,29], an application is given on how to apply similarity measures in NDYB-numbers based on multi attribute decision making problem under medical diagnosis. We will use the proposed similarity measures to diagnose disease by correlating patients' data based on symptoms and symptom-generated data for some diseases.

For this; let $P = \{P_1, P_2, \dots, P_r\}$ be the set of patients, $\{D_1, D_2, , \dots, D_m\}$ the set of diseases, and $S = \{S_1, S_2, \dots, , S_p\}$the set of symptoms. Now we can give the following algorithm;

Algorithm:

Step 1: Give the relation table $(x_{sk})_{rxp}$ between the Patient and their Symptoms; (The value of patient P_s in the table due to symptom S_k ($x_{sk} =< (a_{sk}, b_{sk,}c_{sk}, d_{sk}); w_{sk}^1, w_{sk}^2, \dots, w_{sk}^n) >$ is an NDYB number.) ($s = 1,2, \dots, r, k = 1,2, \dots, p$)

Step 2: Give the relation table $(n_{ik})_{mxp}$ between the Disease D_i and its Symptoms S_k; (value in the table depending on the symptom S_k of the disease D_i($k = 1,2,\dots,p, i = 1,2,\dots,m$) ($n_{ik} =< (a_{ik}, b_{ik,} c_{ik}, d_{ik}); w_{ik}^1, w_{ik}^2, \dots, w_{ik}^n >$ is an NDYB number.)

Step 3: Compute the total similarity measures $\bar{S}_{si} = \bar{S}(P_s, D_i)$ based on S_j(j=1,2,…,n) as;

$$\bar{S}_{si} = \frac{1}{p}\sum_{k=1}^{p} S_j(x_{sk}, n_{ik}) \; (s = 1,2,\dots,r, i = 1,2,\dots,m)$$

Step 4: Rank the possible diseases.(if $\bar{S}_{si} is$ biggest similarity measures then D_i is the best choice)

Example 19. Let's assume that $P = \{P_1, P_2, P_3, P_4\}$ } be set of patients, Let $\mathrm{D} = \{\mathrm{D}_1 = \text{Blood pressure}, \mathrm{D}_2 = \text{Bronchitis}, \mathrm{D}_3 = \text{rheumatism}, \mathrm{D}_4 = \text{diabetes}\}$ be set of diseases and $S = \{\mathrm{S}_1 = \text{Sweating}, \mathrm{S}_2 = \text{Heartache}, \mathrm{S}_3 = \text{bone pain}, \mathrm{S}_4 = \text{Hungry}\}$ be set of symptoms.

Step 1 According to the results obtained after patients P_i(i=1,2,3,4) was given medication 4 times in a day (08:00, 12:00, 16:00, 20:00) and then analyzed the patient P_s(i=1,2,3,4) and the symptom S_k(j=1,2,3,4) the results proposed by Table 1.

Table 1. Situations between the patient and the symptoms

x_{sk}	S_1	S_2
P_1	<(0.0,0.1,0.2,0.3);0.3,0.2,0.5,0.4>	<(0.2,0.3,0.6,0.8);0.3,0.1,0.4,0.7>
P_2	<(0.4,0.5,0.5,0.7);0.1,0.4,0.3,0.6>	<(0.3,0.5,0.5,0.6);0.4,0.5,0.6,0.7>
P_3	<(0.2,0.4,0.5,0.6);0.6,0.4,0.5,0.9>	<(0.1,0.2,0.3,0.5);0.1,0.2,0.4,0.6>
P_4	<(0.3,0.4,0.5,0.6);0.3,0.2,0.6,0.7>	<(0.2,0.4,0.4,0.6);0.8,0.2,0.5,0.4>
	S_3	S_4
P_1	<(0.3,0.4,0.5,0.6);0.2,0.5,0.3,0.6>	<(0.1,0.1,0.4,0.7);0.4,0.2,0.7,0.6>
P_2	<(0.5,0.7,0.9,1.0);0.3,0.7,0.5,0.4>	<(0.3,0.5,0.5,0.8)9;0.3,0.1,0.5,0.7>
P_3	<(0.3,0.5,0.7,0.9);0.7,0.2,0.5,0.6>	<(0.2,0.4,0.5,0.7);0.2,0.6,0.5,0.4>
P_4	<(0.1,0.4,0.5,0.7);0.8,0.4,0.5,0.7>	<(0.1,0.3,0.3,0.6);0.5,0.8,0.3,0.4>

Step 2: The values of *diseases* D_i(i=1,2,3,4) *related to symptoms* S_k(j=1,2,3,4) *based on previous patients* the results given by Table 2. with $(n_{ik})_{4x4}$

Table.2. Situations between the disease and its symptoms

n_{ik}	D_1	D_2
S_1	<(0.5,0.6,0.7,0.8);0.1,0.1,0.1,0.1>	<(0.3,0.4,0.5,0.7);0.6,0.6,0.6,0.6>
S_2	<(0.3,0.5,0.6,0.7);0.3,0.3.0.3,0.3>	<(0.2,0.4,0.5,0.6);0.5,0.5,0.5,0.5>
S_3	<(0.2,0.5,0.6,0.6);0.4,0.4,0.4,0.4>	<(0.2,0.5,0.6,0.9);0.4,0.4,0.4,0.4>
S_4	<(0.4,0.6,0.7,0.7);0.2,0.2,0.2,0.2>	<(0.4,0.7,0.8,0.9);0.7,0.7,0.7,0.7>
	D_3	D_4
S_1	<(0.1,0.2,0.5,0.8);0.3,0.3,0.3,0.3>	<(0.1,0.3,0.5,0.6);0.9,0.9,0.9,0.9>
S_2	<(0.3,0.4,0.6,0.8);0.4,0.4,0.4,0.4>	<(0.2,0.4,0.5,0.6);0.3,0.3,0.3,0.3>
S_3	<(0.2,0.3,0.5,0.7);0.6,0.6,0.6,0.6>	<(0.2,0.3,0.7,0.8);0.5,0.5,0.5,0.5>
S_4	<(0.3,0.4,0.7,0.8);0.8,0.8,0.8,0.8>	<(0.5,0.6,0.7,1.0);0.4,0.4,0.4,0.4>

Step 3: We computed the total similarity measures $\bar{S}_{si}$from Table 1 and Table 2 with $\bar{S}_1(P_s, D_i)$ as Tablo 3;

Table 3. $\bar{S}_1(P_s, D_i)$ for P_s and D_i (i=1,2,3,4 and s=1,2,3,4)

$\bar{S}_1(P_s, D_i)$	D_1	D_2	D_3	D_4	Ranking
P_1	**0.7207**	0.5608	0.6457	0.5513	$D_1 > D_3 > D_2 > D_4$
P_2	0.5599	**0.7161**	0.7142	0.6829	$D_2 > D_3 > D_4 > D_2$
P_3	0.5709	0.6833	0.6309	**0.8022**	$D_4 > D_2 > D_3 > D_1$
P_4	0.6577	0.7376	0.7152	**0.7503**	$D_4 > D_2 > D_3 > D_1$

Step 4: According to the results of Tab*le* 3. with $\bar{S}_1(P_s, D_i)$; P_1 *is* blood pressure patient, P_2 *is* bronchitis,, P_3 diabetes, P_4 *is* diabetes.

Based on $\bar{S}_j(P_s, D_i)$ (i = 1,2, … ,9) *the disease diagnoses are as in Table* 4

Table 4: The *disease diagnoses based* on $\bar{S}_i(P_i, D_k)$ (i = 1,2, … ,9)

	P_1	P_2	P_3	P_4
$\bar{S}_1(P_s, D_i)$	D_1	D_2	D_4	D_4
$\bar{S}_2(P_s, D_i)$	D_1	D_2	D_4	D_2
$\bar{S}_3(P_s, D_i)$	D_3	D_1	D_2	D_2
$\bar{S}_4(P_s, D_i)$	D_3	D_2	D_4	D_2
$\bar{S}_5(P_s, D_i)$	D_3	D_1	D_2	D_2
$\bar{S}_6(P_s, D_i)$	D_4	$D_1 = D_2 = D_3$	$D_1 = D_2 = D_3$	D_1
$\bar{S}_7(P_s, D_i)$	D_2	D_3	D_2	D_3
$\bar{S}_8(P_s, D_i)$	D_2	D_3	D_2	D_2
$\bar{S}_9(P_s, D_i)$	$D_1 = D_2 = D_3 = D_4$	$D_1 = D_2 = D_3 = D_4$	D_2	$D_1 = D_2 = D_3 = D_4$

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Direct Sum of Neutrosophic submodules of an R-module

Binu R[a], Paul Isaac[b]

[a]Department of Mathematics, Rajagiri School of Engineering and Technology, Kerala
[b]Department of Mathematics, Bharata Mata College, Thrikkakara, Kerala

Abstract

In this chapter, we have introduced the notion of direct sum of neutrosophic submodules of an R-module M and discuss some related properties. We also analyze the direct sum of arbitrary family of neutrosophic submodules and derive some results based on support of a neutrosophic submodule.

Keywords: Module, Neutrosophic set, Neutrosophic submodule, Support, Direct sum

1. Introduction

In classical set theory, the membership grades of elements in a set is assessed in binary terms 0 and 1. According to the bivalent condition, an element either belongs or does not belong to the set. As an extension, fuzzy set theory permits the gradual assessment of the membership of elements in a set. A fuzzy set A in X is characterised by a membership function which is associated with each element in X, a real number in the interval $[0, 1]$. In 1965, Lotfi A. Zadeh [14]introduced the concept of vagueness in mathematical modelling. A number of generalisations of the fundamental concept of set theory have come up. As a generalization of fuzzy set theory, intuitionistic fuzzy set theory [1] was proposed by Attanassov in 1986 in which each element is associated with a degree of membership and non membership values. Again as a generalization of fuzzy set and intuitionistic fuzzy set, neutrosophic set was defined with three different types of membership values by Smarandache in 1995. In the real world, the practical problems are related to incomplete, indeterminate and inconsistent information. Neutrosophic set is a powerful tool and the most appropriate frame work for dealing with incomplete, indeterminate and inconsistent information.

The algebraic structure in pure mathematics cloning with uncertainty has been studied by some authors. In 1971, Azriel Rosenfield presented a seminal paper on fuzzy subgroup and W.J. Liu developed the concept of fuzzy normal subgroup and fuzzy sub ring. The direct sum of fuzzy submodules was introduced by Mordeson and Malik [8]. In 2017, Isaac.P, P.P.John [6] identified some algebraic nature of intuitionistic fuzzy submodule of a module. Combining neutrosophic set theory with abstract algebra is an emerging trend in the area of mathematical research. Neutrosophic algebraic structures and its properties give us a

Email addresses: 1984binur@gmail.com (Binu R), pibmct@gmail.com (Paul Isaac)

strong mathematical background to explain applied mathematical concepts in engineering, data mining and economics. Neutrosophy is a new branch of philosophy and logic which studies the origin and features of neutralities in nature. Each proposition in neutrosophic logic is approximated to have the percentage of truth (T), the percentage of indeterminacy (I) and the percentage of falsity (F) [3, 12, 2, 13].

2. Preliminaries

Definition 2.1. *[8] Let R be a commutative ring with unity. A set M with a binary operation $+$ is said to be an R module or a module over the ring R if*

1. *$(M, +)$ is an abelian group*
2. *$\exists$ a map $R \times M \to M$ i.e. $(r, m) \to rm$ (an action of R on M) such that*
   (a) $(r + s)m = rm + sm$
   (b) $(rs)m = r(sm)$
   (c) $r(m + n) = rm + rn$
   (d) $1m = m, 1 \in R$, *$\forall r, s \in R$ and $m, n \in M$.*

Definition 2.2. *[8] Let M be an R module. A submodule is a subgroup N of M which is also an R module i.e, $rn \in N, \forall r \in R, n \in N$.*

Definition 2.3. *[5] Let M_1 and M_2 be the R-submodules of R-module M. Then we define*

$$M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}$$

which is an R-submodule of M containing both M_1 and M_2.

Definition 2.4. *[5] Let M_1 and M_2 be the R-submodules of an R-module M. $M_1 + M_2$ is called direct sum, denoted as $M_1 \oplus M_2$ if any element in $M_1 + M_2$ can be written uniquely as $m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$.*

Theorem 2.1. *[5] Let M_1 and M_2 be the submodules of an R-module M, then $M_1 + M_2$ is direct sum $\Leftrightarrow M_1 \cap M_2 = \{0\}$*

Definition 2.5. *[7] Let μ, η and ν be fuzzy submodules of M, then μ is the direct sum of η and ν if*

1. $\mu = \eta + \nu$
2. $\eta \cap \nu = 1_{\{0\}}$ *where* $1_{\{0\}}(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ *and we write $\mu = \eta \oplus \nu$*

Definition 2.6. *[7] Let μ and η be fuzzy submodules of the R-modules M and N respectively. Consider the direct sum $M \oplus N$. We extend the definition of μ and η to $M \oplus N$ to get μ' and η', fuzzy subsets of $M \oplus N$ as follows*

$$\mu'(m, n) = \begin{cases} \mu(m) & n = 0 \\ 0 & n \neq 0 \end{cases}, \eta'(m, n) = \begin{cases} \eta(n) & m = 0 \\ 0 & m \neq 0 \end{cases}$$

$\forall\ (m,n) \in M \oplus N$. *Then* μ' *and* η' *are fuzzy submodules of* $M \oplus N$. *Moreover*

$$(\mu' \cap \eta')(m,n) = \mu'(m,n) \wedge \eta'(m,n) = \begin{cases} 1 & (m,n) = 0 \\ 0 & (m,n) \neq 0 \end{cases}$$

Therefore $\mu' + \eta'$ *is a direct sum and we denote it by* $\mu \oplus \eta$.

Definition 2.7. *[11] A Neutrosophic set A on the universal set X is defined as*

$$A = \{(x, t_A(x), i_A(x), f_A(x)) : x \in X\}$$

where $t_A, i_A, f_A : X \to (^-0, 1^+)$. *The three components* t_A, i_A *and* f_A *represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval* $(^-0, 1^+)$.
If $t_A, i_A, f_A : X \to [0,1]$, *then A is known as single valued neutrosophic set(SVNS)[9].*

Definition 2.8. *[11] Let A and B be two neutrosophic sets of X. Then A is contained in B, denoted as* $A \subseteq B$ *if and only if* $A(x) \leqslant B(x)\ \forall x \in X$, *this means that*

$$t_A(x) \leq t_B(x), i_A(x) \leq i_B(x), f_A(x) \geq f_B(x)$$

Definition 2.9. *[11] The complement of* $A = \{x, t_A(x), i_A(x), f_A(x) : x \in X\}$ *is denoted by* A^C *and defined as* $A^C = \{x, f_A(x), 1 - i_A(x), t_A(x) : x\}$ *and* $(A^C)^C = A$

Definition 2.10. *[4, 11] Let A and B be two Neutrosophic sets of X*

1. *The union C of A and B is denoted by* $C = A \cup B$ *and defined as*

$$C(x) = A(x) \vee B(x)$$

where $C(x) = \{x, t_C(x), i_C(x), f_C(x) : x \in X\}$ *where*

$$t_C(x) = t_A(x) \vee t_B(x)$$

$$i_C(x) = i_A(x) \vee i_B(x)$$
$$f_C(x) = f_A(x) \wedge f_B(x)$$

2. *The intersection C of A and B is denoted by* $C = A \cap B$ *and is defined as*

$$C(x) = A(x) \wedge B(x)$$

where $C(x) = \{x, t_C(x), i_C(x), f_C(x) : x \in X\}$ *where*

$$t_C(x) = t_A(x) \wedge t_B(x)$$

$$i_C(x) = i_A(x) \wedge i_B(x)$$
$$f_C(x) = f_A(x) \vee f_B(x)$$

Definition 2.11. *[10] Let A and B be neutrosophic sets of a universal set X. Then their sum $A+B$ is a neutrosophic set of X, defined as follows*

$$t_{A+B}(x) = \vee\{t_A(y) \wedge t_B(z) | x = y + z, y, z \in X\}$$

$$i_{A+B}(x) = \vee\{i_A(y) \wedge i_B(z) | x = y + z, y, z \in X\}$$

$$f_{A+B}(x) = \wedge\{f_A(y) \vee f_B(z) | x = y + z, y, z \in X\}$$

Definition 2.12. *For any neutrosophic subset $A = \{(x, t_A(x), i_A(x), f_A(x)) : x \in X\}$, the support A^* of the neutrosophic set A can be defined as*

$$A^* = \{x \in X, t_A(x) > 0, i_A(x) > 0, f_A(x) < 1\}$$

.

3. Direct Sum

Definition 3.1. *Let M be an R module. Let $A \in U^M$ where U^M denotes the neutrosophic power set of R module M. Then a neutrosophic subset $A = (t_A(x), i_A(x), f_A(x))$ in M is called a neutrosophic submodule of M if it satisfies the following*

1. $t_A(0) = 1, i_A(0) = 1, f_A(0) = 0$
2. $t_A(x+y) \geq t_A(x) \wedge t_A(y)$
   $i_A(x+y) \geq i_A(x) \wedge i_A(y)$
   $f_A(x+y) \leq f_A(x) \vee f_A(y), \forall x, y \in M$
3. $t_A(rx) \geq t_A(x)$
   $i_A(rx) \geq i_A(x)$
   $f_A(rx) \leq f_A(x), \forall x \in M, \forall r \in R$

Remark 3.1. We denote the set of all neutrosophic submodules of R module M by $U(M)$.

Remark 3.2. If $A \in U(M)$, then the neutrosophic components of A can be denoted as $\{t_A(x), i_A(x), f_A(x)\}$.

Theorem 3.1. *If $A, B \in U(M)$, then $A + B \in U(M)$.*

Theorem 3.2. *Let A be a neutrosophic set on M. Then $A \in U(M)$ if and only if the following properties are satisfied $\forall\ x, y \in M$, $a, b \in R$*

$$i) \quad t_A(0) = 1, i_A(0) = 1, f_A(0) = 0.$$

$$ii) \quad t_A(ax+by) \geq t_A(x) \wedge t_A(y), \ i_A(ax+by) \geq i_A(x) \wedge i_A(y), \ f_A(ax+by) \leq f_A(x) \vee f_A(y)$$

Theorem 3.3. *Let $A \in U(M)$. Then A^* is an R submodule of M.*

Proof. $A \in U(M)$ and $A^* = (x \in X, t_A(x) > 0, i_A(x) > 0, f_A(x) < 1)$. Let $x, y \in A^*$ and $a, b \in R$. Then

$$t_A(x) > 0, i_A(x) > 0, f_A(x) < 1$$
$$t_A(y) > 0, i_A(y) > 0, f_A(y) < 1$$

Now

$$\begin{aligned} t_A(ax+by) &\geq t_A(ax) \wedge t_A(by) \\ &\geq t_A(x) \wedge t_A(y) \\ &\geq 0 \end{aligned}$$

Similarly $i_A(ax + by) \geq 0$ and $f_A(ax + by) \leq 1$, then $ax + by \in A^*$. Hence A^* is an R submodule of M.

□

[The proof of the theorems 3.1and 3.2 are explained in the paper titled as *Some Characterizations of Neutrosophic submodules of an R-module* which is submitted for the publication by the same authors]

Definition 3.2. *Let X be a non empty set. The neutrosophic point $\hat{N}_{\{0\}}$ in X is defined as $\hat{N}_{\{0\}} = \{(x, t_{\hat{N}_{\{0\}}}, i_{\hat{N}_{\{0\}}}, f_{\hat{N}_{\{0\}}}) : x \in X\}$ where*

$$\hat{N}_{\{0\}}(x) = \begin{cases} (1,0,0) & x = 0 \\ (0,0,1) & x \neq 0 \end{cases}$$

Theorem 3.4. *Let $A \in U(M)$. $A = \hat{N}_{\{0\}}$ if and only if $A^* = \{0\}$.*

Proof. If $A = \hat{N}_{\{0\}}$, then $A^* = (x \in X, t_A(x) > 0, i_A(x) > 0, f_A(x) < 1) = \{0\}$.
Conversely, if $A^* = \{0\}, \Rightarrow t_A(0) > 0, i_A(0) > 0, f_A(x) < 1$ and $t_A(x) = 0,\ i_A(x) = 0$ *and* $f_A(x) = 1\ \forall\ x \neq 0$.Therefore

$$A(x) = \begin{cases} (1,1,0) & x = 0 \\ (0,0,1) & x \neq 0 \end{cases} = \hat{N}_{\{0\}}$$

Hence the proof. □

Definition 3.3. *Let A, B and $C \in U(M)$, then A is said to be the direct sum of B and C if*

1. $A = B + C$
2. $B \cap C = \hat{N}_{\{0\}}$

and we write $A = B \oplus C$.

Definition 3.4. *Let $A_i \in U(M)\ \forall i \in J$, then we say that A is the direct sum of $\{A_i : i \in J\}$ denoted by $\oplus_{i\in J} A_i$ if*

1. $A = \sum_{i\in J} A_i$
2. $A_j \cap \sum_{i\in J-\{j\}} A_i = \hat{N}_{\{0\}}\ \forall j \in J$

Theorem 3.5. *If A, B and $C \in U(M)$ such that $A = B \oplus C$. Then $A^* = B^* \oplus C^*$.*

Proof. Let $x \in A^* \Rightarrow x \in (B \oplus C)^*$

$$\Rightarrow t_{B\oplus C}(x) > 0,\ i_{B\oplus C}(x) > 0 \ and\ f_{B\oplus C}(x) < 1\ \forall x \in M$$

$$\Rightarrow t_{B+C}(x) > 0,\ i_{B+C}(x) > 0 \ and\ f_{B+C}(x) < 1\ \forall x \in M$$

Now

$$\begin{aligned} t_{B+C}(x) &= \vee\{t_B(y) \wedge t_C(z) | x = y+z, y, z \in M\} > 0 \\ &\Rightarrow t_B(y) \wedge t_C(z) > 0\ for\ some\ y, z \in M,\ with\ x = y+z \\ &\Rightarrow \exists\, y, z \in M\ such\ that\ t_B(y) > 0, t_C(z) > 0 \end{aligned}$$

Similarly we can prove that $i_B(y) > 0, i_C(z) > 0$ where $x = y + z$ and
$f_B(y) < 1,\ f_C(z) < 1$ where $x = y + z$
$\Rightarrow \exists\, y, z \in M$ such that $y \in B^*, z \in C^*$ where $x = y + z$
$\Rightarrow A^* \subseteq B^* + C^*$.....(1)
Now $x \in B^* + C^* \Rightarrow \exists\, y \in B^*, z \in C^*$ such that $x = y + z$
$\Rightarrow t_B(y) > 0,\ i_B(y) > 0,\ f_B(y) < 1$ and $t_C(y) > 0,\ i_C(y) > 0,\ f_C(y) < 1$ which is true for all $y \in B^*, z \in C^*$ such that $x = y + z$
$\Rightarrow$

$$\vee\{t_A(y) \wedge t_B(z) | x = y+z, y, z \in M\} > 0$$

$$\vee\{i_A(y) \wedge i_B(z) | x = y+z, y, z \in M\} > 0$$

$$\wedge\ f_A(y) \vee f_B(z) | x = y+z, y, z \in M\} < 1$$

$\Rightarrow t_{B+C}(x) > 0,\ i_{B+C}(x) > 0\ and\ f_{B+C}(x) < 1$
$\Rightarrow t_A(x) > 0,\ i_A(x) > 0\ and\ f_A(x) < 1$ since $A = B \oplus C$
$\Rightarrow x \in A^*$
$\Rightarrow B^* + C^* \subseteq A^*$.....(2)
From (1) and (2), we can conclude $A^* = B^* + C^*$
Now $x \in B^* \cap C^* \Rightarrow x \in B^*\ and\ x \in C^*$
$\Rightarrow t_B(x) > 0,\ i_B(x) > 0,\ f_B(x) < 1$ and $t_C(x) > 0,\ i_C(x) > 0,\ f_C(x) < 1$
$\Rightarrow t_B(x) \wedge t_C(x) > 0,\ i_B(x) \wedge i_C(x) > 0$ and $f_B(x) \wedge f_C(x) < 1$
$\Rightarrow t_{B\cap C}(x) = 1,\ i_{B\cap C}(x) = 0$ and $f_{B\cap C}(x) = 0$ (since $A = B \oplus C \Rightarrow B \cap C = \hat{N}_{\{0\}}$)
$\Rightarrow x = 0 \Rightarrow B^* \cap C^* = \{0\}$
Hence $A^* = B^* \oplus C^*$

□

Remark The converse of the above theorem need not be true as we see in the following example

Example 3.1. *Let $M = R^2 = \{(a,b) : a, b \in R\}$ where R is any ring. Define*

$$A = \{x, t_A(x), i_A(x), f_A(x); x \in M\} \in U(M)$$

$$B = \{x, t_B(x), i_B(x), f_B(x); x \in M\} \in U(M)$$

$$C = \{x, t_C(x), i_C(x), f_C(x); x \in M\} \in U(M)$$

where

$$t_A(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (a,0), a \neq 0 \\ \frac{1}{2} & x = (a,b),\ b \neq 0 \end{cases},\ i_A(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (a,0), a \neq 0 \\ \frac{1}{2} & x = (a,b),\ b \neq 0 \end{cases},\ f_A(x) = \begin{cases} 0 & x = (0,0) \\ \frac{1}{2} & x = (a,0), a \neq 0 \\ \frac{1}{4} & x = (a,b),\ b \neq 0 \end{cases}$$

$$t_B(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (a,0), a \neq 0 \\ 0 & x = (a,b),\ b \neq 0 \end{cases},\ i_B(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (a,0), a \neq 0 \\ 0 & x = (a,b),\ b \neq 0 \end{cases},\ f_B(x) = \begin{cases} 0 & x = (0,0) \\ \frac{1}{2} & x = (a,0), a \neq 0 \\ 1 & x = (a,b),\ b \neq 0 \end{cases}$$

$$t_C(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (0,b), b \neq 0 \\ 0 & x = (a,b),\ a \neq 0 \end{cases},\ i_C(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (0,b), b \neq 0 \\ 0 & x = (a,b),\ a \neq 0 \end{cases},\ f_C(x) = \begin{cases} 0 & x = (0,0) \\ \frac{1}{2} & x = (0,b), b \neq 0 \\ 1 & x = (a,b),\ a \neq 0 \end{cases}$$

Now $A^* = \{x \in R^2 : t_A(x) > 0, i_A(x) > 0, f_A(x) < 1\} = R^2$. Similarly $B^* = (R,0)$ and $C^* = (0,R) \Rightarrow A^* = B^* + C^*,\ B^* \cap C^* = \{0\} \Rightarrow A^* = B^* \oplus B^*$
But $B + C = \{x, t_{B+C}(x), i_{B+C}(x), f_{B+C}(x) : x \in M\}$

$$t_{B+C}(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (a,0), a \neq 0 \\ \frac{1}{4} & x = (a,b),\ b \neq 0 \end{cases}, i_{B+C}(x) = \begin{cases} 1 & x = (0,0) \\ \frac{1}{4} & x = (a,0), a \neq 0 \\ \frac{1}{4} & x = (a,b),\ b \neq 0 \end{cases}, f_{B+C}(x) = \begin{cases} 0 & x = (0,0) \\ \frac{1}{2} & x = (a,0), a \neq 0 \\ \frac{1}{2} & x = (a,b),\ b \neq 0 \end{cases}$$

$$\Rightarrow\ \{t_{B+C}(x), i_{B+C}(x), f_{B+C}(x) : x \in M\} \neq \{t_A(x), i_A(x), f_A(x) : x \in M\}\ \Rightarrow\ A \neq B + C$$

$$\Rightarrow\ A \neq B \oplus C$$

4. Conclusion

Neutrosophic submodule is one of the generalizations of an algebraic structure, module. This chapter has developed a combination of an algebraic structure module with neutrosophic set theory. The algebraic property of direct sum of neutrosophic submodules and its extension to neutrosophic submodules of direct sum are defined.

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Chapter Sixteen

A new distance measure for N-valued Neutrosophic Trapezoidal Numbers based on the centroid points and Their Application to Multi-Criteria Decision-Making.

Davut KESEN and İrfan DELİ

Kilis 7 Aralık University, 79000 Kilis, Turkey

kesen66@gmail.com, irfandeli@kilis.edu.tr

ABSTRACT

One of the special form of neutrosophic multi-set on real number set is N-valued neutrosophic trapezoidal number (NVNT-number). Core of this notion is thatit has a lot of possibility of the same or the different membership functions which are truth-membership functions, indeterminacy-membership functions and falsity-membership functions. In this study, a method for NVNT-numbers based on multi-criteria decision-making problems which isgiven with NVNT-numbers are proposed. Initially, notion of centroid point of NVNT-numbers is introduced. Second, some distance measures under centroid point of NVNT-numbers are proposed. In addition, an algorithm to solve multi-criteria decision-making problems given with proposed concept of NVNT-numbers is developed. Finally, a numerical example of multi-criteria decision-making, in which the ratings of alternatives are given with NVNT-numbers, is proposed to show practicality of the developed algorithm.

Keywords: Neutrosophic sets, N-valued neutrosophic trapezoidal numbers, distance measures, centroid point, multi-criteria decision-making,

1. Introduction

Many theories put forward to deal with problems involving uncertainty in our daily life have lost their importance over time and have been replaced by different theories. Some of the theories are interval mathematic, probability theory, fuzzy set theory [31], intuitionistic fuzzy set theory [1] and neutrosophic set theory [19].Among these theories, the most up-to-date and that has the widest application area is the fuzzy set theory developed by Zadeh [31] in 1965.The theory is constructed with the help of a membership function that takes the values in the interval [0,1] for elements of a universal set X. Intuitionistic fuzzy set theory was constructed by Atanassov [1] in 1986 by adding a non-membership membership function to fuzzy set theory that takes the values in the interval [0,1] for elements of a universal set X. Inintuitionistic fuzzy set theory, the sum of the values of the membership function and non-membership function for each element of the universal set X always remains in the interval [0,1]. This limitation of the membership function and non-membership function creates deficiency for problems involving uncertainty. To overcome this situation, in 1998 Smarandache [19] presented a new set theory called neutrosophic set theory, which includes fuzzy set and intuitionistic fuzzy set theory. Later, single-valued neutrosophic sets, which are special cases of neutrosophic sets, were developed by Wang et al. [26] in 2010. Recently, many author have studied on the neutrosophic sets in [6-9,11,16,17].

Fuzzy sets with single membership value between [0,1] have some disadvantages for solving problems. For instance; as to measurement of amount carbon in weather problem, it is hard to model the data and make a decision by getting results of 4 measurements in a day (09:00, 15:00, 18:00, 23:00). Therefore, the multi-fuzzy sets (N-valued fuzzy sets), which is a different generalization of fuzzy sets, firstly developed on the multi sets of Yager [29] by Miyamoto [14], [15]. In 2018, to model uncertain problems, trapezoidal fuzzy multi-numbers with operation laws by using multi fuzzy sets introduced by Uluçay et al [25].The concept of trapezoidal fuzzy multi-number allows the repeated occurrences of any element on real numbers set R and it is more general when compared to fuzzy numbers. Later, Şahin et al. [20-22] proposed new similarity measures on trapezoidal fuzzy multi-numbers and gave two applications in multi-criteria decision-making problem. Then, Uluçay [24] introduced a decision-making method by defining a new similarity function and a weighted new similarity function on trapezoidal fuzzy multi-numbers. In 2021, some new distance measures on trapezoidal fuzzy multi-numbers and their application to multi-criteria decision-making problems introduced by Deli and Keleş [4]. In fact, the development of the theories is not perfect, and further research and exploration are still needed. This is the reason why this study is written.

2. Preliminary

In this section, we present some basic concepts such as fuzzy sets, trapezoidal fuzzy multi-number, intuitionistic fuzzy multi-sets, intuitionistic fuzzy multi-numbers, neutrosophic sets, neutrosophic multi-sets, neutrosophic multi-numbers and so on.

Definition 2.1 [31] Let X be the universe of discourse. A fuzzy set M defined on X is an object of the form

$$M = \{\langle \mu_M(x)/x \rangle : x \in X\},$$

where $\mu_M : X \rightarrow [0,1]$.

Definition 2.2 [2] Let M and N be two fuzzy numbers. Then, some distance measures between M and N are given as follows;

i. The generalized distance measure $d_r(M,N)$ is defined as;

$$d_q(M,N) = \sqrt[q]{\sum_{i=1}^{n} |\mu_M(x_i) - \mu_N(x_i)|^q}\ , q\geq 1.$$

ii. The Hamming distance measure $d_H(M,N)$ is defined as;

$$d_H(M,N) = \sum_{i=1}^{n} |\mu_M(x_i) - \mu_N(x_i)|$$

iii. The normalized Euclidean distance measure $d_{nE}(M,N)$ is defined as;

$$d_{nE}(M,N) = \frac{1}{n}\sqrt{\sum_{i=1}^{n} |\mu_M(x_i) - \mu_N(x_i)|^2}$$

iv. Supremum distance measure $d_{+\infty}(M,N)$ is defined as;

$$d_{+\infty}(M,N) = \sup|\mu_M(x_i) - \mu_N(x_i)|$$

Definition 2.3 [27] Let $a_1 \leq b_1 \leq c_1 \leq d_1$ such that $a_1, b_1, c_1, d_1 \in R$.A trapezoidal fuzzy number $a =< (a_1, b_1, c_1, d_1); w_a >$is a special fuzzy set on the real number set R, whose membership function $\mu_a: R \to [0, w_a]$can generally be defined as

$$\mu_a(x) = \begin{cases} \frac{(x-a_1)w_a}{b_1-a_1}, & a_1 \leq x < b_1, \\ w_a, & b_1 \leq x < c_1, \\ \frac{(d_1-x)w_a}{d_1-c_1}, & c_1 \leq x < d_1, \\ 0, & otherwise. \end{cases}$$

where$w_a \in [0,1]$ is a constant.

Definition 2.4 [5] Let $a =< (a_1, b_1, c_1, d_1); w_a >$ be a trapezoidal fuzzy number such that $0 \leq \alpha \leq w_a$. Then, α-cut set of a, denoted a_α, is defined as;

$$\begin{aligned} a_\alpha &= \{x; \mu_a(x) \geq \alpha, x\epsilon R\} = [L_a(\alpha), R_a(\alpha)] \\ &= \left[\frac{(w_a - \alpha)a_1 + \alpha b_1}{w_a}, \frac{(w_a - \alpha)d_1 + \alpha c_1}{w_a}\right] \end{aligned}$$

Definition 2.5 [25] Let $w_{\bar{a}}^i \in [0,1] (i \in \{1,2,\ldots,p\})$ and $a, b, c, d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. Then, a trapezoidal fuzzy multi-number (TFM-number) $\bar{a} = \langle [a,b,c,d]; (w_{\bar{a}}^1, w_{\bar{a}}^2, \ldots, w_{\bar{a}}^p)\rangle$

is a special fuzzy multi-set on the real number set $\mathbb{R}$, whose membership functions are defined as;

$$\mu_{\bar{a}}^{i}(x) = \begin{cases} \frac{(x-a)}{(b-a)} w_{\bar{a}}^{i}, & a \leq x < b \\ w_{\bar{a}}^{i}, & b \leq x \leq c \\ \frac{(d-x)}{(d-c)} w_{\bar{a}}^{i}, & c < x \leq d \\ 0, & \text{otherwise,} \end{cases}$$

Definition 2.6 [4] Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \dots w_{\bar{a}_1}^n >$be a trapezoidal fuzzy multi number. Then, integral vector of $\bar{a}_1$, denoted by $\overline{\text{inta}}_1^{\alpha}$, is given as;

$$\overline{\text{inta}}_1^{\alpha} = ([X_1', X_1''], [X_2', X_2''], \dots, [X_n', X_n''])$$

$$= \left(\left[\int_0^{w_{\bar{a}_1}^1} \left(\frac{(w_{\bar{a}_1}^1 - \alpha) a_1 + \alpha b_1}{w_{\bar{a}_1}^1} \right) d\alpha, \int_0^{w_{\bar{a}_1}^1} \left(\frac{(w_{\bar{a}_1}^1 - \alpha) d_1 + \alpha c_1}{w_{\bar{a}_1}^1} \right) d\alpha \right], \right.$$

$$\left[\int_0^{w_{\bar{a}_1}^2} \left(\frac{(w_{\bar{a}_1}^2 - \alpha) a_1 + \alpha b_1}{w_{\bar{a}_1}^2} \right) d\alpha, \int_0^{w_{\bar{a}_1}^2} \left(\frac{(w_{\bar{a}_1}^2 - \alpha) d_1 + \alpha c_1}{w_{\bar{a}_1}^2} \right) d\alpha \right], \dots,$$

$$\left. \left[\int_0^{w_{\bar{a}_1}^n} \left(\frac{(w_{\bar{a}_1}^n - \alpha) a_1 + \alpha b_1}{w_{\bar{a}_1}^n} \right) d\alpha, \int_0^{w_{\bar{a}_1}^n} \frac{(w_{\bar{a}_1}^n - \alpha) a_1 + \alpha b_1}{w_{\bar{a}_1}^n}) d\alpha \right] \right)$$

$$= \left(\left[\frac{(a_1 + b_1) w_{\bar{a}_1}^1}{2}, \frac{(c_1 + d_1) w_{\bar{a}_1}^1}{2} \right], \left[\frac{(a_1 + b_1) w_{\bar{a}_1}^2}{2}, \frac{(c_1 + d_1) w_{\bar{a}_1}^2}{2} \right], \dots, \left[\frac{(a_1 + b_1) w_{\bar{a}_1}^n}{2}, \frac{(c_1 + d_1) w_{\bar{a}_1}^n}{2} \right] \right)$$

Definition 2.7 [4] Let $\bar{a}_1 =< (a_1, b_1, c_1, d_1); w_{\bar{a}_1}^1, w_{\bar{a}_1}^2, \dots w_{\bar{a}_1}^n >$ and $\bar{a}_2 =< (a_2, b_2, c_2, d_2); w_{\bar{a}_2}^1, w_{\bar{a}_2}^2, \dots w_{\bar{a}_2}^n >$ be two trapezoidal fuzzy multi numbers having the integral vector $\overline{\text{inta}}_1^{\alpha} = ([X_1', X_1''], [X_2', X_2''], \dots, [X_n', X_n''])$ and $\overline{\text{inta}}_2^{\alpha} = ([Y_1', Y_1''], [Y_2', Y_2''], \dots, [Y_n', Y_n''])$ respectively. Then, generalized distance measure between $\bar{a}_1$ and $\bar{a}_2$ based on α-cut sets, denoted by $\bar{d}_{1_r}(\bar{a}_1, \bar{a}_2)(r \geq 1)$, is defined as;

$$\bar{d}_r(\bar{a}_1, \bar{a}_2) = \sqrt[r]{\frac{1}{2n} \sum_{i=1}^{n} ((X_i' - Y_i')^r + (X_i'' - Y_i'')^r} \qquad 34$$

Definition 2.8 [30] Let X be a space of discourse, a trapezoidal neutrosophic set Hin X is defined as follow:

$$H = \{ \langle y, T_H(y), I_H(y), F_H(y) \rangle | y \in Y \}$$

where $T_H(y) \subset [0,1], I_H(y) \subset [0,1]$ and $F_H(y) \subset [0,1]$ are three trapezoidal fuzzy numbers such that

$T_H(y) = \left(t_H^1(y), t_H^2(y), t_H^3(y), t_H^4(y)\right)$:Y→ [0,1],$I_H(y) = (i_H^1(y), i_H^2(y), i_H^3(y), i_H^4(y))$:Y→ [0,1] and $F_H(y) = (f_H^1(y), f_H^2(y), f_H^3(y), f_H^4(y))$:Y → [0,1] with the condition $0 \leq t_H^4(y) + i_H^4(y) + f_H^4(y) \leq 3, y \in Y$.

For convenience, the three trapezoidal fuzzy numbers are denoted by $T_H(y) = (a, b, c, d)$, $I_H(y) = (e, f, g, h)$ and $F_H(y) = (i, j, k, l)$. Thus, a trapezoidal neutrosophic numbers is denoted by m= $\langle(a, b, c, d), (e, f, g, h), (i, j, k, l)\rangle$.

Definition 2.9 [30] Let $\tilde{n} = \langle(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\rangle$ be a trapezoidalfuzzyneutrosophicnumber.

i) Centroid point of the truth membership function of trapezoidal fuzzy neutrosophic number ñ is

$$O^T = \left(x^T(\tilde{n}), y^T(\tilde{n})\right) = \frac{1}{3}\left(\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)}\right], \left[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_1 + a_2)}\right]\right).$$

ii) Centroid point of the indeterminacy membership function of trapezoidal fuzzy neutrosophic number ñ is

$$O^I = \left(x^I(\tilde{n}), y^I(\tilde{n})\right) = \frac{1}{3}\left(\left[b_1 + b_2 + b_3 + b_4 - \frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)}\right], \left[1 + \frac{b_3 - b_2}{(b_4 + b_3) - (b_1 + b_2)}\right]\right).$$

iii) Centroid point of the falsity membership function of trapezoidal fuzzy neutrosophic number ñ is

$$O^F = \left(x^F(\tilde{n}), y^F(\tilde{n})\right) = \frac{1}{3}\left(\left[c_1 + c_2 + c_3 + c_4 - \frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)}\right], \left[1 + \frac{c_3 - c_2}{(c_4 + c_3) - (c_1 + c_2)}\right]\right).$$

Finally, centroid point of trapezoidal fuzzy neutrosophic number ñ is

$$O(x(\tilde{n}), y(\tilde{n})) = \left(\frac{x^T(\tilde{n}) + x^I(\tilde{n}) + x^F(\tilde{n})}{3}, \frac{y^T(\tilde{n}) + y^I(\tilde{n}) + y^F(\tilde{n})}{3}\right)$$

Definition 2.11 [3] Let $\eta_{A_1}^i, \vartheta_{A_1}^i, \theta_{A_1}^i \in [0,1] (i \in \{1,2, \dots, p\})$ and $a, b, c, d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. Then, an N-valued neutrosophic trapezoidal number(NVNT-number)

$A_1 = \langle[a, b, c, d]; (\eta_{A_1}^1, \eta_{A_1}^2, \dots, \eta_{A_1}^P), (\vartheta_{A_1}^1, \vartheta_{A_1}^2, \dots, \vartheta_{A_1}^P), (\theta_{A_1}^1, \theta_{A_1}^2, \dots, \theta_{A_1}^P)\rangle$ is a neutrosophic multi-set on the real number set $\mathbb{R}$, whose truth-membership functions, indeterminacy-membership functions and falsity-membership functions are defined as, respectively.

$$T^{i}_{A_1}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\eta^{i}_{A_1}, & a \le x < b \\ \eta^{i}_{A_1}, & b \le x \le c \\ \frac{(d-x)}{(d-c)}\eta^{i}_{A_1}, & c < x \le d \\ 0, & \text{otherwise,} \end{cases}, \quad I^{i}_{\tilde{\alpha}}(x) = \begin{cases} \frac{(b-x)+\vartheta^{i}_{A_1}(x-a)}{(b-a)}, & a \le x < b \\ \vartheta^{i}_{A_1}, & b \le x \le c \\ \frac{(x-c)+\vartheta^{i}_{A_1}(d-x)}{(d-c)}, & c < x \le d \\ 1, & \text{otherwise,} \end{cases}$$

and

$$F^{i}_{A_1}(x) = \begin{cases} \frac{(b-x)+\theta^{i}_{A_1}(x-a)}{(b-a)}, & a \le x < b \\ \theta^{i}_{A_1}, & b \le x \le c \\ \frac{(x-c)+\theta^{i}_{A_1}(d-x)}{(d-c)}, & c < x \le d \\ 1, & \text{otherwise,} \end{cases}$$

Note that the set of all NVNT-numbers on$\mathbb{R}$ will be denoted by Λ.

Definition 2.12 [3] Let $\overline{A}_1 = \langle [a_1, b_1, c_1, d_1]; \left(\eta^{1}_{\overline{A}_1}, \eta^{2}_{\overline{A}_1}, \dots, \eta^{P}_{\overline{A}_1}\right), \left(\vartheta^{1}_{\overline{A}_1}, \vartheta^{2}_{\overline{A}_1}, \dots, \vartheta^{P}_{\overline{A}_1}\right), \left(\theta^{1}_{\overline{A}_1}, \theta^{2}_{\overline{A}_1}, \dots, \theta^{P}_{\overline{A}_1}\right) \rangle$,

$\overline{A}_2 = \langle [a_2, b_2, c_2, d_2]; \left(\eta^{1}_{\overline{A}_2}, \eta^{2}_{\overline{A}_2}, \dots, \eta^{P}_{\overline{A}_2}\right), \left(\vartheta^{1}_{\overline{A}_2}, \vartheta^{2}_{\overline{A}_2}, \dots, \vartheta^{P}_{\overline{A}_2}\right), \left(\theta^{1}_{\overline{A}_2}, \theta^{2}_{\overline{A}_2}, \dots, \theta^{P}_{\overline{A}_2}\right) \rangle \in \Lambda$ and $\gamma \neq 0$ be any real number. Then,

i. $\overline{A}_1 + \overline{A}_2 = \langle [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \left(\frac{\eta^{1}_{\overline{A}_1} + \eta^{1}_{\overline{A}_2}}{1 + \eta^{1}_{\overline{A}_1}\eta^{1}_{\overline{A}_2}}, \frac{\eta^{2}_{\overline{A}_1} + \eta^{2}_{\overline{A}_2}}{1 + \eta^{2}_{\overline{A}_1}\eta^{2}_{\overline{A}_2}}, \dots, \frac{\eta^{p}_{\overline{A}_1} + \eta^{p}_{\overline{A}_2}}{1 + \eta^{p}_{\overline{A}_1}\eta^{p}_{\overline{A}_2}}\right)$

$$\left(\frac{\vartheta^{1}_{\overline{A}_1}\vartheta^{1}_{\overline{A}_2}}{1+\left(1-\vartheta^{1}_{\overline{A}_1}\right)\left(1-\vartheta^{1}_{\overline{A}_2}\right)}, \frac{\vartheta^{2}_{\overline{A}_1}\vartheta^{2}_{\overline{A}_2}}{1+\left(1-\vartheta^{1}_{\overline{A}_1}\right)\left(1-\vartheta^{2}_{\overline{A}_2}\right)}, \dots, \frac{\vartheta^{p}_{\overline{A}_1}\vartheta^{p}_{\overline{A}_2}}{1+\left(1-\vartheta^{p}_{\overline{A}_1}\right)\left(1-\vartheta^{p}_{\overline{A}_2}\right)}\right),$$

$$\left(\frac{\theta^{1}_{\overline{A}_1}\theta^{1}_{\overline{A}_2}}{1+\left(1-\theta^{1}_{\overline{A}_1}\right)\left(1-\theta^{1}_{\overline{A}_2}\right)}, \frac{\theta^{2}_{\overline{A}_1}\theta^{2}_{\overline{A}_2}}{1+\left(1-\theta^{1}_{\overline{A}_1}\right)\left(1-\theta^{2}_{\overline{A}_2}\right)}, \dots, \frac{\theta^{p}_{\overline{A}_1}\theta^{p}_{\overline{A}_2}}{1+\left(1-\theta^{p}_{\overline{A}_1}\right)\left(1-\theta^{p}_{\overline{A}_2}\right)}\right)\rangle$$

ii. $\overline{A}_1 . \overline{A}_2 = \langle [a_1 . a_2, b_1 . b_2, c_1 . c_2, d_1 . d_2]; \left(\frac{\eta^{1}_{\overline{A}_1}\eta^{1}_{\overline{A}_2}}{1+\left(1-\eta^{1}_{\overline{A}_1}\right)\left(1-\eta^{1}_{\overline{A}_2}\right)},\right.$

$$\frac{\eta^2_{\bar{A}_1}\eta^2_{\bar{A}_2}}{1+\left(1-\eta^2_{\bar{A}_1}\right)\left(1-\eta^2_{\bar{A}_2}\right)}, \dots, \frac{\eta^2_{\bar{A}_1}\eta^2_{\bar{A}_2}}{1+\left(1-\eta^2_{\bar{A}_1}\right)\left(1-\eta^2_{\bar{A}_2}\right)}\Bigg),$$

$$\left(\frac{\vartheta^1_{\bar{A}_1}+\vartheta^1_{\bar{A}_2}}{1+\vartheta^1_{\bar{A}_1}\vartheta^1_{\bar{A}_2}}, \frac{\vartheta^2_{\bar{A}_1}+\vartheta^2_{\bar{A}_2}}{1+\vartheta^2_{\bar{A}_1}\vartheta^2_{\bar{A}_2}}, \dots, \frac{\vartheta^p_{\bar{A}_1}+\vartheta^p_{\bar{A}_2}}{1+\vartheta^p_{\bar{A}_1}\vartheta^p_{\bar{A}_2}}\right),$$

$$\left(\frac{\theta^1_{\bar{A}_1}+\theta^1_{\bar{A}_2}}{1+\theta^1_{\bar{A}_1}\theta^1_{\bar{A}_2}}, \frac{\theta^2_{\bar{A}_1}+\theta^2_{\bar{A}_2}}{1+\theta^2_{\bar{A}_1}\theta^2_{\bar{A}_2}}, \dots, \frac{\theta^p_{\bar{A}_1}+\theta^p_{\bar{A}_2}}{1+\theta^p_{\bar{A}_1}\theta^p_{\bar{A}_2}}\right)\rangle$$

iii. $\gamma\bar{A}_1 = \langle[\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1];$

$$\left(\frac{(1+\eta^1_{\bar{A}_1})^\gamma-(1-\eta^1_{\bar{A}_1})^\gamma}{(1+\eta^1_{\bar{A}_1})^\gamma+(1-\eta^1_{\bar{A}_1})^\gamma}, \frac{(1+\eta^2_{\bar{A}_1})^\gamma-(1-\eta^2_{\bar{A}_1})^\gamma}{(1+\eta^2_{\bar{A}_1})^\gamma+(1-\eta^2_{\bar{A}_1})^\gamma}, \dots, \frac{(1+\eta^p_{\bar{A}_1})^\gamma-(1-\eta^p_{\bar{A}_1})^\gamma}{(1+\eta^p_{\bar{A}_1})^\gamma+(1-\eta^p_{\bar{A}_1})^\gamma}\right),$$

$$\left(\frac{2(\vartheta^1_{\bar{A}_1})^\gamma}{(2-\vartheta^1_{\bar{A}_1})^\gamma+(\vartheta^1_{\bar{A}_1})^\gamma}, \frac{2(\vartheta^2_{\bar{A}_1})^\gamma}{(2-\vartheta^2_{\bar{A}_1})^\gamma+(\vartheta^2_{\bar{A}_1})^\gamma}, \dots, \frac{2(\vartheta^p_{\bar{A}_1})^\gamma}{(2-\vartheta^p_{\bar{A}_1})^\gamma+(\vartheta^p_{\bar{A}_1})^\gamma}\right),$$

$$\left(\frac{2(\theta^1_{\bar{A}_1})^\gamma}{(2-\theta^1_{\bar{A}_1})^\gamma+(\theta^1_{\bar{A}_1})^\gamma}, \frac{2(\theta^2_{\bar{A}_1})^\gamma}{(2-\theta^2_{\bar{A}_1})^\gamma+(\theta^2_{\bar{A}_1})^\gamma}, \dots, \frac{2(\theta^p_{\bar{A}_1})^\lambda}{(2-\theta^p_{\bar{A}_1})^\lambda+(\theta^p_{\bar{A}_1})^\lambda}\right)\rangle.$$

iv. $\bar{A}_1{}^\gamma = \langle[a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma];\left(\frac{2(\eta^1_{\bar{A}_1})^\gamma}{(2-\eta^1_{\bar{A}_1})^\gamma+(\eta^1_{\bar{A}_1})^\gamma}, \frac{2(\eta^2_{\bar{A}_1})^\gamma}{(2-\eta^2_{\bar{A}_1})^\gamma+(\eta^2_{\bar{A}_1})^\gamma}, \dots, \frac{2(\eta^p_{\bar{A}_1})^\gamma}{(2-\eta^p_{\bar{A}_1})^\gamma+(\eta^p_{\bar{A}_1})^\gamma}\right)$

$$\left(\frac{(1+\vartheta^1_{\bar{A}_1})^\gamma-(1-\vartheta^1_{\bar{A}_1})^\gamma}{(1+\vartheta^1_{\bar{A}_1})^\gamma+(1-\vartheta^1_{\bar{A}_1})^\gamma}, \frac{(1+\vartheta^2_{\bar{A}_1})^\gamma-(1-\vartheta^2_{\bar{A}_1})^\gamma}{(1+\vartheta^2_{\bar{A}_1})^\gamma+(1-\vartheta^2_{\bar{A}_1})^\gamma}, \dots, \frac{(1+\vartheta^p_{\bar{A}_1})^\gamma-(1-\vartheta^p_{\bar{A}_1})^\gamma}{(1+\vartheta^p_{\bar{A}_1})^\gamma+(1-\vartheta^p_{\bar{A}_1})^\gamma}\right),$$

$$\left(\frac{(1+\theta^1_{\bar{A}_1})^\gamma-(1-\theta^1_{\bar{A}_1})^\gamma}{(1+\theta^1_{\bar{A}_1})^\gamma+(1-\theta^1_{\bar{A}_1})^\gamma}, \frac{(1+\theta^2_{\bar{A}_1})^\gamma-(1-\theta^2_{\bar{A}_1})^\gamma}{(1+\theta^2_{\bar{A}_1})^\gamma+(1-\theta^2_{\bar{A}_1})^\gamma}, \dots, \frac{(1+\theta^p_{\bar{A}_1})^\gamma-(1-\theta^p_{\bar{A}_1})^\gamma}{(1+\theta^p_{\bar{A}_1})^\gamma+(1-\theta^p_{\bar{A}_1})^\gamma}\right)\rangle.$$

Definition 2.13 [3] Let

$\overline{A_j} = \langle(a_j, b_j, c_j, d_j); \left(\eta^1_{\bar{A}_j}, \eta^2_{\bar{A}_j}, \dots, \eta^P_{\bar{A}_j}\right), \left(\vartheta^1_{\bar{A}_j}, \vartheta^2_{\bar{A}_j}, \dots, \vartheta^P_{\bar{A}_j}\right), \left(\theta^1_{\bar{A}_j}, \theta^2_{\bar{A}_j}, \dots, \theta^P_{\bar{A}_j}\right)\rangle \in \Lambda$ ($j \in \{1,2,\dots,n\}$) be a collection of NVNT-numbers. Then,

i. weighted arithmetic operator of NVNT-numbers, denoted by NVNTNa_w, is defined as;

$$\mathrm{NVNTNa}_w(\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n) = \sum_{j=1}^{n} w_j \bar{A}_j$$

$$= \left\langle \left[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j\right]; \left(\frac{\prod_{j=1}^{n}(1+\eta_{\bar{A}_j}^{1})^{w_j} - \prod_{j=1}^{n}(1-\eta_{\bar{A}_j}^{1})^{w_j}}{\prod_{j=1}^{n}(1+\eta_{\bar{A}_j}^{1})^{w_j} + \prod_{j=1}^{n}(1-\eta_{\bar{A}_j}^{1})^{w_j}}, \right.\right.$$

$$\left. \frac{\prod_{j=1}^{n}(1+\eta_{\bar{A}_j}^{2})^{w_j} - \prod_{j=1}^{n}(1-\eta_{\bar{A}_j}^{2})^{w_j}}{\prod_{j=1}^{n}(1+\eta_{\bar{A}_j}^{2})^{w_j} + \prod_{j=1}^{n}(1-\eta_{\bar{A}_j}^{2})^{w_j}}, \ldots, \frac{\prod_{j=1}^{n}(1+\eta_{\bar{A}_j}^{p})^{w_j} - \prod_{j=1}^{n}(1-\eta_{\bar{A}_j}^{p})^{w_j}}{\prod_{j=1}^{n}(1+\eta_{\bar{A}_j}^{p})^{w_j} + \prod_{j=1}^{n}(1-\eta_{\bar{A}_j}^{p})^{w_j}}\right),$$

$$\left(\frac{2\prod_{j=1}^{n}(\vartheta_{\bar{A}_j}^{1})^{w_j}}{\prod_{j=1}^{n}(2-\vartheta_{\bar{A}_j}^{1})^{w_j} + \prod_{j=1}^{n}(\vartheta_{\bar{A}_j}^{1})^{w_j}}, \frac{2\prod_{j=1}^{n}(\vartheta_{\bar{A}_j}^{2})^{w_j}}{\prod_{j=1}^{n}(2-\vartheta_{\bar{A}_j}^{2})^{w_j} + \prod_{j=1}^{n}(\vartheta_{\bar{A}_j}^{2})^{w_j}}, \ldots, \right.$$

$$\left. \frac{2\prod_{j=1}^{n}(\vartheta_{\bar{A}_j}^{p})^{w_j}}{\prod_{j=1}^{n}(2-\vartheta_{\bar{A}_j}^{p})^{w_j} + \prod_{j=1}^{n}(\vartheta_{\bar{A}_j}^{p})^{w_j}}\right), \left(\frac{2\prod_{j=1}^{n}(\theta_{\bar{A}_j}^{1})^{w_j}}{\prod_{j=1}^{n}(2-\theta_{\bar{A}_j}^{1})^{w_j} + \prod_{j=1}^{n}(\theta_{\bar{A}_j}^{1})^{w_j}}, \right.$$

$$\left.\left. \frac{2\prod_{j=1}^{n}(\theta_{\bar{A}_j}^{2})^{w_j}}{\prod_{j=1}^{n}(2-\theta_{\bar{A}_j}^{2})^{w_j} + \prod_{j=1}^{n}(\theta_{\bar{A}_j}^{2})^{w_j}}, \ldots, \frac{2\prod_{j=1}^{n}(\theta_{\bar{A}_j}^{p})^{w_j}}{\prod_{j=1}^{n}(2-\theta_{\bar{A}_j}^{p})^{w_j} + \prod_{j=1}^{n}(\theta_{\bar{A}_j}^{p})^{w_j}}\right)\right\rangle$$

ii. weighted geometric operator of NVNT-numbers, denoted by NVNTNg_w, is defined as

$$\mathrm{NVNTNg}_w(\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n) = \prod_{j=1}^{n} \bar{A}_j^{w_j}$$

$$= \left\langle \left[\prod_{j=1}^{n} a_j^{w_j}, \prod_{j=1}^{n} b_j^{w_j}, \prod_{j=1}^{n} c_j^{w_j}, \prod_{j=1}^{n} d_j^{w_j}\right]; \left(\frac{2\prod_{j=1}^{n}(\eta_{\bar{A}_j}^{1})^{w_j}}{\prod_{j=1}^{n}(2-\eta_{\bar{A}_j}^{1})^{w_j} + \prod_{j=1}^{n}(\eta_{\bar{A}_j}^{1})^{w_j}}, \right.\right.$$

$$\left. \frac{2\prod_{j=1}^{n}(\eta_{\bar{A}_j}^{2})^{w_j}}{\prod_{j=1}^{n}(2-\eta_{\bar{A}_j}^{2})^{w_j} + \prod_{j=1}^{n}(\eta_{\bar{A}_j}^{2})^{w_j}}, \ldots, \frac{2\prod_{j=1}^{n}(\eta_{\bar{A}_j}^{p})^{w_j}}{\prod_{j=1}^{n}(2-\eta_{\bar{A}_j}^{p})^{w_j} + \prod_{j=1}^{n}(\eta_{\bar{A}_j}^{p})^{w_j}}\right)$$

$$\left(\frac{\prod_{j=1}^{n}(1+\vartheta_{\bar{A}_j}^{1})^{w_j} - \prod_{j=1}^{n}(1-\vartheta_{\bar{A}_j}^{1})^{w_j}}{\prod_{j=1}^{n}(1+\vartheta_{\bar{A}_j}^{1})^{w_j} + \prod_{j=1}^{n}(1-\vartheta_{\bar{A}_j}^{1})^{w_j}}, \frac{\prod_{j=1}^{n}(1+\vartheta_{\bar{A}_j}^{2})^{w_j} - \prod_{j=1}^{n}(1-\vartheta_{\bar{A}_j}^{2})^{w_j}}{\prod_{j=1}^{n}(1+\vartheta_{\bar{A}_j}^{2})^{w_j} + \prod_{j=1}^{n}(1-\vartheta_{\bar{A}_j}^{2})^{w_j}}, \ldots, \right.$$

$$\left. \frac{\prod_{j=1}^{n}(1+\vartheta_{\bar{A}_j}^{p})^{w_j} - \prod_{j=1}^{n}(1-\vartheta_{\bar{A}_j}^{p})^{w_j}}{\prod_{j=1}^{n}(1+\vartheta_{\bar{A}_j}^{p})^{w_j} + \prod_{j=1}^{n}(1-\vartheta_{\bar{A}_j}^{p})^{w_j}}\right), \left(\frac{\prod_{j=1}^{n}(1+\theta_{\bar{A}_j}^{1})^{w_j} - \prod_{j=1}^{n}(1-\theta_{\bar{A}_j}^{1})^{w_j}}{\prod_{j=1}^{n}(1+\theta_{\bar{A}_j}^{1})^{w_j} + \prod_{j=1}^{n}(1-\theta_{\bar{A}_j}^{1})^{w_j}}\right.$$

$$\frac{\prod_{j=1}^{n}(1+\theta_{\bar{A}_j}^{2})^{w_j}-\prod_{j=1}^{n}(1-\theta_{\bar{A}_j}^{2})^{w_j}}{\prod_{j=1}^{n}(1+\theta_{\bar{A}_j}^{2})^{w_j}+\prod_{j=1}^{n}(1-\theta_{\bar{A}_j}^{2})^{w_j}},\ldots,\frac{\prod_{j=1}^{n}(1+\theta_{\bar{A}_j}^{p})^{w_j}-\prod_{j=1}^{n}(1-\theta_{\bar{A}_j}^{p})^{w_j}}{\prod_{j=1}^{n}(1+\theta_{\bar{A}_j}^{p})^{w_j}+\prod_{j=1}^{n}(1-\theta_{\bar{A}_j}^{p})^{w_j}}\Bigg)\Bigg\rangle$$

where $w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the weight vector of $\bar{A}_j$, $(j \in \{1,2,\ldots,n\})$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 2.14 [3] Let

$A_i = \langle (a_i, b_i, c_i, d_i); \left(\eta_{A_i}^1, \eta_{A_i}^2, \ldots, \eta_{A_i}^P\right), \left(\vartheta_{A_i}^1, \vartheta_{A_i}^2, \ldots, \vartheta_{A_i}^P\right), \left(\theta_{A_i}^1, \theta_{A_i}^2, \ldots, \theta_{A_i}^P\right)\rangle$ be a collection of NVTN-numbers and $I_n = \{1,2,\ldots,n\}$. The positive ideal solution r^+ and negative solution r^- of NVNT-numbers are given as;

$$r^+ = \langle \max_{i\in I_n}\{a_i\}, \max_{i\in I_n}\{b_i\}, \max_{i\in I_n}\{c_i\}, \max_{i\in I_n}\{d_i\}); (\max_{i\in I_n}\left\{\eta_{A_i}^1\right\}, \max_{i\in I_n}\left\{\eta_{A_i}^2\right\}, \ldots, \max_{i\in I_n}\left\{\eta_{A_i}^P\right\}),$$
$$(\min_{i\in I_n}\{(\vartheta_{A_i}^1)\}, \min_{i\in I_n}\{(\vartheta_{A_i}^2)\}, \ldots, \min_{i\in I_n}\{(\vartheta_{A_i}^P)\}), (\min_{i\in I_n}\{(\theta_{A_i}^1)\}, \min_{i\in I_n}\{(\theta_{A_i}^2)\}, \ldots, \min_{i\in I_n}\{(\theta_{A_i}^P)\})\rangle$$

and

$$r^- = \langle \min_{i\in I_n}\{a_i\}, \min_{i\in I_n}\{b_i\}, \min_{i\in I_n}\{c_i\}, \min_{i\in I_n}\{d_i\}); (\min_{i\in I_n}\left\{\eta_{A_i}^1\right\}, \min_{i\in I_n}\left\{\eta_{A_i}^2\right\}, \ldots, \min_{i\in I_n}\left\{\eta_{A_i}^P\right\}),$$

$$(\max_{i\in I_n}\{(\vartheta_{A_i}^1)\}, \max_{i\in I_n}\{(\vartheta_{A_i}^2)\}, \ldots, \max_{i\in I_n}\{(\vartheta_{A_i}^P)\}), (\max_{i\in I_n}\{(\theta_{A_i}^1)\}, \max_{i\in I_n}\{(\theta_{A_i}^2)\}, \ldots, \max_{i\in I_n}\{(\theta_{A_i}^P)\})\rangle$$

respectively.

3. A New Distance Measure for NVNT-Number

In this section, we introduce a new distance measure for N-valued trapezoidal fuzzy neutrosophic number based on centroid points.

Definition 3.1 Let

$A_1 = \langle (a_1, b_1, c_1, d_1); \left(\eta_{A_1}^1, \eta_{A_1}^2, \ldots, \eta_{A_1}^P\right), \left(\vartheta_{A_1}^1, \vartheta_{A_1}^2, \ldots, \vartheta_{A_1}^P\right), \left(\theta_{A_1}^1, \theta_{A_1}^2, \ldots, \theta_{A_1}^P\right)\rangle$ be a NVTN-number.

i) Centroid point of the truth membership functions of A_1 is;

$$(O^{T(1)}, O^{T(2)}, \ldots, O^{T(P)}) = \left(\left(x^T\left(\eta_{A_1}^1\right), y^T\left(\eta_{A_1}^1\right)\right), \left(x^T\left(\eta_{A_1}^2\right), y^T\left(\eta_{A_1}^2\right)\right), \ldots, \left(x^T\left(\eta_{A_1}^P\right), y^T\left(\eta_{A_1}^P\right)\right)\right)$$

Where

$$x^T\left(\eta^i_{A_1}\right)=\frac{\int_{a_1}^{b_1} x f^L_{\eta^i_{A_1}}dx+\int_{b_1}^{c_1} x.\eta^1_{A_i}dx+\int_{c_1}^{d_1} x f^R_{\eta^i_{A_1}}dx}{\int_{a_1}^{b_1} f^L_{\eta^i_{A_1}}dx+\int_{b_1}^{c_1}\eta^1_{A_i}dx+\int_{c_1}^{d_1} f^R_{\eta^i_{A_1}}dx} \text{ and } y^T\left(\eta^i_{A_1}\right)=\frac{\int_0^{\eta^1_{A_i}} y(g^L_{\eta^i_{A_1}}-g^R_{\eta^i_{A_1}})dy}{\int_0^{\eta^1_{A_i}}(g^L_{\eta^i_{A_1}}-g^R_{\eta^i_{A_1}})dy}.$$

$f^L_{\eta^i_{A_1}}=\frac{(x-a_1)}{(b_1-a_1)}\eta^i_{A_1}$, $f^R_{\eta^i_{A_1}}=\frac{(d_1-x)}{(d_1-c_1)}\eta^i_{A_1}$ and $g^L_{\eta^i_{A_1}}=\frac{a_1.\eta^i_{A_1}+x(b_1-a_1)}{\eta^i_{A_1}}$, $g^R_{\eta^i_{A_1}}=\frac{d_1.\eta^i_{A_1}-x.(d_1-c_1)}{\eta^i_{A_1}}$ are inverse functions of $f^L_{\eta^i_{A_1}}$ and $f^R_{\eta^i_{A_1}}$ respectively.

ii) Centroid points of the indeterminacy membership functions of A^i_1 are;

$$(O^{I(1)},O^{I(2)},\dots,O^{I(P)})=\left(\left(x^I(\vartheta^1_{A_1}),y^I(\vartheta^1_{A_1})\right),\left(x^I(\vartheta^2_{A_1}),y^I(\vartheta^2_{A_1})\right),\dots,\left(x^I(\vartheta^P_{A_1}),y^I(\vartheta^P_{A_1})\right)\right)$$

Where

$$x^I(\vartheta^i_{A_1})=\frac{\int_{a_1}^{b_1} x f^L_{\vartheta^i_{A_1}}dx+\int_{b_1}^{c_1} x.\vartheta^i_{A_1}dx+\int_{c_1}^{d_1} x f^R_{\vartheta^i_{A_1}}dx}{\int_{a_1}^{b_1} f^L_{\vartheta^i_{A_1}}dx+\int_{b_1}^{c_1}\vartheta^i_{A_1}dx+\int_{c_1}^{d_1} f^R_{\vartheta^i_{A_1}}dx}, \text{ and} y^I(\vartheta^i_{A_1})=\frac{\int_0^{\vartheta^i_{A_1}} y(g^L_{\vartheta^i_{A_1}}-g^R_{\vartheta^i_{A_1}})dy}{\int_0^{\vartheta^i_{A_1}}(g^L_{\vartheta^i_{A_1}}-g^R_{\vartheta^i_{A_1}})dy}.$$

$f^L_{\vartheta^i_{A_1}}=\frac{(b_1-x)+\vartheta^i_{A_1}(x-a_1)}{(b_1-a_1)}$, $f^R_{\vartheta^i_{A_1}}=\frac{(x-c_1)+\vartheta^i_{A_1}(d_1-x)}{(d_1-c_1)}$ and $g^L_{\vartheta^i_{A_1}}=\frac{b_1-a_1.\vartheta^i_{A_1}-x(b_1-a_1)}{1-\vartheta^i_{A_1}}$, $g^R_{\vartheta^i_{A_1}}=\frac{d_1.\vartheta^i_{A_1}-c_1-x(d_1-c_1)}{\vartheta^i_{A_1}-1}$ the inverse functions of $f^L_{\vartheta^i_{A_1}}$ and $f^R_{\vartheta^i_{A_1}}$ respectively.

iii) Centroid points of the falsify membership functions of A^i_1 are;

$$(O^{F(1)},O^{F(2)},\dots,O^{F(P)})=\left(\left(x^F(\theta^1_{A_1}),y^F(\theta^1_{A_1})\right),\left(x^F(\theta^2_{A_1}),y^F(\theta^2_{A_1})\right),\dots,\left(x^F(\theta^P_{A_1}),y^F(\theta^P_{A_1})\right)\right)$$

Where

$$x^F(\theta^i_{A_1})=\frac{\int_{a_1}^{b_1} x f^L_{\theta^i_{A_1}}dx+\int_{b_1}^{c_1} x.\theta^i_{A_1}dx+\int_{c_1}^{d_1} x f^R_{\theta^i_{A_1}}dx}{\int_{a_1}^{b_1} f^L_{\theta^i_{A_1}}dx+\int_{b_1}^{c_1}\theta^i_{A_1}dx+\int_{c_1}^{d_1} f^R_{\theta^i_{A_1}}dx} \text{ and } y^F(\theta^i_{A_1})=\frac{\int_0^{\theta^i_{A_1}} y(g^L_{\theta^i_{A_1}}-g^R_{\theta^i_{A_1}})dy}{\int_0^{\theta^i_{A_1}}(g^L_{\theta^i_{A_1}}-g^R_{\theta^i_{A_1}})dy}$$

$f^L_{\theta^i_{A_1}}=\frac{(b_1-x)+\theta^i_{A_1}(x-a_1)}{(b_1-a_1)}$, $f^R_{\theta^i_{A_1}}=\frac{(x-c_1)+\theta^i_{A_1}(d_1-x)}{(d_1-c_1)}$ and $g^L_{\theta^i_{A_1}}=\frac{b_1-a_1.\theta^i_{A_1}-x(b_1-a_1)}{1-\theta^i_{A_1}}$, $g^R_{\theta^i_{A_1}}=\frac{d_1.\theta^i_{A_1}-c_1-x(d_1-c_1)}{\theta^i_{A_1}-1}$ the inverse functions of $f^L_{\theta^i_{A_1}}$ and $f^R_{\theta^i_{A_1}}$ respectively.

By computing integrals given above, we get following result;

Result 3.1 Let

$A_1 = \langle (a_1, b_1, c_1, d_1); \left(\eta^1_{A_1}, \eta^2_{A_1}, \ldots, \eta^P_{A_1}\right), \left(\vartheta^1_{A_1}, \vartheta^2_{A_1}, \ldots, \vartheta^P_{A_1}\right), \left(\theta^1_{A_1}, \theta^2_{A_1}, \ldots, \theta^P_{A_1}\right)\rangle$ be a NVTN-number.

i) Centroid points of the truth membership functions of A^i_1 are;

$$(O^{T(1)}, O^{T(2)}, \ldots, O^{T(P)}) = \left(\left(x^T\left(\eta^1_{A_1}\right), y^T\left(\eta^1_{A_1}\right)\right), \left(x^T\left(\eta^2_{A_1}\right), y^T\left(\eta^2_{A_1}\right)\right), \ldots, \left(x^T\left(\eta^P_{A_1}\right), y^T\left(\eta^P_{A_1}\right)\right)\right)$$

Where for i=1,2,...,P;

$x^T\left(\eta^i_{A_1}\right) = \frac{({c_1}^2+{d_1}^2-{a_1}^2-{b_1}^2+c_1d_1-a_1b_1)}{3(c_1+d_1-a_1-b_1)}$ and $y^T\left(\eta^i_{A_1}\right) = \frac{\eta^1_{A_i}.(2b_1+a_1-d_1-2c_1)}{3(b_1+a_1-d_1-c_1)}$

ii) Centroid points of the indeterminacy membership functions of A^i_1 are;

$$(O^{I(1)}, O^{I(2)}, \ldots, O^{I(P)}) = \left(\left(x^I(\vartheta^1_{A_1}), y^I(\vartheta^1_{A_1})\right), \left(x^I(\vartheta^2_{A_1}), y^I(\vartheta^2_{A_1})\right), \ldots, \left(x^I(\vartheta^P_{A_1}), y^I(\vartheta^P_{A_1})\right)\right)$$

Where for i=1,2,...,P;

$x^I(\vartheta^i_{A_1}) = \frac{\left(\vartheta^i_{A_1}+2\right)({a_1}^2-{d_1}^2)+\left(\vartheta^i_{A_1}-1\right)(a_1b_1+{b_1}^2-{c_1}^2-c_1d_1)}{3.[\left(\vartheta^i_{A_1}+1\right)(a_1-d_1)+\left(\vartheta^i_{A_1}-1\right)(b_1-c_1)]}$ and $y^I(\vartheta^i_{A_1}) = \frac{\vartheta^i_{A_1}[a_1+2b_1-2c_1-d_1-3(b_1-c_1)/\vartheta^i_{A_1}]}{3[a_1+b_1-c_1-d_1-\frac{2(b_1-c_1)}{\vartheta^i_{A_1}}]}$

iii) Centroid points of the falsify membership functions of A^i_1 are;

$$(O^{F(1)}, O^{F(2)}, \ldots, O^{F(P)}) = \left(\left(x^F(\theta^1_{A_1}), y^F(\theta^1_{A_1})\right), \left(x^F(\theta^2_{A_1}), y^F(\theta^2_{A_1})\right), \ldots, \left(x^F(\theta^P_{A_1}), y^F(\theta^P_{A_1})\right)\right)$$

Where for i=1,2,...,P;

$x^F(\theta^i_{A_1}) = \frac{\left(\theta^i_{A_1}+2\right)({a_1}^2-{d_1}^2)+\left(\theta^i_{A_1}-1\right)(a_1b_1+{b_1}^2-{c_1}^2-c_1d_1)}{3.[\left(\theta^i_{A_1}+1\right)(a_1-d_1)+\left(\theta^i_{A_1}-1\right)(b_1-c_1)]}$ and $y^F(\theta^i_{A_1}) = \frac{\theta^i_{A_1}[a_1+2b_1-2c_1-d_1-3(b_1-c_1)/\theta^i_{A_1}]}{3[a_1+b_1-c_1-d_1-\frac{2(b_1-c_1)}{\theta^i_{A_1}}]}$

Definition 3.2 Let

$A_1 = \langle (a_1, b_1, c_1, d_1); \left(\eta^1_{A_1}, \eta^2_{A_1}, \ldots, \eta^P_{A_1}\right), \left(\vartheta^1_{A_1}, \vartheta^2_{A_1}, \ldots, \vartheta^P_{A_1}\right), \left(\theta^1_{A_1}, \theta^2_{A_1}, \ldots, \theta^P_{A_1}\right)\rangle$ be a NVTN-number. Centroid point of A_1, denoted by $C(A_1)$, is defined as;

$$C(A_1) = ((r^1_{A_1}, s^1_{A_1}), (r^2_{A_1}, s^2_{A_1}), \ldots, (r^P_{A_1}, s^P_{A_1}))$$

Where

$$(r^i_{A_1}, s^i_{A_1}) = \left(\frac{x^T(A^i_1)+x^I(A^i_1)+x^F(A^i_1)}{3}, \frac{y^T(A^i_1)+y^I(A^i_1)+y^F(A^i_1)}{3}\right) \text{ (i=1,2,...,P)}$$

Lemma 3.1 [28] Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. Then,

$$|x_1y_1 + x_2y_2 + \cdots + x_ny_n| \leq \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2}.\sqrt{{y_1}^2 + {y_2}^2 + \cdots + {y_n}^2}$$

Definition 3.3 Let
$A_1 = \langle [a_1, b_1, c_1, d_1]; \left(\eta^1_{A_1}, \eta^2_{A_1}, \ldots, \eta^P_{A_1}\right), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \ldots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \ldots, \theta^P_{A_1})\rangle$,
$A_2 = \langle [a_2, b_2, c_2, d_2]; \left(\eta^1_{A_2}, \eta^2_{A_2}, \ldots, \eta^P_{A_2}\right), (\vartheta^1_{A_2}, \vartheta^2_{A_2}, \ldots, \vartheta^P_{A_2}), (\theta^1_{A_2}, \theta^2_{A_2}, \ldots, \theta^P_{A_2})\rangle$ and $A_3 = \langle [a_3, b_3, c_3, d_3]; \left(\eta^1_{A_3}, \eta^2_{A_3}, \ldots, \eta^P_{A_3}\right), (\vartheta^1_{A_3}, \vartheta^2_{A_3}, \ldots, \vartheta^P_{A_3}), (\theta^1_{A_3}, \theta^2_{A_3}, \ldots, \theta^P_{A_3})\rangle$
be three $\mathrm{NVTN} -$ numbers and $C(A_1) = \left((r^1_{A_1}, s^1_{A_1}), (r^2_{A_1}, s^2_{A_1}), \ldots, (r^P_{A_1}, s^P_{A_1})\right)$, $C(A_2) = \left((r^1_{A_2}, s^1_{A_2}), (r^2_{A_2}, s^2_{A_2}), \ldots, (r^P_{A_2}, s^P_{A_2})\right)$ and $C(A_3) = \left((r^1_{A_3}, s^1_{A_3}), (r^2_{A_3}, s^2_{A_3}), \ldots, (r^P_{A_3}, s^P_{A_3})\right)$ be centroid point of A_1, A_2 and A_3 respectively.
Distance between A_1 and A_2 is given as;

$$d_q(A_1, A_2) = \sqrt[q]{\frac{1}{2P}\sum_{i=1}^{P} \left(\left|r^i_{A_1} - r^i_{A_2}\right|^q + \left|s^i_{A_1} - s^i_{A_2}\right|^q\right)}$$

By changing value of q, we get some special cases of the distance between A_1 and A_2 as follows;

Case 1: If q=1, we get Hamming distance between A_1 and A_2 as follows;

$$d_1(A_1, A_2) = \frac{1}{2P}\sum_{i=1}^{P} \left(\left|r^i_{A_1} - r^i_{A_2}\right| + \left|s^i_{A_1} - s^i_{A_2}\right|\right)$$

Case 2: If q=2, we get Euclidean distance between A_1 and A_2 as follows;

$$d_2(A_1, A_2) = \sqrt{\frac{1}{2P}\sum_{i=1}^{P} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right)}$$

Case 3: If q→+∞, we get 1. Chebyshev distance measure between A_1 and A_2 as follows;

$$d_{+\infty}(A_1, A_2) = \max\left\{\frac{\left|r^i_{A_1} - r^i_{A_2}\right| + \left|s^i_{A_1} - s^i_{A_2}\right|}{2P}\right\}$$

Teorem 3.1 Let $A_1 = \langle [a_1, b_1, c_1, d_1]; \left(\eta^1_{A_1}, \eta^2_{A_1}, \ldots, \eta^P_{A_1}\right), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \ldots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \ldots, \theta^P_{A_1})\rangle$,
$A_2 = \langle [a_2, b_2, c_2, d_2]; \left(\eta^1_{A_2}, \eta^2_{A_2}, \ldots, \eta^P_{A_2}\right), (\vartheta^1_{A_2}, \vartheta^2_{A_2}, \ldots, \vartheta^P_{A_2}), (\theta^1_{A_2}, \theta^2_{A_2}, \ldots, \theta^P_{A_2})\rangle$ and
$A_3 = \langle [a_3, b_3, c_3, d_3]; \left(\eta^1_{A_3}, \eta^2_{A_3}, \ldots, \eta^P_{A_3}\right), (\vartheta^1_{A_3}, \vartheta^2_{A_3}, \ldots, \vartheta^P_{A_3}), (\theta^1_{A_3}, \theta^2_{A_3}, \ldots, \theta^P_{A_3})\rangle$
be three $\mathrm{NVTN} -$ numbers and $O^{(i)}(r^i_{A_1}, s^i_{A_1})$, $O^{(i)}(r^i_{A_2}, s^i_{A_2})$ and $O^{(i)}(r^i_{A_3}, s^i_{A_3})$ be centroid point of A_1, A_2 and A_3 respectively. Distance between A_1 and A_2 has following conditions;

i) $d_q(A_1, A_2) \geq 0$
ii) $d_q(A_1, A_2) = 0 \Leftrightarrow A_1 = A_2$

iii) $d_q(A_1, A_2) = d_q(A_2, A_1)$

iv) $d_q(A_1, A_2) + d_q(A_2, A_3) \geq d_q(A_1, A_3)$

Proof The teorem is proven for q=2.

i) By using basic mathematical laws, we can get ;

$$(r^i_{A_1} - r^i_{A_2})^2 \geq 0, (s^i_{A_1} - s^i_{A_2})^2 \geq 0$$

$$\Rightarrow (r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2 \geq 0$$
$$\Rightarrow \sum_{i=1}^{p} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right) \geq 0$$
$$\Rightarrow \frac{1}{2P} \sum_{i=1}^{p} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right) \geq 0$$
$$\Rightarrow \sqrt{\frac{1}{2P} \sum_{i=1}^{p} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right)} \geq 0$$
$$\Rightarrow d_2(A_1, A_2) \geq 0$$

ii) $d_2(A_1, A_2) = \sqrt{\frac{1}{2P} \sum_{i=1}^{p} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right)} = 0$

$$\Rightarrow \frac{1}{2P} \sum_{i=1}^{p} \left(\left(r^i_{A_1} - r^i_{A_2}\right)^2 + \left(s^i_{A_1} - s^i_{A_2}\right)^2\right) = 0$$
$$\Rightarrow \sum_{i=1}^{p} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right) = 0$$

$$\Rightarrow (r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2 = 0.$$

Therefore we get;

$$(r^i_{A_1} - r^i_{A_2})^2 = 0 \Leftrightarrow r^i_{A_1} = r^i_{A_2},$$
$$(s^i_{A_1} - s^i_{A_2})^2 = 0 \Leftrightarrow s^i_{A_1} = s^i_{A_2}.$$

That means

$$A_1 = A_2$$

iii)

$$d_2(A_1, A_2) = \sqrt{\frac{1}{2P} \sum_{i=1}^{p} \left((r^i_{A_1} - r^i_{A_2})^2 + (s^i_{A_1} - s^i_{A_2})^2\right)}$$
$$= \sqrt{\frac{1}{2P} \sum_{i=1}^{p} \left((r^i_{A_2} - r^i_{A_1})^2 + (s^i_{A_2} - s^i_{A_1})^2\right)}$$
$$= d_2(A_2, A_1)$$

iv) By using Cauchy-Schwarz inequality given in Lemma 3.1;

$$(r^i_{A_1} + s^i_{A_1})^2 + (r^i_{A_2} + s^i_{A_2})^2 = ({r^i_{A_1}}^2 + {r^i_{A_2}}^2) + ({s^i_{A_1}}^2 + {s^i_{A_2}}^2) + 2.(r^i_{A_1} s^i_{A_1} + r^i_{A_2} s^i_{A_2})$$

$$\leq ({r^i_{A_1}}^2 + {r^i_{A_2}}^2) + ({s^i_{A_1}}^2 + {s^i_{A_2}}^2) + 2.\sqrt{{r^i_{A_1}}^2 + {r^i_{A_2}}^2}\sqrt{{s^i_{A_1}}^2 + {s^i_{A_2}}^2}$$

$$= \left(\sqrt{{r^i_{A_1}}^2 + {r^i_{A_2}}^2} + \sqrt{{s^i_{A_1}}^2 + {s^i_{A_2}}^2}\right)^2$$

Taking square roots gives;

$$\sqrt{(r^i_{A_1}+s^i_{A_1})^2+(r^i_{A_2}+s^i_{A_2})^2} \le \sqrt{{r^i_{A_1}}^2+{r^i_{A_2}}^2}+\sqrt{{s^i_{A_1}}^2+{s^i_{A_2}}^2}$$

Now we can compute

$$\sqrt{(r^i_{A_1}-r^i_{A_3})^2+(s^i_{A_1}-s^i_{A_3})^2}$$

$$=\sqrt{\left(\left(r^i_{A_1}-r^i_{A_2}\right)+\left(r^i_{A_2}-r^i_{A_3}\right)\right)^2+\left(\left(s^i_{A_1}-s^i_{A_2}\right)+\left(s^i_{A_2}-s^i_{A_3}\right)\right)^2}$$

$$\le\sqrt{(r^i_{A_1}-r^i_{A_2})^2+(s^i_{A_1}-s^i_{A_2})^2}+\sqrt{(r^i_{A_2}-r^i_{A_3})^2+(s^i_{A_2}-s^i_{A_3})^2}$$

Therefore we get;

$$d_2(A_1,A_2)+d_2(A_2,A_3)\ge d_2(A_1,A_3)$$

4. An approach to MCDM problems under for N-valued Neutrosophic Trapezoidal Numbers

Definition 4.1 [3] Let $X=\{x_1,x_2,\dots,x_m\}$ be a set of alternatives, $U=\{u_1,u_2,\dots,u_n\}$ be the set of criteria. Assume that the evaluating value of criteria $u_j$with respect to alternative x_i, be represented by a NVNT-numbers

$A_{ij}=$

$\langle[a_{ij},b_{ij},c_{ij},d_{ij}];\left(\eta^1_{A_{ij}},\eta^2_{A_{ij}},\eta^3_{A_{ij}},\dots,\eta^P_{A_{ij}}\right),\left(\vartheta^1_{A_{ij}},\vartheta^2_{A_{ij}},\vartheta^3_{A_{ij}},\dots,\vartheta^P_{A_{ij}}\right),(\theta^1_{A_{ij}},\theta^2_{A_{ij}},\theta^3_{A_{ij}},\dots,\theta^p_{A_{ij}})\rangle_{m\times n}$

where $0\le a_{ij}\le b_{ij}\le c_{ij}\le d_{ij}\le 1$, $0\le$

$\eta^1_{A_{ij}},\eta^2_{A_{ij}},\eta^3_{A_{ij}},\dots,\eta^P_{A_{ij}},\vartheta^1_{A_{ij}},\vartheta^2_{A_{ij}},\vartheta^3_{A_{ij}},\dots,\vartheta^P_{A_{ij}},\theta^1_{A_{ij}},\theta^2_{A_{ij}},\theta^3_{A_{ij}},\dots,\theta^p_{A_{ij}}\le 1$, and $\left(A_{ij}\right)_{mxn}=$

$\left((a_{ij},b_{ij},c_{ij},d_{ij});\left(\eta^1_{A_{ij}},\eta^2_{A_{ij}},\eta^3_{A_{ij}},\dots,\eta^p_{A_{ij}}\right),\left(\vartheta^1_{A_{ij}},\vartheta^2_{A_{ij}},\vartheta^3_{A_{ij}},\dots,\vartheta^p_{A_{ij}}\right),(\theta^1_{A_{ij}},\theta^2_{A_{ij}},\theta^3_{A_{ij}},\dots,\theta^p_{A_{ij}})\right)_{mxn}$,

(i=1,2,…,m) and (j=1,2,…,n) be the decision matrix given by experts based on Table 1.

Then,

$$(A_{ij})_{mxn}=\begin{pmatrix}A_{11} & A_{12} & \cdots & A_{1n}\\ A_{21} & A_{22} & \cdots & A_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ A_{m1} & A_{m2} & \cdots & A_{mn}\end{pmatrix}$$

is called a NVNT-numbers multi-criteria decision matrix of the decision maker.

Table 1 [3]. The linguistic values of the NVNT-numbers for the evaluation matrix

Linguistic values	NVNT-number values
Very Poor(VP)	$\langle(0.56,0.60,0.65,0.68);(0.8,0.7,0.6),(0.3,0.4,0.3),(0.7,0.2,0.3)\rangle$
Poor(P)	$\langle(0.86,0.88,0.90,1.00);(0.9,0.5,02),(0.7,0.6,0.4),(0.3,0.1,0.2)\rangle$
Medium (M)	$\langle(0.18,0.23,0.30,0.35);(0.6,0.4,0.1),(0.5,0.4,0.3),(0.2,0.4,0.3)\rangle$
Good (G)	$\langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle$
Very Good(VG)	$\langle(0.00,0.12,0.15,0.20);(0.2,0.4,0.2),(0.3,0.4,0.6),(0.2,0.2,0.5)\rangle$

Also, assume that w= $(w_1, w_2, \dots, w_n)$ be weight vector of the criteria set U given by experts.

Algorithm

Step 1: Create an evaluation matrix $(A_{ij})_{m\times n}$ based on Table 1.

Step 2: For all i $(i = 1,2,\dots, m)$ find the aggregation values according to $NVNTNa_w$ operator, in order to obtain the ultimate performance value corresponding to the alternative $A_i(i = 1,2,\dots, m)$ as;

$$A_i = NVNTNa_w(A_{i1}, A_{i2},\dots, A_{in})\ (i = 1,2,\dots, m)$$

Step 3: Find the centroid points of A_i $(i = 1,2,\dots, m)$ for truth, indeterminacy and falsity memberships according to Result 3.1;

$$(O^{T(1)}, O^{T(2)}, \dots, O^{T(P)}) = \left(\left(x^T(A_i^1), y^T(A_i^1)\right), \left(x^T(A_i^2), y^T(A_i^2)\right), \dots, \left(x^T\left(A_i^P\right), y^T\left(A_i^P\right)\right)\right)$$

$$(O^{I(1)}, O^{I(2)}, \dots, O^{I(P)}) = \left(\left(x^I(A_i^1), y^I(A_i^1)\right), \left(x^I(A_i^2), y^I(A_i^2)\right), \dots, \left(x^I\left(A_i^P\right), y^I\left(A_i^P\right)\right)\right)$$

$$(O^{F(1)}, O^{F(2)}, \dots, O^{F(P)}) = \left(\left(x^F(A_i^1), y^F(A_i^1)\right), \left(x^F(A_i^2), y^F(A_i^2)\right), \dots, \left(x^F\left(A_i^P\right), y^F\left(A_i^P\right)\right)\right)$$

Step 4: Find the centroid point of $A_i(i = 1,2, \dots, m)$givenin Definition 3.2;

$$C(A_i) = ((r_{A_i}^1, s_{A_i}^1), (r_{A_i}^2, s_{A_i}^2), \dots, (r_{A_i}^P, s_{A_i}^P))$$

Step 5: Find the distances between A_i $(i = 1,2,\dots,m)$ and positive-negative ideal solution based on Definition 3.3 as;

$$d_2(A_i, r^+), d_2(A_i, r^-)(i = 1,2,\dots,m)$$

Where r^+ and r^- are the positive ideal solution and negative ideal solution of A_i (i=1,2,…,m) respectively. That is,

$$r^+ = \langle \max_{i\in\{1,2,\dots,m\}}\{a_i\}, \max_{i\in\{1,2,\dots,m\}}\{b_i\}, \max_{i\in\{1,2,\dots,m\}}\{c_i\}, \max_{i\in\{1,2,\dots,m\}}\{d_i\}\Big);$$
$$(\max_{i\in\{1,2,\dots,m\}}\left\{\eta^1_{A_i}\right\}, \max_{i\in\{1,2,\dots,m\}}\left\{\eta^2_{A_i}\right\}, \dots, \max_{i\in\{1,2,\dots,m\}}\left\{\eta^P_{A_i}\right\}),$$
$$(\min_{i\in\{1,2,\dots,m\}}\{(\vartheta^1_{A_i})\}, \min_{i\in\{1,2,\dots,m\}}\{(\vartheta^2_{A_i})\}, \dots, \min_{i\in\{1,2,\dots,m\}}\{(\vartheta^P_{A_i})\}),$$
$$(\min_{i\in\{1,2,\dots,m\}}\{(\theta^1_{A_i})\}, \min_{i\in\{1,2,\dots,m\}}\{(\theta^2_{A_i})\}, \dots, \min_{i\in\{1,2,\dots,m\}}\{(\theta^P_{A_i})\})\rangle$$

and

$$r^- = \langle \min_{i\in\{1,2,\dots,m\}}\{a_i\}, \min_{i\in\{1,2,\dots,m\}}\{b_i\}, \min_{i\in\{1,2,\dots,m\}}\{c_i\}, \min_{i\in\{1,2,\dots,m\}}\{d_i\}\Big);$$
$$(\min_{i\in\{1,2,\dots,m\}}\left\{\eta^1_{A_i}\right\}, \min_{i\in\{1,2,\dots,m\}}\left\{\eta^2_{A_i}\right\}, \dots, \min_{i\in\{1,2,\dots,m\}}\left\{\eta^P_{A_i}\right\}),$$
$$(\max_{i\in\{1,2,\dots,m\}}\{(\vartheta^1_{A_i})\}, \max_{i\in\{1,2,\dots,m\}}\{(\vartheta^2_{A_i})\}, \dots, \max_{i\in\{1,2,\dots,m\}}\{(\vartheta^P_{A_i})\}),$$
$$(\max_{i\in\{1,2,\dots,m\}}\{(\theta^1_{A_i})\}, \max_{i\in\{1,2,\dots,m\}}\{(\theta^2_{A_i})\}, \dots, \max_{i\in\{1,2,\dots,m\}}\{(\theta^P_{A_i})\})\rangle$$

Step 6: Calculate the score value $s(A_i)$ of the A_i $(i = 1,2,\dots,m)$ defined as;

$$s(A_i) = \frac{d_2(A_i, r^-)}{d_2(A_i, r^+) + d_2(A_i, r^-)}$$

Step 7: Rank all the alternatives A_i $(i = 1,2,\dots,m)$and select the best one, in accordance with score of each A_i $(S(A_i))$. The smaller the $S(A_i)$, the better the alternatives A_i .

5. Application

In order to show usefulness of proposed method, we give following application adopted from Kesen [13].

Suppose that a factory administration is aim to hire a technician for newly-established section of the factory. The administration doesn't know exactly who is suitable for that position since there are a lot ofalternatives. After a short consideration, the administration managed to shrink the alternatives' list and five alternatives left for choosing which is$X = \{x_1, x_2, x_3, x_4, x_5\}$. The administration will choose a technician from alternatives' list according to four attributes:

1. Work experience (c_1)
2. Tendency to team work (c_2)
3. Reference (c_3)
4. Education backround (c_4)

Weight vector of the attributes is $w = (0.3,0.2,0.4,0.1)$. The administration considers the alternatives in the context of the linguistic terms given in Table 1. The process of finding the best choice is given as follows:

Step 1: Alternatives and attributes evaluated by the administrationand results of the evaluation are presented in decision matrix $(A_{ij})_{5\times 4}$ as;

$$(A_{ij})_{5x4} = \begin{pmatrix} \langle(0.56,0.60,0.65,0.68);(0.8,0.7,0.6),(0.3,0.4,0.3),(0.7,0.2,0.3)\rangle \\ \langle(0.18,0.23,0.30,0.35);(0.6,0.4,0.1),(0.5,0.4,0.3),(0.2,0.4,0.3)\rangle \\ \langle(0.86,0.88,0.90,1.00);(0.9,0.5,0.2),(0.7,0.6,0.4),(0.3,0.1,0.2)\rangle \\ \langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle \\ \langle(0.00,0.12,0.15,0.20);(0.2,0.4,0.2),(0.3,0.4,0.6),(0.2,0.2,0.5)\rangle \end{pmatrix}$$

$$\begin{matrix} \langle(0.18,0.23,0.30,0.35);(0.6,0.4,0.1),(0.5,0.4,0.3),(0.2,0.4,0.3)\rangle \\ \langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle \\ \langle(0.56,0.60,0.65,0.68);(0.8,0.7,0.6),(0.3,0.4,0.3),(0.7,0.2,0.3)\rangle \\ \langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle \\ \langle(0.00,0.12,0.15,0.20);(0.2,0.4,0.2),(0.3,0.4,0.6),(0.2,0.2,0.5)\rangle \end{matrix}$$

$$\begin{matrix} \langle(0.00,0.12,0.15,0.20);(0.2,0.4,0.2),(0.3,0.4,0.6),(0.2,0.2,0.5)\rangle \\ \langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle \\ \langle(0.86,0.88,0.90,1.00);(0.9,0.5,02),(0.7,0.6,0.4),(0.3,0.1,0.2)\rangle \\ \langle(0.18,0.23,0.30,0.35);(0.6,0.4,0.1),(0.5,0.4,0.3),(0.2,0.4,0.3)\rangle \\ \langle(0.56,0.60,0.65,0.68);(0.8,0.7,0.6),(0.3,0.4,0.3),(0.7,0.2,0.3)\rangle \end{matrix}$$

$$\begin{matrix} \langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle \\ \langle(0.56,0.60,0.65,0.68);(0.8,0.7,0.6),(0.3,0.4,0.3),(0.7,0.2,0.3)\rangle \\ \langle(0.86,0.88,0.90,1.00);(0.9,0.5,0.2),(0.7,0.6,0.4),(0.3,0.1,0.2)\rangle \\ \langle(0.18,0.23,0.30,0.35);(0.6,0.4,0.1),(0.5,0.4,0.3),(0.2,0.4,0.3)\rangle \\ \langle(0.72,0.78,0.80,0.85);(0.9,0.8,0.3),(0.2,0.1,0.5),(0.5,0.3,0.4)\rangle \end{matrix}$$

Step 2: For all i $(i = 1,2,\ldots,5)$, the aggregation values according to $NVNTNa_w$ operator are computed, in order to obtain the ultimate performance value corresponding to the alternative $x_i (i = 1,2,\ldots,5)$ as;

$$A_1 = NVNTNa_w(A_{11}, A_{12}, A_{13}, A_{14})$$
$$= \langle(0.312,0.374,0.425,0.469);(0.661,0.554,0.309),(0.357,0.352,0.367),(0.332,0.277,0.294)\rangle$$

$A_2 = NVNTNa_w(A_{21}, A_{22}, A_{23}, A_{24})$
$= \langle 0.542, 0.597, 0.635, 0.683); (0.834, 0.703, 0.278), (0.279, 0.179, 0.411), (0.402, 0.315, 0.357)\rangle$

$A_3 = NVNTNa_w(A_{31}, A_{32}, A_{33}, A_{34})$
$= \langle 0.740, 0.768, 0.800, 0.872); (0.867, 0.589, 0.379), (0.513, 0.514, 0.357), (0.432, 0.133, 0.236)\rangle$

$A_4 = NVNTNa_w(A_{41}, A_{42}, A_{43}, A_{44})$
$= \langle 0.558, 0.615, 0.650, 0.700); (0.845, 0.714, 0.242), (0.268, 0.155, 0.432), (0.387, 0.328, 0.368)\rangle$

$A_5 = NVNTNa_w(A_{51}, A_{52}, A_{53}, A_{54})$
$= \langle 0.184, 0.282, 0.315, 0.361); (0.469, 0.523, 0.302), (0.288, 0.352, 0.518), (0.291, 0.208, 0.444)\rangle$

Step 3: Centroid points of A_i $(i = 1,2,\ldots,5)$ for truth, indeterminacy and falsity memberships are computed as;

For A_1;

$(O_{A_1}{}^{T(1)}, O_{A_1}{}^{T(2)}, O_{A_1}{}^{T(3)})$=((0.394,0.274),(0.394,0.230),(0.394,0.128))

$(O_{A_1}{}^{I(1)}, O_{A_1}{}^{I(2)}, O_{A_1}{}^{I(3)}) = ((0.388, 0.260), (0.388, 0.253), (0.388, 0.276))$

$(O_{A_1}{}^{F(1)}, O_{A_1}{}^{F(2)}, O_{A_1}{}^{F(3)}) = ((0.388, 0.225)(0.387, 0.169)(0.387, 0.185))$

For A_2;

$(O_{A_2}{}^{T(1)}, O_{A_2}{}^{T(2)}, O_{A_2}{}^{T(3)})$=((0.614,0.337), (0.614,0.28),(0.614,0.112))

$(O_{A_2}{}^{I(1)}, O_{A_2}{}^{I(2)}, O_{A_2}{}^{I(3)}) = ((0.611, 0.190), (0.611, 0.102), (0.612, 1.376))$

$(O_{A_2}{}^{F(1)}, O_{A_2}{}^{F(2)}, O_{A_2}{}^{F(3)}) = ((0.612, 0.885), (0.6110.245), (0.612, 0.361))$

For A_3;

$(O_{A_3}{}^{T(1)}, O_{A_3}{}^{T(2)}, O_{A_3}{}^{T(3)})$=((0.797,0.346),(0.797,0.235),(0.797,0.151))

$(O_{A_3}{}^{I(1)}, O_{A_3}{}^{I(2)}, O_{A_3}{}^{I(3)}) = ((0.810, 0.038), (0.810, 0.039), (0.812, 0.572))$

$(O_{A_3}{}^{F(1)}, O_{A_3}{}^{F(2)}, O_{A_3}{}^{F(3)}) = ((0.811, -0.236), (0.816, 0.073), (0.814, 0.155))$

For A_4;

$(O_{A_4}{}^{T(1)}, O_{A_4}{}^{T(2)}, O_{A_4}{}^{T(3)})$=((0.630,0.337),(0.630,0.285),(0.630,0.097))

$(O_{A_4}{}^{I(1)}, O_{A_4}{}^{I(2)}, O_{A_4}{}^{I(3)}) = ((0.628, 0.190), (0.627, 0.088), (0.628, -0.300))$

$(O_{A_4}{}^{F(1)}, O_{A_4}{}^{F(2)}, O_{A_4}{}^{F(3)}) = ((0.628, 1.922), (0.628, 0.323), (0.628, 0.672))$

For A_5;

$(O_{A_5}{}^{T(1)}, O_{A_5}{}^{T(2)}, O_{A_5}{}^{T(3)})$=((0.283,0.181),(0.283,0.202),(0.283,0.116))

$(O_{A_5}{}^{I(1)}, O_{A_5}{}^{I(2)}, O_{A_5}{}^{I(3)}) = ((0.265,0.511),(0.266,-0.196),(0.268,0.109))$

$(O_{A_5}{}^{F(1)}, O_{A_5}{}^{F(2)}, O_{A_5}{}^{F(3)}) = ((0.265,0.554),(0.264,0.151),(0.268,0.048))$

Step 4: Centroid point of A_i $(i = 1,2,\ldots,5)$ is computed as;

$C(A_1) = ((0.390,0.253),(0.390,0.217),(0.390,0.196))$

$C(A_2) = ((0.612,0.471),(0.612,0.210),(0.612,0.616))$

$C(A_3) = ((0.806,0.049),(0.808,0.116),(0.808,0.293))$

$C(A_4) = ((0.629,0.816),(0.629,0.232),(0.629,0.156))$

$C(A_5) = ((0.271,0.415),(0.271,0.052),(0.273,0.091))$

Step 5: Distances between A_i $(i = 1,2,\ldots,5)$and positive ideal-negative ideal computed as;

$d_2(A_1, r^+)$=0.300, $d_2(A_1, r^-)$=0.187
$d_2(A_2, r^+)$=0.212, $d_2(A_2, r^-)$=0.385
$d_2(A_3, r^+)$=0.086, $d_2(A_3, r^-)$=0.403
$d_2(A_4, r^+)$=0.269 $d_2(A_4, r^-)$=0.415
$d_2(A_5, r^+)$=0.396, $d_2(A_5, r^-)$=0.151

Step 6: Score value $(s(A_i))$ of the A_i $(i = 1,2,\ldots,5)$ is computed as;

$s(A_1)$ =0.383, $s(A_2)$ =0.645, $s(A_3)$ =0.824, $s(A_4)$ =0,606, $s(A_5)$ =0.275

Step 7: Ranking of all the alternatives A_i $(i = 1,2,\ldots,5)$ is given as following;

$$x_5 > x_1 > x_4 > x_2 > x_3$$

6. Conclusion

Neutrosophic numbers can be applied to many more areas to model and solve problems containing many uncertainties. For example, studies can be applied on computer science, decision-making problems, business and economics problems, which contain ambiguous statements by their nature. For this reason, neutrosophic numbers and their operations can be extended by using different applications and techniques. As for multi-valued neutrosophic numbers, it can be applied in solving problems with uncertain, imprecise, incomplete and inconsistent information that exist in scientific and engineering situations.

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Neutrosophic Algebraic Structures and Their Applications

<u>Neutrosophic Logic</u> is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionistic logic, etc. University of New Mexico (UNM)'s website on neutrosophic theories and their applications is: http://fs.unm.edu/neutrosophy.htm.

From Classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures

In 2019 Smarandache generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) {whose operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations and axioms are totally false} and on 2020 he continued to develop them [2,3,4].

The NeutroAlgebras & AntiAlgebras are a *new field of research*, which is inspired from our real world.

In classical algebraic structures, all operations are 100% well-defined, and all axioms are 100% true, but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some operations or some laws.

Using the process of *NeutroSophication* of a classical algebraic structure we produce a NeutroAlgebra, while the process of *AntiSophication* of a classical algebraic structure produces an AntiAlgebra.

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