NEUTROSOPHIC SETS AND SYSTEMS

{Special Issue: Mediterranean Conference on Three Decades of Neutrosophic and Plithogenic Theories and Applications (Me-CoNeT 2024)}

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Information for Authors and Suscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their inter actions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).
According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of]-0, 1+[.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.
Neutrosophic Statistics is a generalization of the classical statistics.
What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, igno- rance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:
http://fs.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science: http://fs.unm.edu/ScienceLibrary.htm.

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Preface: Neutrosophic Sets and Systems {Special Issue: Mediterranean Conference on Three Decades of Neutrosophic and Plithogenic Theories and Applications (MeCoNeT 2024)}, Vol. 73, 2024.

This volume contains the proceedings of the Mediterranean Conference on Neutrosophic Theory (MeCoNeT 2024), held at the Accademia Peloritana dei Pericolanti of the University of Messina on September 24-25, 2024. The event was organized by the MIFT Department (Mathematics, Computer Science, Physics, and Earth Sciences) of the University of Messina, marking the first international congress on neutrosophic theories outside the Americas. This milestone has firmly established the Mediterranean region as a key hub for research in the rapidly growing field of neutrosophic theory.

Neutrosophic theory, first introduced by Professor Florentin Smarandache in the mid-1990s, extends classical and fuzzy logic by incorporating the concept of indeterminacy. Its innovative approach offers precise mathematical tools to model uncertainty, contradiction, and incomplete information. With applications spanning artificial intelligence, decision-making, data analysis, economics, and biology, neutrosophic theory continues to find increasing relevance in various scientific fields.

The MeCoNeT 2024 conference drew over 100 participants from more than 15 countries, with more than 50 scientific contributions selected through a rigorous peer review process. The hybrid format of the event—featuring in-person sessions at the historical Accademia Peloritana dei Pericolanti and online parallel sessions—allowed for broad international participation. The conference thus offered an ideal platform for sharing interdisciplinary research and addressing contemporary challenges in mathematics and beyond.

We were privileged to welcome a distinguished lineup of keynote speakers, who shared their expertise on the latest advancements in neutrosophic theory and its applications. These speakers included Prof. Farkhanda Afzal (National University of Sciences and Technology, Pakistan), Prof. Bhimraj Basumatary (Bodoland University, India), Prof. Said Broumi (Hassan II University, Morocco), Prof. Paulraj Gnanachandra (Ayya Nadar Janaki Ammal College, India), Prof. Saeid Jafari (Mathematical and Physical Science Foundation, Denmark), Prof. Hovik Matevossian (Russian Academy of Sciences, Moscow), Prof. Giovanni Molica Bisci (University of Urbino, Italy), and Prof. Florentin Smarandache (University of New Mexico, USA).

We would like to extend our heartfelt thanks to the members of the MeCoNeT 2024 Scientific Committee for their essential role in organizing and reviewing the contributions to this event. The committee members are: Mohamed Abdel-Basset (Zagazig University, Egypt), Reneta P. Barneva (State University of New York, Fredonia, USA), Bhimraj Basumatary (Bodoland University, India), Valentin E. Brimkov (State University of New York, Buffalo, USA), Said Broumi (Hassan II University, Morocco), Mario De Salvo (University of Messina, Italy), Matteo Gorgone (University of Messina, Italy), Saeid Jafari (Mathematical and Physical Science Foundation, Denmark), Hovik Matevossian (Russian Academy of Sciences, Moscow, Russia), Francesco Oliveri (University of Messina, Italy), Patrizia Rogolino (University of Messina, Italy), and Ahmed A. Salama (Port Said University, Egypt).

We are also indebted to the University of Messina, one of Italy's oldest academic institutions, founded in 1548, and recognized globally for its contributions to scientific research and international collaboration. The university plays a leading role in advancing knowledge across various fields, including mathematics and computational sciences. Hosting MeCoNeT 2024 further strengthens its position as a central hub for global academic discourse on mathematical uncertainty.

Our sincere gratitude extends to the Accademia Peloritana dei Pericolanti, established in 1729, which has long been a pillar of academic and cultural life in Messina. Renowned for promoting research across both the humanities and sciences, the Accademia continues to be a beacon of scholarly activity. Its symbolic emblem, a ship navigating the turbulent waters of the Strait of Messina, represents the intellectual challenges faced by researchers as they strive for knowledge and truth.

Additionally, the MIFT Department of the University of Messina—comprising the disciplines of mathematics, computer science, physics, and earth sciences—played a pivotal role in organizing MeCoNeT 2024. The department is at the forefront of interdisciplinary research, fostering collaboration across fields to address complex scientific challenges, such as those explored during this conference.

Additionally, special thanks go to our institutional sponsors, including Regione Calabria, the Città Metropolitana di Reggio Calabria, the Comune di Messina, the Istituto Nazionale di Alta Matematica "Francesco Severi" (INDAM), and Calabria Formazione, whose generous contributions were instrumental in the success of the conference.

The opening session of the conference was marked by a keynote address from Prof. Smarandache, titled "Three Decades of Neutrosophic and Plithogenic Theories with their Applications (1995-2024)". This compelling presentation provided a comprehensive overview of the theory's evolution and the breadth of its applications. Throughout the two-day event, 8 keynote lectures and over 50 research presentations were delivered, covering a wide range of topics that highlighted the diversity and depth of current research in neutrosophic theory.

A highlight of the conference was the symbolic donation of four volumes of Professor Smarandache's works, alongside 870 digital books on neutrosophic topics, to the Accademia Peloritana dei Pericolanti, received by Prof. Giovanni Cupaiolo on behalf of the institution. This gesture solidified the scientific and cultural connection between the father of neutrosophic theory and this historic institution in Messina.

The MeCoNeT 2024 conference was followed closely by local Italian media from the moment of its announcement to the conclusion of the event. Several notable local outlets provided coverage, emphasizing both the importance of the conference and the success it achieved.

Before the event, prominent media platforms such as StrettoWeb introduced the upcoming congress, highlighting Messina's role as host for this groundbreaking international gathering on neutrosophic theory. The announcement stressed that this was the first conference of its kind to be held outside of the Americas, underscoring the significance of this milestone for the region. Articles on CalabriaPost, TempoStretto and Il Reggino described the event as pivotal for advancements in mathematical theories related to uncertainty and contradiction, helping raise awareness within the scientific community and the general public.

StrettoWeb:


Messina, ecco il Congresso Internazionale sulla teoria Neutrofisica: i più grandi esperti mondiali sullo Stretto

MeCoNeT 2024
Mediterranean Conference on Neutrosophic Theory
Sala dell'Accademia dei Pericolanti
Edificio Centrale dell'Università di Messina
24-25 September 2024

Messina, here is the International Congress on Neutrophysical Theory: the world's leading experts on the Strait" Messina will host a prestigious international event that will bring together experts in neutrosophic theory from around the world. This conference marks a historic moment as it is the first to be held outside of the Americas, further cementing the city's role in global scientific discussions."

Il Reggino:


A Messina il MeCoNeT 2024, il primo congresso internazionale sulla "matematica dell'incertezza"

In programma a fine settembre. Sarà presieduto dal reggino Giorgio Nordo

In Messina, MeCoNeT 2024, the first international congress on the "mathematics of uncertainty"

"The MeCoNeT 2024 conference will place Messina at the forefront of mathematical research into uncertainty. It is an unprecedented occasion that will see leading scholars convene to explore the applications of neutrosophic theory, highlighting the city's growing importance on the world stage."

PianaInforma:


Tutto pronto a Messina per il MeCoNeT 2024, l'appuntamento internazionale sulla matematica dell'incertezza

All set in Messina for MeCoNeT 2024: the international meeting on the mathematics of uncertainty

"Messina is preparing to welcome an international cohort of researchers for the MeCoNeT 2024 conference. With a focus on neutrosophic theory and mathematical uncertainty, the event promises to bring new insights and foster collaboration among global experts."

TempoStretto.it:


MeConNeT, a Messina evento internazionale sulla matematica dell'incertezza

MeCoNeT: an international event on the mathematics of uncertainty in Messina

"Messina is set to become a global reference point in the mathematical study of uncertainty, thanks to MeCoNeT 2024. The hybrid format will allow researchers from all corners of the globe to participate, showcasing the city's ability to host major international events."

CalabriaPost:


TUTTO PRONTO A MESSINA PER IL MeCoNeT 2024, L'APPUNTAMENTO INTERNAZIONALE SULLA MATEMATICA DELL'INCERTEZZA

All set in Messina for MeCoNeT 2024: the international meeting on the mathematics of uncertainty

"Messina is poised to become a focal point of scientific discussions with the arrival of MeCoNeT 2024. The conference will gather international scholars to delve into neutrosophic theory, a growing field that addresses uncertainty and complexity in mathematics."

After the event, media outlets reported on the success of the conference, highlighting the participation of over 100 scholars from around the world. Coverage by PianaInforma and Progetto Touring underlined how MeCoNeT 2024 helped position Messina as a global hub for research in mathematical theories of uncertainty. Articles recounted the depth and diversity of contributions, from keynote speeches to research presentations.

TempoStretto.it:


MeCoNeT 2024: a Messina evento internazionale sulla matematica dell'incertezza

MeCoNeT 2024: international event in Messina on the mathematics of uncertainty

"The success of MeCoNeT 2024 has affirmed Messina's place as a leader in hosting scientific events of global significance. Over 100 scholars, with contributions spanning the latest discoveries in neutrosophic theory, have made this a truly international event."

PianaInforma:


Successo del Congresso MeCoNeT 2024: Un evento internazionale dedicato alla matematica dell'incertezza

Successful MeCoNeT 2024 conference: An international event dedicated to the mathematics of uncertainty

"The success of MeCoNeT 2024 marks a turning point for Messina, which has now positioned itself as a center of excellence for research on mathematical uncertainty. This event brought together some of the brightest minds in the field, showcasing the city's ability to host prestigious international conferences."

CalabriaPost:


SUCCESSO DEL CONGRESSO MeCoNeT 2024: UN EVENTO INTERNAZIONALE DEDICATO ALLA MATEMATICA DELL'INCERTEZZA

Successful MeCoNeT 2024 conference: An international event dedicated to the mathematics of uncertainty

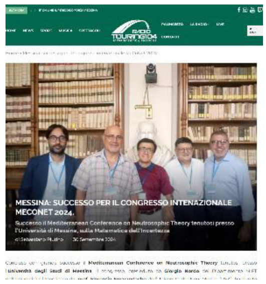
"The MeCoNeT 2024 conference was a clear success, confirming the increasing global importance of neutrosophic theory. The city of Messina, in particular, has demonstrated its growing relevance in the international scientific community."

StrettoWeb:

[Success for MeCoNeT 2024, the international conference on Neutrosophic Theory held at the University of Messina]

"The overwhelming success of the MeCoNeT 2024 conference underscores the growing international recognition of neutrosophic theory. The event saw participation from top global scholars, further strengthening the ties between Messina and cutting-edge scientific research."

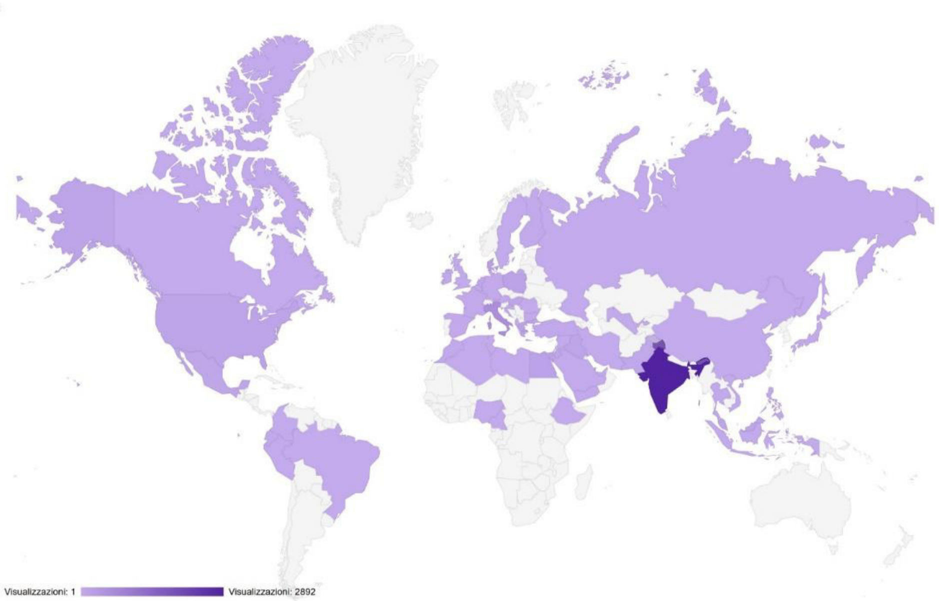
Progetto Touring:



[Success for MeCoNeT 2024: An international conference on the mathematics of uncertainty]

"MeCoNeT 2024 concluded with widespread acclaim, positioning Messina as a significant player in global academic circles. This conference was a unique opportunity to bring together different perspectives and foster collaboration in the field of mathematical uncertainty."

These articles and quotes demonstrate the extensive media coverage that the MeCoNeT 2024 conference received from prominent local outlets, both before and after the event. This attention not only amplified the conference's impact but also solidified Messina's role as a key hub for international scientific discussions on neutrosophic theory and mathematical uncertainty, reaching both the academic community and the broader public.



Based on the official statistics from the MeCoNeT 2024 website (www.meconet.org), the conference announcement sparked remarkable global interest, resonating strongly within the scientific community. Over the course of just three months, the site attracted visitors from 58 countries across multiple continents, illustrating the far-reaching impact of the event.

This impressive engagement—spanning Asia, Europe, the Americas, and parts of Africa—demonstrates the growing global curiosity around neutrosophic theory and its cutting-edge developments. The international response, as reflected in the data, underscores MeCoNeT 2024’s importance as a pivotal platform for advancing discourse on uncertainty, contradiction, and incomplete information.

The diverse geographic participation highlights the broad scientific appeal of neutrosophic theory across various domains. The hybrid format of the conference further contributed to this wide reach, allowing scholars from around the world to engage both in-person and virtually, ensuring a truly global exchange of ideas.

The MeCoNeT 2024 conference was an overwhelming success, bringing together over 100 scholars from 15 countries and fostering vibrant discussions on the latest developments in neutrosophic theory. The hybrid format, combining in-person sessions at the prestigious Accademia Peloritana dei Pericolanti with parallel online sessions, enabled global participation and collaboration. The diverse range of keynote speeches, research presentations, and interdisciplinary contributions reflected the growing importance of neutrosophic theory in addressing real-world challenges. This event not only solidified Messina's role as a key hub for academic discourse but also underscored the expanding influence of neutrosophic research on a global scale.

We hope that the research presented in this volume will continue to advance the understanding and application of neutrosophic theory and serve as a valuable reference for researchers in this expanding field.

September 2024

Giorgio Nordo
Florentin Smarandache

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University of New Mexico

Three Decades of Neutrosophic and Plithogenic Theories with their Applications (1995 - 2024)

= *plenary lecture* =

Florentin Smarandache[1]

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Zadeh introduced the **degree of membership/truth** (T) in 1965 and defined the fuzzy set. *Atanassov* introduced the **degree of nonmembership/falsehood** (F) in 1986 and defined the intuitionistic fuzzy set.

Smarandache introduced the **degree of indeterminacy/neutrality** (I) as independent component in 1995 (published in 1998) and he defined the neutrosophic set on three components: (T, I, F) = (Truth, Indeterminacy, Falsehood), where in general T, I, F are subsets of the interval [0, 1]; in particular T, I, F may be intervals, hesitant sets, single-values, etc.; *Indeterminacy* (or *Neutrality*), as independent component from the truth and from the falsehood, is the main distinction between Neutrosophic Theories and other classical and fuzzy theory or fuzzy extension theories:
https://fs.unm.edu/Indeterminacy.pdf
See F. Smarandache, Neutrosophy / Neutrosophic probability, set, and logic", Proquest, Michigan, USA, 1998.
https://arxiv.org/ftp/math/papers/0101/0101228.pdf
https://fs.unm.edu/eBook-Neutrosophics6.pdf;
reviewed in Zentralblatt für Mathematik (Berlin, Germany):
https://zbmath.org/?q=an:01273000
And cited by Denis Howe in The Free Online Dictionary of Computing, England, 1999.
Neutrosophic Set and Logic are generalizations of classical, fuzzy, and intuitionist fuzzy set and logic:
https://arxiv.org/ftp/math/papers/0404/0404520.pdf
https://arxiv.org/ftp/math/papers/0303/0303009.pdf

Etymology

The words "neutrosophy" and "neutrosophic" were coined/invented by F. Smarandache in his 1998 book.
Neutrosophy: A branch of philosophy, introduced by F. Smarandache in 1980, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.
Neutrosophy considers a proposition, theory, event, concept, or entity <A> in relation to its opposite <antiA>, and with their neutral <neutA>.
Neutrosophy (as dynamic of opposites and their neutrals) is an extension of the Dialectics and Yin Yang (which are the dynamic of opposites only).

Neutrosophy is the basis of neutrosophic set, neutrosophic logic, neutrosophic measure, neutrosophic probability, neutrosophic statistics etc.
https://arxiv.org/ftp/math/papers/0010/0010099.pdf

Neutrosophic Set is a Generalization of Intuitionist Fuzzy Set, Inconsistent Intuitionist Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionist Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision.

https://arxiv.org/ftp/arxiv/papers/1911/1911.07333.pdf
https://fs.unm.edu/Raspunsatan.pdf

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionist logic, etc. The main idea of NL is to characterize each logical statement in a 3D-Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^-0, 1^+[$ with not necessarily any connection between them.

For all engineering, technical, administrative and other practical applications the classical unit interval [0, 1] should be used.

While Neutrosophic Probability and Statistics are generalizations of classical and imprecise probability and classical statistics respectively.

The Most Important Books and Papers on the Advancement of Neutrosophics

1980s - Foundation of **Paradoxism** that is an international movement in science and culture based on excessive use of contradictions, antitheses, oxymoron, and paradoxes [Smarandache]. During three decades (1980-2020) hundreds of authors from tens of countries around the globe contributed papers to 15 international paradoxist anthologies: https://fs.unm.edu/a/paradoxism.htm

1995-1998 – Smarandache extended the paradoxism (*based on opposites*) to a new branch of philosophy called **Neutrosophy** (*based on opposites and their neutral/indeterminacies*), that gave birth to many scientific branches, such as: neutrosophic logic, neutrosophic set, neutrosophic probability and statistics, neutrosophic algebraic structures, and so on with multiple applications in all fields.

Neutrosophy is also an extension of the *Dialectics*, the *Yin-Yang* ancient Chinese philosophy, the *Manichaeism*, and in general of the *Dualism.*
https://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf

Introduced the neutrosophic set/logic/probability/statistics; introduces the single-valued neutrosophic set (pp. 7-8);

https://arxiv.org/ftp/math/papers/0101/0101228.pdf (fourth edition)
https://fs.unm.edu/eBook-Neutrosophics6.pdf (online sixth edition)

Single Valued Neutrosophic Sets

https://fs.unm.edu/SingleValuedNeutrosophicSets.pdf

Indeterminacy in Neutrosophic Theories and their Applications.
https://fs.unm.edu/Indeterminacy.pdf

1998, 2019 - Extended **Nonstandard Neutrosophic Logic, Set, Probability based on NonStandard Analysis**

https://arxiv.org/ftp/arxiv/papers/1903/1903.04558.pdf https://fs.unm.edu/AdvancesOfStandardAndNonstandard.pdf

Improved Definition of NonStandard Neutrosophic Logic and Introduction to **Neutrosophic Hyperreals** (Third version), arXiv, Cornell University, New York City, USA, https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf,
https://fs.unm.edu/NonStandardAnalysis-Imamura-proven-wrong.pdf

2002 – Introduction of **corner cases of sets / probabilities / statistics / logics**, such as:
- Neutrosophic intuitionistic set (different from intuitionist fuzzy set), neutrosophic paraconsistent set, neutrosophic faillibilist set, neutrosophic paradoxist set, neutrosophic pseudo-paradoxist set, neutrosophic tautological set, neutrosophic nihilist set, neutrosophic dialetheist set, neutrosophic trivialist set;
- Neutrosophic intuitionistic probability and statistics, neutrosophic paraconsistent probability and statistics, neutrosophic faillibilist probability and statistics, neutrosophic paradoxist probability and statistics, neutrosophic pseudo-paradoxist probability and statistics, neutrosophic tautological probability and statistics, neutrosophic nihilist probability and statistics, neutrosophic dialetheist probability and statistics,neutrosophic trivialist probability and statistics;
- Neutrosophic paradoxist logic (or paradoxism), neutrosophic pseudo-paradoxist logic (or neutrosophic pseudo-paradoxism), neutrosophic tautological logic (or neutrosophic tautologism):

https://arxiv.org/ftp/math/papers/0301/0301340.pdf
https://fs.unm.edu/DefinitionsDerivedFromNeutrosophics.pdf

2003 – Introduction by Kandasamy and Smarandache of **Neutrosophic Numbers** (a+bI, where I = *literal indeterminacy*, $I^2 = I$, which is different from the *numerical indeterminacy* I = real set), I-Neutrosophic Algebraic Structures and **Neutrosophic Cognitive Maps**

https://arxiv.org/ftp/math/papers/0311/0311063.pdf
https://fs.unm.edu/NCMs.pdf

2005 - Introduction of **Interval Neutrosophic Set/Logic**

https://arxiv.org/pdf/cs/0505014.pdf
https://fs.unm.edu/INSL.pdf

2006 – Introduction of **Degree of Dependence and Degree of Independence between the Neutrosophic Components** T, I, F.
For single valued neutrosophic logic, the sum of the components is: $0 \leq t+i+f \leq 3$ when all three components are independent; $0 \leq t+i+f \leq 2$ when two components are dependent, while the third one is independent from them; $0 \leq t+i+f \leq 1$ when all three components are dependent.
When three or two of the components T, I, F are independent, one leaves room for background incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).
If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).
In general, the sum of two components x and y that vary in the unitary interval [0, 1] is: $0 \leq x + y \leq 2 - d°(x, y)$, where $d°(x, y)$ is the degree of dependence between x and y, while $d°(x, y)$ is the degree of independence between x and y.
Degrees of Dependence and Independence between Neutrosophic Components T, I, F are independent components, leaving room for *incomplete information* (when their superior sum < 1), *paraconsistent and contradictory information* (when the superior sum > 1), or *complete information* (sum of components = 1).
For software engineering proposals the classical unit interval [0, 1] is used.

https://doi.org/10.5281/zenodo.571359
https://fs.unm.edu/eBook-Neutrosophics6.pdf (p. 92)
https://fs.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf

2007 – The Neutrosophic Set was extended [Smarandache, 2007] to **Neutrosophic Overset** (when some neutrosophic component is > 1), since he observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1;
and to **Neutrosophic Underset** (when some neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0;

and to and to **Neutrosophic Offset** (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0).
Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively **Neutrosophic Over-/Under-/Off- Logic / Measure / Probability / Statistics** etc.

https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf
https://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf
https://fs.unm.edu/SVNeutrosophicOverset-JMI.pdf
https://fs.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf
https://fs.unm.edu/NSS/DegreesOf-Over-Under-Off-Membership.pdf

2007 – Smarandache introduced the **Neutrosophic Tripolar Set** and **Neutrosophic Multipolar Set** and consequently the **Neutrosophic Tripolar Graph** and **Neutrosophic Multipolar Graph**
https://fs.unm.edu/eBook-Neutrosophics6.pdf (p. 93)
https://fs.unm.edu/IFS-generalized.pdf

2009 – Introduction of **N-norm and N-conorm**
https://arxiv.org/ftp/arxiv/papers/0901/0901.1289.pdf
https://fs.unm.edu/N-normN-conorm.pdf

2013 - Development of **Neutrosophic Measure** and **Neutrosophic Probability** (*chance that an event occurs, indeterminate chance of occurrence, chance that the event does not occur*)
https://arxiv.org/ftp/arxiv/papers/1311/1311.7139.pdf
https://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf

2013 – Smarandache **Refined / Split the Neutrosophic Components** (T, I, F) into Neutrosophic SubComponents
(T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ...):
https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf
https://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

2014 – Introduction of the **Law of Included Multiple-Middle** (as extension of the Law of Included Middle)
(<A>; <neutA_1>, <neutA_2>, …, <neutA_n>; <antiA>)

https://fs.unm.edu/LawIncludedMultiple-Middle.pdf
and the **Law of Included Infinitely-Many-Middles** (2023)
https://fs.unm.edu/NSS/LawIncludedInfinitely1.pdf

(<A>; <neutA_1>, <neutA_2>, …, <neut$A_{infinity}$>; <antiA>)

2014 - Development of **Neutrosophic Statistics** (*indeterminacy* is introduced into classical statistics with respect to any data regarding the sample / population, probability distributions / laws / graphs / charts etc., with respect to the individuals that only partially belong to a sample / population, and so on):

https://fs.unm.edu/NS/NeutrosophicStatistics.htm
Neutrosophic Numbers used in Neutrosophic Statistics
https://fs.unm.edu/NS/AppurtenanceInclusionEquations-revised.pdf
2015 **-** Extension of the Analytical Hierarchy Process (AHP) to **α-Discounting Method for Multi-Criteria**
Decision Making (α-D MCDC)
https://fs.unm.edu/ScArt/AlphaDiscountingMethod.pdf
https://fs.unm.edu/ScArt/CP-IntervalAlphaDiscounting.pdf
https://fs.unm.edu/ScArt/ThreeNonLinearAlpha.pdf
https://fs.unm.edu/alpha-DiscountingMCDM-book.pdf
2015 - Introduction of **Neutrosophic Precalculus** and **Neutrosophic Calculus**

https://arxiv.org/ftp/arxiv/papers/1509/1509.07723.pdf
https://fs.unm.edu/NeutrosophicPrecalculusCalculus.pdf

2015 – **Refined Neutrosophic Numbers** ($a+ b_1I_1 + b_2I_2 + \ldots + b_nI_n$), where $I_1, I_2, \ldots, I_n$ are SubIndeterminacies of Indeterminacy I.

2015 – **(t,i,f)-Neutrosophic Graphs**.

2015 - **Thesis-AntiThesis-NeutroThesis**, and NeutroSynthesis, Neutrosophic Axiomatic System, neutrosophic dynamic systems, symbolic neutrosophic logic, (t, i, f)-Neutrosophic Structures, I-Neutrosophic Structures, Refined Literal Indeterminacy, Quadruple Neutrosophic Algebraic Structures, Multiplication Law of SubIndeterminacies, and Neutrosophic Quadruple Numbers of the form a + bT + cI + dF, where T, I, F are literal neutrosophic components, and a, b, c, d are real or complex numbers:

https://arxiv.org/ftp/arxiv/papers/1512/1512.00047.pdf
https://fs.unm.edu/SymbolicNeutrosophicTheory.pdf

2015 – Introduction of the **SubIndeterminacies** of the form $I_0^k = \frac{k}{0}$, for $k \in \{0, 1, 2, \ldots, n-1\}$, into the ring of modulo integers Z_n - called natural neutrosophic indeterminacies (Vasantha-Smarandache)
https://fs.unm.edu/MODNeutrosophicNumbers.pdf

2015 – Introduction of **Neutrosophic Crisp Set and Topology** (Salama & Smarandache) https://fs.unm.edu/NeutrosophicCrispSetTheory.pdf

2016 - **Addition, Multiplication, Scalar Multiplication, Power, Subtraction, and Division of Neutrosophic Triplets (T, I, F)**
https://fs.unm.edu/CR/SubstractionAndDivision.pdf

2016 – Introduction of **Neutrosophic Multisets** (as generalization of classical multisets)
https://fs.unm.edu/NeutrosophicMultisets.htm

2016 – Introduction of **Neutrosophic Triplet Structures** and m-valued refined neutrosophic triplet structures [Smarandache - Ali].
https://fs.unm.edu/NeutrosophicTriplets.htm

2016 – Introduction of **Neutrosophic Duplet Structures**
https://fs.unm.edu/NeutrosophicDuplets.htm

2017 **-** 2020 - **Neutrosophic Score, Accuracy, and Certainty Functions** form a total order relationship on the set of (single-valued, interval-valued, and in general subset-valued) neutrosophic triplets (T, I, F); and these functions are used in MCDM (Multi-Criteria Decision Making): **https://fs.unm.edu/NSS/TheScoreAccuracyAndCertainty1.pdf**

2017 - In biology Smarandache introduced the **Theory of Neutrosophic Evolution: Degrees of Evolution, Indeterminacy or Neutrality, and Involution** (as extension of *Darwin's Theory of Evolution*):
https://fs.unm.edu/neutrosophic-evolution-PP-49-13.pdf
https://fs.unm.edu/V/NeutrosophicEvolution.mp4

https://fs.unm.edu/NeutrosophicEvolution.pdf

2017 - Introduction by F. Smarandache of **Plithogeny** (as generalization of Yin-Yang, Manichaeism, Dialectics, Dualism, and Neutrosophy), and **Plithogenic Set** / **Plithogenic Logic as generalization of MultiVariate Logic** / **Plithogenic Probability and Plithogenic Statistics as generalizations of MultiVariate Probability and Statistics** (as generalization of fuzzy, intuitionistic fuzzy, neutrosophic set/logic/probability/statistics):

https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf
https://fs.unm.edu/Plithogeny.pdf

2017 - Enunciation of the Law that: **It Is Easier to Break from Inside than from Outside a Neutrosophic Dynamic System** (Smarandache - Vatuiu):

https://fs.unm.edu/EasierMaiUsor.pdf

2018 - 2023 **-** Introduction of new types of soft sets: **HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, SuperHyperSoft Set, TreeSoft Set**:

https://fs.unm.edu/TSS/NewTypesSoftSets-Improved.pdf
https://fs.unm.edu/TSS/SuperHyperSoftSet.pdf
https://fs.unm.edu/NSS/IndetermSoftIndetermHyperSoft38.pdf
(with *IndetermSoft Operators* acting on *IndetermSoft Algebra*)
https://fs.unm.edu/TSS/

2018 – Introduction to **Neutrosophic Psychology** (*Neutropsyche, Refined Neutrosophic Memory: conscious, aconscious, unconscious, Neutropsychic Personality, Eros / Aoristos / Thanatos, Neutropsychic Crisp Personality*):
https://fs.unm.edu/NeutropsychicPersonality-ed3.pdf

2019 - **Theory of Spiral Neutrosophic Human Evolution** (Smarandache - Vatuiu):
https://fs.unm.edu/SpiralNeutrosophicEvolution.pdf

2019 - Introduction to **Neutrosophic Sociology** (*NeutroSociology*) [neutrosophic concept, or (T, I, F)-concept, is a concept that is T% true, I% indeterminate, and F% false]:
https://fs.unm.edu/Neutrosociology.pdf

2019 - **Refined Neutrosophic Crisp Set**
https://fs.unm.edu/RefinedNeutrosophicCrispSet.pdf

2019-2024 - Introduction of sixteen new types of topologies: **NonStandard Topology, Largest Extended NonStandard Real Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet Topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology, NonStandard Neutrosophic Topology, NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperTopology,** and **Neutrosophic SuperHyperTopology**:

https://fs.unm.edu/TT/RevolutionaryTopologies.pdf
https://fs.unm.edu/TT/

2019 - Generalization of the classical Algebraic Structures to NeutroAlgebraic Structures (or **NeutroAlgebras**) {whose **operations and axioms are partially true, partially indeterminate, and partially false} as extensions of Partial Algebra, and to AntiAlgebraic Structures (or AntiAlgebras) {whose operations and axioms are totally false}.**

https://fs.unm.edu/NA/NeutroAlgebra.htm
https://fs.unm.edu/NA/NeutroAlgebra.pdf

And, in general, he extended any classical Structure, in no matter what field of knowledge, to a NeutroStructure and an AntiStructure:

https://fs.unm.edu/NA/NeutroStructure.pdf

As alternatives and generalizations of the **Non-Euclidean Geometries** he introduced in 2021 the **NeutroGeometry & AntiGeometry.** While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth **Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.), and the NeutroGeometry results from the partial negation of one or more axioms [and no** total negation of no axiom] from any geometric axiomatic system.

https://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf
https://fs.unm.edu/NG/
2019-2022 **-** Extension of HyperGraph to **SuperHyperGraph and Neutrosophic SuperHyperGraph**

https://fs.unm.edu/NSS/n-SuperHyperGraph.pdf

2020 - Introduction to **Neutrosophic Genetics:** https://fs.unm.edu/NeutrosophicGenetics.pdf
2021 - Introduction to Neutrosophic Number Theory (Abobala)

https://fs.unm.edu/NSS/FoundationsOfNeutrosophicNumberTheory10.pdf

2021 - As alternatives and generalizations of the Non-Euclidean Geometries, Smarandache introduced in 2021 the NeutroGeometry & AntiGeometry. While the Non-Euclidean Geometries resulted from the total negation of only one specific axiom (Euclid's Fifth Postulate), the AntiGeometry results from the total negation of any axiom and even of more axioms from any geometric axiomatic system (Euclid's, Hilbert's, etc.), and the NeutroGeometry results from the partial negation of one or more axioms [and no total negation of no axiom] from any geometric axiomatic system:

https://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf
Real Examples of NeutroGeometry and AntiGeometry:
https://fs.unm.edu/NSS/ExamplesNeutroGeometryAntiGeometry35.pdf
2021 - Introduction of Plithogenic Logic as a generalization of MultiVariate Logic
https://fs.unm.edu/NSS/IntroductionPlithogenicLogic1.pdf

2021 - Introduction of Plithogenic Probability and Statistics as generalizations of MultiVariate Probability and Statistics respectively

https://fs.unm.edu/NSS/PlithogenicProbabilityStatistics20.pdf

2021 - Introduction of the AH-Isometry f(x+yI) = f(x) + I[f(x+y) - f(x)] and foundation of the Neutrosophic Euclidean Geometry (by Abobala & Hatip).
https://fs.unm.edu/NSS/AlgebraicNeutrosophicEuclideanGeometry10.pdf
and extension to n-Refined AH-Isometry (Smarandache & Abobala, 2024)
https://fs.unm.edu/NSS/RefinedLiteral21.pdf

2016 - 2022 SuperHyperAlgebra & Neutrosophic SuperHyperAlgebra
https://fs.unm.edu/SuperHyperAlgebra.pdf

2022 - SuperHyperFunction, SuperHyperTopology
https://fs.unm.edu/NSS/SuperHyperFunction37.pdf

2022 - 2023 Neutrosophic Operational Research (Smarandache - Jdid)
https://fs.unm.edu/NeutrosophicOperationsResearch.pdf

2023 - Symbolic Plithogenic Algebraic Structures built on the set of Symbolic Plithogenic Numbers of the form $a_0 + a_1P_1 + a_2P_2 + \ldots + a_nP_n$ where the multiplication $P_i \cdot P_j$ is based on the prevalence order and absorbance law.
https://fs.unm.edu/NSS/SymbolicPlithogenicAlgebraic39.pdf

2023 - Foundation of Neutrosophic Cryptology (Merkepci-Abobala-Allouf)
https://fs.unm.edu/NeutrosophicCryptography1.pdf
https://fs.unm.edu/NeutrosophicCryptography2.pdf
https://fs.unm.edu/NSS/2OnANovelSecurityScheme.pdf

2023 - The MultiNeutrosophic Set (a neutrosophic set whose elements' degrees T, I, F are evaluated by multiple sources):
https://fs.unm.edu/NSS/MultiNeutrosophicSet.pdf

2023 - The MultiAlist System of Thought (an open dynamic system of many opposites, with their neutralities or indeterminacies, formed by elements from many systems):
https://fs.unm.edu/NSS/MultiAlistSystemOfThought.pdf

2023 - Appurtenance Equation, Inclusion Equation, & Neutrosophic Numbers used in Neutrosophic Statistics.
https://fs.unm.edu/NS/AppurtenanceInclusionEquations-revised.pdf

2024 - SuperHyperStructure and Neutrosophic SuperHyperStructure
https://fs.unm.edu/SHS/

2024 - Zarathustra & Neutrosophy
https://fs.unm.edu/Zoroastrianism.pdf

The Principles of (Partial Locality, Partial Indeterminacy, Partial NonLocality) and (Multi Locality, Multi Indeterminacy, Multi NonLocality)

https://fs.unm.edu/nss8/index.php/111/article/view/4858/2043

Neutrosophy Transcends Binary Oppositions in Mythology and Folklore
https://fs.unm.edu/NSS/NeutrosophyTranscendsBinary4.pdf

Neutrosophy means: Common Parts to Uncommon Things and Uncommon Parts to Common Things
https://fs.unm.edu/NSS/NeutroMeans1.pdf

2024 - Upside-Down Logics: Falsification of the Truth & Truthification of the False
https://fs.unm.edu/Upside-DownLogics.pdf

2024 - Neutrosophic (and fuzzy-extensions) TwoFold Algebra

https://fs.unm.edu/NeutrosophicTwoFoldAlgebra.pdf

Applications in:

Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Genetics, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences etc. [Xindong Peng and Jingguo Dai, A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017, Artificial Intelligence Review, Springer, 18 August 2018; **https://fs.unm.edu/BibliometricNeutrosophy.pdf**]

Neutrosophic Researchers:

There are about 7,500 neutrosophic researchers, within 90 countries around the globe, that have produced about 4,000 articles and books, and over 70 PhD and MSc theses, within more than three decades. Many neutrosophic researchers got specialized into various fields of neutrosophics, plithogenics, NeutroAlgebra and AntiAlgebra, NeutroGeometry and AntiGeometry, new types of topologies, new types of soft sets, SuperHyperStructures, etc.

References

University of New Mexico *(USA)* web sites:

https://fs.unm.edu/neutrosophy.htm
https://fs.unm.edu/NSS/Articles.htm

https://fs.unm.edu/CR/CR-Articles.htm
https://fs.unm.edu/NCML/Articles.htm *(Spanish)*
https://fs.unm.edu/NK/Articles.htm *(Arabic, Turkish, French)*

Other journals:

Neutrosophic Optimization and Intelligent Systems (NOIS)
https://sciencesforce.com/index.php/nois
Plithogenic Logic and Computation (PLC)
https://sciencesforce.com/index.php/plc
HyperSoft Set Methods in Engineering (HSSE)
https://sciencesforce.com/index.php/hsse
Information Sciences with Applications (ISWA)
https://sciencesforce.com/index.php/iswa
Neutrosophic Systems And Application (NSWA)
https://sciencesforce.com/index.php/mawa/index
Uncertainty Discourse and Applications (UDA)
https://uda-journal.com/journal

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A Review on Recent Development of Neutro-Topology

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Abstract: Smarandache proposed NeutroAlgebra and AntiAlgebra. NeutroAlgebras and AntiAlgebras are a new research topic based on real-world scenarios. He investigated the concepts of neutro- and anti-structure. He demonstrated using NeutroAlgebra concepts that just because a statement is completely true in a classical Algebra does not imply that it is also completely true in a NeutroAlgebra or AntiAlgebra. It is determined by the operations and axioms on which it is based (whether they are completely true, partially true, totally false, or partially or completely indeterminate). This study examines the concepts of Generalised regular Neutro-Topological space and its properties.

Keywords: Neutro-Topology; NeutroClosed sets; NeutroOpen sets; GR-NeutroInterior.

1. Introduction

Topology is a significant subject of Mathematics, hence it is surprising that topology's appreciation was delayed in the history of Mathematics. Topology is the study of space characteristics that are unaffected by continuous deformation.

A key idea in mathematics, set theory, dates back to the work of Russian mathematician George Cantor (1877). We were able to investigate a variety of mathematical ideas thanks to set theory. However, there are a lot of unknowns in our life. The traditional logic of mathematics is frequently insufficient to resolve these difficulties. Then the idea of fuzzy sets was introduced by Zadeh [1]. It is a development of the traditional idea of a set. In his paper, he presented a hypothesis according to which fuzzy sets are sets with imprecise boundaries. In both directions, gradual changes from membership to non membership can be expressed using fuzzy sets. It offers meaningful representations of vague notions in everyday language in addition to a powerful and meaningful way to quantify uncertainties. a value in the discourse universe that indicates the fuzzy set's degree of membership. Real values in the closed range of 0 to 1 are used to represent these membership classifications. Chang [2] discovered and popularized the theory of fuzzy topological spaces. The concepts for creating fuzzy topological spaces were provided by Lowen [3]. He provided the idea of fuzzy compression and two new functions, which allowed for the evident observation of further relationships between fuzzy topological spaces and topological spaces. A unique fuzzy topological space called the product spaces was discussed by Cheng-Ming [4]. He established a type of fuzzy points neighbourhood formation, such as the Q-neighbourhood, which is a crucial idea in fuzzy topological spaces. He also demonstrated how each fuzzy topological space is isomorphic topologically by a specific space of topology.

Atanassov [5] introduced the concept of intuitionistic fuzzy sets as an extension of sets with better applicability. Coker [6] developed the idea of intuitionistic smooth fuzzy topological spaces using the concept of intuitionistic fuzzy sets. The definitions of the intuitionistic smooth fuzzy topological spaces were first presented by Samanta and Mondal [7].

Smarandache [8] introduced the concept of a neutrosophic set for the first time. These concepts have three different degrees: T for membership, I for uncertainty, and F for non-membership. In other words, a situation is treated in neutrosophy in accordance with its trueness, falsity, and uncertainty. As a result, neutrosophic sets and logic enable us to make sense of a variety of uncertainties in our daily lives. On this topic, numerous studies have been conducted. Sahin et al. recently discovered some operations for neutrosophic sets with interval values; Neutrosophic multigroups and applications were researched by Ulucay et al [9]; Q-neutrosophic soft expert set and its application were introduced by Hassan et al [10]. The acquisition of neutrosophic soft expert sets was introduced by Sahin et al [11]; Interval-valued refined neutrosophic sets and their applications were researched by Ulucay et al [12]. Neutosophic set importance on deep transfer learning techniques was obtained by Khalifa et al. [13]; Generalised Hamming similarity measure based on neutrosophic quadraple numbers and its applications were researched by Kargin et al. [14]; In order to assess the quality of online education, Sahin et al. [15] obtain Hausdorff Measures on generalised set valued neutrosophic quadraple numbers and decision-making applications. The foundation for a wide family of novel mathematical ideas, including both their crisp and fuzzy counterparts, was laid by neutrosophy. The concepts of neutrosophic crisp set and neutrosophic crisp topological space were first developed by Salama et al. and Alblowi [16]. Neutron structures and antistructures are defined by Smarandache [17]. An algebraic structure can be divided into three regions, similar to neutrosophic logic: A, the set of elements that satisfy the conditions of the algebraic structure, the truth region; Neutro A, the set of elements that do not meet the conditions of the algebraic structure, the uncertainty region; and anti-A, the set of elements that do not satisfy the conditions of the algebraic structure, the inaccuracy region. By eliminating neutrosophic sets and neutrosophic numbers, the structure of neutrosophic logic has been translated to the structure of classical algebras. The academic world has seen a rise in interest in neutrosophic set theory research in recent years. As a result, it is possible to generate neutro-algebraic structures, which are more broadly structured than classical algebras. Additionally, the region of elements that do not conform to any of the classical algebras is also considered to have anti-algebraic structures. Recent research includes studies on neutro-algebra by Smarandache et al. [18], the neutrosophic triplet of BI-algebras by Razaei et al. [19], neutro-bck-algebra by Smarandache et al. [20], and neutro-hypergroups by Ibrahim et al. [21].

In this paper, we introduce new Generalization of Regular Neutro-open (briefly, GRN-open) sets and Generalised regulat Anti-open set. This new class shows stronger properties in general topological spaces that mean GRN-open sets exists in between the class of regular open sets and the class of open sets. Also, we investigate GRN-neighbourhood, GRN-interior and GRN-closure properties.

2. Preliminaries

Definition 2.1. The NeutroSophication of the Law [22]

1. Let X be a non-empty set and $*$ be a binary operation. For some elements (a, b) ∈ (X, X), (a∗b) ∈ X (degree of well defined (T)) and for other elements (x, y), (p, q) ∈ (X, X); [x∗y is indeterminate (degree of indeterminacy (I)), or p∗q ∉ X (degree of outer-defined (F)], where (T, I, F) is different from (1,0,0) that represents the Classical Law, and from (0,0,1) that represents the Anti Law.

2. In Neutro Algebra, the classical well-defined for binary operation $*$ is divided into three regions: degree of well-defined (T), degree of indeterminacy (I) and degree of outer-defined (F) similar to neutrosophic set and neutrosophic logic.

Definition 2.2. [23]

Let X be the non-empty set and τ be a collection of subsets of X. Then τ is said to be a Neutro Topology on X and the pair (X, τ) is said to be a Neutro Topological space, if at least one of the following conditions hold good:

1. $[(\emptyset_N \in \tau, X_N \notin \tau)\ or\ (X_N \in \tau, \emptyset_N \notin \tau)\ ]$ or $[\emptyset_N, X_N \in_{\sim} \tau]$
2. For some n elements $a_1, a_2, \ldots a_n \in \tau, \bigcap_{i=1}^{n} a_i \in \tau$ [degree of truth T] and for other n elements $b_1, b_2, \ldots b_n \in \tau, p_1, p_2, \ldots p_n \in \tau$; [($\bigcap_{i=1}^{n} b_i \notin \tau$) [degree of falsehood F] or ($\bigcap_{i=1}^{n} p_i$ is indeterminate (degree of indeterminacy I)], where n if finite; ; [where (T, I, F) is different from (1,0,0) that represents the Classical Axiom, and from (0,0,1) that represents the Anti Axiom].
3. For some n elements $a_1, a_2, \ldots a_n \in \tau, \bigcup_{i=1} a_i \in \tau$ [degree of truth T] and for other n elements $b_1, b_2, \ldots b_n \in \tau$, $p_1, p_2, \ldots p_n \in \tau$; [($\bigcup_{i=I} b_i \notin \tau$) [degree of falsehood F] or ($\bigcup_{i=I} p_i$ is indeterminate (degree of indeterminacy I)], where n is finite; [where (T, I, F) is different from (1,0,0) that represents the Classical Axiom, and from (0,0,1) that represents the Anti Axiom].

Definition 2.3. [23]

Let X be the non-empty set and τ be a collection of subsets of X. Then τ is said to be an Anti Topology on X and the pair (X, τ) is said to be an Anti Topological space, if at least one of the following conditions hold good:

1. $\emptyset_N, X_N \notin \tau$
2. For n elements $a_1, a_2, \ldots, a_n \in \tau, \bigcap_{i=1}^{n} a_i \notin \tau$ [degree of falsehood F] where n is finite.
3. For some n elements $a_1, a_2, \ldots, a_n \in \tau, \bigcup_{i=1}^{n} a_i \notin \tau$ [degree of falsehood F] where n is finite.

Remark 2.1. [23]

The symbol "∈_~" will be used for situations where it is an unclear appurtenance (not sure if an element belongs or not to a set). For example, if it is not certain whether "a" is a member of the set P, then it is denoted by a ∈_~ P.

Main Works

3. GR-NeutroOpen sets and their properties

We introduce GR-NeutroOpen sets and investigate some of relationships between existed classes.

Definition 3.1. A NeutroSubset M of space P is called Generalized Regular Neutrosphic Open (briefly, GR-NeutroOpen) set if M = NeuInt(g-NeuCl(M)). We denote the class of sets as GRNO(P).

Firstly we have to prove the existence of new class GR-NeutroOpen sets in topological spaces.

Theorem 3.1. Every regular NeutroOpen set is GR-NeutroOpen set.

Proof. Let M be a regular NeutroOpen set in P. To prove that M is GR- NeutroOpen in P.

We know that

M ⊆ g-NeuCl(M) ⊆ NeuCl(M) that is NeuInt(M) ⊆ NeuInt(g-NeuCl(M)) ⊆ NeuInt(cl(M)).

As M is regular NeutroOpen, M = NeuInt(cl(M)) and NeuInt(M) = M.

Hence M ⊆ NeuInt(g-NeuCl(M)) ⊆ NeuInt(NeuCl(M)) = M,

Thus NeuInt(g-NeuCl(M)) = M. Therefore M is GR- NeutroOpen in P.

The converse of above theorem need not be true.

Example 3.1. Let P = {1,2,3,4} with the topology on it τ = {P,Ø,{1},{2},{1,2},{2,3},{1,2,3}}, then sets {2}, {1,2} are NeutroOpen sets but not regular NeutroOpen sets in P.

Theorem 3.2. Every GR-NeutroOpen set is NeutroOpen set.

Proof. Let M be a GR- NeutroOpen set in P. That is M = NeuInt(g- NeuCl(M)). As interior of any subset of P is an NeutroOpen set, therefore M is a NeutroOpen in P.

The converse of above theorem need not be true.

Example 3.2. Let P = {1,2,3,4} with the topology on it

τ ={P,Ø,{1},{2},{1,2},{2,3},{1,2,3}}.

Then the set {1,2,3} is NeutroOpen set but not GR- NeutroOpen in P.

Remark 3.1. From Theorem 3.2, we know that every GR- NeutroOpen set is a NeutroOpen set but not conversely. We know that every NeutroOpen set is semi- NeutroOpen but not conversely. Hence every GR- NeutroOpen set is a semi- NeutroOpen set but not conversely.

Remark 3.2. From Theorem 3.2, we know that every NeutroOpen set is a NeutroOpen set but not conversely. We know that every NeutroOpen set is g- NeutroOpen but not conversely. Hence every GR- NeutroOpen set is a g- NeutroOpen set but not conversely.

Theorem 3.3. Intersection of two GR-NeutroOpen sets is a GR- NeutroOpen set in topological spaces.

Proof. Let M and N be two GR- NeutroOpen sets in space P. To prove that M ∩ N is GR-NeutroOpen set in space P, that is to prove that M∩N = NeuInt(g-NeuCl(M∩N)). As M and N are GR-

NeutroOpen sets in P,M = NeuInt(g-NeuCl(M)), N = NeuInt(g-NeuCl (N)). We know that M∩N ⊆ M, g-NeuCl(M∩N) ⊆ g-NeuCl(M) also M∩N ⊆ N, g-NeuCl(M∩N) ⊆ g-NeuCl(N). Which implies NeuInt(g-NeuCl (M ∩ N)) ⊆ NeuInt(g-NeuCl (M)) and NeuInt(g-NeuCl (M ∩ N)) ⊆ NeuInt(g-NeuCl N)). This implies NeuInt(g-NeuCl(M ∩ N))∩ NeuInt(g-NeuCl (M∩N)) ⊆ NeuInt(g-NeuCl (M))∩ NeuInt(g-NeuCl (N)) That is NeuInt(g-NeuCl (M ∩N)) ⊆ NeuInt(g-NeuCl (M))∩ NeuInt(g-NeuCl (N)) = M∩N...(i) M∩N = NeuInt(M)∩NeuInt(N) = NeuInt(M∩N) [M = NeuInt(M) and N = NeuInt(N) because of if M and N are NeutroOpen sets, then every NeutroOpen is NeutroOpen in P] NeuInt(M∩N) ⊆ NeuInt(g-cl(A∩B)). M∩N ⊆ NeuInt(g-NeuCl(M ∩N))...(ii) From (i) and (ii), M ∩N = NeuInt(g-NeuCl (M ∩N)). Hence M ∩N is GR- NeutroOpen set in P.

Remark 3.3. The union of two GR- NeutroOpen sets is generally not a GR- NeutroOpen set in topological spaces.

Example 3.3. Let P = {1,2,3,4} with topology on it

τ = {P,Ø,{1},{2},{1,2},{2,3},{1,2,3}}.If M = {1,2} and

N = {2,3} are GR-open sets in P but M ∩ N = {1,2,3} is not GR- NeutroOpen set in P.

Theorem 3.4. If M is a GR- NeutroOpen then NeuInt(M) = M.

Proof. Let M is GR-NeutroOpen. To prove NeuInt(M) = M. We know that every GR- NeutroOpen set is NeutroOpen, that is M is NeutroOpen set then NeuInt(M) = M. The converse of above theorem need not be true.

Example 3.4. Let P = {1,2,3,4} with topology on it τ = {P,Ø,{1},{2},{1,2},{1,2,3}}, then GRNO(P) = {P,Ø,{1},{2},{1,2}}. Then the Neutro-set M = {1,2,3}, Note that NeuInt(M) = {1,2,3} is not a GR-NeutroOpen set, but it is NeutroOpen set of P.

Theorem 3.5. If M is g-closed and NeutroOpen in P, then M is GR- NeutroOpen in P.

Proof. Let M is g-closed and NeutroOpen in P. To prove that M is GR- NeutroOpen i.e. to prove M = NeuInt(g-NeuCl (M)). Now g-NeuCl(M) = M, because M is g- NeutroOpen set. As NeuInt(g-NeuCl(M)) = NeuInt(M) this implies NeuInt(g-NeuCl (M)) = M, because M is NeutroOpen set. Then M is GR- NeutroOpen in P.

Remark 3.4. Complement of a GR-NeutroOpen set need not be GR- NeutroOpen set.

Example 3.5. Let P={1,2,3,4} with topology on it τ={P,Ø,{1},{2},{1,2},{2,3},{1,2,3}}. Note that {1,2} is a

GR- NeutroOpen set. But P − {1,2} = {3} is not a GR- NeutroOpen set in P.

4. GR-NeutroCNeutroClosed sets and their properties

We introduce GR-NeutroClosed sets and investigate some of their properties.

Definition 4.1. A subset M of space P is called Generalized Regular Neutrosophic Closed (briefly, GR-NeutroClosed) set if P − M is GR- NeutroClosed in P. Then its family is denoted as GRNC(P).

This new class of sets properly lies between the class of regular NeutroClosed sets and the class of NeutroClosed sets

Theorem 4.1. A subset M of P is GR- NeutroClosed if and only if M = NeuCl(g-NeuInt(M)).

Proof. (i) Suppose M is GR- NeutroClosed. To prove M = NeuCl(g-Neu¬Int(M)). As M is GR-NeutroClosed, P − Mis GR-NeutroOpen in P, which implies P−M=NeuInt(g-NeuCl(P−M)). P−M = NeuInt(P−g-NeuInt(M)). [because g-NeuCl(P − M) = P − g-NeuCl(M))] = P − NeuCl(g-NeuInt(M)). So (P−M)c = [P−NeuCl(g-NeuInt(M))]c. That is M = NeuCl(g-NeuInt(M)). (ii) Suppose M = NeuCl(g-Int(M)). To prove M is GR- NeutroClosed, [That is to prove P−M is GR-NeutroOpen set]. That is P−M = NeuInt(g¬NeuCl(M). Now given M = NeuCl(g-NeuInt(M)). P − M = P − NeuCl(g--NeuInt(M)). P − M = NeuInt(g-NeuCl(P − M)).implies that P − M is GR-NeutroOpen set that is M is GR- NeutroClosed in P.

Theorem 4.2. Every regular NeutroClosed set is GR- NeutroClosed set.

Proof. Let M be a regular NeutroClosed set in space P. Then Mc is a regular NeutroOpen set. By Theorem 3.1, Mc is GR- NeutroOpen set. Therefore M is a GR- NeutroClosed set in P.

The converse of above theorem need not be true.

Example 4.1. From Example 3.1, the set {3,4} and {1,3,4} are GR- NeutroClosed sets but not regular NeutroClosed in P.

Theorem 4.3. Every GR- NeutroClosed set is NeutroClosed set.

Proof. Let M be a GR- NeutroClosed set in P. Then Mc is a GR- NeutroOpen in P. By Theorem 3.2, Mc is an NeutroOpen set in P. Therefore M is a NeutroClosed set in P.

The converse of above theorem need not be true.

Example 4.2. From Example 3.1, the set {4} is NeutroClosed set but not GR- NeutroClosed set in P.

Remark 4.1. From Theorem 4.3, we have, every GR- NeutroClosed set is a NeutroClosed set but not conversely. Also, every NeutroClosed set is semi- NeutroClosed set but not conversely. Hence every GR- NeutroClosed set is a semi- NeutroClosed set but not conversely.

Remark 4.2. From Theorem 4.3, we have, every GR- NeutroClosed set is a NeutroClosed set but not conversely. Every NeutroClosed set is NeutroClosed but not conversely. Hence every GR-NeutroClosed set is NeutroClosed set but not conversely.

Remark 4.3. From Theorem 4.3, we know that every GR- NeutroClosed set is a NeutroClosed set but not conversely. It is clear that every NeutroClosed set is g- NeutroClosed but not conversely. Hence every GR- NeutroClosed set is a g- NeutroClosed set but not conversely.

Remark 4.4. The following example shows that GR- NeutroClosed sets are independent of ir-NeutroClosed sets, s-NeutroClosed sets and regular semi- NeutroOpen (=regular semi-NeutroClosed) sets.

Example 4.3. Let P = {1,2,3,4,5} with topology on it

τ = {P,Ø,{1},{1,4},{2,3},{1,2,3},{1,2,3,4}}. Then

NeutroClosed sets in P are P,Ø,{5},{4,5},{1,4,5},{2,3,5}, {2,3,4,5}.

GR- NeutroClosed sets in P are P,Ø,{4,5},{1,4,5},{2,3,5}, {2,3,4,5}.

π- NeutroClosed sets in P are P,Ø,{5},{1,4,5},{2,3,5}.

s- NeutroClosed sets in P are P,Ø,{5},{1,4,5},{2,3,5}.

regular semi-NeutroOpen sets in P are P,Ø,{1,4},{2,3},{1,4 ,5},{2,3,5}.

Theorem 4.4. Union of two GR- NeutroClosed sets is a GR- NeutroClosed set in topological spaces.

Proof. Let M and N be two GR- NeutroClosed sets in P. To prove that M ∪N = NeuCl(g-NeuInt(M ∪ N)). As M and N are GR- NeutroClosed sets in P, M = NeuCl(g-NeuInt(M)), N = NeuCl(g-NeuInt(N)). We know thatM ⊆ M∪N, g-NeuInt(M) ⊆ g-NeuInt(M∪N) also N ⊆ M∪N, g-¬NeuInt(N) ⊆ g-NeuInt(M ∪N). Which implies NeuCl(g-NeuInt(M)) ⊆ NeuCl(g-NeuInt (M ∪ N)) and NeuCl(g-NeuInt(N)) ⊆ NeuCl(g-NeuInt (M ∪ N)). This implies NeuCl(g-NeuInt (M)) ∪ NeuCl(g-NeuInt (N)) ⊆ NeuCl(g-NeuInt (M ∪N)) ∪ NeuCl(g-NeuInt (M∪N)). That is

NeuCl(g-NeuInt (M))∪ NeuCl(g-NeuInt (N)) ⊆ NeuCl(g-NeuInt (M ∪N))...(i)

M ∪N = NeuCl(M)∪ NeuCl(N) = NeuCl(M ∪N) [M = NeuCl(M) and N = NeuCl(N) and M, N are NeutroClosed sets, because every GR- NeutroClosed is NeutroClosed set] NeuCl(M∪N) ⊇ NeuCl(g-NeuInt (M ∪ N)) i.e. M ∪N ⊇ NeuCl(g-NeuInt (M ∪N))...(ii)

From(i) and (ii), M ∪ N = NeuCl(g-NeuInt (M ∪N)). Hence M ∪N is GR- NeutroClosed set in P. Hence A∪ B is GR- NeutroClosed in X.

Remark 4.5 The intersection of two GR- NeutroClosed sets in topological spaces is generally not a GR- NeutroClosed set.

Example 4.4. From Example 3.1, then sets M = {1,4} and N = {3,4} are GR- NeutroClosed sets in P but M ∩N = {4} is not GR- NeutroClosed set in P.

Theorem 4.5. If M is a GR- NeutroClosed if and only if NeuCl(M) = M.

Proof. If M is GR- NeutroClosed. To prove NeuCl(M) = M. We know that every GR- NeutroClosed set is NeutroClosed set i.e. M is NeutroClosed then NeuCl(M) = M.

The converse of above theorem need not be true.

Example 4.5. Let P = {1,2,3,4} with topology on it τ = {P,Ø,{1},{2},{1,2},{1,2,3}}. Then GRNC(P)={P,Ø,{3,4} ,{1,3,4},{2,3,4}}. Then the set M = {4}. Note that NeuCl(M)= {4} is not a GR- NeutroClosed set, but it is a NeutroClosed set of P.

Theorem 4.6. If M is g-NeutroOpen and NeutroClosed in P, then M is GR- NeutroClosed set in P.

Proof. Let M is g-NeutroOpen and NeutroClosed set in P. To prove that

M is GR- NeutroClosed set i.e. to prove M NeuCl(g-NeuInt (M)). Now g-NeuInt(M) = M, because M is g-NeutroOpen set. As NeuCl(g-NeuInt (M)) = NeuCl(M) this implies NeuCl(g-NeuInt (M)) = M, because M is NeutroClosed set. Then Mis GR- NeutroClosed set in P.

5. GR-NeutroNeighbourhoods and GR-NeutroInterior

Definition 5.1. (i) Let P be a topological space and x ∈ P, A subset N of P is said to be a GR-NeutroNeighbourhood (briefly, GR-NeuNhd) of x if and only if there exists a GR-NeutroOpen set G such that x ∈ G ⊆ N.

(ii) The collection of all GR-NeutroNeighbourhood of x ∈ P is GR-NeutroNeighbourhood system at x and is denoted by GR-N(x).

Definition 5.2. Let M be a subset of P. A point x ∈ M is said to be GR-NeutroInterior point of M if and only if P is a GR-NeutroNeighborhood of x. The set of all GR-NeutroInterior points of M is called the GR-NeutroInterior of M and is denoted as GR-int(M).

Theorem 5.1. If M is a subset of P, then GR-NeutroInt(M) = ∪{G : G is GR-NeutroOpen set, G ⊆ M}.

Proof. Let M be a subset of P. x ∈ GR-NeuInt(A) implies that x is a GR-NeutroInterior point of P i.e. M is a GR-NeuNhd of point x. Then there exists a GR-NeutroOpen set G such that x ∈ G ⊆ A

implies that x ∈ ∪{G : G is GR- NeutroOpen set, G ⊆ M}. Hence

GR-NeuInt(M) = ∪{G : G is GR- NeutroOpen set, G ⊆ M}.

Theorem 5.2. Let P be a topological space and M ⊆ P, then show that M is GR- NeutroOpen set if and only if GR-NeuInt(M) = M.

Proof. Let M be a GR- NeutroOpen set in P. Then clearly the largest GR- NeutroOpen set contained in M, is itself M. Hence GR-NeuInt(M) = M.

Conversely, suppose that M ⊆ P and GR-NeuInt(M) = M. Since GR-NeuInt(M) is a GR-NeutroOpen set in P, it follows that M is a GR- NeutroOpen set in P.

Theorem 5.3. Let M and N are subset of P. Then

1. GR-NeuInt(P) = P and GR-NeuInt(Ø) = Ø.
2. GR-NeuInt(M) ⊆ M.
3. If N is any GR- NeutroOpen set contained in M, then N ⊆ GR-NeuInt(M).
4. If M ⊆ N, then GR-NeuInt(M) ⊆ GR-NeuInt(N).
5. GR-NeuInt(GR-NeuInt(M)=GR-NeuInt(M).

Proof. (1) Since P and Ø are GR- NeutroOpen sets, by Theorem 5.3, GR-NeuInt(P) = ∪{G : G is GR-NeutroOpen set, G ⊆ P} = P ∪ { all GR- NeutroOpen sets } = P. That is GR-NeuInt(P) = P. Since Ø is the only GR-NeutroOpen set contained in Ø, GR-NeuInt(Ø) = Ø.

(2) Let x ∈ GR-NeuInt(A) implies that x is a GR-NeutroInterior point of M. That is Mis a GR-NeuNhd of x i.e. x ∈ M. Thus x ∈ GR-int(A) implies x ∈ A. Hence GR-NeuInt(M) ⊆ M.

(3) Let N be any GR- NeutroOpen set such that N ⊆ M. Let x ∈ N. Since N is a GR-NeutroOpen set contained in M, x is a GR-NeuInterior point of M. That is x ∈ GR-NeuInt(M). Hence N ⊆ GR-NeuInt(M).

(4) Let M and N be subsets of P such that M ⊆ N. Let x ∈ GR-NeuInt(M). Since GR-NeuInt(M) ⊆ M and M ⊆ N, we have GR-NeuInt(M) ⊆ N. Now GR-NeuInt(M) is a GR- NeutroOpen set and GR-NeuInt(N) is the largest GR- NeutroOpen set contained in N, we have to find GR-NeuInt(M) ⊆ GR-NeuInt(N).

(5) Since GR-NeuInt(M) is a GR-NeutroOpen set in P, it follows that GR-NeuInt(GR-NeuInt(M))=GR-NeuInt(M).

Theorem 5.4. If M and N are subsets of P, then GR-NeuInt(M) ∪ GR-NeuInt(N) ⊆ GR-NeuInt(M ∪N).

Proof. We know that M ⊆ M ∪ N and N ⊆ M ∪ N. We have, by Theorem 5.5(iv), GR-NeuInt(A) ⊆ GR- NeuInt(M ∪N) and GR- NeuInt (N) ⊆ GR- NeuInt (M ∪ N). This implies GR- NeuInt (M)∪ GR-NeuInt (N) ⊆ GR- NeuInt(M ∪N).

Theorem 5.5. Let M and N are subsets of P, then GR- NeuInt(M)∩ GR- NeuInt (N)=GR- NeuInt (M ∩N).

Proof. We know that M ∩N ⊆ M and M ∩N ⊆ N. We have, by Theorem 5.5(iv), GR- NeuInt (M ∩ N) ⊆ GR-NeuInt(M) and GR- NeuInt(M ∩ N) ⊆ GR- NeuInt(N).

This implies GR- NeuInt(M ∩ N) ⊆ GR- NeuInt (M)∩ GR- NeuInt(N)...(i)

Again, let x ∈ GR- NeuInt(M)∩GR- NeuInt(N). Then x ∈ GR- NeuInt(M) and x ∈ GR- NeuInt(N).

Hence x is a NeutroInterior point of each of NeutroSets M and N. It follows that M and N are GR-NeuNhd of x, so that their intersection M∩N is also a GR-NeuNhd of x. Hence x ∈ GR NeuInt (M ∩ N). Thus x ∈ GR- NeuInt(M)∩ GR- NeuInt (N) implies that x ∈ GR- NeuInt (M ∩ N). Therefore GR-NeuInt(M)∩ GR- NeuInt(N) ⊆ GR- NeuInt (M ∩N)...(ii)

From (i) and (ii), we get GR- NeuInt (M)∩ GR- NeuInt (N)=GR- NeuInt(M∩ N).

6. GRN-closure and their properties

Using the GR-NeutroClosed sets we can introduce the concept of GR-NeutroClosure operator in topological spaces.

Definition 6.1. Let M be a subset of a space P. We define the GR-NeutroClosure of M to be the intersection of all GR-NeutroClosure sets containing M. Mathematically, GR-cl(M) = ∩ {F M ⊆ F ∈ GRC(P)}.

Theorem 6.1. Let P be any topological space and M ⊆ P, then show that M is GR-NeutroClosure set if and only if GR-cl(M) = M.

Proof. Let M be a GR-NeutroClosed set in P. Then clearly the smallest GR-NeutroClosed set contained in M, is itself M. Hence GR-NeuCl(M) = M.

Conversely, suppose that M ⊆ P and GR-NeuCl(M) = M. Since GR-NeuCl(M) is a GR-NeutroOpen set in P, it follows that M is a GR-NeutroClosed set in P.

Theorem 6.2. Let M and N are subset of P. Then

GR-NeuCl(P) = P and GR-cl(∅) = ∅.

M ⊆ GR-NeuCl(M).

If N is any GR-NeuClosed set contained in M, then GR-NeuCl(M) ⊆ N.

If M ⊆ N, then GR-NeuCl(M) ⊆ GR-NeuCl(N).

GR-NeuCl(GR-NeuCl(A))=GR-NeuCl(M)

Proof. (1) Obviously.

(2) By the definition of GR-NeuClosure of M, it is obvious that M ⊆ GR-NeuCl(M).

(3) Let N be any GR-NeutroClosed set containing M. Since GR-NeuCl(M) is the intersection of all GR-NeutroClosed sets containing M i.e GR-NeuCl(M) is contained in every GR-NeutroClosed set containing M. Hence GR-NeuCl(M) ⊆ N.

(4) Let M and N are NeutroSubsets of P such that M ⊆ N. By the definition of GR-NeutroClosure, GR-NeuCl(N) = ∩{F N ⊆ F ∈ GRC(P)}. If N ⊆ F ∈ GRNC(P), then GR-NeuCl(N) ⊆ F. Since M ⊆ N, M ⊆ N ⊆ F ∈ GRNC(P), we have GR-NeuCl(M) ⊆ F. Therefore GR-NeuCl(M) ⊆ ∩{F N ⊆ F ∈ GRNC(P)}=GR-NeuCl(P). That is GR-NeuCl(M) ⊆ GR-NeuCl(N).

Since GR-NeuCl(M) is a GR-NeutroClosed set in P. It follows that GR-NeuCl(GR-NeuCl(P)) = P.

Theorem 6.3. Let M and N are subsets of P, then GR-NeuCl(M∪ N)= GR-cl(M)∪ GR-NeuCl(N).

Proof. Let M and N are subsets of P. Clearly M ⊆ M ∪N and N ⊆ M ∪ N. We have by the Theorem 6.3(iv), GR-NeuCl(M) ⊆ GR-NeuCl(M ∪N) and GR-NeuCl(N) ⊆ GR-NeuCl(M ∪N). This implies GR-NeuCl(M)∪ GR-NeuCl(N) ⊆ GR-NeuCl(M ∪ N)...(i).

Now to prove that GR-NeuCl(M ∪ N) ⊆ GR-NeuCl(M)∪ GR-NeuCl(N). Let x ∈ GR-NeuCl(M ∪N) and x ∉ GR-NeuCl(M)∪ GR-NeuCl(N). Then there exists GR-NeutroClosed sets M1 and N1 with M ⊆ M1, N ⊆ N1 and x ∉ M1 ∪N1. We have M ∪ N ⊆ M1 ∪N1 and M1 ∪N1 is a GR-NeutroClosed set by Theorem 6.3, such that x ∉ M1 ∪N1. Thus x ∉ GR-NeuCl(M ∪ N) which is contradiction to x ∈ GR-NeuCl(M ∪ N).

Hence GR-NeuCl(M ∪N) ⊆ GR-NeuCl(M)∪ GR-NeuCl(N)...(ii).

From (i) and (ii), we have GR-NeuCl(M ∪N)= GR-NeuCl(M)∪ GR-NeuCl(N).

Theorem 6.4. Let M and N are subsets of P, then GR- NeuCl (M∩ N) ⊆ GR- NeuCl (M)∩ GR- NeuCl (N).

Proof. Let M and N are subsets of P. Clearly M∩N ⊆ M and M ∩N ⊆ N. We have, by Theorem 6.3(iv), GR- NeuCl (M ∩N) ⊆ GR- NeuCl(M) and GR- NeuCl (M ∩N) ⊆ GR- NeuCl(N). This implies GR- NeuCl(M ∩N) ⊆ GR- NeuCl(M)∩ GR- NeuCl (N).

Remark 6.1. In general GR- NeuCl (M)∩ GR- NeuCl (N) $\neq$ GR- NeuCl (M∩ N), as seen from the following example.

Example 6.1. Consider P = {1,2,3,4}, topology on it τ= {P,Ø,{1},{2},{1,2},{2,3},{1,2,3}}, M = {2,3}, and N= {3,4}, M∩N = {3}, GR- NeuCl (M) = {2,3,4}, GR- NeuCl (N) = {3,4}, GR- NeuCl (M ∩ N) = {3} and GR- NeuCl (M)∩ GR- NeuCl (N) = {3,4}. Therefore GR- NeuCl (M)∩ GR- NeuCl (N) Z GR- NeuCl(M ∩N).

Theorem 6.5. Let M be a subset of P and x ∈ P. Then x ∈ GR- NeuCl (M) if and only if V ∩M ≠ Ø for every GR-NeutroOpen set V containing x.

Conclusion

In this study, new Generalization of Regular Neutro-open sets and Generalized regular Neutro-open set has been studied. Some properties of Regular Neutro-open sets and are studied. Also, properties of GRN-neighbourhood, GRN-interior and GRN-closure properties are investigated. Hope this work will give more benefits for further studies of Neutro-Topology.

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A Note on Generalized Heptagonal Neutrosophic Sets

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Abstract: The concept of a Generalized Heptagonal Neutrosophic set is introduced and its properties are examined in this paper. We also discussed about the interior and closure operators of the Generalized Heptagonal neutrosophic set.

Keywords: neutrosophic Set, Heptagonal neutrosophic set, Generalized Heptagonal neutrosophic set, GHN-interior and GHN-closure operator.

1. Introduction

Introduction and study of fuzzy set theory were done by Zadeh [12]. Atanassov[4] introduced an intuitionistic fuzzy set theory. Coker [5] created later intuitionistic fuzzy topology. Florentine Smarandache [6] established the concept of Neutrosophic Fuzzy set theory in 1999. Truth, falsehood, and indeterminacy are the three components on which he defined the neutrosophic set. Salama and et al. [1-2] derived the neutrosophic topological spaces by transforming the idea of neutrosophic crisp set in 2012. Many scientists working in the fields of partitioned, quadripartitioned, pentapartitioned, heptapartitioned, etc. and they have developed neutrosophic topological spaces recently. Kungumaraj E and et al[8] invented the heptagonal neutrosophic set and heptagonal neutrosophic topological spaces in 2023. Radha R and Gayathri Devi R K [9] introduced the Generalized Quadripartitioned Neutrosophic Set in 2022. We establish and further investigate the notion of a Generalized Heptagonal Neutrosophic set in this study. We also discussed about the Generalized Heptagonal Neutrosophic set's interior and closure operators.

2. Preliminaries

Definition 2.1. [6] Let a non-empty fixed set be X. An element of the form A = {⟨x, αA(x), βA(x), γA(x)⟩: x ∈ X} is known as a neutrosophic set (NS), where αA(x), βA(x), γA(x) represent the degrees of membership, indeterminacy and non-membership respectively, of each element x ∈ X to the set A accordingly.

A neutrosophic set A = {⟨x, αA(x), βA(x), γA(x)⟩: x ∈ X} can be identified as an ordered triple ⟨αA(x), βA(x), γA(x)⟩ in]−0, 1 +[on X.

Definition 2.2. [3] A family Ţ of neutrosophic subsets in X that meets the following axioms is a neutrosophic topology (NT) on a non-empty set X.

(Axiom 1) 0N, 1N ∈ Ţ

(Axiom 2) G1 ∩ G2 ∈ Ţ for any G1 , G2 ∈ Ţ

(Axiom 3)∪Gi ∈ Ţ ∀{Gi : i∈ J} ⊆ Ţ

The pair (X, Ţ) is used to represent a neutrosophic topological space Ţ over X.

Definition 2.3. [8] A heptagonal neutrosophic number S is briefed as

S = < [(p, q, r, s, t, u, v); μ], [(p ′, q′ , r′ , s′ , t′ , u′ , v′); ℰ] ,[(p ″, q″, r″, s″, t″, u″, v″); η] > where μ, ℰ, η ∈ [0, 1], where α : R ⇒ [0, μ] denotes the truth membership function, β : R⇒ [ℰ, 1] denotes the indeterminacy membership function and γ : R ⇒ [η, 1] denotes the falsity membership function.

Using ranking technique, heptagonal neutrosophic number is changed as,

$$\lambda = \frac{(p+q+r+s+t+u+v)}{7}$$

$$\mu = \frac{(p'+q'+r'+s'+t'+u'+v')}{7}$$

$$\delta = \frac{(p''+q''+r''+s''+t''+u''+v'')}{7}$$

Then the Heptagonal Neutrosophic set HNS takes the form

A_{HN} = <x; λA_{HN} (x), μA_{HN} (x), δA_{HN} (x) >

Definition 2.4. [8] Assume that X is a non-void set and A_{HN} and B_{HN} be a HNS of the form A_{HN} = <x; λA_{HN} (x), μA_{HN} (x), δA_{HN} (x) >, B_{HN} =<x; λB_{HN} (x), μB_{HN} (x), δB_{HN} (x)>, then their heptagonal neutrosophic number operations may be defined as

- **Inclusive:**
  (i) $A_{HN} \subseteq B_{HN} \Rightarrow \lambda A_{HN}(x) \leq \lambda B_{HN}(x), \mu A_{HN}(x) \geq \mu B_{HN}(x), \delta A_{HN}(x) \geq \delta B_{HN}(x)$, for all x∈X.
  (ii) $B_{HN} \subseteq A_{HN} \Rightarrow \lambda B_{HN}(x) \leq \lambda A_{HN}(x), \mu B_{HN}(x) \geq \mu A_{HN}(x), \delta B_{HN}(x) \geq \delta A_{HN}(x)$, for all x∈X.
- **Union and Intersection:**
  (i) $A_{HN} \cup B_{HN}$ = {<x; $(\lambda A_{HN}(x) \vee \lambda B_{HN}(x), \mu A_{HN}(x) \wedge \mu B_{HN}(x), \delta A_{HN}(x) \wedge \delta B_{HN}(x))$>}
  (ii) $A_{HN} \cap B_{HN}$ = {<x; $(\lambda A_{HN}(x) \wedge \lambda B_{HN}(x), \mu A_{HN}(x) \vee \mu B_{HN}(x), \delta A_{HN}(x) \vee \delta B_{HN}(x))$>}
- **Complement:**

  Assume that X is a non-void set and A_{HN} = <x; λA_{HN}(x), μA_{HN}(x), δA_{HN}(x)> be the HNS, then the complement of A is represented by A'_{HN} and it takes the form

  A'_{HN} = <x; δA_{HN} (x), 1−μA_{HN} (x), λA_{HN} (x) > for all x∈X.
- **Universal and Empty set:**

  Let A_{HN} = <x; λA_{HN} (x), μA_{HN} (x), δA_{HN} (x) > be a HNS and the universal set I_A and the null set O_A of A_{HN} is defined by
  (i) I_{HN} = <x; (1, 0, 0)> for all x∈X.
  (ii) O_{HN} = <x; (0, 1, 1)> for all x∈X.

Definition 2.5. [8] A family Ʈ of heptagonal neutrosophic subsets in X that meets the following axioms is a heptagonal neutrosophic topology (HNT) on a non-empty set X.

(HNT1) $I_{HN}(x)$, $O_{HN}(x)$ ∈ Ʈ

(HNT2) ∪ A_i∈ Ʈ , ∀{A_i : i∈ J} ⊆ Ʈ

(HNT3) $A_1 \cap A_2$∈ Ʈ for any A_1, A_2 ∈ Ʈ

The heptagonal neutrosophic topological space Ʈ over X is represented as a pair (X, Ʈ). All the sets in Ʈ are known as heptagonal neutrosophic open set of X and its respective complements are said to be heptagonal neutrosophic closed set of X.

Definition 2.6. [8] Let A be a HNS in HNTS X. Then,

- HN-int(A_{HN}) = ⋃{G_{HN}: G_{HN} is a HNOS in X and G_{HN} ⊆ A_{HN}} is referred to heptagonal neutrosophic interior of A. HN-int(A_{HN}) is the largest HN-open subset contained in A_{HN}.
- HN-cl(A_{HN}) = ⋂{K_{HN}: K_{HN} is a HNCS in X and A_{HN} ⊆ K_{HN}} is referred to heptagonal neutrosophic closure of A. HN-cl(A_{HN}) is the smallest HN-closed subset containing A_{HN}.

Definition 2.7. [8] Let (X_{HN},Ʈ) and (Y_{HN} ,σ)are two non-empty Heptagonal neutrosophic topological spaces.A map f : X_{HN}→ Y_{HN} is called a heptagonal neutrosophic continuous function, if each heptagonal neutrosophic open set A_{HN} in Y_{HN} has an inverted image f^{-1}(A_{HN}) that is also a heptagonal neutrosophic open in X_{HN}.

3. Generalized Heptagonal neutrosophic set

we characterize and define a new category of generalized sets in Heptagonal neutrosophic topological spaces in this section.

Definition 3.1: Assume that X is a non-void set and consider A_{HN} be the HNS after ranking technique in definition 2.3, then a generalized heptagonal neutrosophic set GHN(A) is of the form GHN(A) = < x; λGHNA(x), μGHNA(x), δGHNA(x)> where λGHNA(x) is the truth membership degree, μGHNA(x) is the indeterminacy degree and δGHNA(x) is the false membership degree values respectively of each element x∈X to the set A satisfying the condition λGHNA(x) ∧ μGHNA(x) ∧ δGHNA(x) ≤ 0.5.

Example 3.2: Consider X = {x,y} and A_{HN}, B_{HN} ∈ HN(X).

A_{HN} = { < x; (λ: 0.85,0.65,0.55,0.78,0.92,0.63,0.38), (μ: 0.75,0.95,0.63,0.48,0.56,0.88,0.78), (δ: 0.25,0.36,0.45,0.45,0.42,0.72,0.62)>, < y; (λ:0.83,0.65,0.72,0.98,0.66,0.53,0.92), (μ:0.73,0.53,0.45,0.38,0.92,0.75,0.63), (δ:0.45,0.35,0.25,0.35,0.85,0.65,0.15)>} and

B_{HN} = { < x; (λ:0.86,0.73,0.62,0.52,0.93,0.45,1), (μ:0.43,0.39,0.26,0.59,0.58,0.93,0.32), (δ:0.55,0.73,0.62,0.52,0.95,0.89,0.44)>, < y; (λ:0.73,0.62,0.51,0.42,0.33,0.29,0.19), (μ:0.82,0.92,1,0.61,0.54,0.76,0.46), (δ:0.19,0.23,0.63,0.52,0.95,0.82,1)>}

By Ranking Technique, (Definition 2.3)
A_{HN} = { < x; (λ:0.68), (μ:0.72), (δ:0.47)>, < y; (λ:0.76), (μ:0.63), (δ:0.44)>}
B_{HN} = { < x; (λ:0.73), (μ:0.50), (δ:0.67)>, < y; (λ:0.44), (μ:0.73), (δ:0.62)>}

A_{HN}	x	y	B_{HN}	x	y
λ	0.68	0.76	λ	0.73	0.44
μ	0.72	0.63	μ	0.50	0.73
δ	0.47	0.44	δ	0.67	0.62
$\lambda \wedge \mu \wedge \delta$	**0.47**	**0.44**	$\lambda \wedge \mu \wedge \delta$	**0.50**	**0.44**
A_{HN} is a Generalized Heptagonal NS			**B_{HN} is a Generalized Heptagonal NS**		

Definition 3.3: Generalized Heptagonal neutrosophic Set operations

Assume that X is a non-void set and GHN(U) and GHN(V) are HNS of the form GHN(U) = < a; λGHNU(a), μGHNU(a), δGHNU(a) >,

GHN(V) = < a; λGHNV(a), μGHNV(a), δGHNV(a)>, then the generalized heptagonal neutrosophic number operations may be defined as

- **Inclusive:**

GHN(U)⊆GHN(V) ⇒ λGHNU(a)≤λGHNV(a), μGHNU(a)≥μGHNV(a), δGHNU(a)≥δGHNV(a), for all a∈X.

Union and Intersection:

GHN(U)⋃GHN(V) = {<a; (λGHNU(a)∨λGHNV(a), μGHNU(a)∧μGHNV(a), δGHNU(a)∧δGHNV(a))>}

GHN(U)⋂GHN(V) = {<a; (λGHNU(a)∧λGHNV(a), μGHNU(a)∨μGHNV(a), δGHNU(a)∨δGHNV(a))>}

Complement:

Assume that X is a non-void set and GHN(A) be the GHNS of the form <a; λGHNA(a), μGHNA(a), δGHNA(a)>, then its complement is represented by GHN(A′) and is defined by

GHN(A′) = < a; (δGHNA(a), 1−μGHNA(a), λGHNA(a)) > for all a∈X.

Universal and Empty set:

The universal set and the null set of GHNS over X is defined by

(i) GHN(Ia) = <a; (1,0,0)> for all a∈X.

(ii) GHN(Oa) = <a; (0,1,1)> for all a∈X.

For simplicity, we consider the GHNS after the ranking technique in the subsequent examples:

Example 3.4:

(i) Consider Y = {p,q,r} and the Generalized heptagonal neutrosophic sets
GHN(A) = {<y: (p,0.52, 0.42, 0.67),(q,0.31,0.75,0.44),(r,0.26,0.68,0.88)>}
GHN(B) = {<y: (p,0.66, 0.33,0.48),(q,0.55,0.66,0.33),(r,0.48,0.32,0.64)>}
Here GHN(A) ⊆ GHN(B), since

{<y: (p, 0.52 ≤ 0.66, 0.42 ≥ 0.33, 0.67 ≥ 0.48),(q, 0.31 ≤ 0.55,0.75 ≥ 0.66,0.44 ≥ 0.33), (r, 0.26 ≤ 0.48,0.68 ≥ 0.32,0.88 ≥ 0.64)>}

(ii) Consider Y = {p,q,r} and the Generalized heptagonal neutrosophic sets
GHN(C) = {<y: (p,0.43,0.52,0.68),(q,0.91,0.43,0.26),(r,0.85,0.69,0.37)>}
GHN(D) = {<y: (p,0.74,0.68,0.18),(q,0.39,0.45,0.77),(r,0.14,0.52,0.88)>}

GHN(C)⋃GHN(D) = {<y: (p, 0.43∨0.74, 0.52∧0.68, 0.68∧0.18), (q, 0.91∨0.39, 0.43∧0.45, 0.26∧0.77), (r, 0.85∨0.14, 0.69∧0.52, 0.37∧0.88)>} = {<x: (p,0.74,0.52,0.18),(q,0.91,0.43,0.26),(r,0.85,0.52,0.37)>}
GHN(C)⋂GHN(D) = {<y: (p, 0.43∧0.74, 0.52∨0.68, 0.68∨0.18), (q, 0.91∧0.39, 0.43∨0.45, 0.26∨0.77), (r, 0.85∧0.14, 0.69∨0.52, 0.37∨0.88)>} = {<x: (p,0.43,0.68,0.68),(q,0.39,0.45,0.77),(r,0.14,0.69,0.88)>}
GHN(A′) = {<y: (p,0.68,0.48,0.43),(q,0.26,0.57,0.91),(r,0.37,0.31,0.85)>}

GHN(B′) = {<y: (p,0.18,0.32,0.74),(q,0.77,0.55,0.39),(r,0.88,0.48,0.14)>}

Definition 3.5: A family Ʈ of Generalized heptagonal neutrosophic sets adhering to the following axioms is called a Generalized heptagonal neutrosophic topology on a non-empty set Y.

i) GHN(I_X), GHN(O_X) ∈Ʈ.

ii) For any sub collection of the elements of Ʈ, whose union is contained in Ʈ.

iii) For any finite sub collection of the elements of Ʈ, whose intersection is contained in Ʈ.

The pair (Y, Ʈ) is called a Generalized heptagonal neutrosophic topological space (GHNTS) over Y.

Remark 3.6:

1. Every member of Ʈ is referred to be a GHN-open set in X.
2. The set GHN(A) is referred to be a GHN-closed set in X if GHN(A′) is open in Ʈ.

Example 3.7: Consider the Generalized heptagonal neutrosophic sets with Y = {s, t, u}
GHN(D) = {<y: (s,0.25, 0.45, 0.65), (t,0.50, 0.60, 0.70), (u, 0.35, 0.25, 0.15)>}
GHN(E) = {<y: (s,0.33, 0.44, 0.55), (t,0.55, 0.56, 0.57), (u, 0.48, 0.18, 0.12)>}

Here GHN(D)⋃GHN(E) = GHN(E) and GHN(D)⋂GHN(E) = GHN(E)

Hence Ʈ $= \{GHN(Ix), GHN(D), GHN(E), GHN(Ox)\}$ forms a Generalized Heptagonal Neutrosophic Topological space.

Definition 3.8: Let GHN(A) be a GHNS in GHNTS X. Then,

- GHN-int(A) = ⋃ {GHN(F) ; where GHN(F) is GHNO in X and GHN(F) ⊆ GHN(A)} is said to be a generalized heptagonal neutrosophic interior of A. GHN-int(A) is the largest GHN-open subset contained in GHN(A).
- GHN-cl(A) = ⋂ {GHN(K) ; where GHN(K) is GHNC in X and GHN(A) ⊆ GHN(K)} is said to be a generalized heptagonal neutrosophic closure of A. GHN-int(A) is the smallest GHN-closed subset containing GHN(A).

Example 3.9: Consider X = {s,t} and the Generalized heptagonal neutrosophic sets

GHN(F_1) = {<x: (s, 0.4, 0.3, 0.5), (t, 0.1, 0.2, 0.5)>}
GHN(F_2) = {<x: (s, 0.4, 0.4, 0.5), (t, 0.4, 0.3, 0.4)>}

Here GHN(F_1)⋃GHN(F_2) = GHN(F_2) and GHN(F_1)⋂GHN(F_2) = GHN(F_1)

Ʈ $= \{GHN(I_x), GHN(F_1), GHN(F_2), GHN(O_x)\}$ is a Generalized Heptagonal NeutrosophicTopological space, then

GHN(F_1) and GHN(F_2) are GHN-open sets of X, GHN(F_1') and GHN(F_2') are GHN-closed sets of X.

Consider the GHN sets

GHN(A) {<x: (s, 0.3, 0.3, 0.6), (t, 0.3, 0.2, 0.5)>},
GHN(B) {<x: (s, 0.6, 0.7, 0.3), (t, 0.5, 0.8, 0.3)>},
GHN(C) {<x: (s, 0.4, 0.6, 0.5), (t, 0.3, 0.6, 0.9)>} and
GHN(D) {<x: (s, 0.5, 0.4, 0.4), (t, 0.9, 0.4, 0.3)>}

GHN – interior operator		GHN – closure operator	
GHN-int(F_1) = $GHN(F_1)$	GHN-int(A) = $GHN(O_x)$	GHN-cl(F_1) = $GHN(I_x)$	GHN-cl(A) = $GHN(I_x)$
GHN-int(F_2) = $GHN(F_2)$	GHN-int(B) = $GHN(O_x)$	GHN-cl(F_2) = $GHN(I_x)$	GHN-cl(B) = $GHN(I_x)$
GHN-int(F_1') = $GHN(O_x)$	GHN-int(C) = $GHN(O_x)$	GHN-cl(F_1') = $GHN(F_1')$	GHN-cl(C) = $GHN(I_x)$
GHN-int(F_2') = $GHN(O_x)$	GHN-int(D) = $GHN(O_x)$	GHN-cl(F_2') = $GHN(F_2')$	GHN-cl(D) = $GHN(I_x)$

Proposition 3.10:

Consider (X, Ţ) be a GHNTS. Therefore, for every two generalized heptagonal neutrosophic subsets GHN(M) and GHN(N) of a GHNTS X we have

(i) GHN-int(M)⊆M
(ii) GHN(M) is a GHNO set in X if and only if GHN-int(M) = M
(iii) GHN-int(GHN-int(M)) = GHN-int(M)
(iv) If M ⊆ N then GHN-int(M) ⊆ GHN-int(N)
(v) GHN-int(M⋂N) = GHN-int(M)⋂GHN-int(N)
(vi) GHN-int(M)⋃GHN-int(N) ⊆ GHN-int(M⋃N)

Proof.

(i) Follows from Definition 3.8.

(ii) GHN(M) is a GHNO set in X. Then M ⊆GHN-int(M) and by using (i) we get GHN-int(M) = M. Conversely assume that GHN-int(M) = M. By using Definition 3.8, GHN(M) is a GHNO set in X. Thus (ii) is proved.

(iii) By using (ii), GHN-int(GHN-int(M)) = GHN-int(M). This proves (iii).

(iv) Since M⊆N, by using (i), GHN-int(M)⊆M⊆N. That is GHN-int(M)⊆N. By (iii), GHN-int(GHN-int(M)) ⊆ GHN-int(N). Thus GHN-int(M) ⊆ GHN-int(N). Thus (iv) is proved.

(v) Since M⋂N⊆M and M⋂N⊆N, by using (iv), GHN-int(M⋂N) ⊆ GHN-int(M) and GHN-int(M⋂N) ⊆ GHN-int(N). This implies that
GHN-int(M⋂N) ⊆ GHN-int(M)⋂ GHN-int(N) ---(1).
By(i), GHN-int(M)⊆M and GHN-int(N)⊆N. This implies that
GHN-int(M)⋂GHN-int(N) ⊆ M⋂N.
Now by (iv), GHN-int(GHN-int(M)⋂GHN-int(N)) ⊆ GHN-int(M⋂N)
By (1), GHN-int(GHN-int(M))⋂GHN-int(GHN-int(N)) ⊆ GHN-int(M⋂N).
By (iii), GHN-int(M)⋂GHN-int(N) ⊆ GHN-int(M⋂N) -----(2).
From (1) and (2), GHN-int(M⋂N)=GHN-int(M)⋂GHN-int(N). Thus (v) is proved.

(vi) Since M⊆M⋃N and M⊆M⋃N, by (iv), GHN-int(M)⊆GHN-int(M⋃N) and GHN-int(N)⊆GHN-int(M⋃N). This implies that,
GHN-int(M)⋃GHN-int(N)⊆GHN-int(M⋃N). Thus (vi) is proved.

Proposition 3.11: Consider (X, Ʈ) be a GHNTS. Therefore, for every two generalized heptagonal neutrosophic subsets P and Q of a GHNTS X we have

(i) P ⊆ GHN-cl(P)
(ii) P is GHNC set in X if and only if GHN-cl(P) = P
(iii) GHN-cl(GHN-cl(P)) = GHN-cl(P)
(iv) If P ⊆ Q then GHN-cl(P)⊆ GHN-cl(Q)
(v) GHN-cl(P⋂Q) ⊆ GHN-cl(P)⋂GHN-cl(Q)
(vi) GHN-cl(P)⋃GHN-cl(Q) = GHN-cl(P⋃Q)

Proof.

I. Proceed from the definition 3.8.
II. Consider P as a GHNC set in X. Then P contains GHN-cl(P). Now by using (i), we get P = GHN-cl(P). Conversely assume that P = GHN-cl(P). By using Definition 3.8, P is a GHNC set in X. Thus (ii) is proved.
III. By using (ii), GHN-cl(GHN-cl(P)) = GHN-cl(P). This (iii) is proved.
IV. By using (i), Q ⊆ GHN-cl(Q) and since P ⊆ Q, we have P ⊆ GHN-cl(Q). But GHN-cl(P) is the smallest closed set containing P, hence GHN-cl(P) ⊆ GHN-cl(Q). Thus (iv) is proved.
V. Since P⋂Q ⊆ P and P⋂Q ⊆ Q, by using (iv), GHN-cl(P⋂Q) ⊆ GHN-cl(P) and
GHN-cl(P⋂Q) ⊆ GHN-cl(Q). This implies that
GHN-cl(P⋂Q) ⊆ GHN-cl(P)⋂GHN-cl(Q). Thus (v) is proved.
VI. Since P ⊆ P⋃Q and Q ⊆ P⋃Q, by (iv), GHN-cl(P) ⊆ GHN-cl(P⋃Q) and GHN-cl(Q) ⊆ GHN-cl(P⋃Q). This implies that,
GHN-cl(P) ⋃ GHN-cl(Q) ⊆ GHN-cl(P⋃Q) ------(1)
By(i), P ⊆ GHN-cl(P) and Q ⊆ GHN-cl(Q). This implies that
P⋃Q ⊆ GHN-cl(P) ⋃ GHN-cl(Q).
Now by (iv), GHN-cl (P⋃Q) ⊆ GHN-cl((GHN-cl(P) ⋃ GHN-cl(Q)).
By (1), GHN-cl(P⋃Q) ⊆ GHN-cl(GHN-cl(P)) ⋃ GHN-cl(GHN-cl(Q)).
By (iii), GHN-cl(P⋃Q) ⊆ GHN-cl(P) ⋃ GHN-cl(Q) -----(2).

From (1) and (2), GHN-cl(P⋃Q) = GHN-cl(P) ⋃ GHN-cl(Q).
Thus (vi) is proved.

Proposition 3.12: Consider (X, Ʈ) be a GHNTS. Therefore, for every generalized heptagonal neutrosophic subset U in a GHNTS X we have.

(i) (GHN-int(U))′ = GHN-cl(U′)
(ii) (GHN-cl(U))′ = GHN-int(U′)

Proof.

(i) By definition 3.8, GHN-int (U) = ⋃{ S: S is a GHNO set in X and S ⊆ U}
Taking complement on both sides,
(GHN-int(U))′ = ⋂{ S′: S′ is a GHNC set in X and U′⊆ S′}
Now, replace S′ by L, we get
(GHN-int(U))′ = ⋂{ L : L is a GHNC set in X and U′ ⊆ L}
From the definition 3.8, (GHN-int(A_{HN}))′ = GHN-cl(U′). Thus (i) is proved.
(ii) From (i) Let U′ be the GHNS

We write, (GHN-int(U'))' = GHN-cl(U)
Taking complement on both sides we get
GHN-int(U') = (GHN-cl(U))'. Thus (ii) is proved.

Conclusion

The basic operations of generalized heptagonal neutrosophic sets are demonstrated in this article with suitable examples. Further explanations of the concepts of Generalized Heptagonal neutrosophic interior and closure are provided in order to support the GHN topology. The properties of GHN-closed and GHN-open sets of GHN topologies are explained with similar examples. Furthermore, based on GHN topological spaces, continual functions, connectivity, and compact can be developed in the future.

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Heptagonal Neutrosophic Quotient Mappings

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Abstract: In 2023, [6] Kungumaraj et al. presented the Heptagonal Neurosophic Number and Heptagonal Neurosophic Topology. Heptagonal neutrosophic numbers are essential because they provide a powerful tool for representing and managing uncertainty in decision-making processes across various domains, offering a more nuanced and versatile approach compared to traditional fuzzy or intuitionistic fuzzy sets. Heptagonal neutrosophic methods differ from other neutrosophic methods primarily in the number of parameters they consider and their applications. Heptagonal neutrosophic numbers consider seven parameters, namely truth, falsity, indeterminacy, neutral, anti-neutral, extra-neutral, and pseudo-neutrality. By de-neutrosophication technique, heptagonal neutrosophic numbers transformed into a crisp neutrosophic values for better outcomes.
The major goal of this research is to investigate the concepts of Heptagonal Neutrosophic (HN) quotient mappings as well as Heptagonal neutrosophic strongly quotient maps and HN*-quotient maps in Heptagonal neutrosophic topological spaces. We provide examples of the fundamental concepts and subsequently we also proved their characterizations.

Keywords: Heptagonal neutrosophic number**;** HN-irresolute map; HN-open map; HN-quotient map; HN strongly quotient map; HN*-quotient map.

1. Introduction

The fuzzy set theory was introduced and studied by Zadeh [11]. An intuitionistic fuzzy set theory was introduced by Atanassov [3]. Later intuitionistic fuzzy topology was developed by Coker [4]. Neutrosophic Fuzzy set theory was introduced by Smarandache [5] in 1999. He defined the neutrosophic set on three components (truth, falsehood, indeterminacy). The Neutrosophic crisp set concept was converted into neutrosophic topological spaces by Salama et al. in [1-2]. In recent years, neutrosophic topological spaces was developed by many scientists in the field of triangular, quadripartitioned, pentapartitioned, heptapartitioned etc. Recently, heptagonal neutrosophic set and heptagonal neutrosophic topological spaces was developed by Kungumaraj E and et al in 2023[6].

Quotient mappings have applications across various areas of mathematics, including algebraic topology, differential geometry, and geometric group theory. Neutrosophic Quotient mappings was first introduced by T. Nandhini and M. Vigneshwaran[8] in 2019. Later Mohana Sundari M and etal[7] and Radha R and etal [9] introduced respectively the Quadripartitioned Neutrosophic Mappings and Pentapartitioned Neutrosophic Quotient Mappings. Recently, Subasree R and etal [10] investigated about the Heptagonal Neutrosophic Semi-open Sets in Heptagonal Neutrosophic Topological Spaces.

In this paper, we presented the following in section-wise, Section 2 provides background information that will help readers understand the study better. Section 3 introduces the concept of heptagonal neutrosophic quotient mappings, HN-strongly quotient maps and HN*-quotient maps along with their key features and instances are given. Section 4 looks at these maps' characterizations as well as the compositions of two of them. The study's final conclusions with illustrations are presented in Section 5 of the conclusion, along with some suggestions for more research.

2. Preliminaries

Definition 2.1. [5] Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in X\}$ where $\alpha_A(x)$, $\beta_A(x)$, $\gamma_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element $x \in X$ to the set A.

A Neutrosophic set $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified as an ordered triple $\langle \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle$ in $]-0, 1+[$ on X.

Definition 2.2. [1] A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfies the following axioms:

(NT1) $0_N, 1_N \in \tau$

(NT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(NT3) $\cup G_i \in \tau\ \forall \{G_i : i \in J\} \subseteq \tau$

The pair (X, τ) is used to represent a neutrosophic topological space τ over X.

Definition 2.3. [6] A heptagonal neutrosophic number S is defined and described as

$S = < [(p, q, r, s, t, u, v); \mu],[(p', q', r', s', t', u', v'); \mathscr{E}],[(p'', q'', r'', s'', t'', u'', v''); \eta] >$ where $\mu, \mathscr{E}, \eta \in [0, 1]$. The truth membership function $\alpha : R \Rightarrow [0, \mu]$, the indeterminacy membership function $\beta : R \Rightarrow [\mathscr{E}, 1]$, the falsity membership function $\gamma : R \Rightarrow [\eta, 1]$.

Using deneutrosophication technique, heptagonal neutrosophic number is changed as,

$$\lambda = \frac{(p + q + r + s + t + u + v)}{7}$$

$$\mu = \frac{(p' + q' + r' + s' + t' + u' + v')}{7}$$

$$\delta = \frac{(p'' + q'' + r'' + s'' + t'' + u'' + v'')}{7}$$

Then the Heptagonal Neutrosophic set HNS takes the crisp form

$A_{HN} = <x; \lambda A_{HN}(x), \mu A_{HN}(x), \delta A_{HN}(x)>$

Definition 2.4. [6] Let X be a non-empty set and A_{HN} and B_{HN} are HNS of the form $A_{HN} = <x; \lambda A_{HN}(x), \mu A_{HN}(x), \delta A_{HN}(x)>$, $B_{HN} = <x; \lambda B_{HN}(x), \mu B_{HN}(x), \delta B_{HN}(x)>$, then their heptagonal neutrosophic number operations may be defined as

- **Inclusive:**
  (i) $A_{HN} \subseteq B_{HN} \Rightarrow \lambda A_{HN}(x) \le \lambda B_{HN}(x), \mu A_{HN}(x) \ge \mu B_{HN}(x), \delta A_{HN}(x) \ge \delta B_{HN}(x)$, for all $x \in X$.
  (ii) $B_{HN} \subseteq A_{HN} \Rightarrow \lambda B_{HN}(x) \le \lambda A_{HN}(x), \mu B_{HN}(x) \ge \mu A_{HN}(x), \delta B_{HN}(x) \ge \delta A_{HN}(x)$, for all $x \in X$.
- **Union and Intersection:**
  (i) $A_{HN} \cup B_{HN} = \{<x; (\lambda A_{HN}(x) \vee \lambda B_{HN}(x), \mu A_{HN}(x) \wedge \mu B_{HN}(x), \delta A_{HN}(x) \wedge \delta B_{HN}(x))>\}$
  (ii) $A_{HN} \cap B_{HN} = \{<x; (\lambda A_{HN}(x) \wedge \lambda B_{HN}(x), \mu A_{HN}(x) \vee \mu B_{HN}(x), \delta A_{HN}(x) \vee \delta B_{HN}(x))>\}$
- **Complement:**
  Let X be a non-empty set and $A_{HN} = <x; \lambda A_{HN}(x), \mu A_{HN}(x), \delta A_{HN}(x)>$ be the HNS, then its complement is denoted by A'_{HN} and is defined by
  $A'_{HN} = <x; \delta A_{HN}(x), 1 - \mu A_{HN}(x), \lambda A_{HN}(x)>$ for all $x \in X$.
- **Universal and Empty set:**

Let A_{HN} = <x; $\lambda A_{HN}(x)$, $\mu A_{HN}(x)$, $\delta A_{HN}(x)$ > be a HNS and the universal set I_A and the null set O_A of $A_{HN}$is defined by

(i) I_{HN} = <x: (1,0,0)> for all $x \in X$.

(ii) O_{HN} = <x: (0,1,1)> for all $x \in X$.

Definition 2.5. [6] A Heptagonal neutrosophic topology (HNT) on a non-empty set X is a family τ of heptagonal neutrosophic subsets in X satisfies the following axioms:

(HNT1) $I_{HN}(x)$, $O_{HN}(x) \in \tau$

(HNT2)$\bigcup A_i \in \tau$, $\forall\{A_i : i \in J\} \subseteq \tau$

(HNT3) $A_1 \cap A_2 \in \tau$ for any A_1 , $A_2 \in \tau$

The pair (X, τ) is used to represent a heptagonal neutrosophic topological space τ over X. The sets in τ are called heptagonal neutrosophic open set of X. The complement of heptagonal neutrosophic open sets are called heptagonal neutrosophic closed set of X.

Definition 2.6. [6] Let A be a HNS in HNTS X. Then,

- HNint(A_{HN}) = $\bigcup\{G_{HN}$: G_{HN} is a HNOS in X and $G_{HN} \subseteq A_{HN}\}$ is called a heptagonal neutrosophic interior of A. It is the largest HN-open subset contained in A_{HN}.
- HNcl(A_{HN}) = $\bigcap\{K_{HN}$: K_{HN} is a HNCS in X and $A_{HN} \subseteq K_{HN}\}$ is called a heptagonal neutrosophic closure of A. It is the smallest HN-closed subset containing A_{HN}.

Definition 2.7.[6]Let (X_{HN},τ) and (Y_{HN} ,σ)are two non-empty Heptagonal neutrosophic topological spaces.A map f: $X_{HN} \rightarrow Y_{HN}$ is called a heptagonal neutrosophic continuous function if the inverse image $\boldsymbol{f^{-1}}$(A_{HN}) of each heptagonal neutrosophic open set A_{HN} in Y_{HN} is heptagonal neutrosophic open in X_{HN}.

3. Heptagonal Neutrosophic Quotient Mappings

In this section, we define a new class of sets in Heptagonal Neutrosophic topological spaces and the quotient mappings.

Definition 3.1: Let A_{HN} be a HNS of a HNTS (X_{HN},τ). Then A_{HN} is said to be

(i) Heptagonal Neutrosophic pre-open [written HN-preO] set of X, if
$A_{HN} \subseteq$ HNint (HNcl(A_{HN})).

(ii) Heptagonal Neutrosophic semi-open [written HN-SO] set of X, if
$A_{HN} \subseteq$ HNcl (HNint(A_{HN})).

(iii) Heptagonal Neutrosophic α-open [written HN-αO] set of X, if
$A_{HN} \subseteq$ HNint (HNcl(HNint (A_{HN}))).

Example 3.2: Let X = {x,y}.Consider

A_{HN} = { < x; (λ:0.8,0.8,0.8,0.8,0.8,0.8,0.8), (μ: 0.3,0.3,0.3,0.3,0.3,0.3,0.3), (δ: 0.5,0.5,0.5,0.5,0.5,0.5,0.5)>, < y; (λ:0.6,0.6,0.6,0.6,0.6,0.6,0.6), (μ:0.5,0.5,0.5,0.5,0.5,0.5,0.5), (δ:0.9,0.9,0.9,0.9,0.9,0.9,0.9)>}

B_{HN} = { < x; (λ: 0.9,0.9,0.9,0.9,0.9,0.9,0.9), (μ:0.2,0.2,0.2,0.2,0.2,0.2,0.2), (δ:0.4,0.4,0.4,0.4,0.4,0.4,0.4)>, < y; (λ: 0.8,0.8,0.8,0.8,0.8,0.8,0.8), (μ: 0.3,0.3,0.3,0.3,0.3,0.3,0.3), (δ:0.7,0.7,0.7,0.7,0.7,0.7,0.7)>} and

C_{HN} = { < x; (λ: 0.5,0.5,0.5,0.5,0.5,0.5,0.5), (μ: 0.3,0.3,0.3,0.3,0.3,0.3,0.3), (δ: 0.5,0.5,0.5,0.5,0.5,0.5,0.5)>, < y; (λ: 0.4,0.4,0.4,0.4,0.4,0.4,0.4), (μ: 0.5,0.5,0.5,0.5,0.5,0.5,0.5), (δ: 0.9,0.9,0.9,0.9,0.9,0.9,0.9)>}

After de-neutrosophication technique in definition 2.3,

A_{HN} = { < x; (λ:0.8), (μ: 0.3), (δ: 0.5)>, < y; (λ:0.6), (μ:0.5), (δ:0.9)>}

B_{HN} = { < x; (λ: 0.9), (μ:0.2), (δ:0.4)>, < y; (λ: 0.8), (μ: 0.3), (δ:0.7)>} and

C_{HN} = { < x; (λ: 0.5), (μ: 0.3), (δ: 0.5)>, < y; (λ: 0.4), (μ: 0.5), (δ: 0.9)>}

τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, $A_{HN}∪B_{HN}$, $A_{HN}∩C_{HN}$, $B_{HN}∪C_{HN}$, $A_{HN}∩B_{HN}$, $A_{HN}∩C_{HN}$, $B_{HN}∩C_{HN}$} be the Heptagonal Neutrosophic topological space.

Consider the other HNS after ranking technique,

D_{HN} = { < x; (λ:0.9), (μ: 0.2), (δ: 0.4)>, < y; (λ:0.7), (μ:0.3), (δ:0.8)>}

E_{HN} = { < x; (λ: 0.6), (μ:0.4), (δ:0.6)>, < y; (λ: 0.3), (μ: 0.6), (δ:0.7)>} and

F_{HN} = { < x; (λ: 0.8), (μ: 0.3), (δ: 0.5)>, < y; (λ: 0.6), (μ: 0.5), (δ: 0.7)>}

Then the HN pre-O sets of X are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, E_{HN}, F_{HN}, E'_{HN}}

HN semi-O sets of X are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, F_{HN}}

HN α-O sets of X are {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}, D_{HN}, F_{HN}}

Remark 3.3: HN α-open set is the smallest set contained in both HN Pre open sets and HN semiopen sets of X.

Definition 3.4. Let (X_{HN},τ) and (Y_{HN},σ)are two non-empty Heptagonal neutrosophic topological spaces. A map *f* : X_{HN}→ Y_{HN} is called a

(i) HN pre-continuous function, if f^{-1}(A_{HN}) is HN-pre open in (X_{HN},τ), for each HN open set A_{HN} in (Y_{HN} ,σ).
(ii) HN semi-continuous function, if f^{-1} (A_{HN}) is HN-semi open in (X_{HN},τ), for each HN open set A_{HN} in (Y_{HN} ,σ).
(iii) HN α-continuous function, if f^{-1}(A_{HN}) is HN-α open in (X_{HN},τ), for each HN open set A_{HN} in (Y_{HN} ,σ).

Example 3.5: Let X = {x,y} and τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}}, then (X_{HN},τ) be a Heptagonal neutrosophic topological space with

A_{HN} = { < x; (λ:0.8), (μ: 0.3), (δ: 0.5)>, < y; (λ:0.6), (μ:0.5), (δ:0.9)>}

B_{HN} = { < x; (λ: 0.9), (μ:0.2), (δ:0.4)>, < y; (λ: 0.8), (μ: 0.3), (δ:0.7)>}

C_{HN} = { < x; (λ: 0.5), (μ: 0.3), (δ: 0.5)>, < y; (λ: 0.4), (μ: 0.5), (δ: 0.9)>} and

Let Y = {p,q} and σ = {I_{HN}, O_{HN}, U_{HN}, V_{HN}, W_{HN}}, then (Y_{HN},σ) be a Heptagonal neutrosophic topological space with

U_{HN} = { < p; (λ:0.9), (μ: 0.2), (δ: 0.4)>, < q; (λ:0.7), (μ:0.3), (δ:0.7)>}

V_{HN} = { < p; (λ: 0.6), (μ:0.4), (δ:0.6)>, < q; (λ: 0.3), (μ: 0.6), (δ:0.8)>}

W_{HN} = { < p; (λ: 0.8), (μ: 0.3), (δ: 0.5)>, < q; (λ: 0.6), (μ: 0.5), (δ: 0.7)>}

Define a map *f* : X_{HN}→ Y_{HN} by *f*(x) = p , *f*(y) = q, then *f* is HN pre-continuous map.

Definition 3.6. Let (X_{HN},τ) and (Y_{HN} ,σ)are two non-empty Heptagonal neutrosophic topological spaces.A map *f* : X_{HN}→ Y_{HN} is called a

(i) HN irresolute map, if f^{-1}(A_{HN}) is HN open in (X_{HN},τ), for each HN open set A_{HN} in (Y_{HN} ,σ).
(ii) HN pre-irresolute map, if f^{-1}(A_{HN}) is HN-pre open in (X_{HN},τ), for each HN–pre open set A_{HN} in (Y_{HN} ,σ).
(iii) HN semi-irresolute map, if f^{-1}(A_{HN}) is HN-semi open in (X_{HN},τ), for each HN–semi open set A_{HN} in (Y_{HN} ,σ).
(iv) HN α-irresolute map, if f^{-1}(A_{HN}) is HN-α open in (X_{HN},τ), for each HN–α open set A_{HN} in (Y_{HN} ,σ).

Example 3.7: Let X = {x,y,z} and τ = {I_{HN}, O_{HN}, A_{HN}}, then (X_{HN},τ) be a Heptagonal neutrosophic topological space with

A_{HN} = { < x; (λ:0.5), (μ: 0.5), (δ: 0.5)>, < y; (λ:0.5), (μ:0.5), (δ:0.5)>, < z; (λ:0.5), (μ:0.5), (δ:0.5)>}

Let Y = {p,q,r} and σ = {I_{HN}, O_{HN}, U_{HN}, V_{HN}, W_{HN}}, then (Y_{HN},σ) be a Heptagonal neutrosophic topological space with

U_{HN} = { < p; (λ:0.3), (μ: 0.3), (δ: 0.3)>, < q; (λ:0.3), (μ:0.3), (δ:0.3)>, < r; (λ:0.3), (μ:0.3), (δ:0.3)>}

Define a map *f* : X_{HN}→ Y_{HN} by *f*(x) = p, *f*(y) = q and *f*(z) = r then *f* is HN pre-irresolute map.

Definition 3.8. Let (X_{HN},τ) and (Y_{HN} ,σ) are two non-empty Heptagonal neutrosophic topological spaces.A map *f* : X_{HN}→ Y_{HN} is called a

(i) HN open map, if *f* (A_{HN}) is HN open in (Y_{HN},σ), for each HN open set A_{HN} in (X_{HN},τ).
(ii) HN pre-open map, if *f* (A_{HN}) is HN-pre open in (Y_{HN},σ), for each HN–pre open set A_{HN} in (X_{HN},τ).
(iii) HN semi-open map, if *f* (A_{HN}) is HN-semi open in (Y_{HN},σ), for each HN–semi open set A_{HN} in (X_{HN},τ).
(iv) HN α-open map, if *f* (A_{HN}) is HN-α open in (Y_{HN},σ), for each HN–α open set A_{HN} in (X_{HN},τ).

Example 3.9: Let X = {x,y} and τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}}, then (X_{HN},τ) be a Heptagonal neutrosophic topological space with

A_{HN} = { < x; (λ:0.8), (μ: 0.3), (δ: 0.5)>, < y; (λ:0.6), (μ:0.5), (δ:0.9)>}
B_{HN} = { < x; (λ: 0.9), (μ:0.2), (δ:0.4)>, < y; (λ: 0.8), (μ: 0.3), (δ:0.7)>}
C_{HN} = { < x; (λ: 0.5), (μ: 0.3), (δ: 0.5)>, < y; (λ: 0.4), (μ: 0.5), (δ: 0.9)>} and
Let Y = {p,q} and σ = {I_{HN}, O_{HN}, U_{HN}, V_{HN}, W_{HN}}, then (Y_{HN},σ) be a Heptagonal neutrosophic topological space with
U_{HN} = { < p; (λ:0.9), (μ: 0.2), (δ: 0.4)>, < q; (λ:0.7), (μ:0.3), (δ:0.7)>}
V_{HN} = { < p; (λ: 0.6), (μ:0.4), (δ:0.6)>, < q; (λ: 0.3), (μ: 0.6), (δ:0.8)>}
W_{HN} = { < p; (λ: 0.8), (μ: 0.3), (δ: 0.5)>, < q; (λ: 0.6), (μ: 0.5), (δ: 0.7)>}
Define a map *f* : X_{HN}→ Y_{HN} by *f*(x) = p, *f*(y) = q and *f*(z) = r, then *f* is a HN pre-open map.

Definition 3.10. Let (X_{HN},τ) and (Y_{HN} ,σ) are two non-empty Heptagonal neutrosophic topological spaces.A map *f* : X_{HN}→ Y_{HN} is called a

(i) HN quotient map, if *f* is both HN-continuous and $\boldsymbol{f^{-1}}$(A_{HN}) is HN open in (X_{HN},τ), implies A_{HN} is HN open set in (Y_{HN} ,σ).
(ii) HN pre-quotient map, if *f* is both HN-pre continuous and $\boldsymbol{f^{-1}}$(A_{HN}) is HN open in (X_{HN},τ), implies A_{HN} is HN-pre open set in (Y_{HN} ,σ).
(iii) HN semi-quotient map, if *f* is both HN-semi continuous and $\boldsymbol{f^{-1}}$(A_{HN}) is HN open in (X_{HN},τ), implies A_{HN} is HN-semi open set in (Y_{HN} ,σ).
(iv) HN α-quotient map, if *f* is both HN-α continuous and $\boldsymbol{f^{-1}}$(A_{HN}) is HN open in (X_{HN},τ), implies A_{HN} is HN-α open set in (Y_{HN} ,σ).

Example 3.11: Let X = {p,q} and τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}}, then (X_{HN},τ) be a Heptagonal neutrosophic topological space with
A_{HN} = { < p; (λ:0.8), (μ: 0.3), (δ: 0.5)>, < q; (λ:0.6), (μ:0.5), (δ:0.9)>}
B_{HN} = { < p; (λ: 0.9), (μ:0.2), (δ:0.4)>, < q; (λ: 0.8), (μ: 0.3), (δ:0.7)>}
C_{HN} = { < p; (λ: 0.5), (μ: 0.3), (δ: 0.5)>, < q; (λ: 0.4), (μ: 0.5), (δ: 0.9)>} and
Let Y = {r,s} and σ = {I_{HN}, O_{HN}, U_{HN}, V_{HN}, W_{HN}}, then (Y_{HN},σ) be a Heptagonal neutrosophic topological space with
U_{HN} = { < r; (λ:0.9), (μ: 0.2), (δ: 0.4)>, < s; (λ:0.7), (μ:0.3), (δ:0.7)>}

V_{HN} = { < r; (λ: 0.6), (μ:0.4), (δ:0.6)>, < s; (λ: 0.3), (μ: 0.6), (δ:0.8)>}
W_{HN} = { < r; (λ: 0.8), (μ: 0.3), (δ: 0.5)>, < s; (λ: 0.6), (μ: 0.5), (δ: 0.7)>}
Define a map *f* : X_{HN}→ Y_{HN} by *f*(p) = r, *f*(q) = s.
Here *f* is a HN pre-quotient map.

Definition 3.12. Let (X_{HN},τ) and (Y_{HN},σ) are two non-empty heptagonal neutrosophic topological spaces.A map f: X_{HN}→ Y_{HN} is called a

(i) HN-strongly quotient map, provided A_{HN} is HN-open set in (Y_{HN} ,σ) if and only if f^{-1}(A_{HN}) is HN open in (X_{HN},τ)
(ii) HN-strongly pre-quotient map, provided A_{HN} is HN-open set in (Y_{HN} ,σ) if and only if f^{-1}(A_{HN}) is HN-pre open in (X_{HN},τ).
(iii) HN-strongly semi-quotient map, provided A_{HN} is HN-open set in (Y_{HN} ,σ) if and only if f^{-1}(A_{HN}) is HN-semi open in (X_{HN},τ).
(iv) HN-strongly α-quotient map, provided A_{HN} is HN-open set in (Y_{HN} ,σ) if and only if f^{-1}(A_{HN}) is HN-α open in (X_{HN},τ).

Example 3.13: Let X = {p,q,r} and τ = {I_{HN}, O_{HN}, A_{HN}}, then (X_{HN},τ) be a Heptagonal neutrosophic topological space with
A_{HN} = { < p; (λ:0.5), (μ: 0.6), (δ: 0.4)>, < q; (λ:0.4), (μ:0.5), (δ:0.2)>, < r; (λ:0.7), (μ:0.6), (δ:0.9)>}
Let Y = {a,b,c} and σ = {I_{HN}, O_{HN}, U_{HN}}, then (Y_{HN},σ) be a Heptagonal neutrosophic topological space with
U_{HN} = { < a; (λ:0.5), (μ: 0.5), (δ: 0.5)>, < b; (λ:0.5), (μ:0.5), (δ:0.5)>, < c; (λ:0.5), (μ:0.5), (δ:0.5)>}
Define a map *f* : X_{HN}→ Y_{HN} by *f*(p) = a, *f*(q) = b, *f*(r) = c.
Here *f* is a HN strongly pre-quotient map.

Definition 3.14. Let (X_{HN},τ) and (Y_{HN} ,σ) are two non-empty Heptagonal neutrosophic topological spaces.A map *f* : X_{HN}→ Y_{HN} is called a

(i) HN*-quotient map, if *f* is both HN-irresolute and f^{-1}(A_{HN}) is HN open in (X_{HN},τ), implies A_{HN} is HN open set in (Y_{HN} ,σ).
(ii) HN-semi*-quotient map, if *f* is both HN-semi irresolute and f^{-1}(A_{HN}) is HN-semi open in (X_{HN},τ), implies A_{HN} is HN open set in (Y_{HN} ,σ).
(iii) HN-pre*-quotient map, if *f* is both HN-pre irresolute and f^{-1}(A_{HN}) is HN-pre open in (X_{HN},τ), implies A_{HN} is HN open set in (Y_{HN} ,σ).
(iv) HN-α*-quotient map, if *f* is both HN-$\boldsymbol{\alpha}$ irresolute and f^{-1}(A_{HN}) is HN-α open in (X_{HN},τ), implies A_{HN} is HN open set in (Y_{HN} ,σ).

Example 3.15: Let X = {p,q} and τ = {I_{HN}, O_{HN}, A_{HN}, B_{HN}, C_{HN}}, then (X_{HN},τ) be a Heptagonal neutrosophic topological space with
A_{HN} = { < p; (λ:0.8), (μ: 0.3), (δ: 0.5)>, < q; (λ:0.6), (μ:0.5), (δ:0.9)>}
B_{HN} = { < p; (λ: 0.9), (μ:0.2), (δ:0.4)>, < q; (λ: 0.8), (μ: 0.3), (δ:0.7)>}
C_{HN} = { < p; (λ: 0.5), (μ: 0.3), (δ: 0.5)>, < q; (λ: 0.4), (μ: 0.5), (δ: 0.9)>} and
Let Y = {r,s} and σ = {I_{HN}, O_{HN}, U_{HN}, V_{HN}, W_{HN}}, then (Y_{HN},σ) be a Heptagonal neutrosophic topological space with
U_{HN} = { < r; (λ:0.9), (μ: 0.2), (δ: 0.4)>, < s; (λ:0.7), (μ:0.3), (δ:0.7)>}
V_{HN} = { < r; (λ: 0.6), (μ:0.4), (δ:0.6)>, < s; (λ: 0.3), (μ: 0.6), (δ:0.8)>}
W_{HN} = { < r; (λ: 0.8), (μ: 0.3), (δ: 0.5)>, < s; (λ: 0.6), (μ: 0.5), (δ: 0.7)>}
Define a map *f* : X_{HN}→ Y_{HN} by *f*(p) = r, *f*(q) = s. Here *f* is a HN pre*-quotient map.

4. Characterizations of Heptagonal Neutrosophic Quotient Mappings

In this section, we characterize the Heptagonal Neutrosophic Quotient Mappings and derive some of the results and the composition of two maps.

Theorem 4.1: Let (XHN,τ) and (YHN ,σ) are two non-empty Heptagonal neutrosophic topological spaces. If f : XHN→ YHN is surjective, HN-continuous and HN-open map, then f is a HN-quotient map.

Proof:

We need only to prove that f^{-1}(AHN) is HN-open in XHN implies AHN is a HN–open set in YHN. Let f^{-1}(AHN) is open in XHN. Since f is HN–open map, then $f(f^{-1}$(AHN)) is a HN–open set in YHN. Hence AHN is a HN–open set in YHN, as f is surjective $f(f^{-1}$(AHN)) = AHN. Thus f is a HN–quotient map.

Theorem 4.2: Let (X_{HN},τ) and (Y_{HN} ,σ) are two non-empty Heptagonal neutrosophic topological spaces and f : X_{HN}→ Y_{HN} is a surjective map, then

(i) Every HN-quotient map is HN-pre quotient map.
(ii) Every HN-quotient map is HN-semi quotient map.
(iii) Every HN-quotient map is HN-α quotient map.

Proof.

Let f : X_{HN}→ Y_{HN} be a HN-quotient map and since f is a HN-continuous function, we have for every HN open set A_{HN} in (Y_{HN} ,σ), f^{-1}(A_{HN}) is HN-open in (X_{HN},τ) and thus, f^{-1}(A_{HN}) is HN-pre open in (X_{HN},τ), because "Every HN open set is HN-pre open set". This implies f is a HN-pre continuous function. Now, Let f^{-1}(A_{HN}) is HN-open in (X_{HN},τ), Since f is a HN-quotient map, A_{HN} is HN-open set in (Y_{HN} ,σ) and therefore A_{HN} is a HN-pre open set in (Y_{HN} ,σ). Hence f is a HN-pre quotient map.

Let f : X_{HN}→ Y_{HN} be a HN-quotient map and since f is a HN-continuous function, we have for every HN open set A_{HN} in (Y_{HN} ,σ), f^{-1}(A_{HN}) is HN-open in (X_{HN},τ) and thus, f^{-1}(A_{HN}) is HN-semi open in (X_{HN},τ), because "Every HN open set is HN-semi open set". This implies f is a HN-semi continuous function. Now, Let f^{-1}(A_{HN}) is HN-open in (X_{HN},τ), Since f is a HN-quotient map, A_{HN} is HN-open set in (Y_{HN} ,σ) and therefore A_{HN} is a HN-semi open set in (Y_{HN} ,σ). Hence f is a HN-semi quotient map.

Let f : X_{HN}→ Y_{HN} be a HN-quotient map and since f is a HN-continuous function, we have for every HN open set A_{HN} in (Y_{HN} ,σ), f^{-1}(A_{HN}) is HN-open in (X_{HN},τ) and thus, f^{-1}(A_{HN}) is HN-α open in (X_{HN},τ), because "Every HN open set is HN-α open set". This implies f is a HN- α continuous function. Now, Let f^{-1}(A_{HN}) is HN-open in (X_{HN},τ), Since f is a HN-quotient map, A_{HN} is HN-open set in (Y_{HN} ,σ) and therefore A_{HN} is a HN- α open set in (Y_{HN} ,σ). Hence f is a HN- α quotient map.

Theorem 4.3: Let (X_{HN},τ) and (Y_{HN} ,σ) are two non-empty Heptagonal neutrosophic topological spaces. A surjective map f: X_{HN}→ Y_{HN} is a HN-α quotient map if and only if f is both HN-semi quotient map and HN-pre quotient map.

Proof.

Let f: X_{HN}→ Y_{HN} be a HN-α quotient map. Since f is a HN-α continuous, we have f^{-1}(A_{HN}) is HN-α open in (X_{HN},τ), for every HN-open set A_{HN} in (Y_{HN} ,σ). We know that, "Every HN-α open set is both HN-semi and HN-pre open set". Then f^{-1}(A_{HN}) is both HN-semi open and HN-pre open in (X_{HN},τ) and therefore f is both HN-semi continuous as well as HN-pre continuous function.

Now, for every HN-α open set A_{HN} in (Y_{HN} ,σ), we have $f^{-1}(A_{HN})$ is HN open in (X_{HN},τ) and since "Every HN-α open set is both HN-semi and HN-pre open set". This implies, for every HN-semi open set A_{HN} in (Y_{HN} ,σ), we have $f^{-1}(A_{HN})$ is HN open in (X_{HN},τ) and for every HN-pre open set A_{HN} in (Y_{HN} ,σ), we have $f^{-1}(A_{HN})$ is HN open in (X_{HN},τ). Hence it proves *f* is both HN-semi quotient map and HN-pre quotient map.

Conversely, let *f* be a HN-semi quotient map and HN-pre quotient map. Since *f* is both HN-semi continuous and HN-pre continuous function, $f^{-1}(A_{HN})$ is both HN-semi open and HN-pre open in (X_{HN},τ), for every HN-open set A_{HN} in (Y_{HN} ,σ). Therefore, $f^{-1}(A_{HN})$ is also a HN-α open set. Thus, *f* is a HN-α continuous function. Now, Since *f* is a neutrosophic semi-quotient map and a pre-quotient map, for every $f^{-1}(A_{HN})$ is HN-open in (X_{HN},τ) implies A_{HN} is a HN-semi open and HN-pre open respectively in (Y_{HN} ,σ), so that A_{HN} is a HN-α open in (Y_{HN} ,σ). Hence, *f* is a HN-α quotient map.

Theorem 4.4: Let (X_{HN},τ), (Y_{HN},σ) and (Z_{HN},ω) are three non-empty heptagonal neutrosophic topological spaces. A surjective map ϕ: $X_{HN}\rightarrow Y_{HN}$ is an onto HN-open and HN-pre irresolute map and ψ : $Y_{HN}\rightarrow Z_{HN}$ be a HN-pre quotient map, then $\psi_o\phi$: $X_{HN}\rightarrow Z_{HN}$ is a HN-pre quotient map.

Proof.

To Prove: $\psi_o\phi$ is HN-precontinuous

Let A_{HN} be any HN-open set in (Z_{HN},ω) and since ψ be a HN-pre quotient map, then $\psi^{-1}(A_{HN})$ is a HN-pre open in (Y_{HN},σ).Also since ϕ is HN-pre irresolute, we have $\phi^{-1}(\psi^{-1}(A_{HN}))$ is a HN-pre open set in (X_{HN},τ) which implies $(\psi_o\phi)^{-1}(A_{HN})$ is a HN-pre open set in (X_{HN},τ). Hence $\psi_o\phi$ is a HN-precontinuous function.

To Prove: $(\psi_o\phi)^{-1}(B_{HN})$ is a HN-open set in (X_{HN},τ) implies B_{HN} is HN-pre open set in (Z_{HN},ω).

Let $\phi^{-1}(\psi^{-1}(B_{HN}))$ is a HN-pre open set in (X_{HN},τ) and since ϕ is an onto and HN-open map, we have $\phi^{-1}(\psi^{-1}(B_{HN}))$ is HN-open in (Y_{HN},σ). Since ψ be a HN-pre quotient map, we have $\psi^{-1}(B_{HN})$ is HN-open in (Y_{HN},σ). Thus B_{HN} is a HN-pre open set in (Z_{HN},ω). Hence $\psi_o\phi$ is a HN-pre quotient map.

Corollary 4.5: Let (X_{HN},τ), (Y_{HN},σ) and (Z_{HN},ω) are three non-empty heptagonal neutrosophic topological spaces. A surjective map ϕ: $X_{HN}\rightarrow Y_{HN}$ is an onto HN-open and HN-semi irresolute map and ψ : $Y_{HN}\rightarrow Z_{HN}$ be a HN-semi quotient map, then $\psi_o\phi$: $X_{HN}\rightarrow Z_{HN}$ is a HN-semi quotient map.

Corollary 4.6: Let (X_{HN},τ), (Y_{HN},σ) and (Z_{HN},ω) are three non-empty heptagonal neutrosophic topological spaces. A surjective map ϕ: $X_{HN}\rightarrow Y_{HN}$ is an onto HN-open and HN-α irresolute map and ψ : $Y_{HN}\rightarrow Z_{HN}$be a HN-α quotient map, then $\psi_o\phi$: $X_{HN}\rightarrow Z_{HN}$ is a HN-α quotient map.

Theorem 4.7: Let (X_{HN},τ) and (Y_{HN} ,σ) are two non-empty Heptagonal neutrosophic topological spaces. A surjective map f: $X_{HN}\rightarrow Y_{HN}$ is a HN-strongly pre quotient map and HN-strongly semi quotient map, then f is a HN-strongly α quotient map.

Proof.

Let A_{HN} be a HN-open set in (Y_{HN},σ) and Since f is HN-strongly semi-quotient and HN-strongly pre-quotient, then $f^{-1}(A_{HN})$is HN semi-open as well as HN pre-open. Hence, $f^{-1}(A_{HN})$ is HN-α open in (X_{HN},τ).

Conversely, Let $f^{-1}(A_{HN})$be a HN-αopen set in (X_{HN},τ). Since f is HN strongly semiquotient, for any $f^{-1}(A_{HN})$ is HN-semiopen in (X_{HN},τ). then A_{HN} is HN-open in (Y_{HN},σ). Therefore, it follows that A_{HN} is HN open in (Y_{HN} ,σ) if and only if $f^{-1}(A_{HN})$ is HN-αopen in (X_{HN},τ). So f is a HN-strongly α-quotient map.

Theroem 4.8: Every HN* – quotient map is HN-strongly quotient map.

Proof:

Let f: XHN→ YHN is a HN* – quotient map. To prove f is HN-strongly quotient map.

Let AHN be any HN-open set in (YHN,σ), since f is a HN* – quotient map, f is HN-irresolute and then f^{-1}(AHN) is HN–open in (XHN,τ). This means that AHN is open in (YHN,σ) implies f^{-1}(AHN) is HN–open in (XHN,τ).

Conversely, if f^{-1}(AHN) is HN-open in (XHN,τ) and since f is HN*–quotient map, AHN is an open set in (YHN,σ). This means that f^{-1}(AHN) is HN–open in (XHN,τ) implies AHN is open in (YHN,σ). Hence f is a HN-strongly quotient map.

Corollary 4.9:

(i) Every HN-semi* quotient map is HN-strongly semi quotient map.
(ii) Every HN-pre* quotient map is HN-strongly pre quotient map.
(iii) Every HN-α* quotient map is HN-strongly α quotient map.

Theorem 4.10: The composition of two HN-semi* quotient maps are again HN-semi* quotient.

Proof:

Let (X_{HN},τ), (Y_{HN},σ) and (Z_{HN},ω) are three non-empty heptagonal neutrosophic topological spaces. A surjective map p: $X_{HN}\rightarrow Y_{HN}$ and q : $Y_{HN}\rightarrow Z_{HN}$ be two HN-semi* quotient maps, then $q \circ p$: $X_{HN}\rightarrow Z_{HN}$ is also a HN- semi* quotient map.

First to prove: $q \circ p$: XHN → ZHN is a HN- semi irresolute map.

Let BHN be a HN–semi open set in (ZHN,ω). Since q is HN-semi*quotient, q^{-1}(BHN) is a HN–semi open set in (YHN,σ). Since p is HN-semi*quotient, $p^{-1}(q^{-1}$(BHN)) is HN–semi open in XHN. That is $(q \circ p)^{-1}$ (BHN) is HN–semi open in XHN. Hence $q \circ p$ is HN-semi irresolute.

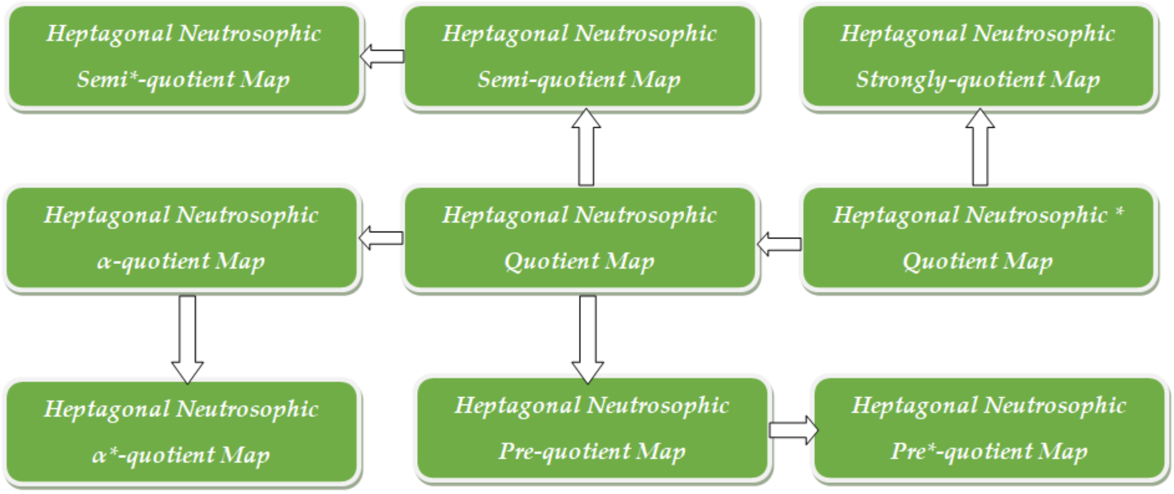
To Prove: $(q \circ p)^{-1}$(BHN) is HN–semi open in XHN implies BHN is open in ZHN.

Let $(q \circ p)^{-1}$ (BHN) is HN-semi open in XHN. That is, $p^{-1}(q^{-1}$(BHN)) is HN-semi open in XHN. Since p is HN-semi*quotient, q^{-1}(BHN) is open in YHN, and hence q^{-1}(BHN) is HN-semi open in YHN. Since q is HN-semi* quotient, BHN is open in ZHN. This implies that $(q \circ p)^{-1}$ (BHN) is HN-semi open in XHN implies BHN is open in ZHN. Hence $q \circ p$ is HN-semi* quotient map.

Corollary 4.11:

(i) The composition of two HN-pre* quotient maps are again HN-pre* quotient.
(ii) The composition of two HN-α* quotient maps are again HN-α* quotient.

Remark 4.12: A brief illustration of this article is as follows:



Conclusion

In this article, we have introduced and studied the concept of Heptagonal Neutrosophic quotient mappings and its characterization. Further, it can be extended in the field of homeomorphism, compactness and connectness and the same can be studied further.

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The Role of Lacunary Statistical Convergence for Double sequences in Neutrosophic Normed Spaces

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Abstract: This paper introduces and explores the concept of lacunary statistical convergence of double sequence within the framework neutrosophic normed spaces. Neutrosophic normed spaces extend classical normed spaces by incorporating neutrosophic numbers, which account for the inherent uncertainty, indeterminacy, and vagueness present in real - world data. The study begins by defining lacunary statistical convergence for double sequences in this extended context and proceeds to establish fundamental theorems and properties related to this new notion. In addition, we present a new idea in this context: statistical completeness. We demonstrate that, while neutrosophic normed space is statistically complete, it is not complete.

Keywords: Neutrosophic Normed Spaces; Lacunary Statistical Convergence and Cauchyness; Statistical Completeness.

1. Introduction

Fuzzy theory has been a hot topic of study in a number of scientific domains in the last few years. Many studies have been published on this theory since Zadeh originally put it forth in 1965. Saadati and Park introduced the idea of intuitionistic fuzzy normed space initially. Smarandache introduced the concept of neutrosophic sets as an extension of the intuitionistic fuzzy set. The requirement can be met when the component sum is equal to one by using neutronosophic set operators. While indeterminacy is treated by neutrosophic operators on the same plane as truth-membership and falsehood-nonmembership, intuitionistic fuzzy operators disregard indeterminacy and may produce different results.

Using the idea of density of positive natural numbers, Fast and Steinhaus separately created statistical convergence in 1951. Mursaleen and Edely have defined and studied double sequence statistical convergence. Karakus et al.'s recent study examined statistical convergence in intuitionistic fuzzy normed space. Rough statistical convergence and statistical $\triangle^m$ convergence were recently established in neutrosophic normed spaces by Jeyaraman and Jenifer.

Lacunary statistical convergence was first proposed by Fridy and Orhan. The lacunary statistical Cauchy and convergence for double sequences in neutrosophic normed space will be examined in this paper. The results presented in this paper contribute to the growing field of neutrosophic mathematics and provide a deeper understanding of convergence behavior in spaces characterized by uncertainty and indeterminacy.

2. Preliminaries

Some of the fundamental notions and definitions that are needed in the following sections are presented in this section.

Let $\mathfrak{J}$ represent a subset of the natural number set $\mathbb{N}$. Next, we define the asymptotic density of $\mathfrak{J}$, represented by $\delta(\mathfrak{J})$, as follows: $\delta(\mathfrak{J}) = \lim_{n} \frac{1}{n} |\{k \leq n : k \in \mathfrak{J}\}|$. The cardinality of the contained set is indicated by the vertical bars. The sequence of numbers $\mathfrak{x} = (\mathfrak{x}_k)$ is statistically convergent to ℓ if, for any $\epsilon > 0$, the set $\mathfrak{J}(\epsilon) = \{k \leq n : |\mathfrak{x}_k - \ell| > \epsilon\}$ has asymptotic density zero, that is, $\lim_{n} \frac{1}{n} |\{k \leq n : |\mathfrak{x}_k - \ell| \geq \epsilon\}| = 0$.In this instance, $\mathfrak{st} - \lim \mathfrak{x} = \ell$ is written. It should be noted that while the converse may not always be true, any convergent sequence approaches the same limit statistically.

If, for any $\epsilon > 0$, the set $\{(j,k), j \leq m \text{ and } k \leq n : |\mathfrak{x}_{jk} - \ell| \geq \epsilon\}$ has double natural density zero, then the real double sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ is statistically convergent to the number ℓ. We indicate the set of all statistically convergent double sequences by $\aleph_2$ in this instance, and the set of all limited statistically convergent double sequences by$\aleph_2^{\infty}$. In this example, we write $\mathfrak{st}_2 - \lim \mathfrak{x} = \ell$.

Definition 2.1 Let $(\mathfrak{X}, \tau, \varphi, \omega *, \diamond, \circledast)$ be an $\mathfrak{NNS}$. Here, $\mathfrak{X}$ is a vector space, $*$ is a continuous t-norm, $\diamond$ and $\circledast$ are continuous t-conorm, and τ, φ and ω are fuzzy sets on $\mathfrak{X} \times (0, \infty)$ satisfy the following conditions. For every $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}$ and $\zeta, \lambda > 0$,

(i) $\tau(\mathfrak{x}, \lambda) + \nu(\mathfrak{x}, \lambda) + \omega(\mathfrak{x}, \lambda) \leq 1$;
(ii) $\tau(\mathfrak{x}, \lambda) > 0$;
(iii) $\tau(\mathfrak{x}, \lambda) = 1$ iff $\mathfrak{x} = 0$;
(iv) $\tau(\alpha\mathfrak{x}, \lambda) = \tau\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(v) $\tau(\mathfrak{x}, \lambda) * \tau(\mathfrak{y}, \zeta) \leq \tau(\mathfrak{x} + \mathfrak{y},)$;
(vi) $\tau(\mathfrak{x}, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(vii) $\lim_{\lambda \to \infty} \tau(\mathfrak{x}, \lambda) = 1$ and $\lim_{\lambda \to 0} \tau(\mathfrak{x}, \lambda) = 0$;
(viii) $\nu(\mathfrak{x}, \lambda) < 1$;
(ix) $\nu(\mathfrak{x}, \lambda) = 0$ iff $\mathfrak{x} = 0$;
(x) $\nu(\alpha\mathfrak{x}, \lambda) = \nu\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(xi) $\nu(\mathfrak{x}, \lambda) \diamond \nu(\mathfrak{y}, \zeta) \geq \nu(\mathfrak{x} + \mathfrak{y}, \lambda + \zeta)$;
(xii) $\nu(\mathfrak{x}, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(xiii) $\lim_{\lambda \to \infty} \nu(\mathfrak{x}, \lambda) = 0$ and $\lim_{\lambda \to 0} \nu(\mathfrak{x}, \lambda) = 1$;
(xiv) $\omega(\mathfrak{x}, \lambda) < 1$;
(xv) $\omega(\mathfrak{x}, \lambda) = 0$ iff $\mathfrak{x} = 0$;
(xvi) $\omega(\alpha\mathfrak{x}, \lambda) = \omega\left(\mathfrak{x}, \frac{\lambda}{|\alpha|}\right)$ for each $\alpha \neq 0$;
(xvii) $\omega(\mathfrak{x}, \lambda) \circledast \omega(\mathfrak{y}, \zeta) \geq \omega(\mathfrak{x} + \mathfrak{y}, \lambda + \zeta)$;
(xviii) $\omega(\mathfrak{x}, \cdot) : (0, \infty) \to [0,1]$ is continuous;
(ixx) $\lim_{\lambda \to \infty} \omega(\mathfrak{x}, \lambda) = 0$ and $\lim_{\lambda \to 0} \omega(\mathfrak{x}, \lambda) = 1$.
In this case, (μ, ν, ω) is called an $\mathfrak{NNS}$.

Definition 2.2. Let a $\mathfrak{NNS}$ be $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$. According to the $\mathfrak{NN}$ (μ, ν, ω), $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be convergent to $\ell \in \mathfrak{X}$ if, $\forall\ \epsilon > 0$ and $\lambda > 0$, $\exists\ k_0 \in \mathbb{N} : \mu(\mathfrak{x}_k - \ell, \lambda) > 1 - \epsilon$, $\nu(\mathfrak{x}_k - \ell, \lambda) < \epsilon$ and $\omega(\mathfrak{x}_k - \ell, \lambda) < \epsilon\ \forall\ k \geq k_0$. In this instance, we write $\mathfrak{x}_k \xrightarrow{(\mu,\nu,\omega)} \ell$ as $k \to \infty$ or $(\mu, \nu, \omega) - \lim \mathfrak{x} = \ell$.

Definition 2.3. Let a $\mathfrak{NNS}$ be $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$. Then, for every $\epsilon > 0$ and $\lambda > 0$, $\exists\, k_0 \in \mathbb{N}$ such that $\mu(\mathfrak{x}_k - \mathfrak{x}_\ell, \lambda) > 1 - \epsilon$, $\nu(\mathfrak{x}_k - \mathfrak{x}_\ell, \lambda) < \epsilon$ and $\omega(\mathfrak{x}_k - \mathfrak{x}_\ell, \lambda) < \epsilon$ $\forall$ $k, \ell \geq k_0$. This indicates that $\mathfrak{x} = (\mathfrak{x}_k)$ is a Cauchy sequence with respect to the $\mathfrak{NN}$ (μ, ν, ω).

Remark 2.4 [13]. The real normed linear space $(\mathfrak{X}, \|\cdot\|)$ has the following properties: $\mu(\mathfrak{x}, \lambda) := \frac{\lambda}{\lambda + \|\mathfrak{x}\|}$, $\nu(\mathfrak{x}, \lambda) := \frac{\|\mathfrak{x}\|}{\lambda + \|\mathfrak{x}\|}$ and $\omega(\mathfrak{x}, \lambda) := \frac{\|\mathfrak{x}\|}{\lambda}$for all $\mathfrak{x} \in \mathfrak{X}$ and $\lambda > 0$.Subsequently, $\mathfrak{x}_n \xrightarrow{\|\cdot\|} \mathfrak{x}$ iff $\mathfrak{x}_n \xrightarrow{(\mu,\nu,\omega)} \mathfrak{x}$.

3. Lacunary Statistical Convergence ($\mathfrak{L}\mathcal{S}t\mathfrak{C}$) of double sequences in $\mathfrak{NNS}$

The idea of $\mathfrak{L}\mathcal{S}t\mathfrak{C}$ sequences in $\mathfrak{NNS}$ is examined in this section. First, let's define what we mean by θ-density:

Definition 3.1 A $\mathfrak{L}\mathcal{S}$ is an ascending integer sequence $\theta = (\mathfrak{J}_\mathfrak{r})$ such that $\mathfrak{h}_\mathfrak{r} := \mathfrak{J}_\mathfrak{r} - \mathfrak{J}_{\mathfrak{r}-1} \to \infty$ as $\mathfrak{r} \to \infty$ and $\mathfrak{J}_0 = 0$ are considered.
In this study, the intervals identified by θ will be represented as $I_\mathfrak{r} := (\mathfrak{J}_{\mathfrak{r}-1}, \mathfrak{J}_\mathfrak{r}]$, and the $\mathfrak{J}_\mathfrak{r}/\mathfrak{J}_{\mathfrak{r}-1}$ ratio will be shortened to $\mathfrak{Q}_\mathfrak{r}$. Allow N to $\subseteq \mathbb{N}$. Assuming the limit exists, the θ -density of $\mathfrak{J}$ is given by the number $\delta_\theta(N) = \lim_{\mathfrak{r}} \frac{1}{\mathfrak{h}_\mathfrak{r}} |\{\mathfrak{J} \in I_\mathfrak{r} : \mathfrak{J} \in N\}|$.

Definition 3.2 Consider the θ. If, for each $\epsilon > 0$, the set $\mathfrak{J}(\epsilon)$ has θ-density zero, where $\mathfrak{J}(\epsilon) := \{k \in I_\mathfrak{r} : |\mathfrak{x}_k - \ell| \geq \epsilon\}$, then a sequence $\mathfrak{x} = (\mathfrak{x}_k)$ is said to be $\aleph_\theta$-convergent to the number ℓ. $\aleph_\theta - \lim \mathfrak{x} = \ell$ or $\mathfrak{x}_k \to \ell(\aleph_\theta)$ is written in this instance.

Now we define the $\aleph_\theta$-convergence of double sequences with respect to $\mathfrak{NNS}$.

Definition 3.3 Let θ be a $\mathfrak{L}\mathcal{S}$ and $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$. Then, $\forall \epsilon > 0$and $\lambda > 0$,

$$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) \leq 1 - \epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon\}\big) = 0$$

or equivalently

$$\delta_\theta\,(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) > 1 - \epsilon,\ \nu(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon\}) = 1.$$

Here, we write $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$ or $\mathfrak{x}_{jk} \xrightarrow{(\mu,\nu,\omega)} \ell(\aleph_\theta)$, where ℓ is referred to as $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x}$, and We signify the collection of all $\aleph_\theta$-convergent sequences with regard to the $\mathfrak{NN}(\mu, \nu, \omega)$ by $\aleph_\theta^{(\mu,\nu,\omega)}$.

Lemma 3.4 Consider a $\mathfrak{NNS}$ $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$. Let θ be a $\mathfrak{L}\mathcal{S}$. Then, $\forall\, \epsilon > 0$ and $\lambda > 0$, the statements that follow are comparable:

$\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$.

$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) \leq 1 - \epsilon\}\big) = \delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \ell, \lambda) \geq \epsilon\}\big) = 0$.

$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell, \lambda) > 1 - \epsilon, \nu(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \ell, \lambda) < \epsilon\}\big) = 1$.

$\delta_\theta\big(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - \ell, \lambda) > 1 - \epsilon\}\big) = \delta_\theta\big(\{k \in : \nu(x_{jk} - \ell, \lambda) < \epsilon\}\big) = \delta_\theta\big(\{k \in : \omega(x_{jk} - \ell, \lambda) < \epsilon\}\big) = 1$.

$\aleph_\theta - \lim \mu(\mathfrak{x}_{jk} - \ell, \lambda) = 1$, $\aleph_\theta - \lim \nu(\mathfrak{x}_{jk} - \ell, \lambda) = 0$ and $\aleph_\theta - \lim\ \omega(\mathfrak{x}_{jk} - \ell, \lambda) = 0$.

Theorem 3.5 Let θ be a $\mathfrak{L}\mathcal{S}$ and $(\mathfrak{X}, \mu, \nu, \omega, *, \diamond, \circledast)$ be a $\mathfrak{NNS}$. $\aleph_\theta^{(\mu,\nu,\omega)}$-limit is unique if $\mathfrak{x} = (\mathfrak{x}_{jk})$ is $\mathfrak{L}\mathcal{S}t\mathfrak{C}$ with regard to the $\mathfrak{NN}(\mu, \nu, \omega)$.

Proof. Assume that $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell_1$ and $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell_2$. Consider $\epsilon > 0$ and choose $> 0 : (1 - \mathfrak{d}) * (1 - \mathfrak{d}) > 1 - \epsilon$, $\mathfrak{d} \diamond \mathfrak{d} < \epsilon$ and $\mathfrak{d} \circledast \mathfrak{d} < \epsilon$. Next, define the following sets as follows for every $\lambda > 0$:

$\mathfrak{J}_{\mu,1}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell_1,\lambda) \leq 1 - \mathfrak{d}\}$,
$\mathfrak{J}_{\mu,2}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell_2,\lambda) \leq 1 - \mathfrak{d}\}$,
$\mathfrak{J}_{\nu,1}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\mathfrak{x}_{jk} - \ell_1,\lambda) \geq \mathfrak{d}\}$,
$\mathfrak{J}_{\nu,2}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(\mathfrak{x}_{jk} - \ell_2,\lambda) \geq \mathfrak{d}\}$.
$\mathfrak{J}_{\omega,1}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\mathfrak{x}_{jk} - \ell_1,\lambda) \geq \mathfrak{d}\}$,
$\mathfrak{J}_{\omega,2}(\mathfrak{d},\lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \omega(\mathfrak{x}_{jk} - \ell_2,\lambda) \geq \mathfrak{d}\}$

We have to use Lemma 3.1 since $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell_1$.

$\delta_\theta\left(\mathfrak{J}_{\mu,1}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{J}_{\nu,1}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{J}_{\omega,1}(\epsilon,\lambda)\right) = 0$ for all $\lambda > 0$.

Additionally, using $\aleph_{\mu,\nu,\omega}^\theta - \lim \mathfrak{x} = \ell_2$, we get

$\delta_\theta\left(\mathfrak{J}_{\mu,2}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{J}_{\nu,2}(\epsilon,\lambda)\right) = \delta_\theta\left(\mathfrak{J}_{\omega,2}(\epsilon,\lambda)\right) = 0$ for all $\lambda > 0$.

Let's now

$\mathfrak{J}_{\mu,\nu,\omega}(\epsilon,\lambda) = \left(\mathfrak{J}_{\mu,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\mu,2}(\epsilon,\lambda)\right) \cap \left(\mathfrak{J}_{\nu,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\nu,2}(\epsilon,\lambda)\right) \cap \left(\mathfrak{J}_{\omega,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\omega,2}(\epsilon,\lambda)\right)$.

Next, note that $\delta_\theta\left(\mathfrak{J}_{\mu,\nu,\omega}(\epsilon,\lambda)\right) = 0$ which suggests

$\delta_\theta\left(\mathbb{N} \setminus \mathfrak{J}_{\mu,\nu,\omega}(\epsilon,\lambda)\right) = 1$. If $k \in \mathbb{N} \setminus \mathfrak{J}_{\mu,\nu,\omega}(\epsilon,\lambda)$, hence, there are three scenarios that could occur.

(a) $k \in \mathbb{N} \setminus \left(\mathfrak{J}_{\mu,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\mu,2}(\epsilon,\lambda)\right)$ and

(b) $k \in N \setminus \left(\mathfrak{J}_{\nu,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\nu,2}(\epsilon,\lambda)\right)$.

(c) $k \in N \setminus \left(\mathfrak{J}_{\omega,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\omega,2}(\epsilon,\lambda)\right)$.

We first consider that $k \in N \setminus \left(\mathfrak{J}_{\mu,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\mu,2}(\epsilon,\lambda)\right)$. Then we have

$\mu(\ell_1 - \ell_2,\lambda) \geq \mu\left(\mathfrak{x}_k - \ell_1,\frac{\lambda}{2}\right) * \mu\left(\mathfrak{x}_k - \ell_2,\frac{\lambda}{2}\right) > (1 - \mathfrak{d}) * (1 - \mathfrak{d})$.

$\mu(\ell_1 - \ell_2,\lambda) > 1 - \epsilon$ since $(1 - \mathfrak{d}) * (1 - \mathfrak{d}) > 1 - \epsilon$.

Since $\epsilon > 0$ was random, we obtain $\mu(\ell_1 - \ell_2,\lambda) = 1$ for any $\lambda > 0$, which results in $\ell_1 = \ell_2$.

As an alternative, we can write

$\nu(\ell_1 - \ell_2,\lambda) \leq \nu\left(\mathfrak{x}_k - \ell_1,\frac{\lambda}{2}\right) \diamond \nu\left(\mathfrak{x}_k - \ell_2,\frac{\lambda}{2}\right) < \mathfrak{d} \diamond \mathfrak{d}$ if $k \in N \setminus \left(\mathfrak{J}_{\nu,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\nu,2}(\epsilon,\lambda)\right)$.

Using the knowledge that $\mathfrak{d} \diamond \mathfrak{d} < \epsilon$, we can now observe that $\nu(\ell_1 - \ell_2,\lambda) < \epsilon$.
Thus, for any $\lambda > 0$, $\nu(\ell_1 - \ell_2,\lambda) = 0$, suggesting that $\ell_1 = \ell_2$.

Also, if $k \in N \setminus \left(\mathfrak{J}_{\omega,1}(\epsilon,\lambda) \cup \mathfrak{J}_{\omega,2}(\epsilon,\lambda)\right)$, after which we could write

$\omega(\ell_1 - \ell_2,\lambda) \leq \omega\left(\mathfrak{x}_k - \ell_1,\frac{\lambda}{2}\right) \circledast \omega\left(\mathfrak{x}_k - \ell_2,\frac{\lambda}{2}\right) < s \circledast s$.

We can observe that $\omega(\ell_1 - \ell_2,\lambda) < \epsilon$ by using the information that $\mathfrak{d} \circledast \mathfrak{d} < \epsilon$.
Thus, for any $\lambda > 0$, $\omega(\ell_1 - \ell_2,\lambda) = 0$ implies $\ell_1 = \ell_2$.

Hence, we deduce that $\aleph_\theta^{(\mu,\nu,\omega)}$-limit is unique in each case.

Hereby, the theorem's proof is concluded.

Theorem 3.6 Let θ be any $\mathfrak{LS}$ and $(\mathfrak{X},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNS}$. If $(\mu,\nu,\omega) - \lim \mathfrak{x} = \ell$, then $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$.

Proof. Let $(\mu,\nu,\omega) - \lim \mathfrak{x} = \ell$. Then for every $\epsilon > 0$ and $\lambda > 0$, there is a number $k_0 \in \mathbb{N}$ such that $\mu(\mathfrak{x}_k - \ell,\lambda) > 1 - \epsilon$ and $\nu(\mathfrak{x}_k - \ell,\lambda) < \epsilon$ and $\omega(\mathfrak{x}_k - \ell,\lambda) < \epsilon$ for all $k \geq k_0$.

Hence the set $\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell,\lambda) \leq 1 - \epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \ell,\lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \ell,\lambda) \geq \epsilon\}$ possesses certain number of terms. Given that each finite subset of $\mathbb{N}$ has a density of zero,

$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \ell,\lambda) \leq 1 - \epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \ell,\lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \ell,\lambda) \geq \epsilon\}) = 0$,

that is, $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$. This concludes the theorem's proof.

Example 3.7 Let $(\mathfrak{X},\|\cdot\|)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$, $a \diamond b = \min\{a + b, 1\}$ and $a \circledast b = \min\{a + b, 1\}$ for all $a,b \in [0,1]$. For all $x \in \mathbb{R}$ and $\lambda > 0$, consider $\mu(\mathfrak{x},\lambda) := \frac{\lambda}{\lambda + \|\mathfrak{x}\|}$, $\nu(\mathfrak{x},\lambda) := \frac{\|\mathfrak{x}\|}{\lambda + \|\mathfrak{x}\|}$ and $\omega(\mathfrak{x},\lambda) := \frac{\|\mathfrak{x}\|}{\lambda}$. Then $(\mathfrak{X},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{NNS}$.

Now we define a sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ by

$$\mathfrak{x}_{jk} = \begin{cases} (j,k);\text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 0;\text{otherwise.} \end{cases}$$

Let for $\epsilon > 0,\ \lambda > 0$.

$\mathfrak{J}_\mathfrak{r}(\epsilon,\lambda) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \frac{\lambda}{\lambda+\|\mathfrak{x}_{jk}\|} \le 1-\epsilon \text{ or } \frac{\|\mathfrak{x}_{jk}\|}{\lambda+\|\mathfrak{x}_{jk}\|} \ge \epsilon, \frac{\|\mathfrak{x}_{jk}\|}{\lambda} \ge \epsilon \right\}$,
$= \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \|\mathfrak{x}_{jk}\| \ge \frac{\epsilon\lambda}{1-\epsilon} > 0 \right\}$,
$= \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \|\mathfrak{x}_{jk}\| = (j,k) \right\}$,
$= \left\{(j,k) \in \mathbb{N} \times \mathbb{N}: j_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le j \le j_\mathfrak{r}, k_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N}\right\}$,
and so, we get

$$\frac{1}{\mathfrak{h}_\mathfrak{r}}|\mathfrak{J}_\mathfrak{r}(\epsilon,\lambda)| \le \frac{1}{\mathfrak{h}_\mathfrak{r}}\left|\left\{(j,k) \in \mathbb{N} \times \mathbb{N}: j_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le j \le j_\mathfrak{r}, k_r - \left[\sqrt{\mathfrak{h}_\mathfrak{r}}\right] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N}\right\}\right| \le \frac{\sqrt{\mathfrak{h}_\mathfrak{r}}}{\mathfrak{h}_\mathfrak{r}},$$

which implies that $\lim_{\mathfrak{r}} \frac{1}{\mathfrak{h}_\mathfrak{r}}|\mathfrak{J}_\mathfrak{r}(\epsilon,\lambda)| = 0$. Hence
$\delta_\theta(\mathfrak{J}_\mathfrak{r}(\epsilon,\lambda)) = \lim \frac{\sqrt{\mathfrak{h}_\mathfrak{r}}}{\mathfrak{h}_\mathfrak{r}} = 0$ as $\mathfrak{r} \to \infty$ implies that $\mathfrak{x}_{jk} \to 0(\aleph_\theta)$.
On the other hand $\mathfrak{x}_{jk} \nrightarrow 0$, since

$$\mu(\mathfrak{x}_{jk},\lambda) = \frac{\lambda}{\lambda+\|\mathfrak{x}_{jk}\|} = \begin{cases} \frac{\lambda}{\lambda+\|jk\|};\text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 1;\text{otherwise.} \end{cases} \le 1, \text{ and}$$

$$\nu(\mathfrak{x}_{jk},\lambda) = \frac{\|\mathfrak{x}_{jk}\|}{\lambda+\|\mathfrak{x}_{jk}\|} = \begin{cases} \frac{\|jk\|}{\lambda+\|jk\|};\text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 0\quad ;\text{otherwise.} \end{cases} \ge 0, \text{ also}$$

$$\omega(\mathfrak{x}_{jk},\lambda) = \frac{\|\mathfrak{x}_{jk}\|}{\lambda+\|\mathfrak{x}_{jk}\|} = \begin{cases} \frac{\|jk\|}{\lambda+\|jk\|};\text{for } j_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le j \le j_\mathfrak{r}, k_r - [\mathfrak{h}_\mathfrak{r}] + 1 \le k \le k_\mathfrak{r}, \mathfrak{r} \in \mathbb{N} \\ 0\quad ;\text{otherwise.} \end{cases} \ge 0.$$

This completes the proof.

4. Lacunary statistically($\mathfrak{L}\mathcal{S}t$) Cauchy double sequences in $\mathfrak{N}\mathfrak{N}\mathfrak{S}$

This section introduces a new notion of statistical completeness and defines lacunary statistically Cauchy double sequences with regard to a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$.

Definition 4.1 Let θ be a $\mathfrak{L}\mathcal{S}$ and $(\mathfrak{X},\mu,\nu,\omega,*,\diamond,\circledast)$ be a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$. Then, $\forall\ \epsilon > 0$ and $\lambda > 0$, $\exists\ n = n(\epsilon)$ and $m = m(\epsilon)$ such that
$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N}: \mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda) \le 1-\epsilon \ or\ \nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda), \omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda) \ge \epsilon\}) = 0.$
This indicates that the sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ is $\mathfrak{L}\mathcal{S}t-$ Cauchy (or $\aleph_\theta$-Cauchy) with regard to the (μ,ν,ω).

Theorem 4.2 Consider a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ $(\mathfrak{X},\mu,\nu,\omega,*,\diamond,\circledast)$ with any $\mathfrak{L}\mathcal{S}$ θ. If a sequence $\mathfrak{x} = (\mathfrak{x}_{jk})$ is $\aleph_\theta$-Cauchy with regard to the (μ,ν,ω), then it is $\aleph_\theta$-convergent.

Proof. Let $\mathfrak{x} = (\mathfrak{x}_k)$ be $\aleph_\theta$-convergent to ℓ with respect to the $\mathfrak{N}\mathfrak{N}(\mu,\nu,\omega)$, i.e., $\aleph_\theta^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \ell$. Then
$\delta_\theta\left(\left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \mu\left(\mathfrak{x}_{jk} - \ell,\frac{\lambda}{2}\right) \le 1-\epsilon \ or\ \nu\left(\mathfrak{x}_{jk} - \ell,\frac{\lambda}{2}\right) \ge \epsilon, \omega\left(\mathfrak{x}_{jk} - \ell,\frac{\lambda}{2}\right) \ge \epsilon\right\}\right) = 0.$
Specifically, for $k = N$
$\delta_\theta\left(\left\{(j,k) \in \mathbb{N} \times \mathbb{N}: \mu\left(\mathfrak{x}_{mn} - \ell,\frac{\lambda}{2}\right) \le 1-\epsilon \ or\ \nu\left(\mathfrak{x}_{mn} - \ell,\frac{\lambda}{2}\right) \ge \epsilon, \omega\left(\mathfrak{x}_{mn} - \ell,\frac{\lambda}{2}\right) \ge \epsilon\right\}\right) = 0.$
Since
$\mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn},\lambda) = \mu\left(\mathfrak{x}_{jk} - \ell - \mathfrak{x}_{mn} + \ell,\frac{\lambda}{2}+\frac{\lambda}{2}\right) \ge \mu\left(\mathfrak{x}_{jk} - \ell,\frac{\lambda}{2}\right) * \mu\left(\mathfrak{x}_{mn} - \ell,\frac{\lambda}{2}\right)$
and since

$\nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq \nu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) \diamond \nu\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right)$, $\omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq \omega\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) \circledast \omega\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right)$,

we have
$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq 1 - \epsilon \ or \ \nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \geq \epsilon\}) = 0,$
that is, with regard to the $\mathfrak{NN}$ (μ, ν, ω), $\mathfrak{x}$ is $\aleph_\theta$-Cauchy.

In contrast, let $\mathfrak{x} = (\mathfrak{x}_{jk})$ be $\aleph_\theta$-Cauchy, but with respect to the $\mathfrak{S}$ (μ, ν, ω), it is not $\aleph_\theta$-convergent.

Consequently, N exists such that $\delta_\theta(\mathfrak{A}(\epsilon, \lambda)) = 0,$
(3)
$\delta_\theta(\mathfrak{B}(\epsilon, \lambda)) = 0, \quad i.e. \ \delta_\theta(\mathfrak{B}^C(\epsilon, \lambda)) = 1;$
(4)
where
$\mathfrak{A}(\epsilon, \lambda) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq 1 - \epsilon \text{ or } \nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \geq \epsilon, \omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \geq \epsilon\},$
$\mathfrak{B}(\epsilon, \lambda) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) > \frac{1-\epsilon}{2}, \ \nu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) < \frac{\epsilon}{2} \text{ and } \omega\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) < \frac{\epsilon}{2}\right\}.$
Since
$\mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \geq 2\mu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) > 1 - \epsilon,$
$\nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq 2\nu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) < \epsilon$ and
$\omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq 2\omega\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) < \epsilon$
if $\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) > \frac{1-\epsilon}{2}$, $\nu\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) < \frac{\epsilon}{2}$ and $\omega\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) < \frac{\epsilon}{2}$.
Therefore,
$\delta_\theta(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) > 1 - \epsilon, \ \nu(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) < \epsilon \text{ and } \omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) < \epsilon\}) = 0,$
since $\mathfrak{x}$ was $\aleph_\theta$-Cauchy with respect to $\mathfrak{NN}$ (μ, ν, ω), $\delta_\theta(\mathfrak{A}(\epsilon, \lambda)) = 1$, which defies (3).
Hence, with regard to (μ, ν, ω), $\mathfrak{x}$ must be $\aleph_\theta$-convergent.

Definition 4.3 If all of the Cauchy sequences in $(\mathfrak{X}, \tau, \varphi, \omega, *, \diamond, \circledast)$, then the $\mathfrak{NNS}$ $(\mathfrak{X}, \tau, \varphi, \omega, *, \diamond, \circledast)$ is considered complete.

Definition 4.4 If every $\aleph_\theta$- Cauchy sequence in relation to $\mathfrak{NN}$ (τ, φ, ω), is $\aleph_\theta$- convergent in relation to $\mathfrak{NN}$ (τ, φ, ω), then a $\mathfrak{NNS}$ $(\mathfrak{X}, \tau, \varphi, \omega, *, \diamond, \circledast)$ is statistically complete ($\aleph_\theta$- complete).

Theorem 4.5 Let any $\mathfrak{L}\mathcal{S}$ be represented by θ. In that case, any $\mathfrak{NNS}$ $(\mathfrak{X}, \tau, \varphi, \omega, *, \diamond, \circledast)$ is $\aleph_\theta$-complete, but not necessarily complete.

Proof. Given a $\mathfrak{NN}$ (τ, φ, ω), let $\mathfrak{x} = (\mathfrak{x}_{jk})$ be $\aleph_\theta$- Cauchy but not $\aleph_\theta$- convergent.
Assuming $\epsilon > 0$ and $\lambda > 0$, select $\mathfrak{d} > 0$: $(1 - \epsilon) * (1 - \epsilon) > 1 - \mathfrak{d}$, $\epsilon \diamond \epsilon < \mathfrak{d}$ and $\epsilon \circledast \epsilon < \mathfrak{d}$.
Now,
$\tau(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \geq \tau\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) * \tau\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right) > (1 - \epsilon) * (1 - \epsilon) > 1 - \mathfrak{d},$
$\varphi(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq \varphi\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) * \varphi\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right) < \epsilon \diamond \epsilon < \mathfrak{d}$ and
$\omega(\mathfrak{x}_{jk} - \mathfrak{x}_{mn}, \lambda) \leq \omega\left(\mathfrak{x}_{jk} - \ell, \frac{\lambda}{2}\right) * \omega\left(\mathfrak{x}_{mn} - \ell, \frac{\lambda}{2}\right) < \epsilon \circledast \epsilon < \mathfrak{d}$, as $\mathfrak{x}$ is not $\aleph_\theta$- convergent.
As a result, $\delta_\theta(\mathfrak{H}^C(\epsilon, \lambda)) = 0,$ where
$\mathfrak{H}(\epsilon, \lambda) = \left\{(j,k) \in \mathbb{N} \times \mathbb{N} : \varphi_{\mathfrak{x}_{jk} - \mathfrak{x}_{mn}}(\epsilon) \leq 1 - \mathfrak{r}, \omega_{\mathfrak{x}_{jk} - \mathfrak{x}_{mn}}(\epsilon) \leq 1 - \mathfrak{r}\right\}.$

Consequently, $\delta_\theta(\mathfrak{H}(\epsilon,\lambda)) = 1$, which is a contradiction, since $\mathfrak{x}$ was $\aleph_\theta$- Cauchy with regard to $\mathfrak{N}\mathfrak{N}$ (τ,φ,ω). Thus, $\mathfrak{x}$ needs to be $\aleph_\theta$- convergent in relation to $\mathfrak{N}\mathfrak{N}$ (τ,φ,ω). As a result, each $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ is $\aleph_\theta$- complete.

We can observe from the following example that a $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ is not complete in general.

Example 4.6 For $\mathfrak{X} = (0,1]$, let $\tau(\mathfrak{x},\lambda) := \frac{\lambda}{\lambda+\|\mathfrak{x}\|}$, $\varphi(\mathfrak{x},\lambda) := \frac{\|\mathfrak{x}\|}{\lambda+\|\mathfrak{x}\|}$ and $\omega(\mathfrak{x},\lambda) := \frac{\|\mathfrak{x}\|}{\lambda}$. When the sequence $\left(\frac{1}{n}\right)$ is Cauchy sequence with respect to $\mathfrak{N}\mathfrak{N}$ (τ,φ,ω) but not convergent with respect to $\mathfrak{N}$ (τ,φ,ω), then $(\mathfrak{X},\tau,\varphi,\omega,min,max,max)$ is $\mathfrak{N}\mathfrak{N}\mathfrak{S}$ but not complete.

This concludes the theorem's proof.

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Plithogenic Hypersoft based Plithogenic Cognitive Maps in Sustainable Industrial Development

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Abstract: Plithogenic Cognitive Maps (PCM) is a comprehensive decision tool discussed in diverse dimensions. The development of PCM models is indispensable in the domain of decision-making. This research work proposes theoretical framework of Plithogenic hypersoft sets (PHS) based PCM. This is a novel approach to make optimal decisions by considering several aspects simultaneously. The objective of this integrated approach is to make concurrent investigations on the interassociational impacts between the factors of the problem taken into account. The PHS based PCM is applied to a decision - making scenario of sustainable industrial development to demonstrate the significance of this PHS based PCM. The future directions and extensions of this work are also presented in this paper.

Keywords: Plithogenic Cognitive Maps, Plithogenic Hypersoft sets, Plithogenic accuracy function, sustainable industrial development.

1. Introduction

Smarandache [1] developed Plithogenic sets to generalize the representations ranging from crisp to neutrosophic forms. The development of this comprehensive kind of set is highly flexible in dealing with the attributes of different kinds. Plithogeny based applications are in increasing scales at recent times demonstrating a vivid picture of the efficacy of Plithogenic sets. The theory of hypersoft sets is developed to deal concurrently with varied attributes and it is discussed with Plithogenic sets. Smarandache is the pioneer in formulating both the theoretical aspects of Hypersoft sets and Plithogenic Hypersoft sets (PHS). Researchers have applied PHS to different decision-making situations involving complex patterns. Plithogenic Hypersoft sets are categorized as advanced kinds of sets as they are competent in handling intricated problems to design optimal solutions to the ranking kind of decision-making problems.

Cognitive Map structures are yet another robust decision tools applied to deal with the problems demanding intensive study on the factors and their relationships subjected to the context of the problem. As Plithogenic embraced theories give a holistic approach, Martin and Smarandache [2] extended the notion of cognitive maps to Plithogenic cognitive maps (PCM) and applied to distinct scenarios of decision making. Researchers have modified the theoretical framework of Plithogenic Cognitive Maps to formulate various versions of PCM models such as Plithogenic sub cognitive maps, induced PCM and many other. The decision-making approach of PCM is highly flexible in incorporating the requirements of the decision-makers. PCM models are initially developed only with the factors pertaining to the problem and then later, the attributes to which the factors are subjected are also considered together in developing the decision models. However, in these developments only the factor relationship subjected to only one attribute or the attribute values is considered for making analysis. This is identified as one of the limitations in factor considerations and hence this research work proposes the idea of integrating Plithogenic hypersoft sets to study the relationship between the factors by considering attributes in a concurrent manner. The objective of this research is to develop a highly comprehensive PCM decision model based on PHS representations to derive solutions to the complicated circumstance encompassing several factors and attributes demanding synchronized focus. The proposed model prototype is applied in analyzing the factors contributing to the sustainable industrial development.

The remaining contents of the paper are segmented into the following sections. The literature review is presented in section 2. The theoretical framework of PHS based PCM is outlined in section 3. The methodology of the proposed model is described in section 4. The application of the proposed framework in the industrial context is discoursed in section 5. The findings from the model are discussed in section 5. The summary of the paper is presented with future directions.

2. Review of Works

This section sketches out the contributions made in the domains of plithogenic cognitive maps and plithogenic hypersoft sets. The research gaps and the motivation behind this work is also outlined. Smarandache [18] developed the concept of hypersoft sets and then later extended to plithogenic hypersoft sets. Smarandache [21] laid a clear distinction between plithogenic sets and plithogenic hypersoft sets. Martin and Smarandache [8] discussed the applications of combined plithogenic hypersoft sets. Gayen et al [8] briefed on plithogenic hypersoft subgroups. Basumatary et al [16] discussed plithogenic neutrosophic hypersoft topological groups. Martin et al [10] introduced the concepts of extended plithogenic hypersoft sets and applied in Covid-19 decision making. Martin and Smarandache [11] illustrated the notion of concentric plithogenic hypergraph based on plithogenic hypersoft sets. Ahmad et al [12] developed a multi-criteria decision-making model based on plithogenic hypersoft sets. Rana et al [13] applied plithogenic fuzzy whole hypersoft sets and generalized plithogenic whole hypersoft sets in multi-attribute decision making with special reference to medical diagnosis. Majid et al [17] constructed a decision model for dam site selection using plithogenic multipolar fuzzy hypersoft sets. Dhivya and Lancy [19-20] conceptualized near plithogenic hypersoft sets and also focused on the notion of strong continuity. The research works on plithogenic hypersoft sets clearly state the applications of these sets in decision making.

Plithogenic cognitive maps are yet another exemplary for the extensive implication of Plithogenic sets. Martin and Smarandache [2] developed PCM for making optimal decisions by considering different factors and their respective contradiction degrees. Martin et al [3] conceptualized the notion of new plithogenic sub cognitive maps considering the mediating effects of the factors of medical diagnosis. Priya and Martin [4] introduced induced PCM with combined connection matrix to study the hitches of online learning system. Priya et al [5] applied PCM in making investigations on the spiritual intelligence of youth. Sujatha et al [6] developed a modified version of PCM approach in analyzing the novel corona virus. Angel et al [7] discoursed PCM model with linguistic contradiction degrees. The PCM decision models discuss the factors associated with the problem considering the degree of contradiction. However, these models determine the inter associational impacts between the factors considering only the factors one at a time. This is identified as one of the limitations of the PCM model. This research work proposes a PCM based decision model with plithogenic hypersoft sets to determine the aggregate inter associational impacts between the factors. This is a novel initiative of this research work and the integration of PHS into PCM framework unveils a comprehensive decision approach.

3. Theoretical Framework of PHS based PCM

PCM is generally considered to be a generalized representation of cognitive map structures. The PCM is also characterized as a directed graph with plithogenic weights considering different attributes in examining the interassociational impacts between factors of the study. Let us consider a problem of determining the interassociational impacts between the factors say F1, F2 and F3 considering the attributes say A1 with attribute values a11,a12,a13 and A2 with attribute values say a21,a22,a23 respectively.

The Plithogenic connection matrix is constructed with each of the cell values representing the interassociational impacts between the factors considering the Plithogenic hypersoft set representations of the form a11× a23, where a11 and a23 are the dominant attribute values with respect to the attributes A1 and A2.

Let us consider the reflection of the factors with respect to the attribute values as given in Table 1

Table 1 Association of Factors and Attribute values

	F1	**F2**	**F3**
a11	0.8	0.7	0.9
a12	0.6	0.5	0.7
a13	0.3	0.2	0.2
a21	0.4	0.3	0.3
a22	0.6	0.7	0.6
a23	0.9	0.9	0.8

The contradiction degrees between the dominant attribute value and other attribute values are presented in Table 2.

Table 2 Contradiction Degrees between the Attribute Values

<table>
<tr><td rowspan="2">a11</td><td>a11</td><td>a12</td><td>a13</td></tr>
<tr><td>0</td><td>1/3</td><td>2/3</td></tr>
<tr><td rowspan="2">a23</td><td>a21</td><td>a22</td><td>a23</td></tr>
<tr><td>2/3</td><td>1/3</td><td>0</td></tr>
</table>

The PHS based Plithogenic cognitive connection matrix representing the associational impacts between the factors are presented in Table 3

Table 3 Plithogenic Connection Matrix

	F1	**F2**	**F3**
F1	0	(a11,a23)	(a12,a22)
F2	(a13,a21)	0	(a11,a22)
F3	(a12,a22)	(a13,a23)	0

This PHS based connection matrix is represented for each of the attribute values in specific as follows.

A1	F1	F2	F3
F1	0	a11	a12
F2	a13	0	a11
F3	a12	a13	0

A2	F1	F2	F3
F1	0	a23	a22
F2	a21	0	a22
F3	a22	a23	0

A1	F1	F2	F3
F1	0	0.7	0.7
F2	0.3	0	0.9
F3	0.6	0.2	0

A2	F1	F2	F3
F1	0	0.9	0.6
F2	0.4	0	0.6
F3	0.6	0.9	0

The modified plithogenic connection matrices for each of the attribute values is obtained using the given formulae based on Plithogenic accuracy function stated by (1)

d(aij, Fg) +d(aik,Fg)* c(aij,aik) ………………..(1)

The modified Plithogenic connection matrix with respect to A1

A1	F1	F2	F3
F1	0	0.7	0.99
F2	0.83	0	0.9
F3	0.87	0.67	0

The cell (1,2) bears a11 in the plithogenic connection matric with respect to A1. The equivalent numerical value is 0.7. In this case the dominant attribute value is same as the cell value, whereas the equivalent numerical value for the cell (1,3) is computed using (1)

0.7 + 0.9* (1/3) ~ 0.99

The modified Plithogenic connection matrices are determined and the cognitive approach is applied to determine the interassociational impacts between the factors with respect to each of the dominant attribute values subjected to the attributes considered for study.

Let us consider the factor F1 to be in ON state

Let X = (1,0,0). This initial vector is passed onto the modified connection matrix and by performing the iterative procedure, the final state vector and the respective graphical representation is obtained in Fig.1.

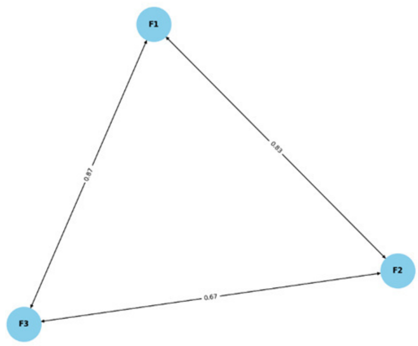

Fig.1 Interassociational Impacts between the Factors

4. Methodology

This section presents the sequential steps to be followed in developing PHS based PCM.

Step 1 : The framework begins with the formulation of decision problem with the consideration of concepts say F1,F2,..Fg and the attributes say A1,A2,..An pertaining to it.

Step 2 : The dominant attribute values say A1i, A2i ,.. Ani for each of the attributes are identified and the connection matrix with plithogenic hypersoft set representations is constructed.

Step 3: The connection matrix with respect to each of the attribute Aj is determined and then the modified plithogenic connection matrix is formulated using the plithogenic accuracy function of the form d(aij, Fg) +d(aik,Fg)* c(aij,aik). In this case d (aij, Fg) represents the degree of association between the attribute values and the factors, d(aik, Fg) id with respect to the dominant attribute values and c(aij, aik) is the degree of contradictions.

Step 4 : An initial state vector of the form (1 0 0 0 0 0..0) is considered and passed onto the connection matrix. The obtained vector is updated by assigning 1 to the values greater than 1 and -1 to the values lesser than 1.

Step 5 : The above step is repeated until the fixed point is obtained. The process the truncated after the vectors get converge to a state vector.

The above algorithmic procedure is presented graphically in Fig 2.

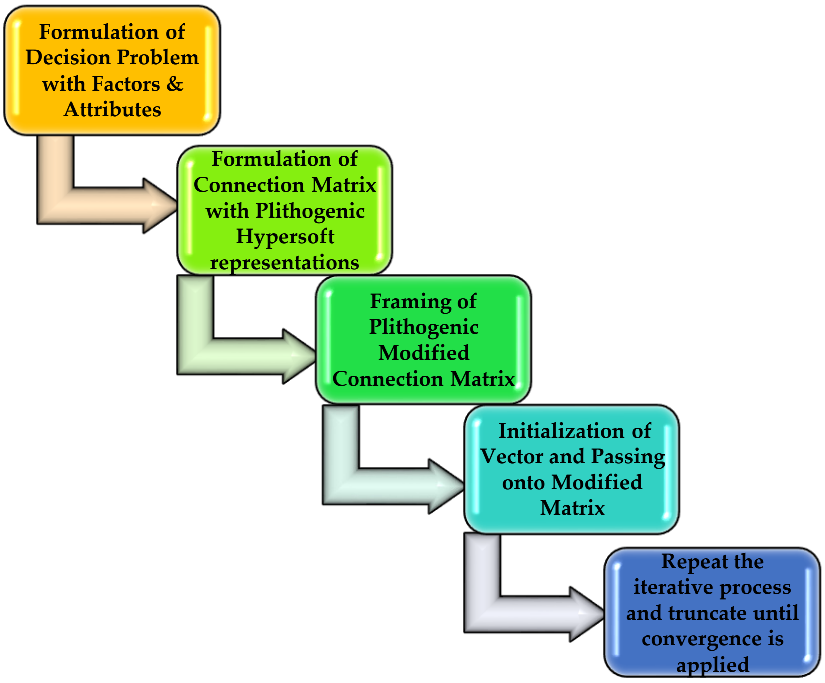

Fig 2: PHS based PCM Framework

5. Application of the Proposed Approach in Sustainable Industrial Development

This section applies the proposed framework in analyzing the factors contributing to the sustainable industrial development. Researchers have discussed about the strategies of institutionalizing sustainability in industrial development and few have focused on the modelling of sustainability. Dyrdonova [22] has developed a modelling framework of promoting sustainability in industries. Agbasi et al [23] considered soft computing and index-based methods in modelling the quality aspects of groundwater. The aforementioned recent research works state the significance of developing suitable models for studying the sustainability in industrial sectors. This section emphasizes that, the industrial sectors taking efforts in building the sustainability have to consider few significant attributes of sustainability into consideration. The attributes and the respective attribute values are presented in Table 4

Table 4 Attributes and their corresponding attribute values

Attributes	Attribute Values		
Efficiency in Resource Optimizations A1	Foundational a11	Enhanced a12	**Optimal a13**
Consistency in ensuring green initiatives A2	Primary a21	Standardized a22	**Integrated a23**
Economic Viability A3	Initial a31	Sustainable a32	**Profitable a33**
Innovative endeavours A4	Adoptive a41	Developmental a42	**Pioneering a43**
Compliance with environmental regulations A5	Baseline a51	Certified a52	**Exemplary a53**

The dominant attribute values are as highlighted in the above table.
Each of the attribute values represent the levels.

The factors considered in this decision-making problem are presented in Fig 3.

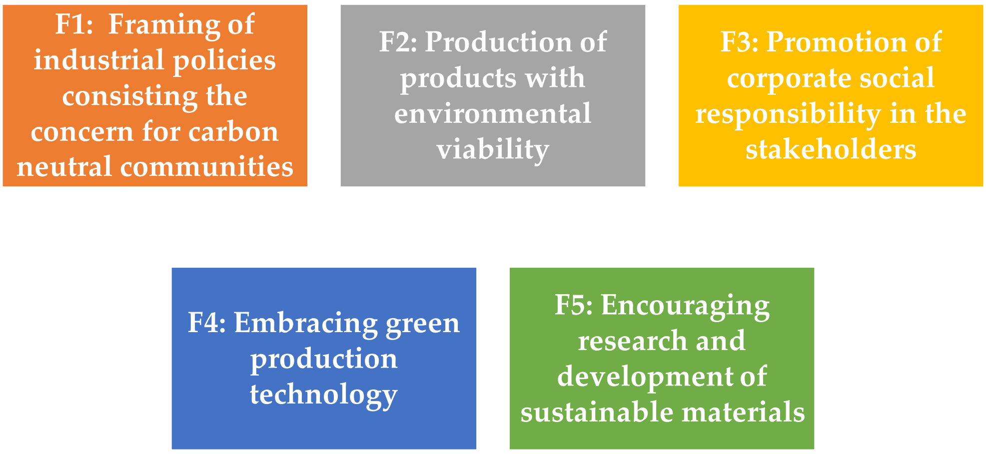

Fig 3: Factors considered for the decision-making problem

The sub factors for each of the factors are presented in Fig.4

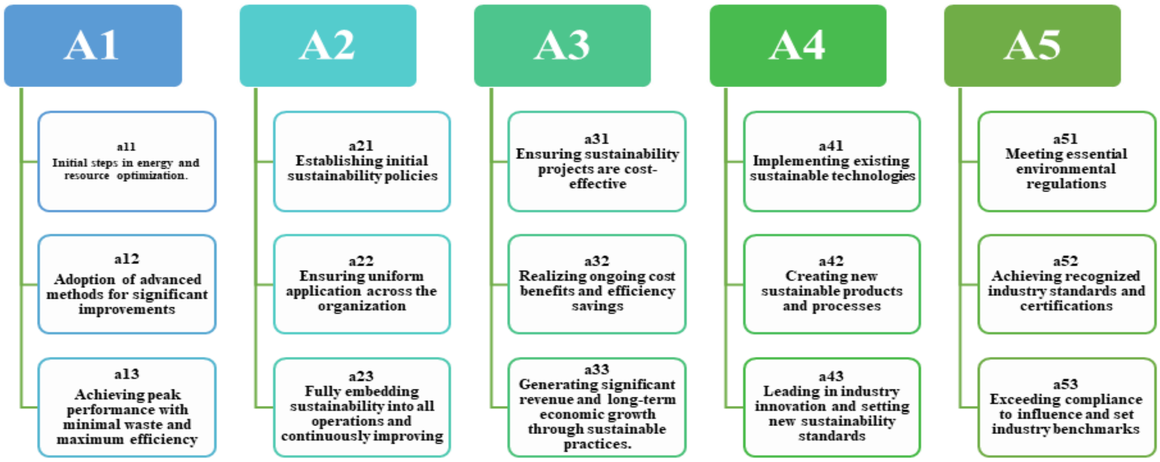

Fig 4. Sub-Factors considered for the decision-making problem

The experts are asked to give the possible associational impacts based on their perception, knowledge and experience and are presented in Table 5

Table 5 Expert's Opinion on Interassociational Impacts between the Factors

	F1	F2	F3	F4	F5
F1	0	(a12,a21,a32,a43, a52)	(a11,a22,a31,a42, a51)	(a13,a21,a33,a41, a52)	(a12,a23,a32,a43, a53)
F2	(a12,a21,a32,a 43,a52)	0	(a11,a21,a31,a42, a51)	(a12,a22,a32,a43, a52)	(a13,a23,a33,a41, a53)
F3	(a11,a22,a31,a 42,a51)	(a11,a21,a31,a42, a51)	0	(a12,a22,a32,a43, a52)	(a13,a23,a33,a41, a53)
F4	(a13,a21,a33,a 41,a52)	(a12,a22,a32,a43, a52)	(a12,a22,a32,a43, a52)	0	(a13,a23,a33,a41, a53)
F5	(a12,a23,a32,a 43,a53)	(a13,a23,a33,a41, a53)	(a13,a23,a33,a41, a53)	(a13,a23,a33,a41, a53)	0

Based on expert's opinion, the association between the attribute values and the factors are presented in Table 6.

Table 6 Attribute Values & Factors based on Expert's Opinion

Attribute values	F1	F2	F3	F4	F5
a11	0.3	0.4	0.3	0.5	0.4
a12	0.6	0.7	0.5	0.7	0.6
a13	0.9	0.9	0.7	0.9	0.8
a21	0.4	0.3	0.4	0.4	0.3
a22	0.7	0.6	0.6	0.7	0.5
a23	0.9	0.8	0.8	0.9	0.7
a31	0.3	0.4	0.3	0.4	0.5
a32	0.6	0.6	0.5	0.6	0.7
a33	0.8	0.8	0.7	0.8	0.9

a41	0.5	0.5	0.4	0.5	0.4
a42	0.7	0.7	0.6	0.8	0.7
a43	0.9	0.9	0.8	0.9	0.9
a51	0.4	0.3	0.2	0.5	0.4
a52	0.7	0.7	0.5	0.7	0.6
a53	0.9	0.9	0.7	0.9	0.8

These below values in Table 7-11 indicate the alignment of the factors with the attribute values

Table 7 Alignment of A1 Factor with the Attribute Values

A1	F1	F2	F3	F4	F5
F1	0	1	0.77	0.9	0.87
F2	0.9	0	0.77	1	0.8
F3	0.6	0.7	0	1	0.8
F4	0.9	1	0.73	0	0.8
F5	0.9	0.9	0.7	0.9	0

Table 8 Alignment of A2 Factor with the Attribute Values

A2	F1	F2	F3	F4	F5
F1	0	0.83	0.87	1	0.7
F2	1	0	0.93	1	0.7
F3	1	0.83	0	1	0.7
F4	1	0.87	0.87	0	0.7
F5	0.9	0.8	0.8	0.9	0

Table 9 Alignment of A3 Factor with the Attribute Values

A3	F1	F2	F3	F4	F5
F1	0	0.87	0.77	0.8	1
F2	0.87	0	0.77	0.87	0.9
F3	0.83	0.93	0	0.87	0.9
F4	0.8	0.87	0.73	0	0.9
F5	0.87	0.8	0.7	0.8	0

Table 10 Alignment of A4 Factor with the Attribute Values

A4	F1	F2	F3	F4	F5
F1	0	0.9	0.87	1	0.9
F2	0.9	0	0.87	0.9	1
F3	1	1	0	0.9	1
F4	1	0.9	0.8	0	1
F5	0.9	1	0.93	1	0

Table 11 Alignment of A5 Factor with the Attribute Values

A5	F1	F2	F3	F4	F5
F1	0	1	0.67	1	0.8
F2	1	0	0.67	1	0.8
F3	1	0.9	0	1	0.8
F4	1	1	0.73	0	0.8
F5	0.9	0.9	0.7	0.9	0

The contradiction degree between the factors and the dominant attribute values are given in the table 12

Table 12 Contradiction Degrees Between the Factors & The Dominant Attribute Values

	F1	F2	F3	F4	F5
A1(a13)	0.6	0.3	0.5	0.7	0.2
A2(a23)	0.4	0.2	0.5	0.5	0.6
A3(a33)	0.7	0.5	0.3	0.2	0.4
A4(a43)	0.3	0.5	0.4	0.1	0.5
A5(a53)	0.5	0.4	0.6	0.2	0.7

Let us consider the instantaneous vector X = (1 0 0 0 0) with the first factor in ON position.

The vector is passed on to the modified plithogenic connection matrix of A1 to find the fixed point of the system

X*P(A1) = (0.6 1 0.885 0.97 0.896) → (1 1 0.89 0.97 0.9) = X1

X1*P(A1) = (0.96 1 0.885 1 0.896) → (1 1 0.89 1 0.9) = X2

X2*P(A1) = (0.96 1 0.885 1 0.896) → (1 1 0.89 1 0.9) = X3

X2 = X3

Hence the fixed point is obtained.

Similarly, this instantaneous vector is passed on to each of the modified plithogenic connection matrix to obtain the fixed point.

6. Discussions

The similar approach is followed for all the other factors and the following table 13 is obtained as follows.

Table 13 Associational Impacts between the Factors

On Position of the Factors	Attribute	Associational Impacts				
		F1	F2	F3	F4	F5
F1 (10000)	A1 (a13)	1	1	0.89	1	0.9
	A2 (a23)	1	0.9	0.94	1	0.88
	A3 (a33)	1	0.94	0.84	0.85	1
	A4 (a43)	1	1	0.96	1	1
	A5 (a53)	1	1	0.89	1	0.94
F2 (01000)	A1 (a13)	0.96	1	0.89	1	0.87
	A2 (a23)	1	1	0.97	1	0.88
	A3 (a33)	0.96	1	0.84	0.9	0.98
	A4 (a43)	1	1	0.96	1	1
	A5 (a53)	1	1	0.89	1	0.94
F3 (00100)	A1 (a13)	0.96	1	1	1	0.87
	A2 (a23)	1	0.9	1	0.88	1
	A3 (a33)	0.95	0.97	1	0.9	0.97
	A4 (a43)	1	1	1	1	1
	A5 (a53)	1	1	1	1	0.94
F4 (00010)	A1 (a13)	0.96	1	0.89	1	0.87
	A2 (a23)	1	0.9	0.94	1	0.88
	A3 (a33)	0.95	0.94	0.81	1	0.97
	A4 (a43)	1	1	0.96	1	1
	A5 (a53)	1	1	0.89	1	0.94
F5 (00001)	A1 (a13)	0.96	0.99	0.88	0.99	1
	A2 (a23)	0.99	0.88	0.93	0.99	1
	A3 (a33)	0.96	0.92	0.81	0.84	1
	A4 (a43)	1	1	0.96	1	1
	A5 (a53)	0.99	0.98	0.88	0.99	1

The above table values shall be graphically represented in Fig.5-9 to visualize the associational impacts with respect to the on position of the factors

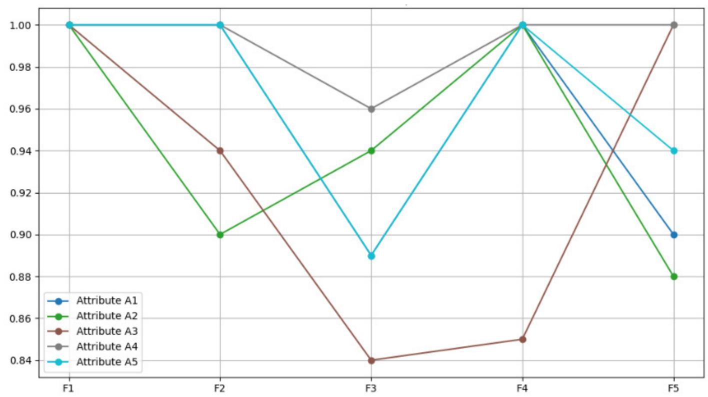

Fig.5 Visualization of the Associational Impacts with respect to ON position of F1

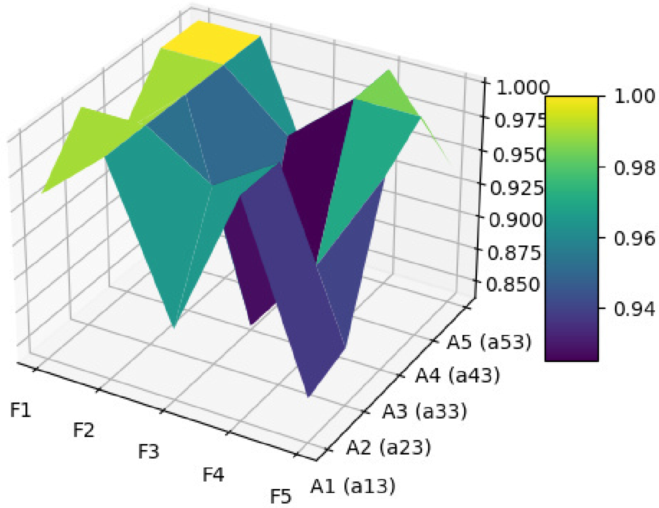

Fig.6 Visualization of the Associational Impacts with respect to ON position of F2

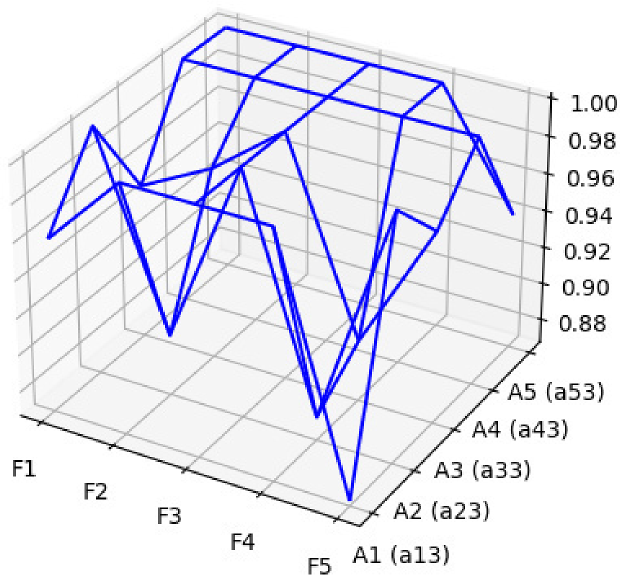

Fig.7 Visualization of the Associational Impacts with respect to ON position of F3

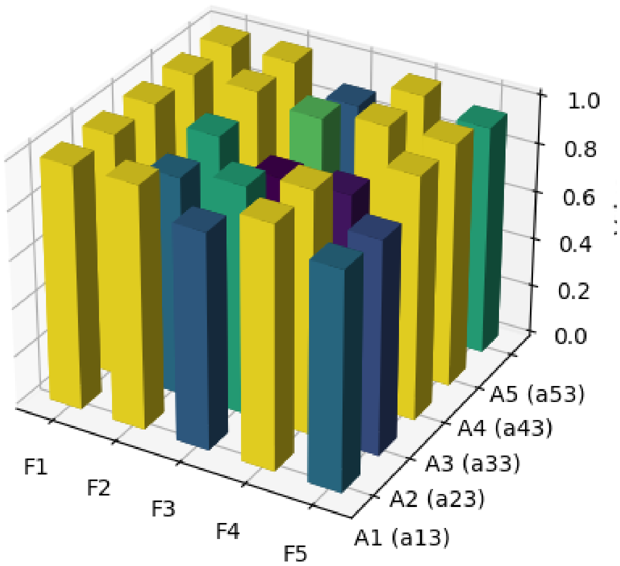

Fig.8 Visualization of the Associational Impacts with respect to ON position of F4

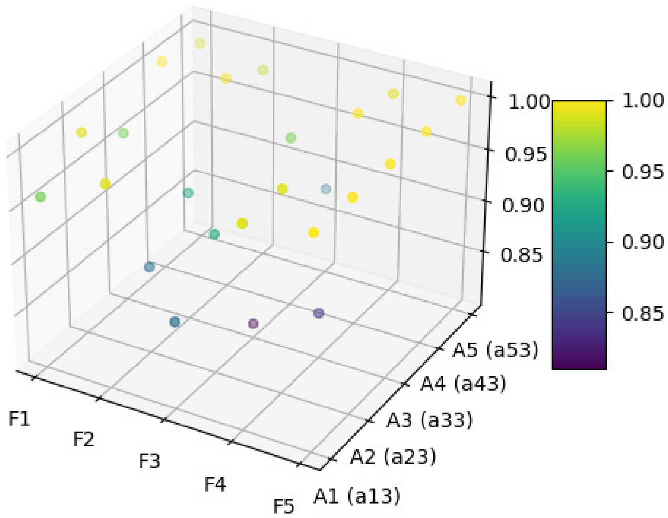

Fig.9 Visualization of the Associational Impacts with respect to ON position of F5

From the above table 13 and the figures 5-9, the interassociational impacts between the factors are determined by considering the attributes and the attribute values. The aggregate impacts between the factors with respect to the dominant attribute values are obtained using plithogenic hypersoft sets representation. The modified connection matrix obtained using plithogenic accuracy function facilitates in considering the degrees of contradiction between the attribute values and the feasibility of the attribute values with respect to the factors. The vector stating the aggregate impacts shall be determined by adding the vectors which will reflect the cumulative effects of one factor over another. The cumulative associational impacts of the factors F1,F2,F3,F4 and F5 with respect to each of the attributes A1 (a13),A(a23),A3(a33),A4(a43) and A5(a53) is presented in Fig.10

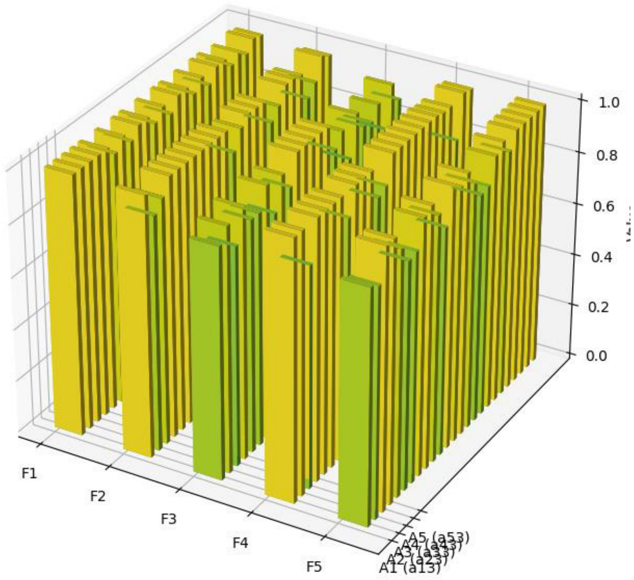

Fig.10 Aggregate Associational Impacts

Conclusion

This research work proposes a novel approach of PCM integrating plithogenic hypersoft sets representation. This novel initiative discloses new avenues of developing decision models. The proposed model is well substantiated with a decision-making problem on promoting the sustainability of industrial development by considering factors of different dimensions confined to attributes and their respective attribute values. This approach shall be extended with different kind plithogenic hypersoft sets. This new kind of decision-making framework is more flexible and robust in handling complex relationships.

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Exploring Topological characteristics of Neutrosophic Banach Spaces

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Abstract: The open mapping theorem, the closed graph theorem, and other topological characteristics of neurosophic Banach spaces are examined in this paper. Moreover, in neutrosophic normed spaces, the closedness characteristics of the sum of two linear operators has been studied.

Keywords: Neutrosophic normed space, Neutrosophic Banach space, Closed linear operator, First category, Second category.

1. Introduction

Zadeh [23], laid the foundation for fuzzy mathematics in 1965. This concept takes standard set theory to a higher level of analysis. After then, the idea received a number of proven improvements, and the reasoning has been used in a variety of scientific and engineering fields, including the study of approximation theory [1], linear systems [6] [17] and matrix theory. Several authors investigated at the theory with its topological aspects from their own angle and came to some important basic results that seem important when examining the idea in connection to different other generalized spaces.

In 1992, Felbin [9] presented a new idea of fuzzy norms on linear spaces. Xiao and Zhu [22] expanded the concept of fuzzy norm by studying the topological properties of fuzzy normed linear spaces. Another fuzzy norm was established by Bag and Samanta [4]. Bag and Samanta [5] developed weak fuzzy boundedness, weak fuzzy continuity, strong fuzzy boundedness, fuzzy continuity, sequential fuzzy continuity, and the fuzzy norm of linear operators with respects to an associated fuzzy norm. Atanassov [2] developed the idea of an intuitionistic fuzzy set in 1984. He did this by designating a new kind of membership function that indicates how much an item does not belong in a particular set. Park [15] defined the notion of Intuitionistic Fuzzy Metric Space with the help of continuous t-norms and continuous. Amazing work was done on intuitionistic fuzzy topological spaces by sadati and park [18]. Many authors have since published their own works in the literature (see [12] [14] [16]). Among them are those who have made multiple important contributions to convergence theory and proposed convergent sequence spaces within the intuitionistic fuzzy normed space framework. Some important topological findings in fuzzy Banach spaces were studied in 2005 by Saadati and Vaezpour [19]. In 1998, Smarandache[21] developed the ideas of neutrosophic logic and Neutrosophic Set. Kirisci and Simsek [13] founded the concept of Neutrosophic Metric Spaces which addresses membership, non-membership and neutralness. The aim of this study is to investigate topological characteristics of neutrosophic Banach space, building on the results of [19] earlier studies published in [19]. Furthermore, the information provides some unresearched results.

2. Preliminaries

Definition 2.1[10]: A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if following conditions hold:

(i) $\mathfrak{z} * \tilde{\varsigma} = \tilde{\varsigma} * \mathfrak{z}$ for all $\mathfrak{z}, \tilde{\varsigma} \in [0,1]$;

(ii) $*$ is continuous;

(iii) $\mathfrak{z} * 1 = \mathfrak{z}$, for all $\mathfrak{z} \in [0, 1]$;

(iv) $*$ is associative;

If $\mathfrak{z} \leq \tilde{\varsigma}$ and $\mathfrak{d} \leq \vartheta$, with $\mathfrak{z}, \tilde{\varsigma}, \mathfrak{d}, \vartheta \in [0,1]$, then $\mathfrak{z} * \mathfrak{d} \leq \tilde{\varsigma} * \vartheta$.

Definition 2.2[10]: A binary operation $\odot$: $[0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-conorm if it holds the followings assertions:

(i) $\mathfrak{z} \odot \tilde{\varsigma} = \tilde{\varsigma} \odot \mathfrak{z}$ for all $\mathfrak{z}, \tilde{\varsigma} \in [0,1]$;

(ii) $\odot$ is continuous;

(iii) $\mathfrak{z} \odot 0 = 0$;

(iv) $\odot$ is associative;

(v) If $\mathfrak{z} \leq \tilde{\varsigma}$ and $\mathfrak{d} \leq \vartheta$, with $\mathfrak{z}, \tilde{\varsigma}, \mathfrak{d}, \vartheta \in [0,1]$, then $\mathfrak{z} \odot \mathfrak{d} \leq \tilde{\varsigma} \odot \vartheta$.

Definition 2.3: The 6-tuple $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ is said to be a Neutrosophic Normed Linear Space [*NNLS*], if $\widetilde{\mathcal{N}}$ is a vector space over a field $\mathbb{R}$, $\star$ is a continuous t-norm, $\diamond$ is a continous t-conorm, and $\eta, \upsilon, \varsigma$ are functions from $\widetilde{\mathcal{N}} \times \mathbb{R} \rightarrow [0,1]$ meets the following conditions for every $\mathfrak{e}, \mathfrak{w} \in \widetilde{\mathcal{N}}$ and $\sigma, \mathfrak{k} \in \mathbb{R}$

(n1) $0 \leq \eta(\mathfrak{e}, \mathfrak{k}) \leq 1$; $0 \leq \upsilon(\mathfrak{e}, \mathfrak{k}) \leq 1$; $0 \leq \varsigma(\mathfrak{e}, \mathfrak{k}) \leq 1$;

(n2) $\eta(\mathfrak{e}, \mathfrak{k}) + \upsilon(\mathfrak{e}, \mathfrak{k}) + \rho(\mathfrak{e}, \mathfrak{k}) \leq 3$;

(n3) $\eta(\mathfrak{e}, \mathfrak{k}) > 0$;

(n4) $\eta(\mathfrak{e}, \mathfrak{k}) = 1 \Leftrightarrow \mathfrak{v} = 0$;

(n5) $\eta(\sigma\mathfrak{e}, \mathfrak{k}) = \eta\left(\mathfrak{e}, \frac{\mathfrak{k}}{|\sigma|}\right)$ for $\sigma \neq 0$;

(n6) $\eta(\mathfrak{e}, \sigma) \star \eta(\mathfrak{w}, \mathfrak{k}) \leq \eta(\mathfrak{e} + \mathfrak{w}, \sigma + \mathfrak{k})$;

(n7) $\eta(\mathfrak{e}, \mathfrak{k}) : (0, \infty) \rightarrow [0,1]$ is continuous;

(n8) $\lim_{\mathfrak{k} \rightarrow \infty} \eta(\mathfrak{e}, \mathfrak{k}) = 1$ and $\lim_{\mathfrak{k} \rightarrow 0} \eta(\mathfrak{e}, \mathfrak{k}) = 0$;

(n9) $\upsilon(\mathfrak{e}, \mathfrak{k}) < 1$;

(n10) $\upsilon(\mathfrak{e}, \mathfrak{k}) = 0 \Leftrightarrow \mathfrak{v} = 0$;

(n11) $\upsilon(\sigma\mathfrak{e}, \mathfrak{k}) = \upsilon\left(\mathfrak{e}, \frac{\mathfrak{k}}{|\sigma|}\right)$ for $\sigma \neq 0$;

(n12) $\upsilon(\mathfrak{e}, \sigma) \diamond \upsilon(\mathfrak{w}, \mathfrak{k}) \geq \upsilon(\mathfrak{e} + \mathfrak{w}, \sigma + \mathfrak{k})$;

(n13) $\upsilon(\mathfrak{e}, \tau) : (0, \infty) \rightarrow [0,1]$ is continuous;

(n14) $\lim_{\mathfrak{k} \rightarrow \infty} \upsilon(\mathfrak{e}, \mathfrak{k}) = 0$ and $\lim_{\mathfrak{k} \rightarrow 0} \upsilon(\mathfrak{e}, \mathfrak{k}) = 1$;

(n15) $\varsigma(\mathfrak{e}, \mathfrak{k}) < 1$;

(n16) $\varsigma(\mathfrak{e}, \mathfrak{k}) = 0 \Leftrightarrow \mathfrak{e} = 0$;

(n17) $\varsigma(\sigma\mathfrak{e}, \mathfrak{k}) = \varsigma\left(\mathfrak{e}, \frac{\mathfrak{k}}{|\sigma|}\right)$ for $\sigma \neq 0$;

(n18) $\varsigma(\mathfrak{e}, \sigma) \diamond \varsigma(\mathfrak{w}, \mathfrak{k}) \geq \varsigma(\mathfrak{e} + \mathfrak{w}, \sigma + \mathfrak{k})$;

(n19) $\varsigma(\mathfrak{e}, \mathfrak{k}) : (0, \infty) \rightarrow [0,1]$ is continuous;

(n20) $\lim_{\mathfrak{k} \rightarrow \infty} \varsigma(\mathfrak{e}, \mathfrak{k}) = 0$ and $\lim_{\mathfrak{k} \rightarrow 0} \varsigma(\mathfrak{e}, \mathfrak{k}) = 1$.

Remark 2.4 [15]:

(i) For every pair $0 < \mathfrak{z}_1, \mathfrak{z}_2 < 1$ we can find $0 < \tilde{\varsigma}_1, \tilde{\varsigma}_2 < 1$ such that $\mathfrak{z}_1 \leq \mathfrak{z}_2 * \tilde{\varsigma}_2$ and $\mathfrak{z}_2 \geq \tilde{\varsigma}_1 \diamond \mathfrak{z}_1$.

(ii) For every $0 < \mu < 1$ we can find $0 < \tilde{\varsigma}_1, \tilde{\varsigma}_2 < 1$ such that $\mu \leq \mathfrak{z}_1 * \mathfrak{z}_2$ and $\mathfrak{z}_1 \diamond \mathfrak{z}_2 \geq \mu$.

Definition 2.5: In neutrosophic normed linear space, the open ball $\mathcal{B}_{\mathfrak{v}}(\varepsilon,\mathfrak{k})$ centred at $\mathfrak{e}$ is defined as $\mathcal{B}_{\mathfrak{e}}(\mathfrak{r},\mathfrak{k}) = \{ \mathfrak{e} \in \widetilde{\mathcal{N}} : 1-\eta(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) < \mathfrak{r}, \upsilon(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) > \varepsilon$ and $\varsigma(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) > \mathfrak{r} \}$ where $0< \mathfrak{r} < 1$ and $\mathfrak{k} > 0$. Similarly, the closed ball centred at $\mathfrak{e}$ is defined as

$\mathcal{B}_{\mathfrak{e}}[\mathfrak{r},\mathfrak{k}] = \{ \mathfrak{e} \in \widetilde{\mathcal{N}} : 1-\eta(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) \le \mathfrak{r}, \upsilon(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) \ge \varepsilon$ and $\varsigma(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) \ge \mathfrak{r} \}$ where $0< \mathfrak{r} < 1$ and $\mathfrak{k} > 0$.

Remark 2.6 [15]: Every open ball is an open set in neutrosophic normed linear space.

Lemma 2.7: Let $(\eta,\upsilon,\varsigma)$ be neutrosophic norm on $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ then
(i) $\eta(\mathfrak{e},\mathfrak{k})$ is a non decreasing, $\upsilon(\mathfrak{e},\mathfrak{k})$ is non a non increasing and $\varsigma(\mathfrak{e},\mathfrak{k})$ is decreasing but not strictly with respect to $\mathfrak{k}$ for each $\mathfrak{e} \in \widetilde{\mathcal{N}}$.
(ii) $\eta(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) = \eta(\mathfrak{w}-\mathfrak{e},\mathfrak{k})$, $\upsilon(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) = \upsilon(\mathfrak{w}-\mathfrak{e},\mathfrak{k})$ and $\varsigma(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) = \varsigma(\mathfrak{w}-\mathfrak{e},\mathfrak{k})$.
Proof: Let $\mathfrak{k}_1 < \mathfrak{k}_2$, and $\theta = \mathfrak{k}_2 - \mathfrak{k}_1$ or $\mathfrak{k}_2 = \theta + \mathfrak{k}_1$, then

(i) $\eta(\mathfrak{e},\mathfrak{k}_1) = \eta(\mathfrak{e},\mathfrak{k}_1) \star 1 = \eta(\mathfrak{e},\mathfrak{k}_1) \star \eta(0,\theta) \le \eta(\mathfrak{e}+0,\mathfrak{k}_1+\theta) = \eta(\mathfrak{e},\mathfrak{k}_2)$ .... (1)
$\Longrightarrow \eta(\mathfrak{e},\mathfrak{k}_1) \le \eta(\mathfrak{e},\mathfrak{k}_2)$.
$\upsilon(\mathfrak{e},\mathfrak{k}_1) = \upsilon(\mathfrak{e},\mathfrak{k}_1) \diamond 0 = \upsilon(\mathfrak{e},\mathfrak{k}_1) \star \upsilon(0,\theta) \ge \upsilon(\mathfrak{e}+0,\mathfrak{k}_1+\theta) = \upsilon(\mathfrak{e},\mathfrak{k}_2)$ .... (2)
$\Longrightarrow \upsilon(\mathfrak{e},\mathfrak{k}_1) \ge \upsilon(\mathfrak{e},\mathfrak{k}_2)$ and
$\varsigma(\mathfrak{e},\mathfrak{k}_1) = \varsigma(\mathfrak{e},\mathfrak{k}_1) \diamond 0 = \varsigma(\mathfrak{e},\mathfrak{k}_1) \star \varsigma(0,\theta) \ge \varsigma(\mathfrak{e}+0,\mathfrak{k}_1+\theta) = \varsigma(\mathfrak{e},\mathfrak{k}_2)$ .... (3)
$\Longrightarrow \varsigma(\mathfrak{e},\mathfrak{k}_1) \ge \varsigma(\mathfrak{e},\mathfrak{k}_2)$.

(ii) $\eta(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) = \eta(-(\mathfrak{w}-\mathfrak{e}),\mathfrak{k}) = \eta\left(\mathfrak{w}-\mathfrak{e},\frac{\mathfrak{k}}{|-1|}\right) = \eta(\mathfrak{w}-\mathfrak{e},\mathfrak{k})$ .... (4)
$\upsilon(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) = \upsilon(-(\mathfrak{w}-\mathfrak{e}),\mathfrak{k}) = \upsilon\left(\mathfrak{w}-\mathfrak{e},\frac{\mathfrak{k}}{|-1|}\right) = \upsilon(\mathfrak{w}-\mathfrak{e},\mathfrak{k})$ .... (5)
$\varsigma(\mathfrak{e}-\mathfrak{w},\mathfrak{k}) = \varsigma(-(\mathfrak{w}-\mathfrak{e}),\mathfrak{k}) = \varsigma\left(\mathfrak{w}-\mathfrak{e},\frac{\mathfrak{k}}{|-1|}\right) = \varsigma(\mathfrak{w}-\mathfrak{e},\mathfrak{k})$. .... (6)

Definition 2.8 A point $\mathfrak{e} \in (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is said to be an interior point if there exist an open ball centred at $\mathfrak{e}$ is contained in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$.

Definition 2.9 Let $\mathcal{I} \subseteq (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ then the interior of the set $\mathcal{I}$ of all the interior point of $\mathcal{I}$ with regard to neutrosophic norm $(\eta,\upsilon,\varsigma)$.

Definition 2.10 A set $\mathcal{I} \subseteq (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is said to be nowhere dense in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ if the closure of $\mathcal{I}$ has no interior point.

Definition 2.11 A neutrosophic normed space $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is called first category if $\widetilde{\mathcal{N}} = \cup_i^\infty \mathcal{I}_i$, for each i, $\mathcal{I}_i$ is nowhere dense in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$.

A neutrosophic normed space which is not first category is said to be second category.

Definition 2.12 $(\mathfrak{e}_n)$ is said to be a Cauchy sequence in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ if for all $0 < \varepsilon < 1$ there exist $\mathfrak{m} \in \mathbb{N}$ such that $1-\eta(\mathfrak{e}_i-\mathfrak{e}_j,\mathfrak{k}) \le \varepsilon, \upsilon(\mathfrak{e}_i-\mathfrak{e}_j,\mathfrak{k}) < \varepsilon$ and $\varsigma(\mathfrak{e}_i-\mathfrak{e}_j,\mathfrak{k}) < \varepsilon$ for all $i,j \ge \mathfrak{m}$ and $\mathfrak{k} > 0$.

Definition 2.13 A neutrosophic normed linear space $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is said to be complete, if every Cauchy sequence $(\mathfrak{e}_n)$ in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ converges in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$.

Definition 2.14 Let $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ and $(\widetilde{\mathcal{M}},\eta,\upsilon,\varsigma,\star,\diamond)$ are neutrosophic normed linear space. The linear operator $\Psi : \mathfrak{C} \to \widetilde{\mathcal{M}}$, where $\mathfrak{C} \subseteq \widetilde{\mathcal{N}}$ is closed $\Leftrightarrow$ it satisfies the following condition that $\Psi(\mathfrak{e}_n) \to \Psi(\mathfrak{e})$ whenever $\mathfrak{e}_n \to \mathfrak{e}$ and $\mathfrak{e} \in \mathfrak{C}$ for all n.

3. Main Results

Theorem 3.1 A complete neutrosophic normed linear space $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is of second category space.

Proof: Suppose the statement is not true. i.e $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ is not of second category and hence $\widetilde{\mathcal{N}} = \bigcup_{i=1}^{\infty}(\mathfrak{F}_i,\eta,\upsilon,\varsigma,\star,\diamond)$ for each i, $(\mathfrak{F}_i,\eta,\upsilon,\varsigma,\star,\diamond)$ is nowhere dense in $\widetilde{\mathcal{N}} = (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$.

Now for $i = 1$, $\mathfrak{F}_1$ is nowhere dense in $\widetilde{\mathcal{N}}$. So the closure of $\mathfrak{F}_1$ is not an open set, this implies $\overline{\mathfrak{F}_1}^c$ contains an interior point, let $\mathfrak{e}_1 \in \overline{\mathfrak{F}_1}^c$ such that for $\mathfrak{k} > 0$ and every $0<\mathfrak{r}_1 < \frac{1}{2}$ there exists a ball centred at $\mathfrak{e}_1$, $\mathcal{B}_1 = \mathcal{B}_{\mathfrak{e}_1}(\mathfrak{r}, \mathfrak{k}) = \{\xi \in \mathfrak{N} : 1-\eta(\mathfrak{e}_1-\xi,\mathfrak{k}) < \varepsilon, \upsilon(\mathfrak{e}_1-\xi,\mathfrak{k}) < \varepsilon$ and $\varsigma(\mathfrak{e}_1-\xi,\mathfrak{k}) < \varepsilon\} \subseteq \overline{\mathfrak{F}_1}^c$. Again $\overline{\mathfrak{F}_2}^c$ is not open in $\widetilde{\mathcal{N}}$ therefore $\overline{\mathfrak{F}_2}^c \cap \mathcal{B}_{\mathfrak{e}_1}\left(\frac{\mathfrak{r}_1}{2}, \mathfrak{k}\right) = \emptyset$; where $\mathfrak{r}_1 < \frac{\mathfrak{r}}{2}$, on the other hand, $\overline{\mathfrak{F}_2}^c$ intersect the ball $\mathcal{B}_{\mathfrak{e}_1}\left(\frac{\mathfrak{r}_1}{2}, \mathfrak{k}\right)$. Now let $\overline{\mathfrak{F}_2}^c \cap \mathcal{B}_{\mathfrak{e}_1}(\mathfrak{r}_2, \mathfrak{k})$ contains a ball $\mathcal{B}_2 = \mathcal{B}_{\mathfrak{e}_2}\left(\frac{\mathfrak{r}_1}{2}, \mathfrak{k}\right)$ where $\mathfrak{r}_2 < \frac{\mathfrak{r}_1}{2}$, continuing this process of forming the ball $\mathcal{B}_n = \mathcal{B}_{\mathfrak{e}_n}(\mathfrak{r}_n, \mathfrak{k})$, we shall have $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$

Where $\mathfrak{r}_{n+1} < \frac{\mathfrak{r}_n}{2}$ and $\mathfrak{r}_n < \frac{1}{2^n}$. We get $(\mathfrak{e}_n)$ of the centres of the balls $\mathcal{B}_n$. Now we show that $(\mathfrak{e}_n)$ is a Cauchy sequence. Let $n_{\mathfrak{r}} \in \mathbb{N}$ and let $n > m > n_{\mathfrak{r}} \Rightarrow \mathcal{B}_n \subseteq \mathcal{B}_m$, take $\varpi \in \mathcal{B}_m$, then

$$1-\eta\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n},\ \upsilon\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \text{ and } \varsigma\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \tag{3.1.1}$$

$$1-\eta\left(\mathfrak{e}_m - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^m},\ \upsilon\left(\mathfrak{e}_m - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^m} \text{ and } \varsigma\left(\mathfrak{e}_m - \varpi, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^m}. \tag{3.1.2}$$

Now,
$$\eta(\mathfrak{e}_n - \mathfrak{e}_m, \mathfrak{k}) = \eta(\mathfrak{e}_n - \varpi + \varpi - \mathfrak{e}_m, \mathfrak{k})$$
$$\geq \eta\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) \star \eta\left(\varpi - \mathfrak{e}_m, \frac{\mathfrak{k}}{2}\right) > \left(1-\frac{1}{2^n}\right) \star \left(1-\frac{1}{2^m}\right) > 1-\mathfrak{r}' \tag{3.1.3}$$

$$\upsilon(\mathfrak{e}_n - \mathfrak{e}_m, \mathfrak{k}) = \upsilon(\mathfrak{e}_n - \varpi + \varpi - \mathfrak{e}_m, \mathfrak{k})$$
$$\leq \upsilon\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) \diamond \upsilon\left(\varpi - \mathfrak{e}_m, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \diamond \frac{1}{2^m} < \mathfrak{r}' \tag{3.1.4}$$
$$\varsigma(\mathfrak{e}_n - \mathfrak{e}_m, \mathfrak{k}) = \varsigma(\mathfrak{e}_n - \varpi + \varpi - \mathfrak{e}_m, \mathfrak{k})$$
$$\leq \varsigma\left(\mathfrak{e}_n - \varpi, \frac{\mathfrak{k}}{2}\right) \diamond \varsigma\left(\varpi - \mathfrak{e}_m, \frac{\mathfrak{k}}{2}\right) < \frac{1}{2^n} \diamond \frac{1}{2^m} < \mathfrak{r}'. \tag{3.1.5}$$

Since for every n and m we can $0<\mathfrak{r}' < 1$ such that $\left(1-\frac{1}{2^n}\right) \star \left(1-\frac{1}{2^m}\right) > 1-\mathfrak{r}'$ and $\frac{1}{2^n} \diamond \frac{1}{2^m} < \mathfrak{r}'$.

Thus from equations (3.1.3), (3.1.4) and (3.1.5), we conclude that $(\mathfrak{e}_n)$ is a Cauchy sequence with respect to neutrosophic norm $(\eta,\upsilon,\varsigma)$, let $(\mathfrak{e}_n)$ converges at $\mathfrak{e} \in \widetilde{\mathcal{N}}$. $\mathfrak{e}$ lies in some $\overline{\mathfrak{F}_\iota}$, because $\widetilde{\mathcal{N}}$ is complete, $\mathfrak{e} \in \overline{\mathfrak{F}_t}$ for a particular t, therefore $\overline{\mathfrak{F}_t}$ contains some open ball $\mathcal{B}_{\mathfrak{e}}(\mathfrak{e}, \mathfrak{k})$ which contradict that $\overline{\mathfrak{F}_t}$ is nowhere dense in $\widetilde{\mathcal{N}}$. Thus theorem is concluded.

Theorem: 3.2: Let $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ and $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$ be Neutrosophic Banach spaces and Ψ be a continuous linear oprator from $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ onto $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$. Then Ψ is an open mapping.

Proof.

Step- I. Let $\mathfrak{P}$ be a ball centred at 0 in $\widetilde{\mathcal{N}} = (\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$, we shall show that $0 \in int(\Psi(\mathfrak{P}))$, let $\mathfrak{F}$ is a neighbourhood of 0 such that $\mathfrak{F}+\mathfrak{F} \subset \mathfrak{P}$, now $\widetilde{\mathcal{M}} = \Psi(\widetilde{\mathcal{N}})$, since Ψ is a surjective mapping, then by Theorem (3.1) we obtain that if $\widetilde{\mathcal{M}} = \bigcup_{n\geq 1}\Psi(\mathfrak{F}_n)$ then there exist $\wp_0 \in \mathbb{N}$ such that

$int(\Psi(\overline{\mathfrak{F}_0}))$ is empty, therefore $0 = \Psi(0) \in \text{int}(\Psi(\overline{\mathfrak{Z}})) - \text{int}(\Psi(\overline{\mathfrak{Z}})) \subseteq \Psi(\overline{\mathfrak{Z}}) - \Psi(\overline{\mathfrak{Z}}) = \Psi(\mathfrak{Z}) - \Psi(\mathfrak{z}) = \Psi(\mathfrak{Z}-\mathfrak{z}) \subset \Psi(\overline{\mathfrak{Z}})$. This shows that $\Psi-$image of the neighbourhood of 0 belongs to $\widetilde{\mathcal{N}}$ contains a neighbourhood of 0 in $\widetilde{\mathcal{M}}$. ρ_n

Step II. Let $0 \in \mathfrak{P}$ and $\mathfrak{P}$ is open then $\mathfrak{P}$ contains a ball $\mathcal{B}_0(\delta, \mathfrak{k}_0)$ for some $0 < \delta < 1$ and $\mathfrak{k}_0 > 0$, a sequence $(\mathfrak{e}_n)$ can be find, where $0 < \mathfrak{e}_n < 1$, such that $\mathfrak{e}_n \to 0$ as $n \to \infty$ and by remark (2.4) we have

$$\lim_n[(1-\mathfrak{e}_1) \star (1-\mathfrak{e}_2) \ldots (1-\mathfrak{e}_n)] > 1-\delta. \tag{3.2.1}$$

However, if we construct a sequence of neighborhoods $\mathcal{B}_0(\mathfrak{e}_n, \tau_n) = \mathcal{B}_n$ (say), where $\tau_n = \frac{\mathfrak{k}_0}{2^n}$ then by step-I, $\Psi(\mathcal{B}_n)$ contains a neighbourhood $\mathfrak{W}_n = \mathcal{B}_0(\lambda_n, \rho_n) \subset \Psi(\mathcal{B}_n)$, where

$0 < \lambda_n < 1$ and $\mathfrak{k}_0 > 0$, we have $\tau_n \to 0$ as $n \to \infty$ this lead us to choose λ_n and ρ_n such that $\lambda_n, \rho_n \to 0$ as $n \to \infty$. We get $\mathfrak{W}_1 \subset int(\Psi(\mathfrak{P}))$, take $\hbar \in \mathfrak{W}_1$ then $\hbar \in \Psi(\mathcal{B}_1)$, we now form a ball centred at $\hbar$, $\mathcal{B}_\hbar(\lambda_2, \rho_2)$ such that $\mathcal{B}_\hbar(\lambda_2, \rho_2) \cap \Psi(\mathcal{B}_1) \neq \emptyset$, there there exist $\mathfrak{d}_1 \in \mathcal{B}_1$ such that $\Psi(\mathfrak{d}_1) \in \Psi(\mathcal{B}_1)$.

$$\eta_1(\hbar - \Psi(\mathfrak{d}_1)), \rho_2) > 1-\lambda_2,\ \upsilon_1(\hbar - \Psi(\mathfrak{d}_1)), \rho_2) < \lambda_2 \text{ and } \varsigma_1(\hbar - \Psi(\mathfrak{d}_1)), \rho_2) < \lambda_2 \tag{3.2.2}$$

$\Rightarrow \mathscr{k} - \Psi(\mathfrak{d}_1) \in \mathfrak{W}_2$ and $\mathfrak{W}_2 \subset \Psi(\mathcal{B}_2)$ then there exists $\mathfrak{d}_2 \in \mathcal{B}_2$ such that $\Psi(\mathfrak{d}_2) \in \Psi(\mathcal{B}_2)$ and $\eta_1(\mathscr{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2)), \rho_3) > 1 - \lambda_3$, $\upsilon_1(\mathscr{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2)), \rho_3) < \lambda_3$ and

$$\varsigma_1(\mathscr{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2)), \rho_3) < \lambda_3. \quad (3.2.3)$$

$\Rightarrow \mathscr{k} - \Psi(\mathfrak{d}_1) - \Psi(\mathfrak{d}_2) \in \mathfrak{W}_3$, keeping up this procedure, we get a sequence $(\mathfrak{d}_n)$ such that $\mathfrak{d}_n \in \mathcal{B}_n$ and $\eta_1(\mathscr{k} - \sum_{i=}^{n-1} \Psi(\mathfrak{d}_i), \rho_n) > 1 - \lambda_n$, $\upsilon_1(\mathscr{k} - \sum_{i=}^{n-1} \Psi(\mathfrak{d}_i), \rho_n) < \lambda_n$ and

$$\varsigma_1(\mathscr{k} - \sum_{i=}^{n-1} \Psi(\mathfrak{d}_i), \rho_n) < \lambda_n. \quad (3.2.4)$$

Now we demonstrate that $(\mathfrak{G}_n)$ to be a Cauchy sequence, where $\mathfrak{G}_n = \sum_{i=1}^{n} \mathfrak{d}_i$. When $n \to \infty$, $\tau_n \to 0$, this implies that there exist some $n_0 \in \mathbb{N}$ such that o< $\tau_n < \mathfrak{k}'$ for all $n \geq n_0$ where $\mathfrak{k}' = \min\{\mathfrak{k}_1, \mathfrak{k}_2, \dots \mathfrak{k}_i\}$. Let $\mathfrak{p} > \mathfrak{q} > n_0$ and then

$$\eta(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) = \eta(\textstyle\sum_{j=\mathfrak{q}+1}^{\mathfrak{q}+i} \mathfrak{d}_j, \mathfrak{k}) \geq \eta(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) > 1 - \mathrm{e} \quad (3.2.5)$$

$$\upsilon(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) = \upsilon(\textstyle\sum_{j=\mathfrak{q}+1}^{\mathfrak{q}+i} \mathfrak{d}_j, \mathfrak{k}) \leq \upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) < \mathrm{e} \quad (3.2.6)$$

$$\varsigma(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) = \varsigma(\textstyle\sum_{j=\mathfrak{q}+1}^{\mathfrak{q}+i} \mathfrak{d}_j, \mathfrak{k}) \leq \varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) < \mathrm{e}. \quad (3.2.7)$$

Since, we had (see remark (2.4) and lemma (2.7))

$$\begin{aligned}\eta(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) &\geq \eta(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}') \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}') \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}') \\ &\geq \eta(\mathfrak{d}_{\mathfrak{q}+1}, \tau_{\mathfrak{q}+1}) \star \eta(\mathfrak{d}_{\mathfrak{q}+2}, \tau_{\mathfrak{q}+2}) \star \dots \star \eta(\mathfrak{d}_{\mathfrak{q}+i}, \tau_{\mathfrak{q}+i}) \\ &> (1 - \mathrm{e}_{\mathfrak{q}+1}) \star (1 - \mathrm{e}_{\mathfrak{q}+1}) \star \dots \star (1 - \mathrm{e}_{\mathfrak{q}+i}) > 1 - \mathrm{e} \quad (3.2.8)\end{aligned}$$

$$\begin{aligned}\upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) &\leq \upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}') \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}') \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}') \\ &\leq \upsilon(\mathfrak{d}_{\mathfrak{q}+1}, \tau_{\mathfrak{q}+1}) \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+2}, \tau_{\mathfrak{q}+2}) \diamond \dots \diamond \upsilon(\mathfrak{d}_{\mathfrak{q}+i}, \tau_{\mathfrak{q}+i}) \\ &< \mathrm{e}_{\mathfrak{q}+1} \diamond \mathrm{e}_{\mathfrak{q}+1} \diamond \dots \diamond \mathrm{e}_{\mathfrak{q}+i} < \mathrm{e} \quad (3.2.9)\end{aligned}$$

$$\begin{aligned}\varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}_1) \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}_2) \diamond \dots \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+i}, \mathfrak{k}_i) &\leq \varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \mathfrak{k}') \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \mathfrak{k}') \diamond \dots \diamond \varsigma(\eta, \mathfrak{k}') \\ &\leq \varsigma(\mathfrak{d}_{\mathfrak{q}+1}, \tau_{\mathfrak{q}+1}) \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+2}, \tau_{\mathfrak{q}+2}) \diamond \dots \diamond \varsigma(\mathfrak{d}_{\mathfrak{q}+i}, \tau_{\mathfrak{q}+i}) \\ &< \mathrm{e}_{\mathfrak{q}+1} \diamond \mathrm{e}_{\mathfrak{q}+1} \diamond \dots \diamond \mathrm{e}_{\mathfrak{q}+i} < \mathrm{e}. \quad (3.2.10)\end{aligned}$$

Thus, from equations (3.2.5), (3.2.6) and (3.2.7) we obtain that

$\eta(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) \to 1, \upsilon(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) \to 0$ and $\varsigma(\mathfrak{G}_n - \mathfrak{G}_m, \mathfrak{k}) \to 0$ for every $\mathfrak{k} > 0$, hence $(\mathfrak{G}_n)$ is a Cauchy sequence. Let $(\mathfrak{G}_n)$ converges to $\mathfrak{d} \in \widetilde{\mathcal{N}}$, since $\widetilde{\mathcal{N}}$ is Banach space, which implies

$\mathfrak{d} = \sum_{j\geq 1} \mathfrak{d}_j$, now $\rho_n \to 0$, so for a fix $\rho > 0$ we can find n_0 such that $\rho > \rho_n$ for $n > n_0$, it follows

$$\eta_1(\mathscr{k} - \Psi(\mathfrak{G}_{n-1})), \rho) > \eta_1(\mathscr{k} - \Psi(\mathfrak{G}_{n-1})), \rho_n) > 1 - \lambda_n \quad (3.2.11)$$

$$\upsilon_1(\mathscr{k} - \Psi(\mathfrak{G}_{n-1})), \rho) < \upsilon_1(\mathscr{k} - \Psi(\mathfrak{G}_{n-1})), \rho_n) < \lambda_n \quad (3.2.12)$$

$$\varsigma_1(\mathscr{k} - \Psi(\mathfrak{G}_{n-1})), \rho) < \varsigma_1(\mathscr{k} - \Psi(\mathfrak{G}_{n-1})), \rho_n) < \lambda_n. \quad (3.2.13)$$

$\Longrightarrow \mathscr{k} = \Psi(\mathfrak{G}_{n-1})$ as $n \to \infty$ or $\mathscr{k} = \Psi(\sum_{i=}^{n-1} \mathfrak{d}_i)$ and $n \to \infty$, but it is known that $\mathfrak{d} = \lim_{n\to\infty} \sum_{i=}^{n-1} \mathfrak{d}_i$ was in $\mathcal{B}_0(\delta, \mathfrak{k}_0)$. Because $\eta(\mathfrak{d}, \mathfrak{k}_0) \geq \lim_{n\to\infty} \eta(\sum_{i=}^{n} \mathfrak{d}_i, \mathfrak{k}_0) \geq \lim_{n\to\infty} [\eta(\mathfrak{d}_1, \tau_1) \star \eta(\mathfrak{d}_2, \tau_2) \star \dots \star \eta(\mathfrak{d}_n, \tau_n)$

$$> \lim_{n\to\infty} (1 - \mathrm{e}_1) \star (1 - \mathrm{e}_2) \star \dots \star (1 - \mathrm{e}_n) > 1 - \delta \quad (3.2.14)$$

$$\begin{aligned}\upsilon(\mathfrak{d}, \mathfrak{k}_0) &\leq \lim_{n\to\infty} \upsilon(\sum_{i=}^{n} \mathfrak{d}_i, \mathfrak{k}_0) \leq \lim_{n\to\infty} [\upsilon(\mathfrak{d}_1, \tau_1) \diamond \upsilon(\mathfrak{d}_2, \tau_2) \diamond \dots \diamond \upsilon(\mathfrak{d}_n, \tau_n) \\ &< \lim_{n\to\infty} \mathrm{e}_1 \diamond \mathrm{e}_2 \diamond \dots \diamond \mathrm{e}_n < \delta \quad (3.2.15)\end{aligned}$$

$\in$

$$\begin{aligned}\varsigma(\mathfrak{d}, \mathfrak{k}_0) &\leq \lim_{n\to\infty} \varsigma(\sum_{i=}^{n} \mathfrak{d}_i, \mathfrak{k}_0) \leq \lim_{n\to\infty} [\varsigma(\mathfrak{d}_1, \tau_1) \diamond \varsigma(\mathfrak{d}_2, \tau_2) \diamond \dots \diamond \varsigma(\mathfrak{d}_n, \tau_n) \\ &< \lim_{n\to\infty} \mathrm{e}_1 \diamond \mathrm{e}_2 \diamond \dots \diamond \mathrm{e}_n < \delta. \quad (3.2.16)\end{aligned}$$

Since, $\mathfrak{k}_0 = \sum_{n=1}^{\infty} \tau_n = \sum_{n=1}^{\infty} \frac{\mathfrak{k}_0}{2^n} = \mathfrak{k}_0 \sum_{n=1}^{\infty} \frac{1}{2^n}$. Thus it is shown that $\Psi -$ image of $\mathcal{B}_0(\delta, \mathfrak{k}_0)$ contains a neighbourhood of $0 = \Psi(0) \in \widetilde{\mathcal{M}}$.

Step-III We will demonstrate that $\Psi(\mathcal{F})$ is open in $\widetilde{\mathcal{M}}$ if $\mathcal{F}$ is open in $\widetilde{\mathcal{N}}$. Let $\mathscr{k} = \Psi(\mathfrak{d})$ this implies $\mathfrak{d} \in \mathcal{F}$, since $\mathcal{F}$ is open, therefore $\mathcal{F}$ contains $\mathcal{B}_{\mathfrak{d}}(\mathfrak{r}, \mathfrak{k})$, a neighbourhood of $\mathfrak{d}$. We have already proved that if $\mathfrak{d}_0 \in \mathcal{B}_0(\delta, \mathfrak{k}_0) \subset \mathfrak{P}$ then $\Psi(\mathfrak{d}_0) \in \mathfrak{W}_1 \subseteq int\Psi(\mathfrak{P})$. Hence if $\mathfrak{d} \in \mathcal{F}$and $\mathcal{B}_{\mathfrak{d}}(\mathfrak{r}, \mathfrak{k}) \subset \mathcal{F}$ then $\Psi(\mathfrak{d}) \in int(\Psi(\mathcal{F}))$. Consequently $\Psi(\mathcal{F})$ is open.

Example 3.3: Let $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}} = \mathbb{R}$, and $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ $(\widetilde{\mathcal{M}}, \eta_1, \upsilon_1, \varsigma_1, \star, \diamond)$ be Neutrosophic Banach spaces. Also $\eta, \upsilon, \varsigma$ are defined by $\eta(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{|\mathfrak{d}|+\mathfrak{k}'}$, $\upsilon(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{k}|}{|\mathfrak{d}|+\mathfrak{k}}$ and $\varsigma(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{\mathfrak{k}}$.

Similarly for $\widetilde{\mathcal{M}}$ the norms are by $\eta_1(\mathfrak{d},\mathfrak{k}) = \frac{|\mathfrak{d}|}{|\mathfrak{d}|+\mathfrak{k}}$, $\upsilon_1(\mathfrak{d},\mathfrak{k}) = \frac{|\mathfrak{k}|}{|\mathfrak{d}|+\mathfrak{k}}$ and $\varsigma_1(\mathfrak{d},\mathfrak{k}) = \frac{|\mathfrak{d}|}{\mathfrak{k}}$.

The continuous linear operator Ψ from$(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ onto $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$ defined by $\Psi(\mathfrak{d}) = \frac{\mathfrak{d}}{2}$. Then Ψ is an open mapping

Proof.

Let $\mathfrak{P}$ be an open set in $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$, the image of $\mathfrak{P}$ under Ψ is $\Psi(\mathfrak{P}) = \{\frac{\mathfrak{d}}{2}/\mathfrak{d} \in \mathfrak{P}\}$.

To Prove Ψ is an open map, it is enough to show that $\Psi(\mathfrak{P})$ is open.

Let $p \in \Psi(\mathfrak{P})$. Therefore, there exists $\mathfrak{d} \in \mathfrak{P}$ such that $p = \Psi(\mathfrak{d}) = \frac{\mathfrak{d}}{2}$.

Since $\mathfrak{d} \in \mathfrak{P}$ and $\mathfrak{P}$ is open, therefore there exist $0< \mathfrak{r} < 1$ such that open ball centered at $\mathfrak{d}$, with respect to the norm η,υ,ς, $\mathcal{B}_{\eta,\upsilon,\varsigma}(\mathfrak{d},\mathfrak{r})$ (say) is contained in $\mathfrak{P}$.

$\mathcal{B}_{\eta,\upsilon,\varsigma}(\mathfrak{d},\mathfrak{r}) \subset \mathfrak{P}$.

Since, $\Psi(\mathcal{B}_{\eta}(\mathfrak{d},\mathfrak{r})) = \mathcal{B}_{\eta_1,\upsilon_1,\varsigma_1}\left(\frac{\mathfrak{d}}{2},\frac{\mathfrak{r}}{2}\right)$ and $\mathcal{B}_{\eta}(\mathfrak{d},\mathfrak{r}) \subset \mathfrak{P}$

$\Longrightarrow \Psi(\mathcal{B}_{\eta,\upsilon,\varsigma}(\mathfrak{d},\mathfrak{r})) \subset \Psi(\mathfrak{P})$

$\Longrightarrow \mathcal{B}_{\eta_1,\upsilon_1,\varsigma_1}\left(\frac{\mathfrak{d}}{2},\frac{\mathfrak{r}}{2}\right) \subset \Psi(\mathfrak{P})$

$\Longrightarrow \mathcal{B}_{\eta_1,\upsilon_1,\varsigma_1}\left(p,\frac{\mathfrak{r}}{2}\right) \subset \Psi(\mathfrak{P})$.

Therefore there exists a open ball centered at p is contained in $\Psi(\mathfrak{P})$.

Since p belongs to $\Psi(\mathfrak{P})$ is arbitrary, there every point of $\Psi(\mathfrak{P})$ in an interior point.

Hence $\Psi(\mathfrak{P})$ is open. Therefore Ψ is an open mapping.

Theorem 3.4: Let $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ and $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$ be Neutrosophic Banach space and Ψ be a linear oprator from $(\widetilde{\mathcal{N}},\eta,\upsilon,\varsigma,\star,\diamond)$ to $(\widetilde{\mathcal{M}},\eta_1,\upsilon_1,\varsigma_1,\star,\diamond)$. Moreover if for every sequence $(\mathfrak{d}_n)$ of the elements of $\widetilde{\mathcal{N}}$ converges to $\mathfrak{d} \in \widetilde{\mathcal{N}}$, the sequence $\Psi(\mathfrak{d}_n)$ converges to $\mathscr{k} \in \widetilde{\mathcal{M}}$ with the property $\mathscr{k} = \Psi(\mathfrak{d})$ then Ψ is continuous

Proof: We firstly define Neutrosophic norm (γ,ϑ,β) on $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ defined as

$\gamma(((\mathfrak{d},\mathscr{k}),\mathfrak{k}) = \gamma(\mathfrak{d},\mathfrak{k}) \star \gamma_1(\mathscr{k},\mathfrak{k})$

$\vartheta(((\mathfrak{d},\mathscr{k}),\mathfrak{k}) = \vartheta(\mathfrak{d},\mathfrak{k}) \diamond \vartheta_1(\mathscr{k},\mathfrak{k})$ and

$\beta(((\mathfrak{d},\mathscr{k}),\mathfrak{k}) = \beta(\mathfrak{d},\mathfrak{k}) \diamond \beta(\mathscr{k},\mathfrak{k})$. Let $(\mathfrak{d}_n)$ be a Cauchy sequence in $\widetilde{\mathcal{N}}$, then we prove that $\Psi(\mathfrak{d}_n)$ will be Cauchy in $\widetilde{\mathcal{M}}$, for this, it is enough to show that $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ is complete with respect to Neutrosophic norms (γ,ϑ,β). Let $(\mathfrak{d}_n,\mathscr{k}_n) \in \widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ be a Cauchy, then for all $0 < \mathfrak{r} < 1$ and $\mathfrak{k} > 0$ there exists a $n_0 \in \mathbb{N}$ such that $\gamma((\mathfrak{d}_n,\mathscr{k}_n) - (\mathfrak{d}_m,\mathscr{k}_m),\mathfrak{k}) > 1 - \mathfrak{r}$, $\vartheta((\mathfrak{d}_n,\mathscr{k}_n) - (\mathfrak{d}_m,\mathscr{k}_m),\mathfrak{k}) < \mathfrak{r}$ and $\beta((\mathfrak{d}_n,\mathscr{k}_n) - (\mathfrak{d}_m,\mathscr{k}_m),\mathfrak{k}) < \mathfrak{r}$ for all $n,m > n_0$.

$$\begin{aligned}\gamma((\mathfrak{d}_n,\mathscr{k}_n) - (\mathfrak{d}_m,\mathscr{k}_m),\mathfrak{k}) &= \gamma((\mathfrak{d}_n - \mathfrak{d}_m) - (\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) \\ &= \gamma((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) \star \gamma((\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) > 1 - \mathfrak{r}, \qquad (3.4.1)\end{aligned}$$

$\Rightarrow \gamma((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) > 1 - \mathfrak{r}_1$ and $\gamma((\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) > 1 - \mathfrak{r}_2$.

$$\begin{aligned}\vartheta((\mathfrak{d}_n,\mathscr{k}_n) - (\mathfrak{d}_m,\mathscr{k}_m),\mathfrak{k}) &= \vartheta((\mathfrak{d}_n - \mathfrak{d}_m) - (\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) \\ &= \vartheta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) \diamond \vartheta((\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) < \mathfrak{r} \qquad (3.4.2)\end{aligned}$$

$\Rightarrow \vartheta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) < \mathfrak{r}_1$ and $\vartheta((\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) < \mathfrak{r}_2$.

$$\begin{aligned}\beta((\mathfrak{d}_n,\mathscr{k}_n) - (\mathfrak{d}_m,\mathscr{k}_m),\mathfrak{k}) &= \beta((\mathfrak{d}_n - \mathfrak{d}_m) - (\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) \\ &= \beta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) \diamond \beta((\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) < \mathfrak{r}. \qquad (3.4.3)\end{aligned}$$

$\Rightarrow \beta((\mathfrak{d}_n - \mathfrak{d}_m),\mathfrak{k}) < \mathfrak{r}_1$ and $\beta((\mathscr{k}_n - \mathscr{k}_m),\mathfrak{k}) < \mathfrak{r}_2$ for every $0 < \mathfrak{r} < 1$, we can find $0 < \mathfrak{r}_1 < 1$ and $0 < \mathfrak{r}_2 < 1$, such that $(1 - \mathfrak{r}_1) \star (1 - \mathfrak{r}_2) > 1 - \mathfrak{r}$ and $\mathfrak{r}_1 \diamond \mathfrak{r}_2 < \mathfrak{r}$. Consequently, for equations (3.3.1) (3.32) and (3.3.3) we get that $(\mathfrak{d}_n)$ converges at $\mathfrak{d} \in \widetilde{\mathcal{N}}$ and $(\mathscr{k}_n)$ converges at $\mathscr{k} \in \widetilde{\mathcal{M}}$, this implies that $(\mathfrak{d}_n,\mathscr{k}_n)$ converges at $(\mathfrak{d},\mathscr{k}) \in \widetilde{\mathcal{N}} \times \widetilde{\mathcal{M}}$ which establish the property that the sequence $\Psi(\mathfrak{d}_n)$ is Cauchy in $\widetilde{\mathcal{M}}$ whenever $(\mathfrak{d}_n)$ is a Cauchy sequence in $\widetilde{\mathcal{N}}$. Hence Ψ is continuous.

Remark 3.5 [15] Every open ball is an open set in neutrosophic normed space.

Example 3.6: Let $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}} = \mathbb{R}$, and $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ $(\widetilde{\mathcal{M}}, \eta, \upsilon, \varsigma, \star, \diamond)$ be Neutrosophic Banach spaces. Also $\eta, \upsilon, \varsigma$ are defined by $\eta(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{|\mathfrak{d}|+\mathfrak{k}}$, $\upsilon(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{k}|}{|\mathfrak{d}|+\mathfrak{k}}$ and $\varsigma(\mathfrak{d}, \mathfrak{k}) = \frac{|\mathfrak{d}|}{\mathfrak{k}}$ The continuous linear oprator Ψ from $(\widetilde{\mathcal{N}}, \eta, \upsilon, \varsigma, \star, \diamond)$ onto $(\widetilde{\mathcal{M}}, \eta_1, \upsilon_1, \varsigma_1, \star, \diamond)$ defined by $\Psi(\mathfrak{d}) = \frac{\mathfrak{d}}{8}$. If $(\mathfrak{d}_n)$ convergesat $\mathfrak{d} \in \widetilde{\mathcal{N}}$ imples the sequence $\Psi(\mathfrak{d}_n)$ converges to $\mathcal{k} \in \widetilde{\mathcal{M}}$ with the property $\mathcal{k} = \Psi(\mathfrak{d})$. then Ψ is continuous.

Theorem 3.7: Let $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$ are two Neutrosophic normed linear spaces. If $\Psi_1, \Psi_2 : \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M}}$ are two linear operators, Ψ_1 is closed and Ψ_2 is bounded then $\Psi_1 + \Psi_2$ is closed with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$.

Proof: Let $(\mathfrak{d}_n)$ be a sequence in $\widetilde{\mathcal{N}}$ such that $\mathfrak{d}_n \longrightarrow \mathfrak{d}$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$, i.e. for every $0 < \delta < 1$ and $\mathfrak{k} > 0$ there exists $n_0 \in \mathbb{N}$ such that fir all $n \geq n_0$ we have $\eta(\mathfrak{d}_n) - \mathfrak{d}, \mathfrak{k}) > 1 - \delta, \upsilon(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) < \delta$ and $\varsigma(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) < \delta$. (3.7.1)

Now, by hypothesis $\Psi_1(\mathfrak{d}_n) \longrightarrow \Psi(\mathfrak{d})$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$ and $\mathfrak{d} \in \widetilde{\mathcal{N}}$.

Therefore, for every $0 < \lambda < 1$ and $\mathfrak{k} > 0$ there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$

$\eta(\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \mathfrak{k}) > 1 - \lambda, \upsilon(\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \mathfrak{k}) < \lambda$ and $\varsigma(\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \mathfrak{k}) < \lambda$. (3.7.2)

Futhermore, Ψ_2 is bounded therefore there exists $K > 0$, such that $\|\Psi_2\| \leq K$. Now we prove that $(\Psi_1 + \Psi_2)(\mathfrak{d}_n) \longrightarrow (\Psi_1 + \Psi_2)(\mathfrak{d})$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$ and $\mathfrak{d} \in \widetilde{\mathcal{N}}$. Let $n_{\mathfrak{r}} = \max\{n_0, n_1\}$ and for every $0 < \lambda, \delta < 1$ there exists $0 < \mathfrak{r} < 1$ such that

$(1-\lambda) \star (1 - \delta) > 1 - \mathfrak{r}$ and $\lambda \diamond \delta < \mathfrak{r}$, then for every $n \geq n_{\mathfrak{r}}$ we get

$$\begin{aligned}
\eta((\Psi_1 + \Psi_2)(\mathfrak{d}_n) - (\Psi_1 + \Psi_2)(\mathfrak{d}), \mathfrak{k}) &= \eta((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}) + \Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \mathfrak{k}) \\
&\geq \eta((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \star \eta((\Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \\
&\geq (1 - \lambda) \star \eta(\mathfrak{d}_n - \mathfrak{d}, \tfrac{\mathfrak{k}}{2\|\Psi_2\|}) \geq (1 - \lambda) \star \eta(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) \\
&> (1 - \lambda) \star (1 - \delta) > 1 - \mathfrak{r}, \qquad (3.7.3)
\end{aligned}$$

$$\begin{aligned}
\upsilon((\Psi_1 + \Psi_2)(\mathfrak{d}_n) - (\Psi_1 + \Psi_2)(\mathfrak{d}), \mathfrak{k}) &= \upsilon((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}) + \Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \mathfrak{k}) \\
&\leq \upsilon((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \diamond \upsilon((\Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \\
&\leq (1 - \lambda) \diamond \upsilon(\mathfrak{d}_n - \mathfrak{d}, \tfrac{\mathfrak{k}}{2\|\Psi_2\|}) \leq (1 - \lambda) \diamond \upsilon(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) \\
&< \lambda \diamond \delta < \mathfrak{r} \qquad (3.7.4)
\end{aligned}$$

$$\begin{aligned}
\varsigma((\Psi_1 + \Psi_2)(\mathfrak{d}_n) - (\Psi_1 + \Psi_2)(\mathfrak{d}), \mathfrak{k}) &= \varsigma((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}) + \Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \mathfrak{k}) \\
&\leq \varsigma((\Psi_1(\mathfrak{d}_n) - \Psi_1(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \diamond \varsigma((\Psi_2(\mathfrak{d}_n) - \Psi_2(\mathfrak{d}), \tfrac{\mathfrak{k}}{2}) \\
&\leq (1 - \lambda) \diamond \varsigma(\mathfrak{d}_n - \mathfrak{d}, \tfrac{\mathfrak{k}}{2\|\Psi_2\|}) \leq (1 - \lambda) \diamond \varsigma(\mathfrak{d}_n - \mathfrak{d}, \mathfrak{k}) \\
&< \lambda \diamond \delta < \mathfrak{r}. \qquad (3.7.5)
\end{aligned}$$

We use $\tau = \frac{\mathfrak{k}}{2K}$ in the above equations. Now equations (3.7.3), (3.7.4) and (3.7.5) simultaneously conclude that $(\Psi_1 + \Psi_2)(\mathfrak{d}_n) \longrightarrow (\Psi_1 + \Psi_2)(\mathfrak{d})$ with respect to neutrosophic norm $(\eta, \upsilon, \varsigma)$. Also it should to noted that Ψ_1 is closed, then we obtain $\mathfrak{d} \in \widetilde{\mathcal{N}}$, by the definition (2.12).

Conclusion

In this paper, we have developed open mapping and closed graph theorem in neutrosophic Banach space and we have presented some suitable examples that support our main results. We hope that the result proved in this paper will form new connection for those who are working in the in neutrosophic Banach space and this work opens a new path for researchers in the concerned field.

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New multi-criteria decision making technique based on neutrosophic QUALIFLEX method

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Abstract: The field of Multicriteria Decision Making (MCDM) has advanced significantly, with methods like QUALIFLEX playing a key role by integrating concordance and discordance measures for robust decision outcomes. This paper introduces an innovative extension, neutrosophic QUALIFLEX, which incorporates neutrosophic sets to better handle uncertainty and indeterminacy. Neutrosophic logic manages truth, indeterminacy, and falsity simultaneously, offering a nuanced approach compared to traditional models. The scope of this article is the application of neutrosophic QUALIFLEX to staff selection in business, a scenario often plagued by high uncertainty. Preliminary results show that neutrosophic QUALIFLEX provides superior performance, delivering more accurate and reliable rankings of candidates. This method addresses the limitations of conventional MCDM approaches, improving decision-making accuracy and expanding the theoretical foundations of MCDM for future research and applications in complex environments. This contribution to the literature bridges a critical gap by addressing the limitations of conventional MCDM methods in dealing with ambiguous and indeterminate information. The integration of neutrosophic sets into the QUALIFLEX framework not only improves decision-making accuracy but also expands the theoretical foundations of MCDM, paving the way for future research and application in complex decision environments.

Keywords: Multi-criteria decision-making method; net concordance/discordance indices; QUALIFLEX method; neutrosophic logic; neutrosophic QUALIFLEX method.

1. Introduction

The limitations of current models in addressing multi-dimensional real business problems using a single criterion have led to the development of multi-criteria decision making. Multiple-Criteria Decision Making (MCDM) is widely recognized as one of the most significant scientific approaches utilized by experts. When a decision maker analyzes more than one characteristic, the concept of MCDM is introduced, which plays a significant role in everyday decisions in organizations and human societies.

Practically, MCDM is used to deal with structuring, decision-making, and planning steps when the domain possesses manifold criteria to reach an optimum solution based on the deciders' preferences. As a result, decisions made fall into the categories of unstructured or semi-structured decisions, necessitating support for the decision-maker through the development of suitable multi-criteria models.

Outranking methods are a vital subset of MCDM techniques, designed to handle complex decision-making scenarios involving multiple, often conflicting criteria. They are used to evaluate and rank alternatives based on multiple criteria through pairwise comparisons. Unlike traditional aggregation methods, outranking methods assess the degree to which one alternative is preferred over another by analyzing concordance (agreement) and discordance (disagreement) among the criteria. This approach allows for a more nuanced assessment, accommodating thresholds and indifference zones to capture significant differences in performance. These techniques are especially useful in complicated decision-making scenarios that involve subjective judgements and qualitative criteria, since they provide a formal framework for efficiently managing ambiguity and competing objectives.

One of the most prominent outranking methods is QUALIFLEX (Qualitative Flexible Multiple Criteria Method), which is a MCDM technique that ranks alternatives based on qualitative and quantitative criteria by evaluating the ordinal rankings of alternatives for each criterion [1-4]. Unlike traditional aggregation methods, QUALIFLEX considers all possible permutations of alternatives and assesses each permutation through concordance and discordance measures, which quantify the agreement and disagreement among the criteria rankings. A global concordance index is calculated for each permutation, representing the overall support from the criteria. The permutation with the highest global concordance index is selected as the optimal ranking, providing a comprehensive ordering of alternatives that aligns best with the individual criterion rankings. Quite recently, several extensions have been developed to enhance the QUALIFLEX method [5-6].

One of the primary disadvantages of QUALIFLEX lies in its deterministic nature and sensitivity to precise rankings of alternatives for each criterion. This approach can be ill-suited for handling situations where data is uncertain, imprecise, or subject to ambiguity. Furthermore, QUALIFLEX lacks inherent mechanisms for probabilistic analysis, making it challenging to integrate uncertainty quantification into the decision-making process. As a result, the method may struggle to accurately reflect preferences and provide reliable rankings when faced with indeterminate information or dynamic changes in criteria evaluations.

In many cases, it is in fact difficult for decision-makers to definitely express their preference in solving MCDM problems with inaccurate, uncertain, or incomplete information. The primary goal of decision-makers is to handle uncertainties, particularly in ambiguous circumstances when the outcome is not simply true or untrue. As a result, new approaches for finding effective answers are emerging, including fuzzy logic, intuitionistic fuzzy logic, interval-valued fuzzy, and, more recently, neutrosophic logic. In this context, the first approximate method that was proposed was fuzzy logic theory [7]. Its objective is to deal with the concept of partial truth, where the truth value might be either true or false.

Smarandache began using non-standard analysis using a tri-component logic/set/probability theory in 1995, inspired by philosophical considerations. As a result, he created the neutrosophic logic theory, arguing that fuzzy logic cannot demonstrate indeterminacy on its own [8]. Neutrosophic logic proposes three functions, truth-membership, indeterminacy-membership, and falsity-membership, to address constraints in classical logic when confronted with incomplete, imprecise, or contradictory data. This innovative paradigm acknowledges not only true and false values but also a third domain of indeterminacy, where items can possess both truth and falsity characteristics simultaneously. The ability of neutrosophic logic to capture and formalize this inherent complexity makes it a crucial tool in various domains, including artificial intelligence, decision sciences, engineering, and philosophy.

The NL overcomes the constraints of fuzzy logic (FL) and intuitionistic fuzzy logic (IFL) by taking into account truth, intermediacy, and falsity membership degrees, as well as its capacity to discriminate between relative truth and absolute truth, and relative falsity and absolute falsity. As a result, multiple studies were inspired to propose several MCDM techniques under neutrosophic conditions [9-14].

In the field of staff selection, the related bibliography shows a plethora of MCDM methods as a valuable tool for addressing the issue of selecting appropriate personnel. Many researchers suggest utilizing decision support system tools in the personnel selection procedure to improve the judgments of decision-makers. Scholars in [15] apply MCDM methods for staff selection, while an aggregating function is used in [16]. The Analytic Hierarchy Process (AHP) technique divides the problem into a top-down hierarchical structure to improve decision-makers' judgments [17]. Fuzzy methods are provided to enhance decision-makers' decisions during the personnel selection process due to vague and imprecise information [18]. In the field of neutrosophic logic, scholars in [19] present a definition of neutrosophic parameterized (NP) soft set and its operations, applying their method to an illustrative example of a staff selection problem. Recently, authors in [13] studied a real case study of academic staff selection and proposed an innovative conceptual framework based on neutrosophic Delphi (N-Delphi) and neutrosophic Analytic Hierarchy Process (N-AHP).

Previous research on neutrosophic MCDM techniques in staff selection has not included the use of neutrsophic QUALIFLEX principles for selecting appropriate employees, motivating us to present this study. This study introduces the neutrosophic QUALIFLEX method, integrating neutrosophic logic with QUALIFLEX to better handle uncertainty and indeterminacy. Applied to staff selection, this method demonstrates superior performance in delivering accurate and reliable rankings compared to traditional MCDM approaches.

The main objectives of our current research are the following:

- Filling the observed research gap by introducing a novel multi-criteria decision-making (MCDM) method, termed neutrosophic QUALIFLEX, which integrates neutrosophic sets to better handle uncertainty and indeterminacy in decision-making processes?
- Application of the neutrosophic QUALIFLEX method to the problem of staff selection in business, showcasing its practical utility in a real-world decision-making scenario characterized by high levels of uncertainty.
- Contribution to the theoretical foundations of MCDM by integrating neutrosophic sets into the QUALIFLEX framework, addressing the limitations of conventional methods in handling ambiguous and indeterminate information.

The remainder of this article is organized as follows: Section 2 outlines basic definitions regarding neutrosophic logic and the methods involved in developing our methodology while section 3 demonstrates the practical utility of the proposed method by applying it to a real-world problem of staff selection in business, providing a detailed step-by-step implementation. Section 4 interprets the findings, discussing the advantages of the proposed method. Finally, section 5 summarizes the key contributions of the paper and suggests directions for future research to further explore and apply the neutrosophic QUALIFLEX method in various complex decision environments.

2. Materials and Methods

In this part, we will briefly discuss the approaches used in our integrated methodology. We will then present a brief outline of the theoretical considerations used to design our technique and ensure its robustness. Finally, we will define the notation utilized in our study, as well as discuss the methodologies employed in our hybrid multi-criteria decision-making analysis for staff recruitment.

2.1 QUALIFLEX method

Paelinck proposed the QUALIFLEX approach in 1975 [1-4], and it is based on Jacquet-Lagreze's permutation method [20]. QUALIFLEX evaluates each conceivable ranking of the current m alternatives. In other words, the ranking of alternatives is compared to the number of $m!$ permutations and only the most relevant ones are chosen for the final ranking. It involves determining the degree of agreement (concordance) and disagreement (discordance) among criteria to form an overall preference index for each pair of alternatives. By combining these indices, QUALIFLEX constructs a comprehensive ranking from the most to least preferred alternatives, accommodating both qualitative and quantitative data.

The main features of the above technique are [4]:

- Evaluates agreement (concordance) and disagreement (discordance) among criteria.
- Compares pairs of alternatives across all criteria to establish preference.
- Combines concordance and discordance indices into an overall preference index.
- Generates a complete ranking from most to least preferred alternatives.
- Accommodates both qualitative and quantitative data.

The QUALIFLEX technique, while effective for multi-criteria decision-making, has drawbacks, particularly in terms of indeterminacy. It has significant degrees of indeterminacy and uses binary criteria, which may not accurately reflect real-world uncertainty. Subjectivity in appraising metrics, as well as difficulties in measuring uncertainty, provides additional complications. Furthermore, its strict reliance on outranking relations and computational complexity may impede adaptation and practical implementation, thus restricting its usefulness in dynamic decision-making situations.

2.2 Neutrosophic logic

Neutrosophic Logic (NL) is an extension of classical and fuzzy logic, introduced by Smarandache in the late 20th century. It provides a framework for dealing with indeterminate, imprecise, and inconsistent information by incorporating a third truth value called "indeterminacy." In neutrosophic logic, a concept A is T% true, I% indeterminate, and F% false, with $(T, I, F) \subset]\!]^-0, 1^+[\![^3$, where $]\!]^-0, 1^+[\![$ is an interval of hyperreals.

Definition 1 [21] Let $\mathcal{X}$ be a space of points (objects), with a generic element in $\mathcal{X}$ denoted by x. A single-valued neutrosophic set (SVNS) $\mathcal{A}$ in $\mathcal{X}$ is characterized by truth membership function T_A, indeterminacy membership function I_A, and falsity membership function F_A. For each point x in $\mathcal{X}$, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$.

Then, a simplification of the neutrosophic set $\mathcal{A}$, which is a a subclass of neutrosophic sets, is denoted by $\mathcal{A} = \{\langle x, \mathbf{T(x), I(x), F(x)} \rangle \mid x \in \mathcal{X} \}$ (1)

2.3 Neutrosophic QUALIFLEX method

The proposed neutrosophic QUALIFLEX (n-QUALIFLEX) method is an advanced multi-criteria decision-making (MCDM) technique that integrates the principles of neutrosophic logic with the QUALIFLEX method. Neutrosophic logic, which extends traditional fuzzy logic, allows for the handling of indeterminacy and uncertainty by incorporating truth, indeterminacy, and falsity membership functions [22]. By combining these elements, the n-QUALIFLEX method aims to enhance the flexibility and robustness of decision-making processes, enabling more nuanced and comprehensive evaluations in scenarios where information is imprecise or incomplete.

This subsection explains how to apply the n-QUALIFLEX approach, as outlined below. Our goal was to propose a new methodological framework based on the "traditional" QUALIFLEX method [1-4], expanded with the suggested net concordance and discordance indices, and integrated with neutrosophic logic, as outlined below:

Step 1: Determine the decision aim. This research seeks to evaluate different candidates for staff selection and choose the best that fits the criteria for the desired position.

Step 2: Classify the candidates into three degrees: strong, average and weak, to quantify the initially evaluated degrees of the candidates.

Step 3: Conduct a preliminary evaluation. Experts are invited to examine applicants for the staff selection dilemma. Truth, Indeterminacy, and Falsity describe the degree of acknowledgment of the candidate's performance, ranging from high to poor. The specific neutrosophic value is calculated as follows: Assume that experts chose Truth, Indeterminacy, and Falsity. Then the neutrosophic value of this indicator is <a/t, b/t, c/t>.

Step 4: Construct the neutrosophic decision matrix. For each alternative and each criterion, determine the corresponding neutrosophic number. A neutrosophic number is represented as (T,I,F) where T denotes the degree of truth, I denotes the degree of indeterminacy, and F denotes the degree of falsity.

Step 5: Calculate the concordance and discordance sets.

Step 6: Calculate the concordance and discordance indices.

Step 7: Construct the Outranking Matrix. Using the net concordance and discordance scores, construct an outranking matrix that indicates the preference relations between each pair of alternatives.

Step 8: Determine the final ranking**.** Analyze the outranking matrix to determine the overall ranking of the alternatives. The alternative that outranks the most others is considered the best choice.

3. Results

In this section, the aforementioned methods will be applied. First, follow Step 1 to organize the candidates' evaluation criteria listed in Table 1.

Table 1. Evaluation criteria for a candidate Y_k

Criteria	**Truth (T)**	**Indeterminacy (I)**	**Falsity (F)**
C_1: Personality			
C_2: Intelligence			
C_3: Experience			

According to Step 2, preliminarily classify candidates into three categories: strong, average, and weak. This will help quantify the initial evaluation of each candidate (Table 2).

Table 2. Candidate's degrees with neutrosophic information.

Criteria	**D_1 (Strong)**	**D_2 (Average)**	**D_3 (Weak)**
C_1	(1.0, 0.0, 0.0)	(0.6, 0.4, 0.0)	(0.2, 0.4, 0.4)
C_2	(1.0, 0.0, 0.0)	(0.6, 0.4, 0.0)	(0.2, 0.4, 0.4)
C_3	(1.0, 0.0, 0.0)	(0.6, 0.4, 0.0)	(0.2, 0.4, 0.4)

Next, based on Steps 3 and 4, we provide Table 1 to five experts for parallel preliminary evaluation of three candidates. The evaluation results are shown in Table 3.

Table 3. Experts' evaluation of candidates.

Criteria	**Y_1**			**Y_2**			**Y_3**		
	T	I	F	T	I	F	T	I	F
C_1	5/5	0/5	0/5	3/5	2/5	0/5	3/5	2/5	0/5
C_2	3/5	2/5	0/5	3/5	2/5	0/5	5/5	0/5	0/5
C_3	3/5	2/5	0/5	5/5	0/5	0/5	1/5	2/5	2/5

From Table 3, the indicator degrees of candidate Y_k (k = 1, 2, 3) can be expressed with the following neutrosophic information:

Y_1: {(C_1, 1.0, 0.0, 0.0), (C_2, 0.6, 0.4, 0.0), (C_3, 0.6, 0.4, 0.0)}

Y_2 :{(C_1, 0.6, 0.4, 0.0), (C_2, 0.6, 0.4, 0.0), (C_3, 1.0, 0.0, 0.0)}
Y_3: {(C_1, 0.6, 0.4, 0.0), (C_2, 1.0, 0.0, 0.0), (C_3, 0.2, 0.4, 0.4)}

According to Step 4, assume that the weight of each element C_j is $w_j = 1/3$ for j = 1, 2, 3.

In order to follow Step 5 and 6 we have to define the concordance /discordance indices.

Given the fact that we have three alternatives ψ_1, ψ_2 and $\psi_3 \in$ Y, three criteria C_1, C_2 and C_3 and the evaluation Table as shown in Table 3, there are 3! possible permutations (comprehensive rankings):

Per_1 : $\psi_1 > \psi_2 > \psi_3$
Per_2 : $\psi_2 > \psi_1 > \psi_3$
Per_3 : $\psi_2 > \psi_3 > \psi_1$
Per_4 : $\psi_3 > \psi_2 > \psi_1$
Per_5 : $\psi_3 > \psi_1 > \psi_2$
Per_6 : $\psi_1 > \psi_3 > \psi_2$

One index is computed for each pair (c_j, Per_k), that, for our example, gives a total of 18 concordance/discordance indices. For example for the pair (c_1, Per_1), we have for the criterion c_1: $\psi_1 > \psi_2$, $\psi_2 \approx \psi_3$, $\psi_1 > \psi_3$ according to Equation (6) and for the Per_1: $\psi_1 > \psi_2$, $\psi_1 > \psi_3$, $\psi_2 > \psi_3$.

Given the three alternatives ψ_1, ψ_2, $\psi_3 \in$ Y and three criteria C_1,C_2,C_3 calculate the concordance and discordance indices for each pair of alternatives across all criteria. This involves comparing the neutrosophic evaluations of each alternative.

For each permutation of alternatives, compute the concordance indices (CI) and discordance indices (DI). The indices reflect how much one alternative is preferred over another considering both the degree of truth, indeterminacy, and falsity.

For instance, for the permutation Per_1: $\psi_1 > \psi_2 > \psi_3$

Calculate CI and DI for each pair (ψ_1, ψ_2), (ψ_1, ψ_3) and (ψ_2, ψ_3) across all criteria.

Use the following neutrosophic relations:

1. Concordance: If the truth value of ψ_a is greater than ψ_b and the falsity is lesser.
2. Discordance: If the truth value of ψ_a is lesser than ψ_b or the indeterminacy is higher.

In order to proceed to Step 6, compile the concordance and discordance indices into a neutrosophic outranking matrix. This matrix shows the relative preference of each alternative over the others (Table 4).

Table 4. Neutrosophic outranking matrix.

	Criteria		
Permutation	**C_1**	**C_2**	**C_3**
Per_1	CI, DI	CI, DI	CI, DI
Per_2	CI, DI	CI, DI	CI, DI
Per_3	CI, DI	CI, DI	CI, DI
Per_4	CI, DI	CI, DI	CI, DI
Per_5	CI, DI	CI, DI	CI, DI
Per_6	CI, DI	CI, DI	CI, DI

For each permutation, calculate the net concordance and discordance scores by summing up the individual indices.

Now we can proceed to Step 7 and construct the outranking matrix. The net concordance/discordance scores that are proposed in this study are given below:

Net $CI_{Per_k} = \sum_{j=1}^{n} CI_{j,Per_k}$ (2)

Net $DI_{Per_k} = \sum_{j=1}^{n} DI_{j,Per_k}$ (3)

Determine the ranking of each permutation based on the highest net concordance and the lowest net discordance by utilizing Equations (2) and (3) as in Table 5.

Table 5. Net scores for permutations.

Permutation	Net CI	Net DI
Per_1	2.0	1.6
Per_2	2.0	1.2
Per_3	2.0	1.6
Per_4	2.0	1.6
Per_5	2.0	1.6
Per_6	2.0	1.6

Table 5 shows the net concordance and discordance scores for each permutation, with Per_2 having the highest net concordance and lowest net discordance, indicating its preferred status.

For example, we will explain how the Net CI and Net DI values in Table 5 are calculated for Per_1.

Let's revisit the neutrosophic values from Table 3 that express the neutrosophic information regarding the evaluation of each candidate Yi in relation to each criterion Cj from the panel of experts.

Y_1: {(C_1, 1.0, 0.0, 0.0), (C_2, 0.6, 0.4, 0.0), (C_3, 0.6, 0.4, 0.0)}

Y_2 :{(C_1, 0.6, 0.4, 0.0), (C_2, 0.6, 0.4, 0.0), (C_3, 1.0, 0.0, 0.0)}

Y_3: {(C_1, 0.6, 0.4, 0.0), (C_2, 1.0, 0.0, 0.0), (C_3, 0.2, 0.4, 0.4)} and the fact that the weight of each criterion is equal, i.e. $w_1=w_2=w_3=1/3$.

We can now proceed to the calculation of CI and DI values as follows:

Concordance Index (CI):

1. *For pair (ψ_1,ψ_2):*
   - Compare each criterion and check if ψ_1 is preferred over ψ_2.
   - C1: 1.0>0.6→ Concordance
   - C2: 0.6=0.6→ Ex aequo (or Tie)
   - C3: 0.6<1.0→ Discordance
   - *CI (ψ_1,ψ_2) = w_1 = 1/3*
2. *For pair (ψ_1,ψ_3):*
   - Compare each criterion and check if ψ_1 is preferred over ψ_3.
   - C1: 0.6=0.6→ Concordance
   - C2: 0.6<1.0→ Discordance
   - C3: 1.0>0.2→ Concordance
   - *CI (ψ_2,ψ_3) = w_1+w_3= 2/3*
3. *For pair (ψ_2,ψ_3):*
   - Compare each criterion and check if ψ_2 is preferred over ψ_3.
   - C1: 0.6=0.6→ Concordance
   - C2: 0.6<1.0→ Discordance
   - C3: 1.0>0.2→ Concordance
   - *CI (ψ_2,ψ_3) = w_1+w_3= 2/3*

Discordance Index (DI):

1. *For pair (ψ_1,ψ_2):*
   - Compare each criterion
   - Discordance on C3: $\frac{|0.6-1.0|}{d_{max}}$
   - Assuming d_{max} for C3 is 1.0
   - $DI(\psi_1,\psi_2) = \frac{0.4}{1} = 0.4$

We should note here that assuming that d_{max} is equal to 1.0, we take into consideration the normalization of the discordance values so that they belong to a standardized range (typically between 0 and 1).

In the same way we find that $DI(\psi_1,\psi_3) = 0.4$ and $DI(\psi_2,\psi_3) = 0.8$

Using equation (2) : $NetCI_{Per1}=CI(\psi_1,\psi_2)+CI(\psi_1,\psi_3)+CI(\psi_2,\psi_3) = 2$

Using equation (3) : $NetDI_{Per1}=DI(\psi_1,\psi_2)+DI(\psi_1,\psi_3)+DI(\psi_2,\psi_3) = 1.6$

These calculations should be repeated for each permutation of the alternatives.

At this point it is useful to highlight the meaning and usefulness of utilizing the net concordance/discordance scores as a novel approach in our methodology. They are based on the seminal work from B. Roy when he and his team developed the ELECTRE method for MCDM problems [23-24]. Although such a principle is supported by substantial evidence from real-world decision-making circumstances, the way it is applied in existing MCDM approaches allows for only partial and restricted use. Instead our manuscript suggests a possible generalization of this principle under the concepts of net concordance/ discordance scores. These scores reflect the overall agreement/ disagreement for a permutation of alternatives, encapsulating the principle of concordance/discordance by providing a holistic measure of cumulative preference strength in decision-making.

More specifically, the net concordance score, which measures the overall agreement or support for a specific set of options, showing how consistently one alternative is chosen over another across all parameters. A high net concordance score demonstrates strong and consistent preference, while a low score indicates inconsistent preference. On the other hand, the net discordance score indicates total disagreement or conflict in the ratings, showing how much one choice is less valued.

A high net discordance score suggests a significant amount of disagreement, while a low score indicates less conflict and more reliable ratings. These ratings work together to assist in choosing the most favored and trustworthy option when dealing with uncertainty, striking a balance between strong support and minimal disagreement.

Lastly, in order to apply Step 8, and based on the results obtained from Table 5, the final ranking is $Per_2 > Per_1 = Per_3 = Per_4 = Per_5 = Per_6$, with ***$Per_2$*** being the most preferred order of alternatives.

4. Applications

It is worth noting that the results discussed in Section 3 demonstrate that Per2 aligns more closely with the defined criteria. Per1, Per3, Per4, Per5, and Per6 received identical scores, indicating that these permutations perform similarly and do not differ substantially according to the analyzed criteria. The slight superiority of Per2, as shown in Table 5, was only observed by applying our method. This could be explained by the utility of the proposed QUALIFLEX method which delves deeper by comparing candidates through concordance and discordance indices at a more granular level. Despite having identical neutrosophic values, Y_2's superior performance in terms of lower discordance can be explained by how the QUALIFLEX method compares candidates relative to each other rather than just looking at their individual scores. The method captures subtle nuances, such as lower conflict in evaluating candidate Y_2 versus candidate Y_1, especially in Experience (C_3), and less disagreement in the overall evaluations. This allows the QUALIFLEX method to rank alternative

Y_2 higher than alternative Y_1, showing that relative preference and stability in decision-making can emerge even from seemingly identical initial evaluations.

More specifically:

Per_2 (where $Y_2 > Y_1 > Y_3$) has the lowest discordance index (Net DI = 1.2), while Per_1 (where $Y_1 > Y_2 > Y_3$) has a higher discordance index (Net DI = 1.6).

1. This difference in discordance arises not from the neutrosophic values themselves, but from how the expert evaluations process uncertainty, especially when comparing candidates directly.
2. Even though Y_1 and Y_2 have identical neutrosophic scores, when experts compare them, Y_2's evaluation is seen as more consistent, with less disagreement across the criteria, particularly in Experience (C_3), where Y_2 performs very well.
3. The neutrosophic QUALIFLEX method synthesizes these evaluations in a way that identifies conflict or agreement between criteria, helping determine which candidate is more stable or reliable overall.

This interpretation of the findings demonstrates the accuracy and efficacy that characterizes our methodology, even in cases where ranking alternatives is a challenging task due to their "equal" performance when dealing with raw data. It also proves our method's efficiency in handling indeterminacy, a common challenge in traditional MCDM methods, leading to more nuanced and accurate rankings. Unlike traditional methods, which might have penalized Y_2 for having some level of indeterminacy, the neutrosophic QUALIFLEX method allowed us to view uncertainty as a manageable factor. This provided Y_2 with an advantage, as the method recognized that indeterminacy leaves room for possible positive interpretation, while falsehood is a definitive negative judgment.

In addition its robustness stems from its ability to handle both qualitative and quantitative data, as well as its flexibility in managing non-strictly deterministic criteria. Thus, it is better equipped to handle conflicting and indeterminate information, leading to decisions that are more reliable and reflective of the actual situation.

Previous research on classic QUALIFLEX approaches has acknowledged their limits in dealing with uncertain and imprecise data, resulting in less accurate rankings when faced with indeterminate information [25-28]. The results of this investigation support these observations, suggesting that the neutrosophic QUALIFLEX approach effectively addresses these constraints. By incorporating neutrosophic logic, which enables the simultaneous evaluation of truth, indeterminacy, and falsehood, the proposed technique offers a more nuanced approach to alternative evaluations.

The results indicate that the neutrosophic QUALIFLEX method not only provides more reliable rankings under uncertain conditions but also offers a flexible framework adaptable to various decision-making contexts. In this context, the results indicate that the neutrosophic QUALIFLEX method provides superior performance in delivering accurate and reliable rankings of candidates, as evidenced by its application in the staff selection problem.

In this context, the suggestion of the net concordance/discordance scores in our methodology offers the following significant advantages over the "traditional" QUALIFLEX MCDM method:

1. *Balanced decision making:* By integrating concordance and discordance scores, our method reduces the potential of bias from too favorable or negative assessments, resulting in more balanced or dependable conclusions.
2. *Holistic evaluation:* The net scores include the level of agreement and disagreement among criteria, offering a more comprehensive perspective of how one choice compares to others.
3. *Enhanced differentiation:* Our approach provides for stronger separation between closely competing options by taking into account both their strengths and drawbacks.

4. *Scalability and adaptability:* The net concordance/discordance framework is scalable and adaptable to a variety of decision-making scenarios, making it an effective tool for multi-criteria evaluation.

This supports the working hypothesis that neutrosophic logic can significantly improve decision-making processes in complex environments. The enhanced capability to handle ambiguous and indeterminate information confirms the theoretical advantages proposed by the integration of neutrosophic sets into MCDM techniques.

Conclusions

The field of multi-criteria decision analysis has established itself as a fundamental area in business research. The rapid growth of this field has resulted in the creation of a new methodological framework for analyzing decision-making problems.

Our research presents the neutrosophic QUALIFLEX method as an innovative approach to multi-criteria decision-making that adeptly manages uncertainty and indeterminacy. By integrating neutrosophic logic with the QUALIFLEX framework, the method provides significant improvements over traditional approaches, offering more accurate and reliable rankings.

In this study, the neutrosophic QUALIFLEX framework was utilized to evaluate the best alternative for staff selection in business under uncertainty, demonstrating its effectiveness in addressing indeterminacy, ambiguity, and inconsistency. By incorporating expert opinions and constructing a neutrosophic decision matrix, the approach facilitated a comprehensive assessment of alternatives, leading to robust and nuanced decision-making. The findings highlight the practical value of the neutrosophic QUALIFLEX method through an illustrative example of staff selection, showing that it provides reliable rankings under uncertain conditions and is adaptable to various decision-making contexts.

Future research could expand our method in order to provide the ranking of alternatives under consideration by calculating different weights of the selected criteria. Furthermore, it is important to compare the proposed approach with alternative decision-making methods in order to ensure consistent results and validate its accuracy.

In a managerial viewpoint, the practical implications of this method in real-world scenarios, the potential for integration with decision support systems, and interdisciplinary applications are expected to highlight its versatility and relevance. Policymakers and practitioners can leverage the neutrosophic QUALIFLEX method to make more informed and transparent decisions, ultimately contributing to better outcomes in complex decision environments.

Overall, the neutrosophic QUALIFLEX framework serves as a valuable tool for making accurate and reliable decisions in uncertain circumstances, underscoring the significance of neutrosophic logic in multi-criteria decision-making processes.

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Improved Method of Interval valued neutrosophic matrix composition and Its Application in Medical Diagnosis

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Abstract. Neutrosophic sets are an important branch of topology that addresses issues with inconsistent, indeterminate, and uncertain situations in everyday life. Interval valued neutrosophic sets, known as subclasses of neutrosophic sets, have been specifically designed to address problems with a set of numbers in the real unit of interval rather than just a single number. Interval valued neutrosophic (IVN) matrix plays an influential role in decision making problems with indeterminant and inconsistent information. The objective of the paper is to discuss the concepts and operations of IVN matrix theory and propose a newly concept of an improved method of IVN matrix composition and apply it in the field of medical diagnosis. Finally, a decision-making algorithm based on IVN matrix composition has been proposed to solve problems with disease diagnosis from the manifestation of different symptoms in individuals and successfully applied in the field of medical diagnosis.

Keywords: Interval valued neutrosophic matrix, Interval valued neutrosophic matrix complement, improved method of Interval valued neutrosophic matrix composition.

1. Introduction

A common problem in various fields, including business, finance, engineering, health care and social sciences is uncertainty. Fuzzy sets[23], intuitionistic fuzzy sets[2] , neutrosophic sets[19], vague soft sets, Interval valued neutrosophic sets [22]are some approaches that can be used as mathematical tools to prevent concerns when handling ambiguous data. To make computations in operations on fuzzy sets easier Fuzzy matrix theory [21] was first introduced by Thomason, who described the convergence of powers of fuzzy matrix.

In 1999, Smarandache introduced a new theory so called the neutrosophic set [19] with three independent membership functions (truth (T), indeterminacy (I), and falsity (F)) to deal with indeterminate and inconsistent data that each range from zero and one. Neutrosophic set is a general framework that encompasses various concepts such as classical, fuzzy, interval valued, and intuitionistic fuzzy sets. In the year 2014, the neutrosophic matrix[8], which is a representation of the neutrosophic set, was firstly given by Kandasamy and Smarandache, and they also discussed the properties of square neutrosophic matrices and super neutrosophic matrices, quasi neutrosophic matrices. In 2004, The subclass of neutrosophic sets, the Interval valued neutrosophic sets[22] introduced and set theoretic operators and various properties of operators of Interval valued neutrosophic sets discussed by Wang, Praveen Madiraju, Yanqing Zhang, Rajshkhar Sunderraman. The relationship between the interval neutrosophic set and other sets is classical set ⊆ Fuzzy set ⊆ Intuitionistic fuzzy set ⊆ Interval valued Intuitionistic fuzzy set ⊆ Interval valued neutrosophic set. Therefore, interval valued neutrosophic sets represent the generalization of other sets. Using these concepts varies authors applied these concepts in decision making problems [1,4,6,9,11,21].

In 2021, Interval valued neutrosphic matrix [8] was first introduced by Faruk Karaaslan, Khizar Hayat, and Chiranjibe Jana. Also, the determinant and adjoint of the interval valued neutrosophic matrix defined and various properties related to the adjoint operator discussed. Based on [20] on improved method of interval valued neutrosophic composition method is proposed and applied in the field of medical diagnosis.

The article is organized as follows section 2 presents basic definitions, algebraic operations of interval valued neutrosophic matrix and section 3 presents the improved method of interval valued neutrosophic matrix composition in decision making problem with real time application. Section 4 gives conclusion of this research paper.

2. Preliminaries

Definition 2.1: Any matrix $\hat{A}$ in $M_{m\times n}$ is called an interval valued neutrosophic matrix (IVN Matrix in short) it can be written in the form $M_{m\times n}(J) = \left\{[\langle[T_{ij}^L,T_{ij}^U],[I_{ij}^L,I_{ij}^U],[F_{ij}^L,F_{ij}^U]\rangle]_{m\times n}:\langle[T_{ij}^L,T_{ij}^U],[I_{ij}^L,I_{ij}^U],[F_{ij}^L,F_{ij}^U]\rangle \in \mathrm{N(J)}\right\}$ where $\mathrm{J} = \mathrm{I}^2\times\mathrm{I}^2\times\mathrm{I}^2$.

Definition 2.2: Consider IVN Matrices $\hat{A} = [\langle\left[T_{A_{ij}}^L,T_{A_{ij}}^U\right],\left[I_{A_{ij}}^L,I_{A_{ij}}^U\right],\left[F_{A_{ij}}^L,F_{A_{ij}}^U\right]\rangle]_{m\times n}$ $\hat{B} = [\langle\left[T_{B_{ij}}^L,T_{B_{ij}}^U\right],\left[I_{B_{ij}}^L,I_{B_{ij}}^U\right],\left[F_{B_{ij}}^L,F_{B_{ij}}^U\right]\rangle]_{m\times n}$ and $\hat{C} = [\langle\left[T_{C_{ij}}^L,T_{C_{ij}}^U\right],\left[I_{C_{ij}}^L,I_{C_{ij}}^U\right],\left[F_{C_{ij}}^L,F_{C_{ij}}^U\right]\rangle]_{m\times n}$.

Then, addition between two IVN matrices $\hat{B}$ and $\hat{C}$ is denoted by $\hat{B}+\hat{C}$, whose truth membership, indeterminacy membership, and false membership functions are related to $\hat{B}+\hat{C}$ is defined as

$$\hat{B}+\hat{C} = \left[\langle\left[T_{B_{ij}}^L+T_{C_{ij}}^L,\ T_{B_{ij}}^U+T_{C_{ij}}^U\right],\ \left[I_{B_{ij}}^L+I_{C_{ij}}^L, I_{B_{ij}}^U+I_{C_{ij}}^U\right],\left[F_{B_{ij}}^L+F_{C_{ij}}^L, F_{B_{ij}}^U+F_{C_{ij}}^U\right]\rangle\right]$$

Where

$$T_{B_{ij}}^L+T_{C_{ij}}^L = T_{B_{ij}}^L \vee T_{C_{ij}}^L,\ T_{B_{ij}}^U+T_{C_{ij}}^U = T_{B_{ij}}^U \vee T_{C_{ij}}^U$$
$$I_{B_{ij}}^L+I_{C_{ij}}^L = I_{B_{ij}}^L \wedge I_{C_{ij}}^L,\quad I_{B_{ij}}^U+I_{C_{ij}}^U = I_{B_{ij}}^U \wedge I_{C_{ij}}^U$$
$$F_{B_{ij}}^L+F_{C_{ij}}^L = F_{B_{ij}}^L \wedge F_{C_{ij}}^L,\ F_{B_{ij}}^U+F_{C_{ij}}^U = F_{B_{ij}}^U \wedge F_{C_{ij}}^U.$$

Example: 2.3

$$\hat{A} = \begin{bmatrix}\langle[0.2,0.6],[0.2,0.3],[0.3,0.6]\rangle & \langle[0.5,0.8],[0.3,0.5],[0.4,0.6]\rangle & \langle[0.2,0.6],[0.4,0.5],[0.1,0.2]\rangle\\ \langle[0.2,0.6],[0.3,0.4],[0.5,0.6]\rangle & \langle[0.5,0.6],[0.1,0.3],[0.5,0.6]\rangle & \langle[0.1,0.4],[0.5,0.9],[0.1,0.5]\rangle\\ \langle[0.3,0.6],[0.3,0.5],[0.5,0.9]\rangle & \langle[0.2,0.7],[0.1,0.3],[0.3,0.5]\rangle & \langle[0.5,0.9],[0.2,0.3],[0.3,0.4]\rangle\end{bmatrix}$$

$$\hat{B} = \begin{bmatrix}\langle[0.2,0.3],[0.2,0.4],[0.4,0.8]\rangle & \langle[0.3,0.6],[0.2,0.4],[0.4,0.6]\rangle & \langle[0.3,0.6],[0.3,0.6],[0.3,0.6]\rangle\\ \langle[0.1,0.7],[0.3,0.5],[0.7,0.8]\rangle & \langle[0.2,0.4],[0.1,0.3],[0.3,0.6]\rangle & \langle[0.2,0.4],[0.4,0.6],[0.2,0.4]\rangle\\ \langle[0.4,0.6],[0.3,0.5],[0.6,0.9]\rangle & \langle[0.2,0.5],[0.1,0.2],[0.3,0.6]\rangle & \langle[0.2,0.3],[0.3,0.4],[0.2,0.3]\rangle\end{bmatrix}$$

$$\hat{A}+\hat{B} = \begin{bmatrix}\langle[0.2,0.6],[0.2,0.3],[0.3,0.6]\rangle & \langle[0.5,0.8],[0.3,0.5],[0.4,0.6]\rangle & \langle[0.2,0.6],[0.3,0.6],[0.1,0.2]\rangle\\ \langle[0.2,0.7],[0.3,0.4],[0.5,0.6]\rangle & \langle[0.5,0.6],[0.1,0.3],[0.3,0.6]\rangle & \langle[0.1,0.4],[0.4,0.6],[0.1,0.4]\rangle\\ \langle[0.4,0.6],[0.3,0.5],[0.5,0.9]\rangle & \langle[0.2,0.7],[0.1,0.2],[0.3,0.5]\rangle & \langle[0.5,0.9],[0.2,0.3],[0.2,0.3]\rangle\end{bmatrix}$$

Definition 2.4: For any two IVN matrices $\hat{A}$ and $\hat{B}$, the product operations between IVN matrices $\hat{A}$ and $\hat{B}$ denoted by $\hat{A}\hat{B}$, whose truth membership, indeterminacy membership, and false membership functions are related to $\hat{A}\hat{B}$ is defined as

$\hat{A}\hat{B} = [\langle [T^L_{AB_{ij}}, T^U_{AB_{ij}}], [I^L_{AB_{ij}}, I^U_{AB_{ij}}], [F^L_{AB_{ij}}, F^U_{AB_{ij}}] \rangle]_{m\times n}$ Where

$$T^L_{AB_{ij}} = \bigvee_{k=1}^{m} (T^L_{A_{ik}} \wedge T^L_{B_{kj}}), \quad T^U_{AB_{ij}} = \bigvee_{k=1}^{m} (T^U_{A_{ik}} \wedge T^U_{B_{kj}})$$

$$I^L_{AB_{ij}} = \bigvee_{k=1}^{m} (I^L_{A_{ik}} \vee I^L_{B_{kj}}), \quad I^U_{AB_{ij}} = \bigvee_{k=1}^{m} (I^U_{A_{ik}} \vee I^U_{B_{kj}})$$

$$F^L_{AB_{ij}} = \bigvee_{k=1}^{m} (F^L_{A_{ik}} \vee F^L_{B_{kj}}), \quad F^U_{AB_{ij}} = \bigvee_{k=1}^{m} (F^U_{A_{ik}} \vee F^U_{B_{kj}}$$

Example: 2.5 For the above example 3.2 , the IVN Matrix $\hat{A}\hat{B}$ is given by

$$\hat{A}\hat{B} = \begin{bmatrix} \langle[0.2,0.6],[0.2,0.3],[0.3,0.6]\rangle & \langle[0.5,0.8],[0.3,0.5],[0.4,0.6]\rangle & \langle[0.2,0.6],[0.3,0.6],[0.1,0.2]\rangle \\ \langle[0.2,0.7],[0.3,0.4],[0.5,0.6]\rangle & \langle[0.5,0.6],[0.1,0.3],[0.3,0.6]\rangle & \langle[0.1,0.4],[0.4,0.6],[0.1,0.4]\rangle \\ \langle[0.4,0.6],[0.3,0.5],[0.5,0.9]\rangle & \langle[0.2,0.7],[0.1,0.2],[0.3,0.5]\rangle & \langle[0.5,0.9],[0.2,0.3],[0.2,0.3]\rangle \end{bmatrix}$$

Definition 2.6: For any two IVN matrices $\hat{A}$, the transpose of the IVN matrix $\hat{A}$ is denoted A^t , whose truth membership, indeterminacy membership, and false membership functions are related to $\hat{B} + \hat{C}$ is defined as $\hat{A}^t = [\langle [T^L_{A_{ji}}, T^U_{A_{ji}}], [I^L_{A_{ji}}, I^U_{A_{ji}}], [F^L_{A_{ji}}, F^U_{A_{ji}}] \rangle]_{m\times n}$.

Example: 2.7: For the above example 3.2, the IVN Matrix $\hat{A}^t$ is given by

$$\hat{A}^t = \begin{bmatrix} \langle[0.2,0.6],[0.2,0.3],[0.3,0.6]\rangle & \langle[0.5,0.8],[0.3,0.5],[0.4,0.6]\rangle & \langle[0.2,0.6],[0.3,0.6],[0.1,0.2]\rangle \\ \langle[0.2,0.7],[0.3,0.4],[0.5,0.6]\rangle & \langle[0.5,0.6],[0.1,0.3],[0.3,0.6]\rangle & \langle[0.1,0.4],[0.4,0.6],[0.1,0.4]\rangle \\ \langle[0.4,0.6],[0.3,0.5],[0.5,0.9]\rangle & \langle[0.2,0.7],[0.1,0.2],[0.3,0.5]\rangle & \langle[0.5,0.9],[0.2,0.3],[0.2,0.3]\rangle \end{bmatrix}$$

Definition 2.8: For any IVN matrix $\hat{A}$, the power of the IVN matrix $\widehat{\mathrm{A}}$, whose truth membership, indeterminacy membership, and false membership functions are related to $\widehat{\mathrm{A}}$ are defined as

$$\hat{A}^k = [\langle [T^{L\,(k)}_{A_{ij}}, T^{U\,(k)}_{A_{ij}}], [I^{L\,(k)}_{A_{ij}}, I^{U\,(k)}_{A_{ij}}], [F^{L\,(k)}_{A_{ij}}, F^{U\,(k)}_{A_{ij}}] \rangle] .$$

Definition 2.9: IVN-unit matrix is defined as $\hat{A}^0 = I = [\langle [T^L_{I_{ij}}, T^U_{I_{ij}}], [I^L_{I_{ij}}, I^U_{I_{ij}}], [F^L_{I_{ij}}, F^U_{I_{ij}}] \rangle]_{m\times n}$

where

$$\langle [T^L_{I_{ij}}, T^U_{I_{ij}}], [I^L_{I_{ij}}, I^U_{I_{ij}}], [F^L_{I_{ij}}, F^U_{I_{ij}}] \rangle = \begin{cases} \langle[1,1],[0,0],[0,0]\rangle, & i = j \\ \langle[0,0],[1,1],[1,1]\rangle, & i \neq j \end{cases}$$

Definition 2.10: For any two IVN matrices $\hat{A}$ and $\hat{B}$, the IVN matrix $\hat{B}$ is said to smaller than the IVN-matrix $\hat{B}$ denoted by $\hat{A} \leq \hat{B}$ and whose truth membership, indeterminacy membership, and false membership functions are defined as $\langle [T^L_{A_{ij}}, T^U_{A_{ij}}], [I^L_{A_{ij}}, I^U_{A_{ij}}], [F^L_{A_{ij}}, F^U_{A_{ij}}] \rangle \leq \langle [T^L_{B_{ij}}, T^U_{B_{ij}}], [I^L_{B_{ij}}, I^U_{B_{ij}}], [F^L_{B_{ij}}, F^U_{B_{ij}}] \rangle$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

4. Some operations in IVN matrices:

Definition 3.1: Let$\hat{A} = [\langle [T^L_{A_{ij}}, T^U_{A_{ij}}], [I^L_{A_{ij}}, I^U_{A_{ij}}], [F^L_{A_{ij}}, F^U_{A_{ij}}] \rangle]_{m\times n}$
$\hat{B} = [\langle [T^L_{B_{ij}}, T^U_{B_{ij}}], [I^L_{B_{ij}}, I^U_{B_{ij}}], [F^L_{B_{ij}}, F^U_{B_{ij}}] \rangle]_{m\times n}$ and
$\hat{C} = [\langle [T^L_{C_{ij}}, T^U_{C_{ij}}], [I^L_{C_{ij}}, I^U_{C_{ij}}], [F^L_{C_{ij}}, F^U_{C_{ij}}] \rangle]_{m\times n}$ then IVN complement of a matrix $\hat{A}$ denoted by $\hat{A}^c$ whose truth membership, indeterminacy membership, and false membership functions are defined as $\hat{A}^c = [\langle [F^L_{A_{ij}}, F^U_{A_{ij}}], [1 - I^U_{A_{ij}}, 1 - I^L_{A_{ij}}], [T^L_{A_{ij}}, T^U_{A_{ij}}] \rangle]_{m\times n}$.

Example 3.2: Let $\hat{A}$ be a IVN Matrix defined by

$$\hat{A}=\begin{bmatrix}\langle[0.2,0.7],[0.1,0.2],[0.3,0.4]\rangle & \langle[0.3,0.7],[0.2,0.3],[0.4,0.5]\rangle & \langle[0.2,0.6],[0.4,0.5],[0.1,0.2]\rangle\\ \langle[0.1,0.6],[0.2,0.3],[0.4,0.5]\rangle & \langle[0.5,0.9],[0.1,0.2],[0.5,0.6]\rangle & \langle[0.1,0.4],[0.5,0.9],[0.1,0.5]\rangle\\ \langle[0.3,0.9],[0.2,0.6],[0.5,0.9]\rangle & \langle[0.2,0.6],[0.1,0.2],[0.3,0.4]\rangle & \langle[0.5,0.9],[0.2,0.3],[0.3,0.4]\rangle\end{bmatrix}$$

Then the Complement of IVN Matrix $\hat{A}$ is

$$\hat{A}^{C}=\begin{bmatrix}\langle[0.3,0.4],[0.8,0.9],[0.2,0.7]\rangle & \langle[0.4,0.5],[0.7,0.8],[0.3,0.7]\rangle & \langle[0.1,0.2],[0.5,0.6],[0.2,0.6]\rangle\\ \langle[0.4,0.5],[0.7,0.8],[0.1,0.6]\rangle & \langle[0.5,0.6],[0.8,0.9],[0.5,0.9]\rangle & \langle[0.1,0.5],[0.1,0.5],[0.1,0.4]\rangle\\ \langle[0.5,0.9],[0.4,0.8],[0.3,0.9]\rangle & \langle[0.3,0.4],[0.8,0.9],[0.2,0.6]\rangle & \langle[0.3,0.4],[0.7,0.8],[0.5,0.9]\rangle\end{bmatrix}$$

Definition 3.2: The difference of two IVN matrix $\hat{A}$ and $\hat{B}$ denoted by $\widehat{C}=\widehat{A}-\widehat{B}$ whose truth membership, indeterminacy membership, and false membership functions are defined as

$\widehat{C}=[\langle\left[T_{C_{ij}}^{L},T_{C_{ij}}^{U}\right],\left[I_{C_{ij}}^{L},I_{C_{ij}}^{U}\right],\left[F_{C_{ij}}^{L},F_{C_{ij}}^{U}\right]\rangle]_{m\times n}$ where

$T_{C_{ij}}^{L}=\wedge[T_{A_{ij}}^{L},T_{B_{ij}}^{L}]$, $T_{C_{ij}}^{U}=\wedge[T_{A_{ij}}^{U},T_{B_{ij}}^{U}]$

$I_{C_{ij}}^{L}=\vee[I_{A_{ij}}^{L},1-I_{B_{ij}}^{U}]$, $T_{C_{ij}}^{U}=\vee[I_{A_{ij}}^{U},1-I_{B_{ij}}^{L}]$

$F_{C_{ij}}^{U}=\vee[F_{A_{ij}}^{L},F_{B_{ij}}^{L}]$, $T_{C_{ij}}^{U}=\vee[F_{A_{ij}}^{U},F_{B_{ij}}^{U}]$

Example 3.4: Consider the IVN Matrices $\hat{A}$ and $\hat{B}$

$$\hat{A}=\begin{bmatrix}\langle[0.45,0.80],[0.15,0.35],[0.1,0.20]\rangle & \langle[0.45,0.80],[0.15,0.30],[0.20,0.30]\rangle & \langle[0.55,0.85],[0.20,0.30],[0.15,0.25]\rangle\\ \langle[0.50,0.80],[0.2,0.40],[0.25,0.35]\rangle & \langle[0.50,0.80],[0.20,0.35],[0.25,0.45]\rangle & \langle[0.60,0.85],[0.25,0.35],[0.20,0.30]\rangle\\ \langle[0.60,0.80],[0.15,0.30],[0.15,0.30]\rangle & \langle[0.55,0.65],[0.10,0.25],[0.20,0.35]\rangle & \langle[0.60,0.75],[0.15,0.25],[0.15,0.25]\rangle\end{bmatrix}$$

6 exam level matrix B are givenexam level matrix B

$$\hat{B}=\begin{bmatrix}\langle[0.25,0.35],[0.50,0.60],[0.35,0.65]\rangle & \langle[0.30,0.45],[0.55,0.65],[0.35,0.55]\rangle & \langle[0.30,0.40],[0.60,0.70],[0.40,0.65]\rangle\\ \langle[0.30,0.40],[0.50,0.60],[0.40,0.75]\rangle & \langle[0.35,0.50],[0.55,0.65],[0.40,0.60]\rangle & \langle[0.35,0.45],[0.60,0.70],[0.50,0.70]\rangle\\ \langle[0.20,0.35],[0.60,0.70],[0.35,0.70]\rangle & \langle[0.25,0.45],[0.65,0.75],[0.35,0.70]\rangle & \langle[0.20,0.40],[0.70,0.80],[0.45,0.75]\rangle\end{bmatrix}$$

Then the difference $C=\widehat{A}-\widehat{B}$ is given by

$$\widehat{C}=\begin{bmatrix}\langle[0.25,0.35],[0.40,0.50],[0.35,0.65]\rangle & \langle[0.30,0.45],[0.35,0.65],[0.35,0.55]\rangle & \langle[0.30,0.40],[0.30,0.40],[0.40,0.65]\rangle\\ \langle[0.30,0.40],[0.40,0.50],[0.40,0.75]\rangle & \langle[0.35,0.50],[0.35,0.45],[0.40,0.60]\rangle & \langle[0.35,0.45],[0.40,0.30],[0.50,0.70]\rangle\\ \langle[0.20,0.35],[0.30,0.40],[0.35,0.70]\rangle & \langle[0.25,0.45],[0.25,0.35],[0.35,0.70]\rangle & \langle[0.20,0.40],[0.20,0.30],[0.45,0.75]\rangle\end{bmatrix}$$

Definition 3.5: Improved method of IVN matrix composition denoted by $A@B$ and defined as

$$A@B=\begin{cases}\bigvee_{k=1}^{m}\dfrac{T_{A_{ij}}^{L}+T_{B_{jk}}^{L}}{2},\bigvee_{k=1}^{m}\dfrac{T_{A_{ij}}^{U}+T_{B_{jk}}^{U}}{2},\\ \bigwedge_{k=1}^{m}\dfrac{I_{A_{ij}}^{L}+I_{B_{jk}}^{L}}{2},\bigwedge_{k=1}^{m}\dfrac{I_{A_{ij}}^{U}+I_{B_{jk}}^{U}}{2},\\ \bigwedge_{k=1}^{m}\dfrac{F_{A_{ij}}^{L}+F_{B_{jk}}^{L}}{2},\bigwedge_{k=1}^{m}\dfrac{F_{A_{ij}}^{U}+F_{B_{jk}}^{U}}{2},\end{cases}$$

Decision making algorithm using improved method of IVN matrix composition:

In this section, we put forward a decision making algorithm using IVN matrix composition.

Algorithm:

Step:1

Input the IVN matrix A (Patient-symptom Matrix)and B (Symptom –disease Matrix) of order $m\times n$.

Step:2

Write the complement of each of IVN Matrix A^{c} and B^{c}.

Step:3

Compute the IVN composition (Patient-symptom disease Matrix) matrix $C = A@B$ and(Patient-symptom non disease Matrix) $D = A^c@B^c$.

Step:4

compute score matrix

$$S(C,D) = \frac{\hat{C}+\hat{D}}{2} = \left[\left\langle \left[\frac{T^L_{B_{ij}}+T^L_{C_{ij}}}{2}, \frac{T^U_{B_{ij}}+T^U_{C_{ij}}}{2}\right] \left[\frac{I^L_{B_{ij}}+I^L_{C_{ij}}}{2}, \frac{I^U_{B_{ij}}+I^U_{C_{ij}}}{2}\right] \left[\frac{F^L_{B_{ij}}+F^L_{C_{ij}}}{2}, \frac{F^U_{B_{ij}}+F^U_{C_{ij}}}{2}\right]\right\rangle\right].$$

Convert Interval valued neutrosophic value into crisp value by using $\frac{4+T^L+T^U-I^L-I^U-F^L-F^U}{6}$-----(i).

Step:5

After calculating the patient P_i's maximum score, it is decided that the patient has the illness D_i.

Example: 3.6: Suppose three patients $P = \{P_1, P_2, P_3\}$ represents persons Arun, Ram, Atul with symptoms $P = \{P_1, P_2, P_3\}$ represents symptoms headache, temperature, body pain. Let the possible disease relate to their symptoms $D = \{D_1, D_2, D_3\}$ be viral fever, typoid, malaria.

Now, consider a collection of an approximate description of patient symptoms in the hospital as follows.

Let Patient-symptom relationship Matrix A is given by

$$A = \begin{matrix} & S_1 & S_2 & S_3 \\ P_1 & \langle[0.3,0.7],[0.4,0.5],[0.1,0.2]\rangle & \langle[0.3,0.7],[0.1,0.2],[0.5,0.6]\rangle & \langle[0.2,0.6],[0.3,0.5],[0.3,0.7]\rangle \\ P_2 & \langle[0.1,0.5],[0.3,0.7],[0.2,0.3]\rangle & \langle[0.6,0.8],[0.1,0.3],[0.1,0.2]\rangle & \langle[0.2,0.5],[0.5,0.6],[0.2,0.5]\rangle \\ P_3 & \langle[0.5,0.8],[0.2,0.6],[0.3,0.5]\rangle & \langle[0.2,0.6],[0.1,0.5],[0.3,0.6]\rangle & \langle[0.7,0.9],[0.2,0.3],[0.1,0.2]\rangle \end{matrix}$$

6 exam level matrix B are givenexam level matrix B are giv Symptom –disease Matrix B is given by

$$B = \begin{matrix} & D_1 & D_2 & D_3 \\ S_1 & \langle[0.1,0.5],[0.3,0.5],[0.1,0.2]\rangle & \langle[0.1,0.5],[0.2,0.5],[0.5,0.6]\rangle & \langle[0.5,0.8],[0.1,0.2],[0.2,0.3]\rangle \\ S_2 & \langle[0.1,0.4],[0.4,0.7],[0.8,0.9]\rangle & \langle[0.5,0.7],[0.1,0.2],[0.2,0.7]\rangle & \langle[0.8,0.9],[0.5,0.6],[0.3,0.8]\rangle \\ S_3 & \langle[0.5,0.9],[0.1,0.5],[0.2,0.4]\rangle & \langle[0.3,0.8],[0.2,0.6],[0.3,0.4]\rangle & \langle[0.8,0.9],[0.1,0.2],[0.6,0.9]\rangle \end{matrix}$$

The complement of the given IVN matrix A is given by

$$A^c = \begin{bmatrix} \langle[0.1,0.2],[0.5,0.6],[0.3,0.7]\rangle & \langle[0.5,0.6],[0.8,0.9],[0.3,0.7]\rangle & \langle[0.3,0.7],[0.5,0.7],[0.2,0.6]\rangle \\ \langle[0.2,0.3],[0.3,0.7],[0.1,0.5]\rangle & \langle[0.1,0.2],[0.7,0.9],[0.6,0.8]\rangle & \langle[0.2,0.5],[0.4,0.5],[0.2,0.5]\rangle \\ \langle[0.3,0.5],[0.4,0.8],[0.5,0.8]\rangle & \langle[0.3,0.6],[0.5,0.9],[0.2,0.6]\rangle & \langle[0.1,0.2],[0.7,0.8],[0.7,0.9]\rangle \end{bmatrix}$$

The complement of the given IVN matrix B is given by

$$B^c = \begin{bmatrix} \langle[0.1,0.2],[0.5,0.7],[0.1,0.3]\rangle & \langle[0.5,0.6],[0.5,0.8],[0.1,0.5]\rangle & \langle[0.2,0.3],[0.8,0.9],[0.5,0.8]\rangle \\ \langle[0.8,0.9],[0.3,0.6],[0.1,0.4]\rangle & \langle[0.2,0.7],[0.8,0.9],[0.5,0.7]\rangle & \langle[0.3,0.8],[0.4,0.5],[0.8,0.9]\rangle \\ \langle[0.2,0.4],[0.5,0.9],[0.5,0.9]\rangle & \langle[0.3,0.4],[0.4,0.8],[0.3,0.8]\rangle & \langle[0.6,0.9],[0.8,0.9],[0.8,0.9]\rangle \end{bmatrix}$$

The IVN Matrix $D = A^\circ B$ is Th

$$A^\circ B = \begin{bmatrix} \langle[0.35,0.75],[0.20,0.45],[0.10,0.20]\rangle & \langle[0.40,0.70],[0.10,0.20],[0.30,0.40]\rangle & \langle[0.55,0.80],[0.20,0.35],[0.15,0.25]\rangle \\ \langle[0.35,0.70],[0.25,0.50],[0.15,0.25]\rangle & \langle[0.55,0.75],[0.10,0.25],[0.15,0.40]\rangle & \langle[0.70,0.85],[0.20,0.40],[0.20,0.30]\rangle \\ \langle[0.60,0.90],[0.15,0.40],[0.15,0.30]\rangle & \langle[0.50,0.85],[0.10,0.35],[0.20,0.30]\rangle & \langle[0.75,0.90],[0.15,0.25],[0.25,0.40]\rangle \end{bmatrix}$$

The IVN Matrix $E = A^{c\circ} B^c$ is

$$E = \begin{bmatrix} \langle[0.65,0.75],[0.50,0.65],[0.20,0.55]\rangle & \langle[0.35,0.65],[0.50,0.70],[0.20,0.60]\rangle & \langle[0.45,0.80],[0.60,0.70],[0.40,0.75]\rangle \\ \langle[0.45,0.55],[0.40,0.70],[0.10,0.50]\rangle & \langle[0.35,0.45],[0.40,0.65],[0.10,0.50]\rangle & \langle[0.40,0.70],[0.55,0.70],[0.30,0.65]\rangle \\ \langle[0.55,0.75],[0.40,0.75],[0.15,0.50]\rangle & \langle[0.40,0.65],[0.45,0.80],[0.30,0.65]\rangle & \langle[0.35,0.55],[0.45,0.70],[0.50,0.75]\rangle \end{bmatrix}$$

FThe The

The score matrix $S(C, D)$ is

$$S = \begin{bmatrix} \langle[0.35,0.75],[0.50,0.65],[0.20,0.55]\rangle & \langle[0.35,0.65],[0.50,0.70],[0.30,0.60]\rangle & \langle[0.45,0.80],[0.60,0.70],[0.40,0.75]\rangle \\ \langle[0.45,0.55],[0.40,0.70],[0.10,0.50]\rangle & \langle[0.35,0.45],[0.40,0.65],[0.10,0.50]\rangle & \langle[0.40,0.70],[0.55,0.70],[0.30,0.65]\rangle \\ \langle[0.55,0.75],[0.40,0.75],[0.15,0.50]\rangle & \langle[0.40,0.65],[0.45,0.8],[0.30,0.65]\rangle & \langle[0.35,0.55],[0.45,0.7],[0.5,0.75]\rangle \end{bmatrix}$$

Ccccc

The conversion of IVN value into crisp value by using equation (i) gives

$$S(C,D) = \begin{matrix} & D_1 & D_2 & D_3 \\ P_1 & \mathbf{0.6125} & 0.5917 & 0.6000 \\ P_2 & 0.6 & 0.6083 & \mathbf{0.6125} \\ P_3 & \mathbf{0.667} & 0.6042 & 0.5958 \end{matrix}$$

The above matrix indicates evident that the patients $\{P_1, P_3\}$ are suffering from the disease $\{D_1\}$ and $\{P_2\}$ is from $\{D_3\}$.

Conclusion

In this paper we have defined an improved IVN matrix composition method and successfully applied it in the field of medical diagnosis. Also, Interval valued neutrosophic matrix provides effective solutions to various decision-making issues and it can be applied to multi criteria decision making methods.

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A Neutrosophic Approach for Breast Mass Detection in digital Mammogram Images

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Abstract: Breast cancer is the most common type of cancer that affect women and yet there is no substantive cure. On early diagnosis of breast lesions with this size using mammogram would help the affected tumor patients have a good survival rate. Detecting cancer cells is a challenging task and hence we propose a single valued neutrosophic approach for segmenting breast tumor lesions in mammogram images. In this neutrosophic approach, the intensity values are represented as truth, indeterminacy and falsity membership in where the indeterminacy are noise, and the breast regions are the true and falsity values. The computations of these memberships are then pre-processed using α-mean and β-enhancement to minimize the indeterminacy. Finally, gamma clustering techniques are used to localize and segment the tumor from the background breast tissue region. Experiments are conducted on publicly available datasets and the proposed method is evaluated and achieves a better segmentation performance.

Keywords: Breast Mass detection; Neutrosophic domain; Alpha-mean; Beta enhancement; Gamma Clustering

1. Introduction

Cancer is an abnormal cell growth that invades all parts of the body bringing health issues. Early detection of tumour or benign cells helps us to save life when given proper treatment. Automatic diagnosis of tissue mass has been a great area of research for a few decades [1]. Breast cancer is developed in breast tissues are very common in women and the risk factors are obesity, alcohol consumption, genetics, early or late menopause etc [2]. In many medical cases, breast tumours are usually located in milk ducts tubes or in the milk secreting gland lobules. Studies have shown that on early detection it can be cured effectively.

Mammography is a low dose X-ray image capture technique used to screen breast for tumour detection. Patients with lumps or mass in breast are recommended for this screening. These lumps can be either benign or malignant. Hence the proposed work motivates to concentrate on localizing these breast lesions or mass. To detects these, the approaches have been categorized into model-based and non-model-based. In a model-based approach, techniques such as shape-based template matching, de-formable model, and deep learning model are involved. Al-Antari et al. [3] uses YOLO (You-Only-Look-Once) to detected the breast mass and FrCN (full resolution convolutional network) to segment the mass. Ahmed et al. [4] uses a hybrid method based on Deeplab-V3 and

Mask-RCNN for breast mass segmentation. Dhungel et al. [5] proposed a SSVM (structured support vector machine) model for locating breast lesion.

In non-model based, Kuo et al. [6] propose the enhancement of the suspected regions of breast mass in the mammographic image, then identified and located the breast mass by PSO. StojiC et al. and Amutha et al. [7] enhanced the edge details on the mammographic image using the mathematical morphology method. Liu et al. [8] uses the image template matching with a bright circular image template. By this method, region of interest is highlighted from the background to detect the mass effectively.

In a traditional fuzzy set, the membership of the set A defined on universe x $\mu_A(x) \in [0, 1]$. If $\mu_A(x)$ can hold values from 0 to 1, in image domain the intensity values classify it is an object or background in an image [9]. The uncertainty parameters such as noise or image acquisition negligence error are avoided, thus degrading the performance on using algorithms. To overcome this issue, a Neutrosophic set (NS) an extended version of fuzzy logic is introduced that deals these uncertainity or neutralities. The spatial images are converted into a set with degree to truth, falsity, and indeterminacy membership. Medical domain is the field where there is indeterminacy, unknown, hidden parameters, imprecision, high conflict between sources of information, non-exhaustive or non-exclusive elements of the frame of discernment so neutrosophy could be applied [10]. In the proposed application, indeterminacy is the medical information such as imprecise, partial, and vague or imperfect lab results or degradation of the X-ray equipment.

Our proposed work employs the positives of neutrosophic sets along with gamma clustering to localize the breast masses. This above approach leads to less computation time and effort claiming with best results. The rest of the paper is organized as follows: The proposed methodology is explained in Section 2. Section 3 explains the results of the experiment carried out using the proposed method and the conclusion of this work is discussed in Section 4.

2. Proposed Work

The proposed architecture consists of the following steps: (i) Neutrosophic domain (ii) Alpha Mean operation (iii) Beta enhancement operation (iv) Gamma Clustering. The system focuses on localization of the lesions and the boundary information related to it. Figure 1 is a sample input mammogram X-ray image.

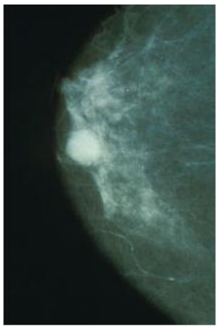

Fig 1: Sample Input X-ray image

2.1 Neutrosophic Domain

2.1.1 Neutrosophic Set (NS)

Neutrosophy derived on the concept of knowledge of neutral thought. It is defined as the study of origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra [9]. It reveals and explore the world is full of indeterminacy.

Definition: Neutrosophic set (NS): Consider Z to be a universe of discourse and in where Neutrosophic set (S) is a part of it. In mathematical terms, an element z in set A is written as z (t, i, f) and represented in NS logic using:

$$S = \{[z, (T_S(z), I_S(z), F_S(z))] z \in Z\} \quad (1)$$

where $T_S(z)$, $I_S(z)$ and $F_S(z)$ are the neutrosophic components and are real standard or non-standard sets of $]0^-, 1^+[$ and is defined using:

$$n_{sup} = t_{sup} + i_{sup} + f_{sup} \quad (2)$$

where $sup\_T = t_{sup}$, $sup\_I = i_{sup}$, $sup\_F = f_{sup}$

$$n_{inf} = t_{inf} + i_{inf} + f_{inf} \quad (3)$$

where $inf\_T = t_{inf}$, $inf\_I = i_{inf}$, $inf\_F = F_{inf}$,

$$so, 0^- \leq T_S(z) + I_S(z) + F_S(z) \leq 3^+$$

In Equations (1) - (3), T, I and F are defined as the degree of the true, indeterminate and false membership function of set A respectively [11]. An element x(t, i, f) belongs to set A and is represented in the following way: t% true, i% indeterminacy, and f% false. In this t varies in T, i varies in I, and f varies in F domain [12].

2.1.2 Images in NS domain

For an image processing application, initially the X-ray image from spatial domain is represented as a neutrosophic image as represented as follows.

Definition: Neutrosophic image (NI): Let's consider Z to be a universe of the discourse and size of the image window is represented as W = w * w where w1, w2 indicates the rows and columns in spatial domain respectively. Thus, W holds the values of image intensity pixels range from 0 to 255 where W ⊆ Z [13]. As per the Equation 4, the neutrosophic image is characterized by T, I and F membership sets [13]. For every input X-ray image with size M*N, each pixel $Image_X(m, n)$ is transformed to $Image_{NS}(m, n)$ in neutrosophic domain. In $Image_{NS}(m, n)$ the bright pixels represent the true domain $T_X(m, n)$, indeterminate pixels as $I_X(m, n)$ and dark pixels as $F_X(m, n)$. This representation of NS image is depicted as Fig 2.

$$Image_{ns}(m, n) = \{T_X(m, n), I_X(m, n), F_X(m, n)\} \quad (4)$$

$$T_X(m, n) = \frac{\overline{g(m, n)} - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \quad (5)$$

$$\text{where } \overline{g(m, n)} = \frac{1}{w \times w} \sum_{x=m-w/2}^{m+w/2} \sum_{y=n-w/2}^{n+w/2} g(x, y)$$

where the g(x,y) in Equation (5) is the input X-ray image and window size w=3,5 or 7. On execution w=3 shows good performance results when compared to other values. In the equation, $\bar{g}_{min}$ and $\bar{g}_{max}$ are the minimum and maximum local mean intensity value of $\overline{g(m, n)}$.

$$I_X(m, n) = \frac{\delta(i,j) - \delta_{min}}{\delta_{max} - \delta_{min}} \quad (6)$$

where $\delta(m, n) = abs(g(m, n) - \overline{g(m, n)})$. The minimum of $\delta(m, n)$ is computed as δ_{min} and the maximum as δ_{max}.

$$F_X(m, n) = 1 - T_X(m, n) \quad (7)$$

where, g(m, n) – Input X-ray image's local mean value.
δ(m, n) – the absolute difference between pixel intensity g(m, n) and mean value g(m, n)

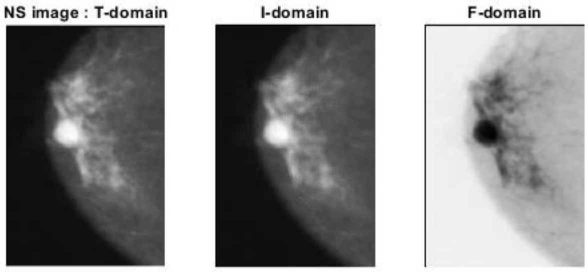

Fig 2: Images in NS domain

2.2 Alpha-mean operation

Various factor leads to uncertainty during the image acquisition errors stages such as intrinsically imperfect lab observations and the quantitative errors in measures. To reduce the indeterminacy degree $Image_{NS}$ (m, n) leads to better segmentation process. To maximize the entropy of true and false domain α-mean is used.

Definition: The α-mean operation for the input $Image_{NS} = Image_X(T_X, I_X, F_X)$ and the transformation is defined using Equation (8) – (11) [14].

$$Image_{NS}(\alpha) = Image_X(T_X(\alpha), I_X(\alpha), F_X(\alpha)) \quad (8)$$

The True α-mean set

$$\overline{T_X}(\alpha) = \begin{cases} T_X & if \quad I_X < \alpha \\ \overline{T_X} & if \quad I_X \geq \alpha \end{cases} \quad (9)$$

where $\overline{T_X}(m, n)=\frac{1}{w \times w}\sum_{x=m-w/2}^{m+w/2}\sum_{y=n-w/2}^{n+w/2} T_X(x, y)$

The False α-mean set

$$\overline{F_X}(\alpha) = \begin{cases} F_X & if \quad I_X < \alpha \\ \bar{F}_X & if \quad I_X \geq \alpha \end{cases} \quad (10)$$

where $\overline{F_X}(m, n)=\frac{1}{w \times w}\sum_{x=m-w/2}^{m+w/2}\sum_{y=n-w/2}^{n+w/2} F_X(x, y)$

The α value ranges within [0 1] and on experimenting many trial and error cases promising outcomes are shown when α=0.9.

The Indeterminate α-mean set

$$\overline{I_X}(\alpha) = \overline{I_X}(m, n)=\frac{\bar{\delta}(x, y)- \bar{\delta}_{min}}{\bar{\delta}_{max}- \bar{\delta}_{min}} \quad (11)$$

where $\delta(m, n)=abs(\overline{T_X}(m, n)- \overline{\overline{T_X}}(m, n))$

where $\overline{\overline{T_X}}(m, n)=\frac{1}{w \times w}\sum_{x=m-w/2}^{m+w/2}\sum_{y=n-w/2}^{n+w/2} \overline{T_X}(x, y)$

At the end of α-mean operation, the indeterminate pixel intensity is reduced thus enhancing the true and false domain as depicted in Figure 3.

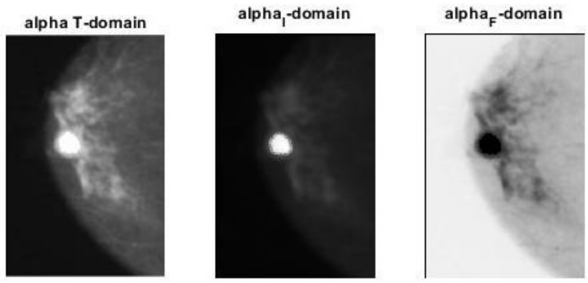

Fig 3: Alpha mean operation of NS image

2.3 Beta-enhancement operation

Definition: The β-enhancement operation for the input $Image_{NS}(\alpha)= Image_X(T_X(\alpha), I_X(\alpha), F_X(\alpha))$ is defined as Equation (12) – (22) [14].

$$Image`_{NS}(\beta) = Image_X (T_X`(\beta), I_X`(\beta), F_X`(\beta)) \tag{12}$$

The True beta mean set is calculated as follows:

$$\grave{T}_X(\beta) = \begin{cases} T_X & if \quad I_X < \beta \\ \grave{T}_X & if \quad I_X \geq \beta \end{cases} \tag{13}$$

where,

$$\grave{T}_\beta(\grave{m}, n)= \begin{cases} 2T_X{}^2(m, n) & \text{if} \quad T_X(m, n) \leq 0.5 \\ 1\text{-}2(1\text{-}T_X(m, n))^2 & \text{if} \quad T_X(m, n) > 0.5 \end{cases}$$

The False beta mean set is calculated as follows:

$$F_X\grave{}(\beta) = \begin{cases} F_X & \text{if} \quad I_X < \beta \\ \grave{F}_X & \text{if} \quad I_X \geq \beta \end{cases} \tag{14}$$

where,

$$\grave{F}_\beta(\grave{m}, n)= \begin{cases} 2F_X{}^2(m, n) & \text{if} \quad F_X(m, n) \leq 0.5 \\ 1\text{-}2(1\text{-}F_X(m, n))^2 & \text{if} \quad F_X(m, n) > 0.5 \end{cases}$$

On experimentation, β value also ranges within the interval 0 to 1 and hence multiple values are tested and an optimized results are yielded for X-ray images when β =0.85.

The Indeterminate mean set is calculated as follows:

$$\grave{I}_X(\beta) = \grave{I}_X(m, n)=\frac{\grave{\delta}(m,n) - \grave{\delta}_{min}}{\grave{\delta}_{max} - \grave{\delta}_{min}} \tag{15}$$

$$\text{Where, } \grave{\delta}(m,n) = abs(\grave{T}_X(m,n) - \overline{\grave{T}_X}(m, n))$$

$$\overline{\grave{T}_X}(m, n) = \frac{1}{w \times w} \sum_{x=m-w/2}^{m+w/2} \sum_{y=n-w/2}^{n+w/2} \grave{T}_X(x, y)$$

At the end of the β-enhancement operation the noises are reduced, thus retaining more edge information for segmentation as shown in Figure 4.

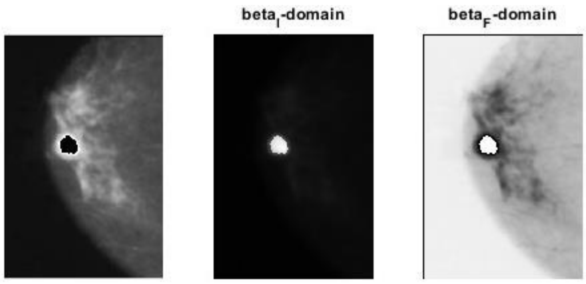

Fig 4: Beta enhancement operation of NS image

2.4 Gamma Clustering Algorithm

The final step involves segmentation of the breast masses using K-means clustering technique. As the name signifies, it groups the pixels of same intensity values as objects in the enhanced beta true images into K groups. The grouping is done using Equation 16 [24]:

$$Z = \sum_{j=1}^{k} \sum_{i=1}^{M_j} \|S_i - P_j\| \quad (16)$$

where P_j and M_j are the center and the total count of pixels of the j[th] cluster and k represents total cluster numbers [14].

This classic clustering technique is further enhanced by data distribution with Gamma factor as it deals with skewed data. Minimize the value of Z using the following Equation 17

$$P_j = \frac{1}{M_j} \sum_{S_i \in C_j} S_i \quad (17)$$

where S={s_i, i=1, 2, 3…….t} , s_i represents the sample in the d-dimensional space and $C = \{C_1, C_2, C_3 \ldots\ldots.. C_n\}$ represents the segmented clusters that satisfies the Equation 18.

$$D = \cup_{i=1}^{n} C_i \quad (18)$$

The clustered data are represented in different color regions for differentiation purpose as depicted in Figure 5

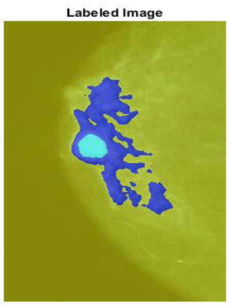

Fig. 5: Gamma-clustering operation (Clustered output)

Algorithm 1: Breast Tumor Detection
Input: Input X-ray data from open-source dataset *Output: Localize the mass regions in breast.*

i.	*Input the image from the open-source dataset.*
ii.	*Transform the input mammography image (Image$_X$) to Neutrosophic domain (Image$_{NS}$) using Equation (4) where T_X, I_X and F_X are calculated using Equation (5), (6) and (7) respectively.*
iii.	*Reduce the indeterminacy using alpha mean operation on each neutrosophic set images from (T_X, I_X, F_X) to ($T_X(\alpha)$, $I_X(\alpha)$, $F_X(\alpha)$) using Equation (8)-(11).*
iv.	*Enhance the true image using beta enhancement operation on each neutrosophic alpha set images from ($T_X(\alpha)$, $I_X(\alpha)$, $F_X(\alpha)$) to ($T_X\grave{}(\beta)$, $I_X\grave{}(\beta)$, $F_X\grave{}(\beta)$) using Equation (12)-(15).*
v.	*Segment the resultant image using Gamma K-means Clustering*
vi.	*Draw the boundary box on the all the blobs of the segmented image to determine the masses.*

3. Evaluation

3.1 Datasets

There are many open datasets available in the market such as Breast Imaging Reporting and Data System (BI-RADS), Emory BrEast imaging Dataset (EMBED), Mammographic Image Analysis Society (MIAS), INbreast database etc. Our proposed method is worked on samples from MIAS and INbreast database. In MIAS, each image had the dimensions 1024×1024, and 90 images out of 322 contained tumor masses. In INbreast database holds about 410 images captured from 115 patients, out of which 90 cases with malignant masses and others are benign, calcifications or other types of masses.

3.2 Performance metrics

3.2.1 Entropy

Entropy calculates the intensity level distribution thereby quantifies the image information content. For better outcomes, a good clarity image with higher entropy value is needed at the end of enhancement stage. For a neutrosophic image entropy is defined as the addition of entropies of T, I and F domain using Equation 19 [14].

$$\mathrm{En_{NS}} = \mathrm{En_T} + \mathrm{En_I} + \mathrm{En_F} \tag{19}$$

$$En_T = -\sum_{i=\min\{T_X\}}^{\max\{T_X\}} p_T(i)\ln p_T(i)$$

$$En_I = -\sum_{i=\min\{I_X\}}^{\max\{I_X\}} p_I(i)\ln p_I(i)$$

$$En_F = -\sum_{i=\min\{F_X\}}^{\max\{F_X\}} p_F(i)\ln p_F(i)$$

where En_T, En_I and En_F - entropies of T, I and F domain

$p_T(i)$, $p_I(i)$ and $p_F(i)$ – probabilities of elements in T, I and F

3.2.2 Dice similarity co-efficient (DICE)

DICE is a statistical metric to validate the segmentation approach. It is a simple approach and is computed by the extent of overlapping pixel values between images that takes the segmentation area and the background with respect to the ground truth segmentation [15]. The Dice coefficients are defined using Equation 20:

$$\mathrm{DICE} = \frac{2 \times \mathrm{P_{TP}}}{2 \times \mathrm{P_{TP}} + \mathrm{P_{FP}} + \mathrm{P_{FN}}} \tag{20}$$

where P_{TP} is the number of true positive, N_{FP} and N_{FN} is the is the number of false positive and false negative respectively. The positive and negative term refers to the pixels belonging to the mass tissue area and background area in comparison with ground truth [16].

3.2.3 Boundary localization error (BLE)

To detect the accuracy of the localization of breast mass region, the difference between the predicted and the closest actual ground truth (top, side, height, and width of boundary boxes) are calculated. The learning objective is formulated using Equation 21 [17]:

$$L_i^{box} = ||\hat{z}_i^{box} - z_i^{box}||^2 \quad (21)$$

where $z_i^{box}$is the original or manual annotated coordinate position of the mass regions and z_i^{box} is the outcome of the proposed work coordinates including top, side, height and width.

3.2.4 Accuracy

The accuracy of the segmentation approach on mammography X-ray images using single valued neutrosophic approach is calculated using the Equation (22):

$$\text{Accuracy} = \frac{\text{TP+TN}}{\text{TP+TN+FP+FN}} \quad (22)$$

In the above equation, TP defines the number of true positive images with all detected masses and TN is the number of true negative i.e. predicting the non-mass region by comparing them manually. FP is the number of false positives i.e. non-mass region as mass. FN is the number of false negatives i.e. falsely neglecting a mass region.

3.3 Experimental Results

To refine the input images for better robustness and effectiveness, the neutrosophic images, T, I, F are enhanced using alpha and beta operations using Equation 4 – 15. At each stage, the entropy is calculated using Equation 19 to ensure how much detailed information is extracted. The values are tabulated in Table 1.

Table 1: Entropy values of Images in each pre-processing stage

Methods	**DICE**	**BLE**	**ACCURACY**
Image Template [1]	0.914	82.4%	87.27
PSO algorithm [6]	0.931	83.2%	86.63
Deep structured learning [5]	0.955	87.5%	89.01
Adaptive threshold [3]	0.85	71.7%	72.2
Proposed Neutrosophic	**0.972**	**92.8%**	**92.38**

The tabulated values derive the idea that after beta enhancement the images shows well defined edges and corners for better segmentation. For segmentation, metrics such as DICE, BLE and accuracy are used to compare with the other existing approaches available in market. For comparison purposes, BLE with the relative error measure ($\boldsymbol{L_i^{box} \leq 0.05}$) is only considered in the Table 2.

Table 2: Comparison of Breast detection methods

Techniques	Entropy value
Input X-ray Images	3.5521
Neutrosophic Image	3.7521
Alpha mean operation	4.8575
Beta enhancement operation	5.3424

As per the tabulated results, the proposed approach shows better results in terms of DICE, BLE and accuracy. As the indeterminate pixel value decreases using neutrosophic domain, better the outcomes in mass detection.

Conclusions

Detection of mass regions both benign or tumor on early stage would be a great help for good survival rate and thus proposed system focusses on breast lesion detection. To refine a medical X-ray input image for mass detection, this work focuses on neutrosophic image sets where the indeterminacy is minimized and maximizing the truth value. It is done using alpha-mean and beta enhancement methods to promote a well-defined edge for better segmentation results. Then, the gamma clustering technique is used segmentation. The above results demonstrate that the proposed neutrosophic method consistently outperform the state-of-the-art methods.

In future work, the detected mass will be classified as benign or tumor based on the mass intensity. It also focuses on developing a graphical user interface automatic tool that allows doctors for diagnosis usage.

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Neutrosophic Industry 5.0 Inventory Model with Technology Driven Demand and Costs Parameters

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Abstract: In this age of technology, the manufacturing sectors are embracing the elements of industry 5.0 to setup a robust kind of production process. This research work proposes a novel neutrosophic production inventory model encompassing the cost parameters of technology inaddition to the conventional inventory costs. In this model the demand is expressed as function of technology of the form $\alpha e^{-\beta t} + \gamma t$ with the coefficients α, β and γdealing with the initial demand, decrease in demand over time and increase in demand with the adoption of new technology. The neutrosophic model developed in this work addresses the decision circumstances of indeterminacy in addition to uncertainty. The primary objective of this paper is to introduce the notion of technology driven demand and new types of costs associated with technology in a neutrosophic modelling environment. The proposed neutrosophic model is simulated and sensitively analyzed to draw inferences of the parameters over the production quantity q(t). The efficiency of this neutrosophic model is determined on making comparative analysis with crisp data sets. The neutrosophic model possesses high degree of flexibility and applicability in technology dominant manufacturing firms facilitating the decision makers to design optimal solutions.

Keywords: Neutrosophic sets, Inventory, Technology driven demand, Optimization

1. Introduction

Industry 5.0[1] is an advancement of industry 4.0 and it primarily focusses on the integration of digital technologies such as IoT (Internet of Things), Artificial Intelligence and big data analytics to evolve intelligent factory frameworks. Industry 4.0 is much concerned on automation whereas Industry 5.0[2] is the next evolution emphasizing on collaborative endeavours between humans and machines. The future manufacturing systems will be certainly embedded into the framework of industry 5.0[3] with key factors such as human-machine collaboration, customization, personalization, sustainability, resilience,flexibility and ethical responsibility.The incorporation of digital oriented technology into production firm costs the manufacturing sectors and this will forman integral part of product production in near future. Inventory modelling is primarily concerned on stock management together with costs optimization. In general, the inventory total cost comprisescosts of ordering, production, holding, shortages, backlog, deterioration, rework, remanufacture and many other related to product production and distribution. However, the modifications in the production system are reflected in the total inventory costs. The developments in the production technology and the augmentations in a production framework incur costs of various kinds and that have to be considered in optimizing the inventory costs.

The production system is expected to be customer centered and function with sustainable consciousness. However, as the demands of the customers are varying with respect to time and with reference to advancements in technology, the production system embeds the facets of technology into their production framework and so as the relevant technology costs with the usual inventory management costs. Inventory modeling with technology driven demand is the key factor of industry 5.0 embraced production firms. The cost parameters considered in modelling may not be of deterministic in nature at all instances as there may exist few indeterminacies in such considerations. To resolve such cases, the theory of neutrosophy is incorporated into inventory modeling to evolve a comprehensive inventory model with neutrosophic parametric representations. Neutrosophic theory developed by Smarandache deals with three- valued function with truth, indeterminacy and falsity membership representations. The objective of this research work is to formulate a neutrosophic inventory model with technology driven demand and technology associated cost parameters. This work also performs a comparative analysis between neutrosophic model and deterministic model to showcase the efficacy of neutrosophic representations of the parameters. The attempt of considering technology dependent demand is a novel aspect of this research work and this has certainly more scope in discussing the nature of the demand under different conditions.

The remaining contents are presented in the following sections. Section 2 consists of the detailed literature review of the works associated with neutrosophic inventory modelling. Section 3 consists of basic definitions of neutrosophy. Section 4 discusses the key aspects of Industry 5.0 and Inventory 5.0. Section 5 develops the inventory model. Section 6 validates the proposed model with numerical examples considering both crisp and neutrosophic representations of the parameters. Section 7 compares the illustrations and draws inferences. Section 8 describes the industrial implications and the last section concludes the work with the ideas and scope of further extension.

2.Literature Review

This section outlines the state of art of neutrosophic inventory models focusing the robustness of neutrosophic representations of inventory parameters. Kar et al[4] modelled inventory model with space constraints using neutrosophic geometric programming. The multi-objective inventory model putforth effectively handled the uncertainty and indeterminacies in inventory management. Mullai and Broumi[5] devised neutrosophic based inventory model without shortages to determine the demand and supply flow in an inventory system. Das and Islam[6] developed fuzzy integrated multi-item inventory model with neutrosophic hesitant fuzzy programming approach to address the challenges of deterioration, space constraints with demand dependent production costs. Mullai and Surya[7] developed neutrosophic inventory backorder problem using triangular neutrosophic numbers considering uncertainty in demand and lead times. Mullai et al[8] formulated a single valued neutrosophic based inventory model with neutrosophic random variable to make accurate estimations of inventory decisive parameters. Islam et al[9] applied neutrosophic hesitant fuzzy programming approach in inventory modeling considering deterioration and space constraints with and without shortages. Mondal et al[10] presented a neutrosophic optimization approach considering inventory policies for seasonal items with logistic-growth demand rates. Rajeswari et al[11] developed reusable based inventory model with octagonal fuzzy neutrosophic numbers. Jdid et al[12] framed a static inventory model with neutrosophic logic and also devised neutrosophic based inventory model with safety reserve.

Sugapriya et al[13] modelled an effective container inventory model with bipolar neutrosophic representations. Garg et al proposed a model for container inventory using trapezoidal bipolar neutrosophic number. Bhavani et al[14] developed a neutrosophic based inventory system with particle swarm optimization algorithm for handling discounts, deterioration items. Sen and Chakrabarthi[15] proposed an industrial based production inventory model with deterioration using neutrosophic fuzzy optimizing approaches. Sarkar and Srivastava[16] presented a multi-item multi-objective neutrosophic model considering sustainability cost parameters. Das and Islam [17]sketched out a neutrosophic based programming approach for optimizing multi-objective shortage follow inventory (SFI) model with ramp demand. Surya and Mullai[18] presented Neutrosophic multi-item inventory control models. Jayanthi [19]developed a neutrosophic fuzzy geometric approach in overage management. Rajeswari [20]employed neutrosophic approaches for optimizing efficient inventory systems. Barman et al[21] modelled rework-based inventory model with neutrosophic representations. Jdid[22] devised neutrosophic static inventory model with economic indicators. Pattnaik et al[23] designed neutrosophic inventory model to handle overage items. Bhavani and Mahapatra[24] proposed an inventory model with generalized triangular neutrosophic cost parameters. Mohanta et al[25] developed neutrosophic inventory model with two-level trade credit policy to handle perishable products. Mohanta[26] framed neutrosophic integrated smart manufacturing-oriented inventory model. Kalaiarasi and Swathi[27] proposed neutrosophic inventory model with quick returns. Miriam et al[28] discussed a neutrosophic rework warehouse inventory model for product distribution considering quality aspects. Dubey et al[29] presented a survey on neutrosophic based inventory problems. Supakar et al [30]developed neutrosophic inventory model and integrated with artificial bee colony algorithm to discuss green production inventory system. Moorthy et al[31] applied neutrosophic logic in inventory management. Surya et al[32] developed neutrosophic inventory model to handle decay items and price dependent demand. Kar et al[33] formulated multi-objective perishable multi-item green neutrosophic inventory models. Das and Islam[34] developed a multi-item inventory model with quadratic demand patterns and with neutrosophic Pythagorean hesitant fuzzy programming. Dubey and Kumar[35] modelled cost effective neutrosophic inventory model.Martin et al[36] designed a neutrosophic based industry 4.0 inventory model integrating neutrosophic logic with inventory cost parameters. These inventory models based on neutrosophy are modelled using different neutrosophic representations. However, neutrosophic trapezoidal interval valued fuzzy numbers are used in inventory optimization. Also, industry 4.0 based neutrosophic inventory model is more industrial centered and this contribution has motivated the authors to explore industry 5.0. Also, the demand nature discussed in the aforementioned inventory models are conventional. This research work has attempted to represent the demand as technology dependent. This work proposes a neutrosophic inventory model integrating the aspects of both industry and inventory 5

3.Preliminaries

This section presents the basic definitions of neutrosophic sets.

3.1 Neutrosophic set[37]

A neutrosophic set is characterized independently by a truth-membership function (x) , an indeterminacy-membership function $\beta(x)$, and a falsity-membership function $\gamma(x)$ and each of the function is defined from $X \rightarrow [0,1]$

3.2 Single valued Neutrosophic set(SVNS)[38]

A SVNS is denoted and defined as $\widetilde{A_N} = \{x, T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)/x\epsilon X\}$, where for each generic point x in X, $T_{\widetilde{A_N}}(x)$ called truth membership function, $I_{\widetilde{A_N}}(x)$ called indeterminacy membership function and $F_{\widetilde{A_N}}(x)$ called falsity membership function in [0,1] and $0 \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3$. For continuous SNVS$\widetilde{A_N} = \int_{A_N} \langle T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle / x_i, x_i \epsilon X$.

3.3 Neutrosophic number[38]

Let x be a generic element of a non empty set x. A neutrosophic number $\widetilde{A_N}$ in X is defined as $\widetilde{A_N} = \{$ x, $\langle T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)\rangle / x\epsilon X\}, \forall T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x) and F_{\widetilde{A_N}}(x)$ belongs to $]0^-, 1^+[$ where $T_{\widetilde{A_N}}: X \rightarrow]0^-, 1^+[I_{\widetilde{A_N}} : X \rightarrow]0^-, 1^+[and F_{\widetilde{A_N}}: X \rightarrow]0^-, 1^+[$ are functions of truth – membership, indeterminacy membership and falsity – membership in $\widetilde{A_N}$respectively with

$0^{-} \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^{+}$.

3.4 Interval – valued neutrosophic set[38]

Let X be a nonempty set. Then an interval – valued neutrosophic set [IVNS] $\widetilde{A_N}^{IV}$ of X is defined as:

$\widetilde{A_N}^{IV} = \{\langle x; [T^L_{\widetilde{A_N}}, T^U_{\widetilde{A_N}}], [I^L_{\widetilde{A_N}}, I^U_{\widetilde{A_N}}], [F^L_{\widetilde{A_N}}, F^U_{\widetilde{A_N}}]\rangle : x \epsilon X\}$ where

$[T^L_{\widetilde{A_N}}, T^U_{\widetilde{A_N}}], [I^L_{\widetilde{A_N}}, I^U_{\widetilde{A_N}}] and [F^L_{\widetilde{A_N}}, F^U_{\widetilde{A_N}}] \subset [0,1]$ $\forall x \epsilon X$ $T^L_{\widetilde{A_N}} = \inf(T_{\widetilde{A_N}})$, $T^U_{\widetilde{A_N}} = \sup(T_{\widetilde{A_N}})$; $I^L_{\widetilde{A_N}} = \inf(I_{\widetilde{A_N}})$, $I^U_{\widetilde{A_N}} = \sup(I_{\widetilde{A_N}})$; and $F^L_{\widetilde{A_N}} = \inf(F_{\widetilde{A_N}})$, $F^U_{\widetilde{A_N}} = \sup(F_{\widetilde{A_N}})$

3.5 Neutrosophic Trapezoidal Interval Valued Number[37]

A single valued trapezoidal Neutrosophic number $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a special neutrosophic set on the real number set R, whose truth – membership, indeterminacy – membership, and a falsity – membership are given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{(x-a_1)w_{\tilde{a}}}{b_1 - a_1}, & (a_1 \leq x < b_1) \\ w_{\tilde{a},} & (b_1 \leq x \leq c_1) \\ \dfrac{(d_1 - x)w_{\tilde{a}}}{d_1 - c_1}, & (c_1 < x \leq d_1) \\ 0, & otherwise \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} (b_1 - x + u_{\tilde{a}}(x - a_1))/(b_1 - a_1), & (a_1 \leq x < b_1) \\ u_{\tilde{a}} & (b_1 \leq x \leq c_1) \\ \dfrac{x - c_1 + u_{\tilde{a}}(d_1 - x)}{(d_1 - c_{1)}}, & (c_1 < x \leq d_1) \\ 1, & otherwise \end{cases}$$

and

$$\lambda_{\tilde{a}}(x) = \begin{cases} (b_1 - x + y_{\tilde{a}}(x - a_1))/(b_1 - a_1), & (a_1 \leq x < b_1) \\ y_{\tilde{a}} & (b_1 \leq x \leq c_1) \\ \dfrac{x - c_1 + y_{\tilde{a}}(d_1 - x)}{(d_1 - c_{1)}}, & (c_1 < x \leq d_1) \\ 1, & otherwise \end{cases}$$

respectively.

3.6 Operational Laws on IVTrNeNs[38]

Let $\widetilde{a_{N_1}}^{IV} = \langle [\{(a_1, b_1, c_1, d_1); u_{\widetilde{a_{N_1}}^{IV}}\}, \{(e_1, f_1, g_1, h_1); v_{\widetilde{a_{N_1}}^{IV}}\}, \{(l_1, m_1, n_1, p_1); w_{\widetilde{a_{N_1}}^{IV}}\}] \rangle$ and $\widetilde{a_{N_2}}^{IV} = \langle [\{(a_2, b_2, c_2, d_2); u_{\widetilde{a_{N_2}}^{IV}}\}, \{(e_2, f_2, g_2, h_2); v_{\widetilde{a_{N_2}}^{IV}}\}, \{(l_2, m_2, n_2, p_2); w_{\widetilde{a_{N_2}}^{IV}}\}] \rangle$ be two IVTrNeNs with twelve components, where $u_{\widetilde{a_{N_1}}^{IV}} = [u_{\widetilde{a_{N_1}}^{IV}}{}^L, u_{\widetilde{a_{N_1}}^{IV}}{}^U]$; $u_{\widetilde{a_{N_2}}^{IV}} = [u_{\widetilde{a_{N_2}}^{IV}}{}^L, u_{\widetilde{a_{N_2}}^{IV}}{}^U]$;

$v_{\widetilde{a_{N_1}}^{IV}} = [v_{\widetilde{a_{N_1}}^{IV}}{}^L, v_{\widetilde{a_{N_1}}^{IV}}{}^U]$; $v_{\widetilde{a_{N_2}}^{IV}} = [v_{\widetilde{a_{N_2}}^{IV}}{}^L, v_{\widetilde{a_{N_2}}^{IV}}{}^U]$; $w_{\widetilde{a_{N_1}}^{IV}} = [w_{\widetilde{a_{N_1}}^{IV}}{}^L, w_{\widetilde{a_{N_1}}^{IV}}{}^U]$;

$w_{\widetilde{a_{N_2}}^{IV}} = [w_{\widetilde{a_{N_2}}^{IV}}{}^L, w_{\widetilde{a_{N_2}}^{IV}}{}^U]$, then the following operations hold:

Addition of IVTrNeNs:

$$\widetilde{a_{N_1}}^{IV} + \widetilde{a_{N_2}}^{IV} = \langle \begin{bmatrix} \{(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2); u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\} \\ \{(e_1+e_2, f_1+f_2, g_1+g_2, h_1+h_2); v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}} \\ \{(l_1+l_2, m_1+m_2, n_1+n_2, p_1+p_2); w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}} \end{bmatrix} \rangle$$

Negative of IVTrNeNs:

$$-\widetilde{a_{N_2}}^{IV} = \langle\left[\{-d_2,-c_2,-b_2,-a_2), u_{\widetilde{a_{N_2}}^{IV}}\right\}, \{-h_2,-g_2,-f_2,-e_2), v_{\widetilde{a_{N_2}}^{IV}}\}, \{-p_2,-n_2,-m_2,-l_2), w_{\widetilde{a_{N_2}}^{IV}}\}]\rangle$$

Subtraction of IVTrNeNs:

$$\widetilde{a_{N_1}}^{IV} - \widetilde{a_{N_2}}^{IV} = \langle\begin{bmatrix} \left\{(a_1-d_2, b_1-c_2, c_1-b_2, d_1-a_2); u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right\} \\ \{(e_1-h_2, f_1-g_2, g_1-f_2, h_1-e_2);\ v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}} \\ \{(l_1-p_2, m_1-n_2, n_1-m_2, p_1-l_2);\ w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}} \end{bmatrix}\rangle$$

Scalar multiplication of SVTrNeN:

$$\lambda\widetilde{a_{N_1}}^{IV} = \begin{cases} \langle\left\{(\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); u_{\widetilde{a_{N_1}}^{IV}}\right\}, \left\{(\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1); v_{\widetilde{a_{N_1}}^{IV}}\right\}, \left\{(\lambda l_1, \lambda m_1, \lambda n_1, \lambda p_1); w_{\widetilde{a_{N_1}}^{IV}}\right\}\rangle \ if \lambda > 0 \\ \langle\left\{(\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1); u_{\widetilde{a_{N_1}}^{IV}}\right\}, \left\{(\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1); v_{\widetilde{a_{N_1}}^{IV}}\right\}, \left\{(\lambda e_1, \lambda f_1, \lambda g_1, \lambda h_1); w_{\widetilde{a_{N_1}}^{IV}}\right\}\rangle \ if \lambda < 0 \end{cases}$$

Multiplication of SVTrNeN:

$$\widetilde{a_{N_1}}^{IV}.\ \widetilde{a_{N_2}}^{IV} = \begin{cases} \langle(a_1.a_2, b_1.b_2, c_1.c_2, d_1.d_2); (e_1.e_2, f_1.f_2, g_1.g_2, h_1.h_2); (l_1.l_2, m_1.m_2, n_1.n_2, p_1.p_2) \\ \left[u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right],\ [v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_2}^{IV}}], \left[u_{\widetilde{a_{N_1}}^{IV}} \vee u_{\widetilde{a_2}^{IV}}\right] if d_1 > 0, d_2 > 0, h_1 > 0, h_2 > 0, p_1 > 0, p_2 > 0\rangle \\ \langle(a_1.d_2, b_1.c_2, c_1.b_2, d_1.a_2); (e_1.h_2, f_1.g_2, g_1.f_2, h_1.e_2); (l_1.p_2, m_1.n_2, n_1.m_2, p_1.l_2) \\ \left[u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right],\ [v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_2}^{IV}}], \left[u_{\widetilde{a_{N_1}}^{IV}} \vee u_{\widetilde{a_2}^{IV}}\right] if d_1 < 0, d_2 > 0, h_1 < 0, h_2 > 0, p_1 < 0, p_2 > 0\rangle \\ (d_1.d_2, c_1.c_2, b_1.b_2, a_1.a_2); (h_1.h, g_1.g_2, f_1.f_2, e_1.e_2); (p_1.p_2, n_1.n_2, m_1.m_2, l_1.l_2) \\ \langle\left[u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right],\ [v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_2}^{IV}}], \left[u_{\widetilde{a_{N_1}}^{IV}} \vee u_{\widetilde{a_2}^{IV}}\right] if d_1 < 0, d_2 < 0, h_1 < 0, h_2 < 0, p_1 < 0, p_2 < 0\rangle \end{cases}$$

Inverse of SVTrNeN:

$$\boldsymbol{s(\widetilde{a_{N_1}}^{IV})^{-1}} = \frac{\mathbf{1}}{\widetilde{\boldsymbol{a_{N_1}}}^{\boldsymbol{IV}}}\begin{cases} \langle\left(\frac{1}{d_1}, \frac{1}{c_1}, \frac{1}{b_1}, \frac{1}{a_1}\right); \left(\frac{1}{h_1}, \frac{1}{g_1}, \frac{1}{f_1}, \frac{1}{e_1}\right); \left(\frac{1}{p_1}, \frac{1}{n}, \frac{1}{m_1}, \frac{1}{l_1}\right); \boldsymbol{u_{\widetilde{a_{N_1}}^{IV}}, v_{\widetilde{a_{N_1}}^{IV}}, w_{\widetilde{a_{N_1}}^{IV}}}\rangle, \\ \boldsymbol{if a_1 > 0, b_1 > 0, c_1 > 0, d_1 > 0, e_1 > 0, f_1 > 0, g_1 > 0, h_1 > 0, l_1 > 0, m_1 > 0, n_1 > 0, p_1 > 0} \\ \langle\left(\frac{1}{a_1}, \frac{1}{b_1}, \frac{1}{c_1}, \frac{1}{d_1}\right); \left(\frac{1}{e_1}, \frac{1}{f_1}, \frac{1}{g_1}, \frac{1}{h_1}\right); \left(\frac{1}{l_1}, \frac{1}{m_1}, \frac{1}{n_1}, \frac{1}{p_1}\right); \boldsymbol{u_{\widetilde{a_{N_1}}^{IV}}, v_{\widetilde{a_{N_1}}^{IV}}, w_{\widetilde{a_{N_1}}^{IV}}}\rangle \\ \boldsymbol{if a_1 < 0, b_1 < 0, c_1 < 0, d_1 < 0, e_1 < 0, f_1 < 0, g_1 < 0, h_1 < 0, l_1 < 0, m_1 < 0, n_1 < 0, p_1 < 0} \end{cases}$$

Division of SVTrNeN:

$$\frac{\widetilde{a_{N_1}}^{IV}}{\widetilde{a_{N_2}}^{IV}} = \begin{cases} \langle\left(\frac{a_1}{d_1}, \frac{b_1}{c_1}, \frac{c_1}{b_1}, \frac{d_1}{a_1}\right), \left(\frac{e_1}{h_1}, \frac{f_1}{g_1}, \frac{g_1}{f_1}, \frac{h_1}{e_1}\right); \left(\frac{l_1}{p_1}, \frac{m}{n_1}, \frac{n_1}{m_1}, \frac{p_1}{l_1}\right); \left[u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right], \left[v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}}\right], \left[w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}}\right]\rangle, \\ if d_1 > 0, d_2 > 0, h_1 > 0, h_2 > 0, p_1 > 0, p_2 > 0, \\ \langle\left(\frac{d_2}{d_1}, \frac{c_2}{c_1}, \frac{b_2}{b_1}, \frac{a_2}{a_1}\right), \left(\frac{h_2}{h_1}, \frac{g_2}{g_1}, \frac{f_2}{f_1}, \frac{e_2}{e_1}\right); \left(\frac{p_2}{p_1}, \frac{n_2}{n_1}, \frac{m_2}{m_1}, \frac{l_2}{l_1}\right); \left[u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right], \left[v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}}\right], \left[w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}}\right]\rangle, if \\ d_1 < 0, d_2 > 0, h_1 < 0, h_2 > 0, p_1 < 0, p_2 > 0, \\ \langle\left(\frac{d_2}{a_1}, \frac{c_2}{b_1}, \frac{b_2}{c_1}, \frac{a_2}{d_1}\right), \left(\frac{h_2}{e_1}, \frac{g_2}{f_1}, \frac{f_2}{g_1}, \frac{e_2}{h_1}\right); \left(\frac{p_2}{l_1}, \frac{n_2}{m_1}, \frac{m_2}{n_1}, \frac{l_2}{p_1}\right); \left[u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}}\right], \left[v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}}\right], \left[w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}}\right]\rangle, if \\ d_1 < 0, d_2 < 0, h_1 < 0, h_2 < 0, p_1 < 0, p_2 < 0. \end{cases}$$

Where $u_{\widetilde{a_{N_1}}^{IV}} \wedge u_{\widetilde{a_2}^{IV}} = \left[\min\left(u_{\widetilde{a_{N_1}}^{IV}}{}^L,\ u_{\widetilde{a_{N_2}}^{IV}}{}^L\right), \min\left(u_{\widetilde{a_{N_1}}^{IV}}{}^U,\ u_{\widetilde{a_{N_2}}^{IV}}{}^U\right)\right]$,

$v_{\widetilde{a_{N_1}}^{IV}} \vee v_{\widetilde{a_{N_2}}^{IV}} = [max\left(v_{\widetilde{a_{N_1}}^{IV}}{}^{L},\ v_{\widetilde{a_{N_2}}^{IV}}{}^{L}\right), max(v_{\widetilde{a_{N_1}}^{IV}}{}^{U},\ v_{\widetilde{a_{N_2}}^{IV}}{}^{U})$ and

$w_{\widetilde{a_{N_1}}^{IV}} \vee w_{\widetilde{a_{N_2}}^{IV}} = [max\left(w_{\widetilde{a_{N_1}}^{IV}}{}^{L},\ w_{\widetilde{a_{N_1}}^{IV}}{}^{L}\right), max(w_{\widetilde{a_{N_1}}^{IV}}{}^{U},\ w_{\widetilde{a_{N_1}}^{IV}}{}^{U})]$.

3.7 Score and Accuracy functions of IVTrNeNs[38]

The score function concept is used to find comparison between two IVTrNeNs. Greater of score function value demonstrate the greater of IVTrNeN. According the base of the score and accuracy functions of an IVTrNeN$\widetilde{a_N}^{IV}$canbe defined asfollows:

$$S(\widetilde{a_{N_1}}^{IV}) = \frac{1}{12}\left((8 + (a_1 + b_1 + c_1 + d_1) - (e_1 + f_1 + g_1 + h_1) - (l_1 + m_1 + n_1 + p_1)\right) \times (2 + u_{\widetilde{a_{N_1}}^{IV}}{}^{L} + u_{\widetilde{a_{N_2}}^{IV}}{}^{U} - v_{\widetilde{a_{N_1}}^{IV}}{}^{L} - v_{\widetilde{a_{N_1}}^{IV}}{}^{U} - w_{\widetilde{a_{N_1}}^{IV}}{}^{L} - w_{\widetilde{a_{N_1}}^{IV}}{}^{U}$$

$S(\widetilde{a_{N_1}}^{IV})\epsilon[0,1]$. The accuracy function $A(\widetilde{a_{N_1}}^{IV}) = \frac{1}{4}(a_1 + b_1 + c_1 + d_1 - l_1 - m_1 - n_1 - p_1) \times (2 + u_{\widetilde{a_{N_1}}^{IV}}{}^{L} + u_{\widetilde{a_{N_2}}^{IV}}{}^{U} - v_{\widetilde{a_{N_1}}^{IV}}{}^{L} - v_{\widetilde{a_{N_1}}^{IV}}{}^{U} - w_{\widetilde{a_{N_1}}^{IV}}{}^{L} - w_{\widetilde{a_{N_1}}^{IV}}{}^{U})$

4. Industry 5.0 and Inventory 5.0

Industry 5.0

Industry 5.0 is an enhanced version of industry 4.0 which focuses more on collaboration of both human and machine to evolve a more technology centered scenario. The integration of digital technologies with industrial settingis the prime highlight of this advanced version. The key characteristic features of industry 5.0 are presented pictorially in Fig. 1.

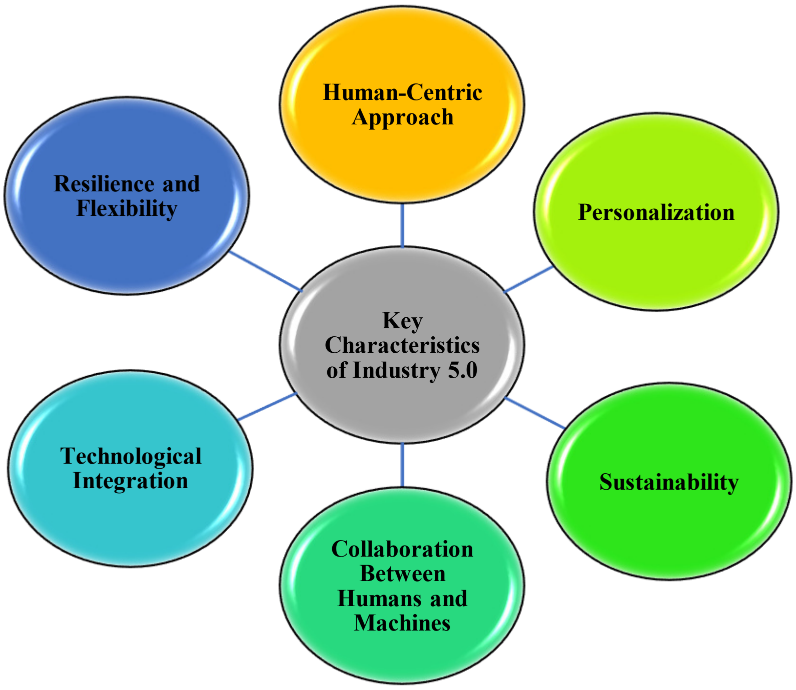

Fig.1. Key Characteristics of Industry 5.0

Human-Centric Approach: Industry 5.0 provides a setting to harmonize the associations between the human and the machines. The competency of the human beings is utilized in resolving the problems by making optimal decisions with the assistance of the machines in handling the tedious tasks.

Personalization: The advanced technical augmentation facilitates customization suiting to the needs of the customers. This is made possible with the collaborative roles of both human and machines enabling a flexible manufacturing process.

Sustainability: Industry 5.0 is more sustainable centric as it facilitates circular economy contributing to waste reduction and environmental sustainability. Resource optimization, remanufacturing and recycling are some of the pivotal roles of this enhanced version.

Collaboration Between Humans and Machines: Industry 5.0 envisages a synergetic relationship between human workers and machines. The working of human with robots in a parallel manner increases the productivity together with the leverage of manpower expertise in the required circumstances.

Technological Integration: Industry 5.0 is circumscribed with advanced technology comprising the approaches of AI, machine learning, data analytics to evolve a more intelligent production system. These well-developed systems are more adaptative in nature and capable of responding to the dynamic needs of the production.

Resilience and Flexibility: Industry 5.0 is potent in adapting to the disruptions occurring in the manufacturing systems thereby ensuring a more flexible and adaptable manufacturing system.

Inventory 5.0

Inventory 5.0 shall be characterized as the enhancement of inventory management in align wit the key components of industry 5.0. The integration of digital technology facets with inventory handling leverages the applications of real-time data analytics, AI and automation making the inventory system more flexible and sustainable.

The key components of Inventory 5.0 are presented in Fig.2 and it explicitly exhibits the attributes of this advanced inventory system.

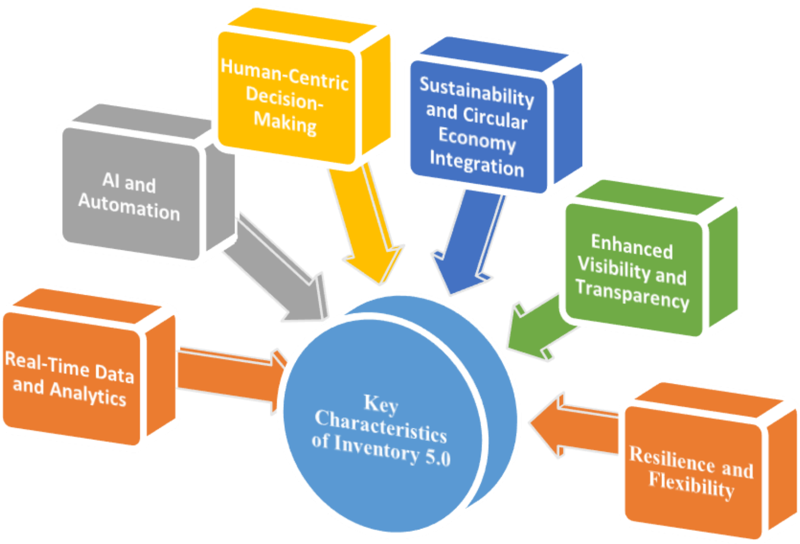

Fig.2. Key Characteristics of Inventory 5.0

On comparing the key components of industry 5.0 and inventory 5.0, it is highly evident that these two phenomena are closely related and there are functioning points of opportunities in align with one another. As both the advanced versions are relatively associated with digital technologies attributing to resilience, flexibility and sustainability, the integration of inventory 5.0 with industry 5.0 may extremely benefit the manufacturing systems.

5. Model Development

This section presents the development of an inventory model characterizing the manufacturing system with technology dependence. Let us consider a technology-oriented inventory system which considers demand to be technology driven and incorporates technology associated cost parameters with other conventional cost parameters. The demand is expressed of the form $\alpha e^{-\beta t} + \gamma t$, where

$\alpha e^{-\beta t}$: Represent the decrease demand over time as initial technology adoption occurs. Here, α is the initial demand due to technologyand β is the rate at which the demand decreases.

γt: Represent the increase demand over time due to continuous market penetration and adoption of new technologies. Here, γ is the coefficient that indicates how demand increases with time.

The model is developed with the assumptions of P > D; no shortages are allowed; D is technology driven.

The notations used in this model are presented as follows,

$\alpha -$ Technology adoption rate coefficient
$\beta -$ Innovation level coefficient
$\gamma -$ Market penetration coefficient
$t -$ Time
P – Production
D – Demand
T_c– Technology Acquisition Costs
$M_c -$ Maintenance and Upgrade Costs
$T_{RC} -$ Training Costs
$I_c -$ Integration Costs
$C_c -$ Cyber security Costs
$D_c -$ Depreciation Costs
T – Total Production time
C_1 – Constant carrying cost per unit time
C_3 – Fixed Ordering cost
$C_2 -$ Purchase Costs
TC – Total Cost
TAC – Total average cost

Let us consider a system of differential equations with technology driven demand

For $0 \le t \le t_1$

$$\frac{dq(t)}{dt} = P - (\alpha e^{-\beta t} + \gamma t)$$

For $t_1 \le t \le T$

$$\frac{dq(t)}{dt} = (\alpha e^{-\beta t} + \gamma t)$$

$\alpha e^{-\beta t}$: Represent the decrease demand over time as initial technology adoption occurs. Here, α is the initial demand due to technology,and β is the rate at which the demand decreases.

γt: Represent the increase demand over time due to continuous market penetration and adoption of new technologies. Here, γ is the coefficient that indicates how demand increases with time.

For $0 \le t \le t_1$

The differential equation is

$$\frac{dq(t)}{dt} = P - (\alpha e^{-\beta t} + \gamma t) \quad (1)$$

This can be solved by integrating both sides with respect to t

$$\int_0^t dq(t) = \int_0^t (P - \alpha e^{-\beta t} - \gamma t)dt$$

$$q(t) - q(0) = Pt + \frac{\alpha e^{-\beta t}}{\beta} - \frac{\gamma t^2}{2} - \frac{\alpha}{\beta} + C_1$$

Given the initial condition $q(0) = 0$ (2)

That implies $C_1 = 0$

Therefore $q(t) = Pt + \frac{\alpha e^{-\beta t}}{\beta} - \frac{\gamma t^2}{2} - \frac{\alpha}{\beta}$ (3)

For $t_1 \le t \le T$

The differential equation is

$\frac{dq(t)}{dt} = (\alpha e^{-\beta t} + \gamma t)$ (4)

This can be solved by integrating both sides with respect to t

$$\int_t^T dq(t) = -\int_t^T (\alpha e^{-\beta t} + \gamma t) dt$$

We use the boundary condition $q(T) = 0$ (5)

That implies $C_2 = 0$

$q(t) = -\frac{\alpha e^{-\beta T}}{\beta} + \frac{\alpha e^{-\beta t}}{\beta} + \frac{\gamma T^2}{2} - \frac{\gamma t^2}{2}$ (6)

Let $q(t_1) = I_{max}$ (7)

Equation (3) and (6) we get

$I_{max} = Pt_1 + \frac{\alpha e^{-\beta t_1}}{\beta} - \frac{\gamma t_1{}^2}{2} - \frac{\alpha}{\beta}$

$t_1 = \frac{I_{max}}{P-\alpha}$ (8)

$$I_{max} = -\frac{\alpha e^{-\beta T}}{\beta} + \frac{\alpha e^{-\beta t_1}}{\beta} + \frac{\gamma T^2}{2} - \frac{\gamma t_1{}^2}{2}$$

$T - t_1 = \frac{I_{max}}{\alpha}$ (9)

$I_{max} = \frac{\alpha(P-\alpha)T}{P}$ (10)

Holding Cost = $C_1 \left[\int_0^{t_1} q(t)dt + \int_{t_1}^{T} q(t)dt\right]$

$$= C_1\left(\int_0^{t_1} (P-\alpha)t_1 dt + \int_{t_1}^{T} \alpha(T-t_1)dt\right)$$

$$= C_1\left((P-\alpha)\frac{t_1{}^2}{2} + \alpha\frac{(T-t_1)^2}{2}\right)$$

Holding Cost $= \frac{C_1}{2}\left(1 - \frac{\alpha}{P}\right)\alpha T^2$ [Using (8), (9) and (10)]

Total Cost = Purchase Costs +Fixed Ordering cost + Holding Cost + Technology Acquisition Costs + Maintenance and Upgrade Costs + Training Costs + Integration Costs + Cyber security Costs + Depreciation Costs

Total Average Cost (T) TC= $\frac{1}{T}\left(C_2 + C_3 + \frac{C_1}{2}\left(1 - \frac{\alpha}{P}\right)\alpha T^2 + T_c + M_c + T_{RC} + I_c + C_c + D_c\right)$ ------- (11)

Differentiate with respect to T

$T = \sqrt{\frac{2(C_2+C_3+T_c+M_c+T_{RC}+I_c+C_c+D_c)}{C_1(1-\frac{\alpha}{P})\alpha}}$ ----------------- (12)

6. Illustration

Let us consider an inventory system with the following parameters.

$$\alpha = 0.5, \beta = 0.1, \gamma = 0.05, P = 1000\ units, T_c = 2000, M_c = 1500, T_{Rc} = 1000, I_c = 500,$$
$$C_c = 300, D_c = 800, C_1 = 2, C_2 = 10000, C_3 = 500.$$

By using the above expressions of T, TC and t_1, the following values are obtained.

$$T = 182.25, t_1 = 0.091, TC = 33199.23$$

Neutrosophic Trapezoidal Fuzzy Numbers for Parameters:

$\alpha = (0.45,0.5,0.55,0.6; [0.9,1], [0.1,0.15], [0,0.05])$

$\beta = (0.08,0.1,0.12,0.14; [0.85,0.95], [0.05,0.1], [0,0.02])$

$\gamma = (0.04,0.05,0.06,0.07; [0.8,0.9], [0.1,0.2], [0,0.05])$

$P = (950,1000,1050,1100; [0.95,1], [0.05,0.1], [0,0.05])$

$T_c = (1900,2000,2100,2200; [0.9,0.95], [0.05,0.1], [0,0.05])$

$M_c = (1425,1500,1575,1650; [0.9,0.95], [0.05,0.1], [0,0.05])$

$T_{RC} = (950,1000,1050,1100; [0.85,0.95], [0.05,0.1], [0,0.05])$

$I_c = (475,500,525,550; [0.85,0.95], [0.05,01], [0,0.05])$

$C_c = (285,300,315,330; [0.8,0.9], [0.1,0.15, [0,0.05])$

$D_c = (760,800,840,880; [0.85,0.9], [0.1,0.15], [0,0.05])$

$C_1 = (1.8,2.,2.2,2.4; [0.95,1], [0.05,0.1], [0,0.05])$

$C_2 = (9500,10000,10500,11000; [0.9,0.95], [0.05,0.1], [0,0.05])$

$C_3 = (475,500,525,550; [0.9,0.95], [0.05,0.1], [0,0.05])$

I_{max}= (180,200,220,240; [0.9,0.95], [0.05,0.1], 0,0.05])

$T = (148.04,165.49,186.76,212.38; [0.8,0.9], [0.05,0.95], [0,0.05])$

$t_1 = (0.164,0.191,0.22,0.25; [0.9,0.95], [0.05,0.1], 0,0.05])$

$TC = (24642.38,30319,38518.65,50715.82; [0.8,0.9], [0.05,0.95], [0,0.05])$

Results and Discussion

The above formulated inventory model is validated with both crisp and interval valued trapezoidal neutrosophic numbers and the results obtained are presented in the Table 1.

Table 1. Result of both crisp and interval valued trapezoidal neutrosophic numbers

Results	Crisp data input	Interval valued trapezoidal neutrosophic data input
T	182.25	(148.04,165.49,186.76,212.38; [0.8,0.9], [0.05,0.95], [0,0.05])
t_1	0.091	(0.164,0.191,0.22,0.25; [0.9,0.95], [0.05,0.1], 0,0.05])
TC	33199.23	(24642.38,30319,38518.65,50715.82; [0.8,0.9], [0.05,0.95], [0,0.05])

8. Sensitivity Analysis

The parameter α are varied and the respective changes on T and TC are determined with both crisp and neutrosophic input.

Case (1) Variations of the parameter α with Crisp data input

The values obtained on varying α are presented in Table 2.

Table 2. Variation of α with Crisp data

α	T	Total Average Cost
0.1	407.451349	81.482121
0.2	288.126020	115.227358
0.3	235.265676	141.117058
0.4	203.756243	162.939793
0.5	182.254241	182.163114
0.6	166.382922	199.539710
0.7	154.048360	215.516737
0.8	144.106257	230.385555
0.9	135.871481	244.348554
1.0	128.905456	257.553101

The respective graphical representations are presented in Fig.3.

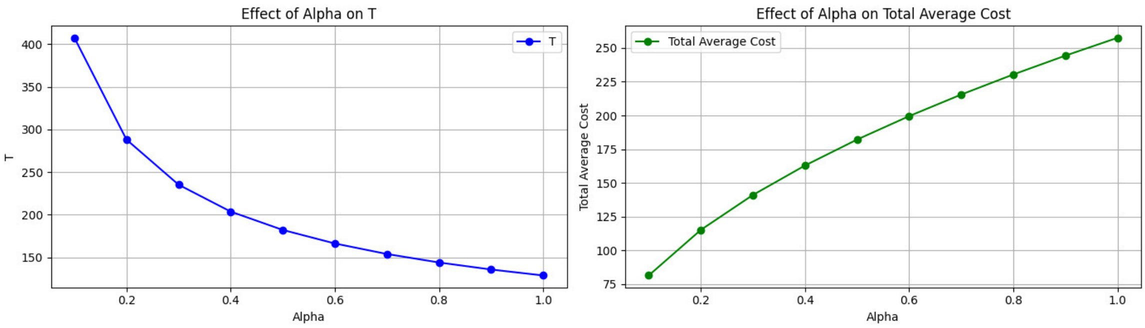

Fig.3. Graphical representations on Effect of α on T and Total Averaege cost using Crisp data

Case (2) Variations of the parameter α with Neutrosophic data input are presented in Table 3.

Table 3. Variation of α with Neutrosophic data

Parameter Value	α	T	Total Average Cost
0	0.400	194.754022	151.260034
1	0.425	188.941646	155.913218
2	0.450	183.620697	160.431261
3	0.475	178.725620	164.825278
4	0.500	174.202484	169.104937

The respective graphical representations are presented in Fig.4.

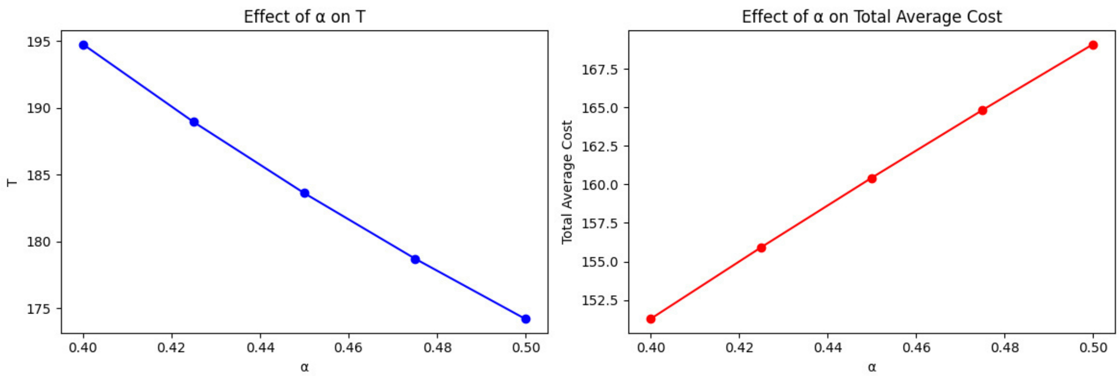

Fig.4. Graphical representations on Effect of α on T and Total Averaege cost using Neutrosophic data

In both the figures[3,4] the effects of α on T and TC are determined. However, the parameter α has positive correlation with TAC and negative correlation with T. The graphical representation of the effect of α is more vivid in terms of neutrosophic data input in comparison with the crisp data input. This shows the efficacy of neutrosophic data representations. In a similar fashion the other cost parametric effects shall be determined.

Case (3) Variations of the parameter α ,β, γ on q(t):

The Sensitivity Analysis for α are presented in the below Table 4.

Table 4. Sensitivity Analysis for α

$q\alpha$=0.45	$q\alpha$=0.5	$q\alpha$=0.55	$q\alpha$=0.6
100.964620	100.959595	100.954570	100.949545
201.929185	201.919185	201.909185	201.899186
302.893689	302.878765	302.863841	302.848916
403.858129	403.838329	403.818530	403.798731
...	...	...	...
9590.881255	9590.572780	9590.264304	9589.955828
9691.825314	9691.514913	9691.204513	9690.894112
9792.769036	9792.456730	9792.144424	9791.832118
9893.712421	9893.398229	9893.084036	9892.769844
9994.655466	9994.339406	9994.023347	9993.707287

The graphical representation of Sensitivity Analysis for α is represented in fig.5.

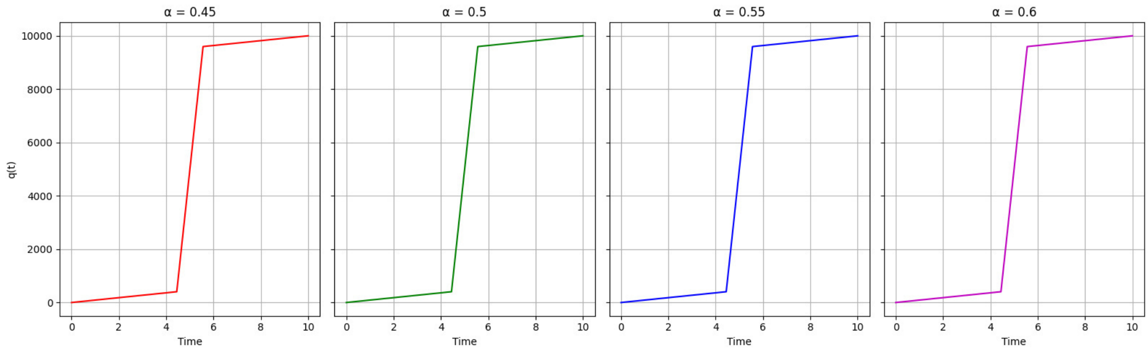

Fig.5. Graphical representation of Sensitivity Analysis for α

The Sensitivity Analysis for β are presented in the below Table 5.

Table 5. Sensitivity Analysis for β

$q\beta$ = 0.08	$q\beta$ =0.1	$q\beta$ = 0.12	$q\beta$ = 0.14
100.959544	100.959595	100.959646	100.959696
201.918983	201.919185	201.919386	201.919587
302.878314	302.878765	302.879214	302.879661
403.837533	403.838329	403.839122	403.839910
...	...	...	...
9590.308101	9590.572780	9590.808201	9591.018079
9691.246139	9691.514913	9691.753712	9691.966374
9792.183854	9792.456730	9792.698904	9792.914343
9893.121245	9893.398229	9893.643775	9893.861983

qβ = 0.08	**qβ =0.1**	**qβ = 0.12**	**qβ = 0.14**
9994.058312	9994.339406	9994.588323	9994.809291

The graphical representation of Sensitivity Analysis for β is represented in fig.6.

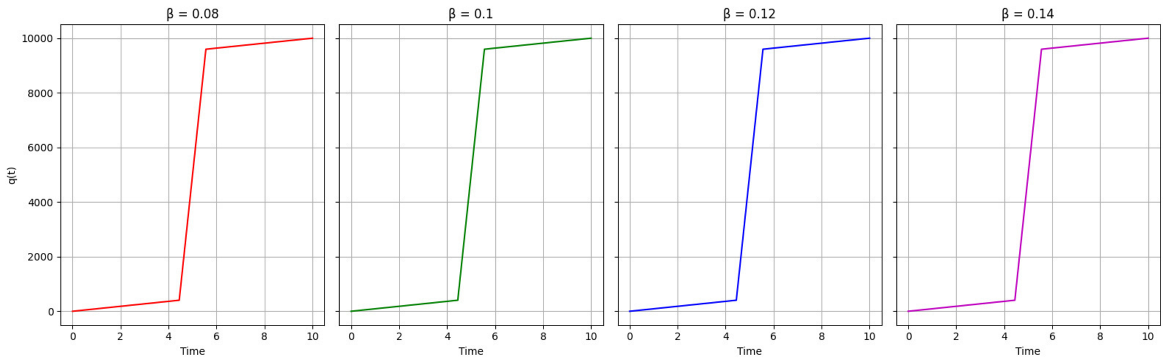

Fig.6. Graphical representation of Sensitivity Analysis for β

The Sensitivity Analysis for β are presented in the below Table 6.

Table 6. Sensitivity Analysis for γ

qγ = 0.04	**qγ = 0.05**	**qγ = 0.06**	**qγ = 0.07**
100.959646	100.959595	100.959544	100.959493
201.919389	201.919185	201.918981	201.918777
302.879224	302.878765	302.878306	302.877847
403.839146	403.838329	403.837513	403.836697
…	…	…	…
9591.033192	9590.572780	9590.112367	9589.651955
9691.985069	9691.514913	9691.044757	9690.574601
9792.936732	9792.456730	9791.976728	9791.496726
9893.888179	9893.398229	9892.908279	9892.418329
9994.839406	9994.339406	9993.839406	9993.339406

The graphical representation of Sensitivity Analysis for γ is represented in fig.7.

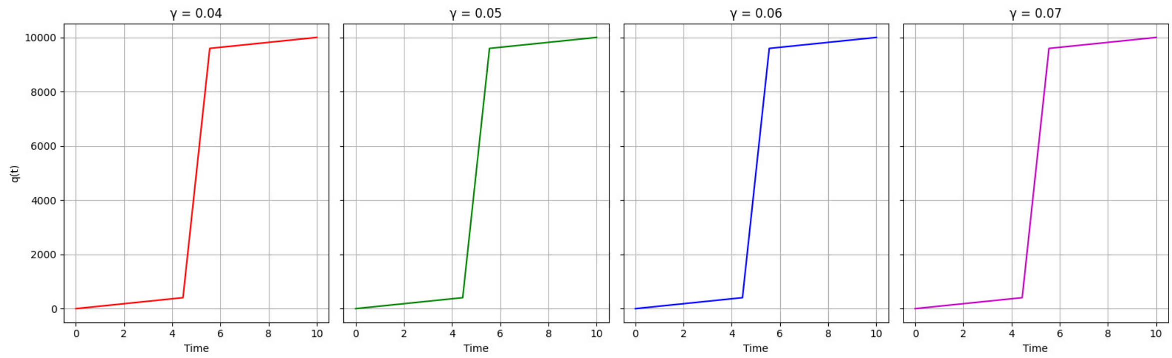

Fig.7. Graphical representation of Sensitivity Analysis for γ

From the above Table [4,5,6] values and respective graphical representations the following inferences on the trend patterns are obtained

The increase in α values causes slight decrease in q(t) over time and then increases rapidly.

The system behaviour is dampened by the impacts of variations in the values of α over a period of time.

The increase in β values causes marginalincrease in q(t) over time with very small variations initially.

The parameter β influences the long-term behaviour and resilience of the system.

The increase in γ values causes decrease in the values of q(t) over a period of time.

The system behaviour is dampened by the impacts of variations in the values of γover a period of time.

All the three parameters α, β and γ causes significant impacts on q(t) enhances the resilience of the system behaviour. The changes in the parameters enables to comprehend the behaviour of the system with respect to output maximization with assurance of the system stability.

9. Industrial Implications

The neutrosophic model with technology dependent demand pattern is more concerned with industry 5.0 and this has high association with industries especially embracing the advanced versions of human -machine collaboration. The discussion of inventory modelling with technology-oriented demand facilitates the decision makers to gain more insights on the costs associated with technology and their influences over the total inventory costs. The variation of the parameters brings the picturization of the demand impacts on inventory decisions. This model also supports manufacturing system with human-machine interferences to devise suitable strategies for cost minimization and profit maximizations. Also, the customer's demand depending on the technology advancements is well reflected in this modelling. The neutrosophic parameter representations model is studied by the influence of the parameters. This neutrosophic model proposed in this work shall be extended by studying the demand patterns influenced by the parameters of sustainability. This work provides room for developing more research ideas of industry and inventory 5.0provides a suitable framework for the managerial to optimize the production based on the demand pattern discussed in this model.

Conclusion

The novel inventory model proposed in this research work with neutrosophic representations is more feasible and comprehensive in nature. The illustration of the model with both crisp and neutrosophic data lays a clear picture of the efficiency of neutrosophic representations over crisp forms. The sensitivity analysis also favours neutrosophic representations. The dynamic behaviour of the model is studied by the influence of the

parameters. This neutrosophic model proposed in this work shall be extended by studying the demand patterns influenced by the parameters of sustainability. This work provides room for developing more research ideas of industry and inventory 5.0

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Neutrosophic Logic to Navigate Uncertainty of Security Events in Mexico

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Abstract: A significant degree of uncertainty surrounds security events in Mexico because of the interaction of multiple socioeconomic and political elements. Conventional probabilistic and logical approaches frequently fall short of properly handling this uncertainty. To effectively navigate and manage security incidents in Mexico, this study suggests applying neutrosophic logic, an extension of classical logic that allows for indeterminacy and partial knowledge. Neutrosophic logic provides a more adaptable framework for making decisions in uncertain contexts by simultaneously handling true, false, and indeterminate values. In this study, we have created a neutrosophic-based model that includes metrics like social unrest, economic instability, and crime rates to provide a thorough evaluation of the security environment. When the model is put to the test using actual data from different parts of Mexico, it outperforms conventional techniques in terms of prediction and interpretation of security occurrences. The findings demonstrate how neutrosophic logic helps security analysts and policymakers make more thoughtful and nuanced judgments by offering insightful information about the ambiguous nature of security occurrences. The ability to foresee and address possible risks is much improved when neutrosophic logic is incorporated into security event analysis. This not only improves overall security management but also creates new opportunities for its application in other sectors where uncertainty is crucial.

Keywords: Neutrosophic Logic, Security Events, Uncertainty, Mexico, Decision-Making, Predictive Analysis, Crime Rates, Socio-Economic Factors, Indeterminacy

1. Introduction

Mexico's security landscape has long been fraught with complexities, shaped by a convergence of organized crime, drug trafficking, violence, political corruption, and social instability. In recent decades, the country has faced escalating security threats, particularly from powerful drug cartels, transnational criminal organizations, and increasing levels of violent crime. The unpredictability of these threats presents a daunting challenge to government agencies, security forces, and policymakers who must continuously adapt to an evolving and uncertain environment [1]. Traditional approaches to security management, which often rely on deterministic models and binary logic, struggle to effectively handle the complexity and uncertainty intrinsic to these events. The crux of the problem lies in the uncertainty surrounding security events. Security analysts and decision-makers are frequently confronted with incomplete, contradictory, or imprecise information [2].

For example, intelligence reports may be based on conflicting sources, and the outcomes of potential interventions may be difficult to predict due to rapidly changing circumstances. In such scenarios, the ability to incorporate and reason with uncertainty is crucial for making informed decisions. Traditional binary logic—where a statement is either true or false—fails to account for the nuances of real-world data, where information may be partially true, indeterminate, or evolving [3].

To address this gap, neutrosophic logic offers a novel approach. Introduced by Florentin Smarandache in the late 1990s, neutrosophic logic extends classical and fuzzy logic by introducing three distinct components: truth (T), indeterminacy (I), and falsity (F). These parameters allow for a more flexible interpretation of reality, where propositions are not simply true or false, but can also include elements of uncertainty or contradiction. This is particularly relevant in security contexts, where data can often be incomplete, unreliable, or fluid. Neutrosophic logic provides the tools to model these ambiguities, allowing for better decision-making under uncertain conditions. In the context of Mexico's security challenges, neutrosophic logic can be a powerful tool for navigating the complex, uncertain, and rapidly changing landscape. By offering a more comprehensive framework for understanding and responding to security threats, neutrosophic logic can assist in areas such as crime prediction, risk assessment, and strategy formulation. This logic allows decision-makers to model not just the known elements of a security event, but also the unknowns and contradictions that often accompany them. This paper aims to explore the application of neutrosophic logic to security events in Mexico, focusing on how it can provide a structured approach to handling uncertainty. We will begin by examining Mexico's current security environment, highlighting the unpredictable nature of crime and violence in the country. Next, we will introduce the principles of neutrosophic logic and discuss its theoretical foundations. Finally, we will present practical examples of how this logic can be applied to real-world security scenarios in Mexico, offering insights into how it can improve decision-making and policy formulation in the face of uncertainty [4].

1.1. Security Events

The incidents or occurrences that pose a risk to the safety, stability, and well-being of a nation, its citizens, and its institutions are referred to as security events. These events can range from organized crime, terrorism, and political unrest to cyber-attacks, natural disasters, and civil disobedience. Security events may involve both internal and external threats, such as domestic violence, insurgencies, cross-border conflicts, or cyber warfare. In the context of Mexico, for instance, security events often include cartel violence, kidnappings, organized crime, extortion, and widespread corruption. However, security events aren't limited to criminal activity; they can also include mass protests, economic instability leading to civil unrest, and even natural disasters like earthquakes or hurricanes, which can destabilize regions and strain government resources [5].

The significance of security events lies in their profound impact on a nation's stability, governance, and quality of life. These events, such as violence, terrorism, organized crime, and political unrest, directly threaten public safety, erode trust in institutions, and destabilize economies. In countries like Mexico, the persistence of security events, particularly driven by drug cartels and organized crime, affects social cohesion, disrupts daily life, and hinders economic growth by deterring investment and tourism. Security issues also undermine the rule of law, leading to impunity and weakening democratic governance. Addressing these events is critical for protecting citizens, preserving social order, and ensuring long-term national development [6]. A secure environment enables not only the physical safety of individuals but also promotes political stability, economic prosperity, and international standing, all essential for the betterment of a country. Determinate factors are the underlying causes or contributors that shape and influence the occurrence of security events. These factors can vary based on geographical, socio-economic, political, and historical contexts, but they generally include elements such as crime, corruption, poverty, governance failures, and social inequality. Understanding these determinate factors is crucial for identifying the root causes of security issues and developing effective strategies to prevent and mitigate violence. For the current study, they are considered as determinate factors throughout this paper. These are as follows[7], [8], [9]:

- Drug Trafficking and Organized Crime (Df1)
- Corruption and Weak Governance (Df2)
- Poverty and Social Inequality (Df3)
- Weak Rule of Law and Impunity (Df4)
- Availability of Firearms (Df5)
- Territorial Control and Fragmentation of Cartels (Df6)
- Migration and Human Trafficking (Df7)
- Political Instability and Social Unrest (Df8)

The rest of the paper is divided into 5 sections. Section 2 and Section 3 present extensive literature and methodology respectively. Section 4 is focused on results and discussion whereas Section 5 presents the conclusion of the present work.

2. Literature Survey

Many studies focus on analyzing the root causes of violence and insecurity in Mexico. For instance, Astorga (2020) explores the roles of political corruption, social inequality, and the weakness of the rule of law as key contributors to the rise of organized crime in the country. Similarly, Buscaglia (2018) provides a comprehensive analysis of how the lack of government control in certain regions has allowed criminal organizations to flourish, leading to greater uncertainty about security outcomes [10].

However, despite the extensive body of work on the causes of security issues in Mexico, there is relatively little research on how to manage the uncertainty that these security threats produce. This gap presents an opportunity to apply advanced logical frameworks, such as neutrosophic logic, to better understand and address the unpredictable nature of security events in the country.

Neutrosophic logic has gained traction in recent years for its applications in decision-making processes, particularly in fields that involve uncertainty, such as engineering, medicine, and economics. One notable example is the use of neutrosophic sets in multi-criteria decision-making (MCDM) frameworks. These frameworks are often employed in situations where decision-makers must choose between multiple alternatives based on incomplete or uncertain information. In a study by Ye, neutrosophic logic was applied to assess risks in industrial processes, demonstrating its effectiveness in environments characterized by uncertain and indeterminate conditions. Other research has focused on using neutrosophic logic to improve decision-making in medical diagnosis, where conflicting data can lead to ambiguous results. These applications suggest that neutrosophic logic could similarly be applied to the complex decision-making processes involved in security events in Mexico, where risks must be assessed under conditions of uncertainty and incomplete information [11]. The potential of neutrosophic logic for risk assessment has been demonstrated in several studies. For instance, Smarandache applied neutrosophic models to security risk analysis in infrastructure systems, proving its effectiveness in handling indeterminate situations where traditional logic failed to provide conclusive outcomes. Given the complexity of security threats in Mexico, this approach could be highly relevant, as it would allow for more comprehensive risk assessments that take into account not only known threats but also the uncertainties surrounding those threats. Indeterminacy, or the presence of uncertain, ambiguous, or incomplete factors, plays a significant role in the perpetuation of violence, particularly in complex environments like Mexico's security landscape. Indeterminate factors make it difficult to predict or fully understand the dynamics behind violent events, complicating efforts to devise effective responses [12]. For instance, inconsistent or unreliable intelligence reports, fluctuating alliances between criminal organizations, and blurred lines between corrupt officials and law enforcement contribute to an indeterminate environment. Additionally, the role of external influences, such as illegal arms trafficking and international drug markets, adds another layer of unpredictability to the situation. These indeterminate factors exacerbate the uncertainty faced by policymakers and law enforcement, making it harder to implement sustainable security solutions. The unpredictability of violence in Mexico, driven by a combination of socio-economic, political, and international variables, underscores the need for flexible decision-making frameworks like neutrosophic logic that can accommodate and navigate such indeterminacies.

Many researchers have emphasized the assessments happening in institutions for the successful implementation of educational models. Black and Wiliam analyze various assessment techniques and their effects on learning [13].

They highlight that the effectiveness of assessments depends on their design, implementation, and students' perceptions. The impact of assessment techniques is indeterminate due to differences in assessment types, teacher practices, and student responses to assessments.

There are as follows as are termed as indeterminate or uncertain throughout the study [14], [15], [16], [17], [18], [19]:

- Exposure to Violence (Idf1)
- Family Dynamics (Idf2)
- Socioeconomic Conditions (Idf3)
- Brain Abnormalities (Idf4)
- Immediate Stressors (Idf5)
- Provocation (Idf6)
- Substance Abuse (Idf7)

3. Methodology

The proposed solution to indeterminacy uses the concept of Neutrosophic Cognitive Map (NCMs). It is a technique in Neutrosophy introduced by W. B. Vasantha Kandasamy [20]. The concept of Neutrosophic logic introduced by Florentine Smarandache [21], which is a merger of the fuzzy logic together with the inclusion of indeterminacy. When data under scrutiny contains concepts which are indeterminate, we are not able to formulate mathematical expression. Presentation of Neutrosophic logic by Florentine Smarandache [22] has put forward a panacea to this problem. It is the reason Neutrosophy has been introduced as an additional notion for evaluation educational models. Fuzzy theory evaluates the existence or non-existence of associateship but it has failed to attribute the indeterminate relations among concepts and most data collected in educational setup has much indeterminate and uncertain concepts. Therefore we have employed Neutrosophic Cognitive Maps (NCMs) in place of Fuzzy Cognitive Maps (FCMs) to show the significance of indeterminate factors of violence in Mexico.

3.1. Neutrosophic Concepts

3.1.1. Neutrosophic Sets:

Neutrosophic sets are an extension of classical sets and fuzzy sets that handle uncertainty, indeterminacy, and contradiction in data. A classical set only allows elements to be either fully part of the set or not (0 or 1). Fuzzy sets extend this by allowing partial membership (values between 0 and 1), but neutrosophic sets go further by introducing three components:

1. Truth (T): The degree to which a statement is true.
2. Indeterminacy (I): The degree to which it is uncertain or indeterminate.
3. Falsity (F): The degree to which a statement is false.

In a neutrosophic set, each element has a degree of truth, indeterminacy, and falsity, all ranging from 0 to 1, and they don't have to sum to 1 (unlike fuzzy sets). This flexibility allows for more nuanced representation of uncertainty and contradictory information.

3.1.2. Neutrosophic Cognitive Maps (NCMs)

3.2. FCM Adjacency Matrix

An **FCM Adjacency Matrix** is a mathematical tool used to represent the relationships between concepts in a Fuzzy Cognitive Map (FCM). It is a square matrix where each row and column corresponds to a concept in the system, and the entries in the matrix reflect the strength and type of influence one concept has on another. The values in the matrix typically range from -1 to 1, where positive values indicate a positive influence, negative values signify a negative influence, and zero means there is no direct influence between the concepts. This matrix helps to model complex systems by showing how changes in one concept can affect others, enabling the analysis and simulation of dynamic behaviors. The FCM adjacency matrix is particularly useful for studying decision-making processes, forecasting, and understanding the interactions in systems with interdependent factors, such as social, economic, or environmental systems.

Let’s consider a system with 3 concepts: C1C_1C1 (Economic Growth), C2C_2C2 (Employment Rate), and C3C_3C3 (Public Spending). The FCM adjacency matrix could look like this:

$$\begin{pmatrix} 0 & 0.6 & -0.2 \\ 0.7 & 0 & 0.3 \\ 0 & -0.4 & 0 \end{pmatrix}$$

- w_12=0.6 means that "Economic Growth" has a positive influence on "Employment Rate".
- w_13 means that "Economic Growth" negatively affects "Public Spending".
- w_21=0.7 means that "Employment Rate" positively affects "Economic Growth".
- w_23=0.3 means that "Employment Rate" positively influences "Public Spending".
- w_32=−0.4 means that "Public Spending" negatively influences "Employment Rate"
- Diagonal entries are typically 0 because a concept does not directly influence itself.

3.3. NCM Adjacency Matrix

The **NCM Adjacency Matrix** is an extension of the Fuzzy Cognitive Map (FCM) adjacency matrix, used to represent the relationships between concepts in a **Neutrosophic Cognitive Map (NCM)**. Unlike FCMs, where each connection between concepts is represented by a single value (indicating positive or negative influence), the NCM adjacency matrix incorporates three components to account for the uncertainty and indeterminacy in relationships. Each entry in the NCM adjacency matrix consists of three values: **Truth (T)**, **Indeterminacy (I)**, and **Falsity (F)**.

Example:

Consider a system with 3 concepts: C1C_1C1 (Economic Stability), C2C_2C2 (Inflation Rate), and C3C_3C3 (Employment Levels). The NCM adjacency matrix could look like this:

$$\begin{pmatrix} (0,0,0) & (0.6,0.2,0.1) & (0.4,0.3,0.1) \\ (0.5,0.3,0.2) & (0,0,0) & (0.7,0.1,0.1) \\ (0.3,0.4,0.2) & (0.6,0.2,0.2) & (0,0,0) \end{pmatrix}$$

- The relationship between "Economic Stability" and "Inflation Rate" is (0.6,0.2,0.1)(0.6, 0.2, 0.1)(0.6,0.2,0.1), meaning there is a 60% positive influence, 20% uncertainty, and 10% negative influence.
- The relationship between "Inflation Rate" and "Employment Levels" is (0.7,0.1,0.1)(0.7, 0.1, 0.1)(0.7,0.1,0.1), indicating a strong positive influence with little uncertainty and falsity.

Neutrosophic Cognitive Maps (NCMs) are an advanced extension of Fuzzy Cognitive Maps (FCMs), designed to model complex systems where uncertainty, indeterminacy, and contradiction play a significant role. Like FCMs, NCMs consist of concepts representing variables within a system, connected by edges that depict the influence between these concepts. However, in NCMs, each connection is characterized by three components: truth (T), indeterminacy (I), and falsity (F), allowing for a more nuanced understanding of relationships. This structure helps capture not just the positive or negative influence between concepts but also the uncertainty and incomplete information that might exist in real-world scenarios. NCMs are particularly useful for dynamic, evolving systems—such as social, economic, or environmental models—where information is often ambiguous or contradictory, providing a more flexible and realistic framework for analysis and decision-making.

In the following NCM diagram, each node represents a concept such as "Economic Stability" or "Inflation Rate," and each directed edge between nodes represents the influence one concept has on another. The relationships between nodes are described by three values: Truth (T),

Indeterminacy (I), and Falsity (F), are displayed as labels on the edges.

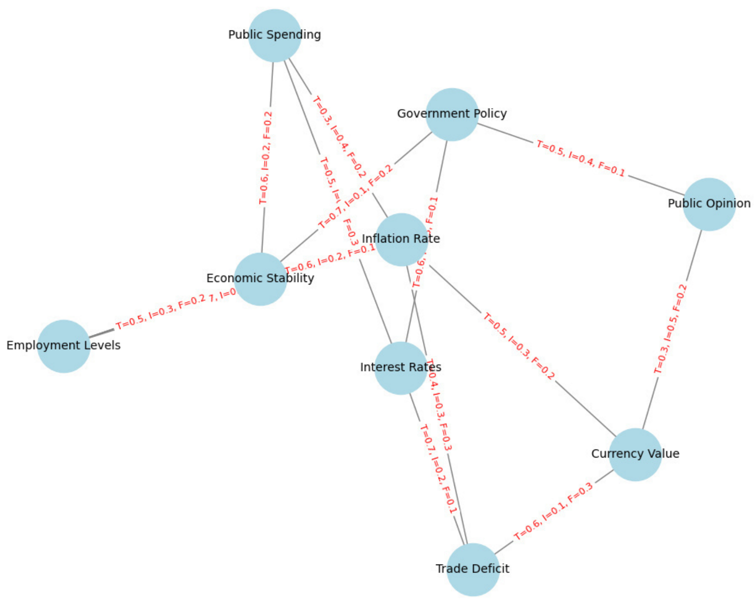

Figure 1: Neutrosophic Cognitive Maps for Economic Stability

4. Results

After taking all these factors & their relationship to violence, we must model the condition of violence based on the concept of Neutrosophic Cognitive Maps. This is the graph for mapping the condition of factors related to violence. Nodes denote factors and edges denote the relationship from node to node or from node to factors. Some nodes have representation like *Df*1, *Df*2, *Df*3, ………, and *Df*7 known as Determinate Factors, and other nodes have representation like *Idf*1, *Idf*2, *Idf*3, ………*I*df8 known as Indeterminate Factors. The edges have weight "1" represent known or determinate edges and the edges having representation "*I*" represent Indeterminate edges. The graph contains two types of edges, the first type of edge is the complete edge as we have seen in the normal graph, this edge shows determinacy and another type of edge is a dotted edge which represents indeterminacy. We took Blue color for showing Determinate Factors and Red color for showing Indeterminate Factors respectively as we have in Figure 2. The straight simple line shows the relationship between Determinate Factors and dotted lines show the relationship between Indeterminate Factors. The edges have

weight "*I*" representing known or determinate edges and the edges having representation "*I*" represent Indeterminate edges. We have formulated the adjacency matrix given in Table 1 for the graph(Figure 2).

4.1 Determinate and Indeterminate Factors| for Affecting Violence

The literature review identified numerous factors that contribute to shaping the conditions of violence. Table 1 presents the factors categorized as determinate, while Table 2 highlights those considered indeterminate.

Table 1. Determinate Factors

S. No.	Determinate Factors	Presentations
1	Drug Trafficking and Organized Crime	Df1
2	Corruption and Weak Governance	Df2
3	Poverty and Social Inequality	Df3
4	Weak Rule of Law and Impunity	Df4
5	Availability of Firearms	Df5
6	Territorial Control and Fragmentation of Cartels	Df6
7	Migration and Human Trafficking	Df7

Table 2. Indeterminate Factors

S. No.	Indeterminate Factors	Presentations
1	Exposure to Violence	Idf1
2	Family Dynamics	Idf2
3	Socioeconomic Conditions	Idf3

S. No.	Indeterminate Factors	Presentations
4	Brain Abnormalities	Idf4
5	Immediate Stressors	Idf5
6	Provocation	Idf6
7	Substance Abuse	Idf7
8	Political Instability and Social Unrest	Df8

4.2 NCM graph of factors

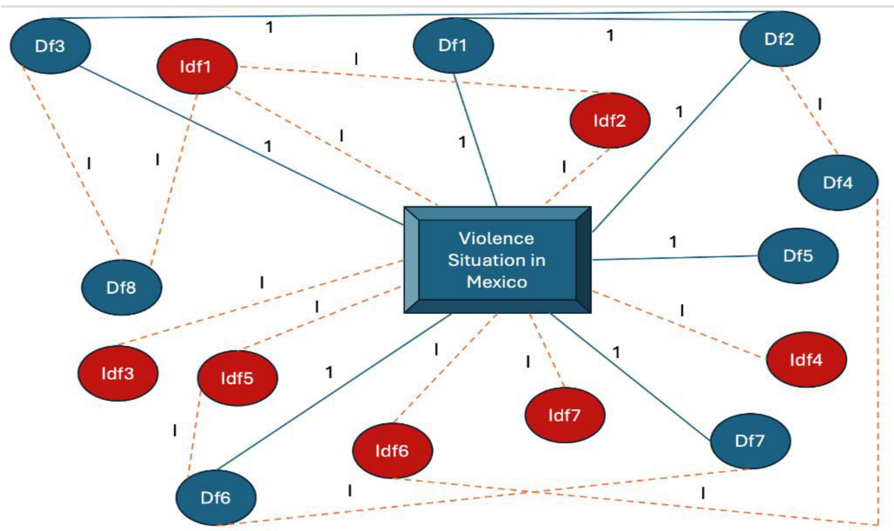

Figure 2: Neutrosophic Cognitive Map based on determinate and indeterminate factors affecting violence in Mexico

4.3 Adjacency matrix

Table 3. AdjecencyMarix

	VSI_ M	Df 1	Idf 1	Df 2	Idf 2	Df 3	Idf 3	Df 4	Idf 4	Df 5	Idf 5	Df 6	Idf 6	Df 7	Idf 7	Df 8
VSI_ M	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
Df1	**1**	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
Idf1	**0**	0	0	0	I	0	0	0	0	0	0	0	0	0	0	I
Df2	**1**	0	0	0	0	0	0	I	0	0	0	0	0	0	0	0
Idf2	**0**	0	I	0	0	0	0	0	0	0	0	0	0	0	0	0
Df3	**1**	0	0	1	0	0	0	0	0	0	0	0	0	0	0	I
Idf3	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df4	**1**	0	0	I	0	0	0	0	0	0	0	I	0	0	0	0
Idf4	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df5	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Idf5	**0**	0	0	0	0	0	0	0	0	0	0	0	I	1	0	0

	VSI_M	**Df 1**	**Idf 1**	**Df 2**	**Idf 2**	**Df 3**	**Idf 3**	**Df 4**	**Idf 4**	**Df 5**	**Idf 5**	**Df 6**	**Idf 6**	**Df 7**	**Idf 7**	**Df 8**
Df6	**1**	0	0	0	0	0	0	0	0	I	0	0	0	I	0	0
Idf6	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df7	**1**	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
Idf7	**0**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Df8	**1**	0	I	0	0	I	0	0	0	0	0	0	0	0	0	0
Idf8	**0**	0	0	I	0	I	0	0	0	0	I	0	0	0	0	0

Now, this Neutrosophy Adjacency Matrix will be evaluated to show the significance of factors causing violence. We will take vector FA as an "ON" state i.e. the state vector is given as input. Based on the methodology mentioned above following iterations are carried out. These iterations are performed till we obtain a limit cycle.

This limit cycle is also referred to as a constant state vector. The Limit cycle shows hidden patterns that are used in mapping inferences. These inferences are used to show the joint effect of interacting knowledge. To get the current result we used NCM which shows that when a crime is in an ON state all the factors. All these factors show that they have a direct effect on crime.

4.4 Iterative Process

Now, this Neutrosophy Adjacency Matrix will be evaluated to show the significance of factors causing violence. We will take vector FA as an "ON" state i.e. the state vector is given as input. Violence in Mexico (VM) = (100000000000000000) and the combined system is VM1*N(E) The symbol → denotes that the resultant vector is updated and threshold. Based on the methodology mentioned above following iterations are carried out. These iterations are performed till we obtain a limit cycle. This limit cycle is also referred to as a constant state vector. The first iteration starts by keeping the critical thinking state as ON rest all other factors are considered null at this time. So, the Initial State becomes,

$$S(0) = [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

The iterations for both FCM and NCM are done separately. The further iterations are carried out using the formula mentioned below. Iterations for FCM:

Iteration 1: $S(1)=S(0)\times WS(1)=S(0)\times W$

Iteration 2: $S(2)=S(1)\times WS(2)=S(1)\times W$

Iteration 1: $S(1)=S(0)\times WFCMS(1)=S(0)\times WFCM$

Iteration 2: $S(2)=S(1)\times WFCMS(2)=S(1)\times WFCM$

Combining the intermediate results of iteration, we get the final state vector (1,1,0,1,0,0,0,1,0,1, I,1,0,1,0,0). This clearly shows the presence of I mentioning the importance of indeterminate factors in crime analysis which was absent in the final result obtained by the FCM.

Conclusion

The crime situation in Mexico has seen an unprecedented change due to prevailing uncertainty surrounding security events. This is due to the complex mix of social, economic, and political factors. Traditional methods struggle to handle this uncertainty effectively. Therefore, this study proposes the use of neutrosophic logic, which can deal with uncertainty by considering true, false, and indeterminate values present in any security event captioning. By creating a model based on neutrosophic logic, we can assess factors like social unrest, economic instability, and crime rates more accurately. Our results show that this approach predicts and interprets security events better than traditional methods, helping analysts and policymakers make better decisions in uncertain situations.

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A Comparison of Reading Strategies Among Spanish Speaking University Students in Different Programs of English for Specific Purposes Based on An Indeterminate Likert Scale

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Abstract. This study examines the relationships of reading strategies among different English for Specific Purposes (ESP) programs. Ninety-seven students in four ESP programs in Ecuador were surveyed regarding their reading strategies. The results found that there were not significant differences among the four strategies. These findings were discussed regarding whether differences in reading strategies could potentially influence teaching effectiveness, learning outcomes, and English language applications. The survey was administered to a random sample obtained from four Ecuadorian universities for 97 ESP students. For the survey, we applied an Indeterminate Likert Scale. Unlike the classic Likert Scale, the indeterminate variant asks the respondent to express a degree of measurement regarding their opinions for each of the elements of satisfaction, dissatisfaction, and neutrality. This allows capturing the opinion of the interviewee in the most reliable way possible, obtaining more accuracy than in the crisp Likert Scale. These data were statistically processed with the help of a Chi-square test for contingency tables.

Keywords: English for Specific Purposes (ESP), Reading Strategies, Indeterminate Likert Scale, triple refined indeterminate neutrosophic set (TRINS), refined neutrosophic set, Chi-Square test.

1 Introduction

Different reading strategies have been researched with many different groups of participants, but no known research has evaluated reading strategies employed by different groups of English for Specific Purposes (ESP) students. Differences in reading strategies could potentially influence teaching effectiveness, learning outcomes, and English language applications. Many studies have been conducted about ESP and specifically to what students need to do in their vocations or jobs. Likewise, many studies have focused on reading strategies, however, no known studies have compared reading strategies among Spanish-speaking students in different programs of ESP.

Therefore, the key research question is: Do Spanish-speaking university students in various programs of English for Specific Purposes differ in their reading strategies and if there are any differences in their reading strategies, what are those specific differences? The answer to this research question is important because such information could be valuable in helping such specialized programs to be more effective and efficient in teaching students specific skills and for helping teachers to be more precise and effective in their teaching. Additionally, this area of inquiry is also significant because it could help students to learn more efficiently and to use reading concepts and strategies more effectively in the area of applied English for Specific Purposes.

Research has found that students often vary in the reading strategies they use to learn. The key question is whether such variance is also occurring with ESP students, independent of their knowledge of general English. Reading is the ability or activity of reading materials. This activity can be performed silently or audibly verbalized. Reading is a mechanism for assisting in the transmission of information immediately and over time. Several studies have attempted to provide more precise measures of reading strategy activities. This research includes need assessment and evaluation. Some of them found that such activities produce better results and outcomes. The research of some authors focused on real-world applications with police officers and with physicians while others had similar findings with bankers. Research also found similar results with second-year medical students and ESP learning, while others found the same outcomes with aerospace engineers.

In regards to metacognition and reading strategies, we note that monitoring and awareness of comprehension processes are two key elements of skilled reading. Such awareness and monitoring processes are often referred to as metacognition in the literature, which can be thought of as the knowledge of the readers' cognition about reading and the self-control mechanisms they exercise when monitoring and regulating text comprehension.

From the perspective of metacognition, researchers have attempted to evaluate differences between unskilled readers and skilled readers in the area of reading comprehension. Thus, reading strategies have received a large amount of research attention regarding their effects on reading comprehension and regarding the most effective reading strategies.

In an attempt to find the most effective reading strategies, many researchers have developed their own comprehensive reading strategies instruments to use when evaluating this aspect in students. In addition, such research has provided teacher educators and practicing teachers with practical suggestions for helping struggling readers increase their awareness and use of reading strategies while reading. However, there are relatively few instruments to measure students' awareness and perceived use of reading strategies while reading for academic purposes. This assertion leads us to think that there is limited available research because of the shortage of standardized reading strategies surveys and because of the large investments of time and money necessary to evaluate the validity and reliability of such instruments.

ESP relates directly to what students need to do in their vocations or jobs. ESP is important because it helps to increase vocational learning and training throughout the world. As globalization is spreading, it is stated that knowledge of English has become the greatest need. It is not just the politician, the business leader, and the academic professor who needs to speak to international colleagues and clients: It is also the hotel receptionist, the nurse, and the site fireman. ESP can be, but not necessarily be, concerned with a specific discipline and it does not have to be focused on a specific ability range or particular age group. Rather, ESP can be viewed as an approach to teaching.

Two distinct perspectives regarding language for specific purposes are indicated. One perspective suggests that English has a common foundation of words that all learners should know. The other perspective suggests that all language is already for specific purposes, and therefore, specialization must begin at an early age.

Whether specialization begins early or late in the life of an individual, there are numerous methods of identifying vocabulary for specific purposes. However, there is a lack of a systematic way of identifying vocabulary for specific purposes. The lack of such a systematic methodology could have caused variations in outcomes. Therefore, these variations may have caused many researchers to produce very different and unique conclusions in their research findings.

In terms of theoretical background, some researchers have suggested that ESP is just technical vocabulary. Other researchers have suggested that ESP is more than technical vocabulary and that it is important to understand that vocabulary is a necessary part of any ESP course. Partly this is because specific technical words are used to describe particular features of the vocabulary specialization and that teaching and vocabulary development are ongoing processes. These concerns are important to this particular research study because much of the study of ESP is based on definitions of technical words and vocabulary, which may be perceived to be perceptually different in different countries and cultures.

In this study, we carried out a comparison between the results of the different universities. To do this we surveyed the selected students, where each of them expressed their opinion on an

Indeterminate Likert Scale [1-3]. Each element is based on a triple refined indeterminate neutrosophic set (TRINS) where instead of the triple of elements that are part of a Single-Valued Neutrosophic Set, two more elements are added, which are: "Indeterminacy leaning towards negative membership" and "Indeterminacy leaning towards positive membership". The respondent must assign a value to each of these elements, according to their feeling about each element of the scale. This way of surveying is more complex, however, more accurate because it allows capturing contradictory feelings, but which are more faithful to what the respondent feels about what is being asked.

TRINS are values that are part of Smarandache's Refined Neutrosophic Sets theory, where each of the elements of the original triple, which are truthfulness, indeterminacy, and falseness, can be divided into more specific components to obtain more accuracy in the results [4-12]. In TRINS, the indeterminacy element is partitioned into three elements, two more in addition to indeterminacy. This allows two more types of indeterminacy to be taken into account, which makes the study carried out more exhaustive.

The results obtained are classified into several nominal results and finally placed in contingency tables to study the independence of the relationship between different aspects, with the help of a Chi-square test.

Thus, to carry out the study, in section 2 of Materials and Methods, we recall the basic notions of Indeterminate Likert Scales. Section 3 contains the results obtained from the study. We finish with the conclusions of the work.

2 Materials and Methods

In this section, we recall the basic notions of the Indeterminate Likert Scale.

Definition 1 ([4-12]). The *Single-Valued Neutrosophic Set* (SVNS) N over U is $A = \{< x; T_A(x), I_A(x), F_A(x) > : x \in U\}$, where $T_A: U \to [0,1]$, $I_A: U \to [0,1]$, and $F_A: U \to [0,1]$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2 ([4-12]). The *refined neutrosophic logic* is defined such that: a truth T is divided into several types of truths: $T_1, T_2, \dots, T_p$, I into various indeterminacies: $I_1, I_2, \dots, I_r$ and F into various falsities: $F_1, F_2, \dots, F_s$, where all $p, r, s \geq 1$ are integers, and $p + r + s = n$.

Definition 3 ([13-17]). A *triple refined indeterminate neutrosophic set* (TRINS) A in X is characterized by positive $P_A(x)$, indeterminacy $I_A(x)$, negative $N_A(x)$, positive indeterminacy $I_{P_A}(x)$ and negative indeterminacy $I_{N_A}(x)$ membership functions. Each of them has a weight $w_m \in [0,1]$ associated with it. For each $x \in X$, there are $P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x) \in [0,1]$, $w_P^m(P_A(x)), w_{I_P}^m(I_{P_A}(x)), w_I^m(I_A(x)), w_{I_N}^m(I_{N_A}(x)), w_N^m(N_A(x)) \in [0,1]$ and $0 \leq P_A(x) + I_{P_A}(x) + I_A(x) + I_{N_A}(x)(x) + N_A(x) \leq 5$. Therefore, a TRINS A can be represented by $A = \{\langle x; P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x)\rangle | x \in X\}$.

Let A and B be two TRINS in a finite universe of discourse, $X = \{x_1, x_2, \dots, x_n\}$, which are denoted by:

$A = \{\langle x; P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x)\rangle | x \in X\}$ and $B = \{\langle x; P_B(x), I_{P_B}(x), I_B(x), I_{N_B}(x), N_B(x)\rangle | x \in X\}$,

Where $P_A(x_i), I_{P_A}(x_i), I_A(x_i), I_{N_A}(x_i), N_A(x_i), P_B(x_i), I_{P_B}(x_i), I_B(x_i), I_{N_B}(x_i), N_B(x_i) \in [0,1]$, for every $x_i \in X$. Let w_i (i = 1,2,...,n) be the weight of an element x_i (i = 1,2,...,n), with $w_i \geq 0$ (i = 1,2,...,n) and $\sum_{i=1}^{n} w_i = 1$.

The generalized TRINS weighted distance is ([13-17]):

$$d_\lambda(A,B) = \left\{\frac{1}{5}\sum_{i=1}^{n} w_i \left[|P_A(x_i) - P_B(x_i)|^\lambda + \left|I_{P_A}(x_i) - I_{P_B}(x_i)\right|^\lambda + |I_A(x_i) - I_B(x_i)|^\lambda + \left|I_{N_A}(x_i) - I_{N_B}(x_i)\right|^\lambda + |N_A(x_i) - N_B(x_i)|^\lambda\right]\right\}^{1/\lambda} \quad (1)$$

Where λ > 0.

The Indeterminate Likert Scale is formed by the following five elements:

- Negative membership,
- Indeterminacy leaning towards negative membership,
- Indeterminate membership,
- Indeterminacy leaning towards positive membership,
- Positive membership.

These values substitute the classical Likert scale with values:

- Strongly disagree,
- Disagree,
- Neither agree or disagree,
- Agree,
- Strongly agree.

Respondents are asked to give their opinion on a scale of 0-5 about their agreement in each of the possible degrees, which are "Strongly disagree", "Disagree", "Neutral", "Agree", "Strongly agree", for this, they were provided with a visual scale like the one shown in Figure 1.

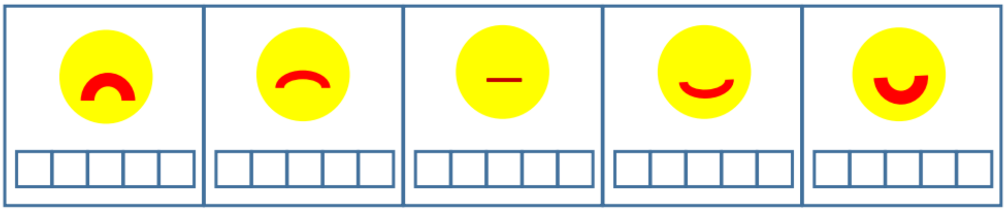

Figure 1. Graphic representation of the proposed Indeterminate Likert Scale. Source: [18].

3 Results of the Study

This research study examines the relationships of reading strategies among four different programs of English for Specific Purposes. 97 students enrolled in four English for Specific Purposes (ESP) programs in two public universities and two private universities in Ecuador were surveyed regarding their reading strategies.

There were a total of 65 items on the surveys completed by the 97 volunteer students in four reading strategy categories. These four categories were:

1. Organizing Reading and Planning, with the number of survey items to be 7.
2. Actions Undertaken While Reading, with the number of survey items to be 18.
3. Evaluation after Reading, with the number of survey items to be 11.
4. Dealing with Problems, with the number of survey items to be 10.

The question to answer is: Are there significant statistical differences between the four ESP groups? Table 1 summarizes the number of students for every university.

Table 1. Summary of Participants.

Summary of Participants		
A public university in Guayaquil	17	18%
A private university in Guayaquil	23	24%
A private university in Quito	32	33%
A public university in Riobamba	25	26%
TOTAL:	97	100%

A total of ninety-seven (97) ESP students were surveyed. Here are the numbers of students responding at each university: 17 surveys at a public university in Guayaquil, representing 18% of respondents, 23 surveys at a private university in Guayaquil, representing 24% of respondents, 32 surveys at a private university in Quito, representing 33% of respondents, and 25 surveys at a public university in Riobamba representing 26% of respondents.

The procedure to be followed to process the survey data was as follows:

1. Students filled out each item of the survey according to the Indeterminate Likert Scale, where they had to specify a degree of satisfaction, dissatisfaction, or neutrality, out of 5 points for each of the 5 elements, as shown in the example in Figure 2.

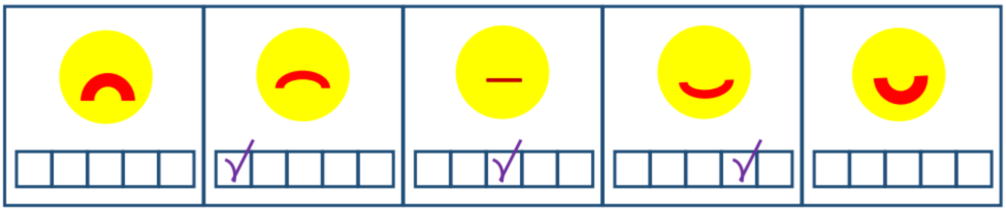

Figure 2. Example of the graphic use of the proposed Indeterminate Likert Scale. Source: [18].

2. They are aggregated to using the arithmetic mean of the TRINS corresponding to each student for each aspect to be measured, and normalized when dividing by 5. This results in a value $\bar{x}_{ij}$ that corresponds to the evaluation of the i-th student concerning the j-th aspect (strategy), where $i \in \{1, 2, \dots, 97\}$, $j \in \{1, 2,3,4\}$.

The strategies identified with the j indices are:

1. Organizing reading and planning,
2. Actions undertaken while reading,
3. Evaluation after reading,
4. Dealing with problems.

3. Let the elements of the following scale be based on TRINS associated with linguistic values, see Table 2:

Table 2. Linguistic scale with associated TRINS ideal values.

Linguistic Value	TRINS Associated
Strongly disagree (V_{-2})	(0, 0,0,0,1)
Disagree (V_{-1})	(0, 0,0,1,0)
Neutral (V_0)	(0, 0,1,0,0)
Agree (V_1)	(0, 1,0,0,0)
Strongly agree (V_2)	(1, 0,0,0,0)

4. For each $\bar{x}_{ij}$ we calculate the minimum $A_{ij} = min_{k\in\{-2,-1,0,1,2\}}\{d_2(\bar{x}_{ij}, V_k)\}$, using distance Equation 1. Thus, the nominal value of the evaluation of the i-th student for the j-th aspect $arg\ A_{ij}$ is taken as the linguistic value associated with it.

5. With these nominal values, the study is carried out using Contingency Tables and applying the Chi-square test as indicated below.

Below we show the results of applying the above procedure:

The use of the Chi-Square test is a method for evaluating possible statistical differences among groups. The null hypothesis is:

H_0: There is no independence between the two populations,

H_a: There is independence between the two populations.

For the rows, the five possible answers were A_1 (strongly disagree), A_2 (disagree), A_3 (neutral), A_4 (agree), and A_5 (strongly agree).

For this study, all responses to the survey from the four question categories were used. These responses are B_1 (Organizing reading and planning), B_2 (Actions undertaken while reading), B_3 (Evaluating after reading), and B_4 (Dealing with problems).

For the columns, the four elements are C_1 (A Public University in Guayaquil), C_2 (A Private University in Quito), C_3 (A Private University in Guayaquil), and C_4 (A Public University in Riobamba).

This study presented variables called First Language (Spanish) and Second Language (English) and the responses were calculated using a separate contingency table. Those results are presented below in Tables 3-10.

Table 3. Contingency table for First Language – First Category

FIRST LANGUAGE				
FIRST CATEGORY: ORGANISING READING AND PLANNING				
	C_1	C_2	C_3	C_4
A_1	0	2	0	0
A_2	1	3	3	2
A_3	2	10	4	4
A_4	6	12	6	8
A_5	8	5	10	11

Table 4. Contingency table for Second Language – First Category

SECOND LANGUAGE				
FIRST CATEGORY: ORGANISING READING AND PLANNING				
	C_1	C_2	C_3	C_4
A_1	0	3	1	2
A_2	2	6	4	7
A_3	3	10	10	8
A_4	6	9	4	5
A_5	6	4	4	3

Table 5. Contingency table for First Language – Second Category

FIRST LANGUAGE				
SECOND CATEGORY: ACTIONS UNDERTAKEN WHILE READING				
	C_1	C_2	C_3	C_4
A_1	0	1	1	1
A_2	1	5	3	2
A_3	4	10	5	7
A_4	7	10	7	8
A_5	5	6	7	7

Table 6. Contingency table for Second Language – Second Category

SECOND LANGUAGE				
SECOND CATEGORY: ACTIONS UNDERTAKEN WHILE READING				
	C_1	C_2	C_3	C_4
A_1	1	2	1	2
A_2	2	7	4	6

SECOND LANGUAGE				
SECOND CATEGORY: ACTIONS UNDERTAKEN WHILE READING				
A_3	5	11	9	9
A_4	6	8	5	5
A_5	3	4	4	3

Table 7. Contingency table for First Language – Third Category

FIRST LANGUAGE				
THIRD CATEGORY: EVALUATION AFTER READING				
	C_1	**C_2**	**C_3**	**C_4**
A_1	0	2	1	1
A_2	2	6	2	1
A_3	4	10	4	4
A_4	5	10	7	10
A_5	6	4	9	9

Table 8. Contingency table for Second Language – Third Category

SECOND LANGUAGE				
THIRD CATEGORY: EVALUATION AFTER READING				
	C_1	**C_2**	**C_3**	**C_4**
A_1	0	4	0	2
A_2	2	7	3	5
A_3	5	11	10	11
A_4	6	7	5	4
A_5	4	3	5	3

Table 9. Contingency table for First Language – Fourth Category

FIRST LANGUAGE				
FOURTH CATEGORY: DEALING WITH PROBLEMS				
	C_1	**C_2**	**C_3**	**C_4**
A_1	0	1	0	1
A_2	1	5	2	1
A_3	3	10	4	5
A_4	6	12	7	9
A_5	7	4	10	9

Table 10. Contingency table for Second Language – Fourth Category

SECOND LANGUAGE				
FOURTH CATEGORY: DEALING WITH PROBLEMS				
	C_1	**C_2**	**C_3**	**C_4**
A_1	0	2	1	2
A_2	2	8	3	4
A_3	4	11	10	9
A_4	6	9	5	7
A_5	5	2	4	3

The results of applying the Chi-square test to the above contingency tables yielded p-values of 0.35532, 0.52974, 0.98541, 0.99568, 0.99568, 0.57146, 0.52244, and 0.69014, respectively. All of these numbers are greater than 0.05, therefore the dependence hypothesis is rejected.

Conclusion

English for Specific Purposes (ESP) is an English learning paradigm that can be very useful for Ecuadorian university students to master this language. In this work we evaluate four reading strategies that are used in ESP by students, these are: "Organizing reading and planning", "Actions undertaken while reading", "Evaluation after reading", and "Dealing with problems". Students from four Ecuadorian universities were selected for a sample of 97 evaluated. They were asked to fill out a questionnaire that evaluated each of the previous points. An Indeterminate Likert Scale was used and processed to apply a Chi-square test in contingency tables, where the evaluation given by the students to the university was compared for each of the strategies and Spanish and English. It was concluded that the results obtained are independent of the university; therefore ESP has the same effect for all higher education centers evaluated.

The best results were obtained for the strategies regarding the first language, with approximately 68, 59, 62, and 66 percentages for "Agree" and "Strongly agree", respectively. However, the same strategies for the English language gave lower percentages of satisfaction 42, 39, 38, and 42, respectively. Nevertheless, when these last percentages are added to the obtained percentages for "Neutral" the results exceed 50%. In general, there is no significant difference between the opinions of the students for each strategy of the same language.

The use of the Indeterminate Likert Scale allowed us to capture students' opinions more reliably, including the contradictions that may exist in opinions. This scale allows for feelings and conflicting opinions that are part of being human.

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TEC 21 Model and Critical Thinking: An NCM-based Neutrosophic Analysis in Higher Education

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Abstract: With a growing emphasis on developing transverse competencies in students, many educational institutions are trying to revolutionize the traditional system of education. One among others is TEC de Monterrey in Mexico which refines its educational models so that the students can be nourished with the skills that prepare them not only for the present but for years to come. Among skill development transverse competencies are considered a key to the successful career development of a student that emphasizes on critical thinking aspect. So mainly the the educational models try to inculcate or enhance the critical thinking of students. Towards this goal, TEC introduced the TEC21 model that is oriented towards critical thinking skill development of the students. The current work in this regard utilizes real-time student data collected using the eOpen instrument and takes into consideration experts' opinions to analyze the TEC21 model. The determinate factors are taken from the TEC21 model and indeterminate factors are identified from the literature. As real-time data especially educational data contain much uncertain, indeterminate, and unknown information, therefore this research introduces the use of Neutrosophic Logic to address the uncertainty of the data. Through the use of neutrosophy, we have shown how indeterminate factors when considered for analysis give a better representation of information. We have also compared neutrosophic cognitive maps with used earlier fuzzy cognitive maps to show their effectiveness in this regard. Overall system developed using NCM gives a better analysis of educational models like TEC21. This is the only work now that has utilized neutrosophy to understand and analyze the TEC21 model.

Keywords: Educational Innovation, Higher Education, TEC21 Model, Neutrosophy, Data Science, Critical Thinking

1. Introduction

The need of society pushes towards the development of not only physical infrastructure but also a lot of development is needed towards the development of the human mind for efficient and effective utilization of resources. Keeping this in mind prominent educational institution TEC de Monterrey keeps on enhancing its educational models to better prepare its students for a better tomorrow. In light of this TEC has proposed an educational model TEC21 which is mainly focused on developing transverse competencies among students [8]. Though this model is in its development phase, here in the present work we try to analyze the model through a very popular approach of neutrosophic cognitive maps (NCMs) [9] [17]. NCMs are the extension of Fuzzy Cognitive Maps with an addition that allows us to quantify the uncertainty that holds a very important position when it comes to modeling uncertain, unknown, and indeterminate factors. Neutrosophy is a branch of philosophy that deals with indeterminacy, contradiction, and incomplete information. The neutrosophic set theory allows for the representation of indeterminate, imprecise, and inconsistent information, which can be useful in situations where traditional binary logic and crisp sets may not fully capture the complexities of the analysis. In a nutshell, we can say that NCM analysis explicitly considers the degree of indeterminacy or ambiguity associated with each factor whereas traditional FCM analysis typically treats factors as either present or absent, without explicitly accounting for degrees of uncertainty. Overall, NCM analysis provides a more flexible and nuanced approach to analyzing the external environment, particularly in situations where information is incomplete or uncertain.

It can help decision-makers better understand the complexities of the external environment and make more informed decisions in response to changes in these factors. Therefore in the present work we have used NCMs to analyze the effect of uncertain and indeterminate factors on the currently running TEC21 model in Tecnologico de Monterrey, Monterrey, Mexico to show how we can take into consideration unknown factors in analyzing any situation. We'll use neutrosophic numbers to represent the degrees of truth, indeterminacy, and falsehood for each factor.

The motivation for employing NCMs for analyzing the TEC 21 model lies in its ability to handle the uncertainties and complexities of the model's external environment. The TEC 21 model operates within a dynamic educational landscape where factors such as technology, policies, and societal trends are constantly evolving. NCM offers a more comprehensive approach by not only considering the presence or absence of these factors but also their degrees of truth, indeterminacy, and falsehood. This nuanced analysis can help in identifying potential risks and opportunities related to the TEC 21 model, enabling organizations to make more informed decisions and develop adaptive strategies. Furthermore, employing a novel approach like NCM can contribute to research and innovation in the field of educational technology, leading to new insights and methodologies for analyzing and improving educational models.

1.1. TEC 21 Model

TEC de Monterrey has always focussed on nourishing young minds to make them aware of what the current world requires. It has always focussed on training our young people and future professionals to have a strong emphasis on addressing the current problems that society demands to address. Tecnologico de Monterrey is known for leading in educational models and innovations. Their new model focuses on practical education, blending theory with skills. Starting in 2019, this model is being gradually implemented in all courses, aligning with Education 4.0. It offers students engaging projects that develop new skills, competencies, and knowledge. The TEC21 Educational Model, implemented on 26 campuses, aims to enhance student´s competitiveness through comprehensive training and active learning activities. It fosters entrepreneurship, leadership, and innovation competencies, with dedicated spaces like libraries, learning commons, and InnovAction Gym for collaborative work. Challenge-based learning is central to the TEC21 model, addressing industry needs and developing digital-based solutions. This model emphasizes the importance of incentives and proper distribution channels for industry collaboration. Academic institutions need to immerse students in real challenges to achieve better results. The TEC21 model focuses on developing competencies for work and lifelong learning, including disciplinary and transversal competencies necessary for professional practice in various sectors. This goal of TEC has enabled it from time to time to come up with different strategies to refine its education models. In light of this TEC has generated a differentiating strategy in its new TEC21 Educational Model. The new model has its focus on building transverse competencies that future professionals must develop in both the disciplinary and personal aspects. The model's main aim is towards the development of these competencies through four major pillars that promote student transformation. These major pillars are: A) Challenging and interactive learning experiences, B) Flexibility in the teaching-learning process, C) The building of a memorable university experience, and, D) Inspiring innovative teachers [8]. The TEC21 Educational Model integrates the purposes of the vision, defines and links the actors and components that participate in the teaching-learning process and takes advantage of opportunities to offer students a comprehensive education of international quality. Its objective is to provide comprehensive training and improve the competitiveness of students in their professional field by enhancing the skills of future generations to develop the competencies required to enable them to become leaders who will face the challenges and opportunities of the 21st century. In August 2019, the Tecnologico de Monterrey (Tec Mty) began to implement the Tec21 Educational Model [6] [7]. Since its inception, many studies have been conducted to ascertain the objectives of this model. The present work in this regard has focused mainly on the influence of uncertainties and indeterminacy on the successful implementation of this model in TEC de Monterrey. The TEC21 model at Tecnológico de Monterrey is an innovative educational model focused on developing competencies through challenge-based learning. The model emphasizes the following key factors and these are the well-defined and measurable aspects of the TEC21 model. For the current study, they are considered as determinate factors throughout this paper.

These are as follows [8].

- Critical Thinking Skills (C1)
- Problem-Solving Ability (C2)
- Creativity (C3)
- Collaboration (C4)
- Communication Skills (C5)
- Self-Regulation (C6)
- Reflective Thinking (C7)

The rest of the paper is divided into 5 sections. Section 2 and Section 3 present extensive literature surveys and methodology respectively. Section 4 is focused on results and discussion whereas Section 5 presents the conclusion of the present work.

2. Literature Survey

With the growing need to bring evolution to education, there have always been efforts to propose new methodologies for refining the education systems across the globe. These efforts are put not only by the academicians but also a lot more is done by the researchers who continuously study the scenarios and come up with different strategies to propose new methods to refine the education system by taking insights from the real-time data. In light of this many researchers have studied the newly introduced TEC21 model in TEC de Monterrey, Monterrey, Mexico. Hugo A López et al. [10] studied the model and provided a way to integrate Education 4.0 with the present model to prepare students for the needs of Industry 4.0. Their research presented a case study of a Capstone project developed with undergraduate engineering students. Through this, they contributed towards showing that a suitable educational framework is needed for making students capable of addressing the demands of Industry 4.0. Guillermo Gándara Fierro et al. [11] have analyzed this model by taking a case study based on the first semester results. Through this work, they proposed the possibility of helping the community's transformations. Under the new Tec21 Educational Model, Cintia Smith et al. [12] designed a strategy for the "Citizenship and Technology" course for the students. The design was proposed in a mixed-model scheme through which they detected areas of opportunity that signaled the need to make important adjustments in the activities design. They claimed that these changes in the strategy are needed to fulfill the following four major goals. These goals are to generate evidence of competence development; make adjustments to class dynamics for the correct teaching process, simplification of the evaluation mechanisms, and reconceptualization of special guests profile. Their study forms the basis for showing that the TEC21 model is not static but is open to changes. As the focus of the model is on the development of transverse competencies among students, there is a clear need for the methods for its development. Helena Belchior-Rocha et al. [13] have studied the importance of transverse competencies among the students enrolled in higher education courses. The authors in their latest work have stated that the acquisition of transversal competencies is affected by social, economic, technological, and political changes in the surrounding environment. Jean Cushen [14] evaluated the significance of the development of transversal competencies among the students of higher education. Through a comprehensive evaluation of its importance in the industry, the authors concluded that transversal competencies are expected to play a definitive role in future work scenarios. They analyzed the decisions and impacts surrounding the integration of transversal competencies into higher education assessments. They also focussed on the changes that higher education leaders must make. Jesús García-Álvarez et al. [15] provided a systematic review of the transversal competencies for employability in university graduates from an employer's perspective, with consideration of the importance of the topic in the cross-national context. The authors claimed the importance of these competencies through data collected from published articles from Scopus and Web of Science journals. For this purpose, the authors classified 41 transversal competencies into five dimensions. Hanesová et al. [16] focussed their research mainly on the development of transverse competencies together with taking into consideration the changing the forms of education so that they lead to the development of these competencies. The author's main objective was to design a new framework for mastering transversal competencies in a higher education environment.

Later they proposed that their framework could be updated based on processes of critical thinking and reflection. The extensive literature survey has given us more insights and data that influence the determinate factors in a significant manner. Like authors [1] Deci and Ryan discussed how intrinsic and extrinsic motivations drive student´s behaviors and learning outcomes. They emphasize that motivation is influenced by various internal and external factors, making it a complex and variable aspect of education. This is considered indeterminate as motivation fluctuates based on personal interests, external rewards, and situational factors, introducing uncertainty into its measurement and influence on educational outcomes. On the other hand, Hattie highlights the significant impact of different teaching methods on student achievement [2]. The effectiveness of these methods varies depending on context, student needs, and implementation strategies. This is considered an indeterminate factor in current work as the variability in the teaching method´s effectiveness, dependent on numerous contextual factors, introduces uncertainty in their impact on student learning. Garrison and Kanuka explored the transformative potential of blended learning, which combines online and face-to-face instruction [3]. The impact of technological tools varies based on their integration, student engagement, and technological proficiency. This factor is indeterminate as the effectiveness of technological tools is uncertain due to differences in implementation quality, technological accessibility, and student´s adaptability to technology. The renowned researcher Fraser discusses the development and importance of classroom environment instruments, emphasizing how physical and psychological classroom conditions affect student learning and behavior [4]. They emphasized that the classroom environment is influenced by dynamic interactions between students and teachers, as well as the physical setup, making its impact on learning unpredictable. Therefore this plays an important role in the successful implementation of any educational model. This can be considered as an indeterminate factor for this study.

Many researchers have emphasized the assessments happening in institutions for the successful implementation of educational models. Black and Wiliam analyzed various assessment techniques and their effects on learning [5]. They highlighted that the effectiveness of assessments depends on their design, implementation, and student´s perceptions. The impact of assessment techniques is indeterminate due to differences in assessment types, teacher practices, and student responses to assessments. Therefore, all these factors are termed as indeterminate or uncertain throughout the study. Below we mention five such factors I1-I5.

- Student Motivation (I1) [1]
- Teaching Methods (I2) [2]
- Technological Tools (I3) [3]
- Classroom Environment (I4) [4]
- Assessment Techniques (I5) [5]

The indeterminate factors I1-I5 are mathematically represented by the neutrosophic sets and systems. Netrosophic theory is not limited to the field of mathematics but it is spreading its wings in various other fields. Researchers around the globe have employed neutrosophic techniques to solve several problems prevailing in the current scenario i.e. in [18] [19] it is being used to solve the problem in multi-criteria decision-making. Authors in [20] have used neutrosophic sets in understanding and enhancing the supply chain sustainability in the current scenario. The proposed approach claims to be efficient in solving decision-making problems while meeting the supply chain sustainability requirement. Authors in [21] used IoT and Fog computing to propose a healthcare system for the prediction and diagnosis of diseases. For this purpose, they introduced a neutrosophic multi-criteria decision-making technique. The above work by prominent researchers proved that the application of neutrosophic theory in various fields of research is the need of the hour and our research methodology for addressing the uncertainty in educational models is based on this.

3. Methodology

The proposed solution to indeterminacy uses the concept of a Neutrosophic Cognitive Map (NCM). It is a technique in Neutrosophy introduced by W. B. Vasantha Kandasamy [23]. The concept of Neutrosophic logic was introduced by Florentine Smarandache [22] [23], which is a merger of fuzzy logic together with the inclusion of indeterminacy.

When data under scrutiny contains indeterminate concepts, we are not able to formulate mathematical expressions. Presentation of Neutrosophic logic by Florentine Smarandache [24] [25] has put forward a panacea to this problem. It is the reason Neutrosophy has been introduced as an additional notion for the evaluation of educational models. Fuzzy theory evaluates the existence or non-existence of associateship but it has been less efficient to attribute the indeterminate relations among concepts but most data collected in the educational setup has many indeterminate and uncertain concepts. Therefore we have employed Neutrosophic Cognitive Maps (NCMs) in place of Fuzzy Cognitive Maps (FCMs) to evaluate the TEC21 model. Earlier research for evaluating the TEC21 model has not included indeterminacy which is a part and parcel of real life. Hence when it comes to assessing the development of transverse competencies and critical thinking by educational models, indeterminacy needs to be considered. Contemplating the importance of indeterminacy we propose to use NCM in evaluating the TEC21 model [8]. Now indeterminacy has been introduced in Fuzzy Cognitive Maps (FCMs) and the generalized structure so obtained is referred to as Neutrosophic Cognitive Maps (NCMs) by W. B. Vasantha Kandasamy [23]. NCM is a neutrosophic directed graph (a directed graph with a dotted edge representing indeterminacy) with concepts represented as nodes of the directed graph and relationship or indeterminacy as the edge of the graph. Let us suppose C_1, C_2,……, and C_n are n nodes from the Neutrosophic vector space V. The edges of the graph are represented by (x_1,x_2,…..,x_n) where x_i's can be'0' or '1' or 'I' (I shows indeterminacy) where $x_i = 1$ indicates the ON state of the node whereas $x_i = 0$ indicates the OFF state and $x_i = I$ indicates the indeterminate state of a node in that situation. Suppose C_i and C_j are two nodes in this model (NCM), a directed edge from C_i to C_j represents the relationship of C_i and C_j. The edges of the directed graph in NCM are weighted having value in set {-1, 0, 1, I}. When e_{ij} is the weight assigned to the directed edge from C_i to C_j then if the value of e_{ij} is '0' it shows C_i does not affect C_j, it is '1' representing an increase (or decrease) of C_i leads to an increase (or decrease) of C_j, when it is '-1' representing increase (or decrease) of C_i leads decrease (or increase) of C_j and when the value is 'I' it shows effect of C_i on C_j is indeterminate. These NCMs are called simple NCMs. Let N (E) be a matrix defined as N (E) = (e_{ij}) then N (E) is called a Neutrosophic adjacency matrix.

3.1. Neutrosophic Concepts
3.1.1. Neutrosophic Sets:

A neutrosophic set is a generalization of a classical set in which the membership of an element is characterized by three functions: truth membership (T), indeterminacy membership (I), and falsehood membership (F) [25] [26].

It can be said that a neutrosophic set A is defined as:

$$A=\{x|\mu_A(x)=(T(x), I(x), F(x))\},$$

where x is an element of the set and $\mu_A(x)$ represents the membership values of x in set A. It's used to model uncertain survey responses which very much occurs while carrying out surveys for educational research.

3.1.2. Neutrosophic Membership:

Neutrosophic membership values indicate the degree to which an element belongs to a neutrosophic set, taking into account truth, indeterminacy, and falsehood. For an element x in a neutrosophic set A, the neutrosophic membership value is represented as $\mu_A(x)=(T(x), I(x), F(x))$ where $T(x)+I(x)+F(x)\leq 1$ and $T(x)$, $I(x)$, and $F(x)$ are the degrees of truth, indeterminacy, and falsehood, respectively [25]. This can be understood from the following diagram. For example Neutrosophic membership values for a single day with a condition very hot with a bit of uncertainty, membership values = [0.8, 0.1, 0.1] is represented in Figure 1.

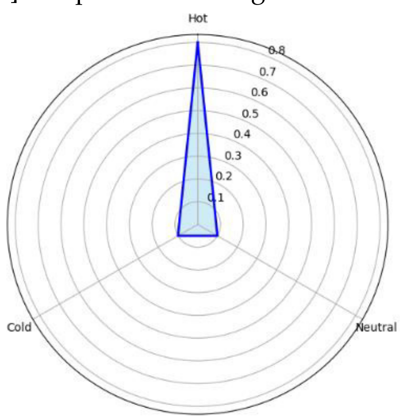

Figure 1 Neutrosophic Membership

3.1.3. Neutrosophic Cognitive Maps (NCMs):

An extension of fuzzy cognitive maps incorporating neutrosophic logic to model complex systems with uncertainties. It can be used to represent and analyze relationships between factors influencing critical thinking. This helps in modeling not only the determinate factors but also indeterminate factors that are also taken into consideration [23]. We can say that in the educational context, the nodes of NCM represent factors like "Innovative Teaching Methods," "Student Engagement," etc., with weighted edges reflecting the strength and uncertainty of their influence.

4. Results

4.1. FCM Adjacency Matrix

The FCM adjacency matrix captures the causal relationships between the determinate factors. Values are assigned based on literature and expert opinions, typically ranging between 0 and 1, where 0 means no influence and 1 means maximum influence. Relationships among Factors in FCM can be understood like C1 (Critical Thinking Skills) has a strong influence on C2 (Problem-Solving Ability) (0.4) and C7 (Reflective Thinking) (0.3). But on the other hand has a moderate influence on C3 (Creativity) (0.3), C4 (Collaboration) (0.2), and C6 (Self-Regulation) (0.2). Likewise, all values are assigned by taking into consideration the expert's opinions and extensive literature survey. These can be seen in the Table 1 and corresponding FCM is shown in Figure 2.

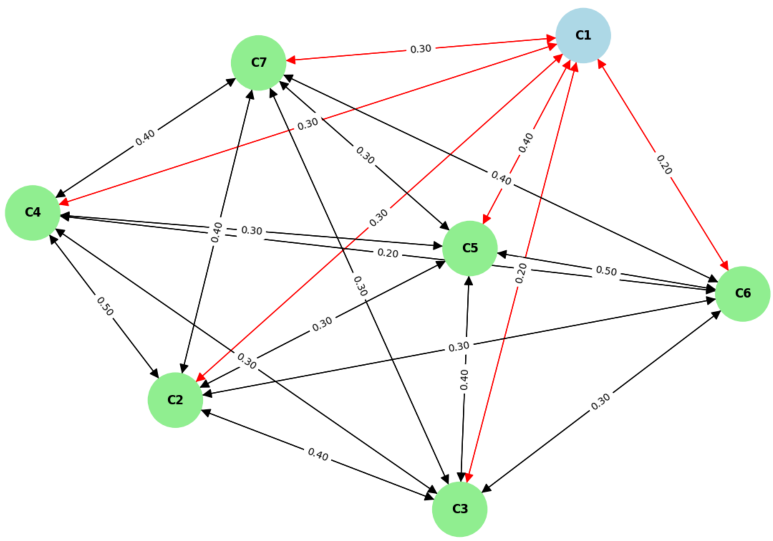

Figure 2 FCM: Factors Relationship to Critical Thinking (C1)

Table 1 Fuzzy Adjacency Matrix: Showing Relationship Among Determinate Factors

Determinate Factors	C1	C2	C3	C4	C5	C6	C7
C1	0	0.4	0.3	0.2	0.1	0.2	0.3
C2	0.3	0	0.5	0.4	0.2	0.3	0.4
C3	0.2	0.4	0	0.3	0.4	0.3	0.2
C4	0.3	0.5	0.3	0	0.4	0.2	0.3
C5	0.4	0.3	0.4	0.3	0	0.5	0.3
C6	0.2	0.3	0.3	0.2	0.5	0	0.4
C7	0.3	0.4	0.3	0.4	0.3	0.4	0

4.2. NCM Adjacency Matrix

The NCM adjacency matrix includes both determinate and indeterminate factors. Indeterminate factors are denoted as 'I' (values between 0 and 1) in the matrix. Relationships among factors in NCM can be understood as C1 (Critical Thinking Skills), Influences C2 (Problem-Solving Ability) (0.4), C3 (Creativity) (0.3), C4 (Collaboration) (0.2), C5 (Communication Skills) (0.1), C6 (Self-Regulation) (0.2), C7 (Reflective Thinking) (0.3), and indeterminate factors I1 (Student Motivation) (0.2**)**, I2 (Teaching Methods) (0.1), I3 (Technological Tools)(0.2), I4 (Classroom Environment) (0.1), I5 (Assessment Techniques) (0.2). The NCM graph shown in Figure 3 shows all the nodes whether determinate or indeterminate. All the nodes are in the same color except C1 as it is the primary node that represents critical thinking in the system. This is done to emphasize its central role. In the graphs, each node represents a factor, and the edges represent the relationships between these factors, with the weight of the edges indicating the strength of these relationships. The directed edges indicate the influence from one factor to another. For example, an edge from C1 to C2 with a weight of 0.4 means that C1 influences C2 with a strength of 0.4. There are some thicker edges representing stronger relationships among nodes. For instance, C2 has a strong influence on C3 (weight 0.5) compared to its influence on C1 (weight 0.3). In a uniform color scheme, the emphasis can be placed on the edges connecting to and from C1 to show its influence. As per NCM in Figure 3 the corresponding neutrosophic adjacency matrix is formed in Table 2.

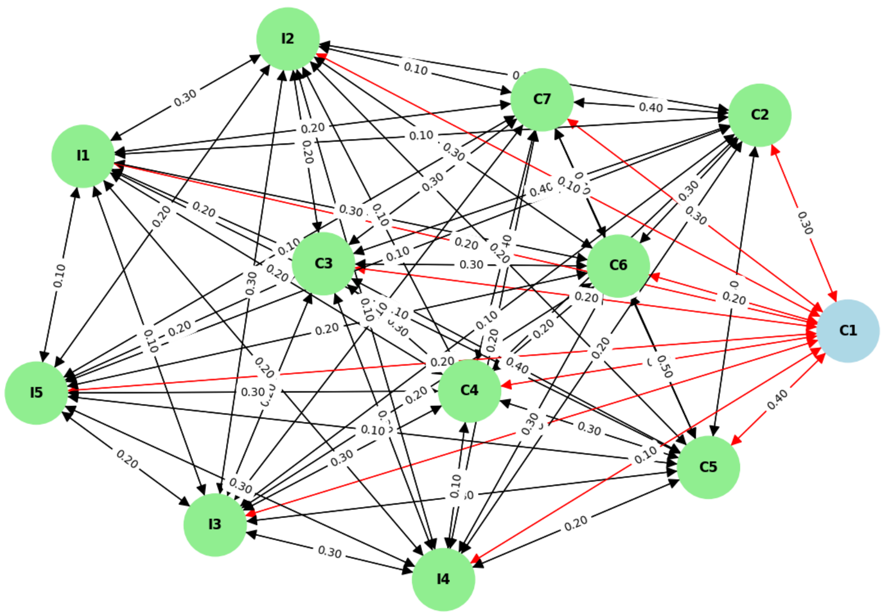

Figure 3 NCM: Factors Relationship to Critical Thinking (C1)

Table 2 Neutrosophy Adjacency Matrix: Showing Relationships Among Determinate and Indeterminate Factors

Determinate & Indeterminate Factors	C1	C2	C3	C4	C5	C6	C7	I1	I2	I3	I4	I5
C1	0	0.4	0.3	0.2	0.1	0.2	0.3	0.2	0.1	0.2	0.1	0.2
C2	0.3	0	0.5	0.4	0.2	0.3	0.4	0.1	0.3	0.1	0.2	0.1
C3	0.2	0.4	0	0.3	0.4	0.3	0.2	0.3	0.2	0.2	0.3	0.2

C4	0.3	0.5	0.3	0	0.4	0.2	0.3	0.2	0.1	0.3	0.1	0.3
C5	0.4	0.3	0.4	0.3	0	0.5	0.3	0.1	0.2	0.3	0.2	0.1
C6	0.2	0.3	0.3	0.2	0.5	0	0.3	0.3	0.3	0.2	0.3	0.2
C7	0.3	0.4	0.3	0.4	0.3	0.4	0	0.2	0.2	0.1	0.2	0.1
I1	0.2	0.1	0.2	0.2	0.1	0.3	0.2	0	0.3	0.1	0.2	0.1
I2	0.1	0.3	0.2	0.1	0.2	0.3	0.1	0.3	0	0.3	0.1	0.2
I3	0.2	0.1	0.2	0.3	0.3	0.2	0.1	0.1	0.3	0	0.3	0.2
I4	0.1	0.2	0.3	0.1	0.2	0.3	0.2	0.2	0.2	0.3	0	0.3
I5	0.2	0.1	0.2	0.3	0.1	0.2	0.1	0.1	0.1	0.2	0.3	0

4.3. Iterative Process

The first iteration starts by keeping the critical thinking state as ON rest all other factors are considered null at this time. So the Initial State becomes S(0)=[1,0,0,0,0,0,0,0,0,0,0,0] . The iterations for both FCM and NCM are done separately. The further iterations are carried out using the formula mentioned below. Iterations for FCM:

Iteration 1: S(1)=S(0)×WS(1)=S(0)×W
Iteration 2: S(2)=S(1)×WS(2)=S(1)×W
Iteration 1: S(1)=S(0)×WFCMS(1)=S(0)×WFCM
Iteration 2: S(2)=S(1)×WFCMS(2)=S(1)×WFCM

4.3.1. Iterations Results for FCM:

Initial State Vector V=(1,0,0,0,0,0,01,0,0,0,0,0,0)

1. **State after iteration 1:** (0.4,0.3,0.2,0.3,0.4,0.2,0.30.4,0.3,0.2,0.3,0.4,0.2,0.3)
2. **State after iteration 2:** (0.56,0.63,0.65,0.61,0.57,0.65,0.610.56,0.63,0.65,0.61,0.57,0.65,0.61)
3. **State after iteration 3:** (0.803,0.748,0.767,0.783,0.778,0.766,0.7830.803,0.748,0.767,0.783,0.778,0.766,0.783)
4. **State after iteration 4:** (0.924,0.925,0.93,0.926,0.929,0.93,0.9260.924,0.925,0.93,0.926,0.929,0.93,0.926)
5. **State after iteration 5:** (0.977,0.974,0.976,0.975,0.975,0.976,0.9750.977,0.974,0.976,0.975,0.975,0.976,0.975)
6. **State after iteration 6:** (0.994,0.991,0.992,0.992,0.992,0.992,0.9920.994,0.991,0.992,0.992,0.992,0.992,0.992)
7. **State after iteration 7:** (0.998,0.997,0.997,0.997,0.997,0.997,0.9970.998,0.997,0.997,0.997,0.997,0.997,0.997)
8. **State after iteration 8:** (0.999,0.999,0.999,0.999,0.999,0.999,0.9990.999,0.999,0.999,0.999,0.999,0.999,0.999)
9. **State after iteration 9:** (1.000,0.999,0.999,0.999,0.999,0.999,0.9991.000,0.999,0.999,0.999,0.999,0.999,0.999)

4.3.2. Iterations Results for NCM:

Initial State Vector V=(1,0,0,0,0,0,0,0,0,0,0,0,01,0,0,0,0,0,0,0,0,0,0,0,0)

Iterations:

1. **State after iteration 1:** (0.4,0.3,0.2,0.3,0.4,0.2,0.3,0.2,0.1,0.2,0.1,0.20.4,0.3,0.2,0.3,0.4,0.2,0.3,0.2,0.1,0.2,0.1,0.2)
2. **State after iteration 2:** (0.8793,0.7827,0.7821,0.7982,0.7686,0.7951,0.8003,0.7824,0.7488,0.7821,0.7488,0.78210.8793, 0.7827,0.7821,0.7982,0.7686,0.7951,0.8003,0.7824,0.7488,0.7821,0.7488,0.7821)
3. **State after iteration 3:** (0.9352,0.8685,0.8675,0.8836,0.8541,0.8801,0.8856,0.8679,0.8324,0.8675,0.8324,0.86750.9352, 0.8685,0.8675,0.8836,0.8541,0.8801,0.8856,0.8679,0.8324,0.8675,0.8324,0.8675)
4. **State after iteration 4:** (0.9518,0.8926,0.8910,0.9071,0.8776,0.9042,0.9090,0.8913,0.8578,0.8910,0.8578,0.89100.9518, 0.8926,0.8910,0.9071,0.8776,0.9042,0.9090,0.8913,0.8578,0.8910,0.8578,0.8910)
5. **State after iteration 5:** (0.9661,0.9161,0.9154,0.9256,0.9066,0.9227,0.9271,0.9155,0.8870,0.9154,0.8870,0.91540.9661, 0.9161,0.9154,0.9256,0.9066,0.9227,0.9271,0.9155,0.8870,0.9154,0.8870,0.9154)
6. **State after iteration 6:** (0.9743,0.9299,0.9295,0.9358,0.9216,0.9332,0.9372,0.9296,0.9020,0.9295,0.9020,0.92950.9743, 0.9299,0.9295,0.9358,0.9216,0.9332,0.9372,0.9296,0.9020,0.9295,0.9020,0.9295)
7. **State after iteration 7:** (0.9791,0.9375,0.9372,0.9417,0.9302,0.9395,0.9433,0.9373,0.9115,0.9372,0.9115,0.93720.9791, 0.9375,0.9372,0.9417,0.9302,0.9395,0.9433,0.9373,0.9115,0.9372,0.9115,0.9372)
8. **State after iteration 8:** (0.9820,0.9419,0.9417,0.9453,0.9352,0.9433,0.9469,0.9418,0.9170,0.9417,0.9170,0.94170.9820, 0.9419,0.9417,0.9453,0.9352,0.9433,0.9469,0.9418,0.9170,0.9417,0.9170,0.9417)
9. **State after iteration 9:** (0.9839,0.9447,0.9445,0.9474,0.9383,0.9456,0.9491,0.9446,0.9205,0.9445,0.9205,0.94450.9839, 0.9447,0.9445,0.9474,0.9383,0.9456,0.9491,0.9446,0.9205,0.9445,0.9205,0.9445)
10. **State after iteration 10:** (0.9852,0.9464,0.9463,0.9489,0.9403,0.9472,0.9506,0.9463,0.9226,0.9463,0.9226,0.94630.9852 ,0.9464,0.9463,0.9489,0.9403,0.9472,0.9506,0.9463,0.9226,0.9463,0.9226,0.9463)

The results are stabilized after the 9th iteration in the case of FCM and after the 10th in the case of NCM. When we compare the results obtained we can compare and contrast the values on four very important factors convergence patterns, degree of variability, interpretation of influence, and implications for educational analysis. If we talk about convergence patterns we can say that in the case of FCM, the results after iteration 9 show that all values have converged very closely to 1 (either 1.000 or 0.999).

This indicates a high level of agreement or certainty in the relationships between the factors while in the case of NCM, the results after iteration 10 show values ranging from approximately 0.9226 to 0.9852 demonstrating that while the system has reached a steady state, there remains a greater diversity in the influence values, reflecting the inclusion of indeterminate components. On grounds of the degree of variability, we can conclude that FCM shows a lack of variability (values very close to 1) suggesting that the factors are strongly and uniformly interrelated, with minimal uncertainty while in the case of NCM, the presence of values less than 1 and the variation among them (0.9226 to 0.9852) indicate that the factors have varying degrees of influence on one another, capturing the inherent uncertainties and indeterminate relationships.

That is the main reason that forms the basis for the use of neutrosophy for educational data analysis purposes. If we talk about the interpretation of influence we can notice that FCM suggests a highly deterministic model where each factor strongly influences the others. This deterministic nature may overlook the nuanced, real-world complexities of educational data. On the other hand, by incorporating indeterminate factors, the NCM results provide a more realistic representation of the educational environment. The varied influence values reflect the complex and uncertain nature of the relationships between the factors. The comparison based on implications for educational analysis forms the basis of conducting this research. It says that close-to-unity values indicate that FCM may be suitable for contexts where relationships are well-defined and less subject to variability. But in the case of educational data, we have a lot of variability. Therefore, the broader range of values suggests that NCM is better suited for analyzing educational models like TEC21 for the development of transverse competencies or critical thinking, where indeterminacies and uncertainties are prevalent. This makes NCM a more robust tool for understanding and managing the complexities inherent in educational data and can be utilized to assess the development of critical thinking skills of the student using TEC21 model.

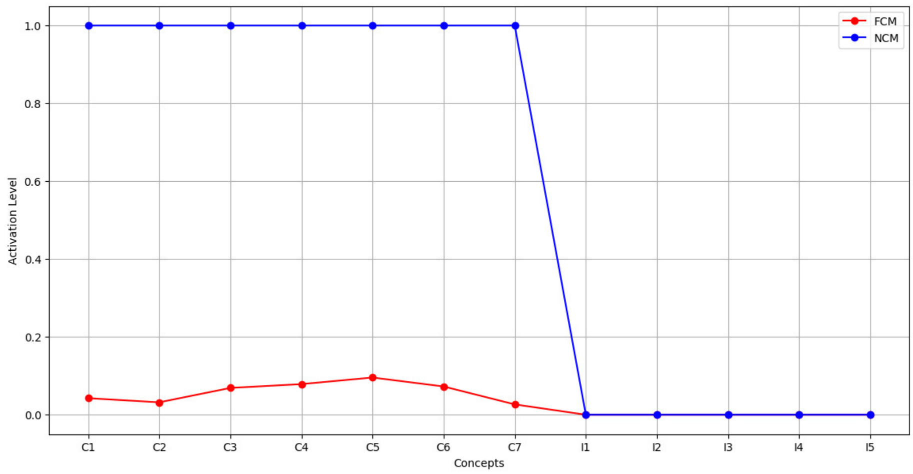

Figure 4 FCM vs NCM: Iterative Results Analysis

The graph in Figure 4 illustrates the activation levels of various concepts after iterative processes in both Fuzzy Cognitive Maps (FCM) and Neutrosophic Cognitive Maps (NCM). The red line represents the FCM results, which show a consistent convergence of activation levels close to 1, indicating stable and deterministic relationships between concepts. The blue line represents the NCM results, which exhibit higher variability and slower convergence, reflecting the inclusion of indeterminate factors and capturing the uncertainties within the system. This comparison highlights that while FCM provides a stable but simplistic view, NCM offers a more nuanced and realistic representation of complex, uncertain environments for assessing the critical thinking skills of the students in the TEC21 model.

Conclusion

In conclusion, the TEC21 model implemented by TEC de Monterrey represents a significant advancement in fostering critical thinking skills among students. By utilizing real-time data collected through the eOpen instrument, this research highlights the importance of addressing the inherent uncertainties and indeterminacies in educational data for determining the critical thinking skills of the students. The application of Neutrosophic Logic in analyzing the TEC21 model has proven to be highly effective, offering a more nuanced and comprehensive representation of the data. The comparative analysis between Neutrosophic Cognitive Maps (NCM) and Fuzzy Cognitive Maps (FCM) further demonstrates the superior capability of NCM in capturing and managing the indeterminate factors that influence educational outcomes. This pioneering work underscores the potential of neutrosophy in enhancing the analysis of educational models also to assess critical thinking skills, particularly in environments characterized by significant uncertainty. By using this innovative approach, educational institutions can better prepare students with the critical thinking skills necessary for their future success.

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Multi Attribute Neutrosophic Optimization Technique forOptimal Crop Selection in Ariyalur District

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Abstract. Ranking crops is a vital part of sustainable farming practices. A deliberate strategy that incorporates multi-criteria decision-making is essential for achieving sustainability in agriculture. The process of choosing crops involves a lot of unknowns and unpredictable elements. The neutrosophic set, characterized by the three independent degrees of truth (T), falsity (F), and indeterminacy (I), is more adept at handling incomplete data. The current study combines single-valued multi-criteria neutrosophic programming with the TOPSIS approach to examine the crop selection process in the Ariyalur district. This study incorporated expert advice and a thorough literature analysis to identify the fundamental criteria for developing sustainable crop planning for important crops in the Ariyalur area. In order to rate the crops and achieve sustainability for agricultural production, eleven significant criteria were selected based on environmental, social, economic, and soil nutrient concerns. In this study, the relative importance of criteria and alternatives is assessed by consolidating the views of different decision-makers into a unified opinion through a single-valued neutrosophic set-based weighted averaging operator. The current approach will greatly enhance the self-sufficiency of the agricultural sector and boost the country's GDP. Additionally, it will support the Ministry of Agriculture and other stakeholders in formulating regulations related to crop harvesting methods.

Keywords: Crop selection; TOPSIS approach; Neutrosophic set; Multi criteria optimization; Group Decision.

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1. Introduction

Agriculture is an essential part of the global economy, providing food, feed, and fibre for an expanding population. Undoubtedly, the primary source of livelihood in India is agriculture and its related sectors, especially in the country's vast rural areas. It also makes a substantial contribution to the GDP. Some countries around the world, especially India, are encountering heightened difficulties in meeting their food demands due to population growth. Food consumption often rises annually in tandem with a country's population growth; as a result, sufficient crop production that meets sustainability standards and yield rates is necessary. Sustainable agriculture is crucial for comprehensive rural development because it promotes rural employment, ensures food security, and supports environmentally friendly practices such as sustainable management of natural resources, soil conservation, and biodiversity preservation. Crop selection is important in agriculture because it influences the sustainability, profitability and productivity of farming operations. Because of fluctuating socioeconomic conditions and limited resources that differ from state to state and region to region in India, choosing crop patterns is even more difficult. The minimum support price (MSP) of crops is just insufficient to support farmers' economically viable growth, which presents another challenge for them in realising crop value. This study employs a multi-attribute decision-making (MADM) approach to examine the crop selection system in Ariyalur district, Tamil Nadu, India, with the aim of achieving sustainable and profitable agriculture. Numerous fields, including operation research, urban planning, natural science, and management science, have used MADM technique with either numerical or descriptive attribute values. When it comes to crop selection, it entails assessing and choosing the optimal crops to plant based on a variety of attributes, including yield, cost, environmental effect, climatic resilience, market demand, etc.

Classical MADM techniques, such as TOPSIS [1], VIKOR [2], PROMETHEE [3], and ELECTRE [4], use crisp numbers to represent the weights of each attribute and alternative ratings. However, due to attribute complexity and ambiguity, MADM problems' attribute values are not always described with precise numerical values. To overcome such an issue, Zadeh [5] introduced fuzzy set theory. It is highly effective for decision-making in MADM scenarios where information is imprecise. To determine the most suitable crop for the land, F. Sari and F. Koyunch [6] integrated AHP and TOPSIS with GIS (geographical information systems). AHP and fuzzy TOPSIS techniques were employed by Weilun Huanga and Qi Zhang [7] to determine the best economic crop in the minority region. The fuzzy TOPSIS approach was proposed by Singh, R.K., and Mallick, J. [8] as a means of selecting the vegetable cash crop for sustainable agriculture within the green chamber. To select the optimal biomass crop type for bio-energy production, Cobuloglu H. I. and Buyuktahtak I. E. [9] introduced a new stochastic analytical hierarchy process (AHP) capable of managing ambiguous data and discovering the relative importance of criteria in the MCDM process. Various authors have created fuzzy-MCDM techniques for addressing the plant location selection (PLS) problem, drawing on principles from fuzzy set theory. For the PLS problem in a linguistic environment, Yong [10] suggested the TOPSIS method. The location of the facility was chosen by Ertugrul and Karakasoglu [11] using the fuzzy TOPSIS and fuzzy AHP approaches. The degree of belonging constitutes the only element in the fuzzy set. The most effective way to solve the

MADM problem requires keeping in mind that membership and non-membership functions are equally important. Therefore, Atanassov [12] proposed the intuitionistic fuzzy set (IFS), defined by belongings and non-belongings degrees simultaneously. To solve the PLS problem, Pankaj Gupta et al. [13] suggested an expanded VIKOR technique using intuitionistic trapezoidal fuzzy parameters. To address the challenge of choosing R & D projects, Wan et al. [14] created a novel approach for handling multi-attribute group decision-making (MAGDM) problems, including incomplete attribute weight information and Atanassov's interval-valued intuitionistic fuzzy values. Abbas Mardani et al. [15] introduced an innovative combined approach that applies the step-wise weight assessment ratio analysis (SWARA) and the complex proportional assessment (COPRAS) method within the context of IFSs to determine the optimal biomass crop type for sustainable production of bio-fuel. In IFSs, the sum of a vague parameter's degrees of belonging and non-belonging does not amount to one. Consequently, an intuitionistic fuzzy set has some degree of incompleteness or indeterminacy. It is unable to effectively handle every kind of uncertainty in various real-world physical challenges, including those with indeterminate data.

To address ambiguous or indeterminate data often encountered in practical situations, Smarandache [16] introduced the idea of a neutrosophic set (NS) within a philosophical context. However, applying NS directly in scientific and engineering fields proves to be challenging. Wang et al. [17] created the single-valued neutrosophic set (SVNS), a subclass of NS, to address challenges. SVNSs have been utilised by neumorous academics to create decision-making models. [18–21]. Using the weighted correlation coefficient of SVNSs, Ye [22] investigated the MCDM problem. MCDM problem under interval neutrosophic set information were examined by Zhang et al. [23]. Chi and Liu [24] developed an enhanced TOPSIS method for addressing multi-attribute decision-making (MADM) problems involving interval neutrosophic sets. A TOPSIS model was introduced by Nancy and Grag [25] to evaluate the MCDM when there was insufficient weight data available for SVNSs. An innovative weighted aggregated sum product assessment (WASPAS) framework using SVNS was developed by Arunodaya Raj Misra et al. [26] to identify the biomass crop option that is most optimal for producing biofuel. For professional selection, Abdel-Basset M. et al. [27] introduced a novel hybrid neutrosophic MCDM framework that combines neutrosophic analytical network processes (ANP) and TOPSIS using bipolar neutrosophic numbers. To manage uncertainty in SVNS data, Shahzaib et al. [28] created a hybrid averaging and geometric aggregation operator utilising a sine trigonometric function. Additionally, the approach is used for agricultural land selection in order to demonstrate its efficacy. In order to define the available water resources in the agricultural sector based on possibility measurements in a generalised single-valued non-linear bipolar neutrosophic environment, Garai T., and Garg H. [29] devised a multi-criterion water resource management technique. To address multi-attribute decision-making issues in a multi-valued neutrosophic environment, Dongsheng Xu and Lijuan Peng [30] suggested a novel approach based on TOPSIS and TODIM. The benefits of TOPSIS include being simpler, easier to understand, and more computationally efficient, according to an assessment of the literature. Consequently, choosing agricultural crops may benefit from the current study's combination of the TOPSIS technique with SVNSs. The primary goal of this research is to evaluate and rank the most suitable crops grown in the Ariyalur district using a TOPSIS-based (MAGDM) approach.

This study is structured as follows: Section 2 reviews preliminary research on SVNSs; Section 3 covers the study area; Section 4 details the methodology employed; Section 5 illustrates the application of the methodology; and Section 6 summarizes the conclusions drawn from the study.

2. **Preliminary**

To advance the paper, certain fundamental definitions of a single valued neutrosophic set are given in this section.

2.1. *Single Valued Neutrosophic Set*

A single valued neutrosophic set (SVNS) P, over the universe of discourse U is represented as

$$P = \{(u, T_P(u), I_P(u), F_P(u))/u \in U\}$$

where $T_P(u), I_P(u), F_P(u)$ are values in the range [0,1], and the sum $T_P(u) + I_P(u) + F_P(u)$ satisfies $0 \leq T_P(u) + I_P(u) + F_P(u) \leq 3$.

2.2. *Complement*

The complement of a SVNS P, represented as c(P), is defined by: $T_{c(P)}(u) = F_P(u)$, $I_{c(P)}(u) = 1–I_P(u)$, $F_{c(P)}(u) = T_P(u)$, for all u in U.

2.3. *Neutrosophic Operators*

If P and Q be two SVNSs then

(i) $P \oplus Q = (T_P(u) + T_Q(u) - T_P(u).T_Q(u), I_P(u) + I_Q(u), F_P(u) + F_Q(u))$

(ii) $P \otimes B = (T_P(u).T_Q(u), I_P(u) + I_Q(u) - I_P(u).I_Q(u), F_P(u) + F_Q(u) - F_P(u).F_Q(u))$

(iii) $P \cup Q = (\max(T_P(u), T_Q(u)), \max(I_P(u), I_Q(u)), \min(F_P(u), F_Q(u)))$ or

$P \cup Q = (\max(T_P(u), T_Q(u)), \min(I_P(u), I_Q(u)), \min(F_P(u), F_Q(u)))$

(iv) $P \cap B = (\min(T_P(u), T_Q(u)), \min(I_P(u), I_Q(u)), \max(F_P(u), F_Q(u)))$ or

$P \cap Q = (\min(T_P(u), T_Q(u)), \max(I_P(u), I_Q(u)), \max(F_P(u), F_Q(u)))$

Study Area

Appropriate crop selection is crucial to establishing sustainable agriculture and increasing agricultural profitability. This paper presents a TOPSIS-based neutrosophic programming approach to crop planning in the Ariyalur district, one of the Cauvery Delta regions. The Ariyalur District covers a total area of 193,338 hectares. 94,725 hectares are the net area that has been planted. Of which, 58441 ha are rainfed and 36284 ha are under irrigation. Rainfall averages 954 mm per year. This district grows a variety of crops, and the majority of its residents work in agriculture. Since 70% of the population makes their living from agriculture and related industries, agriculture remains the most important component of the district's economy. In order to increase production, the Agriculture Department has implemented several development schemes and disseminated pertinent technology, stepping up its efforts to attain a higher growth rate in the sector. The department's objectives and policies focus on sustaining agricultural production stability and promoting sustainable growth to meet the food demands of a growing population, while also providing raw materials for agro-based industries, thereby generating employment for the rural population. The Ariyalur district's soil is made up of ferruginous red clay and limestone. The colour typically varies from red at the top to yellow at the lower horizon, with a loamy texture. These soils are of medium depth with good drainage, characterized by low levels of organic content, nitrogen, and phosphorus, yet generally containing ample amounts of potash and lime. The pH range is 6.5 to 8.0, and they are free from salt and calcium carbonate formation. This district is used to develop sugarcane, groundnuts, maize, cotton, and paddy crops. The primary irrigation sources in this district include open wells, canals, tanks, and tube wells. Bore wells and tube wells contribute significantly to irrigation. The alternatives that were examined in this study were sugarcane, maize, groundnut, cotton, brinjal, tomato, chilli, and onion.

4. Multi Attribute TOPSIS Based Neutrosophic Technique

Hwang and Yoon introduced the TOPSIS method in 1981. This technique determines the optimal solution as the one that is closest to the positive ideal solution and farthest from the negative ideal solution. In terms of determining the ideal response, the negative approach maximizes the cost criterion while decreasing the benefit criteria, and the positive approach maximises the benefit criteria while minimising the cost criteria. Numerous studies have used TOPSIS to address MCDM problems in research. The next set of steps presents the computing procedures for the TOPSIS-based neutrosophic technique.

Step 1: Assume the alternatives A_i and the criteria C_j, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Additionally, assume that the decision makers have provided $w = \{w_1, w_2, \ldots, w_n\}$ as the weight vector for the criteria $C_1, C_2, \ldots, C_n$. Also, create the decision matrix that follows, which shows the values corresponding to the alternatives and criteria for MADM problems.

$$D = [d_{ij}]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mn} \end{bmatrix} \end{array}$$

Step 2: Use professional advice or another technically sound method to determine which criteria are most important.

Step 3: Convert each value assigned to the alternatives according to the attributes in the decision matrix, as defined in step one, into a single-valued neutrosophic number. Following such conversion, the resulting matrix is defined as follows:

$$(d_{ij})_{m\times n} = (T_{ij}, I_{ij}, F_{ij})_{m\times n}$$

$$D = [d_{ij}]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} (T_{11}, I_{11}, F_{11}) & (T_{12}, I_{12}, F_{12}) & \cdots & (T_{1n}, I_{1n}, F_{1n}) \\ (T_{21}, I_{21}, F_{21}) & (T_{22}, I_{22}, F_{22}) & \cdots & (T_{2n}, I_{2n}, F_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{m1}, I_{m1}, F_{m1}) & (T_{m2}, I_{m2}, F_{m2}) & \cdots & (T_{mn}, I_{mn}, F_{mn}) \end{bmatrix} \end{array}$$

Step 4: Convert the decision matrix mentioned in step 3 into the normalized single valued neutrosophic decision matrix $\left(\widetilde{d_{ij}}\right)_{m\times n}$, where $\left(\widetilde{d_{ij}}\right) = d_{ij}$ for benefit criteria C_j and $\left(\widetilde{d_{ij}}\right) = d_{ij}^c$ for cost criteria.

Step 5: In a collective decision-making scenario, decision-makers are not of equal importance. Therefore, at this stage, the weight of the decision-maker is established. Assume that there exist r decision makers and that the linguistic word mentions their importance and expresses it in terms of neutrosophic numbers. Then, the weight of t^{th} decision-maker is specified by:

$$\varphi_t = \frac{1 - \sqrt{\frac{\{(1-T_s(u))^2 + (I_s(u))^2 + (F_s(u))^2\}}{3}}}{\sum_{t=1}^{r}\left(1 - \sqrt{\frac{\{(1-T_s(u))^2 + (I_s(u))^2 + (F_s(u))^2\}}{3}}\right)}, \text{and} \sum_{t=1}^{r} \varphi_t = 1 \tag{1}$$

Step 6: Create an aggregated neutrosophic decision matrix by applying the single-value neutrosophic weighted averaging (SVNWA) operator as described:

$$[d_{ij}]_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} (AT_{11}, AI_{11}, AF_{11}) & (AT_{12}, AI_{12}, AF_{12}) & \cdots & (AT_{1n}, AI_{1n}, AF_{1n}) \\ (AT_{21}, AI_{21}, AF_{21}) & (AT_{22}, AI_{22}, AF_{22}) & \cdots & (AT_{2n}, AI_{2n}, AF_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (AT_{m1}, AI_{m1}, AF_{m1}) & (AT_{m2}, AI_{m2}, AF_{m2}) & \cdots & (AT_{mn}, AI_{mn}, AF_{mn}) \end{bmatrix} \end{array} \quad (2)$$

where $AT_{ij} = 1 - \prod_{t=1}^{r}\left(1 - T_{ij}^{(t)}\right)^{\varphi t}$, $AI_{ij} = \prod_{t=1}^{r}\left(I_{ij}^{(t)}\right)^{\varphi t}$ and $AF_{ij} = \prod_{t=1}^{r}\left(F_{ij}^{(t)}\right)^{\varphi t}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 7: Every decision-maker in a group setting has a different viewpoint on every criterion. Gathering a thorough assessment of each attribute's significance from all decision-makers is necessary to obtain the grouped opinion on the chosen attribute. Consequently, the SVNWA operator is used to determine the aggregated weights for the criteria at this phase as follows: $w_j = \left(1 - \prod_{t=1}^{r}\left(1 - T_j^{(t)}\right)^{\varphi t}, \prod_{t=1}^{r}\left(I_j^{(t)}\right)^{\varphi t}, \prod_{t=1}^{r}\left(F_j^{(t)}\right)^{\varphi t}\right)$, where $j = 1, 2, \ldots, n$.

Step 8: Compute the weighted aggregated neutrosophic decision matrix using the two neutrosophic sets multiplication approach as follows:

$$(d_{ij})_{m\times n} \times w_j = (AT_{ij}\cdot w_j, AI_{ij}\cdot w_j, AF_{ij}\cdot w_j)$$

$$= \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} (AT_{11}\cdot w_1, AI_{11}\cdot w_1, AF_{11}\cdot w_1) & (AT_{12}\cdot w_2, AI_{12}\cdot w_2, AF_{12}\cdot w_2) & \cdots & (AT_{1n}\cdot w_n, AI_{1n}\cdot w_n, AF_{1n}\cdot w_n) \\ (AT_{21}\cdot w_1, AI_{21}\cdot w_1, AF_{21}\cdot w_1) & (AT_{22}\cdot w_2, AI_{22}\cdot w_2, AF_{22}\cdot w_2) & \cdots & (AT_{2n}\cdot w_n, AI_{2n}\cdot w_n, AF_{2n}\cdot w_n) \\ \vdots & \vdots & \ddots & \vdots \\ (AT_{m1}\cdot w_1, AI_{m1}\cdot w_1, AF_{m1}\cdot w_1) & (AT_{m2}\cdot w_2, AI_{m2}\cdot w_2, AF_{m2}\cdot w_2) & \cdots & (AT_{mn}\cdot w_n, AI_{mn}\cdot w_n, AF_{mn}\cdot w_n) \end{bmatrix} \end{array} \quad (3)$$

where $AT_{ij}\cdot w_j = \left(1 - \prod_{t=1}^{r}\left(1 - T_{ij}^{(t)}\right)^{\varphi t}\right) \times \left(1 - \prod_{t=1}^{r}\left(1 - T_j^{(t)}\right)^{\varphi t}\right)$

$AI_{ij}\cdot w_j = \prod_{t=1}^{r}\left(I_{ij}^{(t)}\right)^{\varphi t} + \prod_{t=1}^{r}\left(I_j^{(t)}\right)^{\varphi t} - \prod_{t=1}^{r}\left(I_{ij}^{(t)}\right)^{\varphi t}\cdot\prod_{t=1}^{r}\left(I_j^{(t)}\right)^{\varphi t}$

$AF_{ij}\cdot w_j = \prod_{t=1}^{r}\left(F_{ij}^{(t)}\right)^{\varphi t} + \prod_{t=1}^{r}\left(F_j^{(t)}\right)^{\varphi t} - \prod_{t=1}^{r}\left(F_{ij}^{(t)}\right)^{\varphi t}\cdot\prod_{t=1}^{r}\left(F_j^{(t)}\right)^{\varphi t}$ $\forall\ i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 9: Determine the relative neutrosophic positive ideal solution A_N^+ and negative ideal solution A_N^- for the attributes of benefit and cost as described below:

$$A_N^+ = \left(d_{1.w}^+, d_{2.w}^+, \ldots, d_{j.w}^+\right); \quad A_N^- = \left(d_{1.w}^-, d_{2.w}^-, \ldots, d_{j.w}^-\right)$$

where $d_{j.w}^+ = \left((AT_j.w)^+, (AI_j.w)^+, (AF_j.w)^+\right); \quad d_{j.w}^- = \left((AT_j.w)^-, (AI_j.w)^-, (AF_j.w)^-\right)$

$$(AT_j.w)^+ = \left\{\left(\max_i\{(AT_{ij}.w_j) \mid j \in j_1\}\right), \left(\min_i\{(AT_{ij}.w_j) \mid j \in j_2\}\right)\right\}$$

$$(AI_j.w)^+ = \left\{\left(\min_i\{(AI_{ij}.w_j) \mid j \in j_1\}\right), \left(\max_i\{(AI_{ij}.w_j) \mid j \in j_2\}\right)\right\}$$

$$(AF_j.w)^+ = \left\{ \left(\min_i \{(AF_{ij}.w_j) \mid j \in j_1\} \right), \left(\max_i \{(AF_{ij}.w_j) \mid j \in j_2\} \right) \right\}$$

$$(AT_j.w)^- = \left\{ \left(\min_i \{(AT_{ij}.w_j) \mid j \in j_1\} \right), \left(\max_i \{(AT_{ij}.w_j) \mid j \in j_2\} \right) \right\}$$

$$(AI_{j.w})^- = \left\{ \left(\max_i \left\{ \left(AI_{ij.w_j}\right) \mid j \in j_1 \right\} \right), \left(\min_i \left\{ \left(AI_{ij.w_j}\right) \mid j \in j_2 \right\} \right) \right\}$$

$$(AF_{j.w})^- = \left\{ \left(\max_i \left\{ \left(AF_{ij.w_j}\right) \mid j \in j_1 \right\} \right), \left(\min_i \left\{ \left(AF_{ij.w_j}\right) \mid j \in j_2 \right\} \right) \right\}$$

Step 10: Calculate the closeness coefficient for each alternative based on its distance from the relative neutrosophic positive and negative ideal solutions.

$$C_i^* = \frac{D_i^-}{D_i^+ + D_i^-}$$

where,

$$D_i^+ = \sqrt{\frac{1}{3n} \left\{ \sum_{j=1}^n \left[\left(AT_{ij}.w_j - (AT_j.w)^+\right)^2 + \left(AI_{ij}.w_j - (AI_j.w)^+\right)^2 + \left(AF_{ij}.w_j - (AF_j.w)^+\right)^2 \right] \right\}}$$

$$D_i^- = \sqrt{\frac{1}{3n} \left\{ \sum_{j=1}^n \left[\left(AT_{ij}.w_j - (AT_j.w)^-\right)^2 + \left(AI_{ij}.w_j - (AI_j.w)^-\right)^2 + \left(AF_{ij}.w_j - (AF_j.w)^-\right)^2 \right] \right\}}$$

Step 11: Rank the alternatives according to their closeness coefficients from highest to lowest.

Step 12: Identify the best alternatives in accordance with the closeness coefficients. The best alternative is the one with the maximum closeness coefficient.

5. **Model Implementation**

Crop selection has a significant role in creating sustainable agriculture. In this study, to improve the sustainable agriculture of Ariyalur district, which plays a major role in the agriculture sector, the above-mentioned method is used to rank the crops (sugarcane, paddy, cotton, groundnut, maize, brinjal, tomato, chilli, onion) cultivated there on the basis of criteria (production, profitability, water availability, seed growth, soil texture, precipitation, irrigation, crop demand, price of crop, expenditure and fertilizer). Based on the advice of agriculture field specialists and a comprehensive assessment of the literature, these criteria were developed. An alternative and attribute-based questionnaire is used to gather data for the current study from the agriculture experts in the Ariyalur district. The agriculture experts provided the complete data in the predetermined format. The weight of the attributes and the weight of the alternatives based on the attributes are collected from the four-decision maker in the form of linguistic terms. The linguistic terms' rates are expressed as a single-valued neutrosophic set. The linguistic terms for attributes and importance of decision-makers are shown in Table 1. Also, the linguistic terms of alternatives are mentioned in Table 2 in SVNS rating

TABLE 1. Linguistic terms of decision maker and Attributes.

Linguistics Term	**SVNN**
Very important (VI)	(0.95, 0.05, 0.05)
Important (I)	(0.85, 0.15, 0.15)
Medium (M)	(0.50, 0.40, 0.45)
Unimportant (UI)	(0.30, 0.60, 0.70)
Very unimportant (VUI)	(0.05, 0.85, 0.95)

TABLE 2. Alternatives' Linguistic Terms.

Linguistics Term	**SVNN**
Extremely high (EH)	(1, 0, 0)
Very high (VH)	(0.95, 0.10, 0.05)
High (H)	(0.85, 0.15, 0.15)
Medium (M)	(0.65, 0.40, 0.35)
Low (L)	(0.20, 0.75, 0.80)
Very low (VL)	(0.10, 0.85, 0.90)
Extremely low (EL)	(0.05, 0.90, 0.95)

format. In this present study, the linguistic term for (DM_1, DM_2, DM_3, DM_4) is (VI, I, VI, M) respectively.

Using Eq. (1), the decision makers' weights are ascertained as follows:

$$\varphi_1 = \frac{1 - \sqrt{\frac{\{(1-T_1(x))^2 + (I_1(x))^2 + (F_1(x))^2\}}{3}}}{\sum_{t=1}^{4}\left(1 - \sqrt{\frac{\{(1-T_s(x))^2 + (I_s(x))^2 + (F_s(x))^2\}}{3}}\right)}$$

$$= \frac{1 - \sqrt{\frac{\{(0.05^2 + 0.05^2 + 0.05^2\}}{3}}}{4 - \sqrt{0.0025} - \sqrt{0.0025} - \sqrt{0.0025} - \sqrt{0.2042}}$$

$$\varphi_1 = 0.2888$$

Similarly, $\varphi_2 = 0.2577$, $\varphi_3 = 0.288$, $\varphi_4 = 0.1662$. Therefore (0.288, 0.2577, 0.288, 0.1662) is the four decision makers' weight vectors.

The linguistic term of every alternative $A_i (i = 1, 2, \ldots, 9)$ in relation to every criterion $C_j (j = 1, 2, \ldots, 11)$, as well as the linguistic terms for the weights of the criteria provided by the four decision makers (DM_1, DM_2, DM_3, DM_4) are displayed in Table 3. Using Table 1 & Table 2, convert the linguistic terms of the alternatives and attributes in Table 3 into SVNNs. Then, as stated in Step 3, generate the single-valued neutrosophic decision matrix. Additionally, apply Step 4 to normalize the decision matrix. After these conversions, Step 6 is used to define the aggregated neutrosophic decision matrix in the following manner:

$$d_{11} = \left(1 - \prod_{t=1}^{4}\left(1 - T_{11}^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4}\left(I_{11}^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4}\left(F_{11}^{(t)}\right)^{\varphi_t}\right)$$

TABLE 3. Weight for alternatives and attributes in linguistic terms.

Alter	D.M	Prod	Prof	W.A	S.G	S.T	P	I.T	C.D	P.C	E	F
Sugarcane	1	H	H	H	H	VH	L	VH	VH	H	M	VH
	2	H	M	M	H	M	L	H	H	L	H	H
	3	M	M	M	H	VH	L	VH	M	L	H	H
	4	M	M	M	M	H	M	H	M	M	M	VH
Paddy	1	H	M	VH	H	VH	L	VH	VH	H	H	VH
	2	M	M	M	M	M	L	M	M	L	M	H
	3	M	M	L	L	VH	L	M	L	M	VH	M
	4	VL	VL	VL	H	M	VL	H	VL	VL	VH	VH
Cotton	1	H	H	M	H	VH	M	VH	VH	H	M	H
	2	M	M	VL	H	H	L	H	M	L	H	H
	3	VH	L	VL	H	H	VL	H	M	L	M	H
	4	L	M	M	M	L	VL	VH	VH	VL	VH	M
Groundnut	1	M	M	M	H	VH	M	VH	H	H	H	VH
	2	M	M	M	H	H	L	H	H	M	M	H
	3	M	M	M	M	M	VL	H	H	M	H	H
	4	M	M	M	M	H	L	VH	H	M	H	VH
Maize	1	M	H	M	H	VH	M	VH	M	H	M	H
	2	M	M	M	H	VH	M	H	M	M	M	H
	3	H	M	L	H	H	VL	H	H	L	M	H
	4	M	M	VL	M	M	L	VH	VH	M	H	H
Brinjal	1	M	H	M	H	H	M	H	H	H	M	G
	2	M	M	M	M	H	M	H	M	M	M	H
	3	H	L	M	M	H	VL	H	H	H	H	H
	4	VL	L	L	M	M	VL	H	M	M	VH	M
Tomato	1	M	H	M	H	H	M	M	VH	H	M	H
	2	L	M	M	H	H	L	H	H	H	M	H
	3	H	M	M	H	H	VL	M	H	M	M	H
	4	M	M	M	M	H	VL	H	M	VL	M	VH
Chilli	1	VH	VH	M	H	M	M	M	VH	H	M	H
	2	H	H	H	H	VH	L	H	H	H	M	H
	3	M	M	M	M	H	VL	H	M	M	H	H
	4	M	M	L	M	M	VL	VH	M	M	H	M
Onion	1	M	H	M	M	H	M	H	VH	H	H	H
	2	M	H	M	H	H	H	H	H	M	M	H
	3	H	M	M	M	H	VL	M	H	H	M	H
	4	L	L	VL	L	M	VL	VH	L	M	M	H
weight	1	VI	I	VI	VI	VI	VI	VI	VI	I	I	VI
	2	I	VI	I	I	I	I	I	I	VI	I	I
	3	VI	VI	VI	VI	VI	VI	VI	VI	VI	I	I
	4	VI	I	I	VI	VI	VI	VI	VI	I	I	VI

$$AT_{11} = 1 - \prod_{t=1}^{4}\left(1 - T_{11}^{(t)}\right)^{\varphi_t}$$

$$= 1 - \left((1-0.85)^{0.288} \times (1-0.85)^{0.2577} \times (1-0.65)^{0.288} \times (1-0.65)^{0.1662}\right) = 0.7796$$

$$AI_{11} = \prod_{t=1}^{4} \left(I_{11}^{(t)}\right)^{\varphi_t} = (0.15)^{0.288} \times (0.15)^{0.2577} \times (0.4)^{0.288} \times (0.4)^{0.1662} = 0.2342$$

$$AF_{11} = \prod_{t=1}^{4} \left(F_{11}^{(t)}\right)^{\varphi_t} = (0.15)^{0.288} \times (0.15)^{0.2577} \times (0.35)^{0.288} \times (0.35)^{0.1662} = 0.2204$$

Similarly, all the values of $(AT_{ij}, AI_{ij}, AF_{ij})$ will be determined. Consequently, the following is the final aggregated neutrosophic decision matrix:

Now the aggregated weight of the attributes is defined by using Step7 as follows:

TABLE 4. Aggregated neutrosophic decision matrix.

Alter	Prod	Prof	W.A	S.G	S.T	P	I.T	C.D	P.C	E	F
S	(0.7796, 0.2342, 0.2204)	(0.7257, 0.3016, 0.2742)	(0.7257, 0.3016, 0.2742)	(0.8273, 0.1766, 0.1727)	(0.9009, 0.1529, 0.0991)	(0.3027, 0.6756, 0.6973)	(0.9203, 0.1188, 0.0.0797)	(0.8393, 0.2084, 0.1607)	(0.5694, 0.4250, 0.4306)	(0.4504, 0.5637, 0.5495)	(0.9089, 0.1247, 0.0911)
P	(0.6791, 0.3418, 0.3209)	(0.5905, 0.4534, 0.4095)	(0.7033, 0.3645, 0.2967)	(0.6977, 0.3071, 0.3023)	(0.8859, 0.18, 0.1141)	(0.1842, 0.7658, 0.8158)	(0.9203, 0.228, 0.1736)	(0.8393, 0.3645, 0.2967)	(0.5694, 0.402, 0.397	(0.4504, 0.6613 0.6688	0.9089 0.1655 0.1162
Co	(0.8204, 0.2246, 0.1796)	(0.652, 0.3614, 0.348)	(0.4139, 0.6036, 0.5861)	(0.8273, 0.1766, 0.1727)	(0.8556, 0.1744, 0.1444)	(0.3348, 0.6624, 0.6652)	(0.9089, 0.1248, 0.0911)	(0.8554, 0.2131, 0.1446)	(0.4962, 0.4817, 0.5038)	(0.4932, 0.5332, 0.5068)	(0.8273, 0.1766, 0.1727)
GN	(0.65, 0.4, 0.35)	(0.65, 0.4, 0.35)	(0.65, 0.4, 0.35)	(0.7796, 0.2342, 0.2204)	(0.8604, 0.1771, 0.1396)	(0.3477, 0.6488, 0.6523)	(0.9089, 0.1248, 0.0911)	(0.85, 0.15, 0.15)	(0.7258, 0.3016, 0.2742)	(0.3535, 0.6379, 0.6465)	(0.9089, 0.1248, 0.0911)
M	(0.7258, 0.3016, 0.2742)	(0.7258, 0.3016, 0.2742)	(0.4803, 0.5434, 0.5095)	(0.8273, 0.1766, 0.1727)	(0.9052, 0.1415, 0.0948)	(0.4729, 0.5518, 0.5271)	(0.9089, 0.1248, 0.0911)	(0.8015, 0.2395, 0.1985)	(0.652, 0.3614, 0.348)	(0.5984, 0.4441, 0.4016)	(0.85, 0.15, 0.15)
B	(0.6791, 0.3418, 0.3209)	(0.6008, 0.4013, 0.3992)	(0.5984, 0.4441, 0.4016)	(0.7258, 0.3016, 0.2742)	(0.8273, 0.1766, 0.1727)	(0.4625, 0.5634, 0.5375)	(0.85, 0.15, 0.15)	(0.7851, 0.2274, 0.2149)	(0.7851, 0.2274, 0.2149)	(0.4804, 0.5434, 0.5196)	(0.8273, 0.1766, 0.1727)
T	(0.6606, 0.3546, 0.3394)	(0.7258, 0.3016, 0.2742)	(0.65, 0.4, 0.35)	(0.8273, 0.1766, 0.1727)	(0.85, 0.15, 0.15)	(0.3348, 0.6624, 0.6652)	(0.7556, 0.264, 0.2444)	(0.8741, 0.1571, 0.1259)	(0.7421, 0.2655, 0.2579)	(0.65, 0.4, 0.35)	(0.875, 0.1403, 0.125)
Ch	(0.8393, 0.2084, 0.1607)	(0.8393, 0.2084, 0.1607)	(0.6772, 0.3449, 0.3228)	(0.78, 0.2342, 0.2204)	(0.8339, 0.211, 0.1661)	(0.3348, 0.6624, 0.6652)	(0.8405, 0.186, 0.1595)	(0.8393, 0.2084, 0.1607)	(0.78, 0.2342, 0.2204)	(0.4905, 0.5322, 0.5095)	(0.8273, 0.1766, 0.1727)
O	(0.6854, 0.3348, 0.3146)	(0.7471, 0.26, 0.2529)	(0.5905, 0.4534, 0.4095)	(0.6772, 0.3449, 0.3228)	(0.8273, 0.1766, 0.1727)	(0.5679, 0.4375, 0.4321)	(0.8405, 0.186, 0.1595)	(0.8556, 0.1744, 0.1444)	(0.7851, 0.2274, 0.2149)	(0.5559, 0.4794, 0.4441)	(0.85, 0.15, 0.15)

$$w_1 = \left(1 - \prod_{t=1}^{4}\left(1 - T_1^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4}\left(I_1^{(t)}\right)^{\varphi_t}, \prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t}\right)$$

where

$$1 - \prod_{t=1}^{4}\left(1 - T_1^{(t)}\right)^{\varphi_t} = 1 - (1-0.95)^{0.288} \times (1-0.85)^{0.2577} \times (1-0.95)^{0.288} \times (1-0.95)^{0.1662}$$
$$= 0.9336$$

$$\prod_{t=1}^{4}\left(I_1^{(t)}\right)^{\varphi_t} = 0.05^{0.288} \times 0.15^{0.2577} \times 0.05^{0.288} \times 0.05^{0.1662} = 0.0664$$

$\prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t} = 0.05^{0.288} \times 0.15^{0.2577} \times 0.05^{0.288} \times 0.05^{0.1662} = 0.0664$

Thus, $w_1 = (0.9336, 0.0664, 0.0664)$. Similarly, the weights of the criteria (profitability, water availability, seed growth, soil texture, precipitation, irrigation, crop demand, price of crop, expenditure and fertilize) will be determined, and the final aggregated weight of the criteria is as follows:

$$W = \begin{bmatrix} (0.9336, 0.0664, 0.0664); & (0.9176, 0.0824, 0.0824) \\ (0.9203, 0.0797, 0.0797); & (0.9336, 0.0664, 0.0664) \\ (0.9336, 0.0664, 0.0664); & (0.9336, 0.0664, 0.0664) \\ (0.9336, 0.0664, 0.0664); & (0.9336, 0.0664, 0.0664) \\ (0.9176, 0.0824, 0.0824); & (0.85, 0.15, 0.15) \\ (0.9089, 0.0911, 0.0911) & \end{bmatrix}$$

The weighted aggregated neutrosophic decision matrix is computed using Step 8 in the following manner after locating the aggregated neutrosophic decision matrix and the attributes' aggregated weights:

$d_{11} \times w_1 = (\mathrm{AT}_{11} \cdot w_1,\ \mathrm{AI}_{11} \cdot w_1,\ \mathrm{AF}_{11} \cdot w_1)$

where,

$$\mathrm{AT}_{11} \cdot w_1 = \left(1 - \prod_{t=1}^{4}\left(1 - T_{11}^{(t)}\right)^{\varphi_t}\right) \times \left(1 - \prod_{t=1}^{4}\left(1 - T_1^{(t)}\right)^{\varphi_t}\right)$$

$$= 0.7796 \times 0.9336 = 0.7278$$

$$\mathrm{AI}_{11} \cdot w_1 = \prod_{t=1}^{4}\left(I_{11}^{(t)}\right) + \prod_{t=1}^{4}\left(I_1^{(t)}\right)^{\varphi_t} - \prod_{t=1}^{4}\left(I_{11}^{(t)}\right) \times \prod_{t=1}^{4}\left(I_1^{(t)}\right)^{\varphi_t}$$

$$= 0.2342 + 0.0664 - 0.2342 \times 0.0664 = 0.2851$$

$$\mathrm{AF}_{11} \cdot w_1 = \prod_{t=1}^{4}\left(F_{11}^{(t)}\right) + \prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t} - \prod_{t=1}^{4}\left(F_{11}^{(t)}\right) \times \prod_{t=1}^{4}\left(F_1^{(t)}\right)^{\varphi_t}$$

$$= 0.2204 + 0.0664 - 0.2204 \times 0.0664 = 0.2722$$

Similarly, all the values of $(\mathrm{AT}_{ij} \cdot w_j, \mathrm{AI}_{ij} \cdot w_j, \mathrm{AF}_{ij} \cdot w_j)$ will be determined, and the final weighted aggregated neutrosophic decision matrix is presented in Table 5.

Using Step 9, the relative positive and negative ideal solutions are generated from Table 5 as follows:

$$A_N^+ = \begin{bmatrix} (0.7836, 0.261, 0.2164); & (0.7702, 0.2736, 0.2298) \\ 0.6679, 0.3573, 0.3321); & (0.7724, 0.2313, 0.2276) \\ (0.8451, 0.1985, 0.1549); & (0.5302, 0.4749, 0.4698) \\ (0.8592, 0.1773, 0.1408); & (0.8161, 0.2065, 0.1839) \\ (0.7204, 0.2910, 0.2796); & (0.1408, 0.8279, 0.8592) \\ (0.8261, 0.2045, 0.1739) & \end{bmatrix} \quad A_N^- = \begin{bmatrix} (0.6068, 0.4399, 0.3932); & (0.5418, 0.4985, 0.4582) \\ 0.3809, 0.6352, 0.6191); & (0.6322, 0.3884, 0.3678) \\ (0.7724, 0.2634, 0.2276); & (0.1719, 0.7813, 0.8281) \\ (0.7054, 0.3128, 0.2946); & (0.6566, 0.4067, 0.3434) \\ (0.4553, 0.5244, 0.5447); & (0.2975, 0.66, 0.7025) \\ (0.7519, 0.2516, 0.2481) & \end{bmatrix}$$

Using the above positive and negative ideal solutions, and Step 10 the distance measures are determined as follows:

$$D_1^+ = \sqrt{\frac{1}{33}\left(\sum_{j=1}^{n}\left[\left(\mathrm{AT}_{1j} \cdot w_j - (\mathrm{AT}_j \cdot w)^+\right)^2 + \left(\mathrm{AI}_{1j} \cdot w_j - (\mathrm{AI}_j \cdot w)^+\right)^2 + \left(\mathrm{AF}_{1j} \cdot w_j - (\mathrm{AF}_j \cdot w)^+\right)^2\right]\right)} = 0.1012$$

Table 5. Weighted aggregated neutrosophic decision matrix.

Alter	Prod	Prof	W.A	S.G	S.T	P	I.T	C.D	P.C	E	F
S	(0.7278, 0.2851, 0.2722)	(0.666, 0.3591, 0.334)	(0.6679, 0.3573, 0.3321)	(0.7724, 0.2313, 0.2276)	(0.8411, 0.2092, 0.1589)	(0.2826, 0.6972, 0.7174)	(0.8592, 0.1773, 0.1408)	(0.7836, 0.261, 0.2164)	(0.5225, 0.4724, 0.4775)	(0.2104, 0.7668, 0.7896)	(0.8261, 0.2045, 0.1739)
P	(0.6341, 0.3855, 0.3659)	(0.5418, 0.4985, 0.4582)	(0.6473, 0.4152, 0.3527)	(0.6514, 0.3531, 0.3486)	(0.8271, 0.2345, 0.1729)	(0.1719, 0.7813, 0.8281)	(0.7715, 0.2792, 0.2285)	(0.6566, 0.4067, 0.3434)	(0.5533, 0.4512, 0.4467)	(0.1408, 0.8279, 0.8592)	(0.8032, 0.2415, 0.1968)
Co	(0.7659, 0.2761, 0.2341)	(0.5983, 0.4141, 0.4017)	(0.3809, 0.6352, 0.619)	(0.7724, 0.2313, 0.2276)	(0.7988, 0.2293, 0.2012)	(0.3126, 0.6848, 0.6874)	(0.8486, 0.1829, 0.1514)	(0.7986, 0.2654, 0.2014)	(0.4553, 0.5244, 0.5447)	(0.2309, 0.7468, 0.7806)	(0.7519, 0.2516, 0.2481)
GN	(0.6068, 0.4399, 0.3932)	(0.5964, 0.4495, 0.4036)	(0.5982, 0.4479, 0.4018)	(0.7278, 0.2851, 0.2722)	(0.8033, 0.2317, 0.1967)	(0.3246, 0.6721, 0.6754)	(0.8486, 0.1829, 0.1514)	(0.7935, 0.2065, 0.2065)	(0.666, 0.3591, 0.3340)	(0.1757, 0.8105, 0.8243)	(0.8261, 0.2045, 0.1739)
M	(0.6776, 0.348, 0.3224)	(0.666, 0.3591, 0.3340)	(0.4421, 0.5798, 0.5486)	(0.7724, 0.2313, 0.2276)	(0.8451, 0.1985, 0.1549)	(0.4415, 0.5815, 0.5585)	(0.8486, 0.1829, 0.1514)	(0.7483, 0.29, 0.2517)	(0.5983, 0.4141, 0.4017)	(0.2723, 0.6904, 0.7277)	(0.7723, 0.2275, 0.2275)
B	(0.6341, 0.3855, 0.3659)	(0.5513, 0.4506, 0.4487)	(0.5507, 0.4884, 0.4493)	(0.6776, 0.348, 0.3224)	(0.7724, 0.2313, 0.2276)	(0.4317, 0.5924, 0.5683)	(0.7935, 0.2065, 0.2065)	(0.733, 0.2787, 0.267)	(0.7204, 0.2910, 0.2796)	(0.2142, 0.7531, 0.7858)	(0.7519, 0.2516, 0.2481)
T	(0.6168, 0.3975, 0.3832)	(0.666, 0.3591, 0.3340)	(0.5982, 0.4479, 0.4018)	(0.7724, 0.2313, 0.2276)	(0.7935, 0.2065, 0.2065)	(0.3126, 0.6848, 0.6874)	(0.7054, 0.3128, 0.2946)	(0.8161, 0.2131, 0.1839)	(0.6809, 0.3260, 0.3191)	(0.2975, 0.66, 0.7025)	(0.7953, 0.2186, 0.2047)
Ch	(0.7836, 0.261, 0.2164)	(0.7702, 0.2736, 0.2298)	(0.6232, 0.3971, 0.3768)	(0.7278, 0.2851, 0.2722)	(0.7785, 0.2634, 0.2215)	(0.3126, 0.6848, 0.6874)	(0.7847, 0.2401, 0.2153)	(0.7836, 0.261, 0.2164)	(0.7153, 0.2973, 0.2847)	(0.2259, 0.7474, 0.7741)	(0.7519, 0.2516, 0.2481)
O	(0.6399, 0.379, 0.3601)	(0.6855, 0.321, 0.3145)	(0.5434, 0.497, 0.4566)	(0.6322, 0.3884, 0.3678)	(0.7724, 0.2313, 0.2276)	(0.5302, 0.4749, 0.4698)	(0.7847, 0.2401, 0.2153)	(0.7988, 0.2293, 0.2012)	(0.7204, 0.2910, 0.2796)	(0.2531, 0.7138, 0.7469)	(0.7725, 0.2275, 0.2275)

$$D_1^- = \sqrt{\frac{1}{33}\left(\sum_{j=1}^{n}\left[\left(\mathrm{AT}_{1j}\cdot w_j - (\mathrm{AT}_j\cdot w)^-\right)^2 + \left(\mathrm{AI}_{1j}\cdot w_j - (\mathrm{AI}_j\cdot w)^-\right)^2 + \left(\mathrm{AF}_{1j}\cdot w_j - (\mathrm{AF}_j\cdot w)^-\right)^2\right]\right)} = 0.1375$$

Similarly, all the relative neutrosophic distance measures will be calculated. After calculating all D_i^+, D_i^-, the closeness coefficient is determined by using the formula mentioned in step 10 as follows:

$$C_1^* = \frac{D_1^-}{D_1^+ + D_1^-} = \frac{0.1375}{0.1012+0.1375} = 0.576$$

The distance measures and the closeness coefficients of all the alternatives are mentioned in the Table 6. Additionally, Figure 1 illustrates the graphical representation of alternatives versus the closeness coefficient. From Table 6, and Figure 1, it is clear that the closeness coefficient of the alternative chilli is greater than the other alternatives. Therefore, chilli is the most desirable crop. Also, the order preference of all the alternatives is Chilli > Onion > Sugarcane > Groundnut > Brinjal > Maize > Tomato > Cotton > Paddy.

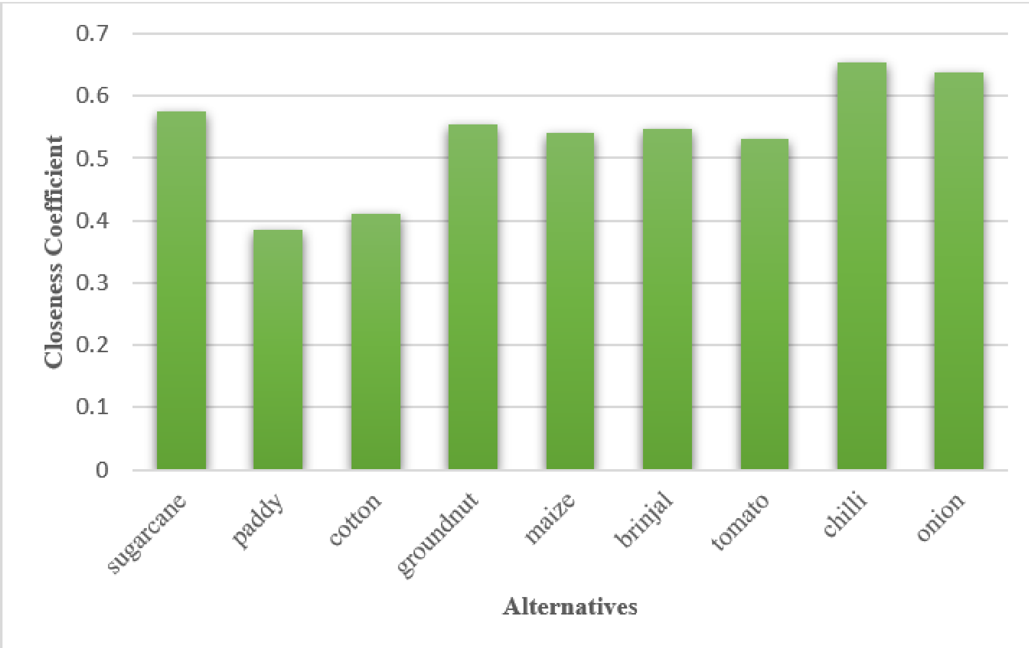

Figure 1. Alternatives vs Closeness coefficients

Table 6. Closeness Coefficients of the Alternatives.

Alternatives	D^+	D^-	C^*
Sugarcane	0.1012	0.1375	0.576
Paddy	0.1573	0.0986	0.3853
Cotton	0.1456	0.1019	0.4117
Groundnut	0.1035	0.1286	0.5541
Maize	0.1037	0.1218	0.5401
Brinjal	0.1053	0.1276	0.5479
Tomato	0.1121	0.1271	0.5314
Chilli	0.0814	0.1535	0.6535
Onion	0.0885	0.1553	0.637

6. Conclusion

Crop selection for sustainable agriculture is a complicated procedure. It presents a number of difficulties, particularly when it is accomplished while taking into consideration a wide range of sustainability-influencing criteria. The TOPSIS-based SVNS is used in this study for Ariyalur district farmers to rank the nine crops—sugarcane, paddy, cotton, groundnut, maize, brinjal, tomato, chilli, and onion—based on the criteria of production, profitability, water availability, seed growth, soil texture, precipitation, irrigation, crop demand, crop price, expenditure, and fertilizer. From the closeness coefficient of this study, it is concluded that chilli is the most optimal crop to cultivate in Ariyalur district. A neutrosophic-based multi-criteria approach is crucial because multiple uncertainties and instabilities frequently impact the crop selection process. This strategy can assist policymakers and farmers in creating all-encompassing policies that support sustainable farming methods, which the world desperately needs. Subsequent studies could look into ways to mitigate and adapt to the consequences of climate change, which would eventually lead to more sustainable farming methods and better decision-making.

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A Novel Approach to Apple Leaf Disease Detection UsingNeutrosophic Logic-Integrated EfcientNetB0

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Abstract. Detecting diseases in apple leaves accurately and efficiently is vital for maintaining healthy crops and ensuring optimal yield. This paper introduces a novel approach that integrates Neutrosophic Logic with the EfficientNetB0 model to enhance the classification of apple leaf diseases. The proposed method significantly improves precision, recall, and F1-scores across multiple disease classes, demonstrating its robustness and effectiveness compared to traditional techniques.

Keywords: Apple, Disease Detection, EfcientNetB0

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1. Introduction

Apple orchards worldwide are susceptible to various diseases that can dramatically reduce fruit quality and yield. Early and accurate disease detection is crucial for effective disease management. Traditional methods of disease detection are often reliant on manual inspection, which is time-consuming and prone to human error. Advances in deep learning have opened new avenues for automating disease detection, offering the potential for more accurate and faster identification of plant diseases.

Neutrosophic Logic (NL) is a generalization of fuzzy logic introduced by Florentin Smarandache in 1998 [1]. NL is designed to handle indeterminacy and uncertainty explicitly, providing a framework to describe and process data that is not only true or false but also indeterminate. In NL, every statement has three degrees: truth (T), indeterminacy (I), and falsity (F), each of which is represented by a value in the range [0, 1]. The use of NL allows systems to better handle real-world data where uncertainty and partial truth are common, making it suitable for applications in image processing, decision-making, and artificial intelligence.

A Neutrosophic Set $\mathcal{Z}$ in a universe of discourse U is defined by three membership functions for each element $x \in U$:

- **Truth-Membership Function** $T_{\mathcal{Z}}(x)$: Indicates the degree to which x belongs to the disease class.
- **Indeterminacy-Membership Function** $I_{\mathcal{Z}}(x)$: Represents the degree of uncertainty or indeterminacy associated with the classification of x.
- **Falsity-Membership Function** $F_{\mathcal{Z}}(x)$: Shows the degree to which x does not belong to the disease class.

These membership functions satisfy the following constraints:

$$T_{\mathcal{Z}}(x), I_{\mathcal{Z}}(x), F_{\mathcal{Z}}(x) \in [0,1], \tag{1}$$

$$0 \leq T_{\mathcal{Z}}(x) + I_{\mathcal{Z}}(x) + F_{\mathcal{Z}}(x) \leq 3. \tag{2}$$

The neutrosophic set for each image x can be represented as:

$$\mathcal{Z}(x) = \left(T_{\mathcal{Z}}(x), I_{\mathcal{Z}}(x), F_{\mathcal{Z}}(x)\right). \tag{3}$$

This representation enables a nuanced classification of apple leaf images into disease categories by capturing the complexity and uncertainty associated with disease symptoms.

In the classification process, the final decision for each image is typically based on the highest truth-membership value among the disease categories:

$$\text{Class}(x) = \arg\max_i T_{\mathcal{Z}_i}(x). \tag{4}$$

The indeterminacy-membership function $I_{\mathcal{Z}}(x)$ provides insights into the uncertainty of the classification, helping to understand the level of ambiguity present.

Additionally, neutrosophic logic operations such as union and intersection can be applied to integrate results from multiple classifiers, further enhancing classification accuracy and robustness:

- **Union**:

$$T_{\mathcal{Z}\cup\mathcal{B}}(x) = \max\left(T_{\mathcal{Z}}(x), T_{\mathcal{B}}(x)\right), \tag{5}$$

$$I_{\mathcal{Z}\cup\mathcal{B}}(x) = \max\left(I_{\mathcal{Z}}(x), I_{\mathcal{B}}(x)\right), \tag{6}$$

$$F_{\mathcal{Z}\cup\mathcal{B}}(x) = \min\left(F_{\mathcal{Z}}(x), F_{\mathcal{B}}(x)\right). \tag{7}$$

- **Intersection**:

$$T_{\mathcal{Z}\cap\mathcal{B}}(x) = \min\left(T_{\mathcal{Z}}(x), T_{\mathcal{B}}(x)\right), \tag{8}$$

$$I_{\mathcal{Z}\cap\mathcal{B}}(x) = \min\left(I_{\mathcal{Z}}(x), I_{\mathcal{B}}(x)\right), \tag{9}$$

$$F_{\mathcal{Z}\cap\mathcal{B}}(x) = \max\left(F_{\mathcal{Z}}(x), F_{\mathcal{B}}(x)\right). \tag{10}$$

This paper proposes an enhanced methodology for classifying apple leaf diseases by incorporating Neutrosophic Logic into the EfficientNetB0 deep learning model. The integration of Neutrosophic Logic allows the model to better handle uncertainties and ambiguities in image data, resulting in improved classification performance.

2. **Related Work**

Various machine learning and deep learning approaches have been explored for plant disease detection. Traditional methods such as SVMs and Random Forests have shown some success but often struggle with large, complex datasets. Recent research has focused on convolutional neural networks (CNNs) due to their superior performance in image recognition tasks. However, most CNN-based models do not explicitly handle uncertainty, which can limit their effectiveness in real-world scenarios.

Our approach builds on these foundations by integrating Neutrosophic Logic with EfficientNetB0, providing a framework that can manage uncertainty and improve the robustness of disease classification. Neutrosophic Theory, introduced by Smarandache, extends classical set theory by incorporating three components: truth (T), indeterminacy (I), and falsity (F). This theory has been successfully applied in fields like medical diagnosis, image processing, and information fusion, yet its application in agricultural disease classification remains limited. Our research leverages Neutrosophic Theory to enhance the reliability and interpretability of soybean disease classification. In recent years, neutrosophic logic has gained traction in various domains, particularly in enhancing the performance of machine learning and image processing systems. Several notable contributions have been made in this field:

- Salama *et al.* (2015) reviewed algorithms for recommender systems in e-Learning platforms utilizing neutrosophic systems. Their work highlights the versatility of neutrosophic logic in handling uncertainty within social network-based e-Learning systems [4].
- Ansari *et al.* (2013) proposed a neutrosophic classifier as an extension of traditional fuzzy classifiers. This approach aimed to improve classification accuracy by incorporating neutrosophic logic, which allows for better handling of ambiguous and incomplete data [5].
- Zhang *et al.* (2010) introduced a neutrosophic approach to image segmentation using the watershed method. Their method demonstrated significant improvements in segmenting images where traditional methods struggled with ambiguity [6].
- Zhang *et al.* (2018) discussed new inclusion relations of neutrosophic sets and explored their applications along with related lattice structures. Their research expanded the theoretical foundations of neutrosophic sets, providing a deeper understanding of their properties and applications [7].

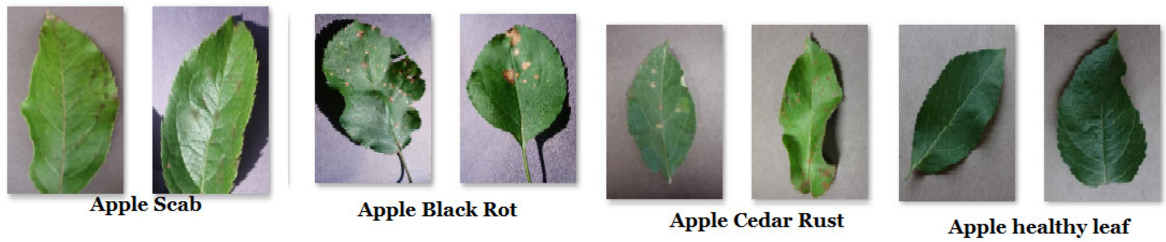

Figure 1. Four classes of Apple Leaves

- Mondal *et al.* (2016) examined the role of neutrosophic logic in data mining, presenting new trends in its application. Their work emphasized the importance of neutrosophic logic in managing uncertainty and extracting meaningful insights from complex datasets [8].
- Sengur and Guo (2011) applied neutrosophic sets combined with wavelet transformation for color texture image segmentation. Their approach demonstrated improved performance in segmenting textured images compared to conventional methods [9].
- Akbulut *et al.* (2017) developed a novel neutrosophic weighted extreme learning machine to address imbalanced datasets. This method aimed to enhance classification performance by integrating neutrosophic logic with extreme learning machines [10].
- Kraipeerapun *et al.* (2007) explored ensemble neural networks utilizing interval neutrosophic sets and bagging. Their approach showed promise in improving neural network performance by incorporating interval neutrosophic sets for better decision-making [11].
- Kavitha *et al.* (2012) designed an ensemble intrusion detection system using neutrosophic logic classifiers. Their work focused on handling uncertainty in intrusion detection, thereby enhancing the system's robustness and accuracy [12].

3. Methodology

3.1. *Image Acquisition*

High-resolution images of apple leaves were obtained from the Plant Village dataset [13]. This dataset includes over ten thousand images classified into four categories: healthy, apple scab, apple rust, and apple black rot (Figure 1). Each image was preprocessed to ensure consistency in size and quality, using standard augmentation techniques such as rotation, scaling, and flipping to enhance the dataset's variability.

3.2. *Model Architecture*

The EfficientNetB0 model, renowned for its efficiency and scalability, serves as the foundational architecture for our classification system. To augment its performance, we have integrated Neutrosophic Logic into the model. This integration enables the model to handle and interpret uncertain information more effectively, thus enhancing its decision-making capabilities.

Neutrosophic Logic is incorporated by assigning three distinct membership values—truth, indeterminacy, and falsity—to the features extracted from input images. These values allow the model to assess the presence of disease symptoms with varying levels of confidence and to handle cases where the data is ambiguous. The mathematical formulation of the proposed system combines neural network operations with Neutrosophic Logic principles. For an input image x, the EfficientNetB0 network generates feature vectors $\mathbf{f}(x)$. These feature vectors are then evaluated using Neutrosophic Logic, which assigns three membership values for each disease class i:

- **Truth-Membership Value**: $T_i(x)$
- **Indeterminacy-Membership Value**: $I_i(x)$
- **Falsity-Membership Value**: $F_i(x)$

The final classification decision is derived from the Neutrosophic inference mechanism. For each disease class i, the decision is based on the difference between the truth-membership and falsity-membership values:

$$C(x) = \arg\max_i \left(T_i(x) - F_i(x)\right) \tag{11}$$

where $C(x)$ denotes the selected class for the image x, and i indexes the disease classes.

To further refine the classification process, the overall confidence score $S_i(x)$ for each class i can be computed as:

$$S_i(x) = T_i(x) - F_i(x) + \lambda \cdot I_i(x) \tag{12}$$

Here, λ is a weight factor that adjusts the influence of the indeterminacy-membership value on the final confidence score. The value of λ is chosen to balance the trade-off between certainty and uncertainty.

In summary, the proposed system leverages Neutrosophic Logic to enhance the classification process by explicitly modeling and integrating uncertainty. This approach ensures a robust classification performance by not only focusing on the most probable class but also accounting for the inherent ambiguity in the data.

3.3. *Training and Evaluation*

The model was trained using a stratified k-fold cross-validation approach to ensure robustness. The Adam optimizer was employed with a learning rate of **0.001**. The model's performance was evaluated using precision, recall, F1-score, and support metrics, focusing on

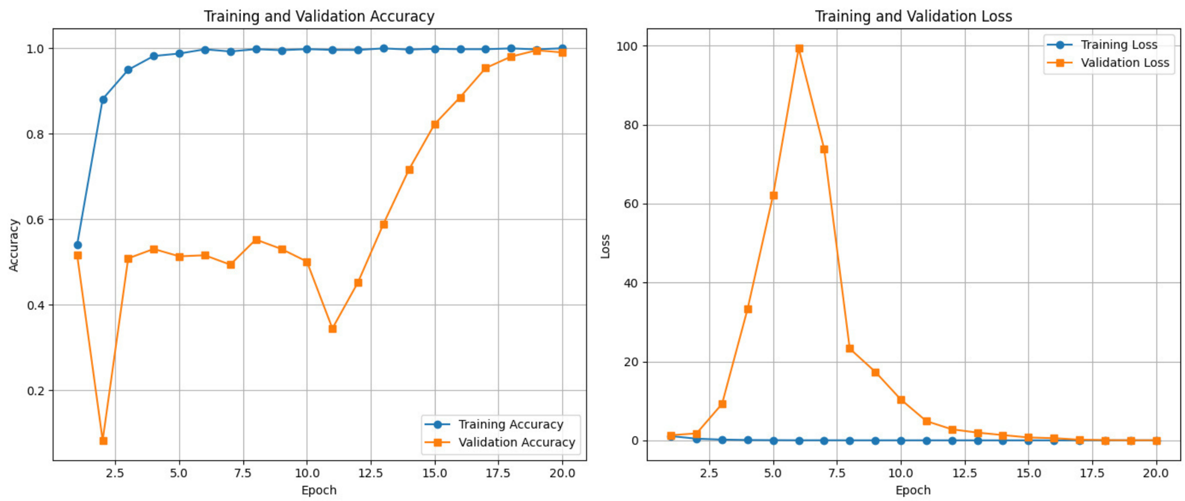

Figure 2. Training and Evaluation

its ability to correctly classify the four different disease classes. The training results indicate that integrating Neutrosophic Logic with the EfficientNetB0 model led to significant enhancements in accuracy throughout the epochs. The model effectively managed uncertainties and progressively improved its performance, showcasing its capability to tackle challenges in plant disease classification. By the end of the training, the model achieved an impressive accuracy of **99.51** percentage and a validation loss of **0.0142**. These results highlight the model's robust learning ability and its strong potential for practical applications in agricultural diagnostics.

4. Results

The proposed model demonstrated high effectiveness in classifying apple leaf diseases. Table 1 shows the performance metrics for each class. The precision, recall, and F1-score for each class were all consistently high, indicating the model's robustness in handling varying degrees of disease severity and types.

Table 1. Classification Report

Class	Precision	Recall	F1-Score	Support
Class 1	0.94	0.94	0.94	1045
Class 2	0.94	0.94	0.94	1025
Class 3	0.92	0.90	0.91	970
Class 4	0.94	0.94	0.94	1035
Macro Avg	**0.93**	**0.93**	**0.93**	**4075**

As seen in Table 1, the model maintained a high precision of **0.94** for most classes, indicating that the majority of predictions made were correct. The recall values, also around **0.94**, show

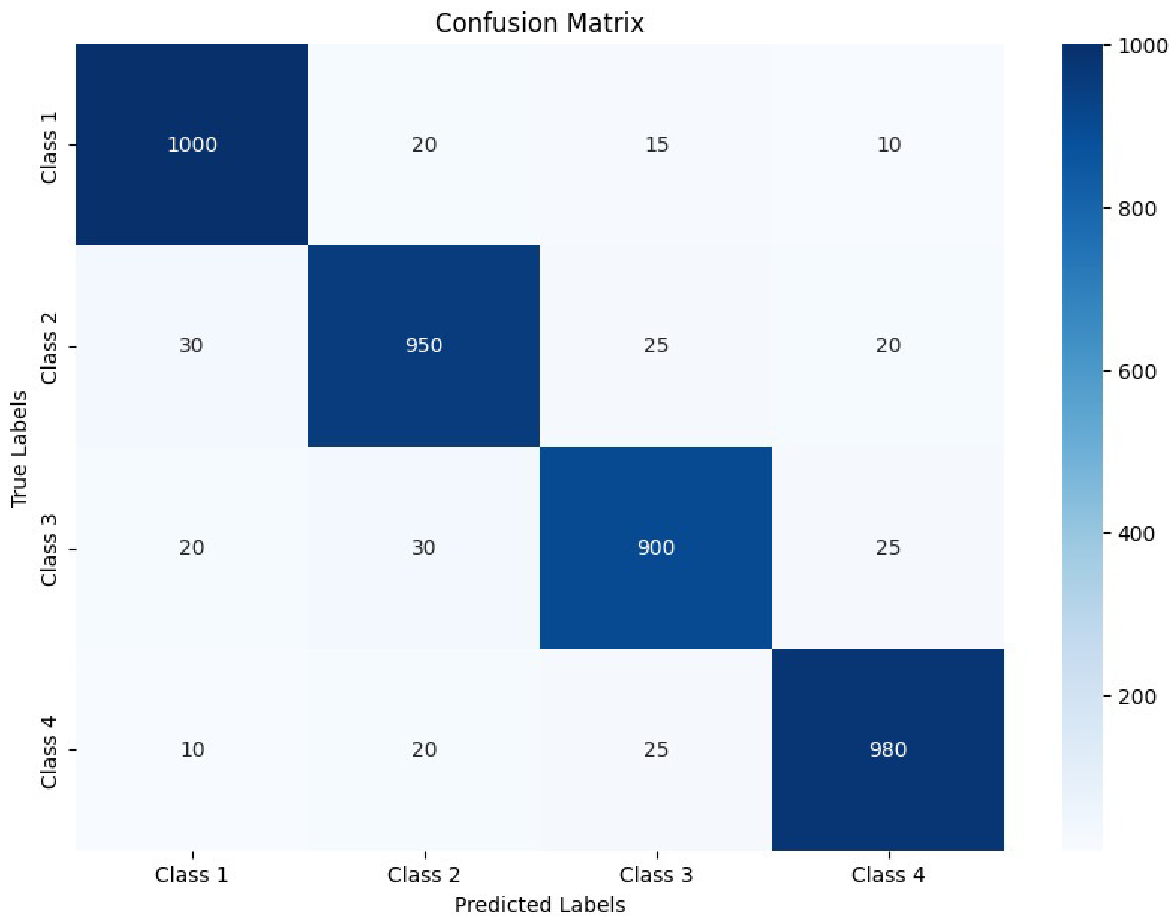

FIGURE 3. Confusion Matrix for Apple Leaf Disease Classification

that the model successfully identified most of the actual disease cases. The F1-scores indicate a strong balance between precision and recall, reflecting the model's effectiveness in managing the inherent complexities of disease classification. The overall macro average of **0.93** for precision, recall, and F1-score demonstrates the consistency of the model's performance across different classes.

5. Discussion

The integration of Neutrosophic Logic into the EfficientNetB0 model is crucial for managing the uncertainties present in real-world data. Traditional deep learning models often assume that input data is completely reliable, which is rarely the case in practical scenarios. By adopting Neutrosophic Logic, our model can process uncertain and incomplete information more effectively. This capability is particularly important in agricultural applications, where factors such as varying light conditions, leaf overlap, and inconsistent disease manifestation can introduce significant ambiguity.

The Neutrosophic approach not only enhances the model's ability to differentiate between similar disease symptoms but also improves its robustness against noise in the input data. This innovation provides a significant advantage over conventional methods, which may not perform well when faced with such uncertainties. The results demonstrate that incorporating Neutrosophic Logic into deep learning models can lead to more accurate and reliable disease detection, making it a valuable addition to the toolkit for modern agricultural management.

6. **Conclusion**

This study presents an innovative approach to apple leaf disease classification by integrating Neutrosophic Logic with EfficientNetB0. The proposed method demonstrates significant improvements in handling uncertainty and achieving high classification accuracy. Future research will explore the application of this approach to other types of crops and diseases, as well as further optimizing the model's efficiency.

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On Neutrosophic Normed Spaces of I-Convergence DiferenceSequences Defned by Modulus Function

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Abstract. In this paper, we introduce the neutrosophic I-convergent difference sequence spaces $I_{(\Delta)}^{(\mathcal{Y})}(f)$ and $I_{(\Delta)}^{0(\mathcal{Y})}(f)$ defined by modulus function. Also, we define an open ball $B(x, \epsilon, \gamma)(f)$ in neutrosophic norm space defined by modulus function. Furthermore, We construct new topological spaces and look into various topological aspects in neutrosophic I-convergent difference sequence spaces $I_{(\Delta)}^{(\mathcal{Y})}(f)$ and $I_{(\Delta)}^{0(\mathcal{Y})}(f)$ defined by modulus function

Keywords: Difference sequence spaces; Ideal; Filter; I-convergence; Modulus function; Neutrosophic normed spaces.

1. Introduction

Dr. Florentin Smarandache created the idea of neutrosophy in the 1990s as a reaction to these difficulties, providing a broader perspective on uncertainty. Neutrosophy aims to analyse and portray systems, phenomena, and concepts that involve ambiguity, incompleteness, and indeterminacy.

The Neutrosophic Set [1], a mathematical structure that expands on the idea of fuzzy sets [2] and conventional crisp sets, is one of the core ideas of Neutrosophy. Three different elements can occur in a neutrosophic set's membership: truth, falsity, and indeterminacy. Neutosophic sets are used in a variety of real-world contexts, including decision-making, artificial intelligence, medical diagnosis, image processing, and pattern identification. By considering the interplay between truth, falsity, and indeterminacy, neutrosophic sets offer a more robust and flexible approach to modeling real-world uncertainties, making it a valuable tool for addressing complex and contradictory data.

In 2006, [10] F. Samarandache and W.B. Vasantha Kanasamy introduced the concept of neutrosophic algebraic structures. Mahapatra and Bera [7] were the first to introduce the neutrosophic soft linear space. Neutrosophic soft norm linear space, metric, convexity [11], and Cauchy sequence were examined by Bera and Mahapatra [8]. The purpose of the current paper is to change the intuitionistic fuzzy normed space of the structure into neutrosophic normed space. The Cauchy sequence has been studied on neutrosophic normed space in an attempt to investigate some beautiful results in this structure.

Mursaleen [14] introduced and presented the concept of statistical convergence with regard to the intuitionistic fuzzy normed (Saadati and Park [15]). Khan [12] recently defined I-convergence and I-Cauchy sequence in intuitionistic fuzzy normed. Kirişci and Şimşek [4] investigated the statistical convergence in neutrosophic normed space.. Since neutrosophic normed space is a generalisation of intuitionistic fuzzy normed (IFNS), this statistical convergence is an important area for research. This piqued our interest in studying I-convergence in neutrosophic normed space. For further detail on ideal and statistical convergence, see [13, 16–18]. Other important aspects of the neutrosophic norm can be found in [4, 5, 7, 8].

Kizmaz [20] developed the concept of difference sequence spaces by studying the difference sequence spaces $X = l_\infty(\Delta), c(\Delta), c_0(\Delta)$.
Some novel sequence spaces were introduce by means of varius matrix transformation in [19, 21, 22] and [23–25]. As seen below, Kizmaz [20] defines the difference sequence spaces using the difference matrix.

$$X(\Delta) = \{\zeta = \zeta_n : \Delta\zeta \in X\}$$

for $X = c,\ l_\infty,\ c_0$, where $\Delta\zeta_n = \zeta_n - \zeta_{n+1}$ and Δ shows the difference matrix $\Delta = (\Delta_{nm})$ defined by

$$\Delta_{nm} = \begin{cases} (-1)^{n-m}, & \text{if } n \leq m \leq n+1 \\ 0\ , & \text{if } 0 \leq m < n. \end{cases} \tag{1.1}$$

Definition 1.1. [28] A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if the following conditions are met,
(a) $f(\zeta) = 0 \Longleftrightarrow \zeta = 0$,
(b) $f(\zeta_1 + \zeta_2) \leq f(\zeta) + f(\zeta_2)$,
(c) f is non-decreasing, and
(d) f is continuous from the right at zero.
Since $|f(\zeta_1) - f(\zeta_2)| \leq f(|\zeta_1 - \zeta_2|)$, condition (4) implies that f is continuous on $\mathbb{R}^+ \cup \{0\}$.

Moreover, from condition (2) we have $f(n\zeta) \leq n\ f(\zeta)$, $\forall n \in \mathbb{N}$, and so $f(\zeta) = f(n\zeta(\frac{1}{n}))$. Hence $\frac{1}{n}f(\zeta) \leq f(\frac{\zeta}{n})\ \forall n \in \mathbb{N}$.

It is possible for the modulus function to be either bounded or unbounded. Consider the following example: $f(\zeta) = \frac{\zeta}{1+\zeta}$, then $f(\zeta)$ is bounded. If $f(\zeta) = \zeta d\ ,0 < d < 1$, then the modulus function $f(\zeta)$ is unbounded.

In this study, we present the neutrosophic I-convergent difference sequence spaces $I_{(\Delta)}^{(\mathcal{Y})}(f)$ and $I_{(\Delta)}^{0(\mathcal{Y})}(f)$ defined by modulus function and investigate some of its topological properties.

Definition 1.2. [5] A binary operation $\star$ on $[0,1]$ is referred to as CTN if (a) $\star$ is associative, commutative and continuous, (b) $\mu = \mu \star 1$ for any $\mu \in [0,1]$ and (c) for each $\mu_1, \mu_2, \mu_3, \mu_4 \in [0,1]$, if $\mu_3 \geq \mu_1$ and $\mu_4 \geq \mu_2$ then $\mu_3 \star \mu_4 \geq \mu_1 \star \mu_2$.

A binary operation $\circ$ on $[0,1]$ is referred to as CTCN if (a) $\circ$ is associative, commutative and continuous, (b) $\mu = \mu \circ 0$ for any $\mu \in [0,1]$ and (c) for each $\mu_1, \mu_2, \mu_3, \mu_4 \in [0,1]$, if $\mu_3 \geq v_1$ and $\mu_4 \geq \mu_2$ then $\mu_3 \circ \mu_4 \geq \mu_1 \circ \mu_2$.

Definition 1.3. [1] Let $X \neq \phi$ and $\mathcal{Y} \subset X$ Then,

$$\mathcal{Y}_{NS} = \{< \zeta, \mathcal{U}(\zeta), \mathcal{V}(\zeta), \mathcal{W}(\zeta) >: \zeta \in X\},$$

where $\mathcal{U}(\zeta), \mathcal{V}(\zeta), \mathcal{W}(\zeta) : X \rightarrow [0,1]$, $\mathcal{U}(\zeta) =$ Truth, $\mathcal{V}(\zeta) =$ Indeterminacy, and $\mathcal{W}(\zeta) =$ Falsehood respectively.

$$0 \leq \mathcal{U}(\zeta) + \mathcal{V}(\zeta) + \mathcal{W}(\zeta) \leq 3.$$

The components of neutrosophic are $\mathcal{U}(\zeta), \mathcal{V}(\zeta)$ and $\mathcal{W}(\zeta)$ independent of each other.

Definition 1.4. [4,27] Assume X is a real vector space, $\star$ and $\diamond$ are CTN and CTCN, respectively, and $\mathcal{Y} = \{< \zeta, \mathcal{U}(\zeta), \mathcal{V}(\zeta), \mathcal{W}(\zeta) >: \zeta \in X\}$ be a neutrosophic set s.t $\mathcal{Y} : X \times (0,\infty) \rightarrow [0,1]$. The four-tuple $(X, \mathcal{Y}, \star, \diamond)$ is called a neutosophic normed space (NNS) if the subsequent terms holds; $\forall\ \zeta, y \in X$ and $s, r > 0$

(i) $0 \leq \mathcal{U}(\zeta, s) \leq 1$, $0 \leq \mathcal{V}(\zeta, s) \leq 1$, $0 \leq \mathcal{W}(\zeta, s) \leq 1$, $s \in R^+$,

(ii) $\mathcal{U}(\zeta, s) + \mathcal{V}(\zeta, s) + \mathcal{W}(\zeta, s) \leq 3$, for $s \in R^+$,

(iii) $\mathcal{U}(\zeta, s) = 1$ iff $\zeta = 0$

(iv) $\mathcal{U}(\lambda\zeta, s) = \mathcal{U}(\zeta, \frac{s}{|\lambda|})$,

(v) $\mathcal{U}(\zeta, s) \star \mathcal{U}(y, r) \leq \mathcal{U}(\zeta + y, s + r)$,

vi) $\mathcal{U}(\zeta, \star)$ is continuous non-decreasing function

(vii) $\lim_{s \to \infty} \mathcal{U}(\zeta, s) = 1$

(viii) $\mathcal{V}(\zeta, s) = 0$ iff $\zeta = 0$

(ix) $\mathcal{V}(\lambda\zeta, s) = \mathcal{V}(\zeta, \frac{s}{|\lambda|})$,

(x) $\mathcal{V}(\zeta, s) \diamond \mathcal{V}(y, s) \geq \mathcal{V}(\zeta + y, s + r)$,

(xi) $\mathcal{V}(\zeta, \diamond)$ is continuous non-increasing function,

(xii) $\lim_{s \to \infty} \mathcal{V}(\zeta, s) = 0$,

(xiii) $\mathcal{W}(\zeta, s) = 0$ iff $\zeta = 0$

(xiv) $\mathcal{W}(\lambda\zeta, s) = \mathcal{W}(\zeta, \frac{s}{|\lambda|})$,

(xv) $\mathcal{W}(\zeta, s) \diamond \mathcal{W}(y, s) \geq \mathcal{W}(\zeta + y, s + r)$,

(xvi) $\mathcal{W}(\zeta, \diamond)$ is continuous non-increasing function,

(xvii) $\lim_{s \to \infty} \mathcal{W}(\zeta, s) = 0$,

(xviii) If $s \leq 0$, then $\mathcal{U}(\zeta, s) = 0$, $\mathcal{V}(\zeta, s) = 1$, $\mathcal{W}(\zeta, s) = 1$.

In such case, $\mathcal{Y} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is called a neutrosophic normed (NN).

Example 1.1. [27] Suppose $(X, \| \cdot \|)$ be a normed space, where $||y|| = |y|, \forall y \in \mathbb{R}$. Give the function as $\zeta \circ y = \zeta + y - \zeta y$ and define $\zeta \star y = \min(\zeta, y)$, For $s > ||y||$,

$$\mathcal{U}(\zeta, s) = \frac{s}{s + ||\zeta||}, \ \mathcal{V}(\zeta, s) = \frac{\zeta}{s + ||\zeta||}, \ \mathcal{W}(\zeta, s) = \frac{||\zeta||}{s} \tag{1.2}$$

$\forall\ \zeta, y \in X$ and $s > 0$.

If we take $s \leq ||\zeta||$, then

$$\mathcal{U}(\zeta, s) = 0, \mathcal{V}(\zeta, s) = 1 \text{ and } \mathcal{W}(\zeta, s) = 1.$$

Hence, $(X, \mathcal{Y}, \circ, \star)$ is neutrosophic norm space s.t $\mathcal{Y} : X \times R^{+} \to [0, 1]$.

Definition 1.5. [5, 29] Let $(X, \mathcal{Y}, \star, \diamond)$ be a NNS. A sequence $x = \{x_n\}$ in X is said to convergent to α_1 with regard to NN-$\mathcal{Y}$ $\iff$ for each $\gamma > 0$, $\epsilon \in (0,1)$, $\exists\ N \in \mathbb{N}$ s.t

$$\mathcal{U}(x_n - \alpha_1, \gamma) > 1 - \epsilon, \mathcal{V}(x_n - \alpha_1, \gamma) < \epsilon, \mathcal{W}(x_n - \alpha_1, \gamma) < \epsilon,\ \forall\ n \in N.$$

i.e, $\gamma > 0$, we have

$$\lim_{n\to\infty} \mathcal{U}(x_n - \alpha_1, \gamma) = 1, \lim_{n\to\infty} \mathcal{V}(x_n - \alpha_1, \gamma) = 0 \text{ and } \lim_{n\to\infty} \mathcal{W}(x_n - \alpha_1, \gamma) = 0.$$

We specify $\mathcal{Y} - \lim x_n = \alpha_1$.

Theorem 1.1. *Let $(X, \mathcal{Y}, \star, \diamond)$ be a NNS. Then, a sequence $x = \{x_n\}$ in X is convergent to $\alpha \in X$ if and only if* $\lim_{n\to\infty} \mathcal{U}(x_n - \alpha, \gamma) = 1$, $\lim_{n\to\infty} \mathcal{V}(x_n - \alpha, \gamma) = 0$ *and* $\lim_{n\to\infty} \mathcal{W}(x_n - \alpha, \gamma) = 0$.

Definition 1.6. [3, 6, 9] Assemblage of subsets $I \subseteq 2^{\mathbb{N}}$ is known as an ideal in $\mathbb{N}$ if I satisfies these condition;

(1) $\emptyset \in I$

(2) $\mathcal{H}, \mathcal{K} \in I \Rightarrow \mathcal{H} \cup \mathcal{K} \in I$, (additive);

(3) $\mathcal{H} \in I, \mathcal{K} \subseteq \mathcal{H} \Rightarrow \mathcal{K} \in I$. (hereditary);

If $I \neq 2^{\mathbb{N}}$, then $I \subseteq 2^{\mathbb{N}}$ is called nontrivial [13]. A nontrivial ideal $I \subseteq 2^{\mathbb{N}}$ is said to be admissible if I includes every singleton subset of $\mathbb{N}$.

If there isn't a non-trivial ideal $K \neq I$, then I is the maximum non-trivial ideal such that $I \subset K$.

Definition 1.7. [27] Assemblage of subsets $F \subseteq 2^{\mathbb{N}}$ is known as a filter in $\mathbb{N}$ if I satisfies these condition:

(1) $\emptyset \notin F$,

(2) For $\mathcal{H}, \mathcal{K} \in F \implies \mathcal{H} \cap \mathcal{K} \in F$,

(3) If $\mathcal{H} \in F$ and $\mathcal{K} \supset \mathcal{H}$ implies $\mathcal{K} \in F$.

Definition 1.8. [27] Suppose $\{x_n\}$ be a sequence in $(X, \mathcal{Y}, \star, \diamond)$. A sequence $\{x_n\}$ is said to be ideally convergent to α with regard to NN-$\mathcal{Y}$, if, for every $\epsilon > 0$ and $\gamma > 0$

$$P = \{n \in \mathbb{N} : \mathcal{U}(x_n - \alpha, \gamma) \leq 1 - \epsilon \text{ or } \mathcal{V}(x_n - \alpha, \gamma) \geq \epsilon, \mathcal{W}(x_n - \alpha, \gamma) \geq \epsilon\} \in I \tag{1.3}$$

It is denoted by $I_{\mathcal{Y}} - \lim x_n = \alpha$ or $x_n \to \alpha$.

Definition 1.9. [27] Suppose $\{x_n\}$ be a sequence in $(X, \mathcal{Y}, \star, \diamond)$. A sequence $\{x_n\}$ is said to ideally Cauchy sequence with regard to NN-$\mathcal{Y}$, if, for every $\epsilon > 0$ and $\gamma > 0$, $\exists\ k \in \mathbb{N}$ s.t

$$Q = \{n \in \mathbb{N} : \mathcal{U}(x_n - x_k, \gamma) \leq 1 - \epsilon \text{ or } \mathcal{V}(x_n - x_k, \gamma) \geq \epsilon,\ \mathcal{W}(x_n - x_k, \gamma) \geq \epsilon\} \in I.$$

2. Main Results

In this study, we created and examined various topological aspects of neutrosophic ideal convergent difference sequence spaces defined by modulus function, a variant of ideal convergent sequence spaces. Let ω be the space containing all real sequences.

$$I^{0(\mathcal{Y})}_{(\Delta)}(f) = \Big\{x = \{x_n\} \in \omega : \{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n, \gamma)\Big) \geq \epsilon\} \in I\Big\} \tag{2.1}$$

$$I^{(\mathcal{Y})}_{(\Delta)}(f) := \Big\{x = \{x_n\} \in \omega : \{n \in \mathbb{N} : \text{for some } \gamma \in \mathbb{R}\ ,\ f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon\} \in I\Big\} \tag{2.2}$$

We describe an open ball with a radius $\epsilon \in (0,1)$ and a center at x with regard to the neutrosophic $\gamma > 0$ parameter, indicated by $B(x,\epsilon,\gamma)$ as follows:

$$B(x,\epsilon,\gamma) = \Big\{y = \{y_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f) : \{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \Delta y_n, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \Delta y_n, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n - \Delta y_n, \gamma)\Big) \geq \epsilon, \} \in I\Big\} \tag{2.3}$$

Theorem 2.1. *The inclusion relation* $I^{0(\mathcal{Y})}_{(\Delta)}(f) \subset I^{(\mathcal{Y})}_{(\Delta)}(f)$ *holds.*

The inverse of an inclusion relation is not true. To defend our claim, take a look at the examples below.

Example 2.1. Suppose $(\mathbb{R}, \|.\|)$ be a normed space s.t $\|x\| = \sup_k |x_k|$, and $x_1 * x_2 = \min\{x_1, x_2\}$ and $x_1 \diamond x_2 = \max\{x_1, x_2\}$, $\forall x_1, x_2 \in (0,1)$. For $\beta > ||x||$, now define norms $\mathcal{Y} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ on $\mathbb{R}^2 \times (0,\infty)$ as follows;

$$\mathcal{U}(x,\beta) = \frac{\beta}{\beta + \|x\|}\ ,\ \mathcal{V}(x,\beta) = \frac{\|x\|}{\beta + \|x\|} \text{ and } \mathcal{W}(x,\beta) = \frac{\|x\|}{\beta}.$$

Then $(\mathbb{R}, \mathcal{Y}, \star, \diamond)$ is a NNS. Consider the sequence (x_k)=$\{1\}$. It is easy to observe that $(x_k) \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ and $x_k \xrightarrow{I_{(\mathcal{Y})}} 1$, but $x_k \notin I^{0(\mathcal{Y})}_{(\Delta)}(f)$.

Lemma 2.1. Let $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$. Then $\forall\ \epsilon \in (0,1)$ and $\gamma > 0$, the following claims are equivalent ,

(a) $I^{(\mathcal{Y})}_{(\Delta)}(f)$–$\lim(x) = \alpha$,

(b) $\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) \leq 1-\epsilon$ or $f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) \geq \epsilon\Big\} \in I$,

(c) $\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) > 1-\epsilon$ and $f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) < \epsilon\Big\} \in F(I)$

(d) $I-\lim\ f\Big(\mathcal{U}(\Delta x_n-\alpha,\gamma)\Big)=1,\ I-\lim\ f\Big(\mathcal{V}(\Delta x_n-\alpha,\gamma)\Big)=0$ and $I-\lim\ f\Big(\mathcal{W}(\Delta x_n-\alpha,\gamma)\Big)=0.$

Theorem 2.2. *The spaces* $I^{(\mathcal{Y})}_{(\Delta)}(f)$ *and* $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ *are linear spaces.*

Proof. We know that $I^{0(\mathcal{Y})}_{(\Delta)}(f)\subset I^{(\mathcal{Y})}_{(\Delta)}(f)$. Then we show the outcome for $I^{(\mathcal{Y})}_{(\Delta)}(f)$. The proof of linearity of the space $I^{0(\mathcal{Y})}_{(\Delta)}(f)$ follows similarly.
Let $\{x_k\},\{y_k\}\in I^{(\mathcal{Y})}_{(\Delta)}(f)$ and α_1,α_2 be scalars. The proof is trivial for $\alpha_1=0$ and $\alpha_2=0$. Now we take $\alpha_1\neq 0$ and $\alpha_2\neq 0$. For a given $\epsilon>0$, take $r>0$ s.t $(1-\epsilon)*(1-\epsilon)>(1-r)$ and $\epsilon\diamond\epsilon<r$.

$$P_1=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\geq\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\geq\epsilon\Big\}\in I,$$

$$P_2=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\geq\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\geq\epsilon\Big\}\in I.$$

Now, we take the complement of P_1 and P_2

$$P_1^c=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon\Big\}\in F(I),$$

$$P_2^c=\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta x_n-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon\Big\}\in F(I);$$

Consequently, set $P=\mathcal{P}_1\cup\mathcal{P}_2$ produces $P\in I$. Thus, P^c is a set that is not empty in $\mathcal{F}(I)$. We'll illustrate this for each $\{x_n\},\{y_n\}\in I^{(\mathcal{Y})}_{(\Delta)}(f)$.

$$P^c\subset\Big\{n\in\mathbb{N}:f\Big(\mathcal{U}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)>1-r \text{ and } f\Big(\mathcal{V}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r, f\Big(\mathcal{W}(\mu\Delta x_n+\nu\Delta y_n)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big)<r\Big\}$$

Let $i\in\mathcal{P}^c$. In this case,

$$f\Big(\mathcal{U}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)<\epsilon$$

$$f\Big(\mathcal{U}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)>1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon, f\Big(\mathcal{W}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)<\epsilon$$

Consider

$$\begin{aligned} f\Big(\mathcal{U}(\mu\Delta x_i+\nu\Delta y_i)-(\mu\alpha_1+\nu\alpha_2,\gamma)\Big) &\geq f\Big(\mathcal{U}(\mu\Delta x_i-\mu\alpha_1,\frac{\gamma}{2})\Big)\star f\Big(\mathcal{U}(\nu\Delta y_i-\nu\alpha_2,\frac{\gamma}{2})\Big)\\ &= f\Big(\mathcal{U}(\Delta x_i-\alpha_1,\frac{\gamma}{2|\mu|})\Big)\star f\Big(\mathcal{U}(\Delta y_i-\alpha_2,\frac{\gamma}{2|\nu|})\Big)\\ &>(1-\epsilon)*(1-\epsilon)>1-r \end{aligned}$$

$$\implies f\Big(\mathcal{U}(\mu\Delta x_i + \nu\Delta y_i) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) > 1 - r$$

and

$$f\Big(\mathcal{V}(\mu\Delta x_i + \nu\Delta y_i) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) \leq f\Big(\mathcal{V}(\mu\Delta x_i - \mu\alpha_1, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{V}(\nu\Delta y_i - \nu\alpha_2, \frac{\gamma}{2})\Big)$$
$$= f\Big(\mathcal{V}(\Delta x_i - \alpha_1, \frac{\gamma}{2|\mu|})\Big) \diamond f\Big(\mathcal{V}(\Delta y_i - \alpha_2, \frac{\gamma}{2|\nu|})\Big)$$
$$< \epsilon \diamond \epsilon < r$$

$$\implies f\Big(\mathcal{V}(\mu\Delta x_i + \nu\Delta y_i) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) < r$$

and

$$f\Big(\mathcal{W}(\mu\Delta x_i + \nu\Delta y_i) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) \leq f\Big(\mathcal{W}(\mu\Delta x_i - \mu\alpha_1, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{W}(\nu\Delta y_i - \nu\alpha_2, \frac{\gamma}{2})\Big)$$
$$= f\Big(\mathcal{W}(\Delta x_i - \alpha_1, \frac{\gamma}{2|\mu|})\Big) \diamond f\Big(\mathcal{W}(\Delta y_i - \alpha_2, \frac{\gamma}{2|\nu|})\Big)$$
$$< \epsilon \diamond \epsilon < r$$

$$\implies f\Big(\mathcal{W}(\mu\Delta x_i + \nu\Delta y_i) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) < r$$

Thus

$P^c \subset \Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\mu\Delta x_n + \nu\Delta y_n) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) > 1 - r$ and $f\Big(\mathcal{V}(\mu\Delta x_n + \nu\Delta y_n) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) < r, f\Big(\mathcal{W}(\mu\Delta x_n + \nu\Delta y_n) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) < r\Big\}$

Since $P^c \in \mathcal{F}(I)$, Thus By the properties of $F(I)$ we have,

$\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\mu\Delta x_n + \nu\Delta y_n) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) > 1 - r$ and $f\Big(\mathcal{V}(\mu\Delta x_n + \nu\Delta y_n) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) < r, f\Big(\mathcal{W}(\mu\Delta x_n + \nu\Delta y_n) - (\mu\alpha_1 + \nu\alpha_2, \gamma)\Big) < r\Big\} \in \mathcal{F}(I)$. Hence $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is a linear space. □

Theorem 2.3. *Every closed ball $B^c(x, \epsilon, \gamma)$ is an open in $I^{(u,v,w)}_{(\Delta)}(f)$, where neutrosophic parameter $\gamma > 0$ with centre at x and radius $0 < \epsilon < 1$.*

Proof. Suppose that $B(x, \gamma, \epsilon)$ is an open ball with a radius of $0 < \epsilon < 1$ and a neutrosophic parameter $\gamma > 0$, with its centre at $x = (x_n) \in I^{(\mathcal{Y})}_{(\Delta)}(f)$.

$$B(x, \epsilon, \gamma)(f) = \big\{y \in I^{(\mathcal{Y})}_{(\Delta)}(f) : f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) \leq 1 - \epsilon \text{ or } f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) \geq \epsilon,$$
$$f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) \geq \epsilon, \in I\big\}$$

Then

$$B^c(x, \epsilon, \gamma)(f) = \big\{y \in I^{(\mathcal{Y})}_{(\Delta)}(f) : f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) > 1 - \epsilon \text{ and } f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) < \epsilon,$$
$$f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) < \epsilon, \in F(I)\big\}$$

suppose $y \in \mathcal{B}^c(x, \gamma, \epsilon)$. Then, For

$f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) > 1-\epsilon$ and $f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) < \epsilon,$
so there exists $\gamma_0 \in (0, \gamma)$ such that
$f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma_0)\Big) > 1-\epsilon$ and $f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma_0)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma_0)\Big) < \epsilon.$

Let $\epsilon_0 = f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma_0)\Big)$, we have $\epsilon_0 > 1-\epsilon$. Then $\exists\, p \in (0,1)$ such that $\epsilon_0 > 1-p > 1-\epsilon$. For $\epsilon_0 > 1-p$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1)$ such that $\epsilon_0 * \epsilon_1 > 1-p$, $(1-\epsilon_0) \diamond (1-\epsilon_2) < p$. and $(1-\epsilon_0) \diamond (1-\epsilon_3) < p$.

Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$. Then $(1-p) < \epsilon_0 * \epsilon_1 \le \epsilon_0 * \epsilon_4$ and $(1-\epsilon_0) \diamond (1-\epsilon_4) \le (1-\epsilon_0) \diamond (1-\epsilon_2) < p$. Consider the closed ball $B^c(y, \gamma-\gamma_0, 1-\epsilon_4)$ and $B^c(x, \gamma, \epsilon)$.
We prove that $B^c(y, \gamma-\gamma_0, 1-\epsilon_4) \subset \mathcal{B}^c(x, \gamma, \epsilon)$. Let $z = \{z_n\} \in \mathcal{B}^c(y, \gamma-\gamma_0, 1-\epsilon_4)$. Then $f\Big(\mathcal{U}(\Delta y - \Delta z, \gamma-\gamma_0)\Big) > \epsilon_4$ and $f\Big(\mathcal{V}(\Delta y - \Delta z, \gamma-\gamma_0)\Big) < 1-\epsilon_4, f\Big(\mathcal{W}(\Delta y - \Delta z, \gamma-\gamma_0)\Big) < 1-\epsilon_4$
Therefore

$$\begin{aligned} f\Big(\mathcal{U}(\Delta x - \Delta z, \gamma)\Big) &\ge f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma_0)\Big) \star f\Big(\mathcal{U}(\Delta y - \Delta z, \gamma-\gamma_0)\Big) \\ &\ge \epsilon_0 \star \epsilon_4 \ge \epsilon_0 \star \epsilon_1 \\ &> (1-p) > \ (1-\epsilon) \\ f\Big(\mathcal{V}(\Delta x - \Delta z, \gamma)\Big) &\le f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma_0)\Big) \diamond f\Big(\mathcal{V}(\Delta y - \Delta z, \gamma-\gamma_0)\Big) \\ &\le (1-\epsilon_0) \diamond (1-\epsilon_4) \le \epsilon_0 \diamond \epsilon_2 \\ &< p < \ \epsilon \end{aligned}$$

and

$$\begin{aligned} f\Big(\mathcal{W}(\Delta x - \Delta z, \gamma)\Big) &\le f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma_0)\Big) \diamond f\Big(\mathcal{W}(\Delta y - \Delta z, \gamma-\gamma_0)\Big) \\ &\le \epsilon_0 \diamond \epsilon_4 \le \epsilon_0 \diamond \epsilon_3 \\ &< p < \ \epsilon \end{aligned}$$

Therefore the set $\Big\{f\Big(\mathcal{U}(\Delta x - \Delta z, \gamma)\Big) > 1-\epsilon$ and $f\Big(\mathcal{V}(\Delta x - \Delta z, \gamma)\Big) < \epsilon$, $f\Big(\mathcal{W}(\Delta x - \Delta z, \gamma)\Big) < \epsilon\Big\} \in \mathcal{F}(I)$.
$\implies z = (z_n) \in \mathcal{B}^c(x, \gamma, \epsilon)$
$\implies B^c\big(y, \gamma-\gamma_0, 1-\epsilon_4\big) \subset \mathcal{B}^c\big(x, \gamma, \epsilon\big)$. □

Remark 2.2. It is clear that $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is a neutrosophic normed space with respect to neutrosophic norms $\mathcal{Y} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$. Define

Now define a collection $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ of a subset of $I^{(\mathcal{Y})}_{(\Delta)}(f)$ as follows:
$\tau^{(\mathcal{Y})}_{(\Delta)}(f) = \Big\{P \subset I^{(\mathcal{Y})}_{(\Delta)}(f) :$ for every $x = (x_n) \in P\ \exists\ \gamma > 0$ and $\epsilon \in (0,1)$ s.t $\mathcal{B}^c(x, \gamma, \epsilon) \subset P\Big\}$.
Then $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ is a topolopgy on $I^{(\mathcal{Y})}_{(\Delta)}(f)$

Theorem 2.4. *The topology $\tau_{(\Delta)}^{(\mathcal{Y})}(f)$ on the space $I_{(\Delta)}^{(\mathcal{Y})}(f)$ is first countable.*

Proof. For every $x = \{x_n\} \in I_{(\Delta)}^{(\mathcal{Y})}(f)$, suppose the set $\mathcal{B} = \{B^c(x, \frac{1}{n}, \frac{1}{n})\} : n = 1, 2, 3, 4, ...\}$,which is a local countable basis at $x \in I_{(\Delta)}^{(\mathcal{Y})}(f)$. As a result, the topology $\tau_{(\Delta)}^{(\mathcal{Y})}(f)$ on the space $I_{(\Delta)}^{0(\mathcal{Y})}(f)$ is first countable. □

Theorem 2.5. *The spaces $I_{(\Delta)}^{(\mathcal{Y})}(f)$ and $I_{(\Delta)}^{0(\mathcal{Y})}(f)$ are Hausdorff spaces.*

Proof. We know that $I_{(\Delta)}^{0(\mathcal{Y})}(f) \subset I_{(\Delta)}^{(\mathcal{Y})}(f)$, We will only show the solution for $I_{(\Delta)}^{(\mathcal{Y})}(f)$
Let $x = (x_n), y = (y_n) \in I_{(\Delta)}^{(\mathcal{Y})}(f)$ such that $x \neq y$. Then
$0 < f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big) < 1$,$0 < f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big) < 1$ and $0 < f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big) < 1$

Putting $\epsilon_1 = f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big)$, $\epsilon_2 = f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big)$, $\epsilon_3 = f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big)$
and $\epsilon = \max\{\epsilon_1, 1-\epsilon_2, 1-\epsilon_3\}$. Then for each $\epsilon_0 \in (\epsilon, 1)$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0, 1)$ such that $\epsilon_4 * \epsilon_4 \geq \epsilon_0$, $(1-\epsilon_5) \diamond (1-\epsilon_5) \leq (1-\epsilon_0)$ and $(1-\epsilon_6) \diamond (1-\epsilon_6) \leq (1-\epsilon_0)$.

Once again putting $\epsilon_7 = \max\{\epsilon_4, 1-\epsilon_5, 1-\epsilon_6, \}$, think about the closed balls. $B^c(x, 1-\epsilon_7, \frac{\gamma}{2})$ and $B^c(y, 1-\epsilon_7, \frac{\gamma}{2})$ respectively centred at x and y.
Then it is obvious that $B^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathcal{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$.
If possible let $z = \{z_n\} \in \mathcal{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathcal{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2})$. Then we have,

$$\begin{aligned}\epsilon_1 &= f\Big(\mathcal{U}(\Delta x - \Delta y, \gamma)\Big)\\ &\geq f\Big(\mathcal{U}(\Delta x - \Delta z, \frac{\gamma}{2})\Big) \star f\Big(\mathcal{U}(\Delta z - \Delta y, \frac{\gamma}{2})\Big)\\ &> \epsilon_7 \star \epsilon_7 \geq \epsilon_4 \star \epsilon_4 \geq \epsilon_0 > \epsilon_1\end{aligned} \tag{2.4}$$

$$\begin{aligned}\epsilon_2 &= f\Big(\mathcal{V}(\Delta x - \Delta y, \gamma)\Big)\\ &\leq f\Big(\mathcal{V}(\Delta x - \Delta z, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{V}(\Delta z - \Delta y, \frac{\gamma}{2})\Big)\\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \leq (1-\epsilon_5) \diamond (1-\epsilon_5)\\ &\leq (1-\epsilon_0) < \epsilon_2\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\epsilon_3 &= f\Big(\mathcal{W}(\Delta x - \Delta y, \gamma)\Big)\\ &\leq f\Big(\mathcal{W}(\Delta x - \Delta z, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{W}(\Delta z - \Delta y, \frac{\gamma}{2})\Big)\\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \leq (1-\epsilon_6) \diamond (1-\epsilon_6)\\ &\leq (1-\epsilon_0) < \epsilon_3\end{aligned} \tag{2.6}$$

We have a contradiction from equations (2.4), (2.5) and (2.6). Therefore, $B^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathcal{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$. Hence the space $I^{(\mathcal{Y})}_{(\Delta)}(f)$ is a Hausdorff space. □

Theorem 2.6. *Suppose $\tau^{(\mathcal{Y})}_{(\Delta)}(f)$ be a topology on a neutrosophic norm spaces $I^{(\mathcal{Y})}_{(\Delta)}(f)$, then a sequence $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ such that (x_n) is Δ-convergent to Δx_0 with regard to NN-$(\mathcal{Y})$,if and only if $f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \longrightarrow 1$,$f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \longrightarrow 0$ and $f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. Let $B(x_0, \gamma, \epsilon)$ be an open ball with centre $x_0 \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ and radius $\epsilon \in (0,1)$ with $\gamma > 0$, i.e.

$$B(x_0, \epsilon, \gamma)(f) = \Big\{x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f) : \{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \leq 1-\epsilon \text{ or } f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \geq \epsilon, f\Big(\mathcal{W}(\Delta x_0 - \Delta y_n, \gamma)\Big) \geq \epsilon, \} \in I\Big\} \tag{2.7}$$

Consider a sequence $\{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ is Δ-convergent to Δx_0 with respect to neutrosophic norm $(\mathcal{Y})$, then for $\epsilon \in (0,1)$,$\gamma > 0$ $\exists$ $n_0 \in \mathbb{N}$ such that $\{x_n\} \in B^c(x_0, \gamma, \epsilon)$, $\forall\, n \geq n_0$. Thus

$$\Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > 1-\epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\Big\} \in F(I).$$

So

$$1-f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > \epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\ \forall n \geq n_0.$$

$$f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 1, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0 \text{ as } n \to \infty$$

Conversely, if $\forall\ \gamma > 0$,

$$f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 1, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) \to 0 \text{ as } n \to \infty$$

Then for each $\epsilon \in (0,1)$, $\exists\, n_0 \in \mathbb{N}$ s.t.

$$1-f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > \epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\ \forall n \geq n_0.$$

So,

$$f\Big(\mathcal{U}(\Delta x_n - \Delta x_0, \gamma)\Big) > 1-\epsilon, f\Big(\mathcal{V}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \Delta x_0, \gamma)\Big) < \epsilon\ \forall\ n \geq n_0.$$

Hence $\{x_n\} \in B^c(x_0, \gamma, \epsilon)(f)$, $\forall\ n \geq n_0$. This proves that a sequence (x_n) is Δ-convergent to Δx_0 with regard to the NN-$(\mathcal{Y})$. □

Theorem 2.7. *Let $x = \{x_n\} \in \omega$ be a sequence. If $\exists$ a sequence $y = \{y_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$ such that $f(\Delta(x_n)) = f(\Delta(y_n))$ for almost all n relative to neutrosophic I, then $x \in I^{(\mathcal{Y})}_{(\Delta)}(f)$.*

Proof. Consider $f(\Delta(x_n)) = f(\Delta(y_n))$ for almost all n relative to neutrosophic I. Then $\{n \in \mathbb{N} : f(\Delta(x_n)) \neq f(\Delta(y_n))\} \in I$. This implies. $\{n \in \mathbb{N} : f(\Delta(x_n)) = f(\Delta(y_n))\} \in \mathcal{F}(I)$. Therefore for $n \in \mathcal{F}(I)$ $\forall\ \gamma > 0$,

$$f\Big(\mathcal{U}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) = 1,\ f\Big(\mathcal{V}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) = 0 \text{ and } f\Big(\mathcal{W}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) = 0$$

Since $\{y_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$, let (y_n) is Δ-convergent to α. Then for any $\epsilon \in (0,1)$ and $\gamma > 0$,

$$A_1 = \Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) > 1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) < \epsilon, f\Big(\mathcal{W}(\Delta y_n - \alpha, \frac{\gamma}{2}) < \epsilon\Big) \in \mathcal{F}(I).$$

Consider the set,

$$A_2 = \Big\{n \in \mathbb{N} : f\Big(\mathcal{U}(\Delta x_n - \alpha, \frac{\gamma}{2})\Big) > 1-\epsilon \text{ and } f\Big(\mathcal{V}(\Delta x_n - \alpha, \frac{\gamma}{2})\Big) < \epsilon, f\Big(\mathcal{W}(\Delta x_n - \alpha, \frac{\gamma}{2}) < \epsilon\Big).$$

We show that $A_1 \subset A_2$. So for $n \in A_1$ we have,

$$\begin{aligned} f\Big(\mathcal{U}(\Delta x_n - \alpha, \gamma)\Big) &\geq f\Big(\mathcal{U}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) \star f\Big(\mathcal{U}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) \\ &> 1 \star (1-\epsilon) = 1-\epsilon \end{aligned}$$

$$\begin{aligned} f\Big(\mathcal{V}(\Delta x_n - \alpha, \gamma)\Big) &\leq f\Big(\mathcal{V}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{V}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) \\ &< 0 \diamond \epsilon = \epsilon \end{aligned}$$

and

$$\begin{aligned} f\Big(\mathcal{W}(\Delta x_n - \alpha, \gamma)\Big) &\leq f\Big(\mathcal{W}(\Delta x_n - \Delta y_n, \frac{\gamma}{2})\Big) \diamond f\Big(\mathcal{W}(\Delta y_n - \alpha, \frac{\gamma}{2})\Big) \\ &< 0 \diamond \epsilon = \epsilon \end{aligned}$$

$\implies$ $n \in A_2$ and hence $A_1 \subset A_2$. Since $A_1 \in \mathcal{F}(I)$, therefore $A_2 \in \mathcal{F}(I)$. Hence $x = \{x_n\} \in I^{(\mathcal{Y})}_{(\Delta)}(f)$. □

3. Conclusions

In the current study, using the concept of difference sequence and modulus function, we extend the intriguing idea of I-convergence to the context of neutrosophic norm spaces via difference sequences by modulus function. Also, we have introduce the new notion of I-convergent difference sequence in neutrosophic normed spaces by modulus function and some fundamental properties are examined.

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Fermatean Fuzzy α-Homeomorphism in Fermatean Fuzzy Topological Spaces

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Abstract. A relatively new development in fuzzy set theory, Fermatean fuzzy sets (FFSs) were developed for working with higher-level, uncertain data. They provide an even more robust structure for characterizing degree of membership and degree of non-membership, extending on the notions of Pythagorean fuzzy sets (PFSs) and Intuitionistic fuzzy sets (IFSs). There are multiple fields of mathematics and applications that depend extensively on topological generalization of open sets. Topological spaces can be described and analyzed with the aid of homeomorphisms, which provide an empirical way to identify whether two spaces are identical in a topological sense with respect to their essential features. In this study we introduce and investigate the concept of FF α-irresolute, FF α open and clossed mapping, FF α-homeomorphism and, FF α^*-homeomorphism.

Keywords: FF α-open sets; FF α-contiunity: FF α-irresolute; FF α-homeomorphism and, FF α^*-homeomorphism.

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1. Introduction

The way in which imperfection and ambiguity are dealt with in mathematical models and decision-making methods was entirely rewritten by Zadeh L A [15] by adoption of fuzzy sets in 1965, which include AV that lies between 0 and 1, in contrast with classical sets, having binary members that either belong to or do not. This adaptability provides fuzzy sets a stronger basis for dealing with circumstances that exist in real life which are characterized by uncertainty and ambiguities.

In 1983 Atanassov K [4] introduced Intuitionistic fuzzy sets as an extension of traditional fuzzy sets with sum of AV and NAV is less than 1. Pythagorean fuzzy sets introduced by Yager R R [14] extend the flexibility and applicability of fuzzy logic systems, allowing for better modeling of uncertainty in complex scenarios taking sum of squares of AV and NAV is less than 1.

In order to tackle confusion and unpredictability in decision-making processes more effectively, Senapati T [13] introduced FFS in 2020, an extension of Intuitionistic fuzzy sets (IFSs) and

Pythagorean fuzzy sets (PFSs). AVs and $NAVs$ of FFSs reveal their dependence on greater powers with sum of cubes is less than 1. Buyukozkan G [5] have done a sysytametic review on FFSs.

Revathy A [12] applied FF normalised Bonferroni mean operator in MCDM for selection of electric bike and also investigated generalization of FF sets and applied FF PROMETHEE II method for decision-making in [3]. Ibrahim H Z [10] introduced and applied n, m-rung orthopair fuzzy sets in MCDM. Kakati [11] used FF Archimedean Heronian Mean-Based Model for MCDM.

The increase in degree of uncertainty and complexity in many scientific and engineering sectors require FF topology. It provides enhanced and flexible estimation and evaluation methods in topological spaces under ambiguity, that provides beneficial application tools and theoretical foundations for additional mathematical research. Ibrahim H Z [9] introduced the concept of FF topological space and studied FF continuity FF points and study some types of separation axioms. A FFT is further extension in fuzzy topology that makes it possible for even more complicated topological space representations includes the construction of open sets employing FFSs. Farid [6] used FF CODAS approach with topology. α-open sets give a more adaptable design which facilitates the enhancement of vague attributes. Alshami T m [2] has worked on soft α open sets in soft topological spaces. In circumstances where conventional continuity is inappropriately rigid, α-continuity, which is established using α-open sets, can be more flexible than classical continuity. Ajay D [1] coined Pythagorean α-continuity. Granados [8] derived some results in Pythagorean neutrosophic topological spaces. Gonul B N [7] coined Fermatean neutrosophic topological spaces and an application of neutrosophic kano method.

Bijective, continuous mappings with continuous inverses that maintain the topological structure of spaces are known as classical homeomorphisms. However, classical homeomorphisms are not adequate to capture the complexity of fuzzy topological spaces, where uncertainty and fuzziness are inherent. In generalized topology, α-irresolute functions and α-homeomorphisms serve as helpful tools to facilitate the investigation of spaces and functions that struggle to fit into the traditional structure. Their applications vary considerably and include things from the study of dynamic systems and complicated systems to the development of innovative mathematical theories related to approximation and fixed point theorems. An in-depth knowledge of topological characteristics in more extended or unpredictable settings has been rendered possible by these concepts. These concepts are important to the exploration of fuzzy topological spaces, wherein FFSs are utilized to model ambiguity. They tend to be valuable in circumstances in which the exact form of the data is unclear or unreliable, such as decision-making, machine learning, and optimization.

The topological framework of Fermatean fuzzy spaces remains intact under mappings which preserve the fuzzy structure of the spaces by using Fermatean fuzzy α-homeomorphisms, that generalize these classical ideas. This generalization serves as crucial for implementing decisions from classical topology to more complex and uncertain situations. α-homeomorphism is an approach which can be employed in topological studies to examine different characteristics in fuzzy settings and modify classical conclusions. This may result in novel findings in both theoretical and applied mathematics. More generalized forms of continuity such as FF α-irresolute functions, FF α-homeomorphism and other topological characteristics have been introduced and investigated in our proposed study.

The contribution of this article is as follows:

- Preliminaries are given in Section 2.
- FF α-irresolute functions are studied in Section 3.
- In Section 4, FF α-homeomorphism are introduced and their properties are investigated.
- Conclusion and Future work is given in Section 5.

The short form and expansion used in this study are given below.

2. Preliminaries

This section conveys some of the essential concepts utilised in this study.

Definition 2.1 ([13]). A set $\mathcal{S} = \{\langle a, \alpha_S(a), \beta_S(a)\rangle : a \in A\}$ in the universe A is called FFs if $0 \leq (\alpha_S(a))^3 + (\beta_S(a))^3 \leq 1$ where $\alpha_S(a) : A \to [0,1]$, $\beta_S(a) : A \to [0,1]$ and $\pi = \sqrt[3]{1 - (\alpha_S(a))^3 - (\beta_S(a))^3}$ are degree of AV, NAV and indeterminacy of a in S and its complement is $\mathcal{S}^c = (\beta_S, \alpha_S)$.

Definition 2.2 ([13]). If $\mathcal{F}_1 = (\mu_{\mathcal{F}_1}, \nu_{\mathcal{F}_1})$ and $\mathcal{F}_2 = (\mu_{\mathcal{F}_2}, \nu_{\mathcal{F}_2})$ are two FFSs then $\mathcal{F}_1 \cup \mathcal{F}_2 = (max\{\mu_{\mathcal{F}_1}, \mu_{\mathcal{F}_2}\}, min\{\nu_{\mathcal{F}_1}, \nu_{\mathcal{F}_2}\})$ and $\mathcal{F}_1 \cap \mathcal{F}_2 = (min\{\mu_{\mathcal{F}_1}, \mu_{\mathcal{F}_2}\}, max\{\nu_{\mathcal{F}_1}, \nu_{\mathcal{F}_2}\})$.

Definition 2.3 ([9]). A FF topological space (FFTS) ia s pair $(\mathcal{S}, \tau)$ if

(1) $1_{\mathcal{S}} \in \tau$

(2) $0_{\mathcal{S}} \in \tau$

(3) for any $\mathcal{F}_1, \mathcal{F}_2 \in \tau$ we have $\mathcal{F}_1 \cup \Gamma_2 \in \tau$

(4) $\mathcal{F}_1 \cap \mathcal{F}_2 \in \tau$ where τ is the family of FFS of non-empty set $\mathcal{S}$.

Acronyms	Expansion
AV	Association value
NAV	Non association value
FS	Fuzzy set
IFS	Intuitionistic fuzzy Set
PFS	Pythagorean fuzzy set
IVFFS	Interval valued Fermatean fuzzy set
FFS	Fermatean fuzzy set
FFTS	Fermatean fuzzy topological space
$FFOS$	Fermatean fuzzy open set
$FF\alpha OS$	Fermatean fuzzy α open set
$FFPOS$	Fermatean fuzzy pre open set
$FFSOS$	Fermatean fuzzy semi open set
$FF\alpha C$	Fermatean fuzzy α continuous function
$FF\alpha I$	Fermatean fuzzy α irresolute function
$FF\alpha O$	Fermatean fuzzy α open mapping
$FF\alpha Cl$	Fermatean fuzzy α closed mapping
$FFPC$	Fermatean fuzzy pre continuous function
$FFSC$	Fermatean fuzzy semi continuous function
$FFS\alpha C$	Fermatean fuzzy strongly α continuous function
FFH	Fermatean fuzzy homeomorphism
$FF\alpha H$	Fermatean fuzzy α homeomorphism
$FF\alpha^* H$	Fermatean fuzzy α^* homeomorphism

The $FFOS$ and $FFCS$ are members of τ and τ^c respectively. FF interior of $\mathcal{F}$ denoted by $int\ \mathcal{F}$ is the union of all FFOSs contained in $\mathcal{F}$ and FF closure of $\mathcal{F}$ denoted by $cl\ \mathcal{F}$ is the intersection of all FFCSs containing $\mathcal{F}$.

Definition 2.4. [[3]] A FFS $\mathcal{F} = (\mu_{\mathcal{S}}, \nu_{\mathcal{S}})$ of a FFTS $(\mathcal{S}, \tau)$ is

(1) a $FFSOS$ if $\mathcal{F} \subseteq cl\,(int\,(\mathcal{F}))$.

(2) a $FFPOS$ if $\mathcal{F} \subseteq int\,(cl\,(\mathcal{F}))$.

(3) a $FF\alpha OS$ if $\mathcal{F} \subseteq int\,(cl\,(int\,(\mathcal{F})))$.

Their complements are $FFSCS$, $FFPCS$ and $FF\alpha CS$ respectively. The FF α-closure of $\mathcal{F}$, $cl_\alpha\,(\mathcal{F})$ is the intersection of all FF α-closed super sets of $\mathcal{F}$ and the FF α-interior of $\mathcal{F}$, $int_\alpha\,(\mathcal{F})$ is the union of all FF α-open subsets of $\mathcal{F}$.

Definition 2.5. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FFSC$ if the inverse image of each $FFOS$ in (B, τ_B) is a $FFSOS$ in (A, τ_A).

Definition 2.6. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FFPC$ if the inverse image of each $FFOS$ in (B, τ_B) is a $FFPOS$ in $(A, \tau_A$

Definition 2.7. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FF\alpha C$ if the inverse image of each $FFOS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).

Definition 2.8. A mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is called FF open mapping if $f(U)$ is $FFOS$ in (B, τ_B) whenever U is $FFOS$ in (A, τ_A).

Definition 2.9. A mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is called FF closed mapping if $f(U)$ is $FFCS$ in (B, τ_B) whenever U is $FFCS$ in (A, τ_A).

Definition 2.10. A FF bijection $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is called FFH if f is both FF open and FF closed mapping.

3. **Fermatean Fuzzy α-irresolute functions**

Definition 3.1. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FF\alpha$-irresolute($FF\alpha I$) if the inverse image of each $FF\alpha OS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).

Definition 3.2. Let (A, τ_A) and (B, τ_B) be FFTS. A function $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a FF strongly α-continuous ($FFS\alpha C$) function if the inverse image of each $FFSOS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).

Theorem 3.3. *Every $FF\alpha I$ is a $FFPC$.*

Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is a $FF\alpha I$. Let us take a $FFOS$, U in (B, τ_B) then U is a $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Since every $FF\alpha OS$ is $FFPOS$, $f^{-1}(U)$ is a $FFPOS$ in (A, τ_A). Hence f is a $FFPC$.

Remark 3.4. The converse of the above theorem need not be true as shown by the following example.

Example 3.5. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.8, 0.7), (a_2, 0.6, 0.63)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.45, 0.63), (ab_2, 0.8, 0.7)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B, τ_B). Since $int\left(cl\left(f^{-1}(B_1)\right)\right) = 1_A$, $f^{-1}(B_1) \subseteq int\left(cl\left(f^{-1}(B_1)\right)\right)$. Hence f is a $FFPC$. But B_1 is a $FF\alpha OS$ in (B, τ_B) and $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = 0_A$ imply $f^{-1}(B_1) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Therefore $f^{-1}(B_1)$ is not a $FF\alpha OS$ in (A, τ_A). Hence f is not $FF\alpha I$.

Theorem 3.6. *Every $FF\alpha I$ is a $FF\alpha C$.*

Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \to (B, \tau_B)$ is a $FF\alpha I$. Let us take a $FFOS$, U in (B, τ_B) then U is a $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Hence f is a $FF\alpha C$.

Remark 3.7. The converse of the above theorem need not be true as shown by the following example.

Example 3.8. Let $A = \{a_1, a_2, a_3\}$, $\tau_A = \{0_A, 1_A, A_1, A_2, A_1 \cap A_2, A_1 \cup A_2\}$ where $A_1 = \{(a_1, 0.9, 0.6), (a_2, 0.4, 0.7), (a_3, 0.6, 0.5)\}$, $A_2 = \{(a_1, 0.6, 0.9), (a_2, 0.7, 0.4), (a_3, 0.5, 0.6)\}$ and $B = \{b_1, b_2, , b_3\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.9, 0.6), (b_2, 0.7, 0.4), (b_3, 0.6, 0.5)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_1$, $f(a_2) = b_2$ and $f(a_3) = b_3$. B_1 is a $FFOS$ in (B, τ_B). $f^{-1}(B_1) = B_1$ and $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1 \cup A_2$ imply $f^{-1}(B_1) \subseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Hence f is a $FF\alpha C$. Consider a FFS, $B_2 = \{(b_1, 0.9, 0.5), (b_2, 0.7, 0.3), (b_3, 0.6, 0.5)\}$ in (B, τ_B). Since $B_2 \subseteq int(cl(int(B_2)))$, B_2 is a $FF\alpha OS$ in (B, τ_B). Also $f^{-1}(B_2) = B_2$ and $int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right) = A_1 \cup A_2$ imply $f^{-1}(B_2) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right)$. Therefore $f^{-1}(B_2)$ is not a $FF\alpha OS$ in (A, τ_A). Hence f is not $FF\alpha I$.

Theorem 3.9. *Every $FF\alpha I$ is a $FFSC$.*

Proof. Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \to (B, \tau_B)$ is a $FF\alpha I$. Let us take a $FFOS$, U in (B, τ_B) then U is a $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Since every $FF\alpha OS$ is $FFSOS$, $f^{-1}(U)$ is a $FFSOS$ in (A, τ_A). Hence f is a $FFSC$.

Remark 3.10. The converse of the above theorem need not be true as shown by the following example.

Example
3.11. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.6, 0.69), (a_2, 0.23, 0.63)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.58, 0.63), (b_2, 0.68, 0.67)\}$ be two FFTSs. Define a FF mapping $f : (A, \tau_A) \to (B, \tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B, τ_B). Since $cl\left(int\left(f^{-1}(B_1)\right)\right) = 1_A$, $f^{-1}(B_1) \subseteq cl\left(int\left(f^{-1}(B_1)\right)\right)$. Hence f is a $FFSC$. But B_1 is a $FF\alpha OS$ in (B, τ_B) and $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1$ imply $f^{-1}(B_1) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Therefore $f^{-1}(B_1)$ is not a $FF\alpha OS$ in (A, τ_A). Hence f is not $FF\alpha I$.

Theorem 3.12. *Every $FFS\alpha C$ is a $FF\alpha I$.*

Proof. Let (A, τ_A) and (B, τ_B) be two FFTSs and let $f : (A, \tau_A) \to (B, \tau_B)$ is a $FFS\alpha C$. Consider a $FF\alpha OS$, B_1 in (B, τ_B). Since every $FF\alpha OS$ is $FFSOS$ and f is $FFS\alpha C$, $f^{-1}(B_1)$

is $FF\alpha OS$ in (A,τ_A). Thus the inverse image of $FF\alpha OS$ (B,τ_B) is $FF\alpha OS$ in (A,τ_A). Hence f is $FF\alpha I$.

Remark 3.13. The converse of the above theorem need not be true as shown by the following example.

Example 3.14. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.63, 0.68), (a_2, 0.73, 0.73)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.73, 0.73), (b_2, 0.63, 0.68)\}$ be two FFTSs. Define a FF mapping $f : (A,\tau_A) \to (B,\tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B,τ_B) and so it is $FF\alpha OS$ in (B,τ_B). $f^{-1}(B_1) = \{(a_1, 0.63, 0.68), (a_2, 0.73, 0.73)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1$, $f^{-1}(B_1) \subseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Hence $f^{-1}(B_1)$ is a $FF\alpha OS$ in (A,τ_A). Therefore f is a $FF\alpha I$. Consider a FFS, $B_2 = \{(b_1, 0.73, 0.73), (b_2, 0.68, 0.63)\}$ in (B,τ_B). Since $cl(int(B_2)) = B_1^C$, $B_2 \subseteq cl(int(B_2))$. Therefore B_2 is a $FFSOS$ in (B,τ_B). $f^{-1}(B_2) = \{(a_1, 0.68, 0.63), (a_2, 0.73, 0.73)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right) = A_1$, $f^{-1}(B_2) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right)$. Therefore $f^{-1}(B_2)$ is a not a $FF\alpha OS$ in (A,τ_A). Hence f is not a $FFS\alpha C$.

Theorem 3.15. *Every $FFS\alpha C$ is a $FF\alpha C$.*

Proof. The proof follows from the theorems 3.12 and 3.6.

Remark 3.16. The converse of the above theorem need not be true as shown by the following example.

Example 3.17. Let $A = \{a_1, a_2\}$, $\tau_A = \{0_A, 1_A, A_1\}$ where $A_1 = \{(a_1, 0.58, 0.69), (a_2, 0.79, 0.79)\}$ and $B = \{b_1, b_2\}$, $\tau_B = \{0_B, 1_B, B_1\}$ where $B_1 = \{(b_1, 0.79, 0.79), (b_2, 0.58, 0.69)\}$ be two FFTSs. Define a FF mapping $f : (A,\tau_A) \to (B,\tau_B)$ such that $f(a_1) = b_2$ and $f(a_2) = b_1$. B_1 is a $FFOS$ in (B,τ_B). $f^{-1}(B_1) = \{(a_1, 0.58, 0.69), (a_2, 0.79, 0.79)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right) = A_1$, $f^{-1}(B_1) \subseteq int\left(cl\left(int\left(f^{-1}(B_1)\right)\right)\right)$. Hence $f^{-1}(B_1)$ is a $FF\alpha OS$ in (A,τ_A). Therefore f is a $FF\alpha C$. Consider a FFS, $B_2 = \{(b_1, 0.79, 0.79), (b_2, 0.69, 0.58)\}$ in (B,τ_B). Since $cl(int(B_2)) = B_1^C$, $B_2 \subseteq cl(int(B_2))$. Therefore B_2 is a $FFSOS$ in (B,τ_B). $f^{-1}(B_2) = \{(a_1, 0.68, 0.63), (a_2, 0.73, 0.73)\}$. Since $int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right) = A_1$, $f^{-1}(B_2) \nsubseteq int\left(cl\left(int\left(f^{-1}(B_2)\right)\right)\right)$. Therefore $f^{-1}(B_2)$ is a not a $FF\alpha OS$ in (A,τ_A). Hence f is not a $FFS\alpha C$.

Theorem 3.18. *If $f : (A,\tau_A) \to (B,\tau_B)$ be a mapping from a FFTS A to a FFTS B. Then the following are equivalent*

(1) *f is $FF\alpha I$.*

(2) $f^{-1}(V)$ *is* $FF\alpha CS$ *in* A *for each* $FF\alpha CS$ V *in* B.

(3) $f(cl_\alpha U) \subseteq cl_\alpha(f(U))$ *for each* FFS U *in* A.

(4) $cl_\alpha f^{-1}(V) \subseteq f^{-1}(cl_\alpha V)$ *or each* FFS V *in* B.

(5) $f^{-1}(int_\alpha V) \subseteq int_\alpha\left(f^{-1}V\right)$ *for each* FFS V *in* B.

Proof. (1) $\Longrightarrow$ (2): Consider a $FF\alpha CS$, V in B. Then V^c is $FF\alpha OS$ in B. Since f is $FF\alpha I$, $f^{-1}(V^c) = \left(f^{-1}(V)\right)^c$ is $FF\alpha OS$ in A. Hence $f^{-1}(V)$ is a $FF\alpha CS$ in A. Hence (1) $\Longrightarrow$ (2).
(2) $\Longrightarrow$ (3): Let U be a FFS in A. Then we have $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(cl_\alpha(f(U)))$ $cl_\alpha(f(U))$ is $FF\alpha CS$ in B. Then by (2), $f^{-1}(cl_\alpha(f(U)))$ is $FF\alpha CS$ in A. $cl_\alpha U \subseteq f^{-1}(cl_\alpha(f(U)))$ implies $f(cl_\alpha(U)) \subseteq f\left(f^{-1}(cl_\alpha(f(U)))\right) = cl_\alpha(f(U))$. Thus $f(cl_\alpha U) \subseteq cl_\alpha(f(U))$. Hence (2) $\Longrightarrow$ (3).
(3) $\Longrightarrow$ (4): For a FFs V in B, let $f^{-1} = U$. Then by (3), $f\left(cl_\alpha\left(f^{-1}(V)\right)\right) \subseteq cl_\alpha\left(f^{-1}(V)\right) \subseteq cl_\alpha V$ and $cl_\alpha\left(f^{-1}(V)\right) \subseteq f^{-1}\left(f\left(cl_\alpha\left(f^{-1}(V)\right)\right)\right) \subseteq cl_\alpha V$. Hence $cl_\alpha\left(f^{-1}(V)\right) \subseteq f^{-1}(cl_\alpha V)$. Hence (3) $\Longrightarrow$ (4).
(4) $\Longrightarrow$ (5): We have $int_\alpha(V) = [cl_\alpha(V^c)]^c$. Then $f^{-1}(int_\alpha(V)) = f^{-1}[cl_\alpha(V^c)]^c = [f^{-1}(cl_\alpha(V^c))]^c \subseteq [cl_\alpha\left(f^{-1}(V^c)\right)]^c = [int_\alpha\left(f^{-1}(V)\right)^c]^c \subseteq int_\alpha\left(f^{-1}(V)\right)$.
(5) $\Longrightarrow$ (1): Let V be any $FF\alpha OS$ in B. Then $V = int_\alpha V$. By (5), $f^{-1}(int_\alpha V) = f^{-1}(V) \subseteq int_\alpha f^{-1}(V)$. We have $int_\alpha f^{-1}(V) \subseteq f^{-1}(V)$. Therefore $int_\alpha f^{-1}(V) = f^{-1}(V)$. Thus $f^{-1}(V)$ is a $FF\alpha OS$ in A and so f is $FF\alpha I$. Hence (5) $\Longrightarrow$ (1).

Lemma 3.19. *Let* $f : (A, \tau_A) \to (B, \tau_B)$ *be a mapping and* U_α *be a family of* FFS *of* B. *Then*

(1) $f^{-1}\left(\bigcup U_\alpha\right) = \bigcup f^{-1}(U)$

(2) $f^{-1}\left(\bigcap U_\alpha\right) = \bigcap f^{-1}(U)$

Lemma 3.20. *Let* $f : A_i \to B$ *be a mapping and* U, V *are* $FFSs$ *of* B_1 *and* B_2 *respectively then* $(f_1 \times f_2)^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)$.

Definition 3.21. Let $(A_i, \tau_i)_{i\in\Omega}$ be a family of FFTSs. Then their product is (A, τ_A) where $A = \prod_{i\in\Omega} A_i$ and τ_A is the initial FFT on A generated by the family of FF projection maps $P_i : A \to (A_i, \tau_i)_{i\in\Omega}$.

Lemma 3.22. *Let* $f : (A, \tau_A) \to (B, \tau_B)$ *and* $g : A \to A \times B$ *be two mappings. If* U *and* V *are* $FFSs$ *of* A *and* B *respectively then* $g^{-1}(1_A \times V) = \left(1_A \cap f^{-1}(V)\right)$.

Lemma 3.23. *Let* A *and* B *be FFTSs then* (A, τ_A) *is product related to* (B, τ_B) *if for any* FFS X *in* A, Y *in* B *whenever* $X \nsubseteq U^c, Y \nsubseteq V^c$ *implies* $X \times Y \subseteq U^c \times 1_A \cup 1_B \times V^c$ *then there exists* $A_1 \in \tau_A$, $B_1 \in \tau_B$ *such that* $X \subseteq A_1^c$ *and* $Y \subseteq B_1^c$ *and* $A_1^c \times 1_A \cup 1_B \times B_1^c = A^c \times 1_A \cup 1_B \times B^c$

Lemma 3.24. *Let (A, τ_A) and (B, τ_B) be FFTSs such that (A, τ_A) is product related to (B, τ_B). Then the product $U \times V$ of a $FF\alpha OS$ U in A and a $FF\alpha OS$ V in B is a $FF\alpha OS$ in FF product space $X \times Y$.*

Theorem 3.25. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be a function and assume that (A, τ_A) is product related to (B, τ_B). If the mapping $g : A \to A \times B$ of f is $FF\alpha I$ then so f.*

Proof. Let V be a $FF\alpha OS$ in B. Then by Lemma 3.23, $f^{-1}(V) = 1_A \times f^{-1}(V) = g^{-1}(1_B \times V)$. Now $1_B \times V$ is a $FF\alpha OS$ in $A \times B$. Since g is $FF\alpha I$, $g^{-1}(1_B \times V)$ is $FF\alpha OS$ in A. Thus f is $FF\alpha I$.

Theorem 3.26. *If a function $f : A \to \prod B_i$ is $FF\alpha I$, then $P_i \circ f : A \to B_i$ is $FF\alpha I$, where P_i is the projection of $\prod B_i$ onto B.*

Proof. Let V_i be any $FF\alpha OS$ of B_i. Since P_i is FF continuous and $FFOS$, it is $FF\alpha OS$. Now $P_i : \prod B_i \to B_i$, $P_i^{-1}(V_i)$ is $FF\alpha OS$ of $\prod B_i$. So P_i is $FF\alpha I$. $(P_i \circ f)^{-1}(V_i) = f^{-1}\left(P_i^{-1}(V_i)\right)$, since f is $FF\alpha I$ and $P_i^{-1}(V_i)$ is $FF\alpha OS$, $P_i^{-1}(V_i)$ is $FF\alpha OS$, $f^{-1}(P_i(V_i))$ is $FF\alpha OS$. Thus $P_i \circ f$ is $FF\alpha I$.

Theorem 3.27. *If $f_i : A_i \to B_i$ $i = 1, 2$ are $FF\alpha I$ and A_1 is product related to A_2 then $f_1 \times f_2 : A_1 \times A_2 \to B_1 \times B_2$ is $FF\alpha I$.*

Proof. Let $X = \bigcup (U_i \times V_i)$ where U_i and V_i, $i = 1, 2$ are $FF\alpha OS$ of B_1 and B_2 respectively. Since B_1 is product related to B_2 , by Lemma 3.24, $X = \bigcup (U_i \times V_i)$ is $FF\alpha OS$ of $B_1 \times B_2$. By Lemma 3.19 and Lemma. 3.20 we get $(f_1 \times f_2)^{-1}(X) = (f_1 \times f_2)^{-1}(\bigcup (U_i \times V_i)) = \bigcup \left(f_1^{-1}(U_i) \times f_2^{-1}(V_i)\right)$. Since f_1 and f_2 are $FF\alpha I$, $(f_1 \times f_2)^{-1}(X)$ is a $FF\alpha OS$ in $f_1 \times f_2$ and so $f_1 \times f_2$ is $FF\alpha I$.

Theorem 3.28. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha I$ if and only if for every FF projection, $P_{(\lambda,\mu)}$ in A and $FF\alpha OS$ V in B such that $f\left(P_{(\lambda,\mu)}\right) \in V$, there exists a $FF\alpha OS$ U in A such that $P_{(\lambda,\mu)} \in U$ and $f(U) \subseteq V$.*

Proof. Consider a $FF\alpha I$ f, a FF projection $P_{(\lambda,\mu)}$ in A and a $FF\alpha OS$ V in B such that $f\left(P_{(\lambda,\mu)}\right) \in V$. Then $P_{(\lambda,\mu)} \in f^{-1}(V) = int_\alpha f^{-1}(V)$. Let $U = int_\alpha f^{-1}(V)$. Then U is a $FF\alpha OS$ in A which contains FF projection, $P_{(\lambda,\mu)}$ and $f(U) = int_\alpha f^{-1}(V) \subseteq f\left(f^{-1}(V)\right) = V$. Conversely, let V be a $FF\alpha OS$ in B and $P_{(\lambda,\mu)}$ be a FF projection in A such that $P_{(\lambda,\mu)} \in f^{-1}(V)$. By assumption there exists $FF\alpha OS$ U in A such that $P_{(\lambda,\mu)} \in U$ and $f(U) \subseteq V$. Hence $P_{(\lambda,\mu)} \in U \subseteq f^{-1}(V)$ and $P_{(\lambda,\mu)} \in U = int_\alpha U \subseteq int_\alpha\left(f^{-1}(V)\right)$. Since $P_{(\lambda,\mu)}$ is an arbitrary FF projection $f^{-1}(V)$ is the union of all FF projection containing in $f^{-1}(V)$, thus we get $f^{-1}(V) = int_\alpha\left(f^{-1}(V)\right)$. Hence f is a $FF\alpha I$.

Definition 3.29. Let (A, τ_A) be a FFTS and U be any FFS in A. U is called FF dense set if $cl\,(U) = 1_A$ and no where FF dense set if $int\,(cl\,(U)) = 0_A$.

Theorem 3.30. *If a function $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is $FF\alpha I$ then $f^{-1}\,(V)$ is a $FF\alpha CS$ in A for any no where FF dense set V in B.*

Proof. Let V be any no where FF dense set in B. Then $int\,(cl\,(V)) = 0_B$. Now $[int\,(cl\,(V))]^c = 1_B$ implies $cl\,([cl\,(V)]^c) = 1_B$ and so $cl[int\,(V^c)] = 1_B$. Since $int 1_B = 1_B$, $int\,(cl\,(int\,(V^c))) = int 1_B = 1_B$. Hence $V^c \subseteq int\,(cl\,(int\,(V^c))) = 1_B$. Then V^c is $FF\alpha OS$ in B. Since f is $FF\alpha I$, $f^{-1}\,(V^c)$ is $FF\alpha OS$ in A. Hence $f^{-1}\,(V)$ $FF\alpha CS$ in A.

Theorem 3.31. *Consider the two functions $f : (A, \tau_A) \rightarrow (B, \tau_B)$ and $g : (B, \tau_B) \rightarrow (C, \tau_C)$. Then the following hold.*

(1) *If f is $FF\alpha I$ and g is $FF\alpha I$ then $g \circ f$ is $FF\alpha I$.*

(2) *If f is $FF\alpha I$ and g is $FFS\alpha C$ then $g \circ f$ is $FFS\alpha C$.*

(3) *If f is $FF\alpha I$ and g is $FF\alpha C$ then $g \circ f$ is $FF\alpha I$.*

Proof.

(1) Let V be any $FF\alpha OS$ in (C, τ_C). Since g is $FF\alpha I$, $g^{-1}\,(V)$ is a $FF\alpha OS$ in (B, τ_B). We have $(g \circ f)^{-1}\,(V) = f^{-1}\left(g^{-1}\,(V)\right)$. Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\,(V)\right)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FF\alpha I$.

(2) Let V be any $FFSOS$ in (C, τ_C). Since g is $FFS\alpha C$, $g^{-1}\,(V)$ is $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\,(V)\right)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FFS\alpha C$.

(3) Let V be any $FFOS$ in (C, τ_C). Since g is $FFS\alpha C$, $g^{-1}\,(V)$ is $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}\,(V)\right)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FF\alpha I$.

The Figure 1 illustrate the relation between the various FF mappings.

4. **Fermatean fuzzy α-Homeomorphism in FFTS**

Definition 4.1. A mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is called FF α-open($FF\alpha O$) if $f(U)$ is $FF\alpha OS$ in (B, τ_B) whenever U is $FF\alpha OS$ in (A, τ_A).

Definition 4.2. A mapping $f : (A, \tau_A) \rightarrow (B, \tau_B)$ is called FF α-closed($FF\alpha O$) if $f(U)$ is $FF\alpha CS$ in (B, τ_B) whenever U is $FF\alpha CS$ in (A, τ_A).

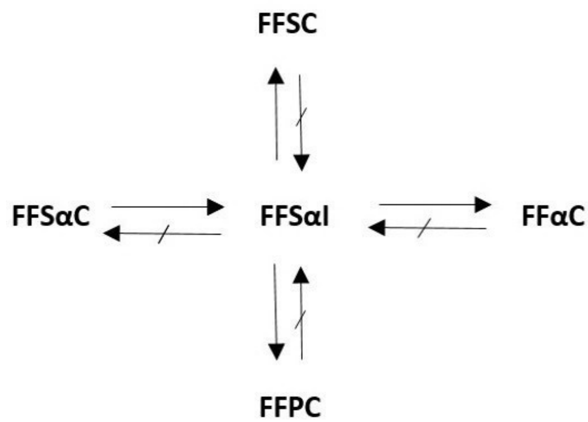

Figure 1. Relation between FF mappings

Theorem 4.3. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha O$ if and only if for each $a \in A$ and each $FFOS$ $U \subseteq (A, \tau_A)$ contained a, there exists a $FF\alpha OS$ $W \subseteq (B, \tau_B)$ such that $W \subseteq f(U)$.*

Proof. Obvious.

Theorem 4.4. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha C$ if and only if $cl_\alpha (f(U)) \subseteq f(U)$ for each FFS, $U \subseteq (A, \tau_A)$.*

Proof. Obvious.

Theorem 4.5. *A mapping $f : (A, \tau_A) \to (B, \tau_B)$ be a $FF\alpha O$ and $U \subseteq (A, \tau_A)$ is a $FFCS$ containing $f^{-1}(W)$, then there exists a $FF\alpha CS$ $H \subseteq (B, \tau_B)$ containing W such that $f^{-1}(H) \subseteq U$.*

Proof. Let $H = [f(U^c)]^c$. Since $f^{-1}(W) \subseteq U$ we have $f(U^c) \subseteq W^c$. Since f is $FF\alpha O$, H is $FF\alpha CS$ and $f^{-1}(H) = [f^{-1}(f(U^c))]^c \subseteq (U^c)^c = U$.

Theorem 4.6. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be $FF\alpha c$. If $W \subseteq (B, \tau_B)$ and $U \subseteq (A, \tau_A)$ is a $FFOS$ containing $f^{-1}(W)$, then there exists a $FF\alpha OS$, $H \subseteq (B, \tau_B)$ containing W such that $f^{-1}(H) \subseteq U$.*

Proof. Let $H = [f(U^c)]^c$. Since $f^{-1}(W) \subseteq U$, we have $f(U^c) \subseteq W^c$. Since f is $FF\alpha C$, then H is $FF\alpha OS$ and $f^{-1}(H) = [f^{-1}(f(U^c))]^c \subseteq (U^c)^c = U$.

Corollary 4.7. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be $FF\alpha O$. Then*

(1) $f^{-1}(cl(int(cl(V)))) \subseteq cl\left(f^{-1}(V)\right)$ *for each FFS $V \subseteq B$.*

(2) $f^{-1}(cl(V)) \subseteq cl\left(f^{-1}(V)\right)$ *for each* $FFPOS$ $V \subseteq B$.

Proof.

(1) $cl\left(f^{-1}(V)\right)$ is a $FFCS$ in (A,τ_A) containing $f^{-1}(V)$ for a FFS $V \subseteq B$.By Theorem.4.5, 4.6 there exists a $FF\alpha CS$, $H \subseteq B$, $V \subseteq B$ such that $f^{-1}(H) \subseteq cl\left(f^{-1}(V)\right)$. Thus $f^{-1}(cl(int(cl(V)))) \subseteq f^{-1}(cl(int(cl(H)))) \subseteq f^{-1}(H) \subseteq f^{-1}(V)$.

(2) Similar to (1).

Theorem 4.8. *Let* $f:(A,\tau_A)\rightarrow(B,\tau_B)$ *be* $FFPC$ *and* $(FF\alpha O)$ *then the inverse image of each* $FFPOS$ *is* $FFPOS$.

Proof. Let V is a $FFPOS$ in (B,τ_B). So $f^{-1}(V) \subseteq f^{-1}(cl(int(V)) \subseteq int\left(cl\left(f^{-1}(int(cl(V)))\right)\right) \subseteq int\left(cl\left(f^{-1}(cl(V))\right)\right)$. Since f is $FF\alpha O$ by Corollary 4.7 we have $f^{-1}(V) \subseteq int\left(cl\left(f^{-1}(cl(V))\right) \subseteq int\left(cl\left(f^{-1}(cl(V))\right)\right) \subseteq int\left(cl\left(f^{-1}(V)\right)\right) = int\left(cl\left(f^{-1}(V)\right)\right)$. Therefore $f^{-1}(H)$ is $FFPOS$ in (A,τ_A).

Definition 4.9. A FF bijection $f:(A,\tau_A)\rightarrow(B,\tau_B)$ is called FF α-homeomorphism $(FF\alpha H)$ if f is both $FF\alpha O$ and $FF\alpha C$.

Example 4.10. Let $A=\{a_1,a_2\}$, $\tau_A=\{0_A,1_A,A_1\}$ where $A_1=\{(a_1,0.6,0.8),(a_2,0.9,0.3)\}$ and $B=\{b_1,b_2\}$, $\tau_B=\{0_B,1_B,B_1\}$ where $B_1=\{(b_1,0.5,0.85),(b_2,0.86,0.45)\}$ be two FFTSs. Define a FF mapping $f:(A,\tau_A)\rightarrow(B,\tau_B)$ such that $f(a_1)=b_1$ and $f(a_2)=b_2$. Then f is $FF\alpha H$.

Theorem 4.11. *Every* FFH *is* $FF\alpha H$.

Proof. Let $f:(A,\tau_A)\rightarrow(B,\tau_B)$ be FFH. Then f is bijective, FFC and FFO mapping. Let V be a $FFOS$ in (B,τ_B). As f is FFC, $f^{-1}(V)$ is $FFOS$ in (A,τ_A). Since every $FFOS$ is $FF\alpha OS$, $f^{-1}(V)$ is $FF\alpha OS$ in (A,τ_A) which implies f is $FF\alpha C$. Assume U is $FFOS$ in (A,τ_A). Also since f is FFO , $f(U)$ is $FFOS$ in (B,τ_B) which implies f is $FF\alpha O$. Hence f is $FF\alpha H$.

Remark 4.12. Every $FF\alpha H$ need not be FFH as given in the following example.

Example 4.13. Let $A=\{a_1,a_2\}$, $\tau_A=\{0_A,1_A,A_1\}$ where $A_1=\{(a_1,0.6,0.8),(a_2,0.9,0.3)\}$ and $B=\{b_1,b_2\}$, $\tau_B=\{0_B,1_B,B_1\}$ where $B_1=\{(b_1,0.5,0.85),(b_2,0.86,0.45)\}$ be two FFTSs. Define a FF mapping $f:(A,\tau_A)\rightarrow(B,\tau_B)$ such that $f(a_1)=b_1$ and $f(a_2)=b_2$. Then f is $FF\alpha H$.

Proposition 4.14. *For a* FF *bijective map* $f:(A,\tau_A)\rightarrow(B,\tau_B)$ *the following are equivalent.*

(1) f is $FF\alpha O$.

(2) f is $FF\alpha Cl$.

(3) $f^{-1} : (B, \tau_B) \to (A, \tau_A)$ is $FF\alpha C$.

Proof. (i) $\implies$ (ii): Let U be $FFCS$ in (A, τ_A). Then U^c is $FFOS$ in (A, τ_A). Since f is $FF\alpha O$, $f(U^c)$ is $FF\alpha OS$ in (B, τ_B) which implies $[f(U)]^c$ is $FF\alpha OS$ in (B, τ_B). So $f(U)$ is $FF\alpha CS$ in (B, τ_B). Hence f is $FF\alpha Cl$.
(ii) $\implies$ (iii): let U be $FFCS$ in (A, τ_A). Since f is $FF\alpha Cl$, $f(U)$ is $FF\alpha CS$ in (B, τ_B). Since f is FF bijective $f(U) = \left(f^{-1}\right)^{-1}(U)$, f^{-1} $FF\alpha Cl$.
(iiI) $\implies$ (i): Let U be $FFOS$ in (A, τ_A) and hence $FF\alpha OS$ in (A, τ_A). $\left(f^{-1}\right)^{-1}(U)$ is $FF\alpha OS$ in (B, τ_B). Therefore $f(U)$ is $FF\alpha OS$ in (B, τ_B). Hence f is $FF\alpha O$.

Theorem 4.15. $f : (A, \tau_A) \to (B, \tau_B)$ *be Ff bijective and* $FF\alpha C$ *then the following are equivalent*

(1) f *is* $FF\alpha O$.

(2) f *is* $FF\alpha H$.

(3) f *is* $FF\alpha Cl$.

Proof. (i) $\implies$ (ii): Since f is FF bijective, $FF\alpha C$ and $FF\alpha O$, by definition f is $FF\alpha H$.
(ii) $\implies$ (iii): let U be $FFCS$ in (A, τ_A). Then U^c is $FFOS$ in (A, τ_A). By hypothesis, $f(U^c) = [f(U)]^c$, is $FF\alpha OS$ in (B, τ_B). Therefore $f(V)$ is $FF\alpha CS$ in (B, τ_B). Thus f is $FF\alpha Cl$.
(iii) $\implies$ (i): let U be $FFOS$ in (A, τ_A). Then U^c is $FFCS$ in (A, τ_A). By hypothesis, $f(U^c) = [f(U)]^c$, is $FFCS$ in (B, τ_B). Therefore $f(U)$ is $FF\alpha OS$ in (B, τ_B). Thus f is $FF\alpha O$.

Definition 4.16. A FF bijection $f : (A, \tau_A) \to (B, \tau_B)$ is said to be $FF\alpha^* H$ homeomorphism $(FF\alpha^* H)$ if f is both f^{-1} are $FF\alpha I$.

Theorem 4.17. *Let* $f : (A, \tau_A) \to (B, \tau_B)$ *and* $g : (B, \tau_B) \to (C, \tau_C)$ *are* $FF\alpha^* H$ *then* $g \circ f$ *is a* $FF\alpha^* H$.

Proof. Let U be a $FF\alpha OS$ in (C, τ_C). Since g is $FF\alpha I$, $g^{-1}(U)$ is $FF\alpha OS$ in (B, τ_B). Since f is $FF\alpha I$, $f^{-1}\left(g^{-1}(U)\right) = (g \circ f)^{-1}(U)$ is $FF\alpha OS$ in (A, τ_A). Therefore $g \circ f$ is $FF\alpha I$. Let V be $FF\alpha OS$ in (A, τ_A). $(g \circ f)(V) = g[f(V)]$ is $FF\alpha OS$ in (C, τ_C). Thus $(g \circ f)(V)$ is $FF\alpha OS$ in (C, τ_C). Therefore $(g \circ f)^{-1}$ is $FF\alpha I$. Also $(g \circ f)$ is a FF bijection. Hence $(g \circ f)$ is $FF\alpha^* H$.

Theorem 4.18. *Every $FF\alpha^*H$ is $FF\alpha H$.*

Proof. Let $f:(A,\tau_A)\to(B,\tau_B)$ be a $FF\alpha^*H$. Then f and f^{-1} are $FF\alpha I$. Let U is $FFOS$ in (B,τ_B). Then U is $FF\alpha OS$ in (B,τ_B). Since f is $FF\alpha I$, $f^{-1}(U)$ is $FF\alpha OS$ in (A,τ_A). Thus U is $FF\alpha OS$ in (B,τ_B) implies $f^{-1}(U)$ is $FF\alpha CS$ in (A,τ_A). Therefore f is $FF\alpha C$. Since f^{-1} is $FF\alpha I$, $\left(f^{-1}\right)^{-1}(U)=f(U)$ is $FF\alpha OS$ in (B,τ_B). Thus U is $FF\alpha OS$ implies $f^{-1}(U)$ is $FF\alpha OS$. Therefore f is $FF\alpha O$. Hence f is $FF\alpha H$.

5. **Conclusion and Future work**

In this study we have introduced generalised FF continuity such as $FF\alpha I$,FF α open and clossed mapping, $FF\alpha H$ and $FF\alpha^*H$. Topological equivalence can be interpreted more generally with the help of α-homeomorphisms, whereby spaces may be considered identical under more kinds of instances. When studying spaces with complicated or unpredictable topologies, such fractals or particular functions of spaces, this is particularly insightful. The FF compactness, FF connectedness and FF separation axioms will be studied in future. These ideas are currently being developed as part of current studies, and there is a demand for additional research into their philosophical foundations and practical significance, especially in sectors where uncertainty in data is an important issue.

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Enhanced Neutrosophic Set and Machine Learning Approach for Breast Cancer Prediction

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Abstract: Breast cancer is the most prevalent type of cancer that affects women worldwide and poses a serious risk to female mortality. In order to lower death rates and enhance treatment results, early detection is critical. Neutrosophic Set Theory (NST) and machine learning (ML) approaches are integrated in this study to provide a novel hybrid methodology (NS-ML) that improves breast cancer diagnosis. Using the Wisconsin Diagnostic Breast Cancer (WDBC) dataset, the research transforms these data into Neutrosophic (N) representations to effectively capture uncertainties. When trained on the N-dataset instead of traditional datasets, ML algorithms such as Decision Tree (DT), Random Forest (RF), and Adaptive Boosting (AdaBoost) perform better. Notably, N-AdaBoost models achieve outstanding results with 99.12% accuracy and 100% precision, highlighting the efficacy of NS in enhancing diagnostic reliability.

Keywords: Neutrosophic sets; Machine Learning; Uncertainity handling; Breast cancer; Classification.

1. Introduction

Women globally have the risk of breast cancer which as a risk factor of cancer due to abnormal growth of cells in breast tissue [1]. Countries like Belgium and the Netherlands reported high rates of incidence, affecting thousands of women, while Barbados and Fiji had notable mortality rates due to the disease [2]. In 2020, it caused over 2 million new cases and led to 6,85,000 deaths globally [3]. By 2030, global cases are expected to rise significantly, impacting approximately 27 million people [4].

Breast cancer involves various tumor types, categorized as malignant or benign, with malignant tumors posing a higher risk due to their rapid spread. Factors contributing to its increasing incidence include lack of awareness, economic disparities, inadequate healthcare access, and screening challenges [5]. Developing independent apps for accurate cancer diagnosis is crucial to overcome these challenges. By analyzing variables and pinpointing the most important elements for a precise diagnosis across several models, ML has shown to be extremely successful in the diagnosis of breast cancer. Several models have been presented in earlier research, using various methods and techniques to identify cancers. ML, particularly integrated with NS to handle data uncertainty, presents an effective approach for enhancing breast cancer models for forecasting.

The main contributions of this study are:

a) Transforming the WDBC into an N-dataset.

b) Model training employing ML classification techniques, such as AdaBoost, DT, and RF.

c) Evaluating the performance of these models.

d) Comparing the performance between the original dataset and the N-dataset.

The remaining sections of the paper are organized as follows: The section 2 summarizes the literature on breast cancer prognosis. The study's materials and procedures are described in depth in Section 3. The study's findings are presented in Section 4. The methodology and results are covered in Section 5, and the paper's conclusions are given in Section 6.

2. Related work

Breast cancer is a serious worldwide health issue, and increasing survival rates requires early identification. Advanced ML techniques, such as Eagle Strategy Optimization (ESO), Gravitational Search Optimization (GSO), and their combined approach, are used to improve classification accuracy on the WDBC dataset. By prioritizing informative features and reducing computational complexity, the approach shows promising results in improving diagnostic precision and efficiency [6]. Breast cancer, characterized by complex development involving various cell types, remains a significant challenge worldwide. Advances in understanding pathogenesis and genetic factors have led to improved prevention and treatment strategies. Effective screening and research into drug-resistant mechanisms have enhanced patient outcomes and quality of life [7]. Over recent decades, significant advancements in breast cancer research have revolutionized treatment approaches, leading to better outcomes. Early detection through improved awareness and screening methods has enabled curative treatments such as surgery and radiation therapy. Ongoing research aims to further enhance diagnostic and therapeutic strategies [8].

Breast cancer is a multifactorial illness with a wide range of subtypes and symptoms. Understanding this diversity is crucial for developing targeted treatment approaches. Research focuses on genetic mutations, micro environmental factors, and epigenetic changes to improve personalized treatment strategies [9]. Technological advancements in mammographic screening and therapeutic interventions have transformed breast cancer management. Innovations in surgical techniques and radiotherapy have improved disease control and cosmetic outcomes. Clinical trials exploring combination therapies and gene-expression profiling aim to enhance treatment selection and patient outcomes [10]. Developing accurate prediction models tailored to specific populations, such as Cuban women, is crucial for effective breast cancer management. ML-based model that achieves high accuracy in estimating breast cancer risk for Cuban women, outperforms existing models. The potential for early diagnosis to enhance patient outcomes and save healthcare expenditures is highlighted by this approach [11]. To select and classify features in multidimensional breast cancer datasets, a new approach called the Rat Swarm Optimizer (RSO) hybridization with Levy Flight-based Cuckoo Search Optimization Algorithm (H-RS-LVCSO) was presented. By merging hybrid adaptive LVCSO with moment invariant wavelet feature extraction, this method greatly improves accuracy, precision, and execution speed. The research contributes to advancing breast cancer classification through innovative Feature Selection (FS) and classification techniques [12].

Evaluating various ML algorithms for breast cancer diagnosis, highlights RF's robustness in handling high-dimensional data and nonlinear decision boundaries. The approach demonstrates high accuracy in distinguishing between healthy individuals and those with breast cancer, showcasing its potential for accurate early detection [13]. The application of ensemble data mining techniques enhances the precision of breast cancer diagnosis by combining Rotation Forest with feature selection based on genetic algorithms. The approach optimizes input variable selection and employs robust classification methods, achieving high accuracy rates and demonstrating the effectiveness of ensemble methods in medical diagnostics [14]. Particle swarm optimization (PSO) was used for FS in data mining techniques to create a predictive model for breast cancer recurrence. The study demonstrates how Particle Swarm Optimization (PSO) enhances classification performance by assessing classifiers such as Naive Bayes (NB), K-Nearest Neighbor (KNN), and the rapid Decision Tree learner (REPTree). The results highlight the effectiveness of feature selection (FS) techniques in optimizing predictive models for breast cancer recurrence [15].

Cardiotocography (CTG) data uncertainty is crucial for classifying fetal heart rate in the biomedical field. The proposed Interval Neutrosophic Rough Neural Network (IN-RNN) framework, utilizing the backpropagation algorithm, enhances RNN's performance through NST. The experimental results indicate exceptional performance, with scores around 95%. Using WEKA application, the framework was compared with algorithms like Neural Network (NN), decision tables, and nearest neighbors, confirming its efficiency. The Receiver Operating Characteristic (ROC) curve displays high and acceptable area-under-curve values for the pathologic, normal, and suspicious states. The IN-RNN framework estimates uncertainty boundaries based on membership, truth, and indeterminacy values, with performance metrics indicating its effectiveness in classifying CTG data [16]. Differentiating COVID-19 from other lung illnesses, like bacterial and viral pneumonia, has become more difficult due to the COVID-19 pandemic. To differentiate between these diseases, a neutrosophic method was put forth, which involved grouping data into sets labeled True (T), False (F), and Indeterminacy (I) to improve feature extraction. Alpha-mean and beta-enhancement preprocessing is applied to chest X-ray pictures in order to decrease indeterminacy and boost opacity detection. Then, in a transfer learning setup, these improved images are examined using ResNet-50, VGG-16, and XGBoost, yielding an accuracy of 97.33% [17].

Decision-making (DM) is naturally challenging because of the uncertain and ambiguous nature of environments, particularly when multiple attributes are considered. The Multi-Polar Interval-Valued Neutrosophic Set (MPIVNS) and the Hypersoft Set (HS) framework were combined to address these issues. New aggregate operators, distance metrics, and similarity measures created especially for MPIVN-HSs are presented. These tools are essential for resolving complex attribute-based decision-making problems. The research utilizes the KNN algorithm to improve decision processes, showcasing practical applications in areas like site selection and beyond. The study significantly advances fields relying on language-based DM, including Artificial Intelligence (AI) and sentiment analysis [18]. The KNN algorithm is a popular non-parametric supervised classifier that assigns class labels to unknown samples based on their nearest neighbors in a training set using distance metrics. While effective, efforts have extended KNN to enhance its accuracy. Neutrosophic KNN, which integrates NST to improve classification. NST computes a final membership $U = T + I - F$ for class labeling, similar to fuzzy KNN, and assigns T, I, and F memberships using a supervised Neutrosophic C-Means (NCM) algorithm. Extensive experiments on synthetic and real-world datasets validate the method's efficacy compared to traditional KNN, fuzzy KNN, and weighted KNN approaches [19].

NST, especially single-valued NS (SVNSs), improves handling of imprecision and uncertainty in medical applications. By integrating NST with fuzzy techniques, more effective solutions for medical image processing, DM, and information fusion are achieved. These methods have shown efficacy in de-noising, clustering, and segmenting medical images. Neutrosophic logic (NL) offers a framework for modeling vagueness and uncertainty, making it ideal for dealing with incomplete or inconsistent information. The importance of NS is emphasized in various medical applications and proposes a framework for leveraging NS to enhance medical image processing and diagnosis [20].

Developing decision support tools for healthcare facility maintenance and asset renewal is challenging due to uncertainties and subjectivity in DM. In order to reduce subjectivity, Neutrosophic Logic (NL), Multi-Attribute Utility Theory (MAUT), and the Analytic Network Process (ANP) were integrated to assess hospital building assets according to their criticality and performance inequalities. ML algorithms, such as DT, KNN, and NB, automate and standardize the prioritization process. Applying the model to healthcare institutions in Canada showed a notable improvement in prediction performance, outperforming the previous model by about 11%. With the help of this framework, hospital asset renewal will be prioritized in a way that is impartial, automated, and consistent, guaranteeing effective resource allocation [21].

A hybrid fuzzy Multi-Criteria Decision-Making (MCDM) methodology utilizing Single-Valued Neutrosophic Fuzzy Sets (SVNFS), Best-Worst Method (BWM), and VIKOR is proposed for assessing cybersecurity risks targeting Connected and Autonomous Vehicles (CAVs). Expert opinions on cyber-attack likelihood and severity are integrated to rank threat-agent categories, identifying insider attackers as posing the greatest risk. This approach addresses subjectivity in opinions and incorporates criteria weights based on the consequences of cyber-attacks, offering a flexible framework applicable beyond CAV cybersecurity to other complex decision contexts with uncertain data [22]. The decision support system (DSS) utilizes the CRITIC and CRADIS models within SVNS to prioritize hydrogen technologies for decarbonizing Iran's oil refining industry. It assesses blue, green, and low-carbon hydrogen technologies across environmental, economic, social, and reliability criteria, identifying solar renewable energy as optimal due to its clean energy conversion and geographical suitability. This study enhances DM under uncertainty, suggesting future research explore broader qualitative factors and stakeholder perspectives [23]. The research focuses on advancing strategic DM in the planning of historic pedestrian bridge remediation through an innovative algorithm based on Rough NS (RNS). This novel approach integrates Rough Sets (RS) and NS theories within a MCDM model. A key contribution is the introduction of a new RN symmetric cross entropy measure and its weighted variant, specifically designed to address uncertainties and the challenge of unknown criteria weights inherent in complex DM processes. By incorporating the VIKOR method, the model enables effective prioritization of bridge remediation efforts by providing robust and reliable rankings. Case studies validate the model's efficacy, demonstrating its practical utility compared to traditional methods in real-world scenarios [24].

A novel approach to Multi-Attribute Decision-making (MADM) was introduced by integrating RS, NS, and Grey System Theory (GST). The RN Grey Relational Analysis (RNGRA) method addresses indeterminate and inconsistent data using RNS, characterized by T, I, and F-membership degrees. Attribute weights are partially known and determined via an information entropy method. The Accumulated Geometric Operator (AGO) converts RN numbers into SVN numbers. The method employs the Hamming distance to calculate the NGR coefficient for assessing reliability and unreliability. Finally, a RN relational degree is established to rank alternatives, with a numerical example provided to demonstrate the method's effectiveness and applicability [25].

A RN TOPSIS method was presented for Multi-Attribute Group DM (MAGDM), effectively handling uncertainty, indeterminacy, and inconsistency in data. By evaluating alternatives and features using RNS, which are distinguished by T, I, and F-membership degrees, the method enhances the conventional TOPSIS technique. Individual opinions are aggregated into a group consensus using the RN weighted averaging operator. The distance between each alternative and the positive and negative Rough Neutrosophic (RN) ideal solutions is estimated using the Euclidean distance. A numerical example demonstrates the method's practicality and efficiency, making it applicable in pattern recognition, AI, and medical diagnosis [26].

3. Materials and Methods

The materials and methods used in the study are described in detail in this section.

3.1. Proposed methodology

The objective is to advance breast cancer prediction by integrating NS with ML algorithms. The WDBC dataset (I_D) was initially retrieved from the UCI ML Repository and then carefully preprocessed (O_{Dpp}) to ensure its quality and consistency.

$I_D = WDBC$ (1)

$O_{Dpp} = f_{pp}(I_D)$ (2)

The dataset was then transformed into an N-representation (O_N), where each data point was characterized not only by its specific attributes but also by degrees of T, I, and F. This approach offers a more nuanced depiction of uncertainty and variability inherent in medical datasets.

$O_N = f_{T,I,F}(O_{Dpp})$ (3)

Following the transformation, the N-dataset was split ($O_{N_{(s)}}$) into training and testing subsets.

$$O_{N_{(s)}} = s(f_{O_N}) = s\ (X_{train}, X_{test}, y_{train}, y_{test})\ (4)$$

The N- dataset was normalized ($O_{N_{(Nor)}}$) to a range of 0 to 1 using Min-Max Scaler.

$O_{N_{(Nor)}} = f_{Nor}\ (O_{N_{(s)}})$ (5)

The normalized N - training dataset was employed to train ML classifiers ($O_{N_{(ML)}}$) such as DT, RF, and AdaBoost. These classifiers were selected based on their capacity to handle intricate feature interactions and identify subtle patterns necessary for an accurate diagnosis of breast cancer.

$$O_{N_{(ML)}} = f_{ML}\ (O_{N_{(Nor)}}) = O_{ML}\ (DT_{O_{N_{(Nor)}}}, RF_{O_{N_{(Nor)}}}, AB_{O_{N_{(Nor)}}})\ (6)$$

The main performance metrics ($O_{N_{Metrics}}$) for the classifier, which include accuracy, precision, recall, and F1 score, were used to evaluate its performance and provide a thorough examination of its ability to predict.

$$\boldsymbol{O_{N_{Metrics}} = O_{N_{(ML)}}(Metrics_{A_{cc},P_r,R_c,F_1})}\ (7)$$

Finally, the study conducted a Comparative Analysis (O_{CA}) between the N-dataset and the original dataset to evaluate how integrating NS enhances the accuracy and reliability of breast cancer prediction models.

$O_{CA} = CA\ (I_D;\ O_N)$ (8)

The workflow is depicted in Figure 1.

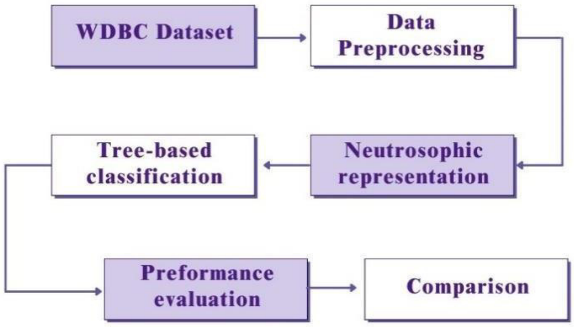

Fig. 1: NS-ML framework for Breast Cancer Prediction

3.2. Dataset description

A popular dataset for ML and statistical analysis, the WDBC dataset was collected by the University of Wisconsin Hospitals in Madison and is particularly useful for predicting and classifying breast cancer. Derived from digital photographs of breast tumors, it has attributes including radius, texture, area, perimeter, smoothness, compactness, concavity, concave spots, symmetry, and fractal dimension. With 212 instances classified as malignant (indicating presence of cancer) and 357 as benign (indicating absence of cancer), this dataset serves as a robust resource for developing accurate models that distinguish between malignant and benign breast cancer cases based on these comprehensive tumor characteristics.

3.3. Preprocessing of data

The breast cancer dataset was preprocessed for effective ML analysis. First, it was divided into features attributes describing tumor and a categorical target variable that distinguishes between benign and malignant cases. This categorical target was encoded into numerical values to make it easier for the models to understand. All features were normalized using Min-Max scaling in order to guarantee accurate predictions. This preprocessing process improved the machine learning models' ability to predict breast cancer diagnosis and maintained data consistency.

3.4. Neutrosophic Sets

Let X be a point or object-containing space, and let x be any element belongs to X. Three unique membership functions define a NS, A within X: T, I and F membership often referred as $T_A(X)$, $I_A(X)$, and $F_A(X)$. These functions assign real values within the interval [0, 1] indicating the degree to which x pertains to the subsets of T, I, and F, respectively:

$$T_A\ maps\ X\ to\ the\ interval\ ]\,0^-, 1^+[$$

$$I_A\ maps\ X\ to\ the\ interval\ ]\,0^-, 1^+[$$

$$F_A\ maps\ X\ to\ the\ interval\ ]\,0^-, 1^+[$$

The sum of $T_A(X)$, $I_A(X)$, $F_A(X)$ for every X, the falls ranges from 0 to 3. This flexibility enables NS to effectively represent and manage uncertainty, ambiguity, and contradiction within sets, making them valuable in contexts where handling incomplete or uncertain information is crucial [26].

3.5. Neutrosophic dataset formation

To address the uncertainties inherent in the WDBC dataset for binary classification, An N- dataset was introduced as an inclusive and generalized solution. This dataset goes beyond conventional and high-risk categories by incorporating a degree of neutrality. Thus, the N-dataset is defined as $< T_A, I_A, F_A >$, where each element of the set $X = \{x_1, x_2, \ldots, x_n\}$ is specified as follows:

$$\forall\, x\,(t, i, f) \in < T_A, I_A, F_A >$$

where, *t, i, and f*, respectively, are the real numbers for T, I, and F.

In order to better capture the uncertainty in the data, the model is being developed by adding N-components to the original dataset. The first step in the method is to compute the mean vectors for the training set as a whole (ρ^{all}), the positive class (ρ^+), and the negative class (ρ^-) which is provided in Eq. (1).

$$\rho^{all} = \sum_{k=1}^{n^{all}} x_k \; ; \; \rho^+ = \sum_{k=1}^{n^+} x_k \; ; \; \rho^- = \sum_{k=1}^{n^-} x_k \qquad (1)$$

The following Eqs. (2), (3) & (4) are u sed to compute the T, I, and F components for a given sample (x):

$$T = 1 - \frac{||x - \rho^+||}{\max\,(||X_{train} - \rho^+)} \qquad (2)$$

$$I = 1 - \frac{||x - \rho^{all}||}{\max\,(||X_{train} - \rho^{all})} \qquad (3)$$

$$F = 1 - \frac{||x - \rho^-||}{\max\,(||X_{train} - \rho^-)}$$

$$\frac{||x - \rho^-||}{\max\,(||X_{train} - \rho^-)}$$ (4)

These formulas are applied to each sample in the training and testing datasets, yielding features that measure the degrees of T, I, and F. Consequently, an N-dataset is produced, which enhances the ML algorithm's capacity to identify data with inherent uncertainty. This strategy considerably improves the classifier's performance in processing ambiguous and complex biomedical data by utilizing the capabilities of NST. The algorithm for the formation of O_N is provided below.

INPUT: The WDBC dataset (I_D)
OUTPUT: Neutrosophic dataset (O_N)

Step 1: Mean Vector Computation
Step 1.1: Positive class (ρ^+)

$$\rho^+ = \sum_{k=1}^{n^+} x_k$$

where $y_k = 1$ and n^+ is the no. of positive samples in X_{train}
Step 1.2: Negative class (ρ^-)

$$\rho^- = \sum_{k=1}^{n^-} x_k$$

where $y_k = 0$ and n^- is the no. of negative samples in X_{train}
Step 1.3: Overall mean (ρ^+)

$$\rho^+ = \sum_{k=1}^{n^+} x_k$$

where $y_k = 1$ and n^+ is the no. of positive samples in X_{train}

Step 2: Calculate the N-components (t, i, f) for each sample.

For each sample x_i in both X_{train} and X_{test}

$$T_i = 1 - \frac{||x_i - \rho^+||}{\max(||X_{train} - \rho^+)}$$

$$I_i = 1 - \frac{||x_i - \rho^{all}||}{\max(||X_{train} - \rho^{all})}$$

$$F_i = 1 - \frac{||x_i - \rho^-||}{\max(||X_{train} - \rho^-)}$$

Step 3: Group the N-components

$$f_{T_i, I_i, F_i} = o_N$$

3.6. *Classification algorithms*

3.6.1. Decision Tree

A hierarchical supervised learning model called a DT makes predictions by gradually dividing the data into groups based on the values of its features. It creates a tree-like structure with internal nodes representing decision points, branches indicating possible outcomes, and leaf nodes delivering final predictions. The tree is built recursively, starting from the root and progressing downwards. At each internal node, the algorithm chooses a feature and threshold that best separates the data, typically using metrics like Gini impurity or entropy. This process continues, refining predictions at each level, until reaching leaf nodes. In order to classify new data, branches are followed depending on feature values as they go from the root of the tree to the leaf. This approach effectively breaks complex decisions into simpler steps, making DT both powerful and interpretable for various prediction and classification tasks [28]. The Gini Impurity and Entropy is calculated using the following Eqs. (5) and (6)

Gini Impurity

$$G = 1 - \sum_{j=1}^{C} p_j{}^2 \tag{5}$$

Entropy:

$$H = -\sum_{j=1}^{C} p_j \log(p_j) \tag{6}$$

where p_j represents the probability of class j in the node, and C is the number of classes

3.6.2. Random Forest

An ensemble learning method called RF combines several DTs to reduce overfitting and improve prediction accuracy. Using a bootstrap sampling of the original dataset, it creates a large number of trees. In order to reduce correlations between the different trees and introduce variety through feature bagging, a random selection of features is chosen at each node for splitting. For classification tasks, the model aggregates predictions by majority voting, while for regression, it uses averaging. By leveraging the collective wisdom of many diverse trees, RF effectively reduces variance and enhances generalization. This method excels in handling complex, high-dimensional datasets and is widely adopted in ML for its robust performance, ability to capture non-linear relationships, and resilience against overfitting. Additionally, the model's capacity to provide feature importance rankings and handle missing values further contributes to its popularity across various domains [29].

3.6.3. Adaptive Boosting

Boosting is a ML technique that combines multiple weak learners to form a strong predictive model. AdaBoost, developed by Freund and Schapire [30], exemplifies this approach and remains widely used in various fields. In AdaBoost, weak learners are trained iteratively on weighted distributions of training data, with weights adjusted based on their performance. Each weak hypothesis receives a weight αt proportional to its accuracy, thereby minimizing errors. By assigning greater weight to more accurate learners, the final model aggregates these weak learners into a robust overall predictor.

3.7. Performance Evaluation

The proposed methods were compared using metrics including recall, accuracy, precision, and F1-score. The percentage of correctly identified subjects is called accuracy. The precision measure shows the percentage of successfully diagnosed positive subjects out of all predicted positive subjects. The recall metric assesses how well the model can detect positive samples. A fair assessment of the model's performance is provided by the F1-score, which is the harmonic mean of precision and recall.

4. Results

4.1. Experimental setup

Three distinct ML tree-based classifiers were utilized to predict breast cancer using the WDBC dataset. Prior to training, the dataset was transformed into an N-dataset to enhance the models' robustness in handling uncertainties. Several tree-based algorithms were trained on this N-dataset, and standard metrics were employed to evaluate the predictive performance of the methods. Furthermore, comparisons were made between results obtained from the N-dataset and the original dataset. All these experiments were efficiently executed on Google Colab, leveraging GPU acceleration to manage the computational complexity of ML tasks effectively.

4.2. Experimental results

The table provides a comparative analysis of ML algorithm performance using both the N-dataset and the original dataset. N-DT, N-RF, and N-AdaBoost denote models trained with the N-dataset using DT, RF, and AdaBoost algorithms respectively.

Table 2. Comparative evaluation of N-DT and DT model

Metrics	**N-DT**	**DT**
Accuracy	**93.86**	90.35
Precision	**91.66**	84.61
Recall	**93.62**	93.62
F1 score	**92.63**	88.88

Table 3. Comparative evaluation of N-RF and RF model

Metrics	N-RF	RF
Accuracy	**96.49**	96.49
Precision	**95.74**	97.77
Recall	**95.74**	93.61
F1 score	**95.74**	95.65

Table 4. Comparative evaluation of N-AdaBoost and AdaBoost model

Metrics	N-AdaBoost	AdaBoost
Accuracy	**99.12**	98.24
Precision	**100.00**	100.00
Recall	**97.87**	95.74
F1 score	**98.92**	97.82

These metrics provide a detailed comparison of ML algorithms using both the N-dataset and the original dataset. According to Table 2, N-DT demonstrate improvements in accuracy, precision, and F1 score, but slightly lower than RF and AdaBoost in precision and F1 score. Table 3 shows that N-RF exhibits notable enhancements in precision and F1 score, indicating improved capability to accurately classify positive instances while maintaining overall performance metrics. Table 4 highlights performance of N-AdaBoost, achieving 99.12% accuracy, perfect precision of 100.00%, 97.87% recall, and a F1 score of 98.92%. These results underscore AdaBoost's effectiveness in handling uncertainties inherent in biomedical data. In contrast, when using the original dataset, DT, RF, and AdaBoost shows lower precision and F1 scores compared to their performance with the N-dataset. The comparison of the results is displayed in Figure 2. Overall, integrating NS theory enhances the predictive capabilities of these algorithms for breast cancer diagnosis.

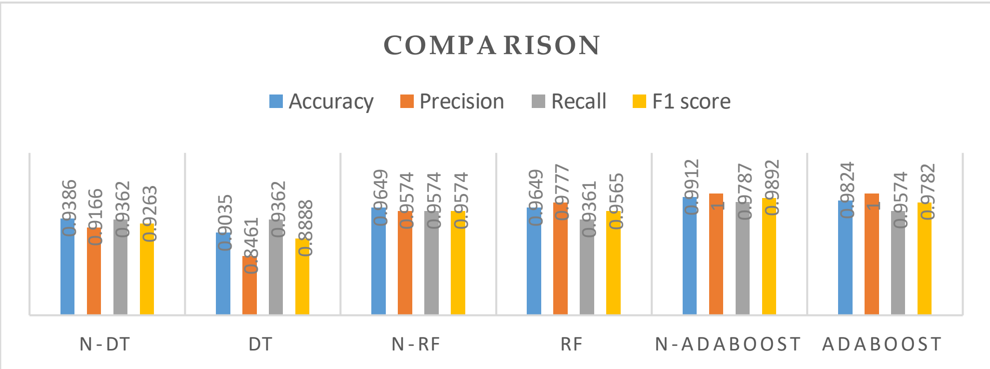

Figure 2. Comparison of N-tree based and conventional ML classifiers

5. Discussion

The proposed hybrid approach, NS-ML aimed at improving the prediction of breast cancer diagnosis. The research findings highlight that N-AdaBoost models, achieve superior accuracy and precision in detecting breast cancer. This underscores the effectiveness of integrating NST into ML models for biomedical applications, particularly in enhancing the reliability and accuracy of breast cancer diagnosis.

Conclusion

This research aims at advancing breast cancer prediction by integrating NS with ML techniques. Dataset was collected from WDBC dataset from the UCI ML Repository. The dataset was transformed into an N-representation, enriching each data point with degrees of T, I, and F to capture the complexities of medical datasets. Subsequently, the N-dataset was partitioned into training and testing subsets for training tree-based classifiers such as DT, RF, and AdaBoost. Predictive performance was measured using evaluation metrics, which demonstrated the models' capacity to recognize trends in breast cancer. Comparative analysis between the N-datasets and original datasets demonstrated improved performance metrics for DT, RF, and AdaBoost with the N-dataset. Notably, N-AdaBoost models demonstrated enhanced reliability of breast cancer diagnosis utilizing NS with scores of 99.12% accuracy, 100.00% precision, 97.87% recall, and an F1 score of 98.92%.

As a future work, the NS-ML approach can be expanded to incorporate Deep Learning (DL) and Neural Network (NN) architectures, aiming to enhance their capability in handling complex biomedical data. This integration will leverage NST to effectively model uncertainties and variability, thereby improving accuracy in tasks like image-based diagnostics and genomic analysis. Developing new neural network structures that integrate N-elements will be crucial for capturing intricate patterns in biomedical data.

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Aczel-Alsina power average aggregation operators of Singlevalued Neutrosophic under confdence levels and theirapplication in multiple attribute decision making

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Abstract. In decision making scenarios, dealing with imprecise information through extensions of fuzzy sets is crucial. Among these extensions, single valued neutrosophic set (SVNS) are especially effective at managing and interpreting such imprecise data. In the current study, decision makers confidence levels, derived from their familiarity with the assessed objects, are combined with the primary data within a neutrosophic framework. This paper focuses on developing innovative confidence single valued neutrosophic (SVN) aggregation operators (AO) that utilize the recently developed Aczel-Alsina (AA) operational laws and power AO (PAO) to capture the interrelationships among aggregated single valued neutrosophic numbers (SVNN). Specifically, it introduces new confidence SVNAA power average AO, namely, confidence SVNAA power weighted and ordered weighted average AO, which integrate the decision maker familiarity with the aggregated arguments. To evaluate the effectiveness of the proposed operators, we perform a comprehensive examination of their desirable properties. Also, we use these suggested operators to establish a innovative approach for SVN multi attribute decision making problems (MADM). A demonstrative example of strategic suppplier selection is provided to validate the proposed approach and highlight its practicality and effectiveness.

Keywords: Single valued neutrosophic sets; Aczel-Alsina; Power aggregation operator; Confidence levels; Average AO; Multiple attribute decision making.

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1. Introduction

In our complex world, managing uncertainty, which arises from navigating imprecision and incomplete data, is essential across various fields, including science, business, and decision making (DM). To address imprecise, contradictory, and incomplete information. Lotfi Zadeh [1] introduced fuzzy set (FS) theory with the membership degree (MD), which is denoted by $\Re$.

Atanassov [2] further developed FS theory with intuitionistic FS (IFS) theory, which incorporates both $\Re$ and non-membership degree (NMD) which is represented by $\hbar$. In IFS, the degree of hesitation is commonly computed as 1-$\Re - \hbar$ which potentially leading to the loss of some uncertainty. As an extension of IFS, Atanassov and Gargov [3] proposed interval valued intuitionistic FS (IVIFS) theory, allowing $\Re$ and $\hbar$ to range over intervals. Further, the generalization of IFS includes pythagorean FS (PyFS) [4], fermatean FS (FFS) [5]and q-rung orthopair FS (q-ROFS) [6]. Cuong et al. [7] introduced picture FS (PFS), by including an dependent neutral MD (∂) along with $\Re$ and $\hbar$. Additionally, the expansion of PFS encompasses spherical FS (SFS) [8], cubical FS (CFS) [9]and T-spherical FS (T-SFS) [10]. Building on IFS, Smarandache introduced the concept of neutrosophic sets (NS) by adding an independent degrees known as the indeterminacy MD (∂) alongside the truth MD ($\Re$) and falsity MD ($\hbar$) over a non-standard interval. Subsequently, Wang et al. [12] developed the single valued neutrosophic set (SVNS) based on the standard real interval [0,1], making it more suitable for computational ease in DM scenarios.

A variety of AO have emerged in the context of different fuzzy environments to enable the combination of evaluated objects, allowing for more effective DM and analysis. Liu and Chen [13] utilized the IF Heronian mean aggregation operator, which is based on Archimedean norms, to aggregate multiple decision matrices in a group DM problem, facilitating the evaluation of multiple perspectives and opinions. Shit and Ghorai [14] proposed FF Dombi AO to solve a MADM. Qiyas et al. [15] investigated Yager operators under PF environment. Ullah et al. [16] explored T-SFS in Hamacher AO.

In recent years, various operators have been designed to integrate confidence and ordering information into the preference components of aggregated data, allowing for more comprehensive and nuanced analysis of complex DM scenarios. In this manner, Dejian Yu [17] developed IF aggregation under confidence levels (CL). Tahir Mahmood et al. [18] established confidence level induced AO based on IF rough sets information. K Rahman et al. [19] proposed confidence based generalized IF AO for group DM. Harish Garg [20] introduced the induced PyF AO and its application to DM process. Manish Kumar [21] made a study on the confidence based q-ROF AO with numerical examples and discussed their applicability in a DM problem. Tanuja Puntaua and Komal [22] introduced the confdence PF AO and applied with a group DM problem. Muhammad Kamran et al. [23] developed CL AO based on SVN rough sets.

Inspired by the literature, the goals of this paper are presented below:

- The confidence an expert has in their assessment significantly impacts the evaluation process and the reliability of the DM outcome.
- There will be an opportunity to incorporate the experts confidence in the evaluated SVN aggregating objects during the DM process.

- Integrating the experts CLs into the exact information within the SVN environment is essential.
- It has been noted that no research has explored the integration of experts CLs with SVN Aczel-Alsina power aggregation operator.
- These objectives have inspired us to develop new confidence-based SVN operators.

The primary contributions of our study are summarized below:

- We propose utilizing SVNAAP weighted and ordered weighted average AO combined with decision makers CLs.
- The essential characteristics of the AO are analyzed.
- We have created SVNMADM approach based on the introduced operators.
- This DM approach has been implemented in the supplier selection process.
- The outcomes are then compared with those obtained from existing SVN average operators documented in the literature.

2. **Preliminaries**

Here, we review some basic definitions related to SVNS within the context of a universal set $\ddot{X}'$.

Definition 2.1. *[12] A SVNS $\breve{\mathbb{C}}$, on the universal set $\ddot{X}'$ is of the form $\breve{\mathbb{C}} = \{\langle \phi, \Re_{\breve{\mathbb{C}}}(\phi), \partial_{\breve{\mathbb{C}}}(\phi), \hbar_{\breve{\mathbb{C}}}(\phi) | \phi \in \ddot{X}' \rangle\}$ where $\Re_{\breve{\mathbb{C}}} : \ddot{X}' \to [0,1]$ represent the truth membership function, $\partial_{\breve{\mathbb{C}}} : \ddot{X}' \to [0,1]$ represent the indeterminacy membership function and $\hbar_{\breve{\mathbb{C}}} : \ddot{X}' \to [0,1]$ represent the falsity membership function and $\Re_{\breve{\mathbb{C}}}(\phi), \partial_{\breve{\mathbb{C}}}(\phi), \hbar_{\mathcal{S}}(\phi) \in [0,1]$ such that $0 \leq \Re_{\breve{\mathbb{C}}}(\phi) + \partial_{\breve{\mathbb{C}}}(\phi) + \hbar_{\breve{\mathbb{C}}}(\phi) \leq 3$. Now we denote the triplets $\breve{\mathbb{C}} = (\Re_{\breve{\mathbb{C}}}, \partial_{\breve{\mathbb{C}}}, \hbar_{\breve{\mathbb{C}}})$ as an single valued neutrosophic numbers (SVNN) for simplicity.*

Definition 2.2. *[24] Let $\breve{\mathbb{C}} = \langle \Re_{\breve{\mathbb{C}}}, \partial_{\breve{\mathbb{C}}}, \hbar_{\breve{\mathbb{C}}} \rangle \in \breve{\mathbb{C}}$ be a SVNN, then the score function $\ddot{\mathbb{M}}$ of $\breve{\mathbb{C}}$ is defined as*

$$\ddot{\mathbb{M}}(\breve{\mathbb{C}}) = \frac{2 + \Re_{\breve{\mathbb{C}}} - \partial_{\breve{\mathbb{C}}} - \hbar_{\breve{\mathbb{C}}}}{3} \in [0,1] \tag{1}$$

$$\ddot{\mathbb{M}}(\breve{\mathbb{C}}) = \tfrac{2+0.9-0.7-0.8}{3} = 0.467 \in [0,1]$$

Definition 2.3. *[24] Let $\breve{\mathbb{C}} = \langle \Re_{\breve{\mathbb{C}}}, \partial_{\breve{\mathbb{C}}}, \hbar_{\breve{\mathbb{C}}} \rangle \in \breve{\mathbb{C}}$ be a SVNN, then the accuracy function $\ddot{\mathbb{M}}$ of $\breve{\mathbb{C}}$ is defined as*

$$\ddot{\mathbb{L}}(\breve{\mathbb{C}}) = \Re_{\breve{\mathbb{C}}} - \hbar_{\breve{\mathbb{C}}} \in [-1,1] \tag{2}$$

Definition 2.4. *[24] Let $\breve{\mathbb{C}}_1 = \langle \Re_{\breve{\mathbb{C}}_1}, \partial_{\breve{\mathbb{C}}_1}, \hbar_{\breve{\mathbb{C}}_1} \rangle$ and $\breve{\mathbb{C}}_2 = \langle \Re_{\breve{\mathbb{C}}_2}, \partial_{\breve{\mathbb{C}}_2}, \hbar_{\breve{\mathbb{C}}_2} \rangle$ be any two SVNNs and $\ddot{\mathbb{M}}(\breve{\mathbb{C}}_j)$ and $\ddot{\mathbb{L}}(\breve{\mathbb{C}}_j)$ for $j = 1, 2$ be their respective score and accuracy values, then we arrive at the following results.*

(1) *If $\ddot{\mathbb{M}}(\breve{\mathbb{C}}_1) > \ddot{\mathbb{M}}(\breve{\mathbb{C}}_2)$, then $\breve{\mathbb{C}}_1 \succ \breve{\mathbb{C}}_2$;*

(2) *If* $\ddot{\mathbb{M}}(\breve{\mathbb{C}}_1) < \ddot{\mathbb{M}}(\breve{\mathbb{C}}_2)$, *then* $\breve{\mathbb{C}}_1 \prec \breve{\mathbb{C}}_2$;

(3) *If* $\ddot{\mathbb{M}}(\breve{\mathbb{C}}_1) = \ddot{\mathbb{M}}(\breve{\mathbb{C}}_2)$ *then*

- *If* $\ddot{\mathbb{L}}(\breve{\mathbb{C}}_1) > \ddot{\mathbb{L}}(\breve{\mathbb{C}}_2)$, *then* $\breve{\mathbb{C}}_1 \succ \breve{\mathbb{C}}_2$;
- *If* $\ddot{\mathbb{L}}(\breve{\mathbb{C}}_1) < \ddot{\mathbb{L}}(\breve{\mathbb{C}}_2)$, *then* $\breve{\mathbb{C}}_1 \prec \breve{\mathbb{C}}_2$;
- *If* $\ddot{\mathbb{L}}(\breve{\mathbb{C}}_1) = \ddot{\mathbb{L}}(\breve{\mathbb{C}}_2)$, *then* $\breve{\mathbb{C}}_1 \sim \breve{\mathbb{C}}_2$.

Definition 2.5. *[25] Let* $\breve{\mathbb{C}}_1 = \langle \Re_{\breve{\mathbb{C}}_1}, \partial_{\breve{\mathbb{C}}_1}, \hbar_{\breve{\mathbb{C}}_1} \rangle$ *and* $\breve{\mathbb{C}}_2 = \langle \Re_{\breve{\mathbb{C}}_2}, \partial_{\breve{\mathbb{C}}_2}, \hbar_{\breve{\mathbb{C}}_2} \rangle$ *be any two SVNNs, then the Euclidean distance between them is defined as follows:*

$$\mathfrak{D}(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2) = \sqrt{\frac{1}{3}\{|\Re_{\breve{\mathbb{C}}_1} - \Re_{\breve{\mathbb{C}}_2}|^2 + |\partial_{\breve{\mathbb{C}}_1} - \partial_{\breve{\mathbb{C}}_2}|^2 + |\hbar_{\breve{\mathbb{C}}_1} - \hbar_{\breve{\mathbb{C}}_2}|^2\}} \quad (3)$$

Definition 2.6. *[26] A PAO of dimension n is mapping PAO:* $Q^\rho \to Q$, *according to the following formula.*

$$PAO(\breve{t}_1, \breve{t}_2, \ldots, \breve{t}_\rho) = \frac{\sum_{j=1}^{\rho}(1 + \ddot{E}(\breve{t}_j))\breve{t}_j}{\sum_{j=1}^{\rho} 1 + \ddot{E}(\breve{t}_j)} \quad (4)$$

where $\ddot{E}(\breve{t}_j) = \sum_{h=1, h\neq j}^{\rho} supp\ (\breve{t}_j, \breve{t}_h)$ *and* $(j = 1, 2, \ldots, \rho; h = 1, 2, \ldots, r)$ *which provides the relationship between* $\breve{t}_j$ *and* $\breve{t}_h$ *which must follow the conditions:*

(1) $supp(\breve{t}_j, \breve{t}_h) \in [0, 1]$;

(2) $supp(\breve{t}_j, \breve{t}_h) = supp(\breve{t}_h, \breve{t}_j)$;

(3) $supp(\breve{t}_j, \breve{t}_h) \geq supp(\breve{t}_s, \breve{t}_t)$ *if* $|\breve{t}_j - \breve{t}_h| < |\breve{t}_s - \breve{t}_t|$.

Definition 2.7. *[27] A TN is a function* $\mathfrak{E} : [0, 1]\times$ *[0,1]* $\to$ *[0,1] that fulfills the properties of symmetry, monotonicity, and associativity, and includes an identity element, i.e., for all* $\breve{\ddot{p}}, \breve{\ddot{q}}, \breve{\ddot{n}} \in$ *[0,1]:*

(1) $\mathfrak{E}(\breve{\ddot{p}}, \breve{\ddot{q}}) = \mathfrak{E}(\breve{\ddot{q}}, \breve{\ddot{p}})$;

(2) $\mathfrak{E}(\breve{\ddot{p}}, \breve{\ddot{q}}) \leq \mathfrak{E}(\breve{\ddot{p}}, \breve{\ddot{n}})$ *if* $\breve{\ddot{q}} < \breve{\ddot{n}}$;

(3) $\mathfrak{E}(\breve{\ddot{p}}, \mathfrak{E}(\breve{\ddot{q}}, \breve{\ddot{n}})) = \mathfrak{E}(\mathfrak{E}(\breve{\ddot{p}}, \breve{\ddot{q}}), \breve{\ddot{n}})$;

(4) $\mathfrak{E}(\breve{\ddot{p}}, 1) = \breve{\ddot{p}}$.

The following is a list of some well-known TNs.

(1) *Minimum TN:* $\mathfrak{E}_M(\breve{\ddot{p}}, \breve{\ddot{q}}) = min(\breve{\ddot{p}}, \breve{\ddot{q}})$;

(2) *Product TN:* $\mathfrak{E}_P(\breve{\ddot{p}}, \breve{\ddot{q}}) = \breve{\ddot{p}} \cdot \breve{\ddot{q}}$;

(3) *Lukasiewicz TN:* $\mathfrak{E}_L(\breve{\ddot{p}}, \breve{\ddot{q}}) = max(\breve{\ddot{p}} + \breve{\ddot{q}} - 1, 0)$;

(4) *Drastic TN:* $\mathfrak{E}_D(\breve{\ddot{p}}, \breve{\ddot{q}}) = \begin{pmatrix} \breve{\ddot{p}}, & if\ \breve{\ddot{q}}=1 \\ \breve{\ddot{q}}, & if\ \breve{\ddot{p}}=1 \\ 0, & otherwise \end{pmatrix}$;

(5) *Nilpotent minimum:*

$$\mathfrak{E}_{nM}(\breve{\ddot{p}}, \breve{\ddot{q}}) = \begin{pmatrix} min(\breve{\ddot{p}}, \breve{\ddot{q}}) & if\ \breve{\ddot{p}} + \breve{\ddot{q}} > 1 \\ 0 & otherwise \end{pmatrix}.$$

Definition 2.8. *[27] A TCN is a function $\mathfrak{O} : [0,1]\times [0,1] \rightarrow [0,1]$ that fulfills the properties of symmetry, monotonicity, and associativity, and includes an identity element, i.e., for all $\breve{\ddot{p}}, \breve{\ddot{q}}, \breve{\ddot{n}} \in [0,1]$:*

1. $\mathfrak{O}(\breve{\ddot{p}}, \breve{\ddot{q}}) = \mathfrak{O}(\breve{\ddot{q}}, \breve{\ddot{p}})$;
2. $\mathfrak{O}(\breve{\ddot{p}}, \breve{\ddot{q}}) \leq \mathfrak{O}(\breve{\ddot{p}}, \breve{\ddot{n}})$ *if* $\breve{\ddot{q}} < \breve{\ddot{n}}$;
3. $\mathfrak{O}(\breve{\ddot{p}}, \mathfrak{O}(\breve{\ddot{q}}, \breve{\ddot{n}})) = \mathfrak{O}(\mathfrak{O}(\breve{\ddot{p}}, \breve{\ddot{q}}), \breve{\ddot{n}})$;
4. $\mathfrak{O}(\breve{\ddot{p}}, 0) = \breve{\ddot{p}}$.

The following is a list of some well-known TCNs.

1. *Maximum TCN:* $\mathfrak{O}_M(\breve{\ddot{p}}, \breve{\ddot{q}}) = max(\breve{\ddot{p}}, \breve{\ddot{q}})$;
2. *Probabilistic sum TCN:* $\mathfrak{O}_P(\breve{\ddot{p}}, \breve{\ddot{q}}) = \breve{\ddot{p}} + \breve{\ddot{q}} - \breve{\ddot{p}} \cdot \breve{\ddot{q}}$;
3. *Bounded sum:* $\mathfrak{O}_L(\breve{\ddot{p}}, \breve{\ddot{q}}) = min(\breve{\ddot{p}} + \breve{\ddot{q}}, 1)$;
4. *Drastic TCN:* $\mathfrak{O}_D(\breve{\ddot{p}}, \breve{\ddot{q}}) = \begin{pmatrix} \breve{\ddot{p}}, & if\ \breve{\ddot{q}}=0 \\ \breve{\ddot{q}}, & if\ \breve{\ddot{p}}=0 \\ 1, & otherwise \end{pmatrix}$;
5. *Nilpotent minimum:*
   $\mathfrak{O}_{nM}(\breve{\ddot{p}}, \breve{\ddot{q}}) = \begin{pmatrix} max(\breve{\ddot{p}}, \breve{\ddot{q}}) & if\ \breve{\ddot{p}} + \breve{\ddot{q}} < 1 \\ 1 & otherwise. \end{pmatrix}$.

Definition 2.9. *[28] AA in early 1982 introduced the concepts of TN and TCN classes for functional equations. The AATN can be defined as follows:*

$$\mathfrak{E}_A^{\phi}(\breve{\ddot{p}}, \breve{\ddot{q}}) = \begin{cases} \mathfrak{E}_D(\breve{\ddot{p}}, \breve{\ddot{q}}), & if\ \alpha = 0 \\ min(\breve{\ddot{p}}, \breve{\ddot{q}}), & if\ \alpha = \infty \\ e^{-\{(-Ln\breve{\ddot{p}})^{\alpha} + (-Ln\breve{\ddot{q}})^{\alpha}\}^{\frac{1}{\alpha}}}, & otherwise. \end{cases}$$

and the AATCN can be defined as follows:

$$\mathfrak{O}_A^{\phi}(\breve{\ddot{p}}, \breve{\ddot{q}}) = \begin{cases} \mathfrak{O}_D(\breve{\ddot{p}}, \breve{\ddot{q}}); & if\ \alpha = 0 \\ max(\breve{\ddot{p}}, \breve{\ddot{q}}); & if\ \alpha = \infty \\ e^{-\{(-Ln(1-\breve{\ddot{p}}))^{\alpha} + (-Ln(1-\breve{\ddot{q}}))^{\alpha}\}^{\frac{1}{\alpha}}}, & otherwise. \end{cases}$$

such that $\mathfrak{E}_A^0 = \mathfrak{E}_D, \mathfrak{E}_A^1 = \mathfrak{E}_P, \mathfrak{E}_A^{\infty} = min, \mathfrak{O}_A^0 = \mathfrak{O}_D, \mathfrak{O}_A^1 = \mathfrak{O}_P, \mathfrak{O}_A^{\infty} = max$. *The TN* $\mathfrak{E}_A^{\alpha}$ *and TCN* $\mathfrak{O}_A^{\alpha}$ *are combained to one another for each* $\alpha \in [0, \infty]$. *The class of AATN is strictly increasing, and the class of AATCN is strictly decreasing. The following is the AATN and AATCN operational laws in connection with SVN theory.*

Definition 2.10. *[24] Let* $\breve{\mathbb{C}}_j = \langle \Re_{\breve{\mathbb{C}}_j}, \partial_{\breve{\mathbb{C}}_j}, \hbar_{\breve{\mathbb{C}}_j} \rangle$, $j = 1, 2$ *be two SVNNs,* $\alpha \geq 1$ *and* $K > 0$. *Then, the AATN and AATCN operations of SVNN are defined as:*

1. $\breve{\mathbb{C}}_1 \oplus \breve{\mathbb{C}}_2 = \langle 1 - e^{-\{(-Ln(1-\Re_{\breve{\mathbb{C}}_1}))^{\alpha} + (-Ln(1-\Re_{\breve{\mathbb{C}}_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, e^{-\{(-Ln(\hbar_{\breve{\mathbb{C}}_1}))^{\alpha} + (-Ln(\hbar_{\breve{\mathbb{C}}_2}))^{\alpha}\}^{\frac{1}{\alpha}}}, e^{-\{(-Ln(\partial_{\breve{\mathbb{C}}_1}))^{\alpha} + (-Ln(\partial_{\breve{\mathbb{C}}_2})))^{\alpha}\}^{\frac{1}{\alpha}}} \rangle$;

(2) $\breve{\mathbb{C}}_1 \otimes \breve{\mathbb{C}}_2 = \langle e^{-\{(-Ln(\Re_{\breve{\mathbb{C}}_1}))^\alpha + (-Ln(\Re_{\breve{\mathbb{C}}_2}))^\alpha\}^{\frac{1}{\alpha}}}, 1 - e^{-\{(-Ln(1-\hbar_{\breve{\mathbb{C}}_1}))^\alpha + (-Ln(1-\hbar_{\breve{\mathbb{C}}_2}))^\alpha\}^{\frac{1}{\alpha}}}, 1 - e^{-\{(-Ln(1-\partial_{\breve{\mathbb{C}}_1}))^\alpha + (-Ln(1-\partial_{\breve{\mathbb{C}}_2})))^\alpha\}^{\frac{1}{\alpha}}} \rangle$;

(3) $K \cdot \breve{\mathbb{C}}_1 = \langle 1 - e^{-\{K(-Ln(1-\Re_{\breve{\mathbb{C}}_1}))^\alpha\}^{\frac{1}{\alpha}}}, e^{-\{K(-Ln(\hbar_{\breve{\mathbb{C}}_1}))^\alpha\}^{\frac{1}{\alpha}}}, e^{-\{K(-Ln(\partial_{\breve{\mathbb{C}}_1}))^\alpha\}^{\frac{1}{\alpha}}} \rangle$;

(4) $\breve{\mathbb{C}}_1^K = \langle e^{-\{K(-Ln(\Re_{\breve{\mathbb{C}}_1}))^\alpha\}^{\frac{1}{\alpha}}}, 1 - e^{-\{K(-Ln(1-\hbar_{\breve{\mathbb{C}}_1}))^\alpha\}^{\frac{1}{\alpha}}}, 1 - e^{-\{K(-Ln(1-\partial_{\breve{\mathbb{C}}_1}))^\alpha\}^{\frac{1}{\alpha}}} \rangle$.

Definition 2.11. *[24] Let* $\breve{\mathbb{C}} = \{\langle \phi, \Re_{\breve{\mathbb{C}}}(\phi), \partial_{\breve{\mathbb{C}}}(\phi), \hbar_{\breve{\mathbb{C}}}(\phi) | \phi \in X \rangle\}$ $\breve{\mathbb{C}}_1 = \{\langle \phi, \Re_{\breve{\mathbb{C}}_1}(\phi), \partial_{\breve{\mathbb{C}}_1}(\phi), \hbar_{\breve{\mathbb{C}}_1}(\phi) | \phi \in X \rangle\}$ *and* $\breve{\mathbb{C}}_2 = \{\langle \phi, \Re_{\breve{\mathbb{C}}_2}(\phi), \partial_{\breve{\mathbb{C}}_2}(\phi), \hbar_{\breve{\mathbb{C}}_2}(\phi) | \phi \in X \rangle\}$ *be any three SVNS, and their set operators are defined as*

(1) $\breve{\mathbb{C}}_1 \subseteq \breve{\mathbb{C}}_2 \Leftrightarrow \Re_{\breve{\mathbb{C}}_1}(\phi) \leq \Re_{\breve{\mathbb{C}}_2}(\phi), \partial_{\breve{\mathbb{C}}_1}(\phi) \leq \partial_{\breve{\mathbb{C}}_2}(\phi)$ *and* $\hbar_{\breve{\mathbb{C}}_1}(\phi) \geq \hbar_{\breve{\mathbb{C}}_2}(\phi) \forall \phi \in X$;

(2) $\breve{\mathbb{C}}_1 \cup \breve{\mathbb{C}}_2 = \{\langle \phi, \{\mathfrak{O}_A\{\Re_{\breve{\mathbb{C}}_1}(\phi), \Re_{\breve{\mathbb{C}}_2}(\phi)\}\}, \{\mathfrak{E}_A\{\partial_{\breve{\mathbb{C}}_1}(\phi), \partial_{\breve{\mathbb{C}}_2}(\phi)\}\}, \{\mathfrak{E}_A\{\hbar_{\breve{\mathbb{C}}_1}(\phi), \hbar_{\breve{\mathbb{C}}_2}\}(\phi)\} | \phi \in X \rangle\}$;

(3) $\breve{\mathbb{C}}_1 \cap \breve{\mathbb{C}}_2 = \{\langle \phi, \{\mathfrak{E}_A\{\Re_{\breve{\mathbb{C}}_1}(\phi), \Re_{\breve{\mathbb{C}}_2}(\phi)\}\}, \{\mathfrak{O}_A\{\partial_{\breve{\mathbb{C}}_1}(\phi), \partial_{\breve{\mathbb{C}}_2}(\phi)\}\}, \{\mathfrak{O}_A\{\hbar_{\breve{\mathbb{C}}_1}(\phi), \hbar_{\breve{\mathbb{C}}_2}(\phi)\}\} | \phi \in X\}\rangle\}$;

(4) $\breve{\mathbb{C}}^c = \{\langle \phi, \hbar_{\breve{\mathbb{C}}}(\phi), \partial_{\breve{\mathbb{C}}}(\phi), \Re_{\breve{\mathbb{C}}}(\phi) | \phi \in X \rangle\}$.

Theorem 2.1. *Let* $\breve{\mathbb{C}}_1 = \langle \Re_{\breve{\mathbb{C}}_1}, \partial_{\breve{\mathbb{C}}_1}, \hbar_{\breve{\mathbb{C}}_1} \rangle$ *and* $\breve{\mathbb{C}}_2 = \langle \Re_{\breve{\mathbb{C}}_2}, \partial_{\breve{\mathbb{C}}_2}, \hbar_{\breve{\mathbb{C}}_2} \rangle$ *be any two SVNNs. Then,*

(1) $\breve{\mathbb{C}}_1 \oplus \breve{\mathbb{C}}_2 = \breve{\mathbb{C}}_2 \oplus \breve{\mathbb{C}}_1$,
(2) $\breve{\mathbb{C}}_1 \otimes \breve{\mathbb{C}}_2 = \breve{\mathbb{C}}_2 \oplus \breve{\mathbb{C}}_1$,
(3) $\Lambda(\breve{\mathbb{C}}_1 \oplus \breve{\mathbb{C}}_2) = \Lambda\breve{\mathbb{C}}_1 \oplus \Lambda\breve{\mathbb{C}}_2, \Lambda \geq 0$,
(4) $\Lambda_1\breve{\mathbb{C}}_1 \oplus \Lambda_2\breve{\mathbb{C}}_1 = (\Lambda_1 + \Lambda_2)\breve{\mathbb{C}}_1, \Lambda_1, \Lambda_2 \geq 0$,
(5) $\breve{\mathbb{C}}_1^\Lambda \otimes \breve{\mathbb{C}}_2^\Lambda = (\breve{\mathbb{C}}_1 \otimes \breve{\mathbb{C}}_2)^\Lambda, \Lambda \geq 0$,
(6) $\breve{\mathbb{C}}_1^{\Lambda_1} \otimes \breve{\mathbb{C}}_2^{\Lambda_2} = (\breve{\mathbb{C}}_1)^(\Lambda_1 + \Lambda_2), \Lambda_1, \Lambda_2 \geq 0$.

Proof. Straightforward. □

3. Proposed Confidence SVN Aczel-Alsina power aggregation operator

The current section defines a series of SVN Aczel-Alsina power averaging operators that incorporate CLs with the evaluated SVNNs.

3.1. *Confidence SVN Aczel-Alsina power average aggregation operator*

In this part, we built the confidence SVN weighted and ordered weighted Aczel-Alsina power averaging AO. Additionally, we investigate several fundamental aspects of these proposed operators.

By employing the fundamental operations of AA aggregation tools, we derived appropriate methodologies, including CSVNAAPWAAO, with reliable properties while considering SVNNs. Additionally, we applied a weighted suppport degree throughout our article, using the following equation: $\mathcal{Z}_j = \frac{\tau_j(1+\breve{U}(\breve{\mathbb{C}}_j))}{\sum_{j=1}^{\rho}\tau_j(1+\breve{U}(\breve{\mathbb{C}}_j))}$ where the suppport of $\breve{\mathbb{C}}_j$ is denoted by $\breve{U}(\breve{\mathbb{C}}_j) = \sum_{h=1,h\neq j}^{\rho} \text{supp}(\breve{\mathbb{C}}_j, \breve{\mathbb{C}}_h),\ j = 1,2,\ldots,\rho, h = 1,2,\ldots,r$ and the associated weight vector of $\breve{\mathbb{C}}_j$ is $\tau = (\tau_1, \tau_2, \ldots, \tau_\rho,)^T, j = 1,2,\ldots,\rho, \tau_j > 0, and \sum_{j=1}^{\rho}\tau_j = 1$.

Definition 3.1. *Let $\breve{\mathbb{C}}_j = (\Re_j, \hbar_j, \partial_j)(j = 1,2,\ldots,\rho)$ be a set of SVNNs and $\eta_j$be the CL of $\breve{\mathbb{C}}_j$ with $0 \leq \eta_j \leq 1$. Let $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_\rho)^T$ be the weight vectors for SVNNs with the condition $\sum_{j=1}^{\rho}\mathcal{Z}_j = 1$. Then, the mapping CSVNAAPWAAO: $b^\rho \to b$ operator is given as follows: CSVNAAPWAAO $\{(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)\} = \oplus_{j=1}^{\rho}\mathcal{Z}_j(\eta_j, \breve{\mathbb{C}}_j)$*

$$= \mathcal{Z}_1(\eta_1, \breve{\mathbb{C}}_1) \oplus \mathcal{Z}_2(\eta_2, \breve{\mathbb{C}}_2) \oplus \cdots \oplus \mathcal{Z}_\rho(\eta_\rho, \breve{\mathbb{C}}_\rho). \tag{5}$$

Theorem 3.1. *The aggregated value of the SVNNs $\breve{\mathbb{C}}_j$ for $j = 1,2,\ldots,\rho$ with respect to the weight vector $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_\rho)^T$ and the CL η_j such that $0 \leq \eta_j \leq 1$ obtained using the CSVNAAPWAAO Equation 5 is also a SVNN and is given by CSVNAAPWAAO $\{(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)\} =$*

$$= \langle 1 - e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln(1-\Re_{\breve{\mathbb{C}}_j}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln\hbar_{\breve{\mathbb{C}}_j})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln\partial_{\breve{\mathbb{C}}_j})^\alpha)\}^{\frac{1}{\alpha}}} \rangle \tag{6}$$

Proof. By mathematical induction the proof as follows:

(1) For $\rho = 2$, we have CSVNAAPWAAO $((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2)) = \mathcal{Z}(\breve{\mathbb{C}}_1, \eta_1) \oplus \mathcal{Z}(\breve{\mathbb{C}}_2, \eta_2)$. By operational laws, we get $\mathcal{Z}_1(\breve{\mathbb{C}}_1, \eta_1) = \langle 1 - e^{-\{(\eta_1 \mathcal{Z}_1(-Ln(1-\Re_{\breve{\mathbb{C}}_1}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_1 \mathcal{Z}_1(-Ln\hbar_{\breve{\mathbb{C}}_1})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_1 \mathcal{Z}_1(-Ln\partial_{\breve{\mathbb{C}}_1})^\alpha)\}^{\frac{1}{\alpha}}} \rangle$. analogously, for $\mathcal{Z}_2(\breve{\mathbb{C}}_2, \eta_2) = \langle 1 - e^{-\{(\eta_2 \mathcal{Z}_2(-Ln(1-\Re_{\breve{\mathbb{C}}_2}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_2 \mathcal{Z}_2(-Ln\hbar_{\breve{\mathbb{C}}_2})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_2 \mathcal{Z}_2(-Ln\partial_{\breve{\mathbb{C}}_2})^\alpha)\}^{\frac{1}{\alpha}}} \rangle$. CSVNAAPWAAO $((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2)) = \mathcal{Z}(\breve{\mathbb{C}}_1, \eta_1) \oplus \mathcal{Z}(\breve{\mathbb{C}}_2, \eta_2) = \langle 1 - e^{-\{(\eta_1 \mathcal{Z}_1(-Ln(1-\Re_{\breve{\mathbb{C}}_1}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_1 \mathcal{Z}_1(-Ln\hbar_{\breve{\mathbb{C}}_1})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_1 \mathcal{Z}_1(-Ln\partial_{\breve{\mathbb{C}}_1})^\alpha)\}^{\frac{1}{\alpha}}} \rangle \oplus \langle 1 - e^{-\{(\eta_2 \mathcal{Z}_2(-Ln(1-\Re_{\breve{\mathbb{C}}_2}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_2 \mathcal{Z}_2(-Ln\hbar_{\breve{\mathbb{C}}_2})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_2 \mathcal{Z}_2(-Ln\partial_{\breve{\mathbb{C}}_2})^\alpha)\}^{\frac{1}{\alpha}}} \rangle = \langle 1 - e^{-\{(\eta_1 \mathcal{Z}_1(-Ln(1-\Re_{\breve{\mathbb{C}}_1}))^\alpha) + (\eta_2 \mathcal{Z}_2(-Ln(1-\Re_{\breve{\mathbb{C}}_2}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_1 \mathcal{Z}_1(-Ln\hbar_{\breve{\mathbb{C}}_1})^\alpha) + (\eta_2 \mathcal{Z}_2(-Ln\hbar_{\breve{\mathbb{C}}_2})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{(\eta_1 \mathcal{Z}_1(-Ln\partial_{\breve{\mathbb{C}}_1})^\alpha) + (\eta_2 \mathcal{Z}_2(-Ln\partial_{\breve{\mathbb{C}}_2})^\alpha)\}^{\frac{1}{\alpha}}} \rangle = \langle 1 - e^{-\{\sum_{j=1}^{2}(\eta_j \mathcal{Z}_j(-Ln(1-\Re_{\breve{\mathbb{C}}_j}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{2}(\eta_j \mathcal{Z}_j(-Ln\hbar_{\breve{\mathbb{C}}_j})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{2}(\eta_j \mathcal{Z}_j(-Ln\partial_{\breve{\mathbb{C}}_j})^\alpha)\}^{\frac{1}{\alpha}}} \rangle$. Hence, this true for j=2.

(2) Now, supppose that this will be true for j=k. Then we have the following equation: CSVNAAPWAAO $\{(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_k, \eta_k)\} = \langle 1 - e^{-\{\sum_{j=1}^{k}(\eta_k \mathcal{Z}_k(-Ln(1-\Re_{\breve{\mathbb{C}}_k}))^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{k}(\eta_k \mathcal{Z}_k(-Ln\hbar_{\breve{\mathbb{C}}_k})^\alpha)\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{k}(\eta_k \mathcal{Z}_k(-Ln\partial_{\breve{\mathbb{C}}_k})^\alpha)\}^{\frac{1}{\alpha}}} \rangle$.

Now, we have to show that it also hods for j=k+1 as follows CSVNAAPWAAO $\{(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_k, \eta_k), (\breve{\mathbb{C}}_{k+1}, \eta_{k+1})\} = \langle 1 -$

$e^{-\{\sum_{j=1}^{k}(\eta_k \mathcal{Z}_k(-Ln(1-\Re_{\breve{\mathbb{C}}_k}))^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{k}(\eta_k \mathcal{Z}_k(-Ln\hbar_{\breve{\mathbb{C}}_k})^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{k}(\eta_k \mathcal{Z}_k(-Ln\partial_{\breve{\mathbb{C}}_k})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle \oplus$
$\langle 1 - e^{-\{\sum_{j=1}^{k+1}(\eta_{k+1} \mathcal{Z}_{k+1}(-Ln(1-\Re_{\breve{\mathbb{C}}_{k+1}}))^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{k+1}(\eta_{k+1} \mathcal{Z}_{k+1}(-Ln\hbar_{\breve{\mathbb{C}}_{k+1}})^{\alpha})\}^{\frac{1}{\alpha}}},$
$e^{-\{\sum_{j=1}^{k+1} k(\eta_{k+1} \mathcal{Z}_{k+1}(-Ln\partial_{\breve{\mathbb{C}}_{k+1}})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle = \langle 1 - e^{-\{\sum_{j=1}^{k+1}(\eta_j \mathcal{Z}_j(-Ln(1-\Re_{\breve{\mathbb{C}}_j}))^{\alpha})\}^{\frac{1}{\alpha}}},$
$e^{-\{\sum_{j=1}^{k+1}(\eta_j \mathcal{Z}_j(-Ln\hbar_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{k+1}(\eta_j \mathcal{Z}_j(-Ln\partial_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle$

which is true for j=k+1. □

Example 3.1. Supppose $\breve{\mathbb{C}}_1 = ((0.3, 0.4, 0.2), 0.4), \breve{\mathbb{C}}_2 = ((0.9, 0.8, 0.6), 0.9)$ $\breve{\mathbb{C}}_3 = ((0.7, 0.5, 0.2), 0.6)$ and $\breve{\mathbb{C}}_4 = ((0.9, 0.2, 0.2), 0.7)$ are four SVN numbers along their CL. If we take $\alpha = 3$ and $\tau_j = (0.2, 0.1, 0.3, 0.4)^T$ then, CSVNAAPAAO can be utilized to aggregate the three SVNNs as follows: $D(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2) = 0.476, D(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_3) = 0.238, D(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_4) = 0.365, D(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_3) = 0.311, D(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_4) = 0.416, D(\breve{\mathbb{C}}_3, \breve{\mathbb{C}}_4) = 0.208$. $supp(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2) = 0.524 supp(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_3) = 0.762 supp(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_4) = 0.635, supp(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_3) = 0.689 supp(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_4) = 0.584 supp(\breve{\mathbb{C}}_3, \breve{\mathbb{C}}_4) = 0.792$ then, $\mathcal{Z}_1 = 0.192$, $\mathcal{Z}_2 = 0.092$, $\mathcal{Z}_3 = 0.32$ and $\mathcal{Z}_4 = 0.39$. By using Equation 6, we get CSVNAAPAAO$(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2, \breve{\mathbb{C}}_3) = \langle 0.813, 0.338, 0.268\rangle$. Employing SVNs allows us to readily demonstrate that the proposed CSVNAAPWAAO fulfills the properties of idempotency, boundedness, and monotonicity, as explained below:

Property 3.1.1. The CSVNAAPWAAO is idempotent. i.e., If $(\breve{\mathbb{C}}_j, \eta_j) = (\breve{\mathbb{C}}, d)$ for all j, then CSVNAAPWAAO $((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)) = \eta\breve{\mathbb{C}}$.

Proof. Since $\breve{\mathbb{C}}_j = \langle \Re_{\breve{\mathbb{C}}_j}, \hbar_{\breve{\mathbb{C}}_j}, \partial_{\breve{\mathbb{C}}_j}\rangle$, $j = 1, 2, \ldots, \rho$ be the set of SVNNs we can get the following equation: $CSVNAAPWAA((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)) = \langle 1 - e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln(1-\Re_{\breve{\mathbb{C}}_j}))^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln\hbar_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln\partial_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle = \langle 1 - e^{-\{(d(-Ln(1-\Re_{\breve{\mathbb{C}}}))^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{(d(-Ln\hbar_{\breve{\mathbb{C}}})^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{(d(-Ln\partial)^{\alpha})\}^{\frac{1}{\alpha}}}\rangle = \langle \Re_{\breve{\mathbb{C}}}, \hbar_{\breve{\mathbb{C}}}, \partial_{\breve{\mathbb{C}}}\rangle = (\eta, \breve{\mathbb{C}})$ □

Property 3.1.2. The CSVNAAPWAAO is boundedness. i.e., For a collection of SVNNs $\breve{\mathbb{C}}_j$ for all $j = 1, 2, \ldots, \rho$ and $\breve{\mathbb{C}}^- = min(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)$ and $\breve{\mathbb{C}}^+ = max(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)$. Then $\breve{\mathbb{C}}^- \leq CSVNAAPWAA((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)) \leq \breve{\mathbb{C}}^+$.

Proof. Consider $\breve{\mathbb{C}}_j = \langle \Re_{\breve{\mathbb{C}}_j}, \hbar_{\breve{\mathbb{C}}_j}, \partial_{\breve{\mathbb{C}}_j}\rangle$, $j = 1, 2, \ldots, \rho$, be the set of SVNNs. Let $\breve{\mathbb{C}}^- = min(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho) = \langle \Re^-_{\breve{\mathbb{C}}_j}, \hbar^-_{\breve{\mathbb{C}}_j}, \partial^-_{\breve{\mathbb{C}}_j}\rangle$ and $\breve{\mathbb{C}}^+ = max(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)\breve{\mathbb{C}}_j = \langle \Re^+_{\breve{\mathbb{C}}_j}, \hbar^+_{\breve{\mathbb{C}}_j}, \partial^+_{\breve{\mathbb{C}}_j}\rangle$. We have $\Re^-_{\breve{\mathbb{C}}} = \min_j \Re_{\breve{\mathbb{C}}_j}, \hbar^-_{\breve{\mathbb{C}}} = \max_j \hbar_{\breve{\mathbb{C}}_j}, \partial^-_{\breve{\mathbb{C}}} = \max_j \partial_{\breve{\mathbb{C}}_j}, \Re^+_{\breve{\mathbb{C}}} = \max_j \Re_{\breve{\mathbb{C}}_j}, \hbar^+_{\breve{\mathbb{C}}} = \min_j \hbar_{\breve{\mathbb{C}}_j}$ and $\partial^+_{\breve{\mathbb{C}}} = \min_j \partial_{\breve{\mathbb{C}}_j}$. Hence there we have the following subsequent inequalities:

$\langle 1 - e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln(1-\Re^-_{\breve{\mathbb{C}}_j}))^{\alpha})\}^{\frac{1}{\alpha}}}\rangle \leq \langle 1 - e^{-\{\sum_{j=1}^{\rho}(\eta_j \mathcal{Z}_j(-Ln(1-\Re_{\breve{\mathbb{C}}_j}))^{\alpha})\}^{\frac{1}{\alpha}}}\rangle \leq \langle 1 -$

$e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln(1-\Re^{+}_{\breve{\mathbb{C}}_j}))^{\alpha})\}^{\frac{1}{\alpha}}}\rangle$; $\langle e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln\hbar^{-}_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}} \leq e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln\hbar_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}} \leq e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln\hbar^{+}_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle$;$\langle e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln\partial^{-}_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}} \leq e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln\partial_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}} \leq e^{-\{\sum_{j=1}^{\rho}(\eta_j\mathcal{Z}_j(-Ln\partial^{+}_{\breve{\mathbb{C}}_j})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle$. Therefore, $\breve{\mathbb{C}}^{-} \leq CSVNAAPWAA((\breve{\mathbb{C}}_1,\eta_1),(\breve{\mathbb{C}}_2,\eta_2),\dots,(\breve{\mathbb{C}}_\rho,\eta_\rho)) \leq \breve{\mathbb{C}}^{+}$. □

Property 3.1.3. The CSVNAAPWAAO is monotonicity. i.e., for any two SVNNs $\breve{\mathbb{C}}_j = \langle\Re_{\breve{\mathbb{C}}_j},\hbar_{\breve{\mathbb{C}}_j},\partial_{\breve{\mathbb{C}}_j}\rangle$ and $\breve{\mathbb{C}}'_j = \langle\Re'_{\breve{\mathbb{C}}_j},\hbar'_{\breve{\mathbb{C}}_j},\partial'_{\breve{\mathbb{C}}_j}\rangle$ such that $\breve{\mathbb{C}}_j \leq \breve{\mathbb{C}}'_j$ for all $j = 1,2,\dots,\rho$. Then CSVNAAPWAAO$(\breve{\mathbb{C}}_1,\breve{\mathbb{C}}_2,\dots,\breve{\mathbb{C}}_\rho) \leq$ CSVNAAPWAAO$(\breve{\mathbb{C}}'_1,\breve{\mathbb{C}}'_2,\dots,\breve{\mathbb{C}}'_\rho)$.

Proof. Due to $\breve{\mathbb{C}}_j \leq \breve{\mathbb{C}}'_j$ for all $j = 1,2,\dots,\rho$, there exists $\oplus_{j=1}^{\rho}\mathcal{Z}_j(\breve{\mathbb{C}}_j,\eta_j) \leq \oplus_{j=1}^{\rho}\mathcal{Z}_j(\breve{\mathbb{C}}'_j,\eta_j)$. Thus CSVNAAPWAAO$(\breve{\mathbb{C}}_1,\breve{\mathbb{C}}_2,\dots,\breve{\mathbb{C}}_\rho) \leq$ CSVNAAPWAAO$(\breve{\mathbb{C}}'_1,\breve{\mathbb{C}}'_2,\dots,\breve{\mathbb{C}}'_\rho)$ is true. □

3.1.2. *CSVN Aczel-Alsina power ordered weighted average aggregation operator*

In this part, a novel CSVNAAPOWAAO. This operator considers the ordered weights associated with the aggregated SVNNs.

Definition 3.2. *Let $\breve{\mathbb{C}}_j = (\Re_j,\hbar_j,\partial_j)(j = 1,2,\dots,\rho)$ be a set of SVNNs and $\eta_j$be the CL of $\breve{\mathbb{C}}_j$ with $0 \leq \eta_j \leq 1$. Let $\mathcal{Z} = (\mathcal{Z}_1,\mathcal{Z}_2,\dots,\mathcal{Z}_\rho)^T$ be the weight vectors for SVNNs with the condition $\sum_{j=1}^{\rho}\mathcal{Z}_j = 1$. Then, the mapping CSVNAAPOWAAO: $b^\rho \to b$ operator is given as follows: CSVNAAPOWAAO $\{(\breve{\mathbb{C}}_1,\eta_1),(\breve{\mathbb{C}}_2,\eta_2),\dots,(\breve{\mathbb{C}}_\rho,\eta_\rho)\} = \oplus_{j=1}^{\rho}\mathcal{Z}_j(\eta_{\vec{\sigma}(j)},\breve{\mathbb{C}}_{\vec{\sigma}(j)})$*

$$= \mathcal{Z}_1(\eta_{\vec{\sigma}(1)},\breve{\mathbb{C}}_{\vec{\sigma}(1)}) \oplus \mathcal{Z}_2(\eta_{\vec{\sigma}(2)},\breve{\mathbb{C}}_{\vec{\sigma}(2)}) \oplus \dots \oplus \mathcal{Z}_\rho(\eta_{\vec{\sigma}(\rho)},\breve{\mathbb{C}}_{\vec{\sigma}(\rho)}) \tag{7}$$

where, $(\vec{\vec{\sigma}}(1),\vec{\sigma}(2),\dots,\vec{\sigma}(\rho))$ is the permutation of $(1,2,\dots,\rho)$ with $\breve{\mathbb{C}}_{\vec{\sigma}(j-1)} \leq \breve{\mathbb{C}}_{\vec{\sigma}(j)}$ for all $j = 1,2,\dots,\rho$.

Theorem 3.2. *The aggregated value of the SVNNs $\breve{\mathbb{C}}_j$ for $j = 1,2,\dots,\rho$ with respect to the weight vector $\mathcal{Z} = (\mathcal{Z}_1,\mathcal{Z}_2,\dots,\mathcal{Z}_\rho)^T$ and the CL η_j such that $0 \leq \eta_j \leq 1$ obtained using the CSVNAAPOWAAO Equation 7 is also a SVNN and is given by CSVNAAPWAAO $\{(\breve{\mathbb{C}}_1,\eta_1),(\breve{\mathbb{C}}_2,\eta_2),\dots,(\breve{\mathbb{C}}_\rho,\eta_\rho)\} =$*

$$= \langle 1-e^{-\{\sum_{j=1}^{\rho}(\eta_{\vec{\sigma}(j)}\mathcal{Z}_j(-Ln(1-\Re_{\breve{\mathbb{C}}_{\vec{\sigma}(j)}}))^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{\rho}(\eta_{\vec{\sigma}(j)}\mathcal{Z}_j(-Ln\hbar_{\breve{\mathbb{C}}_{\vec{\sigma}(j)}})^{\alpha})\}^{\frac{1}{\alpha}}}, e^{-\{\sum_{j=1}^{\rho}(\eta_{\vec{\sigma}(j)}\mathcal{Z}_j(-Ln\partial_{\breve{\mathbb{C}}_{\vec{\sigma}(j)}})^{\alpha})\}^{\frac{1}{\alpha}}}\rangle \tag{8}$$

Proof. The proof of Theorem 3.2 follows the same approach as Theorem 3.1, so it is omitted here. □

Example 3.2. Supppose $\breve{\mathbb{C}}_1 = ((0.3, 0.4, 0.2), 0.4), \breve{\mathbb{C}}_2 = ((0.9, 0.8, 0.6), 0.9)$ $\breve{\mathbb{C}}_3 = ((0.7, 0.5, 0.2), 0.6)$ and $\breve{\mathbb{C}}_4 = ((0.9, 0.2, 0.2), 0.7)$ are four SVN numbers along their CL. If we take $\alpha = 3$ and $\tau_j = (0.2, 0.1, 0.3, 0.4)^T$ then, CSVNAAPAAO can be utilized to aggregate the three SVNNs as follows: $D(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2) = 0.455, D(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_3) = 0.545, D(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_4) = 0.387$,$D(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_3) = 0.238, D(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_4) = 0.311, D(\breve{\mathbb{C}}_3, \breve{\mathbb{C}}_4) = 0.476$. $supp(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2) = 0.545 supp(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_3) = 0.455 supp(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_4) = 0.613, supp(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_3) = 0.762 supp(\breve{\mathbb{C}}_2, \breve{\mathbb{C}}_4) = 0.689 supp(\breve{\mathbb{C}}_3, \breve{\mathbb{C}}_4) = 0.524$ then, $\mathcal{Z}_1 = 0.188$, $\mathcal{Z}_2 = 0.108$, $\mathcal{Z}_3 = 0.296$ and $\mathcal{Z}_4 = 0.407$. By using Equation 8, we get CSVNAAPAAO$(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2, \breve{\mathbb{C}}_3) = \langle$0.841,0.418,0.33$\rangle$. Employing SVNs

allows us to readily demonstrate that the proposed CSVNAAPOWAAO fulfills the properties of idempotency, boundedness, and monotonicity, as explained below.:

Property 3.1.4. The CSVNAAPOWAAO is idempotent. i.e., If $(\breve{\mathbb{C}}_j, \eta_j) = (\breve{\mathbb{C}}, d)$ for all j, then CSVNAAPOWAAO $((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)) = d\breve{\mathbb{C}}$.

Proof. The proof provided is analogous to that of Property 3.1.1 □

Property 3.1.5. The CSVNAAPOWAAO is boundedness. i.e., For a collection of SVNNs $\breve{\mathbb{C}}_j$ for all $j = 1, 2, \ldots, \rho$ and $\breve{\mathbb{C}}^- = min(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)$ and $\breve{\mathbb{C}}^+ = max(\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)\breve{\mathbb{C}}_j$. Then $\breve{\mathbb{C}}^- \leq CSVNAAPOWAA((\breve{\mathbb{C}}_1, \eta_1), (\breve{\mathbb{C}}_2, \eta_2), \ldots, (\breve{\mathbb{C}}_\rho, \eta_\rho)) \leq \breve{\mathbb{C}}^+$.

Proof. The proof provided is analogous to that of Property 3.1.2 □

Property 3.1.6. The CSVNAAPWAAO is monotonicity. i.e., for any two SVNNs $\breve{\mathbb{C}}_j = \langle \Re_{\breve{\mathbb{C}}_j}, \hbar_{\breve{\mathbb{C}}_j}, \partial_{\breve{\mathbb{C}}_j} \rangle$ and $\breve{\mathbb{C}}'_j = \langle \Re'_{\breve{\mathbb{C}}_j}, \hbar'_{\breve{\mathbb{C}}_j}, \partial'_{\breve{\mathbb{C}}_j} \rangle$ such that $\breve{\mathbb{C}}_j \leq \breve{\mathbb{C}}'_j$ for all $j = 1, 2, \ldots, \rho$. Then CSVNAAPWAAO$(\breve{\mathbb{C}}_1, \breve{\mathbb{C}}_2, \ldots, \breve{\mathbb{C}}_\rho) \leq$ CSVNAAPWAAO$(\breve{\mathbb{C}}'_1, \breve{\mathbb{C}}'_2, \ldots, \breve{\mathbb{C}}'_\rho)$.

Proof. The proof provided is analogous to that of Property 3.1.3 □

4. Evaluation of SVNMADM using proposed operators

This part illustrates how the proposed operators are applied by solving an SVNMADM model.

This section, presents a procedure for solving SVNMADM problems using the proposed operators.

Step 1 Let $\breve{\ddot{S}} = (\breve{\ddot{S}}_1, \breve{\ddot{S}}_2, \ldots, \breve{\ddot{S}}_\kappa)$ be a finite number of alternatives, and $\breve{\ddot{C}} = (\breve{\ddot{C}}_1, \breve{\ddot{C}}_2, \ldots, \breve{\ddot{C}}_\rho)$ be the set of attributes. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_\rho)^T$ be the weight vector of attributes, where $\tau_j \geq 0$, j= 1,2,…,ρ such that $\sum_{j=1}^{\rho} \tau_j = 1$. The SVN decision matrix $D = [\breve{\ddot{S}}_{ij}]_{\kappa\times\rho}$ evaluates the alternatives under each attribute, where $\Re_{ij}, \hbar_{ij}, \partial_{ij}$ indicates the truth, falsity and indeterminacy membership function respectively.

Step 2 To normalize an SVN decision matrix with cost type attribute, use the following Equation. 9

$$D = [\breve{\ddot{S}}_{ij}]_{\kappa\times\rho} = \begin{cases} \langle \Re_{ij}, \hbar_{ij}, \partial_{ij} \rangle & \text{if benefit type} \\ \langle \partial_{ij}, \hbar_{ij}, \Re_{ij} \rangle & \text{if cost type} \end{cases} \tag{9}$$

Step 3 Utilize the suggested operators to aggregate the evaluations for each attribute over all alternatives.

Step 4 Choose the best alternative by ranking the options based on their score values.

5. Numerical illustration

Let's consider the practical example of an MADM problem from [29], which involves an organization strategic suppppliers under suppply chain risk management in which the five suppliers called the alternatives $\breve{\ddot{S}}_i (i = 1, 2, 3, 4, 5)$ are assessed based on four different attributes $\breve{\ddot{C}}_j (j = 1, 2, 3, 4)$ namely e technology level, service level, risk managing ability and enterprise environment with respect to the weighting vector $\tau = (0.2, 0.1, 0.3, 0.4)^T$.

Step 1 The decision matrix for the MADM problem, featuring confidence-induced SVN preference values evaluated by a decision expert, is presented in Table 1.

Step 2 Since all the attribute are beneficial, there is no need to normalize the confidence SVN decision matrix.

Table 1. Confidence SVN decision matrix evaluated by a decision expert

	$\breve{\ddot{C}}_1$	$\breve{\ddot{C}}_2$	$\breve{\ddot{C}}_3$	$\breve{\ddot{C}}_4$
$\breve{\ddot{S}}_1$	((0.5, 0.8, 0.1),0.3)	((0.6, 0.3, 0.3),0.3)	((0.3, 0.6, 0.1),0.1)	((0.5, 0.7, 0.2),0.1)
$\breve{\ddot{S}}_2$	((0.7, 0.2, 0.1),0.8)	((0.7, 0.2, 0.2),0.8)	((0.7, 0.2, 0.4),0.7)	((0.8, 0.2, 0.1),0.4)
$\breve{\ddot{S}}_3$	((0.6, 0.7, 0.2),0.9)	((0.5, 0.7, 0.3),0.9)	((0.5, 0.3, 0.1),0.2)	((0.6, 0.3, 0.2),0.7)
$\breve{\ddot{S}}_4$	((0.8, 0.1, 0.3),0.7)	((0.6, 0.3, 0.4),0.5)	((0.3, 0.4, 0.2),0.8)	((0.5, 0.6, 0.1),0.8)
$\breve{\ddot{S}}_5$	((0.6, 0.4, 0.4),0.6)	((0.4, 0.8, 0.1),0.8)	((0.7, 0.6, 0.1),0.4)	((0.5, 0.8, 0.2),0.6)

Step 3 Combine all the attribute, each with its own distinct confidence SVN preference value

for each alternative using CSVNAAPWAAO Equation 6 to get the overall SVN $\breve{\mathbb{C}}_i$ of the corresponding $\ddot{\breve{S}}_i$ as $\ddot{\breve{\mathbb{C}}}_1 = \langle 0.241, 0.751, 0.472\rangle$, $\ddot{\breve{\mathbb{C}}}_2 = \langle 0.59, 0.414, 0.332\rangle$, $\ddot{\breve{\mathbb{C}}}_3 = \langle 0.385, 0.616, 0.407\rangle$, $\ddot{\breve{\mathbb{C}}}_4 = \langle 0.2491, 0.386, 0.364\rangle$, $\ddot{\breve{\mathbb{C}}}_5 = \langle 0.368, 0.7, 0.392\rangle$. Combine all the attribute, each with its own distinct confidence SVN preference value for each alternative using CSVNAAPOWAAO Equation 8 to get the overall SVN $\ddot{\breve{\mathbb{C}}}_i$ of the corresponding $\ddot{\breve{S}}_i$ as $\ddot{\breve{\mathbb{C}}}_1 = \langle 0.257, 0.7, 0.52\rangle$, $\ddot{\breve{\mathbb{C}}}_2 = \langle 0.535, 0.414, 0.331\rangle$, $\ddot{\breve{\mathbb{C}}}_3 = \langle 0.347, 0.558, 0.343\rangle$, $\ddot{\breve{\mathbb{C}}}_4 = \langle 0.489, 0.384, 0.386\rangle$, $\ddot{\breve{\mathbb{C}}}_5 = \langle 0.405, 0.732, 0.351\rangle$.

Step 4 Calculate the score values using Equation 1 corresponding to the SVN $\ddot{\breve{\mathbb{C}}}_i$ obtained in Step 3 based on the CSVNAAPWAAO respectively are $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_1) = 0.339$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_2) = 0.587$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_3) = 0.454$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_4) = 0.581$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_5) = 0.427$. Based on CSVNAAPWAAO, the score value .Thus, we have $\ddot{\breve{S}}_2 > \ddot{\breve{S}}_4 > \ddot{\breve{S}}_3 > \ddot{\breve{S}}_5 > \ddot{\breve{S}}_1$. Hence the best alternative is $\ddot{\breve{S}}_2$.

Calculate the score values using Equation 1 corresponding to the SVN $\ddot{\breve{\mathbb{C}}}_i$ obtained in Step 3 based on the CSVNAAPOWAAO respectively are $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_1) = 0.347$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_2) = 0.587$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_3) = 0.482$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_4) = 0.573$, $\ddot{\mathbb{M}}(\ddot{\breve{\mathbb{C}}}_5) = 0.441$. Based on CSVNAAPOWAAO, the score value .Thus, we have $\ddot{\breve{S}}_2 > \ddot{\breve{S}}_4 > \ddot{\breve{S}}_3 > \ddot{\breve{S}}_5 > \ddot{\breve{S}}_1$. Hence the best alternative is $\ddot{\breve{S}}_2$.

6. Comparative analysis

In this discussion we compare the overall ranking results achieved with the proposed CSVNAAPWAAO and CSVNAAPOWAAO for the demonstrative example presented in Section 5 against the existing outcomes based on the SVN weighted Bonferroni power average aggregation operator (SVNWBPWAAO). From the Table 2, we observe that the top-ranked

TABLE 2. Comparison of the existing operators with the proposed operators

Method	Operator	Ranking	Best
Guiwu Wei and Zuopeng Zhang [29]	SVNWBPWAAO	$\ddot{\breve{S}}_2 > \ddot{\breve{S}}_4 > \ddot{\breve{S}}_3 > \ddot{\breve{S}}_5 > \ddot{\breve{S}}_1$	$\ddot{\breve{S}}_2$
Proposed	CSVNAAPWAAO	$\ddot{\breve{S}}_2 > \ddot{\breve{S}}_4 > \ddot{\breve{S}}_3 > \ddot{\breve{S}}_5 > \ddot{\breve{S}}_1$	$\ddot{\breve{S}}_2$

alternatives for the proposed operators are the same as those for the existing operators, while the least favorable alternatives remain unchanged for the CSVNAAPWAAO. However, the proposed operators, which incorporate CLs into SVNs, provide a more refined ranking of the alternatives, reflecting the decision maker's subjective familiarity. Additionally, the comparison is visually illustrated in Figure 1.

7. Conclusion

This paper presents the development of confidence SVN Aczel-Alsina power average AO, specifically CSVNAAPWAAO and CSVNAAPOWAAO. The fundamental properties of these

FIGURE 1. Graphical comparison of SVNWBPAAO and CSVNAAPWAAO

proposed AO have also been proven. An important feature of these operators is that they take into account not only the assessed arguments of the decision experts but also the experts' confidence in their assessments. In addition, we developed a SVNMADM method employing the proposed operators and applied it to a real world problem of choosing a supplier system based on four attributes, thereby confirming the validity of our proposal. We also compared our findings with the existing SVNWBPAAO and CSVNAAPWAAOs to emphasize the potential of the proposed operators. Additionally, the results were presented graphically for enhanced clarity.

In the future, this proposed concept can be applied to develop SVN geometric AO and to create a variety of AO for SVNs by integrating probabilistic information, and additional factors.

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The Study of Neutrosophic Algebraic Structures and itsApplication in Medical Science

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Abstract. Neutrosophic exact sequence and its construction is presented as algebraic structure in this article. A mathematical construction known as a neutrosophic exact sequence expands the implications of an exact sequence from classical algebra to the domain of neutrosophic sets. This work introduces a full description of neutrosophic exact sequence of R-modules as a structural preserving tools and discusses the fundamental features. Additionally we examine the role of beta level sets in neutrosophic sets and investigate the neutrosophic split exact sequence.

Keywords: Neutrosophic set, Neutrosophic submodule, Neutrosophic exact sequence, Neutrosophic homomorphism, neutrosophic split exact sequence

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1. Introduction

A mathematical paradigm known as the neutrosophic set, put forth by Florentine Smarandache [20,22,24], addresses the ambiguity and indeterminacy of contradicting information. Hierarchical component membership, which generalises the contemporary ideas of fuzzy sets [26] and intuitionistic fuzzy sets [2], can be used to manage the prescribed set. It classifies each element of a set using three distinct types of membership grades: Truth, Indeterminacy, and Falsity. A few experts have examined the algebraic structure associated with uncertainty in pure mathematics. Abstract algebra was among the first few disciplines to use the concept of a neutrosophic set for research. Neutrosophic algebraic structures and their application to advanced neutrosophical models were first introduced by W. B. Vasantha Kandasamy and Florentin Smarandache [11]. Vidan Cetkin [7,14] combined the neutrosophic set theory and algebraic structures for the creation of neutrosophic subgroups and neutrosophic submodules.

Homomorphism is a structural persistence map between two algebraic structures, that serves a purpose akin to or equivalent to continuous functions and inflexible geometry motions. Exact sequences serves as a comprehensive representation of module homomorphisms, including their images and kernels. In the field of mathematical sciences, there is a rapid growth toward the consolidation of the neutrosophic set hypothesis with algebraic structures. Compared to traditional crisp set-based structures, algebraic neutrosophic set-based structures have greater expressive capacity in all dimension. The algebraic structure module [1] also sums up the idea of abelian group which is a module over $\mathbb{Z}$ and the idea of vector space over a field is generalised by the module concept over a ring.. A sequence of morphisms between modules that are exact is one in which the image of one morphism is the kernel of the next. In the area of abstract algebra, this study is an exploratory evaluation using the neutrosophic set and the exact module sequence. Exact sequences are beneficial for unifying module homomorphisms with their images and kernels introduced by Hurewicz [9] and Kelley [12]. A.K. Sahni's [17] study on fuzzy exact sequences over semirings is one of the generalisations in exact sequences. We introduced the notion of exact sequence of R-submodules in neutrosophic set, the most generalized form of the three valued logic in this study. The structure of the proposed study is as follows:Section 1 provides a brief summary of related studies conducted in the past, and Section 2 gives a general overview of the core findings that will be needed to understand the sessions to follow. Section 3 explains the exact sequence of neutrosophic modules along with direct sum of neutrosophic submodules and its algebraic properties followed by neutrosophic split exact sequences. The final session conclusion outlines the significance, extent, and possible follow-up research for the current study.

2. Materials and Methods

(1) Construction of neutrosophic exact sequence
(2) Derivation of the relation connecting neutrosophic exact sequence and direct sum of neutrosophic submodules
(3) Study of neutrosophic split exact sequence of an R-module
(4) Analyze the concept of restriction map, β level sets of neutrosophic submodules with the neutrosophic exact sequence.

3. Preliminaries

We provide some of the fundamental concepts and conclusions in this session, which are necessary for a better comprehension of the sessions that follow.

Definition 3.1. [8, 10, 16, 23] A pair of module homomorphisms $M \xrightarrow{f} N \xrightarrow{g} P$ is said to be exact at N if $Im\ f = ker\ g$. A sequence of module homomorphisms

$$....... \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} ....$$

is exact provided that $Im\ f_i = ker\ f_{i+1}\ \forall i$.

Definition 3.2. *[18, 19, 21, 25] Any element ω in the universal set Σ has the form $\Omega = \{(\omega, \varphi_\Omega(\omega), \chi_\Omega(\omega), \psi_\Omega(\omega)) : \omega \in \Sigma\}$ where $\varphi_\Omega, \chi_\Omega, \psi_\Omega : \Sigma \to [0, 1]$, then Ω is referred as a single valued nuetrosophic set on Σ. The percentage of truth, indeterminacy, and non-membership value are represented by the three components, $\varphi_\Omega, \chi_\Omega$ and ψ_Ω.*

Definition 3.3. *[3, 4, 18] Let Ω and Υ be two neutrosophic sets on Σ. Then Ω is contained in Υ, denoted as $\Omega \subseteq \Upsilon$ if and only if $\Omega(\omega) \leqslant \Upsilon(\omega)\ \forall \omega \in \Sigma$, this means that*

$$\varphi_\Omega(\omega) \leq \varphi_\Upsilon(\omega), \chi_\Omega(\omega) \leq \chi_\Upsilon(\omega), \psi_\Omega(\omega) \geq \psi_\Upsilon(\omega)$$

Remark 3.1. *[21]*

(1) *U^Σ denotes the set of all neutrosophic subsets of Σ or neutrosophic power set of Σ.*

Definition 3.4. *[4, 18] For any neutrosophic subset $\Omega = \{(\omega, \varphi_\Omega(\omega), \chi_\Omega(\omega), \psi_\Omega(\omega)) : \omega \in \Sigma\}$ of Σ, the support Ω^* of the neutrosophic set Ω can be defined as*

$$\Omega^* = \{\omega \in \Sigma, \varphi_\Omega(\omega) > 0, \chi_\Omega(\omega) > 0, \psi_\Omega(\omega) < 1\}.$$

Definition 3.5. *[3, 6, 18] For any $\omega \in \Sigma$, the neutrosophic point $\hat{N}_{\{\omega\}}$ is defined as*

$$\hat{N}_{\{\omega\}}(s) = \{s, \varphi_{\hat{N}_{\{\omega\}}}(s), \chi_{\hat{N}_{\{\omega\}}}(s), \psi_{\hat{N}_{\{\omega\}}}(s) : s \in \Sigma\}$$

where

$$\hat{N}_{\{\omega\}}(s) = \begin{cases} (1, 1, 0) & \omega = s \\ (0, 0, 1) & \omega \neq s \end{cases}$$

Remark 3.2. *Let Σ be a non empty set. The neutrosophic point $\hat{N}_{\{0\}}$ in Σ is $\hat{N}_{\{0\}}(\omega) = \{\omega, \varphi_{\hat{N}_{\{0\}}}(\omega), \chi_{\hat{N}_{\{0\}}}(\omega), \psi_{\hat{N}_{\{0\}}}(\omega) : \omega \in \Sigma\}$ where*

$$\hat{N}_{\{0\}}(\omega) = \begin{cases} (1, 1, 0) & \omega = 0 \\ (0, 0, 1) & \omega \neq 0 \end{cases}$$

Definition 3.6. *[3, 4, 13, 18] Let Σ and Π be two non empty sets and $g : \Sigma \to \Pi$ be a mapping. Let Ω and Υ be neutrosophic subsets of Σ and Π respectively. Then the image of Ω under the map g is denoted by $g(\Omega)$ and is defined as $g(\Omega) = \{\upsilon, \varphi_{g(\Omega)}(\upsilon), \chi_{g(\Omega)}(\upsilon), \psi_{g(\Omega)}(\upsilon) : \upsilon \in \Pi\}$ where*

$$\varphi_{g(\Omega)}(\upsilon) = \begin{cases} \vee\varphi_\Omega(\omega) : \omega \in g^{-1}(\upsilon)) & if\ g^{-1}(\upsilon) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$\chi_{g(\Omega)}(\upsilon) = \begin{cases} \vee\chi_\Omega(\omega) : \omega \in g^{-1}(\upsilon)) & if\ g^{-1}(\upsilon) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$\psi_{g(\Omega)}(\upsilon) = \begin{cases} \wedge\psi_\Omega(\omega) : \omega \in g^{-1}(\upsilon)) & if\ g^{-1}(\upsilon) \neq \emptyset \\ 1 & otherwise \end{cases}$$

Furthermore, the inverse of g, denoted by $g^{-1} : \Pi \to \Omega$ *is defined by* $g^{-1}(\Upsilon) = \{\omega, \varphi_{g^{-1}(\Upsilon)}(\omega), \chi_{g^{-1}(\Upsilon)}(\omega), \psi_{g^{-1}(\Upsilon)}(\omega) : g(\omega) \in \Upsilon\}$ *where* $\varphi_{g^{-1}(\Upsilon)}(\omega) = \varphi_\Upsilon(g(\omega)),\ \chi_{g^{-1}(\Upsilon)}(\omega) = \chi_\Upsilon(g(\omega)),\ \psi_{g^{-1}(\Upsilon)}(\omega) = \psi_\Upsilon(g(\omega))$

Definition 3.7. [7] Let M be an R module. Let $\Omega \in U^M$. Then a neutrosophic subset $\Omega = \{\omega, \varphi_P(\omega), \chi_P(\omega),$
$\psi_\Omega(\omega) : \omega \in M\}$ in M satisfies the following conditions, It is referred to as a neutrosophic submodule of M

(1) $\varphi_\Omega(0) = 1, \chi_\Omega(0) = 1, \psi_\Omega(0) = 0$
(2) $\varphi_\Omega(\omega + \upsilon) \geq \varphi_\Omega(\omega) \wedge \varphi_\Omega(\upsilon)$
$\chi_\Omega(\omega + \upsilon) \geq \chi_\Omega(\omega) \wedge \chi_\Omega(\upsilon)$
$\psi_\Omega(\omega + \upsilon) \leq \psi_\Omega(\omega) \vee \psi_\Omega(\upsilon),\ \forall \omega, \upsilon \in M$
(3) $\varphi_\Omega(r\omega) \geq \varphi_\Omega(\omega),\ \chi_\Omega(r\omega) \geq \chi_\Omega(\omega),\ \psi_\Omega(r\omega) \leq \psi_\Omega(\omega), \forall\ \omega \in M, \forall\ r \in R$.

Remark 3.3. *$U(M)$ is the space of all neutrosophic submodules of R-module M.*

Definition 3.8. *[4, 5, 18] A homomorphism Γ of M into N is called a weak neutrosophic homomorphism of Ω onto Υ if $\Gamma(\Omega) \subseteq \Upsilon$. If Γ is a* ***weak neutrosophic homomorphism*** *of Ω onto Υ, then Ω is weakly homomorphic to Υ and we write $\Omega \sim \Upsilon$. A homomorphism Γ of M into N is called a* ***neutrosophic homomorphism*** *of Ω onto Υ if $\Gamma(\Omega) = \Upsilon$ and we represent it as $\Omega \approx \Upsilon$.*

Definition 3.9. *[3, 15] Let Ω, Υ and $\Psi \in U(M)$, then Ω is said to be the direct sum of neutrosophic submodules Υ and Ψ, we write $\Omega = \Upsilon \oplus \Psi$, if*

(1) $\Omega = \Upsilon + \Psi$
(2) $\Upsilon \cap \Psi = \hat{N}_{\{0\}}$

Definition 3.10. *[18] Let $\Omega \in U^\Sigma$. If for all $\beta \in [0, 1]$, the β-level sets of of Ω, can be denoted and defines as $\Omega_\beta = \{\omega \in \Sigma : \varphi_\Omega(\omega) \geq \beta, \chi_\Omega(\omega) \geq \beta, \psi_\Omega(\omega) \leq \beta\}$ and the strict β level sets of Ω can be denoted and defined as $\Omega^*_\beta = \{\omega \in \Sigma : \varphi_\Omega(\omega) > \beta, \chi_\Omega(\omega) > \beta, \psi_\Omega(\omega) < \beta\}$.*

14. Neutrosophic Exact Sequences

This section discusses the notion of exact sequence in the field of neutrosophic algebra and its algebraic properties.

Definition 4.1. *Let M_i be an arbitrary family of R-modules and $\Omega_i \in U(M_i)$. Suppose that* $\ldots\ldots \xrightarrow{g_{i-1}} M_{i-1} \xrightarrow{g_i} M_i \xrightarrow{g_{i+1}} M_{i+1} \xrightarrow{g_{i+2}} \ldots\ldots$ *is an exact sequence of R-modules. Then the sequence* $\ldots\ldots \xrightarrow{g_{i-1}} \Omega_{i-1} \xrightarrow{g_i} \Omega_i \xrightarrow{g_{i+1}} \Omega_{i+1} \xrightarrow{g_{i+2}} \ldots\ldots$ *of neutrosophic R-modules is said to be neutrosophic exact sequence if* $\forall\ i$

(1) $g_{i+1}(\Omega_i) \subseteq \Omega_{i+1}$
(2) $(g_i(\Omega_{i-1}))^* = ker(g_{i+1})$ *or*
$$\begin{cases} g_i(\Omega_{i-1})(\omega) > 0 & if\ \omega \in ker\ g_{i+1} \\ g_i(\Omega_{i-1})(\omega) = 0 & if\ \omega \notin ker\ g_{i+1} \end{cases}$$

Remark 4.1. *For convenience we have denoted the $\hat{N}_{\{0\}} \in U(M)$ by 0.*

Theorem 4.1. *Let $\Omega, \Upsilon \in U(M)$ be such that $\Omega \oplus \Upsilon$ is the direct sum of neutrosophic R-modules of M so that $\Omega^* \oplus \Upsilon^*$ is a direct sum of R submodules of M. Then the sequence*

$$0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$$

is exact sequence considering $\Omega \in U(\Omega^*)$ and $\Upsilon \in U(\Upsilon^*)$.

Proof. Consider the exact sequence $0 \to \Omega^* \xrightarrow{I} \Omega^* \oplus \Upsilon^* \xrightarrow{\pi} \Upsilon^* \longrightarrow 0$ where I and π are injection and projection mappings respectively.
we have to prove that the sequence $0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$ is an exact sequence. Let $\omega \in \Omega^* + \Upsilon^*$. Then $I(\Omega)(\omega) = \{\omega, \varphi_{I(\Omega)}(\omega), \chi_{I(\Omega)}(\omega), \psi_{I(\Omega)}(\omega)\}$ where

$$\varphi_{I(\Omega)}(\omega) = \begin{cases} \vee(\varphi_\Omega(\eta) : \eta \in \Omega^*, I(\eta) = \omega) & I^{-1}(\omega) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \varphi_\Omega(\omega) & if\ \omega \in \Omega^* \\ 0 & if\ \omega \notin \Omega^* \end{cases}$$

$$\chi_{I(\Omega)}(\omega) = \begin{cases} \vee(\chi_\Omega(\eta) : \eta \in \Omega^*, I(\eta) = \omega) & I^{-1}(\omega) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \chi_\Omega(\omega) & if\ \omega \in \Omega^* \\ 0 & if\ \omega \notin \Omega^* \end{cases}$$

$$\Psi_{I(\Omega)}(\omega) = \begin{cases} \wedge(\Psi_\Omega(\eta) : \eta \in \Omega^*, I(\eta) = \omega) & I^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \Psi_\Omega(\omega) & if\ \omega \in \Omega^* \\ 1 & if\ \omega \notin \Omega^* \end{cases}$$

Also $\xi = \Omega + \Upsilon$ *and* $\xi(\omega) = \{\omega, \varphi_\xi(\omega), \chi_\xi(\omega), f_\xi(\omega) : \omega \in \Omega^* + \Upsilon^*\}$ where

$$\begin{aligned} \varphi_\xi(\omega) &= \vee\{\varphi_\Omega(\upsilon) \wedge \varphi_\Upsilon(\rho) : \upsilon, \rho \in M, \omega = \upsilon + \rho\} \\ &= \varphi_\Omega(\omega) \text{ if } \omega \in \Omega^* \end{aligned}$$

$$\begin{aligned} \chi_\xi(\omega) &= \vee\{\chi_\Omega(\upsilon) \wedge \chi_\Upsilon(\rho) : \upsilon, \rho \in M, \omega = \upsilon + \rho\} \\ &= \chi_\Omega(\omega)\ if\ \omega \in \Omega^* \end{aligned}$$

$$\begin{aligned} \Psi_\xi(\omega) &= \wedge\{\Psi_\Omega(\upsilon) \vee \Psi_\Upsilon(\rho) : \upsilon, \rho \in M, \omega = \upsilon + \rho\} \\ &= \Psi_\Omega(\omega)\ if\ \omega \in \Omega^* \end{aligned}$$

(Note that $\Omega \oplus \Upsilon$ is a direct sum and $\omega = \upsilon + \rho$ where $\omega \in P^*$, then the only option is $\omega = \omega + 0\ or\ \omega = \upsilon + \rho;\ \upsilon, \rho \in \Omega^*$. But in the second case $t_\Upsilon(\rho) = i_\Upsilon(\rho) = f_\Upsilon(\rho) = 0.)$ From the above two derivations, $I(\Omega) \subseteq \Omega + \Upsilon$.

Now we consider, $\omega \in \Upsilon^*$,

$\pi(\Omega + \Upsilon)(\omega) = \{\omega, t_{\pi(\Omega+\Upsilon)}(\omega), i_{\pi(\Omega+\Upsilon)}(\omega), f_{\pi(\Omega+\Upsilon)}(\omega))\}$ where

$$\begin{aligned} \varphi_{\pi(\Omega+\Upsilon)}(\omega) &= \vee\{\varphi_{\Omega+\Upsilon}(\eta) : \eta \in \Omega^* + \Upsilon^*,\ \pi(\eta) = \omega\} \\ &= \vee\{\varphi_{\Omega+\Upsilon}(r + \omega) : r \in \Omega^*\} \\ &\quad (since\ \pi : \Omega^* \oplus \Upsilon^* \to \Upsilon^* \\ &\quad is\ the\ projection\ map) \\ &= \vee\{\varphi_\Omega(r) \wedge \varphi_\Upsilon(\omega) : r \in \Omega^*\} \\ &\quad (Property\ of\ direct\ Sum) \\ &= \varphi_\Upsilon(\omega)\ [\varphi_\Omega(r) = 1\ with\ r = 0] \end{aligned}$$

Similarly, $\chi_{\pi(\Omega+\Upsilon)}(\omega) = \chi_\Upsilon(\omega),\ \Psi_{\pi(\Omega+\Upsilon)}(\omega) = \Psi_\Upsilon(\omega)$. So $\pi(\Omega + \Upsilon) = \Upsilon$

Now $I(\Omega)(\omega) = \{\omega, \varphi_\Omega(\omega), \chi_\Omega(\omega), \Psi_\Omega(\omega)) : \omega \in \Omega^*\}$ and 0 otherwise. $\Rightarrow (I(\Omega))^* = ker\ \pi$

$\therefore 0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$ is an exact sequence of neutrosophic R-modules. Hence the proof. □

Definition 4.2. *Let Ω, Υ and $\Omega \oplus \Upsilon \in U(M)$. Then the exact sequence $0 \to \Omega \xrightarrow{I} \Omega \oplus \Upsilon \xrightarrow{\pi} \Upsilon \longrightarrow 0$ of an R-modules is called a neutrosophic split exact sequence of R-modules.*

Theorem 4.2. *Let $M \xrightarrow{g} N \xrightarrow{h} S$ be a sequence of an R-modules exact at N. Let $\Omega \in U(M),\ \Upsilon \in U(N)$ and $\xi \in U(S)$. Then the sequence of neutrosophic submodules $\Omega \xrightarrow{g} \Upsilon \xrightarrow{h}$*

ξ is exact at Υ only if $\Omega^* \xrightarrow{g^{'}} \Upsilon^* \xrightarrow{h^{'}} \xi^*$ is exact at Υ^*, where $g^{'}$ and $h^{'}$ are restrictions of g and h to Ω^* and Υ^* respectively.

Proof. Suppose the sequence of neutrosophic submodules $\Omega \xrightarrow{g} \Upsilon \xrightarrow{h} \xi$ is exact at Υ. Then by definition $g(\Omega) \subseteq \Upsilon,\ h(\Upsilon) \subseteq \xi$ *and* $(g(\Omega))^* = ker\ h$ Now consider the sequence $\Omega^* \xrightarrow{g^{'}} \Upsilon^* \xrightarrow{h^{'}} \xi^*$ and we claim that this sequence is exact at Υ^*. Now consider

$$\begin{aligned}
\omega \in (g(\Omega))^* \quad &\Leftrightarrow \quad \varphi_{g(\Omega)}(\omega) > 0,\ \chi_{g(\Omega)}(\omega) > 0,\ \psi_{g(\Omega)}(\omega) < 1 \\
&\Leftrightarrow \quad \vee\{\varphi_{\Omega}(\eta) : \eta \in M, g(\eta) = \omega\} > 0, \\
&\qquad \vee\{\chi_{\Omega}(\eta) : \eta \in M, g(\eta) = \omega\} > 0 \ and \\
&\qquad \wedge\{\psi_{\Omega}(\eta) : \eta \in M, g(\eta) = \omega\} < 1 \\
&\Leftrightarrow \quad \exists\ \eta \in M \ such\ that \\
&\qquad \{\omega = g(\eta) : \varphi_{\Omega}(\eta) > 0, \chi_{\Omega}(\eta) > 0,\ \psi_{\Omega}(\eta) < 1\} \\
&\Leftrightarrow \quad \omega \in g(\Omega^*)
\end{aligned}$$

Thus we get $(g(\Omega))^* = g(\Omega^*)$. Similarly we can prove $(h(\Upsilon))^* = h(\Upsilon^*)$.Therefore $g^{'}(\Omega^*) = g(\Omega^*) = (g(\Omega))^* \subseteq \Upsilon^*$ Now $(g(\Omega))^* = ker\ h \Rightarrow (g^{'}(\Omega))^* = ker\ h^{'}$.
Thus the sequence $\Omega^* \xrightarrow{g^{'}} \Upsilon^* \xrightarrow{h^{'}} \xi^*$ is exact at Υ^*. □

Remark 4.2. *The converse of the above theorem need not be true as we see in the following example.*

Example 4.1. Let M be an R-module, N and L are submodules of M such that $N \oplus L$ is a direct sum. Define $\Omega \in U(N),\ \Upsilon \in U(L)$ *and* $\xi \in U(N \oplus L)$ as follows

$$\varphi_{\Omega}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.8 & \omega \in N - \{0\} \end{cases};$$

$$\chi_{\Omega}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.8 & \omega \in N - \{0\} \end{cases};$$

$$\psi_{\Omega}(\omega) = \begin{cases} 0 & \omega = 0 \\ 0.1 & \omega \in N - \{0\} \end{cases}$$

$$\varphi_{\Upsilon}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.5 & \omega \in L - \{0\} \end{cases};$$

$$\chi_{\Upsilon}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.5 & \omega \in L - \{0\} \end{cases};$$

$$\psi_{\Upsilon}(\omega) = \begin{cases} 0 & \omega = 0 \\ 0.3 & \omega \in L - \{0\} \end{cases}$$

$$\varphi_{\xi}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.3 & \omega \in N \oplus L - \{0\} \end{cases};$$

$$\chi_{\xi}(\omega) = \begin{cases} 1 & \omega = 0 \\ 0.3 & \omega \in N \oplus L - \{0\} \end{cases}$$

$$\psi_{\xi}(\omega) = \begin{cases} 0 & \omega = 0 \\ 0.5 & \omega \in N \oplus L - \{0\} \end{cases}$$

Then $\Omega^* = N,\ \chi^* = L\ and\ \xi^* = N \oplus L$. So $N^* \xrightarrow{I} N \oplus L \xrightarrow{\pi} L^*$ is exact at $N \oplus L$
$\Rightarrow\ \Omega^* \xrightarrow{I} \xi^* \xrightarrow{\pi} \chi*$ is exact at ξ^*.
Now, $\varphi_{I(\Omega)}(\omega) = \vee\{\varphi_{\Omega}(z) : z \in N, I(z) = \omega\} = \varphi_{\Omega}(\omega)$, $\chi_{I(\Omega)}(\omega) = \vee\{\chi_{\Omega}(z) : z \in N, I(z) = \omega\} = \chi_{\Omega}(\omega)$ and $\psi_{I(\Omega)}(\omega) = \wedge\{\psi_{\Omega}(z) : z \in N, I(z) = \omega\} = \psi_{\Omega}(\omega)$
$\Rightarrow\ i(\Omega) = \Omega\ but\ \Omega \nsubseteq \chi$. Therefore the sequence $\Omega \xrightarrow{I} \xi \xrightarrow{\pi} \chi$ is not exact.

Theorem 4.3. *Let $M \xrightarrow{g} N \xrightarrow{h} S$ be a sequence of an R-modules exact at N and let $\Omega \in U(M),\ \Upsilon \in U(N)$ and $\xi \in U(S)$ be such that $\Omega \xrightarrow{g} \Upsilon \xrightarrow{h} T$ is a sequence of neutrosophic submodules exact at χ. Then $g(\Omega^*_\beta) \subseteq ker\ h\ \forall\ \beta \in [0, 1]$.*

Proof. Consider the strict β level subsets $\Omega^*_\beta, \Upsilon^*_\beta, \xi^*_\beta$ of Ω, Υ *and* ξ respectively. Then
$\omega \in g(\Omega^*_\beta) \Rightarrow \exists\ \theta\ such\ that\ \omega = g(\theta)$

$\Rightarrow \varphi\,\Omega(\theta) > \beta,\ \chi_\Omega(\theta) > \beta\ and\ \psi_\Omega(\beta) < \beta$

$\Rightarrow\ \vee\{\varphi_\Omega(\theta) : \omega = g(\theta)\} > \beta$

$\Rightarrow\ \vee\{\chi_\Omega(\theta) : \omega = g(\theta)\} > \beta$

$\Rightarrow\ \wedge\{\psi_\Omega(\theta) : \omega = g(\theta)\} < \beta$

$\Rightarrow\ It\ follows\ that\ g(\Omega)(\omega) > \beta$

$\Rightarrow\ \omega \in (g(\Omega))^*_\beta$

$\Rightarrow\ g(\Omega^*_\beta) \subseteq (g(\Omega))^*_\beta$

Similarly it can be prove that $h(\Upsilon^*_\beta) \subseteq (h(\Upsilon))^*_\beta$.

Now $\omega \in g(\Omega^*_\beta) \subseteq (g(\Omega))^*_\beta \Rightarrow g(\Omega)(\omega) > \beta \Rightarrow g(\Omega)(\omega) > 0$

$\Rightarrow\ \omega \in ker\ h$ [Since by definition 4.1, $g(\Omega)(\omega) > 0\ if\ and\ only\ if\ \omega \in ker\ h$]

Hence $g(\Omega^*_\beta) \subseteq ker\ h\ \forall\ \beta \in [0, 1]$. □

5. Conclusion

This paper reviews a comprehensive range of traditional and more recent findings around how classical algebra and neutrosophic sets interact. Neutrosophic exact sequences constitute an important development in the mathematical modelling of uncertainty, enhancing exact sequence theoretical foundations and real-world applications. Direct sum in neutrosophic domain is used to examine the fundamental properties of the exact sequence of neutrosophic submodules, and examples have been constructed to strengthen the study. Additionally, this work analyzes how kernels and restriction maps influence the exact sequence of neutrosophic submodules. The main objective of this study is to expand, generalize, and offer a new perspective on the theory of exact sequences within neutrosophic algebra. The current study is beneficial for shortest path problems based on neutrosophic weighted automata and neutrosophic algorithms that utilize nuetrosophic algebraic structures. On the basis of this approach, future research has produced category theory and proper exact sequences in the neutrosophic domain.

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Neutrosophic Soft Cubic Refined Sets

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Abstract. This paper presents a novel approach to the Neutrosophic Soft Cubic Refined Set (NSCRS), which serves as the foundation for combining the theory of soft sets, the already-existing Neutrosophic cubic set, and the Neutrosophic refined set and Neutrosophic Soft Cubic Set. Further introduced are the internal neutrosophic soft cubic refined set (INSCRS) and external neutrosophic soft cubic refined set (ENSCRS), along with their various properties, including P-order, P-union, P-intersection, and R-order, R-union, and R-intersection neutrosophic soft cubic refined set.

Keywords: Neutrosophic Soft Cubic Refined Set (NSCRS); Neutrosophic Cubic Refined Set (NCRS); internal neutrosophic soft cubic refined set (INSCRS), external neutrosophic soft cubic refined set (ENSCRS)

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1. Introduction

Zadeh [25] introduced Fuzzy sets, which address ambiguity related to imprecise states, perceptions, and preferences. Turksen [19] expanded fuzzy set to an interval valued fuzzy set after Zadeh. There are numerous real-world uses for interval valued fuzzy sets. Several authors have provided its applications in the medical field, including Sambuc [16],Kohout [12],Mukherjee and Sarkar [14]. The concept of cubic sets was first presented by Jun et al. [10] as a combination of fuzzy sets and interval valued fuzzy sets. Later, he [11] expanded on this idea by introducing the concept of neutrosophic cubic sets, which are sets that are neutrosophic in nature. Molodtsov [13] Soft Sets introduce a different approach by combining set theory with parameterized information. A soft set is defined by a collection of parameterized sets, which allows for the handling of uncertainties and variations in the data.Researchers are presenting alternative ideas on a daily basis, attesting to the significance of uncertainties that cannot be characterized by traditional mathematics. Several significant findings on this subject include fuzzy sets, intuitionistic fuzzy sets, ambiguous sets, interval-valued fuzzy sets, rough sets, and classical probability theory. However, Molodtsov pointed out that each of these theories has inherent problems. In 1999, Smarandache [17] proposed the notion of neutrosophic sets, which are a generalization of fuzzy sets, classical sets, and their fuzzy counterparts, by incorporating an independent indeterminacy-membership function. The neutrosophic set allows for the explicit quantification of indeterminacy and the complete independence of truth-membership (T), indeterminacy-membership (I), and false-membership (F).The idea of interval neutrosophic sets, which are generalizations of fuzzy sets, interval valued fuzzy sets, and neutrosophic sets more flexible, was first presented by Wang et al. [20] after Smarandache [17]. It was discovered that interval neutrosophic sets had advantages over neutrosophic sets. The concept of the neutrosophic soft cubic set was introduced and some of its characteristics were examined in different fields by R.A.Cruz and F.N.Irudhayam [1]- [5]

Yager [24] first presented a novel theory known as the theory of bags, or multisets. Subsequently, multisets were first introduced as helpful structures in a variety of computer science and mathematics domains, including database queries, by Blizard [6] and Calude et al. [9]. Smarandache [18] has presented the definition and applications of n-valued refined neutrosophic logic. N-valued interval neutrosophic sets were later introduced by Said Broumi et al. [15], which underlined their use in medical diagnosis. Anjan Mukherjee [26]introduced refined soft set and discussed the several operations between refined soft sets and soft sets.The issue of determining the membership function is just not there in soft set theory. This facilitates the theory's practical application and makes it convenient. Numerous domains could benefit from the application of soft set theory.

In the realm of advanced mathematical theories, Neutrosophic Soft Cubic Refined Sets represent a sophisticated extension of classical set theory, integrating concepts from neutrosophy, soft set theory, and cubic set. This innovative framework addresses the limitations of traditional sets by incorporating the flexibility of neutrosophy, which deals with the uncertainty and imprecision inherent in real-world data. Neutrosophic Soft Cubic Refined Sets combine the principles of neutrosophy, which allows for the representation of truth, falsity, and indeterminacy in a more nuanced manner, with the soft set theory's ability to handle vague and uncertain information. The cubic aspect introduces a three-dimensional approach to managing these uncertainties, adding another layer of depth to the analysis and interpretation of data. In practical terms, this refined set model enhances the ability to represent and process complex information that may not fit neatly into binary or fuzzy frameworks. It offers a robust tool for addressing real-world problems where data is imprecise, incomplete, or conflicting. This approach finds applications across various fields, including decision-making, information retrieval, and data analysis, providing a more comprehensive and adaptable methodology for managing uncertainty. Overall, Neutrosophic Soft Cubic Refined Sets offer a powerful and flexible approach for tackling complex problems where traditional methods fall short, paving the way for more accurate and insightful analysis in uncertain environments.A general approach to addressing uncertainty problems is soft set theory. This paper discusses thefundamental aspects of the Neutrosophic soft cubic refined set.

2. Preliminaries

Definition 2.1. [11]
Let X be a non-empty set. By a cubic set, we mean a structure $\Xi = \{\langle A(x), \mu(x)\rangle | x \in X\}$ in which A is an interval valued fuzzy set (IVF) and $\mu_(x)$ is a fuzzy set. It is denoted by $\langle A, \mu\rangle$.

Definition 2.2. [17]
Let X be an universe. Then a neutrosophic set(NS) λ is an object having the form $\lambda = \{\langle x : T(x), I(x), F(x)\rangle : x\varepsilon X\}$ where the functions $T, I, F : X \rightarrow]0, 1^+[$ define respectively the degree of truth, the degree of indeterminacy, and the degree of falsehood of the element $x\varepsilon X$ to the set λ with the condition
$-0 \leq T(x) + I(x) + F(x) \leq 3+$.

Definition 2.3. [20]
Let X be a non-empty set. An interval neutrosophic set (INS) A in X is characterized by the truth-membership function A_T, the indeterminacy-membership function A_I and the falsity-membership function A_F. For each point $x \in X, A_T(x), A_I(x), A_F(x) \subseteq [0, 1]$.

Definition 2.4. [23] Let E be a universe. A n-valued neutrosophic sets on E can be defined as follows:
$A =< x, (T^1_A(x), T^2_A(x), ..., T^p_A(x)), (I^1_A(x), I^2_A(x), ..., I^p_A(x)), (F^1_A(x), F^2_A(x), , F^p_A(x)) >: x \in X$
where
$T^1_A(x), T^2_A(x), ..., T^p_A(x), I^1_A(x), I^2_A(x), ..., I^p_A(x), F^1_A(x), F^2_A(x), ..., F^p_A(x) : E \longrightarrow [0, 1]$ such that
$0 \leq T^i_A(x) + I^i_A(x) + F^i_A(x) \leq 3$ for i=1, 2,,p for any $x \in X$,
Here, $(T^1_A(x), T^2_A(x), , T^p_A(x)), (I^1_A(x), I^2_A(x), , I^p_A(x)) and (F^1_A(x), F^2_A(x), , F^p_A(x))$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively. Also, P is called the dimension of n-valued neutrosophic sets (NVNS) A.

3. NEUTROSOPHIC CUBIC REFINED SET

In this section, Neutrosophic cubic refined set, complement, union, and intersection are defined in this paper related to Neutrosophic soft cubic refined set.

Definition 3.1. Let X be a universe. A neutrosophic cubic refined set (NCRS) on X is defined as $A = (\mathcal{A}, \Lambda)$ where

$$\begin{aligned}\mathcal{A} &= \{< x, ([T_A^{-1}(x), T_A^{+1}(x)], [T_A^{-2}(x), T_A^{+2}(x)], \ldots, [T_A^{-p}(x), T_A^{+p}(x)]) \\ &\qquad ([I_A^{-1}(x), I_A^{+1}(x)], [I_A^{-2}(x), I_A^{+2}(x)], \ldots, [I_A^{-q}(x), I_A^{+q}(x)]), \\ &\qquad ([F_A^{-1}(x), F_A^{+1}(x)], [F_A^{-2}(x), F_A^{+2}(x)], \ldots, [F_A^{-r}(x), F_A^{+r}(x)]) >: x \in X\}\end{aligned}$$

$$\Lambda = \{< x, (t_A^1(x), t_A^2(x), \ldots, t_A^p(x)), (i_A^1(x), i_A^2(x), \ldots, i_A^q(x)), \quad (f_A^1(x), f_A^2(x), \ldots, f_A^r(x)) >: x \in X\}$$

where $T_A^{-l}(x), T_A^{+l}(x) I_A^{-m}(x), I_A^{+m}(x), F_A^{-n}(x), F_A^{+n}(x), t_A^l(x), i_A^m(x), f_A^n(x) \in [0,1]$, such that $0 \leq T_A^{+l}(x) + I_A^{+m}(x) + F_A^{+n}(x) \leq 3$, $0 \leq t_A^l(x) + i_A^m(x) + f_A^n(x) \leq 3$ for all $l = 1, 2, \ldots, p; m = 1, 2, \ldots, q; n = 1, 2, \ldots, r$.

In our study, we focus only on the case p=q=r ,where p is called the dimension of neutrosophic cubic refined set (NCRS). The set of all neutrosophic cubic refined sets on X is denoted by NCRS(X).

Definition 3.2. The complement of $A = (A, \wedge)$ is denoted by A^c and is defined as

$A^c = (A^c, A^c)$

$A^c = \{< x, ([1 - T_A^{+1}(x), 1 - T_A^{-1}(x))], [1 - T_A^{+2}(x), 1 - T_A^{-2}(x))], \ldots, [1 - T_A^{+p}(x), 1 - T_A^{-p}(x))]), ([1 - I_A^{+1}(x), 1 - I_A^{-1}(x))], [1 - I_A^{+2}(x), 1 - I_A^{+2}(x))], \ldots, [1 - I_A^{+q}(x), 1 - I_A^{-q}(x))]), ([1 - F_A^{+1}(x), 1 - F_A^{-1}(x))], [1 - F_A^{+2}(x), 1 - F_A^{-2}(x))], \ldots, [1 - F_A^{+p}(x), 1 - F_A^{-r}(x))]) >: x \epsilon X\}$

$\wedge^c = \{< x, (1 - T_\lambda^1(x), 1 - T_\lambda^2(x), \ldots, 1 - T_\lambda^p(x)), (1 - I_\lambda^1(x), 1 - I_\lambda^2(x), ldots, 1 - I_\lambda^p(x)), (1 - F_\lambda^1(x), 1 - F_\lambda^2(x), \ldots, 1 - F_\lambda^p(x)) >: x \epsilon X\}$

Definition 3.3. Let $A = (\mathcal{A}, \Lambda)$,$B = (\mathcal{B}, \Psi)$ be two neutrosophic cubic refined set.Then,union of A and B,denoted by $A \cup B = (\mathcal{A} \cup \mathcal{B}, \Lambda \cup \Psi)$,is defined as

$$\begin{aligned}\mathcal{A} \cup \mathcal{B} = \Big\{ &< x, \big([T_A^{-1}(x) \vee T_B^{-1}(x), T_A^{+1}(x) \vee T_B^{+1}(x)], [T_A^{-2}(x) \vee T_B^{-2}(x)], [T_A^{+2}(x) \vee T_B^{+2}(x)], \ldots, \\ &[T_A^{-p}(x) \vee T_B^{-p}(x), T_A^{+p}(x) \vee T_B^{+p}(x)]\big), \big([I_A^{-1}(x) \vee I_B^{-1}(x), I_A^{+1}(x) \vee I_B^{+1}(x)], \\ &[I_A^{-2}(x) \vee I_B^{-2}(x)], [I_A^{+2}(x) \vee I_B^{+2}(x)], \ldots, [I_A^{-p}(x) \vee I_B^{-p}(x), I_A^{+p}(x) \vee I_B^{+p}(x)]\big), \\ &\big([F_A^{-1}(x) \vee F_B^{-1}(x), F_A^{+1}(x) \vee F_B^{+1}(x)], [F_A^{-2}(x) \vee F_B^{-2}(x)], [F_A^{+2}(x) \vee F_B^{+2}(x)], \ldots, \\ &[F_A^{-p}(x) \vee F_B^{-p}(x), F_A^{+p}(x) \vee F_B^{+p}(x)]\big)\Big\} \\ \Lambda \cup \Psi = \Big\{ &< x, \big([t_A^1(x) \vee t_B^1(x), [t_A^2(x) \vee t_B^2(x)], \ldots, t_A^p(x) \vee t_B^p(x)]\big), \\ &\big([i_A^1(x) \vee i_B^1(x), [i_A^2(x) \vee i_B^2(x)], \ldots, i_A^p(x) \vee i_B^p(x)]\big), \\ &\big([f_A^1(x) \vee f_B^1(x), [f_A^2(x) \vee f_B^2(x)], \ldots, f_A^p(x) \vee f_B^p(x)]\big)\Big\}\end{aligned}$$

Definition 3.4. Let $A = (\mathcal{A}, \Lambda)$, $B = (\mathcal{B}, \Psi)$ be two neutrosophic cubic refined sets. Then intersection of A and B,denoted by $A \cap B$,is defined as

$$\mathcal{A} \cap \mathcal{B} = \Big\{ < x, \big([T_A^{-1}(x) \wedge T_B^{-1}(x), T_A^{+1}(x) \wedge T_B^{+1}(x)], [T_A^{-2}(x) \wedge T_B^{-2}(x)], [T_A^{+2}(x) \wedge T_B^{+2}(x)], \ldots, [T_A^{-p}(x) \wedge T_B^{-p}(x), T_A^{+p}(x) \wedge T_B^{+p}(x)]\big), \big([I_A^{-1}(x) \wedge I_B^{-1}(x), I_A^{+1}(x) \wedge I_B^{+1}(x)], [I_A^{-2}(x) \wedge I_B^{-2}(x)], [I_A^{+2}(x) \wedge I_B^{+2}(x)], \ldots, [I_A^{-p}(x) \wedge I_B^{-p}(x), I_A^{+p}(x) \wedge I_B^{+p}(x)]\big), \big([F_A^{-1}(x) \wedge F_B^{-1}(x), F_A^{+1}(x) \wedge F_B^{+1}(x)], [F_A^{-2}(x) \wedge F_B^{-2}(x)], [F_A^{+2}(x) \wedge F_B^{+2}(x)], \ldots, [F_A^{-p}(x) \wedge F_B^{-p}(x), F_A^{+p}(x) \wedge F_B^{+p}(x)]\big) \Big\}$$

$$\Lambda \cap \Psi = \Big\{ < x, ([t_A^1(x) \wedge t_B^1(x), [t_A^2(x) \wedge t_B^2(x)], \ldots, t_A^p(x) \wedge t_B^p(x)]), ([i_A^1(x) \wedge i_B^1(x), [i_A^2(x) \vee i_B^2(x)], \ldots, i_A^p(x) \wedge i_B^p(x)]), ([t_A^1(x) \wedge f_B^1(x), [f_A^2(x) \wedge f_B^2(x)], \ldots, f_A^p(x) \wedge f_B^p(x)]) \Big\}$$

4. NEUTROSOPHIC SOFT CUBIC REFINED SET

In this section presents the fundamental definitions of Neutrosophic soft cubic refined set, Complement, Union, Intersection , P-order , R-order, P-uion, R union ,INSCR and ENSCRS.

Definition 4.1. Let W be an initial universe set. Let $NSCRS(W)$ denote the set of all neutrosophic cubic refined sets and E be the set of parameters. Let $N \subset E$ then the pair (P, N) is termed to be the neutrosophic soft cubic refined set over W where P is a mapping given by $P : N \to NCR(W)$.

Example 4.2. Trisha wants to buy a washing machine so she can wash her clothes. She must assess a special washing machine that meets the requirements of a regular machine. Assume that $W = \{w1, w2, w3, w4\}$ represents a variety of machine brands and $E = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of parameters, where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ represent Expensive, Beautiful, Cheap, Very expensive. Let $N = \{\alpha_2, \alpha_3\} \subseteq E$ Then, the neutrosophic soft cubic refined set, $(P, N) = \{P(\alpha_i) = \{\langle w, A_{\alpha_i}), \Lambda_{\alpha_i}\rangle : w \in W\} \, \alpha_i \in N\}, i = 1, 2, 3, 4 \in W$ *is represented in tabular form* Table 1

Example 4.3. From the above example for $W = \{w_1\}$ and $P(\alpha_1)$ be neutrosophic refined cubic sets in W is also represented as $(P, N) = \{P(\alpha_1) = \{\langle w_1, A_{\alpha_1}, \Lambda_{\alpha_1}\rangle : w_1 \in W\} \, \alpha_1 \in N\}$
$A\alpha_1 = \{< w_1, \{([0.1, 0.2], [0.3, 0.6], [0.6, 0.8]),$
$([0, 0.5], [0.4, 0.7], [0.4, 0.5]), ([0.2, 0.3][0, 0.5][0.4, 0.6]) >\}\}$
$\Lambda e_1 = \{< w_1, (0.1, 0.3, 0.7)(0.3, 0.5, 0.4)(0.1, 0.3, 0.5) >\}$

Definition 4.4. The complement of a neutrosophic soft cubic refined set $(P, N) = \{\{\langle w, A_{\alpha_i}, \Lambda_{\alpha_i}\rangle : w \in W\} \, \alpha_i \in N\}$ is denoted by $(P, N)^c$ where $P^c : N \to NCR(W)$ and

$(P,N)^c = \{(\{\langle w, A^c_{\alpha_i}, \Lambda^c_{\alpha_i}\rangle : w \in W\}\ \alpha_i \in N\}$

$$\begin{aligned} A^c_{\alpha_i} =& \{< w, ([1-T^{+1}_{A_{\alpha_i}}(w), 1-T^{-1}_{A_{\alpha_i}}(w))], [1-T^{+2}_{A_{\alpha_i}}(w), 1-T^{-2}_{A_{\alpha_i}}(w))], \ldots, [1-T^{+p}_{A_{\alpha_i}}(w), 1-T^{-p}_{A_{\alpha_i}}(w))]), \\ & ([1-I^{+1}_{A_{\alpha_i}}(w), 1-I^{-1}_{A_{\alpha_i}}(w))], [1-I^{+2}_{A_{\alpha_i}}(w), 1-I^{+2}_{A_{\alpha_i}}(w))], \ldots, [1-I^{+q}_{A_{\alpha_i}}(w), 1-I^{-q}_{A_{\alpha_i}}(w))]), \\ & ([1-F^{+1}_{A_{\alpha_i}}(w), 1-F^{-1}_{A_{\alpha_i}}(w))], [1-F^{+2}_{A_{\alpha_i}}(w), 1-F^{-2}_{A_{\alpha_i}}(w))], \ldots, [1-F^{+p}_{A_{\alpha_i}}(w), 1-F^{-r}_{A_{\alpha_i}}(w))]) >: w \in W\} \\ \Lambda {\alpha_i}^c =& \{< w, (1-t^1_{A_{\alpha_i}}(w), 1-t^2_{A_{\alpha_i}}(w), \ldots, 1-t^p_{A_{\alpha_i}}(w)), (1-i^1_{A_{\alpha_i}}(w), 1-i^2_{A_{\alpha_i}}(w), \\ & \ldots, 1-i^p_{A_{\alpha_i}}(w)), (1-f^1_{A_{\alpha_i}}(w), 1-f^2_{A_{\alpha_i}}(w), \ldots, 1-f^p_{A_{\alpha_i}}(w)) >: w \in W\} \end{aligned}$$

Definition 4.5. Let (P , N) be neutrosophic soft cubic refined set over W.

(i) (P , N) is called absolute or universal neutrosophic soft cubic refined set W if $P(e) = \hat{W}$ for all $\alpha \in$ E. We denote it by W.

(ii) (P , N) is called null or empty neutrosophic soft cubic set U if $P(e) = \hat{\Phi}\}$ for all $\alpha \in$ E. We denote it by Φ.

Obviously $\Phi^c = W$ and $W^c = \Phi$.

Definition 4.6. Let (P,N) and (Q,M) be two neutrosophic soft cubic refined set. Then the union of(P,N) and (Q,M) is denoted by $P(\alpha_i) \cup Q(\alpha_i)$ and is defined by $(H,C) = (H(\alpha_i)$ where $C = N \cup Mand(H,C) = (P,N) \cup (Q,M))$ $P(\alpha_i) \cup Q(\alpha_i) = H_{\alpha_i} =$

Table 1. The tabular representation of (P,N)

N	α_1	α_2	α_3
w_1	([0.1,0.2],[0.3,.6],[0.6,.8]) ([0,0.5],[0.4,0.7],[0.4,0.5]) ([0.2,0.3][0,0.5][0.4,0.6]) ((0.1,0.3,0.7)(0.3,0.5,0.4)(0.1,0.3,0.5)	([0,0.5],[0.2,0.6],[0,0.4]) ([0,0.5],[0.2,0.6],[0.4,0.5]) ([0,0.7][0.3,0.7][0.3,0.5]) (0.7,0.3,0.2)(0.3,0.5,0.4)(0.8,0.3,0.1)	([0.1,0.3][0.3,0.5],[0,0.7]) ([0.0,0.3][1,0.3],[0,0.5]) ([0,0.6][0.4,0.5][0.3,0.4]) ([0,0.6][0.4,0.5][0.3,0.4])(0.1,0.1,0.9)
w_2	([0.2,0.4],[0.2,0.3][0,0.5]) ([0.6,0.7][0,0.5][0.1,0.5])) ([0.3,0.6][0.2,0.7][0,0.3]) (0.6,0.4,0.3)(0.2,0.3,0.7)(0.4,0.3,0.1)	([0.4,0.6][0.1,0.5][0.2,0.3]) ([0.4,0.7][0.4,0.6][0.3,0.4]) ([0.2,0.3][0.3,0.5][0.5,0.7]) (0.4,0.1,0.7)(0.3,0.3,0.5)(0.6,0.4,0.2)	([0.2,0.3][0.4,0.5][0.1,0.2]) ([0.3,0.5][0.2,0.5][0.2,.3]) ([0.4,0.5][0.2,0.5][0,0.3]) (0.3,0.2,0.6)(0.4,0.5,0.1)(0.1,0.1,0.7)
w_3	([0.1,0.4],[0.1,0.3][.6,.8]) ([.5,.6][0,.5][.1,.5])) ([0.4,.6][.2,.6][0,.3]) (0.4,0.4,0.5)(0.1,0.3,0.6)(0.6,0.3,0.1)	([.5,.6][0,.6][.2,.3]) ([0.3,.6][.4,.6][.3,.4]) ([.1,.3][.2,.3][.5,.6]) (0.4,0.1,0.6)(0.3,0.4,0.5)(0.8,0.3,0.1)	([.3,.4][0.3,0.5][.1,.2]) ([0.1,.5][.3,.5][0.1,.3]) ([.4,.5][.3,.5][0.2,.3]) (0.3,0.1,0.4)(0.5,0.5,0.2)(0.1,0.2,0.6)
w_4	([0.2,0.4],[0.2,0.3][0,.5][.7,.8]) ([.6,.8][0,.5][.1,.5])) ([0.3,.6][.2,.8][0,.3]) (0.6,0.4,0.3)(0.1,0.3,0.8)(0.6,0.3,0.1)	([.5,.6][0,.6][.1,.3]) ([0.3,.8][0.3,.6][.3,.4]) ([.1,.3][.2,.3][.5,.8]) (0.4,0.1,0.8)(0.3,0.4,0.5)(0.8,0.3,0.1)	([.2,.5][0.4,.5][.2,.3]) ([0.3,.5][.2,.5][0,.3]) ([0.2,.5][.1,.5][0.2,.3]) (0.4,0.2,0.6)(0.8,0.5,0.1)(0.1,0.1,0.9)

$\{\langle w, (A_{\alpha_i} \cup B_{\alpha_i}), (\Lambda_{\alpha_i} \cup \Psi_{\alpha_i})\rangle : w \in W\}\ \alpha_i \in N \cap M where$

$$
\begin{aligned}
A_{\alpha_i} \cup B_{\alpha_i} =& \Big\{ < w, \big([T^{-1}_{A_{\alpha_i}}(w) \vee T^{-1}_{B_{\alpha_i}}(w), T^{+1}_{A_{\alpha_i}}(w) \vee T^{+1}_{B_{\alpha_i}}(w)], [T^{-2}_{A_{\alpha_i}}(w) \vee T^{-2}_{B_{\alpha_i}}(w)], [T^{+2}_{A_{\alpha_i}}(w) \vee T^{+2}_{B_{\alpha_i}}(w)], \dots, \\
& [T^{-p}_{A_{\alpha_i}}(w) \vee T^{-p}_{B_{\alpha_i}}(w), T^{+p}_{A_{\alpha_i}}(w) \vee T^{+p}_{B_{\alpha_i}}(w)]\big), \big([I^{-1}_{A_{\alpha_i}}(w) \vee I^{-1}_{B_{\alpha_i}}(w), I^{+1}_{A_{\alpha_i}}(w) \vee I^{+1}_{B_{\alpha_i}}(w)], \\
& [I^{-2}_{A_{\alpha_i}}(w) \vee I^{-2}_{B_{\alpha_i}}(w)], [I^{+2}_{A_{\alpha_i}}(w) \vee I^{+2}_{B_{\alpha_i}}(w)], \dots, [I^{-p}_{A_{\alpha_i}}(w) \vee I^{-p}_{B_{\alpha_i}}(w), I^{+p}_{A_{\alpha_i}}(w) \vee I^{+p}_{B_{\alpha_i}}(w)]\big), \\
& \big([F^{-1}_{A_{\alpha_i}}(w) \vee F^{-1}_{B_{\alpha_i}}(w), F^{+1}_{A_{\alpha_i}}(w) \vee F^{+1}_{B_{\alpha_i}}(w)], [F^{-2}_{A_{\alpha_i}}(w) \vee F^{-2}_{B_{\alpha_i}}(w)], [F^{+2}_{A_{\alpha_i}}(w) \vee F^{+2}_{B_{\alpha_i}}(w)], \dots, \\
& [F^{-p}_{A_{\alpha_i}}(w) \vee F^{-p}_{B_{\alpha_i}}(w), F^{+p}_{A_{\alpha_i}}(w) \vee F^{+p}_{B_{\alpha_i}}(w)]\big)\Big\} \\
\Lambda_{\alpha_i} \cup \Psi_{\alpha_i} =& \Big\{ < w, \big([t^{1}_{A_{\alpha_i}}(w) \vee t^{1}_{B_{\alpha_i}}(w), [t^{2}_{A_{\alpha_i}}(w) \vee t^{2}_{B_{\alpha_i}}(w)], \dots, t^{p}_{A_{\alpha_i}}(w) \vee t^{p}_{B_{\alpha_i}}(w)]\big), \\
& \big([i^{1}_{A_{\alpha_i}}(w) \vee i^{1}_{B_{\alpha_i}}(w), [i^{2}_{A_{\alpha_i}}(w) \vee i^{2}_{B_{\alpha_i}}(w)], \dots, i^{p}_{A_{\alpha_i}}(w) \vee i^{p}_{B_{\alpha_i}}(w)]\big), \\
& \big([f^{1}_{A_{\alpha_i}}(w) \vee f^{1}_{B_{\alpha_i}}(w), [f^{2}_{A_{\alpha_i}}(w) \vee f^{2}_{B_{\alpha_i}}(w)], \dots, f^{p}_{A_{\alpha_i}}(w) \vee f^{p}_{B_{\alpha_i}}(w)]\big)\Big\}
\end{aligned}
$$

Definition 4.7. Let (P, N) and (Q, M) be two neutrosophic soft cubic refined set. Then the intersection of(P, N) and (Q, M) is denoted by $P(\alpha_i) \cap Q(\alpha_i)$ and is defined by $(H, C) = (H_{\alpha_i}$ where $C = N \cup M and (H, C) = (P, N) \cap (Q, M))$

$$P(\alpha_i) \cap Q(\alpha_i) = H_{\alpha_i} = \{\langle w, (A_{\alpha_i} \cap B_{\alpha_i}), (\Lambda_{\alpha_i} \cap \Psi_{\alpha_i})\rangle : w \in W\}\ \alpha_i \in N \cap M\, where$$

$$
\begin{aligned}
A_{\alpha_i} \cap B_{\alpha_i} =& \Big\{ < w, \big([T^{-1}_{A_{\alpha_i}}(w) \wedge T^{-1}_{B_{\alpha_i}}(w), T^{+1}_{A_{\alpha_i}}(w) \wedge T^{+1}_{B_{\alpha_i}}(w)], [T^{-2}_{A_{\alpha_i}}(w) \wedge T^{-2}_{B_{\alpha_i}}(w)], [T^{+2}_{A_{\alpha_i}}(w) \wedge T^{+2}_{B_{\alpha_i}}(w)], \dots, \\
& [T^{-p}_{A_{\alpha_i}}(w) \wedge T^{-p}_{B_{\alpha_i}}(w), T^{+p}_{A_{\alpha_i}}(w) \wedge T^{+p}_{B_{\alpha_i}}(w)]\big), \big([I^{-1}_{A_{\alpha_i}}(w) \wedge I^{-1}_{B_{\alpha_i}}(w), I^{+1}_{A_{\alpha_i}}(w) \wedge I^{+1}_{B_{\alpha_i}}(w)], \\
& [I^{-2}_{A_{\alpha_i}}(w) \wedge I^{-2}_{B_{\alpha_i}}(w)], [I^{+2}_{A_{\alpha_i}}(w) \wedge I^{+2}_{B_{\alpha_i}}(w)], \dots, [I^{-p}_{A_{\alpha_i}}(w) \wedge I^{-p}_{B_{\alpha_i}}(w), I^{+p}_{A_{\alpha_i}}(w) \wedge I^{+p}_{B_{\alpha_i}}(w)]\big), \\
& \big([F^{-1}_{A_{\alpha_i}}(w) \wedge F^{-1}_{B_{\alpha_i}}(w), F^{+1}_{A_{\alpha_i}}(w) \wedge F^{+1}_{B_{\alpha_i}}(w)], [F^{-2}_{A_{\alpha_i}}(w) \wedge F^{-2}_{B_{\alpha_i}}(w)], [F^{+2}_{A_{\alpha_i}}(w) \wedge F^{+2}_{B_{\alpha_i}}(w)], \dots, \\
& [F^{-p}_{A_{\alpha_i}}(w) \wedge F^{-p}_{B_{\alpha_i}}(w), F^{+p}_{A_{\alpha_i}}(w) \wedge F^{+p}_{B_{\alpha_i}}(w)]\big)\Big\} \\
\Lambda_{\alpha_i} \cap \Psi_{\alpha_i} =& \Big\{ < w, ([t^{1}_{A_{\alpha_i}}(w) \wedge t^{1}_{B_{\alpha_i}}(w), [t^{2}_{A_{\alpha_i}}(w) \wedge t^{2}_{B_{\alpha_i}}(w)], \dots, t^{p}_{A_{\alpha_i}}(w) \wedge t^{p}_{B_{\alpha_i}}(w)]), \\
& ([i^{1}_{A_{\alpha_i}}(w) \wedge i^{1}_{B_{\alpha_i}}(w), [i^{2}_{A_{\alpha_i}}(w) \vee i^{2}_{B_{\alpha_i}}(w)], \dots, i^{p}_{A_{\alpha_i}}(w) \wedge i^{p}_{B_{\alpha_i}}(w)]), \\
& ([t^{1}_{A_{\alpha_i}}(w) \wedge f^{1}_{B_{\alpha_i}}(w), [f^{2}_{A_{\alpha_i}}(w) \wedge f^{2}_{B_{\alpha_i}}(w)], \dots, f^{p}_{A_{\alpha_i}}(w) \wedge f^{p}_{B_{\alpha_i}}(w)])\Big\}
\end{aligned}
$$

Definition 4.8. A neutrosophic soft cubic refined set (P, N) is contained in the other neutrosophic soft cubic refined set (P, M), denoted by $(P, N) \subseteq (P, M)$
i.e.$(P, N) \subseteq (P, M)$, if and only if $A_{\alpha_i}(w) \subseteq B_{\alpha_i}(w)$ and $\Lambda_{\alpha_i}(w) \subseteq \Psi_{\alpha_i}(w)$ where

$$\begin{aligned} A_{\alpha_i}(w) \subseteq B_{\alpha_i}(w) \Rightarrow & T^{-k}_{A_{\alpha_i}}(w) \leq T^{-k}_{B_{\alpha_i}}(w), T^{+k}_{A_{\alpha_i}}(w) \leq T^{+k}_{B_{\alpha_i}}(w), \\ & I^{-k}_{A_{\alpha_i}}(w) \geq I^{-k}_{B_{\alpha_i}}(w), I^{+k}_{A_{\alpha_i}}(w) \geq I^{+k}_{B_{\alpha_i}}(w), \\ & F^{-k}_{A_{\alpha_i}}(w) \geq F^{-k}_{B_{\alpha_i}}(w), F^{+k}_{A_{\alpha_i}}(w) \geq F^{+k}_{B_{\alpha_i}}(w), \\ \Lambda_{\alpha_i}(w) \subseteq \Psi_{\alpha_i}(w) \Rightarrow & t^{k}_{A_{\alpha_i}}(w) \leq t^{k}_{B_{\alpha_i}}(w), i^{k}_{A_{\alpha_i}}(w) \geq i^{k}_{B_{\alpha_i}}(w), \\ & f^{k}_{A_{\alpha_i}}(w) \geq f^{k}_{B_{\alpha_i}}(w), \ for\ all\ w \in W, i = 1, 2, \ldots, p. \end{aligned}$$

Definition 4.9. Let W be an universal set. A neutrosophic soft cubic refined set (P, N) in W is said to be

- truth-internal (briefly, $T-$internal)NSCRS if the following inequality is valid

$$(\forall\ w \in W, \alpha_i \in N)(T^{-k}_{A_{\alpha_i}}(w) \leq t^{k}_{A_{\alpha_i}}(w) \leq T^{+k}_{A_{\alpha_i}}(w)), \tag{1}$$

- indeterminacy-internal (briefly, $I-$internal)NSCRS if the following inequality is valid

$$(\forall\ w \in W, \alpha_i \in N)(I^{-k}_{A_{\alpha_i}}(w) \leq i^{k}_{A_{\alpha_i}}(w) \leq I^{+k}_{A_{\alpha_i}}(w)), \tag{2}$$

- falsity-internal (briefly, $F-$internal)NSCRS if the following inequality is valid

$$(\forall\ w \in W, \alpha_i \in N)(F^{-k}_{A_{\alpha_i}}(w) \leq f^{k}_{A_{\alpha_i}}(w) \leq F^{+k}_{A_{\alpha_i}}(w)), \tag{3}$$

If a neutrosophic soft refined cubic set in W satisfies (1), (2) and (3) we say that (P, M) is an internal neutrosophic soft cubic refined set (INSCRS) in W.

Definition 4.10. Let W be an universal set. A neutrosophic soft cubic refined set (P, N) in W is said to be

- truth-external (briefly, $T-$external)NSCRS if the following inequality is valid

$$(\forall w \in W,\ \alpha_i \in N) t^{k}_{A_{\alpha_i}} \notin [T^{-k}_{A_{\alpha_i}}, T^{+k}_{A_{\alpha_i}}], \tag{4}$$

- indeterminacy-external (briefly, $I-$external)NSCRS if the following inequality is valid

$$(\forall w \in W,\ \alpha_i \in N) i^{k}_{A_{\alpha_i}} \notin [I^{-k}_{A_{\alpha_i}}, I^{+k}_{A_{\alpha_i}}], \tag{5}$$

- falsity-external (briefly, $F-$external)NSCRS if the following inequality is valid

$$(\forall w \in W,\ \alpha_i \in N) f^{k}_{A_{\alpha_i}} \notin [F^{-k}_{A_{\alpha_i}}, F^{+k}_{A_{\alpha_i}}], \tag{6}$$

If a neutrosophic soft refined cubic set in W satisfies (4), (5) and (6) we say that (P, N) is an external neutrosophic soft cubic refined set (ENSCRS) in W.

Theorem 4.11. *Let $(P, N) = \{\{\langle w, A_{\alpha_i}, \lambda_{\alpha_i}\rangle : w \in W\}\, \alpha_i \in N\}$ be a neutrosophic soft cubic refined set in W which is not an $ENSCRS$. Then, there exists at least one $\alpha_i \in N$ for which there exist some $w \in W$ such that $t^{k}_{A_{\alpha_i}}(w) \in [T^{-k}_{A_{\alpha_i}}(w), T^{+k}_{A_{\alpha_i}}(w)]$, $i^{k}_{A_{\alpha_i}}(w) \in [I^{-k}_{A_{\alpha_i}}(w), I^{+k}_{A_{\alpha_i}}(w)]$, $f^{k}_{A_{\alpha_i}}(w) \in [F^{-k}_{A_{\alpha_i}}(w), F^{+k}_{A_{\alpha_i}}(w)]$,* *Proof.* By the definition of an external neutrosophic soft cubic refined set (ENSCRS) we know that

$$t^k_{A_{\alpha_i}} \notin [T^{-k}_{A_{\alpha_i}}, T^{+k}_{A_{\alpha_i}}],$$

$$i^k_{A_{\alpha_i}} \notin [I^{-k}_{A_{\alpha_i}}, I^{+k}_{A_{\alpha_i}}],$$

$$f^k_{A_{\alpha_i}} \notin [F^{-k}_{A_{\alpha_i}}, F^{+k}_{A_{\alpha_i}}],$$

for all $w \in W$, corresponding to each $\alpha_i \in N$. But given that (P, N) is not ENSCRS so far at least one $\alpha_i \in N$ there exists some $w \in W$ such that

$$T^{-k}_{A_{\alpha_i}}(w) \leq t^k_{A_{\alpha_i}}(w) \leq T^{+k}_{A_{\alpha_i}}(w).$$

$$I^{-k}_{A_{\alpha_i}}(w) \leq i^k_{A_{\alpha_i}}(w) \leq I^{+k}_{A_{\alpha_i}}(w).$$

$$F^{-k}_{A_{\alpha_i}}(w) \leq f^k_{A_{\alpha_i}}(w) \leq F^{+k}_{A_{\alpha_i}}(w).$$

Hence the result. □

Theorem 4.12. *Let (P, N) be a neutrosophic soft cubic refined set in W. If (P, N) is both $T-$internal soft cubic refined set and $T-$external soft cubic refined set in W, then* $(\forall\ w \in W,\ \alpha_i \in N)$

$$t^k_{A_{\alpha_i}}(w) \in \left\{T^{-k}_{A_{\alpha_i}}(w)/w \in W,\ \alpha_i \in N\right\} \cup \left\{T^{+k}_{A_{\alpha_i}}(w)/w \in W, \alpha_i \in N\right\} \tag{7}$$

Proof. Consider the definitions 4.9 and 4.10 which implies that

$$T^{-k}_{A_{\alpha_i}}(w) \leq t^k_{A_{\alpha_i}}(w) \leq T^{+k}_{A_{\alpha_i}}(w) \text{ and } t^k_{A_{\alpha_i}}(w) \notin [T^{-k}_{A_{\alpha_i}}(w), T^{+k}_{A_{\alpha_i}}(w)] \text{ for all } w \in W,\ \alpha_i \in N.$$

Then it follows that $t^k_{A_{\alpha_i}}(w) = T^{-k}_{A_{\alpha_i}}(w)$ or $t^k_{A_{\alpha_i}}(w) = T^{+k}_{A_{\alpha_i}}(w)$,
and hence $t^k_{A_{\alpha_i}}(w) \in \left\{T^{-k}_{A_{\alpha_i}}(w)/w \in W,\ \alpha_i \in N\right\} \cup \left\{T^{+k}_{A_{\alpha_i}}(w)/w \in W, \alpha_i \in N\right\}$

Hence proved Similarly, the following Theorems hold for the indeterminate and falsity values.

□

Theorem 4.13. *Let (P, N) be a neutrosophic soft cubic refined set in a non-empty set W. If (P, N) is both $I-$ internal NSCRS and $I-$ external NSCRS, then* $(\forall\ w \in W,\ \alpha_i \in N), i^k_{A_{\alpha_i}}(w) \in \left\{I^{-k}_{A_{\alpha_i}}(w)/w \in W,\ \alpha_i \in N\right\} \cup \left\{I^{+k}_{A_{\alpha_i}}(w)/w \in W, \alpha_i \in N\right\}$

Theorem 4.14. *Let (P, N) be a neutrosophic soft cubic refined set in a non-empty set W. If (P, N) is both $F-$ Internal NSCRS and $F-$ External NSCRS, then* $(\forall\ w \in W,\ \alpha_i \in N), f^k_{A_{\alpha_i}}(w) \in \left\{F^{-k}_{A_{\alpha_i}}(w)/w \in W,\ \alpha_i \in N\right\} \cup \left\{F^{+k}_{A_{\alpha_i}}(w)/w \in W, \alpha_i \in N\right\}$

Definition 4.15. Let $(P,N)=\{P(\alpha_i)=\{\langle w,A_{\alpha_i},\Lambda_{\alpha_i}\rangle : w\in W\}\,\alpha_i\in N\}$ and

$$(Q,M)=\{Q(\alpha_i)=\{\langle w,B_{\alpha_i},\Psi_{\alpha_i}\rangle : w\in W\}\,\alpha_i\in M\}$$

be two neutrosophic soft cubic refined sets in W. Let N and M be any two subsets of E (set of parameters), then we have the following

(1) (P, N)=(Q, M) if and only if the following conditions are satisfied
   (a) $N=M$ and
   (b) $P(\alpha_i)=Q(\alpha_i)$ for all $\alpha_i\in N$ if and only if $A_{\alpha_i}=B_{\alpha_i}$ and $\Lambda_{\alpha_i}=\Psi_{\alpha_i}$ for all $w\in W$ corresponding to each $\alpha_i\in N$.

(2) If (P,N) and (Q,M) are two neutrosophic soft cubic refined sets then we define and denote $P-$ order as $(P,N)\subseteq_P(Q,M)$ if and only if the following conditions are satisfied
   (a) $N\subseteq M$ and
   (b) $P(\alpha_i)\leq_P Q(\alpha_i)$ for all $\alpha_i\in N$ if and only if $A_{\alpha_i}\subseteq B_{\alpha_i}$ and $\Lambda_{\alpha_i}\leq\Psi_{\alpha_i}$ for all $w\in W$ corresponding to each $\alpha_i\in M$.

(3) If (P,N) and (Q,M) are two neutrosophic soft cubic refined sets then we define and denote $R-$ order as $(P,N)\subseteq_R(Q,M)$ if and only if the following conditions are satisfied
   (a) $N\subseteq M$ and
   (b) $P(\alpha_i)\leq_R Q(\alpha_i)$ for all $\alpha_i\in N$ if and only if $A_{\alpha_i}\subseteq B_{\alpha_i}$ and $\Lambda_{\alpha_i}\geq\Psi_{\alpha_i}$ for all $w\in W$ corresponding to each $\alpha_i\in N$.

Definitions of the $P-$union, $P-$intersection, $R-$union and $R-$intersection of neutrosophic soft cubic refined sets as follows:

Definition 4.16. Let (P,N) and (Q,M) be two neutrosophic soft cubic refined sets (NSCRS) in W where N and M are any two subsets of the parameters set E. Then we define $P-$union as $(P,N)\cup_p(Q,M)=(H,C)$ where $C=N\cup M$

$$H(\alpha_i)=\left\{\begin{array}{cl} P(\alpha_i) & \text{if } \alpha_i\in N-M\\ Q(\alpha_i) & \text{if } \alpha_i\in M-N\\ P(\alpha_i)\cup_P Q(\alpha_i) & \text{if } \alpha_i\in N\cap M\end{array}\right\}$$

where $P(\alpha_i)\cup_P Q(\alpha_i)$ is defined as

$$P(\alpha_i)\cup_P Q(\alpha_i)=\{\langle w,\{A_{\alpha_i}\cup B_{\alpha_i}\},(\Lambda_{\alpha_i}\cup\Psi_{\alpha_i})\rangle : w\in W\}\,\alpha_i\in N\cap M$$

Definition 4.17. Let (P,N) and (Q,M) be two neutrosophic soft cubic refined sets (NSCRS) in W where N and M are any subsets of the parameter set E. Then we define P-intersection as $(P,N)\cap_P (Q,M)=(H,C)$ where $C=M\cap N$, $H(\alpha_i)=P(\alpha_i)\cap_P Q(\alpha_i)$ and $\alpha_i\in M\cap N$. Here $P(\alpha_i)\cap_P Q(\alpha_i)$ is defined as

$$P(\alpha_i)\cap_P Q(\alpha_i)=H(\alpha_i)=\{\langle w,(A_{\alpha_i}\cap B_{\alpha_i}),(\lambda_{\alpha_i}\cap\Psi_{\alpha_i})\rangle : w\in W\}\,\alpha_i\in N\cap M$$

Definition 4.18. Let (P,N) and (Q,M) be two neutrosophic soft set cubic sets (NSCS) in W where N and M are any subsets of the parameter set E. Then we define $R-$union of neutrosophic soft cubic refined set as $(P,N)\cup_R (Q,M)=(H,C)$ where $C=N\cup M$,

$$H(\alpha_i)=\left\{\begin{array}{cc} P(\alpha_i) & \text{if } \alpha_i\in N-M\\ Q(\alpha_i) & \text{if } \alpha_i\in M-N\\ P(\alpha_i)\cup_R Q(\alpha_i) & \text{if } \alpha_i\in M\cap N\end{array}\right\}$$

where $P(\alpha_i)\cup_R Q(\alpha_i)$ is defined as

$$P(\alpha_i)\cup_R Q(\alpha_i)=\{\langle w,\{A_{\alpha_i}\cup B_{\alpha_i}\},(\Lambda_{\alpha_i}\cap\Psi_{\alpha_i})\rangle : w\in W\}\,\alpha_i\in M\cap N$$

Definition 4.19. Let (P,N) and (Q,M) be two neutrosophic soft cubic refined sets (NSCRS) in W where N and W are any subsets of the parameter set E. Then we define $R-$intersection of neutrosophic soft cubic refined set as $(P,N)\cap_R (Q,M)=(H,C)$ where $C=N\cap M$, $H(\alpha_i)=P(\alpha_i)\cap_R Q(\alpha_i)$, $H(\alpha_i)=P(\alpha_i)\cap_R Q(\alpha_i)$ and $\alpha_i\in M\cap N$. Here $P(\alpha_i)\cap_R Q(\alpha_i)$ is defined as

$$P(\alpha_i)\cap_R Q(\alpha_i)=H(e_i)=\{\langle w,\{A_{\alpha_i}\cap B_{\alpha_i}\},(\Lambda_{\alpha_i}\cup\Psi_{\alpha_i})\rangle : w\in W\}\,\alpha_i\in N\cap M$$

5. Conclusions

As research and applications continue to evolve, the Neutrosophic Soft Cubic Refined Sets framework stands out for its ability to handle complex data with greater accuracy and insight. Its development marks a significant step forward in the ongoing quest for more effective and adaptable tools in the mathematical and data sciences, promising to contribute meaningfully to the advancement of both theoretical and practical applications. In this paper we study the concept of Neutrosophic Cubic Refined Set (NSCRS) and extended to Neutrosophic Soft Cubic Refined Set (NSCRS), which is the combination of the theory of soft sets, Neutrosophic cubic set, Neutrosophic refined set and Neutrosophic Soft Cubic Set. Further studied the INSCR and ENSCRS, along with their various properties, including P-order, P-union, P-intersection, and R-order, R-union, and R-intersection neutrosophic soft cubic refined set.With the motivation of ideas presented in this paper, one can think of similarity measures, Cartesian products, and relations on Neutrosophic refined cubic soft sets. Further studies on the topology generated by the Neutrosophic refined cubic soft sets. It is hoped that the combinations of Neutrosophic refined cubic soft sets with topologyand rough sets will generate potentially interesting some new research direction.

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Cubic Spherical Neutrosophic Sets for Advanced Decision-Making

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ABSTRACT. In the context of aggregation operators and multiple criteria decision-making, this article presents the idea of cubic spherical neutrosophic sets. The notion of neutrosophic informations are transform into a geometric sphere by determining its center and radius. This advanced geometrical representation extends traditional neutrosophic informations. This study defines and discusses weighted additive and weighted geometric aggregation operators tailored to cubic spherical neutrosophic sets that are vital for handling complex and uncertain information. Practical example, such as evaluating fertilizer brands for coconut farming, illustrate their application in decision-making contexts. By integrating cubic spherical neutrosophic sets into multiple criteria decision-making frameworks, decision makers can effectively manage uncertainty and make informed decisions. When multiple stakeholders are involved in the decision-making process, averaging their decision values may not accurately reflect a true perspective. This multiple criteria decision-making approach overcomes the limitations of traditional averaging method. Theoretical discussions and practical examples contribute to advancing the understanding and application of multiple criteria decision-making, enhancing the reliability of decision support systems.

Keywords: Neutrosophic set; cubic spherical neutrosophic set; cubic spherical neutrosophic aggregation operators; multi criteria decision making.

1. Introduction

In 1998, F. Smarandache [19, 20] introduced and examined neutrosophic sets (NSs) as an extension of Atanassov's [6] theory of Intuitionistic fuzzy sets. Since then, various generalizations of NSs have emerged, finding applications across diverse fields. Particularly in Multiple Criteria Decision Making (MCDM), NSs and their variants play a crucial role. In 2014, Peng et al. [17] proposed an outranking approach tailored for MCDM problems within a simplified neutrosophic framework. They also devised a ranking approach utilizing outranking relations of simplified neutrosophic numbers to address

MCDM issues, demonstrating their method through illustrative examples. In 2015, Majumdar [15] introduced concepts of distance and similarity between NSs, vital for identifying interacting segments in datasets. Moreover, the notion of entropy was proposed to quantify uncertainty in neutrosophic sets, underlining its importance in decision-making. In the same year, Deli, et al. [10] introduced bipolar NSs and associated operations, proposing functions for comparison and aggregation. They also developed operators and functions for a bipolar neutrosophic multiple criteria decision-making approach. Jun Ye [23] introduced trapezoidal NSs, proposing aggregation operators and a decision-making method for problems represented by trapezoidal neutrosophic numbers. In 2016, Biswas et al. [7] introduced a new method for multi attribute group decision-making, extending the preference technique to single-valued neutrosophic settings. Ratings of alternatives per attribute are expressed as single-valued NSs, reflecting decision makers views. A weighted averaging operator based on single-valued NSs aggregates opinions, forming a unified consensus on criteria and alternatives importance. In 2018, Ali et al. [3] explored interval complex NSs and their application in green supplier selection, demonstrating efficiency through real dataset examples from Thuan Yen JSC. Neutrosophic sets and their extensions find diverse applications across various domains such as medical diagnosis [9, 18], medical robotics engineering [16], healthcare services supply chain [5], disaster risk management [1], handling imperfect and incomplete information [2], information processing [8], pattern recognition [4, 9], image segmentation [13], and addressing financial issues [11]. The notion of CSNS was introduced and studied by Gomathi et al. [12] as a geometrical representation of collection of NSs. Recently Krishnaprakash [14] et al. introduced cubic spherical neutrosophic Archimedean triangular norms(ATN) and conorms (ATCN), also applied the concepts in MCDM with an example of selection of electric truck.

This article delves into the introduction of CSNSs, a novel advancement in aggregation operators and multiple criteria decision-making (MCDM). CSNSs offer a transformative approach by converting neutrosophic data into geometric spheres, thereby establishing a geometric representation with discernible centers and radii. This sophisticated representation extends the capabilities of traditional neutrosophic sets, enabling a more comprehensive understanding of uncertain information. The study proceeds to define and explore weighted additive and weighted geometric aggregation operators specifically tailored to CSNSs. These operators play a vital role in navigating the complexities of uncertain information, providing decision-makers with robust tools for informed decision-making. To illustrate the practical application of CSNSs, the article presents examples such as the evaluation of fertilizer brands for coconut farming. Through these examples, decision-makers gain insight into how these innovative methodologies can be effectively employed in decision-making contexts. By integrating CSNSs into MCDM frameworks, decision-makers can effectively manage uncertainty and make informed decisions across diverse domains. Theoretical discussions, coupled with practical examples, contribute to advancing the understanding and application of CSNSs, thereby enhancing the reliability of decision support systems. In essence, this article serves as a catalyst for the adoption of innovative methodologies, ensuring robust decision-making processes amidst uncertainty in various domains. Through the exploration of CSNSs, decision-makers are empowered to navigate complex decision landscapes with confidence and precision.

The following contributions are made in the field of neutrosophic sets and MCDM:

(1) Weighted additive and weighted geometric aggregation operators tailored to cubic spherical neutrosophic sets are defined and discussed.
(2) The applicability of weighted additive and weighted geometric aggregation operators in multiple criteria decision-making (MCDM) contexts are demonstrated.
(3) Development of comprehensive framework for integrating CSNSs into MCDM has been made.
(4) Practical examples to illustrate their effectiveness in real-world decision-making have been provided.
(5) To overcome the limitations of traditional averaging methods when multiple stakeholders are involved in decision-making are illustrated.

2. Preliminaries

Let Γ be the universal set containing elements known as Neutrosophic Sets [19] (NSs). Each $\epsilon_i \in \Gamma$ is defined as $\mathcal{N}_{\epsilon_\imath} = \{\langle \epsilon_\imath, \mathbb{T}(\epsilon_\imath), \mathbb{I}(\epsilon_\imath), \mathbb{F}(\epsilon_\imath)\rangle | \epsilon_\imath \in \Gamma\}$, where $\mathbb{T}(\epsilon_\imath)$, $\mathbb{F}(\epsilon_\imath)$, $\mathbb{I}(\epsilon_\imath) : \Gamma \to [0,1]$ represent the degrees of membership, non-membership, and indeterminacy of ϵ_i. These degrees satisfy $0 \leq \mathbb{T}(\epsilon_i) + \mathbb{I}(\epsilon_\imath) + \mathbb{F}(\epsilon_i) \leq 3$ for all $\epsilon_\imath \in \Gamma$ and $\imath = 1, 2, 3, ..., n$. Let Γ be the universal set containing elements known as Cubic Spherical Neutrosophic sets [12] (CSNSs). Each $\epsilon_\imath \in \Gamma$ is defined as $\delta_{\mathbb{R}} = \{\langle \epsilon_\imath, csn\mathbb{T}(\epsilon_\imath), csn\mathbb{I}(\epsilon_\imath), csn\mathbb{F}(\epsilon_\imath); \mathbb{R}\rangle : \epsilon_\imath \in \Gamma\}$, where $csn\mathbb{T}(\epsilon)$, $csn\mathbb{F}(\epsilon)$, $csn\mathbb{I}(\epsilon), \mathbb{R} : \Gamma \to [0,1]$ represent the degrees of membership, non-membership, indeterminacy and radius of $\delta_{\mathbb{R}}$. These degrees satisfy $0 \leq csn\mathbb{T}(\epsilon_i) + csn\mathbb{I}(\epsilon_i) + csn\mathbb{F}(\epsilon_i) \leq 3$ for all $\epsilon_\imath \in \Gamma$ and $\imath = 1, 2, \ldots, k$. Where the center

$$\langle csn\mathbb{T}(\epsilon_\imath), csn\mathbb{I}(\epsilon_\imath), csn\mathbb{F}(\epsilon_\imath)\rangle = \left\langle \frac{\sum_{\jmath=1}^{k_\imath} \mathbb{T}_{\imath,\jmath}}{k_\imath}, \frac{\sum_{\jmath=1}^{k_\imath} \mathbb{I}_{\imath,\jmath}}{k_\imath}, \frac{\sum_{\jmath=1}^{k_\imath} \mathbb{F}_{\imath,\jmath}}{k_\imath} \right\rangle \tag{1}$$

and the radius

$$\mathbb{R}_\imath = \min\left\{ \max_{1\leq \jmath \leq k_\imath} \sqrt{(csn\mathbb{T}(\epsilon_\imath) - \mathbb{T}_{\imath,\jmath})^2 + (csn\mathbb{I}(\epsilon_\imath) - \mathbb{I}_{\imath,\jmath})^2 + (csn\mathbb{F}(\epsilon_\imath) - \mathbb{F}_{\imath,\jmath})^2}, 1 \right\}. \tag{2}$$

For example, choose $X = \{\alpha, \beta\}$ and $\delta_1, \delta_2 \in NS(X)$ such that $\delta_1 = \{\langle \alpha, 0.88, 0.33, 0.22\rangle, \langle \alpha, 0.77, 0.44, 0.11\rangle, \langle \alpha, 0.55, 0.44, 0.22\rangle, \langle \alpha, 0.66, 0.55, 0.33\rangle\}$ and $\delta_2 = \{\langle \beta, 0.66, 0.22, 0.11\rangle, \langle \beta, 0.88, 0.11, 0.22\rangle, \langle \beta, 0.88, 0.33, 0.11\rangle, \langle \beta, 0.99, 0.44, 0.22\rangle\}$. Then the CSNSs are $\delta_{(R_1)} = \{\langle \alpha, 0.72, 0.44, 0.22; 0.20\rangle : \alpha \in X\}$ and $\delta_{(R_2)} = \{\langle \beta, 0.85, 0.28, 0.17; 0.22\rangle : \beta \in X\}$.

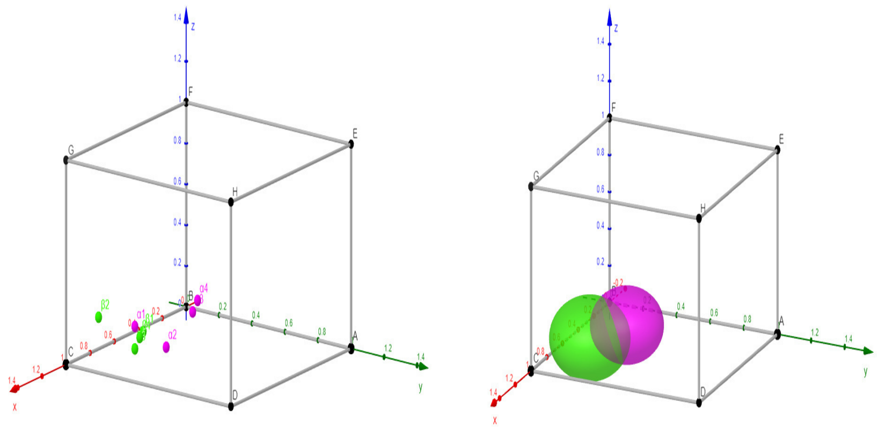

FIGURE 1. The geomentric representation of neutrosophic sets and CSNSs

For any two CSNSs δ_1 and δ_2 defined as $\delta_1 = \langle csn\mathbb{T}_{\delta_1}, csn\mathbb{I}_{\delta_1}, csn\mathbb{F}_{\delta_1}; \mathbb{R}_{\delta_1}\rangle$ and $\delta_2 = \langle csn\mathbb{T}_{\delta_2}, csn\mathbb{I}_{\delta_2}, csn\mathbb{F}_{\delta_2}; \mathbb{R}_{\delta_2}\rangle$, the cosine distance [12] is defined by: $\cos(\delta_1, \delta_2) =$

$$1 - \frac{csn\mathbb{T}_{\delta_1}\cdot csn\mathbb{T}_{\delta_2} + csn\mathbb{I}_{\delta_1}\cdot csn\mathbb{I}_{\delta_2} + csn\mathbb{F}_{\delta_1}\cdot csn\mathbb{F}_{\delta_2}}{\|csn\mathbb{T}_{\delta_1}\|\cdot\|csn\mathbb{T}_{\delta_2}\| + \|csn\mathbb{I}_{\delta_1}\|\cdot\|csn\mathbb{I}_{\delta_2}\| + \|csn\mathbb{F}_{\delta_1}\|\cdot\|csn\mathbb{F}_{\delta_2}\|} \times \frac{|\mathbb{R}_{\delta_1} - \mathbb{R}_{\delta_2}|}{\max(\mathbb{R}_{\delta_1}, \mathbb{R}_{\delta_2})}$$

Lemma 2.1. *[12] Let $\delta_1 = \langle csn\mathbb{T}_{\delta_1}, csn\mathbb{I}_{\delta_1}, csn\mathbb{F}_{\delta_1}; \mathbb{R}_{\delta_1}\rangle$ and $\delta_2 = \langle csn\mathbb{T}_{\delta_2}, csn\mathbb{I}_{\delta_2}, csn\mathbb{F}_{\delta_2}; \mathbb{R}_{\delta_2}\rangle$ be two CSNS over $\mathbb{X}$ and $\alpha > 0$. The subsequent operations are then described as follows:*

(1) $\delta_1 \oplus \delta_2 = \langle csn\mathbb{T}_{\delta_1} + csn\mathbb{T}_{\delta_2} - csn\mathbb{T}_{\delta_1} csn\mathbb{T}_{\delta_2}, csn\mathbb{I}_{\delta_1} csn\mathbb{I}_{\delta_2}, csn\mathbb{F}_{\delta_1} csn\mathbb{F}_{\delta_2};$
$\mathbb{R}_{\delta_1} + \mathbb{R}_{\delta_2} - \mathbb{R}_{\delta_1}\mathbb{R}_{\delta_2}\rangle$.

(2) $\delta_1 \otimes \delta_2 = \langle csn\mathbb{T}_{\delta_1} csn\mathbb{T}_{\delta_2}, csn\mathbb{I}_{\delta_1} + csn\mathbb{I}_{\delta_2} - csn\mathbb{I}_{\delta_1} csn\mathbb{I}_{\delta_2}, csn\mathbb{F}_{\delta_1} + csn\mathbb{F}_{\delta_2} -$
$csn\mathbb{F}_{\delta_1} csn\mathbb{F}_{\delta_2}; \mathbb{R}_{\delta_1}\mathbb{R}_{\delta_2}\rangle$.

(3) $\alpha\delta_1 = \langle 1 - (1 - csn\mathbb{T}_{\delta_1})^{\alpha}, (csn\mathbb{I}_{\delta_1})^{\alpha}, (csn\mathbb{F}_{\delta_1})^{\alpha}; 1 - (1 - \mathbb{R}_{\delta_1})^{\alpha}\rangle$.

(4) ${\delta_1}^{\alpha} = \left\langle csn\mathbb{T}^{\alpha}_{\delta_1}, 1 - (1 - csn\mathbb{I}_{\delta_1})^{\alpha}, 1 - (1 - csn\mathbb{F}_{\delta_1})^{\alpha}; \mathbb{R}^{\alpha}_{\delta_1}\right\rangle$.

3. Cubic Spherical Neutrosophic Aggregation Operators

Definition 3.1. Let δ_ε $(\varepsilon = 1, 2, ..., \lambda)$ be a CSNSs. Then the cubic spherical neutrosophic weighted

(1) arithmetic operator is $CSNWAO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) = \sum_{\varepsilon=1}^{\lambda} \omega'_\varepsilon \delta_\varepsilon$,

(2) geometric operator is $CSNWGO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) = \prod_{\varepsilon=1}^{\lambda} \omega'_\varepsilon \delta_\varepsilon$,

where $\omega'_\epsilon = (\omega'_1, \omega'_2, ..., \omega'_\lambda)^T$ is the weight vector of $\delta_\varepsilon(\varepsilon = 1, 2, ..., \lambda)$, $\omega'_\varepsilon \in [0, 1]$ and $\sum_{\varepsilon=1}^{\lambda} \omega'_\varepsilon = 1$.

Theorem 3.2. *For a CSNS $\delta_\varepsilon(\varepsilon = 1, 2, ..., \lambda)$, we have the following result:*
$CSNWAO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) =$

$$\left\langle 1 - \prod_{\varepsilon=1}^{\lambda}(1 - csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{\lambda}(csn\mathbb{I}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{\lambda}(csn\mathbb{F}_{\delta_\varepsilon})^{\omega'_\varepsilon}; 1 - \prod_{\varepsilon=1}^{\lambda}(1 - \mathbb{R}_{\delta_\varepsilon})^{\omega'_\varepsilon}\right\rangle \quad (3)$$

where $\omega'_\epsilon = (\omega'_1, \omega'_2, ..., \omega'_\lambda)^T$ is the weight vector of $\delta_\varepsilon(\varepsilon = 1, 2, ..., \lambda)$, $\omega'_\varepsilon \in [0, 1]$ and $\sum_{\varepsilon=1}^{\lambda} \omega'_\varepsilon = 1$.

Proof: *Mathematical induction can be used to prove the Theorem.*

Case 1: when $\lambda = 2$, then

$$\omega'_1\delta_1 = \left\langle 1-(1-csn\mathbb{T}_{\delta_1})^{\omega'_1}, (csn\mathbb{I}_{\delta_1})^{\omega'_1}, (csn\mathbb{F}_{\delta_1})^{\omega'_1}; 1-(1-\mathbb{R}_{\delta_1})^{\omega'_1} \right\rangle,$$
$$\omega'_2\delta_2 = \left\langle 1-(1-csn\mathbb{T}_{\delta_2})^{\omega'_2}, (csn\mathbb{I}_{\delta_2})^{\omega'_2}, (csn\mathbb{F}_{\delta_2})^{\omega'_2}; 1-(1-\mathbb{R}_{\delta_2})^{\omega'_2} \right\rangle.$$

Thus,

$$\begin{aligned} &CSNWAO_{\omega'}(\delta_1,\delta_2) = \omega'_1\delta_1 + \omega'_2\delta_2 \\ &= \Big\langle 2-(1-csn\mathbb{T}_{\delta_1})^{\omega'_1}-(1-csn\mathbb{T}_{\delta_2})^{\omega'_2}-(1-(1-csn\mathbb{T}_{\delta_1})^{\omega'_1})(1-(1-csn\mathbb{T}_{\delta_2})^{\omega'_2}), \\ &\quad csn\mathbb{I}_{\delta_1}^{\omega'_1}+csn\mathbb{I}_{\delta_2}^{\omega'_2}, csn\mathbb{F}_{\delta_1}^{\omega'_1}+csn\mathbb{F}_{\delta_2}^{\omega'_2}; 2-(1-cs\mathbb{R}_{\delta_1})^{\omega'_1}-(1-cs\mathbb{R}_{\delta_2})^{\omega'_2} \\ &\quad -(1-(1-cs\mathbb{R}_{\delta_1})^{\omega'_1})(1-(1-cs\mathbb{R}_{\delta_2})^{\omega'_2})\Big\rangle \\ &= \Big\langle 1-(1-csn\mathbb{T}_{\delta_1})^{\omega'_1}(1-csn\mathbb{T}_{\delta_2})^{\omega'_2}, csn\mathbb{I}_{\delta_1}^{\omega'_1}+csn\mathbb{I}_{\delta_2}^{\omega'_2}, csn\mathbb{F}_{\delta_1}^{\omega'_1}+csn\mathbb{F}_{\delta_2}^{\omega'_2}; \\ &\quad 1-(1-\mathbb{R}_{\delta_1})^{\omega'_1}(1-\mathbb{R}_{\delta_2})^{\omega'_2}\Big\rangle \end{aligned}$$

Case 2: when $\lambda = z$, then

$$CSNWAO_{\omega'}(\delta_1,\delta_2,\ldots,\delta_z) = \left\langle 1-\prod_{\varepsilon=1}^{z}(1-csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{z}(csn\mathbb{I}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{z}(csn\mathbb{F}_{\delta_\varepsilon})^{\omega'_\varepsilon}; \right.$$
$$\left. 1-\prod_{\varepsilon=1}^{z}(1-\mathbb{R}_{\delta_\varepsilon})^{\omega'_\varepsilon} \right\rangle$$

Case 3: when $\lambda = z+1$, then

$$\begin{aligned} &CSNWAO_{\omega'}(\delta_1,\delta_2,\ldots,\delta_{z+1}) \\ &= \left\langle 1-\prod_{\varepsilon=1}^{z}(1-csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_\varepsilon}+(1-(1-csn\mathbb{T}_{\delta_{z+1}})^{\omega'_{z+1}})-(1-\prod_{\varepsilon=1}^{z}(1-csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_\varepsilon}) \right. \\ &\quad (1-(1-csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_{z+1}}), \prod_{\varepsilon=1}^{z} csn\mathbb{I}_{\delta_\varepsilon}^{\omega'_\varepsilon} csn\mathbb{I}_{\delta_{z+1}}^{\omega'_{z+1}}, \prod_{\varepsilon=1}^{z} csn\mathbb{F}_{\delta_\varepsilon}^{\omega'_\varepsilon} csn\mathbb{F}_{\delta_{z+1}}^{\omega'_{z+1}}; 1-\prod_{\varepsilon=1}^{z}(1-\mathbb{R}_{\delta_\varepsilon})^{\omega'_\varepsilon}+ \\ &\quad \left.(1-(1-\mathbb{R}_{\delta_{z+1}})^{\omega'_{z+1}})-(1-\prod_{\varepsilon=1}^{z}(1-\mathbb{R}_{\delta_\varepsilon})^{\omega'_\varepsilon})(1-(1-\mathbb{R}_{\delta_\varepsilon})^{\omega'_{z+1}}) \right\rangle \\ &= \left\langle 1-\prod_{\varepsilon=1}^{z+1}(1-csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{z+1} csn\mathbb{I}_{\delta_\varepsilon}^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{z+1} csn\mathbb{F}_{\delta_\varepsilon}^{\omega'_\varepsilon}; 1-\prod_{\varepsilon=1}^{z+1}(1-\mathbb{R}_{\delta_\varepsilon})^{\omega'_\varepsilon} \right\rangle \end{aligned}$$

In light of the aforementioned findings, equation (4) results for every λ, The proof is now complete.

The following characteristics of the $CSNWAO$ operator are evident:

(1) **Idempotency :** Let δ_ε $(\varepsilon = 1,2,\ldots,\lambda)$ be a collection of CSNSs. If δ_ε $(\varepsilon = 1,2,\ldots,\lambda)$ is equal, that is $\delta_\varepsilon = \lambda$ for $\varepsilon = 1,2,\ldots,,\lambda$, then $CSNWAO_{\omega'}(\delta_1,\delta_2,\ldots,\delta_\lambda) = \lambda$.

(2) **Boundedness :** Let δ_ε $(\varepsilon = 1, 2, \ldots, \lambda)$ be a collection of CSNSs,
$A^- = \langle \min_\varepsilon csn\mathbb{T}_{\delta_\varepsilon}, \max_\varepsilon csn\mathbb{I}_{\delta_\varepsilon}, \max_\varepsilon csn\mathbb{F}_{\delta_\varepsilon}; \min_\varepsilon \mathbb{R}_{\delta_\varepsilon}\rangle$ and
$A^+ = \langle \max_\varepsilon csn\mathbb{T}_{\delta_\varepsilon}, \min_\varepsilon csn\mathbb{I}_{\delta_\varepsilon}, \min_\varepsilon csn\mathbb{F}_{\delta_\varepsilon}; \max_\varepsilon \mathbb{R}_{\delta_\varepsilon}\rangle$
for $(\varepsilon = 1, 2, \ldots, \lambda)$, then $A^- \subseteq CSNWAO \subseteq A^+$.

(3) **Monotonity :** Let δ_ε $(\varepsilon = 1, 2, \ldots, \lambda)$ be a collection of CSNSs. If $\delta_\varepsilon \subseteq \delta^*_\varepsilon$ for $\varepsilon = 1, 2, \ldots, , \lambda$, then $CSNWAO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) \subseteq CSNWAO_{\omega'}(\delta^*_1, \delta^*_2, \ldots, \delta^*_\lambda)$.

Theorem 3.3. *For a CSNS $\delta_\varepsilon(\varepsilon = 1, 2, ..., \lambda)$, we have the following result:*
$CSNWGO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) =$

$$\left\langle \prod_{\varepsilon=1}^{\lambda} csn\mathbb{T}_{\delta_\varepsilon}^{\omega'_\varepsilon}, 1 - \prod_{\varepsilon=1}^{\lambda}(1 - csn\mathbb{I}_{\delta_\varepsilon})^{\omega'_\varepsilon}, 1 - \prod_{\varepsilon=1}^{\lambda}(1 - csn\mathbb{F}_{\delta_\varepsilon})^{\omega'_\varepsilon}; \prod_{\varepsilon=1}^{\lambda} \mathbb{R}_{\delta_\varepsilon}^{\omega'_\varepsilon} \right\rangle \tag{4}$$

where $\omega'_\epsilon = (\omega'_1, \omega'_2, ..., \omega'_\lambda)^T$ is the weight vector of $\delta_\varepsilon(\varepsilon = 1, 2, ..., \lambda)$, $\omega'_\varepsilon \in [0, 1]$ and $\sum_{\varepsilon=1}^{\lambda} \omega'_\varepsilon = 1$.

By the similar manner, we will prove Theorem 2.3.

(1) **Idempotency :** Let δ_ε $(\varepsilon = 1, 2, \ldots, \lambda)$ be a collection of CSNSs. Ifδ_ε $(\varepsilon = 1, 2, \ldots, \lambda)$ is equal, that is $\delta_\varepsilon = A$ for $\varepsilon = 1, 2, \ldots, , \lambda$, then $CSNWGO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) = A$.

(2) **Boundedness :** Let δ_ε $(\varepsilon = 1, 2, \ldots, \lambda)$ be a collection of CSNSs,
$A^- = \langle \min_\varepsilon csn\mathbb{T}_{\delta_\varepsilon}, \max_\varepsilon csn\mathbb{I}_{\delta_\varepsilon}, \max_\varepsilon csn\mathbb{F}_{\delta_\varepsilon}; \min_\varepsilon \mathbb{R}_{\delta_\varepsilon}\rangle$ and
$A^+ = \langle \max_\varepsilon csn\mathbb{T}_{\delta_\varepsilon}, \min_\varepsilon csn\mathbb{I}_{\delta_\varepsilon}, \min_\varepsilon csn\mathbb{F}_{\delta_\varepsilon}; \max_\varepsilon \mathbb{R}_{\delta_\varepsilon}\rangle$
for $(\varepsilon = 1, 2, \ldots, \lambda)$, then $A^- \subseteq CSNWGO \subseteq A^+$.

(3) **Monotonity :** Let δ_ε $(\varepsilon = 1, 2, \ldots, \lambda)$ be a collection of CSNSs. If $\delta_\varepsilon \subseteq \delta^*_\varepsilon$ for $\varepsilon = 1, 2, \ldots, , \lambda$, then $CSNWGO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) \subseteq CSNWGO_{\omega'}(\delta^*_1, \delta^*_2, \ldots, \delta^*_\lambda)$.

4. Model for Multi Criteria Decision Making in Cubic Spherical Neutrosophic sets

In this section, we propose a Multi-Criteria Decision Making (MCDM) approach using the cubic spherical neutrosophic CSNWA and CSNWGA operators. When multiple decision makers are involved in the decision-making process, simply averaging decision values may not accurately represent their collective perspective. The cubic spherical neutrosophic approach addresses the limitations of traditional averaging methods. We apply this approach to evaluate the usefulness of emerging technology commercialization.

Let $\bigstar = \{\bigstar_1, \bigstar_2 \ldots \bigstar_\lambda\}$ be a set of alternatives and $\emptyset = \{\emptyset_1, \emptyset_2 \ldots \emptyset_\lambda\}$ be a set of criteria. Suppose $(\delta_{\alpha\varepsilon})_{m\times n} = (csn\mathbb{T}_{\delta_\alpha}(\nu_\varepsilon), csn\mathbb{I}_{\delta_\alpha}(\nu_\varepsilon), csn\mathbb{F}_{\delta_\alpha}(\nu_\varepsilon))_{m\times n}$ is a neutrosopic decision matrix, where $csn\mathbb{T}_{\delta_\alpha}(\nu_\varepsilon)$ is the degree of membership of alternatives $\bigstar_\epsilon$, $csn\mathbb{I}_{\delta_\alpha}(\nu_\varepsilon)$ is the degree of neutral membership of alternatives $\bigstar_\epsilon$, and $csn\mathbb{F}_{\delta_\alpha}(\nu_\varepsilon)$ is the degree non-membership of alternatives $\bigstar_\epsilon$, each alternatives $\bigstar_\epsilon$ satisfy $0 \leq csn\mathbb{T}_{\delta_\alpha}(\nu_\varepsilon) + csn\mathbb{I}_{\delta_\alpha}(\nu_\varepsilon) + csn\mathbb{F}_{\delta_\alpha}(\nu_\varepsilon)) \leq 3$.

We propose the following algorithm to solve MCDM problem with cubic spherical neutrosophic information using cubic spherical neutrosophic CSNWA and CSNWGA operators.

Step 1: Start.

Step 2: Input: The available alternatives.

Step 3: Employ the decision information in the form of a matrix
$(\delta_{\alpha\varepsilon})_{m\times n} = (csn\mathbb{T}_{\delta_\alpha}(\nu_\varepsilon), csn\mathbb{I}_{\delta_\alpha}(\nu_\varepsilon), csn\mathbb{F}_{\delta_\alpha}(\nu_\varepsilon))_{m\times n}$.

$$(\delta_{\alpha\varepsilon})_{m\times n} = \begin{bmatrix} \langle csn\mathbb{T}_{11}, csn\mathbb{I}_{11}, csn\mathbb{F}_{11}\rangle & ... & \langle csn\mathbb{T}_{1\kappa}, csn\mathbb{I}_{1\kappa}, csn\mathbb{F}_{1\kappa}\rangle \\ \langle csn\mathbb{T}_{21}, csn\mathbb{I}_{21}, csn\mathbb{F}_{21}\rangle & ... & \langle csn\mathbb{T}_{2\kappa}, csn\mathbb{I}_{2\kappa}, csn\mathbb{F}_{2\kappa}\rangle \\ . & . & . \\ . & . & . \\ . & . & . \\ \langle csn\mathbb{T}_{\lambda 1}, csn\mathbb{I}_{\lambda 1}, csn\mathbb{F}_{\lambda 1}\rangle & ... & \langle csn\mathbb{T}_{\lambda\kappa}, csn\mathbb{I}_{\lambda\kappa}, csn\mathbb{F}_{\lambda\kappa}\rangle \end{bmatrix}$$

Step 4: For each alternative $\bigstar_\epsilon$, $(\epsilon = 1, 2 \ldots, \lambda)$ construct the cubic spherical neutrosophic set

$$\delta_\alpha = \{\langle \nu_\varepsilon, csn\mathbb{T}_{\delta_\alpha}(\nu_\varepsilon), csn\mathbb{I}_{\delta_\alpha}(\nu_\varepsilon), csn\mathbb{F}_{\delta_\alpha}(\nu_\varepsilon); \mathbb{R}_{\delta_\alpha}(\nu_\varepsilon)\rangle : \nu_\varepsilon \in \nu\}$$

$$(\delta_{\alpha\varepsilon})_{m\times n} = \begin{bmatrix} \langle csn\mathbb{T}_{11}, csn\mathbb{I}_{11}, csn\mathbb{F}_{11}; \mathbb{R}_{11}\rangle & ... & \langle csn\mathbb{T}_{1\kappa}, csn\mathbb{I}_{1\kappa}, csn\mathbb{F}_{1\kappa}; \mathbb{R}_{1\kappa}\rangle \\ \langle csn\mathbb{T}_{21}, csn\mathbb{I}_{21}, csn\mathbb{F}_{21}; \mathbb{R}_{21}\rangle & ... & \langle csn\mathbb{T}_{2\kappa}, csn\mathbb{I}_{2\kappa}, csn\mathbb{F}_{2\kappa}; \mathbb{R}_{2\kappa}\rangle \\ . & . & . \\ . & . & . \\ . & . & . \\ \langle csn\mathbb{T}_{\lambda 1}, csn\mathbb{I}_{\lambda 1}, csn\mathbb{F}_{\lambda 1}; \mathbb{R}_{\lambda 1}\rangle & ... & \langle csn\mathbb{T}_{\lambda\kappa}, csn\mathbb{I}_{\lambda\kappa}, csn\mathbb{F}_{\lambda\kappa}; \mathbb{R}_{\lambda\kappa}\rangle \end{bmatrix}$$

where

$$\langle csn\mathbb{T}(\epsilon_\imath), csn\mathbb{I}(\epsilon_\imath), csn\mathbb{F}(\epsilon_\imath)\rangle = \left\langle \frac{\sum_{\jmath=1}^{k_\imath} \mathbb{T}_{\imath,\jmath}}{k_\imath}, \frac{\sum_{\jmath=1}^{k_\imath} \mathbb{I}_{\imath,\jmath}}{k_\imath}, \frac{\sum_{\jmath=1}^{k_\imath} \mathbb{F}_{\imath,\jmath}}{k_\imath} \right\rangle$$

is the center of $\delta_{\alpha\varepsilon}$ and

$$\mathbb{R}_\imath = \min\left\{ \max_{1\leq \jmath \leq k_\imath} \sqrt{(csn\mathbb{T}(\epsilon_\imath) - \mathbb{T}_{\imath,\jmath})^2 + (csn\mathbb{I}(\epsilon_\imath) - \mathbb{I}_{\imath,\jmath})^2 + (csn\mathbb{F}(\epsilon_\imath) - \mathbb{F}_{\imath,\jmath})^2}, 1 \right\}$$

is the radius of the cubic spherical neutrosophic set $\delta_{\alpha\varepsilon}$ for all $\epsilon = 1, 2, \ldots \lambda$ from the decision matrix $(\delta_{\alpha\varepsilon})_{m\times n}$.

Step 5: Operate $CSNWAO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) =$

$$\left\langle 1 - \prod_{\varepsilon=1}^{\lambda}(1 - csn\mathbb{T}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{\lambda}(csn\mathbb{I}_{\delta_\varepsilon})^{\omega'_\varepsilon}, \prod_{\varepsilon=1}^{\lambda}(csn\mathbb{F}_{\delta_\varepsilon})^{\omega'_\varepsilon}; 1 - \prod_{\varepsilon=1}^{\lambda}(1 - \mathbb{R}_{\delta_\varepsilon})^{\omega'_\varepsilon} \right\rangle$$

and $CSNWGO_{\omega'}(\delta_1, \delta_2, \ldots, \delta_\lambda) =$

$$\left\langle \prod_{\varepsilon=1}^{\lambda} csn\mathbb{T}_{\delta_\varepsilon}^{\omega'_\varepsilon}, 1 - \prod_{\varepsilon=1}^{\lambda}(1 - csn\mathbb{I}_{\delta_\varepsilon})^{\omega'_\varepsilon}, 1 - \prod_{\varepsilon=1}^{\lambda}(1 - csn\mathbb{F}_{\delta_\varepsilon})^{\omega'_\varepsilon}; \prod_{\varepsilon=1}^{\lambda} \mathbb{R}_{\delta_\varepsilon}^{\omega'_\varepsilon} \right\rangle$$

to obtain the overall preference values of the alternative $\bigstar_\epsilon(\epsilon = 1, 2, \ldots \lambda)$.

Step 6: Calculate the cosine distance $\cos(\delta_1, \delta_2) =$

$$1 - \frac{csn\mathbb{T}_{\delta_1} \cdot csn\mathbb{T}_{\delta_2} + csn\mathbb{I}_{\delta_1} \cdot csn\mathbb{I}_{\delta_2} + csn\mathbb{F}_{\delta_1} \cdot csn\mathbb{F}_{\delta_2}}{\|csn\mathbb{T}_{\delta_1}\| \cdot \|csn\mathbb{T}_{\delta_2}\| + \|csn\mathbb{I}_{\delta_1}\| \cdot \|csn\mathbb{I}_{\delta_2}\| + \|csn\mathbb{F}_{\delta_1}\| \cdot \|csn\mathbb{F}_{\delta_2}\|} \times \frac{|\mathbb{R}_{\delta_1} - \mathbb{R}_{\delta_2}|}{\max(\mathbb{R}_{\delta_1}, \mathbb{R}_{\delta_2})}$$

where $\delta_2 = (1, 0, 0; 1)$ is the ideal sphere.

Step 7: The shortest distance value of $\cos(\delta_1, \delta_2)$ is the better alternative $\bigstar_\epsilon$, because it is close to the ideal alternative δ_2.

Step 8: Rank the alternatives $\bigstar_\epsilon, (\epsilon = 1, 2, \ldots \lambda)$ based on the cubic spherical neutrosophic CSNWA and CSNWGA operators evaluations and cosine distance.

Step 9: Output : Best alternative.

Step 10: End.

5. Numerical Example

The coconut industry in India is vital for rural economies, providing employment and contributing to food security and industrial raw materials. Enhanced productivity and diversified product applications continue to strengthen the economic impact of coconuts both nationally and globally. India cultivates several varieties of coconuts, categorized into tall and dwarf types. Some notable varieties include:

Tall Varieties

- **West Coast Tall (WCT)**: Known for its high yield and adaptability to various climatic conditions. It produces around 80-100 nuts per palm annually.
- **East Coast Tall (ECT)**: Another high-yielding variety with an annual production of 70-90 nuts per palm.
- **Tiptur Tall**: Commonly grown in Karnataka, yielding around 60-80 nuts per palm annually.

Dwarf Varieties

- **Chowghat Orange Dwarf (COD)**: Popular for its early bearing and high yield, producing around 50-60 nuts per palm annually.
- **Malayan Yellow Dwarf (MYD)**: Known for its high productivity, yielding 60-70 nuts per palm annually.
- **Gangabondam**: An early bearing variety with an average yield of 40-50 nuts per palm annually.

Hybrid Varieties

- **Chandrasankara (WCT x COD)**: Combines the high yield of tall varieties and the early bearing of dwarf varieties, producing around 100-120 nuts per palm annually.
- **Kerasankara (ECT x MYD)**: Another hybrid with high productivity, yielding 90-110 nuts per palm annually.

The selection of fertilizer for coconut trees is essential for maximizing yield, sustaining tree health, and ensuring the productivity of coconut plantations. Fertilizers provide vital nutrients like nitrogen, phosphorus, potassium, magnesium, and micronutrients, fostering vigorous vegetative growth, enhancing fruit development, and increasing fruit-bearing spikes for higher yields over time. Balanced fertilization also helps coconut trees withstand environmental stressors such as drought, salinity, and temperature fluctuations, improving resilience and reducing susceptibility to diseases and pests. Moreover, proper fertilizer selection contributes to the quality of coconut products, influencing their nutritional composition, flavor profile, and market value, including copra, coconut oil, and coconut water. Sustainable fertilizer practices are crucial for long-term viability, preserving soil fertility, minimizing nutrient runoff, and protecting water quality. Economically, effective fertilizer selection leads to increased farm income, improved livelihoods, and enhanced economic resilience for coconut-growing communities. Overall, the careful selection of fertilizers is integral to the sustainability, productivity, and economic success of coconut plantations, underscoring its critical importance in coconut farming management.

Criteria for Selecting Fertilizer:

- **Beneficiary Criteria:**
  - **Nutrient Content** ($\emptyset_1$) **:** Coconut trees require a balanced supply of essential nutrients such as nitrogen (N), phosphorus (P), potassium (K), magnesium (Mg), and micronutrients like zinc (Zn) and iron (Fe) for healthy growth and fruit development.
  - **Reputation of Brand** ($\emptyset_5$) **:** The reputation and reliability of the fertilizer brand reflect its quality and effectiveness. Farmers often consider the track record of a brand in delivering consistent results and addressing specific crop needs.
  - **Availability** ($\emptyset_6$) **:** Accessibility to the chosen fertilizer is essential for timely application and uninterrupted supply. Factors such as distribution networks, local availability, and logistical considerations influence the suitability of the fertilizer.
- **Non-Beneficiary Criteria:**
  - **Cost** ($\emptyset_2$) **:** Cost-effectiveness plays a significant role in selecting fertilizer. The price of the fertilizer should align with the budget constraints of the coconut farmer while ensuring optimal yield.
  - **Environmental Impact** ($\emptyset_3$) **:** Sustainable farming practices emphasize the importance of minimizing environmental impact. Fertilizers should be chosen based on their potential for leaching, runoff, and contribution to pollution.
  - **Ease of Application** ($\emptyset_4$) **:** The ease of application influences the practicality of fertilizer use. Factors such as application method, frequency, and compatibility with existing farming practices determine the convenience of application.

Methodologies for Selecting Decision Makers in Evaluating Fertilizer Brands for Coconut Farming

The selection of decision makers for evaluating fertilizer brands in coconut farming is a crucial step in agricultural research and decision-making processes. This article presents various methodologies and considerations for identifying suitable decision makers tasked with assessing the best fertilizer options among four brands to optimize coconut yield. The following methods can be considered:

- **Expertise and Experience:** Identifying individuals with expertise and experience in agriculture, particularly in coconut tree cultivation or related crops, such as farmers, agronomists, agricultural researchers, or extension agents knowledgeable about fertilizer selection and its impact on crop yield.
- **Stakeholder Representation:** Ensuring that decision makers represent diverse stakeholders involved in coconut farming, including farmers, agricultural cooperative members, extension officers, representatives from agricultural input suppliers, and agricultural researchers.
- **Diverse Perspectives:** Aim for diversity in decision makers to incorporate a range of perspectives and insights, considering factors such as age, gender, education level, farming practices, and geographic location to ensure a broad representation of views and experiences.
- **Involvement in the Coconut Farming Community:** Selecting decision makers actively engaged in the coconut farming community with a vested interest in improving crop yield and profitability. This may include members of coconut growers' associations, agricultural cooperatives, or local farming communities.
- **Commitment and Availability:** Choosing decision makers committed to actively participating in the selection process, with the time and availability to attend meetings, review information about fertilizer brands, and engage in discussions to make informed decisions.

Once potential decision makers are identified based on these criteria, inviting them to participate in the selection process is essential. Clear communication of the objectives, evaluation criteria, and expected level of involvement is necessary to ensure transparency and collaboration among decision makers. Encouraging open dialogue and collaboration among decision makers will facilitate a thorough and fair evaluation of fertilizer brands.

5.1. *Decision-Maker Evaluation of Fertilizer Brands for Coconut Farming: Assessing Performance Across Six Criteria*

Linguistic terms are crucial in decision-making, allowing qualitative expression of preferences and perceptions. They create a flexible framework for communication, enhancing understanding and consensus-building among stakeholders. By promoting clarity and transparency, linguistic terms enrich decision-making processes, capturing the nuanced nature of human perceptions and preferences.

Linguistic terms	Symbolic Notation	$\mathcal{N} \times 10^{-2}$
Very good	$\star_1$	$(90, 10, 10)$
Good	$\star_2$	$(80, 20, 15)$
Fair	$\star_3$	$(50, 40, 45)$
Bad	$\star_4$	$(35, 60, 70)$
Very bad	$\star_5$	$(10, 80, 90)$

TABLE 1. Linguistic terms for rating of attributes.

Step 1: Four Decision Makers DM1, DM2, DM3 and DM4 evaluates four fertilizer brands $\bigstar_1$, $\bigstar_2$, $\bigstar_3$ and $\bigstar_4$ with six criteria $\emptyset_1$ = Nutrient Content, $\emptyset_2$ = Cost, $\emptyset_3$ = Environmental Impact, $\emptyset_4$ = Ease of Application, $\emptyset_5$ = Reputation of Brand and $\emptyset_6$ = Availability to select the best fertilizer brand for coconut forming. Each evaluators decisions in linguistic phrase are given in Table 2. The neutrosophic numbers that match to the linguistic phrases in Table 1 will be substituted in Table 2. Linguistic terms are replaced with their corresponding neutrosophic number are in Table 3.

Step 2: The normalized decision matrix Table 4 represents the evaluation scores provided by decision makers for each fertilizer brand across the three beneficial $\emptyset_1$, $\emptyset_5$, $\emptyset_6$ and three non beneficial criteria $\emptyset_2$, $\emptyset_3$ and $\emptyset_4$.

Step 3: The process of converting normalized neutrosophic set values into cubic spherical neutrosophic values involves determining their center and radius. Using Equations 1 & 2 we transform the decision makers decisions into cubic spherical neutrosophic numbers, which is represented in Table 5.

Step 4: The weight for each criteria is $\emptyset_1 = 0.0935$, $\emptyset_2 = 0.1594$, $\emptyset_3 = 0.1812$, $\emptyset_4 = 0.2106$, $\emptyset_5 = 0.1812$, $\emptyset_6 = 0.1741$. The Table 6 represents the cubic spherical neutrosophic weighted arithmetic and geometric operators on the calculated CSNSs.

Step 5: The Table 7 illustrates the cosine distances computed between the ideal alternative $\bigstar_{\mathcal{I}}$=(1,0,0;1) and the Cubic Spherical Neutrosophic Weighted Arithmetic Operator (CSNWAO) and the Cubic Spherical Neutrosophic Weighted Geometric Operator (CSNWGO). These distances serve as quantitative measures of alignment between the evaluated operators and the ideal solution for fertilizer selection in coconut farming. Lower cosine distances indicate closer resemblance and alignment with the ideal criteria, suggesting higher suitability for guiding fertilizer selection decisions. Analysis of the results enables stakeholders to refine their decision-making strategies, prioritize options that closely match the ideal criteria, and optimize coconut yield while ensuring sustainable agricultural practices.

Step 6: The ranking of alternatives $\bigstar_\epsilon$ $(\epsilon = 1, 2, 3, 4)$ and the best alternative are given in Table 8

DM's	Brand	$\emptyset_1$	$\emptyset_2$	$\emptyset_3$	$\emptyset_4$	$\emptyset_5$	$\emptyset_6$	DM's	Brand	$\emptyset_1$	$\emptyset_2$	$\emptyset_3$	$\emptyset_4$	$\emptyset_5$	$\emptyset_6$
DM1	$\bigstar_1$	$\star_1$	$\star_2$	$\star_2$	$\star_3$	$\star_1$	$\star_2$	DM2	$\bigstar_1$	$\star_2$	$\star_1$	$\star_2$	$\star_3$	$\star_2$	$\star_1$
	$\bigstar_2$	$\star_2$	$\star_2$	$\star_2$	$\star_1$	$\star_2$	$\star_2$		$\bigstar_2$	$\star_3$	$\star_2$	$\star_2$	$\star_1$	$\star_3$	$\star_2$
	$\bigstar_3$	$\star_2$	$\star_1$	$\star_3$	$\star_4$	$\star_2$	$\star_1$		$\bigstar_3$	$\star_1$	$\star_3$	$\star_2$	$\star_2$	$\star_1$	$\star_3$
	$\bigstar_4$	$\star_3$	$\star_3$	$\star_1$	$\star_2$	$\star_3$	$\star_3$		$\bigstar_4$	$\star_1$	$\star_2$	$\star_3$	$\star_2$	$\star_1$	$\star_2$
DM3	$\bigstar_1$	$\star_3$	$\star_2$	$\star_1$	$\star_2$	$\star_3$	$\star_2$	DM4	$\bigstar_1$	$\star_2$	$\star_1$	$\star_2$	$\star_3$	$\star_2$	$\star_1$
	$\bigstar_2$	$\star_1$	$\star_2$	$\star_3$	$\star_3$	$\star_1$	$\star_1$		$\bigstar_2$	$\star_3$	$\star_3$	$\star_2$	$\star_2$	$\star_3$	$\star_3$
	$\bigstar_3$	$\star_2$	$\star_1$	$\star_1$	$\star_4$	$\star_2$	$\star_1$		$\bigstar_3$	$\star_2$	$\star_1$	$\star_1$	$\star_2$	$\star_2$	$\star_1$
	$\bigstar_4$	$\star_2$	$\star_3$	$\star_3$	$\star_1$	$\star_2$	$\star_3$		$\bigstar_4$	$\star_2$	$\star_2$	$\star_3$	$\star_3$	$\star_2$	$\star_1$

TABLE 2. Linguistic term rating of decision makers

DM's	Brand	$\emptyset_1$ ($\mathcal{N} \times 10^{-2}$)	$\emptyset_2$ ($\mathcal{N} \times 10^{-2}$)	$\emptyset_3$ ($\mathcal{N} \times 10^{-2}$)	$\emptyset_4$ ($\mathcal{N} \times 10^{-2}$)	$\emptyset_5$ ($\mathcal{N} \times 10^{-2}$)	$\emptyset_6$ ($\mathcal{N} \times 10^{-2}$)
DM1	$\bigstar_1$	(90, 10, 10)	(80, 20, 15)	(80, 20, 15)	(50, 40, 45)	(90, 10, 10)	(80, 20, 15)
	$\bigstar_2$	(80, 20, 15)	(80, 20, 15)	(80, 20, 15)	(90, 10, 10)	(80, 20, 15)	(80, 20, 15)
	$\bigstar_3$	(80, 20, 15)	(90, 10, 10)	(50, 40, 45)	(35, 60, 70)	(80, 20, 15)	(90, 10, 10)
	$\bigstar_4$	(50, 40, 45)	(50, 40, 45)	(90, 10, 10)	(80, 20, 15)	(50, 40, 45)	(50, 40, 45)
DM2	$\bigstar_1$	(80, 20, 15)	(90, 10, 10)	(80, 20, 15)	(50, 40, 45)	(80, 20, 15)	(90, 10, 10)
	$\bigstar_2$	(50, 40, 45)	(80, 20, 15)	(80, 20, 15)	(90, 10, 10)	(50, 40, 45)	(80, 20, 15)
	$\bigstar_3$	(90, 10, 10)	(50, 40, 45)	(80, 20, 15)	(80, 20, 15)	(90, 10, 10)	(50, 40, 45)
	$\bigstar_4$	(90, 10, 10)	(80, 20, 15)	(50, 40, 45)	(80, 20, 15)	(90, 10, 10)	(80, 20, 15)
DM3	$\bigstar_1$	(50, 40, 45)	(80, 20, 15)	(90, 10, 10)	(80, 20, 15)	(50, 40, 45)	(80, 20, 15)
	$\bigstar_2$	(90, 10, 10)	(80, 20, 15)	(50, 40, 45)	(50, 40, 45)	(90, 10, 10)	(90, 10, 10)
	$\bigstar_3$	(80, 20, 15)	(90, 10, 10)	(90, 10, 10)	(35, 60, 70)	(80, 20, 15)	(90, 10, 10)
	$\bigstar_4$	(80, 20, 15)	(50, 40, 45)	(50, 40, 45)	90, 10, 10)	(80, 20, 15)	(50, 40, 45)
DM4	$\bigstar_1$	(80, 20, 15)	(90, 10, 10)	(80, 20, 15)	(50, 40, 45)	(80, 20, 15)	(90, 10, 10)
	$\bigstar_2$	(50, 40, 45)	(50, 40, 45)	(80, 20, 15)	(80, 20, 15)	(50, 40, 45)	(50, 40, 45)
	$\bigstar_3$	(80, 20, 15)	(90, 10, 10)	(90, 10, 10)	(80, 20, 15)	(80, 20, 15)	(90, 10, 10)
	$\bigstar_4$	(80, 20, 15)	(80, 20, 15)	(50, 40, 45)	(50, 40, 45)	(80, 20, 15)	(90, 10, 10)

TABLE 3. Neutrosophic values of decision makers rankings

Comparative Analysis and Limitations

Our findings are contrasted with those of Biswas et al. [7], Ye's [22] Gomathi et al. [12], Krishnaprakash et al. [14], and their provided visualization. Table 6 displays the order of rating. The ranking results of the suggested method and the current methods are clearly nearly identical. This confirms even further that the suggested techniques are applicable.

In MCDM, the decision maker's involvement plays a crucial role in determining the weights and preferences associated with different criteria. It has been suggested that the decision maker's influence

DM's	Brand	$\emptyset_1$ ($\mathcal{N}$ $\times 10^{-2}$)	$\emptyset_2$ ($\mathcal{N}$ $\times 10^{-2}$)	$\emptyset_3$ ($\mathcal{N}$ $\times 10^{-2}$)	$\emptyset_4$ ($\mathcal{N}$ $\times 10^{-2}$)	$\emptyset_5$ ($\mathcal{N}$ $\times 10^{-2}$)	$\emptyset_6$ ($\mathcal{N}$ $\times 10^{-2}$)
	$\bigstar_1$	$(90, 10, 10)$	$(15, 20, 80)$	$(15, 20, 80)$	$(45, 40, 50)$	$(90, 10, 10)$	$(80, 20, 15)$
DM1	$\bigstar_2$	$(80, 20, 15)$	$(15, 20, 80)$	$(15, 20, 80)$	$(10, 10, 90)$	$(80, 20, 15)$	$(80, 20, 15)$
	$\bigstar_3$	$(80, 20, 15)$	$(10, 10, 90)$	$(45, 40, 50)$	$(70, 60, 35)$	$(80, 20, 15)$	$(90, 10, 10)$
	$\bigstar_4$	$(50, 40, 45)$	$(45, 40, 50)$	$(10, 10, 90)$	$(15, 20, 80)$	$(50, 40, 45)$	$(50, 40, 45)$
	$\bigstar_1$	$(80, 20, 15)$	$(10, 10, 90)$	$(15, 20, 80)$	$(45, 40, 50)$	$(80, 20, 15)$	$(90, 10, 10)$
DM2	$\bigstar_2$	$(50, 40, 45)$	$(15, 20, 80)$	$(15, 20, 80)$	$(10, 10, 90)$	$(50, 40, 45)$	$(80, 20, 15)$
	$\bigstar_3$	$(90, 10, 10)$	$(45, 40, 50)$	$(15, 20, 80)$	$(15, 20, 80)$	$(90, 10, 10)$	$(50, 40, 45)$
	$\bigstar_4$	$(90, 10, 10)$	$(15, 20, 80)$	$(45, 40, 50)$	$(15, 20, 80)$	$(90, 10, 10)$	$(80, 20, 15)$
	$\bigstar_1$	$(50, 40, 45)$	$(15, 20, 80)$	$(10, 10, 90)$	$(15, 20, 80)$	$(50, 40, 45)$	$(80, 20, 15)$
DM3	$\bigstar_2$	$(90, 10, 10)$	$(15, 20, 80)$	$(45, 40, 50)$	$(45, 40, 50)$	$(90, 10, 10)$	$(90, 10, 10)$
	$\bigstar_3$	$(80, 20, 15)$	$(10, 10, 90)$	$(10, 10, 90)$	$(70, 60, 35)$	$(80, 20, 15)$	$(90, 10, 10)$
	$\bigstar_4$	$(80, 20, 15)$	$(45, 40, 50)$	$(45, 40, 50)$	$(10, 10, 90)$	$(80, 20, 15)$	$(50, 40, 45)$
	$\bigstar_1$	$(80, 20, 15)$	$(10, 10, 90)$	$(15, 20, 80)$	$(45, 40, 50)$	$(80, 20, 15)$	$(90, 10, 10)$
DM4	$\bigstar_2$	$(50, 40, 45)$	$(45, 40, 50)$	$(15, 20, 80)$	$(15, 20, 80)$	$(50, 40, 45)$	$(50, 40, 45)$
	$\bigstar_3$	$(80, 20, 15)$	$(10, 10, 90)$	$(10, 10, 90)$	$(15, 20, 80)$	$(80, 20, 15)$	$(90, 10, 10)$
	$\bigstar_4$	$(80, 20, 15)$	$(15, 20, 80)$	$(45, 40, 50)$	$(45, 40, 50)$	$(80, 20, 15)$	$(90, 10, 10)$

TABLE 4. Normalized neutrosophic values each alternatives $\bigstar_\epsilon$.

Brand	$\emptyset_1$ ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)	$\emptyset_2$ ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)	$\emptyset_3$ ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)
$\bigstar_1$	(75, 23, 21; 39)	(13, 15, 85; 8)	(14, 18, 83; 11)
$\bigstar_2$	(68, 28, 29; 34)	(23, 25, 73; 35)	(23, 25, 73; 35)
$\bigstar_3$	(83, 18, 14; 11)	(19, 18, 80; 46)	(20, 20, 78; 42)
$\bigstar_4$	(75, 23, 21; 39)	(30, 30, 65; 23)	(36, 33, 60; 46)
	$\emptyset_4$ ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)	$\emptyset_5$ ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)	$\emptyset_6$ ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)
$\bigstar_1$	(38, 35, 58; 35)	(75, 23, 21; 39)	(85, 15, 13; 8)
$\bigstar_2$	(20, 20, 78; 42)	(68, 28, 29; 34)	(75, 23, 21; 39)
$\bigstar_3$	(43, 40, 58; 41)	(83, 18, 14; 11)	(80, 18, 19; 46)
$\bigstar_4$	(21, 23, 75; 39)	(75, 23, 21; 39)	(68, 28, 29; 34)

TABLE 5. Cubic spherical neutrosophic representation of each alternatives $\bigstar_\epsilon$.

Brand	CSNWAO ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)	CSNWGO ($\delta_{\mathbb{R}}$ $\times 10^{-2}$)
$\bigstar_1$	(58, 21, 38; 24)	(29, 22, 59; 18)
$\bigstar_2$	(50, 24, 46; 37)	(31, 24, 59; 37)
$\bigstar_3$	(61, 21, 36; 36)	(33, 23, 54; 30)
$\bigstar_4$	(53, 26, 42; 37)	(34, 26, 53; 36)

TABLE 6. CSNWA and CSNWG operators values for fertilizer brands $\bigstar_\epsilon$

	$cos(\bigstar_1, \bigstar_{\mathcal{I}})$	$cos(\bigstar_2, \bigstar_{\mathcal{I}})$	$cos(\bigstar_3, \bigstar_{\mathcal{I}})$	$cos(\bigstar_4, \bigstar_{\mathcal{I}})$
CSNWAO	0.329	0.566	0.47	0.539
CSNWGO	0.585	0.685	0.584	0.625

TABLE 7. Cosine distance between each alternatives $\bigstar_\epsilon$ and ideal sphere $\bigstar_{\mathcal{I}}$

Method	Ranking	Best Brand
$\mathbb{TOPSIS}$ [7]	$\bigstar_2 > \bigstar_3 > \bigstar_4 > \bigstar_1$	$\bigstar_1$
$\mathbb{SNWAA}$ [22]	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{SNWGA}$ [22]	$\bigstar_2 > \bigstar_1 > \bigstar_3 > \bigstar_4$	$\bigstar_4$
$\mathbb{CSNWAAO}$ [12]	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWGAO}$ [12]	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWG}_\rho^A$ [14]	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWG}_\wp^A$ [14]	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWA}_\rho^A$ [14]	$\bigstar_2 > \bigstar_3 > \bigstar_4 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWA}_\wp^A$ [14]	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWAO}$	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$
$\mathbb{CSNWGO}$	$\bigstar_2 > \bigstar_4 > \bigstar_3 > \bigstar_1$	$\bigstar_1$

TABLE 8. Comparative Analysis of Ranking

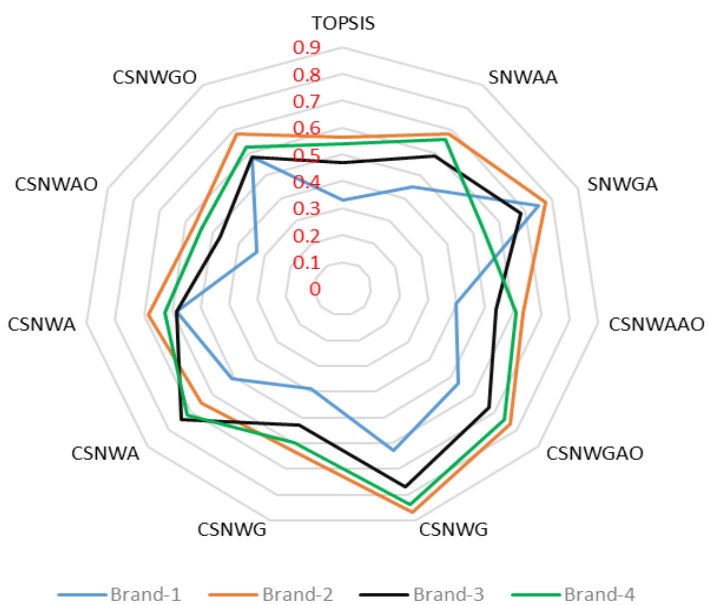

FIGURE 2. Comparative Analysis of Ranking

should be greater than one to enable the creation of a sphere representation in CSNSs. This requirement reflects the need for a significant level of involvement to ensure the meaningful representation of preferences and uncertainties. Understanding the limitations of CSNSs is essential for their effective utilization in MCDM. By addressing constraints such as the eccentricity requirement and the decision maker's involvement, researchers and practitioners can enhance the applicability and reliability of CSNS in real-world decision-making contexts.

6. Conclusion and Future Work

In our investigation, we introduced novel cubic spherical neutrosophic aggregation operators within the agricultural domain, particularly focusing on the intricate task of fertilizer selection for coconut trees. By introducing these operators, we aimed to streamline the evaluation process by considering multiple criteria such as nutrient content, cost, environmental impact, brand reputation, and availability. Through the application of CSNSs, we developed both additive and geometric aggregation operators, providing decision-makers with a comprehensive framework to assess and rank fertilizer alternatives. This innovative approach empowers farmers and agricultural practitioners to make informed decisions tailored to their specific needs and sustainability goals, ultimately contributing to optimized crop yields and environmental conservation in coconut farming and beyond.

Looking forward, our research paves the way for further exploration and refinement of cubic spherical neutrosophic aggregation operators in diverse agricultural contexts. Future endeavors will focus on extending the applicability of these operators to address broader agricultural decision-making challenges, including crop selection, pest management, irrigation strategies, and post-harvest practices. Moreover, we aim to develop tailored decision support systems and tools that cater to the unique requirements of farmers, extension agents, and agricultural stakeholders. By advancing the integration of innovative decision-making methodologies into agricultural practices, we strive to promote sustainable and resilient food systems while empowering farmers to make informed choices for improved productivity and livelihoods.

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Hausdorf Space in Neutrosophic Ideal Topological Spaces: Applications in decision Making

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Abstract. This paper presents the concept of neutrosophic $\mathfrak{I}$–Hausdorffness in the context of neutrosophic ideal topological spaces, along with various related theorems. Additionally, we provide a practical example showcasing the application of neutrosophic ideals in facilitating decision-making in uncertain environnments.

Keywords: Neutrosophic topological space; Neutrosophic ideal; Neutrosophic Hausdorff space, Neutrosophic closure.

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1. Introduction

Smarandache [9] introduced the concept of a neutrosophic set as a generalization of the intuitionistic fuzzy set. Meanwhile, Salama et al. [11–13] introduced the concepts of neutrosophic crisp sets and neutrosophic crisp relations within neutrosophic set theory. Additionally, introduced neutrosophic topology, highlighting several of its characteristics. Also they defined neutrosophic crisp topology and explored some of its properties. Other researchers, such as Wang et al. [14] introduced the concept of a single-valued neutrosophic set. Kim et al. [4] have explored the notions of single-valued neutrosophic partition, single-valued neutrosophic equivalence relation, and single-valued neutrosophic relation.

The introduction of ideals into this framework further refines the structure of neutrosophic topological spaces. An ideal in a topological space is a collection of subsets that is closed under the formation of smaller subsets and finite unions. In neutrosophic ideal topology, ideals are used to define certain "ideal" open sets, providing a way to analyze the space's properties more precisely. Numerous authors [6–8] have conducted research on ideal topological spaces.

In [15] The authors introduced the notion of single-valued neutrosophic ideals sets in Šostak's sense, which is considered as a generalization of fuzzy ideals in Šostak's sense and intuitionistic fuzzy ideals. Also, the concept of single-valued neutrosophic ideal open local function is also introduced for a single-valued neutrosophic topological space. The basic structure, especially a basis for such generated single-valued neutrosophic topologies and several relations between different single-valued neutrosophic ideals and single-valued neutrosophic topologies, are discussed.

In Section 2, we review some definitions and results from the literature and establish certain results needed for subsequent discussions. In Section 3, We introduce the concept of I-Hausdorffness within the context of ideal topological spaces and establish several related results. In Section 4, An example is also presented to demonstrare the applicability of neutrosophic ideals in addressing decision-making challanges in uncertain situations.

2. **Preliminaries**

In this section, we give all basic definitions and results which we need to go through our work. First we give the definition of a neutrosophic set [10, 12].

Definition 2.1. *Let X be a nonempty set and $I = [0,1]$. A neutrosophic set N on X is a mapping defined as $N = \langle T_N, I_N, F_N\rangle : X \to \zeta$, where $\zeta = I^3$ and $T_N, I_N, F_N : X \to I$ such that $0 \leq T_N + I_N + F_N \leq 3$.*

We denote the set of all neutrosophic sets of X by ζ^X and the neutrosophic sets $\langle 0,1,1\rangle$ and $\langle 1,0,0\rangle$ by 0_X and 1_X respectively and $\langle 0,0,0\rangle$ and $\langle 1,1,1\rangle$ by $\bar{0}$ and $\bar{1}$ respectively. Let $(r,s,t),(l,m,n) \in \zeta$, then

- $(r,s,t) \sqcup (l,m,n) = (r \vee l, s \wedge m, t \wedge n)$;
- $(r,s,t) \sqcap (l,m,n) = (r \wedge l, s \vee m, t \vee n)$;
- $(r,s,t) \sqsubseteq (l,m,n) = (r \leq l, s \geq m, t \geq n)$;
- $(r,s,t) \sqsupseteq (l,m,n) = (r \geq l, s \leq m, t \leq n)$.

Definition 2.2. *[10, 12] Let X be a non-empty set and let $N, M \in \zeta^X$ be given by $N = \langle T_N, I_N, F_N\rangle$ and $\mathtt{M} = \langle T_M, I_M, F_M\rangle$. Then*

- *the complement of N denoted by N^c is given by*

$$N^c = \langle 1 - T_N, 1 - I_N, 1 - F_N\rangle$$

- *the union of N and M denoted by $N \sqcup M$ is an neutrosophic set in X given by*

$$N \sqcup M = \langle T_N \vee T_M, I_N \wedge I_M, F_N \wedge F_M\rangle$$

- *the intersection of N and M denoted by $N \sqcap M$ is an neutrosophic set in X given by*
$$N \sqcap M = \langle T_N \wedge T_M, I_N \vee I_M, F_N \vee F_M \rangle$$
- *the product of N and M denoted by $N \times M$ is given by*
$$(N \times M)(x, y) = N(x) \sqcap M(y), \forall\ (x, y) \in X \times Y.$$
- *we say that $N \sqsubseteq M$ if $T_N \leq T_M, I_N \geq I_M, F_N \geq F_M$.*

For an any arbitrary collection $\{N_i\}_{i \in J} \subseteq \zeta^X$ of neutrosophic sets the union and intersection is given by

- $\bigsqcup\limits_{i \in J} N_i = \left\langle \bigvee\limits_{i \in J} T_{N_i}, \bigwedge\limits_{i \in J} I_{N_i}, \bigwedge\limits_{i \in J} F_{N_i} \right\rangle$
- $\bigsqcap\limits_{i \in J} N_i = \left\langle \bigwedge\limits_{i \in J} T_{N_i}, \bigvee\limits_{i \in J} I_{N_i}, \bigvee\limits_{i \in J} F_{N_i} \right\rangle$.

Definition 2.3. *Let X be a nonempty set and $x \in X$. If $r \in (0, 1], s \in [0, 1)$ and $t \in [0, 1)$, then a neutrosophic point $x_{r,s,t}$ in X given by*

$$x_{r,s,t}(y) = \begin{cases} (r, s, t), & \text{if } x = y, \\ (0, 1, 1), & \text{otherwise.} \end{cases}$$

We say $x_{r,s,t} \in N$ if $x_{r,s,t} \sqsubseteq N$. To avoid the ambiguity, we denote the set of all neutrosophic points by $pt(\zeta^X)$.

Definition 2.4. *A neutrosophic set N is said to be quasi coincident with another neutrosophic set M, denoted by NqM if there exists an element $x \in X$ such that $T_N(x) + T_M(x) > 1$ or $I_N(x) + I_M(x) < 1$ or $F_N(x) + F_M(x) < 1$. If M is not quasi coincident with N, then we write $M\bar{q}N$.*

Definition 2.5. *[2] Let X be a nonempty set. Then a neutrosophic set $\tau = \langle T_\tau, I_\tau, F_\tau \rangle : \zeta^X \to \zeta$ is said to be a smooth neutrosophic topology on X if satisfies the following conditions:*

C1 $\tau(0_X) = \tau(1_X) = (1, 0, 0)$

C2 $\tau(N \sqcap M) \sqsupseteq \tau(N) \sqcap \tau(M),\ \forall\ N, M \in \zeta^X$

C3 $\tau(\sqcup_{i \in J} N_i) \sqsupseteq \sqcap_{i \in J} \tau(N_i),\ \ \forall\ N_i \in \zeta^X, i \in J.$

The pair (X, τ) is called a neutrosophic topological space.

Definition 2.6. *[3] Let (X, τ) be smooth neutrosophic topological space. For all $x_{r,s,t} \in pt(\zeta^X)$ and $N \in \zeta^X$, the mapping $Q^\tau_{x_{r,s,t}} : \zeta^X \to \zeta$ defined as follows*

$$Q^\tau_{x_{r,s,t}}(N) = \begin{cases} \bigsqcup\limits_{x_{r,s,t} qM \sqsubseteq N} \tau(M); & \text{if } x_{r,s,t} qN, \\ (0, 1, 1), & \text{otherwise.} \end{cases}$$

The set $Q^\tau = \{Q^\tau_{x_{r,s,t}} : x_{r,s,t} \in pt(\zeta^X)\}$ is called neutrosophic Q-neighborhood system.

Definition 2.7. *[5] A neutrosophic topological space (X,τ) is said to be neutrosophic Hausdorff space if every pair of points $x_{r,s,t}$ and $y_{r',s',t'}$ with $x \neq y$, there exists a Q-neighborhood U and V of $x_{r,s,t}$ and $y_{r',s',t'}$ respectively with $x_{r,s,t}\bar{q}V$ and $y_{r',s',t'}\bar{q}U$ and $U\bar{q}V$.*

Definition 2.8. *[15] A mapping $\mathfrak{I}:\zeta^X \to \zeta^Y$ is called a neutrosophic ideal on X if it satisfies the following conditions:*

(1) $\mathfrak{I}(0_X) = (1,0,0)$.
(2) *If $A \sqsubseteq B$, then $\mathfrak{I}(B) \sqsubseteq \mathfrak{I}(A)$ for each $A,B \in \zeta^X$.*
(3) *If $A,B \in \zeta^X$, then $\mathfrak{I}(A \sqcup B) \sqsupset \mathfrak{I}(A) \sqcap \mathfrak{I}(B)$.*

The triple $(X,\tau,\mathfrak{I})$ is called a Neutrosophic Ideal Topological Space.

If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are two neutrosophic ideals on X, we say that $\mathfrak{I}_1$ is finer than $\mathfrak{I}_2$ (denoted by $\mathfrak{I}_1 \preceq \mathfrak{I}_2$) or $\mathfrak{I}_2$ is coarser than $\mathfrak{I}_1$ if $\mathfrak{I}_2(A) \sqsubseteq \mathfrak{I}_1(A)$.

Definition 2.9. *[15] Let $(X,\tau,\mathfrak{I})$ be a neutrosophic topological space and $A \in \zeta^X$. Then the neutrosophic local function A^* of A is the union of all neutrosophic points $x_{r,s,t}$ such that $Q_{x_{r,s,t}}(U) \sqsupseteq (0,1,1)$ and $\mathfrak{I}(C) \sqsupseteq (0,1,1)$, Then there is at least one $y \in X$ forwhich $U(y) + A(y) - \bar{1}(y) \sqsupseteq C(y)$.*

Theorem 2.10. *[15] Let (X,τ) be neutrosophic topological space and $\mathtt{I}_1$ and $\mathtt{I}_2$ be two neutrosophic Ideals of X. Then for each $\mathtt{A},\mathtt{B} \in \zeta^X$*

(i) *If $\mathtt{A} \sqsubseteq \mathtt{B}$,then $\mathtt{A}^* \sqsubseteq \mathtt{B}^*$*
(ii) *If $\mathtt{I}_1 \sqsubseteq \mathtt{I}_2$ then $\mathtt{A}^*(\mathtt{I}_1,\tau) \sqsupseteq \mathtt{A}^*(\mathtt{I}_2,\tau)$*
(iii) $\mathtt{A}^* = cl(\mathtt{A}^*) \sqsubseteq cl(\mathtt{A})$
(iv) $(\mathtt{A}^*)^* \sqsubseteq \mathtt{A}^*$
(v) $(\mathtt{A}^* \sqcup \mathtt{B}^*) = (\mathtt{A} \sqcup \mathtt{B})^*$
(vi) *If $\mathtt{I}(\mathtt{B}) \sqsupset (0,1,1)$ then $(\mathtt{A} \sqcup \mathtt{B})^* = \mathtt{A}^* \sqcup \mathtt{B}^* = \mathtt{A}^*$*
(vii) $(\mathtt{A} \sqcap \mathtt{B})^* \sqsubseteq (A^* \sqcap B^*)$

Definition 2.11. *[15] Let $(X,\tau,\mathfrak{I})$ be a neutrosophic ideal topological space and $A \in \zeta^X$. Then $C^*_\tau(A) = A \sqcup A^*$. It is clear, C^*_τ is a neutrosophic closure operator and $\tau_{\mathfrak{I}}$ is the neutrosophic topology generatedd by C^*_τ. Note that, if A is neutrosophic closed in $\tau_{\mathfrak{I}}$, then $A^* \sqsubseteq A$.*

3. Neutrosophic $\mathfrak{I}$-Hausdorff space in Neutrosophic ideal topological space

In this section we define $\mathfrak{I}$-Hausdorff space in the context of neutrosophic ideal topological spaces.

Definition 3.1. *Let $(X,\tau,\mathfrak{I})$ be a neutrosophic ideal topological space. Then (X,τ) is said to be neutrosophic $\mathfrak{I}$-Hausdorff with respect to the neutrosophic ideal $\mathfrak{I}$ if for every pair of* *neutrosophic points* $x_{r,s,t}$ *and* $y_{r',s',t'}$ *with* $x_{r,s,t} \neq y_{r',s',t'}$, *there exist* Q-*neighborhoods* U, V *of* $x_{r,s,t}$ *and* $y_{r',s',t'}$ *respectively, there exists* $C_1, C_2 \in \zeta^X$ *with* $\mathfrak{I}(C_1) \sqsupset (0,1,1)$, $\mathfrak{I}(C_2) \sqsupset (0,1,1)$ *such that* $x_{r,s,t}\bar{q}V - C_2$ *and* $y_{r',s',t'}\bar{q}U - C_1$, $U - C_1\bar{q}V - C_2$.

Theorem 3.2. *Let* $(X, \tau, \mathfrak{I})$ *be a neutrosophic ideal topological space. Then* $(X, \tau, \mathfrak{I})$ *is neutrosophic* $\mathfrak{I}$-*Hausdorff space if* $(X, \tau_{\mathfrak{I}})$ *is neutrosophic Hausdorff space.*

Proof. Assume that $(X, \tau_{\mathfrak{I}})$ is Neutrosophic Hausdorff space and $x_{r,s,t} \neq y_{r',s',t'}$. Then there exist Q-neighborhoods U and V of $x_{r,s,t}$ and $y_{r',s',t'}$ with $x_{r,s,t}\bar{q}V$ and $y_{r',s',t'}\bar{q}U$ and $U\bar{q}V$. Since $x_{r,s,t}\bar{q}V$ and $y_{r',s',t'}\bar{q}U$, then we have $x_{r,s,t} \in U^c$ and $y_{r',s',t'} \in V^c$. Also $\tau_{\mathfrak{I}}(U) \sqsupseteq I(0,1,1)$ and $\tau_{\mathfrak{I}}(V) \sqsupseteq (0,1,1)$ implies that $C^*_\tau(U^c) = U^c$ and $C^*_\tau(V^c) = V^c$. Therefore U^c and V^c are Neutrosophic $\mathfrak{I}$-closed sets. Clearly we get that $x_{r,s,t} \notin (U^c)^*$ and $y_{r',s',t'} \notin (V^c)^*$.

Since $x_{r,s,t} \notin (U^c)^*$, there exists a Q-neighborhood U_1 of $x_{r,s,t}$ and there exists $C_1 \in \zeta^X$ with $\mathfrak{I}(C_1) \sqsupseteq (0,1,1)$ such that

$$U_1(k) + U^c(k) - \bar{1}(k) \sqsubseteq C_1(k) \quad \text{for every } k \in X \tag{1}$$

Since $y_{r',s',t'} \notin (V^c)^*$, there exists a Q-neighborhood V_1 of $x_{r',s',t'}$ and there exists $C_2 \in \zeta^X$ with $\mathfrak{I}(C_2) \sqsupseteq (0,1,1)$ such that

$$V_1(k) + V^c(k) - \bar{1}(x) \sqsubseteq C_2(k) \quad \text{for every } k \in X \tag{2}$$

By using $x_{r,s,t}\bar{q}V \implies V(x) + (r,s,t) \sqsubseteq (1,1,1)$ and by (1)

$$V_1(x) + (r,s,t) \sqsubseteq V(x) + V^c(x) \sqsubseteq C_2(x) + \bar{1}(x)$$

Hence $(V_1 - C_2)(x) + (r,s,t) \sqsubseteq \bar{1}(x) \implies x_{r,s,t}\bar{q}(V_1 - C_2)$ Similarly, we can get that $y_{r',s',t'}\bar{q}(U_1 - C_1)$. Since $U\bar{q}V$ and by (1) and (2), $(U_1 - C_1)(x) + (V_1 - C_2)(x) \subseteq U(x) + V(x) \subseteq \bar{1}(x)$ Therefore, (X, τ) is $\mathfrak{I}$-Hausdorff space □

The example of $\mathfrak{I}$-hausdorff space is given below.

Example 3.3. *Let* $X = \{x, y\}$. *Define* $U \in \zeta^X$ *as follows:*

$$U(x) = (0.5, 0.2, 0.3), \quad U(y) = (0.1, 0.0, 0.3).$$

Define $\tau = (\tau_T, \tau_I, \tau_F) : \zeta^X \to \zeta$ *as follows:*

$$\tau(K) = \begin{cases} (1,0,0) & \text{if } K = 0_X \text{ or } 1_X \\ \left(\frac{1}{2}, 0, \frac{1}{2}\right) & \text{if } K = U \\ (0,1,1) & \text{otherwise} \end{cases}$$

Clearly, (X, τ) *is a neutrosophic topological space.*

$_T, \Im_I, \Im_F) : \zeta^X \to \zeta$ as follows:

$$\Im(S) = \begin{cases} (1,0,0) & \text{if } 0_X \sqsubseteq S \sqsubset U \text{ and } S \sqsubseteq \bar{1} \\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) & \text{if } S = U \\ (0,1,1) & \text{otherwise} \end{cases}$$

Hence $\Im$ is a neutrosophic ideal on X.

Let $x_{r,s,t} \neq y_{r,s,t}$ in X. Then the possible Q-neighborhoods of $x_{r,s,t}$ and $y_{r,s,t}$ are U and 1_X.

Case (i): *Suppose U is a Q-neighborhood of $x_{r,s,t}$ and $y_{r,s,t}$. Let $C_1 = U = C_2$. Then, $\Im(C_1) \sqsupseteq (0,1,1)$ and $\Im(C_2) \sqsupseteq (0,1,1)$. Hence, $U - C_2 = (0,0,0) = U - C_1$. Therefore, $x_{r,s,t}\bar{q}U - C_2$ and $y_{r,s,t}\bar{q}U - C_1$. Also, $(U - C_2)\bar{q}(V - C_1)$.*

Case (ii): *Suppose 1_X is a Q-neighborhood of $x_{r,s,t}$ and $y_{r,s,t}$. Let $C_1 = (1,1,1) = C_2$. Clearly, $\Im(C_1) \sqsupseteq (0,1,1)$ and $\Im(C_2) \sqsupseteq (0,1,1)$. Hence $1_X - C_1 = (0,-1,-1) = 1_X - C_2$. Therefore we get that $x_{r,s,t}\bar{q}1_X - C_1$ and $y_{r,s,t}\bar{q}1_X - C_2$. Also, $(1_X - C_2)\bar{q}(1_X - C_1)$.*

Case (iii): *Let U be a Q-neighborhood of $x_{r,s,t}$ and 1_X is the Q-neighborhood of $y_{r,s,t}$. In this case, choose $C_1 = U$ and $C_2 = \bar{1}$. Clearly, $U - C_1 = (0,0,0)$ and $1_X - C_2 = (0,-1,-1)$. Thus $x_{r,s,t}\bar{q}U - C_1$ and $y_{r,s,t}\bar{q}1_X - C_2$. Also, $(1_X - C_2)\bar{q}(U - C_1)$.*

Hence (X, τ) is a neutrosophic $\Im$-Hausdorff space.

Remark 3.4. *If $(X, \tau, \Im)$ is a neutrosophic ideal topolgical space and $A \in \zeta^X$, then*

(1) *$\tau_A : \zeta^A \to \zeta$ by*

$$\tau_A(U) = \{\tau(W) : W \in \zeta^X, A \sqcap W = U\}$$

is a neutrosophic subspace topology [1] inherited from τ. If $x_{r,s,t} \in A$, then the Q-neighborhood U of $x_{r,s,t}$ related to A is defined as $\tau_A(U) \sqsupset (0,1,1)$ and $x_{r,s,t}q_AU$.i.e $T_U(x) + r > T_A(x)$ or $I_U(x) + s < 1$ or $F_U(x) + t < 1$.

(2) *$\Im_A : \zeta^A \to \zeta$ by $\Im_A(U) = \{\Im(W) : W \in \zeta^X, A \sqcap W = U\}$ is neutrosophic ideal on A.*

Theorem 3.5. *Let $(X, \tau, \Im)$ be neutrosophic $\Im$-Hausdorff space and $A \in \zeta^X$. Then (A, τ_A) is $\Im_A$-Hausdorff space, where τ_A is the neutrosophic subspace topology inhirited from τ.*

Proof. Let $x_{r,s,t} \neq y_{r',s',t'}$ in A. Since X is $\Im$-Hausdorff space, there exist Q-neighborhoods U, V of $x_{r,s,t}$ and $y_{r',s',t'}$ respectively, there exists $C_1, C_2 \in \zeta^X$ with $\Im(C_1) \sqsupseteq (0,1,1)$, $\Im(C_2) \sqsupseteq (0,1,1)$ such that $x_{r,s,t}\bar{q}V - C_2$ and $y_{r',s',t'}\bar{q}U - C_1$, $U - C_1\bar{q}V - C_2$.

Let $U_1 = U \sqcap A, V_1 = V \sqcap A$, $D_1 = C_1 \sqcap A$ and $D_2 = C_2 \sqcap A$. Clearly we get that

$$\begin{aligned} \tau_A(U_1) &= \tau_A(U \sqcap A) \sqsupseteq \tau(U) \sqsupset (0,1,1), \\ \tau_A(V_1) &= \tau_A(V \sqcap A) \sqsupseteq \tau(V) \sqsupset (0,1,1), \\ \Im_A(D_1) &= \Im(C_1 \sqcap A) \sqsupseteq \Im(U) \sqsupset (0,1,1) \\ \Im_A(D_1) &= \Im(C_1 \sqcap A) \sqsupseteq \Im(U) \sqsupset (0,1,1). \end{aligned}$$

Since $x_{r,s,t} q U$,

$$\begin{aligned} (r,s,t) + U_1(x) &= (r,s,t) + (A \sqcap U)(x) \\ &= [(r,s,t) + A(x)] \sqcap [(r,s,t) + U(x)] \\ &\sqsupseteq A(x) \sqcap (1,1,1) = (T_A(x), 1, 1). \end{aligned}$$

Therefore $x_{r,s,t} q_A U_1$. Similarly, $y_{r',s',t'} q V$ implies that$y_{r',s',t'} q_A V_1$.
Since $x_{r,s,t} \bar{q} V - C_1$, $(r,s,t) + (V - C_1)(x) \sqsubseteq \bar{1}(x)$. Now,

$$\begin{aligned} (V_1 - D_1)(x) &= (V \sqcap A)(x) - (C_1 \sqcap A)(x) \\ &= (V - C_1)(x) \sqcup (A - A)(x) \\ &= (V - C_1)(x) \sqsubseteq \bar{1}(x) - (r,s,t) \end{aligned}$$

Therefore $x_{r,s,t} \bar{q} V_1 - D_1$. Similarly, $y_{r',s',t'} \bar{q} U - C_2$ implies that $y_{r',s',t'} q_A V_1$. Also,

$$\begin{aligned} (V_1 - D_1)(k) + (U_1 - D_2)(k) &= [(V \sqcap A)(k) - (C_1 \sqcap A)(k)] + [(U \sqcap A)(k) - (C_2 \sqcap A)(k] \\ &\sqsubseteq (V - C_1)(k) + (U - C_2)(k) \sqsubseteq \bar{1}. \end{aligned}$$

Hence, $(V_1 - D_1)\bar{q}(U_1 - D_2)$. Thus (A, τ_A) is neutrosophic $\Im_A$-Hausdorff space. □

If we define the ideal closure of A by
$\Im Cl(A) = \sqcup\{x_{r,s,t}$: every Q-neighborhood $U$of $x_{r,s,t}$, there exits C_1 with $\Im(C_1) \sqsupset (0,1,1), U - C_1 q V\}$,
then the following theorem holds.

Theorem 3.6. *Let $(X, \tau, \Im)$ be ideal topological space. If (X, τ) is $\Im$-Hausdorff space, then $\{x_{r,s,t}\} = \sqcap\{\Im Cl(U_x) : U_x$ is neutrosophic Q-neighborhood of $x_{r,s,t}\}$.*

Proof. Let $x_{r,s,t} \in \zeta^X$. For any $y_{r',s',t'} \neq x_{r,s,t}$, there exist Q-neighborhoods U, V of $x_{r,s,t}$ and $y_{r',s',t'}$ respectively, there exists $C_1, C_2 \in \zeta^X$ with $\Im(C_1) \sqsupseteq (0,1,1)$, $\Im(C_2) \sqsupseteq (0,1,1)$ such that $x_{r,s,t} \bar{q} V - C_2$ and $y_{r',s',t'} \bar{q} U - C_1$, $U - C_1 \bar{q} V - C_2$. Since U and V is a Q-neighborhoods of $x_{r,s,t}$ and $y_{r',s',t'}$ respectively and $U - C_1 \bar{q} V - C_2$, we have $y_{r',s',t'} \notin \Im - Cl(A)$. Therefore $\{x_{r,s,t}\} = \sqcap\{\Im Cl(U_x) : U_x$ is neutrosophic Q-neighborhood of $x_{r,s,t}\}$. □

4. Application

Decision-Making Problem Here's a decision-making problem using the neutrosophic ideal definition:

Problem: Select the best investment option among three alternatives ($A, B,$ and C) using a neutrosophic ideal.

Data:

$$\begin{aligned} A &= (0.7, 0.2, 0.1)(StockMarket) \\ B &= (0.5, 0.3, 0.2)(RealEstate) \\ C &= (0.9, 0.1, 0.0)(GovernmentBonds) \end{aligned}$$

Neutrosophic Ideal (I):

$$\begin{aligned} I(A) &= (0.6, 0.3, 0.1) \\ I(B) &= (0.4, 0.4, 0.2) \\ I(C) &= (0.8, 0.2, 0.0). \end{aligned}$$

Decision-making process:

(1) Compare the neutrosophic ideal values:
$I(A) \sqsubseteq I(C)($ since $0.6 \leq 0.8, 0.3 \geq 0.2,$ and $0.1 \geq 0.0)$
$I(B) \sqsubseteq I(C)($ since $0.4 \leq 0.8, 0.4 \geq 0.2,$ and $0.2 \geq 0.0)$
$I(A) \sqcap I(B) = (0.4, 0.4, 0.2)$

(2) Calculate the neutrosophic ideal of the union:
$I(A \sqcup B) = (0.7, 0.2, 0.1)$

(3) Compare the results:
$I(A \sqcup B) \sqsupset I(A) \sqcap I(B)($ since $0.7 \geq 0.4, 0.2 \leq 0.4,$ and $0.1 \leq 0.2)$

Based on the neutrosophic ideal, the best investment option is C (Government Bonds) since $I(C)$ has the highest truth membership value (0.8) and the lowest falsity membership value (0.0). The combination of A and B (Stock Market and Real Estate) is not better than investing solely in C.

This example illustrates the application of neutrosophic ideals in decision-making under uncertainty. The neutrosophic sets and ideals provide a more comprehensive framework for handling imprecise and uncertain data.

Conclusions

This paper introduces the concept of neutrosophic $\mathfrak{I}$–Hausdorffness within neutrosophic ideal topological spaces, expanding the theoretical framework of neutrosophic set theory. The theorems presented offer deeper insights into the structural properties of these spaces, while the practical example demonstrates the utility of neutrosophic ideals in decision-making processes under uncertainty. This work not only contributes to the mathematical foundation of neutrosophic theory but also highlights its potential for real-world applications in complex, indeterminate environments.

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Topological Aspects of Set-Valued Mappings Defned on Neutrosophic Normed Spaces

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Abstract. Samarandache [43] introduced neutrosophoic sets to generalize the theory of fuzzy sets. In this paper we modify the definition of neutrosophic normed space with help of continuous neutrosophic t-representable norm. Then we study the statistical graphical convergence and pointwise convergence of the sequences of set-valued functions defined on modified version of neutrosophic normed space and give some related theorems. We also introduce neutrosophic upper and lower semi continuities of set valued maps to develop the link between these convergences.

Keywords: NNS; t-norm; t-conorm; t-representable norm; Statistical graph convergence; Set valued maps; neutrosophic set.

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1. Introduction

The landscape of both pure and applied mathematics has been substantially enriched through the insights provided by fuzzy theory. The foundational work was established by L.A. Zadeh in 1965 [38]. In 1975, Kramosil and Michálek introduced the concept of fuzzy metric space [25], which served as a generalization of the conventional metric space. Later, George and Veeramani refined this concept in 1994 [16]. Katsaras proposed the idea of fuzzy normed space in 1984 [21], which was subsequently modified by Bag and Samanta [8]. Saadati and Vaezpour expanded upon this notion, further advancing the field of fuzzy normed spaces [33].

In the realm of fuzzy mathematics, Atanassov introduced intuitionistic fuzzy set theory, broadening the scope beyond traditional fuzzy set theory [2]. Building upon this foundation, Park extended the concept into intuitionistic fuzzy metric spaces [29]. Saadati and Park further generalized the notion to intuitionistic fuzzy normed spaces [34]. Recognizing the need for refinement, additional conditions were incorporated into the concept by Hosseini et al. [18]. In a recent exploration, Jakhar et al. applied fixed point and direct methods to investigate the intuitionistic fuzzy stability of 3-dimensional cubic functional equations [19]. Also Vakeel A. Khan and S. K. Ashadul Rahaman [44] explopred statistical graph and pointwise convergence of sequences of set-valued functions on intuitionistic fuzzy normed spaces.

Beyond intuitionistic fuzzy sets, the emergence of neutrosophic sets marked a significant generalization in dealing with uncertainty and imprecision. Smarandache introduced neutrosophic sets, providing a broader framework for this purpose [43]. Advancing the field, Kirisci and Simsek defined a metric and a norm specifically tailored for neutrosophic sets, while also delving into their topological properties 41, 42. These advancements not only expanded the toolkit for handling complex and uncertain data but also offered fresh perspectives on fuzzy systems.

Jemima and colleagues investigated the convergence patterns within a Kothe sequence space, whereas Al-Marzouki's study delved into the statistical characteristics of the type II Topp Leone inverse exponential distribution [1, 20]. The understanding of sequence set limits is pivotal not just in set-valued analysis [4] but also in variational analysis [32]. In 1902, Painlevé initially proposed the notions of upper and lower limits for sequences of sets, advancing the idea of Kuratowski convergence by aligning the corresponding upper and lower limits [26]. The formalization of upper and lower limits for a sequence of subsets within a metric space $(\mathcal{X}, d)$ was first introduced in [27]. Building upon these foundational ideas, Beer introduced the concept of topological convergence [11]. Later, Kowalczyk utilized the notion of sequence set convergence within a topological space [24]. For those interested in further exploration of sequence set convergence, additional resources include articles such as [12, 37]. informative.

The convergence of function sequences holds paramount significance across both analysis and topology, particularly concerning the convergence of graphs associated with such sequences. When discussing the convergence of sequences of real-valued functions, terms such as pointwise convergence and uniform convergence frequently emerge. In 1983, Beer elucidated the conditions under which topological convergence and uniform convergence of sequences of continuous functions from one metric space to another coincide [10]. In 2008, Grande explored the graph convergence of single-valued functions defined from one topological space to another, drawing comparisons between graph convergence, pointwise convergence, and uniform convergence [17]. For those wishing to delve further into the graph convergence of single-valued functions, additional resources include articles such as [28, 30]. informative.

The convergence of sequences of set-valued functions plays a fundamental role in various mathematical contexts. Attouch extensively examined the graph convergence of sequences of maximal monotone set-valued operators in his seminal work [3]. Aubin and Frankowska introduced the concept of graph convergence for sequences of set-valued functions utilizing the notions of upper and lower limits of sets in the Kuratowski sense [6]. Similarly, Kowalczyk addressed the convergence of sequences of set-valued functions employing Kuratowski limits, while also introducing the equicontinuity of set-valued functions to establish connections between different types of convergence [24]. Delgado and colleagues investigated the interplay between pointwise convergence and graph convergence of sequences of set-valued functions, introducing the concept of outer-semicontinuity for set-valued functions [31]. For further insights and applications concerning the graph convergence of sequences of set-valued functions, interested readers may consult papers and books such as [7, 9, 14].

The primary aim of this article is to conduct an analysis of sequences of set-valued functions originating from a modified variant of intuitionistic fuzzy normed space and extending to another space. This analysis will delve into the phenomena of both graph convergence and pointwise convergence exhibited by these sequences, shedding light on their properties and implications within the framework of fuzzy normed spaces.

2. Preliminaries

Throughout the entirety of this investigation, we will consistently use the symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{Q}$ to represent the sets of natural numbers, real numbers, and rational numbers, respectively. Here are some fundamentals to review:

Consider U and V as two arbitrary non-void sets. A set - valued mapping $\varphi : U \longrightarrow P(V)$ is a mapping from U to power set of V i.e. $P(V)$, such that $\forall\ u \in U$, $\varphi(u) \subseteq V$.
The domain of the function φ is defined as

$$\mathbf{Dom}(\varphi) = \big\{u \in U : \varphi(u) \neq \emptyset\big\}.$$

and the image of the function φ is defined as

$$\mathbf{Im}(\varphi) = \bigcup_{u \in U} \varphi(u)$$

Consider U as an arbitrary non-void set and V be a linear space over the field F. Let $\varphi_1, \varphi_2 : X \longrightarrow P(V)$ to be the set - valued functions. The addition and scalar multiplication of φ_1 and φ_2 are therefore defined below:

$$(\varphi_1 + \varphi_2)(u) = \varphi_1(u) + \varphi_2(u)$$
$$= \Bigg\{v_1 + v_2 : v_1 \in \varphi_1(u) \text{ and } v_2 \in \varphi_2(u)\Bigg\};$$

$$(\alpha\varphi_1)(u) = \alpha\varphi_1(u) = \Bigg\{\alpha v : v \in \varphi_1(u), \alpha \in K\Bigg\}.$$

Definition 2.1. [32] Let U, V be topological spaces and $\varphi_n : U \longrightarrow P(V)$ be a sequence of set - valued functions. Then the pointwise lower limit and pointwise upper limit of the sequence $(\varphi_n)_{n=1}^{\infty}$ are the functions $p-\varphi_n^l$ and $p-\varphi_n^{\mathfrak{u}}$, respectively, defined by

$$(p-\varphi_n^l)(u) = \liminf_{n\to\infty} \varphi_n(u), \ \forall u \in U.$$

and

$$(p-\varphi_n^{\mathfrak{u}})(u) = \limsup_{n\to\infty} \varphi_n(u), \ \forall u \in U$$

If $(p-\varphi_n^{\mathfrak{u}})(u) = (p-\varphi_n^l)(u) = \varphi(u)$, $\forall u \in U$, then $\varphi(u)$ is called pointwise limit and we denote $\lim_{n\to\infty} \varphi_n(u) = \varphi(u)$, $\forall u \in U$ then we say, the sequence $(\varphi_n)_{n=1}^{\infty}$ is pointwise convergent to $\varphi(u)$.

Define the notions as follows:

$$\mathscr{M}_f = \{M \subseteq \mathbb{N} : m^c \text{ is finite }\}$$

$$\mathscr{M}_\infty = \{M' \subseteq \mathbb{N} : M' \text{ is infinite }\}.$$

In general, we denote $\lim_{n\to\infty}$ when n approaches to ∞ along $\mathbb{N}$. Throughout this paper, we will denote $\lim_{n\in M}$ and $\lim_{n\in M'}$ when n approaches to ∞ along the subsets M and M' of $\mathbb{N}$, respectively.

Definition 2.2. [32] Let U, V be topological spaces and $\varphi_n : U \longrightarrow P(V)$ be a sequence of set-valued functions. The graphical lower limit of the sequence $(\varphi_n)_{n=1}^{\infty}$ is the function $Gr(\varphi_n^l) = \liminf_{n\to\infty} (Gr(\varphi_n))$, where $v \in \varphi^l(u)$ if and only if $(u, v) \in Gr(\varphi_n^l)$, i.e., there exists $M \in \mathscr{M}_f$ such that

$$\lim_{n\in M} u_n = u, \lim_{n\in M} v_n = v \text{ for } v_n \in \varphi_n(u_n).$$

and the graphical upper limit of the sequence $(\varphi_n)_{n=1}^{\infty}$ is the function $Gr(\varphi_n^{\mathfrak{u}}) = \limsup_{n\to\infty} (Gr(\varphi_n))$, where $v \in \varphi^{\mathfrak{u}}(u)$ if and only if $(u, v) \in Gr(\varphi_n^{\mathfrak{u}})$, i.e., there exists $M' \in \mathscr{M}_\infty$ such that

$$\lim_{n\in M'} u_n = u, \lim_{n\in M'} v_n = v \ \text{ for } v_n \in \varphi_n(u_n)$$

If $Gr(\varphi_n^{\mathfrak{u}}) = Gr(\varphi_n^l)$, then the limit is known as the graphical limit denoted by $\lim_{n\to\infty} Gr(\varphi_n)$. The sequence $(\varphi_n)_{n=1}^{\infty}$ is graph convergent to a function $\varphi : U \longrightarrow P(V)$ if

$$\limsup_{n\to\infty} (Gr(\varphi_n)) \subseteq Gr(\varphi) \subseteq \liminf_{n\to\infty} (Gr(\varphi_n)).$$

Lemma 2.3. *[39] Consider $(\mathfrak{D}^\star, \leq_{\mathfrak{D}^\star})$ to be partially ordered set, defined as*

$$\mathfrak{D}^\star = \{(\zeta_1, \zeta_2, \zeta_3) : \zeta_1, \zeta_2, \zeta_3 \in [0,1]\},$$

$$(\zeta_1, \zeta_2, \zeta_3) \leq_{\mathfrak{D}^\star} (\eta_1, \eta_2, \eta_3) \text{ if and only if } \zeta_1 \leq \eta_1, \zeta_2 \geq \eta_2, \zeta_3 \geq \eta_3$$

for all $(\zeta_1, \zeta_2, \zeta_3), (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$. Then $(\mathfrak{D}^\star, \leq_{\mathfrak{D}^\star})$ is complete lattice.

$0_{\mathfrak{D}^\star} = (0,1,1)$ *and* $1_{\mathfrak{D}^\star} = (1,0,0)$ *are its units.*

Definition 2.4. [40] In the context of a non-empty set U, a single-valued neutrosophic set M is defined by three essential functions: the truth membership function $T_M(u)$, the indeterminacy-membership function $I_M(u)$, and the false-membership function $F_M(u)$. Therefore, a single-valued neutrosophic set M can be represented as

$$M = \{(u, T_M(u), I_M(u), F_M(u)); u \in U\}$$

where $T_M(u), I_M(u), F_M(u) \in [0,1]$ and $\forall\ u \in U$, they satisfy the condition $0 \leq T_M(u) + I_M(u) + F_M(u) \leq 3$.

In the traditional context, a triangular norm denoted by $*$ on the interval $[0,1]$ refers to a function $* : [0,1]^2 \longrightarrow [0,1]$ that exhibits properties of being increasing, commutative, and associative, satisfying the condition $1 * \zeta = \zeta$ for all $\zeta \in [0,1]$. Conversely, a triangular conorm denoted by $\diamond$ on $[0,1]$ is a function $\diamond : [0,1]^2 \longrightarrow [0,1]$ with similar properties, such as being increasing, commutative, and associative, and satisfying $0 \diamond \zeta = \zeta$ for all $\zeta \in [0,1]$ (refer to [23], [22]). This terminology is utilized within the lattice $(\mathfrak{D}^\star, \leq_{\mathfrak{D}^\star})$.

, these definitions can be extended as follows.

Definition 2.5. [39] A mapping $\Gamma : \mathfrak{D}^\star \times \mathfrak{D}^\star \longrightarrow \mathfrak{D}^\star$ is said to be a neutrosophic t-norm on $\mathfrak{D}^\star$ if it adheres to the following conditions:

(1) $\Gamma(\zeta, 1_{\mathfrak{D}^\star}) = \zeta$ for all $\zeta \in \mathfrak{D}^\star$,

(2) $\Gamma(\zeta_1, \zeta_2) = \Gamma(\zeta_2, \zeta_1)$ for all $\zeta_1, \zeta_2, \in \mathfrak{D}^\star$,

(3) $\Gamma(\zeta_1, \Gamma(\zeta_2, \zeta_3)) = \Gamma(\Gamma(\zeta_1, \zeta_2), \zeta_3)$ for all $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{D}^\star$,

(4) $\zeta_1 \leq_{\mathfrak{D}^\star} \eta_1$ and $\zeta_2 \leq_{\mathfrak{D}^\star} \eta_2$ implies $\Gamma(\zeta_1, \zeta_2) \leq_{\mathfrak{D}^\star} \Gamma(\eta_1, \eta_2)$ for all $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \mathfrak{D}^\star$.

Definition 2.6. [39] A mapping $\Gamma : \mathfrak{D}^\star \times \mathfrak{D}^\star \longrightarrow \mathfrak{D}^\star$ is said to be a neutrosophic t-conorm on $\mathfrak{D}^\star$ if it adheres to the following conditions:

(1) $\Gamma(\zeta, 0_{\mathfrak{D}^\star}) = \zeta$ for all $\zeta \in \mathfrak{D}^\star$,

(2) $\Gamma(\zeta_1, \zeta_2) = \Gamma(\zeta_2, \zeta_1)$ for all $\zeta_1, \zeta_2, \in \mathfrak{D}^\star$,

(3) $\Gamma(\zeta_1, \Gamma(\zeta_2, \zeta_3)) = \Gamma(\Gamma(\zeta_1, \zeta_2), \zeta_3)$ for all $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{D}^\star$,

(4) $\zeta_1 \leq_{\mathfrak{D}^\star} \eta_1$ and $\zeta_2 \leq_{\mathfrak{D}^\star} \eta_2$ implies $\Gamma(\zeta_1, \zeta_2) \leq_{\mathfrak{D}^\star} \Gamma(\eta_1, \eta_2)$ for all $\zeta_1, \zeta_2, \eta_1, \eta_2 \in \mathfrak{D}^\star$.

Definition 2.7. [39] A continuous neutrosophic t-norm Γ defined on $\mathfrak{D}^\star$ is classified as continuous t-representable if there exists both a continuous t-norm denoted by $*$ and a continuous t-conorm represented by $\diamond$ on the interval $[0,1]$ such that,

$$\Gamma(\zeta, \eta) = \big(\zeta_1 * \eta_1,\ \zeta_2 \diamond \eta_2,\ \zeta_3 \diamond \eta_3\big)$$

for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$.

For example, $\Gamma(\zeta, \eta) = \big(\zeta_1\eta_1,\ \min\{\zeta_2 + \eta_2, 1\},\ \min\{\zeta_2 + \eta_2, 1\}\big)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$, is a continuous t-representable norm.

Definition 2.8. [18] Let Ψ, Φ and Π are fuzzy sets from $U \times (0, \infty)$ to $[0, 1]$ such that $0 \leq \Psi(u, r) + \Phi(u, r) + \Pi(u, r) \leq 3$ for all $u \in U$ and $r > 0$. The tuple $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ is called neutrosophic normed space (NNS) if U is a linear space over $F(\mathbb{R}$ or $\mathbb{C})$, Γ is continuous t-representable norm and $\mathfrak{I}_{\Psi,\Phi,\Pi} : U \times (0, \infty) \longrightarrow \mathfrak{D}^\star$ is a mapping such that for all $u, v \in U$ and $r, s > 0$ the following conditions hold:

(a) $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) >_{\mathfrak{D}^\star} 0_{\mathfrak{D}^\star}$,

(b) $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = 1_{\mathfrak{D}^\star}$ if and only if $u = 0$,

(c) $\mathfrak{I}_{\Psi,\Phi,\Pi}(au, r) = \mathfrak{I}_{\Psi,\Phi,\Pi}\big(u, \frac{r}{|a|}\big)$ for any $0 \neq a \in F$,

(d) $\Gamma(\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r), \mathfrak{I}_{\Psi,\Phi,\Pi}(v, s)) \leq_{\mathfrak{D}^\star} \mathfrak{I}_{\Psi,\Phi,\Pi}(u + v, r + s)$,

(e) $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, .) : (0, \infty) \longrightarrow \mathfrak{D}^\star$ is continuous,

(f) $\lim_{r\to\infty} \mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = 1_{\mathfrak{D}^\star}$ and $\lim_{r\to 0} \mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = 0_{\mathfrak{D}^\star}$.

Here, $\mathfrak{I}_{\Psi,\Phi,\Pi}$ is referred to as the neutrosophic norm (NN) on U and

$$\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = \big(\Psi(u, r), \Phi(u, r), \Pi(u, r)\big)$$

Example 2.9. Let $(X, ||.||)$ be a normed linear space and let $\Gamma(\zeta, \eta) = \big(\zeta_1\eta_1,\ \min\{\zeta_2 + \eta_2, 1\},\ \min\{\zeta_2 + \eta_2, 1\}\big)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$. Now let Ψ, Φ and Π are fuzzy sets from $X \times (0, \infty)$ to $[0, 1]$ and define,

$$\mathfrak{I}_{\Psi,\Phi,\Pi}(u, r) = \big(\Psi(u, r), \Phi(u, r), \Pi(u, r)\big) = \Big(\frac{r}{r + ||u||}, \frac{||u||}{r + ||u||}, \frac{||u||}{r}\Big)$$

for all $u \in U$and $r > 0$ then $(U, I_{\Psi,\Phi,\Pi}, \Gamma)$ is neutrosophic normed space.

Definition 2.10. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a NNS. The open ball centered at $u \in U$ of radius $r > 0$ with respect to $\zeta \in (0, 1)$ is the set

$$\mathcal{B}_u(r, \zeta) = \Big\{v \in U : \mathfrak{I}_{\Psi,\Phi,\Pi}(u - v,\ r) >_{\mathfrak{D}^\star} (1 - \zeta, \zeta, \zeta)\Big\}.$$

Consider the set

$$\mathcal{T}_{\mathfrak{I}_{\Psi,\Phi,\Pi}} = \Big\{P \subset U :\ \text{for any } u \in P,\ \text{there exist}\ \zeta \in (0, 1) \text{ and } r > 0 \text{ so that } \mathcal{B}_u(r, \zeta) \subseteq P\Big\}.$$

Then $\mathcal{T}_{\mathfrak{I}_{\Psi,\Phi,\Pi}}$ defines a topology on X, induced by $\mathfrak{I}_{\Psi,\Phi,\Pi}$ and the collection

$$\Big\{\mathcal{B}_x(r, \zeta) : x \in X, r > 0,\ \zeta \in (0, 1)\Big\}$$

is the base for the topology $\mathcal{T}_{\mathfrak{I}_{\Psi,\Phi,\Pi}}$ on U.

Definition 2.11. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a *NNS*. A sequence $(u_n)_{n=1}^{\infty}$ in U is termed as Cauchy if for any $r > 0$ and $\zeta \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{I}_{\Psi,\Phi,\Pi}(u_n - u_m, r) > \mathfrak{D}^{\star}(1-\zeta, \zeta, \zeta)$ for each $m, n \geq n_0$. Additionally, the sequence $(u_n)_{n=1}^{\infty}$ in U is termed as convergent to $u \in U$ if $\lim\limits_{n\to\infty} \mathfrak{I}_{\Psi,\Phi,\Pi}(u_n - u, r) = 1_{\mathfrak{D}^{\star}}$ for every $r > 0$. In this scenario, we denote the limit as $\mathfrak{I}_{\Psi,\Phi,\Pi} - \lim\limits_{n\in\mathbb{N}} u_n = u$.

Definition 2.12. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a *NNS*. A sequence $(u_n)_{n=1}^{\infty}$ in U is termed as statistically convergent to some $u \in U$ with respect to $\mathfrak{I}_{\Psi,\Phi,\Pi}$ if, for every $\zeta \in (0,1)$ and $r > 0$,

$$\delta\big\{k \in \mathbb{N} : \mathfrak{I}_{\Psi,\Phi,\Pi}(u_k - u, s) \not>_{\mathfrak{D}^{\star}} (1-\zeta, \zeta, \zeta)\big\} = 0,$$

or equivalently

$$\delta\big\{k \in \mathbb{N} : \mathfrak{I}_{\Psi,\Phi,\Pi}(u_k - u, r) >_{\mathfrak{D}^{\star}} (1-\zeta, \zeta, \zeta)\big\} = 1.$$

We write the limit as $\mathfrak{I}^{\mathfrak{st}}_{\Psi,\Phi,\Pi} - \lim\limits_{n\in\mathbb{N}} u_n = u$.

3. Main Results

Within this section, we present the concepts of statistical graph convergence and statistical pointwise convergence pertaining to sequences of set-valued functions originating from one neutrosophic normed space and extending to another.

Let's characterize the collections of subsets of $\mathbb{N}$ in the following manner:

$$\mathscr{M} = \{M \subseteq \mathbb{N} : \ \delta(M) = 1\};$$

$$\mathscr{M}^{*} = \{M' \subseteq \mathbb{N} : \ \delta(M') \neq 0\}.$$

Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s with respect to NN_s $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}$, respectively, where U and V are linear spaces over the field of $\mathbb{R}$. Let

$$\mathfrak{D} = \big\{\varphi \mid \varphi : U \longrightarrow P(V) \text{ is a set valued function } \big\} \tag{1}$$

is the collection of all set - valued functions from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$. Now we are going to introduce some definitions:

Definition 3.1. Consider $(\varphi_n)_{n=1}^{\infty}$ as a sequence in $\mathfrak{D}$. The statistical pointwise lower limit and statistical pointwise upper limit of $(\varphi_n)_{n=1}^{\infty}$, denoted by $\mathfrak{st}_p - \varphi_n^{l}$ and $\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}$, respectively, are the set - valued functions from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$, defined at each $u \in U$ by

$$\big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \Big\{w \in V \mid \text{ there exist } M \in \mathscr{M} \text{ and } w_n \in \varphi_n(u)(n \in M) \text{ such that } \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M} w_n = w\Big\}. \tag{2}$$

and

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \Big\{v \in V \mid \text{ there exist } M' \in \mathscr{M}^* \text{ and } v_n \in \varphi_n(u)(n \in M') \text{ such that } \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n = v\Big\} \tag{3}$$

Definition 3.2. Consider $(\varphi_n)_{n=1}^{\infty}$ as a sequence in $\mathfrak{D}$. The statistical graphical lower limit of $(\varphi_n)_{n=1}^{\infty}$, denoted by $\mathfrak{st}_g - \varphi_n^l$, is a set - valued function from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ with its graph $Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$ such that for each $u \in U$,

$$\big(\mathfrak{st}_g - \varphi_n^l\big)(u) = \Big\{w \in V \mid \text{ there exists } M \in \mathscr{M} : \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M} u_n = u, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M} w_n = w, w_n \in \varphi_n(u_n)\Big\}. \tag{4}$$

and the statistical graphical uper limit, denoted by $\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}$, is a set - valued function from $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ to $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ with its graph $Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)$ such that for each $u \in U$,

$$\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)(u) = \Big\{v \in V \mid \text{ there exists } M' \in \mathscr{M}^* : \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M'} u_n = u, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n = v,\ v_n \in \varphi_n(u_n)\Big\} \tag{5}$$

Remark 3.3. From Definition 3.1 and Definition 3.2, it is clear that $(\mathfrak{st}_p - \varphi_n^l)(u)$, $(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}})(u)$, $\big(\mathfrak{st}_g - \varphi_n^l\big)(u)$ and $\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)(u)$ are closed subsets of V, for every $u \in U$. Since $\mathscr{M} \subset \mathscr{M}^*$, we get

$$\mathfrak{st}_p - \varphi_n^l \subseteq \mathfrak{st}_p - \varphi_n^{\mathfrak{u}} \text{ and } \mathfrak{st}_g - \varphi_n^l \subseteq \mathfrak{st}_g - \varphi_n^{\mathfrak{u}}. \tag{6}$$

Now, we itntroduce the definitions of statistical graph convergence and statistical pointwise convergence of a sequence $(\varphi_n)_{n=1}^{\infty}$ of members of $\mathfrak{D}$, by using the concept of lower limit and upper limit introduced above as follows:

Definition 3.4. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s and $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$. Then $(\varphi_n)_{n=1}^{\infty}$ is termed as statistically pointwise convergent, if there exists $\varphi \in \mathfrak{D}$ such that

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^{l}\big)(u) = \varphi(u),\ \forall u \in U$$

.

In such an instance, φ is termed as the statistical pointwise limit of the sequence $(\varphi_n)_{n=1}^{\infty}$, denoted by $st - \lim_n \varphi_n(u) = \varphi(u)$, $\forall u \in U$.

Definition 3.5. Consider $(X, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, T)$ and $(Y, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, T)$ as two NNS_s and $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$. Then $(\varphi_n)_{n=1}^{\infty}$ is termed as statistically graph convergent to some $\varphi \in \mathfrak{D}$ or $Gr(\varphi_n)$ is statistically convergent to $Gr(\varphi)$, if

$$Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big) = Gr\big(\mathfrak{st}_g - \varphi_n^{l}\big) = Gr(\varphi).$$

In such an instance, φ is termed as the statistical graphical limit of the sequence $(\varphi_n)_{n=1}^{\infty}$, denoted by $st - \lim_n Gr(\varphi_n) = Gr(\varphi)$.

Definition 3.6. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ is termed as statistically pointwise bounded if both $\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u)$ and $\big(\mathfrak{st}_p - \varphi_n^{l}\big)(u)$ exist for each $u \in U$.

For a sequence of real or complex numbers to converge in the ordinary sense, it must be bounded. Similarly, we observe that for a sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ to be statistically graph convergent, it must be statistically bounded in terms of the graph. However, it's important to note that statistical boundedness alone is not adequate for statistical graph convergence. Consequently, we introduce the definition of statistical graph boundedness for $(\varphi_n)_{n=1}^{\infty}$ as follows.

Definition 3.7. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ is termed as statistically graph bounded if both $Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big)$ and $Gr\big(\mathfrak{st}_g - \varphi_n^{l}\big)$ exist .

Remark 3.8. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s and let $(\varphi_n)_{n=1}^{\infty}$ represent an infinite sequence in $\mathfrak{D}$. Assuming $\mathscr{M}$ equals $\mathscr{M}_f$ and $\mathscr{M}^*$ equals $\mathscr{M}_\infty$, the concepts of statistical pointwise limits and statistical graphical limits of $(\varphi_n)_{n=1}^{\infty}$ correspond to the pointwise limits and graphical limits of the same sequence. Under these conditions, the statistical pointwise convergence of $(\varphi_n)_{n=1}^{\infty}$ aligns with its pointwise convergence, and the statistical graph convergence of $(\varphi_n)_{n=1}^{\infty}$ aligns with its graph convergence.

Theorem 3.9. *Consider* $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two* NNS_s. *Suppose* $(\varphi_n)_{n=1}^{\infty}$ *and* $(\Omega_n)_{n=1}^{\infty}$ *are two sequences in* $\mathfrak{D}$ *such that* $(\varphi_n)_{n=1}^{\infty}$ *and* $(\Omega_n)_{n=1}^{\infty}$ *are pointwise convergent and graph convergent, respectively. Then* $(\varphi_n)_{n=1}^{\infty}$ *and* $(\Omega_n)_{n=1}^{\infty}$ *are statistically pointwise convergent and statistically graph convergent, respectively.*

Proof. It is evident that $\mathscr{M}_f \subset \mathscr{M}$ and $\mathscr{M}^* \subset \mathscr{M}_\infty$.
Suppose $(\varphi_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$. Then

$$p - \varphi_n^l \subset \mathfrak{st}_p - \varphi_n^l \text{ and}$$

$$\mathfrak{st}_p - \varphi_n^{\mathfrak{u}} \subset p - \varphi_n^{\mathfrak{u}}.$$

Hence, by (6), we get

$$p - \varphi_n^l \subset \mathfrak{st}_p - \varphi_n^l \subseteq \mathfrak{st}_p - \varphi_n^{\mathfrak{u}} \subset p - \varphi_n^{\mathfrak{u}}.$$

Let $(\varphi_n)_{n=1}^{\infty}$ is pointwise convergent. Then

$$(p - \varphi_n^l)(u) = (p - \varphi_n^{\mathfrak{u}})(u) \ \forall u \in U.$$

Therefore

$$(\mathfrak{st}_p - \varphi_n^l)(u) = (\mathfrak{st}_p - \varphi_n^{\mathfrak{u}})(u) \ \forall u \in U.$$

Thus, $(\varphi_n)_{n=1}^{\infty}$ is statistically pointwise convergent.

Let $(\Omega_n)_{n=1}^{\infty}$ be a sequence in $\mathfrak{D}$ such that $(\Omega_n)_{n=1}^{\infty}$ is graph convergent. Similarly, it can be seen that $(\Omega_n)_{n=1}^{\infty}$ is statistically graph convergent. □

The converse of the Theorem 3.9 is not necessarily true. To illustrate this, let's consider the following example:

Example 3.10. Consider $\Gamma(\zeta, \eta) = \big(\zeta_1\eta_1, \ \min(\zeta_2 + \eta_2, 1), \min(\zeta_3 + \eta_3, 1)\big)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3), \eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{D}^\star$. Define fuzzy sets Ψ, Φ and Π on $\mathbb{R} \times (0, \infty)$ by

$$\Psi(u, s) = e^{\left(-\frac{|u|}{s}\right)}, \ \Phi(u, s) = 1 - e^{\left(-\frac{|u|}{s}\right)} \text{ and } \Pi(u, s) = 1 - e^{\left(-\frac{|u|}{s}\right)}$$

for every $u \in \mathbb{R}$ and for all $s \in (0, \infty)$. Then $(\mathbb{R}, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ is a NNS, where $\mathfrak{I}_{\Psi,\Phi,\Pi}(u, s) = \big(\Psi(u, s), \Phi(u, s), \Pi(u, s)\big)$.
Now define $\varphi_n : \mathbb{R} \longrightarrow P(\mathbb{R})$ by

$$\varphi_n(u) = \begin{cases} [0, \frac{1}{2}], & \text{if } n = p \\ [-\frac{1}{2}, 0], & \text{if } n \neq p \end{cases} \quad \text{p is prime.}$$

for each $u \in \mathbb{R}$.
Then

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \big[-\frac{1}{2}, 0\big]$$

for each $u \in \mathbb{R}$.

Also,

$$Gr\big(\mathfrak{st}_g - \varphi_n^{\mathfrak{u}}\big) = Gr\big(\mathfrak{st}_g - \varphi_n^{l}\big) = \big\{(u,v) : u \in \mathbb{R}, -\frac{1}{2} \leq v \leq 0\big\}.$$

Thus, $(\varphi_n)_{n=1}^{\infty}$ is both statistically pointwise convergent and statistically graph convergent. However, on the other hand,

$$(p - \varphi_n^{\mathfrak{u}})(u) = \big[-\frac{1}{2}, \frac{1}{2}\big] \text{ and } (p - \varphi_n^{l})(u) = \{0\}$$

for each $u \in \mathbb{R}$. Thus $(\varphi_n)_{n=1}^{\infty}$ is not a pointwise convergent sequence. Also,

$$Gr(\varphi_n^{\mathfrak{u}}) = \big\{(u,v) : u \in \mathbb{R}, -\frac{1}{2} \leq v \leq \frac{1}{2}\big\}$$

but $Gr(\varphi_n^{l})$, the graphical lower limit of $(\varphi_n)_{n=1}^{\infty}$ does not exist. Hence $(\varphi_n)_{n=1}^{\infty}$ is not a graph convergent sequence.

Theorem 3.11. *Consider $(U, \Im_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \Im_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. Assume $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are two sequences in $\mathfrak{D}$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ for each $u \in U$ and $st - \lim_n Gr(\Omega_n) = Gr(\Omega)$. Then φ and Ω are unique.*

Proof. Let, $\exists\ \Omega^0 \in \mathfrak{D}$ such that

$$st - \lim_n Gr(\Omega_n) = Gr(\Omega^0)$$

. Then

$$Gr\big(\mathfrak{st}_g - \Omega_n^{u}\big) = Gr\big(\mathfrak{st}_g - \Omega_n^{l}\big) = Gr(\Omega^0) = Gr(\Omega).$$

Hence $\Omega = \Omega^0$. Similarly, it can be easily seen that pointwise limit is also unique. □

Proposition 3.12. *Consider $(U, \Im_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \Im_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. Assume $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ are statistically pointwise convergent sequences in $\mathfrak{D}$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ and $st - \lim_n \Omega_n(u) = \Omega(u)$ for each $u \in U$. Then the sum of the sequences $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ is statistically pointwise convergent with $st - \lim_n (\varphi_n + \Omega_n)(u) = (\varphi + \Omega)(u)$ for each $u \in U$.*

Proof. Let $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ be two sequences in $\mathfrak{D}$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ and $st - \lim_n \Omega_n(u) = \Omega(u)$ for each $u \in U$. Then

$$\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^{l}\big)(u) = \varphi(u),\ \forall u \in U$$

and

$$\big(\mathfrak{st}_p - \Omega_n^{u}\big)(u) = \big(\mathfrak{st}_p - \Omega_n^{l}\big)(u) = \Omega(u),\ \forall u \in U.$$

Let v be an arbitrary element of $\varphi(u)$. Since $\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \varphi(u),\ \forall u \in U$, there exist $M_1' \in \mathscr{M}^*$ and $v_n \in \varphi_n(u)$ $(n \in M_1')$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim\limits_{n\in M_1'} v_n = v$. Hence $\lim\limits_{n\in M_1'} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}(v_n - v, r) = 1_{\mathfrak{D}^\star}$ for every $r > 0$, i.e.,

$$\lim_{n\in M_1'} \Psi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) = 1, \lim_{n\in M_1'} \Phi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) = 0, \lim_{n\in M_1'} \Pi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) = 0. \tag{7}$$

Let w be an arbitrary element of $\Omega(u)$. Also, $\big(\mathfrak{st}_p - \Omega_n^u\big)(u) = \Omega(u),\ \forall u \in U$. Hence, there exist $M_2' \in \mathscr{M}^*$ and $w_n \in \Omega_n(u)(n \in M_2)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim\limits_{n\in M_2'} w_n = w$. Hence $\lim\limits_{n\in M_2'} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}(w_n - w, r) = 1_{\mathfrak{D}^\star}$ for every $r > 0$, i.e.,

$$\lim_{n\in M_2'} \Psi_2\bigg(w_n - w,\ \frac{r}{2}\bigg) = 1, \lim_{n\in M_2'} \Phi_2\bigg(w_n - w,\ \frac{r}{2}\bigg) = 0, \lim_{n\in M_2'} \Pi_2\bigg(w_n - w,\ \frac{r}{2}\bigg) = 0.. \tag{8}$$

Because $\big(\mathfrak{st}_p - \varphi_n^{\mathfrak{u}}\big)(u) = \big(\mathfrak{st}_p - \varphi_n^{l}\big)(u)$ and $\big(\mathfrak{st}_p - \Omega_n^{u}\big)(u) = \big(\mathfrak{st}_p - \Omega_n^{l}\big)(u),\ \forall u \in U$, (7) and (8) hold for $n \in M_2'$ and $n \in M_1'$, respectively. Now, in the choice of $M_1', M_2' \in \mathscr{M}^*$, we have the following two possibilities:

(1) $M_1' \cap M_2' = \emptyset$ or $\delta(M_1' \cap M_2') = 0$.

(2) $M_1' \cap M_2' \in \mathscr{M}^*$.

If $M_1' \cap M_2' = \emptyset$ or $\delta(M_1' \cap M_2') = 0$, take $M' = M_1'$ or $M' = M_2'$. If $M_1' \cap M_2' \in \mathscr{M}^*$, put $M' = M_1' \cap M_2'$. Thus for $n \in M'$, we get

$$\begin{aligned}
&\Psi_2\bigg((v_n + w_n) - (v + w),\ r\bigg)\\
&\quad = \Psi_2\bigg((v_n - v) + (w_n - w),\ r\bigg)\\
&\quad \geq \Psi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) * \Psi_2\bigg(w_n - w,\ \frac{r}{2}\bigg).
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{n\in M'} \Psi_2\bigg((v_n + w_n) - (v + w),\ r\bigg)\\
&\quad \geq \lim_{n\in M'} \Psi_2\bigg(v_n - v,\ \frac{r}{2}\bigg) * \lim_{n\in M'} \varphi_2\bigg(w_n - w,\ \frac{r}{2}\bigg)\\
&\quad = 1 * 1\\
&\quad = 1.
\end{aligned}$$

Also for $n \in M'$,

$$\begin{aligned}\Phi_2\Big((v_n + w_n) - (v + w),\ r\Big) \\ &= \Phi_2\Big((v_n - v) + (w_n - w),\ r\Big) \\ &\leq \Phi_2\Big(v_n - v,\ \frac{r}{2}\Big) \diamond \Phi_2\Big(w_n - w,\ \frac{r}{2}\Big).\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\in M'} \Phi_2\Big((v_n + w_n) - (v + w),\ r\Big) \\ &\leq \lim_{n\in M'} \Phi_2\Big(v_n - v,\ \frac{r}{2}\Big) \diamond \lim_{n\in M'} \Phi_2\Big(w_n - w,\ \frac{r}{2}\Big) \\ &= 0 * 0 \\ &= 0.\end{aligned}$$

Similarly, for $n \in M'$,

$$\begin{aligned}\Pi_2\Big((v_n + w_n) - (v + w),\ r\Big) \\ &= \Pi_2\Big((v_n - v) + (w_n - w),\ r\Big) \\ &\leq \Pi_2\Big(v_n - v,\ \frac{r}{2}\Big) \diamond \Pi_2\Big(w_n - w,\ \frac{r}{2}\Big).\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\in M'} \Pi_2\Big((v_n + w_n) - (v + w),\ r\Big) \\ &\leq \lim_{n\in M'} \Pi_2\Big(v_n - v,\ \frac{r}{2}\Big) \diamond \lim_{n\in M'} \Pi_2\Big(w_n - w,\ \frac{r}{2}\Big) \\ &= 0 * 0 \\ &= 0.\end{aligned}$$

Thus $\lim_{n\in M'} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}\big((v_n + w_n) - (v + w), r\big) = 1_{\mathfrak{D}^\star}$ for every $r > 0$ and hence

$$\Big(\mathfrak{st}_p - \big(\varphi_n + \Omega_n\big)^{\mathfrak{u}}\Big)(u) = \big(\varphi + \Omega\big)(u),\ \forall u \in U.$$

Again, since $\big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \varphi(u)$ for each $u \in U$ and $v \in \varphi(u)$, there exist $M_1 \in \mathscr{M}$ and $c_n \in \varphi_n(u)$ $(n \in M_1)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n\in M_1} c_n = v$, i.e., for every $s > 0$,

$$\lim_{n\in M_1} \Psi_2\Big(c_n - v,\ \frac{s}{2}\Big) = 1, \lim_{n\in M_1} \Phi_2\Big(c_n - v,\ \frac{s}{2}\Big) = 0, \lim_{n\in M_1} \Pi_2\Big(c_n - v,\ \frac{s}{2}\Big) = 0.$$

Also, $\big(\mathfrak{st}_p - \Omega_n^l\big)(u) = \Omega(u)$ for each $u \in U$ and $w \in \Omega(u)$. Hence, there exist $M_2 \in \mathscr{M}$ and $d_n \in \Omega_n(x)(n \in M_2)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim\limits_{n\in M_2} d_n = w$, i.e.,

$$\lim_{n\in M_2} \Psi_2\left(d_n - w,\ \frac{s}{2}\right) = 1, \lim_{n\in M_2} \Phi_2\left(d_n - w,\ \frac{s}{2}\right) = 0, \lim_{n\in M_2} \Pi_2\left(d_n - w,\ \frac{s}{2}\right) = 0,$$

for every $s > 0$. Put $M = M_1 \cap M_2$. Clearly, $M \in \mathscr{M}$. Then, similar to above, for every $s > 0$ and $n \in M$, we have

$$\lim_{n\in M} \Psi_2\Big((c_n + d_n) - (v + w),\ s\Big) = 1,$$

$$\lim_{n\in M} \Phi_2\Big((c_n + d_n) - (v + w),\ s\Big) = 0,$$

$$\lim_{n\in M} \phi_2\Big((c_n + d_n) - (v + w),\ s\Big) = 0$$

and thus $\lim\limits_{n\in M} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}\big((c_n + d_n) - (v + w), s\big) = 1_{\mathfrak{D}^\star}$ for every $s > 0$. Since v and w are arbitrary members of $\varphi(u)$ and $\Omega(u)$, respectively, we get

$$\Big(\mathfrak{st}_p - \big(\varphi_n + \Omega_n\big)^l\Big)(u) = \big(\varphi + \Omega\big)(u),\ \forall u \in U.$$

Therefore the sum of the sequence $(\varphi_n)_{n=1}^{\infty}$ and $(\Omega_n)_{n=1}^{\infty}$ is statistically pointwise convergent with $st - \lim\limits_{n}(\varphi_n + \Omega_n)(u) = (\varphi + \Omega)(u)$ for each $u \in U$. □

Lemma 3.13. *consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\phi_2}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS$_s$. Suppose $(\varphi_n)_{n=1}^{\infty}$ is a sequence in $\mathfrak{D}$. Then the following hold:*

(1) $\bigcup\limits_{u\in U} \{u\} \times \big(\mathfrak{st}_p - \varphi_n^l\big)(u) \subseteq Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$,

(2) $\bigcup\limits_{u\in U} \{u\} \times \big(\mathfrak{st}_p - \varphi_n^u\big)(u) \subseteq Gr\big(\mathfrak{st}_g - \varphi_n^u\big)$.

Proof. Part (1) , let $u \in U$ and $v \in \big(\mathfrak{st}_p - \varphi_n^l\big)(u)$. Then there exist $M \in \mathscr{M}$ and $v_n \in \varphi_n(u)$ $(n \in M)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim\limits_{n\in M} v_n = v$, i.e.,

$$\lim_{n\in M} \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}(v_n - v, r) = 1_{\mathfrak{D}^\star},\ \text{for every } r > 0. \tag{9}$$

Now, consider the constant sequence $(u_n)_{n=1}^{\infty} = \{u\}$ in U. Then $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim\limits_{n\in M} u_n = u$, i.e., for every $r > 0$,

$$\lim_{n\in M} \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}(u_n - u, r) = 1_{\mathfrak{D}^\star}.$$

Thus, $v_n \in \varphi_n(u_n) = \varphi_n(u)$ $(n \in M)$ satisfies (9). Therefore $v \in \big(\mathfrak{st}_g - \varphi_n^l\big)(u)$ and hence $(u, v) \in Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$.

Similarly, part (2) can be proved. □

Corollary 3.14. *Consider* $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two* NNS_s *and* $(\varphi_n)_{n=1}^{\infty}$ *be a sequence in* $\mathfrak{D}$*. If statistical pointwise limit* φ *and statistical graphical limit* Ω *of* $(\varphi_n)_{n=1}^{\infty}$ *both exist, then for all elements* u *in* U*, it follows that* $\varphi(u)$ *is a subset of* $\Omega(u)$*.*

It is being established that the presence of a statistical pointwise limit for a sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathfrak{D}$ does not guarantee the existence of the statistical graphical limit, and vice versa. Even in cases where both limits exist, they may not be directly comparable. However, there exists a specific condition under which these limits coincide. To formalize this, we introduce the following theorem:

Theorem 3.15. *Consider* $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two* NNS_s*. Let* $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ *induces the discrete topology* $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ *on* U*. Then a sequence* $(\varphi_n)_{n=1}^{\infty}$ *in* $\mathfrak{D}$ *is statistically pointwise convergent if and only if it is statistically graph convergent and both the statistical pointwise limit and the statistical graphical limit of* $(\varphi_n)_{n=1}^{\infty}$ *are equivalent.*

Proof. Let $(\varphi_n)_{n=1}^{\infty} \in \mathfrak{D}$ such that $(\varphi_n)_{n=1}^{\infty}$ is statistically pointwise convergent. Then there exists $\varphi \in \mathfrak{D}$ such that

$$\left(\mathfrak{st}_p - \varphi_n^u\right)(u) = \left(\mathfrak{st}_p - \varphi_n^l\right)(u) = \varphi(u), \ \forall u \in U.$$

Thus, by using part (1) of Lemma 3.13, we obtain

$$Gr(\varphi) \subset Gr\left(\mathfrak{st}_g - \varphi_n^l\right). \tag{10}$$

Now we claim to show that $Gr\left(\mathfrak{st}_g - \varphi_n^u\right) \subset Gr(\varphi)$.

Let $(u, v) \in Gr\left(\mathfrak{st}_g - \varphi_n^u\right)$. Hence $v \in \left(\mathfrak{st}_g - \varphi_n^u\right)(u)$. Then, there exists $M' \in \mathscr{M}^*$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M'} u_n = u$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n = v$ for $v_n \in \varphi_n(u_n)$. Since $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ is the discrete topology on U, we have $u_n = u$ and $\varphi_n(u_n) = \varphi_n(u)$, for all $n \in M'$. This implies that $v \in \left(\mathfrak{st}_p - \varphi_n^u\right)(u) = \varphi(u)$. Thus $(u, v) \in Gr(\varphi)$ and hence

$$Gr\left(\mathfrak{st}_g - \varphi_n^u\right) \subset Gr(\varphi). \tag{11}$$

Consequently, by (10) and (11), we obtain

$$Gr\left(\mathfrak{st}_g - \varphi_n^l\right) = Gr\left(\mathfrak{st}_g - \varphi_n^u\right) = Gr(\varphi).$$

Thus the sequence $(\varphi_n)_{n=1}^{\infty}$ is statistically graph convergent with $st - \lim_n Gr(\varphi_n) = Gr(\varphi)$.

Conversely, suppose that $(\varphi_n)_{n=1}^{\infty}$ is statistically graph convergent. Then there exists $\varphi \in \mathfrak{D}$ such that

$$Gr\left(\mathfrak{st}_g - \varphi_n^{\mathrm{u}}\right) = Gr\left(\mathfrak{st}_g - \varphi_n^l\right) = Gr(\varphi).$$

Now, by using part (2) of Lemma 3.13, we obtain

$$\left(\mathfrak{st}_p - \varphi_n^{\mathrm{u}}\right)(u) \subset \varphi(u), \ \forall u \in U. \tag{12}$$

Now we claim to show that $\varphi(u) \subset \big(\mathfrak{st}_p - \varphi_n^l\big)(u)$, $\forall u \in U$.

Let $w \in \varphi(u)$. Then $(u, w) \in Gr(\varphi) = Gr\big(\mathfrak{st}_g - \varphi_n^l\big)$. Hence, there exists $M \in \mathscr{M}$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim\limits_{n\in M} u_n = u$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim\limits_{n\in M} w_n = w$ for $w_n \in \varphi_n(u_n)$. Since $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ is discrete topology on U, we have $u_n = u$ and $\varphi_n(u_n) = \varphi_n(u)$, for all $n \in M$. Therefore, $(u, w) \in \big(\mathfrak{st}_p - \varphi_n^l\big)$. Consequently, we get

$$\varphi(u) \subset \big(\mathfrak{st}_p - \varphi_n^l\big)(u), \ \forall u \in U. \tag{13}$$

From (12) and (13), we get

$$\big(\mathfrak{st}_p - \varphi_n^l\big)(u) = \big(\mathfrak{st}_p - \varphi_u^l\big)(u) = \varphi(u), \ \forall u \in U.$$

Thus $(\varphi_n)_{n=1}^{\infty}$ is statistically pointwise convergent with $st - \lim\limits_{n} \varphi(u) = \varphi(u)$ for each $u \in U$. □

Within $\mathfrak{D}$, there exist specific collections of sequences where the statistical graphical limit differs from the statistical pointwise limit, and conversely. Before delving into such cases, let's introduce the concept of semicontinuity of sequences in $\mathfrak{D}$ as follows:

Definition 3.16. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A set–valued function $\varphi : U \longrightarrow P(V)$ is termed as neutrosophic lower semicontinuous ($NLSC$) at $u_0 \in U$ if and only if for any $r > 0$, $\zeta \in (0, 1)$ and $v \in V$ with $\mathcal{B}_v(r, \zeta)$ in V such that $\varphi(u_0) \cap \mathcal{B}_v(r, \zeta) \neq \emptyset$, there exists $\mathcal{B}_{u_0}(r^0, \zeta^0)$ in U for some $r^0 > 0$ and $\zeta^0 \in (0, 1)$ such that $\varphi(w) \cap \mathcal{B}_v(r, \zeta) \neq \emptyset$, for each $w \in \mathcal{B}_{u_0}(r^0, \zeta^0)$.

The set–valued function $\varphi : U \longrightarrow P(V)$ is called $NLSC$ on U, if it is $NLSC$ at every $u \in U$.

Definition 3.17. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A set–valued function $\varphi : U \longrightarrow P(V)$ is termed as neutrosophic upper semicontinuous ($NUSC$) at $u_0 \in U$ if and only if for any $r > 0$ and $\zeta \in (0, 1)$ with $\mathcal{B}_{\varphi(u_0)}(r, \zeta)$ in V, there exists $\mathcal{B}_{u_0}(r^0, \zeta^0)$ in U for some $r^0 > 0$ and $\zeta^0 \in (0, 1)$ such that

$$\varphi\big(\mathcal{B}_{u_0}(r^0, \zeta^0)\big) = \bigcup_{w\in\mathcal{B}_{u_0}(r^0,\zeta^0)} \varphi(w) \subseteq \mathcal{B}_{\varphi(u_0)}(r, \zeta).$$

The set–valued function $\varphi : U \longrightarrow P(V)$ is called $NUSC$ on U, if it is $NUSC$ at every $u \in U$.

Definition 3.18. Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s. A set–valued function $\varphi : U \longrightarrow P(V)$ is termed as neutrosophic continuous (NC) on U, if it is $NUSC$ as well as $NLSC$ on U.

Definition 3.19. Consider $(U, \mathfrak{I}_{\Psi,\Phi,\Pi}, \Gamma)$ as a *NNS* and $C \subset U$. Then the statistical closure of set C with respect to $\mathfrak{I}_{\Psi,\Phi,\Pi}$ denoted by $\overline{C_{st}}$, is defined as

$$\overline{C_{st}} = \Big\{ \mathcal{L} \in U \mid \text{there exist } M \in \mathscr{M} \text{ and } (u_n) \text{ in } E$$
$$\text{such that } \mathfrak{I}_{\Psi,\Phi,\Pi} - \lim_{n \in N} x_n = \mathcal{L} \Big\}.$$

We define C as a statistically closed subset of U if C coincides with its statistical closure, denoted as $\overline{C_{st}}$. It's evident that every closed subset of U is also statistically closed.

For any two topological spaces U and V, we denote $\mathcal{C}(U, P(V))$ as the collection of all continuous set–valued functions from U to V with closed values.

Theorem 3.20. *Consider $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ and $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ as two NNS_s such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ induces the non–discrete topology $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ on U and $\mathcal{C}\big([0,1], P(V)\big)$ is non–trivial, where $[0,1]$ is equipped with the usual topology. Then there exists $\{\varphi, \varphi_n : n \in \mathbb{N}\} \in \mathcal{C}(U, P(V))$ such that $st - \lim_n \varphi_n(u) = \varphi(u)$ for each $u \in U$ but $st - \lim_n Gr(\varphi_n) \neq Gr(\varphi)$.*

Proof. Consider $u_0 \in U$ be a non–isolated point. Then for every neighborhood $\mathbb{B}_{u_0}$ of u_0 in U, we have $\mathbb{B}_{u_0} \neq \{u_0\}$. Without loss of generality, consider the countable collection

$$\mathfrak{B}_{u_0} = \left\{ \mathbb{B}^j_{u_0} : \ \mathbb{B}^j_{u_0} = \mathcal{B}_{u_0}\left(\frac{1}{j}, \frac{1}{j}\right) : j \in \mathbb{N} \right\}$$

of neighborhoods of u_0 in U. For fixed $j \in \mathbb{N}$, choose $\mathbb{B}^j_{u_0} \in \mathfrak{B}_{u_0}$ and let $u_j \in \mathbb{B}^j_{u_0}$ such that $u_j \neq u_0$. Since $\big(U, \mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}\big)$ is completely regular Hausdorff space, there exists a continuous function $f_j : U \longrightarrow [0,1]$ corresponding to $\mathbb{B}^j_{u_0}$ such that $f_j(u_j) = 1$ and

$$f_j(u) = 0, \text{ for all } u \in (\mathbb{B}^j_{u_0})^c \bigcup \{u_0\}.$$

As $\mathcal{C}\big([0,1], P(V)\big)$ is non–trivial, there exists $h \in \mathcal{C}\big([0,1], P(V)\big)$ such that $h(1) \neq h(0)$. Therefore, for every $j \in \mathbb{N}$, $\varphi_j = h \circ f_j \in \mathcal{C}\big(U, P(V)\big)$. Thus, the sequence $\big\{\varphi_j : \mathbb{B}^j_{u_0} \in \mathfrak{B}_{u_0}, j \in \mathbb{N}\big\}$ belongs to $\mathcal{C}\big(U, P(V)\big)$. Let $u \in U$ be arbitrary. If $u = u_0$ or $u \notin \mathbb{B}^j_{u_0}$ for all $j \in \mathbb{N}$, then

$$F_j(u) = h(f_j(u)) = h(0), \text{ for all } j.$$

If $u \neq u_0$ or $u \in \mathbb{B}^j_{u_0}$ for some fixed $j \in \mathbb{N}$, then there is $M_1 \in \mathscr{M}$ such that

$$\varphi_n(u) = h(f_n(u)) = h(0), \text{ for all } n \in M_1.$$

Take $\varphi(u) = h(0)$ for each $u \in U$. Hence the sequence $\big\{\varphi_j : \mathbb{B}^j_{u_0} \in \mathfrak{B}_{u_0}, j \in \mathbb{N}\big\}$ statistically pointwise convergent with $st - \lim_j \varphi_j(u) = \varphi(u)$ for each $u \in U$.

Alternatively, let $u_j \in \mathbb{B}^j_{u_0}$ such that $u_j \neq u_0$ for every $j \in \mathbb{N}$. Consequently, there exists $M_2 \in \mathscr{M}$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{j \in M_2} u_j = u_0$ and $f_j(u_j) = 1$ for each j. This implies $\varphi_j(u_j) =$

$h(1)$ for every j. Suppose $v \in h(1) \setminus h(0)$. Then $(u_{\ j}, v) \in Gr(\varphi_j)$ for every j, and $(u_0, v) \in Gr(\mathfrak{st}_g - \varphi_j^l) \setminus Gr(\varphi)$. Consequently, $st - \lim_j Gr(\varphi_j) \neq Gr(\varphi)$. □

Theorem 3.21. *Consider* $(U, \mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}, \Gamma)$ *and* $(V, \mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2}, \Gamma)$ *as two* NNS_s *such that* $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}$ *induces the non–discrete topology* $\mathcal{T}_{\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1}}$ *on* U *and* $\mathcal{C}\big([0,1], P(V)\big)$ *is non–trivial, where* $[0,1]$ *is equipped with the usual topology. Then there exists* $\{\varphi, \varphi_n : n \in \mathbb{N}\} \in \mathcal{C}(U, P(V))$ *such that* $st - \lim_n Gr(\varphi_n) = Gr(\varphi)$ *but* $st - \lim_n \varphi_n(u) \neq \varphi(u)$ *for each* $u \in U$.

Proof. let $u_0 \in U$ be a non–isolated point and $h_1 \in \mathcal{C}([0,1], P(V))$ such that $h_1(1) \neq h_1(0)$. Then there exist a non constant sequence $(u_n)_{n=1}^{\infty}$ in U and $M \in \mathscr{M}$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M} u_n = u_0$. Now define $h_2 : [0,1] \longrightarrow P(V)$ by

$$h_2(\zeta) = \overline{\bigcup_{\eta \leq \zeta} h_1(\eta)}, \ \zeta \in [0,1].$$

It is certain that $h_2 \in \mathcal{C}([0,1], P(V))$ and $h_2(1) \neq h_2(0)$. Now, for every $\zeta \in [0,1]$, we have $h_2(\zeta) \subset h_2(1)$. let $r > 0$ and $u \in U$. Set $\zeta = \varphi_1(u - u_0, r)$ and $\zeta_n = \varphi_1(u_n - u_0, r)$. Clearly, for each $n \in \mathbb{N}$, $\zeta_{n+1} \geq \zeta_n$ and $\lim_{n \in M} \zeta_n = 1$. Now define the set–valued functions φ, φ_n from U to V as follows:

$$\varphi_n(u) = \begin{cases} h_2\left(\frac{1-\zeta}{1-\zeta_n} \times \frac{\zeta_n}{\zeta}\right), & \text{if } \zeta > \zeta_n, \\ h_2(1), & \text{if } \zeta_n \geq \zeta, \end{cases} \quad n \in \mathbb{N}$$

$$\varphi(u) = h_2(1), \ \text{for all } u \in U.$$

From the definition of h_2, it is clear that $\{\varphi, \varphi_n : n \in \mathbb{N}\} \in \mathcal{C}(U, P(V))$. Let $(u_1, v_1) \in Gr\big(\mathfrak{st}_g - \varphi_n^u\big)$. Then, there exists $M' \in \mathscr{M}^*$ such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M'} u_n^{'} = u_1$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n^{'} = v_1$ for $v_n^{'} \in \varphi_n(u_n^{'}) = h_2(1)$. Hence $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in M'} v_n^{'} = v_1$ for $v_n^{'} \in \varphi(u_1)$. Since $Gr(\varphi)$ is closed, we get $v_1 \in \varphi(u_1)$ and thus $(u_1, v_1) \in Gr(\varphi)$. Therefore,

$$Gr\big(\mathfrak{st}_g - \varphi_n^u\big) \subset Gr(\varphi). \tag{14}$$

Now, let $(u_2, v_2) \in Gr(\varphi)$. Hence $v_2 \in \varphi(u_2) = h_2(1)$. If $u_2 = u_0$, then $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in M} u_n = u_2$ and $\varphi_n(u_n) = h_2(1)$, for all $n \in M$. Since $h_2(1)$ is closed and hence statistically closed, there exists $v_n \in h_2(1) = \varphi_n(u_n)$ such that $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in N} v_n = v_2$. If $u_2 \neq u_0$, there exist $N \in \mathscr{M}$ and a sequence $(u_n^{''})_{n=1}^{\infty}$ in U and such that $\mathfrak{I}_{\Psi_1,\Phi_1,\Pi_1} - \lim_{n \in N} u_n^{''} = u_2$ and $\mathfrak{I}_{\Psi_2,\Phi_2,\Pi_2} - \lim_{n \in N} v_n^{''} = h_2(1)$ for $v_n^{''} \in \varphi_n(u_n^{''})$. Since $v_2 \in h_2(1)$, we get $(u_2, v_2) \in Gr(\mathfrak{st}_g - \varphi_n^l)$. Hence

$$Gr(\varphi) \subset Gr(\mathfrak{st}_g - \varphi_n^l) \tag{15}$$

From (14) and (15), we get $st - \lim_n Gr(\varphi_n) = Gr(\varphi)$.

Alternatively, $\varphi_n(u_0) = h_2(0) \neq h_2(1)$, for every $n \in \mathbb{N}$. Thus $st - \lim_n \varphi_n(u_0) \neq \varphi(u_0)$.

Hence proof of the theorem is completed. □

4. Conclusions

This articles introduces the concepts of statistical pointwise and statistical graphical limits of sequences of set–valued functions defined from a neutrosophic normed space to another, certan theorems about statistical point wise and statistical graphical covergence are proved and lastly the idea of neutrosophic upper and lower semi continuities of set valued maps is given and used to develop link between these convergences.

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The Periodicity of Square Fuzzy Neutrosophic Soft MatricesBased on Minimal Strong Components

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Abstract. In this paper Minimal Strong Component ($\mathcal{MSC}$) of a Fuzzy Neutrosophic Soft Matrix ($\mathcal{FNSM}$) is suggested. By employing the connection of periodicity behaviours of $\mathcal{FNSM}$ with its cut matrices, the periodicity of power sequence of $\mathcal{FNSM}$ is described. Especially the concepts of $\mathcal{MSC}$ is given and the periodicity of a $\mathcal{FNSM}$ by its (Theorem-4.10) on the basis of the above results, the greatest value $\max \sum\limits_{l_i=n} = [l_i]$ of the periodicity of all $\mathcal{SFNSM}$ for a given positive integer n is obtained. So in a case we have clearly resolved the problem of the greatest value of all periodicity $\mathcal{FNSM}$ for a given positive integer n.

Keywords: Fuzzy Neutrosophic Soft Matrix ($\mathcal{FNSM}$), Minimal Strong Component ($\mathcal{MSC}$), Cut Fuzzy Neutrosophic Soft Matrix ($\mathcal{CFNSM}$), Periodicity of Fuzzy Neutrosophic Soft Matrices ($\mathcal{PFNSM}s$)

1. Introduction

The models of real-life problems in almost every field of science like mathematics, physics, operations research, medical sciences, engineering, computer science, artificial intelligence, and management sciences are mostly full of complexities. Many theories have been developed to overcome these uncertainties; one among those theories is fuzzy set theory. Zadeh [1] was the first who gave the concept of a Fuzzy Set ($\mathcal{FS}$) are the generalizations or extensions of crisps sets.

In order to add the concept of nonmembership term to the idea of $\mathcal{FS}$, the concept of an Intuitionistic Fuzzy Set ($\mathcal{IFS}$) was introduced by Atanassov in [2], where he added the thought of nonmembership term to the definition of $\mathcal{FS}$. The $\mathcal{IFS}$ is characterized by a membership function μ and a nonmembership function η with ranges [0,1]. The $\mathcal{IFS}$ is the generalization of a FS. An $\mathcal{IFS}$ can be applied in several fields including modeling, medical diagnosis, and decision-making. In [3] Molodtsov introduced the concept of a Soft Set $\mathcal{SS}$ and developed the fundamental results related to this theory. Basic operations including complement, union, and intersection are also defined on this set. Also he used $\mathcal{SS}$s for applications in games, probability, and operational theories. Maji et. al., [4,5] proposed the Fuzzy Soft Sets ($\mathcal{FSS}s$) and Intuitionistic Fuzzy Soft Set ($\mathcal{IFSS}$) by combining $\mathcal{SS}s$ and $\mathcal{FS}s$ and applied them in decision-making problems.

The concept of neutrosophy was introduced by Smarandache in [6]. A Neutrosophic Set ($\mathcal{NS}$) is characterized by a truth membership function $\mathcal{T}$, an indeterminacy function $\mathcal{I}$, and a falsity membership function $\mathcal{F}$. A $\mathcal{FS}$ is a mathematical framework which generalizes the concept of a classical set, $\mathcal{FS}$, $\mathcal{IFS}$, and $\mathcal{IVFS}$. Broumi et,al., [7] proposed the generalized interval neutrosophic soft set and its decision making problem.

Thomason [8] was the first who gave the concept of a Fuzzy Matrix $\mathcal{FM}$. He discussed the convergence of powers of $\mathcal{FM}$, its play a vital role in scientific development. And he also pointed out that the powers of a $\mathcal{FM}$ either converge or oscillate with a finite period. Li [9,10] discussed the periodicity and index of fuzzy matrices in the general case. In [11] Fan proves that the periodicity of a $\mathcal{FM}$ is the least common multiple (l.c.m.) of periodicity of its cut matrices, and the index of a fuzzy matrix is not greater than the maximum index of its cut matrices. It is also shown that the periodicity set of the power sequences of $\mathcal{FM}s$ of order n is not bounded from above by a power of n for all integers n. Liua and Ji [12,13] have discussed the periodicity of Square Fuzzy Matrices $\mathcal{SFM}s$ based on minimal strong components. Atanassov [14,15] has studied intuitionistic fuzzy index matrix and the index matrix representation of the intuitionistic fuzzy graphs has been studied. Murugadas et.al., [16] presented the periodicity of intuitionistic fuzzy matrix. Manoj Bora et.al., [17] introduced the concepts of Intuitionistic Fuzzy Soft Matrix $\mathcal{IFSM}$ theory and its Application in Medical Diagnosis. Arockiarani and Sumathi [18, 19] proposed Fuzzy Neutrosophic Soft Matrix $\mathcal{FNSM}$ and used them in decision making problems. Kavitha et.al., [20–26] introduced some concepts on priodicity of interval values, on powers of matrices and convergence of matrices usig the notion of $\mathcal{FNSM}$. The idea of monotone interval fuzzy neutrosophic soft eigenproblem and convergence of fuzzy neutrosophic soft circulant matrices are proposed by Murugadas et.al., [27,28]. Uma et.al., [29] presented the concepts of $\mathcal{FNSM}s$ of Type-1 and Type-2.

This paper is organized as follows: In section-2, some basic notions related to this topics are recalled. Section-3 we, discuss the properties of periodicity and index of $\mathcal{FNSM}$. In section-4 we, explain the digraph representaion of Strongly Connected $\mathcal{SC}$ and Minimal Strong Components $\mathcal{MSC}$ by using $\mathcal{SFNSM}$. Section-5 we can frame the algorithm to find the $\mathcal{MSC}$ of $\mathcal{FNSM}$. Section-6 is for conculusion.

2. **Preliminaries**

The following definitions is needed to our study.

Definition 2.1. [6] A Neutrosophic Set $\mathcal{NS}$ A on the universe of discourse X is defined as $A = \{\langle x,\ \mathcal{T}_A(x),\ \mathcal{I}_A(x),\ \mathcal{F}_A(x)\rangle, x \in X\}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} : X \to]^-0, 1^+[$ and

$$^-0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3^+. \ (2.1)$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in Scientific and Engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$. Therefore we can rewrite equation (2.1) as $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3$. In short an element $\widetilde{a}$ in the neutrosophic set A, can be written as $\widetilde{a} = \langle a^{\mathcal{T}}, a^{\mathcal{I}}, a^{\mathcal{F}}\rangle$, where $a^{\mathcal{T}}$ denotes degree of truth, $a^{\mathcal{I}}$ denotes degree of indeterminacy, $a^{\mathcal{F}}$ denotes degree of falsity such that $0 \leq a^{\mathcal{T}} + a^{\mathcal{I}} + a^{\mathcal{F}} \leq 3$.

Example 2.2. Assume that the universe of discourse $X = \{x_1,\ x_2,\ x_3\}$ where $x_1,\ x_2$ and x_3 characterize the quality, reliability, and the price of the objects. It may be further assumed that the values of $\{x_1,\ x_2,\ x_3\}$ are in $[0, 1]$ and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that
$A = \{\langle x_1,\ 0.4,\ 0.5,\ 0.3\rangle, \langle x_2,\ 0.7,\ 0.2,\ 0.4\rangle, \langle x_3, 0.8,\ 0.3,\ 0.4\rangle\}$ where for x_1 the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc.

Definition 2.3. [18] A Fuzzy Neutrosophic Set $\mathcal{FNS}$ A on the universe of discourse X is defined as $A = \{x, \langle \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle, x \in X\}$, where $\mathcal{T}, \mathcal{I}, \mathcal{F} : X \to [0, 1]$ and $0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3$.

Definition 2.4. [3] Let U be the initial universal set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to be the fuzzy neutrosophic soft set over U, where F is a mapping given by $F : A \to P(U)$. Here after we simply consider A as $\mathcal{FNSS}$ over U instead of (F, A).

Definition 2.5. [18] Let $U = \{c_1, c_2, ..., c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, ..., e_m\}$. Let $A \subset E$. A pair (F, A) be a $\mathcal{FNSS}$ over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e);\ e \in A,\ u \in F_A(e)\}$
which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by

$\mathcal{T}_{R_A} : U \times E \to [0,1]$, $\mathcal{I}_{R_A} : U \times E \to [0,1]$ and $\mathcal{F}_{R_A} : U \times E \to [0,1]$ where $\mathcal{T}_{R_A}(u,e) \in [0,1]$, $\mathcal{I}_{R_A}(u,e) \in [0,1]$ and $\mathcal{F}_{R_A}(u,e) \in [0,1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.

If $[(\mathcal{T}_{ij},\ \mathcal{I}_{ij},\ \mathcal{F}_{ij})] = [\mathcal{T}_{ij}(u_i,e_j),\ \mathcal{I}_{ij}(u_i,e_j)\ ,\mathcal{F}_{ij}(u_i,e_j)]$, we define a matrix

$$[\langle \mathcal{T}_{ij}\ ,\mathcal{I}_{ij},\ \mathcal{F}_{ij}\rangle]_{m\times n} = \begin{bmatrix} \langle \mathcal{T}_{11},\ \mathcal{I}_{11},\ \mathcal{F}_{11}\rangle & \cdots & \langle \mathcal{T}_{1n},\ \mathcal{I}_{1n},\ \mathcal{F}_{1n}\rangle \\ \langle \mathcal{T}_{21},\ \mathcal{I}_{21},\ \mathcal{F}_{21}\rangle & \cdots & \langle \mathcal{T}_{2n},\ \mathcal{I}_{2n},\ \mathcal{F}_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle \mathcal{T}_{m1},\ \mathcal{I}_{m1},\ \mathcal{F}_{m1}\rangle & \cdots & \langle \mathcal{T}_{mn},\ \mathcal{I}_{mn},\ \mathcal{F}_{mn}\rangle \end{bmatrix}.$$

Which is called an $m \times n$ $\mathcal{FNSM}$ of the $\mathcal{FNSS}(F_A, E)$ over U.

Definition 2.6. [29] Let $A = (\langle a_{ij}^{\mathcal{T}},\ a_{ij}^{\mathcal{I}},\ a_{ij}^{\mathcal{F}}\rangle),\ B = \langle (b_{ij}^{\mathcal{T}},\ b_{ij}^{\mathcal{I}},\ b_{ij}^{\mathcal{F}}\rangle) \in \mathcal{N}_{m\times n}$. The component wise addition and component wise multiplication is defined as

$A \oplus B = (sup\{a_{ij}^{\mathcal{T}},\ b_{ij}^{\mathcal{T}}\},\ sup\{a_{ij}^{\mathcal{I}},\ b_{ij}^{\mathcal{I}}\},\ inf\{a_{ij}^{\mathcal{F}},\ b_{ij}^{\mathcal{F}}\})$

$A \otimes B = (inf\{a_{ij}^{\mathcal{T}},\ b_{ij}^{\mathcal{T}}\},\ inf\{a_{ij}^{\mathcal{I}},\ b_{ij}^{\mathcal{I}}\},\ sup\{a_{ij}^{\mathcal{F}},\ b_{ij}^{\mathcal{F}}\})$

Definition 2.7. Let $A \in \mathcal{N}_{m\times n},\ B \in \mathcal{N}_{n\times p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n}(a_{ik}^{\mathcal{T}} \wedge\ b_{kj}^{\mathcal{T}}),\ \sum_{k=1}^{n}(a_{ik}^{\mathcal{I}} \wedge\ b_{kj}^{\mathcal{I}}),\ \prod_{k=1}^{n}(a_{ik}^{\mathcal{F}} \vee\ b_{kj}^{\mathcal{F}})\right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n}(a_{ik}^{\mathcal{T}} \wedge b_{kj}^{\mathcal{T}}),\ \bigvee_{k=1}^{n}(a_{ik}^{\mathcal{I}} \wedge b_{kj}^{\mathcal{I}}),\ \bigwedge_{k=1}^{n}(a_{ik}^{\mathcal{F}} \vee b_{kj}^{\mathcal{F}})\right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$.

Where $\sum(a_{ik}^{\mathcal{T}} \wedge\ b_{kj}^{\mathcal{T}})$ means max-min operation and

$\prod_{k=1}^{n}(a_{ik}^{\mathcal{F}} \vee\ b_{kj}^{\mathcal{F}})$ means min-max operation.

Throught this paper, we are following this notation and notions [22, 23].

Let $\mathcal{R} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$ and $\mathcal{P} = (\langle\langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle)$ with their elements in the unit interval $I = [\langle 0,0,1\rangle, \langle 1,1,0\rangle]$.

We discuss some definitions and notations.

- $\mathcal{R} \times \mathcal{P} = \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle \times \langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle = (\bigvee_{m=1}^{n}(\langle\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle \wedge \langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle))$
- $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^{k+1} = \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^{k} \times \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle,\ k = 1, 2, ...$
- $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^{0} = \mathcal{E}$, where $\mathcal{E}$ is the unit $\mathcal{FNSM}$.
- $\mathcal{R} \leq \mathcal{P}$ iff $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle \leq \langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle\ \forall i, j \in \{1, 2, ..., n\}$,

- $\mathbb{N}^+ = \{x | x \ is \ a \ positive \ integer\}$.
- In the sequence $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle, \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^2, \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^3, ..., \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^m, ...$, the number of different matrices is at most l^{n^2} [here, l is the number of all the different elements that occure in $\mathcal{FNSM}$ $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$,] which is neatly finite. Hence, $\exists$ indices $s, t \in \mathbb{N}^+$, $(s \neq t) \ni \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^s = \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^t$.
- Let $\mathcal{H} = \{(s,t) | (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^s = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^t, s \neq t, s, t \in \mathbb{N}^+\}$;
- $\mathcal{D} = \{d | d = |s - t|, (s,t) \in H\}$.
- By the well ordering property (of natural numbers) $\mathcal{D}$ has a least element d. Clearly, $d \geq 1$.
- Let $K = \{k | \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^k = \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^{k+d}, k \in \mathbb{N}^+\}$.
- Truly, K is a nonempty subset of $\mathbb{N}^+$. By well-ordering property, K has a least elemant $k (k \geq 1)$.
- The following definitions, remarks and lemmas are from [22, 23]
- A path in an ordinary directed graph (digraph) is a sequence of distinct vertices $v_1, v_2, ..., v_n \ni$ for $i = \{1, 2, ..., n-1\}$, there is a directed edge in the graph from v_i to v_{i+1}.
- A digraph is Strongly Connected (SC) iff for any two vertices v_i, v_j here v_j is reachable from v_i.
- The $\mathcal{S}$trong $\mathcal{C}$omponents ($\mathcal{SC}$) of a digraph $\mathcal{G}$ are those full subgraphs of $\mathcal{G}$ that are SC and are not properly contained in any other SC subgraph of $\mathcal{G}$.
- A cycle in a digraph is a sequence of vertices $v_1, v_2, ..., v_n \ni$ for $i = \{1, 2, ..., r-1\}$, there is a directed edge from v_i to v_{i+1} and $v_1 = v_n$ and all the other v^s are distinct.

Remark 2.8. An ordinary directed graph is really the same as a Boolean Fuzzy Neutrosophic Soft Matrix $\mathcal{BFNSM}$ and the periodicity of oscillation of a $\mathcal{BFNSM}$ can be determined from its corresponding digraph.

Lemma 2.9. The periodicity of a ($\mathcal{SC}$) is the greatest common divisor (g.c.d.) of the lengths of all cycles in its digraph.

Lemma 2.10. The periodicity of an ordinary digraph is the l.c.m. of the periods of all ($\mathcal{SC}$) in its graph.

3. Properties on Periodicity and Index

In this section, we give an equivalent definition of periodicity and index of a $\mathcal{FNSM}$.

Definition 3.1. If $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ is a $\mathcal{FNSM}$, $m, s \in \mathbb{N}^+$ and $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s}$, then we say that m is a periodicity of $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$, and s starting point of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ corresponding to m.

Proposition 3.2. If s is a initial point corresponding to m; then n is also a first point(pt) corresponding to m $\forall n \in \mathbb{Z}^+, n > s$.

Proof: Multiplying $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{n-s}$ on both sides of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s}$, obtains $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{n+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{n}$. Proof Completes. Next part it is known from Property 3.2 that the periodicity m decides a boundary $\mathcal{T}_m$. Every n with $n \geq \mathcal{T}_m$ is start point corresponding to m, while every n with $n < \mathcal{T}_m$ is not a first point closed to m. We call $\mathcal{T}_m$ an index of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$. Clearly,
$\mathcal{T}_m = \min\{s | (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s}, m \textit{ a given positive integer}\}$.

Definition 3.3. We define the $d = \min\{m | (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{s}$,
$\forall\ s, m \in \mathbb{N}^+\}$ the least periodicity of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.
The natural numbers d exists from the well ordering property.

Proposition 3.4. Every periodicity m of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ is a multiple of d.

Proof: Suppose this property does not hold.
Let as take that $m = nd + p,\ 1 \leq p \leq d,\ r = \max\{\mathcal{T}_m, \mathcal{T}_d\}$.
By known Property 3.2, $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{r} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{r+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{r+nd+p} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{r+p}$, it follows that p is periodicity of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$, a contradiction to the definition of d.

Proposition 3.5. $\mathcal{T}_d$ is an index correponding to every periodicity of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.

Proof: $\mathcal{T}_m$ denote the index corresponding to periodicity m of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.
By Property 3.4, there exists an integer $l \in \mathbb{N}^+$ such that $m = ld$.
Thus $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_d+m} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_d+ld} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_d}$.
By our definition of $\mathcal{T}_m$, we have $\mathcal{T}_d \geq \mathcal{T}_m$. On the other words, for all $m > 0$, we can find $h \in \mathbb{Z}^+$ suth that $\mathcal{T}_m + mh \geq \mathcal{T}_d$. Thus $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_m+d} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_m+d+mh} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_m+mh} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_m}$, so we have $\mathcal{T}_m \geq \mathcal{T}_d$.

Definition 3.6. We said the common index $\mathcal{T}_d$ the index of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$, the least periodicity d the periodicity of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$, denoted by k, d, respectively.

Theorem 3.7. If $\mathcal{N}, \mathcal{H} \in \mathbb{N}^+$, then $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^{N} = \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle^{N+H} \Leftrightarrow N \geq k, d|H$.

Proof: That implies since H is a periodicity of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$, by property 3.4, we get $d|H$. Let $\mathcal{T}_H$ be the index corresponding to H. Since N is also an index of H, by the definition of $\mathcal{T}_H$, we need $N \geq \mathcal{T}_H$. By Property 3.5, $\mathcal{T}_H = \mathcal{T}_d = k$. So we have $N \geq k$.
$\Leftarrow$ Suppose that $H = md,\ m \in \mathbb{Z}^+,\ N \geq k$, then $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+H} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+md} =$

$(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_d+md} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{T}_d} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_k$. We follows that H is a periodicity of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$. By Property 3.2 and $N \geq k$, we obtain that N is $\mathcal{S}$tarting point ($\mathcal{S}pt$) related to H. Thus $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^N = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{N+H}$.

Corollary 3.8. In the sequence $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle), (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^2, ..., (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^n, ...$ the number of different $\mathcal{FNSM}$ is $k+d-1$. The set of the $k+d-1$ different $\mathcal{FNSM}$ is $X = \{(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle), (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^2, ..., (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^k, (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+1},$
$..., (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+d-1}$.

Proof: If $a \in \mathbb{Z}^+,\ a \geq k+d$, then let $a-k = sd+r,\ (s, r \in \mathbb{Z}^+,\ 0 \leq r \leq d-1)$. By Theorem 3.7 $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^a = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{(k+r)+sd} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+r} \in X$. If $a, b, \in \mathbb{Z}^+,\ 1 \leq a \leq b \leq k+d-1$ and $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^a = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^b$, then $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^a = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{a+(b-a)}$. By Theorem 3.7 $a \geq k,\ d|(b-a)$.

Thus $b-a \geq b \Rightarrow b \geq a+d \geq k+d$, contradiction to the assumption of b.

Lemma 3.9. If $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ and $(\langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle)$ are $\mathcal{FNSM}s$, then $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = (\langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle) \Leftrightarrow (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda = (\langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle)_\lambda,\ \forall \lambda \in I$.

From Lemma 3.9 we find if there is a λ and $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda = (\langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle)_\lambda$, then $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) \neq (\langle p_{ij}^{\mathcal{T}}, p_{ij}^{\mathcal{I}}, p_{ij}^{\mathcal{F}}\rangle)$.

Let d_λ, k_λ denote the periodicity and index of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda$ respectively.

Theorem 3.10. $d = [d_\lambda]_{\lambda \in I},\ k = \max_{\lambda \in I}\{k_\lambda\}$. here "[ ]" denotes $l.c.m.$

Proof: Set $N = \max_{\lambda \in I}\{k_\lambda\},\ \mathcal{H} = [d_\lambda]_{\lambda \in I}$. Let $N = k_\lambda + r_\lambda,\ \mathcal{H} = l_\lambda d_\lambda$, where $r_\lambda, l_\lambda \in \mathbb{N}^+$, thus $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{(N+\mathcal{H})} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{(N+\mathcal{H})} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{k_\lambda+r_\lambda+l_\lambda d_\lambda} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{k_\lambda+l_\lambda d_\lambda} \times (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{r_\lambda} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{k_\lambda+r_\lambda} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^N = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^N$.

By Lemma 3.9, we get $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{\mathcal{N}+\mathcal{H}} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^N$.

By Lemma 3.9, we have $N \geq k, d|\mathcal{H}$.

We first proof $K = N$. If $k \neq N$, due to the above discussion, we have $k < N$. By the definition of $N, \exists \lambda \in I \ni k$ is not the index of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda$. Thus $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{k+\mathcal{H}} \neq (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^k$. From Lemma 3.9, we get $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+\mathcal{H}} \neq (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^k$, a contradiction to the definition of k.

Now we prove that $d = \mathcal{H}$. Since $d|\mathcal{H}$, so $d \leq \mathcal{H}$. If $d < \mathcal{H}$, then by the definition of $\mathcal{H}$, there exists λ such that $d_\lambda | d$. By Lemma 3.9, we have $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_\lambda^{k+d} \neq (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_\lambda)^k$. From Lemma 3.9, it follows that $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^{k+d} \neq (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)^k$, a contradiction to the definition of d.

The above Theorem 3.10 is points out the relation of periodicity and index between a $\mathcal{FNSM}$ and its $\mathcal{CFNSM}$. This result provides a new approach to the study of periodicity and index of $\mathcal{FNSM}$.

Corollary 3.11. A $\mathcal{FNSM}$ converges iff each of its cut fuzzy neutrosophic soft matrices converges.

Corollary 3.12. The periodicity of a $\mathcal{FNSM}$ is a prime number p if and only if the periodicity of each of its $\mathcal{CFNSM}s$ are p or 1.

4. **Further Description of $\mathcal{PFNSM}$**

Let $\mathcal{S}$ be a strongly connected digraph, we let $d(\mathcal{S})$ be the periodicity of $\mathcal{S}, h(\mathcal{S})$ the number of all the different vertices of $\mathcal{S}, l(C)$ the length of directed cycle $C, \mathcal{G}_\lambda$ the corresponding digraphs of $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_\lambda$. The strong components of $\mathcal{G}_\lambda$ are called the $\mathcal{SC}$.

Definition 4.1. $\mathcal{G}_\lambda \subseteq \mathcal{G}_\beta$ is that any point $x, y \in \mathcal{G}_\lambda$, if there is a path between x and y in $\mathcal{G}_\lambda$, then this path is retained in $\mathcal{G}_\beta$, where $\lambda, \beta \in I$.
$\mathcal{G}_\lambda \cap \mathcal{G}_\beta,\ \mathcal{G}_\lambda \cup \mathcal{G}_\beta$, stand for the intersection and combination of paths in digraph $\mathcal{G}_\lambda$ and $\mathcal{G}_\beta$ respectively.

Definition 4.2. We say that $\mathcal{S}$ a $\mathcal{SC}$ of a $\mathcal{FNSM}$ $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$, if there is a $\lambda \in I \ni \mathcal{S}$ is a $\mathcal{SC}$ of $\mathcal{CFNSM}$ $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_\lambda$.

Proposition 4.3. If $h(S) = m$, then $l(C) \leq m$, where C is an arbitrary directed cycle of $\mathcal{SC}$ $\mathcal{S}$.
Proof: It is trivial from the definition of directed cycle.

Proposition 4.4. If $h(S) = m$, then $d(\mathcal{S}) \leq m$.
Proof: By referring to Lemma 3.9, Property 4.3 and the properties of $g.c.d$, the proof is clear.

Proposition 4.5. If $\mathcal{S}_1, \mathcal{S}_2$ are $\mathcal{SC}s$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $d(\mathcal{S}_2)|d(\mathcal{S}_1)$ and $h(\mathcal{S}_1) \leq h(\mathcal{S}_2)$.
Proof: From $\mathcal{S}_1 \subseteq \mathcal{S}_2$ we obtain that if C is cycle of $\mathcal{S}_2$ then C is also cycle of $\mathcal{S}$. Hence the cycle set of $\mathcal{S}_2$ includes the cycle set of $\mathcal{S}_1$. By Lemma 2.10 and the properties of $g.c.d.$, we get $d(\mathcal{S}_1)|d(\mathcal{S}_2)$. By the definition of $\subseteq$ we need $h(\mathcal{S}_2) \leq h(\mathcal{S}_1)$.

The l different elements of $\mathcal{FNSM}$ $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ are denoted in an increasing order $\lambda_1 < \lambda_2 < ... < \lambda_l$. Then $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ has l different $\mathcal{CFNSM}$ $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_i} (i = 1, 2, ..., l)$. $\mathcal{G}_i$ denote the corresponding digraph of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_i}$.
Let $\mathcal{Q} = \{\mathcal{S}|\mathcal{S}\ is\ a\ \mathcal{SC}\ of \mathcal{G}_i,\ i = 1, 2, ..., l\}$, $\mathcal{M} = \{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_r|\mathcal{S}_i\ and\ \mathcal{S}_j$ have some common vertices, $\mathcal{S}_i, \mathcal{S}_j \in \mathcal{Q}, i \neq j\}$. Due to the idea of $\mathcal{SC}$ we know that if S_i, S_j are SCs and S_i, S_j have common vertices, then $\mathcal{S}_i, \mathcal{S}_j$ belong to distint $\mathcal{CFNSM}s$. without loss of genarality, we assume that $\mathcal{S}_i \in \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_i}, \mathcal{S}_j \in \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_j}$ and $\lambda_i < \lambda_j$. From the fact that $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_i} \geq \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_j}$ and $\mathcal{S}_i, \mathcal{S}_j$ have common vertices, we conclude that $\mathcal{S}_i \supseteq \mathcal{S}_j$. So $[\mathcal{M}, \subseteq]$ is a partial order set and $\mathcal{M}$ is a subset of $\mathcal{Q}$. The greatest element of $[\mathcal{M}, \subseteq]$. So

$\mathcal{M} \neq \varphi$ and there is a minimal elements in $[\mathcal{M}, \subseteq]$. Next we will explain that the number of set $\mathcal{M}$ is at least one.

Definition 4.6. Any $\mathcal{S} \in \mathcal{M}$ is a Minimal Strong Component $\mathcal{MSC}$ if $\mathcal{T} \in \mathcal{M}$ and $\mathcal{T} \subseteq \mathcal{S} \Rightarrow$ that $\mathcal{T} = \mathcal{S},\ \forall \mathcal{T} \in \mathcal{M}$.

Example 4.7. This illustration clarify the notion of $\mathcal{MSC}$.

$$(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}} \rangle) = \begin{bmatrix} \langle 0.9,0.8,0.1\rangle & \langle 0.5,0.4,0.5\rangle & \langle 0,0,1\rangle & \langle 0.5,0.4,0.5\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \\ \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 0.5,0.4,0.5\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0.5,0.4,0.5\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.1,0.1,0.9\rangle \\ \langle 0.5,0.4,0.5\rangle & \langle 0,0,1\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.1,0.1,0.9\rangle & \langle 0,0,1\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle \end{bmatrix}$$

- Then $\lambda_1 = \langle 0,0,1\rangle, \lambda_2 = \langle 0.1,0.1,0.9\rangle, \lambda_3 = \langle 0.3,0.2,0.7\rangle, \lambda_4 = \langle 0.5,0.4,0.5\rangle,$ $\lambda_5 = \langle 0.9,0.8,0.1\rangle,$
- $\mathcal{G}_i (i = 1, 2, ..., 4)$ can be represented as follows.
- In $\mathcal{G}_{\langle 0.9,0.8,0.1\rangle}$ there is only one $\mathcal{SC}$ $\mathcal{S}_1 = \{v_1\}$.
- In $\mathcal{G}_{\langle 0.5,0.4,0.5\rangle}$ there are two $\mathcal{SC}s$ $\mathcal{S}_2 = \{v_1, v_4\},\ \mathcal{S}_3 = \{v_2, v_3\}$.
- We notice that $\mathcal{S}_3$ is a $\mathcal{SC}$ which has no common vertices with $\mathcal{S}_1$ and $\mathcal{S}_2$. In this sense we say that $\mathcal{S}_3$ is a newly appeared $\mathcal{SC}$.
- In $\mathcal{G}_{\langle 0.3,0.2,0.7\rangle}$ there are two $\mathcal{SC}s$ $\mathcal{S}_4 = \{v_1, v_2, v_3, v_4\}, \mathcal{S}_5 = \{v_5, v_6\}, \mathcal{S}_5$ has no common vertices with $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and $\mathcal{S}_4$. So $\mathcal{S}_5$ is a newly appeared $\mathcal{SC}s$ in $\mathcal{G}_{\langle 0.3,0.2,0.7\rangle}$
- In $\mathcal{G}_{\langle 0.1,0.1,0.9\rangle}$ there is a only one component $\mathcal{S}_6 = \{v_1, v_2, x_3, v_4, v_5, v_6\}$ there is a no newly appeared $\mathcal{SC}s$.
- Noticing that each element of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}} \rangle)_0$ is 1 we claim that $\mathcal{G}_{\langle 0,0,1\rangle}$ is a $\mathcal{SC}s$ by itself. We denote this $\mathcal{SC}s$ as $\mathcal{S}_7$. Clearly, the number of vertices in $\mathcal{S}_7$ is the same as in $\mathcal{S}_6$ and $\mathcal{S}_6 \subseteq \mathcal{S}_7$. Hence we obtain that
- $\mathcal{Q} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6, \mathcal{S}_7\},$
- $\mathcal{M}_1 = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_4, \mathcal{S}_6, \mathcal{S}_7\}$, where $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_4 \subseteq \mathcal{S}_6 \subseteq \mathcal{S}_7,$
- $\mathcal{M}_2 = \{\mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_6, \mathcal{S}_7\}$, where $\mathcal{S}_3 \subseteq \mathcal{S}_4 \subseteq \mathcal{S}_6 \subseteq \mathcal{S}_7,$
- $\mathcal{M}_3 = \{\mathcal{S}_5, \mathcal{S}_6, \mathcal{S}_7\}$, where $\mathcal{S}_5 \subseteq \mathcal{S}_6 \subseteq \mathcal{S}_7$.
- The set of all $\mathcal{MSC}$ of $\mathcal{FNSM}(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}} \rangle)$ is $\Psi = \{\mathcal{S}_1, \mathcal{S}_3, \mathcal{S}_5\}$.

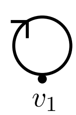


Fig-1. Graph of $G\langle 0.9, 0.8, 0.1\rangle$

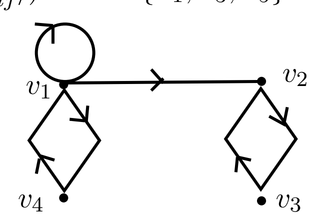


Fig-2.Graph of $G\langle 0.5, 0.4, 0.9\rangle$

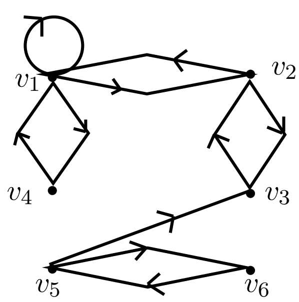


Fig-2.Graph of $G\langle 0.3, 0.2, 0.7\rangle$

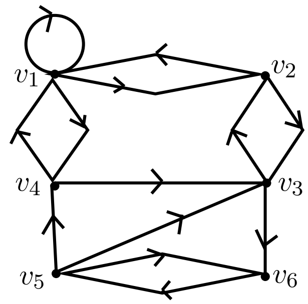


Fig-4 Group of G(0.1, 0.1,0.9)

Lemma 4.8. If $\Psi = \{S | S\ is\ a\ MSC\ of\ \mathcal{FNSM}\}$, then $\sum_{s\in\Psi} d(\mathcal{S}) \leq \sum_{s\in\Psi} h(\mathcal{S}) \leq n$.

Proof: First we take that if $\mathcal{S}, \mathcal{T} \in \Psi, \mathcal{S} \neq \mathcal{T}$, then $\mathcal{S} \cap \mathcal{T} \neq \phi$.

If $\mathcal{S}, \mathcal{T}$ belong to the same $\mathcal{CFNSM}$, by the definition of $\mathcal{SC}$, we have $\mathcal{S} \cap \mathcal{T} \neq \phi$.

If $\langle \alpha_{ij}^{\mathcal{T}}, \alpha_{ij}^{\mathcal{I}}, \alpha_{ij}^{\mathcal{F}}\rangle > \langle \beta_{ij}^{\mathcal{T}}, \beta_{ij}^{\mathcal{I}}, \beta_{ij}^{\mathcal{F}}\rangle,\ S \subseteq \mathcal{G}_{\langle \alpha_{ij}^{\mathcal{T}}, \alpha_{ij}^{\mathcal{I}}, \alpha_{ij}^{\mathcal{F}}\rangle},\ \mathcal{T} \subseteq \mathcal{G}_{\langle \beta_{ij}^{\mathcal{T}}, \beta_{ij}^{\mathcal{I}}, \beta_{ij}^{\mathcal{F}}\rangle}$ and $\mathcal{S} \cap \mathcal{T} \neq \phi$, then $\lambda_\alpha > \lambda_\beta, \mathcal{R}_{\lambda_\alpha} \leq \mathcal{R}_{\lambda_\beta}$. From the definition of the coresponding matrix of an ordinary digraph, we get $\mathcal{G}_{\langle \alpha_{ij}^{\mathcal{T}}, \alpha_{ij}^{\mathcal{I}}, \alpha_{ij}^{\mathcal{F}}\rangle)} \subseteq \mathcal{G}_{\langle \beta_{ij}^{\mathcal{T}}, \beta_{ij}^{\mathcal{I}}, \beta_{ij}^{\mathcal{F}}\rangle}$. By the definition of $\subseteq, \mathcal{S}$ is also strongly connected in digraph $\mathcal{G}_{\langle \beta_{ij}^{\mathcal{T}}, \beta_{ij}^{\mathcal{I}}, \beta_{ij}^{\mathcal{F}}\rangle}$. If $\mathcal{S} \subseteq \mathcal{S}'$, where $\mathcal{S}'$ is a strong component of $\mathcal{G}_{\langle \beta_{ij}^{\mathcal{T}}, \beta_{ij}^{\mathcal{I}}, \beta_{ij}^{\mathcal{F}}\rangle}$. then $\mathcal{S}' \cap \mathcal{T} \supseteq \mathcal{S} \cap \mathcal{T} \neq \phi$. By our known definition of strong component and from the fact that $\mathcal{S}', \mathcal{T}$ are both strong components of $\mathcal{G}_{\langle \beta_{ij}^{\mathcal{T}}, \beta_{ij}^{\mathcal{I}}, \beta_{ij}^{\mathcal{F}}\rangle}$, we have $\mathcal{S}' = \mathcal{T}$. Hence $\mathcal{S} \subseteq \mathcal{S}' = \mathcal{T}$. However, $\mathcal{S} \neq \mathcal{T}$, so $\mathcal{S} \subset \mathcal{T}$, a contradiction to the fact that $\mathcal{T}$ is a MSC. Thus $\mathcal{S} \cap \mathcal{T} = \phi$.

Let $X = \{x_1, x_2, ..., x_n\}$ stand for the set of n different vertices of corresponding digraph of every $\mathcal{CFNSM}$. From the fact that set of all the different vertices of every MSC is included in X and for $\forall \mathcal{S}, \mathcal{T} \in \Psi$ if $\mathcal{S} \neq \mathcal{T}$, then $\mathcal{S} \cap \mathcal{T} = \phi$, we say that $\sum_{s\in\Psi} h(\mathcal{S}) = h(\bigcup_{s\in\Psi} \mathcal{S}) \leq n$ holds. By property 4.4, we get $\sum_{s\in\Psi} d(\mathcal{S}) \leq \sum_{s\in\Psi} h(\mathcal{S}) \leq n$.

Corollary 4.9. The number of $\mathcal{MSC}$ of an arbitrary $\mathcal{SFNSM}$ $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ is not greater than n.

In the below content, we first give the description of periodicity of an arbitrary $\mathcal{SFNSM}$ by using the concept of MSC.

Theorem 4.10. If $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ is a $\mathcal{FNSM}$,
$\Psi = \{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_w\}$, then $d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = [d(S_i)]_{s_i} \in \Psi$.

Proof: Here $\Psi, \mathcal{Q}$ are the same as above. If the number of elements in $\Psi, \mathcal{Q}$ are w and u, respectively, then $w \leq u$. For $\forall \mathcal{T} \in \mathcal{Q} | \Psi$, since $\mathcal{T}$ is not a minimal element, $\exists \mathcal{S} \in \Psi \ni \mathcal{S} \subseteq \mathcal{T}$. By property 4.4, we get

$$d(\mathcal{T}) | d(\mathcal{S}).\ (*)$$

From Lemma 2.10 and Theorem 3.10
$d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = [d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_i})]_{i=1,2,...,l} = [[d(\mathcal{S})]_{s\in\mathcal{G}_i}]_{i=1,2,...,l}$

$= [d(\mathcal{S})]_{\mathcal{S}\in\mathcal{Q}}$
$= [d(\mathcal{S}_1), ..., d(\mathcal{S}_w), d(\mathcal{T}_1), ..., d(\mathcal{T}_h)], \mathcal{T}_j \in \mathcal{Q}|\Psi(j = 1, 2, ..., h).$

By (*), for $\mathcal{T}_j \in \mathcal{Q}\ \Psi$, there exists an $\mathcal{S} \in \{\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_w\}$ satisfying $d(\mathcal{T}_j)|d(\mathcal{S})$. So $d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = [d(\mathcal{S}_1), ..., d(\mathcal{S}_w)] = [d(\mathcal{S}_i)]_{s_i\in\Psi}$.

Theorem 4.11. For given positive integer n, the greastest value of periodicity of all square $\mathcal{FNSM}$ is $\max \sum_{l_i=n} [l_i]$ $(1 \le i \le n)$.

Proof: Let $\mathcal{M} = \{[l_1, ..., l_w]| \sum_{i=1}^{w} l_i \le n,\ l_i \in \mathbb{Z}^+,\ 1 \le w \le n\}$. By Theorem 5.1, for an arbitrary $\mathcal{SFNSM}\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$,
$d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = [d(S_1), ..., d(\mathcal{S}_w)] \in \mathcal{M}$,
$\mathcal{S}_i \in \Psi, 1 \le d(\mathcal{S}_i) \le n, \sum_{i=1}^{w} d(\mathcal{S}_i) \le n.$

On the other hand, for $\forall[l_1, ..., l_w] \in \mathcal{M}$, $\sum_{i=l}^{w} l_i \le n$, we can find a fuzzy neutrosophic soft matrix $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$ such that $d(A) = [l_1, ..., l_w]$. In fact, the periodicity of the $l_i \times l_i$ Block Boolean Fuzzy Neutrosophic Soft Matrix ($\mathcal{BBFNSM}$)

$$A = \begin{bmatrix} \langle 0,0,1\rangle & \vdots & E \\ \cdots & \cdots & \cdots \\ \langle 1,1,0\rangle & \vdots & \langle 0,0,1\rangle \end{bmatrix}$$

is l_i, where $\mathcal{E}$ is the unit matrix. The periodicity of the $\mathcal{SBBFNSM}$

$$A_n = \begin{bmatrix} \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_1 & \langle 0,0,1\rangle & \vdots & \langle 0,0,1\rangle \\ \ddots & & & \\ \langle 0,0,1\rangle & \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_W & \vdots & \langle 0,0,1\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \vdots & E \end{bmatrix}$$

is $[l_1, ..., l_W]$. Hence, the greatest periodicity of square $\mathcal{FNSM}(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ is the greatest value of $\mathcal{M}$.

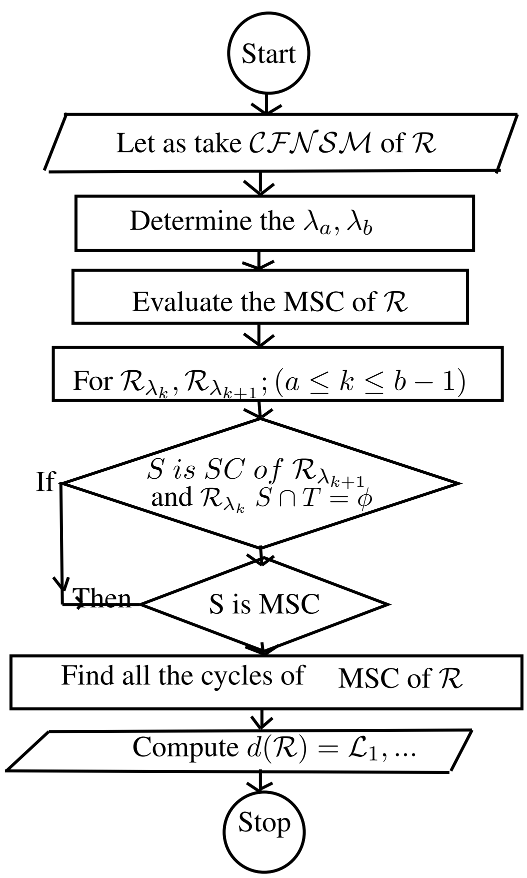
For $\sum_{i=1}^{W} l_i \le n$, we need $[l_1, ..., l_W] \le [l_1, ..., l_W, n - \sum_{i=1}^{W} l_i] \le \max_{\sum l_i=n} [l_1, ..., l_W]$. So $\max_{\sum l_i\le n} [l_1, ..., l_W] = \max_{\sum l_i=n} [l_1, ..., l_W]$.

5. An Algorithm to Find the Period

- According to Theorem 5.1, we can determine the following algorithm to obtain the periodicity d for an arbitrary $\mathcal{CFNSM}$ $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.

- **Step 1:** Compute the $\mathcal{CFNSM}$ according to the different elements of $\mathcal{R} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.
- Let $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_a}$ be the first $\mathcal{CFNSM}$ such that has at least one directed cycle, and $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_a}$ the first $\mathcal{CFNSM}$ such that the number of vertices of all its SCs is n.
- **Step 2:** Determine λ_a, λ_b.
- **Step 3:** Find MSCs of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_a}, ..., (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_b}$.
- From the definition of MSC and Lemma 3.9, we can find MSCs by the following method.
- SC of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_a}$ are MSC of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.
- For $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_k}, (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_{k+1}}(a \leq k \leq b-1)$, if $\mathcal{S}$ is a SC of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_{k+1}}$ and for an arbitary strong component $\mathcal{T}$ if $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_k}$,
$\mathcal{S} \cap \mathcal{T} = \phi$ holds, then $\mathcal{S}$ is a MSC.
- **Step 4:** Find all the directed cycles of MSC of $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.
- **Step 5:** Evaluate $d(\mathcal{R}) = [\mathcal{L}_\infty, \mathcal{L}_\in, ..., \mathcal{L}_{\sqcap_\rangle}]_{i=1,2,...,w}$.

Flowchart to obtain the periodicity d for an arbitrary fuzzy neutrosophic soft matrix $\mathcal{R} = (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$.



Theorem 5.1. The periodicity of square fuzzy neutrosophic soft matrix $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ is $d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = [\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_{u_i}]_{i=1,2,...,W}$, where () stand for the g.c.d.

Proof: All of the different elements and $\mathcal{CFNSM}$ $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)$ are assumed as above. By the definition of $\mathcal{CFNSM}$, we can find $\lambda_a, \lambda_b \in I(\lambda_a \geq \lambda_b)$ satisfying the next conditions.
$(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_a}$ has at least one directed cycle. But if $\lambda_j > \lambda_a \Rightarrow \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_j} \leq \langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_a}$ then $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_j}$ has no cycles. Thus $d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_j}) = 1$.
(ii) $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_b}$ is the first $\mathcal{CFNSM}$ λ that number of different vertices of all its SCs is n. If $\lambda_j < \lambda_b \Rightarrow (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_j} \geq (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_a}$, then the number of different vertices of all its SCs for $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_j}$ is also n.
From the definition of MSC, for $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle_{\lambda_j})$ mentioned in (i) and (ii) $(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_j}$ has no MSC.
Therefore $\Psi = \{\mathcal{S} | \mathcal{S} \in (\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle)_{\lambda_j}, \lambda_a \geq \lambda_j \geq \lambda_b\}$.
Theorem 5.1 we known that $d(\mathcal{R}) = [\mathcal{L}_1, \mathcal{L}_1, ..., \mathcal{L}_{u_i}]_{i=1,2,...,w}$.

Example 5.2. Let $\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle$ be the fuzzy neutrosophic soft matrix mentioned in Example 4.7. Then $\lambda_a = \langle 0.9, 0.8, 0.1\rangle,\ \lambda_b = \langle 0.3, 0.2, 0.7\rangle$
$d(\langle r_{ij}^{\mathcal{T}}, r_{ij}^{\mathcal{I}}, r_{ij}^{\mathcal{F}}\rangle) = [d(\mathcal{S}_1), d(\mathcal{S}_3), , d(\mathcal{S}_5)] = [1, 2, 2] = 2$

6. Conclusion

We have defined the concept of MSC, and obtained the periodicity of fuzzy neutrosophic soft matrices by the periodicity of its MSC. We have also pointed out that the index of $\mathcal{FNSM}$ is the greatest value of indices of its $\mathcal{CFNSM}$. Future scope of this research work could be to investigate the oscillating period index, strongly connected boolean matrix in the freamework of $\mathcal{FNSM}$. We will applied this results for decision making problems.

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Convexity for Interval Valued Neutrosophic Setsand its Application in Decision Making

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Abstract. The notion 'Convexity' is applied in various areas of mathematics particularly in optimization techniques. It is known that this concept is applied in fuzzy sets, which is studied by many authors. This article deals with convexity utilized for neutrosophic and interval valued neutrosophic sets which is a generalization of intuitionistic fuzzy sets. Also some interesting preservation properties of convexity under intersection operators in interval valued neutrosophic sets are discussed. Adding to the discussion, preservation property of digital convexity under intersection using deformation and other techniques are touched upon. Eventually the application of convexity to decision making are illustrated using examples.

Keywords: Convexity; Interval valued neutrosophic sets; Interval valued neutrosophic intersection; Decision making.

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1. Introduction

Concepts like adjacency, connectivity, convexity and concavity, level sets, cut sets are useful in many areas of mathematics [7] [8] [9]. In particular the notion convexity is applied in optimization techniques, image processing, decision making etc., [10] [11] [15] [16]. Fuzzy convexity plays a vital role in applications of real time problems which includes decision making.

Neutrosophic sets which is a generalization of intuitionistic fuzzy sets [2] [18] were introduced by Floretin Smarandache [4], interval valued neutrosophic sets were investigated in [17]. These sets serves as the base for solutions of real time problems. On the other hand, key concepts like convexity and level sets for fuzzy sets, interval valued fuzzy sets serves as important notions in decision making [13].

The fundamental definitions of convexity in neutrosophic sets, digital convexity in neutrosophic sets, neutrosophic digital cut sets, neutrosophic digital level sets etc., are defined and their properties are discussed in [14].

In [13] Pedro Huidobro proved many properties of convexity and explained the importance of the concept convexity for Interval Valued Fuzzy Set (IVFS) and its compatibility with intersection operator. Utility of convexity for IVFS in decision making problems especially by the preservation property under intersection operator is illustrated using examples mentioned in [13].

In this article, as an extension of the convexity concept for IVFS in decision making, convexity for Interval Valued Neutrosophic Set (IVNS) is described and using the preservative property of convexity under intersection in IVNS, the concept is applied to decision making problems to find the optimum solution. In addition discussion on digital convexity upon preservation property under intersection operator is investigated.

2. Preliminaries

Definition 2.1. [17] Let X be a space of points (objects), with a generic elements in X, denoted by x. An interval valued neutrosophic set (IVNS) A in X is characterized by truth- membership $T_A(x)$, indeterminacy $I_A(x)$ and falsity- membership function $F_A(x)$. For each point x in X, we have that $T_A(x) = [infT_A(x), supT_A(x)], I_A(x) = [infI_A(x), supI_A(x)], F_A(x) = [infF_A(x), supF_A(x)] \subseteq [0, 1]$ and $0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3, x \in X$.
We call it "interval" because it is the subclass of a neutrosophic set, that is, we only consider the subunitary interval of [0, 1]. Therefore, All IVNSs are clearly neutrosophic sets.

Definition 2.2. [17] An IVNS A is empty if and only if its $infT_A(x) = supT_A(x) = 0$, $infI_A(x) = supI_A(x) = 1$, and $infF_A(x) = supF_A(x) = 0$ for any $x \in X$.

Definition 2.3. [17] The complement of an IVNS A is denoted by A^c and is defined as $T^c_A(x) = F_A(x)$, $infI^c_A(x) = 1 - supI_A(x)$, $supI^c_A(x) = 1 - infI_A(x)$, $F^c_A(x) = T_A(x)$ for any x in X.

Definition 2.4. [17] An interval valued neutrosophic set A is contained in an IVNS B, A $\subseteq$ B, if and only if $infT_A(x) \leq infT_B(x)$, $supT_A(x) \leq supT_B(x)$, $infI_A(x) \geq infI_B(x)$, $supI_A(x) \geq supI_B(x)$, $infF_A(x) \geq infF_B(x)$, and $supF_A(x) \geq supF_B(x)$ for any x in X.

Definition 2.5. [17] Two IVNSs A and B are equal, written as A = B, if and only if A $\subseteq$ B and B $\subseteq$ A.

Definition 2.6. [17] The intersection of two interval neutrosophic sets A and B is an IVNS C, written as C = A ∩ B, whose truth-membership, indeterminacy-membership, falsity-membership are related to those of A and B by $infT_C(x) = min(infT_A(x), infT_B(x))$
$supT_C(x) = min(supT_A(x), supT_B(x))$
$infI_C(x) = max(infI_A(x), infI_B(x))$
$supI_C(x) = max(supI_A(x), supI_B(x))$
$infF_C(x) = max(infF_A(x), infF_B(x))$
$supF_C(x) = max(supF_A(x), supF_B(x))$

Definition 2.7. [17] The union of two interval neutrosophic sets A and B is an IVNS C, written as C = A ∪ B, whose truth-membership, indeterminacy-membership, falsity-membership are related to those of A and B by
$infT_C(x) = max(infT_A(x), infT_B(x))$
$supT_C(x) = max(supT_A(x), supT_B(x))$
$infI_C(x) = min(infI_A(x), infI_B(x))$
$supI_C(x) = min(supI_A(x), supI_B(x))$
$infF_C(x) = min(infF_A(x), infF_B(x))$
$supF_C(x) = min(supF_A(x), supF_B(x))$

Definition 2.8. [17] An IVNS A is convex if and only if
$infT_A(\lambda x_1 + (1-\lambda)x_2) \geq min(infT_A(x_1), infT_A(X_2))$,
$supT_A(\lambda x_1 + (1-\lambda)x_2) \geq min(supT_A(x_1), supT_A(X_2))$,
$infI_A(\lambda x_1 + (1-\lambda)x_2) \leq max(infI_A(x_1), infI_A(X_2))$,
$supI_A(\lambda x_1 + (1-\lambda)x_2) \leq max(supI_A(x_1), supI_A(X_2))$,
$infF_A(\lambda x_1 + (1-\lambda)x_2) \leq max(infF_A(x_1), infF_A(X_2))$,
$supF_A(\lambda x_1 + (1-\lambda)x_2) \leq max(supF_A(x_1), supF_A(X_2))$
for all x_1 and X_2 in X and all λ in [0, 1].

Theorem 2.9. *[17] If A and B are convex, so is their intersection.*

Definition 2.10. [17] An IVNS A is strongly convex if and only if
$infT_A(\lambda x_1 + (1-\lambda)x_2) > min(infT_A(x_1), infT_A(X_2))$,
$supT_A(\lambda x_1 + (1-\lambda)x_2) > min(supT_A(x_1), supT_A(X_2))$,
$infI_A(\lambda x_1 + (1-\lambda)x_2) < max(infI_A(x_1), infI_A(X_2))$,
$supI_A(\lambda x_1 + (1-\lambda)x_2) < max(supI_A(x_1), supI_A(X_2))$,
$infF_A(\lambda x_1 + (1-\lambda)x_2) < max(infF_A(x_1), infF_A(X_2))$,
$supF_A(\lambda x_1 + (1-\lambda)x_2) < max(supF_A(x_1), supF_A(X_2))$
for all x_1 and X_2 in X and all λ in [0, 1].

Theorem 2.11. *[17] If A and B are strongly convex, so is their intersection.*

Definition 2.12. [14] Let $A_N = \langle \mu_{A_N}, \sigma_{A_N}, \nu_{A_N} \rangle$ be the neutrosophic subset of E which is DN regular and convex, then $A_N^{'}$ is said to be a digital neutrosophic convex (or DN convex) set, if the digital image of A_N is $A_N^{'}$.
The complement $1 - A_N^{'}$ of the DN convex set $A_N^{'}$ is said to be a digital neutrosophic concave (or DN concave) set.

Definition 2.13. [14] Let $\mathrm{A}_N = \langle \mu_{A_N}, \sigma_{A_N}, \nu_{A_N} \rangle$ be the neutrosophic subset of E. Then the level sets of A_N are defined as $A_N^{\omega} = \{P \in E, \mu_{A_N}(P) \geq \omega, \sigma_{A_N}(P) \geq \omega, \nu_{A_N}(P) \leq \omega, \}, \omega \in I$.

3. Decision making based on Interval valued neutrosophic sets

Decision making problems are proposed in this section. Few theories on interval valued fuzzy sets are used in decision making [7] [18] [13]. Decisions are made based on set of goals and set of constraints with symmetry between them [1].

Symmetry representing "and" connective is used. Here are situations in which uncertainty occurs, this situation can be resolved using membership values, which may be included in some interval through specific point is uncertain.

As proposed by Bellman, Zadeh, Yagu and Basson and by the idea used on Interval value fuzzy set [13] the following is proposed:

Definition 3.1. Let $X = \{x_1, x_2, ..., x_n\}$ be the set of alternatives, $G_1, G_2, ...G_m$ the set of goals that can be expressed as Interval valued neutrosophic set with alternatives, and $C_1, C_2, ...C_n$ be the set of constraints which can also be expressed as Interval valued neutrosophic sets on the space of alternatives. Let $\leq$ be an order on [0, 1]. The goals and constraints combine to form a decision D, which is an Interval valued neutrosophic set resulting from Interval valued neutrosophic intersection of the goals and the constraints. Thus $D = G_1 \cap_N ... \cap_N G_P \cap_N C_1 \cap_N ... \cap_N C_m$.
For the alternatives of x, the decision is denoted as D(x) which satisfies the goals and constraints for any $x \in X$. To make a decision we have to find the best alternative.
It is obvious that D(x) depends on the Interval valued neutrosophic intersection operator which satisfies total ordering property [5]. Therefore the decision D is the Interval valued neutrosophic intersection of all goals and constraints depending on the total ordering property.

Example 3.2. A candidate attempting for business has to choose a location in one of the three alternative x_1, x_2, x_3 . He wants to choose a location that minimizes the land cost G and is located near supplies C_1. Let X = $\{x_1, x_2, x_3\}$. To be more precise we take Internal valued neutrosophic set instead of Neutrosophic set. Let us suppose that the membership, non- membership and indeterminant function of Interval valued neutrosophic goal G is $\{\langle x_1, [0.1, 0.3], [0.2, 0.4], [0.2, 0.4] \rangle\} + \{\langle x_2, [0.4, 0.6], [0, 0.1], [0.1, 0.2] \rangle\} +$

$\{\langle x_3, [0.5, 0.7], [0.1, 0.2], [0.1, 0.2]\rangle\}$ and the $T(x_i)$ and $F(x_i)$ of the interval valued neutrosophic constraints C_1 is $\{\langle x_1, [0.4, 0.6], [0.1, 0.2], [0.1, 0.2]\rangle\} + \{\langle x_2, [0.1, 0.2], [0.2, 0.3], [0.4, 0.7]\rangle\} + \{\langle x_3, [0.3, 0.5], [0, 0.1], [0.2, 0.3]\rangle\}$.

If we consider interval neutrosophic intersection, then the membership, indeterminacy and non- membership values of interval valued neutrosophic decision D_N is

$\{\langle x_1, [0.1, 0.3], [0.2, 0.4], [0.2, 0.4]\rangle\} +$

$\{\langle x_2, [0.1, 0.2], [0.1, 0.3], [0.4, 0.7]\rangle\} +$

$\{\langle x_3, [0.3, 0.5], [0.1, 0.2], [0.2, 0.3]\rangle\}$

and the optimal decision would be x_3, since it is the alternative with a maximum value of D(x) with respect to interval valued neutrosophic intersection which satisfies the total ordering.

4. **Extension principle (Interval valued neutrosophic set)**

With the extension principle on IVNS [12], when the Interval valued neutrosophic constraints or goals are defined in n-different spaces, they can be mapped into the same. When we have n functions which maps $(X_1 \times X_2 \times ...X_n)$ to Y, we would assume that if A $\in$ IVNS$(X_1 \times X_2 \times ...X_n)$, then $A(x_1, x_2, ...x_n) = A(x_1) \cap_N A(x_2) \cap_N ... \cap_N A(x_n)$. $\cap_N$ represents IVN intersection. The following example based is on [9]:

Example 4.1. Suppose the same conditions are as in the previous example, and the space Y means a set of the previous business works done by the financial experts, $Y = \{y_1, y_2, y_3, y_4\}$ with the information that y_1 and y_2 were made by x_1, y_3 was supervised by x_2 and y_4 was produced by x_2 and x_3.

Using the above information we defined the following mapping f : Y $\rightarrow$ X defined as $f(y_1) = x_1, f(y_2) = x_1, f(y_3) = x_2$ and $f(y_4) = \{x_2, x_3\}$.

With Interval valued neutrosophic constraint over Y, the impact of each one of works as:

$C_2(Y) = \{\langle y_1, [0.3, 0.6], [0.4, 0.6], [0.4, 0.7]\rangle,$

$\langle y_2, [0.2, 0.4], [0.5, 0.6], [0.5, 0.7]\rangle,$

$\langle y_3, [0.3, 0.5], [0.4, 0.5], [0.4, 0.6]\rangle,$

$\langle y_4, [0.3, 0.4], [0.4, 0.5], [0.4, 0.6]\rangle\}$

$C_2(Y)$ is denoted as above, it is an Interval valued neutrosophic set over Y. Now the extension principle to have $G_1, G_2, ...G_n$ and $C_1, C_2, ...C_n$ as Interval valued neutrosophic set over Y.

Extension principle [12] can be applied for x_1, $[f(C_2)](x_1) = sup_{y_1,y_2} C_2(x) = sup C_2(y_1), C_2(y_2) = \{\langle [0.3, 0.6], [0.5, 0.6], [0.5, 0.7]\rangle$

Similarly, $[f(C_2)](x_2) = \{\langle [0.3, 0.5], [0.4, 0.5], [0.4, 0.6]\rangle$ and

$[f(C_3)](x_3) = \{\langle [0.3, 0.4], [0.4, 0.5], [0.4, 0.6]\rangle$

$\therefore C_2(X) = \{\langle x_1, [0.3, 0.6], [0.5, 0.6], [0.5, 0.7]\rangle,$

$\langle x_2, [0.3, 0.5], [0.4, 0.5], [0.4, 0.6]\rangle,$

$\langle x_3, [0.3, 0.4], [0.4, 0.5], [0.4, 0.6]\rangle\}$.
The decision is $D = G_i \cap C_1 \cap C_2$. (i.e.) the membership, indeterminancy and non- membership degrees for the alternatives in D are
$\{\langle x_1, [0.1, 0.3], [0.5, 0.6], [0.5, 0.7]\rangle, \langle x_2, [0.1, 0.2], [0.4, 0.5], [0.4, 0.7]\rangle,$
$\langle x_3, [0.3, 0.4], [0.4, 0.5], [0.4, 0.6]\rangle\}$

Corollary 4.2. *Now we combine the decision making problems using the following corollary: Let $G_1, G_2, ...G_p$ be the Interval valued neutrosophic goals, $C_1, C_2, ...C_m$ be the Interval valued neutrosophic constraints and $D = G_1 \cap G_2 ... \cap G_p \cap C_1 \cap C_2 ... \cap C_n$ be the resulting decision. If the Interval valued neutrosophic goals and Interval valued neutrosophic constraints are convex Interval valued neutrosophic set, then the resulting decision D is a convex Interval valued neutrosophic set and the set of maximizing decisions of Interval valued neutrosophic set, D is a convex neutrosophic crisp set [3]. If the Interval valued neutrosophic goals and Interval valued neutrosophic constraints are strictly convex Interval valued neutrosophic set, then the resulting decision D is strictly convex Interval valued neutrosophic set and the set of maximizing decisions of D is a singleton or an empty set.*
The following example explains the importance of the convexity for Interval valued neutrosophic set in decision making

Example 4.3. In the previous example we do not mention the relation among the x_1, x_2 and x_3. Here if $x_1 < x_2 < x_3$, it is clear that G, C_1, C_2 and C_3 are strictly convex IVNS with respect to totally order preserving property of the intersection operator in IVNS. Since,
$D = \{\langle x_1, [0.1, 0.3], [0.5, 0.6], [0.5, 0.7]\rangle,$
$\langle x_2, [0.1, 0.2], [0.4, 0.5], [0.4, 0.7]\rangle,$
$\langle x_3, [0.3, 0.4], [0.4, 0.5], [0.4, 0.6]\rangle\}$
Here x_3 is the optimum solution of the decision problem. Moreover if G, C_1, C_2 are strictly convex IVNS we see that D is not only convex IVNS as $D = \{\langle x_1, [0.1, 0.3], [0.5, 0.6], [0.5, 0.7]\rangle,$
$\langle x_2, [0.1, 0.2], [0.4, 0.5], [0.4, 0.7]\rangle,$
$\langle x_3, [0.3, 0.4], [0.4, 0.5], [0.4, 0.6]\rangle\}$ but also strictly convex.

5. **Remarks on convexity for digital neutrosophic sets**

From the above sections, it is understood that the order preserving property is important for utilizing the convexity property of interval valued neutrosophic sets in decision making problems. While considering the digital neutrosophic convexity property [14], it is predicted that the convexity property is not analogous to that of the neutrosophic convexity property , the order preserving property may not be obtained in digital sets [6]. Also in case of interval valued digital sets the same situation occurs . Hence in order to utilize convex digital sets in decision making problems techniques such as deformation, pixel removing to obtain convexity properties etc., may be used.

6. Conclusions

Decision making using interval valued neutrosophic sets, especially convex sets is illustrated in this article. The major operator called interval valued neutrosophic intersection is utilized in order to obtain the optimal solution for the decision making problem which has the structure or model comprising of maximizing or minimization of goals (objective function) and constraints. The concept intersection of goals and constraints forming the optimum solution is illustrated using examples. Extension principle based on interval valued neutrosophic sets is utilized in case of more than one constrain. A comparison on interval valued neutrosophic sets with the convexity property (satisfying the order preserving property) and the interval valued digital neutrosophic convex sets are discussed. Future work may be done on multi attribute decision making problems.

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Numerical Techniques for solving ordinary differential equations with uncertainty

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ABSTRACT. In this article, we derive a numerical solution to the ordinary differential equation with a neutrosophic number as the initial condition.The Adams-Bashforth, Adams-Moulton, and predictorcorrector algorithms are used to solve the differential equation with hexagonal neutrosophic number as the initial condition. The convergence and stability of the methods are also investigated.

Keywords: Neutrosophic number, Adams Bashforth, Adams-Moulton and predictor corrector methods.

1. Introduction

Differential equations may be used to solve various scientific and technological challenges. Analytical approaches for solving differential equations are only relevant to specific types of equations. Many physical problems require numerical approaches to solve differential equations that do not fit into the conventional forms.These strategies are more crucial now that computing devices can significantly reduce numerical effort.

The linear multi-step technique is one of the ways of acquiring an approximation solution to a certain differential equation where the precise or analytical can be determined or not. The linear multi-step technique improves efficiency by combining information from previous approximations. Adams-Bashforth method is a linear multi-step method. Numerical methods are classified as explicit and implicit types.The explicit technique calculates the future system state based on the current system status. The implicit technique derives the future system status from the current and future system states. The explicit type is called the Adams-Bashforth [5] (AB) method which was introduced by John Couch Adams to solve a differential equation modelling capillary action while the implicit type is called the Adams-Moulton (AM) method developed by Ray Moulton [10].

Fuzzy set (FS) [15] theory is one of the efficient tool to represent uncertainty. The study of fuzzy differential equations [7] is fast emerging in many fields. Over the past few years, there has been intense discussion in theoretical aspects and its applications. Intutionistic fuzzy set(IFS) [1] is a generalization of fuzzy set. Classical set and FS were expanded in IFS context.The analytical and numerical solution of differential equations with fuzzy number and IF number have been discussed in [6,8,11,13] and [3]. Neutrosophic logic [12] is a useful technique for dealing with incomplete, uncertain, and inconsistent data, which is an extension of FS and IFS. Many research have been conducted in the theoretical foundations and practical implications of Neutrosophic logic. Neutrosophic number involved in differential equations have been discussed in [2, 14] and [9] . In this research,

- The solution of ODE with hexagonal neutrosophic number as initial condition is discussed.
- Adam-Bashforth , Adam Moulton and Predictior-Corrector methods are applied to solve the ODE
- The stability and convergence criteria is investigated.

The paper is organized as follows: The preliminaries section presents the key concepts that are relevant to this field of study. In the next section we have defined the basic operations and interpolation of neutrosophic number followed by Adam-Bashforth Adam Moulton and Predictior-Corrector methods. Finally, we have highlighted our important findings and proposed areas for further investigation.

2. Preliminaries

Definition 2.1. [4] A neutrosophic set is defined as follows,
S= $\left\{\langle\kappa,\lambda,\mu\rangle\,/\,\langle\kappa,\lambda,\mu\rangle\in[0,1]^3\; and\; 0\leq\kappa(x)+\lambda(x)+\mu(x)\leq 3\right\}$

Definition 2.2. [4] The support of a Neutrosophic set $S=\{\langle x,\kappa_s,\lambda_s,\mu_s\rangle\rangle\}$ is defined as Supp(S)= $\{x\in U/\;\kappa_s\neq 0,\;\lambda_s\neq 1,\;\mu_s\neq 1\}$.

Definition 2.3. [4] A set S is said to be normal if
H(S)= $\{x\in U/\kappa_s=h_1\left(S\right)=1,\lambda_s=h_2\left(S\right)=0,\mu_s=h_3\left(S\right)=0\}$,
where $h_1\left(S\right)=\sup_{x\in U}T_s\left(x\right)$, $h_2\left(S\right)=\inf_{x\in U}I_s\left(x\right)$ and $h_3\left(S\right)=\inf_{x\in U}\mu_s\left(x\right)$.

Definition 2.4. [4] To qualify as a neutrosophic number, a neutrosophic set S on $\mathcal{R}$ must satisfy the following three properties:

(1) The neutrosophic set S should be normal;
(2) $S_{(\theta,\beta,\gamma)}$ should be closed for every $\theta\in(0,1]$, $\beta\in[0,1)$ and $\gamma\in[0,1)$;
(3) The support of S must be bounded.

Definition 2.5. [4] GNHNNA is defined as, $\eta_{GNHNNA}=\Big\{T\left(j_1,j_2,j_3,j_4,j_5,j_6;r,s;\omega\right)_{(n_1,n_2,n_3,n_4)}\,,$
$I\left(o_1,o_2,o_3,o_4,o_5,o_6;r_1,s_1;\rho\right)_{(m_1,m_2,m_3,m_4)}\,,F\left(q_1,q_2,q_3,q_4,q_5,q_6;r_2,s_2;\delta\right)_{(p_1,p_2,p_3,p_4)}\Big\}$.

Where the membership function, $T_{\eta GNHNNA} = \begin{cases} r\left(\frac{x-j_1}{j_2-j_1}\right)^{n_1} & , \text{if } j_1 \leq x \leq j_2 \\ r+(\omega-r)\left(\frac{x-j_2}{j_3-j_2}\right)^{n_2} & , \text{if } j_2 \leq x \leq j_3 \\ \omega & , \text{if } j_3 \leq x \leq j_4 \\ s+(\omega-s)\left(\frac{x-j_5}{j_4-j_5}\right)^{n_3} & , \text{if } j_4 \leq x \leq j_5 \\ s\left(\frac{x-j_6}{j_5-j_6}\right)^{n_4} & , \text{if } j_5 \leq x \leq j_6 \\ 0 & , \text{otherwise.} \end{cases}$

Indeterminacy function, $I_{\eta GNHNNA} = \begin{cases} 1-r_1\left(\frac{x-o_1}{o_2-o_1}\right)^{m_1} & , \text{if } o_1 \leq x \leq o_2 \\ 1-r_1+(r_1-\rho)\left(\frac{x-o_2}{o_3-o_2}\right)^{m_2} & , \text{if } o_2 \leq x \leq o_3 \\ 1-\rho & , \text{if } o_3 \leq x \leq o_4 \\ 1-s_1+(s_1-\rho)\left(\frac{x-o_5}{o_4-o_5}\right)^{m_3} & , \text{if } o_4 \leq x \leq o_5 \\ 1-s_1\left(\frac{x-o_6}{o_5-o_6}\right)^{m_4} & , \text{if } o_5 \leq x \leq o_6 \\ 1 & , \text{otherwise.} \end{cases}$

Non-membership function, $\mu_{\eta_{GNHNNA}} = \begin{cases} 1-r_2\left(\frac{x-q_1}{q_2-q_1}\right)^{p_1} & , \text{if } q_1 \leq x \leq q_2 \\ 1-r_2+(r_2-\delta)\left(\frac{x-q_2}{q_3-q_2}\right)^{p_2} & , \text{if } q_2 \leq x \leq q_3 \\ 1-\delta & , \text{if } q_3 \leq x \leq q_4 \\ 1-s_2+(s_2-\delta)\left(\frac{x-q_5}{q_4-q_5}\right)^{p_3} & , \text{if } q_4 \leq x \leq q_5 \\ 1-s_2\left(\frac{x-q_6}{q_5-q_6}\right)^{p_4} & , \text{if } q_5 \leq x \leq q_6 \\ 1 & , \text{otherwise.} \end{cases}$

where $j_1 < j_2 < j_3 < j_4 < j_5 < j_6$, $o_1 < o_2 < o_3 < o_4 < o_5 < o_6$ and $q_1 < q_2 < q_3 < q_4 < q_5 < q_6$ $\forall\ j_i,\ o_i$ and q_i $(i = 1, ..., 6)$ are real constants and $0 < r, s < \omega$, $1-\rho < r_1, s_1 < 1$ and $1-\delta < r_2, s_2 < 1, \omega,\ \rho,\ \delta \in [0,1]$.

Definition 2.6. The (θ, β, γ)-cut of GNHNNA,

$\eta_{(\alpha,\beta,\gamma)} =$

$\{x \in X/\ T_{\eta_{GNHNNA}} \geq \theta,\ I_{\eta_{GNHNNA}} \leq \beta,\ \mu_{\eta_{GNHNNA}} \leq \gamma\}$.

Let $T_\theta = \{x \in X/\ T_{\eta_{GNHNNA}} \geq \theta\}$ where $\theta \in (0, \omega]$.

If $r \leq s$ then,

$$T_\theta = \begin{cases} \left[j_1 + \left(\frac{\theta}{r}\right)^{\frac{1}{n_1}}(j_2-j_1), j_6 + \left(\frac{\theta}{s}\right)^{\frac{1}{n_4}}(j_5-j_6)\right] & , \text{if } 0 < \theta \leq r \\ \left[j_2 + \left(\frac{\theta-r}{\omega-r}\right)^{\frac{1}{n_2}}(j_3-j_2), j_6 + \left(\frac{\theta}{s}\right)^{\frac{1}{n_4}}(j_5-j_6)\right] & , \text{if } r \leq \theta \leq s \\ \left[j_2 + \left(\frac{\theta-r}{\omega-r}\right)^{\frac{1}{n_2}}(j_3-j_2), j_5 + \left(\frac{\theta-s}{\omega-s}\right)^{\frac{1}{n_3}}(j_4-j_5)\right] & , \text{if } s \leq \theta \leq \omega \\ [j_3, j_4] & , \text{if } \theta = \omega. \end{cases}$$

If $s \leq r$, then,

$$T_\theta=\begin{cases}\left[j_1+\left(\frac{\theta}{r}\right)^{\frac{1}{n_1}}(j_2-j_1),j_6+\left(\frac{\theta}{s}\right)^{\frac{1}{n_4}}(j_5-j_6)\right] & \text{, if } 0<\theta\le s\\ \left[j_1+\left(\frac{\theta}{r}\right)^{\frac{1}{n_1}}(j_2-j_1),j_5+\left(\frac{\theta-s}{\omega-s}\right)^{\frac{1}{n_3}}(j_4-j_5)\right] & \text{, if } s\le\theta\le r\\ \left[j_2+\left(\frac{\theta-r}{\omega-r}\right)^{\frac{1}{n_2}}(j_3-j_2),j_5+\left(\frac{\theta-s}{\omega-s}\right)^{\frac{1}{n_3}}(j_4-j_5)\right] & \text{, if } r\le\theta\le\omega\\ [j_3,j_4] & \text{, if } \theta=\omega.\end{cases}$$

Let $I_\beta=\{x\in X/\ I_{\eta_{GNHNNA}}\le\beta\}$,

where $\beta\in[1-\rho,1)$. If $r_1\le s_1$,

$$I_\beta=\begin{cases}[o_3,o_4] & \text{, if } \beta=1-\rho\\ \left[o_2+\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}(o_3-o_2),o_5+\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}(o_4-o_5)\right] & \text{, if } 1-\rho\le\beta\le1-s_1\\ \left[o_2+\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}(o_3-o_2),o_6+\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(o_5-o_6)\right] & \text{, if } 1-s_1\le\beta\le1-r_1\\ \left[o_1+\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(o_2-o_1),o_6+\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(o_5-o_6)\right] & \text{, if } 1-r_1\le\beta<1.\end{cases}$$

If $s_1\le r_1$, then,

$$I_\beta=\begin{cases}[o_3,o_4] & \text{, if } \beta=1-\rho\\ \left[o_2+\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}(o_3-o_2),o_5+\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}(o_4-o_5)\right] & \text{, if } 1-\rho\le\beta\le1-r_1\\ \left[o_1+\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(o_2-o_1),o_5+\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}(o_4-o_5)\right] & \text{, if } 1-r_1\le\beta\le1-s_1\\ \left[o_1+\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(o_2-o_1),o_6+\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(o_5-o_6)\right] & \text{, if } 1-s_1\le\beta<1.\end{cases}$$

Let $\mu_\gamma=\{x\in X/\mu_{\eta_{GNHNNA}}\le\gamma\}$, where $\gamma\in[1-\delta,1)$. If $r_2\le s_2$, then,

$$\mu_\gamma=\begin{cases}[q_3,q_4] & \text{, if } \gamma=1-\delta\\ \left[q_2+\left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}}(q_3-q_2),q_5+\left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}}(q_4-q_5)\right] & \text{, if } 1-\delta\le\gamma\le1-s_2\\ \left[q_2+\left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}}(q_3-q_2),q_6+\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(q_5-q_6)\right] & \text{, if } 1-s_2\le\gamma\le1-r_2\\ \left[q_1+\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(q_2-q_1),q_6+\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(q_5-q_6)\right] & \text{, if } 1-r_2\le\gamma<1.\end{cases}$$

If $s_2\le r_2$, then

$$\mu_\gamma=\begin{cases}[q_3,q_4] & \text{, if } \gamma=1-\delta\\ \left[q_2+\left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}}(q_3-q_2),q_5+\left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}}(q_4-q_5)\right] & \text{, if } 1-\delta\le\gamma\le1-r_2\\ \left[q_1+\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(q_2-q_1),q_5+\left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}}(q_4-q_5)\right] & \text{, if } 1-r_2\le\gamma\le1-s_2\\ \left[q_1+\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(q_2-q_1),q_6+\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(q_5-q_6)\right] & \text{, if } 1-s_2\le\gamma<1.\end{cases}$$

3. **Interpolation of Neutrosophic Number**

Definition 3.1. We define the upper, middle and lower θ-cuts of $\langle\kappa,\lambda,\mu\rangle\in$ NS, with $\theta\in$ [0,1] by, $\langle\kappa,\lambda,\mu\rangle^\theta$=$\{x\in\Re/T(x)\ge\theta\}$, $\langle\kappa,\lambda,\mu\rangle\,\theta$=$\{x\in\Re/I(x)\le\theta\}$, and $\langle\kappa,\lambda,\mu\rangle_\theta$=$\{x\in\Re/F(x)\le\theta\}$. Where $\Re$ is a real number.

Definition 3.2. The neutrosophic zero in a neutrosophic set is defined by

$$0(\delta) = \begin{cases} (1,0,0), & \text{if } t = 0, \\ (0,0,1), & \text{otherwise.} \end{cases}$$

Definition 3.3. Let $\langle \kappa, \lambda, \mu \rangle$ and $\langle \kappa', \lambda', \mu' \rangle \in$ Neutrosophic Number, and let $\theta \in \mathbb{R}$. We define the following operations:

(1) $\langle \kappa, \lambda, \mu \rangle \bigoplus \langle \kappa', \lambda', \mu' \rangle (x) =$

$$\left(\sup_{x=y+z} \min(\kappa(x), \kappa'(x)), \inf_{x=y+z} \max(\lambda(x), \lambda'(x)), \inf_{x=y+z} \max(\mu(x), \mu'(x)) \right)$$

(2) $\theta \langle \kappa, \lambda, \mu \rangle =$

$$\begin{cases} \langle \theta\kappa, \theta\lambda, \theta\mu \rangle, & \text{if } \theta \neq 0 \\ 0(\delta), & \text{if } \theta = 0 \end{cases}$$

The operations of addition and scalar multiplication for Neutrosophic numbers are defined based on the extension principle as follows:
$[\langle \kappa, \lambda, \mu \rangle \bigoplus \left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]^{\theta}$= $[\langle \kappa, \lambda, \mu \rangle]^{\theta}$ + $[\left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]^{\theta}$, $[\lambda \langle \kappa, \lambda, \mu \rangle]^{\theta}$= λ $[\langle \kappa, \lambda, \mu \rangle]^{\theta}$
$[\langle \kappa, \lambda, \mu \rangle \bigoplus \left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]\theta$= $[\langle \kappa, \lambda, \mu \rangle]\theta$ + $[\left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]\theta$, $[\lambda \langle \kappa, \lambda, \mu \rangle]\theta$= λ $[\langle \kappa, \lambda, \mu \rangle]\theta$
$[\langle \kappa, \lambda, \mu \rangle \bigoplus \left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]_{\theta}$= $[\langle \kappa, \lambda, \mu \rangle]_{\theta}$ + $[\left\langle \kappa^{'}, \lambda^{'}, \mu^{'} \right\rangle]_{\theta}$, $[\lambda \langle \kappa, \lambda, \mu \rangle]_{\theta}$= λ $[\langle \kappa, \lambda, \mu \rangle]_{\theta}$

Definition 3.4. Let $\langle \kappa, \lambda, \mu \rangle$ be an element of the Neutrosophic Set (NS), and let $\theta \in [0,1]$. We define the following sets:
$[\langle \kappa, \lambda, \mu \rangle]_l^-(\theta) = inf\,\{x \in \Re/\kappa(x) \geq \theta\}$, $[\langle \kappa, \lambda, \mu \rangle]_l^+(\theta) = sup\,\{x \in \Re/\lambda(x) \geq \theta\}$,
$[\langle \kappa, \lambda, \mu \rangle]_m^-(\theta) = inf\,\{x \in \Re/\lambda(x) \leq 1-\theta\}$, $[\langle \kappa, \lambda, \mu \rangle]_m^+(\theta) = sup\,\{x \in \Re/\lambda(x) \leq 1-\theta\}$,
$[\langle \kappa, \lambda, \mu \rangle]_u^-(\theta) = inf\,\{x \in \Re/\mu(x) \leq 1-\theta\}$, $[\langle \kappa, \lambda, \mu \rangle]_u^+(\theta) = sup\,\{x \in \Re/\mu(x) \leq 1-\theta\}$.

Remark 3.5. $\langle \kappa, \lambda, \mu \rangle^{\theta}$=$[[\langle \kappa, \lambda, \mu \rangle]_l^-(\theta), [\langle \kappa, \lambda, \mu \rangle]_l^+(\theta)]$,
$\langle \kappa, \lambda, \mu \rangle\,\theta$=$[[\langle \kappa, \lambda, \mu \rangle]_m^-(\theta), [\langle \kappa, \lambda, \mu \rangle]_m^+(\theta)]$, and
$\langle \kappa, \lambda, \mu \rangle_{\theta}$=$[[\langle \kappa, \lambda, \mu \rangle]_u^-(\theta), [\langle \kappa, \lambda, \mu \rangle]_u^+(\theta)]$.

The interpolation problem for Neutrosophic sets can be described as follows: Consider a Neutrosophic set with specified properties, denoted by $\Phi(\xi)$, which encapsulates information at different points. The aim is to approximate the function $\Phi(\xi)$ for each ξ within its domain. Suppose $\langle \kappa_0, \lambda_0, \mu_0 \rangle$, $\langle \kappa_1, \lambda_1, \mu_1 \rangle$, ..., $\langle \kappa_s, \lambda_s, \mu_s \rangle$ represent $s+1$ Neutrosophic fuzzy sets, and let $\xi_0 < \xi_1 < \cdots < \xi_s$ be $s+1$ distinct points in $\mathbb{R}$. A Neutrosophic continuous function $f : I \to N$ that fulfills the following criteria is termed a Neutrosophic polynomial interpolation of the data.

(1) $\varphi(\xi_i) = \langle \kappa_i, \lambda_i, \mu_i \rangle$
(2) If the input data is crisp, then the interpolation φ is a crisp polynomial.

To construct a function φ that satisfies the specified conditions, consider the following approach. For each $\eta = (\eta_0, \eta_1, \ldots, \eta_s) \in \mathbb{R}^{s+1}$, the unique polynomial of degree $\leq s$, denoted by H_η, is defined as follows:

(1) $H_\eta(q) = \eta_q$ for $q = 0, 1, \ldots, s$
(2) $P_\eta(\xi) = \sum_{q=0}^{s} \eta_q \prod_{q \neq r} \frac{\xi - \xi_r}{\xi_q - \xi_r}$

For any $\xi \in \mathbb{R}$, the membership, indeterminacy and non-membership functions $\Phi(\xi)$ can be expressed using the extension principle as follows:

$$\kappa_{\Phi(\xi)}(\delta) = \begin{cases} \sup_{\delta = H^{\eta_0, \eta_1, \ldots, \eta_s}(\xi)} \min_{i=0,1,\ldots,n} \kappa_{\kappa_i}(\eta_i), & \text{if } H^{-1}_{\eta_0, \eta_1, \ldots, \eta_s}(\delta) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

where κ_{κ_i} is the membership function of κ_i.

$$\lambda_{\Phi(\xi)}(\delta) = \begin{cases} \inf_{\delta = H^{\eta_0, \eta_1, \ldots, \eta_s}(\xi)} \max_{i=0,1,\ldots,s} \lambda_{\lambda_q}(\eta_q), & \text{if } H^{-1}_{\eta_0, \eta_1, \ldots, \eta_s}(\delta) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

where λ_{λ_q} is the indeterminacy function of λ_q.

$$\mu_{\Phi(\xi)}(\delta) = \begin{cases} \inf_{\delta = H^{\eta_0, \eta_1, \ldots, \eta_s}(\xi)} \max_{i=0,1,\ldots,s} \mu_{\mu_q}(\eta_q), & \text{if } H^{-1}_{\eta_0, \eta_1, \ldots, \eta_s}(\delta) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}$$

where μ_{μ_q} is the non-membership function of μ_q.

Let $\chi_l^q(\theta) = \langle \kappa, \lambda, \mu \rangle_\theta$, $\chi_m^q(\theta) = \langle \kappa, \lambda, \mu \rangle\, \theta$, and $\chi_u^q(\theta) = \langle \kappa, \lambda, \mu \rangle^\theta$ for any $\theta \in [0, 1]$ and $q = 0, 1, \ldots, s$. The lower, middle, and upper θ-cuts of $\langle \kappa, \lambda, \mu \rangle$ and $\Phi(\xi)$ are denoted by $[\Phi(\xi)]_\theta$, $[\Phi(\xi)]\theta$, and $[\Phi(\xi)]^\theta$, respectively. Therefore,

$$[\Phi(\xi)]_\theta = \left\{ \delta \in \mathbb{R} \mid \kappa_{\Phi(\xi)}(\delta) \geq \theta \right\} = \left\{ \delta \in \mathbb{R} \mid \exists \eta_0, \eta_1, \ldots, \eta_s : \kappa_{\kappa_q}(\eta_q) \geq \theta,\ q = 0, \ldots, s \text{ and } H_{\eta_0, \eta_1, \ldots, \eta_s}(\xi) = \delta \right\} = \left\{ \delta \in \mathbb{R} \mid \exists \eta \in \prod_{q=0}^{s} \chi_u^q(\theta) : H_{\eta_0, \eta_1, \ldots, \eta_s}(\xi) = \delta \right\}$$

$$[\Phi(\xi)]\theta = \left\{ \delta \in \mathbb{R} \mid \lambda_{\Phi(\xi)}(\delta) \leq 1 - \theta \right\} = \left\{ \delta \in \mathbb{R} \mid \exists \eta_0, \eta_1, \ldots, \eta_s : \lambda_{\lambda_q}(\eta_q) \leq 1 - \theta,\ q = 0, \ldots, s \text{ and } H_{\eta_0, \eta_1, \ldots, \eta_s}(\xi) = \delta \right\} = \left\{ \delta \in \mathbb{R} \mid \exists \eta \in \prod_{q=0}^{s} \chi_m^q(\theta) : H_{\eta_0, \eta_1, \ldots, \eta_s}(\xi) = \delta \right\}$$

$$[\Phi(\xi)]^\theta = \left\{ \delta \in \mathbb{R} \mid \mu_{\Phi(\xi)}(\delta) \leq 1 - \theta \right\} = \left\{ \delta \in \mathbb{R} \mid \exists \eta_0, \eta_1, \ldots, \eta_s : \mu_{\mu_q}(\eta_q) \leq 1 - \theta,\ q = 0, \ldots, s \text{ and } P_{\eta_0, \eta_1, \ldots, \eta_s}(\xi) = \delta \right\} = \left\{ \delta \in \mathbb{R} \mid \exists \eta \in \prod_{q=0}^{s} \chi_l^q(\theta) : H_{\eta_0, \eta_1, \ldots, \eta_s}(\xi) = \delta \right\}$$

Finally, for each $\xi \in \mathbb{R}$ and all $\delta \in \mathbb{R}$, $\Phi(\xi)$ is defined in terms of NS by:

$$\Phi(\xi)(\delta) = \left(\sup\left\{ \theta \in (0,1) \mid \exists \eta \in \prod_{i=0}^{s} r_q^u(\theta) : H_\eta(\xi) = \delta \right\},\ 1 - \sup\left\{ \theta \in (0,1) \mid \exists \eta \in \prod_{q=0}^{s} \chi_m^q(\theta) : H_\eta(x) = \delta \right\}, \right.$$
$$\left. 1 - \sup\left\{ \theta \in (0,1) \mid \exists \eta \in \prod_{q=0}^{s} \chi_m^q(\theta) : H_\eta(\xi) = \delta \right\} \right)$$

where $\eta = (\eta_0, \eta_1, \ldots, \eta_s) \in \mathbb{R}^{s+1}$.

The interpolation polynomial can be expressed in terms of level sets as:

$$[\Phi(\xi)]_\theta = \{\eta \in \mathbb{R} \mid \eta = H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi),\ \eta_q \in [\langle \kappa_q, \lambda_q, \mu_q \rangle]_\theta,\ q = 0, \ldots, s\} \text{ for } \theta \in (0,1]$$

$$[\Phi(\xi)]\theta = \{\eta \in \mathbb{R} \mid \eta = H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi),\ \eta_q \in [\langle \kappa_q, \lambda_q, \mu_q \rangle]\theta,\ q = 0, \ldots, s\} \text{ for } \theta \in (0,1]$$
$$[\Phi(\xi)]^\theta = \left\{\eta \in \mathbb{R} \mid \eta = H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi),\ \eta_q \in [\langle \kappa_q, \lambda_q, \mu_q \rangle]^\theta,\ q = 0, \ldots, s\right\} \text{ for } \theta \in (0,1]$$

According to the Lagrange interpolation formula, we have:

$$[\Phi(\xi)]_\theta = \sum_{q=0}^{s} o_q(x)\chi_u^w(\theta), \quad [\Phi(\xi)]\theta = \sum_{q=0}^{s} o_q(x)\chi_m^q(\theta), \quad [\Phi(\xi)]^\theta = \sum_{q=0}^{s} o_q(x)\chi_l^q(\theta)$$

where $o_q(x)$ represents the Lagrange polynomials.

When the data $\langle \kappa, \lambda, \mu \rangle$ is represented as hexagonal neutrosophic numbers, the values of the interpolation polynomial are also hexagonal neutrosophic numbers. Consequently, $\Phi(\xi)$ takes a particularly simple form that is convenient for computation. Define $\chi_u^q(\theta) = [c_u^-(\theta), d_u^+(\theta)]$, $\chi_m^q(\theta) = [c_m^-(\theta), d_m^+(\theta)]$ and $\chi_l^q(\theta) = [c_l^-(\theta), d_l^+(\theta)]$. The upper endpoint of $[\Phi(\xi)]^\theta$ is obtained by solving the following optimization problem:

$$\text{Maximize } H_{\eta_0,\eta_1,\ldots,\eta_s}(\xi) \text{ subject to } c_u^-(\theta) \le \eta_q \le d_u^+(\theta), \quad q = 0, 1, \ldots, s.$$

The optimal solution is:

$$\eta_q = \begin{cases} d_u^-(\theta) & \text{if } o_q(\xi) \ge 0, \\ c_u^+(\theta) & \text{if } o_q(\xi) < 0. \end{cases}$$

Similarly, the lower endpoint is obtained by:

$$\eta_q = \begin{cases} d_u^-(\theta) & \text{if } o_q(\xi) < 0, \\ c_u^+(\theta) & \text{if } o_q(\xi) \ge 0. \end{cases}$$

$[\Phi(\xi)]\theta$ and $[\Phi(\xi)]_\theta$ can be obtained in a similar manner. Hence, if $\langle \kappa_q, \lambda_q, \mu_q \rangle$ is a neutrosophic number for all q, then $\Phi(\xi)$ is also a neutrosophic number for each ξ. Specifically, if

$$\langle \kappa_i, \lambda_i, \mu_i \rangle = \langle t_i^a, t_i^b, t_i^c, t_i^d, t_i^e, t_i^f, i_i^a, i_i^b, i_i^c, i_i^d, i_i^e, i_i^f, f_i^a, f_i^b, f_i^c, f_i^d, f_i^e, f_i^f \rangle,$$

then:

$$\Phi(\xi) =< \tilde{\varphi}^a(\xi), \tilde{\varphi}^b(\xi), \tilde{\varphi}^c(\xi), \tilde{\varphi}^d(\xi), \tilde{\varphi}^e(\xi), \tilde{\varphi}^f(\xi), \varphi^a(\xi), \varphi^b(\xi), \varphi^c(\xi), \varphi^d(\xi), \varphi^e(\xi), \varphi^f(\xi), \underset{\sim}{\varphi}^a(\xi), \underset{\sim}{\varphi}^b(\xi), \underset{\sim}{\varphi}^c(\xi), \underset{\sim}{\varphi}^d(\xi), \underset{\sim}{\varphi}^e(\xi), \underset{\sim}{\varphi}^f(\xi) >$$

where:

$$\tilde{\varphi}^a(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)t_i^a + \sum_{o_q(\xi)<0} o_q(\xi)t_i^f; \quad \tilde{\varphi}^b(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)t_i^b + \sum_{o_q(\xi)<0} o_q(\xi)t_i^e,$$

$$\tilde{\varphi}^c(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)t_i^c + \sum_{o_q(\xi)<0} o_q(\xi)t_i^d,; \quad \tilde{\varphi}^d(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)t_i^d + \sum_{o_q(\xi)<0} o_q(\xi)t_i^c,$$

$$\tilde{\varphi}^e(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)t_i^e + \sum_{o_q(\xi)<0} o_q(\xi)t_i^b; \quad \tilde{\varphi}^f(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)t_i^f + \sum_{o_q(\xi)<0} o_q(\xi)t_i^a.$$

$$\varphi^a(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)i_i^a + \sum_{o_q(\xi)<0} o_q(\xi)i_i^f; \quad \varphi^b(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)i_i^b + \sum_{o_q(\xi)<0} o_q(\xi)i_i^e,$$

$$\varphi^c(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)i_i^c + \sum_{o_q(\xi)<0} o_q(\xi)i_i^d; \quad \varphi^d(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)i_i^d + \sum_{o_q(\xi)<0} o_q(\xi)i_i^c,$$

$$\varphi^e(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)i_i^e + \sum_{o_q(\xi)<0} o_q(\xi)i_i^b; \quad \varphi^f(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)i_i^f + \sum_{o_q(\xi)<0} o_q(\xi)i_i^a.$$

$$\underset{\sim}{\varphi}^a(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)f_i^a + \sum_{o_q(\xi)<0} o_q(\xi)f_i^f; \quad \underset{\sim}{\varphi}^b(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)f_i^b + \sum_{o_q(\xi)<0} o_q(\xi)f_i^e,$$

$$\varphi^c(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)f_i^c + \sum_{o_q(\xi)<0} o_q(\xi)f_i^d; \quad \underset{\sim}{\varphi}^d(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)f_i^d + \sum_{o_q(\xi)<0} o_q(\xi)f_i^c,$$

$$\underset{\sim}{\varphi}^e(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)f_i^e + \sum_{o_q(\xi)<0} o_q(\xi)f_i^b; \quad \underset{\sim}{\varphi}^f(\xi) = \sum_{o_q(\xi)\geq 0} o_q(\xi)f_i^f + \sum_{o_q(\xi)<0} o_q(\xi)f_i^a.$$

4. **Adams-Bashforth Methods**

Now, we are going to solve the Neutrosophic initial value problem $\xi'(\tau) = \varphi(\delta, \xi(\delta))$ using the Adams-Bashforth three-step method. Let the Neutrosophic initial values be $\xi(\delta_{q-1}), \xi(\delta_q), \xi(\delta_{q+1})$, i.e., $\varphi(\delta_{q-1}, \xi(\delta_{q-1})), \varphi(\delta_q, \xi(\delta_q)), \varphi(\delta_{q+1}, \xi(\delta_{q+1}))$, which are represented by hexagonal neutrosophic numbers. The truth membership,

$$\{\tilde{\varphi}^a(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^b(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^c(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^d(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^e(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^f(\delta_{q-1}, \xi(\delta_{q-1})),$$
$$\tilde{\varphi}_a(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_b(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_c(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_d(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_e(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_f(\delta_{q-1}, \xi(\delta_{q-1})),$$
$$\tilde{\varphi}_a(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_b(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_c(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_d(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_e(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_f(\delta_{q-1}, \xi(\delta_{q-1}))\}$$

the indeterminacy,

$$\{\varphi^a(\delta_q,\xi(\delta_q)),\varphi^b(\delta_q,\xi(\delta_q)),\varphi^c(\delta_q,\xi(\delta_q)),\varphi^d(\delta_q,\xi(\delta_q)),\varphi^e(\delta_q,\xi(\delta_q)),\varphi^f(\delta_q,\xi(\delta_q)),$$
$$\varphi_a(\delta_q,\xi(\delta_q)),\varphi_b(\delta_q,\xi(\delta_q)),\varphi_c(\delta_q,\xi(\delta_q)),\varphi_d(\delta_q,\xi(\delta_q)),\varphi_e(\delta_q,\xi(\delta_q)),\varphi_f(\delta_q,\xi(\delta_q)),$$
$$\varphi_a(\delta_q,\xi(\delta_q)),\varphi_b(\delta_q,\xi(\delta_q)),\varphi_c(\delta_q,\xi(\delta_q)),\varphi_d(\delta_q,\xi(\delta_q)),\varphi_e(\delta_q,\xi(\delta_q)),\varphi_f(\delta_q,\xi(\delta_q))\}$$

the falsity,

$$\{\underset{\sim}{\varphi}^a(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^b(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^c(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^d(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^e(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}^f(\delta_{q+1},\xi(\delta_{q+1})),$$
$$\underset{\sim}{\varphi}a(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}b(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}c(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}d(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}e(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim}{\varphi}f(\delta_{q+1},\xi(\delta_{q+1})),$$
$$\underset{\sim a}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim b}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim c}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim d}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim e}{\varphi}(\delta_{q+1},\xi(\delta_{q+1})),\underset{\sim f}{\varphi}(\delta_{q+1},\xi(\delta_{q+1}))\}$$

Also

$$\xi(\delta_{i+2})=\xi(\delta_{q+1})+\int_{\delta_{q+1}}^{\delta_{q+2}}\varphi(\delta,\xi(\delta))d\delta$$

By neutrosophic interpolation of $\varphi(\delta,\xi(\delta_{q-1})),\varphi(\delta,\xi(\delta_q)),\varphi(\delta,\xi(\delta_{q+1}))$, we have:

$$\tilde{\varphi}^a(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r(\delta)\tilde{\varphi}^a(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^f(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^b(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^b(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^e(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^c(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^c(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^d(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^d(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^d(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^c(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^e(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^e(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^b(\delta_j,\xi(\delta_j))$$

$$\tilde{\varphi}^f(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\tilde{\varphi}^f(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\tilde{\varphi}^a(\delta_j,\xi(\delta_j))$$

$$\varphi^a(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r(\delta)\varphi^a(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\varphi^f(\delta_j,\xi(\delta_j))$$

$$\varphi^b(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\varphi^b(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\varphi^e(\delta_j,\xi(\delta_j))$$

$$\varphi^c(\delta,\xi(\delta))=\sum_{r=q-1,o_j(\delta)\geq 0}^{q+1}o_r((\delta)\varphi^c(\delta_j,\xi(\delta_j))+\sum_{r=q-1,o_j(\delta)<0}^{q+1}o_j(\delta)\varphi^d(\delta_j,\xi(\delta_j))$$

$$\varphi^{d}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\varphi^{d}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\varphi^{c}(\delta_j,\xi(\delta_j))$$

$$\varphi^{e}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\varphi^{e}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\varphi^{b}(\delta_j,\xi(\delta_j))$$

$$\varphi^{f}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\varphi^{f}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\varphi^{a}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim a}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim a}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim f}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim b}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim b}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim e}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim c}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim c}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim d}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim d}{\underset{\sim}{\varphi}}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim d}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim c}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim e}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim e}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim b}{\varphi}(\delta_j,\xi(\delta_j))$$

$$\underset{\sim f}{\varphi}(\delta,\xi(\delta)) = \sum_{r=q-1,o_j(\delta)\geq 0}^{q+1} o_r((\delta)\underset{\sim f}{\varphi}(\delta_j,\xi(\delta_j)) + \sum_{r=q-1,o_j(\delta)<0}^{q+1} o_j(\delta)\underset{\sim a}{\varphi}(\delta_j,\xi(\delta_j))$$

For $\delta_{q+1} \leq \delta \leq \delta_{i+2}$, the interpolation polynomials $o_j(\delta)$ are:

$$o_{q-1}(\delta) = \frac{(\delta-\delta_q)(\delta-\delta_{q+1})}{(\delta_{q-1}-\delta_q)(\delta_{q-1}-\delta_{q+1})} \geq 0$$

$$o_{q}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q+1})}{(\delta_{q}-\delta_{q-1})(\delta_{q}-\delta_{q+1})} \leq 0$$

$$o_{q+1}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q})}{(\delta_{q+1}-\delta_{q-1})(\delta_{q+1}-\delta_{q})} \geq 0$$

Therefore, the following results are obtained:

$$\varphi^{a}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{a}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{f}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{a}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{b}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{b}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{e}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{b}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{c}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{c}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{d}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{c}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{d}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{d}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{c}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{d}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{e}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{e}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{b}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{e}(\delta_{q+1},\xi(\delta_{q+1}))$$

$$\varphi^{f}(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^{f}(\delta_{q-1},\xi(\delta_{q-1})) + o_q(\delta)\varphi^{a}(\delta_i,\xi(\delta_i)) + o_{q+1}(\delta)\varphi^{f}(\delta_{q+1},\xi(\delta_{q+1}))$$

Also

$$[\xi(\delta_{q+2})]_\theta = \left[[\xi(\delta_{q+2})]^-_\kappa(\theta), [\xi(\delta_{q+2})]^+_\kappa(\theta)\right]$$

,

$$[\xi(\delta_{q+2})]\theta = \left[[\xi(\delta_{q+2})]^-_\lambda(\theta), [\xi(\delta_{q+2})]^+_\lambda(\theta)\right],$$

, and

$$[\xi(\delta_{q+2})]_\theta = \left[[\xi(\delta_{q+2})]^-_\mu(\theta), [\xi(\delta_{q+2})]^+_\mu(\theta)\right]$$

where

$$[\xi(\delta_{q+1})]^-_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < o \\ [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < o \\ [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^c(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0 < \theta < 1-o \\ [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o < \theta < 1 \end{cases}$$

For the integral $[\xi(\delta_{r+2})]^-_\kappa(\theta)$, the Adams-Bashforth method gives the following expressions:

Case 1: $[\xi(\delta_{s+2})]^-_\kappa(\theta)$ For $0 < \theta < r$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{12}\left[5\varphi_1^T(\delta_{q-1}) - 16\varphi_1^T(\delta_s) + 23\varphi_1^T(\delta_{q+1})\right]$$

where

$$\varphi_1^T(\delta) = \varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{r}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)$$

For $r < \theta < 1$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{12}\left[5\varphi_2^T(\delta_{q-1}) - 16\varphi_2^T(\delta_s) + 23\varphi_2^T(\delta_{q+1})\right]$$

where

$$\varphi_2^T(\delta) = \varphi^b(\delta,\xi(\delta)) + \left(\frac{\theta-r}{1-r}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)$$

Case 2: $[\xi(\delta_{s+2})]^{+}_{\kappa}(\theta)$ For $0 < \theta < r$:

$$[\xi(\delta_{s+2})]^{+}_{\kappa}(\theta) = [\xi(\delta_{q+1})]^{+}_{\kappa}(\theta) + \frac{h}{12}\left[5\varphi_3^T(\delta_{q-1}) - 16\varphi_3^T(\delta_s) + 23\varphi_3^T(\delta_{q+1})\right]$$

where

$$\varphi_3^T(\delta) = \varphi^f(\delta, \xi(\delta)) + \left(\frac{\theta}{r}\right)\left(\varphi^e(\delta, \xi(\delta)) - \varphi^f(\delta, \xi(\delta))\right)$$

For $r < \theta < 1$:

$$[\xi(\delta_{s+2})]^{+}_{\kappa}(\theta) = [\xi(\delta_{q+1})]^{+}_{\kappa}(\theta) + \frac{h}{12}\left[5\varphi_4^T(\delta_{q-1}) - 16\varphi_4^T(\delta_s) + 23\varphi_4^T(\delta_{q+1})\right]$$

where

$$\varphi_4^T(\delta) = \varphi^c(\delta, \xi(\delta)) + \left(\frac{\theta - r}{1 - r}\right)\left(\varphi^d(\delta, \xi(\delta)) - \varphi^e(\delta, \xi(\delta))\right)$$

Case 3: $[\xi(\delta_{s+2})]^{-}_{\chi}(\theta)$ For $0 < \theta < 1 - r$:

$$[\xi(\delta_{s+2})]^{-}_{\chi}(\theta) = [\xi(\delta_{q+1})]^{-}_{\chi}(\theta) + \frac{h}{12}\left[5\varphi_1^\chi(\delta_{q-1}) - 16\varphi_1^\chi(\delta_s) + 23\varphi_1^\chi(\delta_{q+1})\right]$$

where

$$\varphi_1^\chi(\delta) = \varphi^b(\delta, \xi(\delta)) + \left(\frac{1 - \theta - r}{1 - r}\right)\left(\varphi^c(\delta, \xi(\delta)) - \varphi^b(\delta, \xi(\delta))\right)$$

For $1 - r < \theta < 1$:

$$[\xi(\delta_{s+2})]^{-}_{\chi}(\theta) = [\xi(\delta_{q+1})]^{-}_{\chi}(\theta) + \frac{h}{12}\left[5\varphi_2^\chi(\delta_{q-1}) - 16\varphi_2^\chi(\delta_s) + 23\varphi_2^\chi(\delta_{q+1})\right]$$

where

$$\varphi_2^\chi(\delta) = \varphi^a(\delta, \xi(\delta)) + \left(\frac{1 - \theta}{r}\right)\left(\varphi^b(\delta, \xi(\delta)) - \varphi^a(\delta, \xi(\delta))\right)$$

Case 4: $[\xi(\delta_{s+2})]^{+}_{\chi}(\theta)$ For $0 < \theta < 1 - r$:

$$[\xi(\delta_{s+2})]^{+}_{\chi}(\theta) = [\xi(\delta_{q+1})]^{+}_{\chi}(\theta) + \frac{h}{12}\left[5\varphi_3^\chi(\delta_{q-1}) - 16\varphi_3^\chi(\delta_\chi) + 23\varphi_3^\chi(\delta_{q+1})\right]$$

where

$$\varphi_3^\chi(\delta) = \varphi^e(\delta, \xi(\delta)) + \left(\frac{1 - \theta - r}{1 - r}\right)\left(\varphi^d(\delta, \xi(\delta)) - \varphi^e(\delta, \xi(\delta))\right)$$

For $1 - r < \theta < 1$:

$$[\xi(\delta_{s+2})]^{+}_{\chi}(\theta) = [\xi(\delta_{q+1})]^{+}_{\chi}(\theta) + \frac{h}{12}\left[5\varphi_4^\chi(\delta_{q-1}) - 16\varphi_4^\chi(\delta_s) + 23\varphi_4^\chi(\delta_{q+1})\right]$$

where

$$\varphi_4^\chi(\delta) = \varphi^f(\delta, \xi(\delta)) + \left(\frac{1 - \theta}{r}\right)\left(\varphi^e(\delta, \xi(\delta)) - \varphi^f(\delta, \xi(\delta))\right)$$

Case 5: $[\xi(\delta_{s+2})]^{-}_{\mu}(\theta)$ For $0 < \theta < 1 - r$:

$$[\xi(\delta_{s+2})]^{-}_{\mu}(\theta) = [\xi(\delta_{q+1})]^{-}_{\mu}(\theta) + \frac{h}{12}\left[5\varphi_1^\mu(\delta_{q-1}) - 16\varphi_1^\mu(\delta_s) + 23\varphi_1^\mu(\delta_{q+1})\right]$$

where

$$\varphi_1^\mu(\delta) = \varphi^b(\delta, \xi(\delta)) + \left(\frac{1 - \theta - r}{1 - r}\right)\left(\varphi^c(\delta, \xi(\delta)) - \varphi^b(\delta, \xi(\delta))\right)$$

For $1 - r < \theta < 1$:

$$[\xi(\delta_{s+2})]^{-}_{\mu}(\theta) = [\xi(\delta_{q+1})]^{-}_{\mu}(\theta) + \frac{h}{12}\left[5\varphi^{\mu}_{2}(\delta_{q-1}) - 16\varphi^{\mu}_{2}(\delta_{s}) + 23\varphi^{\mu}_{2}(\delta_{q+1})\right]$$

where

$$\varphi^{\mu}_{2}(\delta) = \varphi^{a}(\delta, \xi(\delta)) + \left(\frac{1-\theta}{r}\right)\left(\varphi^{b}(\delta, \xi(\delta)) - \varphi^{a}(\delta, \xi(\delta))\right)$$

Case 6: $[\xi(\delta_{s+2})]^{+}_{\mu}(\theta)$ For $0 < \theta < 1 - r$:

$$[\xi(\delta_{s+2})]^{+}_{\mu}(\theta) = [\xi(\delta_{q+1})]^{+}_{\mu}(\theta) + \frac{h}{12}\left[5\varphi^{\mu}_{3}(\delta_{q-1}) - 16\varphi^{\mu}_{3}(\delta_{s}) + 23\varphi^{\mu}_{3}(\delta_{q+1})\right]$$

where

$$\varphi^{\mu}_{3}(\delta) = \varphi^{e}(\delta, \xi(\delta)) + \left(\frac{1-\theta-r}{1-r}\right)\left(\varphi^{d}(\delta, \xi(\delta)) - \varphi^{e}(\delta, \xi(\delta))\right)$$

For $1 - r < \theta < 1$:

$$[\xi(\delta_{s+2})]^{+}_{\mu}(\theta) = [\xi(\delta_{q+1})]^{+}_{\mu}(\theta) + \frac{h}{12}\left[5\varphi^{\mu}_{4}(\delta_{q-1}) - 16\varphi^{\mu}_{4}(\delta_{s}) + 23\varphi^{\mu}_{4}(\delta_{q+1})\right]$$

where

$$\varphi^{\mu}_{4}(\delta) = \varphi^{f}(\delta, \xi(\delta)) + \left(\frac{1-\theta}{r}\right)\left(\varphi^{e}(\delta, \xi(\delta)) - \varphi^{f}(\delta, \xi(\delta))\right)$$

5. **Adams-Moulton Methods**

Now, we are going to solve the Neutrosophic initial value problem $\xi'(\tau) = \varphi(\delta, \xi(\delta))$ using the Adams-Bashforth three-step method. Let the Neutrosophic initial values be $\xi(\delta_{q-1}), \xi(\delta_q), \xi(\delta_{q+1})$, i.e., $\varphi(\delta_{q-1}, \xi(\delta_{q-1})), \varphi(\delta_q, \xi(\delta_q)), \varphi(\delta_{q+1}, \xi(\delta_{q+1}))$, which are represented by hexagonal neutrosophic numbers, were the truth membership,

$$\{\tilde{\varphi}^{a}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^{b}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^{c}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^{d}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^{e}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}^{f}(\delta_{q-1}, \xi(\delta_{q-1})),$$

$$\tilde{\varphi}_{a}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{b}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{c}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{d}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{e}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{f}(\delta_{q-1}, \xi(\delta_{q-1})),$$

$$\tilde{\varphi}_{a}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{b}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{c}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{d}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{e}(\delta_{q-1}, \xi(\delta_{q-1})), \tilde{\varphi}_{f}(\delta_{q-1}, \xi(\delta_{q-1}))\}$$

the indeterminacy,

$$\{\varphi^{a}(\delta_q, \xi(\delta_q)), \varphi^{b}(\delta_q, \xi(\delta_q)), \varphi^{c}(\delta_q, \xi(\delta_q)), \varphi^{d}(\delta_q, \xi(\delta_q)), \varphi^{e}(\delta_q, \xi(\delta_q)), \varphi^{f}(\delta_q, \xi(\delta_q)),$$

$$\varphi_{a}(\delta_q, \xi(\delta_q)), \varphi_{b}(\delta_q, \xi(\delta_q)), \varphi_{c}(\delta_q, \xi(\delta_q)), \varphi_{d}(\delta_q, \xi(\delta_q)), \varphi_{e}(\delta_q, \xi(\delta_q)), \varphi_{f}(\delta_q, \xi(\delta_q)),$$

$$\varphi_{a}(\delta_q, \xi(\delta_q)), \varphi_{b}(\delta_q, \xi(\delta_q)), \varphi_{c}(\delta_q, \xi(\delta_q)), \varphi_{d}(\delta_q, \xi(\delta_q)), \varphi_{e}(\delta_q, \xi(\delta_q)), \varphi_{f}(\delta_q, \xi(\delta_q))\}$$

the falsity,

$$\{\underset{\sim}{\varphi}^{a}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}^{b}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}^{c}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}^{d}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}^{e}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}^{f}(\delta_{q+1}, \xi(\delta_{q+1})),$$

$$\underset{\sim}{\varphi}a(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}b(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}c(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}d(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}e(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim}{\varphi}f(\delta_{q+1}, \xi(\delta_{q+1})),$$

$$\underset{\sim a}{\varphi}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim b}{\varphi}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim c}{\varphi}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim d}{\varphi}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim e}{\varphi}(\delta_{q+1}, \xi(\delta_{q+1})), \underset{\sim f}{\varphi}(\delta_{q+1}, \xi(\delta_{q+1}))\}$$

For $\delta_{q+1} \leq \delta \leq \delta_{i+2}$, the interpolation polynomials $o_j(\delta)$ are:

$$o_{q-1}(\delta) = \frac{(\delta-\delta_q)(\delta-\delta_{q+1})(\delta-\delta_{q+2})}{(\delta_{q-1}-\delta_q)(\delta_{q-1}-\delta_{q+1})(\delta_{q-1}-\delta_{q+2})} \geq 0$$

$$o_{q}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q+1})(\delta-\delta_{q+2})}{(\delta_{q}-\delta_{q-1})(\delta_{q}-\delta_{q+1})(\delta_{q}-\delta_{q+2})} \leq 0$$

$$o_{q+1}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q})(\delta-\delta_{q+2})}{(\delta_{q+1}-\delta_{q-1})(\delta_{q+1}-\delta_{q})(\delta_{q+1}-\delta_{q+2})} \geq 0$$

$$o_{q+2}(\delta) = \frac{(\delta-\delta_{q-1})(\delta-\delta_{q})(\delta-\delta_{q+1})}{(\delta_{q+1}-\delta_{q-1})(\delta_{q+2}-\delta_{q})(\delta_{q+2}-\delta_{q+1})} \geq 0$$

Therefore, the following results are obtained:

$$\varphi^a(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^a(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^f(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^a(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^a(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^b(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^b(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^e(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^b(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^b(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^c(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^c(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^d(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^c(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^c(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^d(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^d(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^c(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^d(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^d(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^e(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^e(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^b(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^e(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^e(\delta_{q+2},\xi(\delta_{q+2}))$$

$$\varphi^f(\delta,\xi(\delta)) = o_{q-1}(\delta)\varphi^f(\delta_{q-1},\xi(\delta_{q-1}))+o_q(\delta)\varphi^a(\delta_i,\xi(\delta_i))+o_{q+1}(\delta)\varphi^f(\delta_{q+1},\xi(\delta_{q+1}))+o_{q+2}(\delta)\varphi^f(\delta_{q+2},\xi(\delta_{q+2}$$

where

$$[\xi(\delta_{q+1})]^-_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0<\theta<o \\ [\xi(\delta_{q-1})]^-_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o<\theta<1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\kappa(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0<\theta<o \\ [\xi(\delta_{q-1})]^+_\kappa(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^c(\delta,\xi(\delta)) + \left(\frac{\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } o<\theta<1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0<\theta<1-o \\ [\xi(\delta_{q-1})]^-_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o<\theta<1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\lambda(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0<\theta<1-o \\ [\xi(\delta_{q-1})]^+_\lambda(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o<\theta<1 \end{cases}$$

$$[\xi(\delta_{q+1})]^-_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^b(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^c(\delta,\xi(\delta)) - \varphi^b(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0<\theta<1-o \\ [\xi(\delta_{q-1})]^-_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^a(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o<\theta<1 \end{cases}$$

$$[\xi(\delta_{q+1})]^+_\mu(\theta) = \begin{cases} [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^e(\delta,\xi(\delta)) + \left(\frac{1-\theta-o}{1-o}\right)\left(\varphi^d(\delta,\xi(\delta)) - \varphi^e(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 0<\theta<1-o \\ [\xi(\delta_{q-1})]^+_\mu(\theta) + \int_{\delta_{q-1}}^{\delta_{q+1}} \left[\varphi^f(\delta,\xi(\delta)) + \left(\frac{1-\theta}{o}\right)\left(\varphi^e(\delta,\xi(\delta)) - \varphi^f(\delta,\xi(\delta))\right)\right] d\delta, & \text{if } 1-o<\theta<1 \end{cases}$$

For the integral $[\xi(\delta_{r+2})]^-_\kappa(\theta)$, the Adams-Bashforth method gives the following expressions:
Case 1: $[\xi(\delta_{s+2})]^-_\kappa(\theta)$ For $0 < \theta < r$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{24}\left[\varphi_1^T(\delta_{q-1}) - 5\varphi_1^T(\delta_s) + 19\varphi_1^T(\delta_{q+1}) + 9\varphi_1^T(\delta_{q+2})\right]$$

where

$$\varphi_1^T(\delta) = \varphi^a(\delta,\xi(\delta)) + \left(\frac{\theta}{r}\right)\left(\varphi^b(\delta,\xi(\delta)) - \varphi^a(\delta,\xi(\delta))\right)$$

For $r < \theta < 1$:

$$[\xi(\delta_{s+2})]^-_\kappa(\theta) = [\xi(\delta_{q+1})]^-_\kappa(\theta) + \frac{h}{24}\left[\varphi_2^T(\delta_{q-1}) - 5\varphi_2^T(\delta_s) + 19\varphi_2^T(\delta_{q+1}) + 9\varphi_2^T(\delta_{q+2})\right]$$

Other cases can be computed similarly.

6. **Predictor-Corrector Three-Step Approach**

This method utilizes the Adams–Bashforth three-step technique as a predictor and the Adams–Moulton two-step method as a corrector, iterating the process to improve accuracy.

Procedure: To approximate the solution for the given intuitionistic fuzzy initial value problem:

$$\xi'(\delta) = \varphi(\delta,\xi(\delta)), \quad \delta \in I = [\delta_0, T]$$

$$[\xi(\delta_0)]^+_l(\alpha) = \beta_0, \quad [\xi(\delta_1)]^+_l(\alpha) = \beta_1, \quad [\xi(\delta_2)]^+_l(\alpha) = \beta_2$$

$$[\xi(\delta_0)]^+_r(\alpha) = \beta_3, \quad [\xi(\delta_1)]^+_r(\alpha) = \beta_4, \quad [\xi(\delta_2)]^+_r(\alpha) = \beta_5$$

$$[\xi(\delta_0)]^-_l(\alpha) = \beta_6, \quad [\xi(\delta_1)]^-_l(\alpha) = \beta_7, \quad [\xi(\delta_2)]^-_l(\alpha) = \beta_8$$

$$[\xi(\delta_0)]^-_r(\alpha) = \beta_9, \quad [\xi(\delta_1)]^-_r(\alpha) = \beta_{10}, \quad [\xi(\delta_2)]^-_r(\alpha) = \beta_{11}$$

Step 1: Select a positive integer N and set $h = \frac{T-\delta_0}{N}$.

$$[\eta(\delta_0)]^+_l(\alpha) = \beta_0, \quad [\eta(\delta_1)]^+_l(\alpha) = \beta_1, \quad [\eta(\delta_2)]^+_l(\alpha) = \beta_2$$

$$[\eta(\delta_0)]^+_r(\alpha) = \beta_3, \quad [\eta(\delta_1)]^+_r(\alpha) = \beta_4, \quad [\eta(\delta_2)]^+_r(\alpha) = \beta_5$$

$$[\eta(\delta_0)]^-_l(\alpha) = \beta_6, \quad [\eta(\delta_1)]^-_l(\alpha) = \beta_7, \quad [\eta(\delta_2)]^-_l(\alpha) = \beta_8$$

$$[\eta(\delta_0)]^-_r(\alpha) = \beta_9, \quad [\eta(\delta_1)]^-_r(\alpha) = \beta_{10}, \quad [\eta(\delta_2)]^-_r(\alpha) = \beta_{11}$$

Step 2: Initialize with $i = 1$.

Step 3: Calculate $\xi(\delta_{i+2})$ using the Adams–Bashforth three-step predictor.

Step 4: Use the Adams–Moulton two-step corrector for refinement.

Step 5: Continue iterating until the desired accuracy is achieved.

Step 6: Increment i by 1.

Step 7: If $i \leq (N-2)$, repeat from Step 3.

Step 8: The procedure concludes, and $\eta(T)$ serves as an approximation for $\xi(T)$.

7. Convergence and Stability

Consider the exact solutions:

$$[\Phi(\xi_s)]_\theta = \left[[\Phi(\xi_s)]^-_\kappa(\theta) \quad [\Phi(\xi_s)]^+_\kappa(\theta)\right], [\Phi(\xi_s)]_\theta = \left[[\Phi(\xi_s)]^-_\lambda(\theta) \quad [\Phi(\xi_s)]^+_\lambda(\theta)\right]$$

$$[\Phi(\xi_s)]^\theta = \left[[\Phi(\xi_s)]^-_\mu(\theta) \quad [\Phi(\xi_s)]^+_\mu(\theta)\right]$$

Now, let these exact solutions be approximated by the following:

$$[\varphi(\xi_s)]_\theta = \left[[\varphi(\xi_s)]^-_\kappa(\theta) \quad [\varphi(\xi_s)]^+_\kappa(\theta)\right], [\varphi(\xi_s)]_\theta = \left[[\varphi(\xi_s)]^-_\lambda(\theta) \quad [\varphi(\xi_s)]^+_\lambda(\theta)\right]$$

$$[\varphi(\xi_s)]^\theta = \left[[\varphi(\xi_s)]^-_\mu(\theta) \quad [\varphi(\xi_s)]^+_\mu(\theta)\right]$$

at the time points δ_s, where $0 \leq s \leq N$. The grid points are defined as:

$$\delta_0 < \delta_1 < \delta_2 < \cdots < \delta_N = T, \quad k = \frac{T - t_0}{N}, \quad \delta_s = \delta_0 + sk, \quad n = 0, 1, \ldots, N$$

Our objective is to establish the convergence of the proposed methods to the exact solutions. Specifically, we aim to show:

$$d_\infty(\varphi(\xi_s), \varphi(\xi_s)) \to 0 \quad \text{as} \quad k \to 0$$

Theorem 7.1. *For any fixed θ such that $0 \leq \theta \leq 1$, the Adams-Bashforth two-step approximations of converge to the exact solutions $[\chi(\delta)]^\lambda_+(\theta)$, $[\chi(\delta)]^r_+(\theta)$, $[\Phi(\delta)]^l_-(\theta)$, and $[\Phi(\delta)]^r_-(\theta)$, where $[\Phi]^l_+$, $[\Phi]^r_+$, $[\Phi]^l_-$, and $[\Phi]^r_-$ belong to $C^3[t_0, T]$*

Theorem 7.2. *Both the Adams-Bashforth two-step and three-step methods are stable.*

Proof. For the Adams-Bashforth two-step method, the characteristic polynomial is given by $p(\lambda) = \lambda^2 - \lambda$. It is evident that this polynomial satisfies the root condition, which implies that the Adams-Bashforth two-step method is stable.

Similarly, for the Adams-Bashforth three-step method, the characteristic polynomial is $p(\lambda) = \lambda^3 - \lambda^2$. This polynomial also satisfies the root condition, indicating that the three-step method is stable as well.

Conclusion

This study aimed to investigate the numerical solution of an ordinary differential equation with a neutrosophic number as the initial condition. Here we have employed the Adam Bashforth and Adam-Moulton method for finding the solution, and we have also discussed the stability and convergence properties. Finally we can apply Adam Bashforth as predictor and Adam-Moulton as corrector. The numerical solution is an essential component of initial value problems (IVP) and boundary value problems (BVP) with advanced techniques, which plays a significant role in enhancing precision and reliable solutions. In the future, we can develop more numerical techniques to solve IVP and BVP in a neutrosophic environment.

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Prefilters on Neutrosophic Sets

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Abstract. This paper introduces the concept of a prefilter on the collection of neutrosophic sets. The characteristic set and characteristic value of prefilters are introduced and their fundamental properties are explored. The relationships between filters and prefilters are examined, with particular emphasis on how ultrafilters correspond to maximal filters via specific mappings such as i_K and w_K Additionally, the concept of prime prefilters are defined and their basic properties are studied. The compatibility between filters and prefilters is also discussed.

Keywords: Neutrosophic set, Prefilter, Characteristic set, Characteristic value, Prime prefilter.

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1. Introduction

Neutrosophic sets, introduced by Florentin Smarandache in 1998 [2], are a generalization of the classical and fuzzy set theories. Unlike classical set theories where an element either belongs to a set or does not, neutrosophic sets allow for the representation of truth (T), indeterminacy (I), and falsity (F) as independent components. These components are not necessarily complementary, allowing for a more flexible representation of uncertain, inconsistent, and imprecise information. In mathematical terms, a single valued neutrosophic set N in a universe X is characterized by a truth, indeterminacy, falsity membership function for each element in the universe. The values of these functions range independently within the interval $[0, 1]$.

Neutrosophic filters extend the notion of filters in classical set theory and fuzzy set theory to the neutrosophic context. R. Lowen introduced the concept of prefilters in fuzzy set theory in his paper [3]. Also discussed the characterisation of maximal prefilter. In addition he defined the convergence in fuzzy topological spaces. A neutrosophic filter is a special type of neutrosophic set that satisfies certain properties in relation to neutrosophic operations and is used to generalize classical notions of ideal and prime filters.

In this paper we defined prefilter on X which is the collection of neutrosophic sets. Characteristic set and characteristic value of prefilters are defined and their properties are analysed. The filters on X are related to prefilters on X and ultrafilters are related to maximal filters through i_K and w_k mappings. In addition the prime prefilters are defined and basic properties related to prime prefilters are analysed. Also the compatibility between the filters and prefilters on X are analysed.

2. Preliminaries

In this section basic definitions related to neutrosophic sets and filters are provided.

Definition 2.1. [2] A neutrosophic set N for an universe X is defined as $\mathsf{N} = \{\langle x, T_{\mathsf{N}}(x), I_{\mathsf{N}}(x), F_{\mathsf{N}}(x)\rangle : x \in \mathsf{X}\}$, where T_{N}, I_{N}, F_{N} denotes the truth, indeterminacy and falsity membership functions respectively from X to $(0^-, 1^+)$ such that $0^- \leq \sup T_{\mathsf{N}}(x) + \sup I_{\mathsf{N}}(x) + \sup F_{\mathsf{N}}(x) \leq 3^+$. The set of all neutrosophic set over X is denoted by $\mathcal{N}(\mathsf{X})$.

Definition 2.2. [5] A single valued neutrosophic set is a neutrosophic set in which the truth, indeterminacy and falsity membership functions are $T_{\mathsf{N}}, I_{\mathsf{N}}, F_{\mathsf{N}}$ respectively, and they are from X to $[0, 1]$.

Definition 2.3. [5] Let $\mathsf{N}_1, \mathsf{N}_2 \in \mathcal{N}(\mathsf{X})$. Then

$$\mathsf{N}_1 \cup \mathsf{N}_2 = \{\langle x, T_{\mathsf{N}_1}(x) \wedge T_{\mathsf{N}_2}(x), I_{\mathsf{N}_1}(x) \wedge I_{\mathsf{N}_2}(x), F_{\mathsf{N}_1}(x) \vee F_{\mathsf{N}_2}(x)\rangle : x \in \mathsf{X}\},$$

$$\mathsf{N}_1 \cap \mathsf{N}_2 = \{\langle x, T_{\mathsf{N}_1}(x) \vee T_{\mathsf{N}_2}(x), I_{\mathsf{N}_1}(x) \vee I_{\mathsf{N}_2}(x), F_{\mathsf{N}_1}(x) \wedge F_{\mathsf{N}_2}(x)\rangle : x \in \mathsf{X}\}$$

If $T_{\mathsf{N}_1}(x) \leq T_{\mathsf{N}_2}(x), I_{\mathsf{N}_1}(x) \leq I_{\mathsf{N}_2}(x), F_{\mathsf{N}_1}(x) \geq F_{\mathsf{N}_2}(x)$ for all $x \in \mathsf{X}$, then N_1 is a neutrosophic subset of N_2 and is denoted by $\mathsf{N}_1 \subseteq \mathsf{N}_2$

Definition 2.4. [4] If $T_{\mathsf{N}}(x) = 1, I_{\mathsf{N}}(x) = 0, F_{\mathsf{N}}(x) = 0$ for all $x \in \mathsf{X}$ Then N is the neutrosophic universal set and is denoted by 1_{N}

Definition 2.5. [4] If $T_{\mathsf{N}}(x) = 0, I_{\mathsf{N}}(x) = 1, F_{\mathsf{N}}(x) = 1$ for all $x \in \mathsf{X}$ Then N is the neutrosophic empty set and is denoted by 0_{N}

Definition 2.6. [1] A filter $\mathscr{F}$ on a set X is the subsets of X such that

(1) $A \in \mathscr{F}$, for all $B \subset X$, $A \subset B$ then $B \in \mathscr{F}$.
(2) $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$.
(3) $\emptyset \notin \mathscr{F}$.

Definition 2.7. [1] An ultrafilter $\mathscr{U}$ on a set X is a filter such that there is no filter on X which is strictly finer than $\mathscr{F}$.

3. Characteristic set of a prefilter

This section focuses on the concept of a prefilter, including its characteristic set, value, and properties. The relationship between a filter on X and a prefilter on X is explored, and their respective properties are analyzed. Throughout the paper neutrosophic set is a single valued neutrosophic set.

Definition 3.1. A subset $\mathfrak{F}$ of $\mathcal{N}(\mathsf{X})$ is a prefilter iff $\mathfrak{F} \neq \emptyset$ such that

(1) for all N_1, $\mathsf{N}_2 \in \mathfrak{F}$, $\mathsf{N}_1 \cap \mathsf{N}_2 \in \mathfrak{F}$
(2) if $\mathsf{N}_2 \supset \mathsf{N}_1$ and $\mathsf{N}_1 \in \mathfrak{F}$ then $\mathsf{N}_2 \in \mathfrak{F}$
(3) $0_{\mathsf{N}} \notin \mathfrak{F}$.

Definition 3.2. A subset $\mathcal{G}$ of $\mathcal{N}(\mathsf{X})$ is a base for a prefilter iff $\mathcal{G} \neq \emptyset$

(1) for all N_1, $\mathsf{N}_2 \in \mathcal{G}$ there exists $\mathsf{N}_3 \in \mathcal{G}$ such that $\mathsf{N}_3 \subseteq \mathsf{N}_1 \cap \mathsf{N}_2$.
(2) $0_{\mathsf{N}} \notin \mathcal{G}$.

The prefilter $\mathfrak{F}$ generated by $\mathcal{G}$ is defined as $\mathfrak{F} = \{\mathsf{N}_1 : \exists\ \mathsf{N}_2 \in \mathcal{G} \ni \mathsf{N}_1 \supseteq \mathsf{N}_2\}$ and is denoted by $\langle \mathcal{G} \rangle$.

A subset $\mathcal{G}$ of $\mathfrak{F}$ is a base for $\mathfrak{F} \iff$ for all $\mathsf{N}_1 \in \mathfrak{F}$ there exists $\mathsf{N}_2 \in \mathcal{G}$ such that $\mathsf{N}_1 \supseteq \mathsf{N}_2$.

A subset $\mathcal{H}$ of $\mathcal{N}(\mathsf{X})$ is called a generating family or subbase for a prefilter iff the family of finite lower bounds of members of $\mathcal{H}$ is a base for this prefilter. The prefilter generated by a subbase $\mathcal{H}$ is denoted by $\langle \mathcal{H} \rangle$.

Definition 3.3. A prefilter $\mathfrak{F}$ is called a prime filter iff for any $\mathsf{N}_1, \mathsf{N}_2 \in \mathcal{N}(\mathsf{X})$ such that $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathfrak{F}$, then either $\mathsf{N}_1 \in \mathfrak{F}$ or $\mathsf{N}_2 \in \mathfrak{F}$.

For $\mathsf{N} \in \mathcal{N}(\mathsf{X})$, $\dot{\mathsf{N}}$ is a prefilter generated by the neutrosophic set N (i.e.,)

$$\dot{\mathsf{N}} = \{\mathsf{N}' : \mathsf{N}' \supseteq \mathsf{N}\}.$$

For $\mathsf{N} \in \mathcal{N}(\mathsf{X})$ and characteristic value $a = (a_1, a_2, a_3)$

$\mathsf{N}_a = \langle T_{\mathsf{N}_a}(x),\ I_{\mathsf{N}_a}(x),\ F_{\mathsf{N}_a}(x)\rangle \quad x, x \in \mathsf{X}$ where $T_{\mathsf{N}_a}(x) = T_{\mathsf{N}}(x) + a_1$ $I_{\mathsf{N}_a}(x) = I_{\mathsf{N}}(x) + a_2$, $F_{\mathsf{N}_a}(x) = F_{\mathsf{N}}(x) + a_3$. Similarly

$$\begin{aligned} \mathsf{N}_{\dot{a}} \cap 1_{\mathsf{N}} = \{\mathsf{N}_b \cap 1_{\mathsf{N}} |\ & T_{\mathsf{N}_b}(x) \geq T_{\mathsf{N}}(x) + a_1, \\ & I_{\mathsf{N}_b}(x) \geq I_{\mathsf{N}}(x) + a_2, \\ & F_{\mathsf{N}_b}(x) \leq F_{\mathsf{N}}(x) + a_3\} \end{aligned}$$

Let $\mathfrak{F}$ is a prefilter and N be a neutrosophic set

$$\mathcal{C}^{\mathsf{N}}(\mathfrak{F}) = \{a = (a_1, a_2, a_3) : \forall\ \mathsf{N}' \in \mathfrak{F}, \exists\ x \in \mathsf{X}\ \ni \\ T_{\mathsf{N}'}(x) > T_{\mathsf{N}}(x) + a_1, \\ I_{\mathsf{N}'}(x) > I_{\mathsf{N}}(x) + a_2, \\ F_{\mathsf{N}'}(x) < F_{\mathsf{N}}(x) + a_3\}$$

and this subset is called a characteristic set of $\mathfrak{F}$ with respect to N. The characteristic value $c^{\mathsf{N}}(\mathfrak{F})$ of $\mathfrak{F}$ with respect to N is $(\sup a_1, \sup a_2, \inf a_3)$.
Let a set $A = \{a_1, a_2, a_3\}$ is a characteristic set $\iff$ either $A = \emptyset$ or for $a_i, i \in 1, 2, 3$ is one of the following set $\{0\}, [0, c)$ for $c \in I$, $[0, c]$ for $c \in I \setminus \{1\}$.

$$\mathbb{W}(\mathsf{X}) = \text{set of all prefilters.}$$
$$\mathbb{W}^{\mathsf{N}}(\mathsf{X}) = \{\mathfrak{F} \in \mathbb{W}(\mathsf{X}) : \mathcal{C}^{\mathsf{N}}(\mathfrak{F}) \neq \emptyset.\}$$
$$\mathbb{W}^{\mathsf{N}}_{+}(\mathsf{X}) = \{\mathfrak{F} \in \mathbb{W}(\mathsf{X}) : c^{\mathsf{N}}(\mathfrak{F}) > 0.\}$$
$$\mathbb{W}^{\mathsf{N}}_{K}(\mathsf{X}) = \{\mathfrak{F} \in \mathbb{W}(\mathsf{X}) : \mathcal{C}^{\mathsf{N}}(\mathfrak{F}) = K\}$$

where K is some non empty characteristic set. A member of $\mathbb{W}^{\mathsf{N}}_{K}(\mathsf{X})$ will be called $K-$prefilter.

Proposition 3.4. *Let N be a neutrosophic set.*

(1) *If $\mathcal{L}$ is a family of prefilters, then $\mathcal{C}^{\mathsf{N}}(\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F}) = \underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathcal{C}^{\mathsf{N}}(\mathfrak{F})$*
(2) *If $\mathcal{L}$ is a family of prefilters such that $\underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathfrak{F}$ is a subbase for some prefilter then $\mathcal{C}^{\mathsf{N}}(\langle\underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathfrak{F}\rangle) = \underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathcal{C}^{\mathsf{N}}(\mathfrak{F})$*

Proof. 1. For all $\mathcal{G} \in \mathcal{L}$, $\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F} \subset \mathcal{G}$

$$\mathcal{C}^{\mathsf{N}}(\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F}) \supset \underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathcal{C}^{\mathsf{N}}(\mathfrak{F}).$$

Suppose conversely $a \in I \times I \times I \setminus \underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathcal{C}^{\mathsf{N}}(\mathfrak{F})$ then for all $\mathfrak{F} \in \mathcal{L}$ there exists $\mathsf{N}' \in \mathfrak{F}$ such that

$$T_{\mathsf{N}'}(x) \leq \min\{T_{\mathsf{N}}(x) + a_1, 1\},$$
$$I_{\mathsf{N}'}(x) \leq \min\{I_{\mathsf{N}}(x) + a_2, 1\},$$
$$F_{\mathsf{N}'}(x) \geq \max\{F_{\mathsf{N}}(x) + a_3, 0\}$$

and consequently for all $\mathfrak{F} \in \mathcal{L}$ such that $\mathsf{N}_a \cap 1_{\mathsf{N}} \in \mathfrak{F}$ or $\mathfrak{F} \in \mathcal{L}$ such that $\mathsf{N}_{\dot{a}} \cap 1_{\mathsf{N}} \subset \underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F}$.
Since trivially $\mathcal{C}^{\mathsf{N}}(\mathsf{N}_{\dot{a}} \cap 1_{\mathsf{N}}) \subset ([0, a_1), [0, a_2), (a_3, 1])$, thus $a \notin \mathcal{C}^{\mathsf{N}}(\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F})$.
2. The proof is similar to statement 1. □

This section mainly deals with the characteristic set and value of prefilter with respect to 0_{N}.

$\mathbb{F}(\mathsf{X})$ denotes the set of all filters on X. Let K be a non empty characteristic set, then the following mappings can be defined

$$\begin{aligned} &w_K : \mathbb{F}(\mathsf{X}) \to \mathbb{W}_K(\mathsf{X}) \\ &\quad \mathscr{F} \to \{\mathsf{N} : \forall\ k = (k_1,k_2,k_3) \in K, (T_{\mathsf{N}}^{-1}(k_1,1], I_{\mathsf{N}}^{-1}(k_2,1], F_{\mathsf{N}}^{-1}[0,k_3)) \in \mathscr{F}\} \\ &i_K : \bigcup_{K \subset K'} \mathbb{W}_{K'}(\mathsf{X}) \to \mathbb{F}(\mathsf{X}) \\ &\quad \mathfrak{F} \to \{(T_{\mathsf{N}}^{-1}(k_1,1], I_{\mathsf{N}}^{-1}(k_2,1], F_{\mathsf{N}}^{-1}[0,k_3)) : \forall\ \mathsf{N} \in \mathfrak{F}, k = (k_1,k_2,k_3) \in K\} \end{aligned}$$

Proposition 3.5. *Let $K, K^{'}$ be two characteristic sets such that $K^{'} \subset K$, then for all $\mathscr{F} \in \mathbb{F}(\mathsf{X})$*

(i) $w_K(\mathscr{F}) \subset w_{K'}(\mathscr{F})$

(ii) $i_{K'} \circ w_K(\mathscr{F}) = \mathscr{F}$

and for all $\mathfrak{F} \in \bigcup_{K'' \supset K} \mathbb{W}_{K''}$,

(iii) $i_K(\mathfrak{F}) \supset i_{K'}(\mathfrak{F})$

(iv) $w_{K'} \circ i_K(\mathfrak{F}) \supset \mathfrak{F}$.

Furthermore the mappings w_K and i_K are order preserving.

Proof. Given $K^{'} \subset K$, $w_K(\mathscr{F}) = \{\mathsf{N} : \forall\ k \in K(T_{\mathsf{N}}^{-1}(k_1,1], I_{\mathsf{N}}^{-1}(k_2,1], F_{\mathsf{N}}^{-1}[0,k_3)) \in \mathscr{F}\}$. let $\mathsf{N} \in w_K(\mathscr{F}) \implies \forall\ k \in K(T_{\mathsf{N}}^{-1}(k_1,1], I_{\mathsf{N}}^{-1}(k_2,1], F_{\mathsf{N}}^{-1}[0,k_3)) \in \mathscr{F}$ Since $K^{'} \subset K$, $\forall\ k \in K^{'}(T_{\mathsf{N}}^{-1}(k_1,1], I_{\mathsf{N}}^{-1}(k_2,1], F_{\mathsf{N}}^{-1}[0,k_3)) \in \mathscr{F} \implies \mathsf{N} \in w_K^{'}(\mathscr{F})$ From the definition of w_K and $i_{K'}$ its clear that $i_{K'} \circ w_K(\mathscr{F}) = \mathscr{F}$. □

As $\mathbb{W}_K^{\mathsf{N}}(\mathsf{X})$ is inductive, for any neutrosophic set N. Then by Zorn's lemma there exist maximal elements, which is called as maximal K-prefilters. Now if $\mathfrak{F}$ is a maximal $K-$prefilter then since $w_{K'} \circ i_K(\mathfrak{F})$ is also a $K-$prefilter it follows from Proposition 3.5(iv) that $\mathfrak{F} = w_{K'} \circ i_K(\mathfrak{F})$.

Definition 3.6. For a subset A of X, the characteristic function $\chi_A^{\mathsf{N}} : \mathsf{X} \to I \times I \times I$ is a neutrosophic set in X such that $\chi_A^{\mathsf{N}}(x) = \langle T_{\mathsf{N}}^A(x), I_{\mathsf{N}}^A(x), F_{\mathsf{N}}^A(x)\rangle$, where

$$T_{\mathsf{N}}^A : \mathsf{X} \to [0,1] \text{ such that}$$

$$T_{\mathsf{N}}^A(x) = \begin{cases} 1\ \forall\ x \in A \\ 0\ \forall\ x \in A^c \end{cases}$$

$$I_{\mathsf{N}}^{A}:\mathsf{X}\to[0,1] \text{ such that}$$
$$I_{\mathsf{N}}^{A}(x)=1\ \forall\ x\in\mathsf{X}$$
$$F_{\mathsf{N}}^{A}:\mathsf{X}\to[0,1] \text{ such that}$$
$$F_{\mathsf{N}}^{A}(x)=\begin{cases}0\ \forall\ x\in A\\ 1\ \forall\ x\in A^{c}\end{cases}$$

Theorem 3.7. *If $\mathscr{U}$ is an ultrafilter on X, then $w_K(\mathscr{U})$ is a maximal $K-$prefilter, and if $\mathfrak{F}$ is a prime prefilter then for all characteristic sets $K\subset\mathcal{C}(\mathfrak{F})$, $i_K(\mathfrak{F})$ is an ultrafilter on X.*

Proof. Let $\mathscr{U}$ be an ultrafilter on X and let $\mathfrak{F}\in\mathbb{W}_K(\mathsf{X})$ be a maximal $K-$prefilter, finer than $w_K(\mathscr{U})$. Then $i_K(\mathfrak{F})\supset i_K\circ w_K(\mathscr{U})=\mathscr{U}$ and thus $i_K(\mathfrak{F})=\mathscr{U}$. Since $\mathfrak{F}$ is maximal, $\mathfrak{F}=w_K\circ i_K(\mathfrak{F})=w_K(\mathscr{U})$.

Let $\mathfrak{F}$ be a prime prefilter and let $A\subset\mathsf{X}$. Then $\chi_A^{\mathsf{N}}\cup\chi_{A^c}^{\mathsf{N}}=1_{\mathsf{N}}\in\mathfrak{F}$ and $\chi_A^{\mathsf{N}}\in\mathfrak{F}$. Since always $0_{\mathsf{N}}\in K$, $A=([T_{\mathsf{N}}^{A}]^{-1}(0,1],\ [I_{\mathsf{N}}^{A}]^{-1}(0,1],\ [F_{\mathsf{N}}^{A}]^{-1}[0,1))\in i_K(\mathfrak{F})$ which shows that $i_K(\mathfrak{F})$ is an ultrafilter on X. □

Theorem 3.8. *If $\mathfrak{F}$ is a maximal $K-$filter, then it is a prime prefilter.*

Proof. If $\mathfrak{F}\in\mathbb{W}_K(\mathsf{X})$ is a maximal then $i_K(\mathfrak{F})$ is ultrafilter. Let $\mathscr{U}$ be an ultrafilter on X finer than $i_K(\mathfrak{F})$ then $w_K((U))\supset w_K\circ i_K(\mathfrak{F})=\mathfrak{F}$ and since $w_K((U))=\mathfrak{F}$. This in turn implies that $(U)=i_K\circ w_K((U))=i_K(\mathfrak{F})$. Let $\mathsf{N}_1,\mathsf{N}_2$ be a neutrosophic set such that $\mathsf{N}_1\cup\mathsf{N}_2\in\mathfrak{F}$, then for all $k\in K$, $(T^{-1}_{\mathsf{N}_1\cup\mathsf{N}_2}(k_1,1],I^{-1}_{\mathsf{N}_1\cup\mathsf{N}_2}(k_2,1],F^{-1}_{\mathsf{N}_1\cup\mathsf{N}_2}[0,k_3))\in i_K(\mathfrak{F})$.

If K is of the form $([0,k_{0_1}],[0,k_{0_2}],[k_{0_3},1])$ it suffices to remark that for instance $(T^{-1}_{\mathsf{N}_1}(k_{0_1},1]$, $I^{-1}_{\mathsf{N}_1}(k_{0_2},1]$, $F^{-1}_{\mathsf{N}_1}[0,k_{0_3}))\in i_K(\mathfrak{F})$ and since for all $k=(k_1,k_2,k_3)$, $k_0=(k_{0_1},k_{0_2},k_{0_3})$, such that $k_1\leq k_{0_1}$, $k_2\leq k_{0_2}$, $k_3\geq k_{0_3}$, $(T^{-1}_{\mathsf{N}_1}(k_{0_1},1],\ I^{-1}_{\mathsf{N}_1}(k_{0_2},1],\ F^{-1}_{\mathsf{N}_1}[0,k_{0_3}))\subset(T^{-1}_{\mathsf{N}_1}(k_1,1],\ I^{-1}_{\mathsf{N}_1}(k_2,1],\ F^{-1}_{\mathsf{N}_1}[0,k_3))$, thus $(T^{-1}_{\mathsf{N}_1}(k_1,1],\ I^{-1}_{\mathsf{N}_1}(k_2,1],\ F^{-1}_{\mathsf{N}_1}[0,k_3))\in i_K(\mathfrak{F})$ for all $k\in K$.

If K is of the form $([0,k_{0_1}),\ [0,k_{0_2}),\ (k_{0_3},1])$, choose a sequence $(k_{n_1})_{n_1\in\mathbb{N}}$ of increasing numbers in $[0,k_{0_1})$, $(k_{n_2})_{n_2\in\mathbb{N}}$ of increasing numbers in $[0,k_{0_2})$, $(k_{n_3})_{n_3\in\mathbb{N}}$ of decreasing numbers in $(k_{0_3},1]$, such that $\sup_{n_1\in\mathbb{N}}k_{n_1}=k_{0_1}$, $\sup_{n_2\in\mathbb{N}}k_{n_2}=k_{0_2}$, $\inf_{n_3\in\mathbb{N}}k_{n_3}=k_{0_3}$. Then for all $n_i\in\mathbb{N}$ either $(T^{-1}_{\mathsf{N}_1}(k_{n_1},1],I^{-1}_{\mathsf{N}_1}(k_{n_2},1],F^{-1}_{\mathsf{N}_1}[0,k_{n_3}))\in i_K(\mathfrak{F})$ or $(T^{-1}_{\mathsf{N}_2}(k_{n_1},1],I^{-1}_{\mathsf{N}_2}(k_{n_2},1],F^{-1}_{\mathsf{N}_2}[0,k_{n_3}))$ $\in i_K(\mathfrak{F})$. Thus there exists a subsequence $(k'_{n_1})_{n_1\in\mathbb{N}}$ of $(k_{n_1})_{n_1\in\mathbb{N}}$, $(k'_{n_2})_{n_2\in\mathbb{N}}$ of $(k_{n_2})_{n_2\in\mathbb{N}}$, $(k'_{n_3})_{n_3\in\mathbb{N}}$ of $(k_{n_3})_{n_3\in\mathbb{N}}$ such that for all $n_i\in\mathbb{N}$, $(T^{-1}_{\mathsf{N}_1}(k'_{n_1},1],I^{-1}_{\mathsf{N}_1}(k'_{n_2},1],F^{-1}_{\mathsf{N}_1}[0,k'_{n_3}))\in i_K(\mathfrak{F})$.

For any $k=(k_1,k_2,k_3),\in K$ choose $n_i\in\mathbb{N}$ such that $k_1\leq k'_{n_1},k_2\leq k'_{n_2},k_3\leq k'_{n_3}$, $(T^{-1}_{\mathsf{N}_1}(k'_{n_1},1],\ I^{-1}_{\mathsf{N}_1}(k'_{n_2},1],\ F^{-1}_{\mathsf{N}_1}[0,k_{n'_3}))\subset(T^{-1}_{\mathsf{N}_1}(k_1,1],\ I^{-1}_{\mathsf{N}_1}(k_2,1],\ F^{-1}_{\mathsf{N}_1}[0,k_3))$, thus $(T^{-1}_{\mathsf{N}_1}(k_1,1]$, $I^{-1}_{\mathsf{N}_1}(k_2,1],F^{-1}_{\mathsf{N}_1}[0,k_3))\in i_K(\mathfrak{F})$. Consequently in either case $\mathsf{N}_1\in w_K\circ i_K(\mathfrak{F})=\mathfrak{F}$ which proves that $\mathfrak{F}$ is a prime prefilter. □

Corollary 3.9. *If $\mathscr{U}$ is an ultrafilter on X, then for all nonempty characteristic sets K, $w_K(\mathscr{U})$ is a prime filter.*

4. Compatibility of filter and prefilter

In analogy with filters on X one might think that every $K-$filter is equal to the intersection of the family of maximal $K-$prefilters which are finer. Yet this is not the case as is shown by the following counterexample. Let X be arbitrary and let

$$\mathscr{R}(\mathfrak{F}) = \{\mathfrak{P} : \mathfrak{P} \text{ maximal K-prefilter finer than } \mathfrak{F}\}$$

$$\mathscr{R}(\mathcal{G}) = \{\mathfrak{P} : \mathfrak{P} \text{ maximal K-prefilter finer than } \mathcal{G}\}$$

Clearly $\mathscr{R}(\mathfrak{F}) \supset \mathscr{R}(\mathcal{G})$. But if $\mathfrak{P} \in \mathscr{R}(\mathfrak{F})$, then $\mathfrak{P} = w_K \circ i_K(\mathfrak{P}) \supset w_K \circ i_K(\mathfrak{F}) = \mathcal{G}$ so that $\mathfrak{P} \in \mathscr{R}(\mathcal{G})$. Consequently $\mathscr{R}(\mathfrak{F}) = \mathscr{R}(\mathcal{G})$.
Let $\mathscr{P}(\mathfrak{F})$ be the set of all prime prefilters finer than $\mathfrak{F} \in \mathbb{W}(\mathsf{X}.)$ Hence

$$\mathfrak{F} = \bigcap_{\mathcal{G} \in \mathscr{P}(\mathfrak{F})} \mathcal{G}.$$

But as we shall see further on $\mathscr{P}(\mathfrak{F})$ is too large a set for our purposes. The following proposition shows that we can extract a subset of $\mathscr{P}(\mathfrak{F})$ which still contains all the relevant information.

Proposition 4.1. *The set $\mathscr{P}(\mathfrak{F})$ is inductive in the sense that every descending chain of prefilters in it has a lower bound.*

Proof. Let $\mathscr{R} \subset \mathscr{P}(\mathfrak{F})$ be a descending chain. Consider $\mathcal{G}_0 = \underset{\mathcal{G} \in \mathscr{R}}{\cap} \mathcal{G}$. Then clearly $\mathcal{G}_0 \supset \mathfrak{F}$. Next, if $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathcal{G}_0$, then either for all $\mathcal{G} \in \mathscr{R}$ we have $\mathsf{N}_1 \in \mathcal{G}$ and thus $\mathsf{N}_1 \in \mathcal{G}_0$ and we are done, or there exists some $\mathcal{G} \in \mathscr{R}$ such that $\mathsf{N}_1 \notin \mathcal{G}$. Then since $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathcal{G}$ the latter implies that $\mathsf{N}_2 \in \mathfrak{P}$ for all $\mathfrak{P} \in \mathscr{R}$ such that $\mathfrak{P} \supset \mathcal{G}$. And if $\mathfrak{P} \in \mathscr{R}$ such that $\mathfrak{P}$, then since $\mathsf{N}_2 \notin \mathfrak{P}$ we must again have $\mathsf{N}_2 \in \mathfrak{P}$. Thus $\mathsf{N}_2 \in \mathcal{G}_0$. □

It now follows from Zorn's theorem that there exist minimal elements in $\mathscr{P}(\mathfrak{F})$. We denote the family of minimal elements in $\mathscr{P}(\mathfrak{F})$ by $\mathscr{P}_m(\mathfrak{F})$. It then also follows at once that we still have

$$\mathfrak{F} = \bigcap_{\mathcal{G} \in \mathscr{P}_m(\mathfrak{F})} \mathcal{G}.$$

In order to characterize these minimal prime prefilters in $\mathscr{P}_m(\mathfrak{F})$ in a more tangible way we need the following concept.

Definition 4.2. A filter $\mathscr{F}$ on X and a prefilter $\mathfrak{F}$ are said to be compatible iff for all $F^{'} \in \mathscr{F}$ and $\mathsf{N} \in \mathfrak{F}, \mathsf{N}$ does not vanish everywhere on $F^{'}$.

We shall use the notation

$$\mathsf{N}_{F'} : \mathsf{X} \to I \times I \times I : x \to (T_{\mathsf{N}}(x), I_{\mathsf{N}}(x), F_{\mathsf{N}}(x)) \text{ if } x \in F^{'}$$
$$\to 0_{\mathsf{N}} \text{ if } x \notin F^{'}$$

Then since for all $\mathsf{N}, \mathsf{N}^{'} \in \mathfrak{F}$ and $F^{'}, G^{'} \in \mathscr{F}$, $\mathsf{N}_{F'}, \mathsf{N}^{'}_{G'} = (\mathsf{N} \cap \mathsf{N}^{'})_{F' \cap G'}$ it is clear that if $\mathscr{F}$ and $\mathfrak{F}$ are compatible then

$$(\mathfrak{F}, \mathscr{F}) = \langle \{ \mathsf{N}_{F'} : \mathsf{N} \in \mathfrak{F}, F^{'} \in \mathscr{F} \} \rangle$$

is a prefilter.

Theorem 4.3. *Let $\mathfrak{F}$ be a prefilter. Then*

$$\mathscr{P}_m(\mathfrak{F}) = \{ (\mathfrak{F}, \mathscr{U}) : \mathscr{U} \text{ ultrafilter on } \mathsf{X} \text{ and compatible with } \mathfrak{F} \}$$

Proof. Let $\mathscr{U}$ be an ultrafilter on X compatible with $\mathfrak{F}$. To show that $(\mathfrak{F}, \mathscr{U})$ is prime let $\mathsf{N}_1, \mathsf{N}_2$ be neutrosophic sets such that $\mathsf{N}_1 \cup \mathsf{N}_2 \in (\mathfrak{F}, \mathscr{U})$ then there exist $\mathsf{N} \in \mathfrak{F}$ and $U \in \mathscr{U}$ such that $\mathsf{N}_1 \cup \mathsf{N}_2 \geq \mathsf{N}_U$. Let then $A = \{x : T_{\mathsf{N}_1}(x) \geq T_{\mathsf{N}_U}(x), I_{\mathsf{N}_1}(x) \geq I_{\mathsf{N}_U}(x), F_{\mathsf{N}_1}(x) \leq F_{\mathsf{N}_U}(x)\}$ and $B = \{x : T_{\mathsf{N}_2}(x) \geq T_{\mathsf{N}_U}(x), I_{\mathsf{N}_2}(x) \geq I_{\mathsf{N}_U}(x), F_{\mathsf{N}_2}(x) \leq F_{\mathsf{N}_U}(x)\}$. Since $A \cup B = \mathsf{X}$, $A \in \mathscr{U}$. Then since $\mathsf{N}_1 \supseteq \mathsf{N}^{'}_{A \cap U}$ this implies that $\mathsf{N}_1 \in (\mathfrak{F}, \mathscr{U})$. Consequently $(\mathfrak{F}, \mathscr{U})$ is prime. To show that it is minimal let $\mathcal{G} \in \mathscr{P}(\mathfrak{F})$ be such that $(\mathfrak{F}, \mathscr{U}) \supset \mathcal{G} \supset \mathfrak{F}$ and suppose that there exists $\mathsf{N} \in \mathfrak{F}$ and $U \in \mathscr{U}$ such that $\mathsf{N}_U \notin \mathcal{G}$.

Then since $\mathsf{N} \in \mathcal{G}$ and $\mathcal{G}$ is prime, $\mathsf{N}_{U^c} \in \mathcal{G}$ and thus $\mathsf{N}_{U^c} \in (\mathfrak{F}, \mathscr{U})$ which is impossible. Consequently $(\mathfrak{F}, \mathscr{U})$ is minimal and thus $(\mathfrak{F}, \mathscr{U}) \in \mathscr{P}_m(\mathfrak{F})$. To show the converse let $\mathcal{G} \in \mathscr{P}_m(\mathfrak{F})$. Consider the characteristic set $K = \mathcal{C}(\mathcal{G})$ of $\mathcal{G}$. It follows from Theorem 3.7 that $i_K(\mathcal{G})$ is an ultrafilter on X. And it is trivial that $\mathfrak{F}$ is compatible with $i_K(\mathcal{G})$. Now since $\mathscr{P}_m(\mathcal{G}) = \{\mathcal{G}\}$ it follows from the first part of the theorem that $(\mathcal{G}, i_K(\mathcal{G})) = \mathcal{G}$. Since $\mathfrak{P} \subset \mathcal{G}$ it follows that

$$\mathfrak{F} \subset (\mathfrak{F}, i_K(\mathcal{G})) \subset (\mathcal{G}, i_K(\mathcal{G})) = \mathcal{G}.$$

Since $(\mathfrak{F}, i_K(\mathcal{G}))$ is prime, we have $\mathcal{G} = (\mathfrak{F}, i_K(\mathcal{G}))$ □

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$\mathcal{N}_{\alpha b^* g\alpha}$- Closed Sets in Neutrosophic Topological Spaces

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Abstract. In neutrosophic topological spaces, the notion of neutrosophic $\alpha b^* g\alpha$-closed sets, neutrosophic $\alpha b^* g\alpha$-border, and neutrosophic $\alpha b^* g\alpha$-frontier are described and their properties are discussed. The connection between neutrosophic $\alpha b^* g\alpha$-frontier and neutrosophic $\alpha b^* g\alpha$-border are established

Keywords: $\mathcal{N}_{\alpha b^* g\alpha}$-closed sets, $\mathcal{N}_{\alpha b^* g\alpha}$-border, $\mathcal{N}_{\alpha b^* g\alpha}$-frontier.

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1. **Introduction**

Neutrosophic sets is a generalisation of Atanassov's [4] intuitionistic fuzzy sets and Zadeh's [20] fuzzy sets, and were first proposed by Smarandache [17] [18]. It also takes into account the membership functions for falsehood, indeterminacy, and truth. In a number of disciplines, including probability, algebra, control theory, topology, etc., Smarandache introduced the neutrosophic idea in response to fuzzy sets and intuitionistic fuzzy sets' inability to handle indeterminacy-membership functions. Neurosophic set based notions were later introduced by Alblowi et al. [1]. In the past 20 years, numerous scholars have utilised these potent ideas to put forth numerous topological hypotheses. A novel idea in neutrosophic topological spaces was proposed by Salama and Alblowi [14]. It gives a brief overview of neutrosophic topology, which is a generalisation of Chang's [6] and Coker's [5] intuitionistic fuzzy topology.

In the subject of neutrosophic topological spaces, Salama et al., [11–13] presented the generalisation of neutrosophic sets, neutrosophic crisp sets, and neutrosophic closed sets. Salama et al., [13] proposed a few neutrosophic continuous functions as an initial set of continuous functions in neutrosophic topology. In addition, a number of scholars have defined a number of closed sets in neutrosophic topology, including generalised neutrosophic closed sets [7] in neutrosophic topological spaces, neutrosophic α-closed sets [3], neutrosophic αg-closed sets [10], and neutrosophic b-closed sets [8]. Neutrosophic b$^*g\alpha$-closed sets were defined by S. Saranya and M. Vigneshwaran [15]. In order to establish a connection between the operators of neutrosophic interior and neutrosophic closure, Iswarya and Bageerathi [9] introduced a novel notion of neutrosophic frontier operator and neutrosophic semi-frontier operator . $\alpha b^*g\alpha$-closed sets are new closed sets defined in topological spaces by Suthi Keerthana K, Vigneshwaran M, and Vidyarani L [19]. These sets have been used to define various topological functions, such as continuous functions, irresolute functions, and homeomorphic functions with certain separable axioms.

This paper presents the idea of neutrosophic $\alpha b^*g\alpha$-closed sets in neutrosophic topological spaces and examines their characteristics as well as how they relate to other known characters. The concepts in neutrosophic $\alpha b^*g\alpha$-interior, neutrosophic $\alpha b^*g\alpha$-closure, neutrosophic $\alpha b^*g\alpha$-border, and neutrosophic $\alpha b^*g\alpha$-frontier are examined. Neutrosophic topological spaces have a relationship between the neutrosophic $\alpha b^*g\alpha$-border and the neutrosophic $\alpha b^*g\alpha$-frontier, along with associated features.

2. **Preliminaries**

The basic definitions which are used in the next section are referred from the references [14], [3], [7], [16], [10], [2], [15].

3. $\mathcal{N}_{\alpha b^*g\alpha}$- **Closed Sets**

Definition 3.1. Let ${}_{\mathcal{N}}E$ be a subset of a ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X},\tau)$. Then ${}_{\mathcal{N}}E$ is called

- a neutrosophic $\alpha b^*g\alpha$-closed set($\mathcal{N}_{\alpha b^*g\alpha}$-$\mathcal{CS}$) if ${}_{\mathcal{N}}\alpha\text{cl}({}_{\mathcal{N}}E) \subseteq \mathcal{V}$ whenever ${}_{\mathcal{N}}E \subseteq \mathcal{V}$ and $\mathcal{V}$ is a neutrosophic $b^*g\alpha$-open set in $(\mathbb{X},\tau)$.

Example 3.2. Let $\mathbb{X} = \{{}_na, {}_nb, {}_nc\}$ and the $\mathcal{NS}$, ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as
${}_{\mathcal{N}}L = \{\langle {}_nx, ({}_t0.5, {}_i0.3, {}_f0.7), ({}_t0.4, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.2, {}_f0.7)\rangle\, \forall_n x \in \mathbb{X}\}$,
${}_{\mathcal{N}}M = \{\langle {}_nx, ({}_t0.7, {}_i0.3, {}_f0.5), ({}_t0.7, {}_i0.3, {}_f0.6), ({}_t0.7, {}_i0.2, {}_f0.5)\rangle\, \forall_n x \in \mathbb{X}\}$,
Then the $\mathcal{NT}$, $\tau = \{{}_{\mathcal{N}}0, {}_{\mathcal{N}}L, {}_{\mathcal{N}}M, {}_{\mathcal{N}}1\}$, which are $\mathcal{NOS}$ in the ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X},\tau)$.
If ${}_{\mathcal{N}}N = \{\langle {}_nx, ({}_t0.5, {}_i0.8, {}_f0.7), ({}_t0.6, {}_i0.8, {}_f0.7), ({}_t0.5, {}_i0.9, {}_f0.7)\rangle\, \forall_n x \in \mathbb{X}\}$, and
${}_{\mathcal{N}}E = \{\langle {}_nx, ({}_t0.5, {}_i0.3, {}_f0.7), ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.3, {}_f0.7)\rangle\, \forall_n x \in \mathbb{X}\}$
Then the complements of ${}_{\mathcal{N}}L$, ${}_{\mathcal{N}}M$, ${}_{\mathcal{N}}N$ and ${}_{\mathcal{N}}E$ are
${}_{\mathcal{N}}\overline{L} = \{\langle {}_nx, ({}_t0.7, {}_i0.7, {}_f0.5), ({}_t0.7, {}_i0.7, {}_f0.4), ({}_t0.7, {}_i0.8, {}_f0.5)\rangle\, \forall_n x \in \mathbb{X}\}$,
${}_{\mathcal{N}}\overline{M} = \{\langle {}_nx, ({}_t0.5, {}_i0.7, {}_f0.7), ({}_t0.6, {}_i0.7, {}_f0.7), ({}_t0.$ $\mathcal{N}_{\alpha b^*g\alpha}$- $8, {}_f0.7)\rangle\, \forall_n x \in \mathbb{X}\}$,
${}_{\mathcal{N}}\overline{N} = \{\langle {}_nx, ({}_t0.7, {}_i0.2, {}_f0.5), ({}_t0.7, {}_i0.2, {}_f0.6), ({}_t0.7, {}_i0.1, {}_f0.5)\rangle\, {}_n\forall x \in \mathbb{X}\}$ and

${}_{\mathcal{N}}\overline{E} = \{\langle {}_n x, ({}_t 0.7, {}_i 0.7, {}_f 0.5), ({}_t 0.7, {}_i 0.7, {}_f 0.6), ({}_t 0.7, {}_i 0.7, {}_f 0.5)\rangle \, \forall_n x \in \mathbb{X}\}$

Hence ${}_{\mathcal{N}}N$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$, ${}_{\mathcal{N}}\overline{N}$ is a $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}}\overline{E}$ is a $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ of ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$.

$\because {}_{\mathcal{N}\alpha}cl(E) = \{\langle {}_n x, ({}_t 0.5, {}_i 0.3, 0.7), ({}_t 0.6, {}_i 0.3, 0.7), ({}_t 0.5, {}_i 0.3, 0.7)\rangle \, \forall_n x \in \mathbb{X}\}$ which is contained in ${}_{\mathcal{N}}N$. That is ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq {}_{\mathcal{N}}N$.

Remark 3.3. Let ${}_{\mathcal{N}}A$ be a subset of ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, then $\mathcal{N}_{\alpha b^*g\alpha}$- ${}_{\mathcal{N}}int({}_{\mathcal{N}}A)$ is $\mathcal{N}_{\alpha b^*g\alpha}$-open in $(\mathbb{X}, \tau)$.

Theorem 3.4. *In ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, every $\mathcal{NCS}$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$.*

Proof. *Let ${}_{\mathcal{N}}E \subseteq \mathcal{V}$, where $\mathcal{V}$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ in $\mathbb{X}$.*

$\because {}_{\mathcal{N}}E$ is $\mathcal{NCS}$, ${}_{\mathcal{N}}cl({}_{\mathcal{N}}E) = {}_{\mathcal{N}}E \subseteq \mathcal{V}$. But ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq {}_{\mathcal{N}}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$, which implies ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$.

$\therefore {}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$.

The converse is not true.

Example 3.5. Let $\mathbb{X} = \{{}_n a, {}_n b, {}_n c\}$ and the $\mathcal{NS}$, ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}L = \{\langle {}_n x, ({}_t 0.5, {}_i 0.3, {}_f 0.7), ({}_t 0.4, {}_i 0.3, {}_f 0.7), ({}_t 0.5, {}_i 0.2, {}_f 0.7)\rangle \, \forall_n x \in \mathbb{X}\}$,

${}_{\mathcal{N}}M = \{\langle {}_n x, ({}_t 0.7, {}_i 0.3, {}_f 0.5), ({}_t 0.7, {}_i 0.3, {}_f 0.6), ({}_t 0.7, {}_i 0.2, {}_f 0.5)\rangle \, \forall_n x \in \mathbb{X}\}$,

Then the $\mathcal{NT}$, $\tau = \{{}_{\mathcal{N}}0, {}_{\mathcal{N}}L, {}_{\mathcal{N}}M, {}_{\mathcal{N}}1\}$ and the complement of $\mathcal{NS}$ of ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}\overline{L} = \{\langle {}_n x, ({}_t 0.7, {}_i 0.7, {}_f 0.5), ({}_t 0.7, {}_i 0.7, {}_f 0.4), ({}_t 0.7, {}_i 0.8, {}_f 0.5)\rangle \, \forall_n x \in \mathbb{X}\}$,

${}_{\mathcal{N}}\overline{M} = \{\langle {}_n x, ({}_t 0.5, {}_i 0.7, {}_f 0.7), ({}_t 0.6, {}_i 0.7, {}_f 0.7), ({}_t 0.5, {}_i 0.8, {}_f 0.7)\rangle \, \forall_n x \in \mathbb{X}\}$,

If ${}_{\mathcal{N}}E = \{\langle {}_n x, ({}_t 0.5, {}_i 0.3, {}_f 0.7), ({}_t 0.6, {}_i 0.3, {}_f 0.7), ({}_t 0.5, {}_i 0.3, {}_f 0.7)\rangle \, \forall_n x \in \mathbb{X}\}$,

Then ${}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ but it is not a $\mathcal{NCS}$ of ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$.

$\because {}_{\mathcal{N}}cl({}_{\mathcal{N}}E) = {}_{\mathcal{N}}\overline{M}$ which is not equal to ${}_{\mathcal{N}}E$.

Theorem 3.6. *In ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, every $\mathcal{N}_{\alpha}$- $\mathcal{CS}$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$.*

Proof. *Let ${}_{\mathcal{N}}E \subseteq \mathcal{V}$, where $\mathcal{V}$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ in $\mathbb{X}$.*

$\because {}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) = {}_{\mathcal{N}}E$. But ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$.

$\therefore {}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$.

The converse is not true.

Example 3.7. From the example 3.4, the ${}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ but it is not a $\mathcal{N}_{\alpha}$- $\mathcal{CS}$ of ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$. $\because {}_{\mathcal{N}}cl({}_{\mathcal{N}}int({}_{\mathcal{N}}cl(E))) = {}_{\mathcal{N}}\overline{M}$ which is not equal to ${}_{\mathcal{N}}E$.

Theorem 3.8. *In ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, every $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$.*

Proof. *Let ${}_{\mathcal{N}}E \subseteq \mathcal{V}$, where $\mathcal{V}$ is $\mathcal{N}_{*g\alpha}$- $\mathcal{OS}$ in $\mathbb{X}$.*

$\because$ Every $\mathcal{N}_{*g\alpha}$- $\mathcal{OS}$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$, $\mathcal{V}$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$.

$\because {}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$. But ${}_{\mathcal{N}b}cl({}_{\mathcal{N}}E) \subseteq_{\mathcal{N}\alpha} cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$, which implies

${}_{\mathcal{N}b}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$. $\therefore {}_{\mathcal{N}}E$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$.

The converse is not true.

Example 3.9. Let $\mathbb{X} = \{{}_na, {}_nb, {}_nc\}$ and the $\mathcal{NS}$, ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}L = \{\langle {}_nx, ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.2, {}_f0.7)\rangle \, \forall_n x \in \mathbb{X}\}$,

${}_{\mathcal{N}}M = \{\langle {}_nx, ({}_t0.7, {}_i0.3, {}_f0.5), ({}_t0.7, {}_i0.3, {}_f0.6), ({}_t0.7, {}_i0.2, {}_f0.4)\rangle \, \forall_n x \in \mathbb{X}\}$,

Then $\mathcal{NT}$, $\tau = \{{}_{\mathcal{N}}0, {}_{\mathcal{N}}L, {}_{\mathcal{N}}M, {}_{\mathcal{N}}1\}$ and the complement of $\mathcal{NS}$ of ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}\overline{L} = \{\langle {}_nx, ({}_t0.7, {}_i0.7, {}_f0.6), ({}_t0.7, {}_i0.7, {}_f0.5), ({}_t0.7, {}_i0.8, {}_f0.5)\rangle \, \forall_n x \in \mathbb{X}\}$,

${}_{\mathcal{N}}\overline{M} = \{\langle {}_nx, ({}_t0.5, {}_i0.7, {}_f0.7), ({}_t0.6, {}_i0.7, {}_f0.7), ({}_t0.4, {}_i0.8, {}_f0.7)\rangle \, \forall_n x \in \mathbb{X}\}$ and

${}_{\mathcal{N}}N = \{\langle {}_nx, ({}_t0.7, {}_i0.7, {}_f0.6), ({}_t0.7, {}_i0.7, {}_f0.6), ({}_t0.7, {}_i0.3, {}_f0.5)\rangle \, \forall_n x \in \mathbb{X}\}$

If ${}_{\mathcal{N}}E = \{\langle {}_nx, ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.7, {}_f0.7)\rangle \, \forall_n x \in \mathbb{X}\}$

Then ${}_{\mathcal{N}}E$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$ but it is not a $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ of ${}_{\mathcal{NTS}}(\mathbb{X}, \tau)$.

$\because {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) = \{\langle {}_nx, ({}_t0.7, {}_i0.3, {}_f0.6), ({}_t0.7, {}_i0.3, {}_f0.5), ({}_t0.7, {}_i0.7, {}_f0.5)\rangle \, \forall_n x \in \mathbb{X}\}$ which is not contained in ${}_{\mathcal{N}}N$.

Theorem 3.10. *In* ${}_{\mathcal{NTS}}(\mathbb{X}, \tau)$*, every* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *is* $\mathcal{N}_{gs}$- $\mathcal{CS}$.

Proof. *Let* ${}_{\mathcal{N}}E \subseteq \mathcal{V}$*, where* $\mathcal{V}$ *is* $\mathcal{NOS}$ *in* $\mathbb{X}$.

$\because$ *Every* $\mathcal{NOS}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$, $\mathcal{V}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$.

$\because {}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$. *But* ${}_{\mathcal{N}s}cl({}_{\mathcal{N}}E) \subseteq {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$*, which implies* ${}_{\mathcal{N}s}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$. $\therefore {}_{\mathcal{N}}E$ *is* $\mathcal{N}_{gs}$- $\mathcal{CS}$.

The converse is not true.

Example 3.11. Let $\mathbb{X} = \{{}_na, {}_nb, {}_nc\}$ and the $\mathcal{NS}$, ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}L = \{\langle {}_nx, ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.2, {}_f0.7)\rangle \, \forall_n x \in \mathbb{X}\}$,

${}_{\mathcal{N}}M = \{\langle {}_nx, ({}_t0.7, {}_i0.3, {}_f0.5), ({}_t0.7, {}_i0.3, {}_f0.6), ({}_t0.7, {}_i0.7, {}_f0.4)\rangle \, \forall_n x \in \mathbb{X}\}$,

Then $\mathcal{NT}$, $\tau = \{{}_{\mathcal{N}}0, {}_{\mathcal{N}}L, {}_{\mathcal{N}}M, {}_{\mathcal{N}}1\}$ and the complement of $\mathcal{NS}$ of ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}\overline{L} = \{\langle {}_nx, ({}_t0.7, {}_i0.7, {}_f0.6), ({}_t0.7, {}_i0.7, {}_f0.5), ({}_t0.7, {}_i0.8, {}_f0.5)\rangle \, \forall_n x \in \mathbb{X}\}$,

${}_{\mathcal{N}}\overline{M} = \{\langle {}_nx, ({}_t0.5, {}_i0.7, {}_f0.7), ({}_t0.6, {}_i0.7, {}_f0.7), ({}_t0.4, {}_i0.3, {}_f0.7)\rangle \, \forall_n x \in \mathbb{X}\}$ and

${}_{\mathcal{N}}N = \{\langle {}_nx, ({}_t0.7, {}_i0.7, {}_f0.6), ({}_t0.7, {}_i0.7, {}_f0.6), ({}_t0.7, {}_i0.3, {}_f0.5)\rangle \, \forall_n x \in \mathbb{X}\}$

If ${}_{\mathcal{N}}E = \{\langle {}_nx, ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.6, {}_i0.3, {}_f0.7), ({}_t0.5, {}_i0.7, {}_f0.7)\rangle \, \forall_n x \in \mathbb{X}\}$

Here ${}_{\mathcal{N}}E$ is $\mathcal{N}_{gs}$- $\mathcal{CS}$ but it is not a $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ of ${}_{\mathcal{NTS}}(\mathbb{X}, \tau)$.

$\because {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) = \{\langle {}_nx, ({}_t0.7, {}_i0.3, {}_f0.6), ({}_t0.7, {}_i0.3, {}_f0.5), ({}_t0.7, {}_i0.7, {}_f0.5)\rangle \, \forall_n x \in \mathbb{X}\}$ which is not contained in ${}_{\mathcal{N}}N$.

Theorem 3.12. *In* ${}_{\mathcal{NTS}}(\mathbb{X}, \tau)$*, every* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *is* $\mathcal{N}_{gp}$- $\mathcal{CS}$.

Proof. *Let* ${}_{\mathcal{N}}E \subseteq \mathcal{V}$*, where* $\mathcal{V}$ *is* $\mathcal{NOS}$ *in* $\mathbb{X}$.

$\because$ *Every* $\mathcal{NOS}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$, $\mathcal{V}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$. $\because {}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$. *But* ${}_{\mathcal{N}p}cl({}_{\mathcal{N}}E) \subseteq {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$*, which implies* ${}_{\mathcal{N}p}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$.

$\therefore\ {}_{\mathcal{N}}E$ *is* $\mathcal{N}_{gp}$- $\mathcal{CS}$.

The converse is not true.

Example 3.13. Let $\mathbb{X} = \{_na, _nb, _nc\}$ and the $\mathcal{NS}$, ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}L = \{\langle _nx, (_t0.5, _i0.3, _f0.7), (_t0.4, _i0.3, _f0.7), (_t0.5, _i0.2, _f0.7)\rangle\ \forall _nx \in \mathbb{X}\}$,

${}_{\mathcal{N}}M = \{\langle _nx, (_t0.7, _i0.3, _f0.5), (_t0.7, _i0.3, _f0.6), (_t0.7, _i0.2, _f0.5)\rangle\ \forall _nx \in \mathbb{X}\}$,

Then the $\mathcal{NT}$, $\tau = \{_{\mathcal{N}}0, _{\mathcal{N}}L, _{\mathcal{N}}M, _{\mathcal{N}}1\}$ and the complement of $\mathcal{NS}$ of ${}_{\mathcal{N}}L$ and ${}_{\mathcal{N}}M$ are defined as

${}_{\mathcal{N}}\overline{L} = \{\langle _nx, (_t0.7, _i0.7, _f0.5), (_t0.7, _i0.7, _f0.4), (_t0.7, _i0.8, _f0.5)\rangle\ \forall _nx \in \mathbb{X}\}$,

${}_{\mathcal{N}}\overline{M} = \{\langle _nx, (_t0.5, _i0.7, _f0.7), (_t0.6, _i0.7, _f0.7), (_t0.5, _i0.8, _f0.7)\rangle\ \forall _nx \in \mathbb{X}\}$ and

${}_{\mathcal{N}}N = \{\langle _nx, (_t0.4, _i0.8, _f0.7), (_t0.6, _i0.5, _f0.7), (_t0.5, _i0.9, _f0.7)\rangle\ \forall _nx \in \mathbb{X}\}$

If ${}_{\mathcal{N}}E = \{\langle _nx, (_t0.5, _i0.3, _f0.7), (_t0.6, _i0.3, _f0.7), (_t0.5, _i0.2, _f0.7)\rangle\ \forall _nx \in \mathbb{X}\}$

Here ${}_{\mathcal{N}}E$ is $\mathcal{N}_{gp}$- $\mathcal{CS}$ but it is not a $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ of ${}_{\mathcal{NTS}}(\mathbb{X}, \tau)$.

$\because\ {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) = {}_{\mathcal{N}}\overline{M}$ which is not contained in ${}_{\mathcal{N}}N$.

Theorem 3.14. *The union of any two* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$ *is also a* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$.

Proof. *Let* ${}_{\mathcal{N}}E$ *and* ${}_{\mathcal{N}}F$ *be two* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$. *Let* $\mathcal{V}$ *be a* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ *in* $\mathbb{X}$ *s.t* ${}_{\mathcal{N}}E \subseteq \mathcal{V}$ *and* ${}_{\mathcal{N}}F \subseteq \mathcal{V}$. *Then,* ${}_{\mathcal{N}}E \cup {}_{\mathcal{N}}F \subseteq \mathcal{V}$.

$\because\ {}_{\mathcal{N}}E$ *and* ${}_{\mathcal{N}}F$ *are* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$, *implies* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$ *and* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}F) \subseteq \mathcal{V}$. *Now,* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E \cup {}_{\mathcal{N}}F) = {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \cup {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}F) \subseteq \mathcal{V}$. *Thus,* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E \cup {}_{\mathcal{N}}F) \subseteq \mathcal{V}$ *whenever* ${}_{\mathcal{N}}E \cup {}_{\mathcal{N}}F \subseteq \mathcal{V}$, $\mathcal{V}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ *in* $(\mathbb{X}, \tau)$ *implies* ${}_{\mathcal{N}}E \cup {}_{\mathcal{N}}F$ *is a* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$.

Theorem 3.15. *The intersection of any two* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$ *is also a* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$.

Proof. *Let* ${}_{\mathcal{N}}E$ *and* ${}_{\mathcal{N}}F$ *be two* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$. *Let* $\mathcal{V}$ *be a* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ *in* $(\mathbb{X}, \tau)$ *s.t* ${}_{\mathcal{N}}E \subseteq \mathcal{V}$ *and* ${}_{\mathcal{N}}F \subseteq \mathcal{V}$. *Then,* ${}_{\mathcal{N}}E \cap {}_{\mathcal{N}}F \subseteq \mathcal{V}$. $\because\ {}_{\mathcal{N}}E$ *and* ${}_{\mathcal{N}}F$ *are* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$, *implies* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$ *and* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}F) \subseteq \mathcal{V}$. *Now,* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E \cap {}_{\mathcal{N}}F) = {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \cap {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}F) \subseteq \mathcal{V}$. *Thus,*

${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E \cap {}_{\mathcal{N}}F) \subseteq \mathcal{V}$ *whenever* ${}_{\mathcal{N}}E \cap {}_{\mathcal{N}}F \subseteq \mathcal{V}$, $\mathcal{V}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ *in* $(\mathbb{X}, \tau)$ *implies* ${}_{\mathcal{N}}E \cap {}_{\mathcal{N}}F$ *is a* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$.

Theorem 3.16. *Let* ${}_{\mathcal{N}}E$ *be a* $\mathcal{N}_{\alpha b^*g\alpha}$*-closed subset of* $(\mathbb{X}, \tau)$. *If* ${}_{\mathcal{N}}E \subseteq {}_{\mathcal{N}}F \subseteq {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E)$, *then* ${}_{\mathcal{N}}F$ *is also a* $\mathcal{N}_{\alpha b^*g\alpha}$*-closed subset of* $(\mathbb{X}, \tau)$.

Proof. *Let* ${}_{\mathcal{N}}F \subseteq \mathcal{V}$, *where* $\mathcal{V}$ *is* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ *in* $(\mathbb{X}, \tau)$. *Then* ${}_{\mathcal{N}}E \subseteq {}_{\mathcal{N}}F$ *implies* ${}_{\mathcal{N}}E \subseteq \mathcal{V}$. $\because\ {}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$, ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) \subseteq \mathcal{V}$. *Also* ${}_{\mathcal{N}}F \subseteq {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E)$ *implies* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}F) \subseteq {}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E)$. *Thus,* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}F) \subseteq \mathcal{V}$ *and* ${}_{\mathcal{N}}F$ *is* $\mathcal{N}\alpha b^*g\alpha$- $\mathcal{CS}$.

Theorem 3.17. *If a set* ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ *in* $(\mathbb{X}, \tau)$ *iff* ${}_{\mathcal{N}\alpha}cl({}_{\mathcal{N}}E) - {}_{\mathcal{N}}E$ *contains no non-empty* $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$.

Proof. Necessity: *Let $_{\mathcal{N}}F$ be a $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$ in $(\mathbb{X}, \tau)$ such that $_{\mathcal{N}}F \subseteq {}_{\mathcal{N}\alpha}cl(_{\mathcal{N}}E) - {}_{\mathcal{N}}E$. Then $_{\mathcal{N}}F \subseteq \mathbb{X}- {}_{\mathcal{N}}E$. This implies $_{\mathcal{N}}E \subseteq \mathbb{X}- {}_{\mathcal{N}}F$. Now $\mathbb{X}- {}_{\mathcal{N}}F$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$ of $(\mathbb{X}, \tau)$ such that $_{\mathcal{N}}E \subseteq \mathbb{X}- {}_{\mathcal{N}}F$. $\because {}_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$ then $_{\mathcal{N}\alpha}cl(_{\mathcal{N}}E) \subseteq \mathbb{X}- {}_{\mathcal{N}}F$. Thus $_{\mathcal{N}}F \subseteq \mathbb{X}- {}_{\mathcal{N}\alpha}cl(_{\mathcal{N}}E)$. Now $_{\mathcal{N}}F \subseteq_{\mathcal{N}\alpha} cl(_{\mathcal{N}}E)$ $(\mathbb{X} -_{\mathcal{N}\alpha} cl(_{\mathcal{N}}E)) = {}_{\mathcal{N}}0$.*

Sufficiency: *Assume $_{\mathcal{N}\alpha}cl(_{\mathcal{N}}E) - {}_{\mathcal{N}}E$ contains no non-empty $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$. Let $_{\mathcal{N}}E \subseteq \mathcal{V}$, $\mathcal{V}$ is $\mathcal{N}_{b^*g\alpha}$- $\mathcal{OS}$. Suppose $_{\mathcal{N}\alpha}cl(_{\mathcal{N}}E) \nsubseteq \mathcal{V}$, then $_{\mathcal{N}\alpha}cl(E) \cap \mathcal{V}^c$ is a non-empty $\mathcal{N}_{b^*g\alpha}$- $\mathcal{CS}$ of $_{\mathcal{N}\alpha}cl(E) - {}_{\mathcal{N}}E$, which is a contradiction.*

$\therefore N_{\alpha}cl(_{\mathcal{N}}E) \subseteq \mathcal{V}$. Hence $_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- $\mathcal{CS}$.

4. $\mathcal{N}_{\alpha b^*g\alpha}$-Border

Definition 4.1. For any subset $_{\mathcal{N}}E$ of $\mathbb{X}$, the neutrosophic $\alpha b^*g\alpha$-border of $_{\mathcal{N}}E$ is defined by $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = {}_{\mathcal{N}}E \setminus \mathcal{N}_{\alpha b^*g\alpha}\text{-}\ {}_{\mathcal{N}}int(_{\mathcal{N}}E)$

Theorem 4.2. *In $_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, for any subset $_{\mathcal{N}}E$ of $\mathbb{X}$, the following statements are hold.*

(i) $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(\phi)] = N_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(X)] = \phi$

(ii) $_{\mathcal{N}}E = \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E) \cup \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$

(iii) $\mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E) \cap \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = \phi$

(iv) $\mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E) = {}_{\mathcal{N}}E \setminus \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$

(v) $\mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]) = \phi$

(vi) $_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$-open iff $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = \phi$

(vii) $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(\mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E))] = \phi$

(viii) $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)])] = \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$

(ix) $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = {}_{\mathcal{N}}E \cap \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}cl(X \setminus {}_{\mathcal{N}}E)$

Proof. *Statements (i) to (iv) are obvious by the definition of $\mathcal{N}_{\alpha b^*g\alpha}$-border of $_{\mathcal{N}}E$. If possible, let $_{n}x \in \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)])$.*

Then $_{n}x \in \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$, since $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] \subseteq {}_{\mathcal{N}}E$, $_{n}x \in \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]) \subseteq \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E)$.

$\therefore {}_{n}x \in \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E) \cap \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$, which is the contradiction to (iii). Hence (v) is proved. $_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$-open iff $\mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E) = {}_{\mathcal{N}}E$. But $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = {}_{\mathcal{N}}E \setminus \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E)$ implies $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = \phi$.

This proves (vi) & (vii). When $_{\mathcal{N}}E = \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$, then definition of $\mathcal{N}_{\alpha b^*g\alpha}$-border of $_{\mathcal{N}}E$ becomes $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)])] =$

$\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] \setminus \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)])$.

By using (vii), we get the proof of (viii).

Now, $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = {}_{\mathcal{N}}E \setminus \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E) =$

$_{\mathcal{N}}E \cap (\mathbb{X} \setminus \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}int(_{\mathcal{N}}E)) = {}_{\mathcal{N}}E \cap \mathcal{N}_{\alpha b^*g\alpha} - {}_{\mathcal{N}}cl(\mathbb{X} \setminus {}_{\mathcal{N}}E)$.

5. $\mathcal{N}_{\alpha b^* g\alpha}$-Frontier

Definition 5.1. For any subset ${}_{\mathcal{N}}E$ of $\mathbb{X}$, the neutrosophic $\alpha b^* g\alpha$-frontier of ${}_{\mathcal{N}}E$ is defined by
$\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] = \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}cl({}_{\mathcal{N}}E) \setminus \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E)$

Theorem 5.2. *In ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, for any subset ${}_{\mathcal{N}}E$ of $\mathbb{X}$, the following statements are hold.*
(i) $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(\phi)] = \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(X)] = \phi$
(ii) $\mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E) \cap \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] = \phi$
(iii) $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] \subseteq \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}cl({}_{\mathcal{N}}E)$
(iv) $\mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E) \cup \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] = \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}cl({}_{\mathcal{N}}E)$
(v) $\mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E) = {}_{\mathcal{N}}E \setminus \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$
(vi) If ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-closed iff* ${}_{\mathcal{N}}E = \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E) \cup \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$
(vii) ${}_{\mathcal{N}}fr({}_{\mathcal{N}}E) = {}_{\mathcal{N}}fr(\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)])$
(viii) If ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-open, then*
${}_{\mathcal{N}}E \cap \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] = \phi$
(ix) $\mathbb{X} = \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}cl({}_{\mathcal{N}}E) \cup \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}cl(\mathbb{X} \setminus {}_{\mathcal{N}}E)$
(x) If ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-open, then*
$\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(N_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E))] \subseteq \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$
(xi) If ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-closed, then*
$\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(N_{\alpha b^* g\alpha} - {}_{\mathcal{N}}cl({}_{\mathcal{N}}E))] \subseteq \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$
(xii) If ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-open iff then*
$\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(N_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E))] \cap \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E) = \phi$

Proof. *Statements (i) to (vii) are true by the definition of* $\mathcal{N}_{\alpha b^* g\alpha}$*- frontier of* ${}_{\mathcal{N}}E$*. By Remark (3.3), If* ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-open,* ${}_{\mathcal{N}}E = \mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E)$ *and by statement (ii),* ${}_{\mathcal{N}}E \cap \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] = \phi$*. Hence (viii) is proved. (ix) is obvious. Since* $\mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E)$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*-open, then* $\mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E) = {}_{\mathcal{N}}E$*, which implies* $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(\mathcal{N}_{\alpha b^* g\alpha} - {}_{\mathcal{N}}int({}_{\mathcal{N}}E))] \subseteq \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$*. Similarly, (xi) can be proved. By Remark (3.3) and by statement (ii), (xii) is straight forward.*

6. Relationship Between $\mathcal{N}_{\alpha b^* g\alpha}$-Frontier and $\mathcal{N}_{\alpha b^* g\alpha}$-Border

Theorem 6.1. *In ${}_{\mathcal{N}}\mathcal{TS}(\mathbb{X}, \tau)$, for any subset ${}_{\mathcal{N}}E$ of $\mathbb{X}$, the following statements are hold.*
(i) $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}bd({}_{\mathcal{N}}E)] \setminus \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] = \phi$
(ii) $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}bd({}_{\mathcal{N}}E)] \subseteq \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$
(iii) $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr(\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}bd({}_{\mathcal{N}}E)])] = \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}bd({}_{\mathcal{N}}E)]$
(iv) $\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}bd(\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)])] = \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$
(v) If ${}_{\mathcal{N}}E$ *is* $\mathcal{N}_{\alpha b^* g\alpha}$*- open, then*
$\mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)] \cup \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}bd({}_{\mathcal{N}}E)] = \mathcal{N}_{\alpha b^* g\alpha}[{}_{\mathcal{N}}fr({}_{\mathcal{N}}E)]$

(vi) $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)] \cap \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$

(vii) $\overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)]} \cup \overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]} = \overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]}$

(viii) $\overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)]} \cap \overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]} = \overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)]}$

Proof. *Statement (i) to (iv) are obvious by the definitions of $\mathcal{N}_{\alpha b^*g\alpha}$- Frontier and $\mathcal{N}_{\alpha b^*g\alpha}$-border of a set. $\because$ $_{\mathcal{N}}E$ is $\mathcal{N}_{\alpha b^*g\alpha}$- open, then we have a statement from $\mathcal{N}_{\alpha b^*g\alpha}$-border of a set, $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(E)] = \phi$, which implies $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)] \cup \phi = \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)]$. Hence (v) is proved. We know from statement (ii), $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] \subseteq \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)]$ which implies $\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)] \cap \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)] = \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]$. It gives the proof of (vi). By the above statement,*

$\overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]} = \overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)] \cap \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]}$, *and by using De Morgan's law,*

$\overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)] \cap \mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]} =$

$\overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}fr(_{\mathcal{N}}E)]} \cup \overline{\mathcal{N}_{\alpha b^*g\alpha}[_{\mathcal{N}}bd(_{\mathcal{N}}E)]}$, *it gives the proof of (vii).*

Similarly we can prove the statement (viii).

7. Conclusions

The $\mathcal{N}_{\alpha b^*g\alpha}$-closed set in $_{\mathcal{NTS}}$ was defined in this article, and its relationship to other known $_{\mathcal{NS}}$ in $_{\mathcal{NTS}}$ was examined. We also introduced and investigated the properties of $\mathcal{N}_{\alpha b^*g\alpha}$-frontier and $\mathcal{N}_{\alpha b^*g\alpha}$-border of a set. $\mathcal{N}_{\alpha b^*g\alpha}$-frontier of a set in $_{\mathcal{NTS}}$ and found to be connected. A few more functions, including $\mathcal{N}_{\alpha b^*g\alpha}$-continuous, irresolute functions, can be derived from this set. Furthermore, it can be expanded to include the homeomorphism of $_{\mathcal{NTS}}$.

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Neutrosophic Analysis on Internalization of Higher Education in Indian Perspective

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Abstract: The developing nations of the Asian continent, especially India, are trying to elevate the educational standards by inaugurating the portals of international institutional linkages. Collaboration with foreign universities provides opportunities for the aspiring students to confer degrees from world premier institutions by stationing in their host nation. At the same time, the consequential impacts of the institutional linkages beyond the contours of countries need special attention to investigate on the dimensions of social, economic and culture. This chapter aims to make an intensive study on the challenges and impacts of institutional linkages between Institutions belonging to developing and developed nations. In this research work the method of neutrosophic relational maps (NRM) is applied to analyze the association between the attributes of Institutional linkages and the consequential impacts of such a system of cross cross-cultural education with special reference to the Indian nation. The associational impacts based on the opinion of the educational experts are well examined using neutrosophic representations. Based on the findings of this research work, some of the suggestive measures of internalizing higher education and future directions are proposed in this chapter.

Keywords: Neutrosophic sets, Higher Education, Internalization

1. Introduction

Every nation is vesting time in planning for quality of education as it is embedded with the potency of persuading the thought process of mankind. A nation shall contribute to the quality enrichment of education by establishing quality culture in every higher educational institution. It is also possible by empowering the student community with employability potentials both at local and global levels. The educationalists feel that the students must acquire a global exposure and at the same time they also agree that it is not possible to provide such opportunities to all the students. But internalization of higher education can make this happen by dissolving the discernments.

Internalization of higher education refers to the institutional collaboration across the nations to offer twin degree programmes. The primary goal of internalization is to create opportunities for the students to confer their degrees from an international institution without migrating. Generally students migrate to other nations for pursuing professional programmes from their residing countries. Some of the students with good economic backgrounds and recipients of merit scholarships are able to grab such opportunities at ease. But a very large number of aspiring students especially belonging to developing nations like India are disappointed with their unfulfilled dream of education abroad mainly due to the reasons of poor economy, partial scholarships, less sponsorships and limited seats. Internalization of education will definitely facilitate the overcoming of such obstructions by enabling the foreign universities to establish their campuses in the Indian nation. Internalization of higher education in India will alleviate financial burdens, migration problems and at the same time will surge the chances of global exposures. On the other hand, the Indian nation has to make an intense plan on the initiatives to promote the institutional linkages with the foreign universities. Simultaneously the impacts of internalization must also be studied in terms of political, economic, social and cultural dimensions. This chapter studies the relational impacts between the Indian institutional initiatives in strengthening the internalization of higher education and its impacts. As the study involves several factors associated with two different entities, the concept of neutrosophic relational maps is used with linguistic representation of neutrosophic sets.

Vasantha and Smarandache [5] introduced the notion of Neutrosophic relational maps to study the relational impacts of the factors associated with HIV infected women. The neutrosophic representations are highly compatible and comprehensible in making the relational representations. The NRM is applied only to a few decision making problems. Devadoss and Ismail [2] applied to NRM to make a study on the Islamic religious practices. Devadoss and Felix [3] used induced linked NRM to explore the impacts of emotions and personality on physical health. Gaurav, Bhutani, Aggarwal [4] used NRM in studying poverty problems. Savarimuthu and Yuvageswary [6] applied induced neutrosophic cognitive relational maps in studying the acquaintances of rag-pickers. The literature on the applications of NRM is very limited and it has not been applied to study any kinds of problems associated with education. This has influenced the authors to choose this method to study the interrelational impacts. The decision making problem chosen in this chapter has not been explored by the researchers in a mathematical sense. De and Altbach [1] have deliberated on the global trends of internalization. The educational researchers have presented only theoretical descriptions but not have made any kind of analysis by considering any specific case. As the phenomenon of internalization of higher education has recently emerged in the Indian system of education, it is essential to undertake such analytical research.

The contents of this chapter are structured as follows: Section 2 presents the steps involved in NRM. Section 3 describes the factors associated with the decision making problem. Section 4 presents the numerical computations, Section 5 discusses the results with future directions and conclusions.

MATHEMATICAL BACKGROUND

This section presents a brief presentation of the preliminaries of fuzzy and neutrosophic sets.

The theory of Fuzzy sets is developed by Loft.A.Zadeh as an extension of conventional sets.

A *fuzzy set* A of the universal set X is defined as the collection of all ordered pairs of the form $\widetilde{\boldsymbol{A}} = \{(\boldsymbol{x}, \boldsymbol{A}(\boldsymbol{x}), \forall\, \boldsymbol{x} \in \boldsymbol{X}\}$, where A(x) : X → [0,1].

A *fuzzy number* is basically a fuzzy set with the membership function defined from R → [0,1] and it is expected to satisfy the following properties

- Alpha-cut must be a closed interval
- Fuzzy set must be normal

- Support is bounded

The fuzzy sets are extended to neutrosophic sets with the inclusion of the concept of indeterminancy. Smarandache constructed the neutrosophic sets with three components.

A neutrosophic set N defined on universal set X is of the form (T_N, I_N, F_N) where T_N is the truth membership value, I_N is the indeterminate membership value and F_N is the false membership value. Also $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$, where $T_N(x)$, $I_N(x)$, $F_N(x)$: $X \rightarrow [0,1]$.

The *arithmetic operations* between neutrosophic sets and fuzzy sets shall be performed based on component wise.

The fuzzy representations and neutrosophic representations shall be converted into crisp value using the processes of *Defuzzification* and *Deneutrosophication*.

2. Methodology

This section presents the steps involved in neutrosophic relational maps.

Step 1 Decide on the two entities say I and II of decision making

Step 2: Collect the factors associated with the two entities say A1,A2,A3,…,.,An and B1,B2,B3,…, Bm.

Step 3 : Construct the relational matrix R comprising the relational impacts between the factors of the entities I and II of the below form. The values in each cell is of neutrosophic form with truth, indeterminate and falsity values are represented using linguistic variables. The linguistic variables are quantified using fuzzy numbers.

	B1	B2	……	Bm
A1	N11	N12	…….	N1m
A2	N21	N22	…….	N2m
. . . .	. . . .	. . . .	. . . .	. . . .
An	Nn1	Nn2	…….	Nmn

Step 4: The vector of the form (1 0 0 0..0) is multiplied with R and the resulting vector of the form (a1,a2,….ag) is updated using the following threshold values $\begin{cases} -1 \; if \; a_k < 0 \\ 0 \; if \; a_k = 0 \\ 1 \; if \; a_k > 0 \end{cases}$

Step 5 : The updated vector is multiplied with R^T and the resulting vector is again updated.

Step 6 : The above steps are repeated until the updated vectors in Step 4 and 5 remain alike.

DESCRIPTION OF THE DECISION MAKING PROBLEM

This section presents the decision making problem on studying the interrelational impacts between the two entities namely Initiatives to promote Indian Institutional Linkages with Foreign Universities and Consequences of Internationalizing Higher Education. The factors associated with these two entities are described as follows.

Factors associated with the first entity of Initiatives to promote Indian Institutional Linkages with Foreign Universities

The factors under this entity discuss the steps or the action of course that a developing nation like India that has to take up in facilitating the institutional linkages with other nations abroad.

I_1: Infrastructure Augmentation

If the foreign universities are invited to inaugurate their centres in Indian institutions, i.e the host institution, then essentially it the responsibility of the hosting institution to augment the infrastructure. The programmes offered in collaboration with the foreign universities certainly require well advanced research labs, well defined high tech classrooms. Hence it is necessarily very essential to make a wide range of modifications suiting the programmes, faculty and students.

I_2: Faculty exchange programme for adherence to multicultural competency

In the system of internalization of higher education, one of the most vital aspects to be focussed on is the faculty empowerment. The exchange of faculty between the host and the foreign institutions is a part of the system. The cultural awareness of both the nations is equally significant to knowledge gaining and sharing.

I_3: Planning of comprehensive curriculum

If the institutions are in consensus in beginning combined programmes then framing of curriculum is the next immediate step. The curriculum structure and the modalities of implementation must be designed with utmost care. The planning must provide space for flexibility to make learning more compatible.

I_4: Strengthening the portals of alumni networking

Alumni networking has to be enriched by the host institution to receive their physical and financial support in running twin programmes more effectively. The competent alumni who are more suitable shall be rightly channelized for the successful run of such programmes.

I_5: Creating conducive learning environment

India is basically a land of multilingual and multicultural ethnicities. An institution generally embodies students of diverse backgrounds and after the onset of internationalization, the range of diversity gets intensified. With the backdrop of such diversifications, the task of providing unified learning is more challenging. To do so, creating a conducive learning environment is a prerequisite.

I_6: Rapport building programmes for acquiring global citizenship

Internationalization has come into effect with the objective of providing global exposures to the students. To accomplish this objective with ease, several training programmes have to be organized to discuss the aspects of global citizenship.

I_7: Exposures to global level programmes

The faculty and students of the host institution must be provided with the opportunities of familiarizing themselves with the programmes at global level. As internationalization offers education beyond the borders the faculty and the students must be trained to prepare themselves for outsmarting and mastering at global levels.

I_8: Educational support through scholarships

Internationalization is a boon to the students aspiring for abroad education. But still this twin programme system costs high in comparison to the existing higher educational programmes. The host nation must decide on new scholarship schemes to render monetary support to the students belonging to the marginalized. As the quota system is followed in Indian nation, the allocation of funds for promoting such programmes must be rightly channelized to avoid disputes on community discernments.

I_9: Enhancement of capacity through faculty development programmes

In the system of twin programmes, the faculty are expected to play a key role as they have to set a balance between students of two different learning cultures. The faculty must be empowered with the recent advancements and developments of the concepts and technology. The newly emerging techniques must be imparted to them. Adding to it the faculty must be well equipped with new skill sets and robust pedagogy.

I_{10}: International education at affordable costs

The host nation must make essential preparations in providing international education at an optimum cost. The Indian institutions must not consider it as a chance of earning money by causing financial burdens to the students. The aspirations must not be made as an investment of building institutional profit. The host nation must look into such affairs more seriously and the institutions with service motive and good standards must be given the opportunities for international collaborations.

I_{11}: Diverse programmes with need based courses

The programmes designed must be diverse and novel in nature. The twin programmes must be distinct with more viable courses. The planning of such programmes is very essential to make internationalization of education more meaningful.

Factors associated with the second entity of Consequences of Internationalizing Higher Education

The second entity deals with the consequences of internationalizing higher education. Both the positive and negative factors associated with the consequences are sketched out as follows

C_1: Quality learning

One of the positive impacts of internationalization of higher education is quality enhancement. There is no space of uncertainty on quality progression.

C_2: Transition towards global system of education

The international exposure, designing of educational programmes at global level will certainly elevate the existing system of Indian education to global standards. The nation will experience a drastic change after the effective implementation of these twin programmes.

C_3: Overseas employment opportunities

The probability of bagging employment opportunities for the students outside the host nation is very high. There are no territorial confinements for the students who have completed their twin programmes.

C_4: Deficit opportunities for fulfilling the local and national needs

This is one of the consequences with less positive impact to the host nation. Education is meant to fulfil the needs of the nation and it is directly and indirectly connected with the growth of the nation. The twin programmes are aiming to fulfil the global needs but do it have any space or ideas for accomplishing the local or the regional needs? Certainly the space is low and hence this aspect has to be explored.

C_5: High possibilities of global citizenship

The students after completing such programmes will definitely emerge as global citizens. It is indeed a great pride to the hosting nations for creating such a generation with global citizenship.

C_6: Cultural disintegration by westernization of education

This is another consequence with very less positive impacts. The entry of IT companies in the Indian nation has invaded the holistic traditional belief systems. The western culture has distorted the heritage of the Indian value system. Beside profit and economic growth, Indian society is also witnessing such cultural distortions. Will such haphazard be staged in educational institutions after the onset of internationalization? Exchange of western ideas is inevitable but channelization may contribute towards development.

C_7: Challenges of evaluation and assessment

The pattern of evaluation and assessment varies from nation to nation. The written mode of examinations is highly preferred in Indian nations but it is not so in other nations. The existence of certain conflicts in making assessments must be resolved and this may be one of the negative impacts of internationalization.

C_8: Low aspiration level of contributing to the host nation

The students after receiving global exposures may deviate from the attitude of rendering service to their host nation. The prosperity and a good lifestyle in other nations may influence them and make them refrain from their bondage to their mother country

C9: Chances of social disparities caused by quota system

The policies on internationalization must be well delineated to avoid any kind of communal rivalries on preferential ratings to different communities. As the policies are not yet well articulated, the probabilities of social disparities are high and the chances of settling it amicably are scanty.

C10: Adaptation challenges to the new learning environment

The students who have enrolled for twin programmes have to make up their mind to learn the course of international standards in an Indian learning system. It is really a challenging task to attune to the customisation of the global exposure in the Indian context. This is one of the negative impacts of internationalization.

C11: Establishment of rapport with faculty and peer group

Certainly this is one of the inevitable negative consequences of any kind of new learning environment. Kicking off a new sort of relationship with faculty and peer group will be a difficult one but it gets lightened in course of time. But still there are some problems during the period of rapport.

C12: Financial burden in augmenting infrastructure to global standards

One of the negative consequences is huge financial investment both by the host nation and hosting institutions. It is naturally a huge financial burden initially laid to kick start internationalization.

NUMERICAL COMPUTATIONS

This section presents the applications of NRM with linguistic representations to the above set of entities and the associated factors. The objective of the problem is to find the relational impacts of the initiatives of promoting internationalization and the consequential impacts of internationalization. In the previous sections the factors are described but it is also equally essential to estimate the relational impacts between the factors and among the factors. One of the advantages of NRM is to find both inter and intra-relational impacts. The resulting vectors of NRM will present the intra relational impacts between the factors of entity I and the inter impacts between the factors of both entity I and II.

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
I1	(M,L,L)	(VH,L,L)	(L,L,H)	(L,L,H)	(L,L,H)	(L,L,VH)	(L,L,H)	(L,L,H)	(L,L,H)	(L,L,H)	(L,L,H)	(VH,L,L)
I2	(H,VL,L)	(H,L,L)	(L,L,H)	(VL,L,H)	(M,L,L)	(M,L,L)	(M,VL,L)	(M,VL,L)	(M,VL,L)	(M,L,L)	(H,L,L)	(M,L,L)
I3	(VH,VL,VL)	(M,L,L)	(M,L,L)	(L,L,H)	(M,L,L)	(L,L,H)	(M,L,VL)	(M,L,VL)	(M,L,VL)	(M,L,VL)	(L,VL,H)	(M,L,L)

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12
I1	-1	-0.58	-1.2	-1.2	-1.2	-1.4	-1.2	-1.2	-1.2	-1.2	-1.2	-0.58
I2	0.09	-0.75	-1.2	-1.67	-1	-1	-0.16	-0.16	-0.16	-1	-0.75	-1
I3	0.68	-1	-1	-1.2	-1	-1.2	-0.58	-0.58	-0.58	-0.58	-0.41	-1
I4	-1.25	-0.75	-0.75	-1.62	-1	-1.2	-1	-0.58	-0.58	-0.41	-1	-1.25
I5	0.68	-0.75	-1	-1.2	-1.25	-1.25	-0.41	-0.41	-0.41	-1.67	-1	-0.41
I6	-1	-1	-1	-0.925	0.68	0.26	0.68	-0.58	-0.58	-0.58	-0.75	-1.25
I7	-1	-0.58	-0.75	-0.33	0.68	-0.58	0.68	-0.58	-0.58	-0.75	-0.16	-1.25
I8	-1	-1	-1.2	-1.2	-1.2	-1.2	-1.67	-1.67	-1.67	-1.25	-1.25	-0.75
I9	-1	-1.2	-1.2	-1.2	-1.2	-1.42	-1.25	-1.67	-0.75	-1.2	-1.2	-1
I10	-1	-1.2	-1.2	-0.41	-1	-1.2	-0.58	-0.58	-0.58	-0.75	-1	-0.58
I11	-0.75	-0.75	-0.58	-1.25	-0.16	-0.41	-0.16	-1.67	-1.67	-0.16	-0.41	-0.41

By considering the above modified relational matrix, the NRM is applied and the resultant vectors thus obtained are presented in Table 3.2

Table 3.2 Resultant Vectors of NRM

Factors in ON Position	Resultant Vectors
(10000000000)	(11111111111) (111111111111)
(01000000000)	(11111111111) (111111111111)
(00100000000)	(11111111111) (111111111111)
(00010000000)	(11111111111) (111111111111)

(00001000000)	(11111111111) (111111111111)
(00000100000)	(11111111111) (111111111111)
(00000010000)	(11111111111) (111111111111)
(00000001000)	(11111111111) (111111111111)
(00000000100)	(11111111111) (111111111111)
(00000000010)	(11111111111) (111111111111)
(00000000001)	(11111111111) (111111111111)

3. Discussion

The table 3.2 clearly states that each of the initiatives has a positive impact and hence these factors shall be considered in making decisions on internalization of higher education. As internalization is becoming an integral part of the modern educational system, the developing nations must be cautious in handling both the planning part and treating the consequences. Though some of the factors are having negative impacts, they do exist for a very short span of time. For instance in the entity II, the consequential factors C4, C6-C12 do have negative impacts but then the negativity shall be mitigated in course of time by fostering positivity in the minds of the students. The formulation of policies to weed out the commotion causatives will align the internationalization of higher education in the right order.

The instance of taking the Indian nation as a developing nation is mostly apt as it is one of the nations filled with the aspirations of overseas education and employment. Adding to it, this nation is also equally preferred by people of other nationalities to pursue their higher education. There is a high scope of expanding the standards of education in the Indian nation and the contents of this chapter will surely supplement it.

Conclusion

This chapter discusses the status and the effects of internationalizing higher education from an Indian perspective. The method of neutrosophic relational maps with linguistic representation is applied to investigate the relational impacts of the factors associated with two different entities. The theoretical arguments on internationalization is insuffice without mathematical interference. But this research has overcome the limitation by integrating the concepts of NRM with decision making. This research work is only a beginning as it has presented the factors based on predictions and intuitions. There are a lot of opportunities to extend this research work based on the survey from the students after completion of their twin programmes. Neutrosophic statistical techniques shall also be applied to make inferences.

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On Supra Neutrosophic Multiset Topological Spaces

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Abstract: In this article, we present a novel concept of the supra neutrosophic multiset topological space. We describe the behaviour of neutrosophic sets and multiset with in this framework. Additionally, we define the supra neutrosophic open multiset (SNOMS), supra neutrosohic closed multiset (SNCMS), supra neutrosophic interior multiset, and supra neutrosophic closure multiset, and provide examples to illustrate their properties.

Keywords: Supra Neutrosophic Open Multiset, Supra Neutrosophic Closed Multiset, Supra Neutrosophic Interior Multiset, Supra Neutrosophic Closure Multiset, Supra Neutrosophic Multiset Topological sapce.

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1. Introduction

In recent years, multiset and neutrosophic sets have become a subject of great interest for researches. Mathematicans always like to solve a complicated problem in a simple way and to find out the most feasible solution. Neutrosophy has been introduced and studied by Smarandache [11] and developed the neutrsphic topological sapce for and introduced the multiset topological space their properties open bms, closed bms , closure bms, interior and their theorem and their properties are discussed in [8] [7]. The properties of neutrosophic multiset group are establised [9],[10],[3][2]. Then [6] supra topological space establised the [4] and their derived the properties and examples,[5] the pre open set in supra topological sapce. [12] a group of multiset of power whole set in multiset topological space. The application of the neutrosophic of decision making in disease diagnosis in [1].

In this paper we intoduced the new concept for the supra neutrosophic multiset topological space and their properties for supra neutrosophic open multiset, supra neutrosophic closed multiset, supra neutrosophic interior multiset, supra neutrosophic closure multiset, supra neutrosophic multiset topological sapce their discussed properties , theorem and examples.

2. Preliminaries

We define functions T, F and I from X to $[0, 1]$, where T is membership value, F fails membership value, and I is the indeterminacy value. The definition of a neutrosophic multiset was first defined by Smarandache [11] as follows:

Definition 2.1: [11] A neutrosophic multiset is a neutrosophic set where one or more elements are repeated with the same neutrosophic components, or with different neutrosophic components.

Definition 2.2: [11]The empty neutrosophic multiset is denoted by N_θ and defined by $N_\theta = \{< x, (0, 1, 1) >: \forall x \in X\}$ where x can be repeated.

Definition 2.3:[11] Let $A = \{(x, < T_A(x), I_A(x), F_A(x) >) : x \in X\}$ be a neutrosophic multiset on X. Then the complement of A is denoted by A^c and defined by
$A^c = \{(x, < F_{A(x)}, 1 - I_{A(x)}, T_{A(X)} >) : x \in X\}$. Where x can be repeated based on its multiplicity and the corresponding T, F, I values may or may not be equal.

Definition 2.4:[11] Let X be a non-empty set and neutrosophic multisets A and B in the form $A = \{(x, < T_A(x), I_A(x), F_A(x) >) : x \in X\}$ and $B = \{(x, < T_B(x), I_B(x), F_B(x) >) : x \in X\}$, then the operations of maximal union and minimal intersection of NM set relation are defined as follows:

1. $(A \cup B)_{max} = \{(x, < T_{(A\cup B)_{max}}(x), I_{(A\cup B)_{max}}(x), F_{(A\cup B)_{max}}(x) >) : x \in X\}$, where $T_{(A\cup B)max}(x) = \max\{T_A(x), T_B(x)\}, F_{(A\cup B)}(x) = \min\{F_A(x), F_B(x)\}$ and $I_{(A\cup B)})(x) = \min\{I_A(x), I_B(x)\}$.
2. $(A \cap B)_{min} = \{(x, < T_{(A\cap B)_{min}}(x), I_{(A\cap B)_{min}}(x), F_{(A\cap B)_{min}}(x) >) : x \in X\}$,where $T_{(A\cap B)min}(x) = \min\{T_A(x), T_B(x)\}, \quad F_{(A\cap B)}(x) = \max\{F_A(x), F_B(x)\}$ and $I_{(A\cup B)}(x) = \max\{I_A(x), I_B(x)\}$.

Definition 2.5:[11] Let X be neutrosophic multiset and a non-empty family $\mathcal{T}$ subsets of W_X is said to be neutrosophic multiset topological space if the following axioms hold:

(1) $N_\emptyset, W_X \in \mathcal{T}$.

(2) $A \cap B \in \mathcal{T}$, for $A, B \in \mathcal{T}$.

(3) $\bigcup_{i\in\Lambda} A_i \in \mathcal{T}$, $\forall\{A_i : i \in \Lambda\} \subseteq \mathcal{T}$.

In this case, the pair $(W_X, \mathcal{T})$ is called a neutrosophic multiset topological space (NMTS in short) and any neutrosophic multiset in $\mathcal{T}$ is known as an open neutrosophic multiset (ONMS in short) in W_X. The elements of $\mathcal{T}^c$ are called closed neutrosophic multisets, otherwise, a neutrosophic set F is closed if and only if its complement F^c is an open neutrosophic multiset.

Definition 2.6:[11] Let $(W_X, \mathcal{T}_1)$ and $(W_X, \mathcal{T}_2)$ be two neutrosophic multiset topological spaces on W_X. Then $\mathcal{T}_1$ is said to be contained in $\mathcal{T}_2$ that is if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, i.e., $A \in \mathcal{T}_2$ for each $A \in \mathcal{T}_1$. In this case, we also say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.
Definition 2.7. [8] Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topological space of M if τ satisfies the following properties.
(i) The mset M and the empty mset ϕ are in τ.
(ii) The mset union of the elements of any sub collection of τ is τ.
(iii) The mset intersection of the elements of any finite sub collection of τ is in τ".
Definition 2.8.[8] "A sub mset N of M-topological space M in $[X]^w$ is said to be closed if the mset $M \ominus N$ is open. In discrete M-topological space every mset is an open mset as well as a closed mset. In the M-topological space PF(M)$\cup\phi$, every mset is an open mset as well as a closed mset".
Definition 2.9.[8] "Given a suBMS et A of an M-topological space M in $[X]^w$, the Interior of

A is defined as the mset union of all open mset contained in A and its denoted by $Int(A)$. i.e., Int(A)=$\cup\{G \subseteq M : G$is an open mset and $G \subseteq$ A$\}$ and $C_{Int(A)}(x) = max\{C_G(x) : G \subseteq A\}$".

Definition 2.10.[8] "Given a subset A of an M-topological space M in $[X]^w$, the closure of A is defined as the mset intersection of all closed mset containing A and its denoted by $Cl(A)$. i.e., $Cl(A) = \cap\{K \subseteq M : K$is a closed mset and A $\subseteq$ K$\}$ and $C_{Cl(A)}(x) = min\{C_K(x) : A \subseteq K\}$".

Definition 2.11[6] A subfamily τ^* of X is said to be a supra topology on X if ,

(i) X, $\phi \in \tau^*$.

(ii) If $A_i \in \tau^*$ for all $i \in J$, then $\cup A_i \in \tau^*$.

(X, τ^*) is called a supra topological space. The elements of τ^* are called supra open sets in (X, τ^*) and complement of a supra open set is called a supra closed set.

Definition 2.12[6] The supra closure of a set A is denoted by supra cl(A) and defined as supra as supra cl(A)= $\cap\{B : B$ is a supra closed and $A \subseteq B\}$.

The supra interior of a set A is denoted by supra int(A), and defined as supra int(A)= $\cup\{B : B$ is a supra open and $A \supseteq B\}$.

Definition 2.13[6] Let (X, τ) be a topological space and τ^* be a supra topology on X. We call τ^* a supra topology associated with τ if $\tau \subseteq \tau^*$.

3. On Supra Neutrosophic Multiset Topological Spaces

In this section, we define supra neutrosophic multiset topological spaces and their properties are discussed. Throghout this paper we represent supra neutrosophic multiset topological space by SNMTS, neutrosophic closed multiset by SNCMS, supra neutrosophic open multiset by SNOMS,

Definition 3.1: Let X be neutrosophic multiset and a non empty family τ^*_μ neutrosophic submultiset of X is said to be supra neutrosophic multiset topological space if the following axioms hold.

(i) $0_{\mathscr{N}\mathscr{M}}, 1_{\mathscr{N}\mathscr{M}} \in \tau^*_\mu$

(ii) $\bigcup_{i\in\wedge} \mathscr{A}_i \in \tau^*_\mu$ for all $\{\mathscr{A}_i : i \in \wedge\} \in \tau^*_\mu$.

The pair $(X^*_\mu \tau\ )$ is called a supra neutrosophic multiset topological space (SNMSTS in short) and any supra neutrosophic multiset in τ^*_μ is known as supra neutrosophic open multiset (SNOMS in short) . The elements of τ^{*c}_μ are called supra neutrosophic closed multiset (SNCMS in short).

Example 3.2: Let $X = \{a, b, c\}$ and $A = \{(a, <\ 0.3, 0.4, 0.5\ >), (a, <\ 0.3, 0.4, 0.5\ >), (b, < 1, 0, 0.2 >), (b, < 1, 0, 0.2 >), (c, < 0.7, 1, 0 >)\}$ is a neutrosophic multiset.
$\tau^*_\mu = \{0_{\mathscr{N}\mathscr{M}}, 1_{\mathscr{N}\mathscr{M}}, (a, < 0.3, 0.4, 0.5 >), (b, < 1, 0, 0.2 >), (c, < 0.7, 1, 0 >)\}$ is a SNMSTS.

Definition 3.3: Let τ^*_μ be SNMSTS and let $A = \{< x, \mu_{\mathscr{N}\mathscr{M}}, \sigma_{\mathscr{N}\mathscr{M}}, \delta_{\mathscr{N}\mathscr{M}} >: x \in X\}$ be neutrosophic multiset, then the supra neutrosophic interior multiset of A is the union of all supra neutrosophic open multiset (SNOMS) of X contains in A and is defined as
$Sint_{\mathscr{N}\mathscr{M}}(A) = \{< x, \cup_{(max)}\mu_{\mathscr{N}\mathscr{M}}, \cap_{(min)}\sigma_{\mathscr{N}\mathscr{M}}, \cap_{(min)}\delta_{\mathscr{N}\mathscr{M}} >: x \in X\}$

Example 3.4: Let $X = \{a, b\}$ and $A = \{(a, <\ 0.3, 0.4, 0.5\ >), (a, <\ 0.3, 0.4, 0.5\ >), (b, < 1, 0, 0.2 >), (b, < 1, 0, 0.2 >), (c, < 0.7, 1, 0 >)\}$. Then $\tau^*_\mu = \{(a, < 0.3, 0.4, 0.5 >), (0_{\mathscr{N}\mathscr{M}}, 1_{NM}, b, < 0.2, 0.3, 0.4 >), (b, < 0.2, 0.3, 0.4 >)\}$ be a supra neutrosophic multiset topological space.
Let $B = \{a, < 0.1, 0.2, 0.3 >\}$ then $B \subseteq A$, $Sint_{\mathscr{N}\mathscr{M}}(B) = 0_{NM}$.

Theorem 3.5: Let (X, τ_μ^*) be a supra neutrosophic multiset topological space. Let A, B be two NMS on X, then the following property hold:

(i) $Sint_{\mathscr{N}\mathscr{M}}(0_{\mathscr{N}\mathscr{M}}) = 0_{\mathscr{N}\mathscr{M}}$, $Sint_{NM}(1_{\mathscr{N}\mathscr{M}}) = 1_{\mathscr{N}\mathscr{M}}$

(ii) $Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq A$.

(iii) $A \subseteq B \Rightarrow Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq Sint_{\mathscr{N}\mathscr{M}}(B)$.

Proof : (i) Since $0_{\mathscr{N}\mathscr{M}}$ and $1_{\mathscr{N}\mathscr{M}}$ are supra neutrosophic open multiset. We have
$Sint_{\mathscr{N}\mathscr{M}}(0_{\mathscr{N}\mathscr{M}}) = 0_{\mathscr{N}\mathscr{M}}$
$Sint_{\mathscr{N}\mathscr{M}}(1_{\mathscr{N}\mathscr{M}}) = 1_{NM}$
(ii) Let $x \in Sint_{\mathscr{N}\mathscr{M}}(A)$. Since x is an interior element of A, which implies that A is an neighbourhood of x. Thus $x \in A$. Hence $Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq A$.
(iii) Let $x \in Sint_{\mathscr{N}\mathscr{M}}(B)$. Since x is an interior point of A, so A is a neighbourhood of x. Since $A \subseteq B$, so B is also neighbourhood of x, which implies that $x \in Sint_{\mathscr{N}\mathscr{M}}(B)$. Thus $x \in Sint_{\mathscr{N}\mathscr{M}}(A) \Rightarrow x \in Sint_{\mathscr{N}\mathscr{M}}(B)$. Hence $A \subseteq B \Rightarrow Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq Sint_{\mathscr{N}\mathscr{M}}(B)$.

Proposition 3.6: Let (X, τ_μ^*) be a SNMTS. Then

(a) $Sint_{\mathscr{N}\mathscr{M}}(A \cap B) = Sint_{\mathscr{N}\mathscr{M}}(A) \cap Sint_{\mathscr{N}\mathscr{M}}(B)$.

(b) $Sint_{\mathscr{N}\mathscr{M}}(A) \cup Sint_{\mathscr{N}\mathscr{M}}(B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A \cup B)$.

Proof : (a) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ then we have from (iii) $A \subseteq B \Rightarrow Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq Sint_{\mathscr{N}\mathscr{M}}(B)$. $Sint_{\mathscr{N}\mathscr{M}}(A \cap B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A)$ and $Sint_{\mathscr{N}\mathscr{M}}(A \cap B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(B)$. $\Rightarrow$ $Sint_{\mathscr{N}\mathscr{M}}(A \cap B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A) \cap Sint_{\mathscr{N}\mathscr{M}}(B)$. (1)
Now, let $x \in Sint_{\mathscr{N}\mathscr{M}}(A) \cap Sint_{\mathscr{N}\mathscr{M}}(B)$. Hence x is an interior point of each of the sets A and B. It gives the A and B are neigbourhood of x, so their intersection $A \cap B$ is also a neighbourhood of x. Therefore, $x \in Sint_{\mathscr{N}\mathscr{M}}(A \cap B)$.
Let $x \in Sint_{\mathscr{N}\mathscr{M}}(A) \cap Sint_{\mathscr{N}\mathscr{M}}(B)$.
$\Rightarrow x \in Sint_{\mathscr{N}\mathscr{M}}(A \cap B)$ which $\Rightarrow Sint_{\mathscr{N}\mathscr{M}}(A) \cap Sint_{\mathscr{N}\mathscr{M}}(B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A \cap B)$. (2) From (1) and (2) $Sint_{\mathscr{N}\mathscr{M}}(A \cap B) = Sint_{\mathscr{N}\mathscr{M}}(A) \cap Sint_{\mathscr{N}\mathscr{M}}(B)$. **(b)** From $A \subseteq B \Rightarrow Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq Sint_{\mathscr{N}\mathscr{M}}(B)$. We have:
$A \subseteq (A \cup B) \Rightarrow Sint_{\mathscr{N}\mathscr{M}}(A) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A \cup B)$ and
$B \subseteq (A \cup B) \Rightarrow Sint_{\mathscr{N}\mathscr{M}}(B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A \cup B)$ Hence $Sint_{\mathscr{N}\mathscr{M}}(A) \cup Sint_{\mathscr{N}\mathscr{M}}(B) \subseteq Sint_{\mathscr{N}\mathscr{M}}(A \cup B)$.

Definition 3.7: Let (X, τ_μ^*) be a SNMSTS and let $A = \{< x, \mu_{\mathscr{N}\mathscr{M}}, S_{\mathscr{N}\mathscr{M}}, \delta_{\mathscr{N}\mathscr{M}} >: x \in X\}$ be NMS. Then the supra neutrosophic closure multiset of A is the intersection of all supra neutrosophic closed multisets (SNCMS) of X containing in A and is defined as $SCl_{\mathscr{N}\mathscr{M}}(A) = \{< x, \cap\mu_{\mathscr{N}\mathscr{M}}, \cup\sigma_{\mathscr{N}\mathscr{M}}, \cup\delta_{\mathscr{N}\mathscr{M}} >: x \in X\}$.

Proposition 3.8: Let (X, τ_μ^*) be a SNMSTS. Then

(i) $SCl_{\mathscr{N}\mathscr{M}}(0_{\mathscr{N}\mathscr{M}}) = 0_{\mathscr{N}\mathscr{M}}$, $SCl_{\mathscr{N}\mathscr{M}}(1_{\mathscr{N}\mathscr{M}}) = 1_{\mathscr{N}\mathscr{M}}$

(ii) $P \subseteq SCl_{\mathscr{N}\mathscr{M}}(P)$

(iii) P is SNCMS if and only if $P \subseteq SCl_{\mathscr{N}\mathscr{M}}(P)$

(iv) $P \subseteq Q \Rightarrow SCl_{\mathscr{N}\mathscr{M}}(P) \subseteq SCl_{\mathscr{N}\mathscr{M}}(Q)$

Proof : (i) Since $0_{\mathscr{N}\mathscr{M}}$ and $1_{\mathscr{N}\mathscr{M}}$ are supra neutrosophic closed sets. We have $SCl_{\mathscr{N}\mathscr{M}}(0_{\mathscr{N}\mathscr{M}}) = 0_{\mathscr{N}\mathscr{M}}$,$SCl_{\mathscr{N}\mathscr{M}}(1_{\mathscr{N}\mathscr{M}}) = 1_{\mathscr{N}\mathscr{M}}$.
(ii) Since $SCl_{\mathscr{N}\mathscr{M}}(P)$ is the smallest SNCMS containing P, so $P \subseteq SCl_{\mathscr{N}\mathscr{M}}(P)$.
(iii) Since P is SNCMS then P itself is the smallest SNCMS containing P and so $SCl_{\mathscr{N}\mathscr{M}}(P) = P$. Conversely, let $SCl_{\mathscr{N}\mathscr{M}}(P) = P$. Then $SCl_{\mathscr{N}\mathscr{M}}(P)$ is SNCMS and hence P is also SNCMS.
(iv) From (ii), we have $Q \subset SCl_{\mathscr{N}\mathscr{M}}(Q)$. Since $P \subseteq Q$, so we have $P \subseteq SCl_{\mathscr{N}\mathscr{M}}(Q)$. But $SCl_{\mathscr{N}\mathscr{M}}(Q)$ is a SNCMS. So $SCl_{\mathscr{N}\mathscr{M}}(Q)$ is a SNCMS containing P. Since $SCl_{\mathscr{N}\mathscr{M}}(P)$ is the smallest SNCMS containing P so we have $SCl_{\mathscr{N}\mathscr{M}}(P) \subseteq SCl_{\mathscr{N}\mathscr{M}}(Q)$. Hence $P \subseteq Q \Rightarrow SCl_{\mathscr{N}\mathscr{M}}(P) \subseteq SCl_{\mathscr{N}\mathscr{M}}(Q)$.

4. On Supra Neutrosophic Multi Semi-Open Set Topological Spaces

Definition 4.1: Let $\mathcal{A}$ be an NM of an SNMSTS (X, τ_{μ}^{*}), then $\mathcal{A}$ is called a *supra neutrosophic multi semi-open set (SNMSOS)* of NMS if $\exists \mathcal{B} \in \tau_{\mu}^{*}$, such that $\mathcal{A} \leq MN \sim Sint_{\mathscr{N}\mathscr{M}}(MN \sim SCl_{\mathscr{N}\mathscr{M}}(B)))$.

Example 4.2:. Let $X = \{a, b\}$:

$$\mathcal{A} = \left\{ \begin{array}{l} \langle a, 0.8, 0.1, 0.2\rangle, \langle a, 0.7, 0.1, 0.3\rangle, \langle a, 0.6, 0.2, 0.4\rangle, \\ \langle b, 0.7, 0.2, 0.3\rangle, \langle b, 0.6, 0.3, 0.4\rangle, \langle b, 0.4, 0.2, 0.5\rangle \end{array} \right\}$$

$$\mathcal{B} = \left\{ \begin{array}{l} \langle a, 0.9, 0.1, 0.1\rangle, \langle a, 0.8, 0.1, 0.2\rangle, \langle a, 0.7, 0.2, 0.3\rangle, \\ \langle b, 0.8, 0.2, 0.2\rangle, \langle b, 0.7, 0.2, 0.3\rangle, \langle b, 0.5, 0.2, 0.4\rangle \end{array} \right\}$$

Then $\tau_{\mu}^{*} = \{0_{\mathscr{N}\mathscr{M}}, 1_{\mathscr{N}\mathscr{M}}, \mathcal{B}\}$ is a supra neutrosophic multiset topological space.
Then $Sint_{\mathscr{N}\mathscr{M}}(\mathcal{B}) = 1_{\mathscr{N}\mathscr{M}}$, $SInt_{\mathscr{N}\mathscr{M}}(SCl_{\mathscr{N}\mathscr{M}}(\mathcal{B})) = 1_{\mathscr{N}\mathscr{M}}$.
Hence, $\mathcal{B}$ is SNMSOS.

Definition 4.3: Let $\mathcal{A}$ be an NMS of an SNMSTS (X, τ_{μ}^{*}), then $\mathcal{A}$ is called a *supra neutrosophic multi semi-closed set (SNMSCoS)* of X if $\exists \mathcal{B}^{c} \in \tau_{\mu}^{*}$, such that $MN \sim SCl_{\mathscr{N}\mathscr{M}}(MN \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{B})) \subseteq \mathcal{A}$.

Theorem 4.4: The statements below are equivalent:

(i) $\mathcal{A}$ is an SNMCoS;

(ii) $\mathcal{A}^{c}$ is an SNMOS;

(iii) $NM \sim SInt_{\mathscr{N}\mathscr{M}}(NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{A})) \subseteq \mathcal{A}$;

(iv) $NM \sim SCl_{\mathscr{N}\mathscr{M}}(NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{A}^{c})) \supseteq \mathcal{A}^{c}$.

Proof: (i) and (ii) are equivalent, since for an SNMS, $\mathcal{A}$ of an SNMTS (X, τ_{μ}^{*}) such that

$$1_{NM} - NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{A}) = NM \sim SCl_{\mathscr{N}\mathscr{M}}(1_{NM} - \mathcal{A})$$

and

$$1_{NM} - NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{A}) = NM \sim SInt_{\mathscr{N}\mathscr{M}}(1_{NM} - \mathcal{A}).$$

(i) $\Rightarrow$ (iii): By definition 4.3, $\exists$ an SNMCoS, $\mathcal{B}$ such that $NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{B}) \subseteq \mathcal{A} \subseteq \mathcal{B}$; hence,

$$NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{B}) \subseteq \mathcal{A} \subseteq NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{A}) \subseteq \mathcal{B}.$$

Since $NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{B})$ is the largest SNMOS contained in $\mathcal{B}$, we have

$$NM \sim Int_{\mathscr{N}\mathscr{M}}(NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{B})) \subseteq NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{B}) \subseteq \mathcal{A};$$

(iii) $\Rightarrow$ (i) follows by taking $\mathcal{B} = NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{A})$;
(ii) $\Leftrightarrow$ (iv) can similarly be proved. □

Theorem 4.5:
Arbitrary union of SNMSOSs is an SNMSOS;
Proof: Let $\{\mathcal{A}_\alpha\}$ be a collection of SNMSOSs of an SNMSTS (X, τ^*_μ). Then $\exists$ a $\mathcal{B}_\alpha \in \tau^*_\mu$ such that $\mathcal{B}_\alpha \subseteq \mathcal{A}_\alpha \subseteq NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{B}_\alpha)$ for each α. Thus,

$$\bigcup_\alpha \mathcal{B}_\alpha \subseteq \bigcup_\alpha \mathcal{A}_\alpha \subseteq \bigcup_\alpha (NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{B}_\alpha)) \subseteq NM \sim SCl_{\mathscr{N}\mathscr{M}}\left(\bigcup_\alpha \mathcal{B}_\alpha\right)$$

and $\bigcup_\alpha \mathcal{B}_\alpha \in \tau^*_\mu$, this shows that $\bigcup_\alpha \mathcal{A}_\alpha$ is an SNMSOS;
Definition 4.6. An NMS $\mathcal{A}$ of an SNMSTS (X, τ^*_μ) is called a *supra neutrosophic multi regularly open set (SNMROS)* of (X, τ^*_μ) if

$$NM \sim SInt_{\mathscr{N}\mathscr{M}}(NM \sim SCl_{\mathscr{N}\mathscr{M}}(\mathcal{A})) = \mathcal{A}.$$

Then *supra neutrosophic multi regularly closed set (SNMRCoS)* of (X, τ^*_μ) if

$$NM \sim SCl_{\mathscr{N}\mathscr{M}}(NM \sim SInt_{\mathscr{N}\mathscr{M}}(\mathcal{A})) = \mathcal{A}.$$

Theorem 4.7. An NMS, $\mathcal{A}$ of SNMSTS (X, τ^*_μ) is an SNMRO if $\mathcal{A}^c$ is SNMRCo.
Proof: It follows from theorem 4.4. □
Remark 2.8. It is obvious that every SNMROS (SNMRCoS) is an SNMOS (SNMCoS). The converse need not be true.
Example 4.9. Let $X = \{a, b\}$ and

$$\mathcal{A} = \left\{ \begin{array}{l} \langle a, 0.8, 0.1, 0.2\rangle, \langle a, 0.7, 0.1, 0.3\rangle, \langle a, 0.6, 0.2, 0.4\rangle, \\ \langle b, 0.7, 0.2, 0.3\rangle, \langle b, 0.6, 0.3, 0.4\rangle, \langle b, 0.4, 0.2, 0.5\rangle \end{array} \right\}$$

$$\mathcal{B} = \left\{ \begin{array}{l} \langle a, 0.9, 0.1, 0.1\rangle, \langle a, 0.8, 0.1, 0.2\rangle, \langle a, 0.7, 0.2, 0.3\rangle, \\ \langle b, 0.8, 0.2, 0.2\rangle, \langle b, 0.7, 0.2, 0.3\rangle, \langle b, 0.5, 0.2, 0.4\rangle \end{array} \right\}$$

Then $\tau^*_\mu = \{0_{\mathscr{N}\mathscr{M}}, 1_{\mathscr{N}\mathscr{M}}, \mathcal{B}\}$ is a supra neutrosophic multiset topological space.
Then $SCl_{\mathscr{N}\mathscr{M}}(\mathcal{B}) = 1_{\mathscr{N}\mathscr{M}}$, $SInt_{\mathscr{N}\mathscr{M}}(SCl_{\mathscr{N}\mathscr{M}}(\mathcal{B})) = 1_{\mathscr{N}\mathscr{M}}$, which is not SNMROS.
Remark 4.10. The union of any two SNMROSs (SNMRCoS) need not be an SNMROS (SNMRCoS).

Example 4.11. Let $X = \{a, b\}$ and

$$\tau^*_\mu = \{0_{\mathscr{N}\mathscr{M}}, 1_{\mathscr{N}\mathscr{M}}, \mathcal{A}, \mathcal{B}, \mathcal{A} \cup \mathcal{B}\}$$

be a supra neutrosophic multiset topological space, where

$$\mathcal{A} = \left\{ \begin{array}{l} \langle a, 0.4, 0.5, 0.6\rangle, \langle a, 0.3, 0.5, 0.7\rangle, \langle a, 0.2, 0.6, 0.8\rangle, \\ \langle b, 0.7, 0.5, 0.3\rangle, \langle b, 0.6, 0.5, 0.4\rangle, \langle b, 0.4, 0.5, 0.6\rangle \end{array} \right\}$$

$$\mathcal{B} = \left\{ \begin{array}{l} \langle a, 0.6, 0.5, 0.4\rangle, \langle a, 0.7, 0.5, 0.3\rangle, \langle a, 0.8, 0.4, 0.2\rangle, \\ \langle b, 0.3, 0.5, 0.7\rangle, \langle b, 0.4, 0.5, 0.6\rangle, \langle b, 0.6, 0.5, 0.4\rangle \end{array} \right\}$$

$$\mathcal{A} \cup \mathcal{B} = \left\{ \begin{array}{l} \langle a, 0.6, 0.5, 0.4\rangle, \langle a, 0.7, 0.5, 0.3\rangle, \langle a, 0.8, 0.4, 0.2\rangle, \\ \langle b, 0.7, 0.5, 0.3\rangle, \langle b, 0.6, 0.5, 0.4\rangle, \langle b, 0.4, 0.5, 0.6\rangle \end{array} \right\}$$

Here, $SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A}) = \mathcal{B}$, $SInt_{\mathcal{N}\mathcal{M}}(SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A})) = \mathcal{A}$, and $SCl_{\mathcal{N}\mathcal{M}}(\mathcal{B}) = \mathcal{A}^C$, $SInt_{\mathcal{N}\mathcal{M}}(SCl_{\mathcal{N}\mathcal{M}}(\mathcal{B})) = \mathcal{B}$. Then $SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A} \cup \mathcal{B}) = 1_{\mathcal{N}\mathcal{M}}$. Thus, $SInt_{\mathcal{N}\mathcal{M}}(SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A} \cup \mathcal{B})) = 1_{\mathcal{N}\mathcal{M}}$. Hence, $\mathcal{A}$ and $\mathcal{B}$ are SNMROS, but $\mathcal{A} \cup \mathcal{B}$ is not SNMROS.

Theorem 4.12.

The union of any two SNMRCoSs is an SNMRCoS.

Proof:

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two SNMROSs of an SNMTS (X, τ_μ^*). Since $\mathcal{A}_1 \cup \mathcal{A}_2$ is SNMOS (from Remark 3), we have $\mathcal{A}_1 \cup \mathcal{A}_2 \supseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \subseteq SInt_{\mathcal{N}\mathcal{M}}(\mathcal{A}_1 \cup \mathcal{A}_2))$. Now,

$$NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \subseteq SInt_{\mathcal{N}\mathcal{M}}(\mathcal{A}_1 \cup \mathcal{A}_2)) \supseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \sim Int_{\mathcal{N}\mathcal{M}}(\mathcal{A}_1)) = \mathcal{A}_1$$

and

$$NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \sim Int_{\mathcal{N}\mathcal{M}}(\mathcal{A}_1 \cup \mathcal{A}_2)) \supseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \subseteq SInt(\mathcal{A}_2)) = \mathcal{A}_2$$

implies that $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \subseteq SInt_{\mathcal{N}\mathcal{M}}(\mathcal{A}_1 \cup \mathcal{A}_2))$, hence the theorem. □

Theorem 4.13.

(i) The closure of an SNMOS is an SNMRCoS;

(ii) The interior of an SNMCoS is an SNMROS.

Proof:

(i) Let $\mathcal{A}$ be an SNMOS of an SNMTS $(X, \tau^\mu_{\mathcal{N}\mathcal{M}})$. Clearly, $NM \sim Int_{\mathcal{N}\mathcal{M}}(NM \sim SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A})) \subseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A})$. Now, $\mathcal{A}$ is SNMOS implies that $\mathcal{A} \subseteq NM \sim Int_{\mathcal{N}\mathcal{M}}(NM \sim SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A}))$, and hence, $NM \sim SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A}) \subseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \sim Int_{\mathcal{N}\mathcal{M}}(NM \sim SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A})))$. Thus, $NM \sim SCl_{\mathcal{N}\mathcal{M}}(\mathcal{A})$ is SNMRCoS;

(ii) Let $\mathcal{A}$ be an SNMCoS of an SNMTS $(X, \tau\ ^\mu_{\mathcal{N}\mathcal{M}})$. Clearly, $NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \sim Int_{\mathcal{N}\mathcal{M}}(\mathcal{A})) \supseteq NM \sim Int_{\mathcal{N}\mathcal{M}}(\mathcal{A})$. Now, $\mathcal{A}$ is SNMCoS implies that $\mathcal{A} \supseteq NM \sim SCl_{\mathcal{N}\mathcal{M}}(NM \sim Int_{\mathcal{N}\mathcal{M}}(\mathcal{A}))$, and hence, $NM \sim Int_{\mathcal{N}\mathcal{M}}(\mathcal{A}) \supseteq NM \sim Int_{\mathcal{N}\mathcal{M}}(NM \sim SCl_{\mathcal{N}\mathcal{M}}$

Definition 4.14. Let $\phi : (X, \tau_\mu^*) \to (Y, \tau_{\mu 1}^*)$ be a mapping from an SNMSTS (X, τ_μ^*) to another SNMSTS $(Y, \tau_{\mu 1}^*)$. Then ϕ is known as a supra neutrosophic multiset continuous mapping (SNMCM) if $\phi^{-1}(\mathcal{A}) \in \tau_\mu^*$ for each $\mathcal{A} \in \tau_{\mu 1}^*$, or equivalently $\phi^{-1}(\mathcal{B})$ is an SNMCoS of X for each SNMCoS $\mathcal{B}$ of Y.

Example 4.15. Let $X = Y = \{a, b, c\}$ and
$\mathcal{A} = \{\langle a, 0.4, 0.5, 0.6\rangle, \langle a, 0.3, 0.5, 0.7\rangle, \langle a, 0.2, 0.6, 0.8\rangle, \langle b, 0.3, 0.5, 0.4\rangle, \langle b, 0.2, 0.5, 0.6\rangle, \langle b, 0.1, 0.5, 0.7\rangle, \langle c, 0.4, 0.5, 0.6\rangle, \langle c, 0.3, 0.5, 0.7\rangle, \langle c, 0.2, 0.6, 0.8\rangle\}$,
$\mathcal{B} = \{\langle a, 0.6, 0.1, 0.2\rangle, \langle a, 0.5, 0.1, 0.3\rangle, \langle a, 0.4, 0.2, 0.4\rangle, \langle b, 0.3, 0.5, 0.4\rangle, \langle b, 0.2, 0.5, 0.6\rangle, \langle b, 0.1, 0.5, 0.7\rangle, \langle c, 0.4, 0.5, 0.6\rangle, \langle c, 0.3, 0.5, 0.7\rangle, \langle c, 0.2, 0.6, 0.8\rangle\}$.

Then $\tau_\mu^* = \{0_{\mathcal{N}\mathcal{M}}, 1_{\mathcal{N}\mathcal{M}}, \mathcal{A}\}$ and $\tau_{\mu 1}^* = \{0_{\mathcal{N}\mathcal{M}}, 1_{\mathcal{N}\mathcal{M}}, \mathcal{B}\}$ are supra neutrosophic multiset topological spaces. Now, define a mapping $f : (X, \tau_\mu^*) \to (Y, \tau_{\mu 1}^*)$ by $f(a) = f(c) = c$ and $f(b) = b$. Thus, f is SNMCM.

Definition 4.16. Let $\phi : (X, \tau_\mu^*) \to (Y, \tau_{\mu 1}^*)$ be a mapping from an SNMSTS (X, τ_μ^*) to another SNMSTS $(Y, \tau_{\mu 1}^*)$. Then ϕ is called a supra neutrosophic multiset open mapping (SNMoM) if $\phi(\mathcal{A}) \in \tau_{\mu 1}^*$ for each $\mathcal{A} \in \tau_\mu^*$.

Definition 4.17. Let $\phi : (X, \tau^*_\mu) \to (Y, \tau^*_{\mu 1})$ be a mapping from an SNMTS (X, τ^*_μ) to another SNMTS $(Y, \tau^*_{\mu 1})$. Then ϕ is said to be a supra neutrosophic multiset closed mapping (SNMCoM) if $\phi(\mathcal{B})$ is an SNMCoS of Y for each SNMCoS $\mathcal{B}$ of X.

Definition 4.18. Let $\phi : (X, \tau^*_\mu) \to (Y, \tau^*_{\mu 1})$ be a mapping from an SNMTS (X, τ^*_μ) to another SNMTS $(Y, \tau^*_{\mu 1})$. Then ϕ is called a supra neutrosophic multi semi-continuous mapping (SNMSCM) if $\phi^{-1}(\mathcal{A})$ is the SNMSOS of X, for each $\mathcal{A} \in \tau^*_{\mu 1}$.

Definition 4.19. Let $\phi : (X, \tau^*_\mu) \to (Y, \tau^*_{\mu 1})$ be a mapping from an SNMTS (X, τ^*_μ) to another SNMTS $(Y, \tau^*_{\mu 1})$. Then ϕ is called a supra neutrosophic multi semi-open mapping (SNMSOM) if $\phi(\mathcal{A})$ is a SOSNMS for each $\mathcal{A} \in \tau^*_\mu$.

Example 4.20. Let $X = Y = \{a, b, c\}$ and

$\mathcal{A} = \{\langle a, 0.6, 0.1, 0.2\rangle, \langle a, 0.5, 0.1, 0.3\rangle, \langle a, 0.4, 0.2, 0.4\rangle, \langle b, 0.3, 0.5, 0.4\rangle, \langle b, 0.2, 0.5, 0.6\rangle, \langle b, 0.1, 0.5, 0.7\rangle, \langle c, 0.4, 0.5, 0.6\rangle, \langle c, 0.3, 0.5, 0.7\rangle, \langle c, 0.2, 0.6, 0.8\rangle\}$,

$\mathcal{B} = \{\langle a, 0.3, 0.5, 0.4\rangle, \langle a, 0.2, 0.5, 0.6\rangle, \langle a, 0.1, 0.5, 0.7\rangle, \langle b, 0.6, 0.1, 0.2\rangle, \langle b, 0.5, 0.1, 0.3\rangle, \langle b, 0.4, 0.2, 0.4\rangle, \langle c, 0.4, 0.5, 0.6\rangle, \langle c, 0.3, 0.5, 0.7\rangle, \langle c, 0.2, 0.6, 0.8\rangle\}$.

Then $\tau^*_{\mu 1} = \{0_{\mathcal{NM}}, 1_{\mathcal{NM}}, \mathcal{A}\}$ and $\tau^*_{\mu 1} = \{0_{\mathcal{NM}}, 1_{\mathcal{NM}}, \mathcal{B}\}$ are supra neutrosophic multiset topological spaces. Clearly, $\mathcal{A}$ is a semi-open set. Then a mapping $f : (X, \tau^*_\mu) \to (Y, \tau^*_{\mu 1})$ defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Hence, f is SNMSOM.

Definition 4.21. Let $\phi : (X, \tau^*_\mu) \to (Y, \tau^*_{\mu 1})$ be a mapping from an SNMSTS (X, τ^*_μ) to another SNMTS (Y, τ^*_μ). Then ϕ is called a supra neutrosophic multiset semi-closed mapping (SNMSCoM) if $\phi(\mathcal{B})$ is an SNMSCoS for each SNMCoS $\mathcal{B}$ of X.

Remark 4.22. From Remark 1, an SNMCM (SNMOM, SNMCoM) is also an SNMSCM (SNMSOM, SNMSCoM).

Example 4.23. Let $X = Y = \{a, b, c\}$ and

$\mathcal{A} = \{\langle a, 0.4, 0.5, 0.6\rangle, \langle a, 0.3, 0.5, 0.7\rangle, \langle a, 0.2, 0.6, 0.8\rangle, \langle b, 0.3, 0.5, 0.4\rangle, \langle b, 0.2, 0.5, 0.6\rangle, \langle b, 0.1, 0.5, 0.7\rangle, \langle c, 0.4, 0.5, 0.6\rangle, \langle c, 0.3, 0.5, 0.7\rangle, \langle c, 0.2, 0.6, 0.8\rangle\}$

$\mathcal{B} = \{\langle a, 0.4, 0.5, 0.6\rangle, \langle a, 0.3, 0.5, 0.7\rangle, \langle a, 0.2, 0.6, 0.8\rangle, \langle b, 0.4, 0.6, 0.4\rangle, \langle b, 0.3, 0.5, 0.5\rangle, \langle b, 0.2, 0.5, 0.6\rangle, \langle c, 0.6, 0.5, 0.5\rangle, \langle c, 0.4, 0.5, 0.6\rangle, \langle c, 0.2, 0.6, 0.9\rangle\}$.

Then $\tau^*_\mu = \{0_{\mathcal{NM}}, 1_{\mathcal{NM}}, \mathcal{A}\}$ and $\tau^*_{\mu 1} = \{0_{\mathcal{NM}}, 1_{\mathcal{NM}}, \mathcal{B}\}$ are supra neutrosophic multiset topological spaces. Let us define a mapping $f : (X, \tau^*_\mu) \to (Y, \tau^*_{\mu 1})$ by $f(a) = f(c) = c$ and $f(b) = b$. Thus, f is SNMSCM, which is not an SNMCM.

Conclusion

In this paper we established some properties of the supra neutrosophic multi- set topological space such as supra neutrosocphic multiset topological space, supra neutrosophic open multiset,supra neutrosophic closed multiset, supra neutrosophic interior multiset, supra neutrosophic closure multiset and their theorem and properties. Also we have introduced the notion of the supra neutrosophic multiset and examined some properties.

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Comparative Analysis of Neutrosophic, Pythagorean neutrosophic, and Fermatean neutrosophic Soft Matrices in the context of Industrial Accidents: A Case Study

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Abstract: Industrial accidents pose significant risks to human lives, the environment, and economic stability. This study presents a comparative analysis of three plithogenic frameworks—Neutrosophic soft sets, Pythagorean neutrosophic soft sets, and Fermatean neutrosophic soft sets—aiming to model and analyze industrial accidents. The primary objective is to evaluate the effectiveness of these soft sets in handling the uncertainties, complexities, and imprecisions inherent in accident data. Utilizing Dhar's (2021) algorithm for Neutrosophic soft matrices, we introduce a new parameter called a risk score, which consolidates various values for each industry or parameter into a singular magnitude. Our findings indicate that neutrosophic soft sets are more accurate than Pythagorean neutrosophic soft sets and Fermatean neutrosophic soft sets. The study concludes that adopting these advanced frameworks can enhance decision-making processes and safety protocols, thereby mitigating the adverse impacts of industrial accidents.

Keywords: Industrial Accidents; Pythagorean Neutrosophic Soft Sets; Fermatean Neutrosophic Soft Sets; Uncertainty Handling; Safety Protocols.

1. Introduction

Plithogeny is a theory developed by Florentin Smarandache that introduces the notion of a "plithogenic set," which is a union of multiple sets. This allows for the representation of complex ideas by combining simpler ones. Neutrosophic sets, also introduced by Florentin Smarandache, extend the classical notion of sets to accommodate indeterminate or uncertain information. Unlike classical sets, where an element either belongs or does not belong to the set, neutrosophic sets allow for elements to have degrees of membership, non-membership, and indeterminacy simultaneously. This makes them suitable for modeling and analyzing situations where the boundaries between categories are fuzzy or where the information is incomplete or contradictory [1].

The idea of fuzzy sets was introduced by Zadeh in "Fuzzy sets. Information and control (1965)" [2]. Fuzzy sets are required as classical sets do not provide a good representation of the uncertainties encountered in real-life scenarios. Fuzzy sets contain a membership value and a non-membership value ranging from 0 to 1, pertaining to the belongingness and non-belongingness of an element to the universal set. It is not always that the value of the summation of membership and non- membership values is 1; hence a degree of uncertainty was introduced by Atanassov who generalized fuzzy sets and named the new set, intuitionistic fuzzy set [3].

Intuitionistic fuzzy sets contain a hesitation margin to quantify the ambiguity of human nature in different situations. The hesitation degree can take values from 0 to 1. A null hesitation degree indicates that the values of membership and non-membership of an element are certain and are agreed upon by everyone present. But in most cases, a degree of uncertainty exists as in decision- making problems, sales analysis, new product marketing, financial services, etc. Modeling and analyzing real-life problems in various areas such as disaster management, marketing, environment, medicine, economics, social sciences, etc., requires a hesitation margin, but most of the modeling and computation tools use classical sets. Hence theories namely the theory of probability, evidence, fuzzy set, intuitionistic fuzzy set, rough set, etc., were introduced for dealing with uncertainties. Expanding upon these theories by addressing their difficulties, Molodtsov introduced the concept of soft sets [4].

Soft sets have a good potential in solving practical problems in various areas of difficulty. Maji and colleagues, in their works, extensively explored the theory of fuzzy soft sets [5]. Additionally, subsequent research expanded the theory to include intuitionistic fuzzy soft sets [6]. Smarandache further generalized soft sets into hypersoft sets, applying them in decision-making processes [7]. Furthermore, Vellapandi and Gunasekaran investigated a novel decision-making approach utilizing multi soft set logic [8]. These studies collectively contribute to the advancement and application of soft set theories in various domains.

Smarandache introduced the concepts of neutrosophic sets to deal with uncertain and conflicting information present in belief systems [9]. To further extend the capabilities of neutrosophic sets, Smarandache introduced Pythagorean neutrosophic sets, and Senapati & Yager introduced Fermatean neutrosophic sets [10]. It is assumed that these neutrosophic sets produce more accurate results. An application of neutrosophic sets was seen in the paper "Neutrosophic soft matrices and its application in medical diagnosis" by Mamoni Dhar, which uses neutrosophic sets in diagnostic sciences [11]. This paper extends the approach by doing a comparative analysis of neutrosophic sets, Pythagorean neutrosophic sets, and Fermatean neutrosophic sets in analyzing industrial accidents.

1.1 Preliminaries

Some vital definitions for terms and concepts used throughout the paper are listed below:

Classical Set: A classical set is a collection of distinct elements where each element either belongs to the set or does not. Membership is binary and unambiguous.

Example: The set of natural numbers less than 5, $A = \{1,2,3,4\}$.

Fuzzy set: A fuzzy set consists of elements whose membership values range from 0 to 1. If þ(x) represents the degree of membership of x in a set, then þ(x) = 1 implies x is fully in the set, þ(x) = 0 implies x is not in the set at all and 0<þ(x)<1 implies x is partially in the set.

Example: Consider a fuzzy set A consisting of how tall the people are in a universal set of discourse U={P, Q, R, S}. The membership function þ(x) assigns a degree of membership based on their height, let þ (P)=0.9 (P is very tall), þ(Q) = 0.6 (Q is moderately tall), þ(R)=0.7 (R is also moderately tall) and þ(S) = 0.3 (S is less tall). The fuzzy set A can be represented as, A={(P,0.9),(Q,0.6),(R,0.7),(S,0.3)}.

Intuitionistic Fuzzy Set: An intuitionistic fuzzy consists of a membership function θ: U → [0,1] and a non-membership function φ: U → [0,1] to illustrate how much each element x in U belongs and does not belong to a set. The condition is that 0£ θ(x)+ φ(x) £1.

Example: Consider an intuitionistic fuzzy set A representing tall people in a universe U = {P, Q, R, S} where, {θ (P)=0.9, φ(P)=0.05}, {θ(Q)=0.6, φ(Q)=0.3}, {θ(R)=0.7, φ(R)=0.2},{θ(S)=0.3, φ(S)=0.6}. Then intuitionistic fuzzy set A can be represented as, A={(P,(0.9,0.05)),(Q,(0.6,0.3)),(R,(0.7,0.2)),(S,(0.3,0.6))}.

Neutrosophic set: A neutrosophic set consists of a membership function θ: U → [0,1], a non-membership function φ: U → [0,1], and an indeterminacy membership function ϕ: U → [0,1]. These three functions characterize each element x of a universal set U on how much they belong, do not belong and uncertainty of each element in a set. The condition is that 0£ θ(x)+ φ(x)+ ϕ(x) £3.

Example: Consider an intuitionistic fuzzy set A representing tall people in a universe U = {P, Q, R, S} where, {θ (P)=0.9, ϕ(P)=0.1, φ(P)=0.05}, {θ(Q)=0.6, ϕ(Q)=0,2, φ(Q)=0.3}, {θ(R)=0.7, ϕ(R)=0.1, φ(R)=0.2},{θ(S)=0.3, ϕ(S)=0.4, φ(S)=0.6}. Then intuitionistic fuzzy set A can be represented as, A={(P,(0.9,0.1,0.05)),(Q,(0.6,0.2,0.3)),(R,(0.7,0.1,0.2)),(S,(0.3,0.4,0.6))}.

Pythagorean neutrosophic set: A Pythagorean neutrosophic set consists of a membership function θ: U → [0,1], a non-membership function φ: U → [0,1] , and an indeterminacy membership function ϕ: U → [0,1] where each element x in the universal set U must satisfy 0£ $θ(x)^2$+ $φ(x)^2$+ $ϕ(x)^2$ £1.

Fermatean neutrosophic set: A fermatean neutrosophic set consists of a membership function θ: U → [0,1], a non-membership function φ: U → [0,1] and an indeterminacy membership function ϕ: U → [0,1] where each element x in the universal set U must satisfy 0£ $θ(x)^3$+ $φ(x)^3$+ $ϕ(x)^3$ £1.

Soft set: A soft set is a mapping from a certain set of parameters to a subset under the set of universal discourse. If Z is the mapping, then the soft set is represented by, Z: X→ P(U) where X is the set consisting of the parameters and P is the power set of the Universe of discourse, U. (X, Z) is called a soft set over U.

Example: Let the universe of discourse U be a set of cars, U = {car1, car2, car3}. Let the set of parameters consist of the fuel on which the cars run, represented by X, X = {biodiesel(x1), diesel(x2), electric(x3)}. The mapping Z can be defined as follows, Z(x1) = {car1, car3}, Z(x2) = {car2, car3}, and Z(x3) = {car2}. The soft set (X, Z) can be represented as, (X, Z) = {(x1, {car1, car3}), (x2, {car2, car3}), (x3, {car2})}

Soft Matrix: A soft matrix is a representation of soft sets in matrix form, where rows and columns represent different parameters and their corresponding values.

Example: Let the universe of discourse U be a set of cars, U = {car1, car2, car3}. Let the set of parameters consist of the fuel on which the cars run, represented by X, X = {biodiesel(x1), diesel(x2), electric(x3)}. The mapping Z can be defined as follows, Z(x1) = {car1, car3}, Z(x2) = {car2, car3}, and Z(x3) = {car2}. The soft matrix can be represented by,

Table (i)

X/U	**Car1**	**Car2**	**Car3**
Biodiesel(x1)	1	0	1
Diesel(x2)	0	1	1
Electric(x3)	0	1	0

Fuzzy Soft Set: A fuzzy soft set combines the concepts of fuzzy sets and soft sets, where the mapping involves fuzzy sets instead of classical sets, allowing for partial membership.

Example: Let the universe of discourse U be a set of cars, U = {car1, car2, car3}. Let the set of parameters consist of features of the cars, represented by X, X = {comfort(x1), fuel efficiency(x2), safety(x3)}. The fuzzy soft set (X,Z) can be represented in matrix form as,

Table (ii)

X/U	**Car1**	**Car2**	**Car3**
Comfort(x1)	0.8	0.4	0.6
Fuel efficiency(x2)	0.7	0.9	0.5
Safety(x3)	0.5	0.6	0.9

Neutrosophic soft matrix: A neutrosophic soft matrix combines the principle of neutrosophic sets and soft matrices hence allowing a mapping between parameters and subsets to be characterized with a membership value θ: U → [0,1], a non-membership value φ: U → [0,1], and an indeterminacy value ϕ: U → [0,1].

Example: Let M be a neutrosophic soft matrix of a set of cars, U = {car1, car2, car3} based on the parameters X = {comfort(x1), fuel efficiency(x2)}. Each element m_{ij} of M is represented as $m_{ij} = (\theta_{ij}(x), \phi_{ij}(x), \varphi_{ij}(x))$

Table (iii)

X/U	**Car1**	**Car2**	**Car3**
Comfort(x1)	(0.9, 0.05, 0)	(0.7, 0.2, 0)	(0.8, 0.1, 0)
Fuel efficiency(x2)	(0.8, 0, 0.1)	(0.6, 0, 0.3)	(0.7, 0, 0.2)

Pythagorean neutrosophic soft matrix: A pythagorean neutrosophic soft matrix combines the principle of pythagorean neutrosophic sets and soft matrices hence allowing mapping between parameters and subsets to be characterized with a membership value θ: U → [0,1], a non-membership value φ: U → [0,1], and an indeterminacy value ϕ: U → [0,1] where each element x in the universal set U must satisfy 0£ $\theta(x)^2$+ $\varphi(x)^2$+ $\phi(x)^2$ £1.

Fermatean neutrosophic soft matrix: A fermatean neutrosophic soft matrix combines the principle of fearmatean neutrosophic sets and soft matrices hence allowing mapping between parameters and subsets to be characterized with a membership value θ: U → [0,1], a non-membership value φ: U → [0,1], and an indeterminacy value ϕ: U → [0,1] where each element x in the universal set U must satisfy 0£ $\theta(x)^3$+ $\varphi(x)^3$+ $\phi(x)^3$ £1.

Complement of Neutrosophic soft matrix, Pythagorean neutrosophic soft matrix and Fermatean neutrosophic soft matrix: Let M be a soft matrix such that M Î NSM_{ij} or M Î $PNSM_{ij}$ or M Î $FNSM_{ij}$. An element of M denoted by m_{ij} for all i and j values is described as $m_{ij} = (\theta_{ij}(x), \phi_{ij}(x), \varphi_{ij}(x))$ then the complement of M is denoted by M^c where the elements of M^c is given by $m_{ij}^c = (\varphi_{ij}(x), \phi_{ij}(x), \theta_{ij}(x))$ for all i and j values.

Example: Let M be a neutrosophic soft matrix of a set of cars, U = {car1, car2, car3} based on the parameters X = {comfort(x1), fuel efficiency(x2)}.

Table (iv)

X/U	**Car1**	**Car2**	**Car3**
Comfort(x1)	(0.9, 0.05, 0)	(0.7, 0.2, 0)	(0.8, 0.1, 0)
Fuel efficiency(x2)	(0.8, 0, 0.1)	(0.6, 0, 0.3)	(0.7, 0, 0.2)

The complement of M is M^c.

Table (v)

X/U	**Car1**	**Car2**	**Car3**
Comfort(x1)	(0, 0.05, 0.9)	(0, 0.2, 0.7)	(0, 0.1, 0.8)
Fuel efficiency(x2)	(0.1, 0, 0.8)	(0.3, 0, 0.6)	(0.2, 0, 0.7)

Max-min Product of NSM, PNSM, FNSM: Let M and N be two neutrosophic soft matrices, represented as, $M = [\theta_{ij}^M, \phi_{ij}^M, \varphi_{ij}^M]$ and $N = [\theta_{ij}^N, \phi_{ij}^N, \varphi_{ij}^N]$. The max-min product of the two neutrosophic soft matrices M and N, denoted as M∗N is defined as follows:

$(M*N)_{ij} = [\max(\min(\theta_{ij}^M,\theta_{ij}^N)),\min(\max(\phi_{ij}^M,\phi_{ij}^N)),\min(\max(\varphi_{ij}^M,\varphi_{ij}^N))]$

1.2 Motivation

In the realm of mathematical computations involving soft matrices, current research focuses only on neutrosophic soft matrices. This paper seeks to expand this by introducing novel methodologies for soft matrix calculations specifically for Pythagorean neutrosophic and Fermatean neutrosophic matrices. By doing so, it not only enhances the understanding of these matrix types but also compares their results, offering a comprehensive discussion on the reasons behind the observed differences. This comparative analysis is a novel contribution, providing deeper insights into the properties and applications of these matrices.

In the field of chemical engineering, ensuring industrial safety is paramount to preventing accidents and safeguarding human lives and the environment. This paper addresses this critical concern by identifying the most prevalent types of accidents in various industries and the primary causes behind them. Utilizing the innovative concept of a risk score, it quantifies these causes, thereby providing a practical framework for risk assessment and management. This approach not only highlights potential hazards but also offers a systematic method for mitigating risks, ultimately contributing to safer industrial practices.

1.3 Theory

1.3.1. Types of industries:

1. *Petrochemical Industry*

   The petroleum industry manufactures chemicals from crude oil and natural gas. The industry's products are plastics, fertilizers, solvents, and many other substances used as raw materials in different branches of the economy. This industry has many different types of disasters that might take place at any given time. Explosions might happen when dealing with explosive liquids or gases and their leakages or accidental ignition. Because of the presence of highly inflammable materials, these fires become particularly hazardous [16]. Unintentional chemical releases may occur during processing or transportation with considerable health and environmental impacts [36]. Earthquakes or floods, among other natural phenomena, can damage infrastructure, exposing chemicals to humans and the environment [37]. Mistakes in operation together with oversights are the major causes of accidents due to human error in running equipment that uses chemicals like in industries [40]. Design factors, such as faulty equipment and poor design, can lead to breakdowns [28]. Simultaneously, improper maintenance and management failures, also known as maintenance factors, might enhance the probability of accidents [43].

2. *Oil and Gas Industry*

   Oil and gas is the industry that involves the exploration, extraction, refining, and transportation of oil and natural gas to produce fuels, lubricants, and feedstock for petrochemicals. One risk is the rise of accidents in this field. Many of those happen due to high pressure for gas storage or any substances which cause explosions when mixed and exposed to fire. Thus, fire outbreaks might occur once there is oil spillage or any gaseous compound leaks out [47]. One of the primary concerns about pollution is that it causes spills and leaks, leading to vast environmental damage [50]. The safety of the structure is at risk during weather events and natural disasters that are natural hazards [19]. Major accidents can be caused by human error or neglect in operation [51]. Catastrophic failures can happen when there is a failure in the design or equipment of a system [42]. Unsafe conditions leading to an increase in the probability of accidents may be brought about by inadequate maintenance practices referred to as maintenance factors [45].

3. *Power Generation Industry*

   Various sources are used to generate electricity within the power generation industry; for example, fossil fuels, nuclear power, as well as renewable resources like wind power or solar energy among others. Gas explosions or boiler explosions are some of the common accidents that happen in this industry [29]. Fires caused by electrical faults or overheating are also common occurrences within this sector [20]. However, runaway reactions, especially in nuclear power plants, may be catastrophic [22]. Infrastructure suffers damage due to natural hazards like extreme weather, which in turn causes failures [21]. Accidents happen because of monitoring errors or control failures caused by human mistakes [39]. Also causing accidents is the failure of vital parts of a design or faults with some elements of it [41]. Another cause of accidents is poor maintenance practices that compromise safety systems, increasing the chances of their occurrence [46].

4. *Chemicals and Solvents Industry*

   In the field of chemistry, researchers work on creating a myriad of chemicals as well as solvents which are used in multiple ways from industry to home goods. Chemical releases may occur unintentionally during processing procedures or their transportation, thereby risking life and the environment [30]. Due to the flammability of a lot of chemicals, fires can be especially massive [48]. Environmental pollution easily takes place whenever disposal is done incorrectly or in case leakages happen on any type of chemical [31]. Natural crises can generate a failure in the container that will create an incident, just as human mistakes can lead to accidents when handling chemicals incorrectly [39]. Problems with poorly designed storage or handling equipment that have been identified as design factors may trigger equipment failure [28]. Furthermore, lack of prevention or management defaults, also referred to as maintenance factors, improve risks from accidents [44].

5. *Food and Water Industry*

   The food and water sector includes manufacturing, processing, and distribution. These sectors encompass agricultural companies, food processors, and water works among them. Pollution exposes us to great risk because our sources may be poisoned either in the form of food crops or drinking water [33]. Even mishandling induces diseases if not hazardous substances that contaminate these sources directly [24]. Sometimes chemicals are mistakenly discharged while dealing with them during their treatments [35]. Natural disasters, for example, can interrupt supply routes or pollute resources, while wrong processing due to human errors such as handling causes accidents [37]. Malfunctions arise from machinery designs as a type of processing or treatment fault, and lastly, risks can be ramped up due to incompetence in managing resources, otherwise known as maintenance-related factors [38].

1.3.2. Types of accidents

1. *Explosions*

   Violent energy bursts happening when highly pressurized combustible gases, vapors, or dust encounter flames can result in catastrophic damage to property and human fatalities as well. Explosions in this sector mainly arise from equipment failure, which leads to spillage of these dangerous substances. When these escape and combine with oxygen in the atmosphere and encounter ignition sources, vast explosions occur [47]. Accessible gases or dust can make eruptions in power generation or particularly in gas and ordinary fuel plants [29]. Such events are majorly influenced by design factors, for example, ineffective ventilation, mechanical defaults like damaged control valves or pipes, as well as non-examination during repairs and maintenance [28]. Additionally, infrastructure damages might come up because of natural occurrences like earthquakes, which can trigger outflows and then explosions [37].

2. *Fires*

   Many fires are caused every now and then because of flammable substances getting lit. In the petrochemical, oil and gas, chemicals, and solvents industry, there is a possibility of experiencing leaks or spillages of gases or liquids that ignite because they come into contact with hot surfaces, sparks, or static electricity [16]. Fires start in power generation when machines become too hot or electrical systems fail [20]. Human error such as improper handling of materials, design flaws like inadequate fire suppression systems, and maintenance issues like neglected electrical systems are some of the factors that lead to these incidents [41]. It is important to mention that natural hazards, such as lightning strikes, also play a role in setting off fires [19].

3. *Chemical Release*

   The negligent release of harmful substances is dangerous for human health as well as environmental safety. Chemical release may be experienced while producing or storing items within oil, gas, petrochemicals, chemical industries, and solvents alike. This mainly results from equipment breaking down just like when pipes give way, and tanks rupture, among other things such as human errors through mishandling chemicals [30]. Design factors, including inadequate containment measures, and maintenance errors, such as not replacing worn-out seals or gaskets, play significant roles in these incidents [44].

4. *Pollution*

   Pollution is when harmful substances contaminate air, water, or soil, for example, because of industrial activities. In the petroleum and gas exploration, production, and distribution industries, as well as in the chemical and solvent, food, and water industries, these wastes could be spilled, leaked, or dumped carelessly [33]. There are many things that can lead to this, including equipment breakdowns, for example, tanks that spill out their contents, or errors by people who handle materials poorly; but these factors are equated with natural disasters resulting in contaminants being carried away during rainstorms as well [36]. Moreover, one must also consider other contributing aspects like poor equipment design or improper use due to lack of proper attention on examining these pipes during regular intervals over subsequent weeks or months [28].

5. *Disease*

A petrochemical plant may release chemicals into the atmosphere causing pollution as well as illnesses due to poor maintenance practices or mechanical defects [35]. Due to incompetence in design or human error and chemical release at an oil and gas facility leading to emissions which harm people breathing [51]. The food and water industry can have pandemics as a result of harmful microorganisms contaminating foods or water products [24]. Common causes include poor washing habits, poorly processed foods & mismanaged water sources [34]. In order to mitigate dangers and protect the health of the general public, there is a necessity for proper comprehension of the interconnection between these industries, various accidents encountered within them, and the causes of such accidents [32].

1.3.3. Type of factors

1. *Natural Hazards*

Natural catastrophes take the form of disasters resulting from the interaction between the earth (geology) and its life forms (environment), whether human or physical, which cause harm to human projects like buildings, roads, etc., leading to accidents (unintentional harmful events) [36]. For example, there are calamities like floods in Singapore, while lightning has been found responsible for the greatest number of injuries as well as deaths after thunderstorms [37]. During bursts in petrochemical and oil gas industries, earthquakes may lead to breakages of pipelines, tank explosions, among other things due to vibration or breaking apart some vital facilities [17]. Facilities must be protected from floods due to chemical releases or contamination in the chemicals and solvents industry [30]. Power lines and infrastructure in power generation can be damaged by hurricanes, leading to fire outbreaks and power outages [29]. In the food and water industry, natural calamities can invade water supplies and crops leading to pollution and disease outbreaks [24]. Such unwanted determinants always take place unexpectedly, thus requiring strong design plans and preparedness approaches against them [38].

2. *Human Error/Operation Factors*

The term human error and operational factors are used to talk about mistakes or slips in judgment that are made by industrial processes because of workers [39]. Any individual can make a mistake in different ways, including touching materials wrongly, using the wrong methods with machines, or not taking care of protective clothing seriously when working at a construction site [40]. For instance, if someone who works at a petrochemical plant makes a mistake while regulating temperature levels or pressures in the processing plant, it can result in runaway reactions [16]. Proper shutdown procedures must be adhered to in the oil and gas industry, or else risk fires and the release of chemicals [19]. Similarly, errors in monitoring systems can result in overheating of the equipment, hence fires or explosions in the power generation industry [22]. The food and water industry can cause outbreaks of diseases and contamination due to poor hygiene practices or poor handling of raw materials [34]. Avoiding improper training, following safety protocols, and proper supervision will reduce human errors [43].

3. *Design Factor/Equipment Fault*

Industrial systems and equipment suffer from design factors and equipment faults [41]. Inadequate safety margins, poor material selections, or purposes without enough substitution systems can be a list of design mishandling [28]. Conversely, equipment problems can arise from assembly errors, normal wear and tear, or inadequate testing [42]. Poorly designed storage tanks used in the petrochemical industry may not be able to withstand very high pressures, causing explosions or releases of harmful substances [44]. Faulty valves or pipelines can cause leaks and fires in the oil and gas industry [19]. There can be explosions or runaway reactions in cases of turbine or boiler failures that are defective [22]. In the chemicals and solvents sector, uncontrolled reactions might occur from poorly designed reaction vessels [28]. The food and water industry might also experience contamination and pollution because of equipment failures in processing plants [33]. Regular inspections, robust design standards, and testing are crucial to prevent these issues [46].

4. *Maintenance Factor*

Problems with maintenance could happen due to the oversight of regular maintenance procedures, not replacing worn parts, or the use of low-quality repair methods [45]. Lapses in maintenance control systems could occur because of inappropriate preventive maintenance schedules, absence of skilled personnel, or poor documentation practices [43]. Leaks and explosions are consequences of failing to carry out regular inspections and maintain pipelines in the sector that deals with oil chemicals [19]. Furthermore, fires may result from poor servicing of drilling rigs in oil and gas industries, causing either loss or blowout [18]. Insufficient power will be experienced in case there is an interruption within electric systems that are not often maintained during the generation process [20]. This is also related to chemical releases from solvents, but more specifically if storage tanks are not considered under the chemicals category [31]. Contamination and disease outbreaks can result from inadequate upkeep of processing equipment in the food and water sector [24]. It is crucial to implement a thorough maintenance program and effective management strategies for reliability and safety [38].

2. Materials and Methods

2.1. Algorithm

The algorithm proposed in this study is fundamentally based on the formulae developed by Mamoni Dhar in their work on Neutrosophic Soft Sets, as detailed in reference [11]. Their pioneering methods provided a robust foundation for our case study and allowed us to expand these same concepts and algorithms to Pythagorean Neutrosophic Soft Sets and Fermatean Neutrosophic Soft Sets. The algorithm is detailed as a flowchart below:

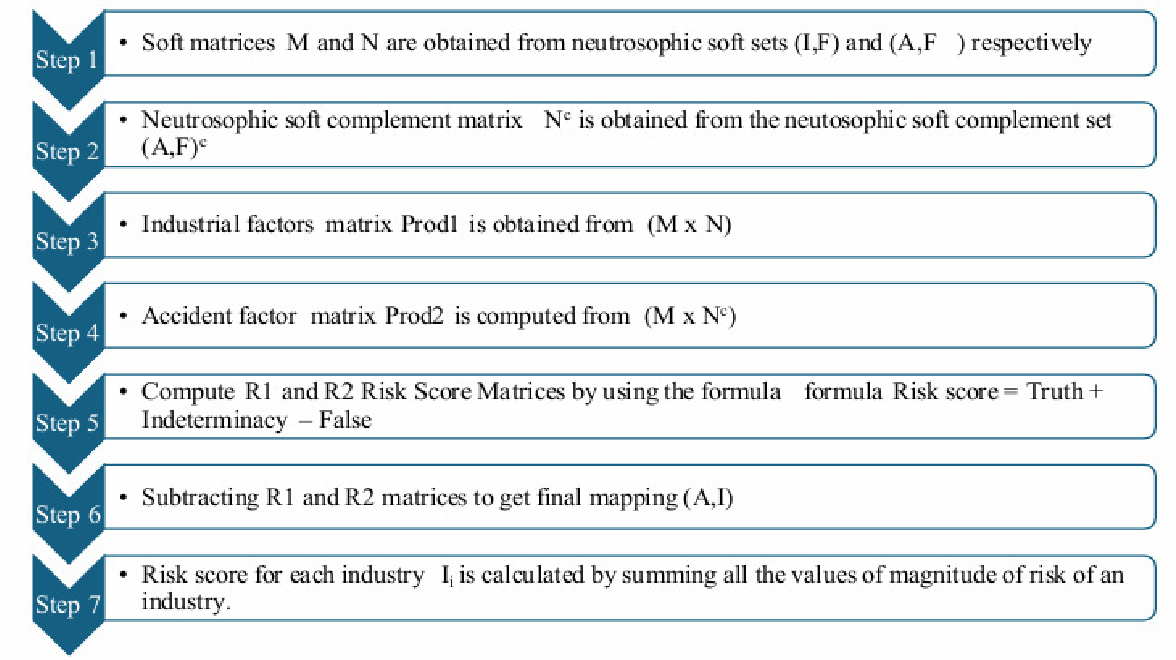

Figure (i)

2.2. Methodology

2.2.1 Neutrosophic Soft Set

Let the overall set of industries be the Universal set 'I' where I = $\{I_1, I_2, I_3, I_4, I_5\}$ such that:

I_1 is the Petrochemical Industry

I_2 is the Oil and Gas Extraction Industry

I_3 is the Energy Generation Industry

I_4 are the Fertilizer and Pharmaceutical Industries

I_5 are the Food and Water Industries

Let the overall type of accident be represented by matrix 'A' where A= $\{A_1, A_2, A_3, A_4, A_5\}$ where:

A_1 is Explosion

A_2 is Fire
A_3 is Chemical release
A_4 is Pollution
A_5 is Disease

The possible factors causing these accidents in the above industries can be represented by matrix 'F' such that F = {F_1, F_2, F_3, F_4} where:

F_1 is Natural Disasters
F_2 is Human Error
F_3 is Design Error
F_4 is Maintenance Error

Consider a neutrosophic soft mapping factoring in the types of industries and the factors that can potentially contribute to their accidents. M is a Mapping such that M_N : F $\rightarrow$ M_N^I

(M_N,F)={{ M_N (F_1)={(I_1, 0.35,0.75,0.4),(I_2, 0.4,0.7,0.3), (I_3, 0.35,0.7,0.5), (I_4, 0.1,0.8,0.2) , (I_5, 0.1,0.8,0.25)},

{M_N (F_2)={(I_1, 0.65,0.4,0.3),(I_2, 0.55,0.3,0.7), (I_3, 0.75,0.3,0.4), (I_4, 0.25,0.8,0.2), (I_5, 0.3,0.6,0.4) },

M_N (F_3)={(I_1, 0.4,0.65,0.65),(I_2, 0.8,0.2,0.5), (I_3, 0.6,0.45,0.6), (I_4, 0.15,0.85,0.25), (I_5, 0.15,0.7,0.5) },

{M_N (F_4)={(I_1, 0.55,0.45,0.3),(I_2, 0.65,0.25,0.6), (I_3, 0.7,0.3,0.45), (I_4, 0.35,0.7,0.15), (I_5, 0.15,0.7,0.5)}}.

Collating the mapping M into a Neutrosophic soft matrix 'M_N' between Universal sets I and F.

$$M_N = \begin{matrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{matrix} \overset{\begin{matrix} F_1 & \quad F_2 & \quad F_3 & \quad F_4 \end{matrix}}{\begin{bmatrix} (0.35,0.75,0.4) & (0.65,0.4,0.3) & (0.4,0.65,0.65) & (0.55,0.45,0.3) \\ (0.4,0.7,0.3) & (0.55,0.3,0.7) & (0.8,0.2,0.5) & (0.65,0.25,0.6) \\ (0.35,0.7,0.5) & (0.75,0.3,0.4) & (0.6,0.45,0.6) & (0.7,0.3,0.45) \\ (0.1,0.8,0.2) & (0.25,0.8,0.2) & (0.15,0.85,0.25) & (0.35,0.7,0.15) \\ (0.1,0.8,0.25) & (0.3,0.6,0.4) & (0.15,0.7,0.5) & (0.15,0.7,0.5) \end{bmatrix}}$$

Consider another neutrosophic soft mapping factoring in the types of accidents and the factors that can potentially contribute them. N is a Mapping such that N_N : A $\rightarrow$ N_N^F

(N_N, A) ={ N_N (A_1)={(F_1, 0.2,0.7,0.2),(F_2, 0.7,0.2,0.6), (F_3, 0.6,0.25,0.75), (F_4, 0.8,0.1,0.5)},
{N_N (A_2)={(F_1, 0.5,0.5,0.6), (F_2, 0.75,0.25,0.4), (F_3, 0.55,0.4,0.55), (F_4, 0.7,0.3,0.45)},
{N_N (A_3)={(F_1, 0.1,0.8,0.8), (F_2, 0.4,0.6,0.5), (F_3, 0.55,0.5,0.45), (F_4, 0.7,0.4,0.35)},
{N_N (A_4)={(F_1, 0.1,0.8,0.7), (F_2, 0.05,0.75,0.4), (F_3, 0.2,0.7,0.3), (F_4, 0.3,0.65,0.2)},
{N_N (A_5)={(F_1, 0.1,0.8,0.7), (F_2, 0.05,0.75,0.4), (F_3, 0.1,0.7,0.4), (F_4, 0.3,0.65,0.2)},

Collating the mapping N_N into a Neutrosophic soft matrix 'N_N' between Universal sets F and A:

$$N_N = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ (0.2,0.7,0.2) & (0.5,0.5,0.6) & (0.1,0.8,0.8) & (0.1,0.8,0.7) & (0.1,0.8,0.7) \\ (0.7,0.2,0.6) & (0.75,0.25,0.4) & (0.4,0.6,0.5) & (0.05,0.75,0.4) & (0.05,0.75,0.4) \\ (0.6,0.25,0.75) & (0.55,0.4,0.55) & (0.55,0.5,0.45) & (0.2,0.7,0.3) & (0.1,0.7,0.4) \\ (0.8,0.1,0.5) & (0.7,0.3,0.45) & (0.7,0.4,0.35) & (0.3,0.65,0.2) & (0.3,0.65,0.2) \end{array}$$

Using Max-Min formulae to calculate product matrix $Prod1_N$

Some examples of the function are given below:

$Prod1_{11}$ = (Max (0.2,0.65,0.4,0.55) , Min(0.75,0.4,0.65,0.45) , Min(0.4,0.6,0.75,0.5)) = (0.65,0.4,0.4)

$Prod1_{12}$ = (Max (0.35,0.65,0.4,0.3) , Min(0.75,0.4,0.65,0.45) , Min(0.6,0.4,0.65,0.45)) = (0.65,0.4,0.4)

$Prod1_{21}$ = (Max (0.2,0.55,0.6,0.65) , Min(0.7,0.3,0.25,0.25) , Min(0.3,0.7,0.75,0.6)) = (0.65,0.25,0.3)

This was continued for all 25 entries…

When combining all the above values, we get $Prod1_N$ matrix

$$Prod1_N = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0.65,0.4,0.4 & 0.65,0.4,0.4 & 0.55,0.45,0.35 & 0.3,0.65,0.3 & 0.3,0.65,0.3 \\ 0.65,0.25,0.3 & 0.65,0.3,0.55 & 0.65,0.4,0.5 & 0.3,0.65,0.5 & 0.3,0.65,0.5 \\ 0.7,0.3,0.5 & 0.75,0.3,0.4 & 0.7,0.4,0.45 & 0.3,0.65,0.4 & 0.3,0.65,0.4 \\ 0.35,0.7,0.2 & 0.35,0.7,0.4 & 0.35,0.7,0.35 & 0.3,0.7,0.2 & 0.35,0.7,0.2 \\ 0.3,0.6,0.25 & 0.3,0.6,0.4 & 0.3,0.6,0.5 & 0.3,0.7,0.4 & 0.3,0.7,0.4 \end{array}$$

And so, applying the formula Risk score = Truth + Indeterminacy – False, we get risk score matrix R_{N1}

$$R_{N1} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0.65 & 0.65 & 0.65 & 0.65 & 0.65 \\ 0.6 & 0.4 & 0.55 & 0.45 & 0.45 \\ 0.5 & 0.65 & 0.65 & 0.55 & 0.55 \\ 0.85 & 0.65 & 0.7 & 0.8 & 0.85 \\ 0.65 & 0.5 & 0.4 & 0.6 & 0.6 \end{array}$$

Next. the complement of Neutrosophic soft matrix N_N is N_N^C.

$$N_N^C = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \\ (0.2,0.7,0.2) & (0.6,0.5,0.5) & (0.8,0.8,0.1) & (0.7,0.8,0.1) & (0.7,0.8,0.1) \\ (0.6,0.2,0.7) & (0.4,0.25,0.75) & (0.5,0.6,0.4) & (0.4,0.75,0.05) & (0.4,0.75,0.05) \\ (0.75,0.25,0.6) & (0.55,0.4,0.55) & (0.45,0.5,0.55) & (0.3,0.7,0.2) & (0.4,0.7,0.1) \\ (0.5,0.1,0.8) & (0.45,0.3,0.7) & (0.35,0.4,0.7) & (0.2,0.65,0.3) & (0.2,0.65,0.3) \end{array}$$

Using Max-Min formulae to calculate product matrix $Prod2_N$

I_1 I_2 I_3 I_4 I_5 A_1 A_2 A_3 A_4 A_5

$$Prod2_N = \begin{bmatrix} 0.6,0.4,0.4 & 0.35,0.4,0.5 & 0.5,0.45,0.4 & 0.4,0.65,0.3 & 0.4,0.65,0.3 \\ 0.75,0.25,0.3 & 0.55,0.3,0.5 & 0.5,0.4,0.3 & 0.4,0.65,0.5 & 0.4,0.65,0.5 \\ 0.75,0.3,0.5 & 0.55,0.3,0.5 & 0.5,0.4,0.4 & 0.2,0.65,0.4 & 0.2,0.65,0.4 \\ 0.35,0.7,0.2 & 0.35,0.7,0.5 & 0.35,0.7,0.2 & 0.25,0.7,0.2 & 0.25,0.7,0.2 \\ 0.3,0.6,0.25 & 0.3,0.6,0.5 & 0.3,0.6,0.25 & 0.3,0.7,0.25 & 0.3,0.7,0.25 \end{bmatrix}$$

Again, applying the formula Risk score = Truth + Indeterminacy – False, we get risk score matrix R_{N2}

$$R_{N2} = \begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ I_1 & 0.6 & 0.25 & 0.55 & 0.75 & 0.75 \\ I_2 & 0.7 & 0.35 & 0.6 & 0.25 & 0.25 \\ I_3 & 0.55 & 0.35 & 0.5 & 0.45 & 0.45 \\ I_4 & 0.85 & 0.55 & 0.85 & 0.75 & 0.75 \\ I_5 & 0.65 & 0.4 & 0.65 & 0.75 & 0.75 \end{matrix}$$

Subtracting R_{N1} and R_{N2} to find magnitude of risk for every specific type of industry gives the final mapping.

$$R_{N1} - R_{N2} = \begin{matrix} & A_1 & A_2 & A_3 & A_4 & A_5 \\ I_1 & 0.05 & 0.4 & 0.1 & -0.1 & -0.1 \\ I_2 & -0.1 & 0.05 & -0.05 & 0.2 & 0.2 \\ I_3 & -0.05 & 0.3 & 0.15 & 0.1 & 0.1 \\ I_4 & 0 & 0.1 & -0.15 & 0.05 & 0.1 \\ I_5 & 0 & 0.1 & -0.25 & -0.15 & -0.15 \end{matrix}$$

The calculation of the final risk score is done by using the following formula and this yields:

$$\text{Risk Score } (I_n) = \sum_{k=1}^{5} (R_{N1} - R_{N2})_{nk}$$

Table (vi)

Industry	Risk score for Neutrosophic Soft Values
I_1	0.35
I_2	0.3
I_3	0.6
I_4	0.1
I_5	-0.45

2.2.2 Pythagorean neutrosophic Soft Set

Repeating the same procedure for Pythagorean neutrosophic soft mapping and creating mappings M_P : F → M_P^I and N_P : A → N_P^F

I_1
I_2
I_3
I_4 F_1 F_2 F_3 F_4
I_5

$$M_P = \begin{bmatrix} (0.4,0.8,0.45) & (0.7,0.45,0.35) & (0.45,0.7,0.7) & (0.6,0.5,0.35) \\ (0.45,0.75,0.35) & (0.6,0.35,0.75) & (0.85,0.25,0.55) & (0.7,0.3,0.65) \\ (0.4,0.75,0.55) & (0.8,0.35,0.45) & (0.65,0.5,0.65) & (0.75,0.35,0.5) \\ (0.15,0.85,0.25) & (0.3,0.85,0.25) & (0.2,0.9,0.3) & (0.4,0.75,0.2) \\ (0.15,0.85,0.3) & (0.35,0.65,0.45) & (0.2,0.75,0.55) & (0.2,0.75,0.55) \end{bmatrix}$$

$$N_P = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} (0.25,0.75,0.25) & (0.55,0.55,0.65) & (0.15,0.85,0.85) & (0.15,0.85,0.75) & (0.15,0.85,0.75) \\ (0.75,0.25,0.65) & (0.8,0.3,0.45) & (0.45,0.65,0.55) & (0.1,0.8,0.45) & (0.1,0.8,0.45) \\ (0.65,0.3,0.8) & (0.6,0.45,0.6) & (0.6,0.55,0.5) & (0.25,0.75,0.35) & (0.15,0.75,0.45) \\ (0.85,0.15,0.55) & (0.75,0.35,0.5) & (0.75,0.45,0.4) & (0.35,0.7,0.25) & (0.35,0.7,0.25) \end{bmatrix} \end{array}$$

Once again, applying the max-min formula and finding Risk score = Truth + Indeterminacy – False, we get risk score matrices R_{P1} and R_{P2} respectively.

$$R_{P1} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.6 & 0.65 & 0.7 & 0.7 & 0.7 \\ 0.65 & 0.45 & 0.6 & 0.5 & 0.5 \\ 0.55 & 0.65 & 0.7 & 0.55 & 0.55 \\ 0.85 & 0.65 & 0.7 & 0.8 & 0.8 \\ 0.8 & 0.55 & 0.55 & 0.4 & 0.4 \end{bmatrix} \end{array}$$

$$R_{P2} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.75 & 0.5 & 0.65 & 0.55 & 0.55 \\ 1 & 0.45 & 0.4 & 0.2 & 0.2 \\ 0.65 & 0.6 & 0.45 & 0.35 & 0.35 \\ 0.85 & 0.65 & 0.7 & 0.75 & 0.75 \\ 0.7 & 0.55 & 0.45 & 0.55 & 0.55 \end{bmatrix} \end{array}$$

Then subtracting R_{P1} and R_{P2} to find magnitude of risk for every specific type of industry gives the final mapping.

$$R_{P1} - R_{P2} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} -0.15 & 0.15 & 0.05 & 0.15 & 0.15 \\ -0.35 & 0 & 0.2 & 0.3 & 0.3 \\ -0.1 & 0.05 & 0.25 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.05 & 0.05 \\ 0.1 & 0 & 0.1 & -0.15 & -0.15 \end{bmatrix} \end{array}$$

Calculating Risk Scores:

Table (vii)

Industry	Risk score for Pythagorean neutrosophic soft values
I_1	0.35
I_2	0.45
I_3	0.6
I_4	0.1
I_5	-0.1

4.3 Fermatean neutrosophic Soft Set

Repeating the same procedure for Fermatean neutrosophic soft mapping and creating mappings M_F : F → M_F^I and N_F : A → N_F^F

$$M_F = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{cccc} F_1 & F_2 & F_3 & F_4 \end{array} \\ \begin{bmatrix} (0.45,0.85,0.5) & (0.75,0.5,0.4) & (0.5,0.75,0.75) & (0.65,0.55,0.4) \\ (0.5,0.8,0.4) & (0.65,0.4,0.8) & (0.9,0.3,0.6) & (0.75,0.35,0.7) \\ (0.45,0.8,0.6) & (0.85,0.4,0.5) & (0.7,0.55,0.7) & (0.8,0.4,0.55) \\ (0.2,0.9,0.3) & (0.35,0.9,0.3) & (0.25,0.95,0.35) & (0.45,0.8,0.25) \\ (0.2,0.9,0.35) & (0.4,0.7,0.5) & (0.25,0.8,0.6) & (0.25,0.8,0.6) \end{bmatrix} \end{array}$$

$$N_F = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} (0.3,0.8,0.3) & (0.6,0.6,0.7) & (0.2,0.9,0.9) & (0.2,0.9,0.8) & (0.2,0.9,0.8) \\ (0.8,0.3,0.7) & (0.85,0.35,0.5) & (0.5,0.7,0.6) & (0.15,0.85,0.5) & (0.15,0.85,0.5) \\ (0.7,0.35,0.85) & (0.65,0.5,0.65) & (0.65,0.6,0.56) & (0.3,0.8,0.4) & (0.2,0.8,0.5) \\ (0.9,0.2,0.6) & (0.8,0.4,0.55) & (0.8,0.5,0.45) & (0.4,0.75,0.3) & (0.4,0.75,0.3) \end{bmatrix} \end{array}$$

Again, applying the max-min formula and finding Risk score = Truth + Indeterminacy – False, we get risk score matrices R_{F1} and R_{F2} respectively.

$$R_{F1} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.75 & 0.75 & 0.75 & 0.75 & 0.75 \\ 0.7 & 0.5 & 0.65 & 0.55 & 0.55 \\ 0.6 & 0.75 & 0.75 & 0.65 & 0.65 \\ 0.95 & 0.75 & 0.8 & 0.9 & 0.9 \\ 0.77 & 0.6 & 0.5 & 0.55 & 0.55 \end{bmatrix} \end{array}$$

$$R_{P2} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.7 & 0.45 & 0.65 & 0.85 & 0.85 \\ 0.8 & 0.45 & 0.7 & 0.85 & 0.85 \\ 0.5 & 0.45 & 0.6 & 0.75 & 0.75 \\ 0.95 & 0.65 & 0.95 & 0.85 & 0.85 \\ 0.75 & 0.5 & 0.75 & 0.85 & 0.85 \end{bmatrix} \end{array}$$

Then subtracting R_{P1} and R_{P2} to find magnitude of risk for every specific type of industry gives the final mapping.

I_1 I_2 I_3 I_4 I_5 A_1 A_2 A_3 A_4 A_5

$$R_{F1} - R_{F2} = \begin{bmatrix} 0.05 & 0.3 & 0.1 & -0.1 & -0.1 \\ -0.1 & 0.05 & -0.05 & -0.3 & -0.3 \\ 0.1 & 0.3 & 0.15 & -0.1 & -0.1 \\ 0 & 0.1 & -0.15 & 0.05 & 0.05 \\ 0 & 0.1 & -0.25 & -0.3 & -0.3 \end{bmatrix}$$

Calculating risk scores

Table (viii)

Industry	**Risk score for Fermatean neutrosophic soft values**
I_1	0.25
I_2	-0.7
I_3	0.35
I_4	0.05
I_5	-0.75

3. Results

3.1. Analysis

3.1.1. Comparative Evaluation of Neutrosophic Frameworks

The values taken for the matrices are arbitrary, based on theoretical knowledge of accidents and the IChemE accident report. From the analysis of the soft matrices, we can see that Neutrosophic analysis is more accurate than Pythagorean neutrosophic and Fermatean neutrosophic.

- The most common accident is fire, and the power generation industry has the highest risk score in all these analyses, with fire being the most likely accident in the power generation industry.
- In the Fermatean neutrosophic analysis, the most likely accident for all industries is fire. Perhaps the reason for this can be theorized that Fermatean neutrosophic analysis considers the severity of the accident over a short period of time, whereas Neutrosophic and Pythagorean neutrosophic analyses consider longer periods of time.
- For example, for the oil and gas industry, only fire is listed as the probable accident in the Fermatean neutrosophic analysis. However, in the Pythagorean neutrosophic analysis, pollution and disease are listed as the most probable accidents. While fire damages more infrastructure and causes severe implications, it can be argued that pollution and disease have long-lasting effects, which are irreversible and have a higher mortality rate.

3.1.2. Risk Scores Across Industries

The food, water and fertilizer, and pharmaceutical industries have the lowest risk scores.

- This might be because these industries work on batch operations, while the power generation industry, oil and gas sector, and petrochemical industry operate on a continuous basis.

- A batch operation is where an industrial plant has a startup and shutdown sequence and does not work for a continuous 24 hours.
- Plants that run continuously do not shut down daily, only when required, such as in emergencies or during maintenance. Plants working continuously tend to have less time for maintenance, hence giving rise to more safety risks.
- Shift changes can cause confusion between day shift and night shift workers. Many accidents occur during the night due to the tiredness of working during the twilight hours.

3.1.3. Human Factors and Operational Risks

Additionally, it is crucial to consider the impact of human factors in the analysis.

- Continuous operations demand high levels of vigilance and can lead to fatigue among workers, thereby increasing the likelihood of accidents.
- Proper training and adequate rest periods are essential to mitigate these risks.
- The inherent nature of industries like power generation and petrochemicals, which handle hazardous materials and high-pressure systems, adds to the complexity and potential danger.
- On the other hand, industries with batch operations can implement thorough inspections and maintenance during downtime, significantly reducing the risk of accidents.

This comprehensive approach highlights the need for industry-specific safety protocols and regular risk assessments to ensure a safer working environment across different sectors.

3.2. *Graphical Representation of Results*

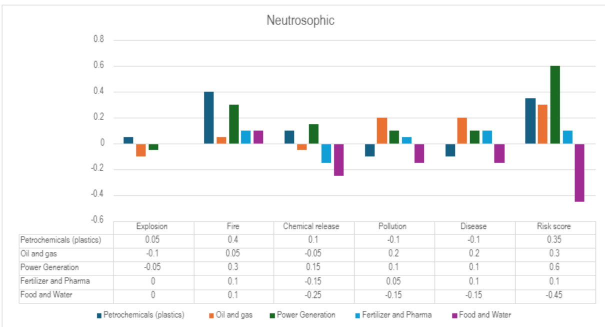

Figure (ii)

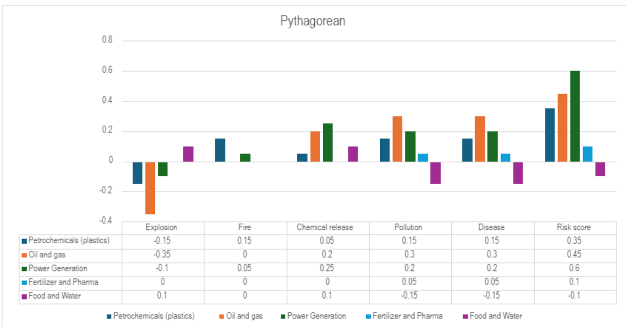

Figure (iii)

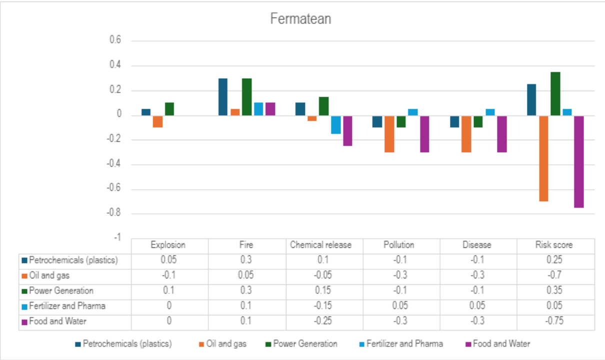

Figure (iv)

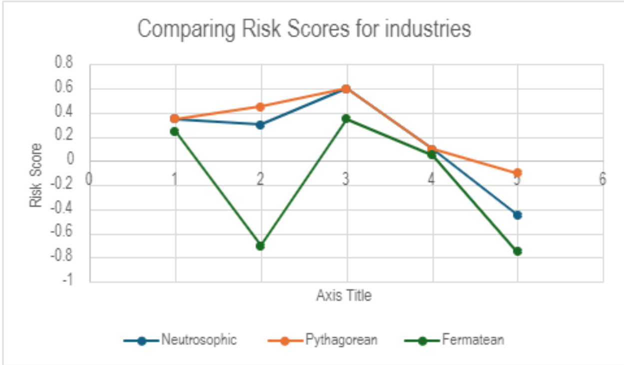

Figure (v)

4. Applications

The findings of this study have significant potential applications in various industrial sectors prone to industrial accidents. By implementing the advanced neutrosophic soft set frameworks, industries can improve their risk assessment and decision-making processes, leading to enhanced safety protocols and reduced accident rates. Specifically, the newly introduced risk score parameter can be utilized by safety engineers and managers to consolidate and analyze complex accident data, providing a more accurate measure of potential hazards. This, in turn, can facilitate better emergency response planning, targeted safety training programs, and optimized resource allocation for accident prevention measures.

5. Conclusions

The comparative analysis of Neutrosophic soft matrix, Pythagorean neutrosophic soft matrix analysis, and Fermatean neutrosophic soft matrix analysis has given rise to the conclusion that the power generation industry is the most dangerous workplace with fire being the most common accident in any industry. The findings of the paper are summarized in the following points.

- Neutrosophic analysis is deemed more accurate than Pythagorean and Fermatean neutrosophic analyses.
- The power generation industry has the highest risk score.
- Fermatean Neutrosophic Analysis emphasizing short-term severity, consistently identifying fire as the most likely accident across industries.
- Pythagorean analysis considers long-term impacts, showing pollution and disease as major concerns, which have irreversible and high mortality rates.
- Power generation, oil and gas, and petrochemical industries operate continuously, leading to higher safety risks due to limited maintenance opportunities and potential worker fatigue.
- Industries like food, water, fertilizer, and pharmaceuticals have lower risk scores due to batch operations allowing for regular maintenance and inspections.
- Continuous operations contribute to worker fatigue and increasing accident likelihood.
- Industries dealing with hazardous materials and high-pressure systems, such as power generation and petrochemicals, face greater complexity and potential dangers.

Using the soft matrix analysis as a basis for various other industries we can identify the most probable accidents and take steps to mitigate the likelihood of the accident.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The values for the neutrosophic, Pythagorean neutrosophic and Fermatean neutrosophic soft matrices are based on the theoretical knowledge and the incident report of the IChemE accident database taken from their e-book named "Learning lessons from major incidents - Improving process safety by sharing experience".

Provided are the references for the e-book

https://www.icheme.org/media/24872/learning-lessons-from-major-incidents.pdf

https://www.icheme.org/media/20722/icheme-lessons-learned-database-rev-11.pdf

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Neutrosophic semi δ-preopen sets and neutrosophic semi δ-pre continuity

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Abstract. The purpose of this paper is to introduce the concepts of neutrosophic semi δ-preopen sets and neutrosophic semi δ-precontinuous mappings in neutrosophic topological spaces and obtain some of their properties and characterizations.

Keywords: Neutrosophic set, neutrosophic topology, neutrosophic semi δ-preopen sets and neutrosophic semi δ-precontinuity.

1. Introduction

As an elaboration of Zadeh's fuzzy sets [19] from 1965 and Atanassav's intuitionistic fuzzy sets [4] from 1983, Smarandache has proposed and described neutrosophic sets.He [13] defined neutrosophic set on a non empty set by considering three components, namely membership, Indeterminacy and non-membership whose sum lies between 0 and 3. Some more properties of neutrosophic sets are presented by Smarandache [13–15], Salama and Alblowi [11] , Lupiáñez [9]. Smarandache's Neutrosophic concepts have wide range of real time applications for the fields of Information systems, Computer science, Artificial Intelligence, Applied Mathematics and Decision making. In 2008, Lupiáñez [9] introduced the neutrosophic topology as a extension of intuitionistic fuzzy topology. Since 2008 many authors such as Lupiáñez [9, 10], Salama et.al. [11, 12] Karatas and Cemil [8], Acikgoz and his coworkers [1], Dhavaseelan et.al. [5], Al-Musaw [2], Dey and Ray [?] and others contributed in neutrosophic topological spaces. Recently, many weak and strong forms of neurosophic open sets such as neutrosophic regular open [3], neutrosophic α-open [3], neutrosophic semi open [3, 7],neutrosophic pre open [3,18],neutrosophic semi pre open [3,16],neutrosophic b-open [6], neutrosophic δ-open sets [17], neutrosophic δ-pre open [17] and weak and strong forms of neurosophic continuity such as neutrosophic semi continuity,and neurosophic precontinuity [3, 18] and neurosophic α-continuity [3], and neurosophic semi pre continuity [3, 16], neutrosophic b-continuity [6] have been investigated by different authors. In this paper we introduce a super class of above classes of neutrosophic open sets and a super class above mentioned neutrosophic continuity of mappings and studied their characterizations and properties.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1. [13] A Neutrosophic set (**NS**) in X is a structure

$$A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$$

where $\mu_A : X \rightarrow]^-0, 1^+[$, $\varpi A : X \rightarrow]^-0, 1^+[$, and $\gamma_A : X \rightarrow]^-0, 1^+[$ denotes the membership, indeterminacy, and non-membership of A satisfies the condition if $^-0 \leq \mu_A(x) + \varpi_A(x) + \gamma_A(x) \leq 3^+, \ \forall \ x \in X$.

In the real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the closed interval [0,1] and sum of membership, indeterminacy, and non-membership degrees of each element of universe of discourse lies between 0 and 3.

Definition 2.2. [12] Let X be a non empty set and the neutrosophic sets A and neutrosophic set B be in the form $A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \varpi_B(x), \gamma_B(x) >: x \in X\}$ and let $\{A_i : i \in J\}$ be a arbitrary family of neutrosophic sets in X. Then:

(a) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, $\varpi_A(x) \leq \varpi_B(x)$, and $\gamma_A(x) \geq \gamma_B(x)$.
(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$.
(c) $A^c = \{< x, \gamma_A(x), \varpi_A(x), \mu_A(x) >: x \in X\}$.
(d) $\cap A_i = \{< x, \wedge\mu_{A_i}(x), \wedge\varpi_{A_i}(x), \vee\gamma_{A_i}(x) >: x \in X\}$.
(e) $\cup A_i = \{< x, \vee\mu_{A_i}(x), \vee\varpi_{A_i}(x), \wedge\gamma_{A_i}(x) >: x \in X\}$.
(f) $\tilde{\mathbf{0}} = \{< x, 0, 0, 1 >: x \in X\}$ and $\tilde{\mathbf{1}} = \{< x, 1, 1, 0 >: x \in X\}$

Definition 2.3. [9, 11] A neutrosophic topology on a non empty set X is a family τ of neutrosophic sets in X, satisfying the following axioms:

(T_1) $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}} \in \tau$

(T_2) $G_1 \bigcap G_2 \in \tau$

(T_3) $G_1 \bigcup G_2 \in \tau$

In this case the pair (X, τ) is called a neutrosophic topological space and each neutrosophic set in τ is known as a neutrosophic open set in X. The complement A^c of a neutrosophic open set A is called a neutrosophic closed set in X.

Definition 2.4. [11] Let (X, τ) be a neutrosophic topological space and A be a neutrosophic set in X. Then the neutrosophic interior and neutrosophic closure of A are defined by:

Cl(A) = ∩ {K: K is a neutrosophic closed set such that A ⊆ K }

Int(A) = ∪ {K: K is a neutrosophic open set such that K ⊆ A }

Definition 2.5. [5] Let $\alpha, \eta, \beta \in [0, 1]$ and $0 \leq \alpha + \eta + \beta \leq 3$. A neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X is a neutrosophic set in X defined by

$$x_{(\alpha,\eta,\beta)}(y) = \begin{cases} (\alpha, \eta, \beta) & \text{if } y = x \\ (0, 0, 1) & \text{if } y \neq x \end{cases} \tag{1}$$

Definition 2.6. [1] Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point in X and $A = \{< x, \mu_A, \varpi_A, \gamma_A >: x \in X\}$ is a neutrosophic set in X. Then $x_{(\alpha,\eta,\beta)} \subseteq A$ if and only if $\alpha \subseteq \mu_A(x)$, $\eta \subseteq \varpi_A$, and $\beta \supseteq \nu_A(x)$.

Definition 2.7. [1] A nuetrosophic point $x_{(\alpha,\eta,\beta)}$ is said to be quasi-coincident (q-coincident, for short) with A, denoted by $x_{(\alpha,\eta,\beta)}qA$ iff $x_{(\alpha,\eta,\beta)} \nsubseteq A^c$. If $x_{(\alpha,\eta,\beta)}$ is not quasi-coincident with A, we denote by $\rceil(x_{(\alpha,\eta,\beta)}qA)$.

Definition 2.8. [1] Two neutrosophic set A and B of X said to be q-coincident (denoted by A_qB) if $A \nsubseteq B^c$.

Lemma 2.9. *[1] For any two neutrosophic sets A and B of X , $\rceil(A_qB) \Leftrightarrow A \subset B^c$. where $\rceil(A_qB)$ A is not q-coincident with B.*

Definition 2.10. [3] A neutrosophic set A of a NTS (X, τ) is called neutrosophic regular open set (resp. neutrosophic regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$).

Remark 2.11. [3] Every neutrosophic regular open (resp. neutrosophic regular closed) set is neutrosophic open (resp. neutrosophic closed), But the converse may not be true.

Definition 2.12. [17] The δ-interior (denoted by δint) of a neutrosophic set A of a NTS (X, τ) is the union of all neutrosophic regular open sets contained in A.

Definition 2.13. [17] The δ-closure (denoted by δcl) of a neutrosophic set A of a NTS (X, τ) is the intersection of all neutrosophic regular closed sets containing A.

Definition 2.14. [3, 6, 17] A neutrosophic set A of a NTS (X, τ) is called neutrosophic semiopen (resp. neutrosophic preopen ,, neutrosophic α-open, neutrosophic semi preopen ,]neutrosophic δ-open , neutrosophic δ-preopen, neutrosophic δ-semiopen , neutrosophic b-open if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(A))$, $A \subseteq int(cl(int(A)))$,$A \subseteq cl(int(cl(A)))$, $A = \delta int(A)$, $A \subseteq int(\delta cl(A))$, $A \subseteq cl(\delta int(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$).

The family of all neutrosophic semiopen (resp. neutrosophic preopen, neutrosophic α-open, neutrosophic semi preopen, neutrosophic δ-open, δ-preopen, neutrosophic δ-semiopen, neutrosophic γ-open) sets of a NTS (X, τ) is denoted by $NSO(X)$ (resp. $NPO(X), N\alpha O(X), NSPO(X), N\delta O(X), N\delta PO(X), N\delta SO(X), NbO(X)$).

Definition 2.15. [3,6,17] A neutrosophic set A in a NTS (X, τ) is called neutrosophic neutrosophic semiclosed (resp. neutrosophic preclosed, neutrosophic α-closed, neutrosophic semi preclosed, neutrosophic δ-preclosed, neutrosophic δ-semiclosed, neutrosophic γ-closed) if $A^c \in NSO(X)$ (resp. $NPO(X), N\alpha O(X), NSPO(X), N\delta O(X), N\delta PO(X), N\delta SO(X), NbO(X)$).

Remark 2.16. [3, 16–18] Every neutrosophic δ-open (resp. neutrosophic δ-closed) set is neutrosophic open (resp. neutrosophic closed), every neutrosophic open (resp. neutrosophic closed) set is neutrosophic α-open (resp. neutrosophic α-closed), every neutrosophic α-open (resp. neutrosophic α-closed) set is neutrosophic semiopen (resp. neutrosophic semiclosed) as well as neutrosophic preopen (resp. neutrosophic preclosed) and every neutrosophic semiopen (resp. neutrosophic semiclosed) set and every neutrosophic preopen (resp. neutrosophic preclosed) set is neutrosophic semi-preopen (resp. neutrosophic semi-preclosed). But the separate converses may not be true.

Remark 2.17. [17] Every neutrosophic preopen (resp. neutrosophic preclosed) set is neutrosophic δ-preopen (resp. neutrosophic δ-preclosed) but the converse may not be true.

Remark 2.18. [6] Every neutrosophic semiopen (resp. neutrosophic semiclosed) and neutrosophic preopen (resp. neutrosophic preclosed) set is neutrosophic b-open (resp.neutrosophic b-closed), and every neutrosophic b-open (resp.neutrosophic b-closed)set is neutrosophic semipreopen (resp.neutrosophic semi-preclosed)but the separate converses may not be true.

Definition 2.19. [1] Consider that f is a mapping from X to Y.

(a) Let A be a neutrosophic set in X with membership function $\mu_A(x)$, indeterminacy function $\varpi_A(x)$ and non-membership function $\sigma_A(x)$. The image of A under f, written as f(A), is a neutrosophic set of Y whose membership function, indeterminacy function and non-membership function are defined as

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \{\mu_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$\varpi_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \{\varpi_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & if f^{-1}(y) = \phi, \end{cases}$$

$$\gamma_{f(A)}(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \{\gamma_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & if f^{-1}(y) = \phi, \end{cases}$$

$\forall\ , y \in Y$. Where $f^{-1}(y) = \{x : f(x) = y\}$.

(b) Let B be a neutrosophic set in Y with membership function $\mu_B(y)$, indeterminacy function $\varpi_B(y)$ and non-membership function $\gamma_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$ is a neutrosophic set of X whose membership function, indeterminacy function and non-membership function are defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\varpi_{f^{-1}(B)}(x) = \varpi_B(f(x))$, and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$, respectively $\forall\ x \in X$.

Definition 2.20. [3,6,17] A mapping $f : (X, \tau) \to (Y, \sigma)$ is called neutrosophic semi continuous (resp. neutrosophic pre continuous , neutrosophic α-continuous,neutrosophic semi pre continuous, neutrosophic δ-pre continuous, neutrosophic b-continuous if $f^{-1}(A) \in NSO(X)$ (resp. $f^{-1}(A) \in NPO(X)$, $f^{-1}(A) \in N\alpha O(X)$, $f^{-1}(A) \in NSPO(X)$, $f^{-1}(A) \in N\delta PO(X)$, $f^{-1}(A) \in NbO(X)$) for each neutrosophic set $A \in\ \sigma$.

Remark 2.21. [3, 16, 17] Every neutrosophic continuous mappings is neutrosophic α-continuous, every neutrosophic α-continuous mapping is neutrosophic semi continuous and neutrosophic pre continuous , every neutrosophic semi continuous (resp. neutrosophic pre continuous) mapping is neutrosophic b-continuous and neutrosophic b-continuous mapping is neutrosophic semi pre continuous but the separate converses may not be true.The concepts of neutrosophic semi continuous and neutrosophic pre continuous mappings are independent.

Remark 2.22. [17] Every neutrosophic pre continuous mapping is neutrosophic δ-pre continuous but the converse may not be true.

3. **Neutrosophic semi δ-preopen sets**

In this section, we introduce the concept of neutrosophic semi δ-preopen set and study some of their properties in neutrosophic topological spaces.

Definition 3.1. A neutrosophic set A in a NTS (X, τ) is called:

(a) neutrosophic semi δ-preopen if there exists a neutrosophic δ-preopen set O such that $O \subseteq A \subseteq \delta cl(O)$.

(b) neutrosophic semi δ-preclosed if there exists a neutrosophic δ-preclosed set F such that $\delta int(F) \subseteq A \subseteq F$.

The family of all neutrosophic semi δ-preopen (resp. neutrosophic semi δ-preclosed) sets of a NTS (X, τ) is denoted by $NS\delta PO(X)$ (resp. $NS\delta PC(X)$).

Remark 3.2. Every neutrosophic δ-semiopen (resp. neutrosophic δ-semiclosed)set is neutrosophic semiopen (resp. neutrosophic semiclosed) but the converse may not be true.

Example 3.3. Let $X = \{a, b\}$ and neutrosophic sets A, B, O are defined as follows:

$$A = \{< a, 0.3, 0.5, 0.7 >, < b, 0.4, 0.5, 0.6 >\}$$

$$B = \{< a, 0.4, 0.5, 0.6 >, < b, 0.3, 0.5, 0.7 >\}$$

$$O = \{< a, 0.5, 0.5, 0.5 >, < b, 0.6, 0.5, 0.4 >\}$$

let $\tau = \{\tilde{0}, A, B, A \cup B, A \cap B, \tilde{1}\}$ be the neutrosophic topology on (X, τ). Then O is neutrosophic semiopen but not neutrosophic δ-semiopen.

Remark 3.4. Every neutrosophic δ-open (resp. neutrosophic δ-closed) set is neutrosophic δ-semiopen (resp. neutrosophic δ-semiclosed) but the converse may not be true. For, in the NTS (X, τ) of example (3.3), the neutrosophic set A is neutrosophic δ-semiopen but not neutrosophic δ-open.

Remark 3.5. The concepts of neutrosophic δ-semiopen and neutrosophic open sets are independent. For, in the NTS (X, τ) of example (3.3), the neutrosophic set O is neutrosophic δ-semiopen but not neutrosophic open and neutrosophic A is neutrosophic open but not neutrosophic δ-semiopen.

Theorem 3.6. *An neutrosophic set $A \in NS\delta PC(X)$ if and only if $A^c \in NS\delta PO(X)$.*

Remark 3.7. Every neutrosophic semi preopen (resp. neutrosophic semi preclosed) set and Every neutrosophic δ-preopen (resp. neutrosophic δ-preclosed) set is neutrosophic semi δ-preopen (resp. neutrosophic semi δ-preclosed). But the separate converse may not be true.

Example 3.8. Let $X = \{a, b\}$ and neutrosophic sets A, B, O, F are neutrosophic sets defined as follows:

$$A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.3, 0.5, 0.7 >\}$$

$$B = \{< a, 0.5, 0.5, 0.5 >, < b, 0.1, 0.5, 0.9 >\}$$

$$O = \{< a, 0.5, 0.5, 0.5 >, < b, 0.9, 0.5, 0.1 >\}$$

$$F = \{< a, 0.5, 0.5, 0.5 >, < b, 0.6, 0.5, 0.4 >\}$$

let $\tau = \{\tilde{0}, A, B, \tilde{1}\}$ be a neutrosophic topology on (X, τ), Then

(a) O is neutrosophic semi δ-preopen (resp. O^c neutrosophic semi preclosed) but not neutrosophic semi preopen (resp. neutrosophic semi preclosed).

(b) F is neutrosophic semi δ-preopen (resp. F^c is neutrosophic semi preclosed) but not neutrosophic δ-preopen (resp. neutrosophic δ-preclosed).

Remark 3.9. It is clear that from remark (2.11), (2.16), (2.17), (2.18), (3.2), (3.4), (3.5) and (3.7) that the following figure of implications is true.

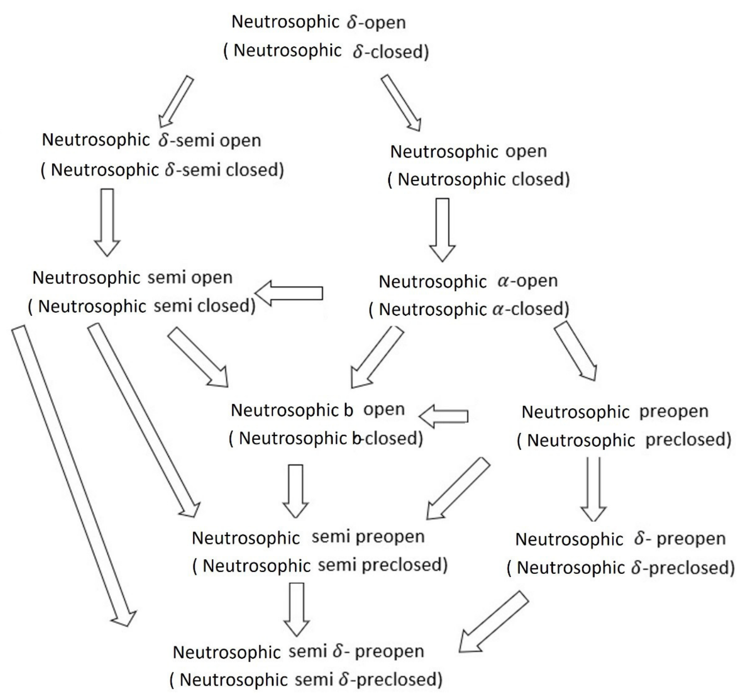

Figure 1

Theorem 3.10. *Let (X, τ) be a NTS . Then*

(a) *Any union of neutrosophic semi δ-preopen sets is neutrosophic semi δ-preopen.*

(b) *Any intersection of neutrosophic semi δ-preclosed sets is neutrosophic semi δ-preclosed.*

Theorem 3.11. *A neutrosophic set $A \in NS\delta PO(X)$ if and only if for every neutrosophic point $x_{(\alpha,\eta,\beta)} \in A$ there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq A$.*

Proof. If $A \in NS\delta PO(X)$ then we may take $O = A$ for every $x_{(\alpha,\eta,\beta)} \in A$.

Conversely. We have $A =_{x_{(\alpha,\eta,\beta)} \in A} \cup \{x_{(\alpha,\eta,\beta)}\} \subseteq_{x_{(\alpha,\eta,\beta)} \in A} \cup O \subseteq A$.The result now follows from the fact that any union of neutrosophic δ-preopen sets is neutrosophic δ-preopen. □

Theorem 3.12. *Let (X, τ) be a NTS .*

(a) *If $A \subseteq O \subseteq \delta cl(A)$ and $A \in NS\delta PO(X)$ then $O \in NS\delta PO(X)$,*

(b) *If $\delta int(B) \subseteq F \subseteq B$ and $B \in NS\delta PC(X)$ then $F \in NS\delta PC(X)$*

Proof. (a) Let $O_1 \in N\delta PO(X)$ such that $O_1 \subseteq A \subseteq \delta cl(O_1)$.
Clearly $O_1 \subseteq O$ and $A \subseteq \delta cl(O_1)$ implies that $\delta cl(A) \subseteq \delta cl(O_1)$. Consequently,
$O_1 \subseteq O \subseteq \delta cl(O_1)$. Hence $O \in NS\delta PO(X)$.
(b) Follows from (a). □

Lemma 3.13. *An neutrosophic set $A \in N\delta PO(X)$ if and only if there exists a neutrosophic set O such that $A \subseteq O \subseteq \delta cl(A)$.*

Proof. **Necessity** If $A \in N\delta PO(X)$ then $A \subseteq int(\delta cl(A))$.
Put $O = int(\delta cl(A))$ then O is a neutrosophic open and $A \subseteq O \subseteq \delta cl(A)$.
Sufficiency Let O be a neutrosophic open set such that $A \subseteq O \subseteq \delta cl(A)$, then $A \subseteq int(O) \subseteq int(\delta cl(A))$. Hence $A \in N\delta PO(X)$. □

Lemma 3.14. *Let Y be a neutrosophic subspace of NTS (X, τ) and A be a neutrosophic set in Y. If $A \in N\delta PO(X)$ then $A \in N\delta PO(Y)$.*

Proof. Since $A \in N\delta PO(X)$, By Lemma (3.13), there exists a neutrosophic set O in (X, τ) such that $A \subseteq O \subseteq \delta cl(A)$. Therefore $A \cap Y \subseteq O \cap Y \subseteq \delta cl(A) \cap Y = \delta cl_Y(A)$. It follows that $A \subseteq O \subseteq \delta cl_Y(A)$. Hence by Lemma (3.13), $A \in N\delta PO(Y)$. □

Theorem 3.15. *Let Y be a neutrosophic subspace of NTS (X, τ) and A be a neutrosophic set in Y. If $A \in NS\delta PO(X)$ then $A \in NS\delta PO(Y)$.*

Proof. Let $O \in N\delta PO(X)$ such that $O \subseteq A \subseteq cl(O)$. Then $O \cap Y \subseteq A \cap Y \subseteq cl(O) \cap Y$. It follows that $O \subseteq A \subseteq cl_Y(O)$. Now by Lemma (3.14), $O \in N\delta PO(Y)$ and hence $A \in NS\delta PO(Y)$. □

Theorem 3.16. *Let X and Y be NTS , such that X is product related to Y.*

(a) *If $A \in N\delta PO(X)$ and $O \in N\delta PO(Y)$, then $A \times O \in N\delta PO(X \times Y)$.*

(b) *N $A \in NS\delta PO(X)$ and $O \in NS\delta PO(Y)$, then $A \times O \in NS\delta PO(X \times Y)$.*

Proof. **(a)** N $A \in N\delta PO(X)$ and $O \in N\delta PO(Y)$. Then $A \times O \subseteq int(\delta cl(A)) \times int(\delta cl(O)) = int(\delta cl(A \times O))$.

(b) Let $F \subseteq A \subseteq \delta cl(F)$ and $\psi \subseteq O \subseteq \delta cl(\psi)$, $F \in N\delta PO(X)$ and $\psi \in N\delta PO(Y)$. Then $F \times \psi \subseteq A \times O \subseteq \delta cl(F) \times \delta cl(\psi) = \delta cl(A \times \psi)$. Now the result follows from (a). □

Definition 3.17. Let (X, τ) be a NTS and A be a neutrosophic set of X. Then the neutrosophic semi δ-preinterior (denoted by $s\delta pint$) and neutrosophic semi δ-preclosure (denoted by $s\delta pcl$) of A respectively defined as follows:
$s\delta pint(A) = \cup\{O : O \subseteq A; O \in NS\delta PO(X)\}$,
$s\delta pcl(A) = \cap\{O : O \supseteq A; O \in NS\delta PC(X)\}$.

The following theorem can be easily verified.

Theorem 3.18. *Let A and O be neutrosophic sets in a NTS (X, τ). Then:*

(a) $s\delta pcl(A) \subseteq cl(A)$
(b) $s\delta pcl(A)$ *is a neutrosophic semi δ-preclosed.*
(c) $A \in NS\delta PC(X) \Leftrightarrow A = s\delta pcl(A)$.
(d) $A \subseteq O \Rightarrow s\delta pcl(A) \subseteq s\delta pcl(O)$.
(e) $int(A) \subseteq s\delta pint(A)$.
(f) $s\delta pint(A)$ *is a neutrosophic semi preopen.*
(g) $A \in NS\delta PO(X) \Leftrightarrow A = s\delta pint(A)$.
(h) $A \subseteq O \Rightarrow s\delta pint(A) \subseteq s\delta pint(O)$.
(i) $s\delta pint(A^c) = (s\delta pcl(A))^c$.

Definition 3.19. Let A be a neutrosophic sets in a NTS (X, τ) and $x_{(\alpha,\eta,\beta)}$ is a neutrosophic point of X. Then A is called:

(a) neutrosophic semi δ-preneighborhood of $x_{(\alpha,\eta,\beta)}$ if there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq A$.
(b) neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ if there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O \subseteq A$.

Theorem 3.20. *An neutrosophic set $A \in NS\delta PO(X)$ if and only if for each neutrosophic point $x_{(\alpha,\eta,\beta)} \in A$, A is a neutrosophic semi δ-preneighborhood of $x_{(\alpha,\eta,\beta)}$.*

Theorem 3.21. *Let A be a neutrosophic sets in a NTS (X, τ). Then a neutrosophic point $x_{(\alpha,\eta,\beta)} \in s\delta pcl(A)$, if and only if every neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ is quasi-coincident with A.*

Proof. **Necessity** Suppose $x_{(\alpha,\eta,\beta)} \in s\delta pcl(A)$ and if possible let there exists a neutrosophic semi δ-pre Q-neighborhood O of $x_{(\alpha,\eta,\beta)}$ such that $\rceil(O_q A)$. Then there exists a neutrosophic set

$O_1 \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O_1 \subseteq O_1 \subseteq O$ which show that $\rceil(O_{1q}A)$ and hence $A \subseteq O_1^c$. As $O_1^c \in NS\delta PC(X)$, $s\delta pcl(A) \subseteq O_1^c$. Since $x_{(\alpha,\eta,\beta)} \in O_1^c$, we obtain that $x_{(\alpha,\eta,\beta)} \notin s\delta pcl(A)$ which is a contradiction.

Sufficiency Suppose every neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ is quasi-coincident with A. If $x_{(\alpha,\eta,\beta)} \notin s\delta pcl(A)$ then there exists a neutrosophic semi δ-preclosed set $O \supseteq A$ such that $x_{(\alpha,\eta,\beta)} \notin O$. So $O^c \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O^c$ and $\rceil(O_q^cA)$ a contradiction. □

Definition 3.22. A mappings $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic δ-preirresolute if $f^{-1}(A) \in NS\delta PO(X)$ for every neutrosophic set $A \in N\delta PO(Y)$.

Theorem 3.23. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic δ-pre irresolute and neutrosophic open mapping, then $f^{-1}(A) \in NS\delta PO(X)$, for every $A \in NS\delta PO(Y)$.*

Proof. Let $A \in NS\delta PO(Y)$. Then there exists a neutrosophic set $O \in N\delta PO(X)$ such that $O \subseteq A \subseteq \delta cl(O)$. Therefore $f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\delta cl(O))$ since f is neutrosophic open and δ-pre irresolute. $f^{-1}(O) \subseteq f^{-1}(A) \subseteq f^{-1}(\delta cl(O)) \subseteq \delta cl(f^{-1}(O))$ and $f^{-1}(O) \in NS\delta PO(X)$. Hence $f^{-1}(A) \in NS\delta PO(X)$. □

4. Neutrosophic semi δ-precontinuous mappings

Definition 4.1. A mappings $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic semi δ-precontinuous if $f^{-1}(A) \in NS\delta PO(X)$ for every neutrosophic set $A \in \sigma$.

Remark 4.2. Every neutrosophic δ-pre continuous (resp. neutrosophic semi precontinuous) mappings is neutrosophic semi δ-precontinuous but the converse may not be true.

Example 4.3. Let $X = \{a, b\}$ and $Y = \{p, q\}$ and neutrosophic sets A, B, O, F are defined as follows:

$$A = \{< a, 0.5, 0.5, 0.5 >, < b, 0.3, 0.5, 0.7 >\}$$

$$B = \{< a, 0.5, 0.5, 0.5 >, < b, 0.1, 0.5, 0.9 >\}$$

$$O = \{< p, 0.5, 0.5, 0.5 >, < q, 0.9, 0.5, 0.1 >\}$$

$$F = \{< p, 0.5, 0.5, 0.5 >, < q, 0.6, 0.5, 0.4 >\}$$

let $\tau_1 = \{\tilde{0}, A, B, \tilde{1}\}$, $\tau_2 = \{\tilde{0}, O, \tilde{1}\}$ and $\tau_3 = \{\tilde{0}, F, \tilde{1}\}$. Then the mapping $f : (X, \tau_1) \to (Y, \tau_2)$ defined by $f(a) = p$, $f(b) = q$ is a neutrosophic semi δ-pre continuous but not neutrosophic semi precontinuous and the mapping $g : (X, \tau_1) \to (Y, \tau_3)$ defined by $g(a) = p$, $g(b) = q$ is a neutrosophic semi δ-precontinuous but not neutrosophic δ-pre continuous.

Remark 4.4. Remark (2.21), (2.22) and (4.2) reveals that the following figure of implications is true.

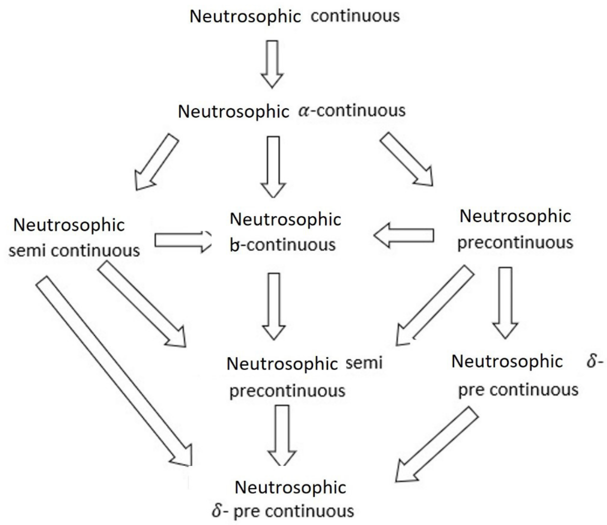

Figure 2

Theorem 4.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping from a NTS (X, τ) to NTS (Y, σ). Then the following statements are equivalent:*

(a) *f is neutrosophic semi δ-precontinuous.*

(b) *for every neutrosophic closed set A in Y, $f^{-1}(A) \in NS\delta PC(X)$.*

(c) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ in X and every neutrosophic open set A such that $f(x_{(\alpha,\eta,\beta)}) \in NS\delta PO(X)$ there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$.*

(d) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neighborhood A of $f(x_{(\alpha,\eta,\beta)}), f^{-1}(A)$ is a neutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$.*

(e) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neighborhood A of $f(x_{(\alpha,\eta,\beta)})$, there is a neutrosophic semi δ-pre neighborhood U of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$.*

(f) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every neutrosophic open set A such that $f(x_{(\alpha,\eta,\beta)})_q A$, there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)_q} O$ and $f(O) \subseteq A$.*

(g) *for every neutrosophic point $x_{(\alpha,\eta,\beta)}$ of X and every Q-neighborhood A of $f(x_{(\alpha,\eta,\beta)}), f^{-1}(A)$ is a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$.*

(h) *for every neutrosophic point* $x_{(\alpha,\eta,\beta)}$ *of* X *and every* Q*-neighborhood* A *of* $f(x_{(\alpha,\eta,\beta)})$, *there is a neutrosophic semi pre* Q*-neighborhood* U *of* $x_{(\alpha,\eta,\beta)}$ *such that* $f(U) \subseteq A$.

(i) $f(s\delta pcl(A)) \subseteq cl(f(A))$, *for every neutrosophic set* A *of* X.

(j) $s\delta pcl(f^{-1}(O)) \subseteq f^{-1}(cl(O))$, *for every neutrosophic set* O *of* Y.

(k) $f^{-1}(int(0)) \subseteq s\delta pint(f^{-1}(O))$, *for every neutrosophic set* O *of* Y.

Proof. $(a) \Rightarrow (b)$ Obvious.

$(a) \Rightarrow (c)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set in Y such that $f(x_{(\alpha,\eta,\beta)}) \in A$. Put $O = f^{-1}(A)$, then by (a), $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$.

$(c) \Rightarrow (a)$ Let A be a neutrosophic open set in Y and $x_{(\alpha,\eta,\beta)} \in f^{-1}(A)$. Then $f(x_{(\alpha,\eta,\beta)}) \in A$. Now by (c) there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O$ and $f(O) \subseteq A$. Then $x_{(\alpha,\eta,\beta)} \in O \subseteq f^{-1}(A)$. Hence by theorem (3.11), $f^{-1}(A) \in NS\delta PO(X)$.

$(a) \Rightarrow (d)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then there is a neutrosophic open set U such that $f(x_{(\alpha,\eta,\beta)}) \in U \subseteq A$. Now $F^{-1}(U) \in NS\delta PO(X)$ and $x_{(\alpha,\eta,\beta)} \in f^{-1}(U) \subseteq f^{-1}(A)$. Thus $f^{-1}(A)$ is a neutrosophic semi pre neighborhood of $x_{(\alpha,\eta,\beta)}$ in X.

$(d) \Rightarrow (e)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then $U = f^{-1}(A)$ is a neutrosophic semi δ-pre neighborhood of $x_{(\alpha,\eta,\beta)}$ and $f(U) = f(f^{-1}(A)) \subseteq A$.

$(e) \Rightarrow (c)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set such that $f(x_{(\alpha,\eta,\beta)}) \in A$. Then A is a neighborhood of $f(x_{(\alpha,\eta,\beta)})$. So there is neutrosophic semi δ-pre neighborhood U of $x_{(\alpha,\eta,\beta)}$ in X such that $x_{(\alpha,\eta,\beta)} \in U$ and $f(U) \subseteq A$. Hence there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)} \in O \subseteq U$ and so $f(O) \subseteq f(U) \subseteq A$.

$(a) \Rightarrow (f)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set in Y such that $f(x_{(\alpha,\eta,\beta)})_q \in A$. Let $O = f^{-1}(A)$. Then $O \in NS\delta PO(X)$, $x_{(\alpha,\eta,\beta)q}O$ and $f(O) = f(f^{-1}(A)) \subseteq A$.

$(f) \Rightarrow (a)$ Let A be a neutrosophic open set in Y and $x_{(\alpha,\eta,\beta)} \in f^{-1}(A)$ clearly $f(x_{(\alpha,\eta,\beta)}) \in A$, choose the neutrosophic point $x^c_{(\alpha,\eta,\beta)}$ defined as

$$x^c_{(\alpha,\eta,\beta)}(z) = \begin{cases} (\beta,\eta,\alpha) & \text{if } z = x \\ (1,0,0) & \text{if } z \neq x \end{cases} \tag{2}$$

Then $f(x_{(\alpha,\eta,\beta)})_q A$ and so by (f), there exists a neutrosophic set $O \in NS\delta PO(X)$, such that $x_{(\alpha,\eta,\beta)q}O$ and $f(O) \subseteq A$.

Now $x^c_{(\alpha,\eta,\beta)q}O$ implies $x_{(\alpha,\eta,\beta)} \in O$.

Thus $x_{(\alpha,\eta,\beta)} \in O \subseteq f^{-1}(A)$. Hence by theorem (3.11) $f^{-1} \in NS\delta PO(X)$.

$(f) \Rightarrow (g)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a Q-neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then there is a neutrosophic open set A_1 in Y such that $f(x_{(\alpha,\eta,\beta)})_q A_1 \subseteq A$. By hypothesis there is a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O$ and $f(O) \subseteq A_1$. Thus $x_{(\alpha,\eta,\beta)q}O \subseteq f^{-1}(A_1) \subseteq f^{-1}(A)$.
Hence $f^{-1}(A)$ is a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$.

$(g) \Rightarrow (h)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a Q-neighborhood of $f(x_{(\alpha,\eta,\beta)})$. Then $U = f^{-1}(A)$ is a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$ and $f(U) = f(f^{-1}(A)) \subseteq A$.

$(g) \Rightarrow (f)$ Let $x_{(\alpha,\eta,\beta)}$ be a neutrosophic point of X and A be a neutrosophic open set such that $f(x_{(\alpha,\eta,\beta)})_q \in A$. Then A is Q-neighborhood of $f(x_{(\alpha,\eta,\beta)})$. So there is a neutrosophic semi δ-pre Q-neighborhood δ of $x_{(\alpha,\eta,\beta)}$ such that $f(U) \subseteq A$. Now U being a neutrosophic semi δ-pre Q-neighborhood of $x_{(\alpha,\eta,\beta)}$. Then there exists a neutrosophic set $O \in NS\delta PO(X)$ such that $x_{(\alpha,\eta,\beta)q}O \subseteq U$. Hence $x_{(\alpha,\eta,\beta)q}O$ and $f(O) \subseteq f(U) \subseteq A$.

$(b) \Leftrightarrow (i)$ Obvious.
$(h) \Leftrightarrow (j)$ Obvious.
$(j) \Leftrightarrow (k)$ Obvious. □

Theorem 4.6. *Let X, X_1, X_2 be a neutrosophic topological spaces and $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection of $X_1 \times X_2$ into X_i. Then if $f : X_1 \times X_2$ is a neutrosophic semi δ- pre continuous mapping, it follows that $p_i o f$ is also a neutrosophic semi δ- pre continuous mapping.*

Theorem 4.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping. If the graph mapping $g : X \to X \times Y$ of f is a neutrosophic semi δ-pre continuous, then f is a neutrosophic semi δ-pre continuous.*

Proof. Let O be a neutrosophic open set of Y. Then $\tilde{1} \times O$ is a neutrosophic open in $X \times Y$. Since g is a neutrosophic semi δ-pre continuous, $g^{-1}(\tilde{1} \times O) \in NS\delta PO(X)$. But $f^{-1}(O) = \tilde{1} \cap f^{-1}(O) = g^{-1}(\tilde{1} \times O)$, $f^{-1}(O) \in NS\delta PO(X)$. Hence f is a neutrosophic semi δ-pre continuous. □

Theorem 4.8. *Let X_i and X_i^* $(i = 1, 2)$ be a neutrosophic topological spaces such that X_1 is product related to X_2. If $f_i : X_i \to X_i^*$ $(i = 1, 2)$ is a neutrosophic semi δ-pre continuous, then $f_1 \times f_2 : X_1 \times X_2 \to X_i^* \times X_2^*$ is a neutrosophic semi δ-pre continuous.*

Proof. Let $A = \cup(A_\alpha \times O_\beta)$ where $A_\alpha's$ and $O_\beta's$ are neutrosophic open sets of X_1^* and X_2^* respectively, be a neutrosophic open sets of $X_1^* \times X_2^*$. We obtain $(f_1 \times f_2)^{-1}(A) = \cup\{f_1^{-1}(A_\alpha) \times f_2^{-1}(O_\beta)\}$. Since f_1 and f_2 are neutrosophic semi δ-pre continuous, $f_1^{-1}(A_\alpha) \in NS\delta PO(X_1)$ and $f_2^{-1}(O_\beta) \in NS\delta PO(X_2)$. Therefore by theorem (3.16)(b), $f_1^{-1}(A_\alpha) \times f_2^{-1}(O_\beta) \in NS\delta PO(X_1 \times X_2)$. Hence by theorem (3.10), $(f_1 \times f_2)^{-1}(A) \in NS\delta PO(X_1 \times X_2)$. □

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On Neutrosophic Crisp g$^{\#}\alpha$ Closed Set Operators

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Abstract. The concept of $g^{\#}\alpha$ closed set in general topological spaces was first introduced by Muthukumaraswamy K et. al.,. Recently, kokilavani V, Tharani K et. al., introduced neutrosophic crisp $g^{\#}\alpha$ closed set in neutrosophic crisp topological space. Now, in this present paper, we introduced and study the neutrosophic crisp topological properties of neutrosophic crisp $g^{\#}\alpha$ interior, neutrosophic crisp $g^{\#}\alpha$ closure, neutrosophic crisp $g^{\#}\alpha$ frontier, neutrosophic crisp $g^{\#}\alpha$ border, neutrosophic crisp $g^{\#}\alpha$ exterior via the concept of neutrosophic crisp $g^{\#}\alpha$ open set.

Keywords: $NCg^{\#}\alpha int(\mathcal{A})$; $NCg^{\#}\alpha cl(\mathcal{A})$; $NCg^{\#}\alpha Fr(\mathcal{A})$; $NCg^{\#}\alpha Br(\mathcal{A})$; $NCg^{\#}\alpha Ext(\mathcal{A})$

1. Introduction

Zadeh [13] proposed the concept of a fuzzy set that provides a degree of membership function in 1965. Chang [3] first proposed the idea of fuzzy topological space, in 1968. Atanassov [2] created the next stage of fuzzy sets in 1983. These sets, known as intuitionistic fuzzy sets provide a degree of membership and a degree of non-membership functions. Coker [4] introduced the idea of intuitionistic fuzzy topological space in intuitionistic fuzzy sets in 1997. Salama and Alblowi [5] defined neutrosophic topological space and many of its applications. The concept of neutrosophic crisp set and neutrosophic set was investigated by Smaradache [7] [10] [11] in 2005. Since the invention of the neutrosophic set, numerous mathematical topics and applications have been developed. The neutrosophic closed sets and neutrosophic continuous functions were introduced by Salama et.al. [6] in 2014. Salama, et, al. [9] proposed an innovative mathematical model called " Neutrosophic crisp sets and Neutrosophic crisp topological spaces ".

Salama, et, al., [8] expand the notion of neutrosophic crisp topological spaces to neutrosophic crisp α-topological spaces in 2016. V. Kokilavani , K.Tharani et. al., [12] presented neutrosophic crisp $g^{\#}\alpha$ closed set in neutrosophic crisp topological space. Riad K. Al-Hamido [1] introduced new operators like neutrosophic crisp frontier, neutrosophic crisp border and neutrosophic crisp exterior using neutrosophic crisp open set in 2023. In this paper, we use the neutrosophic crisp sets to introduce neutrosophic crisp $g^{\#}\alpha$ interior, neutrosophic crisp $g^{\#}\alpha$ closure, neutrosophic crisp $g^{\#}\alpha$ frontier, neutrosophic crisp $g^{\#}\alpha$ border, neutrosophic crisp $g^{\#}\alpha$ exterior and discuss their properties in neutrosophic crisp topological space.

2. Preliminaries

Definition 2.1. [9] Let (X,Γ) be a $NCTS$ on X and $\mathcal{A}$ be a NCS on X. Then the neutrosophic crisp closure of $\mathcal{A}$ (shortly $NC\text{cl}(\mathcal{A})$) and neutrosophic crisp interior (shortly $NC\text{int}(\mathcal{A})$) of $\mathcal{A}$ are defined by

$NC\text{cl}(\mathcal{A})=\cap\ \{\mathcal{C}:\mathcal{A} \subseteq \mathcal{C}\ \&\ \mathcal{C}$ is a $NCCS$ in X}

$NC\text{int}(\mathcal{A})=\cup\ \{\mathcal{F}:\mathcal{F} \subseteq \mathcal{A}\ \&\ \mathcal{F}$ is a $NCOS$ in X}

Definition 2.2. Let $\mathcal{A}$ be a neutrosophic crisp subset, and let $\mathcal{F}$ be a $NCgOS$ in a $NCTS(X,\Gamma)$ where $\mathcal{A} \subseteq \mathcal{F}$ then $\mathcal{A}$ is called neutrosophic crisp g$^{\#}\alpha$-closed set (briefly,$NCg^{\#}\alpha CS$) if $NC\alpha cl(\mathcal{A}) \subseteq \mathcal{F}$ and the complement of a $NCg^{\#}\alpha CS$ is a $NCg^{\#}\alpha OS$ in (X,Γ).

3. Neutrosophic Crisp $g^{\#}\alpha$ Interior

In this section, we introduce neutrosophic crisp $g^{\#}\alpha$ interior and discuss their properties in neutrosophic crisp topological spaces.

Definition 3.1. Let (X,Γ) be a $NCTS$ and let $x \in$ X. A subset $\mathcal{A}$ of X is said to be $NCg^{\#}\alpha$-neighbourhood of x if there exists a $NCg^{\#}\alpha$ open set $\mathcal{F}$ such that $x \in \mathcal{F} \subset \mathcal{A}$.

Definition 3.2. Let (X,Γ) be a $NCTS$ and let $\mathcal{A} \subset$ X. A point $x \in \mathcal{A}$ is said to be $NCg^{\#}\alpha$ interior point of $\mathcal{A}$ if and only if $\mathcal{A}$ is a $NCg^{\#}\alpha$-neighbourhood of x.

Remark 3.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of the $NCTS(X,\Gamma)$. Then the set of all $NCg^{\#}\alpha$ interior points of $\mathcal{A}$ is called the $NCg^{\#}\alpha$ interior of $\mathcal{A}$ and is denoted by $NCg^{\#}\alpha int(\mathcal{A})$.

Theorem 3.1. *If $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha int(\mathcal{A}) = \cup\{\mathcal{F}$: $\mathcal{F}$ is a $NCg^{\#}\alpha$ open, $\mathcal{F} \subset \mathcal{A}\}$.*

Proof. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$.

$$\begin{aligned} \text{Then } x \in NCg^{\#}\alpha int(\mathcal{A}) &\Leftrightarrow x \text{ is a } NCg^{\#}\alpha \text{ interior point of } \mathcal{A}. \\ &\Leftrightarrow \mathcal{A} \text{ is a } NCg^{\#}\alpha \text{ nbhd of point } x. \\ &\Leftrightarrow \text{there exists } NCg^{\#}\alpha \text{ open set } \mathcal{F} \text{ such that } x \in \mathcal{F} \subset \mathcal{A}. \\ &\Leftrightarrow x \in \cup\{\mathcal{F} : \mathcal{F} \text{ is a } NCg^{\#}\alpha \text{ open}, \mathcal{F} \subset \mathcal{A}\} \end{aligned}$$

Hence $NCg^{\#}\alpha int(\mathcal{A}) = \cup\{\mathcal{F} : \mathcal{F} \text{ is a } NCg^{\#}\alpha\ open,\ \mathcal{F} \subset \mathcal{A}\}$. □

Theorem 3.2. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X, \Gamma)$. Then*

(i) $NCg^{\#}\alpha int(X_N) = X_N$ *and* $NCg^{\#}\alpha int(\phi_N) = \phi_N$
(ii) $NCg^{\#}\alpha int(\mathcal{A}) \subset \mathcal{A}$.
(iii) *If $\mathcal{D}$ is any $NCg^{\#}\alpha OS$ contained in $\mathcal{A}$, then $\mathcal{D} \subset NCg^{\#}\alpha int(\mathcal{A})$*
(iv) *If $\mathcal{A} \subset \mathcal{D}$, then $NCg^{\#}\alpha int(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{D})$*
(v) $NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}))= NCg^{\#}\alpha int(\mathcal{A})$

Proof.

(i) Since X_N and ϕ_N are $NCg^{\#}\alpha$ open sets,

$$\begin{aligned} NCg^{\#}\alpha int(X_N) &= \cup\{\mathcal{F} : \mathcal{F} \text{ is a } NCg^{\#}\alpha open, \mathcal{F} \subset X\} \\ &= X \cup \mathcal{F} \text{ is } NCg^{\#}\alpha OS \\ &= X_N \end{aligned}$$

(ie) $NCg^{\#}\alpha int(X_N)$=X_N. Since ϕ_N is the only $NCg^{\#}\alpha OS$ contained in ϕ_N, $NCg^{\#}\alpha int(\phi_N)$=$\phi_N$

(ii) Let $X \in NCg^{\#}\alpha int(\mathcal{A}) \Rightarrow x$ is a interior point of $\mathcal{A}$.

$$\begin{aligned} &\Rightarrow \mathcal{A} \text{ is a nbhd of } x \\ &\Rightarrow x \in \mathcal{A} \end{aligned}$$

Thus, $x \in NCg^{\#}\alpha int(\mathcal{A}) \Rightarrow x \in \mathcal{A}$. Hence, $NCg^{\#}\alpha int(\mathcal{A}) \subset \mathcal{A}$.

(iii) Let $\mathcal{D}$ be any $NCg^{\#}OS$ such that $\mathcal{D} \subset \mathcal{A}$. Let $x \in \mathcal{D}$. Since $\mathcal{D}$ is a $NCg^{\#}OS$ contained in $\mathcal{A}$. x is a $NCg^{\#}\alpha$ interior point of $\mathcal{A}$. (ie) $x \in NCg^{\#}\alpha int(\mathcal{A})$. Hence $\mathcal{D} \subset NCg^{\#}\alpha int(\mathcal{A})$.

(iv) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X, \Gamma)$ such that $\mathcal{A} \subset \mathcal{D}$. Let $x \in NCg^{\#}\alpha int(\mathcal{A})$. Then x is a $NCg^{\#}\alpha$ interior point of $\mathcal{A}$ and so $\mathcal{A}$ is a $NCg^{\#}\alpha$-nbhd of x. Since $\mathcal{D} \supset \mathcal{A}, \mathcal{D}$ is also $NCg^{\#}\alpha$-nbhd of x. $\Rightarrow x \in NCg^{\#}\alpha int(\mathcal{D})$. Thus we have shown that $x \in NCg^{\#}\alpha int(\mathcal{A}) \Rightarrow x \in NCg^{\#}\alpha int(\mathcal{D})$.

(v) Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$. $NCg^{\#}\alpha int(\mathcal{A})$ is $NCg^{\#}\alpha OS$ and hence $NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A})) = NCg^{\#}\alpha int(\mathcal{A})$

Theorem 3.3. *If a neutrosophic crisp subset $\mathcal{A}$ of $NCTS(X,\Gamma)$ is $NCg^{\#}\alpha$open, then $NCg^{\#}\alpha int(\mathcal{A})$=$\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be $NCg^{\#}\alpha$open subset of $NCTS(X,\Gamma)$. We know that $NCg^{\#}\alpha int(\mathcal{A}) \subset \mathcal{A}$. Also, $\mathcal{A}$ is $NCg^{\#}\alpha OS$ contained in $\mathcal{A}$. From Theorem 3.2 (iii) $\mathcal{A} \subset NCg^{\#}\alpha int(\mathcal{A})$. Hence $NCg^{\#}\alpha int(\mathcal{A})$=$\mathcal{A}$. □

Theorem 3.4. *If $\mathcal{A}$ and $\mathcal{D}$ are neutrosophic crisp subset of $NCTS(X,\Gamma)$, then $NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A} \cup \mathcal{D})$.*

Proof. We know that $\mathcal{A} \subset \mathcal{A} \cup \mathcal{D}$ and $\mathcal{D} \subset \mathcal{A} \cup \mathcal{D}$. The result from Theorem 3.2 (iv) that $NCg^{\#}\alpha int(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A}\cup\mathcal{D})$ and also we have $NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A}\cup\mathcal{D})$ This implies that $NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A}\cup\mathcal{D})$. □

Theorem 3.5. *If $\mathcal{A}$ and $\mathcal{D}$ are neutrosophic crisp subset of $NCTS(X,\Gamma)$, then $NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) = NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$.*

Proof. We know that $\mathcal{A} \cap \mathcal{D} \subset \mathcal{A}$ and $\mathcal{A} \cap \mathcal{D} \subset \mathcal{D}$. The result from Theorem 3.2 (iv) that $NCg^{\#}\alpha int(\mathcal{A}\cap\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A})$ and $NCg^{\#}\alpha int(\mathcal{A}\cap\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{D})$. This implies that

$$NCg^{\#}\alpha int(\mathcal{A}\cap\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D}). \qquad (1)$$

Let $x \in NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$. Then $x \in NCg^{\#}\alpha int(\mathcal{A})$ and $x \in NCg^{\#}\alpha int(\mathcal{D})$. Hence x is a $NCg^{\#}\alpha$-int point of each of sets $\mathcal{A}$ and $\mathcal{D}$. It follows that $\mathcal{A}$ and $\mathcal{D}$ is $NCg^{\#}\alpha$-nbhds of x, so that their intersection $\mathcal{A} \cap \mathcal{D}$ is also a $NCg^{\#}\alpha$-nbhds of x. Hence $x \in NCg^{\#}\alpha int(\mathcal{A}\cap\mathcal{D})$. Thus $x \in NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$ implies that $x \in NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D})$. Therefore

$$NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D}) \subset NCg^{\#}\alpha int(\mathcal{A}\cap\mathcal{D}) \qquad (2)$$

From (1) and (2), We get $NCg^{\#}\alpha int(\mathcal{A} \cap \mathcal{D}) = NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha int(\mathcal{D})$. □

Theorem 3.6. *If $\mathcal{A}$ neutrosophic crisp subset of a $NCTS(X,\Gamma)$, then $NCint(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ neutrosophic crisp subset of a $NCTS(X, \Gamma)$.

Let $x \in NCint(\mathcal{A}) \Rightarrow x \in \cup \{\mathcal{F}$: $\mathcal{F}$ is $NCOS$, $\mathcal{F} \subset \mathcal{A}\}$

$\Rightarrow$ there exists an NCOS $\mathcal{F}$ such that $x \in \mathcal{F} \subset \mathcal{A}$.

$\Rightarrow$ there exist a $NCg^{\#}\alpha OS$ $\mathcal{F}$ such that $x \in \mathcal{F} \subset \mathcal{A}$,

as every NCOS is a $NCg^{\#}\alpha OS$ in x.

$\Rightarrow x \in \cup\{\mathcal{F} : \mathcal{F}$ is $NCg^{\#}\alpha OS,\ \mathcal{F} \subset \mathcal{A}\}$

$\Rightarrow x \in NCg^{\#}\alpha int(\mathcal{A})$

Thus $x \in NCint(\mathcal{A}) \Rightarrow x \in NCg^{\#}\alpha int(\mathcal{A})$. Hence $NCint(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$. □

Remark 3.2. If $\mathcal{A}$ is a neutrosophic crisp subset of $NCTS(X, \Gamma)$, then

(i) $NC\alpha int(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$

(ii) $NCg\alpha gint(\mathcal{A}) \subset NCg^{\#}\alpha int(\mathcal{A})$

(iii) $NCg^{\#}\alpha int(\mathcal{A}) \subset NC\alpha gint(\mathcal{A})$

(iv) $NCg^{\#}\alpha int(\mathcal{A}) \subset NCgsint(\mathcal{A})$

4. Neutrosophic Crisp $g^{\#}\alpha$ Closure

Definition 4.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$. We define the $NCg^{\#}\alpha$ closure of $\mathcal{A}$ to be the intersection of all $NCg^{\#}\alpha CS's$ containing $\mathcal{A}$. It denotes, $NCg^{\#}\alpha cl(\mathcal{A})$ $= \cap \{\mathcal{C} : \mathcal{C}$ is a $NCg^{\#}\alpha CS$ and $\mathcal{A} \subset \mathcal{C}\}$.

Theorem 4.1. *If $\mathcal{A}$ and $\mathcal{D}$ are neutrosophic crisp subset of $NCTS(X, \Gamma)$. Then,*

(i) $NCg^{\#}\alpha cl(X_N) = X_N$ *and* $NCg^{\#}\alpha cl(\phi_N) = \phi_N$

(ii) $\mathcal{A} \subset NCg^{\#}\alpha cl(\mathcal{A})$

(iii) *If $\mathcal{D}$ is any $NCg^{\#}\alpha$ closed set containing $\mathcal{A}$, then $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{D}$*

(iv) *If $\mathcal{A} \subset \mathcal{D}$ then $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCg^{\#}\alpha cl(\mathcal{D})$*

(v) $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})$

(vi) $NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) = NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$

(vii) $NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})) = NCg^{\#}\alpha cl(\mathcal{A})$

Proof.

(i) By the definition of $NCg^{\#}\alpha cl(\mathcal{A})$, X is the only $NCg^{\#}\alpha$ closed set containing X.

$\therefore\ NCg^{\#}\alpha cl(X_N) =$ Intersection of all the $NCg^{\#}\alpha$ closed sets containing X.

$= \cap\{X\} = X_N$

That is $NCg^{\#}\alpha cl(X_N)$=X_N.

Consequently,

$$NCg^{\#}\alpha cl(\phi_N) = \text{Intersection of all the } NCg^{\#}\alpha \text{ closed sets containing } \phi$$
$$= \cap\{\phi\} = \phi_N.$$

That is $NCg^{\#}\alpha cl(\phi_N) = \phi_N$.

(ii) By the definition of $NCg^{\#}$ closure of $\mathcal{A}$, it is obvious that $\mathcal{A} \subset NCg^{\#}\alpha cl(A)$.

(iii) Let $\mathcal{D}$ be any $NCg^{\#}\alpha$CS containing A. Since $NCg^{\#}\alpha cl(A)$ is the intersection of all $NCg^{\#}\alpha CS's$ containing $\mathcal{A}$, $NCg^{\#}\alpha cl(A)$ is contained in every $NCg^{\#}\alpha CS$ containing $\mathcal{A}$. Hence in particular, $NCg^{\#}\alpha cl(A) \subset \mathcal{D}$.

(iv) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of (X, Γ) such that $\mathcal{A} \subset \mathcal{D}$. By the definition $NCg^{\#}\alpha cl(D)$ =$\cap$ {$\mathcal{C}$: $\mathcal{D} \subset \mathcal{C} \in NCg^{\#}\alpha C(X)$ }. If $\mathcal{D} \subset \mathcal{C} \in NCg^{\#}\alpha C(X)$, then $NCg^{\#}\alpha cl(\mathcal{D}) \subset \mathcal{C}$. Since $\mathcal{A} \subset \mathcal{D}$, and by the definition, if $\mathcal{D} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$ for any $\mathcal{C} \in NCg^{\#}\alpha C(X)$, we have $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{C}$. Therefore $NCg^{\#}\alpha cl(\mathcal{A}) \subset \cap$ {$\mathcal{C}$: $\mathcal{D} \subset \mathcal{C} \in NCg^{\#}\alpha C(X)$}= $NCg^{\#}\alpha cl(\mathcal{D})$.

(i.e) $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCg^{\#}\alpha cl(\mathcal{D})$.

(v) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of (X, Γ). Clearly $\mathcal{A} \cap \mathcal{D} \subset \mathcal{A}$ and $\mathcal{A} \cap \mathcal{D} \subset \mathcal{D}$. By theorem $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A})$ and $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{D})$. Hence $NCg^{\#}\alpha cl(\mathcal{A} \cap \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})$.

(vi) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of (X, Γ). Clearly $\mathcal{A} \subset \mathcal{A} \cup \mathcal{D}$ and $\mathcal{D} \subset \mathcal{A} \cup \mathcal{D}$. We have $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D})$ and $NCg^{\#}\alpha cl(\mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D})$. Hence,

$$NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) \tag{1}$$

Since $NCg^{\#}\alpha cl(\mathcal{A})$, $NCg^{\#}\alpha cl(\mathcal{D})$ are NCCS's. $NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$ are NCCS. Also, $\mathcal{A} \subset NCg^{\#}\alpha cl(\mathcal{A})$ and $\mathcal{D} \subset NCg^{\#}\alpha cl(\mathcal{D})$, which implies $\mathcal{A} \cup \mathcal{D} \subset NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$. Thus, $NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$ is a NCCS containing $\mathcal{A} \cup \mathcal{D}$. Since, $NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D})$ is the smallest NCCS containing $\mathcal{A} \cup \mathcal{D}$, we have

$$NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) \subset NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D}) \tag{2}$$

from (1) and (2) we have, $NCg^{\#}\alpha cl(\mathcal{A} \cup \mathcal{D}) = NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})$.

Theorem 4.2. *If $\mathcal{A} \subset X$ is $NCg^{\#}\alpha$ closed, then $NCg^{\#}\alpha cl(\mathcal{A})$ =$\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be $NCg^{\#}\alpha$ closed neutrosophic crisp subset of (X, Γ). We know that $\mathcal{A} \subset NCg^{\#}\alpha cl(\mathcal{A})$. Also $\mathcal{A}$ is $NCg^{\#}\alpha$ closed set containing $\mathcal{A}$. By theorem (iii) $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{A}$. Hence, $NCg^{\#}\alpha cl(\mathcal{A}) = \mathcal{A}$.

Theorem 4.3. *If $\mathcal{A}$ is a neutrosophic crisp subset of a space (X,Γ), then $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCcl(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ is a neutrosophic crisp subset of a space (X,Γ). By the definition of Neutrosophic crisp closure,$NCcl(\mathcal{A})$=$\cap$ $\{\mathcal{C}$: $\mathcal{C}$ is NC closed, $\mathcal{A} \subset \mathcal{C}\}$. If $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{C}$ is a neutrosophic crisp closed subset of X, Then $\mathcal{A} \subset \mathcal{C} \in NCg^{\#}\alpha cl(X)$, because every neutrosophic crisp closed set is $NCg^{\#}\alpha$ closed set. That is $NCg^{\#}\alpha cl(\mathcal{A}) \subset \mathcal{C}$. Therefore $NCg^{\#}\alpha cl(\mathcal{A}) \subset \cap \{\mathcal{C} : \mathcal{A} \subset \mathcal{C}$ and $\mathcal{C}$ is a neutrosophic crisp closed in $X\}$= $NCcl(\mathcal{A})$. Hence $NCg^{\#}\alpha cl(\mathcal{A}) \subset NCcl(\mathcal{A})$. □

Remark 4.1. Let $\mathcal{A}$ be any neutrosophic crisp subset of X. Then

(i) $(NCg^{\#}\alpha int(\mathcal{A}))^c = NCg^{\#}\alpha cl(\mathcal{A}^c)$
(ii) $NCg^{\#}\alpha int(\mathcal{A}) = (NCg^{\#}\alpha cl(\mathcal{A}^c))^c$
(iii) $NCg^{\#}\alpha cl(\mathcal{A}) = (NCg^{\#}\alpha int(\mathcal{A}^c))^c$

5. Neutrosophic Crisp $g^{\#}\alpha$ Frontier

Definition 5.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha$ frontier of $\mathcal{A}$ is defined as $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$.

Theorem 5.1. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then*

(i) $NCg^{\#}\alpha Fr(X_N) = \phi_N$ *and* $NCg^{\#}\alpha Fr(\phi_N) = \phi_N$
(ii) $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha Fr(\mathcal{A}^c)$
(iii) $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$
(iv) *If $\mathcal{A}$ is $NCg^{\#}\alpha CS$ in X if and only if $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}$*
(v) *If $\mathcal{A}$ is $NCg^{\#}\alpha OS$ in X, then $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}^c$*
(vi) $(NCg^{\#}\alpha Fr(\mathcal{A}))^c = NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha int(\mathcal{A}^c)$
(vii) $\mathcal{A} \cup NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq NCg^{\#}\alpha cl(\mathcal{A})$
(viii) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha int(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$
(ix) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha cl(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$
(x) $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A}$ - $NCg^{\#}\alpha Fr(\mathcal{A})$
(xi) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$
(xii) $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))) \subseteq NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))$

Proof.

(i) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$.

$$
\begin{aligned}
NCg^{\#}\alpha Fr(X_N) &= NCg^{\#}\alpha cl(X_N) \cap NCg^{\#}\alpha cl({X_N}^c)\\
&= NCg^{\#}\alpha cl(X_N) \cap NCg^{\#}\alpha cl(\phi_N)\\
&= X_N \cap \phi_N\\
&= \phi_N\\
NCg^{\#}\alpha Fr(\phi_N) &= NCg^{\#}\alpha cl(\phi_N) \cap NCg^{\#}\alpha cl({\phi_N}^c)\\
&= NCg^{\#}\alpha cl(\phi_N) \cap NCg^{\#}\alpha cl(X_N)\\
&= \phi_N \cap X_N\\
&= \phi_N
\end{aligned}
$$

(ii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$. Then by definition of $NCg^{\#}\alpha$ frontier,

$$
\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}) &= NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)\\
&= NCg^{\#}\alpha cl(\mathcal{A}^c) \cap NCg^{\#}\alpha cl(\mathcal{A})\\
&= NCg^{\#}\alpha cl(\mathcal{A}^c) \cap (NCg^{\#}\alpha cl(\mathcal{A}^c))^c\\
&= NCg^{\#}\alpha Fr(\mathcal{A}^c) \text{ [But, by Definition 5.1]}
\end{aligned}
$$

(iii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$. Since, $(NCg^{\#}\alpha cl(\mathcal{A}))^c = NCg^{\#}\alpha int(\mathcal{A}^c)$, then $(NCg^{\#}\alpha cl(\mathcal{A}^c))^c = NCg^{\#}\alpha int(\mathcal{A})$

By Definition 5.1, $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$
$= NCg^{\#}\alpha cl(\mathcal{A}) \cap (NCg^{\#}\alpha int(\mathcal{A}))^c$

By using, $\mathcal{A} - \mathcal{D} = \mathcal{A} \cap \mathcal{D}^c$
$= NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$

Hence, $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$

(iv) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$.

By Definition 5.1, $NCg^{\#}\alpha Fr(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$
$\subseteq NCg^{\#}\alpha cl(\mathcal{A})$
$=\mathcal{A}$

Therefore, $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}$

Conversely,

Assume that, $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}$. Then $NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A}$. Since, $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A}$. We conclude that, $NCg^{\#}\alpha cl(\mathcal{A}) = \mathcal{A}$ and hence $\mathcal{A}$ is $NCg^{\#}\alpha CS$.

(v) Let $\mathcal{A}$ be a $NCg^{\#}\alpha OS$ in $NCTS(X,\Gamma)$. Then $\mathcal{A}^c$ is $NCg^{\#}\alpha CS$ in $NCTS(X,\Gamma)$. By the Theorem 5.1 (iv), $NCg^{\#}\alpha Fr(\mathcal{A}^c) \subseteq \mathcal{A}^c$.
and by Theorem 5.1 (ii), $NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq \mathcal{A}^c$.

(vi) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$.

By Defintion 5.1, $(NCg^{\#}\alpha Fr(\mathcal{A}))^c = (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c))^c$
$= (NCg^{\#}\alpha cl(\mathcal{A}))^c \cup (NCg^{\#}\alpha cl(\mathcal{A}^c))^c$
$= (NCg^{\#}\alpha int(\mathcal{A}^c)) \cup (NCg^{\#}\alpha int(\mathcal{A}))$

Hence, $(NCg^{\#}\alpha Fr(\mathcal{A}))^c = (NCg^{\#}\alpha int(\mathcal{A}^c)) \cup (NCg^{\#}\alpha int(\mathcal{A}))$

(vii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$.

By Defintion 5.1, $\mathcal{A} \cup NCg^{\#}\alpha Fr(\mathcal{A}) = \mathcal{A} \cup (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c))$
$= (\mathcal{A} \cup NCg^{\#}\alpha cl(\mathcal{A})) \cap (\mathcal{A} \cup (NCg^{\#}\alpha cl(\mathcal{A}^c))$
$\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$
$\subseteq NCg^{\#}\alpha cl(\mathcal{A}^c)$

Hence, $\mathcal{A} \cup NCg^{\#}\alpha Fr(\mathcal{A}) \subseteq NCg^{\#}\alpha cl(\mathcal{A}^c)$

(viii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$. Then by Definition 5.1,

$$\begin{aligned}
NCg^{\#}\alpha Fr(NCg^{\#}\alpha int(\mathcal{A})) &= NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}))^c \\
&= NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}^c)) \\
&\qquad [(NCg^{\#}\alpha int(\mathcal{A}))^c = NCg^{\#}\alpha cl(\mathcal{A}^c)] \\
&= NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})) \cap NCg^{\#}\alpha cl(\mathcal{A}^c) \\
&\qquad [NCg^{\#}\alpha cl(\mathcal{A}^c) \text{ is } NCg^{\#}\alpha CS] \\
&\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c) \quad [\text{by Theorem 3.2 (ii)}] \\
&= NCg^{\#}\alpha Fr(\mathcal{A}) \quad [\text{again by Definition 5.1}]
\end{aligned}$$

$NCg^{\#}\alpha Fr(NCg^{\#}\alpha int(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

(ix) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$. Then by Definition 5.1,

$$\begin{aligned}
NCg^{\#}\alpha Fr(NCg^{\#}\alpha cl(\mathcal{A})) &= NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}))^c \\
&= NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}^c)) \\
&\qquad [(NCg^{\#}\alpha int(\mathcal{A}))^c = NCg^{\#}\alpha cl(\mathcal{A}^c)] \\
&\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c) \\
&= NCg^{\#}\alpha Fr(\mathcal{A}) \quad [\text{again by Definition 5.1}]
\end{aligned}$$

$NCg^{\#}\alpha Fr(NCg^{\#}\alpha cl(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

(x) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X,\Gamma)$.
$$\begin{aligned}\mathcal{A} - NCg^{\#}\alpha Fr(\mathcal{A}) &= \mathcal{A} \cap (NCg^{\#}\alpha Fr(\mathcal{A}))^c \\ &= \mathcal{A} \cap (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c))^c \quad \text{[by Definition 5.1]} \\ &= \mathcal{A} \cap (NCg^{\#}\alpha int(\mathcal{A}^c) \cup NCg^{\#}\alpha int(\mathcal{A})) \\ &= (\mathcal{A} \cap NCg^{\#}\alpha int(\mathcal{A}^c)) \cup (\mathcal{A} \cap NCg^{\#}\alpha int(\mathcal{A})) \\ &= (\mathcal{A} \cap NCg^{\#}\alpha int(\mathcal{A}^c)) \cup NCg^{\#}\alpha int(\mathcal{A}) \supseteq NCg^{\#}\alpha int(\mathcal{A})\end{aligned}$$
Hence, $NCg^{\#}\alpha int(\mathcal{A}) \subseteq \mathcal{A} - NCg^{\#}\alpha Fr(\mathcal{A})$

(xi) Let $\mathcal{A}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1,
$$\begin{aligned}NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})) &= NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))^c \\ &= NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)) \\ &\quad \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c))^c \\ &\subseteq (NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A})) \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha cl(\mathcal{A}^c))) \\ &\quad \cap (NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}^c) \cup NCg^{\#}\alpha int(\mathcal{A}))) \\ &\subseteq (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)) \\ &\quad \cap (NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A}^c) \cup (NCg^{\#}\alpha cl(NCg^{\#}\alpha int(\mathcal{A})))) \\ &\subseteq (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c)) \cap (NCg^{\#}\alpha cl(\mathcal{A}^c) \\ &\quad \cup NCg^{\#}\alpha cl(\mathcal{A})) \\ &\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c) \\ &= NCg^{\#}\alpha Fr(\mathcal{A})\end{aligned}$$
Therefore, $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A})) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

(xii) Let $\mathcal{A}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1,
$$\begin{aligned}NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))) &= NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))) \\ &\quad \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr((NCg^{\#}\alpha Fr(\mathcal{A})))^c) \\ &\subseteq NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A})) \\ &\quad \cap NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A})^c) \\ &\subseteq NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))\end{aligned}$$
Hence, $NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(NCg^{\#}\alpha Fr(\mathcal{A}))) \subseteq NCg^{\#}\alpha cl(NCg^{\#}\alpha Fr(\mathcal{A}))$

□

Theorem 5.2. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha Fr(\mathcal{A} \cup \mathcal{D}) \subseteq NCg^{\#}\alpha Fr(\mathcal{A}) \cup NCg^{\#}\alpha Fr(\mathcal{D})$*

Proof. Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1

$$\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}\cup\mathcal{D}) &= NCg^{\#}\alpha cl(\mathcal{A}\cup\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}\cup\mathcal{D})^c\\
&= NCg^{\#}\alpha cl(\mathcal{A}\cup\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^c\cap\mathcal{D}^c)\\
&\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^c)\cap NCg^{\#}\alpha cl(\mathcal{D}^c))\\
&\qquad\qquad \text{by Theorem 4.1 (v) and (vi)}\\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^c)))\\
&\qquad \cap((NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{D}^c)))\\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))\cup(NCg^{\#}\alpha cl(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^c)))\\
&\quad \cap((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}^c))\cup(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(\mathcal{D}^c)))\\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cup(NCg^{\#}\alpha cl(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^c)))\\
&\qquad \cap(NCg^{\#}\alpha Fr(\mathcal{D})\cup(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(\mathcal{D}^c)))\\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cup NCg^{\#}\alpha Fr(\mathcal{D}))\cap((NCg^{\#}\alpha cl(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^c))\\
&\qquad \cup(NCg^{\#}\alpha cl(\mathcal{A}))\cap NCg^{\#}\alpha cl(\mathcal{D}^c)))\\
&\subseteq NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D})
\end{aligned}$$

□

Theorem 5.3. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D})\subseteq(NCg^{\#}\alpha Fr(\mathcal{A})\cap NCg^{\#}\alpha Fr(\mathcal{D}))\cup(NCg^{\#}\alpha Fr(\mathcal{D})\cap NCg^{\#}\alpha Fr(\mathcal{A}))$*

Proof. Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 5.1

$$\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D}) &= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})^c\\
&= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}^c\cup\mathcal{D}^c)\\
&\subseteq (NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^c)\cup NCg^{\#}\alpha cl(\mathcal{D}^c))\\
&\qquad\qquad \text{by Theorem 4.1 (v) and (vi)}\\
&= ((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{A}^c)))\\
&\qquad \cup((NCg^{\#}\alpha cl(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cap(NCg^{\#}\alpha cl(\mathcal{D}^c)))\\
&= (NCg^{\#}\alpha Fr(\mathcal{A})\cap NCg^{\#}\alpha cl(\mathcal{D}))\cup(NCg^{\#}\alpha Fr(\mathcal{D})\cap NCg^{\#}\alpha cl(\mathcal{A}))
\end{aligned}$$

□

Corollary 5.4. *Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$, $NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D})\subseteq NCg^{\#}\alpha cl(\mathcal{A})\cup NCg^{\#}\alpha cl(\mathcal{D})$*

Proof.

$$
\begin{aligned}
NCg^{\#}\alpha Fr(\mathcal{A}\cap\mathcal{D}) &= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D}) \cap NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D})^c \\
&= NCg^{\#}\alpha cl(\mathcal{A}\cap\mathcal{D}) \cap NCg^{\#}\alpha cl(\mathcal{A}^c\cup\mathcal{D}^c) \\
&\subseteq (NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})) \cap (NCg^{\#}\alpha cl(\mathcal{A}^c) \cup NCg^{\#}\alpha cl(\mathcal{D}^c)) \\
&\qquad \text{by Theorem 4.1 (v) and (vi)} \\
&= ((NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})) \cap (NCg^{\#}\alpha cl(\mathcal{A}^c))) \\
&\quad \cup ((NCg^{\#}\alpha cl(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})) \cap (NCg^{\#}\alpha cl(\mathcal{D}^c))) \\
&= (NCg^{\#}\alpha Fr(\mathcal{A}) \cap NCg^{\#}\alpha cl(\mathcal{D})) \cup (NCg^{\#}\alpha Fr(\mathcal{D}) \cap NCg^{\#}\alpha cl(\mathcal{A})) \\
&\subseteq NCg^{\#}\alpha cl(\mathcal{A}) \cup NCg^{\#}\alpha cl(\mathcal{D})
\end{aligned}
$$

□

6. **Neutrosophic Crisp $g^{\#}\alpha$ Border**

Definition 6.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$. Then $NCg^{\#}\alpha$ border of $\mathcal{A}$ is defined as $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A})$.

Theorem 6.1. *If $\mathcal{A}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then*

(i) $\mathcal{A}$ *is a* $NCg^{\#}\alpha OS$ *iff* $NCg^{\#}\alpha Br(\mathcal{A}) = \phi_N$

(ii) $NCg^{\#}\alpha Br(X_N) = NCg^{\#}\alpha Br(\phi_N) = \phi_N$

(iii) $\mathcal{A} = NCg^{\#}\alpha int(\mathcal{A}) \cup NCg^{\#}\alpha Br(\mathcal{A})$

(iv) $NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha Br(\mathcal{A}) = \phi_N$

(v) $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(\mathcal{A})) = \phi_N$

(vi) $NCg^{\#}\alpha Br(NCg^{\#}\alpha int(\mathcal{A})) = \phi_N$

(vii) $NCg^{\#}\alpha Br(NCg^{\#}\alpha Br(\mathcal{A})) = NCg^{\#}\alpha Br(\mathcal{A})$

(viii) $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A} \cap NCg^{\#}\alpha cl(\mathcal{A}^c)$

(ix) $NCg^{\#}\alpha Br(\mathcal{A}) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

Proof. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X,\Gamma)$.

(i) Necessity: Suppose $\mathcal{A}$ is $NCg^{\#}\alpha OS$. Then $NCg^{\#}\alpha int(\mathcal{A}) = \mathcal{A}$.
Now, $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A}) = \mathcal{A}$ - $\mathcal{A} = \phi_N$.
Sufficiency: Suppose $NCg^{\#}\alpha Br(\mathcal{A})=\phi_N$. This implies, $\mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A})=\phi_N$. Therefore $\mathcal{A} = NCg^{\#}\alpha int(\mathcal{A})$ and hence $\mathcal{A}$ is $NCg^{\#}\alpha OS$.

(ii) Since ϕ_N and X_N are $NCg^{\#}\alpha OS$, by Theorem 6.1 (i), $NCg^{\#}\alpha Br(\phi_N) =\phi_N$ and $NCg^{\#}\alpha Br(X_N)=\phi_N$.

(iii) Let $x \in \mathcal{A}$. If $x \in NCg^{\#}\alpha int(A)$, then the result is obvious.
If $x \notin NCg^{\#}\alpha int(A)$, then by the definition of $NCg^{\#}\alpha Br(A)$, x $\in NCg^{\#}\alpha Br(A)$. Hence $x \in NCg^{\#}\alpha int(A) \cup NCg^{\#}\alpha Br(A)$ and so $\mathcal{A} \subseteq NCg^{\#}\alpha int(A) \cup$

$NCg^{\#}\alpha Br(A)$.

On the other hand, Since $NCg^{\#}\alpha int(A) \subseteq \mathcal{A}$ and $NCg^{\#}\alpha Br(A) \subseteq \mathcal{A}$, we have $NCg^{\#}\alpha int(A) \cup NCg^{\#}\alpha Br(A) \subseteq \mathcal{A}$

(iv) Suppose $NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha Br(\mathcal{A}) \neq \phi_N$.

Let $x \in NCg^{\#}\alpha int(A) \cap NCg^{\#}\alpha Br(A)$. Then $x \in NCg^{\#}\alpha int(A)$ and $x \in NCg^{\#}\alpha Br(A)$. Since $NCg^{\#}\alpha Br(A) = \mathcal{A} \cap NCg^{\#}\alpha int(A)$, then $x \in \mathcal{A}$. But $x \in NCg^{\#}\alpha int(A)$ and $x \in \mathcal{A}$, there is a contradiction.

Hence, $NCg^{\#}\alpha int(\mathcal{A}) \cap NCg^{\#}\alpha Br(\mathcal{A}) = \phi_N$.

(v) Let $x \in X$ and assume that $x \in NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A))$.

Then $x \in NCg^{\#}\alpha Br(A)$, Since $NCg^{\#}\alpha Br(A) \subseteq \mathcal{A}$, $x \in NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A)) \subseteq NCg^{\#}\alpha int(\mathcal{A})$. Therefore $x \in NCg^{\#}\alpha int(A) \cap NCg^{\#}\alpha Br(A)$, this leads to a contradiction to Theorem 6.1 (iv).

Hence $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A)) = \phi_N$.

(vi) By the Definition 6.1,

$NCg^{\#}\alpha Br(NCg^{\#}\alpha int(A)) = NCg^{\#}\alpha int(A)$ - $NCg^{\#}\alpha int(NCg^{\#}\alpha int(A))$.

But Theorem 3.2 (v) we have, $NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}))= NCg^{\#}\alpha int(\mathcal{A})$.

Hence, $NCg^{\#}\alpha Br(NCg^{\#}\alpha int(\mathcal{A})) = \phi_N$.

(vii) By the Definition 6.1,

$NCg^{\#}\alpha Br(NCg^{\#}\alpha Br(A)) = NCg^{\#}\alpha Br(A)$ - $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(A))$.

By Theorem 6.1 (v), $NCg^{\#}\alpha int(NCg^{\#}\alpha Br(\mathcal{A})) = \phi_N$.

And hence $NCg^{\#}\alpha Br(NCg^{\#}\alpha Br(\mathcal{A})) = NCg^{\#}\alpha Br(\mathcal{A})$

(viii) Since, $NCg^{\#}\alpha Br(\mathcal{A}) = \mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A})$

$$= \mathcal{A} \cap (NCg^{\#}\alpha int(\mathcal{A}))^c$$

$$= \mathcal{A} \cap NCg^{\#}\alpha cl(\mathcal{A})^c$$

(ix) Since, $\mathcal{A} \subseteq NCg^{\#}\alpha cl(\mathcal{A})$, $\mathcal{A}$ - $NCg^{\#}\alpha int(\mathcal{A}) \subseteq NCg^{\#}\alpha cl(\mathcal{A})$ - $NCg^{\#}\alpha int(\mathcal{A})$, that implies, $NCg^{\#}\alpha Br(\mathcal{A}) \subseteq NCg^{\#}\alpha Fr(\mathcal{A})$

□

7. Neutrosophic Crisp $g^{\#}\alpha$ Exterior

Definition 7.1. Let $\mathcal{A}$ be a neutrosophic crisp subset of $NCTS(X, \Gamma)$. Then $NCg^{\#}\alpha$ exterior of $\mathcal{A}$ is defined as $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha int(\mathcal{A}^c)$.

Theorem 7.1. *If $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X, \Gamma)$. Then*

(i) $NCg^{\#}\alpha Ext(X_N) = \phi_N$ *and* $NCg^{\#}\alpha Ext(\phi_N) = X_N$

(ii) $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha cl(\mathcal{A})^c$

(iii) $NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A})) = NCg^{\#}\alpha int(NCg^{\#}\alpha cl(\mathcal{A})) \supseteq NCg^{\#}\alpha int(\mathcal{A})$

(iv) *If* $\mathcal{A} \subseteq \mathcal{D}$*, then* $NCg^{\#}\alpha Ext(\mathcal{D}) \subseteq NCg^{\#}\alpha Ext(\mathcal{A})$

(v) *If* $\mathcal{A}$ *is a* $NCg^{\#}\alpha CS$ *iff* $NCg^{\#}\alpha Ext(\mathcal{A}) = \phi_N$

(vi) $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A}))^c$

(vii) $NCg^{\#}\alpha Ext(\mathcal{A} \cup \mathcal{D}) \subseteq NCg^{\#}\alpha Ext(\mathcal{A}) \cap NCg^{\#}\alpha Ext(\mathcal{D})$

(viii) $NCg^{\#}\alpha Ext(\mathcal{A} \cap \mathcal{D}) \supseteq NCg^{\#}\alpha Ext(\mathcal{A}) \cup NCg^{\#}\alpha Ext(\mathcal{D})$

Proof.

(i) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. By Definition 7.1,
$NCg^{\#}\alpha Ext(X_N) = NCg^{\#}\alpha int(X_N{}^c) = NCg^{\#}\alpha int(\phi_N)$
$NCg^{\#}\alpha Ext(\phi_N) = NCg^{\#}\alpha int(\phi_N{}^c) = NCg^{\#}\alpha cl(X_N)$
Since, ϕ_N and X_N are $NCg^{\#}\alpha OS$, then $NCg^{\#}\alpha int(\phi_N) = \phi_N$, $NCg^{\#}\alpha int(X_N) = X_N$. Hence $NCg^{\#}\alpha Ext(X_N) = \phi_N$ and $NCg^{\#}\alpha Ext(\phi_N) = X_N$

(ii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by definition of $NCg^{\#}\alpha$ Extontier, $(NCg^{\#}\alpha cl(\mathcal{A}))^c = NCg^{\#}\alpha int(\mathcal{A}^c)$, then $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha int(\mathcal{A}^c)$ $= (NCg^{\#}\alpha cl(\mathcal{A}))^c$

(iii) Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then by definition of $NCg^{\#}\alpha$ Extontier,

$$\begin{aligned} NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A})) &= NCg^{\#}\alpha Ext(NCg^{\#}\alpha int(\mathcal{A}^c)) \\ &= NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}^c))^c \\ &= NCg^{\#}\alpha int(NCg^{\#}\alpha cl(\mathcal{A})) \\ &\supseteq NCg^{\#}\alpha int(\mathcal{A}) \end{aligned}$$

(iv) Let $\mathcal{A} \subseteq \mathcal{D}$. Then by Definition 7.1, $NCg^{\#}\alpha Ext(\mathcal{D}) = NCg^{\#}\alpha int(\mathcal{D}^c) \subseteq NCg^{\#}\alpha int(\mathcal{A}^c) = NCg^{\#}\alpha Ext(\mathcal{A})$

(v) Necessity: Let $\mathcal{A}$ be a neutrosophic crisp subset in $NCTS(X, \Gamma)$. Then its complement $\mathcal{A}^c$ is $NCg^{\#}\alpha OS$. By Definition 7.1, $NCg^{\#}\alpha Ext(\mathcal{A}) = NCg^{\#}\alpha int(\mathcal{A}^c)$
Since $\mathcal{A}^c$ is $NCg^{\#}\alpha OS$, $NCg^{\#}\alpha int(\mathcal{A}^c) = \mathcal{A}^c$. Thus, $NCg^{\#}\alpha Ext(\mathcal{A}) = \mathcal{A}^c$.
If $NCg^{\#}\alpha Ext(\mathcal{A}) = \phi_N$, then $\mathcal{A}^c = \phi_N$, which implies $\mathcal{A} = X$.
Hence, $\mathcal{A}$ is $NCg^{\#}\alpha CS$.
Sufficient: If $NCg^{\#}\alpha Ext(\mathcal{A}) = \phi_N$, then $NCg^{\#}\alpha int(\mathcal{A})$. So, $\mathcal{A}^c = \phi_N$. This implies $\mathcal{A} = X$, and X is $NCg^{\#}\alpha CS$. Hence, $\mathcal{A}$ is $NCg^{\#}\alpha CS$.

(vi)
$$\begin{aligned} NCg^{\#}\alpha Ext(NCg^{\#}\alpha Ext(\mathcal{A}))^c &= NCg^{\#}\alpha Ext(NCg^{\#}\alpha int(\mathcal{A}^c))^c \\ &= NCg^{\#}\alpha int((NCg^{\#}\alpha int(\mathcal{A}^c))^c)^c \\ &= NCg^{\#}\alpha int(NCg^{\#}\alpha int(\mathcal{A}^c)) \\ &= NCg^{\#}\alpha int(\mathcal{A}^c) \\ &= NCg^{\#}\alpha Ext(\mathcal{A}) \end{aligned}$$

(vii) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 7.1,

$$\begin{aligned} NCg^{\#}\alpha Ext(\mathcal{A}\cup\mathcal{D}) &= NCg^{\#}\alpha int((\mathcal{A}\cup\mathcal{D})^c) \\ &= NCg^{\#}\alpha int((\mathcal{A})^c\cap(\mathcal{D})^c) \\ &\subseteq NCg^{\#}\alpha int(\mathcal{A})^c\cap NCg^{\#}\alpha int(\mathcal{D})^c \\ &= NCg^{\#}\alpha Ext(\mathcal{A})\cap NCg^{\#}\alpha Ext(\mathcal{D}) \end{aligned}$$

(viii) Let $\mathcal{A}$ and $\mathcal{D}$ be neutrosophic crisp subsets of $NCTS(X,\Gamma)$. Then by Definition 7.1,

$$\begin{aligned} NCg^{\#}\alpha Ext(\mathcal{A}\cap\mathcal{D}) &= NCg^{\#}\alpha int((\mathcal{A}\cap\mathcal{D})^c) \\ &= NCg^{\#}\alpha int((\mathcal{A})^c\cup(\mathcal{D})^c) \\ &\supseteq NCg^{\#}\alpha int(\mathcal{A})^c\cup NCg^{\#}\alpha int(\mathcal{D})^c \\ &= NCg^{\#}\alpha Ext(\mathcal{A})\cup NCg^{\#}\alpha Ext(\mathcal{D}) \end{aligned}$$

□

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Study of the immunoserological patterns of respiratory infections in children under 15 years of age in the Riobamba region based on Refined Neutrosophic Statistics

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Abstract. Acute respiratory infection (ARI) is an important cause of consultation in the emergency room and depending on the general condition of the person; it can become complicated and life-threatening. The general objective of this study is to identify the immunoserology of acute respiratory infections in children under 15 years of age in the city of Riobamba, Ecuador in 2023; it consists of a statistical study in 85 children and adolescents. This research made it possible to identify the presence or absence of acute respiratory infections. The infections found are provoked by viruses, atypical bacteria, or a combination of both pathogens. These diseases may have an atypical origin due to previously unknown or little-known pathogens. Another indetermination is caused by the lack of knowledge of how the individuals studied were infected. That is why we use Refined Neutrosophic Statistics to study the collected data. This theory allows us to make the study more flexible, where the indeterminacy caused by ignorance is included, either about the biological origin of the diseases or about the immunological origin in the transmission of the disease. Specifically, we use the chi-square test for Contingency Tables for data processing.

Keywords: Acute respiratory infection (ARI), immunoserology, neutrosophy, refined neutrosophic statistics, contingency table, chi-square test.

1 Introduction

Acute Respiratory Infections (ARIs) are a group of diseases that occur in the respiratory system caused by various viral and bacterial agents, with an insidious onset that lasts less than two weeks. These types of infections are highly frequent and represent a public health problem.

Most of these infections are mild; however, complications may arise depending on the immune status of the vulnerable person or group, including cases of pneumonia, which can even lead to death. This type of disease is more aggressive in the extreme ages of the population, both in pediatric and elderly patients. The symptoms of ARI can vary and include fever, cough and often sore throat, dyspnea, wheezing, or difficulty breathing.

Among the acute respiratory infections of the lower respiratory tract, pneumonia stands out due to its incidence, severity, high mortality, resource consumption, and the epidemiological changes of the microorganisms causing it. Added to this is the growing bacterial resistance to antimicrobials. Pneumonia is a common and potentially serious infection with a significant prevalence in childhood, causing the highest morbidity and mortality in the world in children under 5 years of age, especially in children in developing countries, whose sociodemographic conditions and health indicators pose risks to their quality of life.

Epidemiological systems have been prepared for common viral and/or bacterial infectious conditions, but changes in lifestyle and health conditions have been determined that lead to new disease conditions, which is why it is necessary to recognize the new pathogens that are part of this infection.

The incidence of respiratory viruses in the 21st century includes H1N1 pmd09, AH3N2, and Influenza B1. In the case of bacterial agents, an increase has been seen in *Pseudomonas aeruginosa*, *Staphylococcus aureus,* and *Streptococcus pneumoniae*.

When the etiological factors of the infection are recognized, this health problem could decrease since there are means for its prevention, through immunizations and viral or antibiotic treatment according to the evidence of the pathogen.

The pediatric population is strongly affected by ARIs. In children under 5 years of age, 95% of ARIs are caused by viruses, and a smaller percentage may present complications such as otitis, sinusitis, and pneumonia. In this context, a group of bacteria with particular microbiological and clinical characteristics are known as "atypical" agents, which include: *Mycoplasma pneumoniae*, *Chlamydophila pneumoniae,* and *Legionella pneumophila*, which should be investigated particularly to achieve a timely diagnosis and adequate antibiotic treatment.

In bacterial pneumonia, the main etiological agent is *Streptococcus pneumoniae*. However, due to the introduction of previous vaccines and the use of molecular biology techniques, the etiological agents detected have varied, and new pathogens have been identified in recent decades.

To identify etiologic agents, tests are performed that have allowed us to identify a broad spectrum of pathogens in the population. This knowledge is used to create effective treatment lines. Due to these findings, serology was the diagnostic method of choice for a long time. This technique is based on the detection of *IgM antibodies* that generally appear 10 days after infection, as well as *IgG antibodies* that can be found approximately 3 weeks after infection. The presence of *IgM antibodies* indicates recent infection, but they can persist for several months.

However, along with the development of science and technology, new methods have been applied in the diagnosis of infections. Immunoserology is a method of diagnosing infectious and viral diseases through direct observation and detection of serological components to investigate antibodies of the *IgA*, *IgG,* and *IgM types*.

In Latin America, ARIs are the main cause of consultation and hospitalization in pediatric patients. Therefore, there is interest in responding to these conditions that cause mortality and illness in children; for this reason, a pertinent response must be taken, using antivirals, antibiotics, and resources for the health care of these patients. In addition, it affects work absenteeism and crises in parents when children become ill, this being a social problem.

The main objective of this article is to identify the immunoserology of acute respiratory infections in children under 15 years of age in Riobamba, Ecuador during the year 2023. Other objectives that we propose are:

- To characterize the population with acute respiratory infections in children under 15 years of age.
- To identify the presence of atypical viruses and bacteria causing acute respiratory infections in children under 15 years of age using the indirect immunofluorescence technique.
- Relate bacterial and viral incidence according to the characterization of the population with acute respiratory infections.

The task we propose is broad since it is not enough to know what is happening with the current pediatric population of Riobamba in terms of the incidence of respiratory diseases, since the behavior of these diseases itself is a question. Some bacteria were believed to be eliminated and have emerged due to the loss of sensitivity to medications. Human populations have spread to areas where there were previously only wild animals and this has caused the infection of humans with zoonotic diseases such as the COVID-19 virus. Contact between humans from different countries due to increased international transportation has caused infection with viruses such as the COVID-19 pandemic itself that hit all countries in the world. So, there are still unknown aspects within this subject, nor can the infection from one population to another be controlled, nor can it even be specified how this occurs. Therefore, Neutrosophy is the appropriate theoretical framework to carry out the study.

Neutrosophy is the branch of philosophy that studies neutrality, but also the indeterminacy caused by the unknown, the contradictory, the paradoxical, the inconsistent, and so on [1, 2]. Neutrosophic Statistics extends classical statistical methods to data or parameters in interval form, or when the population or sample size is not exactly known [1-4]. This is very common in the problem we are studying, which is why the tool chosen for the study is Refined Neutrosophic Statistics [1, 2].

In Refined Neutrosophic Statistics, data are represented as products of real values by interval-like elements that signify various forms of truthfulness, indeterminacy, or falsity [1, 2]. In this case, we want to represent the data in the form of refined neutrosophic numbers, where one part represents the indeterminacy due to ignorance, divided into two types: biological ignorance due to the type of virus, unexplained disease, among others, and on the other hand epidemiological ignorance, due to a component of lack of knowledge of how the disease is transmitted from one individual to another.

This paper is divided into a section of Preliminaries where Refined Neutrosophic Statistics is recalled. The following section contains the details of the study carried out. We conclude the article with the Conclusion section.

2 Refined Neutrosophic Statistics

Definition 1: ([1, 2]) Let X be a universe of discourse. A *Neutrosophic Set* (NS) is characterized by three membership functions, $u_A(x), r_A(x), v_A(x) : X \rightarrow \,]^-0, 1^+[$, which satisfy the condition $^-0 \leq \inf u_A(x) + \inf r_A(x) + \inf v_A(x) \leq \sup u_A(x) + \sup r_A(x) + \sup v_A(x) \leq 3^+$ for all $x \in X$. $u_A(x), r_A(x)$, and $v_A(x)$ are the membership functions of truthfulness, indeterminacy, and falseness of x in A, respectively, and their images are standard or non-standard subsets of $]^-0, 1^+[$.

Definition 2: ([1, 2]) Let X be a universe of discourse. A *Single-Valued Neutrosophic Set* (SVNS) *A* on *X* is a set of the form:

$$A = \{\langle x, u_A(x), r_A(x), v_A(x)\rangle : x \in X\} \qquad (1)$$

Where $u_A, r_A, v_A : X \rightarrow [0,1]$, satisfy the condition $0 \leq u_A(x) + r_A(x) + v_A(x) \leq 3$ for all $x \in X$. $u_A(x), r_A(x)$, and $v_A(x)$ denote the membership functions of truthfulness, indeterminate, and falseness of x in A, respectively. For convenience, a *Single-Valued Neutrosophic Number* (SVNN) will be expressed as $A = (a, b, c)$, where $a, b, c \in [0,1]$ and satisfy $0 \leq a + b + c \leq 3$.

Neutrosophic Statistics extends classical statistics, such that we deal with set values rather than crisp values [1, 2, 5-7].

Neutrosophic Descriptive Statistics is comprised of all techniques to summarize and describe the neutrosophic numerical data characteristics.

Neutrosophic Inferential Statistics consists of methods that allow the generalization from a neutrosophic sampling to a population from which the sample was selected.

Neutrosophic Data is the data that contains some indeterminacy. Similarly to classical statistics, it can be classified as:

- *Discrete neutrosophic data*, if the values are isolated points.
- *Continuous neutrosophic data*, if the values form one or more intervals.

Another classification is the following:

- *Quantitative (numerical) neutrosophic data*; for example a number in the interval (we do not know exactly), 47, 52, 67, or 69 (we do not know exactly);

- *Qualitative (categorical) neutrosophic data*; for example: blue or red (we do not know exactly), white, black or green or yellow (not knowing exactly).

The *univariate neutrosophic data* is a neutrosophic data that consists of observations on a neutrosophic single attribute.

Multivariable neutrosophic data is neutrosophic data that consists of observations on two or more attributes.

A *Neutrosophical Statistical Number* N has the form $N = d + I$, [5-7], where d is called the *determinate part* and I is called the *indeterminate part*.

A *Neutrosophic Frequency Distribution* is a table displaying the categories, frequencies, and relative frequencies with some indeterminacy. Most often, indeterminacies occur due to imprecise, incomplete, or unknown data related to frequency. As a consequence, relative frequency becomes imprecise, incomplete, or unknown too.

Neutrosophic Survey Results are survey results that contain some indeterminacy.

A *Neutrosophic Population* is a population not well determined at the level of membership (i.e. not sure if some individuals belong or do not belong to the population).

A *simple random neutrosophic sample* of size n from a classical or neutrosophic population is a sample of n individuals such that at least one of them has some indeterminacy.

A *stratified random neutrosophic sampling* is the pollster groups of the (classical or neutrosophic) population by strata according to a classification. Then, the pollster takes a random sample (of appropriate size according to a criterion) from each group. If there is some indeterminacy, we deal with neutrosophic sampling.

Additionally, we describe some concepts of interval calculus, which should be useful in this paper.

Given $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ two neutrosophic numbers, some operations between them are defined as follows, [1,2, 5-7]:

$N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I$ (Addition),

$N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I$ (Difference),

$N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$ (Product),

$\frac{N_1}{N_2} = \frac{a_1+b_1 I}{a_2+b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2+b_2)} I$ (Division).

Additionally, given $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ we have the following operations between them:

1. $I_1 \leq I_2$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$.
2. $I_1 + I_2 = [a_1 + a_2, b_1 + b_2]$ (Addition);
3. $I_1 - I_2 = [a_1 - b_2, b_1 - a_2]$ (Subtraction),
4. $I_1 \cdot I_2 = [\min\{a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2\}, \max\{a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2\}]$ (Product),
5. $\frac{I_1}{I_2} = \left[\frac{a_1}{b_1}, \frac{a_2}{b_2}\right]$, always that $0 \notin I_2$ (Division).
6. $\sqrt{I} = \left[\sqrt{a}, \sqrt{b}\right]$, always that $a \geq 0$ (Square root).
7. $I^n = \underbrace{I \cdot I \cdot \cdots \cdot I \cdot I}_{n \text{ times}}$.

Smarandache also defined types of truth, indeterminacy, and falsity in a symbolic way beyond the T, I, and F. He called this refinement, where T is divided into $T_1, T_2, \ldots, T_p$; $I_1, I_2, \ldots, I_q$; $F_1, F_2, \ldots, F_r$, which depend on the problem [8-17]. Specifically, he extended the numbers of the form given in Equation 1, to represent the Refined Neutrosophic Numbers.

Definition 3: ([8-10]) Given $I_1, I_2, \dots, I_q$, with $q \geq 1$, a *Refined Neutrosophic Number* is obtained from the set above as $N_q = a + b_1 I_1 + b_2 I_2 + \cdots + b_q I_q$, where a is the determinate part and $b_j I_j$ ($j = 1,2,\dots,q$) are the indeterminate parts, such that $a, b_1, b_2, \dots, b_q$ are real or complex numbers.

Some properties that are fulfilled are those shown below:

- $mI_k + nI_k = (m + n)I_k$,
- $0I_k = 0$,
- $I_k^n = I_k$,
- I_k / I_k = undefined,
- $I_j I_k$ with $j \neq k$ is defined depending on the problem being addressed.

3 Results

The final eligible population for this study was made up of a total of 400 patients, all under 15 years of age, with an acute respiratory infection, located in Riobamba. A prior estimate is necessary and if it is difficult to obtain it, a proportion of 50% is usually taken, that is, p = q = 0.5. Simple random sampling was used.

The following calculation was made for the study:

N= 400,
p= q= 0.5 → 50% of the population has the characteristic studied,
Z= 2; 95.5% probability that the results obtained in the sample are valid,
E= 5%→ 0.05 valid error allowed.

$$n = \frac{p.q.N.Z^2}{E^2.(N-1)+p.q.Z^2} \qquad (2)$$

To characterize the population with acute respiratory infections, frequencies, relative frequencies, and analysis of these were calculated in the information collected from the population under study, see Table 1.

Table 1. Characteristics of the population studied with Acute Respiratory Infection. Source: Prepared by the authors.

Feature	Total patients (n= 85)
Sex	
Male	34 (40%)
Female	51 (60%)
Age range	
Less than 5 years old	2 (2.4%)
5 to 10 years	33 (38.8%)
11 to 15 years	33 (38.8%)
Over 15 years old	17 (20%)
Influenza Vaccination	
	12 (14%)
	73 (86%)
Poultry farming	
	21 (25%)

	64 (75%)
Pig farming	
	5 (5.8%)
	80 (94.2%)

For the study we carried out we have identified the following variables.

Nominal variable: immunoserology of acute respiratory infections,

Conceptual variable: immunoserology, with which studies are carried out aimed at diagnosing human infectious and viral diseases using direct observation and component detection methodologies.

Operational variable: the process of immunoserology of acute respiratory infections through indirect immunofluorescence for the detection of atypical agents, can determine the relationship between the causal agents of the infections, and the viral or bacterial etiology.

For more details on the variables see Table 2.

Table 2. Operationalization of Variables. Source: Own elaboration.

Aim	Variable	Dimensions	Indicators
To characterize the population with acute respiratory infections in children under 15 years of age.	Sex	Men Women	Frequency
	Age	Years compliments	Average Standard deviation Age groups
	Exposure	Presence of exhibition	Poultry farming Pig farming
	Vaccination	Condition Vaccine	Vaccinated Not vaccinated
To identify the presence of viruses and bacteria causing acute respiratory infections in children under 15 years of age using the indirect immunofluorescence technique.	Etiology of Acute Respiratory Infection	Viral Bacterial Mixed	Incidence of Viral Etiology Incidence of bacterial etiology Incidence of Mixed Etiology (Viral and Bacterial)
Relate bacterial and viral incidence according to the characterization of the population with acute respiratory infections.	Relationship of variables	Bacterial, viral, and mixed incidence according to population characterization variables	Bacterial, viral, and mixed incidence according to sex. Bacterial, viral, and mixed incidence according to age. Bacterial, viral, and mixed incidence according to exposure. Bacterial, viral, and mixed incidence according to vaccination.

The pathogens to be studied are summarized in Table 3.

Table 3. Viruses and bacteria that cause ARIs to be detected in the study using indirect immunofluorescence. Source: Own elaboration.

Pathogen	Type of pathogen
LEGIONELLA PNEUMOPHILA SG 1	Bacteria
MYCOPLASMA PNEUMANIAE	
COXIELLA BURNETTI	
CHLAMYDOPHILA PNEUMONIAE	
ADENOVIRUS	Virus
RESPIRATORY SYNCYTIAL VIRUS	
INFLUENZA A	
INFLUENZA B	
PARAINFLUENZA 1, 2 and 3	

The data obtained is represented by Refined Neutrosophic Numbers of the form $p = \alpha + \beta I_1 + \gamma I_2$, where α is the determined part, βI_1 is the indeterminate part produced by biological or clinical indeterminacy, for example, if there are doubts about the real presence or not of the disease. γI_2 is the indeterminate part obtained from indeterminacy for epidemiological reasons, for example, due to a lack of explanation on how people became infected. We have, $\alpha, \beta, \gamma \in \mathbb{R}$ and in general to convert p to intervals we have $I_1 = [0,1]$ and $I_2 = [0,1]$.

To process the data we de-neutrosophy them using Equation 3 ([1, 2]):

$$\lambda([a,b]) = \frac{a+b}{2} \quad (3)$$

Data are quantities of individuals that fall into certain categories.

The next step is to place the cases in contingency tables, and then apply the chi-square test [18].

The final results are shown in Tables 4-10. Table 4 contains the results of the biological origin of ARI concerning the four age ranges of the patients and the p-value of applying the Chi-square test.

Table 4. Origin of Acute Respiratory Infection as a Function of Age. In parentheses, the value de-neutrosiphied appears. Source: Own elaboration.

Feature	Total patients	Diagnosis				p-value*
		Bacterial	Viral	Mixed	Negative	
Age						**0.29227**
Less than 5 years	4	0	1	1	1	
5 to 10 years	$33 + 6I_1$(36)	$6 + 2I_1$(7)	$3 + 2I_1$(4)	$21 + 2I_1$(22)	1	
11 to 15 years	32	4	2	20	6	
Over 15 years old	$17 + 4I_1$(19)	$2 + 2I_1$(3)	0	$10 + 2I_1$(11)	2	

Table 5 represents the contingency table of biological disease origin versus vaccination. Note the p-value of applying the Chisquare test.

Table 5. Diagnosis of Acute Respiratory Infection based on Immunization in pediatric patients studied. The de-neutrosiphied value appears in parentheses. Source: Own elaboration.

Feature	Total patients	Diagnosis				p-value*
		Bacterial	Viral	Mixed	Negative	
Influenza Vaccination						**0.25635**
Yes	12	0	0	10	2	
No	$68 + 10I_1(73)$	$12 + 4I_1(14)$	$6 + 2I_1(7)$	$52 + 4I_1(54)$	8	

Table 6 relates the raising of poultry or pigs in the patient's home against the origin of the disease and the p-value of the test. This is because some ARIs have a zoonotic origin.

Table 6. Diagnosis of Acute Respiratory Infection according to the condition of living with animals (poultry and pigs) in pediatric patients studied. The de-neutrosiphied values appears in parentheses. Source: Own elaboration.

Feature	Total patients	Diagnosis				p-value*
		Bacterial	Viral	Mixed	Negative	
Poultry farming						**0.15556**
Yes	$20 + 6I_1(23)$	$2 + 2I_1(3)$	$2 + 2I_1(3)$	$16 + 2I_1(17)$	0	
No	$62 + 4I_1 + 2I_2(65)$	$10 + 2I_1 + I_2$ (11.5)	4	$38 + 2I_1 + I_2(39.5)$	10	
Pig farming						**0.81367**
Yes	5	1	0	3	0	
No	$76 + 8I_1 + 2I_2$ (81)	$11 + 4I_1 + I_2(15.5)$	6	$49 + 4I_1 + I_2(52)$	10	

Tables 7-10 contain the relationship of pathogens against the sex of the patient, immunization, presence of poultry raising in the home, and presence of pig raising in the home, respectively.

Table 7. Etiological agents are identified according to the sex of pediatric patients.

Etiological Agent/Total patients	Man	Women	p-value*
Legionella pneumophila	27	39	0.51112
Mycoplasma pneumoniae	3	7	
Coxiella Burnetti	0	3	
Chlamydophila Pneumoniae	0	5	
Adenovirus	1	4	
V. Respiratory syncytial	1	1	
Influenza A	4	4	
Influenza B	19	36	

Etiological Agent/Total patients			
	Man	Women	p-value*
Parainfluenza 1,2 and 3	15	33	

Table 8. Established etiologic agents in pediatric patients with influenza immunization. Source: Elaboration own.

Influenza Vaccination	Yes	No	p-value*
Legionella pneumophila	8	58	
Mycoplasma pneumoniae	3	7	
Coxiella Burnetti	0	3	
Chlamydophila Pneumoniae	0	5	
Adenovirus	0	5	0.57102
V. Respiratory syncytial	1	1	
Influenza A	1	7	
Influenza B	9	46	
Parainfluenza 1,2 and 3	8	40	

Table 9. Etiological agents were identified among groups exposed and not exposed to poultry farming. Source: Prepared by own.

Poultry farming	Yes	No	p-value*
Legionella pneumophila	18	48	
Mycoplasma pneumoniae	0	10	
Coxiella Burnetti	0	3	
Chlamydophila Pneumoniae	1	4	
Adenovirus	1	4	0.59874
V. Respiratory syncytial	0	2	
Influenza A	2	6	
Influenza B	16	39	
Parainfluenza 1,2 and 3	15	33	

Table 10. Etiological agents were identified among groups exposed and not exposed to pig farming. Source: Prepared by own.

Pig farming	Yes	No	p-value*
Legionella pneumophila	4	62	
Mycoplasma pneumonia	1	8	
Coxiella Burnetti	0	3	
Chlamydophila Pneumoniae	0	5	0.96521
Adenovirus	0	5	
V. Respiratory syncytial	0	2	
Influenza A	0	8	

Pig farming	Yes	No	p-value*
Influenza B	2	52	
Parainfluenza 1,2 and 3	3	45	

Conclusion

In this paper, we conducted a study on a group of patients up to 15 years of age from the city of Riobamba, Ecuador during the year 2023, who suffer from Acute Respiratory Infections. 85 patients were analyzed as a random sample from a potential population of 400 children and adolescents. The data were statistically processed using Refined Neutrosophic Statistics. This mathematical tool allowed us to take into account imprecise cases due to uncertainty about the etiology of the disease or uncertainty due to epidemiological factors related to the source of infection. Based on the results obtained, we reached the following conclusions:

1. Pediatric patients in the study are most frequently in the 11-15-year age range with a low influenza vaccination rate.
2. Among the etiological agents of acute respiratory infections analyzed through indirect immunofluorescence are: *Legionella pneumophila sg 1*, *Mycoplasma pneumoniae*, *Coxiella burnetti*, *Chlamydophila pneumoniae*, *Adenovirus*, *Respiratory syncytial virus*, *Influenza a*, *Influenza b* and *Parainfluenza 1, 2 and 3*; it is found that for viral etiology the most contagious per patient are *influenza b* and *parainfluenza*; in addition, in bacterial etiology the most frequent is *Legionella pneumophila sg 1*.
3. The incidence of *Legionella pneumophila SG 1* is extremely important because it is present in all the applied categorizations: all age groups, vaccinated and unvaccinated, as well as its high frequency; and it is associated with bacterial pneumonia. This timely diagnosis can improve the health condition of the pediatric patient and generate appropriate treatment options.
4. In general terms, infections transmitted by patients have a mostly viral rather than bacterial etiology, however, there is a tendency that as the patient's age increases, the bacterial load increases but without becoming greater than the viral load.
5. In the pediatric patients in the study, a smaller proportion of healthy groups are present, in the absence of contagion with the infections in the test. Another minority group is morbidity, associated with mono-infection in respiratory disease. Morbidity is mostly of bacterial rather than viral etiology and is completely absent when patients receive the influenza vaccine. The largest group of pediatric patients present in the study shows a co-infection of approximately 70%, generally including co-infection of viruses, bacteria, or viruses and bacteria. The category of co-infection of viruses and bacteria in the same patient is associated with contagion by viral etiology and then complications due to bacterial superinfections throughout the epidemic.
6. After the application of the flu vaccine, it is mostly in the category of co-infection. This coinfection is most frequent in two and one bacteria, a combination corresponding to *Legionella pneumophila sg 1*, *influenza b,* and *parainfluenza*.
7. The application of the vaccine suppresses some bacterial pathogens (*Coxiella burnetti* and *Chlamydophila pneumoniae*), and a virus (*Adenovirus*) in infected patients, observing up to a co-infection of three viruses and one bacteria. However, no significant relationship is found in the percentage differentiation of respiratory disease infections per patient between individuals who were exposed to the vaccine and those who were not exposed.

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On perserving neutrosophic g-closed sets

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Abstract. In this paper we extend the concepts of a-closed and a-continuous mappings to neutrosophic topological spaces and obtain several results concerning the preservation of neutrosophic g-closed sets. Further more we characterize neutrosophic $T_{\frac{1}{2}}$-spaces in terms of neutrosophic a-continuous and neutrosophic a-closed mappings and obtain some of the basic properties and characterization of these mappings.

Keywords: Neutrosophic g-closed sets; neutrosophic g-open sets; neutrosophic g-continuous;neutrosophic g-irresolute; neutrosophic a-closed and neutrosophic a-continuous mappings.

1. Introduction

In 1965, Zadeh [24] introduced the notion of fuzzy sets. Later, fuzzy topological space was introduced by Chang [5] in 1968 using fuzzy sets. In 1986, Atanassov [4] introduced the notion of intuitionistic fuzzy sets, where the degree of membership and degree of non-membership of an element in a set X are discussed. In 1997, Intuitionistic fuzzy topological spaces were introduced by Coker [6] using intuitionistic fuzzy sets. Neutrality the degree of indeterminacy as an independent concept was introduced by Florentine Smarandache [21]. He [21] defined neutrosophic set on a non empty set by considering three components, namely membership, Indeterminacy and non-membership whose sum lies between 0 and 3. Some more properties of neutrosophic sets are presented by Smarandache [21–23], Salama and Alblowi [18] , Lupiáñez [13]. Smarandache's Neutrosophic concepts have wide range of real time applications for the fields of Information systems, Computer science, Artificial Intelligence, Applied Mathematics and Decision making. In 2008, Lupiáñez [13] introduced the neutrosophic topology as an extension of intuitionistic fuzzy topology. Since 2008 many authors such as Lupiáñez [13,14], Salama et.al. [18,19] Karatas and Cemil [11], Acikgoz and his coworkers [1,2], Dhavaseelan et.al. [7–9], Parimala et.al. [16], Al-Omeri and Jafari [3], and others extended various topological notions to neutrosophic sets and studied in neutrosophic topological spaces. In 1970, Levine [12] introduced the concept of g-closed sets, which play a very important role in general topology and they are now the research topics of many researchers worldwide. In 2018 [8] extended g-closed sets in neutrosophic topology. Recently many authors such as Salmaa [19], Jayanti [10], Al-Omeri and Jafari [3], Mohammed Ali Jaffer and Ramesh [15], Parimala et.al. [16,17] and others sudies neutrosophic g-closed sets and their weak forms in neutrosophic topological spaces. In this paper we introduce the concepts of neutrosophic a - closed and neutrosophic a-continuous mappings using neutrosophic g - closed sets. These definitions enable us to obtain conditions under which maps and inverse maps preserve neutrosophic g - closed sets [9] . We also characterize neutrosophic $T_{\frac{1}{2}}$-spaces in terms of neutrosophic a -continuous and neutrosophic a - closed mappings. Finally some of basic properties of neutrosophic a - continuous and neutrosophic a - closed mappings are investigated.

2. **Preliminaries**

Since we shall require the following known definitions, notations and some properties, we recall them in this section.

Definition 2.1. [21] A Neutrosophic set (**NS**) in X is a structure

$$A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$$

where $\mu_A : X \to]^-0, 1^+[$, $\varpi A : X \to]^-0, 1^+[$, and $\gamma_A : X \to]^-0, 1^+[$ denotes the membership, indeterminacy, and non-membership of A satisfies the condition if $^-0 \leq \mu_A(x) + \varpi_A(x) + \gamma_A(x) \leq 3^+, \ \forall \ x \in X$.

In the real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the closed interval [0,1] and sum of membership, indeterminacy, and non-membership degrees of each element of universe of discourse lies between 0 and 3. The family of all **NS**s over X will be denoted by $\mathbf{N(X)}$.

Definition 2.2. [20] Let X be a non empty set and the neutrosophic sets A and neutrosophic set B be in the form $A = \{< x, \mu_A(x), \varpi_A(x), \gamma_A(x) >: x \in X\}$, $B = \{< x, \mu_B(x), \varpi_B(x), \gamma_B(x) >: x \in X\}$ and let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then:

(a) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$, $\varpi_A(x) \leq \varpi_B(x)$, and $\gamma_A(x) \geq \gamma_B(x)$.

(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$.

(c) $A^c = \{< x, \gamma_A(x), \varpi_A(x), \mu_A(x) >: x \in X\}$.

(d) $\cap A_i = \{< x, \wedge\mu_{A_i}(x), \wedge\varpi_{A_i}(x), \vee\gamma_{A_i}(x) >: x \in X\}$.

(e) $\cup A_i = \{< x, \vee\mu_{A_i}(x), \vee\varpi_{A_i}(x), \wedge\gamma_{A_i}(x) >: x \in X\}$.

(f) $\tilde{\mathbf{0}} = \{< x, 0, 0, 1 >: x \in X\}$ and $\tilde{\mathbf{1}} = \{< x, 1, 1, 0 >: x \in X\}$

Definition 2.3. [13, 18] A neutrosophic topology on a non empty set X is a family τ of neutrosophic sets in X, satisfying the following axioms:

(T_1) $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}} \in \tau$

(T_2) $G_1 \bigcap G_2 \in \tau$

(T_3) $G_1 \bigcup G_2 \in \tau$

In this case the pair (X, τ) is called a neutrosophic topological space and each neutrosophic set in τ is known as a neutrosophic open set in X. The complement A^c of a neutrosophic open set A is called a neutrosophic closed set in X.

Definition 2.4. [18] Let (X, τ) be a neutrosophic topological space and A be a neutrosophic set in X. Then the neutrosophic interior and neutrosophic closure of A are defined by:

Cl(A) = $\cap$ {K: K is a neutrosophic closed set such that A $\subseteq$ K }

Int(A) = $\cup$ {K: K is a neutrosophic open set such that K $\subseteq$ A }

Definition 2.5. [8, 19] A neutrosophic set A of a neutrosophic topological space (X, τ) is called:

(a) neutrosophic g - closed if $Cl(A) \subseteq O$ whenever $A \subseteq O$ and O is neutrosophic open.

(b) neutrosophic g - open if and only if A^c is neutrosophic g - closed.

Remark 2.6. [8, 19] Every neutrosophic closed set is neutrosophic g - closed but its converse may not be true.

Remark 2.7. [8, 19] Every neutrosophic open set is neutrosophic g - open but its converse may not be true.

Theorem 2.8. *[8, 19] A neutrosophic set A of a neutrosophic topological space is neutrosophic g-open if and only if $F \subseteq Int(A)$ whenever F is neutrosophic closed and $F \subseteq A$.*

Theorem 2.9. *[19] Let (X, τ) be a neutrosophic topological space and Ω be the family of all neutrosophic closed sets of X. Then $\Omega = \tau$ if and only if every neutrosophic set of X is neutrosophic g-closed.*

Definition 2.10. [3] A neutrosophic topological space (X, τ) is said to be neutrosophic $T_{\frac{1}{2}}$-space if every neutrosophic g - closed set in X is neutrosophic closed in X.

Definition 2.11. [1] Consider that f is a mapping from X to Y.

(a) Let A be a neutrosophic set in X with membership function $\mu_A(x)$, indeterminacy function $\varpi_A(x)$ and non-membership function $\sigma_A(x)$. The image of A under f, written as f(A), is a neutrosophic set of Y whose membership function, indeterminacy function and non-membership function are defined as

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$\varpi_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\varpi_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & if f^{-1}(y) = \phi, \end{cases}$$

$$\gamma_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\gamma_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & if f^{-1}(y) = \phi, \end{cases}$$

$\forall\ , y \in Y$. Where $f^{-1}(y) = \{x : f(x) = y\}$.

(b) Let B be a neutrosophic set in Y with membership function $\mu_B(y)$, indeterminacy function $\varpi_B(y)$ and non-membership function $\gamma_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$ is a neutrosophic set of X whose membership function, indeterminacy function and non-membership function are respectively defined as:

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)),$$
$$\varpi_{f^{-1}(B)}(x) = \varpi_B(f(x)),$$
$$\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x)).$$

$\forall\ x \in X$.

Definition 2.12. Let (X, τ) and (Y, σ) be two neutrosophic topological spaces and let $f : X \to Y$ be a mapping. Then f is said to be:

(a). neutrosophic continuous [18] if the pre image of each neutrosophic open set if Y is a neutrosophic open set in X.

(b). neutrosophic g -continuous [8, 19] if the pre image of every neutrosophic closed set in Y is a neutrosophic g - closed set in X.

(c). neutrosophic g -irresolute [8] if the pre image of every neutrosophic g -closed set in Y is a neutrosophic g -closed set in X.

(d). neutrosophic closed [8, 19] if the image of each neutrosophic closed set in X is a neutrosophic closed set in Y.

(e). neutrosophic open [8, 19] if the image of each neutrosophic open set of X is a neutrosophic open set in Y.

Remark 2.13. [8,19] Every neutrosophic continuous mapping is neutrosophic g - continuous, but the converse may not be true.

Remark 2.14. [8]Every neutrosophic g - irresolute mapping is neutrosophic g -continuous, but the converse may not be true. The concepts of neutrosophic g -irresolute and neutrosophic continuous mapping are independent.

3. Neutrosophic a-Closed and neutrosophic a-continuous mappings

Definition 3.1. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic a - closed provided that $f(F) \subseteq Int(A)$ whenever F is a neutrosophic closed set in X, A is a neutrosophic g - open set in Y and $f(F) \subseteq A$.

Theorem 3.2. *Every neutrosophic closed mapping is neutrosophic a - closed.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a neutrosophic closed mapping. Let F be neutrosophic closed set in X and A is a neutrosophic g - open set in Y such that $f(F) \subseteq A$. Since f is neutrosophic closed mapping, $f(A)$ is neutrosophic closed set in Y. Now A is neutrosophic g - open and $f(F) \subseteq A \Rightarrow f(F) \subseteq Int(A)$. Hence f is neutrosophic a-closed. □

Remark 3.3. The converse of Theorem 3.2 may not be true.

Example 3.4. Let $X = \{a, b\}$ and $U = \{< a, 0.6, 0.5, 0.3 >, < b, 0.3, 0.5, 0.6 >\}$ be a neutrosophic set on$X$. Let $\tau = \{ \tilde{\mathbf{0}}, \text{U}, \tilde{\mathbf{1}} \}$ be neutrosophic topology on X .Then the mapping $f : (X, \tau) \to (X, \tau)$ defined by $f(a) =$ b and $f(b) =$ a is neutrosophic a -closed but it is not neutrosophic closed.

Definition 3.5. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be neutrosophic a - continuous provided that $Cl(F) \subseteq f^{-1}(O)$ whenever F is neutrosophic g-closed set in X , O is a neutrosophic open set in Y and $F \subseteq f^{-1}(O)$.

Theorem 3.6. *Every neutrosophic continuous mapping is neutrosophic a - continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a neutrosophic continuous mapping. Let O be a neutrosophic open set of Y and F is a neutrosophic g -closed set of X such that $F \subseteq f^{-1}(O)$. Now since f is neutrosophic continuous, $f^{-1}(O)$ is neutrosophic open set in X. Since F is neutrosophic g - closed and $F \subseteq f^{-1}(O) \Rightarrow Cl(F) \subseteq f^{-1}(O)$. Hence f is neutrosophic a - continuous. □

Remark 3.7. The converse of Theorem 3.6 may not be true.

Example 3.8. Let $X = \{a, b\}$ and $U = \{< a, 0.5, 0.5, 0.4 >, < b, 0.4, 0.5, 0.5 >\}$ be a neutrosophic set on$X$. Let $\tau = \{ \tilde{\mathbf{0}}$, U, $\tilde{\mathbf{1}} \}$ be a neutrosophic topology on X . Then the mapping $f : (X, \tau) \to (X, \tau)$ defined by $f(a) =$ b and $f(b) =$ a is neutrosophic a -continuous but it is not neutrosophic continuous.

Theorem 3.9. *If $f : (X, \tau) \to (Y, \sigma)$ is a bijection , then f is neutrosophic a -closed if and only if f^{-1} is neutrosophic a - continuous.*

Proof. Obvious. □

4. **Preserving neutrosophic g-closed sets**

In this section the concepts of neutrosophic a - continuous and neutrosophic a - closed mappings are used to obtain some results on preservation of neutrosophic g - closed sets.

Theorem 4.1. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic g - continuous and neutrosophic a - closed then $f^{-1}(A)$ is a neutrosophic g - closed set in X whenever A is a neutrosophic g - closed set in Y.*

Proof. Suppose that $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic g- continuous and neutrosophic a-closed . Let A be a neutrosophic g-closed set in Y such that $f^{-1}(A) \subseteq O$, where O is neutrosophic open set in X. Then $O^c \subseteq f^{-1}(A^c)$ which implies that $f(O^c) \subseteq Int(A^c) = (Cl(A))^c$. Hence $f^{-1}(Cl(A)) \subseteq O$. Since f is neutrosophic g-continuous and $f^{-1}(Cl(A))$ is neutrosophic g-closed in X, therefore $Cl(f^{-1}(Cl(A))) \subseteq O$ which implies that $Cl(f^{-1}(A)) \subseteq O$. Hence $f^{-1}(A)$ is neutrosophic g -closed set in X. □

Corollary 4.2. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic continuous and neutrosophic a -closed then $f^{-1}(A)$ is a neutrosophic g -closed set in X whenever A is a neutrosophic g -closed set in Y.*

Corollary 4.3. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic continuous and neutrosophic closed then $f^{-1}(A)$ is a neutrosophic g -closed set in X whenever A is a neutrosophic g -closed set in Y.*

Theorem 4.4. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic g - continuous and neutrosophic a -closed then $f^{-1}(A)$ is neutrosophic g - open set in X whenever A is neutrosophic g - open set in Y.*

Proof. Suppose that $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic g - continuous and neutrosophic a -closed mapping. Let A be neutrosophic g - open in Y. Then by Definition 2.5, A^c is neutrosophic g - closed in Y. Hence by Theorem 4.1, $f^{-1}(A^c)$ is neutrosophic g- closed in X . Since $f^{-1}(A^c) = (f^{-1}(A))^c$ for every neutrosophic set A of Y , therefore $(f^{-1}(A))^c$ is neutrosophic g -closed set in X. This means that $f^{-1}(A)$ is neutrosophic g - open set in X. □

Corollary 4.5. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic continuous and neutrosophic a -closed then $f^{-1}(A)$ is a neutrosophic g -open set in X whenever A is a neutrosophic g - open set in Y.*

Corollary 4.6. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic continuous and neutrosophic closed then $f^{-1}(A)$ is a neutrosophic g -open set in X whenever A is a neutrosophic g - open set in Y.*

Theorem 4.7. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic a - continuous and neutrosophic closed mapping then the image of every neutrosophic g - closed set of X is neutrosophic g - closed in Y.*

Proof. Let B be a neutrosophic g - closed set of X, and $f(B) \subseteq O$, where O is a neutrosophic open set in Y. Then $B \subseteq f^{-1}(O)$ and since f is neutrosophic a - continuous, $Cl(B) \subseteq f^{-1}(O)$ which implies that $f(Cl(B)) \subseteq O$. Since f is neutrosophic closed mapping and $Cl(B)$ is neutrosophic closed in X, $f(Cl(B))$ is neutrosophic closed in Y. Thus, we have $Cl(f(B)) \subseteq Cl(f(Cl(B))) = f(Cl(B)) \subseteq O$. Hence $f(B)$ is neutrosophic g - closed in Y. □

Corollary 4.8. *If $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic a - continuous and neutrosophic closed mapping then the image of every neutrosophic g - closed set of X is neutrosophic g - closed in Y.*

5. Characterization of neutrosophic $T_{\frac{1}{2}}$ - spaces

In the following theorems we give some characterizations of a class of neutrosophic $T_{\frac{1}{2}}$-spaces by using the concepts of neutrosophic a - closed and neutrosophic a - continuous mapping.

Theorem 5.1. *A neutrosophic topological space (X, τ) is neutrosophic $T_{\frac{1}{2}}$ - space if and only if every mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a - continuous.*

Proof. Necessity: Let $f : (X, \tau) \to (Y, \sigma)$ and A is a neutrosophic g -closed set of X and $A \subseteq f^{-1}(O)$ where O is a neutrosophic open set of Y. Since X is a neutrosophic $T_{\frac{1}{2}}$ - space, A is neutrosophic closed set in X. Therefore $Cl(A) = A \subseteq f^{-1}(O)$. Hence A is neutrosophic a- continuous.

Sufficiency: Let A be a non empty neutrosophic g - closed set in X and let Y is neutrosophic topological space with the neutrosophic topology $\sigma = \{ \tilde{\mathbf{0}}, \text{A} , \tilde{\mathbf{1}} \}$. Finally let $f : (X, \tau) \to$

(Y, σ) be the identity mapping. By assumption f is neutrosophic a - continuous. Since A is neutrosophic g - closed in X and neutrosophic open in Y and $A \subseteq f^{-1}(A)$, it follows that $Cl(A) \subseteq f^{-1}(A) = A$, because f is the identity mapping . Hence A is neutrosophic closed in X and therefore X is a neutrosophic $T_{\frac{1}{2}}$ - space. □

An analogous argument proves the following result for neutrosophic a- closed mapping.

Theorem 5.2. *A neutrosophic topological space (X, τ) is a neutrosophic $T_{\frac{1}{2}}$ - space if and only if every mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a - closed.*

6. Properties of Neutrosophic a - closed and neutrosophic a - continuous mappings

In this section we investigate some of the properties of neutrosophic a -closed and neutrosophic a -continuous mappings.

Theorem 6.1. *Every neutrosophic g - continuous and neutrosophic a -closed mapping is neutrosophic g -irresolute.*

Proof. Suppose that $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic g - continuous and neutrosophic a - closed mapping and A is a neutrosophic g - closed set in Y. Let $f^{-1}(A) \subseteq O$ where O is a neutrosophic open set in X . Then $O^c \subseteq f^{-1}(A^c)$ which implies that $f(O^c) \subseteq Int(A^c) = (Cl(A))^c$. Hence $f^{-1}(Cl(A)) \subseteq O$. Since f is neutrosophic g -continuous $f^{-1}(Cl(A))$ is neutrosophic g -closed in X. Therefore $Cl(f^{-1}(Cl(A))) \subseteq O$ which implies that $Cl(f^{-1}(A)) \subseteq O$. Hence $f^{-1}(A)$ is a neutrosophic g -closed set in X. Therefore f is neutrosophic g - irresolute. □

Corollary 6.2. *Every neutrosophic continuous and neutrosophic a -closed mapping is neutrosophic g -irresolute.*

Theorem 6.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a mapping for which $f(F)$ is neutrosophic open set in Y for every neutrosophic closed set F of X then f is a neutrosophic a - closed mapping.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping, F a neutrosophic closed set in X, A a neutrosophic g - open set in Y and $f(F) \subseteq A$. By hypothesis $f(F)$ is neutrosophic open in X. Therefore $f(F) = Intf(F) \subseteq Int(A)$. Hence f is neutrosophic a-closed. □

Theorem 6.4. *If $f : (X, \tau) \to (Y, \sigma)$ is a mapping for which $f^{-1}(V)$ is neutrosophic closed in X for every neutrosophic open set V of Y, then f is a neutrosophic a-continuous mapping.*

Proof. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a mapping. Let F is neutrosophic g-closed set in X and V is neutrosophic open set of Y such that $F\subseteq f^{-1}(V)$. By hypothesis $f^{-1}(V)$ is neutrosophic closed in X . Hence $Cl(f^{-1}(V))=f^{-1}(V)$. Therefore $Cl(F)\subseteq Cl(f^{-1}(V))=f^{-1}(V)$. Hence f is neutrosophic a - continuous. □

Remark 6.5. Since the identity mapping on any neutrosophic topological space is both neutrosophic a - continuous and neutrosophic a - closed, it is clear that the converse of Theorem 6.3 and Theorem 6.4 do not hold.

Theorem 6.6. *If $\sigma=\Upsilon$ where Υ denotes the family of all neutrosophic closed sets of Y, then the mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is neutrosophic a - closed if and only if $f(F)$ is a neutrosophic open set in Y, for every neutrosophic closed set F of X.*

Proof. Necessity: Let$f:(X,\tau)\rightarrow(Y,\sigma)$ is neutrosophic a - closed mapping. By Theorem 2.9 every neutrosophic set of Y is neutrosophic g - closed and hence all are neutrosophic g -open. Thus for any neutrosophic closed set F of X, $f(F)$ is neutrosophic g - open in Y. Since f is neutrosophic a - closed, $f(F)\subseteq Int(f(F))$ and then $f(F)=Int(f(F))$. Hence $f(F)$ is neutrosophic open.

Sufficiency: Let F be a neutrosophic closed set of X and A be a neutrosophic g - open set of Y and $f(F)\subseteq A$. By hypothesis $f(F)$ is neutrosophic open in Y and $f(F)=Int(f(F))\subseteq Int(A)$. Hence f is neutrosophic a - closed. □

Theorem 6.7. *If $\sigma=\Upsilon$ where Υ denotes the family of all neutrosophic closed sets of Y then the mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is neutrosophic a - closed if and only if f is neutrosophic closed.*

Proof. Necessity: Let O be a neutrosophic closed set of X. Then by Theorem 6.6 $f(O)$ is neutrosophic open in Y. Since every neutrosophic open set is neutrosophic open , therefore $f(O)$ is neutrosophic open in Y and hence by hypothesis $f(O)$ is neutrosophic closed in Y and therefore $f(O)$ is neutrosophic closed in Y. Hence f is neutrosophic closed.

Sufficiency: Let F be a neutrosophic closed set of X and A be a intuitionistic g-open set of Y and $f(F)\subseteq A$. Since f is neutrosophic closed, $f(F)$ is neutrosophic closed in Y and therefore $(f(F))^c$ is neutrosophic open in Y. By hypothesis$(f(F))^c$ is neutrosophic closed in Y and hence $f(F)$ is neutrosophic open in Y which implies that $f(F)=Int(f(F))\subseteq Int(A)$. Hence f is neutrosophic a -closed. □

Theorem 6.8. *If τ=Ω where , Ω denotes the family of all neutrosophic closed sets of X, then the mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a - continuous if and only if $f^{-1}(O)$ is neutrosophic closed in X for every neutrosophic open set O of Y.*

Proof. Necessity: Let $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a -continuous mapping. By Theorem 2.9 every neutrosophic set of X is neutrosophic g - closed and hence all are neutrosophic g - open. Thus for any neutrosophic open set O of Y, $f^{-1}(O)$ is neutrosophic g -closed in X. Since $f^{-1}(O) \subseteq f^{-1}(O)$ and f is neutrosophic a -continuous then $Cl(f^{-1}(O)) \subseteq f^{-1}(O)$. Hence $f^{-1}(O)$ is neutrosophic closed set in X.

Sufficiency: Let O be a neutrosophic open set of Y and A be a neutrosophic g -closed set of X such that $A \subseteq f^{-1}(O)$ then $Cl(A) \subseteq Cl(f^{-1}(O)) = f^{-1}(O)$ because by hypothesis $f^{-1}(O)$ is neutrosophic closed in X . Hence f is neutrosophic a -continuous. □

Theorem 6.9. *If $\tau = \Omega$ where, Ω denotes the family of all neutrosophic closed sets of X, then the mapping $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a -continuous if and only if it is neutrosophic continuous.*

Proof. Necessity: Let $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a -continuous mapping. Let O is a neutrosophic open set of Y, then by Theorem 6.7 $f^{-1}(O)$ is neutrosophic closed in X and so by hypothesis $f^{-1}(O)$ is neutrosophic open in X . Hence f is neutrosophic continuous.

Sufficiency: Let O be a neutrosophic open set of Y and A be a neutrosophic g-closed set of X such that $A \subseteq f^{-1}(O)$. Since f is neutrosophic continuous, $f^{-1}(O)$ is neutrosophic open in X and thus by hypothesis $f^{-1}(O)$ is neutrosophic pre-closed in X which implies that $Cl(A) \subseteq Cl(f^{-1}(O)) = f^{-1}(O)$. Hence f is neutrosophic a - continuous. □

Theorem 6.10. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic closed and $g : (Y, \sigma) \to (Z, \vartheta)$ is neutrosophic a -closed mapping, then $gof : (X, \tau) \to (Z, \vartheta)$ is neutrosophic a -closed.*

Proof. Let F be a neutrosophic closed set of X and A is neutrosophic g - open set of Z for which $gof(F) \subseteq A$. Since $f : (X, \tau) \to (Y, \sigma)$ is a neutrosophic closed mapping , $f(F)$ is a neutrosophic closed set of Y. Now $g : (Y, \sigma) \to (Z, \vartheta)$ is neutrosophic a - closed mapping, then $g(f(F)) \subseteq Int(A)$. Hence $gof : (X, \tau) \to (Z, \vartheta)$ is neutrosophic a -closed mapping. □

Theorem 6.11. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a -closed and $g : (Y, \sigma) \to (Z, \vartheta)$ is neutrosophic open and neutrosophic g -irresolute then $gof : (X, \tau) \to (Z, \vartheta)$ is neutrosophic a - closed.*

Proof. Let F be a neutrosophic closed set of X and A a neutrosophic g -open set of Z for which $gof(F) \subseteq A$. Then $f(F) \subseteq g^{-1}(A)$. Since g is g - irresolute, $g^{-1}(A)$ is neutrosophic g -open in X and $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a - closed mapping. It follows that $f(F) \subseteq Int(g^{-1}(A))$. Thus $(gof)(F) = g(f(F)) \subseteq g(Int(g^{-1}(A)) \subseteq Int(g(g^{-1}(A))) \subseteq Int(A)$. Hence $gof : (X, \tau) \to (Z, \phi)$ is neutrosophic a - closed. □

Theorem 6.12. *If $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a -continuous and $g : (Y, \sigma) \to (Z, \vartheta)$ is neutrosophic continuous then $gof : (X, \tau) \to (Z, \vartheta)$ is neutrosophic a -continuous.*

Proof. Let A be a neutrosophic g - closed set of X and V a neutrosophic open set of Z for which $A \subseteq (gof)^{-1}(V)$. Now since $g : (Y, \sigma) \to (Z, \vartheta)$ is neutrosophic continuous,$g^{-1}(V)$ is a neutrosophic open set of Y. Since $f : (X, \tau) \to (Y, \sigma)$ is neutrosophic a -continuous, $Cl(A) \subseteq f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$. Hence $gof : (X, \tau) \to (Z, \vartheta)$ is a neutrosophic a -continuous mapping. □

7. Conclusions

We have introduced a-closed and a-continuous mappings in neutrosophic topology. Some basic results related to the preservation of neutrosophic g-closed sets have been established. We also have provided examples where such properties fail to be preserved. Moreover, we obtained some characterizations of neutrosophic $T_{\frac{1}{2}}$-spaces and neutrosophic g-continuity in terms of newly defined mappings. The work presented in this paper may be the starting point for new theoretical studies, but can be utilized to study compactness, connectedness, and separation axioms in neutrosophic topological spaces. Therefore, we believe that it will be necessary to carry out more theoretical research to establish a general framework for decision-making, pattern recognition, and data mining applications which will be the aim of our future research works.

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Schweizer-Sklar power aggregation operators based on complexsingle-valued neutrosophic information using SMART and their application in green supply chain management

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Abstract. A complex single-valued neutrosophic (CSVN) set is a very useful tool to handle the uncertainty, inconsistency, and ambiguity of knowledge in a periodic framework. The Schweizer-Sklar (SS) t-norm (TN) and t-conorm (TCN) enhance the flexibility of a procedure for aggregation depending on its parameters, whereas the power aggregation(PA) operator prevents insufficient impact of excessively high or low arguments on the results. Based on the idea of Schweizer-Sklar power aggregation operators, this paper aims to propose a number of aggregation operators such as complex single-valued neutrosophic Schweizer-Sklar power weighted averaging (CSVNSSPWA), complex single-valued neutrosophic Schweizer-Sklar power ordered weighted averaging (CSVNSSPOWA). Weights determination to solve green supply chain management problems in the single valued neutrosophic context is accomplished through the Simple Multi-Attribute Rating Technique (SMART). Furthermore, an example is given to show the uniqueness of the proposed operator and the difference between the proposed work and previous research.

Keywords: Complex single valued neutrosophic set; Power aggregation operator; Schweizer- Sklar operator; SMART; Green supply chain management.

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1. Introduction

Uncertainty is one of the conditions where certainty of knowledge is low or unclear, or when results and patterns are less identifiable or precise. It is an inherent feature of different areas of expertise such as sciences, mathematics, economy, and decision-making. In general, the multi attribute group decision making (MAGDM) helps to handle decision-making problems as it considers the expert's opinion and available choices. MAGDM problems deal with the identification of the most appropriate choice or set of alternatives regarding some criteria that reflect the views of the decision maker, which are expressed by certain attribute values. To address the uncertainty information, Zadeh [1] introduced the extension of the classical set theory in the form of fuzzy set (FS) theory with the aid of membership degree μ, which lies between 0 and 1.

Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFS), which defines membership and non- membership functions with the restriction that the total of membership and non-membership grades must be ≤ 1. However, it is not able to solve all kinds of uncertain problems or issues and other real life problems especially when dealing with indeterminate data, Smarandache [3] defined neutrosophic set (NS) to deal with the information that is indeterminate or inconsistent in a real environment independently. Mariming Wang et al. [4] proposed the concept of single valued neutrosophic(SVN) based on the real standard interval [0,1] for solving the decision-making problems. This set is a generalization of the following sets; crisp set, fuzzy set(FS), interval-valued fuzzy set(IVFS), intuitionistic fuzzy set(IFS), interval-valued intuitionistic fuzzy set(IVIFS), etc. Table 1 consists of the comparison of CNS with existing methods.

Table 1. Analysis of CNS against existing sets

Models	Truth	Indeterminacy	Falsity	Periodicity
FS	✓	×	×	×
IFS	✓	×	✓	×
NS	✓	✓	✓	×
CFS	✓	×	×	✓
CIFS	✓	×	✓	✓
CNS	✓	✓	✓	✓

Recently, an emerging trend is the use of big data which is often associated with uncertainty and periodicity. However, FS, IFS, IVIFS, and SVN sets are applied to handle uncertain data but cannot address periodical data. To handle this problem, Ramot et al. [5] put forward a groundbreaking idea and named it complex fuzzy set (CFS) as a synthesis of the other two concepts: fuzzy set and complex number. The complex fuzzy set still retains the representation of the uncertainty of information by the amplitude term having a value belonging to the set [0,1] with addition of phase term. The developed complex fuzzy set is a fundamentally different from the concept of fuzzy complex number developed by Buckley [6], Nguyen et al [7] and Zhang et al [8]. Membership functions of complex fuzzy sets have amplitude and phase terms which exhibit wave characteristics such as interference and periodicity. The interference can be constructive or destructive based on the phase term. The concept of the complex intuitionistic fuzzy set (CIFS) was introduced by Alkouri and Saleh [9]. In order to confer even more flexibility to decision-makers, it is recommended that the experts provide their preferences in intervals. In this context, Garg and Rani [10] have presented the concept of complex interval-valued intuitionistic fuzzy set (CIVIFS), its algebraic operators and aggregation, along with the development of a model for handling MAGDM. Ali and Smarandache [11] further put forward the complex neutrosophic set (CNS) improving the prior CFS, CIFS as well as CIVIFS, and used CNS for signal processing. The CNS is defined from the concepts of complex-valued truth, indeterminate and falsehood membership functions.

1.1. *Literature study*

MAGDM is mainly utilized for selecting problems and corresponds to decision space with limited choice and preference order. A aggregation operator(AO) is one of the most used operators in solving multi- attribute decision making problems where its main purpose to combine multiple values into a single numeric value. Grabisch et al. [12] in the "Encyclopedia of Mathematics and its Applications" and Beliakov et at. [13], who has taken work related to aggregation functions who defined averaging operators for Atanassov's IFS. A few years ago, several geometric aggregation operators have been considered by Xu and Yager [14], Wang and Liu [15], and Xu with Da [16]. Aggregation operators have been studied by various authors; in particular, Garg and Arora [17] as well as Xia et al. [18] have focused on Archimedean t-norm and t-conorm. The single-valued neutrosophic set was later widely used in operational research to develop multi- attribute decision-making techniques such as TOPSIS [19], COPRAS [20], VIKOR [21] , WASPAS [22] , EDAS [23] , TODIM [24] and others.

Finally, after some attempts, Ramot et al. [5] discover the new theory called CFS by incorporating the phase term into the supporting grade. Rani and Garg [10]established a MADM method by describing the basic algorithm of CIFS with different power operators. Moreover, Mahmood and Ali [42] derived the complex single valued neutrosophic set. Deschrijver et al. [25] and Klement et al. [26] have explored various types of t-norms and t-conorms. In 2010, Xu and Yager [27] first proposed and defined the abstraction idea of power aggregation. Since the range of the power aggregation operators has been successfully derived, many people has applied it in many fields. The Schweizer-Sklar (SS) t-norm and t-conorm theory was developed by Schweizer and Sklar [28]. This was achieved by adding a parameter p, which is more flexible and helpful in handling complex situations. The Hamacher and Lukasiewicz t-norm information can be obtained easily by using the parameter $p = -1$. Hence, the theory of TODIM and Schweizer-Sklar power aggregation operators can be derived respectively by Zindani et al [29]. T-norms, also known as triangular norms, along with their dual counterparts, t-conorms, play a crucial role in fuzzy systems, particularly in fuzzy sets and its derivatives. Notably, some of the most well-known t-norms and t-conorms, such as Frank's, Archimedean, Einstein, Dombi, and Algebraic t-norms and t-conorms, as well as the Aczel-Alsina (AA) t-norm, Schweizer-Sklar, Yager, and others, have been thoroughly investigated by Klir [30]. In addition, given the interdependency of criteria, it is appropriate to aggregate with a mean function that is either power, Bonferroni or Heronian. The weights of the attribute have also been another concern particularly in cases where they are completely unknown. In these instances, the SMART method can be applied to derive the attribute weights based on the subjective insights of decision experts. Keshavarz-Ghorabaee et al. [44] employed the SMART method for the computation of the weights of criteria.

The main motive of Green supply chain management(GSCM) is to improve the supply chain's environmental capacity sustainability, which has been substantiated in empirical research by a recent review on operations management [33]. GSCM has been widely used as a preventive technique in enhancing the environmental quality of products and policies that meet set environmental standards. The choices on both supply management and environmental issues are shifting towards increasing systematic processes globally. This indicates the need to come up with ways of integrating environmental aspects of supply chain management analyses. Analyzing the literature where the topic of GSCM was discussed, the author stated that the amount of research in this area remains rather limited.

The need to adopt the complex neutrosophic model in the area is to accurately solve and handle uncertainty and ambiguity information within the decision-making of GSCM. The more inclusive classification also helps to explain to investigators, interpreters and educationalists the relevance of stating that a more useful methodology is connected to GSCM. Table 2 reveals a gap in the existing literature on Schweizer-Sklar power aggregation operators.

1.2. *Motivation of the paper*

Here are the motivations behind the research presented in the Schweizer- Sklar power operators under CSVN information:

(1) Enhancing the accuracy of decision-making processes by introducing Schweizer- Sklar power operators for handling complex information.
(2) Contributing to the advancement of decision analytics theory by developing and analyzing new operators within the CSVN framework.
(3) Comparing newly proposed operators with existing ones to validate their effectiveness and demonstrate their superiority.
(4) Providing practical tools and insights for decision-makers in various domains by showcasing the utility of Schweizer- Sklar power operators under CSVN sets.

Table 2. Comparison with the existing Schweizer-Sklar power aggregation operator with different sets

Author	Set	Averaging Operators	DM Solution Method	Application	Weighting Vector	Periodicity Involved
H. Zhang et al (2019) [36]	Neutrosophic Set	SVNSSMM WSVNSSMM	Ranking	Application in Company Investment	Known	No
D. Zindawi et al (2020) [37]	Interval-valued Intuitionistic Fuzzy Set	IVIFSSPWA IVIFSSPWG	TODIM	Material, Personnel, Supplier Selection	Known	No
A. Biwas, N. Deb (2021) [38]	Pythagorean Fuzzy Set	PFSSPA PFSSPWA PFSSPG PFSSPWG	Ranking	Selection of Best Emerging Technology Enterprise	Known	No
Q. Khan et al (2021) [39]	T-Spherical Fuzzy Set	T-SPHFSSPHEM T-SPHFSSGHEM T-SPHFSSPWHEM T-SPHFSSPWGHEM	Ranking	Water Reuse Application	Known	No
U. Khalid (2023) [40]	Interval-valued Pythagorean Fuzzy Set	IVPFSSPG IVPFSSPA IVPFSSPWA	Ranking	Rice Quality Management	Known	No
U. Kalsoom et al (2023) [41]	Complex Interval-valued Intuitionistic Fuzzy Set	CIVIFSSPA CIVIFSSPOA CIVIFSSPG CIVIFSSPOG	Ranking	Selection of Green Suppliers	Unknown	Yes
P. Liu et al (2024) [42]	Complex Intuitionistic Fuzzy Set	CA-IFSSPA CA-IFSSPOA CA-IFSSPG CA-IFSSPOG	Ranking	Application of an Electronic Commerce	Unknown Distributor	Yes
Bai Chungsong (2024) [43]	Cubic Intuitionistic Fuzzy Set	CIFSSPWA CIFSSPWG	Ranking	Diabetes Care	Known	Yes
Proposed	Complex Neutrosophic Set	CSVNSSPWA CSVNSSOWA	Ranking	Green Supply Chain Management	Unknown (SMART)	Yes

1.3. *Contribution of the paper*

(1) To develop a complex single-valued neutrosophic set model utilizing an application of the Schweizer-Sklar t-norm and t-conorm.

(2) To derive complex single-valued neutrosophic Schweizer-Sklar power weighted average (CSVNSSPWA) and complex single-valued neutrosophic Schweizer-Sklar power ordered weighted average (CSVNSSPOWA) operators.

(3) To examine the key properties of these newly proposed operators.

(4) To formulate a multi attribute group decision making (MADM) method for evaluating green supply chain management problem using proposed operators complex neutrosophic information.

(5) To illustrate the effectiveness of the proposed methodology by comparing the newly developed operators with known operators through application example.

1.4. *Layout of the paper*

This paper is organized as follows: some important definitions are recalled in Section 2, namely CSVNs, PA operators and several of Schweizer-Sklar operational laws. Section 3 also presents and discusses the CSVNSSPWA, CSVNSSPOWA operators. In Section 4, a decision-making approach based on CSVNs is suggested and the procedure to evaluate MADM problems is described. Section 5 gives a numerical example to illustrate the decision-making process using the obtained operators and makes a comparison with other operators in order to reveal the advantages and efficiency of the developed methods. Finally, the concluding summary is provided in Section 6.

2. **Preliminary**

This section introduces fundamental concepts of CSVN set within the universal set $\bar{\bar{\mathcal{U}}}$.

Definition 2.1. *[11] A complex single valued neutrosophic (CSVN) $\mathcal{I}$, on the universal set $\bar{\bar{\mathcal{U}}}$ is of the form :*

$$\mathcal{I} = \{\langle \dot{æ}, \mathbb{T}_{\mathcal{I}}(\dot{æ}), \mathbb{I}_{\mathcal{I}}(\dot{æ}), \mathbb{F}_{\mathcal{I}}(\dot{æ}) | \dot{æ} \in \bar{\bar{\mathcal{U}}} \rangle\}$$

where, complex valued truth membership function expressed by $\mathbb{T}_{\mathcal{I}}(\dot{æ}) = \mathcal{P}_{\mathcal{I}}.e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}}}(\dot{æ})}$, complex valued indeterminacy membership function expressed by $\mathbb{I}_{\mathcal{I}}(\dot{æ}) = \mathcal{S}_{\mathcal{I}}.e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}}}(\dot{æ})}$, complex valued falsity membership function expressed by $\mathbb{F}_{\mathcal{I}}(\dot{æ}) = \mathcal{R}_{\mathcal{I}}.e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}}}(\dot{æ})}$. Here, $\mathcal{P}_{\mathcal{I}}$, $\mathcal{S}_{\mathcal{I}}$, $\mathcal{R}_{\mathcal{I}} \in [0,1]$ and $\omega_{\mathcal{P}_{\mathcal{I}}}$, $\omega_{\mathcal{S}_{\mathcal{I}}}$, $\omega_{\mathcal{R}_{\mathcal{I}}} \in [0,1]$ and $i = \sqrt{-1}$ such that $0 \leq \mathcal{P}_{\mathcal{I}} + \mathcal{S}_{\mathcal{I}} + \mathcal{R}_{\mathcal{I}} \leq 3$. If $\bar{\bar{\mathcal{U}}}$ contain single element then, for CSVN set $\mathcal{I}$ on $\bar{\bar{\mathcal{U}}}$, we write $X = ((\mathcal{P}_{\mathcal{I}}, \omega_{\mathcal{P}_{\mathcal{I}}}), (\mathcal{S}_{\mathcal{I}}, \omega_{\mathcal{S}_{\mathcal{I}}}), (\mathcal{R}_{\mathcal{I}}, \omega_{\mathcal{R}_{\mathcal{I}}}))$ and is termed as complex single value neutrosophic number.

Definition 2.2. *Let $\bar{\bar{X}} = ((\mathcal{P}_x, \omega_{\mathcal{P}_x}), (\mathcal{S}_x, \omega_{\mathcal{S}_x}), (\mathcal{R}_x, \omega_{\mathcal{R}_x}))$ and $\bar{\bar{Y}} = ((\mathcal{P}_y, \omega_{\mathcal{P}_y}), (\mathcal{S}_y, \omega_{\mathcal{S}_y}), (\mathcal{R}_y, \omega_{\mathcal{R}_y}))$ be the two complex single valued neutrosophic number (CSVNN) and $\lambda \geq 0$ be a real number. Then we have,*

(1) $\bar{\bar{X}} \subseteq \bar{\bar{Y}}$ *if and only if* $\mathcal{P}_x \leq \mathcal{P}_y$, $\mathcal{S}_x \geq \mathcal{S}_y$, $\mathcal{R}_x \geq \mathcal{R}_y$ *and* $\omega_{\mathcal{P}_x} \leq \omega_{\mathcal{P}_y}$, $\omega_{\mathcal{S}_x} \geq \omega_{\mathcal{S}_y}$, $\omega_{\mathcal{R}_x} \geq \omega_{\mathcal{R}_y}$.
(2) $\bar{\bar{X}} = \bar{\bar{Y}}$ *if* $\bar{\bar{X}} \subseteq \bar{\bar{Y}}$ *and* $\bar{\bar{Y}} \subseteq \bar{\bar{X}}$.
(3) $\bar{\bar{X}}^c = ((\mathcal{R}_x, \omega_{\mathcal{R}_x}), (\mathcal{S}_x, \omega_{\mathcal{S}_x}), (\mathcal{P}_x, \omega_{\mathcal{P}_x}))$.
(4) $\bar{\bar{X}} \cup \bar{\bar{Y}} = \langle max(\mathcal{P}_x, \mathcal{P}_y).e^{i2\pi(max(\omega_{\mathcal{P}_x}, \omega_{\mathcal{P}_y}))}, \quad min(\mathcal{S}_x, \mathcal{S}_y).e^{i2\pi(min(\omega_{\mathcal{S}_x}, \omega_{\mathcal{S}_y}))}, min(\mathcal{R}_x, \mathcal{R}_y).e^{i2\pi(min(\omega_{\mathcal{R}_x}, \omega_{\mathcal{R}_y}))}\rangle$.

(5) $\bar{\bar{X}} \cap \bar{\bar{Y}} = \langle min(\mathcal{P}_x, \mathcal{P}_y).e^{i2\pi(min(\omega_{\mathcal{P}_x}, \omega_{\mathcal{P}_y}))}, max(\mathcal{S}_x, \mathcal{S}_y).e^{i2\pi(max(\omega_{\mathcal{S}_x}, \omega_{\mathcal{S}_y}))}, max(\mathcal{R}_x, \mathcal{R}_y).e^{i2\pi(max(\omega_{\mathcal{R}_x}, \omega_{\mathcal{R}_y}))}\rangle$.

Definition 2.3. *The basic operational laws on these numbers are given by,*

(1) $\bar{\bar{X}} \oplus \bar{\bar{Y}} = \langle(\mathcal{P}_x + \mathcal{P}_y - \mathcal{P}_x\mathcal{P}_y).e^{i2\pi(\omega_{\mathcal{P}_x} + \omega_{\mathcal{P}_y} - \omega_{\mathcal{P}_x}\omega_{\mathcal{P}_y})}, \mathcal{S}_x.\mathcal{S}_y e^{i2\pi(\omega_{\mathcal{S}_x}.\omega_{\mathcal{S}_y})}, \mathcal{R}_x.\mathcal{R}_y e^{i2\pi(\omega_{\mathcal{R}_x}.\omega_{\mathcal{R}_y})}\rangle$,
(2) $\bar{\bar{X}} \otimes \bar{\bar{Y}} = \langle(\mathcal{P}_x.\mathcal{P}_y e^{i2\pi(\omega_{\mathcal{P}_x}.\omega_{\mathcal{P}_y})}, \mathcal{S}_x + \mathcal{S}_y - \mathcal{S}_x\mathcal{S}_y.e^{i2\pi(\omega_{\mathcal{S}_x} + \omega_{\mathcal{S}_y} - \omega_{\mathcal{S}_x}\omega_{\mathcal{S}_y})}, \mathcal{R}_x + \mathcal{R}_y - \mathcal{R}_x\mathcal{R}_y.e^{i2\pi(\omega_{\mathcal{R}_x} + \omega_{\mathcal{R}_y} - \omega_{\mathcal{R}_x}\omega_{\mathcal{R}_y})}\rangle$,
(3) $\lambda\bar{\bar{X}} = \langle(1 - (1 - \mathcal{P}_x)^\lambda)e^{i2\pi(1-(1-\mathcal{P}_x)^\lambda)}, (\mathcal{S}_x)^\lambda e^{i2\pi(\omega_{\mathcal{S}_x})^\lambda}, (\mathcal{R}_x)^\lambda e^{i2\pi(\omega_{\mathcal{R}_x})^\lambda}\rangle$,
(4) $\bar{\bar{X}}^\lambda = \langle(\mathcal{P}_x)^\lambda e^{i2\pi(\omega_{\mathcal{P}_x})^\lambda}, (1 - (1 - \mathcal{S}_x)^\lambda)e^{i2\pi(1-(1-\mathcal{S}_x)^\lambda)}, (1 - (1 - \mathcal{R}_x)^\lambda)e^{i2\pi(1-(1-\mathcal{R}_x)^\lambda)}\rangle$.

Definition 2.4. *[34] For any CSVN $\mathcal{I} = ((\mathcal{P}_\mathcal{I}, \omega_{\mathcal{P}_\mathcal{I}}), (\mathcal{S}_\mathcal{I}, \omega_{\mathcal{S}_\mathcal{I}}), (\mathcal{R}_\mathcal{I}, \omega_{\mathcal{R}_\mathcal{I}}))$, its score and accuracy functions can be explained as follows,*

$$\mathfrak{S}_\mathcal{I} = \frac{|\mathcal{P}_\mathcal{I} - \mathcal{S}_\mathcal{I} - \mathcal{R}_\mathcal{I} + \omega_{\mathcal{P}_\mathcal{I}} - \omega_{\mathcal{S}_\mathcal{I}} - \omega_{\mathcal{R}_\mathcal{I}}|}{3} \tag{1}$$

$$\mathfrak{A}_\mathcal{I} = \frac{\mathcal{P}_\mathcal{I} + \mathcal{S}_\mathcal{I} + \mathcal{R}_\mathcal{I} + \omega_{\mathcal{P}_\mathcal{I}} + \omega_{\mathcal{S}_\mathcal{I}} + \omega_{\mathcal{R}_\mathcal{I}}}{3} \tag{2}$$

Definition 2.5. *Let $\mathcal{I}_1 = ((\mathcal{P}_{\mathcal{I}_1}, \omega_{\mathcal{P}_{\mathcal{I}_1}}), (\mathcal{S}_{\mathcal{I}_1}, \omega_{\mathcal{S}_{\mathcal{I}_1}}), (\mathcal{R}_{\mathcal{I}_1}, \omega_{\mathcal{R}_{\mathcal{I}_1}}))$ and $\mathcal{I}_2 = ((\mathcal{P}_{\mathcal{I}_2}, \omega_{\mathcal{P}_{\mathcal{I}_2}}), (\mathcal{S}_{\mathcal{I}_2}, \omega_{\mathcal{S}_{\mathcal{I}_2}}), (\mathcal{R}_{\mathcal{I}_2}, \omega_{\mathcal{R}_{\mathcal{I}_2}}))$ be any two CSVNs and $\mathfrak{S}_{\mathcal{I}_j}$ and $\mathfrak{A}_{\mathcal{I}_j}$ for j=1,2, representing their respective score and accuracy values, we obtain the following results:*

(1) *If $\mathfrak{S}_{\mathcal{I}_1} > \mathfrak{S}_{\mathcal{I}_2}$, then $\mathcal{I}_1 > \mathcal{I}_2$.*
(2) *If $\mathfrak{S}_{\mathcal{I}_1} < \mathfrak{S}_{\mathcal{I}_2}$, then $\mathcal{I}_1 < \mathcal{I}_2$.*
(3) *If $\mathfrak{S}_{\mathcal{I}_1} > \mathfrak{S}_{\mathcal{I}_2}$, then*
    a. *If $\mathfrak{A}_{\mathcal{I}_1} > \mathfrak{A}_{\mathcal{I}_2}$, then $\mathcal{I}_1 > \mathcal{I}_2$.*
    b. *If $\mathfrak{A}_{\mathcal{I}_1} < \mathfrak{A}_{\mathcal{I}_2}$, then $\mathcal{I}_1 < \mathcal{I}_2$.*

Definition 2.6. *[32] Let $\mathcal{I}_1 = ((\mathcal{P}_{\mathcal{I}_1}, \omega_{\mathcal{P}_{\mathcal{I}_1}}), (\mathcal{S}_{\mathcal{I}_1}, \omega_{\mathcal{S}_{\mathcal{I}_1}}), (\mathcal{R}_{\mathcal{I}_1}, \omega_{\mathcal{R}_{\mathcal{I}_1}}))$ and $\mathcal{I}_2 = ((\mathcal{P}_{\mathcal{I}_2}, \omega_{\mathcal{P}_{\mathcal{I}_2}}), (\mathcal{S}_{\mathcal{I}_2}, \omega_{\mathcal{S}_{\mathcal{I}_2}}), (\mathcal{R}_{\mathcal{I}_2}, \omega_{\mathcal{R}_{\mathcal{I}_2}}))$ be any two CSVN numbers, then the Euclidean distance between them is determined by:*

$$\mathfrak{D}(\mathcal{I}_1, \mathcal{I}_2) = \sqrt{\begin{array}{l}\frac{1}{6}\left(|\mathcal{P}_{\mathcal{I}_1} - \mathcal{P}_{\mathcal{I}_2}|^2 + |\mathcal{S}_{\mathcal{I}_1} - \mathcal{S}_{\mathcal{I}_2}|^2 + |\mathcal{R}_{\mathcal{I}_1} - \mathcal{R}_{\mathcal{I}_2}|^2\right.\\ \left.+|\omega_{\mathcal{P}_{\mathcal{I}_1}} - \omega_{\mathcal{P}_{\mathcal{I}_2}}|^2 + |\omega_{\mathcal{S}_{\mathcal{I}_1}} - \omega_{\mathcal{S}_{\mathcal{I}_2}}|^2 + |\omega_{\mathcal{R}_{\mathcal{I}_1}} - \omega_{\mathcal{R}_{\mathcal{I}_2}}|^2\right)\end{array}} \tag{3}$$

Definition 2.7. *[31] For any set of positive numbers $\mathcal{I}_i$, where i=1,2,..n, the final forms of the power average(PA) operator is given by:*

$$PA(\breve{\mathcal{Y}}_1, \breve{\mathcal{Y}}_2, \ldots, \breve{\mathcal{Y}}_n) = \frac{\sum_{i=1}^{n}(1+\acute{U}(\breve{\mathcal{Y}}_i))\breve{\mathcal{Y}}_i}{\sum_{i=1}^{n} 1+\acute{U}(\breve{\mathcal{Y}}_i)} \quad (4)$$

The value of $\acute{U}(\breve{\mathcal{Y}}_i) = \sum_{j=1, j\neq k}^{n}$ sup $(\breve{\mathcal{Y}}_i, \breve{\mathcal{Y}}_j)$ the support of the $\breve{\mathcal{Y}}_i$ & $\breve{\mathcal{Y}}_j$ under the consideration of some variable properties:

(1) *sup $(\breve{\mathcal{Y}}_i, \breve{\mathcal{Y}}_j) \in [0,1]$;*
(2) *sup $(\breve{\mathcal{Y}}_i, \breve{\mathcal{Y}}_j) =$ sup $(\breve{\mathcal{Y}}_j, \breve{\mathcal{Y}}_i)$;*
(3) *sup $(\breve{\mathcal{Y}}_i, \breve{\mathcal{Y}}_j) \geq$ sup $(\breve{\mathcal{Y}}_s, \breve{\mathcal{Y}}_t)$ if $|\breve{\mathcal{Y}}_i, \breve{\mathcal{Y}}_j| < |\breve{\mathcal{Y}}_s, \breve{\mathcal{Y}}_t|$.*

Inspired by Yager (2001), Xu and Yager (2010) developed the power geometric (PG) operator, which is based on the PA operator and the geometric mean.

Definition 2.8. *[28] Let $\mathcal{I}_1 = ((\mathcal{P}_{\mathcal{I}_1}, \omega_{\mathcal{P}_{\mathcal{I}_1}}), (\mathcal{S}_{\mathcal{I}_1}, \omega_{\mathcal{S}_{\mathcal{I}_1}}), (\mathcal{R}_{\mathcal{I}_1}, \omega_{\mathcal{R}_{\mathcal{I}_1}}))$ and $\mathcal{I}_2 = ((\mathcal{P}_{\mathcal{I}_2}, \omega_{\mathcal{P}_{\mathcal{I}_2}}), (\mathcal{S}_{\mathcal{I}_2}, \omega_{\mathcal{S}_{\mathcal{I}_2}}), (\mathcal{R}_{\mathcal{I}_2}, \omega_{\mathcal{R}_{\mathcal{I}_2}}))$ be any two CSVN numbers.The generalized union and intersection are specified as follows:*
$\mathcal{I}_1 \cup_{\mathfrak{T},\mathfrak{T}^*} \mathcal{I}_2 = \{\langle \dot{æ}, \{\mathfrak{T}^*\{\mathbb{T}_{\mathcal{I}_1}(\dot{æ}), \mathbb{T}_{\mathcal{I}_2}(\dot{æ})\}\}, \{\mathfrak{T}\{\mathbb{I}_{\mathcal{I}_1}(\dot{æ}), \mathbb{I}_{\mathcal{I}_2}(\dot{æ})\}\}, \{\mathfrak{T}\{\mathbb{F}_{\mathcal{I}_1}(\dot{æ}), \mathbb{F}_{\mathcal{I}_2}\}(\dot{æ})\}|\dot{æ} \in X\rangle\}$;
$\mathcal{I}_1 \cap_{\mathfrak{T},\mathfrak{T}^*} \mathcal{I}_2 = \{\langle \dot{æ}, \{\mathfrak{T}\{\mathbb{T}_{\mathcal{I}_1}(\dot{æ}), \mathbb{T}_{\mathcal{I}_2}(\dot{æ})\}\}, \{\mathfrak{T}^*\{\mathbb{I}_{\mathcal{I}_1}(\dot{æ}), \mathbb{I}_{\mathcal{I}_2}(\dot{æ})\}\}, \{\mathfrak{T}^*\{\mathbb{F}_{\mathcal{I}_1}(\dot{æ}), \mathbb{F}_{\mathcal{I}_2}(\dot{æ})\}\}|\dot{æ} \in X\}\rangle\}$; *where $\mathfrak{T}$ and $\mathfrak{T}^*$ denote the t-norm and t-conorm respectively. However, Schwweizer-Sklar t-norm and t-conorm are defined as follows:*

$$\mathfrak{T}(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{x}^\tau + \mathfrak{y}^\tau - 1)^{\frac{1}{\tau}}$$
$$\mathfrak{T}^*(\mathfrak{x}, \mathfrak{y}) = 1 - ((1-\mathfrak{x})^\tau + (1-\mathfrak{y})^\tau - 1)^{\frac{1}{\tau}}$$

where $\tau \leq 0$ and $(\mathfrak{x}, \mathfrak{y}) \in [0,1]$. Furthermore, when $\tau = 0$, then $\mathfrak{T}(\mathfrak{x}, \mathfrak{y}) = \mathfrak{x}\mathfrak{y}$ and $\mathfrak{T}^*(\mathfrak{x}, \mathfrak{y}) = \mathfrak{x} + \mathfrak{y} - \mathfrak{x}\mathfrak{y}$. These are algebraic t-norm and t-conorm respectively.

3. **Formulation of CSVN Schweizer-Sklar Power Aggregation Operators**

This section focuses on determining the Schweizer-Sklar operating rules for CSVN data. We propose a theory for power aggregation operators, including CSVNSSPWA, CSVNSSPOWA operators based on Schweizer-Sklar operational law. Additionally, properties and results for the derived work are outlined.

3.1. *CSVN Schweizer-Sklar power averaging operators*

Definition 3.1. *Let $\mathcal{I}_i = ((\mathcal{P}_{\mathcal{I}_i}, \omega_{\mathcal{P}_{\mathcal{I}_i}}), (\mathcal{S}_{\mathcal{I}_i}, \omega_{\mathcal{S}_{\mathcal{I}_i}}), (\mathcal{R}_{\mathcal{I}_i}, \omega_{\mathcal{R}_{\mathcal{I}_i}}))$,where $i = 1,2$ be a collection of CSVNs. Then the CSVNSSPA operator of dimension n is a mapping CSVNSSPA:$\mathcal{I}^n \to \mathcal{I}$ such that*

$$CSVNSSPA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \frac{\oplus_{i=1}^{n}((1+\mathfrak{U}(\mathcal{I}_i))\mathcal{I}_i)}{\sum_{i=1}^{n}(1+\mathfrak{U}(\mathcal{I}_i))} \tag{5}$$

where $\mathcal{I}$ is the set of all CSVN numbers and $\mathfrak{U}(\mathcal{I}_j) = \sum_{k=1, k\neq i}^{n} Sup(\mathcal{I}_i, \mathcal{I}_k)$.

Theorem 3.1. *For a group of CSVNs $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$, where $i = 1, 2, ..., n$, the value aggregated by the developed CSVNSSPA operator is still a CSVNN and specified by:*

$$CSVNSSPA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$$

$$= \begin{pmatrix} \left[1 - \left(\sum_{i=1}^{n} \Xi_i (1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}\right)^{\frac{1}{\alpha}}\right] \cdot e^{2\pi i \left[1 - \left(\sum_{i=1}^{n} \Xi_i (1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{n} \Xi_i \mathcal{S}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\sum_{i=1}^{n} \Xi_i \omega_{\mathcal{S}_{\mathcal{I}_i}}^{\alpha}\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{n} \Xi_i \mathcal{R}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\sum_{i=1}^{n} \Xi_i \omega_{\mathcal{R}_{\mathcal{I}_i}}^{\alpha}\right]^{\frac{1}{\alpha}}} \end{pmatrix} \tag{6}$$

where $(i = 1, 2, ..., n)$ is a set of integrated weights, $\Xi_i = \frac{1+\mathfrak{U}(\mathcal{I}_i)}{\sum_{i=1}^{n}(1+\mathfrak{U}(\mathcal{I}_i))}$.

Proof. Equation 6 will be proven through the method of mathematical induction.

$$\begin{aligned} CSVNSSPA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) &= \frac{\oplus_{i=1}^{n}((1+\mathfrak{U}(\mathcal{I}_i))\mathcal{I}_i)}{\sum_{i=1}^{n}(1+\mathfrak{U}(\mathcal{I}_i))} \\ &= \oplus_{i=1}^{n} \left[\frac{((1+\mathfrak{U}(\mathcal{I}_i))\mathcal{I}_i)}{\sum_{i=1}^{n}(1+\mathfrak{U}(\mathcal{I}_i))}\right] \\ &= \oplus_{i=1}^{n} \left[\Xi_i \mathcal{I}_i\right] \end{aligned}$$

where, $\Xi_i = \frac{((1+\mathfrak{U}(\mathcal{I}_i))\mathcal{I}_i)}{\sum_{i=1}^{n}(1+\mathfrak{U}(\mathcal{I}_i))}$.
Then, based on the operational rules of CSVN numbers using Schweizer- Skalr operations, we have

$$\Xi_i \mathcal{I}_i = \begin{pmatrix} \left[1 - \left(\Xi_i (1-\mathcal{P}_{\mathcal{I}_i})^{\alpha} - \Xi_i + 1\right)^{\frac{1}{\alpha}}\right] \cdot e^{2\pi i \left[1 - \left(\Xi_i (1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha} - \Xi_i + 1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\Xi_i \mathcal{S}_{\mathcal{I}_i}^{\alpha} - \Xi_i + 1\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\Xi_i \omega_{\mathcal{S}_{\mathcal{I}_i}}^{\alpha} - \Xi_i + 1\right]^{\frac{1}{\alpha}}}, \\ \left[\Xi_i \mathcal{R}_{\mathcal{I}_i}^{\alpha} - \Xi_i + 1\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\Xi_i \omega_{\mathcal{R}_{\mathcal{I}_i}}^{\alpha} - \Xi_i + 1\right]^{\frac{1}{\alpha}}} \end{pmatrix}.$$

When $n = 2$, we have

$$\Xi_1 \mathcal{I}_1 = \begin{pmatrix} \left[1 - \left(\Xi_1 (1-\mathcal{P}_{\mathcal{I}_1})^{\alpha} - \Xi_1 + 1\right)^{\frac{1}{\alpha}}\right] \cdot e^{2\pi i \left[1 - \left(\Xi_1 (1-\omega_{\mathcal{P}_{\mathcal{I}_1}})^{\alpha} - \Xi_1 + 1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\Xi_1 \mathcal{S}_{\mathcal{I}_1}^{\alpha} - \Xi_1 + 1\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\Xi_1 \omega_{\mathcal{S}_{\mathcal{I}_1}}^{\alpha} - \Xi_1 + 1\right]^{\frac{1}{\alpha}}}, \\ \left[\Xi_1 \mathcal{R}_{\mathcal{I}_1}^{\alpha} - \Xi_1 + 1\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\Xi_1 \omega_{\mathcal{R}_{\mathcal{I}_1}}^{\alpha} - \Xi_1 + 1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$\Xi_2\mathcal{I}_2 = \begin{pmatrix} \left[1-(\Xi_2(1-\mathcal{P}_{\mathcal{I}_2})^{\alpha}-\Xi_2+1)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\Xi_2(1-\omega_{\mathcal{P}_{\mathcal{I}_2}})^{\alpha}-\Xi_2+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\Xi_2\mathcal{S}_{\mathcal{I}_2}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_2\omega_{\mathcal{S}_{\mathcal{I}_2}}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}}, \\ \left[\Xi_2\mathcal{R}_{\mathcal{I}_2}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_2\omega_{\mathcal{R}_{\mathcal{I}_2}}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

then,

$$CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2) = \Xi_1\mathcal{I}_1 \oplus \Xi_2\mathcal{I}_2$$

$$= \begin{pmatrix} \left[1-(\Xi_1(1-\mathcal{P}_{\mathcal{I}_1})^{\alpha}-\Xi_1+1)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\Xi_1(1-\omega_{\mathcal{P}_{\mathcal{I}_1}})^{\alpha}-\Xi_1+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\Xi_1\mathcal{S}_{\mathcal{I}_1}^{\alpha}-\Xi_1+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_1\omega_{\mathcal{S}_{\mathcal{I}_1}}^{\alpha}-\Xi_1+1\right]^{\frac{1}{\alpha}}}, \\ \left[\Xi_1\mathcal{R}_{\mathcal{I}_1}^{\alpha}-\Xi_1+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_1\omega_{\mathcal{R}_{\mathcal{I}_1}}^{\alpha}-\Xi_1+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \left[1-(\Xi_2(1-\mathcal{P}_{\mathcal{I}_2})^{\alpha}-\Xi_2+1)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\Xi_2(1-\omega_{\mathcal{P}_{\mathcal{I}_2}})^{\alpha}-\Xi_2+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\Xi_2\mathcal{S}_{\mathcal{I}_2}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_2\omega_{\mathcal{S}_{\mathcal{I}_2}}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}}, \\ \left[\Xi_2\mathcal{R}_{\mathcal{I}_2}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_2\omega_{\mathcal{R}_{\mathcal{I}_2}}^{\alpha}-\Xi_2+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$= \begin{pmatrix} \left[1-\left(\sum_{i=1}^{2}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}-\sum_{i=1}^{2}\Xi_i+1\right)^{\frac{1}{\alpha}}\right]\cdot \\ e^{2\pi i\left[1-\left(\sum_{i=1}^{2}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}-\sum_{i=1}^{2}\Xi_i+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{2}\Xi_i\mathcal{S}_{\mathcal{I}_i}^{\alpha}-\sum_{i=1}^{2}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{2}\Xi_i\omega_{\mathcal{S}_{\mathcal{I}_i}}^{\alpha}-\sum_{i=1}^{2}\Xi_i+1\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{2}\Xi_i\mathcal{R}_{\mathcal{I}_i}^{\alpha}-\sum_{i=1}^{2}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{2}\Xi_i\omega_{\mathcal{R}_{\mathcal{I}_i}}^{\alpha}-\sum_{i=1}^{2}\Xi_i+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

Suppose n=m,

$$CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_m)$$

$$= \begin{pmatrix} \left[1-(\sum_{i=1}^{m}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}-\sum_{i=1}^{m}\Xi_i+1)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\sum_{i=1}^{m}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}-\sum_{i=1}^{m}\Xi_i+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{m}\Xi_i\mathcal{S}_{\mathcal{I}_i}^{\alpha}-\sum_{i=1}^{m}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{m}\Xi_i\omega_{\mathcal{S}_{\mathcal{I}_i}}^{\alpha}-\sum_{i=1}^{m}\Xi_i+1\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{m}\Xi_i\mathcal{R}_{\mathcal{I}_i}^{\alpha}-\sum_{i=1}^{m}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{m}\Xi_i\omega_{\mathcal{R}_{\mathcal{I}_i}}^{\alpha}-\sum_{i=1}^{m}\Xi_i+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

When n= m+1, according to the operational rules of CSVN numbers, we have $\Xi_{m+1}\mathcal{I}_{m+1}$

$$= \begin{pmatrix} \left[1-\left(\Xi_{m+1}(1-\mathcal{P}_{\mathcal{I}_{m+1}})^{\alpha}-\Xi_{m+1}+1\right)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\Xi_{m+1}(1-\omega_{\mathcal{P}_{\mathcal{I}_{m+1}}})^{\alpha}-\Xi_{m+1}+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\Xi_{m+1}\mathcal{S}^{\alpha}_{\mathcal{I}_{m+1}}-\Xi_{m+1}+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_{m+1}\omega^{\alpha}_{\mathcal{S}_{\mathcal{I}_{m+1}}}-\Xi_{m+1}+1\right]^{\frac{1}{\alpha}}}, \\ \left[\Xi_{m+1}\mathcal{R}^{\alpha}_{\mathcal{I}_{m+1}}-\Xi_{m+1}+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\Xi_{m+1}\omega^{\alpha}_{\mathcal{R}_{\mathcal{I}_{m+1}}}-\Xi_{m+1}+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

and

$$CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_m,\mathcal{I}_{m+1}) = CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_m)\oplus(\Xi_{m+1}\mathcal{I}_{m+1})$$

$$= \begin{pmatrix} \left[1-\left(\sum_{i=1}^{m+1}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}-\sum_{i=1}^{m+1}\Xi_i+1\right)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\sum_{i=1}^{m+1}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}-\sum_{i=1}^{m+1}\Xi_i+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{m+1}\Xi_i\mathcal{S}^{\alpha}_{\mathcal{I}_i}-\sum_{i=1}^{m+1}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{m+1}\Xi_i\omega^{\alpha}_{\mathcal{S}_{\mathcal{I}_i}}-\sum_{i=1}^{m+1}\Xi_i+1\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{m+1}\Xi_i\mathcal{R}^{\alpha}_{\mathcal{I}_i}-\sum_{i=1}^{m+1}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{m+1}\Xi_i\omega^{\alpha}_{\mathcal{R}_{\mathcal{I}_i}}-\sum_{i=1}^{m+1}\Xi_i+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

□ Following this, we will investigate the principles of idempotency, monotonicity, and boundedness based on the information given in Equation 6.

Proposition 1. (Idempotency)
For any collection of CSVNs, $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i}e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i}e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i}e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$, where (i=1,2,...,n), if $\mathcal{I}_i = \mathcal{I} = (\mathcal{P}_{\mathcal{I}}e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}}}}, \mathcal{S}_{\mathcal{I}}e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}}}}, \mathcal{R}_{\mathcal{I}}e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}}}})$, then $CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_n) = \mathcal{I}$.

Proof.

$$CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_n)=\begin{pmatrix} \left[1-(\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}-\sum_{i=1}^{n}\Xi_i+1)^{\frac{1}{\alpha}}\right]\cdot \\ e^{2\pi i\left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{n}\Xi_i\mathcal{S}_{\mathcal{I}_i}^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{n}\Xi_i\omega_{\mathcal{S}_{\mathcal{I}_i}}^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{n}\Xi_i\mathcal{R}_{\mathcal{I}_i}^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\sum_{i=1}^{n}\Xi_i\omega_{\mathcal{R}_{\mathcal{I}_i}}^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$=\begin{pmatrix} \left[1-((1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}-1+1)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left((1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}-1+1\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\mathcal{S}_{\mathcal{I}_i}^{\alpha}-1+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\omega_{\mathcal{S}_{\mathcal{I}_i}}^{\alpha}-1+1\right]^{\frac{1}{\alpha}}}, \\ \left[\mathcal{R}_{\mathcal{I}_i}^{\alpha}-1+1\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\omega_{\mathcal{R}_{\mathcal{I}_i}}^{\alpha}-1+1\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$=\begin{pmatrix} [1-((1-\mathcal{P}_{\mathcal{I}_i})^{\alpha})^{\frac{1}{\alpha}}]\cdot e^{2\pi i\left[1-\left((1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\mathcal{S}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\omega_{\mathcal{S}_{\mathcal{I}_i}^{\alpha}}\right]^{\frac{1}{\alpha}}}, \\ \left[\mathcal{R}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}}\cdot e^{2\pi i\left[\omega_{\mathcal{R}_{\mathcal{I}_i}^{\alpha}}\right]^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$=\left(\ [1-(1-\mathcal{P}_{\mathcal{I}_i})].e^{2\pi i[1-(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})]},[\mathcal{S}_{\mathcal{I}_i}].e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}},[\mathcal{R}_{\mathcal{I}_i}].e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}}\ \right)$$

$$=\left(\mathcal{P}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i}}},\mathcal{S}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}},\mathcal{R}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}}\right)$$

$$=\mathcal{I}.$$

□**Proposition 2.** (Monotonicity)

For any collection of CSVNs $\mathcal{I}_i=\left\langle\mathcal{P}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i}}},\mathcal{S}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}},\mathcal{R}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}}\right\rangle$ which satisfies $CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_n)\leq CSVNSSPA(\mathcal{I}'_1,\mathcal{I}'_2,\ldots,\mathcal{I}'_n)$, i=1,2,...,n if $\mathcal{I}_i=\mathcal{I}'_i$.

Proof. Let if $\mathcal{I}_i=\mathcal{I}'_i$, then $\mathcal{P}_{\mathcal{I}_i}\leq\mathcal{P}'_{\mathcal{I}_i}$, $\mathcal{S}_{\mathcal{I}_i}\leq\mathcal{S}'_{\mathcal{I}_i}$, $\mathcal{R}_{\mathcal{I}_i}\leq\mathcal{R}'_{\mathcal{I}_i}$ and $\omega_{\mathcal{P}_{\mathcal{I}_i}}\leq\omega_{\mathcal{P}'_{\mathcal{I}_i}}$, $\omega_{\mathcal{S}_{\mathcal{I}_i}}\leq\omega_{\mathcal{S}'_{\mathcal{I}_i}}$, $\omega_{\mathcal{R}_{\mathcal{I}_i}}\leq\omega_{\mathcal{R}'_{\mathcal{I}_i}}$, thus

$$\mathcal{P}_{\mathcal{I}_i} \leq \mathcal{P}'_{\mathcal{I}_i}$$
$$\Rightarrow \quad 1-\mathcal{P}_{\mathcal{I}_i} \geq 1-\mathcal{P}'_{\mathcal{I}_i}$$
$$\Rightarrow \quad (1-\mathcal{P}_{\mathcal{I}_i})^{\alpha} \geq (1-\mathcal{P}'_{\mathcal{I}_i})^{\alpha}$$
$$\Rightarrow \quad \sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha} \geq \sum_{i=1}^{n}\Xi_i(1-\mathcal{P}'_{\mathcal{I}_i})^{\alpha}$$
$$\Rightarrow \quad \left[\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq \left[\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}'_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$
$$\Rightarrow \quad -\left[\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq -\left[\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}'_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$
$$\Rightarrow \quad \left[1-\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq \left[1-\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}'_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$

also

$$\Rightarrow \quad \left[1-\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq \left[1-\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{P}'_{\mathcal{I}_i}})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$

Further, we derive that,

$$\mathcal{S}_{\mathcal{I}_i} \leq \mathcal{S}'_{\mathcal{I}_i}$$
$$\Rightarrow \quad \mathcal{S}_{\mathcal{I}_i} \geq \mathcal{S}'_{\mathcal{I}_i}$$
$$\Rightarrow \quad (\mathcal{S}_{\mathcal{I}_i})^{\alpha} \geq (\mathcal{S}'_{\mathcal{I}_i})^{\alpha}$$
$$\Rightarrow \quad \sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i})^{\alpha} \geq \sum_{i=1}^{n}\Xi_i(\mathcal{S}'_{\mathcal{I}_i})^{\alpha}$$
$$\Rightarrow \quad \left[\sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq \left[\sum_{i=1}^{n}\Xi_i(\mathcal{S}'_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$
$$\Rightarrow \quad -\left[\sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq -\left[\sum_{i=1}^{n}\Xi_i(\mathcal{S}'_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$
$$\Rightarrow \quad \left[1-\sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq \left[1-\sum_{i=1}^{n}\Xi_i(\mathcal{S}'_{\mathcal{I}_i})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$

also

$$\Rightarrow \quad \left[1-\sum_{i=1}^{n}\Xi_i(\omega_{\mathcal{S}_{\mathcal{I}_i}})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}} \geq \left[1-\sum_{i=1}^{n}\Xi_i(\omega_{\mathcal{S}'_{\mathcal{I}_i}})^{\alpha} - \sum_{i=1}^{n}\Xi_i + 1\right]^{\frac{1}{\alpha}}$$

Similarly,

$$\mathcal{R}_{\mathcal{I}_i} \leq \mathcal{R'}_{\mathcal{I}_i}$$

$$\Rightarrow \quad \left[1-\sum_{i=1}^{n}\Xi_i(\omega_{\mathcal{R}_{\mathcal{I}_i}})^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right]^{\frac{1}{\alpha}} \geq \left[1-\sum_{i=1}^{n}\Xi_i(\omega\mathcal{R'}_{\mathcal{I}_i})^{\alpha}-\sum_{i=1}^{n}\Xi_i+1\right]^{\frac{1}{\alpha}}$$

By this, the final results can be expressed as follows:

$$CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_n) \leq CSVNSSPA(\mathcal{I'}_1,\mathcal{I'}_2,\ldots,\mathcal{I'}_n).$$

□ **Proposition 3.** (Boundedness)

For any collection of CSVNs $\mathcal{I}_i = \left\langle \mathcal{P}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}} \right\rangle$, i=1,2,...,n if $\mathcal{I}_i^- = \langle[\min\mathcal{P}_{\mathcal{I}_i}].e^{2\pi i[\min\omega_{\mathcal{P}_{\mathcal{I}_i}}]}, [\max\mathcal{S}_{\mathcal{I}_i}].e^{2\pi i[\max\omega_{\mathcal{S}_{\mathcal{I}_i}}]}, [\max\mathcal{R}_{\mathcal{I}_i}].e^{2\pi i[\max\omega_{\mathcal{R}_{\mathcal{I}_i}}]}\rangle$
and $\mathcal{I}_i^+ = \langle[\max\mathcal{P}_{\mathcal{I}_i}].e^{2\pi i[\max\omega_{\mathcal{P}_{\mathcal{I}_i}}]}, [\min\mathcal{S}_{\mathcal{I}_i}].e^{2\pi i[\min\omega_{\mathcal{S}_{\mathcal{I}_i}}]}, [\min\mathcal{R}_{\mathcal{I}_i}].e^{2\pi i[\min\omega_{\mathcal{R}_{\mathcal{I}_i}}]}\rangle$ then, $\mathcal{I}_i^- \leq CSVNSSPA(\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_n) \leq \mathcal{I}_i^+$.

Proof. Suppose that $\mathcal{I}_i = \left\langle \mathcal{P}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}} \right\rangle$, i=1,2,...,n be the collection of CSVNs. Let $\mathcal{I}_i^- = min\{\mathcal{I}_1,\mathcal{I}_2,\ldots\mathcal{I}_n\} = \left\langle \mathcal{P}_{\mathcal{I}_i^-}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i^-}}}, \mathcal{S}_{\mathcal{I}_i^-}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i^-}}}, \mathcal{R}_{\mathcal{I}_i^-}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i^-}}} \right\rangle$ and $\mathcal{I}_i^- = max\{\mathcal{I}_1,\mathcal{I}_2,\ldots\mathcal{I}_n\} = \left\langle \mathcal{P}_{\mathcal{I}_i^+}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i^+}}}, \mathcal{S}_{\mathcal{I}_i^+}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i^+}}}, \mathcal{R}_{\mathcal{I}_i^+}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i^+}}} \right\rangle$. Then, we have: $\mathcal{P}_{\mathcal{I}_i^-} = \{min\,\mathcal{P}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i}}}\}$, $\mathcal{S}_{\mathcal{I}_i^-} = \{max\,\mathcal{S}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}}\}$, $\mathcal{R}_{\mathcal{I}_i^-} = \{max\,\mathcal{R}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}}\}$ and $\mathcal{P}_{\mathcal{I}_i^+} = \{max\,\mathcal{P}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{P}_{\mathcal{I}_i}}}\}$, $\mathcal{S}_{\mathcal{I}_i^+} = \{min\,\mathcal{S}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{S}_{\mathcal{I}_i}}}\}$, $\mathcal{R}_{\mathcal{I}_i^+} = \{min\,\mathcal{R}_{\mathcal{I}_i}.e^{2\pi i\omega_{\mathcal{R}_{\mathcal{I}_i}}}\}$. Now,

$$\begin{pmatrix} \left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i^-})^{\alpha}\right)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i^-}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]} \leq \\ \left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i})^{\alpha}\right)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]} \leq \\ \left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\mathcal{P}_{\mathcal{I}_i^+})^{\alpha}\right)^{\frac{1}{\alpha}}\right]\cdot e^{2\pi i\left[1-\left(\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{P}_{\mathcal{I}_i^+}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]} \end{pmatrix}$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i^-})^{\alpha}\right)^{\frac{1}{\alpha}}\cdot e^{2\pi i\left(\sum_{i=1}^{n}\Xi_i(1-\omega_{\mathcal{S}_{\mathcal{I}_i^-}})^{\alpha}\right)^{\frac{1}{\alpha}}} \leq \\ \left(\sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i})^{\alpha}\right)^{\frac{1}{\alpha}}\cdot e^{2\pi i\left(\sum_{i=1}^{n}\Xi_i(\omega_{\mathcal{S}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}} \leq \\ \left(\sum_{i=1}^{n}\Xi_i(\mathcal{S}_{\mathcal{I}_i^+})^{\alpha}\right)^{\frac{1}{\alpha}}\cdot e^{2\pi i\left(\sum_{i=1}^{n}\Xi_i(\omega_{\mathcal{S}_{\mathcal{I}_i^+}})^{\alpha}\right)^{\frac{1}{\alpha}}} \end{pmatrix}$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n} \Xi_i (\mathcal{R}_{\mathcal{I}_i^-})^{\alpha}\right)^{\frac{1}{\alpha}} \cdot e^{2\pi i \left(\sum_{i=1}^{n} \Xi_i (1-\omega_{\mathcal{R}_{\mathcal{I}_i^-}})^{\alpha}\right)^{\frac{1}{\alpha}}} \leq \\ \left(\sum_{i=1}^{n} \Xi_i (\mathcal{R}_{\mathcal{I}_i})^{\alpha}\right)^{\frac{1}{\alpha}} \cdot e^{2\pi i \left(\sum_{i=1}^{n} \Xi_i (\omega_{\mathcal{R}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}} \leq \\ \left(\sum_{i=1}^{n} \Xi_i (\mathcal{R}_{\mathcal{I}_i^+})^{\alpha}\right)^{\frac{1}{\alpha}} \cdot e^{2\pi i \left(\sum_{i=1}^{n} \Xi_i (\omega_{\mathcal{R}_{\mathcal{I}_i^+}})^{\alpha}\right)^{\frac{1}{\alpha}}} \end{pmatrix}$$

From this, it is concluded that $\mathcal{I}_i^- \leq CSVNSSPA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) \leq \mathcal{I}_i^+$. □

3.2. *CSVN Schweizer- Sklar power weighted averaging operator*

Definition 3.2. *For any collection of CSVNs $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$, $(i = 1, 2)$. Then the CSVNSSPWA operator of dimension n is a mapping CSVNSSPA:$\mathcal{I}^n \to \mathcal{I}$ such that*

$$CSVNSSPWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \frac{\oplus_{i=1}^{n} (\breve{\mathbb{W}}_i (1 + \mathfrak{U}(\mathcal{I}_i))\mathcal{I}_i)}{\sum_{i=1}^{n} \breve{\mathbb{W}}_i (1 + \mathfrak{U}(\mathcal{I}_i))} \tag{7}$$

where $\mathcal{I}$ is the set of all CSVN numbers and $\mathfrak{U}(\mathcal{I}_i) = \sum_{k=1,k\neq i}^{n} Sup(\mathcal{I}_i, \mathcal{I}_k)$ and $\breve{\mathbb{W}}_i = (\breve{\mathbb{W}}_1, \breve{\mathbb{W}}_2, \ldots, \breve{\mathbb{W}}_n)^T$ is the weight vector of $\mathcal{I}_i (i = 1, 2, ..., n)$, $\breve{\mathbb{W}}_i \in [0, 1]$, $\sum_{i=1}^{n} \breve{\mathbb{W}}_i = 1$. Suppose when $\breve{\mathbb{W}}_i = \{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}^T$, CSVNSSPWA operator will be reduced to CSVNSSPA operator.

Theorem 3.2. *Let $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$, $(i = 1, 2)$ be a set of CSVN numbers and $\alpha < 0$ and $\breve{\mathbb{W}}_i = (\breve{\mathbb{W}}_1, \breve{\mathbb{W}}_2, \ldots, \breve{\mathbb{W}}_n)^T$ is the weight vector of $\mathcal{I}_i (i = 1, 2, ..., n)$, $\sum_{i=1}^{n} \breve{\mathbb{W}}_i = 1$, $\breve{\mathbb{W}}_i \in [0, 1]$, then the aggregated value obtained using CSVNSSPWA operator is also a CSVN number and can be expressed as follows:*

$$CSVNSSPWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$$

$$= \begin{pmatrix} \left[1 - (\sum_{i=1}^{n} \mathbb{W}_i (1 - \mathcal{P}_{\mathcal{I}_i})^{\alpha})^{\frac{1}{\alpha}}\right] \cdot e^{2\pi i \left[1 - \left(\sum_{i=1}^{n} \mathbb{W}_i (1-\omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{n} \mathbb{W}_i \mathcal{S}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\sum_{i=1}^{n} \mathbb{W}_i \omega_{\mathcal{S}_{\mathcal{I}_i}^{\alpha}}\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{n} \mathbb{W}_i \mathcal{R}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\sum_{i=1}^{n} \mathbb{W}_i \omega_{\mathcal{R}_{\mathcal{I}_i}^{\alpha}}\right]^{\frac{1}{\alpha}}} \end{pmatrix} \tag{8}$$

where $\mathbb{W}_i (i = 1, 2, ..., n)$ is a set of integrated weights, $\mathbb{W}_i = \frac{\breve{\mathbb{W}}_i (1+\mathfrak{U}(\mathcal{I}_i))}{\sum_{i=1}^{n} \breve{\mathbb{W}}_i (1+\mathfrak{U}(\mathcal{I}_i))}$.

Proof. As the proof of Theorem 3.2 parallels that of Theorem 3.1, it is omitted. The following properties hold: □

Proposition 4. (idempotency)

Let $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$,where i=1,2,...,n be a set of CSVN numbers, if $\mathcal{I}_i = \mathcal{I} = (\mathcal{P}_{\mathcal{I}} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}}}}, \mathcal{S}_{\mathcal{I}} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}}}}, \mathcal{R}_{\mathcal{I}} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}}}})$, then $CSVNSSPWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \mathcal{I}$.

Proposition 5. (Boundedness)

Let $\mathcal{I}_i = \left(\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}}\right)$, $i = 1, 2, \ldots, n$ be a set of CSVN numbers. Then, $\mathcal{I}_i^+ = \left\langle\ [\max \mathcal{P}_{\mathcal{I}_i}] e^{2\pi i[\max \omega_{\mathcal{P}_{\mathcal{I}_i}}]}, [\min \mathcal{S}_{\mathcal{I}_i}] e^{2\pi i[\min \omega_{\mathcal{S}_{\mathcal{I}_i}}]}, [\min \mathcal{R}_{\mathcal{I}_i}] e^{2\pi i[\min \omega_{\mathcal{R}_{\mathcal{I}_i}}]},\ \right\rangle$ and $\mathcal{I}_i^- = \left\langle\ [\min \mathcal{P}_{\mathcal{I}_i}] e^{2\pi i[\min \omega_{\mathcal{P}_{\mathcal{I}_i}}]}, [\max \mathcal{S}_{\mathcal{I}_i}] e^{2\pi i[\max \omega_{\mathcal{S}_{\mathcal{I}_i}}]}, [\max \mathcal{R}_{\mathcal{I}_i}] e^{2\pi i[\max \omega_{\mathcal{R}_{\mathcal{I}_i}}]},\ \right\rangle$ then,

$$\mathcal{I}_i^- \leq CSVNSSPWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) \leq \mathcal{I}_i^+$$

Proposition 6. (Monotonicity)

Let $\mathcal{I}_i(i = 1, 2, ...n)$ be any permutation of $\mathcal{I}_i(i = 1, 2, ...n)$. Then, $CSVNSSPWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) \leq CSVNSSPWA(\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_n)$.

3.3. *CSVN Schweizer - Sklar power ordered weighted averaging operator*

Definition 3.3. *For any collection of CSVNs $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$, $(i = 1, 2)$. Then the CSVNSSPOWA operator of dimension n is a mapping CSVNSSPOWA:$\mathcal{I}^n \to \mathcal{I}$ such that*

$$CSVNSSPOWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \frac{\oplus_{i=1}^n (\breve{\mathbb{W}}_i (1 + \mathfrak{U}(\mathcal{I}_{\sigma(i)}))\mathcal{I}_{\sigma(i)})}{\sum_{i=1}^n \breve{\mathbb{W}}_i (1 + \mathfrak{U}(\mathcal{I}_{\sigma(i)}))} \tag{9}$$

where $\sigma(i)$ is the permutation such that $\mathcal{I}_{\sigma(i-1)} \geq \mathcal{I}_{\sigma(i)}$ for any i=1,2,..,n and $\breve{\mathbb{W}}(\mathcal{I}_{\sigma(i)}) = \sum_{h=1,h\neq i}^n Sup(\mathcal{I}_{\sigma(i)}, \mathcal{I}_{\sigma(h)})$ and $\breve{\mathbb{W}}_i = (\breve{\mathbb{W}}_1, \breve{\mathbb{W}}_2, \ldots, \breve{\mathbb{W}}_n)^T$ is the weight vector of $\mathcal{I}_i(i = 1, 2, ..., n), \breve{\mathbb{W}}_i \in [0, 1], \sum_{i=1}^n \breve{\mathbb{W}}_i = 1$. Suppose when $\breve{\mathbb{W}}_i = \{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}^T$, CSVNSSPOWA operator will be reduced to CSVNSSPA operator.

Theorem 3.3. *Let $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$, $(i = 1, 2)$ be a set of CSVN numbers and $\alpha < 0$ and $\breve{\mathbb{W}}_i = (\breve{\mathbb{W}}_1, \breve{\mathbb{W}}_2, \ldots, \breve{\mathbb{W}}_n)^T$ is the weight vector of $\mathcal{I}_i(i = 1, 2, ..., n), \sum_{i=1}^n \breve{\mathbb{W}}_i = 1, \breve{\mathbb{W}}_i \in [0, 1]$, then the aggregated value obtained using*

$$CSVNSSPOWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$$
$$= \begin{pmatrix} \left[1 - (\sum_{i=1}^n \mathfrak{E}_i (1 - \mathcal{P}_{\mathcal{I}_i})^\alpha)^{\frac{1}{\alpha}}\right] e^{2\pi i\left[1 - \left(\sum_{i=1}^n \mathfrak{E}_i (1 - \omega_{\mathcal{P}_{\mathcal{I}_i}})^\alpha\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^n \mathfrak{E}_i \mathcal{S}_{\mathcal{I}_i}^\alpha\right]^{\frac{1}{\alpha}} e^{2\pi i\left[\sum_{i=1}^n \mathfrak{E}_i \omega_{\mathcal{S}_{\mathcal{I}_i}^\alpha}\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^n \mathfrak{E}_i \mathcal{R}_{\mathcal{I}_i}^\alpha\right]^{\frac{1}{\alpha}} e^{2\pi i\left[\sum_{i=1}^n \mathfrak{E}_i \omega_{\mathcal{R}_{\mathcal{I}_i}^\alpha}\right]^{\frac{1}{\alpha}}} \end{pmatrix} \tag{10}$$

where $\mathfrak{E}_i(i = 1, 2, ..., n)$ is a set of integrated weights, $\mathfrak{E}_i = \frac{\breve{\mathbb{W}}_i(1+\mathfrak{U}(\mathcal{I}_{\sigma(i)}))}{\sum_{i=1}^n \breve{\mathbb{W}}_i(1+\mathfrak{U}(\mathcal{I}_{\sigma(i)}))}$.

Proposition 7. (idempotency)
Let $\mathcal{I}_i = (\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}})$,where i=1,2,...,n be a set of CSVN numbers, if $\mathcal{I}_i = \mathcal{I} = (\mathcal{P}_{\mathcal{I}} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}}}}, \mathcal{S}_{\mathcal{I}} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}}}}, \mathcal{R}_{\mathcal{I}} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}}}})$, then $CSVNSSPOWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \mathcal{I}$.

Proposition 8. (Boundedness)
Let $\mathcal{I}_i = \left(\mathcal{P}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{P}_{\mathcal{I}_i}}}, \mathcal{S}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{S}_{\mathcal{I}_i}}}, \mathcal{R}_{\mathcal{I}_i} e^{i2\pi\omega_{\mathcal{R}_{\mathcal{I}_i}}}\right)$, $i = 1, 2, \ldots, n$ be a set of CSVN numbers. Then, $\mathcal{I}_i^+ = \left\langle \ [\max \mathcal{P}_{\mathcal{I}_i}] e^{2\pi i[\max \omega_{\mathcal{P}_{\mathcal{I}_i}}]}, [\min \mathcal{S}_{\mathcal{I}_i}] e^{2\pi i[\min \omega_{\mathcal{S}_{\mathcal{I}_i}}]}, [\min \mathcal{R}_{\mathcal{I}_i}] e^{2\pi i[\min \omega_{\mathcal{R}_{\mathcal{I}_i}}]}, \ \right\rangle$ and $\mathcal{I}_i^- = \left\langle \ [\min \mathcal{P}_{\mathcal{I}_i}] e^{2\pi i[\min \omega_{\mathcal{P}_{\mathcal{I}_i}}]}, [\max \mathcal{S}_{\mathcal{I}_i}] e^{2\pi i[\max \omega_{\mathcal{S}_{\mathcal{I}_i}}]}, [\max \mathcal{R}_{\mathcal{I}_i}] e^{2\pi i[\max \omega_{\mathcal{R}_{\mathcal{I}_i}}]}, \ \right\rangle$ then,

$$\mathcal{I}_i^- \leq CSVNSSPOWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) \leq \mathcal{I}_i^+$$

Proposition 9. (Monotonicity)
Let $\mathcal{I}_i(i = 1, 2, ...n)$ be any permutation of $\mathcal{I}_i(i = 1, 2, ...n)$. Then, $CSVNSSPOWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) \leq CSVNSSPOWA(\mathcal{I}'_1, \mathcal{I}'_2, \ldots, \mathcal{I}'_n)$.

4. Decision-making procedures

In this section, we present a procedure for addressing group decision-making within a complex neutrosophic environment. Suppose $\ddot{\mathfrak{G}} = \{\ddot{\mathfrak{G}}_1, \ddot{\mathfrak{G}}_2, ..., \ddot{\mathfrak{G}}_n\}$ be a set of n alternatives, $\mathfrak{R} = \{\mathfrak{R}_1, \mathfrak{R}_2, ..., \mathfrak{R}_n\}$ be a set of n attributes with weight vectors $\mathbb{W} = (w_1, w_2, ..., w_n)^t$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and $\dot{\dot{E}} = \{\dot{e}_1, \dot{e}_2, ..., \dot{e}_n\}$ with weight vectors $\mathbb{W} = (w_1, w_2, ..., w_n)^t$ such that $w_p \in [0, 1]$ and $\sum_{p=1}^{n} w_p = 1$. Thus, the decision matrix can be expressed as:

$$H^a = [\mathcal{I}_{ij}]_{m\times n} = \begin{array}{c} \\ \ddot{\mathfrak{G}}_1 \\ \ddot{\mathfrak{G}}_2 \\ \cdots \\ \ddot{\mathfrak{G}}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathfrak{R}_1 & \mathfrak{R}_2 & \ldots & \mathfrak{R}_n \end{array} \\ \begin{bmatrix} \mathcal{I}_{11}^a & \mathcal{I}_{12}^a & \ldots & \mathcal{I}_{1n}^a \\ \mathcal{I}_{21}^a & \mathcal{I}_{22}^a & \ldots & \mathcal{I}_{2n}^a \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{I}_{m1}^a & \mathcal{I}_{m2}^a & \ldots & \mathcal{I}_{mn}^a \end{bmatrix} \end{array}$$

The following steps should be taken to identify the best alternative.

Step 1: Normalize the CSVN decision matrix by the following equation in case, the MAGDM problem contains cost and benefit factors as shown in Equation 11.

$$\mathcal{I}_{ij} = \begin{cases} \langle \mathbb{T}_{ij}^a, \mathbb{I}_{ij}^a, \mathbb{F}_{ij}^a \rangle & \text{if benefit type} \\ \langle \mathbb{F}_{ij}^a, \mathbb{I}_{ij}^a, \mathbb{T}_{ij}^a \rangle & \text{if cost type} \end{cases} \tag{11}$$

Step 2: Determine the weight $\mathfrak{D}j$ for each criterion $\mathfrak{R}j$. This process involves using the SMART technique to establish the criteria weights according to the subjective assessments of decision experts. In this approach, the decision maker ranks the criteria from least to most important. There are 10 points for the least important criterion and the most important criterion gets 100 points. The rest of the criteria are assigned point scores in an order based on their relative weights. The weight of each criterion is then obtained by normalizing total points so that their sum is equal to 1.

Step 3: Calculate the overall evaluation score for each alternative as follows:

$$CSVNSSPWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \begin{pmatrix} \left[1 - \left(\sum_{i=1}^{n} \mathbb{W}_i (1 - \mathcal{P}_{\mathcal{I}_i})^{\alpha}\right)^{\frac{1}{\alpha}}\right] \cdot \\ e^{2\pi i \left[1 - \left(\sum_{i=1}^{n} \mathbb{W}_i (1 - \omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\left(\sum_{i=1}^{n} \mathbb{W}_i \mathcal{S}_{\mathcal{I}_i}^{\alpha}\right)^{\frac{1}{\alpha}}\right] \cdot e^{2\pi i \left[\left(\sum_{i=1}^{n} \mathbb{W}_i \omega_{\mathcal{S}_{\mathcal{I}_i}^{\alpha}}\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\left(\sum_{i=1}^{n} \mathbb{W}_i \mathcal{R}_{\mathcal{I}_i}^{\alpha}\right)^{\frac{1}{\alpha}}\right] \cdot e^{2\pi i \left[\left(\sum_{i=1}^{n} \mathbb{W}_i \omega_{\mathcal{R}_{\mathcal{I}_i}^{\alpha}}\right)^{\frac{1}{\alpha}}\right]} \end{pmatrix} \quad (12)$$

$$CSVNSSPOWA(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n) = \begin{pmatrix} \left[1 - \left(\sum_{i=1}^{n} \mathfrak{E}_i (1 - \mathcal{P}_{\mathcal{I}_i})^{\alpha}\right)^{\frac{1}{\alpha}}\right] \cdot \\ e^{2\pi i \left[1 - \left(\sum_{i=1}^{n} \mathfrak{E}_i (1 - \omega_{\mathcal{P}_{\mathcal{I}_i}})^{\alpha}\right)^{\frac{1}{\alpha}}\right]}, \\ \left[\sum_{i=1}^{n} \mathfrak{E}_i \mathcal{S}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\sum_{i=1}^{n} \mathfrak{E}_i \omega_{\mathcal{S}_{\mathcal{I}_i}^{\alpha}}\right]^{\frac{1}{\alpha}}}, \\ \left[\sum_{i=1}^{n} \mathfrak{E}_i \mathcal{R}_{\mathcal{I}_i}^{\alpha}\right]^{\frac{1}{\alpha}} \cdot e^{2\pi i \left[\sum_{i=1}^{n} \mathfrak{E}_i \omega_{\mathcal{R}_{\mathcal{I}_i}^{\alpha}}\right]^{\frac{1}{\alpha}}} \end{pmatrix} \quad (13)$$

where $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; a = 1, 2, \ldots, l$.

Step 4: Determine the aggregate score values for the overall evaluations $\mathcal{I}_i (i = 1, 2, \ldots, m)$.

Step 5: Rank the alternatives.

5. Application to Green Supply Chain Management

In this section, we form a multi attribute group decision making (MAGDM) framework using the proposed CSVNSSPWA, CSVNSSPOWA operators and employed it to assess green suppliers from the perspective of GSCM. In this context, objective is about green suppliers which are known as sustainable or eco-friendly suppliers who provides products and services are less damaging to the environment as compared to the conventional suppliers.

We choose the five green suppliers $\ddot{\mathfrak{G}}_1, \ddot{\mathfrak{G}}_2, \ddot{\mathfrak{G}}_3, \ddot{\mathfrak{G}}_4, \ddot{\mathfrak{G}}_5$ and four attributes as

- $\mathfrak{R}_1$ - product quality factor ,
- $\mathfrak{R}_2$ - environmental factor,
- $\mathfrak{R}_3$ - Delivery factor,

- $\mathfrak{R}_4$ - price factor.

The weight of the decision makers is given by $\mathbb{W} = \{0.35, 0.25, 0.40\}$. Obtain the decision makers input by utilizing the linguistic terms and their corresponding CSVNs outlined in Table 3. Table 4, 5, 6 represents the CSVN information.

TABLE 3. Linguistic terms for CSVNs

Linguistic terms	**CSVNs**
VH - Very high	[(0.6,0.5), (0.2,0.3), (0.4,0.1)]
H - High	[(0.8,0.3), (0.6,0.2), (0.3,0.4)]
MH - Moderately high	[(0.5,0.1), (0.4,0.6), (0.9,0.3)]
M - Medium	[(0.7,0.2), (0.1,0.4), (0.2,0.5)]
ML - Moderately low	[(0.3,0.1), (0.8,0.8), (0.5,0.3)]
L - Low	[(0.4,0.3), (0.6,0.4), (0.2,0.5)]
VL - Very low	[(0.5,0.2), (0.9,0.1), (0.7,0.4)]

TABLE 4. CSVN information in the decision matrix DM_1

	$\mathfrak{R}_1$	$\mathfrak{R}_2$
$\ddot{\mathfrak{G}}_1$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$
$\ddot{\mathfrak{G}}_2$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_3$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_4$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$
$\ddot{\mathfrak{G}}_5$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$
	$\mathfrak{R}_3$	$\mathfrak{R}_4$
$\ddot{\mathfrak{G}}_1$	$[(0.5, 0.1), (0.4, 0.6), (0.9, 0.3)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_2$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$
$\ddot{\mathfrak{G}}_3$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$
$\ddot{\mathfrak{G}}_4$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$
$\ddot{\mathfrak{G}}_5$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$

TABLE 5. CSVN information in the decision matrix DM_2

	$\mathfrak{R}_1$	$\mathfrak{R}_2$
$\ddot{\mathfrak{G}}_1$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_2$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$
$\ddot{\mathfrak{G}}_3$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$	$[(0.5, 0.1), (0.4, 0.6), (0.9, 0.3)]$
$\ddot{\mathfrak{G}}_4$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$
$\ddot{\mathfrak{G}}_5$	$[(0.5, 0.1), (0.4, 0.6), (0.9, 0.3)]$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$
	$\mathfrak{R}_3$	$\mathfrak{R}_4$
$\ddot{\mathfrak{G}}_1$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$
$\ddot{\mathfrak{G}}_2$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_3$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$
$\ddot{\mathfrak{G}}_4$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_5$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$

TABLE 6. CSVN information in the decision matrix DM_3

	$\mathfrak{R}_1$	$\mathfrak{R}_2$
$\ddot{\mathfrak{G}}_1$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$
$\ddot{\mathfrak{G}}_2$	$[(0.5, 0.1), (0.4, 0.6), (0.9, 0.3)]$	$[(0.5, 0.1), (0.4, 0.6), (0.9, 0.3)]$
$\ddot{\mathfrak{G}}_3$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$	$[(0.8, 0.3), (0.6, 0.2), (0.3, 0.4)]$
$\ddot{\mathfrak{G}}_4$	$[(0.5, 0.1), (0.4, 0.6), (0.9, 0.3)]$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$
$\ddot{\mathfrak{G}}_5$	$[(0.3, 0.1), (0.8, 0.8), (0.5, 0.3)]$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$
	$\mathfrak{R}_3$	$\mathfrak{R}_4$
$\ddot{\mathfrak{G}}_1$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$
$\ddot{\mathfrak{G}}_2$	$[(0.5, 0.2), (0.9, 0.1), (0.7, 0.4)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_3$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$	$[(0.6, 0.5), (0.2, 0.3), (0.4, 0.1)]$
$\ddot{\mathfrak{G}}_4$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$	$[(0.4, 0.3), (0.6, 0.4), (0.2, 0.5)]$
$\ddot{\mathfrak{G}}_5$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$	$[(0.7, 0.2), (0.1, 0.4), (0.2, 0.5)]$

Step 1: Construct a matrix (see Table 4, 5, 6) and standardise it based on the varying information types.

$$\mathcal{I}_{ij} = \begin{cases} \langle(\mathcal{P}_{\mathcal{I}}, \omega_{\mathcal{P}_{\mathcal{I}}}), (\mathcal{S}_{\mathcal{I}}, \omega_{\mathcal{S}_{\mathcal{I}}}), (\mathcal{R}_{\mathcal{I}}, \omega_{\mathcal{R}_{\mathcal{I}}})\rangle & \text{if benefit type} \\ \langle(\mathcal{R}_{\mathcal{I}}, \omega_{\mathcal{R}_{\mathcal{I}}}), (\mathcal{S}_{\mathcal{I}}, \omega_{\mathcal{S}_{\mathcal{I}}}), (\mathcal{P}_{\mathcal{I}}, \omega_{\mathcal{P}_{\mathcal{I}}})\rangle & \text{if cost type} \end{cases}$$

Normalization is not required since all the given attributes are of the benefit type.

Step 2: In this step, we use the SMART method to calculate attribute weights. Experts assign scores to each criterion, ranging from 10 for the least important to 100 for the most important. These scores are then normalized by dividing each criterion's score by the total sum of all scores, determining the final attribute weights. Table 7 displays the points allocated by each expert and the corresponding normalized attribute weights.

Table 7. Attribute weights

Attribute	Points assigned by			Sum	Normalized
	DM_1	DM_2	DM_3	of points	weights $\mathfrak{D}j$
$\mathfrak{R}_1$	80	70	90	240	0.26
$\mathfrak{R}_2$	90	60	80	230	0.25
$\mathfrak{R}_3$	70	80	60	210	0.22
$\mathfrak{R}_4$	90	80	80	250	0.27
		Total		**930**	**1**

Step 3: Using the theory of CSVNSSPWA, CSVNSSPOWA operators using Equations 12 and 13, we aggregate our specified decision matrix for $\alpha = -2$. The results are shown in Table 8.

Table 8. Aggregated matrix for CSVNSSPWA and CSVNSSPOWA operators

	CSVNSSPWA	**CSVNSSPOWA**
$\ddot{\mathfrak{G}}_1$	$[(0.6706, 0.3533), (0.2018, 0.216), (0.3005, 0.1698)]$	$[(0.6725, 0.3533), (0.1963, 0.2192), (0.3059, 0.1689)]$
$\ddot{\mathfrak{G}}_2$	$[(0.6514, 0.2487), (0.2261, 0.1975), (0.3045, 0.2715)]$	$[(0.6525, 0.2498), (0.2212, 0.2), (0.3034, 0.267)]$
$\ddot{\mathfrak{G}}_3$	$[(0.6261, 0.2487), (0.1959, 0.2039), (0.2713, 0.2043)]$	$[(0.6331, 0.3249), (0.2047, 0.2161), (0.2694, 0.1996)]$
$\ddot{\mathfrak{G}}_4$	$[(0.6273, 0.2581), (0.1992, 0.2019), (0.2656, 0.292)]$	$[(0.6166, 0.2594), (0.2037, 0.1984), (0.269, 0.2839)]$
$\ddot{\mathfrak{G}}_5$	$[(0.6611, 0.1861), (0.1846, 0.2084), (0.309, 0.364)]$	$[(0.6685, 0.1891), (0.1864, 0.2111), (0.3083, 0.3641)]$

Step 4: The score values for the CSVNSSPWA, CSVNSSPOWA operators are calculated and presented in Table 9.

Table 9. Score values

	CSVNSSPWA	CSVNSSOWA
$\ddot{\mathfrak{G}}_1$	0.0452	0.0452
$\ddot{\mathfrak{G}}_2$	0.0331	0.0298
$\ddot{\mathfrak{G}}_3$	0.0226	0.0228
$\ddot{\mathfrak{G}}_4$	0.0245	0.0263
$\ddot{\mathfrak{G}}_5$	0.0729	0.0708

Step 5: Rank the alternatives are showed in Table 10.

Table 10. Ranking of the alternatives

Operator	Ranking
CSVNSSPWA	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3$
CSVNSSOWA	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3$

5.1. *Parameter analysis on decision results*

This section presents the results of our proposed approaches by varying parametric values. We perform all the proposed methodologies with several values of the parameters to investigate the effects of the parametric values on the score values of the result obtained through the use of the CSVNSSPWA, CSVNSSPOWA operators. Table 11 summarize all the score values that have been obtained for the proposed operators respectively. Integrated results may be different at different parametric values α. The choice $\ddot{\mathfrak{G}}_5$ becomes the top preference for the operators. Figure 1 illustrate the computed score values for the CSVNSSPWA, CSVNSSPOWA operators respectively, as detailed in Table 11.

TABLE 11. Results of score values by the CSVNSSPWA and CSVNSSPOWA operators

Parameters	**CSVNSSPWA**	**CSVNSSPOWA**
$\alpha = -1$	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_1$	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_1$
$\alpha = -2$	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3$	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3$
$\alpha = -5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_5$
$\alpha = -10$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$
$\alpha = -20$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$
$\alpha = -30$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$
$\alpha = -50$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$
$\alpha = -100$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$
$\alpha = -200$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$	$\ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_5$

5.2. *Comparative Analysis*

To highlight the proficiency and prevalence of the proposed operators, it is essential to compare them with existing methods. For this, we choose an idea of Aczel - Alsina aggregation operators for CSVNS was exposed by Areeba Naseem et al. [35]. Based on the preceding discussion, the comparison results are tabulated in Table 12. Figure 2 shows the graphical structure of all computed score values from Table 12.

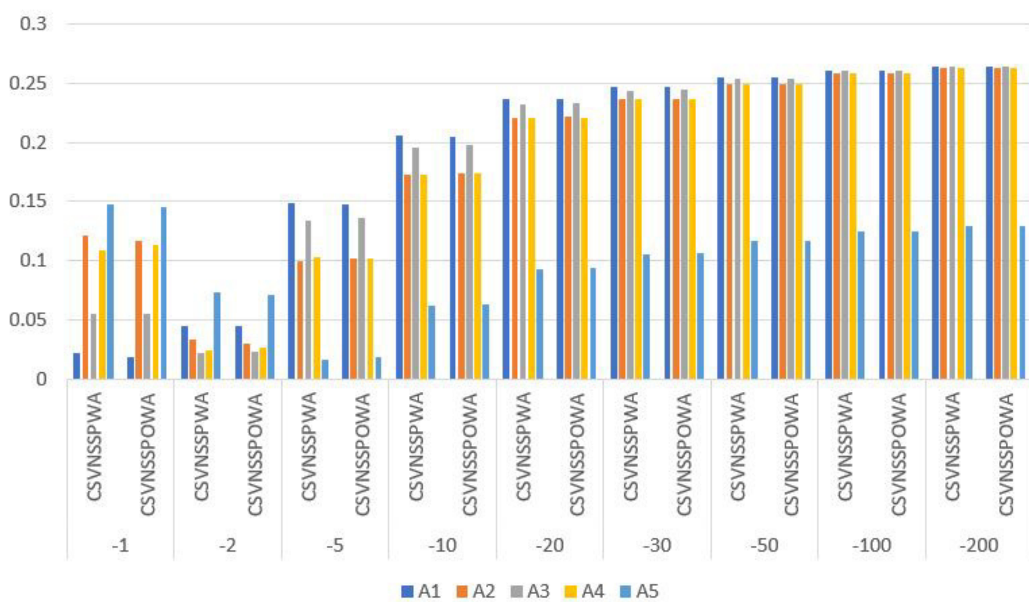

FIGURE 1. The results of the CSVNSSPWA and CSVNSSPOWA operator

TABLE 12. Comparative study

Aggregation operators	Score values	Ranking
CSVNAAWA ($\alpha = 1$) [35]	$\ddot{\mathfrak{G}}_1$= 0.1104, $\ddot{\mathfrak{G}}_2$= 0.2304, $\ddot{\mathfrak{G}}_3$= 0.1569, $\ddot{\mathfrak{G}}_4$= 0.2118, $\ddot{\mathfrak{G}}_5$= 0.2682.	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_1$
CSVNAAOWA ($\alpha = 1$) [35]	$\ddot{\mathfrak{G}}_1$= 0.1127, $\ddot{\mathfrak{G}}_2$= 0.2264, $\ddot{\mathfrak{G}}_3$= 0.1539, $\ddot{\mathfrak{G}}_4$= 0.2092, $\ddot{\mathfrak{G}}_5$= 0.2658.	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3 > \ddot{\mathfrak{G}}_1$
CSVNSSPWA ($\alpha = -2$)	$\ddot{\mathfrak{G}}_1$= 0.0452, $\ddot{\mathfrak{G}}_2$= 0.0331, $\ddot{\mathfrak{G}}_3$= 0.0226, $\ddot{\mathfrak{G}}_4$= 0.0245, $\ddot{\mathfrak{G}}_5$= 0.0729.	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3$
CSVNSSPOWA ($\alpha = -2$)	$\ddot{\mathfrak{G}}_1$= 0.0452, $\ddot{\mathfrak{G}}_2$= 0.0298, $\ddot{\mathfrak{G}}_3$= 0.0228, $\ddot{\mathfrak{G}}_4$= 0.0263, $\ddot{\mathfrak{G}}_5$= 0.0708.	$\ddot{\mathfrak{G}}_5 > \ddot{\mathfrak{G}}_1 > \ddot{\mathfrak{G}}_2 > \ddot{\mathfrak{G}}_4 > \ddot{\mathfrak{G}}_3$

5.3. *Stability Analysis*

In this section, Spearman's rank correlation is used to find out the stability of the proposed operators of different parameters and along with them. Table 13 shows the relationship between all the proposed operators and other parameters.

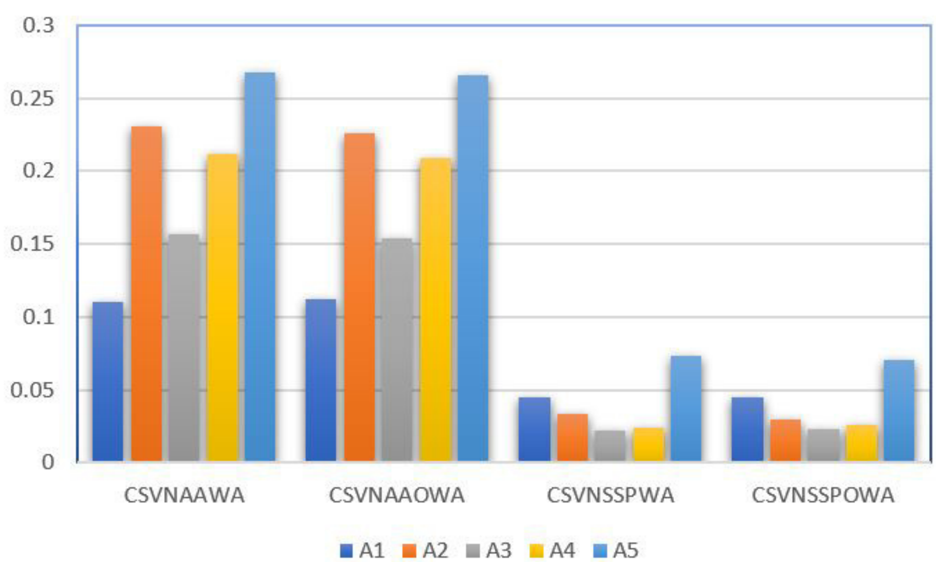

FIGURE 2. Aggregated results by the comparative study

Table 13. Comparative study

Parameter	Aggregation Operator	Spearman's Rank Correlation
$\alpha = -1$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -2$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -5$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -10$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -20$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -30$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -50$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -100$	CSVNSSPWA CSVNSSPOWA	1
$\alpha = -200$	CSVNSSPWA CSVNSSPOWA	1

6. Conclusion

MAGDM is a group decision-making approach that takes multiple criteria or attributes into account. The technique involves evaluating and comparing alternative choices based on these criteria to reach a collective decision. The process is facilitated by using a variety of analytical methods, including decision matrices, pairwise comparison and group decision techniques. Here we present the basic operations for the Schweizer- Sklar t-norms and t-conorms involving CNS information, which is better to fuzzify uncertainty in different fuzzy domains and provide an accurate estimation based on the decision analysis. We examined $\ddot{\mathfrak{G}}_5$ as a flexible supplier of GSCM, considering our operators. In addition, we also explore in detail how different parametric values influence the results of the proposed methodologies. Finally, a comparative technique is used to present the results of existing strategies with the developed methodologies. In the future, we intend to use novel aggregation operators, similarity measures, and new MADM methods for more complex environments. These advancements are expected to address challenging real-world problems effectively.

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Spherical Fermatean Neutrosophic Topology

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Abstract. In this paper, we introduce Spherical Fermatean Neutrosophic Topological Spaces (SFNTS), expanding on neutrosophic sets characterized by Degrees of Membership (DoM), Degrees of Indeterminacy (DoI) and Degrees of Non-Membership (DoN). Fermatean neutrosophic sets in a universe satisfy the conditions where the sum of the cubes of the DoM and DoN is between zero and one and the cube of the DoI is between zero and one. The DoM, DoI and DoN are represented accordingly. We define a Spherical Fermatean Neutrosophic Set (SFNS) as a set where each element consists of an element in the universe, along with its DoM, DoN, DoI and a radius. The DoM, DoN, DoI and the radius are functions mapping the universe to the interval from zero to one. We extend this to a topological framework by defining an SFNTS on a set. We also studied the properties of the SFN closure and SFN interior operators, provided numerical examples and presented a geometric representation of SFNTS. Additionally, we explored the separation of two SFNSs, the intersection of two SFNSs and the overlapping of two SFNSs.

Keywords: Neutrosophic topology; SFNTS; DoN, DoM, Between zero

1. Introduction

Topology [12], as a foundational branch of mathematics, explores the properties of spaces that remain invariant under continuous transformations. It serves as a vital framework for understanding key concepts such as convergence, continuity, and compactness, which are central to various mathematical disciplines, including analysis, geometry, and mathematical physics. Neutrosophic sets [19], introduced by Florentin Smarandache in 1998, generalize traditional fuzzy sets by incorporating three distinct DoM, DoI and DoN. Neutrosophic topology [11] represents a significant advancement in the field by incorporating the concept of neutrosophic sets. Neutrosophic sets extend traditional set theory by allowing for the representation of indeterminacy and uncertainty in membership values. This capability is particularly valuable in contexts where classical binary logic fails to capture the complexity of information, such as in decision-making processes that involve ambiguous or incomplete data. Neutrosophic topological spaces thus provide a more generalized mathematical framework that accommodates these complexities, facilitating the study of continuity, convergence, and related topological properties in a broader context. The geomentric representation of collection of neutrosophic sets namely cubic spherical neutrosophic sets [7, 13] and cubic spherical neutrosophic topology [8]plays vital role in recent research of this field.

Fermatean fuzzy sets (FFS) [18], introduced by Tapan Senapati and Ronald R. Yager, extend traditional fuzzy set theory by allowing for a more nuanced representation of uncertainty through DoM and DoN. A notable extension within this field is the introduction of Fermatean fuzzy topology [10], which integrates Fermatean fuzzy sets into topological structures. This enriched framework allows for more sophisticated modeling of complex systems. The geomentric representation of collection of Fermatean fuzzy sets namely circular Fermatean fuzzy sets [1, 16, 17]. The foundation of spherical Fermatean neutrosophic sets [14] [15] lies in their ability to encapsulate varying degrees of membership that reflect the inherent vagueness and ambiguity present in many real-world scenarios. By utilizing the principles of Fermatean fuzzy sets, these sets enable a more effective representation of uncertainties, thereby facilitating improved decision-making processes. The spherical representation further enhances this framework by allowing for a more comprehensive analysis of complex systems, particularly in fields where traditional binary logic and flat geometric structures are insufficient. Gonul Bilgin et. al. introduced and studied the notion of Fermatean neutrosophic topology [9].

In this paper, we further extend the concept of Fermatean neutrosophic sets [2–5] to introduce Spherical Fermatean Neutrosophic Sets (SFNS). A SFNS is defined by a four-tuple $\langle \aleph, \zeta(\aleph), \varkappa(\aleph), \digamma(\aleph); \kappa \rangle$, where $\zeta(\aleph)$, $\varkappa(\aleph)$, $\digamma(\aleph)$ and $\kappa(\aleph)$ are functions mapping Υ to $[0, 1]$. The additional component κ represents the radius of a sphere with the center at $(\zeta(\aleph), \varkappa(\aleph), \digamma(\aleph))$, encapsulating the DoM, DoI and DoN.

Acronyms Used in the Article:

The following table lists the acronyms used in the article and their meanings:

Acronym	Meaning
DoM	Degrees of Membership
DoI	Degrees of Indeterminacy
DoN	Degrees of Non-membership
NS	Neutrosophic Sets
FNS	Fermatean Neutrosophic Sets
SFNS	Spherical Fermatean Neutrosophic Sets
SFNTS	Spherical Fermatean Neutrosophic Topological Spaces

2. Preliminaries

Definition 2.1. [19] A neutrosophic set A in a universe Υ is defined as:

$$A = \{\langle \aleph, \zeta_A(\aleph), \digamma_A(\aleph), \varkappa_A(\aleph)\rangle | \aleph \in \Upsilon\}$$

where $\zeta_A : \Upsilon \to [0,1]$, $\digamma_A : \Upsilon \to [0,1]$ and $\varkappa_A : \Upsilon \to [0,1]$. Here, $\zeta_A(\aleph)$ represents the DoM, $\digamma_A(\aleph)$ the DoI and $\varkappa_A(\aleph)$ the DoN.

Definition 2.2. [18] A Fermatean Fuzzy Set (FFS) A in the universe Υ is defined as:

$$A = \{\langle \aleph, \zeta_A(\aleph), \varkappa_A(\aleph)\rangle | \aleph \in \Upsilon\}$$

with the conditions $0 \leq \varkappa_A^3(\aleph) + \zeta_A^3(\aleph) \leq 1$ for all $\aleph \in \Upsilon$. The functions $\zeta_A(\aleph)$ and $\varkappa_A(\aleph)$ represent the DoM and DoN respectively.

Definition 2.3. [20] A Fermatean Neutrosophic Set (FNS) A in the universe Υ is defined as:

$$A = \{\langle \aleph, \zeta_A(\aleph), \digamma_A(\aleph), \varkappa_A(\aleph)\rangle | \aleph \in \Upsilon\}$$

with the conditions $0 \leq \zeta_A^3(\aleph) + \varkappa_A^3(\aleph) \leq 1$ and $0 \leq \digamma_A^3(\aleph) \leq 1$. where $\zeta_A : \Upsilon \to [0,1]$, $\digamma_A : \Upsilon \to [0,1]$ and $\varkappa_A : \Upsilon \to [0,1]$. Here, $\zeta_A(\aleph)$ represents the DoM, $\digamma_A(\aleph)$ the DoI and $\varkappa_A(\aleph)$ the DoN. This definition extends the concept of Fermatean fuzzy sets by incorporating an additional DoI.

Definition 2.4. [14,15] A Spherical Fermatean Neutrosophic Set (SFNS) A_κ in the universe Υ is defined as:

$$A_\kappa = \{\langle \aleph, \zeta(\aleph), \digamma(\aleph), \varkappa(\aleph); \kappa\rangle | \aleph \in \Upsilon\}$$

where $\zeta(\aleph)$, $\varkappa(\aleph)$, $\digamma(\aleph)$ and κ are functions mapping Υ to $[0,1]$. The radius κ represents the distance from the center $(\zeta(\aleph), \digamma(\aleph), \varkappa(\aleph))$ to the boundary of the sphere within the cube.

The center of the sphere is

$$< \zeta(\aleph_i), \varkappa(\aleph_i), \digamma(\aleph_i) > = < \frac{\sum_{j=1}^{k_i} \zeta_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} \digamma_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} \varkappa_{i,j}}{k_i} >$$

and the radius

$$\kappa_i = \min\left\{ \max_{1 \leq j \leq k_i} \sqrt{(\zeta(\aleph_i) - \zeta_{i,j})^2 + (\digamma(\aleph_i) - \digamma_{i,j})^2 + (\varkappa(\aleph_i) - \varkappa_{i,j})^2}, 1 \right\}.$$

Definition 2.5. [15] Let $\Delta_{\kappa_1} = \{\langle \aleph, \zeta_{\Delta_1}, \varkappa_{\Delta_1}, \digamma_{\Delta_1}; \kappa_{\Delta_1}\rangle : \aleph \in \Upsilon\}$ and $\Delta_{\kappa_2} = \{\langle \aleph, \zeta_{\Delta_2}, \varkappa_{\Delta_2}, \digamma_{\Delta_2}; \kappa_{\Delta_2}\rangle : \aleph \in \Upsilon\}$ be two SFNSs over the universal set Υ. Then the following operations are defined as follows

1. The union of any two SFN sets $\Delta_{\kappa_1} \cup_{\max} \Delta_{\kappa_2} =$
$\{< \aleph, \max\{\zeta_{\Delta_1}, \zeta_{\Delta_2}\}, \min\{\digamma_{\Delta_1}, \digamma_{\Delta_2}\}, \min\{\varkappa_{\Delta_1}, \varkappa_{\Delta_2}\}; \max\{\kappa_{\Delta_1}, \kappa_{\Delta_2}\} > : \aleph \in \Upsilon\}$.

2. The intersection of any two SFN sets $\Delta_{\kappa_1} \cap_{\min} \Delta_{\kappa_2} =$
$\{< \aleph, \min\{\zeta_{\Delta_1}, \zeta_{\Delta_2}\}, \max\{F_{\Delta_1}, F_{\Delta_2}\}, \max\{\varkappa_{\Delta_1}, \varkappa_{\Delta_2}\}; \min\{\kappa_{\Delta_1}, \kappa_{\Delta_2}\} > : \aleph \in \Upsilon\}$.
3. $\Delta_{\kappa_1} = \Delta_{\kappa_2}$ IFF $\{< \aleph, \zeta_{\Delta_1} = \zeta_{\Delta_2}, F_{\Delta_1} = F_{\Delta_2}, \varkappa_{\Delta_1} = \varkappa_{\Delta_2}; \kappa_{\Delta_1} = \kappa_{\Delta_2} >: \aleph \in \Upsilon\}$.
4. $\Delta_{\kappa_1} \subseteq \Delta_{\kappa_2}$ IFF $\{< \aleph, \zeta_{\Delta_1} \subseteq \zeta_{\Delta_2}, F_{\Delta_1} \supseteq F_{\Delta_2}, \varkappa_{\Delta_1} \supseteq \varkappa_{\Delta_2}; \kappa_{\Delta_1} \subseteq \kappa_{\Delta_2} >: \aleph \in \Upsilon\}$.
5. $\Delta^c_{\kappa_1} = \{< \aleph, \varkappa_{\Delta_1}, F_{\Delta_1}, \zeta_{\Delta_1}; \kappa_{\Delta_1} >: \aleph \in \Upsilon\}$.

Example 2.6. Let $B = \{\langle \aleph, 0.5, 0.3, 0.4\rangle, \langle \aleph, 0.6, 0.2, 0.3\rangle, \langle \aleph, 0.4, 0.5, 0.2\rangle, \langle \aleph, 0.5, 0.4, 0.5\rangle\}$ be a collection of SFNSs. Then the center of the SFNS is ,

$\zeta_\aleph = \frac{0.5+0.6+0.4+0.5}{4} = 0.5$, $F_\aleph = \frac{0.3+0.2+0.5+0.4}{4} = 0.35$, $\varkappa_\aleph = \frac{0.4+0.3+0.2+0.5}{4} = 0.35$,

The radius is given by:

$$\Delta_i = \min\left\{\max_{1\le j\le 4} \sqrt{(\zeta_\aleph - \zeta_{i,j})^2 + (F_\aleph - F_{i,j})^2 + (\varkappa_\aleph - \varkappa_{i,j})^2}, 1\right\}$$

$$\Delta_i = \min\left\{\max\left\{\sqrt{(0.5-0.5)^2 + (0.35-0.3)^2 + (0.35-0.4)^2},\right.\right.$$
$$\sqrt{(0.5-0.6)^2 + (0.35-0.2)^2 + (0.35-0.3)^2},$$
$$\sqrt{(0.5-0.4)^2 + (0.35-0.5)^2 + (0.35-0.2)^2},$$
$$\left.\left.\sqrt{(0.5-0.5)^2 + (0.35-0.4)^2 + (0.35-0.5)^2}\right\}, 1\right\}$$
$$= \min\{\max\{0.07, 0.14, 0.15, 0.15\}, 1\} = 0.15$$

Thus, the SFNS is: $A_\kappa = \langle 0.5, 0.35, 0.35; 0.15\rangle$.

3. SPHERICAL FERMATEAN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, we study the new notion namely spherical Fermatean neutrosophic topology and its characterization. The spherical Fermatean neutrosophic $1_\odot$ and spherical Fermatean neutrosophic $0_\odot$ in Υ as follows $1_\odot = \langle 1, 0, 0; 1\rangle$, and $0_\odot = \langle 0, 1, 1; 0\rangle$.

Proposition 3.1. *Let $\Delta_{\kappa_1} = < \zeta_{\Delta_1}, \varkappa_{\Delta_1}, F_{\Delta_1}; \kappa_{\Delta_1} >$ and $\Delta_{\kappa_2} = < \zeta_{\Delta_2}, \varkappa_{\Delta_2}, F_{\Delta_2}; \kappa_{\Delta_2} >$ be two SFNSs over the universal set Υ. Then the following hold:*

(1) $\Delta_{\kappa_1} \cup \Delta_{\kappa_1} = \Delta_{\kappa_1}$ *and* $\Delta_{\kappa_1} \cap \Delta_{\kappa_1} = \Delta_{\kappa_1}$.
(2) $\Delta_{\kappa_1} \cup 0_\odot = \Delta_{\kappa_1}$ *and* $\Delta_{\kappa_1} \cap 0_\odot = 0_\odot$.
(3) $\Delta_{\kappa_1} \cup 1_\odot = 1_\odot$ *and* $\Delta_{\kappa_1} \cap 1_\odot = \Delta_{\kappa_1}$.
(4) $(\Delta^c_{\kappa_1})^c = \Delta_{\kappa_1}$.

Definition 3.2. Let $\Xi_\odot \in FN(\Upsilon)$, then $\Xi_\odot$ is called a spherical Fermatean neutrosophic topology on Υ, if the following hold

(1) $1_\odot, 0_\odot \in \Xi_\odot$.
(2) $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_\odot \Rightarrow \Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_\odot$.
(3) $\{\Delta_{\kappa_i}; i \in \triangle\} \subseteq \Xi_\odot \Rightarrow \bigcup \Delta_{\kappa_i} \in \Xi_\odot$.

The pair $(\Upsilon, \Xi_\odot)$ is called a Spherical Fermatean Neutrosophic Topological Space (FNTS) over Υ. This generalization involves considering sphere for the DoM, DoI and DoN in the context of topology. Furthermore, the members of $\Xi_{\bigodot}$ are said to be SFN-open sets in Υ. If $\Delta^c_{\kappa_1} \in \Xi_{\bigodot}$, then $\Delta_{\kappa_1} \in FN(\Upsilon)$ is said to be SFN-closed set in Υ.

The SFN-interior of set Δ_{κ_1}, denoted by $\Gamma_{\bigodot}(\Delta_{\kappa_1})$ is defined as the union of all SFN-open subsets of Δ_{κ_1}. Notably, $\Gamma_{\bigodot}(\Delta_{\kappa_1})$ represents the largest SFN-open set over Υ that containing Δ_{κ_1}. The SFN-closure of set Δ_{κ_1}, denoted by $\Delta_{\bigodot}(\Delta_{\kappa_1})$ is defined as the intersection of all SFN-closed supersets of Δ_{κ_1}. Notably, $\Delta_{\bigodot}(\Delta_{\kappa_1})$ represents the smallest SFN-closed set over Υ that contains Δ_{κ_1}.

Let $\Xi_{\bigodot} = \{0_{\bigodot}, 1_{\bigodot}\}$ and $\Psi_{\bigodot} = FN(\Upsilon)$. Then, $(\Upsilon, \Xi_{\bigodot})$ and $(\Upsilon, \Psi_{\bigodot})$ are two trivial FNTS over Υ. Additionally, they are referred to as FN-discrete topological space and FN-indiscreet topological space over Υ, respectively.

Example 3.3. Let Υ=$\{\aleph, \sqsupset\}$ and Δ_1, $\Delta_2 \in SFNS(\Upsilon)$ such that
$\Delta_1 = \{\langle \aleph, 0.7, 0.6, 0.3\rangle, \{\langle \aleph, 0.8, 0.1, 0.4\rangle, \{\langle \aleph, 0.95, 0.05, 0.3\rangle\}$ and
$\Delta_2 = \{\langle \sqsupset, 0.2, 0.1, 0.5\rangle, \langle \sqsupset, 0.4, 0.3, 0.6\rangle, \langle \sqsupset, 0.6, 0.5, 0.4\rangle\}$. Then

(1) The SFNSs are $\Delta_{\kappa_1}= \{\langle \aleph, 0.82, 0.25, 0.33; 0.37\rangle : \aleph \in \Upsilon\}$ and $\Delta_{\kappa_2}= \{\langle \sqsupset, 0.40, 0.30, 0.50; 0.30\rangle : \sqsupset \in \Upsilon\}$.
(2) The union of two SFNSs Δ_{κ_1} and Δ_{κ_2} is $\Delta_{\kappa_1} \cup_{\max} \Delta_{\kappa_2} = \{\langle \aleph, 0.82, 0.25, 0.33; 0.37\rangle : \sqsupset \in \Upsilon\}$.
(3) The intersection of two SFNSs Δ_{κ_1} and Δ_{κ_2} is $\Delta_{\kappa_1} \cap_{\min} \Delta_{\kappa_2} = \{\langle \sqsupset, 0.40, 0.30, 0.50; 0.30\rangle : \sqsupset \in \Upsilon\}$.
(4) The complement of a SFNS Δ_{κ_1} is $\Delta^c_{\kappa_1} = \{\langle \aleph, 0.33, 0.44, 0.82; 0.20\rangle : \aleph \in \Upsilon\}$.
(5) We have that $\Delta_{\kappa_2} \subset \Delta_{\kappa_1}$.
(6) The family $\Xi_{\bigodot} = \{0_{\bigodot}, 1_{\bigodot}, \Delta_{\kappa_1}, \Delta_{\kappa_2}\}$, of SFNSs in Υ is SFNTS.
(7) The geometric representation of Δ_1, Δ_2, Δ_{κ_1}, and Δ_{κ_2} are

Proposition 3.4. *Let $(\Upsilon, \Xi_{1\bigodot})$ and $(\Upsilon, \Xi_{2\bigodot})$ be two SFNTSs over Υ, then $(\Upsilon, \Xi_{1\bigodot} \cap \Xi_{2\bigodot})$ is a SFNTS over Υ.*

Proof. *Let $(\Upsilon, \Xi_{1\bigodot})$ and $(\Upsilon, \Xi_{2\bigodot})$ be two SFNTSs over Υ. It can be seen clearly that $0_{\bigodot}, 1_{\bigodot} \in \Xi_{1\bigodot} \cap \Xi_{2\bigodot}$. If $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_{1\bigodot} \cap \Xi_{2\bigodot}$ then, $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_{1\bigodot}$ and $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in \Xi_{2\bigodot}$. It is given that $\Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_{1\bigodot}$ and $\Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_{2\bigodot}$. Thus, $\Delta_{\kappa_1} \cap \Delta_{\kappa_2} \in \Xi_{1\bigodot} \cap \Xi_{2\bigodot}$. Let $\{\Delta_{\kappa_1 i} : i \in I\} \subseteq \Xi_{1\bigodot} \cap \Xi_{2\bigodot}$. Then, $\Delta_{\kappa_1 i} \in \Xi_{1\bigodot} \cap \Xi_{2\bigodot}$ for all $i \in I$. Thus, $\Delta_{\kappa_1 i} \in \Xi_{1\bigodot}$ and $\Delta_{\kappa_1} i \in \Xi_{2\bigodot}$ for all $i \in I$. So, we have $\bigcap_{i\in I} \Delta_{\kappa_1 i} \in \Xi_{1\bigodot} \cap \Xi_{2\bigodot}$.*

Corollary 3.5. *Let $\{(\Upsilon, \Xi_{\bigodot i}) : i \in I\}$ be a family of SFNTSs over $\mathbb{X}$. Then, $(\Upsilon, \bigcap_{i\in I} \Xi_{\bigodot i})$ is a SFNTS over $\mathbb{X}$.*

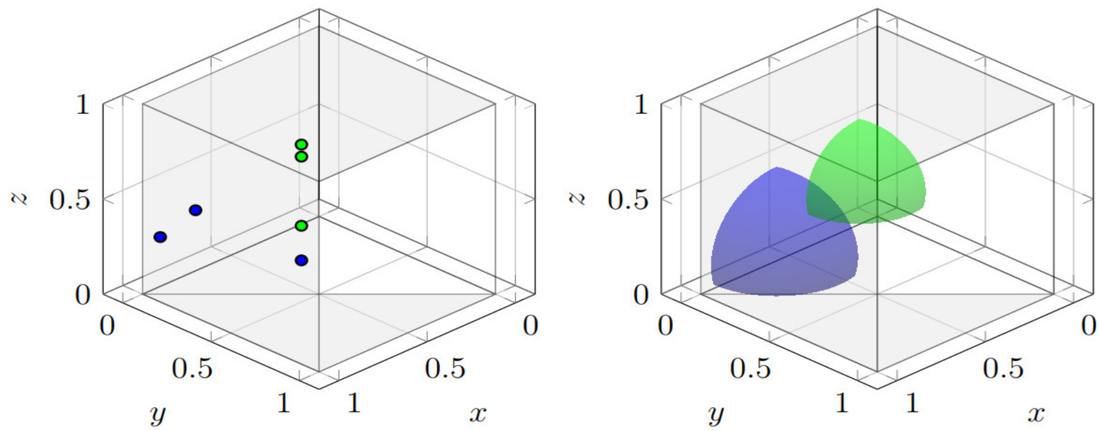


FIGURE 1

Example 3.6. Let $\Upsilon = \{\aleph\}$, and $\Delta_1, \Delta_2, \Delta_3 \in FNS(\Upsilon)$ such that

$\Delta_1 = \{\langle \aleph, 0.7, 0.6, 0.3\rangle, \{\langle \aleph, 0.8, 0.1, 0.4\rangle, \{\langle \aleph, 0.95, 0.05, 0.3\rangle\}$

$\Delta_2 = \{\langle \aleph 0.2, 0.1, 0.5\rangle, \langle \aleph, 0.4, 0.3, 0.6\rangle, \langle \aleph, 0.6, 0.5, 0.4\rangle\}$.

$\Delta_3 = \{\langle \aleph, 0.8, 0.1, 0.4\rangle, \langle \aleph, 0.7, 0.6, 0.3\rangle, \langle \aleph, 0.7, 0.6, 0.3\rangle\}$.

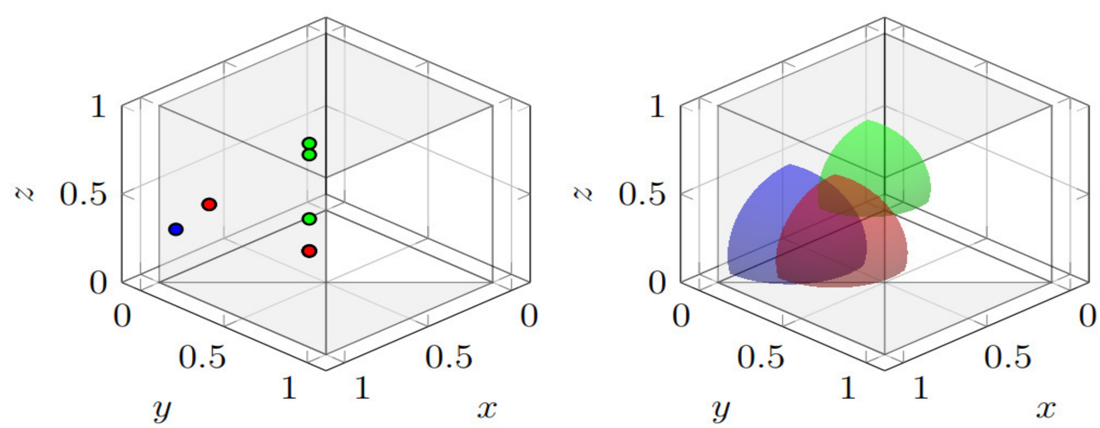
Then the SFNSs are

$\Delta_{\kappa_1} = \{\langle \aleph, 0.82, 0.25, 0.33; 0.37\rangle : \aleph \in \Upsilon\}$,

$\Delta_{\kappa_2} = \{\langle \aleph, 0.40, 0.30, 0.50; 0.30\rangle : \aleph \in \Upsilon\}$,

$\Delta_{\kappa_3} = \{\langle \aleph, 0.73, 0.43, 0.33; 0.35\rangle : \aleph \in \Upsilon\}$ and

The family $\Xi_{\odot} = \{0_{\odot}, 1_{\odot}, \Delta_{\kappa_1}, \Delta_{\kappa_2}\}$, and $\Psi_{\odot} = \{0_{\odot}, 1_{\odot}, \Delta_{\kappa_1}, \Delta_{\kappa_3}\}$, are SFNTSs but their union $\Xi_{\odot} \cup \Psi_{\odot}$ is not a SFNTS, since $\Delta_{\kappa_2} \cup \Delta_{\kappa_3} \notin \Xi_{\odot} \cup \Psi_{\odot}$.



Proposition 3.7. *Let $(\Upsilon, \Xi_{\odot})$ be a SFNTS over Υ. Then*

(1) $0_{\odot}$ *and* $1_{\odot}$ *are SFN closed sets over* Υ.

(2) *The intersection of any number of SFN-closed sets is a SFN-closed set over* Υ.

(3) *The union of any two SFN-closed sets is a SFN-closed set over* Υ.

Proposition 3.8. *Let* $(\Upsilon, \Xi_{\odot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in FNS(\Upsilon)$. *Then*

(1) $\Gamma_{\odot}(0_{\odot}) = 0_{\odot}$ *and* $\Gamma_{\odot}(1_{\odot}) = 1_{\odot}$.
(2) $\Gamma_{\odot}(\Delta_{\kappa_1}) \subseteq \Delta_{\kappa_1}$.
(3) Δ_{κ_1} *is a SFN-open set if and only if* $\Delta_{\kappa_1} = \Gamma_{\odot}(\Delta_{\kappa_1})$.
(4) $\Gamma_{\odot}(\Gamma_{\odot}(\Delta_{\kappa_1})) = \Gamma_{\odot}(\Delta_{\kappa_1})$.
(5) $\Delta_{\kappa_1} \subseteq \Delta_{\kappa_2}$ *implies* $\Gamma_{\odot}(\Delta_{\kappa_1}) \subseteq \Gamma_{\odot}(\Delta_{\kappa_2})$.
(6) $\Gamma_{\odot}(\Delta_{\kappa_1}) \cup \Gamma_{\odot}(\Delta_{\kappa_2}) \subseteq \Gamma_{\odot}(\Delta_{\kappa_1} \cup \Delta_{\kappa_2})$.
(7) $\Gamma_{\odot}(\Delta_{\kappa_1} \cap \Delta_{\kappa_2}) = \Gamma_{\odot}(\Delta_{\kappa_1}) \cap \Gamma_{\odot}(\Delta_{\kappa_2})$.

Proof. *1. and 2. are obvious.*

3. If A *is a SFN-open set over* Υ, *then* A *is itself a SFN-open set over* Υ *which contains* A. *So,* A *is the largest neutrosophic open set contained in* A *and* $\Gamma_{\odot}(A) = A$. *Conversely, suppose that* $\Gamma_{\odot}(A) = A$. *Then* $A \in \Xi_{\odot}$.

4. Let $\Gamma_{\odot}(A) = B$. *Then,* $\Gamma_{\odot}(B) = B$ *from 3. and then,* $\Gamma_{\odot}(\Gamma_{\odot}(A)) = \Gamma_{\odot}(A)$.

5. Suppose that $A \subseteq B$. *As* $\Gamma_{\odot}(A) \subseteq A \subseteq B$. *By definition, we have* $\Gamma_{\odot} \subseteq \Gamma_{\odot}(B)$.

6. It is clear that $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$. *Thus* $\Gamma_{\odot}(A) \subseteq \Gamma_{\odot}(A \cup B)$ *and* $\Gamma_{\odot}(B) \subseteq \Gamma_{\odot}(A \cup B)$. *So we have* $\Gamma_{\odot}(A) \cup \Gamma_{\odot}(B) \subseteq \Gamma_{\odot}(A \cup B)$.

7. It is known that $\Gamma_{\odot}(A \cap B) \subseteq \Gamma_{\odot}(A)$ *and* $\Gamma_{\odot}(A \cap B) \subseteq \Gamma_{\odot}(B)$ *by 5. So that* $\Gamma_{\odot}(A \cap B) \subseteq \Gamma_{\odot}(A) \cap \Gamma_{\odot}(B)$. *Also, from* $\Gamma_{\odot}(A) \subseteq A$ *and* $\Gamma_{\odot}(B) \subseteq B$, *we have* $\Gamma_{\odot}(A) \cap \Gamma_{\odot}(B) \subseteq A \cap B$. *These imply that* $\Gamma_{\odot}(A \cap B) = \Gamma_{\odot}(A) \cap \Gamma_{\odot}(B)$.

Proposition 3.9. *Let* $(\Upsilon, \Xi_{\odot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1}, \Delta_{\kappa_2} \in FNS(\Upsilon)$. *Then*

(1) $\Delta_{\odot}(0_{\odot}) = 0_{\odot}$ *and* $\Delta_{\odot}(1_{\odot}) = 1_{\odot}$.
(2) $\Delta_{\kappa_1} \subseteq \Delta_{\odot}(\Delta_{\kappa_1})$.
(3) Δ_{κ_1} *is a SFN closed set if and only if* $\Delta_{\kappa_1} = \Delta_{\odot}(\Delta_{\kappa_1})$.
(4) $\Delta_{\odot}(\Delta_{\odot}(\Delta_{\kappa_1})) = \Delta_{\odot}(\Delta_{\kappa_1})$.
(5) $\Delta_{\kappa_1} \subseteq \Delta_{\kappa_2}$ *implies* $\Delta_{\odot}(\Delta_{\kappa_1}) \subseteq \Delta_{\odot}(\Delta_{\kappa_2})$.
(6) $\Delta_{\odot}(\Delta_{\kappa_1} \cup \Delta_{\kappa_2}) = \Delta_{\odot}(\Delta_{\kappa_1}) \cup \Delta_{\odot}(\Delta_{\kappa_2})$.
(7) $\Delta_{\odot}(\Delta_{\kappa_1} \cap \Delta_{\kappa_2}) \subseteq \Delta_{\odot}(\Delta_{\kappa_1}) \cap \Delta_{\odot}(\Delta_{\kappa_2})$.

Corollary 3.10. *Let* $(\Upsilon, \Xi_{\bigodot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1} \in FN(\Upsilon)$*. Then* $\Gamma_{\bigodot}(\Delta^c_{\kappa_1}) = (\Delta_{\bigodot}(\Delta_{\kappa_1}))^c$*. and* $\Delta_{\bigodot}(\Delta^c_{\kappa_1}) = (\Gamma_{\bigodot}(\Delta_{\kappa_1}))^c$.

Proposition 3.11. *Let* $(\Upsilon, \Xi_{\bigodot})$ *be a FNTS over* Υ*,* $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ *be a FN set on* Υ*. Then* $(\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}$ *is a SFN-closed set.*

Proof. To prove $(\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}$ is SFN-closed it is enough to prove that $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{\acute{}}$ is SFN-open. If $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{\acute{}} = \phi$ then it is SFN-open. Let $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{\acute{}} \neq \phi$ and $\aleph \in \left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{\acute{}}$ $\Rightarrow \aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'} \Rightarrow \aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2})$ and $\aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'} \Rightarrow \exists$ a SFN-open set $G * B \ni \aleph \in (G * B)$ and $(G * B) \cap (\Delta_{\kappa_1} * \Delta_{\kappa_2}) = \phi \Rightarrow \aleph \in (G * B) \subseteq (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}$ Again $\aleph \notin (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'} \Rightarrow \aleph \in G * B \subseteq ((\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'})^{'}$. Therefore $\aleph \in (G * B) \subseteq \left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{\acute{}}$ and hence $\left((\Delta_{\kappa_1} * \Delta_{\kappa_2}) \cup (\Delta_{\kappa_1} * \Delta_{\kappa_2})^{'}\right)^{\acute{}}$ is SFN-open set.

Proposition 3.12. $(\Upsilon, \Xi_{\bigodot})$ *be a SFNTS over* Υ *and* $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ *be a SFN set over* X *then* $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ *is a SFN-closed IFF* $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}} = \Delta_{\kappa_1} * \Delta_{\kappa_2}$

Proof. If $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ is an SFN-closed set then the smallest SFN-closed super set of $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ is $\Delta_{\kappa_1} * \Delta_{\kappa_2}$ itself. Therefore $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}} = \Delta_{\kappa_1} * \Delta_{\kappa_2}$. Conversely if $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}} = \Delta_{\kappa_1} * \Delta_{\kappa_2}$ then $\overline{\Delta_{\kappa_1} * \Delta_{\kappa_2}}$ being SFN-closed so as $\Delta_{\kappa_1} * \Delta_{\kappa_2}$.

4. Distance between Spherical Fermatean Neutrosophic Sets

Definition 4.1. The distance between two SFNS A_κ and B_κ can be defined using a metric that incorporates the differences in their parameters:

$$D(A_\kappa, B_\kappa) = \sqrt{(\zeta_A(\aleph) - \zeta_B(y))^2 + (\varkappa_A(\aleph) - \varkappa_B(y))^2 + (F_A(\aleph) - F_B(y))^2}$$

where:

$$A_\kappa = \{\langle \aleph, \zeta_A(\aleph), \varkappa_A(\aleph), F_A(\aleph); \kappa_A \rangle | \aleph \in X\}$$

$$B_\kappa = \{\langle y, \zeta_B(y), \varkappa_B(y), F_B(y); \kappa_B \rangle | y \in Y\}$$

$$D = D(A_\kappa, B_\kappa) - (\kappa_A + \kappa_B)$$

Example 4.2. Consider two SFNSs:

$$A_\rho = \langle 0.7, 0.2, 0.1; 0.1 \rangle \quad \text{and} \quad B_\rho = \langle 0.4, 0.4, 0.2; 0.1 \rangle$$

The center of SFNS A_ρ is $(0.7, 0.2, 0.1)$ and for B_ρ is $(0.4, 0.4, 0.2)$, both with radius 0.1.

$$d = \sqrt{(0.7 - 0.4)^2 + (0.2 - 0.4)^2 + (0.1 - 0.2)^2}$$
$$= \sqrt{(0.3)^2 + (-0.2)^2 + (-0.1)^2}$$

$$= \sqrt{0.09 + 0.04 + 0.01} = \sqrt{0.14} \approx 0.374$$

$$D = d - (\rho_A + \rho_B) = 0.374 - (0.1 + 0.1) = 0.374 - 0.2 = 0.174$$

This indicates separation of two SFNSs.

Example 4.3. Consider two SFNSs:

$$A_\rho = \langle 0.5, 0.4, 0.1; 0.1\rangle \quad \text{and} \quad B_\rho = \langle 0.6, 0.3, 0.1; 0.1\rangle$$

The center of SFNS A_ρ is $(0.5, 0.4, 0.1)$ and for B_ρ is $(0.6, 0.3, 0.1)$, both with radius 0.1.
Distance Calculation:

$$d = \sqrt{(0.5 - 0.6)^2 + (0.4 - 0.3)^2 + (0.1 - 0.1)^2}$$

$$= \sqrt{(-0.1)^2 + (0.1)^2 + (0)^2}$$

$$= \sqrt{0.01 + 0.01 + 0} = \sqrt{0.02} \approx 0.1414$$

$$D = d - (\rho_A + \rho_B) = 0.1414 - (0.1 + 0.1) = 0.1414 - 0.2 = -0.0586$$

This indicates intersection of two SFNSs.

Example 4.4. Consider two SFNSs:

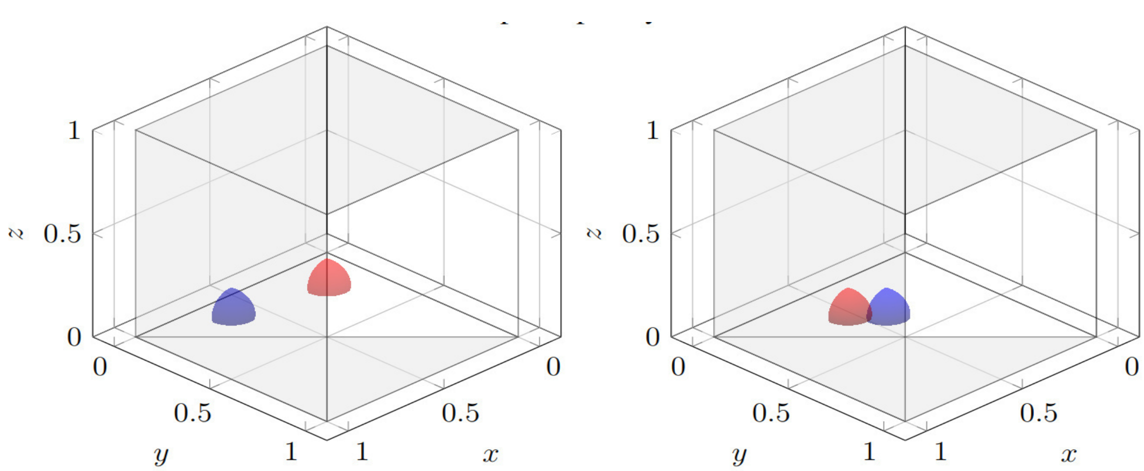
$$A_\rho = \langle 0.6, 0.3, 0.1; 0.1\rangle \quad \text{and} \quad B_\rho = \langle 0.6, 0.3, 0.1; 0.1\rangle$$

The center of both SFNSs is $(0.6, 0.3, 0.1)$ and the radius of both SFNSs is 0.1.
Distance Calculation:

$$d = \sqrt{(0.6 - 0.6)^2 + (0.3 - 0.3)^2 + (0.1 - 0.1)^2} = \sqrt{0 + 0 + 0} = 0$$

$$D = d - (\rho_A + \rho_B) = 0 - (0.1 + 0.1) = 0 - 0.2 = -0.2$$

This indicates that the two SFNSs overlap completely.



Conclusion and Future Work:

In this paper, we introduced SFNTS as an extension of neutrosophic sets and Fermatean fuzzy sets, characterized by Degrees of Membership, Degrees of Indeterminacy, and Degrees of Non-Membership, along with a radius component. We defined the properties of SFN closure and SFN interior operators, providing numerical examples and a geometric representation of SFNTS. Furthermore, we explored the SFN distance between spheres, as well as the separation, intersection, and overlapping of two SFNSs. These foundational concepts and examples demonstrate the potential of SFNTS in modeling and analyzing complex systems that require a nuanced representation of uncertainty and vagueness.

Future research can extend the study of SFNTS by exploring its applications in real-world scenarios, such as decision-making, pattern recognition, and multi-criteria analysis. Additionally, there is significant potential in examining the topological properties of SFNTS, such as continuity, connectedness, compactness, separation axioms, metrizability, homotopy, homology and other fundamental concepts. Investigating the relationships between SFNTS and other topological structures.

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Neutrosophic set connected mappings

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Abstract. In this paper, the concept of neutrosophic connectedness between neutrosophic sets in neutrosophic topological spaces has been introduced. It is shown that a neutrosophic topological space is neutrosophic connected if and only if it is neutrosophic connected between every pair of its nonempty neutrosophic sets. Further, a new class of mappings called neutrosophic set connected mappings has been defined. It is shown that the class of all neutrosophic continuous mappings is a subclass all neutrosophic set-connected mappings. Several properties and characterizations of neutrosophic set-connected mappings in neutrosophic topological spaces have been studied.

Keywords: Neutrosophic sets; neutrosophic connectedness; neutrosophic connectedness between neutrosophic sets and neutrosophic set-connected mappings.

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1. Introduction

The fusion of technology and generalized forms of classical sets is very useful to solve many real world complex problems which involve the vague and uncertain information. A classical set is defined by its characteristic function from universe of discourse to two point set {0,1}. Classical set theory is insufficient to handle the complex problems involving vague and uncertain information. To handle the vagueness and uncertainty of complex problems, Zadeh [18] in 1965, created fuzzy sets as a generalization of classical sets which characterised by membership function from universe of discourse to closed interval [0,1]. After the occurrence of fuzzy sets most of the mathematical structure have been extended to fuzzy sets. In 1968, Chang [4] used fuzzy sets to create fuzzy topological spaces and extended some topological concepts to fuzzy sets. In 1986, Atanassov [3] gave a generalization of fuzzy set called Intuitionistic fuzzy set characterized by a membership degree and a non-membership degree that satisfies the case in which the sum of its membership degree and a non-membership degree is less than or equal to one. In 1997, Coker [5], proposed the notion of intuitionistic fuzzy topological spaces and studied some analog versions of classical topology such as continuity and compactness. In 1999 , Smarandache [15] created the concept of neutrosophic sets as an extension of intuitionistic fuzzy sets and developed the theory of neutrosophy. Smarandache [?, 15],Salama and Alblowi [13] , Lupiáñez [10] and others presented some more properties of neutrosophic sets. In 2008, Lupiáñez [10] introduced the neutrosophic topology as an extension of intuitionistic fuzzy topology. Since 2008 many authors such as Lupiáñez [10, 11], Salama et.al. [13, 14] Karatas and Cemil [9], Acikgoz and his coworkers [1, 2], Dhavaseelan et.al. [6–8], Parimala et.al. [12], and others extended various topological notions to neutrosophic sets and studied in neutrosophic topological spaces. Recently Acikgoz and his coworkers [2] iniated the study of connectedness in neutrosophic topology. Connectedness between sets and set connected mappings are the important topic of research in Topology. Till the date these concepts are not studied in neutrosophic topology. Therefore, to fill this gap, the present paper introduces the concept of connectedness between neutrosophic sets and studied some of its properties in neutrosophic topological spaces. Further, neutrosophic set-connected mappings are defined and some theorems related to its characterizations and properties are established.

2. Preliminaries

Definition 2.1. [15] A Neutrosophic set (**NS**) in X is a structure

$$A = \{< x, \varrho_A(x), \varpi_A(x), \sigma_A(x) >: x \in X\}$$

where $\varrho_A : X \to]^-0, 1^+[$, $\varpi A : X \to]^-0, 1^+[$, and $\sigma_A : X \to]^-0, 1^+[$ denotes the membership, indeterminacy, and non-membership of A satisfies the condition if $^-0 \leq \varrho_A(x) + \varpi_A(x) + \sigma_A(x) \leq 3^+, \ \forall\, x \in X$.

In the real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the closed interval [0,1] and sum of membership, indeterminacy, and nonmembership degrees of each element of universe of discourse lies between 0 and 3. The family of all **NS**s over X will be denoted by $\mathbf{N}(\mathbf{X})$.

Definition 2.2. [15] A neutrosophic set $A = \{< x, \varrho_A(x), \varpi_A(x), \sigma_A(x) >: x \in X\}$ is said to be

(a) Universal neutrosophic set if $\varrho_A(x) = 1$, $\varpi_A(x) = 1$, and $\sigma_A(x) = 0$, $\forall x \in X$. It is denoted by $\tilde{X}$.

(b) Empty neutrosophic set if $\varrho_A(x) = 0$, $\varpi_A(x) = 0$, and $\sigma_A(x) = 1$, $\forall x \in X$. It is denoted by $\tilde{\Phi}$.

Definition 2.3. [15–17] Let $A = \{< x, \varrho_A(x), \varpi_A(x), \sigma_A(x) >: x \in X\}$, $A_1 = \{< x, \varrho_{A_1}(x), \varpi_{A_1}(x), \sigma_{A_1}(x) >: x \in X\}$, and $A_2 = \{< x, \varrho_{A_2}, \varpi_{A_2}(x), \sigma_{A_2}(x) >: x \in X\}$ be **NS**s over X. Then the subset, equality, union, intersection and complement operations over $\mathbf{N(X)}$ are defined as follow:

(a) $A_1 \subset A_2 \Leftrightarrow \varrho_{A_1}(x) \leq \varrho_{A_2}(x)$, $\varpi_{A_1}(x) \leq \varpi_{A_2}(x)$, and $\sigma_{A_1} \geq \sigma_{A_2}$, $\forall\, x \in X$.

(b) $A_1 = A_2 \Leftrightarrow \varrho_{A_1}(x) = \varrho_{A_2}(x)$, $\varpi_{A_1}(x) = \varpi_{A_2}(x)$, and $\sigma_{A_1}(x) = \sigma_{A_2}(x)$, $\forall\, x \in X$.

(c) $A_1 \cup A_2 = \{< x, \varrho_{A_1}(x) \vee \varrho_{A_2}(x),\ \varpi_{A_1}(x) \vee \varpi_{A_2}(x), \sigma_{A_1}(x) \wedge \sigma_{A_2}(x) >: x \in X\}$.

(d) $A_1 \cap A_2 = \{< x, \varrho_{A_1}(x) \wedge \varrho_{A_2}(x),\ \varpi_{A_1}(x) \wedge \varpi_{A_2}(x), \sigma_{A_1}(x) \vee \sigma_{A_2}(x) >: x \in X\}$.

(e) $A^c = \{< x, \sigma_A(x), \varpi_A(x), \varrho_A(x) >: x \in X\}$.

Definition 2.4. [15–17] Let $\{A_i : i \in \Lambda\}$ be an arbitrary family of **NS**s in X. Then

(a) $\cup A_i = \{< x, \vee\varrho_{A_i}(x),\ \vee\varpi_{A_i}(x), \wedge\sigma_{A_i}(x) >: x \in X\}$.

(b) $\cap A_i = \{< x, \wedge\varrho_{A_i}(x),\ \wedge\varpi_{A_i}(x), \vee\sigma_{A_1}(x) >: x \in X\}$.in

Example 2.5. Let $X = \{x_1, x_2, x_3\}$ and **NS**s A, B, C over X are defined as follows:

$$A = \{< x_1, 0.8, 0.2, 0.7 >, < x_2, 0.7, 0.5, 0.6 >, < x_3, 0.6, 0.4, 0.9 >\}$$

$$B = \{< x_1, 0.9, 0.3, 0.4 >, < x_2, 0.8, 0.5, 0.3 >, < x_3, 0.7, 0.5, 0.3 >\}$$

$$C = \{< x_1, 0.6, 0.5, 0.3 >, < x_2, 0.9, 0.4, 0.5 >, < x_3, 0.8, 0.5, 0.6 >\}.$$

Then,

(i) $A \subset B$, but $A \not\subseteq C$.

(ii) $A \cup B = \{< x_1, 0.9, 0.3, 0.4 >, < x_2, 0.8, 0.5, 0.3 >, < x_3, 0.7, 0.5, 0.3 >\}$.

(iii) $B \cap C = \{< x_1, 0.6, 0.3, 0.4 >, < x_2, 0.8, 0.4, 0.5 >, < x_3, 0.7, 0.5, 0.6 >\}$.

(iv) $A^c = \{< x_1, 0.7, 0.2, 0.8 >, < x_2, 0.6, 0.5, 0.7 >, < x_3, 0.9, 0.4, 0.6 >\}$.

Definition 2.6. [10, 13] A subfamily Γ of $\mathbf{N(X)}$ is called a Neutrosophic topology (**NT**) on X if:

(a) $\tilde{\Phi}, \tilde{X} \in \Gamma$.

(b) $G_i \in \Gamma,\ \forall\, i \in \Lambda \Rightarrow \cup_{i\in\Lambda} G_i \in \Gamma$.

(c) $G_1, G_2 \in \Gamma \Rightarrow G_1 \cap G_2 \in \Gamma$.

If Γ is a **NT** on X then the structure (X, Γ) is called a neutrosophic topological space (**NTS**) over X and the members of Γ are called neutrosophic open (**NO**) sets.The complement of **NO** set is called neutrosophic closed (**NC**).

Definition 2.7. [8] Let X be a nonempty set. If r, t, s are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$, then the neutrosophic set $x_{r,t,s}$ defined by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p, \\ (0, 0, 1), & \text{if } x \neq x_p. \end{cases}$$

is called a neutrosophic point. The point $x_p \in X$ is called the support of $x_{r,t,s}$, and r denotes the degree of membership value, t the degree of indeterminacy and s the degree of non-membership value of $x_{r,t,s}$.

Definition 2.8. [1] a nuetrosophic point $x_{r,t,s}$ is said to be quasi-coincident (q-coincident, for short) with F, denoted by $x_{r,t,s}qF$ iff $x_{r,t,s} \nsubseteq F^c$. If $x_{r,t,s}$ is not quasi-coincident with F, we denote by $x_{r,t,s}\tilde{q}F$.

Definition 2.9. [1] a **NS** F is a neutrosophic topological space (X, Γ) is said to be a q-neighbourhood of a neutrosophic point $x_{r,t,s}$ if there exists a neutrosophic open set G such that $x_{r,t,s}qG \subset F$.

Definition 2.10. [1] a **NS** G is said to be quasi-coincident (q-coincident, for short) with F, denoted by GqF if $G \nsubseteq F^c$. If G is not quasi coincident with F, we denote by $G\tilde{q}F$.

Definition 2.11. [1] a nuetrosophic point $x_{r,t,s}$ of a **NTS** (X, Γ)is said to be interior point of a **NS** F if there exists a neutrosophic open q-neighbourhood G of $x_{r,t,s}$ such that $G \subset F$. The union of all interior points of F is called the interior of F and denoted by Int(F).

Definition 2.12. [1] a nuetrosophic point $x_{r,t,s}$ of a **NTS** (X, Γ) is said to be cluster point of a **NS** F if every neutrosophic open q-neighbourhood G of$x_{r,t,s}$ is q-coincident with F. The union of all cluster points of F is called the closure of F. It is denoted by Cl(F).

Definition 2.13. [1] Let (X, Γ) be a **NTS** and $Y \subset X$. Then the family $\Gamma_Y = \{G \cap Y : G \in \Gamma\}$ is called the neutrosophic relative topology on Y and the pair (Y, Γ_Y) is called neutrosophic sub space of (X, Γ).

Lemma 2.14. *Let (Y, Γ_Y) be a neutrosophic subspace of a* **NTS** *(X, Γ) and F be a neutrosophic open set in Y. If $Y \in \Gamma$ then $F \in \Gamma$.*

Lemma 2.15. *Let (X,Γ) be a* **NTS** *and (Y, Γ_Y) be a neutrosophic subspace of (X, Γ), then a neutrosophic closed set F_Y of Y is neutrosophic closed in X if and only if Y is neutrosophic closed in X.*

Definition 2.16. [1] A **NTS** (X,Γ) said to be neutrosophic connected, if $\nexists$ proper neutrosophic open sets U and V in (X,Γ) such that that $A\tilde{q}B$ and $A^c\tilde{q}B^c$.

Theorem 2.17. *[1] A* **NTS** *(X,Γ)is neutrosophic connected if and only if it has no proper neutrosophic clopen (neutrosophic closed and neutrosophic open) set.*

Example 2.18. Let X = {a,b} be universe of discourse and the **NS**s U, A and B on X are defined as follows:

$$U = \{\langle a, 0.4, 0.3, 0.8\rangle, \langle b, 0.3, 04, 0.9\rangle\}$$
$$V = \{\langle a, 0.6, 0.5, 0.5\rangle, \langle b, 0.5, 0.7, 0.8\rangle\}$$

Let Γ = {$\tilde{\Phi}$, $\tilde{X}$, U, V } be a neutrosophic topology on X, then **NTS** (X,Γ) is neutrosophic connected

Definition 2.19. [1] Consider that f is a mapping from X to Y.

(a) Let $A \in \mathbf{N}(\mathbf{X})$ with membership function $\varrho_A(x)$, indeterminacy function $\varpi_A(x)$ and non-membership function $\sigma_A(x)$. The image of A under f, written as f(A), is a neutrosophic set of Y whose membership function, indeterminacy function and non-membership function are defined as

$$\varrho_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\varrho_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$\varpi_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\varpi_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & if f^{-1}(y) = \phi, \end{cases}$$

$$\sigma_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\sigma_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & if f^{-1}(y) = \phi, \end{cases}$$

$\forall\ , y \in Y$. Where $f^{-1}(y) = \{x : f(x) = y\}$.

(b) Let $B \in \mathbf{N}(\mathbf{Y})$ with membership function $\varrho_B(y)$, indeterminacy function $\varpi_B(y)$ and non-membership function $\sigma_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$ is a neutrosophic set of X whose membership function, indeterminacy function and non-membership function are respectively defined as:

$$\varrho_{f^{-1}(B)}(x) = \varrho_B(f(x)),$$
$$\varpi_{f^{-1}(B)}(x) = \varpi_B(f(x)),$$
$$\sigma_{f^{-1}(B)}(x) = \sigma_B(f(x)).$$

$\forall\ x \in X$.

Definition 2.20. [14] A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is said to be neutrosophic continuous if $f^{-1}(G) \in \Gamma$ for every **NS** $G \in \vartheta$.

Example 2.21. Let $X = \{a, b\}$, $Y = \{p, q\}$ and the neutrosophic sets U , V and W are defined as follows:

$$U = \{\langle a, 0.3, 0.5, 0.6\rangle, \langle b, 0.4, 0, 5, 0.6\rangle\}$$

$$V = \{\langle a, 0.4, 0.6, 0.5\rangle, \langle b, 0.5, 0.7, 0.4\rangle\}$$

$$W = \{\langle p, 0.3, 0.5, 0.6\rangle, \langle q, 0.4, 0.5, 0.6\rangle\}$$

Let $\Gamma = \{\tilde{\Phi},\ \tilde{X},$ U, V $\}$ and $\vartheta = \{\tilde{\Phi},\ \tilde{Y},$ W $\}$ be neutrosophic topologies on X and Y respectively. Then the mapping $f : (X, \Gamma) \to (Y, \vartheta)$ defined by f(a)= p and f(b)= q is neutrosophic continuous.

3. **Connectedness between neutrosophic sets**

Definition 3.1. A **NTS** (X ,Γ) is said to be neutrosophic connected between **NS**s A and B if there is no neutrosophic clopen set F in X such that A ⊂ F and F$\tilde{q}$B.

Theorem 3.2. *A* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B if and only if there is no neutrosophic clopen set F in X such that* $A \subset F \subset B^c$.

Proof. Follows from Definition 3.1. □

Example 3.3. Let X = {a,b} be universe of discourse and the **NS**s U, A and B on X are defined as follows:

$$U = \{\langle a, 0.4, 0.5, 0.2\rangle, \langle b, 0.6, 04, 0.7\rangle\}$$

$$A = \{\langle a, 0.6, 0.5, 0.5\rangle, \langle b, 0.5, 04, 0.8\rangle\}$$

$$B = \{\langle a, 0.4, 0.5, 0.7\rangle, \langle b, 0.5, 04, 0.9\rangle\}$$

Let $\Gamma = \{\tilde{\Phi}\ ,\ \tilde{X},$ U $\}$ be a neutrosophic topology on X, then **NTS** (X,Γ) is neutrosophic connected between the **NS**s A and B, but A$\tilde{q}B$.

Theorem 3.4. *If a* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B, then* $A \neq \tilde{\Phi} \neq B$.

Proof. If any **NS** A = $\tilde{\Phi}$ then A is a neutrosophic clopen set over X such that A ⊂ B and A$\tilde{q}$B and hence (X,Γ) can not be neutrosophic connected between **NS**s A and B, which is a contradiction. □

Theorem 3.5. *If a* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B and if* $A \subset C$ *and* $B \subset D$ *then (X,Γ) is neutrosophic connected between* **NS***s C and D.*

Proof. Suppose **NTS** (X,Γ) is not neutrosophic connected between **NS**s C and D then there is a neutrosophic clopen set F over X such that C ⊂ F and F$\tilde{q}$D. Clearly A ⊂ F. Now we claim that F$\tilde{q}$B. If FqB then FqD a contradiction. Consequently, (X,Γ) is not neutrosophic connected between **NS**s A and B. □

Theorem 3.6. *A* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B if and only if (X,Γ) is neutrosophic connected between* **NS***s Cl(A) and Cl(B).*

Proof. Necessity : Follows from Theorem 3.5.

Sufficiency : Suppose **NTS** (X,Γ) is not neutrosophic connected between **NS**s A and B, then there exists neutrosophic clopen set F in X such that A ⊂ F and F $\tilde{q}$ B. Since F is neutrosophic closed, Cl(A) ⊂ Cl(F) = F. Clearly, by Definition 2.10, F $\tilde{q}$ B ⇔ F ⊂ B^c. Therefore F = Int(F) ⊂ Int(B^c) = $(Cl(B))^c$. Hence, F $\tilde{q}$ Cl(B) and (X,Γ) is not neutrosophic connected between **NS**s Cl(A) and Cl(B). □

Theorem 3.7. *If A and B are two* **NS***s in a* **NTS** *(X,Γ) and A q B, then (X,Γ) is neutrosophic connected between A and B.*

Proof. If F is any neutrosophic clopen set over X such that A ⊂ F, then AqB ⇒ FqB. □

Remark 3.8. The converse of Theorem 3.7 need not be true . For **NTS** (X,Γ) of Example 3.3 is neutrosophic connected between A and B but $A\tilde{q}B$.

Theorem 3.9. *If a* **NTS** *(X,Γ) is neither neutrosophic connected between F and* F_0 *nor neutrosophic connected between F and* F_1 *then it is not neutrosophic connected between F and* $F_0 \cup F_1$.

Proof. Since a **NTS** (X,Γ) is not neutrosophic connected between F and F_0, there is a neutrosophic clopen set G_0 in X such that $F \subset G_0$ and $G_0\tilde{q}F_0$. Also since (X,Γ) is not neutrosophic connected between F and F_1 there exists a neutrosophic clopen set G_1 in X such that F ⊂ G_1 and $G_1\tilde{q}F_1$. Put G = $G_0 \cap G_1$. Since any intersection of neutrosophic closed sets is neutrosophic closed, G is neutrosophic closed. Again intersection of finite family of neutrosophic open sets is neutrosophic open, G is neutrosophic open. Therefore G is neutrosophic clopen set over X such that $F \subset G$ and $G\tilde{q}F_0 \cup F_1$. If $GqF_0 \cup F_1$,then GqF_0 or GqF_1 a contradiction. Hence, (X,Γ) is not neutrosophic connected between F and $F_0 \cup F_1$. □

Theorem 3.10. *A* **NTS** *(X,Γ) is neutrosophic connected if and only if it is neutrosophic connected between every pair of its nonempty* **NS***s.*

Proof. Let F and G be a pair of nonempty **NS**s in X. Suppose (X,Γ) is not neutrosophic connected between F and G. Then there is a neutrosophic clopen set H in X such that $F \subset H$ and $G\tilde{q}G$. Since F and G are nonempty it follows that H is a nonempty neutrosophic proper clopen set in X. Hence, (X,Γ) is not neutrosophic connected.

Conversely, suppose that (X,Γ) is not neutrosophic connected. Then there exists a nonempty proper **NS** H in X which is both neutrosophic open and neutrosophic closed. Consequently, (X,Γ) is not neutrosophic connected between H and H^c. Thus, (X,Γ) is not neutrosophic connected between arbitrary pair of its nonempty **NS**s. □

Remark 3.11. If a **NTS** (X,Γ) is neutrosophic connected between a pair of its **NS**s, then it is not necessary that it is neutrosophic connected between each pair of its **NS**s and hence it is not necessarily neutrosophic connected.

Example 3.12. Let X = {a,b} be an universe set, and the neutrosophic sets U,V,A,B,C,D,E over X are defined as follows:

$$U = \{\langle a, 0.2, 0.5, 0.7\rangle, \langle b, 0.3, 0, 5, 0.6\rangle\}$$

$$V = \{\langle a, 0.7, 0.5, 0.2\rangle, \langle b, 0.6, 0.5, 0.3\rangle\}$$

$$A = \{\langle a, 0.4, 0.5, 0.5\rangle, \langle b, 0.2, 0.5, 0.6\rangle\}$$

$$B = \{\langle a, 0.5, 0.5, 0.4\rangle, \langle b, 0.4, 0.5, 0.5\rangle\}$$

$$C = \{\langle a, 0.1, 0.5, 0.8\rangle, \langle b, 0.2, 0.5, 0.7\rangle\}$$

$$D = \{\langle a, 0.3, 0.5, 0.4\rangle, \langle b, 0.3, 0.5, 0.4\rangle\}$$

Let Γ = {$\tilde{\Phi}$, $\tilde{X}$, U, V} be a neutrosophic topology over X. Then the **NTS** (X, Γ) is neutrosophic connected between the **NS**s A and B but it is not neutrosophic connected between C and D. Also the **NTS** (X, Γ) is not neutrosophic connected.

Theorem 3.13. *Let (Y,Γ_Y,E) be a neutrosophic subspace of a* **NTS** *(X,Γ). If (Y,Γ_Y) is neutrosophic connected between the* **NS***s F and G over Y, then* **NTS** *(X,Γ) is neutrosophic connected between F and G.*

Proof. Suppose **NTS** (X,Γ) is not neutrosophic connected between **NS**s F and G,then there is neutrosophic clopen set H in X such that $F \subset H$ and $H\tilde{q}G$. Then $Y \cap H$ is neutrosophic clopen set in Y such that $F \subset H \cap Y$ and $H \cap Y\ \tilde{q}G$. Consequently, (Y,Γ_Y) is not neutrosophic connected between F and G, a contradiction. □

Theorem 3.14. *Let (Y,Γ_Y) be a neutrosophic clopen subspace of a* **NTS** *(X,Γ) and F, G are* **NS***s of Y. If (X,Γ) is neutrosophic connected between F and G then (Y,Γ_Y) is neutrosophic connected between F and G.*

Proof. Suppose (Y,Γ_Y) is not neutrosophic connected between F and G. Then there is neutrosophic clopen set H of (Y,Γ_Y) such that F ⊂ H and $H\tilde{q}G$. Since, (Y,Γ_Y) is neutrosophic clopen in (X,Γ), by Lemma 2.14 and Lemma 2.15 H is neutrosophic clopen set of (X,Γ) such that $F \subset H$ and $H\tilde{q}G$. Consequently, (X,Γ) is not neutrosophic connected between F and G, a contradiction. □

4. Neutrosophic set-connected mappings

Definition 4.1. A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is said to be neutrosophic set-connected provided, if **NTS**(X, Γ) is neutrosophic connected between **NS**s F and G then neutrosophic subspace $(f(X), \vartheta_{f(X)})$ is neutrosophic connected between $f(F)$ and $f(G)$ with respect to neutrosophic relative topology.

Theorem 4.2. *A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is neutrosophic set-connected if and only if $f^{-1}(F, K)$ is a neutrosophic clopen set over X for any neutrosophic clopen set H of ($f(X)$,$\vartheta_{f(X)}$).*

Proof. Necessity : Let f be neutrosophic set-connected and H is a neutrosophic clopen set in $(f(X), \vartheta_{f(X)})$. Suppose $f^{-1}(H)$ is not neutrosophic clopen in (X ,Γ). Then (X ,Γ) is neutrosophic connected between $f^{-1}(H)$ and $(f^{-1}(H))^c$. Therefore, $(f(X), \vartheta_{f(X)})$ is neutrosophic connected between $f(f^{-1}(H))$ and $f((f^{-1}(H))^c)$ because f is neutrosophic set-connected. But, $f(f^{-1}(H)) = H \cap (f(X) = H$ and $f((f^{-1}(H))^c = H^c$ imply that H is not neutrosophic clopen in $(f(X), \vartheta_{f(X)})$, a contradiction. Hence, $f^{-1}(H)$ is neutrosophic clopen in (X ,Γ).

Sufficiency : Let (X ,Γ) be neutrosophic connected between F and G. If $(f(X), \vartheta_{f(X)})$ is not neutrosophic-connected between $f(F)$ and $f(G)$ then there exists a neutrosophic clopen set H in $(f(X), \vartheta_{f(X)})$ such that $f(F) \subset H \subset (f(G))^c$. By hypothesis, $f^{-1}(H)$ is neutrosophic clopen set over X and $F \subset f^{-1}(H) \subset G^c$. Therefore, (X ,Γ) is not neutrosophic connected between F and G. This is a contradiction. Hence, f is neutrosophic set-connected. □

Theorem 4.3. *Every neutrosophic continuous mapping f : (X ,Γ) → (Y,ϑ) is a neutrosophic set-connected.*

Proof. It is obvious. □

Remark 4.4. The converse of Theorem 4.3 need not be true.

Example 4.5. Let $X = \{a, b\}$ and $Y = \{p, q\}$. The neutrosophic sets U and V are defined as follows:

$$U = \{\langle a, 0.3, 0.5, 0.6\rangle, \langle b, 0.4, 0, 5, 0.5\rangle\}$$

$$V = \{\langle p, 0.4, 0.5, 0.6\rangle, \langle q, 0.5, 0.5, 0.4\rangle\}$$

Let $\Gamma = \{\tilde{\Phi},\tilde{X}$, U } and $\vartheta = \{\tilde{\Phi}$, $\tilde{Y}$, V } are neutrosophic topologies on X and Y respectively. Then the mapping $f : (X, \Gamma) \to (Y, \vartheta)$ defined by f(a)=p and f(b)= q is neutrosophic set-connected but it is not neutrosophic continuous.

Theorem 4.6. *Every mapping $f : (X, \Gamma) \to (Y, \vartheta)$ such that $(f(X), \vartheta_{f(X)})$ is neutrosophic connected is neutrosophic set-connected mapping.*

Proof. Let $(f(X), \vartheta_{f(X)})$ be neutrosophic connected. Then by Theorem 2.17, no nonempty proper **NS** of $(f(X), \vartheta_{f(X)})$ which is neutrosophic clopen. Hence, f is neutrosophic set-connected. □

Theorem 4.7. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a neutrosophic set-connected mapping. If (X ,Γ) is neutrosophic connected , then $(f(X), \vartheta_{f(X)})$ is a neutrosophic connected sub space of (Y,ϑ).*

Proof. Suppose $(f(X), \vartheta_{f(X)})$ is not neutrosophic connected in (Y,ϑ). Then by Theorem 2.17, there is a nonempty proper neutrosophic clopen set G of $(f(X), \vartheta_{f(X)})$. Since f is neutrosophic set-connected, $f^{-1}(G)$ is a nonempty proper neutrosophic clopen set over X. Consequently, (X, Γ) is not neutrosophic connected. □

Theorem 4.8. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a neutrosophic set-connected mapping and F be a neutrosophic set over X such that $f(F)$ is neutrosophic clopen set of $(f(X), \vartheta_{f(X)})$. Then $f/F : F \to (Y, \vartheta)$ is a neutrosophic set-connected mapping.*

Proof: Let F be neutrosophic connected between G and H. Then by Theorem 3.13, (X, Γ) is neutrosophic connected between G and H. Since f is neutrosophic set-connected, $(f(X), \vartheta_{f(X)})$ is neutrosophic connected between $f(G)$ and $f(H)$. Now, since $f(F)$ is neutrosophic clopen set of $(f(X), \vartheta_{f(X)})$, it follows by Theorem 3.14 that $f(F)$ is neutrosophic connected between $f(G)$ and $f(H)$.

Theorem 4.9. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a neutrosophic set-connected surjection. Then every neutrosophic clopen set H of (Y, ϑ) is neutrosophic connected if $f^{-1}(H)$ is neutrosophic connected in (X ,Γ). In particular, if (X ,Γ) is neutrosophic connected then (Y,ϑ) is neutrosophic connected.*

Proof. By Theorem 4.8 $f/f^{-1}(H) : f^{-1}(H) \rightarrow (Y, \vartheta)$ is neutrosophic set-connected. And, since $f^{-1}(H)$ is neutrosophic connected by Theorem 4.7, $f/f^{-1}(H)[f^{-1}(H)] = H$ is neutrosophic connected. □

Theorem 4.10. *Let $f : (X, \Gamma) \rightarrow (Y, \vartheta)$ be a surjective neutrosophic set-connected and $g : (Y, \vartheta) \rightarrow (Z, \eta)$ a neutrosophic set-connected mapping. Then $gof : (X, \Gamma) \rightarrow (Z, \eta)$ is neutrosophic set-connected.*

Proof. Let H be a neutrosophic clopen set in g(Y). Then $g^{-1}(H)$ is neutrosophic clopen over Y $= f(X)$ and so f^{-1} (g^{-1}(H)) is neutrosophic clopen in (X ,Γ). Now (gof)(X) = g (Y) and $(gof)^{-1}$ (H) = f^{-1} (g^{-1}(H)) is neutrosophic clopen in (X ,Γ). Hence, gof is neutrosophic set connected. □

Definition 4.11. A mapping $f : (X, \Gamma) \rightarrow (Y, \vartheta)$ is said to be neutrosophic weakly continuous if for each neutrosophic point $x_{r,t,s} \in X$ and each neutrosophic open set G over Y containing $f(x_{r,t,s})$, there exists a neutrosophic open set F over X containing $x_{r,t,s}$ such that f(F) ⊂ Cl(G).

Theorem 4.12. *A mapping $f : (X, \Gamma) \rightarrow (Y, \vartheta)$ is neutrosophic weakly continuous if and only if for each neutrosophic open set H in Y, $f^{-1}(H) \subset Int(f^{-1}(Cl(H)))$.*

Proof. Necessity : Let H be a neutrosophic open set over Y and $x_{r,t,s} \in f^{-1}(H)$, then $f(x_{r,t,s}) \in H$. Therefore, there exists a neutrosophic open set F in X such that $x_{r,t,s} \in F$ and $f(F) \subset Cl(H)$. Hence, $x_{r,t,s} \in F \subset f^{-1}(Cl(H))$ and $x_{r,t,s} \in Int(f^{-1}(Cl(H)))$ since F is neutrosophic open.

Sufficiency : Let $x_{r,t,s} \in X$ and $f(x_{r,t,s}) \in H$. Then $x_{r,t,s} \in f^{-1}(H) \subset Int(f^{-1}(Cl(H)))$. Let $F = Int(f^{-1}(Cl(H)))$ then F is neutrosophic open set containing $x_{r,t,s}$ and $f(F) = f(Int(f^{-1}(Cl(H)))) \subset f(f^{-1}(Cl(H))) \subset Cl(H)$. Hence,f is neutrosophic weakly continuous. □

Theorem 4.13. *If a* **NTS** *space (X ,Γ) is neutrosophic connected and $f : (X, \Gamma) \rightarrow (Y, \vartheta)$ is a neutrosophic weakly continuous surjection, then (Y,ϑ) is neutrosophic connected.*

Proof. Follows from Theorem 4.12 and Theorem 2.17. □

Theorem 4.14. *A mapping $f : (X, \Gamma) \rightarrow (Y, \vartheta)$ is neutrosophic weakly continuous, then $Cl(f^{-1}(H)) \subset (f^{-1}(Cl(H))$ for each neutrosophic open set H over Y*

Proof. Suppose there exists a neutrosophic point $x_{r,t,s} \in Cl(f^{-1}(H))$ but $x_{r,t,s} \notin f^{-1}(Cl(H))$. Then $f(x_{r,t,s}) \notin (Cl(H))$. Therefore , there exists a neutrosophic open q-neighbourhood G of $f(x_{r,t,s})$ such that $G\tilde{q}H$. Since H is neutrosophic open set in Y, we have $H\tilde{q}Cl(G)$. Again, f is neutrosophic weakly continuous, there exists a neutrosophic open set F in X containing $x_{r,t,s}$ such that $f(F) \subset Cl(G)$. Thus, we obtain $f(F)\tilde{q}H$. On the other hand, since $x_{r,t,s} \in Cl(f^{-1}(H))$, we have $Fqf^{-1}(H)$ and hence, $f(F)qH$. Thus we have a contradiction. Hence $Cl(f^{-1}(H)) \subset (f^{-1}(Cl(H))$. □

Theorem 4.15. *If a neutrosophic surjection f : (X ,Γ) → (Y,ϑ) is neutrosophic weakly continuous , then f is neutrosophic set-connected.*

Proof. Let H be any neutrosophic clopen set over Y. Since H is neutrosophic closed, we have Cl(H) = H. Thus, by Theorem 4.12, we obtain $f^{-1}(H) \subset Int(f^{-1}(H))$. This shows that $f^{-1}(H)$ is neutrosophic open set in X. Moreover , by Theorem 4.14, we obtain $Cl(f^{-1}(H)) \subset f^{-1}(H)$. This shows that $f^{-1}(H)$ is a neutrosophic closed set in X. Since f is neutrosophic surjection, by Theorem 4.2, we obtain that f is a neutrosophic set-connected mapping. □

Remark 4.16. The converse of Theorem 4.15 is not true.

Example 4.17. Let $X = \{a, b\}$ and $Y = \{p, q\}$. The neutrosophic sets U and V are defined as follows:

$$U = \{\langle a, 0.3, 0.5, 0.6\rangle, \langle b, 0.4, 0, 5, 0.6\rangle\}$$

$$V = \{\langle p, 0.4, 0.5, 0.5\rangle, \langle q, 0.3, 0.5, 0.5\rangle\}$$

Let $\Gamma = \{\tilde{\Phi}, \tilde{X}, U\}$ and $\vartheta = \{\tilde{\Phi}, \tilde{Y}, V\}$ are neutrosophic topologies on X and Y respectively. Consider a mapping $f : (X, \Gamma) \to (Y, \vartheta)$ defined by f(a)= p and f(b)= q. Clearly,$\tilde{\Phi}$, $\tilde{Y}$ are the only neutrosophic clopen sets of Y and their inverse images Φ, $\tilde{Y}$ are neutrosophi clopen sets in X. Hence By Theorem 4.2 f is neutrosophic set-connected . But it is not neutrosophic weakly continuous.

Definition 4.18. A **NTS** (X, Γ) is said to be neutrosophic extremally disconnected if the closure of every neutrosophic open set of X is neutrosophic open in X.

Theorem 4.19. *Let (Y,ϑ) be a neutrosophic extremally disconnected space . If a mapping* $f : (X, \Gamma) \to (Y, \vartheta)$ *is neutrosophic set-connected, then f is neutrosophic weakly continuous.*

Proof. Let $x_{r,t,s}$ be a neutrosophic point of X and G be any neutrosophic open set in Y containing $f(x_{r,t,s})$. Since (Y,ϑ,) is neutrosophic extremally disconnected, Cl(G) is a neutrosophic clopen set in Y. Thus $Cl(G) \cap f(X)$ is neutrosophic clopen set in the neutrosophic subspace $(f(X), \vartheta_{f(X)})$. Put $f^{-1}(Cl(G) \cap f(X)) = F$. Then, since f is neutrosophic set-connected, it follows from Theorem 4.2 that F is neutrosophic clopen set over X. Therefore, F is a neutrosophic open set containing $x_{r,t,s}$ in X such that $f(F) \subset Cl(G)$. This implies that f is neutrosophic weakly continuous. □

Theorem 4.20. *Let (Y, ϑ) be a neutrosophic extremally disconnected space. A neutrosophic surjection $f : (X, \Gamma) \to (Y, \vartheta)$ is neutrosophic set-connected if and only if f is neutrosophic weakly continuous.*

Proof. It follows from Theorem 4.15 and Theorem 4.19. □

5. Conclusions

Connectedness is an important and major area of topology and it can give many relationships between other scientific areas and mathematical models. The notion of connectedness captures the idea of hanging-togetherness of image elements in an object by assigning a strength of connectedness to every possible path between every possible pair of image elements. This paper, introduces the notion of neutrosophic connectedness between neutrosophic sets in neutrosophic topological spaces. It is shown that a neutrosophic topological space is neutrosophic connected if and only if it is neutrosophic connected between every pair of its nonempty neutrosophic sets. Further two new classes of neutrosophic mappings called neutrosophic set connected and neutrosopic weakly continuous mappings have been introduced . It is shown that the class of neutrosophic set connected (respt. neutrosophic weakly continuous) mappings properly contains the class of all neutrosophic continuous mappings. Several properties and characterizations of neutrosophic set connected and neutrosophic weakly continuous mappings have been studied. Hope that the concepts and results established in this paper will help researcher to enhance and promote the further study on neutrosophic topology to carry out a general framework for the development of information systems.

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On Diminishing Fuzzy Neutrosophic Topological Spaces over Nano Topology

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Abstract. As an extension and also, a necessity to compute the indeterminacy, the concept of diminishing fuzzy sets is protracted to introduce the conception of diminishing fuzzy (single-valued) neutrosophic sets in the standard unit interval. In addition, the axioms of neutrosophic topological spaces are extended to configure the collection, diminishing fuzzy neutrosophic topology, and also, the basic theorems of nano topology is incorporated to the case of diminishing fuzzy neutrosophic set in terms of the approximation space and the boundary region by acquiring the same operators for the quantities, non-membership and indeterminacy are also discussed. Moreover, a nano continuous function from one diminishing fuzzy neutrosophic nano topological space to another is defined with an example. Furthermore, the approximation space for the neighborhood of the diminishing cells in terms of the idea of pixel neighborhood by means of the diminishing fuzzy neutrosophic membership function and also, approximation space for the NFA of the layer word are defined with an illustration.

Keywords: Diminishing fuzzy sets, Fuzzy neutrosophic topology, Nano continuous, Neighborhood, NFA.

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1. Introduction

The imprecision in nature, which is formulated as fuzzy set theory that contemplates both vagueness and uncertainty in terms of grades of membership by addressing the classical notion of general set theory, and the domains of mathematics such as general topology, algebra, operations research, image processing, and so on, had been fuzzified to model various real life applications in a different aspect [21]. Intuitionistic fuzzy set theory generalizes both fuzzy sets and classical sets which consist of membership functions along with non-membership functions [9]. As an extension of intuitionistic fuzzy set theory, neutrosophic set theory is coined employing truth-membership, falsity-membership and indeterminacy-membership constructed over]-0, 1 +[instead of [0, 1] to handle the impreciseness, incompleteness, uncertainty, and indeterminacy that arise in the real-life scenario, which was introduced in 1999 [19]. In 2010, the set-theoretic operations for the neutrosophic sets are defined over the standard unit interval called single valued neutrosophic sets (fuzzy neutrosophic sets) [4] and some of the researchers gave prominence to this study in other domains [1, 16, 18].

In 2011, an infinite array with diminishing cells is termed and denoted by D_r in which each layer is identified with the help of binary strings [7] and application of D_r array can be found in [15]. In 2012, the term nano topology based on the rough set theory concept in terms of lower approximation and upper approximation which represents interior and closure operations of a topology respectively was initiated [10, 11] and the applications in real life scenarios [2, 6, 12, 14, 20]. In 2018, a new hybrid by combining both nano topology and neutrosophic topology was called Neutrosophic nano topology based on the approximation space and the boundary region over the neutrosophic membership grades [13]. In 2020, the D_r array is generalized to the case of diminishing fuzzy sets (DFS) by defining a function based on the positions of the D_r cells and also, proved some of the fundamental theorems of fuzzy topological space for DFS [8].

This paper organizes its objectives as follows: Section 2 presents the basic definitions of diminishing cells with infinite array, single valued neutrosophic sets, fuzzy neutrosophic topology, nano topology and nano neutrosophic topology and some of the basic operations of the fuzzy neutrosophic sets. Section 3 discusses the basic theorems, lemma and proposition of nano neutrosophic topology and neutrosophic topology for diminishing fuzzy neutrosophic sets as well as a nano continuity on diminishing fuzzy neutrosophic topological space and the approximation space for the diminishing cells and the NFA for the layer word. Section 4 ends with conclusion remarks.

2. **Preliminaries**

This section presents the construction of diminishing cells with infinite array, the definitions of single valued neutrosophic set with its operations, fuzzy neutrosophic topology, nano topology and nano neutrosophic topology.

The diminishing cells in an infinite array is constructed by reducing the length and the breadth of the rectangular array recursively by a common ratio r from the source cell in the Euclidean plane along row wise and column wise respectively where $r = \frac{1}{n}, n \geq 2$ and $n \in \mathbb{N}$ and this special type of array is denoted by D_r [8]. A non-deterministic finite accepter (NFA) $\mathcal{M} = \{S, \sigma_n, \Gamma, q_0, q_n\}$ where S is the finite set of internal states, Σ_n is an alphabet, $\Gamma : S \times \Sigma_n^* \to 2^S$ [17]. Let $\mathcal{U} = \{u\}$ be a domain of discourse with a collection of $u \in \mathcal{U}$. A fuzzy set A in $\mathcal{U}$ is defined by $\mu_A : \mathcal{U} \to [0, 1]$, with the grades of membership $\mu_A(u)$ for each $u \in \mathcal{U}$ [21].

Let $U \neq \emptyset$ be a domain of finite set and R be an indiscernible relation over U. Then,

U is divided into disjoint equivalence classes and members belonging to the same equivalence class are indiscernible with one another. The pair (U, R) is called the approximation space corresponding to $X \subset U$. Then, the collection $\mathcal{T}_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$ forms a nano topology, where the lower approximation $L_R(X)$ is the union of all equivalence classes $R(x)$ such that $R(x) \subseteq X$, the upper approximation $U_R(X)$ is the union of all equivalence classes intersecting with X is non empty and the boundary region $B_R(X)$ is the difference between $U_R(X)$ and $L_R(X)$ of X and its complement is called the dual nano topology, denoted by $[\mathcal{T}_R(X)]^c$. Every member belonging to $\mathcal{T}_R(X)$ is known as the nano-open sets whereas the members belonging to $[\mathcal{T}_R(X)]^c$ are referred to as nano closed sets in the nano topological space $(U, \mathcal{T}_R(X))$. The basis for $\mathcal{T}_R(X)$ with respect to X is given by $\beta_R(X) = \{U, L_R(X), B_R(X)\}$ [10]. Let $(U, \mathcal{T}_R(X))$ and $(V, \mathcal{T}_R(Y))$ be two nano topological spaces with respect to the subsets of X and Y respectively. Then, a function $f : (U, \mathcal{T}_R(X)) \to (V, \mathcal{T}_R(Y))$ is nano continuous on U if the inverse image of every nano-open set in V is nano-open in U [11]. Let $\mathcal{G}$ be a graph with a sub-graph $\mathcal{H}$ and $\mathcal{N}(p)$ be a neighborhood of the point of $\mathcal{G}$. Then, the lower approximation $L_R(\mathcal{H})$ on points of $\mathcal{H}$ is defined as the union of all neighborhood points of $\mathcal{G}$ which is contained in $\mathcal{H}$, the upper approximation $U_R(\mathcal{H})$ is the set of all neighborhood points of $\mathcal{H}$ and the boundary region is the difference between $L_R(\mathcal{H})$ and $U_R(\mathcal{H})$ [14]. A single valued neutrosophic set (SVNS) also known as fuzzy neutrosophic set A in $\mathcal{U}$ is defined in terms of the following components: truth-membership function μ_{T_A}, indeterminacy-membership function μ_{I_A} and falsity-membership function μ_{F_A} denoted by $(\mu_{T_A}, \mu_{I_A}, \mu_{F_A})$ where $\mu_{T_A}(x), \mu_{I_A}(x), \mu_{F_A}(x) \in [0, 1]$ for each point $x \in X$. The operations on SVNSs are given by: Complement: $\mu_c(A)(x) = (\mu_{F_A}(x), 1 - \mu_{I_A}(x), \mu_{T_A}(x))$, Union: $\mu_{A\cup B}(x) = (\vee(\mu_{T_A}(x), \mu_{T_B}(x)), \vee(\mu_{I_A}(x), \mu_{I_B}(x)), \wedge(\mu_{F_A}(x), \mu_{F_B}(x)))$, Intersection: $\mu_{A\cap B}(x) = (\wedge(\mu_{T_A}(x), \mu_{T_B}(x)), \wedge(\mu_{I_A}(x), \mu_{I_B}(x)), \vee(\mu_{F_A}(x), \mu_{F_B}(x)))$, Difference: $\mu_{A\ B}(x) = (\wedge(\mu_{T_A}(x), \mu_{F_B}(x)), \wedge(\mu_{I_A}(x), 1 - \mu_{I_B}(x)), \vee(\mu_{F_A}(x), \mu_{T_B}(x)))$, Containment: $\mu_{A\subset B}(x) = (\mu_{T_A}(x) \leq \mu_{T_B}(x), \mu_{F_A}(x) \leq \mu_{F_B}(x), \mu_{F_A}(x) \geq \mu_{F_B}(x))$ for each $x \in X$, where A and B are the two single valued neutrosophic sets [4]. A neutrosophic topology (NT for short) on a $X \neq \emptyset$ is a family $\mathcal{T}_N$ of neutrosophic subsets in X satisfying the following axioms: $(NT_1) 0_N, 1_N \in \mathcal{T}_N$, $(NT_2) G_1 \cap G_2 \in \mathcal{T}_N$, for any $G_1, G_2 \in \mathcal{T}_N$, $(NT_3) \cup G_i \in \mathcal{T}_N \forall \{G_i : i \in J\} \subset \mathcal{T}_N$ and we call $(X, \mathcal{T}_N)$ is called a neutrosophic topological space (NTS for short) and its elements are known as neutrosophic open set (NOS for short) in X. A neutrosophic set F is closed $\Leftrightarrow$ its complement C(F) is neutrosophic open [1].

Let $U \neq \emptyset$ be a set with an equivalence relation R on U. Let F be a neutrosophic set in U with the membership function μ_F, the indeterminacy function σ_F and the non-membership function $\sqsubseteq_F$. The neutrosophic nano lower approximation, neutrosophic nano upper approximation and neutrosophic nano boundary of F in the approximation space (U, R) denoted by $\underline{N}(F), \overline{N}(F)$

and $BN(F)$ are respectively defined as follows: $\underline{N}(F) = \{< x, \mu_{\underline{R}(A)}(x), \sigma_{\underline{R}(A)}(x), \sqsubseteq_{\underline{R}(A)}(x) > |y \in [x]_R, x \in U\}$, $\overline{N}(F) = \{< x, \mu_{\overline{R}(A)}(x), \sigma_{\overline{R}(A)}(x), \sqsubseteq_{\overline{R}(A)}(x) > |y \in [x]_R, x \in U\}$, $BN(F) = \overline{N}(F) - \underline{N}(F)$ where, $\mu_{\underline{R}(A)}(x) = \wedge_{y\in[x]_R}\mu_A(y)$, $\sigma_{\underline{R}(A)}(x) = \wedge_{y\in[x]_R}\sigma_A(y)$, $\sqsubseteq_{\underline{R}(A)}(x) = \vee_{y\in[x]_R}\sqsubseteq_A(y)$, $\mu_{\overline{R}(A)}(x) = \vee_{y\in[x]_R}\mu_A(y)$, $\sigma_{\overline{R}(A)}(x) = \vee_{y\in[x]_R}\sigma_A(y)$, $\sqsubseteq_{\overline{R}(A)}(x) = \wedge_{y\in[x]_R}\sqsubseteq_A(y)$ and if the collection $\mathcal{T}_{N_R}(F) = \{0_N, 1_N, \underline{N}(F), \overline{N}(F), BN(F)\}$ forms a topology then it is said to be a neutrosophic nano topology. We call $(U, \mathcal{T}_{N_R}(F))$ as the neutrosophic nano topological space. Every element in $\mathcal{T}_{N_R}(F)$ is called a neutrosophic nano open set [13].

3. Approximation Space and Diminishing Fuzzy Neutrosophic Sets (DFNS)

In this section, the concept of pixel neighborhood and approximation space induced by neighborhood of a vertex in graph thoery are incorporated in the conceptualization of the diminishing cells in an infinite array and also, the pictorial representation of diminishing fuzzy neutrosophic membership grades for the diminishing cells is presented and the approximation space for the NFA of the layer word is achieved. Furthermore, the definition of diminishing fuzzy set is presented and based on that, diminishing fuzzy neutrosophic set (DFNS) is introduced. Some basic theorems, lemma and propositions of nano neutrosophic topology and neutrosophic topology are extended in terms of diminishing fuzzy neutrosophic sets as well as the notion of nano continuity on diminishing fuzzy neutrosophic topological space (DFNTS), which is studied with an example.

Definition 3.1. Let D_r be a diminishing array and X_{N_r} be a sub-array of D_r array. Let Y_{N_r} be a sub-array of X_{N_r} and $N_{X_{N_r}}$ be a neighborhood of cells in X_{N_r}. Then, we define the following:

- The lower approximation of the neighborhood of the cells in X_{N_r} is defined as $L_{N_{Y_{N_r}}} : X_{N_r} \to X_{N_r}$ such that $L_{N_{Y_{N_r}}} = \bigcup_{(x,y)\in X_{N_r}}\{(x,y)|N_{X_{N_r}} \in N_{Y_{N_r}}\}$
- The upper approximation of the neighborhood of the cells in X_{N_r} is defined as $U_{N_{Y_{N_r}}} : X_{N_r} \to X_{N_r}$ such that $U_{N_{Y_{N_r}}} = \{N_{(x,y)}|(x,y) \in Y_{N_r}\}$
- The boundary region of the neighborhood of the cells in X_{N_r} is defined as $U_{N_{Y_{N_r}}} - L_{N_{Y_{N_r}}}$.

Example 3.2. Without loss of generality, Let $X_{N_r} = \{(2,3),(3,3),(4,3),(2,4),(2,5),(3,4),(4,4),(3,5),(4,5)\}$ be a sub-array of D_r array Then, the neighborhood cells of $(2,3)$ are $\{(3,3),(3,4),(2,4)\}$, $N_{(3,3)} = \{(4,3),(4,4),(3,4),(2,4),(2,3)\}$, $N_{(4,3)} = \{(4,4),(3,4),(3,3)\}$, $N_{(2,4)} = \{(2,3),(3,3),(3,4),(3,5),(2,5)\}$, $N_{(2,5)} = \{(2,4),(3,4),(3,5)\}$, $N_{(3,4)} = \{(2,3),(3,3),(4,3),(4,4),(4,5),(3,5),(2,5),(2,4)\}$, $N_{(4,4)} = \{(4,3),(3,3),(3,4),(3,5),(4,5)\}$, $N_{(3,5)} = \{(2,5),(2,4),(3,4),(4,4),(4,5)\}$, and $N_{(4,5)} = \{(4,4),(3,4),(3,5)\}$. Then, $L_{N_{Y_{N_r}}} = U_{N_{Y_{N_r}}} = Y_{N_r}$ which implies $B_{N_{Y_{N_r}}} = \emptyset$. This is true for all the sub-arrays of sub-array.

Remark 3.3. The lower and upper approximations for the sub-array are equal for every sub-arrays. Thus, the collection $\{X_{N_r}, \emptyset, L_{N_{Y_{N_r}}}\}$ forms a nano topological space for all the sub-arrays of sub-arrays in D_r.

Definition 3.4. The language on diminishing cells is defined as

$$\mathcal{D}_r^{\mathcal{L}}(X_{ij}^r) = \begin{cases} 0^i 1^{j-i}, & i < j \\ 1^j 0^{i-j}, & i > j \\ 1^j, & i = j \end{cases}$$

The right angle path from the cell $X_{k,1}$ to the cell $X_{1,k}$ along the cells $X_{k,2}, X_{k,3}, ...X_{k,k}, X_{k-1,k}, X_{k-2,k}, ...X_{2,k}, k \geq 2, k \in \mathbb{N}$ is called a layer. The concatenation of words on a layer is called a layer word. For example, the cells with words, $X_{21}^{10}, X_{22}^{11}, X_{12}^{01}$ form a layer word 101101.

Definition 3.5. Let D_r be a diminishing array and X_{N_r} be a sub-array of D_r array. $D_r^{L_n}$ be a n^{th} layer of the D_r array and Q'_r be a subset of Q_r, a finite set of internal states. Then, we define the approximation space for NFA as follows:

- The lower approximation of L_{Q_r} is defined as $L_{Q_r} : Q_r \to Q_r$ such that $L_{Q_r}(Q'_r) = \cup_{q_i \in Q_r}\{q_i | \Gamma(q_i, w) \cap Q'_r\}$,
- The lower approximation of U_{Q_r} is defined as $U_{Q_r} : Q_r \to Q_r$ such that $U_{Q_r}(Q'_r) = \cup_{q_i \in Q_r}\{q_i | \Gamma(q_i, w) \subseteq Q'_r \neq \emptyset\}$,
- The boundary region, $B_{Q_r}(Q'_r) = U_{Q_r}(Q'_r) - L_{Q_r}(Q'_r)$.

Example 3.6. Let X_{N_r} be a sub-array of D_r array with a layer 1 and then, the respective layer word is 101101. Let $\mathcal{M} = (\{q_0, q_1, q_2, q_3\}, \{0, 1\}, \Gamma, q_0, \{q_1, q_2, q_3\})$ with the transitions $\Gamma(q_0, 0) = \{q_1\}$, $\Gamma(q_0, 1) = \{q_0\}$, $\Gamma(q_1, 0) = \emptyset$, $\Gamma(q_1, 1) = \{q_2\}$, $\Gamma(q_2, 0) = \{q_3\}$, $\Gamma(q_2, 1) = \{q_2\}$, $\Gamma(q_3, 0) = \emptyset$, and $\Gamma(q_3, 1) = \{q_3\}$ and $Q'_r = \{q_1, q_3\}$

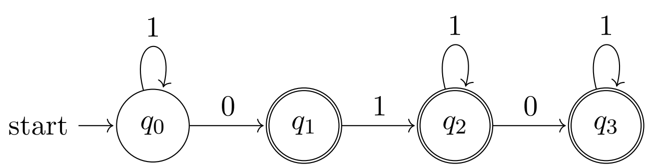

FIGURE 1. Non-deterministic finite automata for layer word with the layer 1

The approximation space of this NFA is obtained as follows

- $L_{Q_r}(Q'_r) = \{q_3\}$
- $U_{Q_r}(Q'_r) = \{q_0, q_2, q_3\}$
- $B_{Q_r}(Q'_r) = \{q_0, q_2\}$

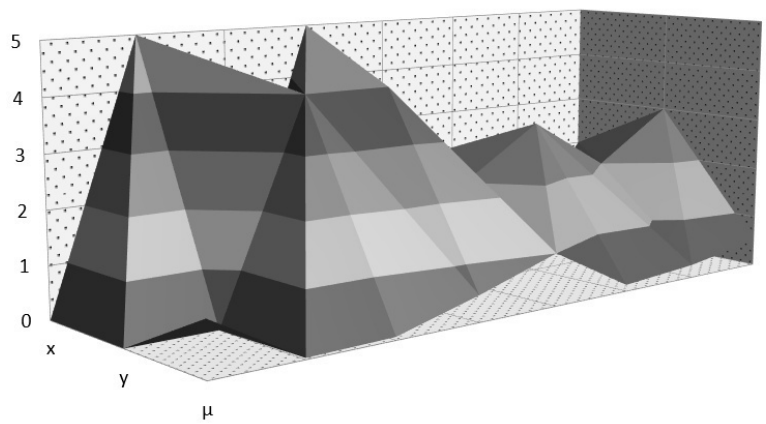

Figure 2. Surface plotting of diminishing fuzzy sets

Proposition 3.7. *Let D_r be a diminishing array and X_{N_r} be a sub-array of D_r array. $D_r^{L_n}$ be a n^{th} layer of the D_r array and Q'_r be a subset of Q_r, a finite set of internal states. Then,*

- $Q'_r \subset L_{Q_r}(Q'_r) \subset U_{Q_r}(Q'_r)$
- $B_{Q_r}(Q'_r) \subset U_{Q_r}(Q'_r) \subset Q_r$
- $L_{Q_r}(Q'_r) \subset B_{Q_r}(Q'_r) \subset U_{Q_r}(Q'_r)$

Remark 3.8. The collection $\mathcal{T}_r = \{\emptyset, Q_r, L_{Q_r}(Q'_r), U_{Q_r}(Q'_r), B_{Q_r}(Q'_r)\}$ forms a topology for the NFA of the layer word with respect to the finite subset of internal states Q'_r.

Definition 3.9. We define a function $\mu_{D_r} : D_r \to [0,1]$ by

$$\mu_{D_r}(x,y) = \begin{cases} 1 & x = y \\ \frac{1}{n^k} & x \neq y, k = |x-y| \end{cases}$$

where $n \in \mathbb{N}, (x,y) \in D_r$ and $n \geq 2$. We call this set as diminishing fuzzy set (DFS) and it is denoted by D_r. The class of all diminishing fuzzy sets (DFSs) is denoted by $\mathcal{D}_r$ which may be a denumerable set. The graphical representation of the diminishing fuzzy set is given in the below figure.

Definition 3.10. The grades of truth-membership, indeterminacy-membership and falsity –membership for diminishing fuzzy neutrosophic (single valued neutrosophic) set based on the measure function of DFS defined as

$$\mu_{T_{D_r}}(x,y) = \begin{cases} 1 & x = y \\ \frac{1}{n^k} & x \neq y, k = |x-y| \end{cases}$$

$$\mu_{I_{D_r}}(x,y) = \begin{cases} 0(\text{or})1 & x = y \\ \frac{n^k}{n^k+1} & x \neq y, k = |x-y| \end{cases}$$

$$\mu_{F_{D_r}}(x,y) = \begin{cases} 0 & x = y \\ \frac{n^k-1}{n^k} & x \neq y, k = |x-y| \end{cases}$$

Then, the diminishing fuzzy neutrosophic set (DFNS) can be written as

$$\mu_{N_{D_r}}(x,y) = \begin{cases} (1,0,0) & x = y \\ (\frac{1}{n^k}, \frac{n^k}{n^k+1}, \frac{n^k-1}{n^k}) & x \neq y, k = |x-y| \end{cases}$$

(or)

$$\mu_{N_{D_r}}(x,y) = \begin{cases} (1,1,0) & x = y \\ (\frac{1}{n^k}, \frac{n^k}{n^k+1}, \frac{n^k-1}{n^k}) & x \neq y, k = |x-y| \end{cases}$$

where $n \in \mathbb{N}, (x,y) \in D_r, n \geq 2$ and $0 \leq \mu_{T_{D_r}}, \mu_{I_{D_r}}, \mu_{F_{D_r}} \leq 1, 0 \leq \mu_{T_{D_r}} + \mu_{F_{D_r}} \leq 1$, $0 \leq \mu_{T_{D_r}} + \mu_{I_{D_r}} + \mu_{F_{D_r}} \leq 2$ (i.e.,) $0 \leq \frac{2n^k+1}{n^k+1}/ \leq 2$.

Note 3.11. In the above definition, truth membership values are independent followed by both falsity and indeterminacy membership values that are dependent. The indeterminacy and the falsehood quantities would share the same operator throughout this paper. For example, for the arithmetic operation union, truth membership would have **max** operator whereas indeterminacy and falsehood membership will take **min** operator.

The 4-neighborhood and 8-neighborhood of the cells in the D_r array with their diminishing fuzzy neutrosophic membership grades are pictorially represented as follows:

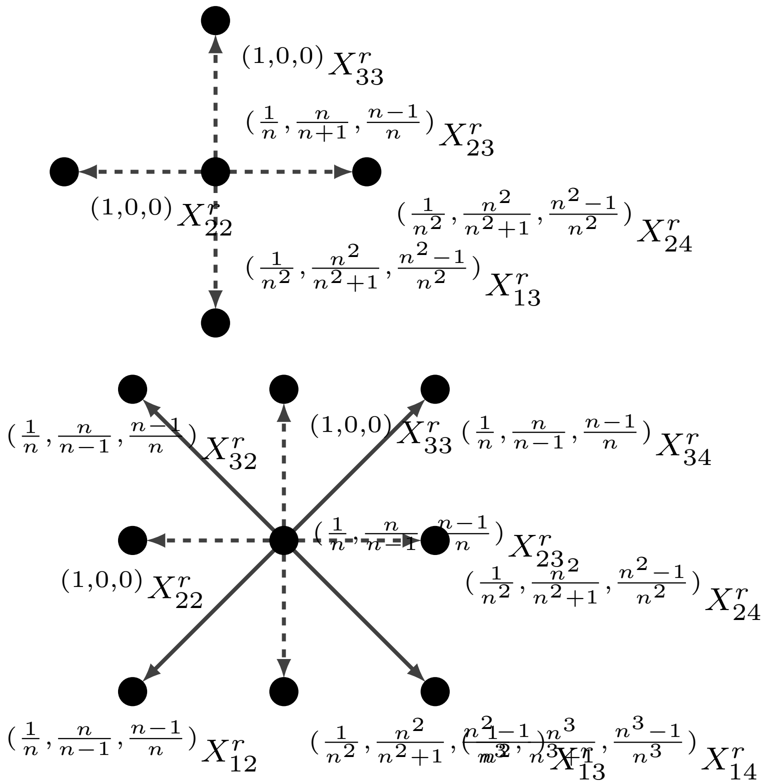

FIGURE 3. 4–Connectivity and 8–connectivity of the diminishing cells in terms of diminishing fuzzy neutosophic sets

Proposition 3.12. *Let X_{N_r} be a sub-array of D_r array with the collection $\mathcal{T}_{N_r}$. Then, $(\mu_{N_{D_r}}, \mathcal{T}_{N_r})$ is a diminishing fuzzy neutrosophic topological space.*

Proof. Without loss of generality, let us take $X_{N_r} = \{(x, y)\}$ be a finite collection of points $(x, y) \in D_r$ with its following diminishing single valued neutrosophic subsets $\mu_{N_{A_r}}(x, y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_2,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_3,y_3)}{(0,1,1)}\}$ with $n \geq 2, k = 1, 2$ for (x_1, y_1) and (x_2, y_2), respectively, whereas $(x_3, y_3) = 0_{N_r}, \mu_{N_{B_r}}(x, y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_2,y_2)}{(0,1,1)}, \frac{(x_3,y_3)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}\}$ with $n \geq 2, k = 1, 3$ for (x_1, y_1) and (x_3, y_3), respectively, whereas $(x_2, y_2) = 0_{N_r}, \mu_{N_{C_r}}(x, y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_2,y_2)}{(0,1,1)}, \frac{(x_3,y_3)}{(0,1,1)}\}$ with $n \geq 2, k = 1$ for (x_1, y_1) and $(x_3, y_3) = (x_2, y_2) = 0_{N_r}, \mu_{N_{D_r}}(x, y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_2,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_3,y_3)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}\}$ with $n \geq 2, k = 1, 2, 3$ for $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), respectively, and we claim that the topology $\mathcal{T}_{N_r}(x, y) = \{0_{N_r}, 1_{N_r}, \mu_{N_{A_r}}(x, y), \mu_{N_{B_r}}(x, y), \mu_{N_{C_r}}(x, y), \mu_{N_{D_r}}(x, y)\}$ is a diminishing fuzzy neutrosophic topology on X_{N_r} where $0_{N_r} = \{(0, 1, 1)\}$ and $1_{N_r} = \{(1, 0, 0)\}$. Then, $\mu_{N_{A_r}} \cap \mu_{N_{B_r}}(x, y) = \mu_{N_{C_r}}(x, y) \in \mathcal{T}_{N_r}, \mu_{N_{A_r}} \cap \mu_{N_{C_r}}(x, y) = \mu_{N_{C_r}}(x, y) \in \mathcal{T}_{N_r}, \mu_{N_{B_r}} \cap \mu_{N_{C_r}}(x, y) = \mu_{N_{C_r}}(x, y) \in \mathcal{T}_{N_r}$, $\mu_{N_{A_r}} \cap \mu_{N_{D_r}}(x, y) = \mu_{N_{A_r}}(x, y) \in \mathcal{T}_{N_r}, \mu_{N_{B_r}} \cap \mu_{N_{D_r}}(x, y) = \mu_{N_{B_r}}(x, y) \in \mathcal{T}_{N_r}$, $\mu_{N_{C_r}} \cap \mu_{N_{D_r}}(x, y) = \mu_{N_{C_r}}(x, y) \in \mathcal{T}_{N_r}$, $\mu_{N_{A_r}} \cup \mu_{N_{B_r}}(x, y) = \mu_{N_{D_r}}(x, y) \in \mathcal{T}_{N_r}$, $\mu_{N_{A_r}} \cup \mu_{N_{C_r}}(x, y) =$

$\mu_{N_{A_r}}(x,y) \in \mathcal{T}_{N_r}$, $\mu_{N_{B_r}} \cup \mu_{N_{C_r}}(x,y) = \mu_{N_{B_r}}(x,y) \in \mathcal{T}_{N_r}$, $\mu_{N_{A_r}} \cup \mu_{N_{D_r}}(x,y) = \mu_{N_{D_r}}(x,y) \in \mathcal{T}_{N_r}$, $\mu_{N_{B_r}} \cup \mu_{N_{D_r}}(x,y) = \mu_{N_{D_r}}(x,y) \in \mathcal{T}_{N_r}, \mu_{N_{C_r}} \cup \mu_{N_{D_r}}(x,y) = \mu_{N_{D_r}}(x,y) \in \mathcal{T}_{N_r}$ and so on. Thus, $\mathcal{T}_{N_r}$ is a diminishing fuzzy neutrosophic topology on X_{N_r}. Hence proved. □

Remark 3.13. Any diminishing fuzzy neutrosophic set in $\mathcal{T}_{N_r}$ is known as the diminishing fuzzy neutrosophic open set (DFNOS) in X_{N_r} and its complement is called as diminishing fuzzy neutrosophic closed set (DFNCS).

Lemma 3.14. *Let X_{N_r} be a subarray of D_r array and R_{N_r} be an equivalence relation on it. Then, (X_{N_r}, R_{N_r}) forms an approximation space for diminishing nano neutrosophic sets.*

Proof. We prove this theorem by considering the grades of membership arbitrarily for the components $\mu_{T_{D_r}}, \mu_{F_{D_r}}, \mu_{I_{D_r}}$. Let us consider the domain of discourse, $X_{N_r} = \{(x,y)\}$. Suppose that the relation X_{N_r}/R_{N_r} is defined as $X_{N_r}/R_{N_r}(x,y) = \{(x_1,y_1),(x_2,y_2)\},(x_1,y_2),(x_2,y_1)\}$. Let $\mu_{N_{A_r}}(x,y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3}7/8)}, \frac{(x_2,y_2)}{(1,0,0)}\}$ with $n \geq 2, k = 1$ for $(x_1,y_1), k = 2$ for $(x_1,y_2), k = 3$ for (x_2,y_1) and (x_2,y_2) whereas $(x_2,y_2) = 1_{N_r}$. Then, we have $\underline{N_r(\mu_{N_{A_r}})}(x,y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}\}$, $\overline{N_r(\mu_{N_{A_r}})}(x,y) = \{\frac{(x_1,y_1)}{(1,0,0)}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(1,0,0)}\}$. If $B_{N_r}(\mu_{N_{A_r}})(x,y) = \overline{N_r(\mu_{N_{A_r}})}(x,y) - \underline{N_r(\mu_{N_{A_r}})}(x,y)$, then $B_{N_r}(\mu_{N_{A_r}})(x,y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(\frac{1}{n},\frac{n}{n-1},\frac{n-1}{n})}\}$. Thus, we obtained the approximation space with respect to the lower approximation space, the upper approximation space and the boundary region using the basic operations of the neutrosophic set. □

Remark 3.15. The fuzzy neutrosophic elements present in the boundary region B_{N_r} may not satisfy the basic definition of DFNS. Since, it involves the complement of the component $\mu_{I_{D_r}}$ yet it satisfies the general neutrosophic set property. Hence, we can call it, the elements of a fuzzy neutrosophic set.

Theorem 3.16. *Let X_{N_r} be a sub-array of D_r array and R_{N_r} be an equivalence relation on it with the collection of diminishing nano fuzzy neutrosophic subsets $\mathcal{T}_{N_{R_r}}$. Then, $(X_{N_r}, \mathcal{T}_{N_{R_r}})$ is a nano topological space for the diminishing fuzzy neutrosophic set.*

Proof. We prove this theorem with the help of lemma 3.5. Let us consider the universe of discourse as the pixel values of the image domain, $X_{N_r} = \{(x,y)\}$. Suppose that the relation X_{N_r}/R_{N_r} is defined as $X_{N_r}/R_{N_r}(x,y) = \{\{(x_1,y_1),(x_2,y_2)\},(x_1,y_2),(x_2,y_1)\}$ and $\mu_{N_{A_r}}(x,y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(1,0,0)}\}$ where $n \geq 2, k = 1$ for

$(x_1, y_1), k = 2$ for $(x_1, y_2), k = 3$ for (x_2, y_1) and $(x_2, y_2) = 1_{N_r}$ with the approximation space and the boundary region $\underline{N_r(\mu_{N_{A_r}})}(x,y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n}1/2)}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}\}$, $\overline{N_r(\mu_{N_{A_r}})}(x,y) = \{\frac{(x_1,y_1)}{(1,0,0)}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(1,0,0)}\}$ and $B_{N_r}(\mu_{N_{A_r}})(x,y) = \{\frac{(x_1,y_1)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}, \frac{(x_1,y_2)}{(\frac{1}{n^2},\frac{n^2}{n^2+1},\frac{n^2-1}{n^2})}, \frac{(x_2,y_1)}{(\frac{1}{n^3},\frac{n^3}{n^3+1},\frac{n^3-1}{n^3})}, \frac{(x_2,y_2)}{(\frac{1}{n},\frac{n}{n+1},\frac{n-1}{n})}\}$ respectively. Then, $\mathcal{T}_{N_{R_r}}(x,y) = \{0_{N_r}, 1_{N_r}, \underline{N_r(\mu_{N_{A_r}})}(x,y), \overline{N_r(\mu_{N_{A_r}})}(x,y), B_{N_r}(\mu_{N_{A_r}})(x,y)\}$ is a nano topology on X_{N_r}. Since, $\underline{N_r(\mu_{N_{A_r}})} \cap \overline{N_r(\mu_{N_{A_r}})}(x,y) = \underline{N_r(\mu_{N_{A_r}})}(x,y) \in \mathcal{T}_{N_{R_r}}, \underline{N_r(\mu_{N_{A_r}})}m \cap B_{N_r}(\mu_{N_{A_r}})(x,y) = \underline{N_r(\mu_{N_{A_r}})}(x,y) \in \mathcal{T}_{N_{R_r}}, \underline{N_r(\mu_{N_{A_r}})} \cup \overline{N_r(\mu_{N_{A_r}})}(x,y) = \overline{N_r(\mu_{N_{A_r}})}(x,y) \in \mathcal{T}_{N_{R_r}}, \underline{N_r(\mu_{N_{A_r}})} \cup B_{N_r}(\mu_{N_{A_r}})(x,y) = B_{N_r}(\mu_{N_{A_r}})(x,y) \in \mathcal{T}_{N_{R_r}}$ and so on. Hence, $(X_{N_r}, \mathcal{T}_{N_{R_r}})$ is a nano topology for the diminishing fuzzy neutrosophic set. □

Remark 3.17. The elements of diminishing fuzzy neutrosophic nano topological space (DFNNTS) are called the diminishing fuzzy neutrosophic nano open sets (DFNNOSs) and their complement is the diminishing fuzzy neutrosophic nano closed sets (DFNNCSs).

Theorem 3.18. *Let X_{N_r} be a sub-array of the D_r array and R_{N_r} be an equivalence relation it with the collection of its diminishing fuzzy neutrosophic subsets, $\mathcal{T}_{N_r}$. Then, $\beta_{N_{R_r}}$ forms a basis for diminishing fuzzy neutrosophic nano topological space.*

Proof. The proof of this theorem is obtained by the lemma 3.6. and the above theorem. Let $\mathcal{T}_{N_r}(x,y) = \{0_{N_r}, 1_{N_r}, \underline{N_r(\mu_{N_{A_r}})}(x,y), \overline{N_r(\mu_{N_{A_r}})}(x,y), B_{N_r}(\mu_{N_{A_r}})(x,y)\}$ is a nano topology on X_{N_r} and $\mu_{N_{A_r}}(x,y) \subset X_{N_r}$. We claim that $\beta_{N_{R_r}} = \{1_{N_r}, \underline{N_r(\mu_{N_{A_r}})}(x,y), B_{N_r}(\mu_{N_{A_r}})(x,y)\}$ is an open base for DFNNTS. Since, $1_{N_r} \cup \underline{N_r(\mu_{N_{A_r}})}(x,y) = 1_{N_r}(x,y)$, $1_{N_r} \cup B_{N_r}(\mu_{N_{A_r}})(x,y) = 1_{N_r}(x,y)$, $\underline{N_r(\mu_{N_{A_r}})} \cup B_{N_r}(\mu_{N_{A_r}})(x,y) = B_{N_r}(\mu_{N_{A_r}})(x,y)$ and $1_{N_r} \cup \underline{N_r(\mu_{N_{A_r}})} \cup B_{N_r}(\mu_{N_{A_r}})(x,y) = 1_{N_r}(x,y)$ which belongs to DFNNTS $(X_{N_r}, \mathcal{T}_{N_r})$. If $1_{N_r} \cap \underline{N_r(\mu_{N_{A_r}})}(x,y) = \underline{N_r(\mu_{N_{A_r}})}(x,y)$ and $\underline{N_r(\mu_{N_{A_r}})}(x,y) \subset 1_{N_r} \cap \underline{N_r(\mu_{N_{A_r}})}(x,y)$, then for every element in $1_{N_r} \cap \underline{N_r(\mu_{N_{A_r}})}$ also belongs to $\underline{N_r(\mu_{N_{A_r}})}$ and if $1_{N_r} \cap B_{N_r}(\mu_{N_{A_r}})(x,y) = B_{N_r}(\mu_{N_{A_r}})(x,y)$ and $B_{N_r}(\mu_{N_{A_r}})(x,y) \subset 1_{N_r} \cap B_{N_r}(\mu_{N_{A_r}})(x,y)$, then for every element in $1_{N_r} \cap B_{N_r}(\mu_{N_{A_r}})$ also belongs to $B_{N_r}(\mu_{N_{A_r}})$. As for $B_{N_r}(\mu_{N_{A_r}})$ and $\underline{N_r(\mu_{N_{A_r}})}$, we have $B_{N_r}(\mu_{N_{A_r}}) \cap \underline{N_r(\mu_{N_{A_r}})}(x,y) = \underline{N_r(\mu_{N_{A_r}})}(x,y)$. Thus, the collection $\beta_{N_{R_r}}$ forms a basis for DFNNTS. □

Definition 3.19. Let X_{N_r} and Y_{N_r} be two domains of discourse with the topologies $\mathcal{T}_{A_{N_r}}$ and $\mathcal{T}_{B_{N_r}}$ respectively, where A_{N_r} and B_{N_r} are the two diminishing single valued neutrosophic subsets of X_{N_r} and Y_{N_r} respectively. A function $f_{N_{D_r}}$ is said to be a nano continuous from $(X_{N_r}, \mathcal{T}_{A_{N_r}})$ to $(Y_{N_r}, \mathcal{T}_{B_{N_r}})$ if the inverse of each $\mathcal{T}_{B_{N_r}}$-diminishing nano fuzzy neutrosophic open set is $\mathcal{T}_{A_{N_r}}$-diminishing nano fuzzy neutrosophic open set.

Example 3.20. Without loss of generality, let $X_{N_r} = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2)\}$ be a universe of discourse. Let $X_{N_r}/R_{N_r}(x,y) = \{\{(x_1,y_1),(x_2,y_2)\},(x_1,y_2),(x_2,y_1)\}$ and $\mu_{N_{A_r}}(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\} \subset X_{N_r}$ be a DFNS with $n = 2, k = 1$ for $(x_1,y_1), k = 3$ for (x_2,y_1). Then $\underline{N_r(\mu_{N_{A_r}})}(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}$, $\overline{N_r(\mu_{N_{A_r}})}(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}$ and $B_r(\mu_{N_{A_r}})(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}$. Then, $\mathcal{T}_{N_{A_r}}(x,y) = \{0_{N_r},1_{N_r},\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}$. Let $X_{N_r}/R'_{N_r}(x,y) = \{\{(x_1,y_1),(x_1,y_2)\},(x_2,y_2),(x_2,y_1)\}$ and $\mu_{N_{B_r}}(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/4,4/5,3/4)},\frac{(x_2,y_2)}{(1,0,0)}\} \subset X_{N_r}$ be a DFNS with $n = 2, k = 1$ for $(x_1,y_1), k = 2$ for (x_1,y_2) and $(x_2,y_2) = 1_{N_r}$ is free. Then $\underline{N_r(\mu_{N_{B_r}})}(x,y) = \{\frac{(x_1,y_1)}{(1/4,4/5,3/4)},\frac{(x_1,y_2)}{(1/4,4/5,3/4)},\frac{(x_2,y_2)}{(1,0,0)}\}, \overline{N_r(\mu_{N_{B_r}})}(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(1,0,0)}\}$ and $B_r(\mu_{N_{B_r}})(x,y) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(0,1,1)}\}$. Then $\mathcal{T}_{N_{B_r}}(x,y) = \{0_{N_r},1_{N_r},\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(1,0,0)}\},\{\frac{(x_1,y_1)}{(1/4,4/5,3/4)},\frac{(x_1,y_2)}{(1/4,4/5,3/4)},\frac{(x_2,y_2)}{(1,0,0)}\},\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(0,1,1)}\}\}$. Let us define a function $f_{N_{D_r}} : \mathcal{T}_{N_{A_r}} \to \mathcal{T}_{N_{B_r}}$ as $f_{N_{D_r}}(0,1,1) = \{\frac{(x_1,y_1)}{(1/4,4/5,3/4)},\frac{(x_1,y_2)}{(1/4,4/5,3/4)},\frac{(x_2,y_2)}{(1,0,0)}\}$, $f_{N_{D_r}}(\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(0,1,1)}\}$, $f_{N_{D_r}}(\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(1,0,0)}\}$, $f_{N_{D_r}}(1,0,0) = (1,0,0)$ and $f_{N_{D_r}}(0,1,1) = (0,1,1)$ Then, $f^(-1)_{N_{D_r}}(\{\frac{(x_1,y_1)}{(1/4,4/5,3/4)},\frac{(x_1,y_2)}{(1/4,4/5,3/4)},\frac{(x_2,y_2)}{(1,0,0)}\}) = (0,1,1)$, $f^(-1)_{N_{D_r}}(\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(0,1,1)}\}) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}$, $f^{-1}_{N_{D_r}}(\{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_1,y_2)}{(1/2,2/3,1/2)},\frac{(x_2,y_2)}{(1,0,0)}\}) = \{\frac{(x_1,y_1)}{(1/2,2/3,1/2)},\frac{(x_2,y_1)}{(1/8,8/9,7/8)}\}$, $f^{-1}_{N_{D_r}}(1,0,0) = (1,0,0)$ and $f^{-1}_{N_{D_r}}(0,1,1) = (0,1,1)$ Clearly, the inverse image of every diminishing nano fuzzy neutrosophic open set in N_{B_r} is the diminishing nano fuzzy neutrosophic open set in N_{A_r}. Hence, $f_{N_{D_r}}$ is a nano continuous on diminishing nano fuzzy neutrosophic topological space.

Remark 3.21. A nano continuous function $f_{N_{D_r}}$ defined on diminishing fuzzy neutrosophic topological space need not be bijective.

Conclusion

The function of diminishing fuzzy sets is extended to the case of fuzzy neutrosophic sets, termed a diminishing fuzzy neutrosophic set. The collections of diminishing fuzzy neutrosophic topology and diminishing fuzzy neutrosophic topology induced by nano topology are determined for the extended fuzzified (neutrosophified) sub-arrays of diminishing cells infinite array. Furthermore, a nano continuous function on the diminishing fuzzy neutrosophic topological spaces is also studied. The approximation space with respect to the conceptualization of pixel neighborhood in the field of image processing for the neighborhoods of diminishing cells and for NFA of the layer word are discussed and the depiction of the neighborhood of arbitrary cells is contrived to correspond to the diminishing fuzzy neutrosophic sets.

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Modified Explicit Scheme for Solving Neutrosophic Fuzzy Heat Equation

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Abstract: In this paper, an explicit finite difference method is developed and applied for the first time to solve the neutrosophic fuzzy heat equation. For this purpose, the triangular neutrosophic number is used in both the neutrosophic exact and numerical solution. In addition, solutions of neutrosophic fuzzy heat equation were observed at the (α, β, γ)-cut with varied time scales. Also, the error between the analytical and numerical solution obtained on the (α, β, γ)-cut is evaluated and illustrated. A good amount of agreement is seen using closed-form and numerical solutions.

Keywords: Explicit finite difference scheme; Intuitionistic fuzzy number; Intuitionistic fuzzy time fractional diffusion equation

1. Introduction

Zadeh (1965) [1] first introduced the idea of fuzzy sets and fuzzy numbers. After that, different extensions of fuzzy set theory have been developed. One of these important extensions is the fuzzy intuitionistic set introduced by Atanassov (1983) [2]. The intuitionistic set introduced the concepts of the degree of non-membership and falsehood, which led to the creation of intuitionistic fuzzy sets (IFS). This new approach differed from classical fuzzy sets and provided a better way to describe incomplete ideas. Neutrosophy is a new tool to handle problems with unclear, uncertain, and inconsistent information. In 2005, Smarandache (2005) [3] introduced the Neutrosophic sets as a generalization concept of IFS. Neutrosophy is a new tool to handle problems with unclear, uncertain, and inconsistent information. In 2005, Smarandache introduced the Neutrosophic sets as a generalization concept of Intuitionistic fuzzy sets. In the Neutrosophic set, the grade of membership of Truth values (T), Indeterminate values (I) and False values (F) has been defined within the non-standard interval -] 0, 1[+. Non-standard intervals of the neutrosophic set theory work well in the concept of philosophy. But in reality, if we deal with real-life problems in science and engineering it is not possible to put the data into the non-standard interval. To address these issues, Wang et al. (2010) [4] developed single-valued Neutrosophic sets by utilizing the standard form of the unit interval [0, 1]. In addition, some researchers have defined single-valued Neutrosophic numbers [5-7]. Similarly, Ye (2015) [8] introduced Trapezoidal Neutrosophic numbers and discussed their application in decision-making. This approach has since led to a lot of research on many real-world problems.

The Fuzzy Neutrosophic Differential Equations (FNDEs) are one of the important tools to model the uncertainty in particular quantities for certain real-world phenomena. They have fundamental applications in various areas, such as chemistry, engineering, physics, and biology. The exact solution of the FNDEs is usually not obtainable or difficult to obtain. So, many mathematicians have resorted to utilizing the approximation or numerical methods. Among the numerical methods, the finite difference method (FDM) stands out as one of the most widely used methods because of its simplicity and universal applicability. The FDM is used to solve the FNDE numerically. Parikh et al. (2022) [9] solve the first-order non-homogeneous fuzzy differential equation with neutrosophic initial conditions. The triangular neutrosophic numbers used in the solution process. After that, Kamal et al. (2023) [10] used a modified FDM to solve second-order differential equation with neutrosophic fuzzy boundary condition.

Fuzzy heat equation is utilized for modelling certain real-life problems in physics, and biology [11-16]. When dealing with fuzzy heat equations and we are not entirely sure about certain factors, neutrosophic fuzzy numbers (NFNs) can be helpful. They give a clearer way to handle uncertainty, which makes it easier to decide on the best numerical methods and parameters for solving the equations. Based on our literature review, there seems to have been no attempt to solve neutrosophic fuzzy heat equations using FDM. Therefore, in this paper, an explicit FDM is developed and applied to solve the neutrosophic fuzzy heat equations with neutrosophic initial conditions.

2. Heat Equation in Neutrosophic Fuzzy environment

We investigate the Neutrosophic fuzzy heat equation (NFHE) along with its corresponding Neutrosophic initial and Neutrosophic boundary conditions [12]:

$$\frac{\partial \tilde{u}(x,t)}{\partial t} = \widetilde{D}(x,t)\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + \tilde{q}(x,t), \qquad 0 < x < l, t > 0$$
$$\tilde{u}(x,0) = \tilde{f}(x), \tilde{u}(0,t) = \tilde{g}(t),\ \tilde{u}(l,t) = \tilde{z}(t), \tag{1}$$

where $\tilde{v}(x,t)$ is a fuzzy Neutrosophic fractional function and α is fractional arbitrary order, $\frac{\partial \tilde{u}(x,t)}{\partial t}$ is the fuzzy Neutrosophic time derivative, $\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2}$ is a fuzzy Neutrosophic Hukuhara derivatives. $\tilde{q}(x,t)$, and $\widetilde{D}(x,t)$ are fuzzy Neutrosophic functions. $\tilde{u}(x,0)$, is the fuzzy Neutrosophic initial conditions while $\tilde{u}(0,t)$ and also $\tilde{u}(l,t)$ are fuzzy Neutrosophic boundary condition with $\tilde{g}(t)$, $\tilde{z}(t)$ are fuzzy Neutrosophic c convex normalized numbers. Finally, the fuzzy functions $\widetilde{D}(x,t)$, $\tilde{q}(x,t)$, and $\tilde{f}(x)$ in Eq. (1) are characterized as follows:

$$\begin{cases} \widetilde{D}(x,t) = \tilde{\theta}_1 b_1(x,t), \\ \tilde{q}(x,t) = \tilde{\theta}_2 b_2(x,t), \\ \tilde{f}(x) = \tilde{\theta}_3 b_3(x). \end{cases} \tag{2}$$

where b_1, b_2, and b_3 represent the crisp functions of the independent variable x, while $\tilde{\theta}_1$,$\tilde{\theta}_2$, and $\tilde{\theta}_3$ denote the fuzzy Neutrosophic convex number.

Now, the NFHE in Eq. (1) is defuzzified by using the single parametric form based on the α,β,γ-cut approach described in section 2 for all $0 \le \alpha + \beta + \gamma \le 3$ as the following [21-25]:

$$[\tilde{u}(x,t)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{u_T}(x,t;\alpha), \overline{u_T}(x,t;\alpha)\right], \left[\underline{u_I}(x,t;\beta), \overline{u_I}(x,t;\beta)\right], \left[\underline{u_F}(x,t;\gamma), \overline{u_F}(x,t;\gamma)\right]\right\} \tag{3}$$

$$\left[\frac{\partial \tilde{u}(x,t)}{\partial t}\right]_{\alpha,\beta,\gamma} = \left\{\left[\frac{\partial \underline{u_T}(x,t;\alpha)}{\partial t}, \frac{\partial \overline{u_T}(x,t;\alpha)}{\partial t}\right], \left[\frac{\partial \underline{u_I}(x,t;\beta)}{\partial t}, \frac{\partial \overline{u_I}(x,t;\beta)}{\partial t}\right], \left[\frac{\partial \underline{u_F}(x,t;\gamma)}{\partial t}, \frac{\partial \overline{u_F}(x,t;\gamma)}{\partial t}\right]\right\} \tag{4}$$

$$\left[\frac{\partial^2\tilde{u}(x,t)}{\partial x^2}\right]^{\alpha,\beta,\gamma} = \left\{\left[\frac{\partial^2\underline{u_T}(x,t;\alpha)}{\partial x^2}, \frac{\partial^2\overline{u_T}(x,t;\alpha))}{\partial x^2}\right], \left[\frac{\partial^2\underline{u_I}(x,t;\beta)}{\partial x^2}, \frac{\partial^2\overline{u_I}(x,t;\beta)}{\partial x^2}\right], \left[\frac{\partial^2\underline{u_F}(x,t;\gamma)}{\partial x^2}, \frac{\partial^2\overline{u_F}(x,t;\gamma)}{\partial x^2}\right]\right\} \quad (5)$$

$$\left[\tilde{D}(x,t)\right]_{\alpha,\beta,\gamma} = \left\{\left[\underline{D_T}(x,t;\alpha), \overline{D_T}(x,t;\alpha)\right], \left[\underline{D_I}(x,t;\beta), \overline{D_I}(x,t;\beta)\right], \left[\underline{D_F}(x,t;\gamma), \overline{D_F}(x,t;\gamma)\right]\right\} \quad (6)$$

$$[\tilde{q}(x,t)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{q_T}(x,t;\alpha), \overline{q_T}(x,t;\alpha)\right], \left[\underline{q_I}(x,t;\beta), \overline{q_I}(x,t;\beta)\right], \left[\underline{q_F}(x,t;\gamma), \overline{q_F}(x,t;\gamma)\right]\right\} \quad (7)$$

$$[\tilde{u}(x,0)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{u_T}(x,0;\alpha), \overline{u_T}(x,0;\alpha)\right], \left[\underline{u_I}(x,0;\beta), \overline{u_I}(x,0;\beta)\right], \left[\underline{u_F}(x,0;\gamma), \overline{u_F}(x,0;\gamma)\right]\right\} \quad (8)$$

$$[\tilde{u}(0,t)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{u_T}(0,t;\alpha), \overline{u_T}(0,t;\alpha)\right], \left[\underline{u_I}(0,t;\beta), \overline{u_I}(0,t;\beta)\right], \left[\underline{u_F}(0,t;\gamma), \overline{u_F}(0,t;\gamma)\right]\right\} \quad (9)$$

$$[\tilde{u}(l,t)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{u_T}(l,t;\alpha), \overline{u_T}(l,t;\alpha)\right], \left[\underline{u_I}(l,t;\beta), \overline{u_I}(l,t;\beta)\right], \left[\underline{u_F}(l,t;\gamma), \overline{u_F}(l,t;\gamma)\right]\right\} \quad (10)$$

$$\left[\tilde{f}(x)\right]_{\alpha,\beta,\gamma} = \left\{\left[\underline{f_T}(x;\alpha), \overline{f_T}(x;\alpha)\right], \left[\underline{f_I}(x;\beta), \overline{f_I}(x;\beta)\right], \left[\underline{f_F}(x;\gamma), \overline{f_F}(x;\gamma)\right]\right\} \quad (11)$$

$$\begin{cases}[\tilde{g}(t)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{g_T}(t;\alpha), \overline{g_T}(t;\alpha)\right], \left[\underline{g_I}(t;\beta), \overline{g_I}(t;\beta)\right], \left[\underline{g_F}(x;\gamma), \overline{g_F}(t;\gamma)\right]\right\} \\ [\tilde{z}(t)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{z_T}(t;\alpha), \overline{z_T}(t;\alpha)\right], \left[\underline{z_I}(t;\beta), \overline{z_I}(t;\beta)\right], \left[\underline{z_F}(x;\gamma), \overline{z_F}(t;\gamma)\right]\right\}\end{cases} \quad (12)$$

where

$$\begin{cases}\left[\tilde{D}(x,t)\right]_{\alpha,\beta,\gamma} = \left\{\left[\left[\underline{\theta_T}(\alpha)_1, \overline{\theta_T}(\alpha)_1\right] b_1(x,t)\right], \left[\left[\underline{\theta_I}(\alpha)_1, \overline{\theta_I}(\alpha)_1\right] b_1(x,t)\right], \left[\left[\underline{\theta_F}(\alpha)_1, \overline{\theta_F}(\alpha)_1\right] b_1(x,t)\right]\right\} \\ [\tilde{q}(x,t)]_{\alpha,\beta,\gamma} = \left\{\left[\left[\underline{\theta_T}(\alpha)_2, \overline{\theta_T}(\alpha)_2\right] b_2(x,t)\right], \left[\left[\underline{\theta_I}(\alpha)_2, \overline{\theta_I}(\alpha)_2\right] b_2(x,t)\right], \left[\left[\underline{\theta_F}(\alpha)_2, \overline{\theta_F}(\alpha)_2\right] b_2(x,t)\right]\right\} \\ \left[\tilde{f}(x)\right]_{\alpha,\beta,\gamma} = \left\{\left[\left[\underline{\theta_T}(\alpha)_3, \overline{\theta_T}(\alpha)_3\right] b_3(x,t)\right], \left[\left[\underline{\theta_I}(\alpha)_3, \overline{\theta_I}(\alpha)_3\right] b_3(x,t)\right], \left[\left[\underline{\theta_F}(\alpha)_3, \overline{\theta_F}(\alpha)_3\right] b_3(x,t)\right]\right\}\end{cases} \quad (13)$$

The membership function is established by applying the Zadeh expansion principle described in [1]

$$\begin{cases}\underline{u_T}(x,t;\alpha) = min\{\tilde{u}(\tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \tilde{u}(x,t;\alpha)\} \\ \overline{u_T}(x,t;\alpha) = max\{\tilde{u}(\tilde{\mu}(\alpha)) | \tilde{\mu}(\alpha) \in \tilde{u}(t,x;\alpha)\}\end{cases} \quad (14)$$

$$\begin{cases}\underline{u_I}(x,t;\beta) = min\{\tilde{u}(\tilde{\mu}(\beta)) | \tilde{\mu}(\beta) \in \tilde{u}(x,t;\beta)\} \\ \overline{u_I}(x,t;\beta) = max\{\tilde{u}(\tilde{\mu}(\beta)) | \tilde{\mu}(\beta) \in \tilde{u}(x,t;\beta)\}\end{cases} \quad (15)$$

$$\begin{cases} \underline{u_F}(x,t;\gamma) = min\{\tilde{u}(\tilde{\mu}(\gamma))|\tilde{\mu}(\gamma) \in \tilde{u}(x,t;\gamma)\} \\ \overline{u_F}(x,t;\gamma) = max\{\tilde{u}(\tilde{\mu}(\gamma))|\tilde{\mu}(\gamma) \in \tilde{u}(x,t;\gamma)\} \end{cases} \quad (16)$$

Now Eq. (1) for $0 < x \le l, t > 0$ and $r \in [0,1]$ is rewritten to obtain the general equation of NFHE as follows:

$$\begin{cases} \dfrac{\partial \underline{u_T}(x,t)}{\partial t} = \left[\underline{\theta_T}(\alpha)_1\right] b_1(x,t) \dfrac{\partial^2 \underline{u_T}(x,t;\alpha)}{\partial x^2} + \left[\underline{\theta_T}(\alpha)_2\right] b_2(x,t) \\ \underline{u_T}(x,0;\alpha) = \left[\underline{\theta_T}(\alpha)_3\right] b_3(x,t) \\ \underline{u_T}(0,t;\alpha) = \underline{g}(t,\alpha), \underline{u_T}(l,t;\alpha) = \underline{z}(t,\alpha) \end{cases} \quad (17)$$

$$\begin{cases} \dfrac{\partial \overline{u_T}(x,t)}{\partial t} = \left[\overline{\theta_T}(\alpha)_1\right] b_1(x,t) \dfrac{\partial^2 \overline{u_T}(x,t;\alpha)}{\partial x^2} + \left[\overline{\theta_T}(\alpha)_2\right] b_2(x,t) \\ \overline{u_T}(x,0;\alpha) = \left[\overline{\theta_T}(\alpha)_3\right] b_3(x,t) \\ \overline{u_T}(0,t;\alpha) = \overline{g}(t,\alpha), \overline{u_T}(l,t;\alpha) = \overline{z}(t,\alpha) \end{cases} \quad (18)$$

$$\begin{cases} \dfrac{\partial \underline{u_I}(x,t,\beta)}{\partial t} = \left[\underline{\theta_I}(\beta)_1\right] b_1(x,t) \dfrac{\partial^2 \underline{u_I}(x,t;\beta)}{\partial x^2} + \left[\underline{\theta_I}(\beta)_2\right] b_2(x,t) \\ \underline{u_I}(x,0;\beta) = \left[\underline{\theta_I}(\beta)_3\right] b_3(x,t) \\ \underline{u_I}(0,t;\beta) = \underline{g}(t,\beta), \underline{u_I}(l,t;\beta) = \underline{z}(t,\beta) \end{cases} \quad (19)$$

$$\begin{cases} \dfrac{\partial \overline{u_I}(x,t,\beta)}{\partial t} = \left[\overline{\theta_I}(\beta)_1\right] b_1(x,t) \dfrac{\partial^2 \overline{u_I}(x,t;\beta)}{\partial x^2} + \left[\overline{\theta_I}(\beta)_2\right] b_2(x,t) \\ \overline{u_I}(x,0;\beta) = \left[\overline{\theta_I}(\beta)_3\right] b_3(x,t) \\ \overline{u_I}(0,t;\beta) = \overline{g}(t,\beta), \overline{u_I}(l,t;\beta) = \overline{z}(t,\beta) \end{cases} \quad (20)$$

$$\begin{cases} \dfrac{\partial \underline{u_F}(x,t,\gamma)}{\partial t} = \left[\underline{\theta_F}(\gamma)_1\right] b_1(x,t) \dfrac{\partial^2 \underline{u_F}(x,t;\gamma)}{\partial x^2} + \left[\underline{\theta_F}(\gamma)_2\right] b_2(x,t) \\ \underline{u_F}(x,0;\gamma) = \left[\underline{\theta_F}(\gamma)_3\right] b_3(x,t) \\ \underline{u_F}(0,t;\gamma) = \underline{g}(t,\gamma), \underline{u_F}(l,t;\gamma) = \underline{z}(t,\gamma) \end{cases} \quad (21)$$

$$\begin{cases} \dfrac{\partial \overline{u_F}(x,t,\gamma)}{\partial t} = \left[\overline{\theta_F}(\gamma)_1\right] b_1(x,t) \dfrac{\partial^2 \overline{u_F}(x,t;\gamma)}{\partial x^2} + \left[\overline{\theta_F}(\gamma)_2\right] b_2(x,t) \\ \overline{u_F}(x,0;\gamma) = \left[\overline{\theta_F}(\gamma)_3\right] b_3(x,t) \\ \overline{u_F}(0,t;\gamma) = \overline{g}(t,\gamma), \overline{u_F}(l,t;\gamma) = \overline{z}(t,\gamma) \end{cases} \quad (22)$$

The Eq. (17) and Eq. (18) present the Neutrosophic lower and upper bounds of membership of Truth values (T) for the general form of NFHE. Also, Eq. (19) and Eq. (20) present the Neutrosophic lower and upper bounds of membership of Indeterminate values (I), while Eq. (21) and Eq. (22) present the Neutrosophic lower and upper bounds of membership of False values (F) for the general form of NFHE.

3. Solution of NFHE by FTCS scheme

In this section, the FTCS scheme is reformatted and applied in Neutrosophic single parametric representation with forward difference approximation for time derivatives and central difference approximation for spatial derivatives to solve the NFHE.

The time derivatives $\frac{\partial \widetilde{u_T}(x,t;\alpha)}{\partial t}, \frac{\partial \widetilde{u_I}(x,t;\alpha)}{\partial t}, \frac{\partial \widetilde{u_F}(x,t;\alpha)}{\partial t}$ are discretizes as follows:

$$\frac{\partial \widetilde{u_T}(x,t;\alpha)}{\partial t} = \begin{cases} \dfrac{\underline{u_T}_i^{n+1}(x,t;\alpha) - \underline{u_T}_i^{n}(x,t;\alpha)}{\Delta t} \\ \dfrac{\overline{u_T}_i^{n+1}(x,t;\alpha) - \overline{u_T}_i^{n}(x,t;\alpha)}{\Delta t} \end{cases} \tag{23}$$

$$\frac{\partial \widetilde{u_I}(x,t;\beta)}{\partial t} = \begin{cases} \dfrac{\underline{u_I}_i^{n+1}(x,t;\beta) - \underline{u_I}_i^{n}(x,t;\beta)}{\Delta t} \\ \dfrac{\overline{u_I}_i^{n+1}(x,t;\beta) - \overline{u_I}_i^{n}(x,t;\beta)}{\Delta t} \end{cases} \tag{24}$$

$$\frac{\partial \widetilde{u_F}(x,t;\gamma)}{\partial t} = \begin{cases} \dfrac{\underline{u_F}_i^{n+1}(x,t;\gamma) - \underline{u_F}_i^{n}(x,t;\gamma)}{\Delta t} \\ \dfrac{\overline{u_F}_i^{n+1}(x,t;\gamma) - \overline{u_F}_i^{n}(x,t;\gamma)}{\Delta t} \end{cases} \tag{25}$$

The spatial derivatives $\frac{\partial^2 \widetilde{u_T}(x,t;\alpha)}{\partial x^2}, \frac{\partial^2 \widetilde{u_I}(x,t;\beta)}{\partial x^2}, \frac{\partial^2 \widetilde{u_F}(x,t;\gamma)}{\partial x^2}$ are discretizes as follows:

$$\frac{\partial^2 \widetilde{u_T}(x,t;\alpha)}{\partial x^2} = \begin{cases} \dfrac{\underline{u_T}_{i+1}^{n}(x,t;\alpha) - 2\underline{u_T}_i^{n}(x,t;\alpha) + \underline{u_T}_{i-1}^{n}(x,t;\alpha)}{\Delta x^2} \\ \dfrac{\overline{u_T}_{i+1}^{n}(x,t;\alpha) - 2\overline{u_T}_i^{n}(x,t;\alpha) + \overline{u_T}_{i-1}^{n}(x,t;\alpha)}{\Delta x^2} \end{cases} \tag{26}$$

$$\frac{\partial^2 \widetilde{u_I}(x,t;\beta)}{\partial x^2} = \begin{cases} \dfrac{\underline{u_I}_{i+1}^{n}(x,t;\beta) - 2\underline{u_I}_i^{n}(x,t;\beta) + \underline{u_I}_{i-1}^{n}(x,t;\beta)}{\Delta x^2} \\ \dfrac{\overline{u_I}_{i+1}^{n}(x,t;\beta) - 2\overline{u_I}_i^{n}(x,t;\beta) + \overline{u_I}_{i-1}^{n}(x,t;\beta)}{\Delta x^2} \end{cases} \tag{27}$$

$$\frac{\partial^2 \widetilde{u_F}(x,t;\gamma)}{\partial x^2} = \begin{cases} \dfrac{\underline{u_F}_{i+1}^{n}(x,t;\gamma) - 2\underline{u_F}_i^{n}(x,t;\gamma) + \underline{u_F}_{i-1}^{n}(x,t;\gamma)}{\Delta x^2} \\ \dfrac{\overline{u_F}_{i+1}^{n}(x,t;\gamma) - 2\overline{u_F}_i^{n}(x,t;\gamma) + \overline{u_F}_{i-1}^{n}(x,t;\gamma)}{\Delta x^2} \end{cases} \tag{28}$$

where i represents a point on a spatial grid, while n signifies a point in time.

Now substitute Eq. (23- 28) into Eq.(17 - 22) respectively to get:

$$\frac{\underline{u_T}_i^{n+1}(x,t;\alpha)-\underline{u_T}_i^{n}(x,t;\alpha)}{\Delta t} = \underline{D_T}(x,t,\alpha)\frac{\underline{u_T}_{i+1}^{n}(x,t;\alpha)-2\underline{u_T}_i^{n}(x,t;\alpha)+\underline{u_T}_{i-1}^{n}(x,t;\alpha)}{\Delta x^2}+\underline{q_T}(x,t,\alpha) \quad (29)$$

$$\frac{\overline{u_T}_i^{n+1}(x,t;\alpha)-\overline{u_T}_i^{n}(x,t;\alpha)}{\Delta t} = \overline{D_T}(x,t,\alpha)\frac{\overline{u_T}_{i+1}^{n}(x,t;\alpha)-2\overline{u_T}_i^{n}(x,t;\alpha)+\overline{u_T}_{i-1}^{n}(x,t;\alpha)}{\Delta x^2}+\overline{q_T}(x,t,\alpha) \quad (30)$$

$$\frac{\underline{u_I}_i^{n+1}(x,t;\beta)-\underline{u_I}_i^{n}(x,t;\beta)}{\Delta t} = \underline{D_I}(x,t,\beta)\frac{\underline{u_I}_{i+1}^{n}(x,t;\beta)-2\underline{u_I}_i^{n}(x,t;\beta)+\underline{u_I}_{i-1}^{n}(x,t;\beta)}{\Delta x^2}+\underline{q_I}(x,t,\beta) \quad (31)$$

$$\frac{\overline{u_I}_i^{n+1}(x,t;\beta)-\overline{u_I}_i^{n}(x,t;\beta)}{\Delta t} = \overline{D_I}(x,t,\beta)\frac{\overline{u_I}_{i+1}^{n}(x,t;\beta)-2\overline{u_I}_i^{n}(x,t;\beta)+\overline{u_I}_{i-1}^{n}(x,t;\beta)}{\Delta x^2}+\overline{q_I}(x,t,\beta) \quad (32)$$

$$\frac{\underline{u_F}_i^{n+1}(x,t;\gamma)-\underline{u_F}_i^{n}(x,t;\gamma)}{\Delta t} = \underline{D_F}(x,t,\gamma)\frac{\underline{u_F}_{i+1}^{n}(x,t;\gamma)-2\underline{u_F}_i^{n}(x,t;\gamma)+\underline{u_F}_{i-1}^{n}(x,t;\gamma)}{\Delta x^2}+\underline{q_F}(x,t,\gamma) \quad (33)$$

$$\frac{\overline{u_F}_i^{n+1}(x,t;\gamma)-\overline{u_F}_i^{n}(x,t;\gamma)}{\Delta t} = \overline{D_F}(x,t,\gamma)\frac{\overline{u_F}_{i+1}^{n}(x,t;\gamma)-2\overline{u_F}_i^{n}(x,t;\gamma)+\overline{u_F}_{i-1}^{n}(x,t;\gamma)}{\Delta x^2}+\overline{q_F}(x,t,\gamma) \quad (34)$$

Now let $\tilde{s}=\frac{\tilde{D}(x,t)\Delta t}{\Delta x^2}$, and from Eq. (29 - 34) we obtain for all $\alpha,\beta,\gamma \in [0,1]$

$$\begin{cases}\underline{u_T}_i^{n+1}(x,t;\alpha)=s\,\underline{u_T}_{i+1}^{n}(x,t;\alpha)+(1-2s)\underline{u_T}_i^{n}(x,t;\alpha)+s\,\underline{u_T}_{i-1}^{n}(x,t;\alpha)\ +\Delta t\,\underline{q_T}(x,t,\alpha)\\ \overline{u_T}_i^{n+1}(x,t;\alpha)=s\,\overline{u_T}_{i+1}^{n}(x,t;\alpha)+(1-2s)\overline{u_T}_i^{n}(x,t;\alpha)+s\,\overline{u_T}_{i-1}^{n}(x,t;\alpha)\ +\Delta t\,\overline{q_T}(x,t,\alpha)\end{cases} \quad (35)$$

$$\begin{matrix}\underline{u_I}_i^{n+1}(x,t;\beta)=s\,\underline{u_I}_{i+1}^{n}(x,t;\beta)+(1-2s)\underline{u_I}_i^{n}(x,t;\beta)+s\,\underline{u_I}_{i-1}^{n}(x,t;\beta)\ +\Delta t\,\underline{q_I}(x,t,\beta)\\ \overline{u_I}_i^{n+1}(x,t;\beta)=s\,\overline{u_I}_{i+1}^{n}(x,t;\beta)+(1-2s)\overline{u_I}_i^{n}(x,t;\beta)+s\,\overline{u_I}_{i-1}^{n}(x,t;\beta)\ +\Delta t\,\overline{q_I}(x,t,\beta)\end{matrix} \quad (36)$$

$$\begin{matrix}\underline{u_F}_i^{n+1}(x,t;\beta)=s\,\underline{u_F}_{i+1}^{n}(x,t;\beta)+(1-2s)\underline{u_F}_i^{n}(x,t;\beta)+s\,\underline{u_F}_{i-1}^{n}(x,t;\beta)\ +\Delta t\,\underline{q_F}(x,t,\beta)\\ \overline{u_F}_i^{n+1}(x,t;\beta)=s\,\overline{u_F}_{i+1}^{n}(x,t;\beta)+(1-2s)\overline{u_F}_i^{n}(x,t;\beta)+s\,\overline{u_F}_{i-1}^{n}(x,t;\beta)\ +\Delta t\,\overline{q_F}(x,t,\beta)\end{matrix} \quad (37)$$

4. Numerical Experiment

Consider NFHE along with its corresponding neutrosophic fuzzy initial and boundary conditions [17].

$$\frac{\partial\tilde{u}(x,t)}{\partial t}=\frac{\partial^2\tilde{u}(x,t)}{\partial x^2}\ ,\quad t>0\ ,\ \ 0\le x\le 1 \quad (38)$$

Conditional on the fuzzy intuitionistic boundary condition $\tilde{u}(1,t) = \tilde{u}(0,t) = 0$ and fuzzy neutrosophic initial

$$\tilde{u}(x,0) = \tilde{\mu}\sin(\pi x)\,,\ 0 < x < 1. \tag{39}$$

In α,β,γ-cut of the neutrosophic fuzzy number will be same as follow:

$$\tilde{\mu}(\alpha,\beta,\gamma) = \left\{ \left[\underline{\mu}(\alpha),\overline{\mu}(\alpha)\right], \left[\underline{\mu}(\beta),\overline{\mu}(\beta)\right], \left[\underline{\mu}(\gamma),\overline{\mu}(\gamma)\right] \right\}$$

$$= \{\, [\alpha\,, 2-\alpha], [1-0.5\beta, 1+0.5\beta], [1-0.25\gamma, 1+0.25\gamma] \,\}$$

The neutrosophic analytical solution of Eq. (38) was obtained in [17]:

$$\begin{cases} \underline{V}(x,t;\alpha) = \underline{\mu}(\alpha)e^{-\pi^2 t}\sin(\pi x) \\ \overline{V}(x,t;\alpha) = \overline{\mu}(\alpha)\, e^{-\pi^2 t}\sin(\pi x) \\ \underline{V}(x,t;\beta) = \underline{\mu}(\beta)e^{-\pi^2 t}\sin(\pi x) \\ \overline{V}(x,t;\beta) = \overline{\mu}(\beta)\, e^{-\pi^2 t}\sin(\pi x) \\ \underline{V}(x,t;\gamma) = \underline{\mu}(\gamma)e^{-\pi^2 t}\sin(\pi x) \\ \overline{V}(x,t;\gamma) = \overline{\mu}(\gamma)\, e^{-\pi^2 t}\sin(\pi x) \end{cases} \tag{40}$$

At $\Delta x = 0.1$ and $\Delta t = 0.01$, we obtained the results as the follows:

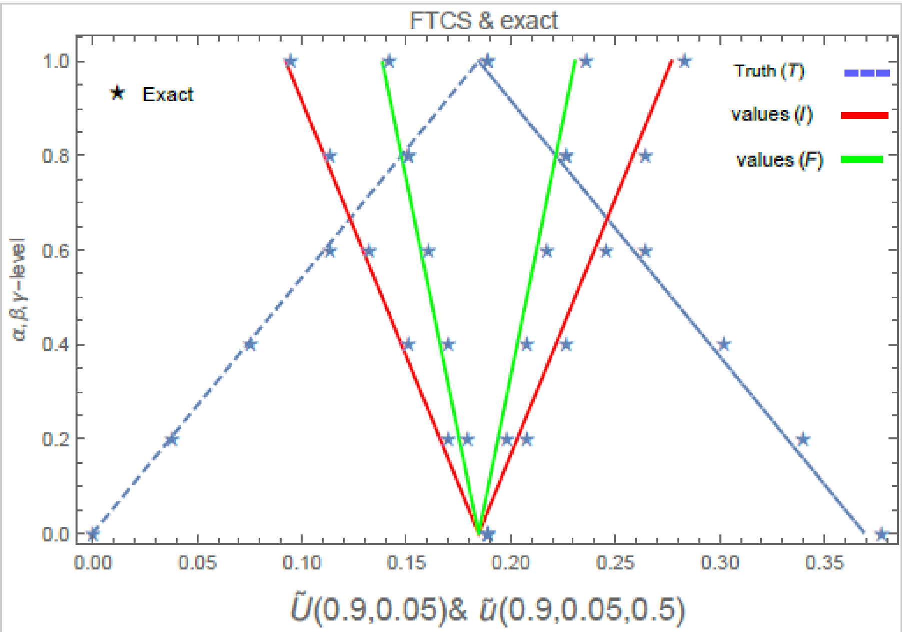

Figure 1: Exact and Neutrosophic fuzzy solution of Eq. (38) by explicit FDM at $x = 0.9, t = 0.05$ and for all $\alpha, \beta, \gamma \in [0,1]$.

The results obtained shows that the proposed explicit FDM methods and exact solution at $x = 0.9, t = 0.05$, and for all $\alpha,\beta,\gamma \in [0,1]$ attaining the triangular fuzzy number shape and thus satisfies the Neutrosophic fuzzy number properties.

Conclusions

In this study, the neutrosophic fuzzy heat equation is examined, incorporating uncertain variables and parameters, where the variables and parameters are considered as neutrosophic fuzzy numbers. An explicit scheme is reformulated and implemented to solve the neutrosophic fuzzy heat equation. The triangular neutrosophic number is used in both the neutrosophic exact and numerical solution on the (α,β,γ)-cut. A numerical example is presented to illustrate the approach. It was found that the results obtained show good agreement with both the exact solution.

Conflicts of Interest: The authors declare no conflict of interest

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Various Degrees of Directed Single Valued Neutrosophic Graphs

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Abstract. This paper introduces the concept of Directed Single Valued Neutrosophic Graphs (DSVNG) and explains how to determine the degree and total degree of vertices, considering both indegree and outdegree, along with the maximum and minimum degrees of a DSVNG. The properties of these aspects are analyzed through examples. Additionally, the paper identifies the effective edges of a DSVNG and explores the effective indegree, effective outdegree of vertices, as well as the effective maximum and minimum degrees in relation to indegree and outdegree with examples.

Keywords: Neutrosophic graphs, Directed fuzzy graphs, Indegree, Outdegree, Effective edges.

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1. Introduction

The Neutrosophic Set [6], introduced by Smarandache, is an effective tool for managing incomplete, indeterminate, and inconsistent information in real-world scenarios. A neutrosophic set is defined by three independent membership degrees: truth (T), indeterminacy (I), and falsity (F), each ranging within the real standard or nonstandard unit interval $(0^-, 1^+)$. When these degrees are confined to the real standard unit interval $[0, 1]$, neutrosophic sets become more applicable to engineering problems. Wang et al. [4] introduced the concept of a Single-Valued Neutrosophic Set (SVNS), a subclass of the neutrosophic set, to facilitate its application. They also proposed interval-valued neutrosophic sets [5], where the truth, indeterminacy, and falsity membership degrees are represented as intervals rather than single real numbers. Neutrosophic sets and their extensions, such as SVNS, interval neutrosophic sets, and simplified neutrosophic sets, have found applications in a broad range of fields, including computer science, engineering, mathematics, medicine, and economics.

A directed fuzzy graph is a mathematical structure that extends the concept of a traditional graph by incorporating the idea of fuzziness into the relationships between vertices. In a directed fuzzy graph, each edge has a direction and is associated with a membership value that indicates the strength or degree of connection between two vertices. This membership value, typically ranging between 0 and 1, allows the graph to represent uncertainty, imprecision, or partial relationships in a network. Directed fuzzy graphs are particularly useful when the connections between elements are not strictly binary but instead possess varying degrees of strength, such as in social networks, decision-making processes.

A Single-Valued Neutrosophic Graph (SVNG) is an advanced generalization of fuzzy and intuitionistic fuzzy graphs, designed to model more complex and uncertain relationships. In an SVNG, each edge is characterized by three independent membership degrees: truth (T), indeterminacy (I), and falsity (F). These degrees are real values within the interval [0, 1], allowing the graph to represent not only the certainty and strength of a relationship (as in fuzzy graphs) but also the degree of indeterminacy or ambiguity. This makes SVNGs a powerful tool for handling incomplete, inconsistent, and uncertain information in various applications, such as decision support systems, knowledge representation, and network analysis. The ability to separately account for truth, indeterminacy, and falsity provides a more flexible environment for modeling real-world problems.

In section 2, basic definitions related to neutrosophic sets and graphs are provided. Section 3 introduces the concept of Directed Single Valued Neutrosophic Graphs (DSVNG) and explains how to determine the degree, total degree of vertices in terms of indegree and outdegree, as well as the maximum and minimum degrees of a DSVNG. Their properties are analyzed with examples. In section 4, the effective edges of a DSVNG are identified, and the effective indegree, effective outdegree of vertices, as well as the effective maximum and minimum degrees of a DSVNG in relation to indegree and outdegree, are explored with examples.

2. **Preliminaries**

In this section basic definitions related to neutrosophic sets and graphs are provided.

Definition 2.1. [2] A neutrosophic set N for an universe X is defined as $\mathsf{N} = \{\langle x, T_{\mathsf{N}}(x), I_{\mathsf{N}}(x), F_{\mathsf{N}}(x)\rangle : x \in \mathsf{X}\}$, where T_{N}, I_{N}, F_{N} denotes the truth, indeterminacy and falsity membership functions respectively from X to $(0^-, 1^+)$ such that $0^- \leq \sup T_{\mathsf{N}}(x) + \sup I_{\mathsf{N}}(x) + \sup F_{\mathsf{N}}(x) \leq 3^+$. The set of all neutrosophic set over X is denoted by $\mathcal{N}(\mathsf{X})$.

Definition 2.2. [4] A single valued neutrosophic set is a neutrosophic set in which the truth, indeterminacy and falsity membership functions are $T_{\mathsf{N}}, I_{\mathsf{N}}, F_{\mathsf{N}}$ respectively, and they are from X to $[0, 1]$.

Definition 2.3. [1] A directed fuzzy graph is a fuzzy graph in which the edges are ordered pair of vertices.

Definition 2.4. [3] A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair $G = (A, B)$ where

(1) The functions $T_A : V \to [0, 1]$,$I_A : V \to [0, 1]$, and $F_A : V \to [0, 1]$, denote the degree of truth, indeterminacy and falsity membership of the element $v_i \in V$, respectively, and

$$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$$

for all $v_i \in V$.

(2) The functions $T_B : E \subseteq V \times V \to [0, 1]$,$I_B : E \subseteq V \times V \to [0, 1]$, and $F_B : E \subseteq V \times V \to [0, 1]$, denote the degree of truth, indeterminacy and falsity membership of the element $(v_i, v_j) \in E$, respectively such that $T_B(v_i, v_j) \leq \min(T_A(v_i), T_A(v_j))$, $I_B(v_i, v_j) \geq \max(I_A(v_i), I_A(v_j))$, $F_B(v_i, v_j) \geq \max(F_A(v_i), F_A(v_j))$ and

$$0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$$

for all $(v_i, v_j) \in E$.

3. Directed Single Valued Neutrosophic Graphs

In this section the concept of Directed Single Valued Neutrosophic Graph is introduced and degree, total degree of vertices with respect to indegree and outdegree, maximum degree and minimum degree of DSVNG are estabilished and their properties are analysed with example.

Definition 3.1. A directed single valued neutrosophic graph(DSVNG) $\vec{G}(A, B)$ is a single valued neutrosophic graph(SVNG) in which the edges are ordered pair of vertices.

Definition 3.2. Let $\vec{G}(A, B)$ be a DSVNG. Then the indegree of a vertex $v_i \in V$ is denoted by $d^-(v_i) = (d^-_T(v_i), d^-_I(v_i), d^-_F(v_i))$ where $d^-_T(v_i) = \sum_{\overrightarrow{v_jv_i} \in E} T_B(\overrightarrow{v_jv_i})$ denotes the indegree of truth membership of vertex v_i. $d^-_I(v_i) = \sum_{\overrightarrow{v_jv_i} \in E} I_B(\overrightarrow{v_jv_i})$ denotes the indegree of indeterminacy membership of vertex v_i. $d^-_F(v_i) = \sum_{\overrightarrow{v_jv_i} \in E} F_B(\overrightarrow{v_jv_i})$ denotes the indegree of falsity membership of vertex v_i.

Definition 3.3. Let $\vec{G}(A, B)$ be a DSVNG. Then the outdegree of a vertex $v_i \in V$ is denoted by $d^+(v_i) = (d^+_T(v_i), d^+_I(v_i), d^+_F(v_i))$ where $d^+_T(v_i) = \sum_{\overrightarrow{v_iv_j} \in E} T_B(\overrightarrow{v_iv_j})$ denotes the outdegree of truth membership of vertex v_i. $d^+_I(v_i) = \sum_{\overrightarrow{v_iv_j} \in E} I_B(\overrightarrow{v_iv_j})$ denotes the outdegree of indeterminacy membership of vertex v_i. $d^+_F(v_i) = \sum_{\overrightarrow{v_iv_j} \in E} F_B(\overrightarrow{v_iv_j})$ denotes the outdegree of falsity membership of vertex v_i.

Definition 3.4. Let $\vec{G}(A, B)$ be a DSVNG. Then the degree of a vertex $v_i \in V$ is $d(v_i) = d^-(v_i) + d^+(v_i) = (d^-_T(v_i) + d^+_T(v_i), d^-_I(v_i) + d^+_I(v_i), d^-_F(v_i) + d^+_F(v_i))$.

Definition 3.5. Let $\overrightarrow{G}(A, B)$ be a DSVNG. Then the total indegree of a vertex $v_i \in V$ is $td^-(v_i) = (td_T^-(v_i), td_I^-(v_i), td_F^-(v_i))$ where

$$td_T^-(v_i) = \sum_{\overrightarrow{v_jv_i} \in E} T_B(\overrightarrow{v_jv_i}) + T_A(v_i)$$
$$td_I^-(v_i) = \sum_{\overrightarrow{v_jv_i} \in E} I_B(\overrightarrow{v_jv_i}) + I_A(v_i)$$
$$td_F^-(v_i) = \sum_{\overrightarrow{v_jv_i} \in E} F_B(\overrightarrow{v_jv_i}) + F_A(v_i)$$

$td_T^-(v_i)$, $td_I^-(v_i)$, $td_F^-(v_i)$ denotes the total indegree of truth, indeterminacy, falsity membership of vertex v_i respectively.

Definition 3.6. Let $\overrightarrow{G}(A, B)$ be a DSVNG. Then the total outdegree of a vertex $v_i \in V$ is $td^+(v_i)$=($td_T^+(v_i)$, $td_I^+(v_i)$, $td_F^+(v_i)$) where

$$td_T^+(v_i) = \sum_{\overrightarrow{v_iv_j} \in E} T_B(\overrightarrow{v_iv_j}) + T_A(v_i)$$
$$td_I^+(v_i) = \sum_{\overrightarrow{v_iv_j} \in E} I_B(\overrightarrow{v_iv_j}) + I_A(v_i)$$
$$td_F^+(v_i) = \sum_{\overrightarrow{v_iv_j} \in E} F_B(\overrightarrow{v_iv_j}) + F_A(v_i)$$

$td_T^+(v_i), td_I^+(v_i), td_F^+(v_i)$ denotes the total outdegree of truth, indeterminacy, falsity membership of vertex v_i respectively.

Definition 3.7. Let $\overrightarrow{G}(A, B)$ be a DSVNG. Then the total degree of a vertex $v_i \in V$ is $td(v_i) = td^-(v_i) + td^+(v_i) = (td_T^-(v_i) + td_T^+(v_i), td_I^-(v_i) + td_I^+(v_i), td_F^-(v_i) + td_F^+(v_i))$.

Definition 3.8. Let $\overrightarrow{G}(A, B)$ be a DSVNG. Then the minimum indegree of $\overrightarrow{G}$ is $\delta^-(\overrightarrow{G}) = (\delta_T^-(\overrightarrow{G}), \delta_I^-(\overrightarrow{G}), \delta_F^-(\overrightarrow{G}))$ where

$$\delta_T^-(\overrightarrow{G}) = \wedge\{d_T^-(v_i) | v_i \in V\}$$
$$\delta_I^-(\overrightarrow{G}) = \wedge\{d_I^-(v_i) | v_i \in V\}$$
$$\delta_F^-(\overrightarrow{G}) = \wedge\{d_F^-(v_i) | v_i \in V\}$$

$\delta_T^-(\overrightarrow{G})$, $\delta_I^-(\overrightarrow{G})$, $\delta_F^-(\overrightarrow{G})$ denotes the minimum indegree of truth, indeterminacy, falsity membership of $\overrightarrow{G}$ respectively.

Definition 3.9. Let $\vec{G}(A, B)$ be a DSVNG. Then the minimum outdegree of $\vec{G}$ is $\delta^+(\vec{G}) = (\delta_T^+(\vec{G}), \delta_I^+(\vec{G}), \delta_F^+(\vec{G}))$ where

$$\delta_T^+(\vec{G}) = \wedge\{d_T^+(v_i)|v_i \in V\}$$
$$\delta_I^+(\vec{G}) = \wedge\{d_I^+(v_i)|v_i \in V\}$$
$$\delta_F^+(\vec{G}) = \wedge\{d_F^+(v_i)|v_i \in V\}$$

$\delta_T^+(\vec{G})$, $\delta_I^+(\vec{G})$, $\delta_F^+(\vec{G})$ denotes the minimum outdegree of truth, indeterminacy, falsity membership of $\vec{G}$ respectively.

Definition 3.10. Let $\vec{G}(A, B)$ be a DSVNG. Then the minimum degree of a vertex $v_i \in V$ is $\delta(\vec{G}) = \delta^-(\vec{G}) + \delta^+(\vec{G}) = (\delta_T^-(\vec{G}) + \delta_T^+(\vec{G}), \delta_I^-(\vec{G}) + \delta_I^+(\vec{G}), \delta_F^-(\vec{G}) + \delta_F^+(\vec{G}))$.

Definition 3.11. Let $\vec{G}(A, B)$ be a DSVNG. Then the maximum indegree of $\vec{G}$ is $\Delta^-(\vec{G}) = (\Delta_T^-(\vec{G}), \Delta_I^-(\vec{G}), \Delta_F^-(\vec{G}))$ where

$$\Delta_T^-(\vec{G}) = \vee\{d_T^-(v_i)|v_i \in V\}$$
$$\Delta_I^-(\vec{G}) = \vee\{d_I^-(v_i)|v_i \in V\}$$
$$\Delta_F^-(\vec{G}) = \vee\{d_F^-(v_i)|v_i \in V\}$$

$\Delta_T^-(\vec{G})$, $\Delta_I^-(\vec{G})$, $\Delta_F^-(\vec{G})$ denotes the maximum indegree of truth, indeterminacy, falsity membership of $\vec{G}$ respectively.

Definition 3.12. Let $\vec{G}(A, B)$ be a DSVNG. Then the maximum outdegree of $\vec{G}$ is $\Delta^+(\vec{G}) = (\Delta_T^+(\vec{G}), \Delta_I^+(\vec{G}), \Delta_F^+(\vec{G}))$ where

$$\Delta_T^+(\vec{G}) = \vee\{d_T^+(v_i)|v_i \in V\}$$
$$\Delta_I^+(\vec{G}) = \vee\{d_I^+(v_i)|v_i \in V\}$$
$$\Delta_F^+(\vec{G}) = \vee\{d_F^+(v_i)|v_i \in V\}$$

$\Delta_T^+(\vec{G})$, $\Delta_I^+(\vec{G})$, $\Delta_F^+(\vec{G})$ denotes the maximum outdegree of truth, indeterminacy, falsity membership of $\vec{G}$ respectively.

Definition 3.13. Let $\vec{G}(A, B)$ be a DSVNG. Then the maximum degree of $\vec{G}$ is $\Delta(\vec{G}) = \Delta^-(\vec{G}) + \Delta^+(\vec{G}) = (\Delta_T^-(\vec{G}) + \Delta_T^+(\vec{G}), \Delta_I^-(\vec{G}) + \Delta_I^+(\vec{G}), \Delta_F^-(\vec{G}) + \Delta_F^+(\vec{G}))$.

Remark 3.14. For any vertex v_i, $\delta_T(\vec{G}) \leq d_T(v_i) \leq \Delta_T(\vec{G})$, $\delta_I(\vec{G}) \leq d_I(v_i) \leq \Delta_I(\vec{G})$ and $\delta_F(\vec{G}) \leq d_F(v_i) \leq \Delta_F(\vec{G})$.

Proposition 3.15. *For any DSVNG $\vec{G}(A,B)$ with n vertices,*

$$\sum_{v_i\in V} d(v_i) = \big(\sum_{v_i\in V}(d_T^+(v_i)+d_T^-(v_i)), \sum_{v_i\in V}(d_I^+(v_i)+d_I^-(v_i)), \sum_{v_i\in V}(d_F^+(v_i)+d_F^-(v_i))\big)$$
$$= \big(\sum_{v_i\neq v_j} T_B^+(\overrightarrow{v_iv_j}) + T_B^-(\overrightarrow{v_jv_i}), \sum_{v_i\neq v_j} I_B^+(\overrightarrow{v_iv_j}) + I_B^-(\overrightarrow{v_jv_i}),$$
$$\sum_{v_i\neq v_j} F_B^+(\overrightarrow{v_iv_j}) + F_B^-(\overrightarrow{v_jv_i})\big).$$

Proof. Proof follows from the definition of indegree and outdegree of DSVNG $\vec{G}(A,B)$. □

Proposition 3.16. *The maximum indegree or outdegree of any vertex in a DSVNG with n vertices is $n-1$.*

Proof. Let $\vec{G}(A,B)$ be a DSVNG. The maximum value given to an edge is 1 and the number of edges incident on a vertex can be atmost $n-1$. Similarly, the maximum indeterminacy and falsity membership value given to an edge is 1 and the number of edges incident on a vertex can be at most $n-1$. Hence, the maximum truth, indeterminacy and falsity membership degree $d_T(v_i)$, $d_I(v_i)$, $d_F(v_i)$ of any vertex v_i in a DSVNG with n vertices is $n-1$. Hence the result. □

Example 3.17. Consider the DSVNG $\vec{G}$ given in Figure 1 and the truth, indeterminacy and falsity membership of vertices and edges of $\vec{G}$ is provided in Table 1, 2.

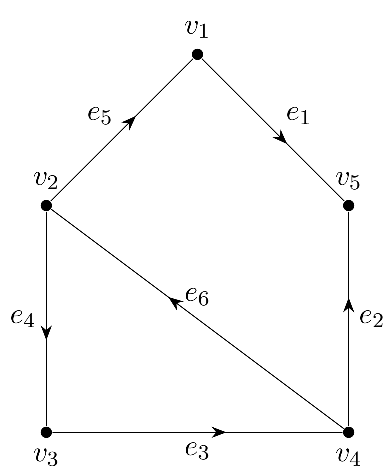

FIGURE 1. DSVNG

TABLE 1. Truth, indeterminacy and falsity membership of vertices

V	v_1	v_2	v_3	v_4	v_5
T_A	0.5	0.5	0.6	0.5	0.3
I_A	0.3	0.7	0.6	0.2	0.4
F_A	0.2	0.3	0.4	0.8	0.7

TABLE 2. Truth, indeterminacy and falsity membership of edges

E	e_1	e_2	e_3	e_4	e_5	e_6
T_B	0.2	0.3	0.4	0.5	0.4	0.4
I_B	0.5	0.5	0.8	0.7	0.7	0.8
F_B	0.7	0.9	0.9	0.6	0.4	0.9

Table 3 provides the degree, indegree, outdegree, total degree, total indegree, total outdegree of vertices of $\vec{G}$.

TABLE 3. Degree and total degree of vertices

(A) Degree

V	v_1	v_2	v_3	v_4	v_5
Indegree of v_i					
d_T^-	0.4	0.4	0.5	0.4	0.5
d_I^-	0.7	0.8	0.7	0.8	1
d_F^-	0.4	0.9	0.6	0.9	1.6
Outdegree of v_i					
d_T^+	0.2	0.9	0.4	0.7	0
d_I^+	0.5	1.4	0.8	1.3	0
d_F^+	0.7	1	0.9	1.8	0
Degree of v_i					
d_T	0.6	1.3	0.9	1.1	0.5
d_I	1.2	2.1	1.3	2.1	1
d_F	1.1	2.4	1.5	2.7	1.6

(B) Total degree

V	v_1	v_2	v_3	v_4	v_5
Total indegree of v_i					
td_T^-	0.9	0.9	1.1	0.9	0.8
td_I^-	1	1.5	1.3	1	0.5
td_F^-	0.6	1.2	1	1.7	2.3
Total outdegree of v_i					
td_T^+	0.7	1.4	1	1.2	0.3
td_I^+	0.8	2.1	1.4	1.5	0.4
td_F^+	0.9	1.3	1.3	2.6	0.7
Total degree of v_i					
td_T	1.6	2.3	2.1	2.1	1.1
td_I	1.8	3.6	2.7	2.5	0.9
td_F	1.5	2.5	2.3	4.3	3

Table 4, 5 provides the minimum degree, indegree, outdegree and maximum degree, indegree, outdegree of $\vec{G}$.

Table 4. Minimum degree of DSVNG.

$\vec{G}$	$\delta^-(\vec{G})$	$\vec{G}$	$\delta^+(\vec{G})$	$\vec{G}$	$\delta(\vec{G})$
$\delta_T^-(\vec{G})$	0.4	$\delta_T^+(\vec{G})$	0	$\delta_T(\vec{G})$	0.4
$\delta_I^-(\vec{G})$	0.7	$\delta_I^+(\vec{G})$	0	$\delta_I(\vec{G})$	0.7
$\delta_F^-(\vec{G})$	0.4	$\delta_F^+(\vec{G})$	0	$\delta_F(\vec{G})$	0.4

Table 5. Maximum degree of DSVNG.

$\vec{G}$	$\Delta^-(\vec{G})$	$\vec{G}$	$\Delta^+(\vec{G})$	$\vec{G}$	$\Delta(\vec{G})$
$\Delta_T^-(\vec{G})$	0.5	$\Delta_T^+(\vec{G})$	0.9	$\Delta_T(\vec{G})$	1.4
$\Delta_I^-(\vec{G})$	1	$\Delta_I^+(\vec{G})$	1.4	$\Delta_I(\vec{G})$	2.4
$\Delta_F^-(\vec{G})$	1.6	$\Delta_F^+(\vec{G})$	1.8	$\Delta_F(\vec{G})$	3.4

4. Effective edges

In this section the effective edges of a Directed Single Valued Neutrosophic Graph is identified and efffective indegree, efffective outdegree of vertex, efffective maximum degree, efffective minimum degree of DSVNG with respect to indegree and outdegree are investigated with example.

Definition 4.1. An edge $\overrightarrow{v_iv_j}$ of a DSVNG $\vec{G}(A,B)$ is called an effective edge if $T_B(\overrightarrow{v_i,v_j}) = T_A(v_i) \wedge T_A(v_j)$, $I_B(\overrightarrow{v_i,v_j}) = I_A(v_i) \vee I_A(v_j)$, and $F_B(\overrightarrow{v_i,v_j}) = F_A(v_i) \vee F_A(v_j)$.

Definition 4.2. The effective indegree of a vertex v_i in $\vec{G}(A,B)$ is defined by $d_E^-(v_i) = (d_{ET}^-(v_i), d_{EI}^-(v_i), d_{EF}^-(v_i))$, where $d_{ET}^-(v_i)$ is the sum of the truth membership values of the effective edges with v_i as the terminal vertex, $d_{EI}^-(v_i)$ is the sum of the indeterminacy membership values of the effective edges with v_i as the terminal vertex, $d_{EF}^-(v_i)$ is the sum of the falsity membership values of the effective edges with v_i as the terminal vertex.

Definition 4.3. The effective outdegree of a vertex v_i in $\vec{G}(A,B)$ is defined by $d_E^+(v_i) = (d_{ET}^+(v_i), d_{EI}^+(v_i), d_{EF}^+(v_i))$, where $d_{ET}^+(v_i)$ is the sum of the truth membership values of the effective edges with v_i as the terminal vertex, $d^+_{EI}(v_i)$ is the sum of the indeterminacy membership values of the effective edges with v_i as the terminal vertex, $d^+_{EF}(v_i)$ is the sum of the falsity membership values of the effective edges with v_i as the terminal vertex.

Definition 4.4. Let $\vec{G}(A, B)$ be a DSVNG. Then the effective degree of a vertex $v_i \in V$ is $d_E(v_i) = d^-_E(v_i) + d^+_E(v_i) = (d^-_{ET}(v_i) + d^+_{ET}(v_i), d^-_{EI}(v_i) + d^+_{EI}(v_i), d^-_{EF}(v_i) + d^+_{EF}(v_i))$.

Definition 4.5. The minimum effective indegree of $\vec{G}$ is $\delta^-_E[\vec{G}] = (\delta^-_{ET}[\vec{G}],\ \delta^-_{EI}[\vec{G}],\ \delta^-_{EF}[\vec{G}])$ where

$$\delta^-_{ET}[\vec{G}] = \wedge\{d^-_{ET}[v_i] \mid v_i \in V\}$$
$$\delta^-_{EI}[\vec{G}] = \wedge\{d^-_{EI}[v_i] \mid v_i \in V\}$$
$$\delta^-_{EF}[\vec{G}] = \wedge\{d^-_{EF}[v_i] \mid v_i \in V\}$$

$\delta^-_{ET}[\vec{G}]$, $\delta^-_{EI}[\vec{G}]$, $\delta^-_{EF}[\vec{G}]$ denotes the minimum effective T-indegree, I-indegree, F-indegree respectively.

Definition 4.6. The minimum effective outdegree of $\vec{G}$ is $\delta^+_E[\vec{G}] = (\delta^+_{ET}[\vec{G}],\ \delta^+_{EI}[\vec{G}],\ \delta^+_{EF}[\vec{G}])$ where

$$\delta^+_{ET}[\vec{G}] = \wedge\{d^+_{ET}[v_i] \mid v_i \in V\}$$
$$\delta^+_{EI}[\vec{G}] = \wedge\{d^+_{EI}[v_i] \mid v_i \in V\}$$
$$\delta^+_{EF}[\vec{G}] = \wedge\{d^+_{EF}[v_i] \mid v_i \in V\}$$

$\delta^+_{ET}[\vec{G}]$, $\delta^+_{EI}[\vec{G}]$, $\delta^+_{EF}[\vec{G}]$ denotes the maximum effective T-indegree, I-indegree, F-indegree respectively.

Definition 4.7. Let $\vec{G}(A, B)$ be a DSVNG. Then the minimum effective degree of $\vec{G}$ is $\delta_E(\vec{G}) = \delta^-_E(\vec{G}) + \delta^+_E(\vec{G}) = (\delta^-_{ET}(\vec{G}) + \delta^+_{ET}(\vec{G}), \delta^-_{EI}(\vec{G}) + \delta^+_{EI}(\vec{G}), \delta^-_{EF}(\vec{G}) + \delta^+_{EF}(\vec{G}))$.

Definition 4.8. The maximum effective indegree of $\vec{G}$ is $\Delta^-_E[\vec{G}] = (\Delta^-_{ET}[\vec{G}], \Delta^-_{EI}[\vec{G}], \Delta^-_{EF}[\vec{G}])$ where

$$\Delta^-_{ET}[\vec{G}] = \vee\{d^-_{ET}[v_i] \mid v_i \in V\}$$
$$\Delta^-_{EI}[\vec{G}] = \vee\{d^-_{EI}[v_i] \mid v_i \in V\}$$
$$\Delta^-_{EF}[\vec{G}] = \vee\{d^-_{EF}[v_i] \mid v_i \in V\}$$

$\Delta^-_{ET}[\vec{G}]$, $\Delta^-_{EI}[\vec{G}]$, $\Delta^-_{EF}[\vec{G}]$ denotes the minimum effective T-indegree, I-indegree, F-indegree respectively.

Definition 4.9. The maximum effective outdegree of $\vec{G}$ is $\Delta^+_E[\vec{G}] = (\Delta^+_{ET}[\vec{G}],\ \Delta^+_{EI}[\vec{G}],\ \Delta^+_{EF}[\vec{G}])$ where

$$\Delta^+_{ET}[\vec{G}] = \vee\{d^+_{ET}[v_i] \mid v_i \in V\}$$
$$\Delta^+_{EI}[\vec{G}] = \vee\{d^+_{EI}[v_i] \mid v_i \in V\}$$
$$\Delta^+_{EF}[\vec{G}] = \vee\{d^+_{EF}[v_i] \mid v_i \in V\}$$

$\Delta^+_{ET}[\vec{G}]$, $\Delta^+_{EI}[\vec{G}]$, $\Delta^+_{EF}[\vec{G}]$ denotes the maximum effective T-indegree, I-indegree, F-indegree respectively.

Definition 4.10. Let $\vec{G}(A, B)$ be a DSVNG. Then the maximum effective degree of $\vec{G}$ is $\Delta_E(\vec{G}) = \Delta^-_E(\vec{G}) + \Delta^+_E(\vec{G}) = (\Delta^-_{ET}(\vec{G}) + \Delta^+_{ET}(\vec{G}), \Delta^-_{EI}(\vec{G}) + \Delta^+_{EI}(\vec{G}), \Delta^-_{EF}(\vec{G}) + \Delta^+_{EF}(\vec{G}))$.

Example 4.11. Consider the DSVNG $\vec{G}$ given in Figure 2 and the truth, indeterminacy and falsity membership of vertices and edges of $\vec{G}$ is provided in Table 6.

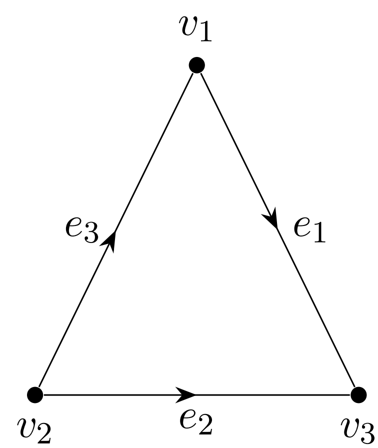

FIGURE 2. DSVNG

TABLE 6. Truth, indeterminacy and falsity membership of vertices and edges

(A) Membership of vertices

$\vec{G}$	v_1	v_2	v_3
T_A	0.5	0.2	0.3
I_A	0.3	0.6	0.4
F_A	0.2	0.7	0.2

(B) Membership of edges

$\vec{G}$	e_1	e_2	e_3
T_B	0.3	0.2	0.2
I_B	0.4	0.6	0.6
F_B	0.2	0.7	0.7

Table 7 provides the effective degree, effective indegree, effective outdegree of vertices of $\vec{G}$. Table 8, 9 provides the minimum effective degree, indegree, outdegree and maximum effective degree, indegree, outdegree of $\vec{G}$.

Table 7. Effective degree of DSVNG

(a) Indegree of DSVNG

$\vec{G}$	v_1	v_2	v_3
Effective indegree			
d^-_{ET}	0.2	0	0.5
d^-_{EI}	0.6	0	1.0
d^-_{EF}	0.7	0	0.9

(b) Outdegree of DSVNG

$\vec{G}$	v_1	v_2	v_3
Effective outdegree			
d^+_{ET}	0.3	0.4	0
d^+_{EI}	0.4	1.2	0
d^+_{EF}	0.2	1.4	0

(c) Degree of DSVNG

$\vec{G}$	v_1	v_2	v_3
Effective degree			
d_{ET}	0.5	0.4	0.5
d_{EI}	1.0	1.2	1.0
d_{EF}	0.9	1.4	0.9

Table 8. Minimum effective degree of DSVNG.

$\vec{G}$	$\delta^-(\vec{G})$	$\vec{G}$	$\delta^+(\vec{G})$	$\vec{G}$	$\delta(\vec{G})$
$\delta^-_{ET}(\vec{G})$	0	$\delta^+_{ET}(\vec{G})$	0	$\delta_{ET}(\vec{G})$	0
$\delta^-_{EI}(\vec{G})$	0	$\delta^+_{EI}(\vec{G})$	0	$\delta_{EI}(\vec{G})$	0
$\delta^-_{EF}(\vec{G})$	0	$\delta^+_{EF}(\vec{G})$	0	$\delta_{EF}(\vec{G})$	0

Table 9. Maximum effective degree of DSVNG.

$\vec{G}$	$\Delta^-(\vec{G})$	$\vec{G}$	$\Delta^+(\vec{G})$	$\vec{G}$	$\Delta(\vec{G})$
$\Delta^-_{ET}(\vec{G})$	0.5	$\Delta^+_{ET}(\vec{G})$	0.4	$\Delta_{ET}(\vec{G})$	0.9
$\Delta^-_{EI}(\vec{G})$	1	$\Delta^+_{EI}(\vec{G})$	1.2	$\Delta_{EI}(\vec{G})$	2.2
$\Delta^-_{EF}(\vec{G})$	0.9	$\Delta^+_{EF}(\vec{G})$	1.4	$\Delta_{EF}(\vec{G})$	2.3

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A Study on the Number of Neutrosophic Crisp Topological Spaces in a Finite Set

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Abstract. The computation of formulae for the number of topological spaces is one of the challenging areas of study. The present work aims to find formulae to compute the number of neutrosophic crisp topological spaces having 2-NCrOSs, 3-NCrOSs, and 4-NCrOSs.

Keywords: Topological spaces; neutrosophic topological spaces, neutrosophic crisp topological spaces.

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1. Introduction

Zadeh [1] introduced the fuzzy set and Atanassov [2] introduced the intuitionistic fuzzy set. By generalizing the crisp and fuzzy counterparts, neutrosophy has established the groundwork for an entire family of new mathematical theories. The concept of a "neutrosophic set" is introduced by Smarandache [3–5]. Later, some possible definitions for basic concepts of the neutrosophic crisp set and its operations have been investigated by Hanafy et al. [6] and Salama [7].

A topology tells how elements of the set are related to each other. From the literature, it is found that the explicit formula for finding the number of topologies in a set is still not obtained. This is one of the fascinating research areas of topology. Let τ_n denotes the number of topologies on a finite set $\mathcal{X}$ with $|S| = n$. Krishnamurty [8] computed a sharper bound namely $2^{n(n-1)}$ for τ_n. Sharp [9] shows that only discrete topology has cardinal greater than $\frac{3}{4}2^n$ and derived bounds for the cardinality of topologies which are connected, nonconnected, non-T_0, and some more. Several authors [**?**, 10–19, 21] also worked in this interesting and difficult research area. Recently Basuma tary et al. [22, 23] started research work on number of neutrosophic topological spaces. They computed some results for finding the number of neutosophic topologies for $k \leq 4$ open sets, the number of neutrosophic clopen topological spaces having small ($k = 2, 3, 4, 5$) open sets with neutrosophic values in M, and the number of neutrosophic bitopological spaces and tritopological spaces. Salama et al. [24] introduced the basic concept of neutosophic crisp topological spaces. In this paper, the formulae for computation of the number of neutrosophic crisp toplogical spaces for small ($k = 2, 3, 4$) open sets are initiated.

2. Materials and Methods

Definition 2.1. [24] Let $\mathcal{X}$ be a non-empty fixed set. A neutrosophic crisp set (NCrS) A is an object having the form $A = \langle A_1, A_2, A_3\rangle$, where A_1, A_2, and A_3 are subsets of $\mathcal{X}$ satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$.

Remark 2.2. [24] A NCrS $A = \langle A_1, A_2, A_3\rangle$ can be identified as an ordered triple $\langle A_1, A_2, A_3\rangle$, where A_1, A_2, and A_3 are subsets of $\mathcal{X}$.

Definition 2.3. [24] $\phi_{\mathcal{N}}$ may be defined in many ways as an NCrS as follows:

(1) $\phi_{\mathcal{N}} = \langle \phi, \phi, \mathcal{X}\rangle$.
(2) $\phi_{\mathcal{N}} = \langle \phi, \mathcal{X}, \mathcal{X}\rangle$.
(3) $\phi_{\mathcal{N}} = \langle \phi, \mathcal{X}, \phi\rangle$.
(4) $\phi_{\mathcal{N}} = \langle \phi, \phi, \phi\rangle$.

$\mathcal{X}_{\mathcal{N}}$ may also be defined in many ways as an NCrS as follows:

(1) $\mathcal{X}_{\mathcal{N}} = \langle \mathcal{X}, \phi, \phi\rangle$.
(2) $\mathcal{X}_{\mathcal{N}} = \langle \mathcal{X}, \mathcal{X}, \phi\rangle$.
(3) $\mathcal{X}_{\mathcal{N}} = \langle \mathcal{X}, \mathcal{X}, \mathcal{X}\rangle$.

Definition 2.4. [24] Let $\mathcal{X}$ be a non-empty set, and the NCrSs A and B be in the form $A = \langle A_1, A_2, A_3\rangle$, $B = \langle B_1, B_2, B_3\rangle$ respectively. Then the following two possible definitions may be considered for subsets ($A \subseteq B$):

(1) $A \subseteq B \iff A_1 \subseteq B_1$, $A_1 \subseteq B_2$, and $A_3 \supseteq B_3$.
(2) $A \subseteq B \iff A_1 \subseteq B_1$, $A_2 \supseteq B_2$, and $A_3 \supseteq B_3$.

Definition 2.5. [24] Let $\mathcal{X}$ is a non-empty set, and the NCrSs A and B be in the form $A = \langle A_1, A_2, A_3\rangle$ and $B = \langle B_1, B_2, B_3\rangle$ respectively. Then,

(1) $A \cap B$ may be defined in two ways:
  (a) $A \cap B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3\rangle$.
  (b) $A \cap B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3\rangle$.

(2) $A \cup B$ may also be defined in two ways:

(a) $A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle$.

(b) $A \cup B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle$.

Definition 2.6. [17] The number of partitions of a finite set with n elements into k blocks, is the Stirling number of the second kind. It is denoted by $S(n,k)$ or $S_{n,k}$. The explicit formula for Stirling numbers of the second kind is

$$S(n,k) = S_{n,k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

Definition 2.7. [25] The set of all neutrosophic crisp subsets of a non-empty finite set $\mathcal{X}$ is called the neutrosophic crisp power set of $\mathcal{X}$.

The notation for the neutrosophic crisp power set of $\mathcal{X}$ is $\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})$ and its cardinality is denoted by $|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})|$.

Proposition 2.8. *[25] A set $\mathcal{X}$ with $|\mathcal{X}| = n$ has*

$$(3.2^n - 4) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(i,3)\binom{n}{j}\right\}$$

neutrosophic crisp subsets.

Corollary 2.9. *[25] If $|\mathcal{X}| = n$, then the cardinality of the power set of NCrS of $\mathcal{X}$ is*

$$|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| = (3.2^n - 4) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}.$$

Definition 2.10. [24] A neutrosophic crisp topology (NCrT) on a non-empty set $\mathcal{X}$ is a family τ^{NC} of neutrosophic crisp subsets in $\mathcal{X}$ satisfying the following axioms

(1) $\phi_{\mathcal{N}}$, $\mathcal{X}_{\mathcal{N}} \in \tau^{NC}$.

(2) $A_1 \cap A_2 \in \tau^{NC}$; for any A_1, $A_2 \in \tau^{NC}$.

(3) $\cup A_j \in \tau^{NC}$; $\forall \{A_j : j \in J\} \subseteq \tau^{NC}$.

In this case, the pair $(\mathcal{X}, \tau^{NC})$ is called a neutrosophic crisp topological space (NCrTS) in $\mathcal{X}$. The elements in τ^{NC} are called neutrosophic crisp open sets (NCrOSs) in $\mathcal{X}$. A NCrS F is closed if and only if its complement F^c is an open NCrS.

3. Neutrosophic Crisp Topological Spaces

Definition 3.1. An NCrT having k-NCrOSs on a non-empty set $\mathcal{X}$ is said to be an NCrT of cardinality k. The number of NCrTs of cardinality k on $\mathcal{X}$ with $|\mathcal{X}| = n$ will be denoted by $\mathcal{T}_{\mathcal{C}r}(n,k)$.

Example 3.2. Let $\mathcal{X} = \{u, v, w\}$ and $\mathcal{A}_1 = \langle\emptyset, \emptyset, \{u\}\rangle$, then $\tau^{NCr} = \{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, \mathcal{A}_1\}$ form an NCrT on $\mathcal{X}$. So, τ^{NCr} is an NCrT of cardinality 3 as it has 3-NCrOSs.

Proposition 3.3. *For a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$,*

(1) $\mathcal{T}_{\mathcal{C}r}(n, 2) = 1$,

(2) $\mathcal{T}_{\mathcal{C}r}(n, k) = 1$, *where $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})|$, the cardinality of the neutrosophic crisp power set on $\mathcal{X}$.*

Proof:

(1) The NCrT having 2-NCrOSs is the indiscrete NCrT which is $\mathcal{T}_{\mathcal{N}} = \{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}\}$. Therefore, $(\mathcal{X}, \mathcal{T}_{\mathcal{N}})$ is the only NCrTS having 2-NCrOSs as $\mathcal{T}_{\mathcal{N}}$ contains only two members $\phi_{\mathcal{N}}$ and $\mathcal{X}_{\mathcal{N}}$. Hence, the number of neutrosophic crisp topological spaces (NCrTSs) having 2-NCrOSs is 1 i.e., $\mathcal{T}_{\mathcal{C}r}(n, 2) = 1$.

(2) The NCrT of cardinality $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})|$ is the discrete NCrT only. Hence, $\mathcal{T}_{\mathcal{C}r}(n, k) = 1$, for $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})|$.

Example 3.4. Let $\mathcal{X} = \{u, v\}$, then, $|\mathcal{X}| = n = 2$. Here, the neutrosophic crisp subsets on $\mathcal{X}$ are

$\phi_{\mathcal{N}}$, $\mathcal{X}_{\mathcal{N}}$, $\mathcal{A}_1 = \langle\emptyset, \emptyset, \{u\}\rangle$, $\mathcal{A}_2 = \langle\emptyset, \{u\}, \emptyset\rangle$, $\mathcal{A}_3 = \langle\{u\}, \emptyset, \emptyset\rangle$,
$\mathcal{A}_4 = \langle\emptyset, \emptyset, \{v\}\rangle$, $\mathcal{A}_5 = \langle\emptyset, \{v\}, \emptyset\rangle$, $\mathcal{A}_6 = \langle\{v\}, \emptyset, \emptyset\rangle$,
$\mathcal{A}_7 = \langle\emptyset, \{u\}, \{v\}\rangle$, $\mathcal{A}_8 = \langle\{u\}, \emptyset, \{v\}\rangle$, $\mathcal{A}_9 = \langle\{u\}, \{v\}, \emptyset\rangle$,
$\mathcal{A}_{10} = \langle\emptyset, \{v\}, \{u\}\rangle$, $\mathcal{A}_{11} = \langle\{v\}, \emptyset, \{u\}\rangle$, $\mathcal{A}_{12} = \langle\{v\}, \{u\}, \emptyset\rangle$.

In this case, the only NCrT having 2-NCrOSs is $\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}\}$ and hence $\mathcal{T}_{\mathcal{C}r}(n, 2) = 1$. Also, the NCrT having $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| = 14$-NCrOSs is

$$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{A}_{12}\}$$

and hence, $\mathcal{T}_{\mathcal{C}r}(n, k) = 1$, for $k = |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| = 14$.

4. **Neutrosophic Crisp Topological Spaces with 3-NCrOSs**

Proposition 4.1. *The number of NCrTs of cardinality 3 on a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$ is given by the formula*

$$\begin{aligned}\mathcal{T}_{\mathcal{C}r}(n, 3) &= |\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| - 2 \\ &= 3(2^n - 2) + 3!\left[\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right].\end{aligned}$$

Proof:

The NCrTs having 3-NCrOSs necessarily consists of a chain containing $\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}$ and any other neutrosophic crisp subset $\mathcal{A}_{\mathcal{N}}$ of $\mathcal{X}$ other than $\phi_{\mathcal{N}}$ and $\mathcal{X}_{\mathcal{N}}$. Clearly, $\phi_{\mathcal{N}} \subset \mathcal{A}_{\mathcal{N}} \subset \mathcal{X}_{\mathcal{N}}$. It is observed that the number of such $\mathcal{A}_{\mathcal{N}}$ is equal to $|\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| - 2$. Since the set $\{\phi_{\mathcal{N}}, \mathcal{A}_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}\}$ form an NCrT and the total number of such NCrTs is $|\mathcal{P}_{\mathcal{N}\mathcal{C}r}(\mathcal{X})| - 2$.

Now, $|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| = (3.2^n - 4) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}$.

Therefore,

$$\begin{aligned}|\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| - 2 &= \left[(3.2^n - 4) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}\right] - 2\\ &= (3.2^n - 6) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}\\ &= 3(2^n - 2) + 3!\left\{\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{T}_{\mathcal{C}r}(n,3) &= |\mathcal{P}_{\mathcal{NC}r}(\mathcal{X})| - 2\\ &= 3(2^n - 2) + 3!\left[\sum_{i=2}^{n}\mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n}\mathcal{S}(j,3)\binom{n}{j}\right].\end{aligned}$$

Example 4.2. Let $\mathcal{X} = \{u, v\}$, then

$$\mathcal{T}_{\mathcal{C}r}(2,3) = 3(2^2 - 2) + 3!\left\{\sum_{i=2}^{2}\mathcal{S}(i,2)\binom{2}{i} + \sum_{j=3}^{2}\mathcal{S}(j,3)\binom{2}{j}\right\}.$$

Clearly, $\sum_{j=3}^{2}\mathcal{S}(j,3)\binom{2}{j} = 0$.

So, $\mathcal{T}_{\mathcal{C}r}(2,3) = 6 + 6\left\{\mathcal{S}(2,2)\binom{2}{2} + 0\right\} = 12$.

Consequently, $\mathcal{T}_{\mathcal{C}r}(2,3) = 12$ and these NCrTs having 3-NCrOSs are listed below

$\{\phi_{\mathcal{N}}, \mathcal{A}_1, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_2, \mathcal{X}_{\mathcal{N}}\}$,$\{\phi_{\mathcal{N}}, \mathcal{A}_3, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_4, \mathcal{X}_{\mathcal{N}}\}$,
$\{\phi_{\mathcal{N}}, \mathcal{A}_5, \mathcal{X}_{\mathcal{N}}\}$,$\{\phi_{\mathcal{N}}, \mathcal{A}_6, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_7, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_8, \mathcal{X}_{\mathcal{N}}\}$,
$\{\phi_{\mathcal{N}}, \mathcal{A}_9, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_{10}, \mathcal{X}_{\mathcal{N}}\}$, $\{\phi_{\mathcal{N}}, \mathcal{A}_{11}, \mathcal{X}_{\mathcal{N}}\}$,$\{\phi_{\mathcal{N}}, \mathcal{A}_{12}, \mathcal{X}_{\mathcal{N}}\}$.

5. **Neutrosophic Crisp Topological Spaces with 4-NCrOSs**

The NCrT having 4-NCrOSs must have the form $\mathcal{T} = \{\phi_{\mathcal{N}}, \mathcal{A}, \mathcal{B}, \mathcal{X}_{\mathcal{N}}\}$, where $\mathcal{A} \neq \mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B}, \mathcal{A} \cup \mathcal{B} \in \mathcal{T}$. To compute the number of NCrTs with exactly 4-NCrOSs, we need to compute formulae for following cases:

Case 1: $\mathcal{A} \cap \mathcal{B} = \phi_{\mathcal{N}}, \mathcal{A} \cup \mathcal{B} = \mathcal{X}_{\mathcal{N}}$
Case 2: $\mathcal{A} \cap \mathcal{B} = \phi_{\mathcal{N}},\ \mathcal{A} \cup \mathcal{B} = \phi_{\mathcal{N}}$
Case 3: ($\mathcal{A} \cap \mathcal{B} = \mathcal{A}$ or $\mathcal{B}$, $\mathcal{A} \cup \mathcal{B} = \phi_{\mathcal{N}}$) or
($\mathcal{A} \cap \mathcal{B} = \phi_{\mathcal{N}}, \mathcal{A} \cup \mathcal{B} = \mathcal{A}$ or $\mathcal{B}$)
Case 4: ($\mathcal{A} \cap \mathcal{B} = \mathcal{A}, \mathcal{A} \cup \mathcal{B} = \mathcal{A}$) or ($\mathcal{A} \cap \mathcal{B} = \mathcal{B}, \mathcal{A} \cup \mathcal{B} = \mathcal{B}$)
Case 5: ($\mathcal{A} \cap \mathcal{B} = \mathcal{A}, \mathcal{A} \cup \mathcal{B} = \mathcal{B}$) or ($\mathcal{A} \cap \mathcal{B} = \mathcal{B}, \mathcal{A} \cup \mathcal{B} = \mathcal{A}$).

Proposition 5.1.

For a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$, the number of NCrTs having 4-NCrOSs satisfying the condition in case 1 is obtained by the formula

$$\mathcal{S}(n,2)(2^n + 1).$$

Proof:

In general, the number of partitions of a non-empty set $\mathcal{X}$ with $|\mathcal{X}| = n$ into two blocks is given by $\mathcal{S}(n,2)$. To obtain $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\mathcal{X}_{\mathcal{N}}$, clearly $\mathcal{A}$ and $\mathcal{B}$ must have the following two forms:

(1) $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\emptyset\rangle$ & $\mathcal{B}=\langle\mathcal{B}_1,\mathcal{B}_2,\emptyset\rangle$,

(2) $\mathcal{A}=\langle\mathcal{A}_1,\emptyset,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\mathcal{B}_1,\emptyset,\mathcal{B}_3\rangle$.

Let us count the ways that they can be chosen.

(1) We have, $\mathcal{A}\cap\mathcal{B}=\langle\mathcal{A}_1\cap\mathcal{B}_1,\mathcal{A}_2\cap\mathcal{B}_2,\emptyset\rangle$, and $\mathcal{A}\cup\mathcal{B}=\langle\mathcal{A}_1\cup\mathcal{B}_1,\mathcal{A}_2\cap\mathcal{B}_2,\emptyset\rangle$. Now, to get $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\mathcal{X}_{\mathcal{N}}$, we must have, $\mathcal{A}_1\cap\mathcal{B}_1=\emptyset$, $\mathcal{A}_1\cup\mathcal{B}_1=\mathcal{X}$ and $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$. This implies that $\mathcal{A}_1$, $\mathcal{B}_1$ is a partition of $\mathcal{X}$ and so, $\mathcal{B}_1=\mathcal{X}-\mathcal{A}_1$. Therefore, $\mathcal{A}_1$, $\mathcal{B}_1$ can be chosen in $\mathcal{S}(n,2)$ ways. Now, if $|\mathcal{A}_1|=i$ then $|\mathcal{B}_1|=n-i$. Since $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$, then the neutrosophic crisp subset $\mathcal{A}_2$ can be chosen out of $n-i$ elements in $\binom{n-i}{k}, k=0,1,2,\ldots,n-i$ ways with $k=0$ representing the empty set. Therefore, $\mathcal{A}_2$ can be chosen in $\sum_{k=0}^{n-i}\binom{n-i}{k}=2^{n-i}$ ways. Similarly, $\mathcal{B}_2$ can be chosen out of $n-(n-i)=i$ elements in $\sum_{k=0}^{i}\binom{i}{k}=2^i$ ways. Hence, the total number of ways is $\mathcal{S}(n,2).2^{n-i}.2^i=\mathcal{S}(n,2).2^n$.

(2) We have, $\mathcal{A}\cap\mathcal{B}=\langle\mathcal{A}_1\cap\mathcal{B}_1,\emptyset,\mathcal{A}_3\cup\mathcal{B}_3\rangle$, and $\mathcal{A}\cup\mathcal{B}=\langle\mathcal{A}_1\cup\mathcal{B}_1,\emptyset,\mathcal{A}_3\cap\mathcal{B}_3\rangle$. Now, to get $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\mathcal{X}_{\mathcal{N}}$, we must have, $\mathcal{A}_1\cap\mathcal{B}_1=\emptyset$, $\mathcal{A}_3\cup\mathcal{B}_3=\mathcal{X}$ and $\mathcal{A}_1\cup\mathcal{B}_1=\mathcal{X}$, $\mathcal{A}_3\cap\mathcal{B}_3=\emptyset$ simultaneously. This shows that $\mathcal{A}_1$ and $\mathcal{B}_1$ is a partition of $\mathcal{X}$ and $\mathcal{A}_3=\mathcal{A}_1^C=\mathcal{B}_1$, $\mathcal{B}_3=\mathcal{B}_1^C=\mathcal{A}_1$. Therefore, we can take $\mathcal{A}_1$ and $\mathcal{B}_1$ or $\mathcal{A}_3$ and $\mathcal{B}_3$ in $\mathcal{S}(n,2)$ ways.

From (i) and (ii), the total number of ways is $\mathcal{S}(n,2)(2^n+1)$.

Hence, the number of NCrTs having 4-NCrOSs satisfying the condition in case 1 is obtained by the formula

$$\mathcal{S}(n,2)(2^n+1).$$

Proposition 5.2.

The number of NCrTs having 4-NCrOSs on a non-empty set $\mathcal{X}$ satisfying the condition in case 2 is obtained by the formula

$$\frac{n(n-1)}{2}+\{\mathcal{S}(n,2)\times 2^n\}+\sum_{i=3}^{n}\left\{\binom{n}{i}\mathcal{S}(i,2)\right\}$$

where $|\mathcal{X}|=n$.

Proof:

To obtain $\mathcal{A}\cap\mathcal{B}=\phi_{\mathcal{N}}$ and $\mathcal{A}\cup\mathcal{B}=\phi_{\mathcal{N}}$, clearly, $\mathcal{A}$ and $\mathcal{B}$ must have the following two forms

(1) $\mathcal{A}=\langle\emptyset,\mathcal{A}_2,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\mathcal{B}_3\rangle$ such that $\mathcal{A}_3\cup\mathcal{B}_3=\mathcal{X}$ and $\mathcal{A}_3\cap\mathcal{B}_3=\emptyset$ and $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$.

(2) $\mathcal{A}=\langle\emptyset,\mathcal{A}_2,\emptyset\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\emptyset\rangle$ such that $\mathcal{A}_2\cap\mathcal{B}_2=\emptyset$.

From (i), $\mathcal{A}\cap\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \mathcal{A}_3\cup\mathcal{B}_3\rangle$, and $\mathcal{A}\cup\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \mathcal{A}_3\cap\mathcal{B}_3\rangle$. Since, $\mathcal{A}_3\cup\mathcal{B}_3 = \mathcal{X}$ and $\mathcal{A}_3 \cap \mathcal{B}_3 = \emptyset$, which implies that $\mathcal{A}_3$ and $\mathcal{B}_3$ is a partition of $\mathcal{X}$ and say $\mathcal{B}_3 = \mathcal{X} - \mathcal{A}_3$. Therefore, $\mathcal{A}_3$ and $\mathcal{B}_3$ can be chosen in $\mathcal{S}(n, 2)$ ways. Now, if $|\mathcal{A}_3| = i$, $|\mathcal{B}_3| = n-i$, $1 \le i \le n-1$, and $\mathcal{A}_2 \cap \mathcal{B}_2 = \emptyset$, then $\mathcal{A}_2$ can be chosen in $\sum_{k=0}^{n-i}\binom{n-i}{k} = 2^{n-i}$ ways, and similarly, $\mathcal{B}_2$ can be chosen out of $n - (n - i) = i$ elements in $\sum_{k=0}^{i}\binom{i}{k} = 2^i$ ways.

Therefore, the total number of ways is $\mathcal{S}(n, 2) \times 2^{n-i} \times 2^i$ i.e., $\mathcal{S}(n, 2) \times 2^n$.

From (ii), $\mathcal{A}\cap\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \emptyset\rangle$, $\mathcal{A}\cup\mathcal{B} = \langle\emptyset, \mathcal{A}_2\cap\mathcal{B}_2, \emptyset\rangle$, and $\mathcal{A}_2\cap\mathcal{B}_2 = \emptyset$. If $|\mathcal{A}_2\cup\mathcal{B}_2| = i$, $2 \le i \le n$, then $\mathcal{A}_2 \cup \mathcal{B}_2$ is chosen in $\binom{n}{i}$ different ways and then it is partitioned into two disjoint blocks: this is done in $\mathcal{S}(i, 2)$ different ways. Therefore, the number of ways for form (ii) is $\sum_{i=2}^{n}\binom{n}{i}\mathcal{S}(i, 2)$.

Hence, the number of NCrTs having 4-NCrOSs satisfying condition in case 2 is obtained by the formula

$$\{\mathcal{S}(n, 2) \times 2^n\} + \sum_{i=2}^{n}\left\{\binom{n}{i}\mathcal{S}(i, 2)\right\}$$

i.e.,

$$\frac{n(n-1)}{2} + \{\mathcal{S}(n, 2) \times 2^n\} + \sum_{i=3}^{n}\left\{\binom{n}{i}\mathcal{S}(i, 2)\right\}.$$

Proposition 5.3.

For a non-empty finite set $\mathcal{X}$ with $|\mathcal{X}| = n$, the number of NCrTs having 4-NCrOSs satsifying conditions in case 3 is obtained by the formula $2(2^n - 2)^2$.

Proof:

There are two forms

(i) $\mathcal{A} = \langle\emptyset, \emptyset, \mathcal{A}_3\rangle$ & $\mathcal{B} = \langle\emptyset, \mathcal{B}_2, \emptyset\rangle$,

(ii) $\mathcal{A} = \langle\mathcal{A}_1, \emptyset, \emptyset\rangle$ & $\mathcal{B} = \langle\emptyset, \mathcal{B}_2, \emptyset\rangle$.

Let us count the ways that they can be chosen.

Clearly, these two forms agree with the conditions in case 3 i.e., for the first kind, we have, $\mathcal{A} \cap \mathcal{B} = \langle\emptyset, \emptyset, \mathcal{A}_3\rangle = \mathcal{A}$ and $\mathcal{A} \cup \mathcal{B} = \langle\emptyset, \emptyset, \emptyset\rangle = \phi_{\mathcal{N}}$, and for the second kind $\mathcal{A} \cap \mathcal{B} = \langle\emptyset, \emptyset, \emptyset\rangle = \phi_{\mathcal{N}}$ and $\mathcal{A} \cup \mathcal{B} = \langle\mathcal{A}_1, \emptyset, \emptyset\rangle = \mathcal{A}$. Now, since $\emptyset \subset \mathcal{A}_3 \subset \mathcal{X}$, $\emptyset \subset \mathcal{B}_2 \subset \mathcal{X}$ such that $|\mathcal{A}_3| = |\mathcal{B}_2| = i, 1 \le i \le n-1$ so, $\mathcal{A}_3$ and $\mathcal{B}_2$ are chosen in $\binom{n}{i}$ different ways. This implies that $\mathcal{A}$ and $\mathcal{B}$ are chosen in $\binom{n}{i}$ different ways. Therefore, the number of ways in this kind is $\left\{\sum_{i=1}^{n-1}\binom{n}{i}\right\} \times \left\{\sum_{i=1}^{n-1}\binom{n}{i}\right\} = \left(\sum_{i=1}^{n-1}\binom{n}{i}\right)^2 = (2^n - 2)^2$.

Similarly, the second kind is computed and is equal to $(2^n - 2)^2$.

Finally, the desired number of ways is $2\left(2^n - 2\right)^2$.

Proposition 5.4.

For a non-empty set $\mathcal{X}$ with $|\mathcal{X}| = n$, the number of NCrTs having 4-NCrOSs satisfying *condition in case 4 is obtained by the formula*

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+2\mathcal{T}_1+6\left(\mathcal{T}_2+\mathcal{T}_3\right),$$

where $\mathcal{T}_k=\sum_{i=k}^{n-1}\left\{\binom{n}{i}\mathcal{S}(i,k)\left(2^{n-i}-1\right)\right\}$, $k=1,2,3.$

Proof:

Let $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\rangle$ and $\mathcal{B}=\langle\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3\rangle$. Then to satisfy the condition ($\mathcal{A}\cap\mathcal{B}=\mathcal{A},\mathcal{A}\cup\mathcal{B}=\mathcal{A}$) or ($\mathcal{A}\cap\mathcal{B}=\mathcal{B},\mathcal{A}\cup\mathcal{B}=\mathcal{B}$), we must have, $\mathcal{A}_1=\mathcal{B}_1,\mathcal{A}_2\subset\mathcal{B}_2,\mathcal{A}_3=\mathcal{B}_3$ or $\mathcal{A}_1=\mathcal{B}_1,\mathcal{B}_2\subset\mathcal{A}_2,\mathcal{A}_3=\mathcal{B}_3$ respectively . Then, we obtain four forms

(1) $\mathcal{A}=\langle\emptyset,\mathcal{A}_2,\emptyset\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\emptyset,\rangle$ such that $\mathcal{A}_2\subset\mathcal{B}_2$ or $\mathcal{B}_2\subset\mathcal{A}_2$,
(2) $\mathcal{A}=\langle\emptyset,\emptyset,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\emptyset,\mathcal{B}_2,\mathcal{A}_3\rangle$ and $\mathcal{A}=\langle\mathcal{A}_1,\emptyset,\emptyset\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\emptyset\rangle$,
(3) $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\mathcal{A}_3\rangle$; exactly one of $\mathcal{A}_i, i=1,2,3$ is $\emptyset$ and $\mathcal{A}_2\subset\mathcal{B}_2$.
(4) $\mathcal{A}=\langle\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\mathcal{A}_3\rangle$ such that all $\mathcal{A}_i, i=1,2,3$ are non-empty and $\mathcal{A}_2\subset\mathcal{B}_2$.

Let us count the ways that they can be chosen.

(1) Let $\mathcal{A}_2\subset\mathcal{B}_2$ and if $|\mathcal{A}_2|=i, 1\leq i\leq n-2$ then $i<|\mathcal{B}_2|=k\leq n-1$. Therefore, $\mathcal{A}_2$ is chosen in $\binom{n}{i}$ ways and $\mathcal{B}_2$ is chosen in $\sum_{j=1}^{(n-i)-1}\binom{n-i}{j}=2^{n-i}-2$ different ways. Since, i varies from 1 to $n-2$, $\mathcal{A}_2$ and $\mathcal{B}_2$ are chosen in $\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}$ different ways. Hence, the neutrosophic crisp subsets $\mathcal{A}$ and $\mathcal{B}$ are chosen in $\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}$ different ways.
(2) Let $|\mathcal{A}_3|=i, 1\leq i\leq n-1$ then $\mathcal{A}_3$ is chosen in $\binom{n}{i}$ different ways then it is partitioned into one block: this is done in $\mathcal{S}(i,1)$ different ways and hence $\mathcal{A}$. Next, in $\mathcal{B}$, $\mathcal{A}_3\cap\mathcal{B}_2=\emptyset$ and so, $\mathcal{B}_2$ is chosen from $n-i$ elements in $\sum_{j=1}^{n-i}\binom{n-i}{j}=2^{n-i}-1$ different ways and hence $\mathcal{B}$. Since i varies from 1 to $n-1$, we obtain $\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1)$ different ways for $\mathcal{A}$ and $\mathcal{B}$.

    Similarly, for $\mathcal{A}=\langle\mathcal{A}_1,\emptyset,\emptyset\rangle$ & $\mathcal{B}=\langle\mathcal{A}_1,\mathcal{B}_2,\emptyset\rangle$, we have $\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1)$ different ways.

(3) We have, $\mathcal{A}_1\cap\mathcal{A}_3=\emptyset$. If $|\mathcal{A}_1\cup\mathcal{A}_3|=i, 2\leq i\leq n-1$ then $\mathcal{A}_1,\mathcal{A}_3$ is chosen in $\binom{n}{i}\mathcal{S}(i,2)$ different ways. Since $\mathcal{A}_1\cap\mathcal{B}_2=\mathcal{A}_3\cap\mathcal{B}_2=\emptyset$, so, $\mathcal{B}_2$ is chosen in $\binom{n-i}{j}, 1\leq j\leq n-i$ different ways. Therefore, $\mathcal{B}_2$ is chosen in $\sum_{j=1}^{n-i}\binom{n-i}{j}=2^{n-i}-1$ different ways. Together $\mathcal{A}$ and $\mathcal{B}$ is chosen in $\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1)$ different ways. It is known that we can arrange three element into three places in six different ways, so, $\mathcal{A}$ has six forms, as three components of $\mathcal{A}$ are the neutrosophic crisp subsets $\mathcal{A}_1,\mathcal{A}_3$ and $\emptyset$.

Hence, the total number of ways to choose $\mathcal{A}$ and $\mathcal{B}$ is

$$6\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1).$$

(4) We have, $\mathcal{A}_1\cap\mathcal{A}_2=\mathcal{A}_1\cap\mathcal{A}_3=\mathcal{A}_2\cap\mathcal{A}_3=\emptyset$. If $|\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3|=i, 3\leq i\leq n-1$ then $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ are chosen in $\binom{n}{i}\mathcal{S}(i,3)$ different ways. Since $\mathcal{A}_1\cap\mathcal{B}_2=\mathcal{A}_3\cap\mathcal{B}_2=\emptyset$, so, $\mathcal{B}_2$ is chosen in $\binom{n-i}{j}, 1\leq j\leq n-i$ different ways. Therefore, $\mathcal{B}_2$ is chosen in $\sum_{j=1}^{n-i}\binom{n-i}{j}=2^{n-i}-1$ different ways. Together $\mathcal{A}$ and $\mathcal{B}$ are chosen in $\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)2^{n-i}-1$ different ways. It is known that we can arrange three elements into the three places in six different ways, so, $\mathcal{A}$ has 6 forms, as three components of $\mathcal{A}$ are different neutrosophic crisp subsets $\mathcal{A}_1,\mathcal{A}_2$ and $\mathcal{A}_3$.

Hence, the total number of ways to choose $\mathcal{A}$ and $\mathcal{B}$ is

$$6\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)(2^{n-i}-1).$$

Hence, we have the total

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1)+$$

$$6\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1)+6\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)(2^{n-i}-1).$$

i.e.,

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+2\mathcal{T}_1+6\left(\mathcal{T}_2+\mathcal{T}_3\right),$$

where $\mathcal{T}_k=\sum_{i=k}^{n-1}\left\{\binom{n}{i}\mathcal{S}(i,k)\left(2^{n-i}-1\right)\right\}$, $k=1,2,3$. This formula gives the number of NCrTs having 4-NCrOSs satisfying condition in case 4.

Proposition 5.5.

For a non-empty set $\mathcal{X}$ with $|\mathcal{X}|=n$, the number of NCrTs having 4-NCrOSs satisfying condition in case 5 is obtained by the formula

$$\sum_{i=1}^{n-1}\binom{n}{i}\left[(2^n-2)+2\left\{\left(\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}\right)+(2^{n-i}-1)\right\}\right]+$$

$$2\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)+\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)^2+2\mathcal{T}_n+\sum_{i=0}^{n-2}\binom{n}{i}\mathcal{T}_{n-i},$$

where $\mathcal{T}_n=\sum_{i=1}^{n-2}\binom{n}{i}\left\{\sum_{k=1}^{n-(i+1)}\binom{n-i}{k}(2^{n-(i+k)}-1)\right\}+$
$\sum_{i=1}^{n-2}\binom{n}{i}\left[\sum_{j=1}^{n-i}\binom{n-i}{j}\left\{\sum_{k=1}^{j}\binom{j}{k}(2^{n-(i+j)}-1)\right\}\right]$
or

$\mathcal{T}_n = \sum_{i=1}^{n-2} \binom{n}{i} \left\{ \sum_{k=1}^{n-(i+1)} \binom{n-i}{k} (2^{n-(i+k)} - 1) \right\} +$
$\sum_{i=1}^{n-2} \binom{n}{i} \left[\sum_{j=1}^{n-i} \left\{ \sum_{k=j}^{n-i} \binom{n-i}{j} \binom{j}{k} \right\} (2^{n-(i+j)} - 1) \right].$

Proof:

Here, the second component must always match to satisfy the conditions in case 5.

For $\mathcal{A} = \langle \emptyset, \emptyset, \mathcal{A}_3 \rangle$ we can choose $\mathcal{B}$ in two forms which are $\mathcal{B} = \langle \mathcal{B}_1, \emptyset, \emptyset \rangle$ and $\mathcal{B} = \langle \mathcal{B}_1, \emptyset, \mathcal{B}_3 \rangle$ such that $\mathcal{B}_3 \subseteq \mathcal{A}_3$. For this kind of $\mathcal{A}$ we have $\binom{n}{i}$ different ways. For each $\mathcal{A}$, we can choose $\mathcal{B} = \langle \mathcal{B}_1, \emptyset, \emptyset \rangle$ in $2^n - 2$ different ways. Next if $\mathcal{B}_3 \subset \mathcal{A}_3$, say $|\mathcal{B}_3| = j < i = |\mathcal{A}_3|$, we can choose $\mathcal{B}$ in $\sum_{j=1}^{i-1} \binom{i}{j} 2^{n-j}$ different ways and if $\mathcal{B}_3 = \mathcal{A}_3$, say $|\mathcal{B}_3| = |\mathcal{A}_3| = i$, then $\mathcal{B}$ can be chosen in $2^{n-i} - 1$ different ways. Similarly, for $\mathcal{A} = \langle \mathcal{A}_1, \emptyset, \emptyset \rangle$, we have same number of choices for $\mathcal{B}$ satisfying conditions in case 5.
Therefore, in this part we have

$$\sum_{i=1}^{n-1} \binom{n}{i} \left[(2^n - 2) + 2 \left\{ \left(\sum_{j=1}^{i-1} \binom{i}{j} 2^{n-j} \right) + (2^{n-i} - 1) \right\} \right]$$

different ways.

For $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \emptyset \rangle$, we can choose $\mathcal{B} = \langle \emptyset, \mathcal{A}_2, \mathcal{B}_3 \rangle$ and $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \emptyset \rangle$. Since $\mathcal{A}_2$ can be chosen in $\binom{n}{i}, i = 1, 2, \ldots, n-1$ different ways then $\mathcal{B}_3$ can be chosen in $2^{n-i} - 1$ different ways for each i and therefore, $\mathcal{B}$. As we have two forms of $\mathcal{B}$ and are symmetric, and i varies from 1 to $n-1$, we have the total $2 \sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1)$.

For $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can choose $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_3 \rangle, \mathcal{A}_3 \subseteq \mathcal{B}_3$ and $\mathcal{B}_1$ is any subset of $\mathcal{X}$ different from $\mathcal{A}_2$ and $\mathcal{B}_3$. Then $\mathcal{A}$ and $\mathcal{B}$ can be chosen in $\sum_{i=1}^{n-2} \binom{n}{i} \left\{ \sum_{k=1}^{n-(i+1)} \binom{n-i}{k} (2^{n-(i+k)} - 1) \right\} +$ $\sum_{i=1}^{n-2} \binom{n}{i} \left[\sum_{j=1}^{n-i} \binom{n-i}{j} \left\{ \sum_{k=1}^{j} \binom{j}{k} (2^{n-(i+j)} - 1) \right\} \right]$ different ways. Let us take $\sum_{i=1}^{n-2} \binom{n}{i} \left\{ \sum_{k=1}^{n-(i+1)} \binom{n-i}{k} (2^{n-(i+k)} - 1) \right\} +$ $\sum_{i=1}^{n-2} \binom{n}{i} \left[\sum_{j=1}^{n-i} \binom{n-i}{j} \left\{ \sum_{k=1}^{j} \binom{j}{k} (2^{n-(i+j)} - 1) \right\} \right] = \mathcal{T}_n$ for further use. Also, for $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \emptyset \rangle$, we have equal number of choices as it is symmetric to $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \mathcal{A}_3 \rangle$. Hence, a total of $2\mathcal{T}_n$ different ways.

For $\mathcal{A} = \langle \emptyset, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can also choose $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \emptyset \rangle$ which can be done in $\sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1)^2$ different ways.

For $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can choose $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_3 \rangle$ such that $\mathcal{A}_1 \subseteq \mathcal{B}_1$, $\mathcal{B}_3 \subseteq \mathcal{A}_3$ and $|\mathcal{A}_2| = i, 0 \leq i \leq n-2$. If $|\mathcal{A}_2| = 0$ i.e., $\mathcal{A}_2 = \emptyset$ then $\mathcal{B}$ can be chosen in $\binom{n}{0} \mathcal{T}_n$ different ways. Further, if $|\mathcal{A}_2| = 1$ then $\mathcal{B}$ can be chosen in $\binom{n}{1} \mathcal{T}_{n-1}$ different ways. Continuing in the similar way for $|\mathcal{A}_2| = n-2$, we have $\binom{n}{n-2} \mathcal{T}_{n-(n-2)}$ i.e., $\binom{n}{n-2} \mathcal{T}_2$ different ways. Thus, for $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$, we can choose $\mathcal{B}$ in $\sum_{i=0}^{n-2} \binom{n}{i} \mathcal{T}_{n-i}$ different ways.

Hence, the number of NCrTs having 4-NCrOSs satisfying conditions in case 5 is obtained by the formula

$$\sum_{i=1}^{n-1}\binom{n}{i}\left[(2^n-2)+2\left\{\left(\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}\right)+(2^{n-i}-1)\right\}\right]+$$
$$2\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)+\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)^2+2\mathcal{T}_n+\sum_{i=0}^{n-2}\binom{n}{i}\mathcal{T}_{n-i}.$$

Example 5.6.

The follwing table gives the number of NCrTs having 4-NCrOSs for $\mathcal{X} \leq 5$. Suppose, $\mathcal{X} =$

TABLE 1. Number of NCrTSs having 4-NCrOSs on $\mathcal{X}$

$\mathcal{A}\cap\mathcal{B}, \mathcal{A}\cup\mathcal{B}$	Number of NCrTSs having 4-NCrOSs on $\mathcal{X}$				
	$\lvert\mathcal{X}\rvert = 1$	$\lvert\mathcal{X}\rvert = 2$	$\lvert\mathcal{X}\rvert = 3$	$\lvert\mathcal{X}\rvert = 4$	$\lvert\mathcal{X}\rvert = 5$
Case 1: $\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}}$, $\mathcal{A}\cup\mathcal{B} = \mathcal{X}_{\mathcal{N}}$	0	5	27	119	495
Case 2: $\mathcal{A}\cap\mathcal{B} = \phi_{\mathcal{N}}$, $\mathcal{A}\cup\mathcal{B} = \phi_{\mathcal{N}}$	0	5	30	137	570
Case 3: $\mathcal{A}\cap\mathcal{B} = \mathcal{A}$, $\mathcal{A}\cup\mathcal{B} = \phi_{\mathcal{N}}$	0	8	72	392	1800
Case 4: $\mathcal{A}\cap\mathcal{B} = \mathcal{A}$, $\mathcal{A}\cup\mathcal{B} = \mathcal{A}$	0	4	48	340	2040
Case 5: $\mathcal{A}\cap\mathcal{B} = \mathcal{A}$, $\mathcal{A}\cup\mathcal{B} = \mathcal{B}$	0	14	216	1958	15240
The total number of NCrTSs having 4-NCrOSs on $\mathcal{X}$	0	36	393	2946	20145

$\{a, b\}$ i.e., $|\mathcal{X}| = 2$, then from Table 1, we have, $\mathcal{T}_{\mathcal{C}r}(2, 4) = 36$. These are

For Case 1:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle, A_6 = \langle\{b\}, \emptyset, \emptyset\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_8 = \langle\{a\}, \emptyset, \{b\}\rangle, A_{11} = \langle\{b\}, \emptyset, \{a\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle\}.$

For Case 2:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\}.$

For Case 3:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\},$
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\},$

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\}$.

For Case 4:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle\}$.

For Case 5:

$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_3 = \{a\}, \emptyset, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_7 = \langle\emptyset, \{a\}, \{b\}\rangle, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{12} = \langle\{b\}, \{a\}, \emptyset\rangle, A_2 = \langle\emptyset, \{a\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_8 = \langle\{a\}, \emptyset, \{b\}\rangle, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_8 = \langle\{a\}, \emptyset, \{b\}\rangle, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{11} = \langle\{b\}, \emptyset, \{a\}\rangle, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle\}$,
$\{\phi_{\mathcal{N}}, \mathcal{X}_{\mathcal{N}}, A_{11} = \langle\{b\}, \emptyset, \{a\}\rangle, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle\}$.

As a result, we have the total $\mathcal{T}_{\mathcal{C}r}(2,4) = 36$.

Conclusion

This paper computes the formulae for the number of neutrosophic crisp topological spaces having 2, 3, and 4 open sets. This work is the foundation for computation of the formulae to find the number of neutrosophic crisp topological spaces.

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Application of Neutrosophic Fourier Transform in solving Heat Equation and Integral Equation

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Abstract. Fourier transform is one of the oldest and well-known technique in the field of mathematics and engineering mathematical works. As the concept of uncertainty has been introduced in the mathematics, most of the works gravitate towards the use Neutrosophic set. So, it is also important to study the Fourier transform in the sense of Neutrosophic set. In this paper we have applied the Neutrosophic Fourier transform in solving heat equation, and integral equation. Where detailed examples are given to clarify each case.

Keywords: Fourier Transform, Neutrosophic Fourier Transform, Neoutrosophic Function.

Abreviation: F.T.= Fourier transform

N.F.T.=Neutrosophic Fourier Transform

F.S.T= Fourier sine transform

F.C.T.= Fourier cosine transform

N.R.N.= Neutrosophic real number

N.C.N.= Neutrosophic complex number

1. Introduction:

Since the world is full of indeterminacy, the Neutrosophic found their place in contemporary research. Thus, Smarandache [1] put forward the definitions of Neutrosophic measure, Neutrosophic real number in standard form, and the considerations for the existence of a division of two Neutrosophic real numbers. He [2,3,4,5] also defined the Neutrosophic complex numbers in standard form and found the root index $n \geq 2$ of a Neutrosophic real and complex number, studying the concept of the Neutrosophic probability, the Neutrosophic statistics, and professor Smarandache [6,8] entered the concept of preliminary calculus of the differential and integral calculus. Madeleine [9] presented results on single-valued Neutrosophic (weak) polygroups. Edalatpanah [10] proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side are represented with triangular Neutrosophic numbers. Chakraborty [11, 12] used pentagonal Neutrosophic numbers in networking problems, and shortest-path problems.

A. Kharal[15] presents a method of multicriteria decision making using neutrosophic sets. A. A. Salama and F. Smarandache [16] used neutrosophic set to introduce new types of neutrosophic crisp sets with three types 1, 2, 3. D. Koundal, S. Gupta and S. Singh [17] demonstrates the use of neutrosophic theory in medical image denoising and segmentation, using which the performance is observed to be much better.

The concepts of Neutrosophic set have been used in different areas of Mathematics. Here we are using this concepts in Fourier integral and Fourier transform. Fourier integral represents a certain type of non periodic functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$. Fourier transform is a mathematical tool used to decompose a signal into its constituent frequency components. It breaks down signals into a combination of sines and cosines, which can be used to analyse the frequency content of a signal. Fourier analysis has been used in digital image and processing of image and for analysis of a single image into a two-dimensional wave form, and more recently has been used for magnetic resonance imaging, angiographic assessment, automated lung segmentation and image quality assessment and Mobile stethoscope [18]. Fourier transforms which is also used in frequency domain representation. Fourier analysis used as time series analysis proved its application in Quantum mechanics; Signal processing, Image Processing and filters, representation, Data Processing and Analysis and many more.

Fourier transforms are obviously very essential to conduct of Fourier spectroscopy, and that alone would justify its importance. Fourier transforms are very vital in other pursuits as well; such as electrical signal analysis, diffraction, optical testing, optical processing, imaging, holography, and also for remote sensing [13, 14]. Thus, knowledge of Fourier transforms can be a springboard to many other fields. The main idea behind Fourier transforms is that a function of direct time can be expressed as a complex valued function of reciprocal space, that is, frequency.

As we all are known that no things in this universe are certain or fully determined, there must need a detailed study of every phenomenon in the universe and try to find the uncertainty amount and then try to find out the actual value. For example, when we want to find the distance between two places, then different people will find different distances. Similarly, if we want to know the age of a person, then we cannot say the actual age of that person i.e., we cannot say accurately that the age of that person is 25 years. Because, during the time of giving the answer, that person's age has increased even by some seconds.

Fourier transform is a mathematical tool used to decompose a signal into its constituent frequency components. It breaks down signals into a combination of sines and cosines, which can be used to analyse the frequency content of a signal. During the transformation in Fourier transform, some informations are lost. It means that, when we extract the components of the signal by F.T., we can not determine the actual frequencies of each of the components of the signal. Thus, there is some indeterminacy, and thus we study F.T. by using the concept of neutrosophy. Here, the idea of neutrosophic set is used in some basics of Fourier integral. Neutrosophic Fourier transform will help to get the actual frequencies which constitutes a signal or a sound and can be filter out the unwnated frequncies in most appropriately.

The Fourier transform is beneficial in differential equations because it can reformulate them as problems which are easier to solve. In this work we shall discuss the applications of neutrosophic Fourier transform in some partial differential equations of neutrosophic sense. We have applied the Neutrosophic Fourier transform in solving heat equation. The solution of heat equation in one dimension represents the temperature of a position at a time of a thin rod or a wire. The concept of N.F.T. in solving heat equation will lead the result in another level.

We have studied neutrosophic fourier transform of the Derivatives of a neutrosophic function.

We have also used the N.F.T. in solving neutrosophic integral equatios. The neutrosophic integral equation is an equation in which an unknown neutrosophic function is to be found lies within in integral sign. It plays an pivotal role in various applications, including physics, engineering, and quantum mechanics, bridging differential equations and broader mathematical analysis. Solution of integral equations with the help of N.F.T. will give more advance results than the classical form.

2. **Preliminaries:**

In this part, definitions of N.R.N. and division of two N.R.N.s are discussed.

2.1. *Neutrosophic Real Number[4]:*

If a number that can be written in the form $p_n + q_n I$, where p_n, q_n are real numbers and I is an indeterminate number such that $I.0 = 0$ and $I^N = I$, for all natural number **N**, is called N.R.N..

Here we denote the N.R.N. by w, and thus we can write $w = p_n + q_n I$ and it is knwon as the standard form of N.R.N..

2.2. *Division of two N.R.N.s[4]:*

Consider that w_1 and w_2 be two N.R.N.s where,
$w_1 = p_{n1} + q_{n1}I$ and $w_2 = p_{n2} + q_{n2}I$
Then to find $(p_{n1} + q_{n1}I) \div (p_{n2} + q_{n2}I)$, let us take

$$\frac{p_{n1} + q_{n1}I}{p_{n2} + q_{n2}I} = x + yI$$

where x and y are real numbers.

$$\Rightarrow p_{n1} + q_{n1}I = (p_{n2} + q_{n2}I)(x + yI)$$
$$\Rightarrow p_{n1} + q_{n1}I = p_{n2}x + (p_{n2}y + q_{n2}x + q_{n2}y)I$$

Equating both sides, we get

$$\begin{aligned} p_{n1} &= p_{n2}x \\ \Rightarrow x &= \frac{p_{n1}}{p_{n2}} \\ \textit{and }\; q_{n1} &= p_{n2}y + q_{n2}x + q_{n2}y \\ \Rightarrow q_{n1} &= p_{n2}y + q_{n2}.\frac{p_{n1}}{p_{n2}} + q_{n2}y \\ \Rightarrow \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}} &= (p_{n2} + q_{n2})y \\ \Rightarrow y &= \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}(p_{n2} + q_{n2})} \end{aligned}$$

provided $p_{n2}(p_{n2} + q_{n2}) \neq 0$ or $p_{n2} \neq 0$ and $p_{n2} \neq -q_{n2}$

Thus

$$\frac{p_{n1} + q_{n1}I}{p_{n2} + q_{n2}I} = \frac{p_{n1}}{p_{n2}} + \frac{p_{n2}q_{n1} - p_{n1}q_{n2}}{p_{n2}(p_{n2} + q_{n2})}.I$$

3. **Application of N.F.T. to Heat Equation:**

The heat equation is a parabolic partial differential equation, describing the distribution of heat in a given space over time. The theory of heat equation was first developed by Joseph Fourier in 1822 for the purpose of modeling how a quantity such as heat diffuses through a given region. In this study we have applied the concept of neutrosophic set in solving the heat equation to minimize the vagueness or uncertainity occured in the result by the classical method.
The general form of one dimensional heat equation in neutrosophic sense is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where $u = u(x,t,I)$ is the neutrosophic temperature in rod at position x at time t and the constant c^2 is called the thermal diffusivity of the rod.

The N.F.S.T. and N.F.C.T. can be applied when the range of the variable selected for exclusion is 0 to ∞. The choice of N.F.S.T. and N.F.C.T. is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. In this connection note the following:

$$\begin{aligned}
\mathscr{F}_s\{\frac{\partial^2 u}{\partial x^2}\} &= \int_0^\infty \frac{\partial^2 u}{\partial x^2} sins(p_n+q_nI)xdx \\
&= \left[\frac{\partial u}{\partial x} sins(p_n+q_nI)x\right]_0^\infty - s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n+q_nI)xdx \\
&= -s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n+q_nI)xdx, \quad \text{if } \frac{\partial u}{\partial x}\to 0 \quad as \quad x\to\infty \\
&= -s(p_n+q_nI)\left\{[ucoss(p_n+q_nI)x]_0^\infty + s(p_n+q_nI)\int_0^\infty usins(p_n+q_nI)xdx\right\} \\
&= s(p_n+q_nI)(u)_{x=0} - s^2(p_n+q_nI)^2\overline{u}_s, \quad u\to 0 \quad as \quad x\to\infty
\end{aligned}$$

$$Thus,\ \mathscr{F}_s\{\frac{\partial^2 u}{\partial x^2}\} = s(p_n+q_nI)u(0,t,I) - s^2(p_n+q_nI)^2\overline{u}_s(s,t,I) \tag{1}$$

Where, $u(x,t,I)$ is a function of three variables x,t and I and $\overline{u}_s(s,t,I)$ is the N.F.S.T. of $u(x,t,I)$ with respect to x.

Again, by using the N.F.C.T., we get the following result:

$$\begin{aligned}
\mathscr{F}_c\{\frac{\partial^2 u}{\partial x^2}\} &= \int_0^\infty \frac{\partial^2 u}{\partial x^2} coss(p_n+q_nI)xdx \\
&= \left[\frac{\partial u}{\partial x} coss(p_n+q_nI)x\right]_0^\infty + s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} sins(p_n+q_nI)xdx \\
&= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} sins(p_n+q_nI)xdx, \quad \text{if } \frac{\partial u}{\partial x}\to 0 \quad as \quad x\to\infty \\
&= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s(p_n+q_nI)\left[[usins(p_n+q_nI)x]_0^\infty - s(p_n+q_nI)\int_0^\infty ucoss(p_n+q_nI)xdx\right] \\
&= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2(p_n+q_nI)^2\overline{u}_c, \quad u\to 0 \quad as \quad x\to\infty
\end{aligned}$$

$$\text{Thus, } \mathscr{F}_c\{\frac{\partial^2 u}{\partial x^2}\} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2(p_n+q_nI)^2\overline{u}_c(s,t,I) \tag{2}$$

Where, $\overline{u}_c$(s,t,I)is the N.F.C.T. of $u(x,t,I)$ with respect to x.

Choice of N.F.S.T. & N.F.C.T. :

From (1) and (2), it can be noted that successful use of a N.F.S.T. in removing a term $\frac{\partial^2 u}{\partial x^2}$ from partial differential equation requires a knowledge of u(s,t,I) and u(0,t,I) while the use of N.F.C.T. for the same purpose requires $u_x(s,t,I)$ and $u_x(0,t,I)$.

4. **Solved Examples based on Application of N.F.S.T. & N.F.C.T. to Heat Equation:**

Using the N.F.S.T. and N.F.C.T., we can solve the boundary value problems of heat equation. The result so obtained will give more accurate result. We have discussed some problems and solutions of them.

Example 1. Let us consider the following equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ x>0,t>0 \text{ subject to conditions}$$
$$u(0,t,I)=0,$$
$$u(x,0,I)=\begin{cases}1+I & \text{if } 0<x<1\\ 0 & \text{if } x>1\end{cases}$$

Since $u(0,t,I) i.e., (u)_{x=0}$ is given, taking the N.F.S.T. of both sides of the given partial differential equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t} sins(p_n+q_nI)xdx=\int_0^\infty \frac{\partial^2 u}{\partial t^2} sins(p_n+q_nI)xdx$$
$$\Rightarrow \frac{d}{dt}\int_0^\infty usins(p_n+q_nI)xdx=\left[\frac{\partial u}{\partial x} sins(p_n+q_nI)x\right]_0^\infty - s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n+q_nI)xdx$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s)=-s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n+q_nI)xdx, \quad \text{if } \frac{\partial u}{\partial x}\to 0 \quad as \quad x\to\infty$$
$$[Here \quad \overline{u}_s(s,t,I) \quad is \quad the \quad N.F.S.T. of\ u(x,t,I)]$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s)=-s(p_n+q_nI)\left[[ucoss(p_n+q_nI)x]_0^\infty + s(p_n+q_nI)\int_0^\infty usins(p_n+q_nI)xdx\right]$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s)=s(p_n+q_nI)u(0,t,I)-s^2(p_n+q_nI)^2\overline{u}_s, \quad u\to 0 \quad as \quad x\to\infty$$
$$\Rightarrow \frac{d}{dt}(\overline{u}_s)=-s^2(p_n+q_nI)^2\overline{u}_s,\ \{as\ u(0,t,I)=0$$
$$\Rightarrow \frac{1}{\overline{u}_s}d\overline{u}_s=-s^2(p_n+q_nI)^2dt$$

Whose solution is given by $\overline{u}_s(s,t,I)=ce^{-s^2(p_n+q_nI)^2t}$ (3)

where, c is an arbitrary constant.

Putting $t=0$ in (1), we get $c=\overline{u}_s(s,0,I)$

Therefore,

$$c = \overline{u}_s(s,0,I) = \int_0^\infty u(x,0,I) sins(p_n + q_nI)xdx$$

$$= \int_0^1 u(x,0,I) sins(p_n + q_nI)xdx + \int_1^\infty u(x,0,I) sins(p_n + q_nI)xdx$$

$$= \int_0^1 (1+I) sins(p_n + q_nI)xdx, \ using \ the \ given \ value \ of \ u(x,0,I)$$

$$= -\left[\frac{1+I}{s(p_n + q_nI)}.coss(p_n + q_nI)x\right]_{x=0}^{2}$$

$$= -\frac{1+I}{s(p_n + q_nI)}.[coss(p_n + q_nI) - 1]$$

$$= \frac{(1+I)\{1 - coss(p_n + q_nI)\}}{s(p_n + q_nI)}$$

$$(1) \Rightarrow \overline{u}_s(s,t,I) = \frac{(1+I)\{1 - coss(p_n + q_nI)\}}{s(p_n + q_nI)}.e^{-s^2(p_n+q_nI)^2t}$$

Now, taking the inverse N.F.S.T., we get

$$u(x,t,I) = \frac{2}{\pi}\int_0^\infty \frac{(1+I)\{1 - coss(p_n + q_nI)\}}{s(p_n + q_nI)}.e^{-s^2(p_n+q_nI)^2t} sins(p_n + q_nI)xds$$

$$\Rightarrow u(x,t,I) = \frac{2(1+I)}{\pi(p_n + q_nI)}\int_0^\infty \frac{\{1 - coss(p_n + q_nI)\}}{s}.e^{-s^2(p_n+q_nI)^2t} sins(p_n + q_nI)xds$$

Which is the required solution.

Example 2. Let $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, if $u(0,t,I) = 0$, $u(x,0,I) = e^{-x+I}$, $x > 0$, $u(x,t,I)$ is bounded where $x > 0, t > 0$.

here,

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad x > 0, t > 0 \tag{4}$$

with the boundary conditions:

$$u(0,t,I) = 0, \quad u(x,t,I) \quad \text{is} \quad \text{bounded} \tag{5}$$

and initial condition: $u(x,0,I) = e^{-x+I} \quad x > 0$ (6)

Since, $u(0,t,I)$ is given, we take the N.F.S.T. of both sides of (1) and obtain

$$\int_0^\infty \frac{\partial u}{\partial t} sins(p_n + q_nI)xdx = 2\int_0^\infty \frac{\partial^2 u}{\partial t^2} sins(p_n + q_nI)xdx$$

$$\Rightarrow \frac{d}{dt}\int_0^\infty usins(p_n + q_nI)xdx = 2\left[\frac{\partial u}{\partial x} sins(p_n + q_nI)x\right]_0^\infty - 2\int_0^\infty \frac{\partial u}{\partial x} s(p_n + q_nI)coss(p_n + q_nI)xdx$$

$$\Rightarrow \frac{d}{dt}(\overline{u}_s) = -2s(p_n + q_nI)\int_0^\infty \frac{\partial u}{\partial x} coss(p_n + q_nI)xdx, \quad \text{if } \frac{\partial u}{\partial x} \to 0 \quad as \quad x \to \infty$$

$[Here \quad \overline{u}_s(s,t,I) \quad is \quad the \quad N.F.S.T. \ of \ u(x,t,I)]$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2s(p_n+q_nI)\left[[u(x,t,I)coss(p_n+q_nI)x]_0^\infty + s(p_n+q_nI)\int_0^\infty u(x,t,I)sins(p_n+q_nI)xdx\right]$$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2s(p_n+q_nI)\left[0 - u(0,t,I) + s(p_n+q_nI)\int_0^\infty u(x,t,I)sins(p_n+q_nI)xdx\right]$$

$$\text{Since} \quad u(x,t,I) \to 0 \quad as \quad x \to \infty$$

$$\Rightarrow \frac{d\overline{u}_s}{dt} = -2s^2(p_n+q_nI)^2\overline{u}_s, \quad as \quad u(0,t,I) = 0$$

$$\Rightarrow \frac{1}{\overline{u}_s}d\overline{u}_s = -2s^2(p_n+q_nI)^2dt$$

$$Integrating, \quad log\overline{u}_s - logc = -2s^2(p_n+q_nI)^2t$$

$$\Rightarrow \overline{u}_s(s,t,I) = ce^{-2s^2(p_n+q_nI)^2t} \tag{7}$$

Taking the N.F.S.T. of both sides of (3), we get

$$\int_0^\infty u(x,0,I)sins(p_n+q_nI)xdx = \int_0^\infty e^{-x+I}sins(p_n+q_nI)xdx$$

$$\Rightarrow \overline{u}_s(s,0,I) = \left[\frac{e^{-x+I}}{1+s^2(p_n+q_nI)^2}\{-sins(p_n+q_nI)x - s(p_n+q_nI)coss(p_n+q_nI)x\}\right]_0^\infty$$

$$\Rightarrow \overline{u}_s(s,0,I) = \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2} \tag{8}$$

Putting $t = 0$ in (4), and using (5), we get

$$c = \overline{u}_s(s,0,I) = \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2}$$

Therefore from (4), we get

$$\overline{u}_s(s,t,I) = \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2}.e^{-2s^2(p_n+q_nI)^2t}$$

Now, taking the inverse N.F.S.T., we get

$$u(x,t,I) = \frac{2}{\pi}\int_0^\infty \frac{e^I.s(p_n+q_nI)}{1+s^2(p_n+q_nI)^2}.e^{-2s^2(p_n+q_nI)^2t}.sins(p_n+q_nI)xds$$

This equation gives the solution of the given heat equation.

Example 3. Let us consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x>0, t>0 \quad \text{subject} \quad \text{to} \quad \text{conditions}$$

$$u(0,t,I) = 0, u(x,t,I) \quad is \quad bounded \quad and$$

$$u(x,0,I) = \begin{cases} x+I & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Since, $u(x,t,I)$ is given, taking the N.F.C.T. of both sides of the given partial differential equation, we have

$$\int_0^\infty \frac{\partial u}{\partial t} coss(p_n+q_nI)xdx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} coss(p_n+q_nI)xdx$$
$$\Rightarrow \frac{d}{dt}\int_0^\infty ucoss(p_n+q_nI)xdx = \left[\frac{\partial u}{\partial x}coss(p_n+q_nI)x\right] + s(p_n+q_nI)\int_0^\infty \frac{\partial u}{\partial x}sins(p_n+q_nI)xdx$$
$$\Rightarrow \frac{d}{dt}\overline{u}_c = -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s(p_n+q_nI)\{[usins(p_n+q_nI)x]_0^\infty - s(p_n+q_nI)\int_0^\infty ucoss(p_n+q_nI)xdx\}$$

[where, we have assumed that $\frac{\partial u}{\partial x} \to 0$ as $x\to\infty$ and $\overline{u}_c(s,t,I)$ is the N.F.C.T. of $u(x,t,I)$.]

$$\Rightarrow \frac{d\overline{u}_c}{dt} = -s^2(p_n+q_nI)^2\overline{u}_c,\ \{\text{ since, } u_x(0,t,I)=0\ and\ u\to 0\ as\ x\to\infty\}$$
$$\Rightarrow \frac{1}{\overline{u}_c}d\overline{u}_c = s^2(p_n+q_nI)^2dt$$
$$Integrating,\quad log\overline{u}_c - logA = -s^2(p_n+q_nI)^2t$$
$$\Rightarrow \overline{u}_c(s,t,I) = Ae^{-s^2(p_n+q_nI)^2t} \tag{9}$$

Putting $t=0$ in (1) we get $A=\overline{u}_c(s,0,I)$.

Hence by the definition of N.F.C.T., we get

$$\begin{aligned} A =&\overline{u}_c(s,0,I)\\ =&\int_0^\infty u(x,0,I)coss(p_n+q_nI)xdx\\ =&\int_0^1 u(x,0,I)coss(p_n+q_nI)xdx + \int_1^\infty u(x,0,I)coss(p_n+q_nI)xdx\\ =&\int_0^1 (x+I)coss(p_n+q_nI)xdx,\quad \text{By using the given value of u(x,0,I)}\\ =&\left[(x+I).\frac{sins(p_n+q_nI)x}{s(p_n+q_nI)}\right]_0^1 - \int_0^1 1.\frac{sins(p_n+q_nI)x}{s(p_n+q_nI)}dx\\ =&\frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \left[\frac{coss(p_n+q_nI)x}{s^2(p_n+q_nI)^2}\right]_0^1 \end{aligned}$$

Thus,

$$A = \frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \frac{coss(p_n+q_nI)-1}{s^2(p_n+q_nI)^2}$$

Therefore, from equation (1) we get,

$$\overline{u}_c(s,t,I) = \left[\frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \frac{coss(p_n+q_nI)-1}{s^2(p_n+q_nI)^2}\right].e^{-s^2(p_n+q_nI)^2t}$$

. Now taking the inverse N.F.C.T., we get

$$u(x,t,I) = \frac{2}{\pi}\int_0^\infty \left[\frac{Isins(p_n+q_nI)}{s(p_n+q_nI)} + \frac{coss(p_n+q_nI)-1}{s^2(p_n+q_nI)^2}\right].e^{-s^2(p_n+q_nI)^2t}coss(p_n+q_nI)xds$$

5. **Convolution or Falting of Neutrosophic Functions:**

The convolution is a mathematical tool for combining two signals to form a third signal. Therefore, in signals and systems, the convolution is very important because it relates the input signal and the impulse response of the system to produce the output signal from the system. Here, we have been discussed the convolution in terms of neutrosophic functions and the convolution of neutrosophic functions is defined as follows:

Definition: The convolution of two neutrosophic functions $\mathscr{F}(x,I)$ and $\mathscr{G}(x,I)$, where $-\infty < x < \infty$ and I is an indterminate number, is denoted and defined as

$$\mathscr{F} * \mathscr{G} = \int_{-\infty}^{\infty} \mathscr{F}(u,I)\mathscr{G}(x-u,I)du$$

or,

$$\mathscr{F} * \mathscr{G} = \int_{-\infty}^{\infty} \mathscr{G}(u,I)\mathscr{F}(x-u,I)du$$

5.1. *The Neutrosophic Convolution Theorem or Neutrosophic Falting Theorem:*

The N.F.T. of the convolutionof two neutrosophic functions $\mathscr{F}(x,I)$ and $\mathscr{G}(x,I)$ is the product of the N.F.T.s of $\mathscr{F}(x,I)$ and $\mathscr{G}(x,I)$ i.e.,

$$\mathscr{F}\{\mathscr{F} * \mathscr{G}\} = \mathscr{F}\{\mathscr{F}(x,I)\}\mathscr{F}\{\mathscr{G}(x,I)\}$$

Proof:

$$\begin{aligned}
L.H.S. =&\mathscr{F}\left\{\int_{-\infty}^{\infty}\mathscr{F}(u,I)\mathscr{G}(x-u,I)du\right\},\text{ By definition of neutrosphic Convolution}\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\left\{\int_{-\infty}^{\infty}\mathscr{F}(u,I)\mathscr{G}(x-u,I)du\right\}dx,\text{ By definition of N.F.T.}\\
=&\int_{-\infty}^{\infty}\mathscr{F}(u,I)\left\{-\int_{\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{G}(x-u,I)dx\right\}du,\text{ Changing the order of integration}\\
=&\int_{-\infty}^{\infty}\mathscr{F}(u,I)\left\{\int_{-\infty}^{\infty}e^{is(p_n+q_nI)(u+v)}\mathscr{G}(v,I)dv\right\}du,\text{ Putting } x-u=v \textit{ so that } dx=dv\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(u,I)\left\{\int_{-\infty}^{\infty}e^{is(p_n+q_nI)v}\mathscr{G}(v,I)dv\right\}du\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(u,I)\left\{\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{G}(x,I)dx\right\}du,\text{ Changing the variable from v to x}\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(u,I)\mathscr{F}\{\mathscr{G}(x,I)\}du\\
=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)u}\mathscr{F}(x,I)dx\mathscr{F}\{\mathscr{G}(x,I)\},\text{ Changing the variable from u to x}\\
=&\mathscr{F}\{\mathscr{F}(x,I)\}\mathscr{F}\{\mathscr{G}(x,I)\}
\end{aligned}$$

The convolution theorem is certainly useful in solving differential equations, but it can also help us to solve integral equations, equations involving an integral of the unknown function, integro-differential equations, those involving both a derivative and an integral of the unknown function. As in this case, we have discussed the theorem by using the neutrosophic functions, it will give more reliable result than the classical form.

6. Neutrosophic Fourier Transforms of the Derivatives of a Neutrosophic function (The Derivative Theorem for N.F.T.):

Theorem 1: If $\mathscr{F}\{\mathscr{F}(x,I)\}=f(s,I)$, then $\mathscr{F}\{\mathscr{F}'(x,I)\}=-is(p_n+q_nI)f(s,I)$.
Or, If $\mathscr{F}(x,I)$ has the N.F.T. $f(s,I)$, then the N.F.T. of $\mathscr{F}'(x,I)$, the derivative of $\mathscr{F}(x,I)$ is $-is(p_n+q_nI)f(s,I)$

Proof:

$$\begin{aligned}
\mathscr{F}\{\mathscr{F}'(x,I)\}=&\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)dx\\
=&e^{is(p_n+q_nI)x}\int_{-\infty}^{\infty}\mathscr{F}'(x,I)dx-\int_{-\infty}^{\infty}\{\frac{d}{dx}e^{is(p_n+q_nI)x}\int\mathscr{F}'(x,I)dx\}dx\\
=&\left[e^{is(p_n+q_nI)x}\mathscr{F}(x,I)\right]_{-\infty}^{\infty}-is(p_n+q_nI)\int_{-\infty}^{\infty}e^{is(p_n+q_nI)x}\mathscr{F}(x,I)dx\\
=&(0-0)-is(p_n+q_nI)f(s,I),\ \{\text{Since } \mathscr{F}(x,I)\to 0 \text{ as } x\to\pm\infty\}\\
=&-is(p_n+q_nI)f(s,I).
\end{aligned}$$

Theorem 2 : $\mathscr{F}\{\mathscr{F}^n(x,I)\} = -is(p_n+q_nI)^n\mathscr{F}\{\mathscr{F}(x,I)\}$ i.e., the N.F.T. of the function $\frac{d^n\mathscr{F}(x,I)}{dx^n}$ is $\{-is(p_n+q_nI)\}^n$ times the N.F.T. of the function $\mathscr{F}(x,I)$, provided that first (n-1) derivatives of $\mathscr{F}(x,I)$ vanish as $x \to \pm\infty$.

Proof: From the theorem 1, we get

$$\begin{aligned}\mathscr{F}\{\mathscr{F}'(x,I)\} &= -is(p_n+q_nI)f(s,I)\\ &= -is(p_n+q_nI)\mathscr{F}\{\mathscr{F}(x,I)\}\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}\mathscr{F}\{\mathscr{F}''(x,I)\} &= \int_{-\infty}^{\infty} e^{-is(p_n+q_nI)x}\mathscr{F}''(x,I)dx\\ &= \Big[e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)\Big]_{-\infty}^{\infty} - is(p_n+q_nI)\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)dx\\ &=(0-0) - is(p_n+q_nI)\int_{-\infty}^{\infty} e^{is(p_n+q_nI)x}\mathscr{F}'(x,I)dx,\ \{\text{Since } \mathscr{F}'(x,I)\to 0 \text{ as } x\to\pm\infty\ \}\\ &=-is(p_n+q_nI)-is(p_n+q_nI)\mathscr{F}\{\mathscr{F}(x,I)\}\\ &=\{-is(p_n+q_nI)\}^2\mathscr{F}\{\mathscr{F}(x,I)\}\end{aligned}$$

Proceeding in this way, we get

$$\mathscr{F}\{\mathscr{F}^n(x,I)\} = \{-is(p_n+q_nI)\}^n\mathscr{F}\{\mathscr{F}(x,I)\}.$$

Theorem 3: To show that $\mathscr{F}\{\int\mathscr{F}(x,I)dx\} = \frac{f(s,I)}{-is(p_n+q_nI)}$, where $f(s,I)$ is N.F.T. of $\mathscr{F}(x,I)$.

Proof:

$$\begin{aligned}\mathscr{F}\left\{\int\mathscr{F}(x,I)dx\right\} &= \int_{-\infty}^{\infty}\left\{e^{is(p_n+q_nI)x}\int\mathscr{F}(x,I)dx\right\}dx\\ &= \left[\int\mathscr{F}(x,I)dx.\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left\{\frac{d}{dx}\int\mathscr{F}(x,I)dx.\int e^{is(p_n+q_nI)x}dx\right\}dx\\ &= (0-0) - \int_{-\infty}^{\infty}\mathscr{F}(x,I).\frac{e^{is(p_n+q_nI)x}}{is(p_n+q_nI)}dx,\ \{\text{Since, } \mathscr{F}(x,I)\to 0\ as\ x\to\pm\infty\}\\ &= -\frac{1}{is(p_n+q_nI)}\int_{-\infty}^{\infty}\mathscr{F}(x,I)e^{is(p_n+q_nI)x}dx\\ &= \frac{f(s,I)}{-is(p_n+q_nI)}\end{aligned}$$

7. **Solution of Integral Equation using N.F.T.**

Before going to the solutions of Integral Equations with the help of N.F.T., let us discuss neutrosophic integral equations.

Neutrosophic Integral Equation:

Definition: A neutrosophic integral equation is an equation in which an unknown neutrosophic function appears under one or more integral signs.

For example, for $a \leq x \leq b$, $a \leq t \leq b$, the equations

$$\int_a^b K(x,t,I)y(t,I)dt = f(x,I) \tag{11}$$

$$y(x,I) - \lambda \int_a^b K(x,t,I)y(t,I)dt = f(x,I) \tag{12}$$

and

$$y(x,I) = \int_a^b K(x,t,I)[y(t,I)]^2 dt \tag{13}$$

Where $y(x,I)$ is the unknown neutrosophic function while $f(x,I)$ and $K(x,t,I)$ are known functions and λ, a and b are constants, are all neutrosophic integral equations.

Linear and Non-Linear Neutrosophic Integral Equations:

Definition: A neutrosophic integral equation is called linear if only linear operators are performed in it upon the unknown neutrosophic function.

A neutrosophic integral equation which is not linear is called as non-linear neutrosophic integral equation.

In this study, we shall discuss only the linear neutrosophic integral equations. In the above examples of neutrosophic integral equations, the equations (1) and (2) are linear while the equation (3) in non-linear neutrosophic integral equation.

The most general type of linear neutrosophic integral equation is defined as

$$g(x,I)y(x,I) = f(x,I) + \lambda \int_a K(x,t,I)y(t,I)dt \tag{14}$$

Where the upper limit may be either variable x or fixed. The neutrosophic functions f, g and K are known functions while y is to be determined; λ is a non-zero real or complex parameter. The function $K(x,t,I)$ is known as the neutrosophic kernel of the neutrosophic integral equation.

Remark: If $g(x,I) \neq 0$, equation (4) is known as linear neutrosophic integral equation of the third kind.

When $g(x,I) = 0$, the equation (4) reduces to

$$f(x,I) + \lambda \int_a K(x,t,I)y(t,I)dt = 0 \tag{15}$$

which is known as the linear neutrosophic integral equation of the first kind.

Again when $g(x,I) = 1$, then equation (1) reduces to

$$y(x,I) = f(x,I) + \lambda \int_a K(x,t,I)y(t,I)dt \tag{16}$$

which is konwn as the linear neutrosophic integral equation of the second kind.

In this work, we shall study the solutions of neutrosophic integral equations with the help of neutrosophic fourier transform (N.F.T.) of the first and second kind only.

The linear neutrosophic integral equation can be divided into two different types depending on the limits of the integrations.

(1) Fredholm Neutrosophic Integral Equation:

Definition 1: A linear neutrosophic integral equation of the form

$$f(x,I)+\lambda\int_a^b K(x,t,I)y(t,I)dt=0 \tag{17}$$

where a, b are both constatnts, $f(x,I)$ and $K(x,t,I)$ are known functions while $y(x,I)$ is unknown function and λ is a non-zero real or complex parameter is called Fredholm neutrosophic integral equation of the first kind.

Definition 2: A linear neutrosophic integral equation of the form

$$y(x,I)=f(x,I)+\lambda\int_a^b K(x,t,I)y(t,I)dt \tag{18}$$

is known as Fredholm neutrosophic integral equation of the second kind.

Definition 3: A linear neutrosophic integral equation of the form

$$y(x,I)=\lambda\int_a^b K(x,t,I)y(t,I)dt \tag{19}$$

is known as a homogeneous Fredholm neutrosphic integral equation of the second kind.

Volterra Neutrosophic Integral Equation:

Definition 1: A linear neutrosophic integral equation of the form

$$f(x,I)+\lambda\int_a^x K(x,t,I)y(t,I)dt=0 \tag{20}$$

is known as Volterra neutrosphic integral equation of the first kind. In this case, the upper limit of the integration is a variable.

Definition 2: A linear neutrosophic integral equation of the form

$$y(x,I)=f(x,I)+\lambda\int_a^x K(x,t,I)y(t,I)dt \tag{21}$$

is known as Volterra neutrosophic integral equation of the second kind.

Definition 3: A linear neutrosophic integral equation of the form

$$y(x,I)=\lambda\int_a^x K(x,t,I)y(t,I)dt \tag{22}$$

is known as a homogeneous Volterra neutrosphic integral equation of the second kind.

Now, we shall solve some neutrosophic integral equations with the help of N.F.T.

Exmaple 1: Let us consider the neutrosophic integral equation

$$\int_0^\infty F(x,I)coss(p_n+q_nI)xdx=\begin{cases}1-s+I,\ 0\le s\le 1\\ 0,\ s>1\end{cases}$$

Here,

$$\int_0^\infty F(x,I)coss(p_n+q_nI)xdx=f_c^N(s,I) \tag{23}$$

where,

$$f_c^N(s,I) = \begin{cases} 1-s+I, \ 0 \le s \le 1 \\ 0, \ s > 1 \end{cases} \tag{24}$$

Then by the definition of $f_c^n(s,I)$ is N.F.C.T. of $F(x,I)$. In order to solve the given neutrosophic integral equation, we must obtain value of $F(x,I)$.
Now, by inversion formula for Neutrosophic Fourier cosine transform, we get

$$\begin{aligned} F(x,I) &= F_c^{-1}\{f_c(s,I)\} \\ &= \frac{2}{\pi}\int_0^\infty f_c^N(s,I)coss(p_n+q_nI)xds \\ &= \frac{2}{\pi}\left[\int_0^1 f_c^N(s,I)coss(p_n+q_nI)xds + \int_1^\infty f_c^N(s,I)coss(p_n+q_nI)xds\right] \\ &= \frac{2}{\pi}\left[\int_0^\infty (1-s+I)coss(p_n+q_nI)xds + 0\right], \ by \end{aligned} \tag{25}$$

$$\begin{aligned} &= \frac{2}{\pi}\left\{\left[(1-s+I)\frac{sins(p_n+q_nI)x}{(p_n+q_nI)x}\right]_0^1 - \int_0^1 (-1)\frac{sins(p_n+q_nI)x}{(p_n+q_nI)x}ds\right\} \\ &= \frac{2}{\pi}.I.\frac{sin(p_n+q_nI)x}{(p_n+q_nI)x} + \frac{2}{\pi(p_n+q_nI)x}\left[\frac{-coss(p_n+q_nI)x}{(p_n+q_nI)x}\right]_0^1 \\ &= \frac{2}{\pi(p_n+q_nI)x}\left[Isin(p_n+q_nI)x + \frac{1-cos(p_n+q_nI)x}{(p_n+q_nI)x}\right] \\ &= \frac{2}{\pi(p_n+q_nI)^2x^2}[I(p_n+q_nI)xsin(p_n+q_nI)x + 1 - cos(p_n+q_nI)x] \end{aligned}$$

Example 2: In the integral equation

$$\int_0^\infty f(x,I)coss(p_n+q_nI)xdx = e^{-s+I}$$

In this case we assume,

$$\int_0^\infty f(x,I)coss(p_n+q_nI)xdx = f_c(s,I) \tag{26}$$

where,

$$f_c(s,I) = e^{-s+I} \tag{27}$$

Then by definition, $f_c(s,I)$ is neutrosophic fourier cosine transform of $f(x,I)$. In order to solve the given neutrosophic integral equation, we must obtain the value of $f(x,I)$.

Now, by inversion formula for N.F.C.T., we get

$$\begin{aligned}
f(x,I) &= F_c^{-1}\{f_c(s,I)\} \\
&= \frac{2}{\pi}\int_0^\infty f_c(s,I)coss(p_n+q_nI)xds \\
&= \frac{2}{\pi}\int_0^\infty e^{-s+I}coss(p_n+q_nI)xds \\
&= \frac{2}{\pi}\left[\frac{e^{-s+I}}{1+(p_n+q_nI)^2x^2}\{-coss(p_n+q_nI)x+(p_n+q_nI)xsins(p_n+q_nI)xds\}\right]_0^\infty \\
&= \frac{2}{\pi\{1+(p_n+q_nI)^2x^2\}}[0-e^I(-1+0)] \\
&= \frac{2e^I}{\pi\{1+(p_n+q_nI)^2x^2\}}
\end{aligned}$$

Example 3: For the integral equation

$$\int_0^\infty F(x,I)sins(p_n+q_nI)xdx = \begin{cases} 1+I, \ 0\le s\le 1 \\ 2-I, \ 1\le s\le 2 \\ 0, \ s>2 \end{cases}$$

Here, we assume,

$$\int_0^\infty F(x,I)sins(p_n+q_nI)xdx = f_s(s,I) \tag{28}$$

where,

$$f_s(s,I) = \begin{cases} 1+I, \ 0\le s\le 1 \\ 2-I, \ 1\le s\le 2 \\ 0, \ s>2 \end{cases} \tag{29}$$

Then by definition, $f_s(s,I)$ is the N.F.S.T. of $F(x,I)$.

Now, by inversion formula for N.F.S.T., we have

$$\begin{aligned}
F(x,I) &= F_s^1\{f_s(s,I)\} \\
&= \frac{2}{\pi}\int_0^\infty f_s(s,I)sins(p_n+q_nI)xds \\
&= \frac{2}{\pi}[\int_0^1 f_s(s,I)sins(p_n+q_nI)xds + \int_1^2 f_s(s,I)sins(p_n+q_nI)xds \\
&+ \int_2^\infty f_s(s,I)sins(p_n+q_nI)xds] \\
&= \frac{2}{\pi}\left[\int_0^1(1+I)sins(p_n+q_nI)xds + \int_1^2(2-I)sins(p_n+q_nI)xds + 0\right]
\end{aligned}$$

$$= \frac{2}{\pi}\left\{(1+I)\left[\frac{-coss(p_n+q_nI)x}{(p_n+q_nI)x}\right]_0^1 + (2-I)\left[\frac{-coss(p_n+q_nI)x}{(p_n+q_nI)x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left[-\frac{(1+I)}{(p_n+q_nI)x}\{cos(p_n+q_nI)x-1\} - \frac{(2-I)}{(p_n+q_nI)x}\{cos2(p_n+q_nI)x - cos(p_n+q_nI)x\}\right]$$

$$= \frac{2}{\pi(p_n+q_nI)x}[cos(p_n+q_nI)x - 2Icos(p_n+q_nI)x - 2cos2(p_n+q_nI)x + I]$$

Conclusion:

In this article, we have discussed applications of Neutrosophic Fourier Transform in solving two important partial different equations. We have studied the solution Heat Equation and Integral Equations by using the Neutrosophic Fourier Transform and given detailed examples.It is observed that the application of Neutrosophic Fourier Transform in these equations give new results which are more fruitful than the results obtained in classical form.

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A Study on $(\lambda - \mu)$ Zweier Sequences and Their Behaviour in Neutrosophic Normed Spaces

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Abstract. Ideal convergence of sequences in neutrosophic normed spaces is defined by Ömer Kişi [12]. This paper defines new sequence spaces using the Zweier matrix and neutrosophic norm. We explore (λ, μ)-Zweier convergence, (λ, μ) ideal convergence of double sequences in neutrosophic norm. We show that a double sequence that is (λ, μ)-Zweier convergent or ideal convergent has a unique limit with respect to neutrosophic norm. Additionally, we prove that (λ, μ)-Zweier ideal convergence is equivalent to (λ, μ)-Zweier ideal Cauchy for a double sequence in neutrosophic normed space.

Keywords: Neutrosophic normed space, Statistical convergence, $(\lambda - \mu)$ Zweier convergence, I-convergence.

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced by Lotfi A. Zadeh [24] in 1965 as a mathematical framework to deal with uncertainty and vagueness in data. In a fuzzy set, each element is assigned a membership value ranging from 0 to 1, indicating the degree of membership of that element in the set. Kaleva, O. and Seikkala, S. (1984) [7] defined fuzzy metric space and Felbin, C. (1992) [4], Bag, T. and Samanta, S. K. (2003) [2] studied fuzzy normed linear space. Krassimir Atanassov [1] generalized the notion of fuzzy set to intuitionistic fuzzy set. Park [19] defined intuitionistic fuzzy metric space In 1995, Florentin Smarandache [23], [22] introduced the concept of neutrosophic sets as an extension of intuitionistic fuzzy sets. Neutrosophic sets aim to handle three types of indeterminacy: membership, non-membership and indeterminacy, represented by the values of truth-membership, falsehood-membership, and indeterminacy-membership respectively. Kirişci, M. and Şimşek, N. [10], [11] defined metric on neutrosophic set and also studied statistical convergence.

Concept of Statistical convergence was given H. Fast [5]. λ statistical convergence was introduced by Mursaleen [16]. λ statistical convergence is the extension of $[V, \lambda]$ summability [14]. Mursaleen et al. [17] defined the concept of (λ, μ) convergence for double sequence. The I-convergence [13] gives a unifying look on several types of convergence related to the statistical convergence. Mursaleen and Mohiuddine [18] defined I-convergence for double sequences in intuitionistic fuzzy normed spaces. In neutrosophic normed spaces, statistically convergent and statistically Cauchy double sequences are defined and studied by Granados C and Dhital [6]. Khan V. A. and Faisal M. [8], Khan V. A. and Ahmad M. [9] defined (λ, μ) Zweier ideal convergence on neutrosophic normed space and studied Cesaro summability in neutrosophic normed spaces respectively.. Zweier-Verfahren is a German word, in which Zwei means two and Verfahren means method. Zweier matrix or operator is denoted as $\mathcal{Z}^{\varrho}$, where $\varrho \neq 1$. Let $x = (\xi_n) \in \omega$ where $\omega = \{x = (\xi_n) : \xi_n \in \mathbb{R}/\mathbb{C}\}$, then $\mathcal{Z}^{\varrho}$ is defined as

$$\mathcal{Z}^{\varrho} = \mathcal{Z}_{ik} = \begin{cases} \varrho, & i = k \\ 1 - \varrho, & i - 1 = k \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Şengönül [21], define new sequence $y = (y_n)$ as $\mathcal{Z}^{\varrho}$ transform of the sequence $x = (\xi_n)$,

$$y_n = \mathcal{Z}^{\varrho}(\xi_n) = \varrho\xi_n + (1 - \varrho)\xi_{n-1} \tag{2}$$

where $\xi_{-1} = 0$ and $\varrho \neq 1$. Matrix

$$\mathcal{Z}^{\varrho} = \begin{bmatrix} \varrho & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots \\ 1-\varrho & \varrho & 0 & 0 & 0 & 0 & 0 & \ldots\ldots \\ 0 & 1-\varrho & \varrho & 0 & 0 & 0 & 0 & \ldots\ldots \\ 0 & 0 & 1-\varrho & \varrho & 0 & 0 & 0 & \ldots\ldots \\ . & . & 0 & 1-\varrho & \varrho & 0 & 0 & \ldots\ldots \\ . & . & . & . & . & . & . & \ldots\ldots \\ . & . & . & . & . & . & . & \ldots\ldots \\ . & . & . & . & . & . & . & \ldots\ldots \end{bmatrix}.$$

Definition 1.1. [20] A double sequence $x = (\xi_{gh})$ is said to be convergent(Pringsheim's Sense) to $\mathfrak{p} > 0$, if for every $\kappa > 0$, there exist $N \in \mathbb{N}$ such that

$$|\ \xi_{gh} - \mathfrak{p}\ | < k, \text{ whenever } g, h \geq N.$$

Definition 1.2. [20] A double sequence $x = (\xi_{gh})$ is said to be Cauchy (Pringsheim's Sense) , if for every $\kappa > 0$ there exist $\mathfrak{N} \in \mathbb{N}$ such that

$$|\ \xi_{gh} - \xi_{pq}\ | < \kappa, \text{ whenever } g \geq p \geq \mathfrak{N} \text{ and } h \geq q \geq \mathfrak{N}.$$

Definition 1.3. Let $\mathcal{K} \subseteq \mathbb{N} \times \mathbb{N}$, if the sequence $\dfrac{|\mathcal{K}(g,h)|}{nm}$ is convergent (Pringsheim's Sense), then we say that $\mathcal{K}$ has double natural density and defined as

$$\delta_2(\mathcal{K}) = (P) \lim_{g,h} \frac{|\mathcal{K}(g,h)|}{gh}$$

where $\mathcal{K}(g,h) = \{(p,q) \in \mathbb{N} \times \mathbb{N} : p \leq g \text{ and } q \leq h\}$. If, $g = h$ Christopher's [3] two-dimensional natural density is obtained.

Definition 1.4. [20] A double sequence $x = (\xi_{gh})$ is said to be statistically convergent to $\mathfrak{p} > 0$, if for every $\kappa > 0$,

$$\mathcal{A} = \{(g,h) : g \leq n, h \leq m \text{ such that } |\xi_{gh} - \mathfrak{p}| \geq \kappa\}$$

has $\delta_2(\mathcal{A}) = 0$.

Definition 1.5. [20] A double sequence $x = (\xi_{gh})$ is said to be statistically Cauchy, if for every $\kappa > 0$, there exist $\mathfrak{N} = \mathfrak{N}(\kappa)$ and $\mathfrak{M} = \mathfrak{M}(\kappa)$ such that for all $g, p \geq \mathfrak{N}$ and $h, q \geq \mathfrak{M}$ the set

$$\mathcal{A} = \{(g,h) : g \leq \mathfrak{N}, h \leq \mathfrak{M} \text{ such that } |\xi_{gh} - \xi_{pq}| \geq \kappa\}$$

has $\delta_2(\mathcal{A}) = 0$.

Definition 1.6. [15] If continuous mapping $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ meets the following requirements, it is called a continuous $t - norm$:

(a) $\star(\mathfrak{y}, \mathfrak{z}) = \star(\mathfrak{z}, \mathfrak{y})$ and $\star\big(\mathfrak{y}, \star(\mathfrak{z}, \mathfrak{c})\big) = \star\big(\star(\mathfrak{y}, \mathfrak{z}), \mathfrak{c}\big)$, for all $\mathfrak{y}, \mathfrak{z}, \mathfrak{c} \in [0,1]$,

(b) $\star(\mathfrak{y}, 1) = \mathfrak{y}$, $\forall\ \mathfrak{y} \in [0,1]$,

(c) $\mathfrak{y} \leq \mathfrak{c}$ and $\mathfrak{z} \leq \mathfrak{d} \Longrightarrow \star(\mathfrak{y}, \mathfrak{z}) \leq \star(\mathfrak{c}, \mathfrak{d})$, for each $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{c}$, $\mathfrak{d} \in [0,1]$.

Definition 1.7. [15] If continuous mapping $\bullet : [0,1] \times [0,1] \longrightarrow [0,1]$ meets the following requirements, it is called continuous $t - conorm$ if:

(a)$\bullet(\mathfrak{y}, \mathfrak{z}) = \bullet(\mathfrak{z}, \mathfrak{y})$ and $\bullet\big(\mathfrak{y}, \bullet(\mathfrak{z}, \mathfrak{c})\big) = \bullet\big(\bullet(\mathfrak{y}, \mathfrak{z}), \mathfrak{c}\big)$, for all $\mathfrak{y}, \mathfrak{z}, \mathfrak{c} \in [0,1]$

(b) $\bullet(\mathfrak{y}, 0) = \mathfrak{y}$, $\forall\ \mathfrak{y} \in [0,1]$,

(c) $\mathfrak{y} \leq \mathfrak{c}$ and $\mathfrak{y} \leq \mathfrak{d} \Longrightarrow \bullet(\mathfrak{y}, \mathfrak{y}) \leq \bullet(\mathfrak{c}, \mathfrak{d})$ for each $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{c}$, $\mathfrak{d} \in [0,1]$.

Definition 1.8. [11] Let V, $\star$, and $\bullet$ be linear space, continuous $t - norm$ and continuous t-conorm respectively. A four tuple of the form $\{V, \phi(z,.), \psi(z,.), \gamma(z,.) : z \in V\}$, is called neutrosophic normed space where ϕ, ψ, and γ are fuzzy sets on $V \times \mathbb{R}^+$ which satisfy the following conditions:

(i) $0 \leq \phi(\nu,p), \psi(\nu,p), \gamma(\nu,p) \leq 1$ for all $p \in \mathbb{R}^+$,

(ii) $0 \leq \phi(\nu,p) + \psi(\nu,p) + \gamma(\nu,p) \leq 3$ for all $p \in \mathbb{R}^+$,

(iii) $\phi(\nu,p) = 1$ (for $p > 0$) if and only if $\nu = 0$,

(iv) $\phi(\alpha\nu, p) = \phi(\nu, \frac{p}{|\alpha|})$ for $\alpha \neq 0$

(v) $\star\big(\phi(\nu, p), \phi(y, q)\big) \leq \phi(\nu + y, p + q)$

(vi) $\phi(\nu, .)$ is continuous and non-decreasing

(vii) $\lim_{p\to\infty} \phi(\nu, p) = 1$,

$(viii)$ $\psi(\nu, p) = 0$ for $(p > 0)$ if and only if $\nu = 0$,

(ix) $\psi(\alpha\nu, p) = \psi(\nu, \frac{p}{|\alpha|})$ for $\alpha \neq 0$,

(x) $\bullet\big(\psi(\nu, p), \psi(y, q)\big) \geq \psi(\nu + y, p + q)$

(xi) $\psi(\nu, .)$ is continuous and non-increasing,

(xii) $\lim_{p\to\infty} \psi(\nu, p) = 0$,

$(xiii)$ $\gamma(\nu, p) = 0$ (for $p > 0$) if and only if $\nu = 0$,

(xiv) $\gamma(\alpha\nu, p) = \gamma(\nu, \frac{p}{|\alpha|})$ if $\alpha \neq 0$,

(xv) $\bullet\big(\gamma(\nu, p), \gamma(y, q)\big) \geq \gamma(\nu + y, p + q)$,

(xvi) $\gamma(\nu, .)$ is continuous and non-increasing,

$(xvii)$ $\lim_{p\to\infty} \gamma(\nu, p) = 0$,

$(xviii)$ If $p \leq 0$ then $\phi(\nu, p) = 0, \psi(\nu, p) = 1,$ and $\gamma(\nu, \mathbf{p}) = 1$

Then $\mathcal{N} = \big(\phi, \psi, \gamma\big)$ is called neutrosophic norm. Throughout the paper we will use usual t-norm and usual t-conorm i.e; $\star(\tau, \varrho) = min\{\tau, \varrho\}$ and $\bullet(\tau, \varrho) = max\{\tau, \varrho\}$.

Example 1.9. 11 Let $\big(V, ||.||\big)$ be a normed space. Let ϕ, ψ, γ be Fuzzy sets on $V \times \mathbb{R}^+$ such that, for $\mathfrak{t} \geq ||\mathfrak{V}||$

$$\phi(\mathfrak{V}, \mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \leq 0 \\ \frac{\mathfrak{t}}{\mathfrak{t}+||\mathfrak{V}||}, & \mathfrak{t} > 0, \end{cases} \quad \psi(\mathfrak{V}, \mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \leq 0 \\ \frac{||\mathfrak{V}||}{\mathfrak{t}+||\mathfrak{V}||}, & \mathfrak{t} > 0, \end{cases} \text{ and } \gamma(\mathfrak{V}, \mathfrak{t}) = \frac{||\mathfrak{V}||}{\mathfrak{t}}$$

for all $\mathfrak{V} \in V$ and $\mathfrak{t} \geq 0$. If $\mathfrak{t} \leq ||\mathfrak{V}||$ then $\phi(\mathfrak{V}, \mathfrak{t}) = 0$, $\psi(\mathfrak{V}, \mathfrak{t}) = 1$ and $\gamma(\mathfrak{V}, \mathfrak{t}) = 1$. Then $\big(V, \mathcal{N}, \star, \bullet\big)$ is neutrosophic normed space.

Now we will discuss about the convergence of the sequence (ς_k) in neutrosophic normed space $\big(V, \mathcal{N}, \star, \bullet\big)$.

Definition 1.10. [11] Let (ξ_g) be sequence in NNS $\big(V, \mathcal{N}, \star, \bullet\big)$ Then (ξ_g) is said to be convergent to $\mathfrak{p} \in V$ if for each $t > 0$ and $\kappa \in$ (0,1) there exists $g_0 \in \mathbb{N}$ such that

$$\phi(\xi_g - \mathfrak{p}, t) > 1 - \kappa,\ \psi(\xi_g - \mathfrak{p}, t) < \kappa \text{ and } \gamma(\xi_n - \mathfrak{p}, t) < c \tag{3}$$

for all $g \geq g_0$.

Definition 1.11. [11] Let (ξ_g) be sequence in NNS $\big(V, \mathcal{N}, \star, \bullet\big)$ Then (ξ_g) is said to be Cauchy if, for each $t > 0$ and $\kappa \in$ (0,1) there exists $g_0 \in \mathbb{N}$ such that

$$\phi(\xi_k - \xi_h, t) > 1 - \kappa,\ \psi(\xi_k - \xi_h, t) < \kappa \text{ and } \gamma(\xi_k - \xi_h, t) < \kappa \tag{4}$$

for all $h, k \geq g_0$.

Definition 1.12. [6] Let $(V, \phi, \psi, \gamma, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{gh})$ in V is said to be convergent statistically to $\mathfrak{p}$ if, for each $\kappa > 0$ and $t > 0$

$$\delta_2\{(g,h) \in \mathbb{N} \times \mathbb{N} : \phi(\xi_{gh} - \mathfrak{p}, t) \leq 1 - \kappa,\ \psi(\xi_{gh} - \mathfrak{p}, t) \geq \kappa \text{ and } \gamma(\xi_{gh} - \mathfrak{p}, t) \geq \kappa\} = 0.$$

Definition 1.13. Let $\Upsilon = (\Upsilon_g)$ and $\mathit{\Pi} = (\mathit{\Pi}_h)$ be two positive non-decreasing sequences and $\lim_{g\to\infty} \Upsilon_g = \infty$, $\lim_{h\to\infty} \mathit{\Pi}_h = \infty$, defined as

$$\Upsilon_{g+1} \leq \Upsilon_g + 1, \qquad \text{where } \Upsilon_1 = 0. \tag{5}$$

$$\mathit{\Pi}_{h+1} \leq \mathit{\Pi}_h + 1, \qquad \text{where } \mu_1 = 0. \tag{6}$$

Let $I_g = [g - \Upsilon_g + 1, g]$ and $I_h = [h - \mathit{\Pi}_h + 1, h]$. Then

$$\delta_{\Upsilon,\mathit{\Pi}}(\mathcal{K}) = \lim_{g,h\to\infty} \frac{1}{\Upsilon_g \mu_h} \Big| \{(i,j) \in \mathbb{N} \times \mathbb{N} : (i,j) \in \mathcal{K}\} \Big|$$

is the $(\Upsilon, \mathit{\Pi})$ density of the set $\mathcal{K} \subseteq \mathbb{N} \times \mathbb{N}$. If $\Upsilon_g = g$ and $\mathit{\Pi}_h = h$, the $(\Upsilon, \mathit{\Pi})$ density reduces to natural double density of $\mathcal{K}$. The generalized double Valée Poussin mean is

$$t_{g,h} = \frac{1}{\Upsilon_g \mathit{\Pi}_h} \sum_{i \in I_g} \sum_{j \in I_h} \xi_{ij}.$$

Throughout the paper we will denote the sequences 5,6 by $\lambda = (\lambda_i)$ and $\mu = (\mu_i)$ respectively.

2. Main Results

Definition 2.1. Let $(V, \phi, \psi, \gamma, \star, \bullet)$ be a NNS and I^2 be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (\xi_{mn})$ in V is said to be I^2- convergent to ξ if, for each $\epsilon > 0$ and $t > 0$ the set

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \phi(\xi_{mn} - \xi, t) \leq 1 - \epsilon,\ \psi(\xi_{mn} - \xi, t) \geq \epsilon \text{ and } \gamma(\xi_{mn} - \xi, t) \geq \epsilon\} \in I^2.$$

Definition 2.2. Let $(V, \phi, \psi, \gamma, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{mn})$ in V is said to be (λ, μ)- Zweier convergent to ξ with respect to neutrosophic norm (ϕ, ψ, γ) if, for every $\epsilon > 0$ and $t > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \leq 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon$$

for all $m, n \geq n_0$ and write it as $(\lambda, \mu) - \lim_{m,n\to\infty} \mathcal{Z}^q \xi_{mn} = \xi$.

Definition 2.3. Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS and I^2 is an admissible ideal. A double sequence $x = (\xi_{mn})$ in V is said to be (λ, μ)- Zweier ideal convergent to ξ with respect to neutrosophic norm (ϕ, ψ, γ), if for every $\epsilon > 0$ and for all $t > 0$, the set

$$\Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \leq 1 - \epsilon, \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon, \text{ and } \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon, \Big\} \in I^2.$$

In short we write $(\phi, \psi, \gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n \to \infty} \mathcal{Z}^q \xi_{mn} = \xi$

Definition 2.4. Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{mn})$ in V is said to be (λ, μ)- Zweier Cauchy sequence with respect to neutrosophic norm (ϕ, ψ, γ), if for every $\epsilon > 0$ and for all $t > 0$, there exists N such that

$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \leq 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon.$$

for all $m, n, k, p \geq N$.

Definition 2.5. Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS and I^2 is an admissible ideal. A double sequence $\xi = (\xi_{mn})$ in V is said to be (λ, μ)-Zweier ideal Cauchy sequence if for each $\epsilon > 0$ and $t > 0$ there exists a positive integer N such that

$$\Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \leq 1 - \epsilon, \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon, \text{ and } \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon \Big\} \in I^2.$$

Lemma 2.6. *Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS and $I^2 \subseteq \mathbb{N} \times \mathbb{N}$ be an admissible ideal. Let $x = (\xi_{mn})$ be a double sequence in V, then the following are equivalent*

(i) $(\phi, \psi, \gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n \to \infty} \mathcal{Z}^q(\xi_{mn}) = \mathcal{L}$.

(ii) $\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big)\leq 1-\kappa\right\}\in I^2,$
$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big)\geq \kappa\right\}\in I^2$ *and*
$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big)\geq \kappa\right\}\in I^2$ *for all* $\kappa>0$ *and* $t>0$.

(iii) $\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big)> 1-\kappa\right\}\in F(I^2),$
$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big)< \kappa\right\}\in F(I^2)$ *and*
$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big)< \kappa\right\}\in F(I^2)$ *for all* $\kappa>0$ *and* $t>0$.

(iv) $(\phi,\psi,\gamma)-I^2_{(\lambda,\mu)}-\lim_{m,n\to\infty}\phi(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t)=1,$
$(\phi,\psi,\gamma)-I^2_{(\lambda,\mu)}-\lim_{m,n\to\infty}\phi(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t)=0$ *and*
$(\phi,\psi,\gamma)-I^2_{(\lambda,\mu)}-\lim_{m,n\to\infty}\phi(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t)=0$ *for all* $\kappa>0$ *and* $t>0$.

Theorem 2.7. *Let $(V,\mathcal{N},\star,\bullet)$ be a NNS and I^2 is an admissible ideal. If a double sequence $x=(\xi_{mn})$ in V is (λ,μ)- Zweier ideal convergent with respect to neutrosophic norm (ϕ,ψ,γ), then its limit is unique.*

Proof. We will prove it by contradiction i.e. Let on contrary that the sequence $x=(\xi_{mn})$ converges to two limits (say) ℓ_1 and ℓ_2. By definition 2.3, for a given $\epsilon>0$, choose $\kappa>0$ such that $(1-\kappa)\star(1-\kappa)>1-\epsilon$ and $\kappa\star\kappa<\epsilon$. Then for $t>0$

$$A_{\phi,1}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)\leq 1-\kappa\right\}\in I^2.$$

$$A_{\phi,2}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)\leq 1-\kappa\right\}\in I^2.$$

$$A_{\psi,1}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)\geq \kappa\right\}\in I^2.$$

$$A_{\psi,2}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)\geq \kappa\right\}\in I^2.$$

$$A_{\gamma,1}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)\geq \kappa\right\}\in I^2.$$

$$A_{\gamma,2}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)\geq\kappa\right\}\in I^2.$$

Now the following set

$$A_{\phi,\psi,\gamma}(\kappa,t)=\Big[A_{\phi,1}(\kappa,t)\cup A_{\phi,2}(\kappa,t)\Big]\cap\Big[A_{\psi,1}(\kappa,t)\cup A_{\psi,2}(\kappa,t)\Big]\cap\Big[A_{\gamma,1}(\kappa,t)\cup A_{\gamma,2}(\kappa,t)\Big]\in I^2$$

$$A_{\phi,\psi,\gamma}(\kappa,t)\in I^2\implies A^{\complement}_{\phi,\psi,\gamma}(\kappa,t)\in F(I^2).$$

If $(i,j)\in A^{\complement}_{\phi,\psi,\gamma}(\kappa,t)$, then there are three cases. Either $(i,j)\in A^{\complement}_{\phi,1}(\kappa,t)\cap A^{\complement}_{\phi,2}(\kappa,t)$ or $(i,j)\in A^{\complement}_{\psi,1}(\kappa,t)\cap A^{\complement}_{\psi,2}(\kappa,t)$ or $(i,j)\in A^{\complement}_{\gamma,1}(\kappa,t)\cap A^{\complement}_{\gamma,2}(\kappa,t)$. Firstly, if $(i,j)\in A^{\complement}_{\phi,1}(\kappa,t)\cap A^{\complement}_{\phi,2}(\kappa,t)$, we have

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)>1-\kappa$$

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)>1-\kappa$$

Now take $(a,b)\in\mathbb{N}\times\mathbb{N}$, such that

$$\phi\left(\mathcal{Z}^q\xi_{ab}-\ell_1,\frac{t}{2}\right)>\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)>1-\kappa$$

and

$$\phi\left(\mathcal{Z}^q\xi_{ab}-\ell_2,\frac{t}{2}\right)>\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)>1-\kappa.$$

Select (a,b) for (m,n) such that the maximum hold i.e. $\max\left\{\phi\Big(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\Big),\psi\Big(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\Big),\gamma\Big(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\Big):m\in\Lambda_i,n\in\Lambda_j\right\}$. Then, we have

$$\phi(\ell_1-\ell_2,t)\geq\phi\Big(\mathcal{Z}^q\xi_{ab}-\ell_1,\frac{t}{2}\Big)\star\phi\Big(\mathcal{Z}^q\xi_{ab}-\ell_2,\frac{t}{2}\Big)>(1-\kappa)\star(1-\kappa)>1-\epsilon$$

Since $\epsilon>0$ was arbitrary, for every $t>0$, we get $\phi\Big(\ell_1-\ell_2,t\Big)=1$, that means $\ell_1=\ell_2$. For other cases if, $(i,j)\in A^{\complement}_{\psi,1}(\kappa,t)\cap A^{\complement}_{\psi,2}(\kappa,t)$ or $(i,j)\in A^{\complement}_{\gamma,1}(\kappa,t)\cap A^{\complement}_{\gamma,2}(\kappa,t)$, in similar fashion we get $\psi(\ell_1-\ell_2,t)>\epsilon$ and $\gamma(\ell_1-\ell_2,t)>\epsilon$ for $t>0$. Hence for each case, we get $\ell_1=\ell_2$.Hence limit is unique. □

Theorem 2.8. *Let $(V,\mathcal{N},\star,\bullet)$ be a NNS. If a double sequence $x=(\xi_{mn})$ in V is (λ,μ)- Zweier convergent with respect to neutrosophic norm (ϕ,ψ,γ), then its limit is unique.*

Proof. Let $(\lambda,\mu)-\lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn}=\xi_1$ and $(\lambda,\mu)-\lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn}=\xi_2$. Given $\epsilon>0$, select $\kappa>0$ such that $(1-\kappa)\star(1-\kappa)>1-\epsilon$ and $\kappa\bullet\kappa<\epsilon$. Then for any $t>0$, there exists $n_1\in\mathbb{N}$ such that

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\xi_1,t\big)>1-\epsilon$$

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\xi_1,t\big)<\epsilon$$

and

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\xi_1,t\big)<\epsilon$$

for all $i,j\geq n_1$. There also exists n_2 such that

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\xi_2,t\big)>1-\epsilon$$

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\xi_2,t\big)<1-\epsilon$$

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\lambda\big(\mathcal{Z}^q\xi_{mn}-\xi_2,t\big)<\epsilon$$

for all $i,j\geq n_2$. Now chose $n_0=max\{n_1,n_2\}$. Then for all $n\geq n_0,$ we have$(a,b)\in\mathbb{N}\times\mathbb{N}$ such that

$$\phi\big(\mathcal{Z}^q\xi_{ab}-\xi_1,t\big)>\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\xi_1,t\big)>1-\kappa$$

. and

$$\phi\big(\mathcal{Z}^q\xi_{ab}-\xi_2,t\big)>\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\xi_2,t\big)>1-\kappa.$$

Hence, we get

$$\phi(\xi_1-\xi_2,t)\geq\phi\big(\mathcal{Z}^q\xi_{mn}-\xi_1,t\big)\star\phi\big(\mathcal{Z}^q\xi_{mn}-\xi_2,t\big)>(1-\kappa)\star(1-\kappa)>1-\epsilon.$$

Since ϵ was arbitrary. Hence we get $\phi(\xi_1-\xi_2,t)=1$ for all $t>0$. Hence $\xi_1=\xi_2$. Similarly, we will prove for ψ and γ that $\psi(\xi_1-\xi_2,t)<\epsilon$ and $\gamma(\xi_1-\xi_2,t)<\epsilon$ for any $\epsilon>0$. Hence we get $\psi(\xi_1-\xi_2,t)=0$ and $\gamma(\xi_1-\xi_2,t)=0$ for $t>0$, this implies that $\xi_1=\xi_2$. □

Theorem 2.9. *Let $(V,\mathcal{N},\star,\bullet)$ be a NNS. (λ,μ)-Zweier convergence of a double sequence $x=(\xi_{mn})$ in V implies (λ,μ)-Zweier ideal convergence of $x=(\xi_{mn})$ with respect to neutrosophic norm and $(\lambda,\mu)-\lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn}=(\phi,\psi,\gamma)-I^2_{(\lambda,\mu)}-\lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn}=\xi$.*

Proof. Let sequence $x = (\xi_{mn})$ is (λ,μ)- Zweier convergent to ξ. Then for $\epsilon > 0$ and $t > 0$, there exists positive integer N such that

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\xi,t\big) > 1-\epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\xi,t\big) < \epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\xi,t\big) < \epsilon$$

for all $i, j \geq N$. Then the set

$$R(\epsilon,t) = \Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Bigg(\mathcal{Z}^q\xi_{mn}-\xi,t\Bigg) \leq 1-\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\xi,t\big) \geq \epsilon, \text{and } \frac{1}{\lambda_i\mu_j}\sum_{n\in\Lambda_i}\sum_{m\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\xi,t\big) \geq \epsilon,\Bigg\}.$$

is contained in the set $Q = \{(1,1),(1,2),(2,1),(2,2),....,(N-1,N-1)\}$. Since I^2 is an admissible ideal. We get, $R(\epsilon,t) \subseteq Q \in I^2$. Hence (λ,μ)-Zweier convergence of a double sequence implies (λ,μ)-Zweier ideal convergence and $(\phi,\psi,\gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn} = \xi$. □

Theorem 2.10. *Let $(V,\mathcal{N},\star,\bullet)$ be a NNS. A double sequence $x = (\xi_{mn})$ in V is (λ,μ)-Zweier ideal convergent if and only if it is (λ,μ)-Zweier ideal Cauchy.*

Proof. Let $x = (\xi_{mn})$ in V is (λ,μ)-Zweier ideal Cauchy sequence but not (λ,μ)-Zweier ideal convergent in V with respect to neutrosophic norm (ϕ,ψ,γ). Now, by the definition of (λ,μ)-Zweier ideal Cauchy for each $\epsilon > 0$ and $t > 0$, there exists positive integer N such that

$$G(\epsilon,t) = \Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big) \leq 1-\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big) \geq \epsilon,$$
$$\text{and } \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big) \geq \epsilon\Bigg\} \in I^2.$$

and

$$H(\epsilon,t)=\Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\leq 1-\epsilon,$$

$$\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\geq\epsilon,\text{ and}$$

$$\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\geq\epsilon\Bigg\}\in F(I^2)$$

Since

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)>1-\epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\epsilon,$$

and

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\epsilon.$$

If

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)>\frac{1-\epsilon}{2},\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\frac{\epsilon}{2},$$

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\frac{\epsilon}{2}.$$

Then, we have

$$\delta_(\lambda,\mu)\Bigg(\Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq 1-\epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon,$$

$$\text{and }\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon\Bigg\}\Bigg)=0$$

That is $G(\epsilon,t)\in F(I^2)$, which is a contradiction. Hence, sequence $\xi=(\xi_{mn})$ is (λ,μ)-Zweier ideal convergent with respect to neutrosophic norm.

Conversely, suppose that $(\phi,\psi,\gamma)-I^2_{(\lambda,\mu)}-\lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn}=\xi$. Choose $r>0$, such that $(1-r)\star(1-r)>1-\epsilon$ and $r\bullet r<\epsilon$.For $t>0$ define

$$W(r,t) = \Bigg\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big) \leq 1-r,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big) \geq r, \text{ and}$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big) \geq r\Bigg\} \in I^2$$

or $W(r,t)^{\complement} \in F(I^2)$. Assume $(k,p) \in W(r,t)^{\complement}$. Then, we get

$$\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big) > 1-r$$

$$\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big) < r$$

$$\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big) < r$$

For every $\epsilon > 0$, we take

$$U(\epsilon,t) = \Bigg\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big) \leq 1-\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big) \geq \epsilon,$$
$$\text{and } \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big) \geq \epsilon\Bigg\} \in I^2.$$

Now we will show that $U(\epsilon,t) \subset W(r,t)$. Let $(u,v) \in U(\epsilon,t)$

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big) \leq 1-\epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big) \geq \epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big) \geq \epsilon.$$

Here we are going to divide it into three cases as follows

Case1 :

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big) \leq 1-\epsilon.$$

Then

$$\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\leq 1-r \implies (u,v)\in W(r,t).$$

Otherwise, if $\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)>1-r$. Then, we have

$$1-\epsilon\geq\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)$$

$$\geq\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\star\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{kp}-\xi,\frac{t}{2}\Big)$$

$$>(1-r)\star(1-r)$$

$$>1-\epsilon.$$

We reach at a contradiction. Hence $U(\epsilon,t)\subset W(r,t)$.

Case2 : Consider

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon.$$

We get

$$\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\geq r \implies (u,v)\in W(r,t).$$

Otherwise, if

$$\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)<r.$$

Then we obtain

$$1-\epsilon\leq\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)$$

$$\leq\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\bullet\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{kp}-\xi,\frac{t}{2}\Big)$$

$$<(1-r)\bullet(1-r)$$

$$<1-\epsilon.$$

Which is a contradiction. Hence, $U(\epsilon,t)\subset W(r,t)$.

Case 3 is similar to the case 2 , in this case we will consider that

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon.$$

And we obtain that, $U(\epsilon,t)\subset W(r,t)$.

In each case we obtain that $U(\epsilon,t)\subset W(r,t)$. This implies that $U(\epsilon,t)\in I^2$. Hence double sequence is a Zweier ideal Cauchy sequence with respect to th neutrosophic norm (ϕ,ψ,γ). □

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More on neutrosophic topology

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Abstract. The aim of this paper is to introduce the notion of neutrosophic singletons and the induced neutrosophic topology. We also study some of its basic properties.

Keywords: Neutrosophic topological space, neutrosophic singleton, induced neutrosophic topology.

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1. Notations and Terminology

In 1965, L. A. Zadeh [16] introduced the concept of *fuzzy sets*, which revolutionized our understanding of set theory. Since then, fuzzy set theory has influenced almost every branch of pure and applied mathematics, as well as fields such as physics, engineering, information theory, and control theory. Three years later, Chang [6] introduced the notion of *fuzzy topological spaces*, sparking numerous research projects that extended classical topological concepts into the fuzzy context.

Nineteen years after the introduction of fuzzy sets, Atanassov [1] proposed the idea of *intuitionistic fuzzy sets*. Subsequently, he and his colleagues [2–5] advanced this concept, yielding several interesting and important results. Eleven years later, Coker [7] introduced the notion of *intuitionistic fuzzy topological spaces*.

Smarandache [14, 15] later introduced the concepts of *neutrosophy* and *neutrosophic sets*. In 2002, Smarandache [14] expanded this framework by proposing the notion of *neutrosophic topology* on the non-standard interval. Notably, Lupiáñez [8–10] established several properties of neutrosophic topological spaces, showing, for instance, that an intuitionistic fuzzy topology is generally not a neutrosophic topology [8]. In 2012, Salama and Alblowi [12] introduced the concepts of *neutrosophic crisp sets* and *neutrosophic topological spaces*. V. L. Nayagam and G. Sivaraman [11] later explored properties of induced topology on intuitionistic fuzzy singletons.

In this paper, we introduce and investigate the concepts of *neutrosophic singletons* and the corresponding *induced neutrosophic topology* along similar lines. We also study some of their fundamental properties. Below, we summarize some well-known notions that will be used in subsequent sections.

Definition 1.1. Let $\mathcal{X}$ be a nonempty fixed set. A neutrosophic set (briefly NS) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in \mathcal{X}$ to the set A.

Remark 1.2. (1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on $\mathcal{X}$.

(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$.

Definition 1.3. Let $\mathcal{X}$ be a nonempty set and the neutrosophic sets A and B in the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in \mathcal{X}\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in \mathcal{X}$;
(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in \mathcal{X}\}$; [Complement of A]
(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in \mathcal{X}\}$;
(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in \mathcal{X}\}$;

Definition 1.4. Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in $\mathcal{X}$. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in \mathcal{X}\}$;
(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in \mathcal{X}\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in $\mathcal{X}$ as follows:

Definition 1.5. $0_N = \{\langle x, 0, 0, 1\rangle : x \in \mathcal{X}\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in \mathcal{X}\}$.

Definition 1.6. [13] A neutrosophic topology (briefly NT) on a nonempty set $\mathcal{X}$ is a family $\mathcal{T}$ of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in \mathcal{T}$,

(ii) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$,

(iii) $\cup G_i \in \mathcal{T}$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \mathcal{T}$.

In this case the ordered pair $(\mathcal{X}, \mathcal{T})$ or simply $\mathcal{X}$ is called a neutrosophic topological space (briefly NTS($\mathcal{X}$)) and each neutrosophic set in $\mathcal{T}$ is called a neutrosophic open set (briefly NOS). The complement $\overline{A}$ of a NOS A in $\mathcal{X}$ is called a neutrosophic closed set (briefly NCS) in $\mathcal{X}$.

Definition 1.7. Let $\mathcal{X}$ be a nonempty set. If r, t, s are real standard or non standard subsets of $]0^-, 1^+[$ then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point or singleton in $\mathcal{X}$ given by

$$x_{r,t,s}(x_p) = \begin{cases} (r,t,s), & \text{if } x = x_p \\ (0,0,1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in \mathcal{X}$ is called the support of $x_{r,t,s}$, where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

From now on, we denote a neutrosophic singleton by $p = (\mu_p, \sigma_p, \gamma_p)$.

Definition 1.8. Let $\mathcal{X}$ be any non-empty set. A neutrosophic singleton p defined on x is said to belong to a neutrosophic set $A = (\mu_A, \sigma_A, \gamma_A)$ ($p \in A$) if $\mu_p \leq \mu_A$, $\sigma_p \leq \sigma_A$ and $\gamma_A \leq \gamma_p$.

Definition 1.9. A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is said to be neutrosophic Hausdorff if for every distinct points $x, y \in \mathcal{X}$, there exist neutrosophic open sets $A = (\mu_A, \sigma_A, \gamma_A)$, $B = (\mu_B, \sigma_B, \gamma_B) \in \tau$ such that $\mu_A(x) = 1$, $\mu_B(y) = 1$ and $A \cap B = 0_N$.

Definition 1.10. A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is said to be neutrosophic compact if for every cover ϱ by neutrosophic open sets $1_N = \cup_{A \in \varrho}$, there exists a finite subcover A_1, A_2, A_3, ...,A_n of ϱ such that $1_N = \cup_{i=1}^{n} A_i$.

Definition 1.11. A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is said to be neutrosophic connected if 1_N can not be written as the union of two neutrosophic open sets $A = (\mu_A, \sigma_A, \gamma_A)$, $B = (\mu_B, \sigma_B, \gamma_B) \in \mathcal{T}$ such that $A \cap B = 0_N$.

2. Induced neutrosophic topology and some properties

Here we introduce the induced neutrosophic topology on neutrosophic singletons.

Definition 2.1. Let $(\mathcal{X}, \mathcal{T})$ be a neutrosophic topological space and $\mathcal{P}(\mathcal{X})$ the collection of all neutrosophic singletons of $\mathcal{X}$. The induced neutrosophic topology $\sigma_{\mathcal{T}}$ on $\mathcal{P}(\mathcal{X})$ is defined as the topology generated by $\mathcal{B}$= $\{\mathcal{V}_A \mid A \in \mathcal{T}\}$, where $\mathcal{V}_A = \{p \in \mathcal{P}(\mathcal{X}) \mid p \in A\}$. We call $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ the induced neutrosophic topological space.

Remark 2.2. It should be mentioned that obviously $\mathcal{B}$ is a basis for the topology $\mathcal{P}(\mathcal{X})$. We know that $1_N \in \tau$ and $\mathcal{V}_{1_N} = \mathcal{P}(\mathcal{X}) \in \mathcal{B}$. Thus for every neutrosophic singleton p, $p \in \mathcal{P}(\mathcal{X}) \in \mathcal{B}$. Moreover, if $\mathcal{V}_A$, $\mathcal{V}_B \in \mathcal{B}$, then it is clear that $\mathcal{V}_A \cap \mathcal{V}_B = \mathcal{V}_{A\cap B}$ and $\mathcal{V}_A \cap \mathcal{V}_B \in \mathcal{B}$.

Example 2.3. Let $\mathcal{X} = \{a, b, c\}$ and

$$\mathcal{T} = \big\{\{(x, y, 0), (r, s, 0), (n, m, 1)\}, \{(0, y, z), (0, r, s), (1, m, d)\}, \{(0, y, 0), (0, s, 0), (1, m, 1)\}\big\},$$

where $z < x < y$. Here $\{(x, y, z), (r, s, t), (n, m, d)\} \in \mathcal{I}^{\mathcal{X}} \times \mathcal{I}^{\mathcal{X}} \times \mathcal{I}^{\mathcal{X}}$. Observe that $(\mathcal{X}, \mathcal{T})$ is a neutrosophic topological space. Take $\sigma_{\mathcal{T}}= (\{\mathcal{V}_{0_N}, \mathcal{V}_{1_N},\ \mathcal{V}_{\{(x,y,0),(r,s,0),(n,m,1)\}}, \mathcal{V}_{\{(0,y,z),(0,r,s),(1,m,d)\}}, \mathcal{V}_{\{(0,y,0),(0,s,0),(1,m,1)\}})$. As it can be seen the induced neutrosophic topology $\sigma_{\mathcal{T}}$ on $\mathcal{P}(\mathcal{X})$ is not necessarily discrete.

Now we introduce the notion of the induced neutrosophic function.

Definition 2.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be any function, where $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} \neq \emptyset$. If the function $i_f : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$ is defined for any neutrosophic singleton $p = (\mu_p, \sigma_p, \gamma_p)$ defined on $x \in \mathcal{X}$ by $i_f(p) = q$, where $q = (\mu_q, \sigma_q, \gamma_q)$ is a neutrosophic singleton defined on $f(x) \in \mathcal{Y}$ with $\mu_q(f(x)) = \mu_p(x)$, $\sigma_q(f(x)) = \sigma_p(x)$ and $\gamma_q(f(x)) = \gamma_p(x)$, then i_f is called the induced neutrosophic function of f.

Lemma 2.5. *If $f : \mathcal{X} \to \mathcal{Y}$ is any function and $i_f : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$ the induced neutrosophic function of f, then the following statements hold:*
a) For any neutrosophic set $A = (\mu_A, \sigma_A, \gamma_A)$ of $\mathcal{Y}$, $\mathcal{V}_{f^{-1}(A)} = i_{f^{-1}}(\mathcal{V}_A)$, where $f^{-1}(A) = (\mu_{f^{-1}(A)}, \sigma_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$, where $\mu_{f^{-1}(A)}(x) = \mu_A(f(x))$, $\sigma_{f^{-1}(A)}(x) = \sigma_A(f(x))$ and $\gamma_{f^{-1}(A)}(x) = \gamma_A(f(x))$.
b) For any neutrosophic set $A = (\mu_A, \sigma_A, \gamma_A)$ of X, if $\mathcal{V}_A = \cup_{A_\gamma} \mathcal{V}_{A_\gamma}$, where A_γ belongs to the neutrosophic sets of $\mathcal{X}$, then $A = \cup A_\gamma$.

Proof. a) $p \in \mathcal{V}_{f^{-1}(A)}$ iff $\mu_p(x) \leq \mu_{f^{-1}(A)}(x)$, $\sigma_p(x) \leq \sigma_{f^{-1}(A)}(x)$ and $\gamma_p(x) \geq \gamma_{f^{-1}(A)}(x)$. This means that $\mu_p(x) \leq \mu_A(f(x))$, $\sigma_p(x) \leq \sigma_A(f(x))$ and $\gamma_p(x) \geq \gamma_A(f(x))$ iff $\mu_{i_{f(p)}}(f(x)) \leq \mu_A(f(x))$, $\sigma_{i_{f(p)}}(f(x)) \leq \sigma_A(f(x))$ and $\gamma_{i_{f(p)}}(f(x)) \geq \gamma_A(f(x))$ iff $i_f(p) \in \mathcal{V}_A$ iff $p \in i_{f^{-1}}(\mathcal{V}_A)$.

b) Let $\mathcal{V}_A = \cup_{A_\gamma} \mathcal{V}_{A_\gamma}$, where $A = (\mu_A, \sigma_A, \gamma_A)$. If we want to show $A = \cup A_\gamma$, it is the same to prove $A(x) = \cup_{A_\gamma}(x)$ for any $x \in \mathcal{X}$. Let us define a neutrosophic singleton p on x such that $\mu_p(x) = \mu_A(x)$, $\sigma_p(x) = \sigma_A(x)$ and $\gamma_p(x) = \gamma_A(x)$. It is obvious that $p \in \mathcal{V}_A$. Thus if $p \in \cup A_\gamma$ implies that there exists a γ such that $p \in \mathcal{V}_{A_\gamma}$. Therefore $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$. Hence $\mu_A(x) \leq \mu_{A_\gamma}(x)$, $\sigma_A(x) \leq \sigma_{A_\gamma}(x)$ and $\gamma_A(x) \geq \gamma_{A_\gamma}(x)$. Observe that $\mu_A(x) \leq sup_{A_\gamma} \mu_{A_\gamma}(x)$, $\sigma_A(x) \leq sup_{A_\gamma} I_{A_\gamma}(x)$ and $\gamma_A(x) \geq inf_{A_\gamma} \gamma_{A_\gamma}(x)$. If $\mu_A(x) \leq sup_{A_\gamma} \mu_{A_\gamma}(x)$, then there exists some A_α in the set of neutrosophic sets of $\mathcal{X}$ such that $\mu_A(x) < \mu_{A_\alpha}(x) \leq sup_{A_\gamma}\mu_{A_\gamma}(x)$. Moreover, define a neutrosophic singleton q on x such that $\mu_q(x) = \mu_{A_\alpha}(x)$, $\sigma_q(x) = \sigma_{A_\alpha}(x)$ and $\gamma_q(x) = \gamma_{A_\alpha}(x)$. Thus $q \in \mathcal{V}_{A_\alpha}$ and obviously $q \in \cup_{A_\gamma}\mathcal{V}_{A_\gamma}$. Here we come to a contradiction since $q \notin \mathcal{V}_A$. Therefore $\mu_A(x) = sup_{A_\gamma}\mu_{A_\gamma}(x)$. By the same token, we obtain $\sigma_A(x) = sup_{A_\gamma}\sigma_{A_\gamma}(x)$ and $\gamma_A(x) = inf_{A_\gamma}\gamma_{A_\gamma}(x)$. This shows that $A = \cup A_\gamma$. □

Theorem 2.6. *A function $f : (\mathcal{P}(\mathcal{X}), \mathcal{T}) \to (\mathcal{Y}, \varrho)$ is a neutrosophic continuous function iff the induced neutrosophic function $i_f : (P(\mathcal{X}), \sigma_\mathcal{T}) \to (P(\mathcal{X}), \sigma_\varrho)$ is continuous.*

Proof. Let f be a neutrosophic continuous function. Now we will prove that i_f is continuous. Suppose $\mathcal{V}_A$ is an open set in σ_ϱ and thus $A \in \varrho$, By statement (a) in Lemma 2.5, $i_{f^{-1}}(\mathcal{V}_A) = \mathcal{V}_{f^{-1}(A)}$. By the fact that f is neutrosophic continuous, then $f^{-1}(A)$ is a neutrosophic open set in $\mathcal{T}$. This means that $i_{f^{-1}}(\mathcal{V}_A)$ is an open set in $\sigma_\mathcal{T}$.
Conversely, let i_f be continuous and $A \in \varrho$. We know that A is neutrosophic open, thus $\mathcal{V}_A$ is an open set in σ_ϱ. Since i_f is continuous , then $i_{f^{-1}}(\mathcal{V}_A)$ is open in $\sigma_\mathcal{T}$. This implies that $i_{f^{-1}}(\mathcal{V}_A) = \cup_{A_\gamma}\mathcal{V}_{A_\gamma}$, where $A_\gamma \in \sigma_\mathcal{T}$. By (a) in Lemma 2.5, this means that $\mathcal{V}_{f^{-1}(A)} = \cup_{A_\gamma}\mathcal{V}_{A_\gamma}$. By (b) in Lemma 2.5, $f^{-1}(A) = \cup A_\gamma$. This shows that $f^{-1}(A) \in \mathcal{T}$. □

Theorem 2.7. *Let $(\mathcal{X}, \mathcal{T})$ be a neutrosophic compact space. The induced neutrosophic topological space $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$ is compact but the converse is not true.*

Proof. Suppose that $(\mathcal{X}, \mathcal{T})$ is a neutrosophic compact space. For some $B_0 \subseteq \mathcal{B}$ and $\mathcal{B}$ is a base for $\sigma_\mathcal{T}$, let $\mathcal{P}(\mathcal{X}) = \cup_{\mathcal{V}_{A \in B_0}}\mathcal{V}_A$. Take any $x \in \mathcal{X}$, $(1_x, 0_x) \in \mathcal{P}(\mathcal{X})$, where 1_x and 0_x are the characteristic functions on $\{x\}$ and on $A \setminus \{x\}$, respectively. Observe that $(1_x, 0_x) \in \mathcal{V}_{A_x}$ for some $\mathcal{V}_{A_x} \in B_0$. Then, we have $1_x \leq T_{A_x}$, $1_x \leq I_{A_x}$ and $0_x \geq F_{A_x}$. This means that $T_{A_x}(x) = 1$, $I_{A_x}(x) = 1$ and $F_{A_x}(x) = 0$. Hence $1_N = \vee_{x \in X} A_x$. Since $(\mathcal{X}, \mathcal{T})$ is a neutrosophic compact space, then $1_N = \vee_i^n A_{x_i}$ for $i = 1, 2, ..., n$. Thus $\mathcal{P}(X) = \vee_i^n A_{x_i}$. This means that $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$ is compact.
The converse is not true. Take $(\mathcal{X}, \mathcal{T})$ such that $\mathcal{T} = \{0_N, 1_N, (1 - \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n})\}$. This space is a neutrosophic topological space. Let $A_n = (1 - \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n})$. An open covering of $\mathcal{P}(\mathcal{X})$ which contains $\mathcal{V}_{\cup A_n}$ and hence $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$ is compact but $(\mathcal{X}, \mathcal{T})$ is not compact. □

Remark 2.8. It is worth-noticing that if we take two neutrosophic singletons which are defined on the same point with different values are distinct in $\mathcal{P}(\mathcal{X})$, then it is impossible to talk about Hausdorffness since we can not separate then by two open sets. Mind that any two neutrosophic singletons defined on distinct points can be separated by open sets. Thus this suggests to introduce the notion of neutrosophic pseudo Hausdorff space on $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$.

Definition 2.9. The induced neutrosophic topological space $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is called neutrosophic pseudo Hausdorff if for any two neutrosophic singletons $p = (\mu_p, \sigma_p, \gamma_p)$, $q = (\mu_q, \sigma_q, \gamma_q)$ defined on distinct points, there exists two disjoint open sets $\mathcal{V}_A$ and $\mathcal{V}_B$ such that $p \in \mathcal{V}_A$ and $q \in \mathcal{V}_B$.

Theorem 2.10. *A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff iff $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is neutrosophic pseudo Hausdorff.*

Proof. Suppose that $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff. Proving $(\Rightarrow)$, let us take two neutrosophic singletons $p = (\mu_p, \sigma_p, \gamma_p)$, $q = (\mu_q, \sigma_q, \gamma_q)$ with distinct supports, i.e., $\mu_p = \{x\} \neq \mu_q = \{y\}$. Since $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff, there exist $A, B \in \mathcal{T}$ such that $\mu_A(x) = 1$, $\mu_B(y) = 1$ such that $A \cap B = 0_N$. By the fact that A and B are neutrosophic subsets, $\sigma_A(x) = 1$, $\sigma_B(x) = 1$ and $\gamma_A(x) = 0$, $\gamma_B(x) = 0$. Therefore, $p \in \mathcal{V}_A$, $q \in \mathcal{V}_B$. By the fact that $A \cap B = 0_N$, then $\mathcal{V}_{A\cap B} = \emptyset$. Thus $\mathcal{V}_A \cap \mathcal{V}_B = \mathcal{V}_{A\cap B} = \emptyset$. So there exist disjoint open sets $\mathcal{V}_A$ and $\mathcal{V}_B$ belonging to $\sigma_{\mathcal{T}}$ such that $p \in \mathcal{V}_A$ and $q \in \mathcal{V}_B$. For proving $(\Leftarrow)$, Let $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ be neutrosophic pseudo Hausdorff. Let x, y be two distinct points of $\mathcal{X}$. Let us define neutrosophic singletons p and q on x and y, respectively with $\mu_p(x) = \mu_q(y) = 1$. Since $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is neutrosophic pseudo Hausdorff, there exist $\mathcal{V}_A$ and $\mathcal{V}_B$ belonging to $\sigma_{\mathcal{T}}$ such that for $A, B \in \mathcal{T}$, $p \in \mathcal{V}_A$ and $q \in \mathcal{V}_B$ and $\mathcal{V}_A \cap \mathcal{V}_B = \emptyset$. It is obvious that by definition of $\mu_p(x) = 1 \leq \mu_A(x)$ and we have $\mu_A(x) = 1$. By the same token, $\mu_B(y) = 1$. So $(V)_{A\cap B} = \mathcal{V}_A \cap \mathcal{V}_B = \emptyset$. This means that $A \cap B = 0_N$. This shows that $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff. □

Theorem 2.11. *The space $(\mathcal{X}, \mathcal{T})$ is neutrosophic connected iff $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is connected.*

Proof. Suppose that $(\mathcal{X}, \mathcal{T})$ is neutrosophic connected. Assume that $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is not connected. Then $\mathcal{P}(\mathcal{X}) = \mathcal{V}_A \cup \mathcal{V}_B$, for some $A, B \in \mathcal{T}$ for which $\mathcal{V}_A \cap \mathcal{V}_B = \emptyset$. It is obvious that $A \cap B = 0_N$. For every $x \in \mathcal{X}$, we have $(1_x, 0_x) \in \mathcal{P}(\mathcal{X})$. This implies that $(1_x, 0_x) \in \mathcal{V}_A$ or $(1_x, 0_x) \in \mathcal{V}_B$. Thus $\mu_A(x) = 1$, $\sigma_A(x) = 1$ and $\gamma_A(x) = 0$ or $\mu_B(x) = 1$, $\sigma_B(x) = 1$ and $\gamma_B(x) = 0$. Therefore $\mu_A \vee \mu_B = 1$, $\sigma_A \vee \sigma_B = 1$ and $\gamma_A \vee \gamma_B = 0$. So we have $A \cup B = 1_N$ which is a contradiction. To prove the converse implication, let $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ be connected and that $(\mathcal{X}, \mathcal{T})$ is not neutrosophic connected.. Thus, for some $A, B \in \mathcal{T}$, $1_N = A \cup B$ and $0_N = A \cap B$. Therefore $1 = \mu_A \vee \mu_B$, $1 = \sigma_A \vee \sigma_B$ and $0 = \gamma_A \wedge \gamma_B$ and $0 = \mu_A \wedge \mu_B$, $0 = \sigma_A \wedge \sigma_B$ and $1 = \gamma_A \vee \gamma_B$. Therefore for any $x \in \mathcal{X}$, $\mu_A(x) = 1$, $\mu_B(x) = 0$, $\sigma_A(x) = 1$, $\sigma_B(x) = 0$, $\gamma_A(x) = 0$, $\gamma_B(x) = 1$ or $\mu_A(x) = 0$, $\mu_B(x) = 1$, $\sigma_A(x) = 0$, $\sigma_B(x) = 1$, $\gamma_A(x) = 1$, $\gamma_B(x) = 0$. Thus, for any neutrosophic singleton $p = (\mu_A, \sigma_A, \gamma_A)$, we have $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$ or $\mu_p(x) \leq \mu_B(x)$, $\sigma_p(x) \leq \sigma_B(x)$ and $\gamma_p(x) \geq \gamma_B(x)$. Therefore, $p \in \mathcal{V}_A$ or $\mathcal{V}_B$. Observe that $\mathcal{V}_A \cap \mathcal{V}_B = \mathcal{V}_{A\cap B} = \emptyset$ which is contradiction to our hypothesis. □

Theorem 2.12. *Let $(\mathcal{X}, \mathcal{T})$ be a neutrosophic topological space and $E \subseteq \mathcal{X}$. The subspace on $(\mathcal{P}(E))$ inherited from $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ equals to the topological space on $\mathcal{P}(E)$ induced by neutrosophic subspace on E inherited from $(\mathcal{X}, \sigma_{\mathcal{T}})$.*

Proof. For any neutrosophic open set $A \in \mathcal{T}$, we prove $\mathcal{V}_A \cap \mathcal{P}(E) = \mathcal{V}_{A|E}$. Take $p = (\mu_p, \sigma_p, \gamma_p) \in \mathcal{V}_A \cap \mathcal{P}(E)$. By the fact that $p \in \mathcal{V}_A$, we have $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$. Also for $p \in \mathcal{P}(E)$, supp $\mu_p = x$ which is in E. Thus $\mu_p(x) \leq (\mu_A \mid E)(x)$, $\sigma_p(x) \leq (\sigma_A \mid E)(x)$ and $\gamma_p(x) \geq (\gamma_A \mid E)(x)$. Therefore $p \in \mathcal{V}_{A|E}$. It follows that $\mathcal{V}_A \cap \mathcal{P}(E) \subseteq \mathcal{V}_{A|E}$. Suppose $p \in \mathcal{V}_{A|E}$. Thus $\mu_p(x) \leq (\mu_A \mid E)(x)$, $\sigma_p(x) \leq (\sigma_A \mid E)(x)$ and $\gamma_p(x) \geq (\gamma_A \mid E)(x)$. If supp $\mu_p = x$ and supp $\sigma_p = x$ does not belong to E, then $(\mu_A \mid E)(x) = 0$ and $\sigma_A \mid E)(x) = 0$ which means that $\mu_A(x) = 0$ and $\sigma_A(x) = 0$. But this is a contradiction to the fact that $p \in \mathcal{V}_{A|E}$. Hence supp $\mu_p = x$ and supp $\sigma_p = x$. Thus $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$. This shows that $p \in \mathcal{V}_A \cap \mathcal{P}(E)$. □

Remark 2.13. Suppose that $W \cap \mathcal{P}(E)$ is an arbitrary open set in $(\mathcal{P}(E), \sigma_\tau \cap \mathcal{P}(E))$ where W is an open set in the induced neutrosophic topology σ_T.Since $W = \cup_{A \in B} \mathcal{V}_A$ for some $B \subseteq T$, then $W \cap \mathcal{P}(E) = (\cup_{A \in B} \mathcal{V}_A) \cap \mathcal{P}(E) = \cup_{A \in B} (\mathcal{V}_A) \cap \mathcal{P}(E))$. Hence by the above theorem, $W \cap \mathcal{P}(E) = \cup_{A \in B} (\mathcal{V}_{A|E}) \in (\mathcal{P}(E), \sigma_{T|E})$. By the same token, one can show that any basic open set $\mathcal{V}_{A|E}$ in $(\mathcal{P}(E), \sigma_{T|E})$ is also a basic open set in $(\mathcal{P}(E), \sigma_T \cap \mathcal{P}(E))$.

Theorem 2.14. *Suppose that $(\mathcal{Y}, \delta)$ and $(\mathcal{Z}, \rho)$ are neutrosophic topological spaces. The induced neutrosophic topological space on $\mathcal{P}(\mathcal{Y} \times \mathcal{Z})$ defined by product neutrosophic topological space $(Y \times \mathcal{Z}, \delta \times \rho)$ is embedded in $(\mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Z}), \sigma_\delta \times \sigma_\rho)$*

Proof. The proof is done by proving that there exists a function $f : (\mathcal{P}(\mathcal{Y} \times \mathcal{Z}), \sigma_{\delta \times \rho}) \rightarrow \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Z}), \sigma_\delta \times \sigma_\rho)$ which is bijective and f and its inverse are continuous. □

Conclusions

In this paper, we introduce and investigate the concept of induced neutrosophic topology through neutrosophic singletons. Additionally, we present several important properties of the induced neutrosophic function. These notions provide fertile ground for further research on neutrosophic separation axioms, neutrosophic weak separation axioms, and neutrosophic generalized open and closed sets.

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(m,a,n)-Fuzzy Neutrosophic Sets and Their Topological Structure

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Abstract. In this paper the concept of (m, a, n)-fuzzy neutrosophic set is introduced and its basic operations are studied. It is shown by examples that the class of (m, a, n)-fuzzy neutrosophic sets properly contains the class of q-rung orthopair neutrosophic sets. Moreover, a topological structure on (m, a, n)-fuzzy neutrosophic sets is created, and the topological notions such as closure, interior, subspace, connectedness, and separation axioms, are extended to (m, a, n)-fuzzy neutrosophic sets and explore their properties in (m, a, n)-fuzzy neutrosophic topological spaces.

Keywords: Neutrosopic sets; (m,a,n)-fuzzy neutrosophic sets; (m,a,n)-fuzzy neutrosophic topology; (m,a,n)-fuzzy neutrosophic separation axioms;(m,a,n)-fuzzy neutrosophic connectedness;(m,a,n)-fuzzy neutrosophic compactness.

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1. Introduction

The traditional methods relying on crisp sets are unsuitable for decision-making involving incomplete and inconsistent information, given the inherent challenges posed by vagueness, hesitancy, and uncertainties. Consequently, researchers have devised various set-theoretic models to tackle real-life problems characterized by uncertainty. These models aim to manage inconsistent information in a more effective manner. Zadeh [28] in 1965 proposed FSs as a generalization of crisp sets. Since the advent of Zadeh's paper [28] paper, many generalizations of FS such as IFS [3], PyFS [26] FFS [18]and q-ROFS [27]and (m,n)-FS [1] have been introduced by imposing certain conditions on PMD and NMD for each member of universe of discourse. Topological structures on these classes of FSs have been studied by different authors [5, 6, 11, 12, 15, 23]. Smarandache [19]defined neutrosophic set on a non empty set by

TABLE 1. Abbreviations and their description

Abbreviation	Description	Abbreviation	Description
PMD	Positive membership degree	$\mho_{\mathcal{H}}(v)$	PMD of v to $\mathcal{H}$
IMD	Indeterminacy degree	$\varpi_{\mathcal{H}}(v)$	IMD of v to $\mathcal{H}$
NMD	Negative membership degree	$\Omega_{\mathcal{H}}(v)$	NMD of v to $\mathcal{H}$
FS	Fuzzy set	IFS	Intuitionistic fuzzy set
PyFS	Pythagorean fuzzy set	FFS	Fermatean fuzzy set
q-ROFS	q-rung orthopair fuzzy set	(m,n)-FS	(m,n)-fuzzy set
NS	Neutrosophic set	PyNS	Pythagorean neutrosophic sets
FNS	Fermatean neutrosophic set	q-RONS	q-rung orthopair neutrosophic set
(m,a,n)-FNS	(m,a,n)- fuzzy neutrosophic set	(m,a,n)-FNS($\mathbb{V}$)	Family of (m,a,n)-FNSs on $\mathbb{V}$
(m,a,n)-FNT	(m,a,n)-fuzzy neutrosophic topology	(m,a,n)-FNTS	(m,a,n)-fuzzy neutrosophic topological space
(m,a,n)-FN point	(m,a,n)-fuzzy neutrosophic point	(m,a,n)-FNC($\mathbb{V}$)	Family of (m,a,n)-FN closed sets on $\mathbb{V}$

considering three components, namely PMD, IMD and NMD whose sum lies between 0 and 3. Some more properties of neutrosophic sets are presented by Smarandache [19–21], Salama and Alblowi [16] , Lupiáñez [14],Wang [25]. Smarandache's Neutrosophic concepts have wide range of real time applications for the fields of Information systems, Computer science, Artificial Intelligence, Applied Mathematics and Decision making. In 2008, Lupiáñez [14] introduced the neutrosophic topology as an extension of intuitionistic fuzzy topology. Recent work in neutrosophic topology can be seen in [2, 8, 9, 16, 17] Recently the concepts of PyFS , FFS, and q-ROFS have been extended to neutrosophic environments and studied their topological structures [4, 22, 24]. In this work we introduce the concept of the *(m,a,n)-fuzzy neutrosophic set* focusing on fundamental properties of this kind of set and on *(m,a,n)-fuzzy neutrosophic topological spaces.* The rest of the paper is formulated as follows: Section 2 contains the necessary mathematical background for the understanding of the paper. The concept of the (m,a,n)-fuzzy neutrosophic set is presented in Section 3 together with basic properties of these sets. In Section 4 the classical notion of topological space is extended to *(m,a,n)-fuzzy neutrosophic topological spaces* together with fundamental properties and concepts like ,connectedness and separation axioms in (m,a,n)-fuzzy neutrosophic topological spaces. The paper closes with the final conclusions and some hints for further research included in Section 5.

2. **Preliminaries**

This section contains some basic definitions and preliminary results which will be needed in the sequel. Throughout this paper $\mathbb{V}$ denotes a universe of discourse and $\mathbb{N}$ refers to the set of all natural numbers.

Definition 2.1. Let $\mathbb{V}$ be a non empty set, $\mho_{\mathcal{H}} : \mathbb{V} \to [0, 1]$ and $\Omega_{\mathcal{H}} : \mathbb{V} \to [0, 1]$. A structure

$$\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$$

is called :

(a) IFS [3] in $\mathbb{V}$ if $0 \leq \mho_{\mathcal{H}}(v) + \Omega_{\mathcal{H}}(v) \leq 1, \ \forall \ v \in \mathbb{V}$.

(b) PyFS [26] in $\mathbb{V}$ if $0 \leq \mho^2_{\mathcal{H}}(v) + \Omega^2_{\mathcal{H}}(v) \leq 1, \ \forall \ v \in \mathbb{V}$.

(c) FFS [18] in $\mathbb{V}$ if $0 \leq \mho^3_{\mathcal{H}}(v) + \Omega^3_{\mathcal{H}}(v) \leq 1, \ \forall \ v \in \mathbb{V}$.

(d) q-ROFS [27] in $\mathbb{V}$ if $0 \leq \mho^q_{\mathcal{H}}(v) + \Omega^q_{\mathcal{H}}(v) \leq 1, \ \forall \ v \in \mathbb{V}$ and $q \in \mathbb{N}$.

(e) (m,n)-FS [1] in $\mathbb{V}$ if $0 \leq \mho^m_{\mathcal{H}}(v) + \Omega^n_{\mathcal{H}}(v) \leq 1, \ \forall \ v \in \mathbb{V}$ and $m, n \in \mathbb{N}$.

Remark 2.2. [1] The definition of (m,n)-FS can be reduced to the definition of:

(i) q-ROFS, if m=n=q.

(ii) FFS, if m=n=3.

(iii) PyFS, if m=n=2.

(iv) IFS, if m=n=1.

Remark 2.3. [1] Let $\mathbb{V}$ be a non empty set:

(i) If $m \geq q$ and $n \geq q$ then every q-ROFS is a(m,n)-FS.

(ii) If $m \geq 3$ and $n \geq 3$ then every FFS is a (m,n)-FS.

(iii) If $m \geq 2$ and $n \geq 2$ then every PyFS is a (m,n)-FS.

(iv) Every IFS is an (m,n)-FS.

Definition 2.4. [19] Let $\mathbb{V}$ be a non empty set. A Neutrosophic set (**NS**) $\mathcal{H}$ in $\mathbb{V}$ is a structure

$$\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \varpi_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$$

where $\mho_{\mathcal{H}} : \mathbb{V} \to]^-0, 1^+[$, $\varpi_{\mathcal{H}} : \mathbb{V} \to]^-0, 1^+[$, and $\Omega_{\mathcal{H}} : \mathbb{V} \to]^-0, 1^+[$ denotes the PMD, IMD, and NMD of $\mathcal{H}$ which satisfies the condition if $^-0 \leq \mho_{\mathcal{H}}(v) + \varpi_{\mathcal{H}}(v) + \Omega_{\mathcal{H}}(v) \leq 3^+, \ \forall \ v \in \mathbb{V}$.

In the real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the closed interval [0,1] and sum of MD, IMD, and NMD of each element of universe of discourse lies between 0 and 3.

Definition 2.5. Let $\mathbb{V}$ be a non empty set and $\mho_{\mathcal{H}} : \mathbb{V} \to [0,1]$,$\varpi_{\mathcal{H}} : \mathbb{V} \to [0,1]$, and $\Omega_{\mathcal{H}} : \mathbb{V} \to [0,1]$. A structure

$$\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \varpi_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$$

is called :

(a) PyNS [4] in $\mathbb{V}$ if $0 \leq \mho^2_{\mathcal{H}}(v) + \Omega^2_{\mathcal{H}}(v) \leq 1$, and $0 \leq \varpi^2_{\mathcal{H}}(v) \leq 1, \ \forall \ v \in \mathbb{V}$.

(b) FNS [22] in $\mathbb{V}$ if $0 \leq \mho^3_{\mathcal{H}}(v) + \Omega^3_{\mathcal{H}}(v) \leq 1$, and $0 \leq \varpi^3_{\mathcal{H}}(v) \leq 1 \ , \ \forall \ v \in \mathbb{V}$.

(c) q-RONS [24] in $\mathbb{V}$ if $0 \leq \mho^q_{\mathcal{H}}(v) + \Omega^q_{\mathcal{H}}(v) \leq 1$, and $0 \leq \varpi^q_{\mathcal{H}}(v) \leq 1 \ , \ \forall \ v \in \mathbb{V}$ and $q \in \mathbb{N}$.

Here $\mho_{\mathcal{H}}(v)$ and $\Omega_{\mathcal{H}}(v)$ are dependent components and $\varpi_{\mathcal{H}}(v)$ is an independent component.

3. (m,a,n)-fuzzy neutrosophic sets

In this section we introduce a new class of orthopair neutrosophic sets which is properly contains the classes of q-RONS, FNS, and PyNS and explores its studied.

Definition 3.1. Let $\mathbb{V}$ be a non empty set. A (m,a,n)-fuzzy neutrosophic set ((m,a,n)-FNS) $\mathcal{H}$ in $\mathbb{V}$ is an object of the form

$$\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \varpi_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$$

where $\mho_{\mathcal{H}} : \mathbb{V} \to [0,1]$,$\varpi_{\mathcal{H}} : \mathbb{V} \to [0,1]$, $\Omega_{\mathcal{H}} : \mathbb{V} \to [0,1]$, $0 \leq \mho^m_{\mathcal{H}}(v) + \Omega^n_{\mathcal{H}}(v) \leq 1$, and $0 \leq \varpi^a_{\mathcal{H}}(v) \leq 1$, $\forall\ v \in \mathbb{V}$ and $m, a, n \in \mathbb{N}$. Here $\mho_{\mathcal{H}}(v)$ and $\Omega_{\mathcal{H}}(v)$ are dependent components and $\varpi_{\mathcal{H}}(v)$ is an independent component.

Theorem 3.2. *If $\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \varpi_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$ is a (m,a,n)-FNS over $\mathbb{V}$ then $0 \leq \mho^m_{\mathcal{H}}(v) + \varpi^a_{\mathcal{H}}(v) + \Omega^n_{\mathcal{H}}(v) \leq 2$, $\forall\ v \in \mathbb{V}$ and $m, a, n \in \mathbb{N}$.*

For simplicity a (m,a,n)-FNS $\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \varpi_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$ over $\mathbb{V}$ will be denoted by $(\mho_{\mathcal{H}}, \varpi_{\mathcal{H}}, \Omega_{\mathcal{H}})$.

Remark 3.3. (m,a,n)-FNS is coincide with:

(i) q-RONS , if m=a=n=q.
(ii) FNS, if m=a=n=3.
(iii) PyNS, if m=a=n=2.

Remark 3.4. Let $\mathbb{V}$ be a non empty set:

(i) If $m \geq q$, $a \geq q$, and $n \geq q$ then every q-RONS is a(m,a,n)-FNS.
(ii) If $m \geq 3$, $a \geq 3$, and $n \geq 3$ then every FNS is a (m,a,n)-FNS.
(iii) If $m \geq 2$, $a \geq 2$, and $n \geq 2$ then every PyNS is a (m,,a,n)-FNS.

Example 3.5. Let $\mathbb{V} = \{v_1, v_2\}$. Then the structure $\mathcal{E} = \{< v_1, 0.8, 0.2, 0.9 >, < v_2, 0, 8, 0.5, 0.7 >\}$ defined over $\mathbb{V}$ is (4,5,6)-FNS but not 4-RONS , and hence not FNS,and PyNS over $\mathbb{V}$.

Definition 3.6. The whole (m,a,n)-FNS and empty (m,a,n)-FNS over universe of discourse $\mathbb{V}$ are defined as follows

$$\tilde{\Phi} = \{< v, 0, 0, 1 >: v \in \mathbb{V}\}.$$

$$\tilde{\mathbb{V}} = \{< v, 1, 1, 0 >: v \in \mathbb{V}\}.$$

Definition 3.7. Let $\mathcal{H} = (\mho_{\mathcal{H}}, \varpi_{\mathcal{H}}, \Omega_{\mathcal{H}})$, $\mathcal{H}_1 = (\mho_{\mathcal{H}_1}, \varpi_{\mathcal{H}_1}, \Omega_{\mathcal{H}_1})$ and $\mathcal{H}_2 = (\mho_{\mathcal{H}_2}, \varpi_{\mathcal{H}_2}, \Omega_{\mathcal{H}_2})$ be three (m,a,n)-FNSs in $\mathbb{V}$. Then the subset, equality, complement, union, and intersection operations over $(m, a, n) - FNS(\mathbb{V})$ are defined as follow:

(a) $\mathcal{H}_1 \subset \mathcal{H}_2 \Leftrightarrow \mho_{\mathcal{H}_1} \leq \mho_{\mathcal{H}_2}$, $\varpi_{\mathcal{H}_1} \geq \varpi_{\mathcal{H}_2}$ and $\Omega_{\mathcal{H}_1} \geq \Omega_{\mathcal{H}_2}$

(b) $\mathcal{H}_1 = \mathcal{H}_2 \Leftrightarrow \mho_{\mathcal{H}_1} = \mho_{\mathcal{H}_2}$, $\varpi_{\mathcal{H}_1} = \varpi_{\mathcal{H}_2}$, and $\Omega_{\mathcal{H}_1} = \Omega_{\mathcal{H}_2}$.

(c) $\mathcal{H}^c = (\Omega_{\mathcal{H}}^{\frac{n}{m}}, 1 - \varpi_{\mathcal{H}}, \mho_{\mathcal{H}}^{\frac{m}{n}})$.

(d) $\mathcal{H}_1 \cup \mathcal{H}_2 = (max\{\mho_{\mathcal{H}_1}, \mho_{\mathcal{H}_2}\}, max\{\varpi_{\mathcal{H}_1}, \varpi_{\mathcal{H}_2}\}, min\{\Omega_{\mathcal{H}_1}, \Omega_{\mathcal{H}_2}\})$

(e) $\mathcal{H}_1 \cap \mathcal{H}_2 = (min\{\mho_{\mathcal{H}_1}, \mho_{\mathcal{H}_2}\}, min\{\varpi_{\mathcal{H}_1}, \varpi_{\mathcal{H}_2}\}, max\{\Omega_{\mathcal{H}_1}, \Omega_{\mathcal{H}_2}\})$

Example 3.8. Let $\mathbb{V} = \{v_1, v_2, v_3\}$ and (m,a,n)-FNS $\mathcal{E}, \mathcal{F}, \mathcal{G}$ over $\mathbb{V}$ (for m=4,a=5,n=6) bedefined as follows:

$$\mathcal{E} = \{< v_1, 0.8, 0.2, 0.9 >, < v_2, 0, 8, 0.5, 0.7 >, < v_3, 0.6, 0.4, 0.9 >\}.$$

$$\mathcal{F} = \{< v_1, 0.9, 0.3, 0.9 >, < v_2, 0.8, 0.5, 0.6 >, < v_3, 0.7, 0.5, 0.8 >\}.$$

$$\mathcal{G} = \{< v_1, 0.6, 0.5, 0.8 >, < v_2, 0.9, 0.4, 0.9 >, < v_3, 0.5, 0.5, 0.9 >\}.$$

Then,

(i) $\mathcal{E} \subset \mathcal{F}$, but $\mathcal{E} \nsubseteq \mathcal{G}$.

(ii) $\mathcal{E} \cup \mathcal{F} = \{< v_1, 0.9, 0.3, 0.9 >, < v_2, 0.8, 0.5, 0.6 >, < v_3, 0.7, 0.5, 0.8 >\}$.

(iii) $\mathcal{F} \cap \mathcal{G} = \{< v_1, 0.6, 0.3, 0.9 >, < v_2, 0.8, 0.4, 0.9 >, < v_3, 0.5, 0.5, 0.9 >\}$.

(iv) $\mathcal{E}^c = \begin{Bmatrix} < v_1, 0.8538149682, 0.8, 0.8617738754 >, \\ < v_2, 0.5856629186, 0.5, 0.8617738754 >, \\ < v_3, 0.8538149682, 0.6, 0.7113786597 > \end{Bmatrix}$.

Theorem 3.9. *Let* $\mathcal{H} = (\mho_{\mathcal{H}}, \varpi_{\mathcal{H}}, \Omega_{\mathcal{H}}), \mathcal{H}_1 = (\mho_{\mathcal{H}_1}, \varpi_{\mathcal{H}_1}, \Omega_{\mathcal{H}_1}), \mathcal{H}_2 = (\mho_{\mathcal{H}_2}, \varpi_{\mathcal{H}_2}, \Omega_{\mathcal{H}_2}) \in (m, a, n) - FNS(\mathbb{V})$. *Then*

(a) $\mathcal{H}^c \in (m, a, n) - FNS(\mathbb{V})$.

(b) $\mathcal{H}_1 \cup \mathcal{H}_2 \in (m, a, n) - FNS(\mathbb{V})$.

(c) $\mathcal{H}_1 \cap \mathcal{H}_2 \in (m, a, n) - FNS(\mathbb{V})$.

The Proof of Theorem 3.9 is obvious and left to the readers.

Theorem 3.10. *Let* $\mathcal{H}_1 = (\mho_{\mathcal{H}_1}, \varpi_{\mathcal{H}_1}, \Omega_{\mathcal{H}_1}), \mathcal{H}_2 = (\mho_{\mathcal{H}_2}, \varpi_{\mathcal{H}_2}, \Omega_{\mathcal{H}_2}), \mathcal{H}_3 = (\mho_{\mathcal{H}_3}, \varpi_{\mathcal{H}_3}, \Omega_{\mathcal{H}_3}) \in (m, a, n) - FNS(\mathbb{V})$. *Then:*

(a) $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}_2 \cup \mathcal{H}_1$.

(b) $\mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{H}_2 \cap \mathcal{H}_1$.

(c) $(\mathcal{H}_1 \cup \mathcal{H}_2) \cup \mathcal{H}_3 = \mathcal{H}_1 \cup (\mathcal{H}_2 \cup \mathcal{H}_3)$.

(d) $(\mathcal{H}_1 \cap \mathcal{H}_2) \cap \mathcal{H}_3 = \mathcal{H}_1 \cap (\mathcal{H}_2 \cap \mathcal{H}_3)$.

(e) $\mathcal{H}_1 \cup (\mathcal{H}_2 \cap \mathcal{H}_3) = (\mathcal{H}_1 \cup \mathcal{H}_2) \cap (\mathcal{H}_1 \cup \mathcal{H}_3)$

(f) $\mathcal{H}_1 \cap (\mathcal{H}_2 \cup \mathcal{H}_3) = (\mathcal{H}_1 \cap \mathcal{H}_2) \cup (\mathcal{H}_1 \cap \mathcal{H}_3)$

(g) $(\mathcal{H}_1 \cup \mathcal{H}_2)^c = \mathcal{H}_2^c \cap \mathcal{H}_1^c$.

(h) $(\mathcal{H}_1 \cap \mathcal{H}_2)^c = \mathcal{H}_2^c \cup \mathcal{H}_1^c$.

Proof. Follows on similar lines as the proofs of Theorem 1 and Theorem 2 [1] □

Definition 3.11. Let $\{\mathcal{H}_i : i \in \Lambda\}$ be an arbitrary family of (m,a,n)-FNS in $\mathbb{V}$. Then:

(a) $\bigcup \mathcal{H}_i = \{< v, \bigvee \mho_{\mathcal{H}_i}(v),\ \bigvee \varpi_{\mathcal{H}_i}(v), \bigwedge \Omega_{\mathcal{H}_i}(v) >: v \in \mathbb{V}\}$.
(b) $\bigcap \mathcal{H}_i = \{< v, \bigwedge \mho_{\mathcal{H}_i}(v),\ \bigwedge \varpi_{\mathcal{H}_i}(v), \bigvee \Omega_{\mathcal{H}_i}(v) >: v \in \mathbb{V}\}$.

4. (m,a,n)-fuzzy neutrosophic topological spaces

Definition 4.1. A subfamily Γ of $(m, a, n) - FNS(\mathbb{V})$ is called a (m,a,n)-fuzzy neutrosophic topology ((m,a,n)-FNT) on $\mathbb{V}$ if:

(a) $\tilde{\Phi}, \tilde{\mathbb{V}} \in \Gamma$.
(b) $\mathcal{H}_i \in \Gamma,\ \forall\ i \in \Lambda \Rightarrow \cup_{i\in\Lambda}\mathcal{H}_i \in \Gamma$.
(c) $\mathcal{H}_1, \mathcal{H}_2 \in \Gamma \Rightarrow \mathcal{H}_1 \cap \mathcal{H}_2 \in \Gamma$.

If Γ is a (m,a,n)-FNTS on $\mathbb{V}$ then the structure $(\mathbb{V}, \Gamma)$ is called a (m,a,n)-fuzzy neutrosophic topological space ((m,a,n)-FNTS) over $\mathbb{V}$ and the members of Γ are called (m,a,n)-fuzzy neutrosophic open ((m,a,n)-FN open) sets. The complement of a (m,a,n)-FN open set is called (m,a,n)-fuzzy neutrosophic closed((m,a,n)-FN closed).

Example 4.2. Let $\mathbb{V} = \{v_1, v_2, v_3\}$ be the universal set. Consider the following (m,a,n)-FNSs for m=3,a=4, and n=5.

$$\mathcal{G}_1 = \{< v_1, 0.8, 0.4, 0.8 >, < v_2, 0.7, 0.3, 0.9 >, < v_3, 0.9, 0.4, 0.6 >\}.$$
$$\mathcal{G}_2 = \{< v_1, 0.9, 0.4, 0.7 >, < v_2, 0.8, 0.4, 0.7 >, < v_3, 0.9, 0.5, 0.5 >\}.$$

Then $\Gamma = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{G}_1, \mathcal{G}_2\}$ is a (2,3,4)-FNT on $\mathbb{V}$.

Definition 4.3. Let $(\mathbb{V}, \Gamma)$ be a (m,a,n)-FNTS with $\Gamma = \{\tilde{\Phi}, \tilde{\mathbb{V}}\}$ then Γ is said to be the indiscrete a (m,a,n)-FNT on $\mathbb{V}$ and $(\mathbb{V}, \Gamma)$ is called the indiscrete (m,a,n)-FNTS. The indiscrete (m,a,n)-FNT is the smallest (m,a,n)-FNT on $\mathbb{V}$.

Definition 4.4. Let $(\mathbb{V}, \Gamma)$ be a (m,a,n)-FNTS with $\Gamma = (m, a, n) - FNS(\mathbb{V})$ then Γ is said to be the discrete (m,a,n)-FNT on $\mathbb{V}$ and $(\mathbb{V}, \Gamma)$ is called the discrete (m,a,n)-FNTS. The discrete (m,a,n)-FNT is the largest (m,a,n)-FNT on $\mathbb{V}$.

Remark 4.5. The union of two (m,a,n)-FNTs over $\mathbb{V}$ is not a (m,a,n)-FNT over $\mathbb{V}$.

Example 4.6. Let $\mathbb{V} = \{v_1, v_2\}$. Consider the following (m,a,n)-FNSs (for m=2,a=3, and n=4)

$$\mathcal{G}_1 = \{< v_1, 0.8, 0.2, 0.4 >, < v_2, 0.6, 0.1, 0.5 >\}$$

$$\mathcal{G}_2 = \{< v_1, 0.7, 0.2, 0.5 >, < v_2, 0.5, 0.1, 0.6 >\}$$

$$\mathcal{H}_1 = \{< v_1, 0.8, 0.4, 0.6 >, < v_2, 0.7, 0.3, 0.7 >\}$$

$$\mathcal{H}_1 = \{< v_1, 0.9, 0.4, 0.5 >, < v_2, 0.8, 0.3, 0.6 >\}.$$

Then $\Gamma_1 = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{G}_1, \mathcal{G}_2\}$ and $\Gamma_2 = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{H}_1, \mathcal{H}_2\}$ are two (m,a,n)-FNTs over $\mathbb{V}$. Now $\Gamma_1 \cup \Gamma_2 = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{H}_1, \mathcal{H}_2\}$, and $\mathcal{G}_1 \cup \mathcal{H}_1 = \{< v_1, 0.8, 0.4, 0.4 >, < v_2, 0.7, 0.3, 0.5 >\}$. Thus $\mathcal{G}_1, \mathcal{H}_1 \in \Gamma_1 \cup \Gamma_2$, but $\mathcal{G}_1 \cup \mathcal{H}_1 \notin \Gamma_1 \cup \Gamma_2$. Therefore $\Gamma_1 \cup \Gamma_2$ is not a (m,a,n)-FNT on $\mathbb{V}$.

However,

Theorem 4.7. *The intersection of two (m,a,n)-FNTs on $\mathbb{V}$ is also a (m,a,n)-FNT on $\mathbb{V}$.*

Proof. Suppose that Γ_1 and Γ_2 are two (m,a,n)-FNTs on $\mathbb{V}$. Since $\Phi, \mathbb{V} \in \Gamma_1$ and $\Phi, \mathbb{V} \in \Gamma_2$, then $\Phi, \mathbb{V} \in \Gamma_1 \cap \Gamma_2$. Let $\mathcal{G}_1, \mathcal{G}_2 \in \Gamma_1 \cap \Gamma_2 \implies \mathcal{G}_1, \mathcal{G}_2 \in \Gamma_1$ and $(\mathcal{G}_1, \mathcal{G}_2 \in \Gamma_2 \implies \mathcal{G}_1 \cap \mathcal{G}_2 \in \Gamma_1$ and $\mathcal{G}_1 \cap \mathcal{G}_2 \in \Gamma_2 \implies \mathcal{G}_1 \cap \mathcal{G}_2 \in \Gamma_1 \cap \Gamma_2$.

Let $\mathcal{G}_i \in \Gamma_1 \cap \Gamma_2, i \in \Lambda,$ *an index set* $\implies \mathcal{G}_i \in \Gamma_1$ and $\mathcal{G}_i \in \Gamma_2, \forall i \in \Lambda \implies \cup_{i\in\Lambda}\mathcal{G}_i \in \Gamma_1$ and $\cup_{i\in\Lambda}\mathcal{G}_i \in \Gamma_2, \implies \cup_{i\in\Lambda}\mathcal{G}_i \in \Gamma_1 \cap \Gamma_2$. Thus $\Gamma_1 \cap \Gamma_2$ satisfies all the requirements to (m,a,n)-FNT on $\mathbb{V}$. □

Definition 4.8. Let $(\mathbb{V}, \Gamma_1)$ and $(\mathbb{V}, \Gamma_2)$ be (m,a,n)-FNTSs. Then:

(a) Γ_2 is called finer than Γ_1 if $\Gamma_2 \subset \Gamma_1$.

(b) Γ_1 is comparable with Γ_2 if either $\Gamma_1 \subset \Gamma_2$ or $\Gamma_2 \subset \Gamma_1$.

Definition 4.9. Let $(\mathbb{V}, \Gamma)$ be a (m,a,n)-FNTS and $\mathcal{H} \in (m, a, n) - FNS(\mathbb{V})$. Then the interior, closure, frontier and exterior of $\mathcal{H}$ denoted respectively by $Int\mathcal{H}$, $Cl(\mathcal{H})$,$Fr(\mathcal{H})$ and $Ext(\mathcal{H})$ are defined as follows:

(a) $Int(\mathcal{H}) = \cup\{\mathcal{K} \in \Gamma : \mathcal{K} \subset \mathcal{H}\}$.

(b) $Cl(\mathcal{H}) = \cap\{\mathcal{F} \in (m, a, n) - FNC(\mathbb{V}) : \mathcal{H} \subset \mathcal{F}\}$.

(c) $Fr(\mathcal{H}) = Cl(\mathcal{H}) \cap Cl(\mathcal{H}^c)$.

(d) $Ext(\mathcal{H}) = Int(\mathcal{H}^c)$.

Theorem 4.10. *Let $(\mathbb{V}, \Gamma)$ be a (m,a,n)-FNTS and $\mathcal{H}, \mathcal{F} \in (m, a, n) - FNS(\mathbb{V})$. Then:*

(a) $Int(\tilde{\Phi}) = \tilde{\Phi}$ *and* $Int(\tilde{\mathbb{V}}) = \tilde{\mathbb{V}}$.

(b) $Int(\mathcal{H}) \subset \mathcal{H}$.

(c) $\mathcal{H} \in \Gamma \Leftrightarrow Int(\mathcal{H}) = \mathcal{H}$.

(d) $Int(Int(\mathcal{H})) = Int(\mathcal{H})$.

(e) $\mathcal{H} \subset \mathcal{F} \Longrightarrow Int(\mathcal{H}) \subset Int(\mathcal{F})$.

(g) $Int(\mathcal{H}) \cup Int(\mathcal{F}) \subset Int(\mathcal{H} \cup \mathcal{F})$.

(h) $Int(\mathcal{H}) \cap Int(\mathcal{F}) = Int(\mathcal{H} \cap \mathcal{F})$.

Proof. Straightforward. □

Theorem 4.11. *Let* $(\mathbb{V}, \Gamma)$ *be a (m,a,n)-FNTS and* $\mathcal{H}, \mathcal{F} \in (m, a, n) - FNS(\mathbb{V})$. *Then:*

(a) $Cl(\tilde{\Phi}) = \tilde{\Phi}$, $Cl(\tilde{\mathbb{V}}) = \tilde{\mathbb{V}}$.

(b) $\mathcal{H} \subset Cl(\mathcal{H})$.

(c) $\mathcal{H} \in (m, a, n) - FNC(\mathbb{V}) \Leftrightarrow Cl(\mathcal{H}) = \mathcal{H}$.

(d) $Cl(Cl(\mathcal{H})) = Cl(\mathcal{H})$.

(e) $\mathcal{H} \subset \mathcal{F} \Longrightarrow Cl(\mathcal{H}) \subset Cl(\mathcal{F})$.

(f) $Cl(\mathcal{H}) \cup Cl(\mathcal{F}) = Cl(\mathcal{H} \cup \mathcal{F})$.

(g) $Cl(\mathcal{H} \cap \mathcal{F}) \subset Cl(\mathcal{H}) \cap Cl(\mathcal{F})$.

Proof. Straightforward. □

Theorem 4.12. *Suppose that* $(\mathbb{V}, \Gamma)$ *is a (m,a,n)-FNTS and* $(\mathcal{H})$ *is a (m,a,n)-FNS over* $\mathbb{V}$. *Then*

(a) $(Int(\mathcal{H}))^c = Cl(\mathcal{H}^c)$.

(b) $(Cl(\mathcal{H}))^c = Int(\mathcal{H}^c)$.

(c) $Fr(\mathcal{H}) = Fr(\mathcal{H}^c)$.

Proof. Obvious □

Definition 4.13. Let $(\mathbb{V}, \Gamma)$ be a (m,a,n)-FNT on $\mathbb{V}$ and let $\mathbb{Y} \subset \mathbb{V}$. Then $\Gamma_{\tilde{\mathbb{Y}}} = \{\mathcal{H} \cap \tilde{\mathbb{Y}} : \mathcal{H} \in \Gamma\}$ is called the relative (m,a,n)-FNT on $\mathbb{Y}$. The pair $(\mathbb{Y}, \Gamma_{\mathbb{Y}})$ is known as the (m,a,n)-FN subspace of the (m,a,n)-FNTS $(\mathbb{V}, \Gamma)$.

Example 4.14. Let $\mathbb{V} = \{v_1, v_2, v_3\}$ and $\Gamma = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{H}_1, \mathcal{H}_2\}$ where $(\mathcal{H}_1), (\mathcal{H}_2)$ be (m,a,n)-FNS (for m=3 a=4,and n=5) over $\mathbb{V}$, defined as follows:

$$\mathcal{H}_1 = \{< v_1, 0.7, 0.3, 0.8 >< v_2, 0.7, 0.2, 0.9 >, < v_3, 0.6, 0.1, 0.5 >\}$$

$$\mathcal{H}_2 = \{< v_1, 0.8, 0.3, 0.7 >, < v_2, 0.9, 0.2, 0.5 >, < v_3, 0.8, 0.1, 0.4 >\}$$

Then $(\mathbb{V}, \Gamma)$ is a (m,a,n)-FNTS. Consider the (m,a,n)-FNS $\mathbb{Y} = \{v_1, v_2\}$. Then $\Gamma_{\mathbb{Y}} = \{\tilde{\Phi}, \tilde{\mathbb{Y}}, (\mathcal{H}_1)_{\mathbb{Y}}, (\mathcal{H}_2)_{\mathbb{Y}}\}$ is a relative (m,a,n)-FNT over $\mathbb{Y}$. Where

$$(\mathcal{H}_1)_Y = \{< v_1, 0.7, 0.3, 0.8 >< v_2, 0.7, 0.2, 0.9 >\}$$

$$(\mathcal{H}_2)_Y = \{< v_1, 0.8, 0.3, 0.7 >, < v_2, 0.9, 0.2, 0.5 >\}$$

Theorem 4.15. *Let $(\mathbb{Y}, \Gamma_{\mathbb{Y}})$ be a (m,a,n)-FN-subspace of a (m,a,n)-FNTS$(\mathbb{V}, \Gamma)$ and $\mathcal{H} \in (m, a, n) - FN(\mathbb{V})$, then:*

(a) $\mathcal{H} \in \Gamma_{\mathbb{Y}} \Leftrightarrow \mathcal{H} = \mathbb{Y} \cap \mathcal{E}$ *for some* $\mathcal{E} \in \Gamma$.

(b) $\mathcal{H} \in (m, a, n) - FNC(\mathbb{Y}) \Leftrightarrow \mathcal{H} = \tilde{\mathbb{Y}} \cap \mathcal{F}$ *for some* $\mathcal{F} \in (m, a, n) - FNC(\mathbb{V})$.

Theorem 4.16. *Let $(\mathbb{Y}, \Gamma_{\mathbb{Y}})$ be a (m,a,n)-FN subspace of a (m,a,n)-FNTS $(\mathbb{V}, \Gamma)$ and $\mathcal{H} \in \Gamma_{\mathbb{Y}}$. If $\mathbb{Y} \in \Gamma$ then $\mathcal{H} \in \Gamma$.*

Theorem 4.17. *Let $(\mathbb{Y}, \Gamma_{\mathbb{Y}})$ be a (m,a,n)-FN subspace of a (m,a,n)-FNTS $(\mathbb{V}, \Gamma)$. Then a (m,a,n)-FNS $\mathcal{H}_{\mathbb{Y}} \in (m, a, n) - FNC(\mathbb{Y}) \Rightarrow \mathcal{H}_{\mathbb{Y}} \in (m, a, n) - FNC(\mathbb{V}) \Leftrightarrow \tilde{\mathbb{Y}} \in (m, a, n) - FNC(\mathbb{V})$.*

Definition 4.18. Let $\zeta, \eta, \varsigma \in [0, 1]$. A (m,a,n)-fuzzy neutrosophic point ((m,a,n)-FNP) $v_{(\zeta,\eta,\varsigma)}$ of $\mathbb{V}$ is a (m,a,n)-FNS in $\mathbb{V}$ defined by

$$v_{(\zeta,\eta,\varsigma)}(w) = \begin{cases} (\zeta, \eta, \varsigma) & \text{if } w = v \\ (0, 0, 1) & \text{if } w \neq v \end{cases}$$

Definition 4.19. Let $v_{(\zeta,\eta,\varsigma)}$ be a (m,a,n)-FNP in $\mathbb{V}$ and $\mathcal{H} = \{< v, \mho_{\mathcal{H}}(v), \varpi_{\mathcal{H}}(v), \Omega_{\mathcal{H}}(v) >: v \in \mathbb{V}\}$ is a (m,a,n)-FNS in $\mathbb{V}$. Then $v_{(\zeta,\eta,\varsigma)} \subseteq \mathcal{H}$ if and only if $\zeta \subseteq \mho_{\mathcal{H}}(v)$, $\eta \subseteq \varpi_{\mathcal{H}}(v)$, and $\varsigma \supseteq \Omega_{\mathcal{H}}(v)$.

Definition 4.20. A (m,a,n)-FNP $v_{(\zeta,\eta,\varsigma)}$ is said to be q-coincident with $\mathcal{H}$, denoted by $v_{(\zeta,\eta,\varsigma)} q\mathcal{H}$ if $v_{(\zeta,\eta,\varsigma)} \nsubseteq \mathcal{H}^c$. If $v_{(\zeta,\eta,\varsigma)}$ is not q-coincident with $\mathcal{H}$, we denote by $\rceil(v_{(\zeta,\eta,\varsigma)} q\mathcal{H})$.

Definition 4.21. Two (m,a,n)-FNSs $\mathcal{H}$ and $\mathcal{G}$ of $\mathbb{V}$ are said to be q-coincident (denoted by $\mathcal{H}q\mathcal{G}$) if $\mathcal{H} \nsubseteq \mathcal{G}^c$.

Lemma 4.22. *If $\mathcal{H}, \mathcal{G} \in (m, a, n) - FN(\mathbb{V})$, then $\rceil(\mathcal{H}q\mathcal{G}) \Leftrightarrow \mathcal{H} \subset \mathcal{G}^c$., where $\rceil(\mathcal{H}q\mathcal{G})$ means $\mathcal{H}$ is not q-coincident with $\mathcal{G}$.*

Definition 4.23. Let $(\mathbb{V}, \Gamma)$ a (m,a,n)-FNTS and $\mathcal{H}$ be a(m,a,n)-FNS over $\mathbb{V}$. Then $\mathcal{H}$ is said to be a (m,a,n)-FN neighbourhood of the (m,a,n)-FNP $v_{(\zeta,\eta,\varsigma)}$over $\mathbb{V}$, if $\exists$ (m,a,n)-FNS $\mathcal{G} \in \Gamma$ such that $v_{(\zeta,\eta,\varsigma)} \in \mathcal{G} \subset \mathcal{H}$.

Theorem 4.24. *Let $(\mathbb{V}, \Gamma)$ be a (m,a,n)-FNTS . Then a (m,a,n)-FNS $\mathcal{F} \in \Gamma$ if and only if $\forall$, (m,a,n)-FN point $v_{(\zeta,\eta,\varsigma)} \in \mathcal{F}$ $\exists$ $\mathcal{G} \in \Gamma$ such that $v_{(\zeta,\eta,\varsigma)} \in \mathcal{G} \subseteq \mathcal{F}$.*

Definition 4.25. Let $(\mathbb{V},\Gamma)$ be a (m,a,n)-FNTS and $\mathcal{H}$ be a(m,a,n)-FNS over $\mathbb{V}$. Then $\mathcal{H}$ is said to be a (m,a,n)-FN neighbourhood of the(m,a,n)-FNS $\mathcal{H}$, if $\exists$ a $\mathcal{G} \in \Gamma$ such that $\mathcal{H} \in \mathcal{G} \subset \mathcal{H}$.

Theorem 4.26. *Let $(\mathbb{V},\Gamma)$ a (m,a,n)-FNTS. A (m,a,n)-FNS $\mathcal{H} \in \Gamma$ if and only if $\forall$ (m,a,n)-FNS $\mathcal{G}$ such that $\mathcal{G} \subset \mathcal{H}$, $\mathcal{H}$ is a (m,a,n)-FN neighbourhood of $\mathcal{G}$.*

Proof. Suppose that(m,a,n)-FNS $\mathcal{H} \in \Gamma$. Thus for each $\mathcal{G} \subset \mathcal{H}$, $\mathcal{H}$ is a (m,a,n)-FN neighbourhood of $\mathcal{G}$. Conversely suppose that for each $\mathcal{G} \subset \mathcal{H}$, $\mathcal{H}$ is a (m,a,n)-FN neighbourhood of $\mathcal{G}$. Since $\mathcal{H} \subset \mathcal{H}$, $\mathcal{H}$ is (m,a,n)-FN neighbourhood of $\mathcal{H}$ itself. Therefore $\exists$ $\mathcal{F} \in \Gamma$ such that $\mathcal{H} \subset \mathcal{F} \subset \mathcal{H} \Rightarrow \mathcal{H} = \mathcal{F} \Rightarrow \mathcal{H} \in \Gamma$. □

Definition 4.27. Let $(\mathbb{V},\Gamma)$ be a (m,a,n)-FNTS. A sub collection $\mathbb{B}$ of Γ is referred as a (m,a,n)-fuzzy neutrosophic basis((m,a,n)-FN basis) for Γ, if every nonempty (m,a,n)-FN open set is the union of certain members of $\mathbb{B}$.

Theorem 4.28. *Let $(\mathbb{V},\Gamma)$ be a (m,a,n)-FNTS. Let $\mathbb{B} = \{\mathcal{G}_i : i \in \Lambda\}$ be a subcollection of(m,a,n)-FNT Γ. Then $\mathbb{B}$ is a (m,a,n)-FN basis for Γ if and only if for any $\mathcal{F} \in \Gamma$ and a (m,a,n)-FN point $v_{(\zeta,\eta,\varsigma)} \in \mathcal{F}$, there exists a $\mathcal{G}_i \in \mathbb{B}$ for some $i \in \Lambda$, such that $v_{(\zeta,\eta,\varsigma)} \in \mathcal{G}_i \subset \mathcal{F}$.*

Now we extended the concept of connectedness to (m,a,n)-FNSs and explores its study in (m,a,n)-FNTSs.

Definition 4.29. A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is said to be (m,a,n)-fuzzy neutrosophic connected ((m,a,n)-FN connected), if $\nexists$ proper (m,a,n)-FN open sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{V}$ such that $\rceil(\mathcal{A}q\mathcal{B})$ and $\rceil(\mathcal{A}^c q\mathcal{B}^c)$.

Example 4.30. Let $\mathbb{V} = \{v_1, v_2\}$ be universe of discourse and the (m,a,n)-FNSs (for m=2, a=3,n=4) $\mathcal{G}$ and $\mathcal{H}$ on $\mathbb{V}$ be defined as follows:

$$\mathcal{G} = \{< v_1, 0.4, 0.3, 0.8 >, < v_1, 0.3, 04, 0.9 >\}$$
$$\mathcal{H} = \{< v_2, 0.6, 0.5, 0.5 >, < v_2, 0.5, 0.7, 0.8 >\}$$

Let $\Gamma = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{G}, \mathcal{H}\}$ be a (m,a,n)-FNT on $\mathbb{V}$, then (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is (m,a,n)-FN connected

Theorem 4.31. *A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is (m,a,n)-fuzzy neutrosophic connected if and only if it has no proper (m,a,n)-fuzzy neutrosophic clopen ((m,a,n)-FN closed and (m,a,n)-FN open) set.*

Theorem 4.32. *A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is (m,a,n)-fuzzy neutrosophic connected if and only if it has no proper (m,a,n)-FN open sets $\mathcal{G}$ and $\mathcal{H}$ providing that $\mho^m_{\mathcal{G}}(v) = \Omega^n_{\mathcal{H}}(v)$, $\Omega^n_{\mathcal{G}}(v) = \mho^m_{\mathcal{H}}(v)$ and $\varpi_{\mathcal{H}}(v) + \varpi_{\mathcal{H}}(v) = 1$.*

Now we define (m,a,n)-FN separation axioms by using the concept of (m,a,n)-FN point, (m,a,n)-FN open sets and (m,a,n)-FN closed sets.

Definition 4.33. A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is said to be (m,a,n)-FN T_0, if for every pair of distinct (m,a,n)-FN points $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ over $\mathbb{V}$, $\exists\ \mathcal{H},\mathcal{G}\in\Gamma$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\mathcal{H}$ but $v_{(\zeta_2,\eta_2,\varsigma_2)}\notin\mathcal{H}$ or $v_{(\zeta_2,\eta_2,\varsigma_2)}\in\mathcal{G}$ but $v_{(\zeta_1,\eta_1,\varsigma_1)}\notin\mathcal{G}$

Example 4.34. All discrete (m,a,n)-FNTSs are (m,a,n)-FN T_0, because for any two distinct (m,a,n)-FN points $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ over $\mathbb{V}$, $\exists$ a (m,a,n)-FN open set $\{v_{(\zeta_1,\eta_1,\varsigma_1)}\}$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\{v_{(\zeta_1,\eta_1,\varsigma_1)}\}$ but $w_{(\zeta_2,\eta_2,\varsigma_2)}\notin\{v_{(\zeta_1,\eta_1,\varsigma_1)}\}$.

Theorem 4.35. *Every (m,a,n)-FN subspace of a (m,a,n)-FN T_0 space is (m,a,n)-FN T_0.*

Proof. Let $(\mathbb{Y},\Gamma_{\mathbb{Y}})$ be a (m,a,n)-FN subspace of a (m,a,n)-FN T_0 space $(\mathbb{V},\Gamma)$. Let $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ be two distinct (m,a,n)-FN points over $\mathbb{Y}$. Then $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ are distinct (m,a,n)-FN points over $\mathbb{V}$. Since $(\mathbb{V},\Gamma)$ is (m,a,n)-FN T_0, $\exists$ a (m,a,n)-FN open set containing one of the (m,a,n)-FN point but not other. Without loss of generality, let $\mathcal{H}\in\Gamma$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\mathcal{H}$ but $w_{(\zeta_2,\eta_2,\varsigma_2)}\notin\mathcal{H}$. Put $\mathcal{H}_{\mathbb{Y}}=\mathcal{H}\cap\mathbb{Y}$. Then $\mathcal{H}_{\mathbb{Y}}\in\Gamma_{\mathbb{Y}}$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\mathcal{H}_{\mathbb{Y}}$ but $w_{(\zeta_2,\eta_2,\varsigma_2)}\notin\mathcal{H}_{\mathbb{Y}}$. Hence $(\mathbb{Y},\Gamma_{\mathbb{Y}})$ is (m,a,n)-FN T_0. □

Definition 4.36. A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is said to be (m,a,n)-FN T_1, if for every pair of distinct (m,a,n)-FN points $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ over $\mathbb{V}$, $\exists\ \mathcal{H},\mathcal{G}\in\Gamma$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\mathcal{H}$ but $v_{(\zeta_2,\eta_2,\varsigma_2)}\notin\mathcal{H}$ and $v_{(\zeta_2,\eta_2,\varsigma_2)}\in\mathcal{G}$ but $v_{(\zeta_1,\eta_1,\varsigma_1)}\notin\mathcal{G}$

Example 4.37. Every discrete (m,a,n)-FNTS is (m,a,n)-FN T_1, because, for any two distinct (m,a,n)-FN point $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ over $\mathbb{V}$, $\exists$ (m,a,n)-FN open sets $\{v_{(\zeta_1,\eta_1,\varsigma_1)}\}$ and $\{w_{(\zeta_2,\eta_2,\varsigma_2)}\}$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\{v_{(\zeta_1,\eta_1,\varsigma_1)}\}$ but $v_{(\zeta_1,\eta_1,\varsigma_1)}\notin\{w_{(\zeta_2,\eta_2,\varsigma_2)}\}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}\in\{w_{(\zeta_2,\eta_2,\varsigma_2)}\}$ but $v_{(\zeta_1,\eta_1,\varsigma_1)}\notin\{w_{(\zeta_2,\eta_2,\varsigma_1)}\}$.

Theorem 4.38. *Every (m,a,n)-FN subspace of a (m,a,n)-FN T_1 space is (m,a,n)-FN T_1.*

Definition 4.39. A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is said to be (m,a,n)-FN T_2 or (m,a,n)-FN Hausdorff, if for every pair of distinct (m,a,n)-FN points $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ over $\mathbb{V}$, $\exists\ \mathcal{H},\mathcal{G}\in\Gamma$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\mathcal{H}$, $w_{(\zeta_2,\eta_2,\varsigma_2)}\in\mathcal{G}$ and $\mathcal{H}\cap\mathcal{G}=\tilde{\Phi}$.

Theorem 4.40. *A (m,a,n)-FNTS $(\mathbb{V},\Gamma)$ is (m,a,n)-FN T_2 if and only if for any two distinct (m,a,n)-FN points $v_{(\zeta_1,\eta_1,\varsigma_1)}$ and $w_{(\zeta_2,\eta_2,\varsigma_2)}$ over $\mathbb{V}$ $\exists\ \ \mathcal{E},\mathcal{F}\in(m,a,n)-FNC(\mathbb{V})$ such that $v_{(\zeta_1,\eta_1,\varsigma_1)}\in\mathcal{E}$ but $w_{(\zeta_2,\eta_2,\varsigma_2)}\notin\mathcal{E}$, $v_{(\zeta_1,\eta_1,\varsigma_1)}\notin\mathcal{F}$ but $w_{(\zeta_2,\eta_2,\varsigma_2)}\in\mathcal{F}$ and $\mathcal{E}\cup\mathcal{F}=\mathbb{V}$.*

Theorem 4.41. *Every (m,a,n)-FN subspace of a (m,a,n)-FN T_2 space is (m,a,n)-FN T_2.*

Definition 4.42. A (m,a,n)-FNTS $(\mathbb{V}, \Gamma)$ is said to be (m,a,n)-FN regular if for every $\mathcal{F} \in (m,a,n)-FNC(\mathbb{V})$ and every (m,a,n)-FN point $v_{(\zeta,\eta,\varsigma)}$ over $\mathbb{V}$ such that $v_{(\zeta,\eta,\varsigma)} \notin \mathcal{F}$, $\exists\, \mathcal{G}, \mathcal{H} \in \Gamma$ such that $v_{(\zeta,\eta,\varsigma)} \in \mathcal{G}$, $\mathcal{F} \subset \mathcal{H}$ and $\mathcal{G} \cap \mathcal{H} = \tilde{\Phi}$.

Theorem 4.43. *Every (m,a,n)-FN subspace of a (m,a,n)-FN regular space is (m,a,n)-FN regular.*

Definition 4.44. A (m,a,n)-FNTS $(\mathbb{V}, \Gamma)$ is said to be (m,a,n)-FN normal if for every pair $\mathcal{E}, \mathcal{F} \in (m,a,n) - FNC(\mathbb{V})$ such that $\mathcal{E} \cap \mathcal{F} = \tilde{\Phi}$, $\exists\, \mathcal{G}, \mathcal{H} \in \Gamma$ such that $\mathcal{E} \subset \mathcal{G}$, $\mathcal{F} \subset \mathcal{H}$ and $\mathcal{G} \cap \mathcal{H} = \tilde{\Phi}$.

Example 4.45. Let $\mathbb{V} = \{v_1, v_2\}$ and(3,4,5)-FNSs $\mathcal{F}_1$ and $\mathcal{F}_2$ be defined as follows:

$$\mathcal{F}_1 = \{< v_1 0.5, 0.3, 0.4 >, < v_2 0.3, 0.2, 0.5 >\}$$

$$\mathcal{F}_2 = \{< v_1, 0.7, 0.3, 0.2 >, < v_2 0.9, 0.2, 0.1 >\}$$

Let $\Gamma = \{\tilde{\Phi}, \tilde{\mathbb{V}}, \mathcal{F}_1, \mathcal{F}_2\}$ be a (3,4,5)-FNT over $\mathbb{V}$. Then the (3,4,5)-FNTS $(\mathbb{L}, \Gamma_1, \Sigma)$ is (3,4,5)-FN normal.

Theorem 4.46. *Every (m,a,n)-FN closed subspace of a (m,a,n)-FN normal space is (m,a,n)-FN normal.*

Proof. Suppose $(\mathbb{Y}, \Gamma_{\mathbb{Y}}, \Sigma)$ be a (m,a,n)-FN closed subspace of a (m,a,n)-FN normal space $(\mathbb{V}, \Gamma)$. Let $\mathcal{F}_1, \mathcal{F}_2 \in (m,a,n)-FNC(\mathbb{Y})$ such that $\mathcal{F}_1 \cap \mathcal{F}_2 = \tilde{\Phi}$. Since $\mathbb{Y} \in (m,a,n)-FNC(\mathbb{V})$, by Theorem 4.17, $\mathcal{F}_1, \mathcal{F}_2 \in (m,a,n)-FNC(\mathbb{V})$. By (m,a,n)-FN normality of $(\mathbb{V}, \Gamma)$, $\exists\, \mathcal{G}_1, \mathcal{G}_2 \in \Gamma$ such that $\mathcal{F}_1 \subset \mathcal{G}_1$, $\mathcal{F}_2 \subset \mathcal{G}_2$ and $\mathcal{G}_1 \cap \mathcal{G}_2 = \tilde{\Phi}$. Put $(\mathcal{G}_1)_{\mathbb{Y}} = \mathcal{G}_1 \cap \mathbb{Y}$ and $(\mathcal{G}_2)_{\mathbb{Y}} = \mathcal{G}_1 \cap \mathbb{Y}$. Then $(\mathcal{G}_1)_{\mathbb{Y}}, (\mathcal{G}_2)_{\mathbb{Y}} \in \Gamma_{\mathbb{Y}}$. Clearly we have $\mathcal{F}_1 \subset \mathcal{G}_1 \Rightarrow \mathcal{F}_1 \cap \mathbb{Y} \subset \mathcal{G}_1 \cap \mathbb{Y} \Rightarrow \mathcal{F}_1 \subset (\mathcal{G}_1)_{\mathbb{Y}}$ and $\mathcal{F}_2 \subset \mathcal{G}_2 \Rightarrow \mathcal{F}_2 \cap \mathbb{Y} \subset \mathcal{G}_2 \cap \mathbb{Y} \Rightarrow \mathcal{F}_2 \subset (\mathcal{G}_2)_{\mathbb{Y}}$. Moreover, $(\mathcal{G}_1)_{\mathbb{Y}} \cap (\mathcal{G}_2)_{\mathbb{Y}} = (\mathcal{G}_1 \cap \mathcal{G}_2) \cap \mathbb{Y} = \tilde{\Phi} \cap \mathbb{Y} = \tilde{\Phi}$. Hence $(\mathbb{Y}, \Gamma_{\mathbb{Y}})$ is (m,a,n)-FN normal. □

5. Conclusions

We introduced the concept of (m,a,n)-fuzzy neutrosophic set as a super class of q-rung orthopair neutrosophic set and we created topological structure on (m,a,n)-fuzzy neutrosophic set. The fundamental topological concepts closure,interior, subspaces,bases, connectedness and separation axioms are extended to (m,a,n)-neutrosophic topological spaces. Examples were also given to illustrate our results. It looks that proper combinations of the theories developed for tackling the existing uncertainty is a promising tool for obtaining better results in a variety of human activities characterized by uncertainty. This is, therefore, a fruitful area for future research.

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A Neutrosophic Solution of Heat Equation by Neutrosophic Laplace Transform

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Abstract. This article aims to study the one-dimensional neutrosophic heat equation. In this study, we discuss the one-dimensional heat equation using neutrosophic numbers and provide numerical example to demonstrate the effectiveness of the neutrosophic Laplace transform method.

Keywords: neutrosophic real number; neutrosophic differential equation; neutrosophic laplace transformation; one-dimensional geometric AH-Isometry.

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1. Introduction

Partial differential equations (PDEs) are instrumental in modeling a wide array of phenomena across physical, biological, and social sciences. They are particularly useful for describing dynamic systems, such as heat conduction, where classical models often encounter uncertainties in variables and parameters, such as initial and boundary conditions or material properties. Neutrosophic differential equations (NDEs), which extend classical differential equations by incorporating neutrosophic numbers to address these uncertainties, offer a more robust framework for handling such imprecise or vague conditions. Neutrosophic set theory, introduced by Smarandache [1] as an extension of fuzzy sets invented by Zadeh (1965), provides a valuable tool for managing uncertainty in mathematical models, leading to more accurate and stable solutions. For example, in modeling heat diffusion, the neutrosophic heat equation accounts for the vagueness inherent in real-world conditions, such as varying ambient temperatures and material impurities. While exact analytical solutions for neutrosophic heat equations can be challenging to derive, numerical methods are employed to obtain practical solutions. The application of neutrosophic differential equations spans various fields, including engineering and medicine, demonstrating their versatility in modeling dynamic systems with inherent uncertainties. As such, neutrosophic calculus enriches the theory of differential equations, offering a more comprehensive approach than interval computations for addressing real-world complexities.

Smarandache proposed neutrosophic logic to represent a mathematical model of uncertainty, inaccuracy, ambiguity, imprecision, vagueness, unknown, incompleteness, inconsistency, redundancy, and contradiction, the concept of neutrosophy being a new branch of philosophy introduced by Smarandache (1998) [2–14]. Also, he introduced the meaning of the standard form of a neutrosophic real number and the circumstances for the division of two neutrosophic real numbers, characterized the standard form of a complex neutrosophic number, and found the root index $n \geq 2$ of a neutrosophic real and complex number (Smarandache, 1998 - 2014) [2–5]. While studying the concept of neutrosophic probability and neutrosophic statistics, professor Smarandache (1998 - 2015) [2–9] entered the concept of provisional calculus, where he first introduced the concepts of the neutrosophic mereo limit, neutrosophic mereo continuity, neutrosophic mereo derivative and neutrosophic mereo integral. Edalatpanah proposed a new simple algorithm for solving linear neutrosophic programming, in which the variables and the Right hand si de represent the Triangular neutrosophic numbers [12]. Mondal et al. (2021b) described the application of the neutrosophic differential equation on mine safety via a single-valued neutrosophic number. Sumathi et al. (2019) discussed the differential equation in a neutrosophic environment and the solution of a second-order linear differential equation with trapezoidal neutrosophic numbers as boundary conditions. Acharya et al. (2023a) used the differential equations in a neutrosophic environment under Hukuhara differentiability to model the amount of glucose distribution and absorption rates in blood. Lathamaheswari et al. (2022c) solved the neutrosophic differential equation by using bipolar trapezoidal neutrosophic number and applied this concept in predicting bacterial reproduction over separate bodies. Parikh et al. (2022d) describe the solution of a first-order linear non-homogeneous fuzzy differential equation with initial conditions in a neutrosophic environment. He also introduced the neutrosophic analytical method and the fourth-order Runge-Kutta numerical method by using triangular neutrosophic numbers.

Recently Alhasan (2021a - 2022b) [15–17] discussed some basics of differential and integral methods based on neutrosophic real numbers. Salamah et al. (2023b) used the concepts of continuity, differentiability, and integrability from real analysis to study the derivative and integration of a neutrosophic real function with one variable depending on the geometry isometry (AH-Isometry), also they studied the neutrosophic differential equation by using one-dimensional geometry AH-Isometry (Salamah et al., 2023c) [18], where they discussed the methods of finding the solution of neutrosophic identical linear differential equation and neutrosophic non-homogeneous linear differential equation. Moreover several researches have made multiple contribution to neutrosophic topology [19–25] and neutrosophic analysis.

The structure of this paper is organized as follow: In the first section, we provides a scientific overview of neutrosophists. The second section covers some basic concepts which will be used in this work. The third section, defined the fuzzy heat equation, its solution by a neutrosophic laplace transform method and an example is illustrated. Lastly, in the fourth section, the conclusion of this article is drawn.

2. Preliminaries

2.1. Definition: Neutrosophic Real Number. [3]

Let η be a neutrosophic real number, then it takes the standard form $\eta = \eta_1 + \eta_2 I$, where $\eta_1, \eta_2 \in R$, and I represent indeterminacy, such that $I.0 = 0$ and $I^n = I \quad \forall n \in Z^+$.

2.2. Definition: Division of two neutrosophic real numbers. [3]

If η and γ are two neutrosophic real numbers where, $\eta = \eta_1 + \eta_2 I, \quad \gamma = \gamma_1 + \gamma_2 I$.

Then, $\dfrac{\eta_1 + \eta_2 I}{\gamma_1 + \gamma_2 I} = \dfrac{\eta_1}{\gamma_1} + \dfrac{\gamma_1\eta_2 - \eta_1\gamma_2}{\gamma_1(\gamma_1 + \gamma_2)}.I$, provided, $\gamma_1 \neq 0$ and $\gamma_1 \neq -\gamma_2$.

2.3. Definition: [18]

Let $R(I) = \{\eta_1 + \eta_2 I; \eta_1, \eta_2 \in R\}$ be the neutrosophic field of reals. Then the one-dimensional isometry (AH-Isometry) is defined as follows

$$\mathcal{T} : R(I) \to R \times R$$
$$\mathcal{T}(\eta_1 + \eta_2 I) = (\eta_1, \eta_1 + \eta_2)$$

Remark: $\mathcal{T}$ is an algebraic isomorphism between two rings, it has the following properties:

(1) $\mathcal{T}$ is bijective.

(2) $\mathcal{T}$ preserves addition and multiplication, i.e.

$$\mathcal{T}[(\eta_1 + \eta_2 I) + (\gamma_1 + \gamma_2 I)] = \mathcal{T}(\eta_1 + \eta_2 I) + \mathcal{T}(\gamma_1 + \gamma_2 I)$$
$$\mathcal{T}[(\eta_1 + \eta_2 I).(\gamma_1 + \gamma_2 I)] = \mathcal{T}(\eta_1 + \eta_2 I).T(\gamma_1 + \gamma_2 I)$$

(3) $\mathcal{T}$ is invertible, i.e.

$$\mathcal{T}^{-1} : R \times R \to R(I)$$
$$\mathcal{T}^{-1}(\eta_1, \eta_2) = \eta_1 + (\eta_2 - \eta_1)I$$

(4) $\mathcal{T}$ preserves distances, i.e.

If $A = \eta_1 + \eta_2 I, \quad \gamma_1 + \gamma_2 I$ are two neutrosophic real numbers, then

$$\mathcal{T}(||\vec{AB}||) = ||\mathcal{T}(\vec{AB})||$$

2.4. **Definition: Neutrosophic Real Function [18]**

Let $f : R(I) \to R(I)$; where $f = f(x)$ and $x = x_1 + x_2 I \in R(I)$, then f is said to be a neutrosophic real function having one neutrosophic variable. A nuetrosophic real function $f(x)$ is written as

$$f(x) = f(x_1 + x_2 I) = f(x_1) + I[f(x_1 + x_2) - f(x_1)]$$

2.5. **Definition: Neutrosophic Laplace Transformation. [18]**

Let $f(x) = f(x_1 + x_2 I) = f(x_1) + [f(x_1 + x_2) - f(x_1)]I$ be a neutrosophic function on $R(I)$, where $x = x_1 + x_2 I$, then Neutrosophic Laplace transformation of $f(X)$ is defined as

$$L\{f(x), s\} = L\{f(x)\} = F(s) = \int_0^{-\infty} e^{-sx} f(x)dx \tag{1}$$

Where $s = s_1 + s_2 I \in R(I)$ and L is the Laplace transformation operator.
Equation (1) can be written as-

$$\Rightarrow F(s_1 + s_2 I) = \int_0^{-\infty} e^{-s_1 x_1} f(x_1)d(x_1) + \Big[\int_0^{-\infty} e^{-(s_1+s_2)(x_1+x_2)} f(x_1 + x_2)d(x_1 + x_2) - \int_0^{-\infty} e^{-s_1 x_1} f(x_1)d(x_1)\Big]I \tag{2}$$

This is the Neutrosophic Laplace transformation of $f(x)$ or $f(x_1 + x_2 I)$

Method of finding solution:

(1) Taking AH-Isometry on both sides of the equation (2), we get

$$F(s_1) = \int_0^{-\infty} e^{-s_1 x_1} f(x_1)d(x_1)$$

$$\Rightarrow F(s_1 + s_2) = \int_0^{-\infty} e^{-(s_1+s_2)(x_1+x_2)} f(x_1 + x_2)d(x_1 + x_2)$$

This $F(s_1)$ and $F(s_1 + s_2)$ are two classical Laplace transformation.

(2) We find $F(s_1)$ and $F(s_1 + s_2)$.

(3) Taking invertible AH-Isometry, then we get the required Neutrosophic Laplace transformation.

$$T^{-1}\big(F(s_1), F(s_1 + s_2)\big) = F(s_1) + \big(F(s_1 + s_2) - F(s_1)\big)I = F(s_1 + s_2 I).$$

2.6. **Definition: Neutrosophic Inverse Laplace Transformation.**

If $F(s)$ be the Laplace transform of a function $f(x)$, then $f(x)$ is called the Neutrosophic Laplace Transform of the function $F(s)$ and is written as $f(x) = L^{-1}[F(s)]$, where L^{-1} is the Inverse Laplace Transformation operator and $x = x_1 + x_2 I \in R(I), \quad s = s_1 + s_2 I \in R(I)$.

3. **Neutrosophic Heat Equation**

The field of partial differential equations holds a unique and fascinating place in mathematics. Problems that involve time 'T' as an independent variable generally lead to parabolic or hyperbolic equations. The simplest parabolic equation, $\frac{\partial U}{\partial T} = k\frac{\partial^2 U}{\partial X^2}$, arises from the theory of heat conduction. Its solution provides, for instance, the temperature 'U' at a distance 'X' units from one end of a thermally insulated bar after 'T' seconds of heat conduction. In reality, information about dynamical systems modeled by partial differential equations, especially in heat equations, is often incomplete or unclear. This uncertainty may manifest in different parts of heat equations, such as initial conditions, boundary conditions, etc. Therefore, we solve the heat equations based on neutrosophic conditions to obtain more accurate results than those based on real conditions.

We now assume that the temperature $U(X,T)$ of bar of constant cross-section and homogeneous material, lying along the axis and completely insulated laterally may be modeled by the one-dimensional heat equation

$$\frac{\partial U}{\partial T} = k\frac{\partial^2 U}{\partial X^2}$$

where k is the diffusivity of the material of the bar.
with the neutrosophic initial condition

$$U(X,T) = f(X,I) \in R(I), \quad 0 \leq X \leq a$$

and the neutrosophic boundary condition

$$[U(0,T)]_{X=0} \in R(I), \quad and \quad T > 0$$

$$[U(a,T)]_{X=a} \in R(I), \quad and \quad T > 0$$

In this work, without loss of generality, we will consider the diffusivity of the material of the bar k=1.

Example:

We consider the following heat equation

$$\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} \tag{3}$$

with initial condition

$$U(X,T) = \sin\frac{\pi X}{a} \in R(I) \tag{4}$$

and boundary condition

$$U(0,T) = 0 \quad \& \quad U(a,T) = 0 \tag{5}$$

where $U = U_1+U_2I, T = T_1+T_2I, X = X_1+X_2I \in R(I)$ and $0 = 0+0.I$ is also a neutrosophic number.

We now find the solution by applying Neutrosophic laplace transform

Taking AH-Isometry on (3), (4) and (5) respectively, we get

$$\frac{\partial U_1}{\partial T} = \frac{\partial^2 U_1}{\partial X^2} \quad \& \quad \frac{\partial U_1 + U_2}{\partial T} = \frac{\partial^2 U_1 + U_2}{\partial X^2} \tag{6}$$

$$U_1(X_1,T_1) = \sin\frac{\pi X_1}{a} \quad \& \quad (U_1+U_2)(X,T) = \sin\frac{\pi X_1}{a} + \sin\frac{\pi X_2}{a} \tag{7}$$

$$U_1(0,T_1) = (U_1+U_2)(0,T) = 0 \quad \& \quad U_1(a,T_1) = (U_1+U_2)(a,T) = 0 \tag{8}$$

Taking Laplace Transform on (6), we get

$$S_1\overline{U_1} - \sin\frac{\pi X_1}{a} = \frac{d^2\overline{U_1}}{dX^2} \tag{9}$$

$$(S_1+S_2)(\overline{U_1+U_2}) - \big(\sin\frac{\pi X_1}{a} + \sin\frac{\pi X_2}{a}\big) = \frac{d^2\overline{U_2}}{dX^2} \tag{10}$$

The A.E. of equation (9) is given by

$$D^2 - S_1 = 0 \Rightarrow D = \pm\sqrt{S_1}$$

Its C.F. is

$$Ae^{\sqrt{s_1}X_1} + Be^{-\sqrt{s_1}X_1}$$

Therefore P.I. of equation (9) is given by

$$\frac{1}{D^2 - S_1}\Big[sin\frac{\pi X_1}{a}\Big] = \frac{1}{\frac{\pi^2}{a^2} + S_1} sin\frac{\pi X_1}{a}$$

Thus the General solution of equation (9) is

$$\overline{U_1}(X,S_1) = Ae^{\sqrt{s_1}X_1} + Be^{-\sqrt{s_1}X_1} + \frac{1}{\frac{\pi^2}{a^2} + S_1} sin\frac{\pi X_1}{a} \tag{11}$$

Now taking Laplace Transform on (8), we get

$$\overline{U_1}(0,S_1) = \overline{(U_1+U_2)}(0,S_1+S_2) = 0 \quad \& \quad \overline{U_1}(a,S_1) = \overline{(U_1+U_2)}(a,S_1+S_2) = 0 \tag{12}$$

When $X = 0$ Then

Equation (11) gives,

$$\overline{U_1}(0,S_1) = A + B \Rightarrow A + B = 0 \tag{13}$$

When $X = a$ Then

Equation (11) gives,

$$\overline{U_1}(a, S_1) = Ae^{\sqrt{s_1}a} + Be^{-\sqrt{s_1}a} + 0 \Rightarrow Ae^{\sqrt{s_1}a} + Be^{-\sqrt{s_1}a} = 0 \tag{14}$$

Solving equation (13) and (14), we get $A = B = 0$.

Now, using these values in (11), the solution of equation (9) is given

$$\overline{U_1(X, S_1)} = \frac{sin\dfrac{\pi X_1}{a}}{\dfrac{\pi^2}{a^2} + S_1}$$

$$U_1(X, T_1) = sin\frac{\pi X_1}{a} L^{-1}\Big[\frac{1}{\dfrac{\pi^2}{a^2} + S_1}\Big]$$

$$U_1(X, T_1) = sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}} \tag{15}$$

Proceeding the same way, the solution of equation (10) gives,

$$(U_1 + U_2)(X, T_1 + T_2) = sin\frac{\pi(X_1 + X_2)}{a} e^{\dfrac{-\pi^2(T_1 + T_2)}{a^2}} \tag{16}$$

Now, taking inverse AH-Isometry on (15) and (16), we get

$$U(X, T) = \mathcal{T}^{-1}\Big[\sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}}, \sin\frac{\pi(X_1 + X_2)}{a} e^{\dfrac{-\pi^2(T_1 + T_2)}{a^2}}\Big]$$

$$U(X, T) =$$

$$\sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}} + \Big[\Big[\sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}} + \sin\frac{\pi(X_1 + X_2)}{a} e^{\dfrac{-\pi^2(T_1 + T_2)}{a^2}}\Big] - \sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}}\Big].I$$

This is the required temperatrure, where $\sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}}$ and $\Big[\Big[\sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}} + \sin\frac{\pi(X_1+X_2)}{a} e^{\dfrac{-\pi^2(T_1 + T_2)}{a^2}}\Big] - \sin\frac{\pi X_1}{a} e^{\dfrac{-\pi^2 T_1}{a^2}}\Big]$ are the determinant and indeterminant temperature.

4. **Conclusions**

In this article, we discussed the neutrosophic heat equation and its solution by taking numerical example. We have solved the practical problem by using one-dimensional geometric AH-Isometry of neutrosophic Laplace transformation. In the future, we will try to find general theoretical solutions so that practical problems can be solved quickly. In addition, we will try to solve the sensible solution to the neutrosophic heat equation by using different methods.

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$\mathfrak{c}$-Continuity, $\mathfrak{c}$-Compact and $\mathfrak{c}$-Separation Axioms via Soft Sets

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Abstract. This paper focuses on the concept of $S\mathfrak{c}$-open sets as a generalization of classical $\mathfrak{c}$-open sets in topology. The reason behind introducing $S\mathfrak{c}$-open sets is to overcome the limitations of traditional open sets in handling uncertainty and vagueness prevalent in decision-making processes. Moreover, the paper makes significant contributions to the discussion of the concepts of soft topological spaces (STS) by utilizing $S\mathfrak{c}$-open sets that investigate the theoretical foundations and mathematical properties of $S\mathfrak{c}$-open sets, exploring their relationships with other soft open sets and soft topological concepts. Overall, the paper aims to provide a comprehensive understanding of STS and their properties and theorems utilizing the concept of $S\mathfrak{c}$-open set and explores the theoretical foundations, mathematical properties, and relationships of these sets while extending their application to domains such as $S\mathfrak{c}$-continuity, $S\mathfrak{c}$-separation axiom and $S\mathfrak{c}$ compactness.

Keywords: $S\mathfrak{c}$-open set; $S\mathfrak{c}$-regular space; $S\mathfrak{c}$-normal space; $S\mathfrak{c}$-compact space; $S\mathfrak{c}^*$-compact space; $S\mathfrak{c}$-continuous function; $S\mathfrak{c}^*$-continuous function.

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1. Introduction

In 1999, Molodtsov introduced soft sets as a mathematical tool for handling uncertainty. Since then, soft set theory has found applications in diverse fields such as smoothness of functions, game theory, operations research, integration theory, probability, and measurement theory [1–14]. The properties and applications of soft set theory have been extensively studied by researchers [4], [10], [12], [14], [18], [20], [23], [24], [25]. In recent years, the combination of soft set theory with fuzzy set theory has led to interesting developments and applications [3], [5], [12], [13], [14], [15] , [17], [21]. This integration has involved redefining operations on soft sets and constructing decision-making methods using these new concepts [6]. The study of soft topological spaces was initiated by Shabir and Naz in 2011 [16]. They introduced soft

topology $\Im$ defined on the collection of soft sets over a set Λ. They established fundamental concepts such as soft open sets, soft closed sets, soft subspaces, soft closures, soft neighborhoods of a point, soft separation axioms, soft regular spaces, and soft normal spaces. They also investigated the properties of these concepts. Hussain and Ahmad further expanded the study of soft topological spaces in 2011 [7]. They focused on properties related to open soft sets, closed soft sets, soft neighborhoods, and soft closures. Additionally, they introduced and discussed the properties of soft interior, soft exterior, and soft boundary within the context of soft topological spaces. The introduction of soft set theory and the subsequent development of soft topological spaces have provided valuable tools for handling uncertainty and vagueness in various mathematical and applied domains. These theories have been successfully applied in numerous fields, and researchers continue to explore their properties and applications.
In 2023, [22] Alqahtani and Saleh introduced a novel type of open sets known as $\mathfrak{c}$-open sets. Their work focused on exploring the relationship between these sets and other types of open sets existing in classical topology. Specifically, a set $\aleph$ in a classical topological space is defined to be $\mathfrak{c}$-open if and only if $cl(\aleph) \setminus \aleph$ is countable set. This concept generalizes the traditional notion of open sets and offers a fresh perspective on openness in classical topology. This research aims to expand the investigation of the concept introduced by Alqahtani and Saleh by incorporating soft sets and $\mathfrak{c}$-open. Fundamental properties and theories are discussed in soft topological spaces via the $\mathfrak{c}$-open sets. This article is organized as follows: it begins by introducing a category of $S\mathfrak{c}$-open sets and outlining their fundamental properties. It then proceeds to explore various concepts such as $S\mathfrak{c}$-regular, $S\mathfrak{c}$-normal, and $S\mathfrak{c}$-T_i-spaces for $i \in 0, 1, 2, 3, 4$, as well as $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact spaces, utilizing $S\mathfrak{c}$-open sets. The article then delves into the examination of concepts such as $S\mathfrak{c}$-continuous, $S\mathfrak{c}^*$-continuous, $S\mathfrak{c}$-homeomorphism, and $S\mathfrak{c}^*$-homeomorphism functions through the perspective of $S\mathfrak{c}$-open sets. Additionally, it reviews essential properties related to $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact spaces, supported by numerous illustrative examples. Finally, the article concludes by providing insights and suggesting potential future research directions.

2. **Preliminaries**

Throughout this paper, we revisit some concepts in soft sets and soft topological spaces and soft topological spaces. Let Λ be an universe set and Π be a set of parameters. Let $\Gamma \subseteq \Pi$ and $P(\Lambda)$ denotes the collection of all subsets of Λ. The family of all soft sets over Λ with set of parameters $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ will be denoted by $SS(\Lambda)_\Gamma$. A pair $(\hat{\aleph}, \Gamma)$ is called a soft set (SS) on Λ, where $\hat{\aleph}$ is a mapping since Γ into P(Λ) is defined by $(\hat{\aleph}, \Gamma) = \{(\gamma, \hat{\aleph}(\gamma)) : \gamma \in \Gamma, \hat{\aleph}(\gamma) \subseteq \Lambda\}$. A relative null soft set (Φ, Γ) is a soft set $(\hat{\aleph}, \Gamma)$ if $\hat{\aleph}(\gamma) = \emptyset$ for all $\gamma \in \Gamma$. Similarly, a relative absolute soft set $(\hat{\Lambda}, \Gamma)$ is a soft set $(\hat{\aleph}, \Gamma)$ if $\hat{\aleph}(\gamma) = \Lambda$ for all $\gamma \in \Gamma$.

We include herein some definitions utilized in this study: [1, 2, 4]

Let $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ be *SSs* over Λ, we define

(1) $(\hat{\aleph}_1, \Gamma_1)$ is a soft subset of $(\hat{\aleph}_2, \Gamma_2)$ if: $\Gamma_1 \subseteq \Gamma_2$ and, $\hat{\aleph}_1(\gamma) \subseteq \hat{\aleph}_2(\gamma)$ for all $\gamma \in \Gamma$. This relationship is denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqsubseteq} (\hat{\aleph}_2, \Gamma_2)$. So, $(\hat{\aleph}_1, \Gamma_1)$ is a soft superset of $(\hat{\aleph}_2, \Gamma_2)$, denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqsupseteq} (\hat{\aleph}_2, \Gamma_2)$, if $(\hat{\aleph}_2, \Gamma_2)$ is a soft subset of $(\hat{\aleph}_1, \Gamma_1)$. Also, $(\hat{\aleph}_1, \Gamma_1)$ is a soft equal of $(\hat{\aleph}_2, \Gamma_2)$ if $(\hat{\aleph}_1, \Gamma_1)$ it is a soft subset and a soft superset of $(\hat{\aleph}_2, \Gamma_2)$.

(2) The soft union of two soft sets $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ over the common universe Λ is denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqcup} (\hat{\aleph}_2, \Gamma_2)$ and is the soft set $(\hat{\aleph}, \Gamma)$, where $\Gamma = \Gamma_1 \cup \Gamma_2$ for all $\gamma \in \Gamma$ defined as: $\hat{\aleph}(\gamma) = \hat{\aleph}_1(\gamma)$, if $\gamma \in \Gamma_1 \setminus \Gamma_2$, $\hat{\aleph}_2(\gamma)$, if $\gamma \in \Gamma_2 \setminus \Gamma_1$, and $\hat{\aleph}_1(\gamma) \cup \hat{\aleph}_2(\gamma)$ if $\gamma \in \Gamma_1 \cap \Gamma_2$. The soft intersection of two soft sets $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ over the common universe Λ is denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqcap} (\hat{\aleph}_2, \Gamma_2)$ and is the soft set $(\hat{\aleph}, \Gamma)$, where $\Gamma = \Gamma_1 \cap \Gamma_2 \neq \phi$ for all $\gamma \in \Gamma$, $\hat{\aleph}(\gamma) = \hat{\aleph}_1(\gamma) \cap \hat{\aleph}_2(\gamma)$.

(3) The complement of a soft set $(\hat{\aleph}, \Gamma)$ is denoted by $(\hat{\aleph}, \Gamma)^c$ and defined by $(\hat{\aleph}, \Gamma)^c$=$(\hat{\aleph}^c, \Gamma)$, where $\hat{\aleph}^c$ is a mapping given by $\hat{\aleph}^c(\gamma) = \Lambda \setminus \hat{\aleph}(\gamma)$ for all $\gamma \in \Gamma$. The soft difference between the two soft sets $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ is the soft set $(\hat{\aleph}, \Gamma)$ where $\Gamma = \Gamma_1 \cup \Gamma_2$ is defined as: $(\hat{\aleph}_1, \Gamma_1) \hat{\setminus} (\hat{\aleph}_2, \Gamma_2) = (\hat{\aleph}_1, \Gamma_1) \hat{\sqcap} (\hat{\aleph}_2, \Gamma_2)^c$.

(4) A soft set $(\hat{\aleph}, \Gamma)$ is said to be a soft countable set, denoted by *S*-countable if $\hat{\aleph}(\gamma)$ is countable.

Definition 2.1. [22] Let $(\Lambda, \Im)$ be a classical topological space. Below are some important definitions used in the study:

(1) An open subset $\aleph$ of a topological space $(\Lambda, \Im)$ is called $\mathfrak{c}$-*open* set if $cl(\aleph) \setminus \aleph$ is a countable set. That is, $\aleph$ is an open set. So, a closed subset $\aleph$ of a topological space $(\Lambda, \Im)$ is called $\mathfrak{c}$-closed set if $\aleph \setminus int(\aleph)$ is a countable set. That is, $\aleph$ is a closed set.

(2) $(\Lambda, \Im)$ is called a $\mathfrak{c}$-*regular* space if for each closed subset $\Omega \subseteq \Lambda$ and each point $\mathfrak{x} \notin \Omega$, there exist disjoint $\mathfrak{c}$-open sets $\aleph_1$ and $\aleph_2$ such that $\mathfrak{x} \in \aleph_1$ and $\Omega \subseteq \aleph_2$. So, $(\Lambda, \Im)$ is called a $\mathfrak{c}$-normal space if for each pair of closed disjoint subsets Ω_1 and Ω_2 of Λ, there exist disjoint $\mathfrak{c}-$open sets $\aleph_1$ and $\aleph_2$ such that $\Omega_1 \subseteq \aleph_1$ and $\Omega_2 \subseteq \aleph_2$.

(3) Let $(\Lambda, \Im)$ be a topology space, we say that Λ is $\mathfrak{c}-T_0$-space, if given $\mathfrak{x}_1, \mathfrak{x}_2 \in \Lambda$, $\mathfrak{x}_1 \neq \mathfrak{x}_2$, then there is either a $\mathfrak{c}$-open set containing $\mathfrak{x}_1$ but not $\mathfrak{x}_2$ or a $\mathfrak{c}$-open set containing $\mathfrak{x}_2$ but not $\mathfrak{x}_1$. So, Λ is $\mathfrak{c}-T_1$-space, if given $\mathfrak{x}_1, \mathfrak{x}_2 \in \Lambda$, $\mathfrak{x}_1 \neq \mathfrak{x}_2$, then there are two $\mathfrak{c}$-open subsets $\aleph_1$ and $\aleph_2$ of Λ, such that $\mathfrak{x}_1 \in \aleph_1$, $\mathfrak{x}_2 \notin \aleph_1$, and $\mathfrak{x}_1 \notin \aleph_2$, $\mathfrak{x}_2 \in \aleph_2$. Moreover, Λ is $\mathfrak{c}-T_2$-space, if given $\mathfrak{x}_1, \mathfrak{x}_2 \in \Lambda$, $\mathfrak{x}_1 \neq \mathfrak{x}_2$, then there are two disjoint $\mathfrak{c}$-open subsets $\aleph_1$ and $\aleph_2$ of Λ, such that $\mathfrak{x}_1 \in \aleph_1$ and $\mathfrak{x}_2 \in \aleph_2$.

(4) Let $(\Lambda, \Im)$ be a topological space, then Λ is $\mathfrak{c}$-compact (resp., $\mathfrak{c}^*$-compact) if for each open (resp., $\mathfrak{c}$-open) cover of Λ has a finite subcover of $\mathfrak{c}$-open (resp., open) sets.

(5) A function $\varphi : (\Lambda, \Im) \to (\zeta, \Im')$ is said to be $\mathfrak{c}$-continuous (resp., $\mathfrak{c}^*$-continuous) if $\varphi^{-1}(\aleph)$ is $\mathfrak{c}$-open (resp., open) in Λ for each open (resp., $\mathfrak{c}$-open) subset $\aleph$ in ζ. A function φ is said to be $\mathfrak{c}$-open function if for each $\mathfrak{c}$-open subset $\aleph \subseteq \Lambda$, we have $\varphi(\aleph)$ is an open subset in ζ. So, φ is said to be $\mathfrak{c}$-closed function if for each $\mathfrak{c}$-closed subset $\hat{\Theta} \subseteq \Lambda$, we have $\varphi(\hat{\Theta})$ is closed subset in ζ. Moreover A bijection function $\varphi : (\Lambda, \Im) \to (\zeta, \Im')$ is said to be $\mathfrak{c}$-homeomorphism (resp., $\mathfrak{c}^*$-homeomorphism) if φ and φ^{-1} are $\mathfrak{c}$-continuous (resp., $\mathfrak{c}^*$-continuous).

Definition 2.2. [2] Let $\hat{\Im}$ be a family of soft sets on Λ, then $\hat{\Im}$ is said to be soft topology (ST) on Λ if null soft set and absolute soft set in $\hat{\Im}$, the union of any members of soft sets in $\hat{\Im}$ belongs to $\hat{\Im}$ and intersection of any two soft sets in $\hat{\Im}$ belongs to $\hat{\Im}$. The triple $(\Lambda, \hat{\Im}, \Gamma)$ is called a soft topological space (STS) on Λ. Every member of $\hat{\Im}$ is called a soft open set, denoted by, S-open. The complement of an S-open set is a soft closed set, denoted by, S-closed.

Definition 2.3. [26], [9] Let $(\hat{\Lambda}, \Gamma)$ and $(\hat{\Lambda}', \Gamma')$ be two SSs, $\sigma : \Lambda \longrightarrow \Lambda'$ and $\alpha : \Gamma \longrightarrow \Gamma'$ be two mappings. Then $\hat{\varphi}_{\sigma\alpha} : (\hat{\Lambda}, \Gamma) \longrightarrow (\hat{\Lambda}', \Gamma')$ is called a soft mapping, S mapping. The image of $(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ under $\hat{\varphi}_{\sigma\alpha}$, $\hat{\varphi}_{\sigma\alpha}\big((\hat{\aleph}, \Gamma)\big) = \big(\hat{\varphi}_{\alpha}(\aleph), \Gamma'\big)$ is a SS in $(\hat{\Lambda}', \Gamma')$ given as, for all $\gamma' \in \Gamma'$, $\hat{\varphi}_{\sigma\alpha}(\aleph)(\gamma') = \hat{\sigma}\big(\bigcup_{\gamma \in \alpha^{-1}(\gamma') \cap \Gamma} \aleph(\gamma)\big)$, if $\alpha^{-1}(\gamma') \cap \Gamma \neq \phi$ and ϕ for otherwise. For $\gamma' \in \Gamma'$, $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma) = (\sigma(\aleph), \alpha(\Gamma))$ is called a soft image of $(\hat{\aleph}, \Gamma)$. The inverse image of $(\hat{\Theta}, \Gamma')$ $\hat{\sqsubseteq}$ $(\hat{\Lambda}', \Gamma)$ under $\hat{\varphi}_{\sigma\alpha}$, $\hat{\varphi}^{-1}_{\hat{\varphi}\alpha}\big((\hat{\Theta}, \Gamma')\big) = \big(\hat{\varphi}^{-1}_{\hat{\varphi}\alpha}(\hat{\Theta}), \Gamma\big)$ is a SS in $(\hat{\Lambda}, \Gamma)$ given as, for all $\gamma \in \Gamma$, $\hat{\varphi}^{-1}_{\hat{\varphi}\alpha}(\hat{\Theta})(\gamma) = \hat{\varphi}^{-1}_{\sigma\alpha}\big(\hat{\Theta}(\alpha(\gamma))\big)$, if $\alpha(\gamma) \in \Gamma'$, ϕ, if $\alpha(\gamma) \notin \Gamma'$. The S mapping, $\hat{\varphi}_{\sigma\alpha}$ is called a

(1) S surjective mapping if σ and α are surjective mappings.
(2) S injective mapping if σ and α are injective mappings.
(3) S bijective mapping if σ and α are bijective mappings.

3. $S\mathfrak{c}$-Open and $S\mathfrak{c}$-Closed Sets in $STSs$

In this section, we introduce the definitions of $S\mathfrak{c}$-open and $S\mathfrak{c}$-closed sets, and discuss the theorems and properties derived from them, supported by relevant counterexamples.

Definition 3.1. A S-open subset $(\hat{\aleph}, \Gamma)$ of a STS $(\Lambda, \hat{\Im}, \Gamma)$ is called soft $\mathfrak{c}$-open set, $S\mathfrak{c}$-open set if $cl((\hat{\aleph}, \Gamma)) \hat{\setminus} (\hat{\aleph}, \Gamma)$ is S-countable set. That is, $(\hat{\aleph}, \Gamma)$ is a S-open set.

Definition 3.2. A S-closed subset $(\hat{\aleph}, \Gamma)$ of a STS $(\Lambda, \hat{\Im}, \Gamma)$ is called soft $\mathfrak{c}$-closed set, $S\mathfrak{c}$-closed set if $(\hat{\aleph}, \Gamma) \hat{\setminus} int(\hat{\aleph}, \Gamma)$ is S-countable set. That is, $(\hat{\aleph}, \Gamma)$ is a S-closed set.

Theorem 3.3. *The complement of any $S\mathfrak{c}$-open (resp., $S\mathfrak{c}$-closed) subset of a STS $(\Lambda, \hat{\Im}, \Gamma)$ is a $S\mathfrak{c}$-closed (resp., $S\mathfrak{c}$-open) set.*

Proof. Let $(\hat{\aleph},\Gamma)$ be any $S\mathfrak{c}$-open subset of a STS $(\Lambda,\hat{\Im},\Gamma)$. Then its complement, $(\hat{\aleph},\Gamma)^c = (\hat{\aleph}^c,\Gamma)$ is S-closed, and satisfy the following:

$$\begin{aligned}(\hat{\aleph},\Gamma)^c \hat{\setminus} int(\hat{\aleph},\Gamma)^c =&(\hat{\aleph},\Gamma)^c \hat{\setminus} (cl(\hat{\aleph},\Gamma))^c\\ &= (\hat{\aleph},\Gamma)^c \hat{\sqcup} cl(\hat{\aleph},\Gamma)\\ &= cl(\hat{\aleph},\Gamma) \hat{\setminus} (\hat{\aleph},\Gamma),\end{aligned}$$

which is S-countable by Definition 3.1. Consequently, $(\hat{\aleph},\Gamma)^c \hat{\setminus} int(\hat{\aleph},\Gamma)^c$ is also S-countable. Therefore, $(\hat{\aleph},\Gamma)^c$ is a $S\mathfrak{c}$-closed set. On the other hand, if $(\hat{\aleph},\Gamma)$ is any $S\mathfrak{c}$-closed subset of a STS $(\Lambda,\hat{\Im},\Gamma)$, following the same method, $(\hat{\aleph},\Gamma)^c$ is a $S\mathfrak{c}$-open set.

Corollary 3.4. *Let $(\Lambda,\hat{\Im},\Gamma)$ be a STS and $(\hat{\aleph},\Gamma) \hat{\in} SS(\Lambda)_\Gamma$, then*

(1) *Every S-clopen (S-closed and S-open) subset of a STS is $S\mathfrak{c}$-clopen ($S\mathfrak{c}$-open and $S\mathfrak{c}$-closed) set.*
(2) *Every S-countable S-closed set is $S\mathfrak{c}$-closed.*

Proof.

(1) Utilizing Definitions 3.1 and 3.2, alongside the properties of S-open and S-closed sets, incorporating their S-interior and S-closure, we can demonstrate it directly.
(2) Suppose $(\hat{\aleph},\Gamma)$ is S-countable and S-closed. Then, we have $cl(\hat{\aleph},\Gamma) = (\hat{\aleph},\Gamma) = bd(\hat{\aleph},\Gamma)$. Consequently,

$$\begin{aligned}(\hat{\aleph},\Gamma) \setminus int(\hat{\aleph},\Gamma) &= cl(\hat{\aleph},\Gamma) \hat{\sqcap} (int(\hat{\aleph},\Gamma))^c\\ &= cl(\hat{\aleph},\Gamma) \hat{\sqcap} cl(\hat{\aleph},\Gamma)^c\\ &= bd(\hat{\aleph},\Gamma)\\ &= (\hat{\aleph},\Gamma).\end{aligned}$$

Since $(\hat{\aleph},\Gamma)$ is S-countable, it follows that $(\hat{\aleph},\Gamma) \setminus int(\hat{\aleph},\Gamma)$ is also S-countable. Therefore, $(\hat{\aleph},\Gamma)$ is $S\mathfrak{c}$-closed.

Remark 3.5. Indeed, every S-countable S-open set may not necessarily be $S\mathfrak{c}$-open.

Example 3.6. Let $\Re$ be a set of real number and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ be soft point (SP). The collection

$$\hat{\Im} = \{(\hat{\aleph},\Gamma) \hat{\sqsubseteq} (\hat{\Lambda},\Gamma) : \hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \in (\hat{\aleph},\Gamma)\} \hat{\sqcup} \{(\Phi,\Gamma)\},$$

is the particular soft point topology on Λ and $(\hat{\mathcal{N}},\Gamma)$ be a S-countable S-open set but $cl(\hat{\mathcal{N}},\Gamma) \hat{\setminus} (\hat{\mathcal{N}},\Gamma) = (\hat{\Re},\Gamma) \setminus (\hat{\mathcal{N}},\Gamma)$ is S-uncountable set.

Proposition 3.7. *In STS, every $S\mathfrak{c}$-open (resp. $S\mathfrak{c}$-closed) set is a S-open (resp. S-closed) set.*

Proof. Obvious, by the Definitions 3.1 and 3.2.

Remark 3.8. In general, every S-open (resp. S-closed) set is not necessarily a $S\mathfrak{c}$-open (resp. $S\mathfrak{c}$-closed) set.

Example 3.9. Let $(\hat{\Lambda},\Gamma)$ be a non-empty SS, and let (I,Γ) be a soft subset such that (I,Γ) $\hat{\sqsubseteq}$ $(\hat{\Lambda},\Gamma)$. We define the soft subset $\hat{\Im}_{(I,\Gamma)}$ of SSs as follows:

$$\hat{\Im}_{(I,\Gamma)} = \{(\mu,\Gamma) \hat{\sqsubseteq} (\hat{\Lambda},\Gamma) : (\mu,\Gamma) \hat{\sqcap} (I,\Gamma) = (\Phi,\Gamma)\} \hat{\sqcup} (\hat{\Lambda},\Gamma).$$

All soft subsets of $(\hat{\Lambda},\Gamma)$ that are soft disjoint from (I,Γ), along with $(\hat{\Lambda},\Gamma)$ itself, constitute $\hat{\Im}_{(I,\Gamma)}$. This SS forms a STS called the excluded set ST on $(\hat{\Lambda},\Gamma)$ with respect to (I,Γ). Consequently, $(\hat{Q},\Gamma)$ is S-open in $(\Re,\hat{\Im}_I,\Gamma)$. However, $cl(\hat{Q},\Gamma) \hat{\setminus} (\hat{Q},\Gamma) = (\hat{\Re},\Gamma) \hat{\setminus} (\hat{Q},\Gamma) = (\hat{IQ},\Gamma)$ is a S-uncountable set. Thus, $(\hat{Q},\Gamma)$ is not $S\mathfrak{c}$-open. Furthermore, $(\hat{IQ},\Gamma)$ exemplifies a S-closed set that is not $S\mathfrak{c}$-closed.

Remark 3.10. An example exists where a $S\mathfrak{c}$-open (resp., $S\mathfrak{c}$-closed) set is not a S-open (resp., S-closed) domain set.

Example 3.11. Consider $(\Re,\hat{\Im}_{\mathcal{U}},\Gamma)$ as a STS on $\Re$ with $\Gamma = \{\gamma_1,\gamma_2\}$. Let $(\hat{\aleph},\Gamma) = \{(\gamma_1,(2,5)),(\gamma_2,(2,5))\} \hat{\sqcup} \{(\gamma_1,(5,9)),(\gamma_2,(5,9))\}$, which is a $S\mathfrak{c}$-open subset in $(\Re,\hat{\Im}_{\mathcal{U}},\Gamma)$. However, $(\hat{\aleph},\Gamma)$ is not S-open domain. Furthermore, $(\hat{\aleph},\Gamma)^c$ is a $S\mathfrak{c}$-closed set, but $(\hat{\aleph},\Gamma)^c$ is not S-closed domain.

Definition 3.12. [27] In a STS, a SS $(\hat{\aleph},\Gamma)$ is considered soft regular open ($S\mathfrak{r}$-open) if it $int(cl(\hat{\aleph},\Gamma)) = (\hat{\aleph},\Gamma)$. Similarly, a SS $(\hat{\aleph},\Gamma)$ is considered soft regular closed ($S\mathfrak{r}$-closed) if $cl(int(\hat{\aleph},\Gamma)) = (\hat{\aleph},\Gamma)$, or if its soft complement is an $S\mathfrak{r}$-open.

Remark 3.13. In general, $S\mathfrak{c}$-open (resp., $S\mathfrak{c}$-closed) sets and $S\mathfrak{r}$-open (resp., $S\mathfrak{r}$-closed) sets are not comparable, as illustrated by the following example.

Example 3.14. By Example 3.9, let $(\hat{\aleph},\Gamma) = \{(\gamma_1,\{1,2,3\}),(\gamma_2,\{2,3\})\}$. Then $int(cl(\hat{\aleph},\Gamma))$ $= int(\hat{\aleph},\Gamma) \hat{\sqcup} (\hat{I},\Gamma) = (\hat{\aleph},\Gamma)$ is $S\mathfrak{r}$-open. However, $cl((\hat{\aleph},\Gamma)) \hat{\setminus} (\hat{\aleph},\Gamma) = (\hat{\aleph},\Gamma) \sqcup(\hat{I},\Lambda) = (\hat{I},\Gamma)$ is S-uncountable set, hence $(\hat{\aleph},\Gamma)$ is not $S\mathfrak{c}$-open.

Moreover, let $(\hat{\aleph}_1,\Gamma) = (\hat{\hat{\Re}},\Gamma)\hat{\setminus}(\hat{\aleph},\Gamma)$. Then

$$\begin{aligned} cl(int((\hat{\aleph}_1,\Gamma))) &= cl(int((\hat{\hat{\Re}},\Gamma) \hat{\setminus} (\hat{\aleph},\Gamma))) \\ &= cl((Q,\Gamma) \hat{\setminus} (\hat{\aleph},\Gamma)) \\ &= (\hat{I},\Gamma) \hat{\sqcup} ((Q,\Gamma) \hat{\setminus} (\hat{\aleph},\Gamma)) \\ &= (\hat{\hat{\Re}},\Gamma) \hat{\setminus} (\hat{\aleph},\Gamma) \\ &= (\hat{\aleph}_1,\Gamma), \end{aligned}$$

s $S\mathfrak{r}$-closed. Also,

$$\begin{aligned}(\hat{\aleph}_1,\Gamma)\,\hat{\setminus}\,int((\hat{\aleph}_1,\Gamma)) &= ((\hat{\hat{\Re}},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\hat{\setminus} int((\hat{\hat{\Re}},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\\ &= ((\hat{\hat{\Re}},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\hat{\setminus}((\hat{Q},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\\ &= (\hat{I},\Gamma),\end{aligned}$$

is S-uncountable set, then $(\hat{\aleph}_1,\Gamma)$ is not $S\mathfrak{c}$-closed set.

Remark 3.15. The definitions of S-open, S-closed, $S\mathfrak{c}$-open, and $S\mathfrak{c}$-closed sets lead to the following diagram:

$$\begin{aligned} S\mathfrak{c}\text{-open set} &\longrightarrow S\text{-open set}\\ S\mathfrak{c}\text{-closed set} &\longrightarrow S\text{-closed set}\end{aligned}$$

Diagram (i)

None of the above implications are reversible.

Proposition 3.16. *A finite soft union of $S\mathfrak{c}$-open sets remains $S\mathfrak{c}$-open.*

Proof. Let $\{(\hat{\aleph}_{\mathrm{j}},\Gamma)\}_{\mathrm{j}\in\mathfrak{J}}$ be a finite collection of $S\mathfrak{c}$-open sets. Then $(\hat{\aleph}_{\mathrm{j}},\Gamma)$ is an S-open set and $cl((\hat{\aleph}_{\mathrm{j}},\Gamma))\,\hat{\setminus}(\hat{\aleph}_{\mathrm{j}},\Gamma)$ is S-countable for all $\mathfrak{j}$. Now, consider $\hat{\bigsqcup}_{\mathrm{j}=1}^{n}(\hat{\aleph}_{\mathrm{j}},\Gamma)$ is S-open, then to show the other condition of $S\mathfrak{c}$-open set. Now, we have, $cl(\hat{\bigsqcup}_{\mathrm{j}=1}^{n}\,(\hat{\aleph}_{\mathrm{j}},\Gamma))\,\hat{\setminus}\,\hat{\bigsqcup}_{\mathrm{j}=1}^{n}\,(\hat{\aleph}_{\mathrm{j}},\Gamma)$ $=\hat{\bigsqcup}_{\mathrm{j}=1}^{n}(cl((\hat{\aleph}_{\mathrm{j}},\Gamma))\,\hat{\setminus}\,(\hat{\aleph}_{\mathrm{j}},\Gamma))$, since the finite soft union of S-countable sets is S-countable. Then, $cl(\hat{\bigsqcup}_{\mathrm{j}=1}^{n}(\hat{\aleph}_{\mathrm{j}},\Gamma))\,\hat{\setminus}\,\hat{\bigsqcup}_{\mathrm{j}=1}^{n}(\hat{\aleph}_{\mathrm{j}},\Gamma)$ is S-countable. Therefore, $\hat{\bigsqcup}_{\mathrm{j}=1}^{n}(\hat{\aleph}_{\mathrm{j}},\Gamma)$ is $S\mathfrak{c}$-open.

Example 3.17. Let $(\hat{\Lambda}_i,\Gamma)=\{\{\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i},\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}_i}\}:\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}\neq\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}_i}, i\in I\}$ be a family of pairwise disjoint soft spaces, where I is an uncountable index set. Let $\hat{\Im}_i$ be a particular soft point topology on $(\hat{\Lambda}_i,\Gamma)$ at $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}$. Let the $\hat{\Im}=\{(\hat{\aleph},\Gamma)\,\hat{\sqsubseteq}\,\hat{\sqcup}_{i\in I}(\hat{\Lambda}_i,\Gamma):(\hat{\aleph},\Gamma)\,\hat{\sqcap}\,(\hat{\Lambda}_i,\Gamma)\,\hat{\sqsubseteq}\,(\hat{\Lambda}_i,\Gamma)$ S-open for all $i\}$ be a ST on the disjoint soft union $(\hat{\Lambda},\Gamma)=\hat{\sqcup}_{i\in I}\,(\hat{\Lambda}_i,\Gamma)$. We call $(\Lambda,\hat{\Im},\Gamma)$ soft topological sum of the $(\hat{\Lambda}_i,\Gamma)$. for all $i\in I$, pick $(\hat{\mu}_i,\Gamma)=\{\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}\}\,\hat{\sqsubseteq}\,(\hat{\Lambda},\Gamma)$, where $(\hat{\mu}_i,\Gamma)=\{\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}\}$ for all i and $(\hat{\mu}_i,\Gamma)=(\Phi,\Gamma)$ for all $i\neq j$ in I. Then, $(\hat{\mu}_i,\Gamma)$ is S-open and $cl((\hat{\mu}_i,\Gamma))\,\hat{\setminus}\,(\hat{\mu}_i,\Gamma)$ is finite. Therefore, $(\hat{\mu}_i,\Gamma)$ is $S\mathfrak{c}$-open in $(\hat{\Lambda},\Gamma)$ for all $i\in I$. However, $cl(\hat{\sqcup}_{i\in I}\,(\hat{\mu}_i,\Gamma))\,\hat{\setminus}\,\hat{\sqcup}_{i\in I}\,(\hat{\mu}_i,\Gamma)$ is S-uncountable. Hence, $\hat{\sqcup}_{i\in I}\,(\hat{\mu}_i,\Gamma)$ is not $S\mathfrak{c}$-open in $(\hat{\Lambda},\Gamma)$.

Corollary 3.18. *A finite soft intersection of $S\mathfrak{c}$-closed sets remains $S\mathfrak{c}$-closed.*

Proof. This is evident from the Theorems 3.16, Theorem 3.3 and by Morgan's Laws.

Theorem 3.19. *A finite soft union of $S\mathfrak{c}$-closed sets remains $S\mathfrak{c}$-closed.*

Proof.

Let $(\hat{\aleph}_{\mathrm{j}},\Gamma)$ be a collection of $S\mathfrak{c}$-closed sets for all $\{j=1,2,...,n\}$, it follows that $(\hat{\aleph}\mathrm{j},\Gamma)$ is S-closed, and the soft difference $(\hat{\aleph}_{\mathrm{j}},\Gamma)\,\hat{\setminus}\,\mathrm{int}((\hat{\aleph}_{\mathrm{j}},\Gamma))$ is S-countable. Considering the union

$\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)$, which is also S-closed, demonstrating the other condition suffices to prove the $S\mathfrak{c}$-closed property.

Claim

$$\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\,\hat{\setminus}\,int\big(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\big)\,\hat{\sqsubseteq}\,\hat{\bigsqcup}_{j=1}^{n}\big((\hat{\aleph}_j,\Gamma)\,\hat{\setminus}\,int((\hat{\aleph}_j,\Gamma))\big).$$

Let $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ \hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\ \hat{\setminus}\ int(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma))$ be arbitrary. Since $\hat{\bigsqcup}_{j=1}^{n}int((\hat{\aleph}_j,\Gamma))\ \hat{\sqsubseteq}\ int(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma))$, then there exists $j'\ \hat{\in}\ \{1,2,3,\ldots,n\}$ such that $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ (\hat{\aleph}_1,\Gamma)_{j'}$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\notin}\ int((\hat{\aleph}_j,\Gamma))$ for all $j\in\{1,2,3,\ldots,n\}$. Then $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ ((\hat{\aleph}_1,\Gamma)_{j'})\ \hat{\setminus}\ int\big((\hat{\aleph}_1,\Gamma)_{j'}\big)$, then $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ \bigsqcup_{j=1}^{n}\big((\hat{\aleph}_j,\Gamma)\ \hat{\setminus}\ int\big((\hat{\aleph}_j,\Gamma)\big)\big)$. Claim is proved.

Since the finite soft union of S-countable sets is S-countable. Then, $\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\ \hat{\setminus}\ int(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma))$ is S-countable set. Therefore, $\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)$ is $S\mathfrak{c}$-closed.

Corollary 3.20. *a finite soft intersections of $S\mathfrak{c}$-open sets remains $S\mathfrak{c}$-open.*

Proof. This is evident from the Theorems 3.3, Theorem 3.19 and by Morgan's Laws.

We suggest referring to soft infra-topological spaces using the term "$S\mathfrak{c}$-open sets," emphasizing their notable characteristic. Furthermore, we can define soft generalized infra-topological spaces as those that may not necessarily encompass the entire absolute soft set $(\hat{\Lambda},\Gamma)$.

Definition 3.21. Let $\hat{\mathfrak{I}}_{S\mathfrak{c}}$ be a collection of $S\mathfrak{c}$-open sets over Λ under a fixed set of parameters Γ, then $\hat{\mathfrak{I}}_{S\mathfrak{c}}$ is said to be infra soft c-topological space ($ISTS_{\mathfrak{c}}$) on Λ if it satisfies the following axioms:

(1) (Φ,Γ), $(\hat{\Lambda},\Gamma)\ \hat{\in}\ \hat{\mathfrak{I}}_{S\mathfrak{c}}$,
(2) Closed under finite soft intersection.

Then, the triple $(\Lambda,\hat{\mathfrak{I}}_{S\mathfrak{c}},\Gamma)$ is called an $ISTS_{\mathfrak{c}}$. Every member of $\hat{\mathfrak{I}}_{S\mathfrak{c}}$ is called an $IS\mathfrak{c}$-open set, and its soft complement is called an $IS\mathfrak{c}$-closed set.

4. Separation Axioms and Compactness are Explored Through a Novel Category of Soft Open Sets.

This section presents the definitions of regular, normal and T_i-spaces for $i = 1,2,3,4$ utilizing $S\mathfrak{c}$-open sets. Furthermore, we define compactness within this category of soft open sets, namely $\mathfrak{c}$-compact space, and $\mathfrak{c}^*$-compact space, and explores their principal properties.

Definition 4.1. A STS $(\Lambda,\hat{\Im},\Gamma)$ on Λ is said to be

(1) $S\mathfrak{c}$-Regular if for each S-closed $(\hat{\mu},\Gamma)$; $(\hat{\mu},\Gamma)\ \hat{\sqsubseteq}\ (\hat{\Lambda},\Gamma)$ and each SP $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\notin}\ (\hat{\mu},\Gamma)$, there exist soft disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1,\Gamma)$ and $(\hat{\aleph}_2,\Gamma)$ such that $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ (\hat{\aleph}_1,\Gamma)$ and $(\hat{\mu},\Gamma)\ \hat{\sqsubseteq}\ (\hat{\aleph}_2,\Gamma)$.

(2) $S\mathfrak{c}$-Normal if for each pair of S-closed disjoint soft subsets $(\hat{\mu}_1, \Gamma)$ and $(\hat{\mu}_2, \Gamma)$ of $(\hat{\Lambda}, \Gamma)$, there exist soft disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ such that $(\hat{\mu}_1, \Gamma) \hat{\sqsubseteq} (\hat{\aleph}_1, \Gamma)$ and $(\hat{\mu}_2, \Gamma) \hat{\sqsubseteq} (\hat{\aleph}_2, \Gamma)$.

(3) $S\mathfrak{c}$-$\hat{T}_0$, if given $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}, \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\Lambda}, \Gamma)$, $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \neq \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$, then there is either a $S\mathfrak{c}$-open set containing $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}$ but not $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$ or a $S\mathfrak{c}$-open set containing $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$ but not $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}$.

(4) $S\mathfrak{c}$-$\hat{T}_1$, if given $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}, \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\Lambda}, \Gamma)$, $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \neq \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$, then there are two $S\mathfrak{c}$-open soft subsets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ of $(\hat{\Lambda}, \Gamma)$, such that $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \hat{\in} (\hat{\aleph}_1, \Gamma)$, $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\notin} (\hat{\aleph}_1, \Gamma)$, and $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \hat{\notin} (\hat{\aleph}_2, \Gamma)$, $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\aleph}_2, \Gamma)$.

(5) $S\mathfrak{c}$-$\hat{T}_2$, if given $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}, \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\Lambda}, \Gamma)$, $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \neq \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$, then there are two soft disjoint $S\mathfrak{c}$-open subsets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ of $(\hat{\Lambda}, \Gamma)$, such that $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \hat{\in} (\hat{\aleph}_1, \Gamma)$ and $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\aleph}_2, \Gamma)$.

(6) $S\mathfrak{c}$-$\hat{T}_3$ if it is a $S\mathfrak{c}$-$\hat{T}_1$ and $S\mathfrak{c}$-regular space.

(7) $S\mathfrak{c}$-$\hat{T}_4$ if it is a $S\mathfrak{c}$-$\hat{T}_1$ and $S\mathfrak{c}$-normal space.

Proposition 4.2. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS on Λ, then any $S\mathfrak{c}$-regular (resp. $S\mathfrak{c}$-normal) space is a S-regular (resp. S-normal) space.*

Proof. Obvious from the definitions 4.1 and 3.2.

Remark 4.3. In general, the converse of Proposition 4.2 does not hold as illustrated by the following examples.

Example 4.4. Consider $(\hat{\Re} \times \hat{\Re}, \hat{\Im}_{\mathcal{U}\times\mathcal{U}}, \Gamma)$ represents the usual soft topology on $\hat{\Re} \times \hat{\Re}$, and $\Gamma = \{\gamma_1, \gamma_2\}$ is a set of parameters. Then, $(\hat{\Re} \times \hat{\Re}, \hat{\Im}_{\mathcal{U}\times\mathcal{U}}, \Gamma)$ is a S-normal. However, if we select $(\hat{\mu_1}, \Gamma)$ and $(\hat{\mu_2}, \Gamma)$ as two disjoint S-closed sets defined by:

$$(\hat{\mu_1}, \Gamma) = \{(\gamma_1, [2,3] \times [2,3]), (\gamma_2, [-2,-3] \times [2,3])\},$$

$$(\hat{\mu_2}, \Gamma) = \{(\gamma_1, [-2,-3] \times [2,3]), (\gamma_2, [2,3] \times [2,3])\}.$$

Can not be separated by two disjoint $S\mathfrak{c}$-open sets. Consequently, $(\hat{\Re}\times\hat{\Re}, \hat{\Im}_{\mathcal{U}\times\mathcal{U}}, \Gamma)$ is S-normal space, which is not $S\mathfrak{c}$-normal space.

Example 4.5. The Niemytzki Plane. Let $L = \{(a,b) \in \Re : b \geq 0\}$, be the upper half-plane with the X-axis. Define a soft set $(\hat{\aleph}, \Gamma)$ over L, where Γ is a set of parameters and $\hat{\aleph}$ is a mapping from Γ into the set of subsets of L. Let $L_1 = \{(a,0) : a \in \Re\}$, i.e., the X-axis, and $\mathsf{L}_2 = L \setminus L_1$. Define the *SS* $(\hat{\aleph}, \Gamma)$ such that for every $(a,0) \in L_1$ and $r \in \Re, r > 0$, the *SS* $(\hat{\aleph}, \Gamma)\big((a,0), r\big)$ is the set of all *SP* of L inside the circle of radius r tangent to L_1 at $(a,0)$. Furthermore, let $(\hat{\aleph}, \Gamma)_i\big((a,0)\big) = (\hat{\aleph}, \Gamma)\big((a,0), \frac{1}{i}\big) \hat{\sqcup} \{(a,0)\}$ for $i \in \mathcal{N}$. For every $(a,b) \in L_2$ and $r > 0$, let $(\hat{\aleph}, \Gamma)\big((a,b), r\big)$ be the set of all *SPs* of L inside the circle of radius r centered at (a,b), and define $(\hat{\aleph}, \Gamma)_i\big((a,b)\big) = (\hat{\aleph}, \Gamma)\big((a,b), \frac{1}{i}\big)$ for $i \in \mathcal{N}$. The Niemytzki Plane over *SS* is S-regular. Let $(\hat{\mu}, \Gamma) = cl\Bigg((\hat{\aleph}, \Gamma)\big((2,2), \frac{1}{2}\big)\Bigg)$ be a S-closed set. Since $(-5,2) \hat{\notin} (\hat{\mu}, \Gamma)$, it cannot be separated by two disjoint $S\mathfrak{c}$-open sets because the smallest $S\mathfrak{c}$-open set containing $(\hat{\mu},\Gamma)$ is L_2. Hence, the Niemytzki Plane is not a $S\mathfrak{c}$-regular space.

Remark 4.6. It is evident from the definitions that every $S\mathfrak{c}$-$\hat{T}_i$-space is an S-$\hat{\mathtt{T}}_{\mathtt{i}}$-space for $i \in 0,1,2$. However, the converse may not hold, as the following examples demonstrate.

Example 4.7. From Example 4.5, The Niemytzki Plane in soft set theory is S-$\hat{T}_3$-space, but is not $S\mathfrak{c}$-$\hat{T}_0$-space.

Proposition 4.8. *Let $(\Lambda,\hat{\Im},\Gamma)$ be a STS on Λ, then any $S\mathfrak{c}$-$\hat{T}_4$-space is a $S\mathfrak{c}$-$\hat{T}_3$-space.*

Proof. Let $(\Lambda,\hat{\Im},\Gamma)$ be a $S\mathfrak{c}$-$\hat{T}_4$-space. Then, it is $S\mathfrak{c}$-normal and $S\mathfrak{c}$-$\hat{T}_1$-space. Let $(\hat{\mu},\Gamma)$ be any S-closed subset of $(\hat{\Lambda},\Gamma)$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ be a SP in $(\hat{\Lambda},\Gamma)$ with $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\notin} (\hat{\mu},\Gamma)$. Since $S\mathfrak{c}$-$\hat{T}_1$-space is S-$\hat{T}_1$-space. Hence, $\{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\}$ is S-closed. Thus, $(\hat{\mu},\Gamma)$ and $\{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\}$ are disjoint S-closed, by $S\mathfrak{c}$-normality of $(\Lambda,\hat{\Im},\Gamma)$, there exists two disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1,\Gamma)$ and $(\hat{\aleph}_2,\Gamma)$ containing $(\hat{\mu},\Gamma)$ and $\{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\}$, respectively. Therefore, $(\Lambda,\hat{\Im},\Gamma)$ is a $S\mathfrak{c}$-$\hat{T}_3$-space

Remark 4.9. In general, the converse of Proposition 4.8 does not hold.

Example 4.10. Let $(\hat{\Re}^2,\hat{\Im},\Gamma)$ be a STS over $\Re^2$ and $(\hat{\Theta},\Gamma)$ be a SS, where $(\hat{\Theta},\Gamma)=\{(\gamma,\hat{\Theta}(\gamma))$: $\hat{\Theta}(\gamma)=(x,y), y\geq 0, \gamma\in\Gamma\}$, the upper half soft plane with the $\hat{X}$-axis. Let $(\hat{\Theta}_1,\Gamma)=\{(\gamma,\hat{\Theta}_1(\gamma))$: $\hat{\Theta}_1(\gamma)=(x,0),\gamma\in\Gamma\}$, i.e., the $\hat{X}$-axis. Let $(\hat{P}',\Gamma)=(\hat{\Theta},\Gamma)\ \hat{\setminus}\ (\hat{\Theta}_1,\Gamma)$ is S-$\hat{T}_2$-space, but is not $S\mathfrak{c}$-$\hat{T}_0$-space. There exist two SPs $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}=\hat{\mathcal{P}}_{(-1,2)}^{\gamma}\neq\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}=\hat{\mathcal{P}}_{(1,2)}^{\gamma'}$ such that there is not $S\mathfrak{c}$-open set containing either $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ but not $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ or containing $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ but not $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, because the smallest $S\mathfrak{c}$-open set containing either $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ or $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ is $(\hat{P}',\Gamma)$.

Based on the previous theorems and examples, we can derive the following diagram:

$$\begin{array}{ccccccccc}
S\mathfrak{c}\text{–}\hat{T}_4 & \rightarrow & S\mathfrak{c}\text{–}\hat{T}_3 & \rightarrow & S\mathfrak{c}\text{–}\hat{T}_2 & \rightarrow & S\mathfrak{c}\text{–}\hat{T}_1 & \rightarrow & S\mathfrak{c}\text{–}\hat{T}_0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S\text{-}\hat{T}_4 & \longrightarrow & S\text{-}\hat{T}_3 & \longrightarrow & S\text{-}\hat{T}_2 & \longrightarrow & S\text{-}\hat{T}_1 & \longrightarrow & S\text{-}\hat{T}_0.
\end{array}$$

Diagram (ii)

None of the above implications is reversible.

Definition 4.11. Let $(\Lambda,\hat{\Im},\Gamma)$ be a STS, then a SS $(\hat{\aleph},\Gamma)$ is a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) set if for any S-open (resp., $S\mathfrak{c}$-open) soft cover of $(\hat{\aleph},\Gamma)$ has a finite soft subcover of $S\mathfrak{c}$-open (resp., S-open) sets. In particular, a STS is said to be a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) space if $(\hat{\Lambda},\Gamma)$ is a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) set.

Theorem 4.12. *Let $(\Lambda,\hat{\Im},\Gamma)$ be a STS, every $S\mathfrak{c}$-compact set (space) is S-compact.*

Proof. Let $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-compact set in STS $(\Lambda, \hat{\Im}, \Gamma)$, then $(\hat{\aleph}, \Gamma)$ is S-open, by definition 4.11, $(\hat{\aleph}, \Gamma)$ is S-compact.

Remark 4.13. The converse does not always hold. In general, every S-compact space which is not $S\mathfrak{c}$-compact.

Example 4.14. Let $(\hat{\aleph}, \Gamma) = \{(\gamma, [-1, 1])\}$ be a SS, and let $(\hat{\aleph}, \hat{\Im}_{\aleph}, \Gamma)$ be a STS over $\aleph$ and parameterize $\Gamma = \{\gamma\}$ generated by $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$, where

$$(\hat{\aleph}_1, \Gamma) = \{(\gamma, (a, 1]) : -1 < a < 0\},$$
$$(\hat{\aleph}_2, \Gamma) = \{(\gamma, [-1, b)]) : 0 < b < 1\}.$$

Then all SSs of the form $\{(\gamma, (a, b)) : a < 0 < b\})$ are S-open sets in $(\hat{\aleph}, \hat{\Im}_{\aleph}, \Gamma)$. Clearly, $(\hat{\aleph}, \Gamma)$ is a S-compact space because any S-open covering of the two SSs $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ which include $\hat{\mathcal{P}}^{\gamma}_{\{1\}}$ and $\hat{\mathcal{P}}^{\gamma}_{\{-1\}}$ will finite S-open cover $(\hat{\aleph}, \Gamma)$, but it is not $S\mathfrak{c}$-compact because if we take

$$(\hat{\aleph}_3, \Gamma) = \{(\gamma, [-1, 0.5))\},$$
$$(\hat{\aleph}_4, \Gamma) = \{(\gamma, (-0.5, 1])\},$$

is an S-open cover for $(\hat{\aleph}, \Gamma)$ which no finite S-subcover of $S\mathfrak{c}$-open due to $(\hat{\aleph}_3, \Gamma)$ and $(\hat{\aleph}_4, \Gamma)$ are not $S\mathfrak{c}$-open sets such that

$$\begin{aligned}
cl(\hat{\aleph}_3, \Gamma)\hat{\setminus}(\hat{\aleph}_3, \Gamma) &= cl(\gamma, [-1, 0.5))\hat{\setminus}(\gamma, [-1, 0.5)) \\
&= (\gamma, [-1, 1])\hat{\setminus}(\gamma, [-1, 0.5)) \\
&= (\gamma, [0.5, 1]), \\
cl(\hat{\aleph}_4, \Gamma)\hat{\setminus}(\hat{\aleph}_4, \Gamma) &= cl(\gamma, (-0.5, 1])\hat{\setminus}(\gamma, (-0.5, 1]) \\
&= (\gamma, [-1, 1])\hat{\setminus}(\gamma, (-0.5, 1]) \\
&= (\gamma, [-1, -0.5]).
\end{aligned}$$

are not S-countable sets.

Theorem 4.15. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS, every S-compact set (space) is $S\mathfrak{c}^*$-compact.*

Proof. By Definition 4.11 any S-compact set(space) is $S\mathfrak{c}^*$-compact.

Remark 4.16. In general, every $S\mathfrak{c}^*$-compact which is not S-compact.

Example 4.17. Consider $(\Re, \hat{\Im}_{\hat{r}}, \Gamma)$, where $\hat{\Im}_{\hat{r}}$ is the right order ST on $\Re$. Then, $(\hat{\Re}, \hat{\Im}_{\hat{r}}, \Gamma)$ is not S-compact space. Alternatively, the collection of $S\mathfrak{c}$-open set in $(\Re, \hat{\Im}_{\hat{r}}, \Gamma)$ is $\{(\Phi, \Gamma), (\hat{\Re}, \Gamma)\}$. Hence, $(\hat{\Re}, \hat{\Im}_{\hat{r}}, \Gamma)$ is $S\mathfrak{c}^*$-compact.

Remark 4.18. From Theorems 4.12, 4.15 and Examples 4.14, 4.17 the following diagram is acquired:

$$S\mathfrak{c}\text{-compact} \longrightarrow S\text{-compact} \longrightarrow S\mathfrak{c}^{*}\text{-compact}.$$

Diagram (iii)

Remark 4.19. None of these relationships can be reversed.

Remark 4.20. In a *STS* $(\Lambda, \hat{\Im}, \Gamma)$, the $S\mathfrak{c}^{*}$-compactness is not hereditary.

Example 4.21. Let $(\Re, \hat{\Im}_{(\gamma,\{2\})}, \Gamma)$ be the included soft point topological space on $\Re$ by $(\gamma, \{2\})$. Then $(\Re, \hat{\Im}_{(\gamma,\{2\})}, \Gamma)$ is $S\mathfrak{c}^{*}$-compact space, because $(\hat{\Re}, \Gamma)$ is $S\mathfrak{c}$-open finite subcover for any S-open cover of $(\hat{\Re}, \Gamma)$. However, $(\hat{\Re}, \Gamma)\hat{\setminus}(\gamma, \{2\})$ is a soft infinite discrete subspace. Hence, $(\hat{\Re}, \Gamma)\hat{\setminus}(\gamma, \{2\})$ with soft discrete topology is not $S\mathfrak{c}^{*}$-compact.

Remark 4.22. The $S\mathfrak{c}^{*}$-compactness is hereditary with respect to $S\mathfrak{c}$-closed subspace.

Theorem 4.23. *Every $S\mathfrak{c}$-closed subset of $S\mathfrak{c}$-compact (resp. $S\mathfrak{c}^{*}$-compact) space is $S\mathfrak{c}$-compact (resp. $S\mathfrak{c}^{*}$-compact).*

Proof. Let $(\hat{\Theta}, \Gamma)$ be a S-closed subset in a *STS* $(\Lambda, \hat{\Im}, \Gamma)$ and $\hat{\Psi} = \{(\hat{\aleph}, \Gamma)_{\zeta} : \zeta \in \Delta\}$ be a S-open (resp., $S\mathfrak{c}$-open) cover of $(\hat{\Theta}, \Gamma)$, where $(\hat{\aleph}, \Gamma)_{\zeta} \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ is S-open (resp., $S\mathfrak{c}$-open) in $(\hat{\Lambda}, \Gamma)$, for each $\zeta \in \Psi$. Since $(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-closed in $(\hat{\Lambda}, \Gamma)$, then $(\hat{\Lambda}, \Gamma)\hat{\setminus}(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-open in $(\hat{\Lambda}, \Gamma)$. However, any $S\mathfrak{c}$-open set is S-open, then $(\hat{\Theta}, \Gamma)^{c} = (\hat{\Lambda}, \Gamma)\hat{\setminus}(\hat{\Theta}, \Gamma)$ is S-open in $(\hat{\Lambda}, \Gamma)$. Thus $\{(\hat{\aleph}, \Gamma)_{\zeta} : \zeta \in \Omega\} \hat{\sqcup} (\hat{\Theta}, \Gamma)^{c}$ is a S-open (resp., $S\mathfrak{c}$-open) cover for $(\hat{\Lambda}, \Gamma)$. Since $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^{*}$-compact), then there exist $\zeta_1, \zeta_2, ..., \zeta_n$ such that $(\hat{\Lambda}, \Gamma) \hat{\sqsubseteq} \left(\hat{\sqcup}_{i=1}^{n}(\hat{\aleph}, \Gamma)_i\right) \hat{\sqcup} (\hat{\Theta}, \Gamma)^{c}$. Thus $(\hat{\Theta}, \Gamma) \hat{\sqsubseteq} \hat{\sqcup}_{i=1}^{n}(\hat{\aleph}, \Gamma)_i$. Therefore, $(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^{*}$-compact).

Corollary 4.24. *If $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}$-compact and $(\hat{\Theta}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ is S-closed, then $(\hat{\Theta}, \Gamma)$ is S-compact.*

Proof. Obviously by 4.12, since any $S\mathfrak{c}$-compact space is S-compact

Theorem 4.25. *A $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^{*}$-compact) subset of $S\mathfrak{c}$-$\hat{T}_2$-space is $S\mathfrak{c}$-closed.*

Proof.

Let $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^{*}$-compact) subset of $S\mathfrak{c}$-$\hat{T}_2$-space over Λ. If $(\hat{\aleph}, \Gamma) = (\hat{\Lambda}, \Gamma)$, then $(\hat{\Lambda}, \Gamma)$ is S-closed and $(\hat{\Lambda}, \Gamma)\hat{\setminus} int(\hat{\Lambda}, \Gamma) = (\Phi, \Gamma)$ is S-countable set. Hence $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}$-closed. If $(\hat{\aleph}, \Gamma) \neq (\hat{\Lambda}, \Gamma)$. Suppose $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}, \hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} SP(\Lambda)_{\Gamma}$, $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \neq \hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ such that $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\in} (\hat{\aleph}, \Gamma)^{c}$ and $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}, \Gamma)$. Then there are two disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}$ and $(\hat{\aleph}_2, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}}$ such that $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}_2, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}}$, $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma'} \hat{\in} (\hat{\aleph}_1, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}$ with $(\hat{\aleph}_2, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}} \hat{\sqcap} (\hat{\aleph}_1, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}} = (\Phi, \Gamma)$. Let $\hat{\Psi} = \{(\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}_i}^{\gamma'}} : i \in I\}$, then $\hat{\Psi}$ is S-open (resp., $S\mathfrak{c}$-open) cover of $(\hat{\aleph}, \Gamma)$. Since $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact (resp.,

$S\mathfrak{c}^*$-compact), then there are $\hat{\mathcal{P}}_{\mathfrak{b}_1}^{\gamma'}, \hat{\mathcal{P}}_{\mathfrak{b}_2}^{\gamma'}, ..., \hat{\mathcal{P}}_{\mathfrak{b}_n}^{\gamma'}$, such that $(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} \hat{\sqcup}_{i=1}^{n} (\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}_i}^{\gamma'}} = (\hat{\mu}, \Gamma)$. Let $(\hat{\nu}, \Gamma) = \hat{\sqcap}_{i=1}^{n} (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_i}^{\gamma}}$, by Corollary 3.20, $(\hat{\nu}, \Gamma)$ is a $S\mathfrak{c}$-open set containing $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, clearly $(\hat{\nu}, \Gamma)$ $\hat{\sqcap}$ $(\hat{\mu}, \Gamma) = (\Phi, \Gamma)$, so that $(\hat{\nu}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)$, hence $\hat{\mathcal{P}}_{\mathfrak{a}_i}^{\gamma}$ is a $S\mathfrak{c}$-interior point of $(\hat{\Lambda}, \Gamma) \hat{\setminus}$ $(\hat{\aleph}, \Gamma)$, Therefore $(\hat{\Lambda}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-open. Hence, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed.

Theorem 4.26. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-regular.*

Proof. Let $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\in} SP(\Lambda)_{\Gamma}$ and $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-closed set not containing $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$. Since $S\mathfrak{c}$-$\hat{T}_2$, for each $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}, \Gamma)$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\in} (\hat{\aleph}, \Gamma)^c$ there are two disjoint $S\mathfrak{c}$-open sets $(\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}$ and $(\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}}$ containing $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, respectively. Suppose $\hat{\Psi} = \{(\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}\colon \hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}, \Gamma)\}$ is a $S\mathfrak{c}$-open cover of $(\hat{\aleph}, \Gamma)$. By Theorem 4.25, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact, and therefore there is a $S\mathfrak{c}$-open finite subcover $\{(\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_1}^{\gamma}}, (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_2}^{\gamma}}, ..., (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_n}^{\gamma}}\} \hat{\sqsubseteq} \hat{\Psi}$. Thus, $(\hat{\nu}, \Gamma) = \hat{\sqcup}_{i}^{n} (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}_i}^{\gamma'}}$ and $(\hat{\mu}, \Gamma)$ $= \hat{\sqcap}_{i}^{n} (\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_i}^{\gamma}}$ are disjoint $S\mathfrak{c}$-open sets containing $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, hence $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-regular.

Corollary 4.27. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-$\mathtt{T}_3$-space.*

Proof. Obviously.

Theorem 4.28. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}^*$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is S-regular.*

Proof. The same of Theorem 4.26.

Corollary 4.29. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}^*$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is S-$\mathtt{T}_3$-space.*

Theorem 4.30. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-normal (resp., S-normal).*

Proof. The same of Theorem 4.26.

5. $S\mathfrak{c}$-Continuous and $S\mathfrak{c}^*$-Continuous Functions

This section introduces two definitions of continuity utilizing $S\mathfrak{c}$-open set namely, $S\mathfrak{c}$-continuous, $S\mathfrak{c}^*$-continuous. Moreover, the concept of $S\mathfrak{c}$-homeomorphism and $S\mathfrak{c}^*$-homeomorphism via the concept of $S\mathfrak{c}$-open sets are studied. In addition, some of their properties with $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact spaces are discussed.

Definition 5.1. A S-function $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \rightarrow (\Delta, \hat{\Im}', \Gamma')$ is said to be $S\mathfrak{c}$-*continuous* (resp., $S\mathfrak{c}^*$-continuous) if $\hat{\varphi}_{\sigma\alpha}^{-1}(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-open (resp., S-open) in $(\hat{\Lambda}, \Gamma)$ for any S-open (resp., $S\mathfrak{c}$-open) subset $(\hat{\Theta}, \Gamma)$ in $(\hat{\Delta}, \Gamma')$.

Theorem 5.2. *Every $S\mathfrak{c}$-continuous function is S-continuous, and every S-continuous function is $S\mathfrak{c}^*$-continuous.*

Proof. Obviously, from the definition 5.1

Remark 5.3. The converse may not be true as shown by the following two examples.

Example 5.4. Let $(\Re, \hat{\Im}_I, \Gamma)$ be the excluded soft set topological space on $\Re$ by I. Then the identity S-function $\hat{\mathcal{I}}_{\sigma\alpha} : (\Re, \hat{\Im}_I, \Gamma) \to (\Re, \hat{\Im}_I, \Gamma)$ is S-continuous function, which is not $S\mathfrak{c}$-continuous, because $(\hat{Q}, \Gamma) \hat{\in} \hat{\Im}_I$ is S-open and $\hat{\mathcal{I}}_{\sigma\alpha}^{-1}(\hat{Q}, \Gamma) = (\hat{Q}, \Gamma)$ is not $S\mathfrak{c}$-open, because $cl(\hat{Q}, \Gamma) \hat{\setminus} (\hat{Q}, \Gamma) = (\hat{\Re}, \Gamma) \hat{\setminus} (\hat{Q}, \Gamma) = (\hat{I}, \Gamma)$ is S-uncountable set.

Example 5.5. In Example 4.17, $(\Re, \hat{\Im}_r, \Gamma')$ is the right order soft topology on the set of all soft real numbers $(\hat{\Re}, \Gamma')$. The family of all $S\mathfrak{c}$-open set in $(\Re, \hat{\Im}_r, \Gamma')$ is $\{(\Phi, \Gamma'), (\hat{\Re}, \Gamma')\}$ only. Consider $(\hat{\Re}, \hat{\Im}_{cof}, \Gamma)$, where $\hat{\Im}_{cof}$ is the finite complement soft topology on the soft set of all soft real numbers $(\hat{\Re}, \Gamma')$. Then the identity S-function $\hat{\mathcal{I}}_{\sigma\alpha} : (\hat{\Re}, \hat{\Im}_{cof}, \Gamma) \to (\hat{\Re}, \hat{\Im}_r, \Gamma')$ is $S\mathfrak{c}^*$-continuous function, which is not S-continuous,

Remark 5.6. It is clear, from the definitions of S-continuous, , and Examples 5.4, 5.5, the following diagram is obtained:

$S\mathfrak{c}$-continuity $\longrightarrow$ S-continuity $\longrightarrow$ $S\mathfrak{c}'$-continuity. **Diagram (iv)**

None of the above implications is reversible.

Definition 5.7. Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ be a S-continuous function. Then, $\hat{\varphi}_{\sigma\alpha}$ is a $S\mathfrak{c}$-open function if for any $S\mathfrak{c}$-open subset $(\hat{\Theta}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$, we have $\hat{\varphi}_{\sigma\alpha}(\hat{\Theta}, \Gamma)$ is a S-open subset in $(\hat{\Delta}, \Gamma')$. Moreover, $\hat{\varphi}_{\sigma\alpha}$ is said to be $S\mathfrak{c}$-closed function if any $S\mathfrak{c}$-closed subset $(\hat{\Sigma}, \Gamma)$ $\hat{\sqsubseteq}$ $(\hat{\Lambda}, \Gamma)$, we have $\hat{\varphi}_{\sigma\alpha}$ $(\hat{\Sigma}, \Gamma)$ is S-closed subset in $(\hat{\Delta}, \Gamma')$.

Theorem 5.8. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ be $S\mathfrak{c}$-continuous (resp., $S\mathfrak{c}^*$-continuous), onto S-function and $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}^*$-compact, then $(\hat{\Delta}, \Gamma')$ is $S\mathfrak{c}^*$-compact.*

Proof. Let $\{(\hat{\Theta}, \Gamma')_\alpha : \alpha \in \Omega\}$ be a $S\mathfrak{c}$-open cover of $(\hat{\Delta}, \Gamma')$. Since $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-continuous (resp., $S\mathfrak{c}^*$-continuous) and any $S\mathfrak{c}$-open set is S-open, then $\hat{\varphi}_{\alpha}^{-1}((\hat{\Theta}, \Gamma')_\alpha)$ is $S\mathfrak{c}$-open (resp., S-open) in $(\hat{\Lambda}, \Gamma)$ for each $\alpha \in \Omega$. Since $(\hat{\Delta}, \Gamma') \hat{\sqsubseteq} \hat{\bigsqcup}_{\alpha\in\Omega}(\hat{\Theta}, \Gamma')_\alpha$, then $(\hat{\Lambda}, \Gamma)$ $= \hat{\varphi}_{\sigma\alpha}^{-1}(\hat{\Delta}, \Gamma') \hat{\sqsubseteq} \hat{\varphi}_{\sigma\alpha}^{-1}(\bigcup_{\alpha\in\Omega}(\hat{\Theta}, \Gamma')_\alpha) = \hat{\bigsqcup}_{\alpha\in\Omega} \hat{\varphi}_{\alpha}^{-1}((\hat{\Theta}, \Gamma')_\alpha)$, that is means $\{\hat{\varphi}_{\alpha}^{-1}((\hat{\Theta}, \Gamma')_\alpha) :$ $\alpha \in \Omega\}$ is $S\mathfrak{c}$-open cover of $(\hat{\Lambda}, \Gamma)$. Then, by the $S\mathfrak{c}^*$-compactness of $(\hat{\Lambda}, \Gamma)$, there exist $\alpha_1, \alpha_2, \cdots \alpha_n \in \Omega$ such that $\hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_1}) \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_2}) \hat{\sqcup} \ldots \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_n}) = (\hat{\Lambda}, \Gamma)$, then $\hat{\varphi}_{\sigma\alpha}[\hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_1}) \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_2}) \hat{\sqcup} \ldots \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_n})] = \hat{\varphi}_{\sigma\alpha}(\hat{\Lambda}, \Gamma)$, thus $\hat{\varphi}_{\sigma\alpha}$ $\big(\hat{\varphi}_{\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_1})\big) \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}\big(\hat{\varphi}_{\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_2})\big) \hat{\sqcup} \ldots \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}\big(\hat{\varphi}_{\sigma\alpha}^{-1}((\hat{\Theta}, \Gamma')_{\alpha_n})\big) = (\hat{\Delta}, \Gamma')$, then $(\hat{\Theta}, \Gamma')_{\alpha_1}$ $\hat{\sqcup}$ $(\hat{\Theta}, \Gamma')_{\alpha_2} \hat{\sqcup} \ldots \hat{\sqcup} (\hat{\Theta}, \Gamma')_{\alpha_n} = (\hat{\Delta}, \Gamma')$. Hence, $\{(\hat{\Theta}, \Gamma')_{\alpha_1} \hat{\sqcup} (\hat{\Theta}, \Gamma')_{\alpha_2} \hat{\sqcup} \ldots \hat{\sqcup} (\hat{\Theta}, \Gamma')_{\alpha_n}\}$ is a finite S-subcover of S-open sets for $(\hat{\Delta}, \Gamma')$. Therefore, $(\Delta, \hat{\Im}', \Gamma')$ is a $S\mathfrak{c}^*$-compact space.

Corollary 5.9. *$S\mathfrak{c}^*$-compactness is a topological property.*

From the previous theorem and the Diagram (*iii*), we have the following corollary:

Corollary 5.10.

i) *$S\mathfrak{c}$-continuous image of S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact) is S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact);*

ii) *$S\mathfrak{c}^*$-continuous image of S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact) is S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact);*

iii) *S-continuous image of $S\mathfrak{c}$-compact is $S\mathfrak{c}$-compact.*

Theorem 5.11. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is onto $S\mathfrak{c}$-continuous function and $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}^*$-compact, then $(\hat{\Delta}, \Gamma')$ is S-compact.*

Proof. Same the proof of Theorem 5.8.

Theorem 5.12. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ be $S\mathfrak{c}$-continuous, $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}^*$-compact, and $(\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-$\hat{T}_2$-space, then $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-closed function.*

Proof. Let $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-closed subset in $(\hat{\Lambda}, \Gamma)$. Since $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}^*$-compact, then from Theorem 5.9, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}^*$-compact. Since the image of a $S\mathfrak{c}^*$-compact space is $S\mathfrak{c}^*$-compact under a $S\mathfrak{c}$-continuous function (see Corollary 5.10). Hence, $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}^*$-compact. Since every $S\mathfrak{c}^*$-compact subspace of a $S\mathfrak{c}$-$\hat{T}_2$-space is $S\mathfrak{c}$-closed (see Theorem 4.25) this implies that $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed. But any $S\mathfrak{c}$-closed set is S-closed. Therefore, $\hat{\varphi}_{\sigma\alpha}$ is a $S\mathfrak{c}$-closed function.

From Theorem 5.12, and Diagram (iv), we have the following corollaries:

Corollary 5.13. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-continuous, $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-compact (resp., S-compact), and $(\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-$\hat{T}_2$-space, then $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-closed function.*

Corollary 5.14. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is S-continuous, $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-compact, and $(\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-$\hat{T}_2$-space, then $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-closed function.*

Definition 5.15. A bijection S-function $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is said to be $S\mathfrak{c}$-*homeomorphism* (resp., $S\mathfrak{c}^*$-homeomorphism) if and $\hat{\varphi}_{\sigma\alpha}$ and $\hat{\varphi}^{-1}_{\alpha}$ are $S\mathfrak{c}$-continuous (resp., $S\mathfrak{c}^*$-continuous).

$$S\mathfrak{c}\text{-homeomorphism} \longrightarrow S\text{-homeomorphism} \longrightarrow S\mathfrak{c}^*\text{-homeomorphism.}$$

Diagram (v)

Proposition 5.16. *Every $S\mathfrak{c}$-homeomorphism function is S-homeomorphism, and every S-homeomorphism function is $S\mathfrak{c}^*$-homeomorphism.*

Proof. It is clear, from the definition 5.15.

Remark 5.17. The converse of Proposition 5.16 may not be hold in general as shown by the following two examples.

Example 5.18. See Example 5.4.

Example 5.19. Consider $(\hat{\Re}, \hat{\Im}_r, \Gamma')$ is the right order soft topology on the soft set of all soft real numbers $(\hat{\Re}, \Gamma')$, and $(\Re, \hat{\Im}_{ID}, \Gamma)$, where $\hat{\Im}_{ID}$ is the indiscrete soft topology on the soft set of all soft real numbers $(\hat{\Re}, \Gamma)$. Then the identity S-function $\hat{\mathcal{I}}_{\sigma\alpha} : (\Re, \hat{\Im}_{ID}, \Gamma) \to (\hat{\Re}, \hat{\Im}_r, \Gamma')$ is $S\mathfrak{c}^*$-homeomorphism function, which is not S-continuous because The family of all $S\mathfrak{c}$-open sets in $(\hat{\Re}, \hat{\Im}_r, \Gamma')$ and $(\Re, \hat{\Im}_{ID}, \Gamma)$ are $\{(\Phi, \Gamma), (\hat{\Re}, \Gamma)\}$ only.

Theorem 5.20. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact soft topological space and let $(\Delta, \hat{\Im}', \Gamma')$ be a $S\mathfrak{c}$-$\hat{T}_2$-space. Then any $S\mathfrak{c}^*$-continuous S-bijection $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is a $S\mathfrak{c}^*$-homeomorphism.*

Proof. Let $(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ be a $S\mathfrak{c}$-closed set. By Theorem 5.9, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact, and therefore, $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact by Corollary 5.10. By Theorem 4.25, we have that $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed, as required.

6. **Conclusions**

This article contributes to expanding the literature on new soft topological properties with a new class of soft open sets. The results show that some soft topological properties, such as compactness, continuity and others, can be generalized, leading to new examples and properties that enhance the understanding of soft topological spaces. On the other hand, the paper introduces the definitions of novel types of soft sets namely, $S\mathfrak{c}$-open and $S\mathfrak{c}$-closed sets and discusses their fundamental properties. So, it presents a study about soft separation axioms defined by these new soft sets. Additionally, the paper defines $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact in *STS* accompanied by properties and counterexamples related to these definitions. Furthermore, it introduces the concept of $S\mathfrak{c}$-continuous, $S\mathfrak{c}$-homeomorphism and $S\mathfrak{c}^*$-homeomorphism. Moreover, it investigate new kinds of regularity and normality in*STSs* using $S\mathfrak{c}$-open(closed) sets. According to the obtained results, many properties of these concepts in *TSs* are still valid for soft topological structures. The aim is to further develop the soft topological concepts utilizing $S\mathfrak{c}$-open and $S\mathfrak{c}$-closed sets, including $S\mathfrak{c}$-paracompact spaces, $S\mathfrak{c}$-connected spaces, as well as hyper $S\mathfrak{c}$-open and hyper $S\mathfrak{c}$-closed sets.

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On the algebra of possibly paraconsistent sets

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Abstract. In this paper we define the notion of *possibly paraconsistent sets*. We introduce algebraic operations on them and we analyze their properties. Moreover, it is shown that our class can be considered as isomorphic to the classes of intuitionistic and weak rough sets (with the assumption that intersection, union and complement are understood in an appropriate manner). Hence, the framework of paraconsistent sets can be treated as a new semantics for three-valued logic. Two less typical operations on possibly paraconsistent sets are studied too. They do not give us de Morgan algebra but rather a bisemilattice with only one absorption law. Finally, we pay attention to the fact that possibly paraconsistent sets can be treated as neutrosophic crisp sets of type 2. As for the exact isomorphism, it should be a matter of further research.

Keywords: possibly paraconsistent sets, neutrosophic sets, intuitionistic sets, weak rough sets

1. Introduction

Assume that our initial universe X is divided into three mutually disjoint parts. Then we can treat each such triple as a new non-classical set (that is, some kind of "data container"). Then we can define various operations on these sets and study their properties.

For example, we may assume that our triple consists of two distinguished sets A_T and A_F such that their intersection is empty while the third component is (obviously but also tacitly) $(A_T \cup A_F)^c$. These are *intuitionistic sets* in the sense of Çoker (see [3] and [4]). We may define their complement, union and intersection in a very natural way that leads to the structure of de Morgan algebra. In fact, intuitionistic sets have more to do with the three-valued logic of Łukasiewicz than with the intuitionistic logic of Brouwer and Heyting. Hence, some authors refuse to call them "intuitionistic". Instead of this, they use the notion of *orthopairs*. For example, in [2] Ciucci analyzed orthopairs in the context of granular computing.

Alternatively, we may assume that our universe consists of $A_1 \subseteq A_2$ and A_2^c (where the last component is not explicitly mentioned). These are *flou sets* of Gentilhomme (see e.g. [12]), known as *double sets* (see [16] for their soft version) or *weak rough sets* too (the last notion was used by Yong-jin in [18] together with the concept of weak rough numbers). Again, we may easily give them a structure of de Morgan algebra. Weak rough sets can be considered as isomorphic to intuitionistic sets.

Another approach is the one that is typical for *neutrosophic crisp sets of type 2*. This is a subclass of the wider algebra of *neutrosophic crisp sets* that has been introduced by Salama and Smarandache (see [13]). In this case we assume that we have three independent and mutually disjoint sets A_1, A_2 and A_3 and that their union is the whole X.

In the rest of the paper we shall go back to the concepts mentioned above. Now we can point out that all these ideas are crisp variants of some "fuzzy" solutions and due to this fact they are used to model the phenomenon of uncertainty and ambiguity. For example, Çoker sets refer to the notion of Atanassov's *intuitionistic fuzzy sets* (see [1]).

In this paper a new semantics is given. While intuitionistic sets sets are based on the supposition that our distinguished sets A_T and A_F have empty intersection, our idea is to consider A and $\sim A$ such that their intersection can be non-empty. Moreover, we assume that $A^c \subseteq \sim A$. In intuitionistic sets we assume that A_F is somewhat *weaker* than the classical complement of A_T. It gathers exactly those elements that are openly *rejected*. They do not belong to A_T. On the other hand, $(A_T \cup A_F)^c$ consists of those elements that are *neutral*. In our approach, $\sim A$ is *stronger* (at least in some sense) than the complement of A. Thus, in $A \cap \sim A$ we have those elements that are ambiguous.

The idea is to establish some appropriate operations in this framework. We prove that possibly paraconsistent sets equipped with these operations are isomorphic with other classes mentioned above. In this sense, our paper fills some gap in understanding of three-valued logics and tripartite division of space.

However, we also study some less typical operations that do not give us de Morgan algebra. This part of our study refers to the concept of bisemilattices. They are much weaker than de Morgan algebras and even weaker than lattices.

2. On the algebra of possibly paraconsistent sets

In this section we define our structures and we investigate their algebraic properties. In particular, we propose somewhat natural partial order together with the operations of union, intersection and complement.

2.1. *Basic notions*

We need to define basic components of our semantics.

Definition 2.1. Let $X \neq \emptyset$. Assume that $A, \sim A \subseteq X$ and $A^c \subseteq \sim A$. Then the ordered pair $\mathcal{A} = (A, \sim A)$ is called a *possibly paraconsistent set* on X.

Example 2.2. Let $X = \{a, b, c, d, e, f, g\}$, $A = \{a, b, g\}$ and $\sim A = \{b, c, d, e, f\}$. Then $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X. In this case $A \cap \sim A = \{b\}$.

Analogously, if $B = \{d, e\}$ and $\sim B = \{a, b, c, f, g\}$, then $\mathcal{B} = (B, \sim B)$ is a possibly paraconsistent set. Note, however, that $\mathcal{B}$ is an intuitionistic set too (because in this case $\sim B = B^c$). Clearly, each intuitionistic set is a possibly paraconsistent set too. Obviously, the converse is not true.

Example 2.3. Let $X = \mathbb{R}_+ \cup \{0\}$ (that is, the set of positive real numbers with zero). Assume that $A = [0, 100)$ and $\sim A = [90, +\infty)$. Then $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set.

Example 2.4. Let $X = \mathbb{N}$ (that is, the set of natural numbers with the assumption that zero is natural). Let $A = 2\mathbb{N}$ (even numbers) while $\sim A = (2\mathbb{N})^c \cup 8\mathbb{N}$ (odd numbers together with multiplicities of 8). Now $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X.

The following observation is simple but in some sense important. It explains the idea of paraconsistency in our framework.

Remark 2.5. Note that if $\mathcal{A}$ is a possibly paraconsistent set on X, then $A \cup \sim A = X$ but $A \cap \sim A$ is not necessarily equal to $\emptyset$.

This shows that the *internal* structure of paraconsistent sets satisfies the law of the excluded middle (when $\sim A$ is treated as an *ersatz* of complement) but it does not satisfy the law of noncontradiction. Contrary to this, in the semantics of intuitionistic sets we have $A_T \cap A_F = \emptyset$ but $A_T \cup A_F$ may be different than X.

Clearly, this internal structure is a different thing than the relationship between possibly paraconsistent (or intuitonistic) sets as such. This will be discussed later.

Let us define some binary algebraic operations on possibly paraconsistent sets.

Definition 2.6. Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. Then we define their:

(1) *Union*: $\mathcal{A} \cup \mathcal{B} = (A \cup B, \sim A \cap \sim B)$.
(2) *Strong union*: $\mathcal{A} \vee \mathcal{B} = (A \cup B, \sim A \cup \sim B)$.
(3) *Intersection*: $\mathcal{A} \cap \mathcal{B} = (A \cap B, \sim A \cup \sim B)$.

Lemma 2.7. *The operations defined in Def. 2.6 return possibly paraconsistent sets.*

Proof: Let us discuss all the cases.

(1) Check $\cup$. We need to ensure that the complement of the left component is contained in the right component. We have $(A \cup B)^c = A^c \cap B^c \subseteq \sim A \cap \sim B$.

(2) Check $\vee$. We have $(A \cup B)^c = A^c \cap B^c \subseteq \sim A \cap \sim B \subseteq \sim A \cup \sim B$.

(3) Check $\cap$. We have $(A \cap B)^c = A^c \cup B^c = \sim A \cup \sim B$.

Remark 2.8. Note that the following pair: $(A \cap B, \sim A \cap \sim B)$ is not necessarily a possibly paraconsistent set. This function can be corrected but this will be done later.

2.2. *About orderings*

Assume that our algebra is of the signature $(X, \cup, \cap)$ where $\cup$ refers to $+$ (that is, join) operation in lattices and other structures, while $\cap$ refers to $\cdot$ (that is, meet).

If so, then we can reconstruct the following two orderings:

Definition 2.9. Assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. We define the following relations:

(1) $\mathcal{A} \subseteq_{\cup} \mathcal{B}$ if and only if $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$.

(2) $\mathcal{A} \subseteq_{\cap} \mathcal{B}$ if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$.

Lemma 2.10. *Both orderings defined in Def. 2.9 are equal and they can be described as* $A \subseteq B$ *and* $\sim B \subseteq \sim A$.

Proof: Let $\mathcal{A} \subseteq_{\cup} \mathcal{B}$. Then $(A \cup B, \sim A \cap \sim B) = (B, \sim B)$. Thus, $A \cup B = B$ and $\sim A \cap \sim B = \sim B$. Hence, $A \subseteq B$ and $\sim B \subseteq \sim A$.

Now let $A \subseteq_{\cap} B$. Thus, $(A \cap B, \sim A \cup \sim B) = (A, \sim A)$. Then $A \cap B = A$ and $\sim A \cup \sim B = \sim A$. Hence, $A \subseteq B$ and $\sim B \subseteq \sim A$.

Now we can use only one symbol $\subseteq$ to denote our partial order (it is easy to check that this relation satisfies all the properties of partial order).

Remark 2.11. What about $\vee$ operator? Assume that it plays the role of join. This assumption is natural because it is based on two classical unions. Then we could write that $\mathcal{A} \subseteq_{\vee} \mathcal{B}$ if and only if $\mathcal{A} \vee \mathcal{B} = \mathcal{B}$. Thus $A \cup B = B$ and $\sim A \cup \sim B = \sim B$. Hence, $A \subseteq B$ and $\sim A \subseteq \sim B$.

2.3. *Complement*

We propose the following definition.

Definition 2.12. Let $\mathcal{A}$ be a possibly paraconsistent set on X. Then the *complement* of $\mathcal{A}$ is defined as $\mathcal{A}^c = (\sim A, A)$.

Of course this complement forms a new possibly paraconsistent set. Contrary to this, $\neg \mathcal{A} = (A^c, (\sim A)^c)$ (another hypothetical candidate for complement) is a possibly paraconsistent set if and only if $\sim A = A^c$.

2.4. *Distinguished sets and algebraic identities*

We can discuss at least three *distinguished* sets, namely: $\tilde{\emptyset} = (\emptyset, X)$, $\tilde{X} = (X, \emptyset)$ and $\overline{X} = (X, X)$.

Lemma 2.13. *Let $\mathcal{A}$ be a possibly paraconsistent set on X. Then the following properties are true:*

(1) $(\tilde{\emptyset})^c = \tilde{X}$ *and* $(\tilde{X})^c = \emptyset$.
(2) $(\overline{X})^c = (X, X)$.
(3) $\mathcal{A} \subseteq \tilde{X}$ *and* $\tilde{\emptyset} \subseteq \mathcal{A}$ *for every possibly paraconsistent set* $\mathcal{A}$.
(4) $\mathcal{A} \subseteq \overline{X}$ *if and only if* $\mathcal{A} = (A, X)$.
(5) $\mathcal{A} \subseteq_{\vee} \overline{X}$ *fo every paraconsistent set* $\mathcal{A}$.
(6) $\mathcal{A} \cap \tilde{\emptyset} = \tilde{\emptyset}$, $\mathcal{A} \cup \tilde{\emptyset} = \mathcal{A}$.
(7) $\mathcal{A} \cap \tilde{X} = \mathcal{A}$, $\mathcal{A} \cup \tilde{X} = \tilde{X}$.
(8) $\mathcal{A} \cap \overline{X} = (A, X)$ *and* $\mathcal{A} \cup \overline{X} = (X, \sim A)$.

Proof:

(1) This is obvious.
(2) This is simple too.
(3) Let $\mathcal{A} = (A, \sim A)$. Then $A \subseteq X$ and $\emptyset \subseteq \sim A$. The second fragment is analogous.
(4) Again, let $\mathcal{A} = (A, \sim A) \subseteq (X, X)$. Then $A \subseteq X$ (it is trivially true for any classical set A) and $X \subseteq \sim A$. This means that $X = \sim A$.
(5) Clearly, both A and $\sim A$ are contained in X.
(6) $\mathcal{A} \cap \tilde{\emptyset} = (A \cap \emptyset, \sim A \cup X) = (\emptyset, X) = \tilde{\emptyset}$. The second case is similar.
(7) Similar to the preceding point.
(8) $\mathcal{A} \cap \overline{X} = (A \cap X, \sim A \cup X) = (A, X)$. The second case is similar.

The next theorem deals with the essential properties of any de Morgan algebra.

Theorem 2.14. *Suppose that $X \neq \emptyset$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are possibly paraconsistent sets on X. Then the following identities are true:*

(1) $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ *and* $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$ *(commutativity).*
(2) $\mathcal{A} \cap \mathcal{A} = \mathcal{A}$ *and* $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$ *(idempotence).*
(3) $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$ *and* $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$ *(de Morgan laws).*
(4) $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$ *and* $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$ *(distributivity laws).*
(5) $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$ *and* $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$ *(absorption laws).*

Proof:

(1) Obvious by the commutativity of classical union and intersection.

(2) Obvious by the idempotence of classical union and intersection.

(3) $(\mathcal{A} \cap \mathcal{B})^c = (A \cap B, \sim A \cup \sim B)^c = (\sim A \cup \sim B, A \cap B) = (\sim A \cup \sim B, A \cap B) = (\sim A, A) \cup (\sim B, B) = \mathcal{A}^c \cup \mathcal{B}^c$. The second case is similar.

(4) $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = \mathcal{A} \cap (B \cup C, \sim B \cap \sim C) = (A \cap (B \cup C)), \sim A \cup (\sim B \cap \sim C) = ((A \cap B) \cup (A \cap C), (\sim A \cup \sim B) \cap (\sim A \cup \sim C)) = (A \cap B, \sim A \cup \sim B) \cup (A \cap C, \sim A \cup \sim C) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$. The second case is similar.

(5) For example, $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = (A, \sim A) \cap (A \cup B, \sim A \cap \sim B) = (A \cap (A \cup B), \sim A \cup (\sim A \cap \sim B)) = (A, \sim A)$ by classical absorption laws. The second case is similar.

Remark 2.15. One can check that the law of the excluded middle is not true. For example, take $X = \{a, b, c, d, e\}$ and $\mathcal{A} = (\{a, c, d\}, \{a, b, d, e\})$. Then $\mathcal{A}^c = (\{a, b, d, e\}, \{a, c, d\})$ and their union is $(X, \{a, d\}) \neq \tilde{X}$. Analogously, take the intersection which is $(\{a, d\}, X) \neq \tilde{\emptyset}$.

3. **Relationship with Çoker's intuitionistic sets**

3.1. *Initial notions*

Some notions are necessary to show the correspondence between possibly paraconsistent and intuitionistic sets in a coherent way.

Definition 3.1. Let $X \neq \emptyset$ and suppose that $\mathcal{A}$ and $\mathcal{B}$ are two possibly paraconsistent sets on X. Then let us identify:

(1) $\mathcal{A}_\sim$ with $\sim A$ (that is, with the right component of $\mathcal{A}$).

(2) $(\mathcal{A} \cap \mathcal{B})_\sim$ and $(\mathcal{A} \vee \mathcal{B})_\sim$ with $\sim A \cup \sim B$.

(3) $(\mathcal{A} \cup \mathcal{B})_\sim$ with $\sim A \cap \sim B$.

(4) $(\mathcal{A}^c)_\sim$ with A.

Definition 3.2. Let $X \neq \emptyset$ and assume that $\mathcal{A}$ is a possibly paraconsistent set on X. Then let us use the following denotation: $\widehat{\mathcal{A}} = A \setminus \mathcal{A}_\sim = \{x \in X : x \notin \mathcal{A}_\sim\}$.

Obviously, $\widehat{\mathcal{A}} \subseteq A$. Moreover, it is clear that $\widehat{\mathcal{A}} = (\mathcal{A}_\sim)^c$.

Example 3.3. Let $X = \{a, b, c, d, e, f, g\}$, $\mathcal{A} = (\{a, c, d, e\}, \{b, c, d, f, g\})$ and $\mathcal{B} = (\{c, e, f\}, \{a, b, d, e, g\})$. Then (among other relationships that can be found):

(1) $\mathcal{A}_\sim = \{b, c, d, f, g\}$, $\widehat{\mathcal{B}} = B \setminus \mathcal{B}_\sim = B \setminus \sim B = \{c, f\}$.

(2) $(\mathcal{A} \cap \mathcal{B})_\sim = X$ and $(\mathcal{A} \cup \mathcal{B})_\sim = \{b, d, g\}$.

(3) $\widehat{\mathcal{A} \cup \mathcal{B}} = (A \cup B) \setminus (\mathcal{A} \cup \mathcal{B})_\sim = \{a, c, d, e, f\} \setminus \{b, d, g\} = \{a, c, e, f\}$.

Note that we write $\widehat{\mathcal{A}}$ and not $\widehat{A}$. This is because the exact form of this set depends not only on A but also on $\sim A$. And this is determined by the internal structure of $\mathcal{A}$. Due to the same reason we write $(\mathcal{A} \cap \mathcal{B})_\sim$ instead of $(A \cap B)_\sim$ or $\sim (A \cap B)$ (the same for union).

3.2. *Transforming function*

Now let us introduce the following function.

Definition 3.4. Let $X \neq \emptyset$. Assume that $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X. Then let $\mathbf{f}$ be a function that assigns to $\mathcal{A}$ an intuitionistic set of the form $\mathbf{f}(\mathcal{A}) = (\widehat{\mathcal{A}}, A^c)$.

We assume that it is visible that $\mathbf{f}(\mathcal{A})$ is an intuitionistic set on X. It means that $\widehat{\mathcal{A}} \cap A^c = \emptyset$. This is true by the very definition of $\widehat{\mathcal{A}}$.

We see that in fact we divided our paraconsistent sets into three parts. The first one contains those elements that are strictly in A. The second consists of those points that are beyond A. These two parts are openly mentioned. We assume tacitly that the third part is the complement of their union, that is $(\hat{\mathcal{A}} \cup A^c)^c$. This is the area of (internal) paraconsistency.

Example 3.5. Let $X = \{a, b, c, d, e, f, g\}$, $\mathcal{A} = (\{a, b\}, \{b, c, d, e, f, g\})$ and $\mathcal{B} = (\{b, c, f, g\}, \{a, d, e, f, g\})$. Then $\mathbf{f}(\mathcal{A}) = (\{a\}, \{c, d, e, f, g\})$

Lemma 3.6. *Let $X \neq \emptyset$, $\mathcal{A}$ be a possibly paraconsistent set on X and $\mathbf{f}$ (defined as above) be our transforming function. Then $\mathbf{f}$ is one-to-one and surjective.*

Proof: Let us analyze two aspects mentioned.

(1) Suppose that $\mathcal{A} \neq \mathcal{B}$ but $\mathbf{f}(\mathcal{A}) = \mathbf{f}(\mathcal{B})$. This means that $\widehat{\mathcal{A}} = \widehat{\mathcal{B}}$ and $A^c = B^c$. The second equality implies that $A = B$. Hence, we can write that $\mathcal{A} = (A, \sim A)$ and $\mathcal{B} = (A, \sim B)$. Suppose that $\sim A \neq \sim B$. But $\sim A = A^c \cup (\widehat{\mathcal{A}})^c = B^c \cup (\widehat{\mathcal{B}})^c = \sim B$.

(2) Assume that we have an intuitionistic set on X of the form $\mathfrak{A} = (A_T, A_F)$ where $A_T \cap A_F = \emptyset$. We are looking for such possibly paraconsistent set $\mathcal{A} = (A, \sim A)$ that $\mathbf{f}(A, \sim A) = (\widehat{\mathcal{A}}, A^c) = (A_T, A_F)$. Thus $A^c = A_F$ and $A = (A_F)^c$. On the other hand $\widehat{\mathcal{A}} = A_T$. Thus $(\widehat{\mathcal{A}})^c = (A_T)^c$. This is important because now we can write that $\sim A = A^c \cup (\widehat{\mathcal{A}})^c = A_F \cup (A_T)^c$.

    Now $A^c = A_F \subseteq A_F \cup (A_T)^c = \sim A$ so our set is indeed a possibly paraconsistent one.

    One can check the reasoning presented above: $\mathbf{f}(\mathcal{A}) = \mathbf{f}(A_F^c, A_F \cup A_T^c) = (\widehat{\mathcal{A}}, ((A_F)^c)^c) = (\{x \in X : x \in (A_F)^c \text{ and } x \notin (A_F \cup (A_T)^c)\}, A_F) = (\{x \in X : x \notin A_F \text{ and } x \notin A_F \text{ and } x \notin (A_T)^c\}, A_F) = (\{x \in X : x \in A_T\}, A_F) = (A_T, A_F)$.

Now we should prove that our function is a homomorphism. First, there is a technical lemma.

Lemma 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly paraconsistent set on X. Then the following properties hold:*

(1) $\widehat{\mathcal{A} \cap \mathcal{B}} = \widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}$.

(2) $\widehat{\mathcal{A} \cup \mathcal{B}} = \widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}$.
(3) $\widehat{\mathcal{A}^c} = (\widehat{\mathcal{A}})^c$.

Where the symbols on the right sides are understood in a classical sense.

Proof:

(1) We have $\widehat{\mathcal{A} \cap \mathcal{B}} = \{x \in X : x \notin (\mathcal{A} \cap \mathcal{B})_\sim\} = \{x \in X : x \notin (\sim A \cup \sim B)\} = \{x \in X : x \notin \sim A \text{ and } x \notin \sim B\}$.

On the other hand: $\widehat{\mathcal{A}} \cap \widehat{\mathcal{B}} = \{x \in X : x \notin \mathcal{A}_\sim\} \cap \{x \in X : x \notin \mathcal{B}_\sim\} = \{x \in X : x \notin \sim A\} \cap \{x \in X : x \notin \sim B\} = \{x \in X : x \notin \sim A \text{ and } x \notin \sim B\} = \widehat{\mathcal{A} \cap \mathcal{B}}$.

(2) $\widehat{\mathcal{A} \cup \mathcal{B}} = \{x \in X : x \notin (\mathcal{A} \cup \mathcal{B})_\sim\} = \{x \in X : x \notin (\sim A \cap \sim B)\} = \{x \in X : x \notin \sim A \text{ or } x \notin \sim B\} = \{x \in X : x \in (\sim A)^c \text{ or } x \in (\sim B)^c\} = \{x \in X : x \in \widehat{\mathcal{A}} \text{ or } x \in \widehat{\mathcal{B}}\} = \widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}$.

(3) $\widehat{\mathcal{A}^c} = \{x \in X : x \notin (\mathcal{A}^c)_\sim\} = \{x \in X : x \notin A\}$. On the other hand $(\widehat{\mathcal{A}})^c = \{x \in X : x \notin \sim A\}^c = \{x \in X : x \notin A\}$. We get the same set.

Having in mind the lemma above, we can prove the essential theorem. It says that our transforming function preserves intersection, union and complement. However, at first we should recall intuitionistic understanding of these operations.

Definition 3.8. (see [3]) Let $X \neq \emptyset$ and assume that $\mathfrak{A} = (A_T, A_F)$ and $\mathfrak{B} = (B_T, B_F)$ are two intuitionistic sets on X. Then we define their:

(1) *Union*: $\mathfrak{A} \cup \mathfrak{B} = (A_T \cup B_T, A_F \cap B_F)$.
(2) *Intersection*: $\mathfrak{A} \cap \mathfrak{B} = (A_T \cup B_T, A_F \cap B_F)$.
(3) *Complement*: $\mathfrak{A}^c = (A_F, A_T)$.

Note that we use (and we shall use) the same symbols (that is $\cup$, $\cap$, ^{c}, $\vee$ and $\wedge$) to denote operations in three frameworks: possibly paraconsistents, intuitionistic and then weak rough sets. The reader should be able to read from the context in which setting we are at a given moment.

Theorem 3.9. *Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. Let $\mathbf{f}$ be a transforming function. Then the following properties are true:*

(1) $\mathbf{f}(\mathcal{A} \cap \mathcal{B}) = \mathbf{f}(\mathcal{A}) \cap \mathbf{f}(\mathcal{B})$.
(2) $\mathbf{f}(\mathcal{A} \cup \mathcal{B}) = \mathbf{f}(\mathcal{A}) \cup \mathbf{f}(\mathcal{B})$.
(3) $\mathbf{f}(\mathcal{A}^c) = (\mathbf{f}(\mathcal{A}))^c$.

Where the intersection, union and complement symbols on the right sides are understood in the sense of Çoker's intuitionistic sets.

Proof: Note that we shall use Lemma 3.7.

(1) We have $\mathbf{f}(\mathcal{A}\cap\mathcal{B}) = \mathbf{f}(A\cap B, \sim A\cup \sim B) = (\widehat{\mathcal{A}\cap\mathcal{B}}, (A\cap B)^c) = (\widehat{\mathcal{A}}\cap\widehat{\mathcal{B}}, A^c\cup B^c) = (\widehat{\mathcal{A}}, A^c)\cap(\widehat{\mathcal{B}}, B^c) = \mathbf{f}(\mathcal{A})\cap\mathbf{f}(\mathcal{B})$.

(2) We have $\mathbf{f}(\mathcal{A}\cup\mathcal{B}) = \mathbf{f}(A\cup B, \sim A\cap \sim B) = (\widehat{\mathcal{A}\cup\mathcal{B}}, (A\cup B)^c) = (\widehat{\mathcal{A}}\cup\widehat{\mathcal{B}}, A^c\cap B^c) = (\widehat{\mathcal{A}}, A^c)\cup(\widehat{\mathcal{B}}, B^c) = \mathbf{f}(\mathcal{A})\cup\mathbf{f}(\mathcal{B})$.

(3) We have $\mathbf{f}(\mathcal{A}^c) = \mathbf{f}(\sim A, A) = (\widehat{\mathcal{A}^c}, (\sim A)^c) = (A^c, \widehat{\mathcal{A}}) = (\widehat{\mathcal{A}}, A^c)^c = (\mathbf{f}(\mathcal{A}))^c$.

Remark 3.10. One can check that:

(1) $\mathbf{f}(\emptyset, X) = (\widehat{\widehat{\emptyset}}, \emptyset^c) = (\emptyset, X)$.

(2) $\mathbf{f}(X, \emptyset) = (\widehat{\widehat{X}}, X^c) = (X, \emptyset)$.

(3) $\mathbf{f}(X, X) = (\widehat{\widehat{X}}, X^c) = (\emptyset, \emptyset)$.

Remark 3.11. Note that we can associate intuitionistic sets with possibly paraconsistent sets in many ways. However, not all of them are injections. For example, assume that $\mathbf{f}(\mathcal{A}) = \mathbf{f}(A, \sim A) = (\widehat{\mathcal{A}}, (\widehat{\mathcal{A}})^c)$. For example, if $X = \{x, y, z, p, q, r\}$, $A = \{x, y, p, q\}$ and $\sim A = \{z, p, q, r\}$, then $\mathbf{f}(\mathcal{A}) = (\{x, y\}, \{z, p, q, r\})$.

Now take $B = \{x, y, z, p\}$ and $\sim B = \{z, p, q, r\}$. Clearly, $\mathcal{A} \neq \mathcal{B}$ (because $A \neq B$). But $\mathbf{f}(\mathcal{B}) = (\{x, y\}, \{z, p, q, r\}) = \mathbf{f}(\mathcal{A})$.

4. Relationship with weak rough sets

Let us recall the definition of weak rough set and essential operations on the objects of this type.

Definition 4.1. (see [18]). Let $X \neq \emptyset$. Let $\mathfrak{A}$ be an ordered pair of the form (A_1, A_2) where $A_1 \subseteq A_2 \subseteq X$. Then we say that $\mathfrak{A}$ is a weak rough set on X. If $\mathfrak{A}$ and $\mathfrak{B}$ are two weak rough sets on X, then we define their:

(1) *Union*: $\mathfrak{A}\cup\mathfrak{B} = (A_1\cup B_1, A_2\cup B_2)$.

(2) *Intersection*: $\mathfrak{A}\cap\mathfrak{B} = (A_1\cap B_1, A_2\cap B_2)$.

(3) *Complement*: $\mathfrak{A}^c = (A_2^c, A_1^c)$.

4.1. *Transforming function*

Definition 4.2. Let $X \neq \emptyset$. Assume that $\mathcal{A} = (A, \sim A)$ is a possibly paraconsistent set on X. Then let $\mathbf{g}$ be a function that assigns to $\mathcal{A}$ a weak rough set of the form $\mathbf{g}(\mathcal{A}) = (\widehat{\mathcal{A}}, A)$.

Example 4.3. (1) Let $X = \mathbb{R}$. Assume that $\mathcal{A} = ([-10, 100], (-\infty, -5]\cup[90, +\infty))$. Then $\mathbf{g}(\mathcal{A}) = ((-5, 90), [-10, 100])$.

(2) Let $X = \{a, b, c, d, e, f, g\}$ and $\mathcal{B} = (\{a, b, d, e\}, \{a, c, f, g\})$. Then $\mathbf{g}(\mathcal{B}) = (\{b, d, e\}, \{a, b, d, e\})$.

Lemma 4.4. *Let $X \neq \emptyset$, $\mathcal{A}$ be a possibly paraconsistent set on X and* $\mathbf{g}$ *(defined as above) be our transforming function. Then* $\mathbf{g}$ *is one-to-one and surjective.*

Proof: We shall prove both properties.

(1) Let $\mathcal{A} \neq \mathcal{B}$ but $\mathbf{g}(\mathcal{A}) = \mathbf{g}(\mathcal{B})$. Then $\widehat{\mathcal{A}} = \widehat{\mathcal{B}}$ and $A = B$. Hence $(\widehat{\mathcal{A}})^c = (\widehat{\mathcal{B}})^c$ and $A^c = B^c$. Then $\sim A = A^c \cup (\widehat{\mathcal{A}})^c = B^c \cup (\widehat{\mathcal{B}})^c = \sim B$. So $\mathcal{A} = \mathcal{B}$ (contradiction).

(2) Assume that we have a weak rough set of the form $\mathfrak{A} = (A_1, A_2)$ where $A_1 \subseteq A_2$. We are looking for such possibly paraconsistent set $\mathcal{A} = (A, \sim A)$ that $\mathbf{g}(A, \sim A) = (\widehat{\mathcal{A}}, A) = (A_1, A_2)$. Thus $\widehat{\mathcal{A}} = A_1$ and $A = A_2$. Then $\sim A = (\widehat{\mathcal{A}})^c = A_1^c$. Finally, we obtain $\mathcal{A} = (A_1^c, A_2)$.

Note that this is indeed a possibly paraconsistent set because $(A_1^c)^c = A_1 \subseteq A_2$.

Remark 4.5. Again, note that it would not be reasonable to define $\mathbf{g}(\mathcal{A})$ as e.g. $(\widehat{\mathcal{A}}, X)$. Obviously, the resulting object is a weak rough set. But this function is not one-to-one (the reader is encouraged to find some simple counter-example).

Clearly, our function is homomorphic.

Theorem 4.6. *Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets. Let* $\mathbf{g}$ *be a wrs-transforming function. Then the following properties are true:*

(1) $\mathbf{g}(\mathcal{A} \cap \mathcal{B}) = \mathbf{g}(\mathcal{A}) \cap \mathbf{g}(\mathcal{B})$.
(2) $\mathbf{g}(\mathcal{A} \cup \mathcal{B}) = \mathbf{g}(\mathcal{A}) \cup \mathbf{g}(\mathcal{B})$.
(3) $\mathbf{g}(\mathcal{A}^c) = (\mathbf{g}(\mathcal{A}))^c$.

Where the intersection, union and complement symbols on the right sides are understood in the sense of weak rough sets.

Proof: For example, $\mathbf{g}(\mathcal{A} \cap \mathcal{B}) = \mathbf{g}(A \cap B, \sim A \cup \sim B) = (\widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}, A \cap B) = (\widehat{\mathcal{A}}, A) \cap (\widehat{\mathcal{B}}, B) = \mathbf{g}(\mathcal{A}) \cap \mathbf{g}(\mathcal{B})$.

Other cases are similar.

5. Bisemilattice of possibly paraconsistent sets

In Remark 2.8 we pointed out that it is not reasonable to define the operation analogous to $\vee$ but with classical intersections instead of unions. However, this flaw can be easily repaired.

Definition 5.1. Let $X \neq \emptyset$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two possibly paraconsistent sets on X. Then we define *strong intersection* as $\mathcal{A} \wedge \mathcal{B} = (A \cap B, (\sim A \cap \sim B) \cup (A \cap B)^c)$.

Obviously, the operation defined above returns possibly paraconsistent set.

Let us consider the following algebra of possibly paraconsistent sets: $(X, \wedge, \vee)$. Let us analyze what is the interplay of these two operators.

Theorem 5.2. *Suppose that $X \neq \emptyset$ and $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets on X. Then the following relationships hold:*

(1) $\mathcal{A} \wedge \mathcal{B} = \mathcal{B} \wedge \mathcal{A}$ *and* $\mathcal{A} \vee \mathcal{B} = \mathcal{B} \vee \mathcal{A}$.
(2) $\mathcal{A} \wedge \mathcal{A} = \mathcal{A} \vee \mathcal{A} = \mathcal{A}$.
(3) $\mathcal{A} \wedge (\mathcal{A} \vee \mathcal{B}) = \mathcal{A}$.
(4) $\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) = (\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C}$.
(5) $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) = (\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}$.

Proof:

(1) Commutativity of both operations is obvious.
(2) Idempotence of both operations is obvious.
(3) We have: $\mathcal{A} \wedge (\mathcal{A} \vee \mathcal{B}) = \mathcal{A} \wedge (A \cup B, \sim A \cup \sim B) = (A \cap (A \cup B), (\sim A \cap (\sim A \cup \sim B)) \cup (A \cap (A \cup B))^c) = (A, \sim A \cup (A^c \cup (A \cup B)^c)) = (A, \sim A \cup (A^c \cup (A^c \cap B^c))) = (A, \sim A \cup A^c) = (A, \sim A)$.

   We used classical absorption laws and the fact that $A^c \subseteq \sim A$.
(4) We have $\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) = \mathcal{A} \vee (B \cup C, \sim B \cup \sim C) = (A \cup (B \cup C), \sim A \cup (\sim B \cup \sim C)) = ((A \cup B) \cup C, (\sim A \cup \sim B) \cup \sim C)$.
(5) Let us calculate:

   $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) = \mathcal{A} \wedge (B \cap C, (\sim B \cap \sim C) \cup (B \cap C)^c) = \mathcal{A} \wedge (B \cap C, (\sim B \cap \sim C) \cup (B^c \cup C^c)) = (A \cap (B \cap C), (\sim A \cap ((\sim B \cap \sim C) \cup (B^c \cup C^c))) \cup (A \cap (B \cap C))^c)) = (A \cap B \cap C, (\sim A \cap (\sim B \cap \sim C)) \cup (\sim A \cap (B^c \cup C^c)) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, (\sim A \cap \sim B \cap \sim C) \cup (\sim A \cap (B^c \cup C^c)) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, X_1 \cup X_2 \cup X_3)$.

   On the other hand:

   $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} = (A \cap B, (\sim A \cap \sim B) \cup (A \cap B)^c) \wedge \mathcal{C} = (A \cap B, (\sim A \cap \sim B) \cup (A^c \cup B^c)) \wedge \mathcal{C} = ((A \cap B) \cap C, (((\sim A \cap \sim B) \cup (A^c \cup B^c)) \cap \sim C) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, ((\sim A \cap \sim B) \cap \sim C) \cup ((A^c \cup B^c) \cap \sim C) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, (\sim A \cap \sim B \cap \sim C) \cup ((A^c \cup B^c) \cap \sim C) \cup (A \cap B \cap C)^c) = (A \cap B \cap C, Y_1 \cup Y_2 \cup Y_3)$.

   Both expressions are similar but not identical. Clearly, left components are the same. Now think that x belongs to the right component of $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$ but it does not belong to the right component of $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}$.

   The second assumption means that $x \notin Y_1$ and $x \notin Y_2$ and $x \notin Y_3$.

   Let us think about the first assumption. We have a logical disjunction of statements (or set-theoretical union of sets). Suppose that $x \in X_1$. Clearly $X_1 = Y_1$. But this gives us a contradiction.

   Now suppose that $x \in X_3$. Clearly, $X_3 = Y_3$. Again, contradiction.

   Now let us think that $x \in X_2$. This means that $x \in \sim A \cap (B^c \cup C^c)$. In particular, it means that $x \in B^c \cup C^c$. Suppose that $x \in B^c$. But at the same time $x \notin Y_3 =$

$(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$. So $x \notin A^c$ and $x \notin B^c$ and $x \notin C^c$. In particular, $x \notin B^c$. Contradiction. The same conclusion appears under the assumption that $x \in C^c$.

The reader is encouraged to use a similar reasoning to prove the second inclusion: namely, that $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C} \subseteq \mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$.

Remark 5.3. What about the second absorption law? Take $X = \{a, b, c, d, e, f, g, h\}$ and define $\mathcal{A} = (\{a, c, d, e\}, \{b, c, d, f, g, h\})$ together with $\mathcal{B} = (\{b, d, f, h\}, \{a, b, c, f, g, h\})$.

Now $\mathcal{A} \wedge \mathcal{B} = (\{d\}, \{a, b, c, e, f, g, h\})$. Then $\mathcal{A} \vee (\mathcal{A} \wedge \mathcal{B}) = (\{a, c, d, e\}, X)$. But this set is different than $\mathcal{A}$. Thus this hypothetical absorption law (namely "$\vee - \wedge$") does not hold.

Let us recall the notion of *bisemilattice*. We say that a structure $(L, +, \cdot)$ is a bisemilattice if and only if both operations are idempotent, commutative and associative. Other requirements (like distributivity, absorption laws or existence of zero and one) are not mandatory.

Bisemilattices were analyzed by several authors (in particular, by Polish algebraists): see [7], [8], [10] or [11].

All the facts proved above allow us to say that the structure $(X, \vee, \wedge)$ (of possibly paraconsistent sets) is a bisemilattice with only one absorption law. Such specific bisemilattices were mentioned e.g. by Dudek in [6]. In this paper he proved that there exists a bisemilattice of this type being not a lattice. In fact, we may use our class as an example too (note that in the definition of lattice we require *two* absorption laws while in our case on of them is not true).

5.1. *Relationship with weak rough sets*

We want to use the function that has been already used: $\mathbf{g}(\widehat{\mathcal{A}}, A)$. It associates each possibly paraconsistent set with some weak rough set. But what about the appropriate operations?

First, let us prove the following lemma:

Lemma 5.4. *Let $X \neq \emptyset$ and $\mathcal{A}$, $\mathcal{B}$ be two possibly paraconsistent sets. Then the following properties hold:*

(1) $\widehat{\mathcal{A} \vee \mathcal{B}} = \widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}$.
(2) $\widehat{\mathcal{A} \cap \mathcal{B}} = (\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}) \cap (A \cap B)$.

Proof:

(1) We have $\widehat{\mathcal{A} \vee \mathcal{B}} = \{x \in X : x \notin (\mathcal{A} \vee \mathcal{B})_{\sim}\} = \{x \in X : x \notin (\mathcal{A} \cap \mathcal{B})_{\sim}\} = \widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}$.
(2) We have $\widehat{\mathcal{A} \cap \mathcal{B}} = \{x \in X : x \notin (\sim A \cap \sim B) \cup (A \cap B)^c\} = \{x \in X : x \notin (\sim A \cap \sim B)$ and $x \notin (A \cap B)^c\} = \{x \in X : x \notin (\sim A \cap \sim B)$ and $x \in A \cap B\} = (\sim A \cap \sim B)^c \cap (A \cap B) = ((\sim A)^c \cup (\sim B)^c) \cap (A \cap B) = (\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}) \cap (A \cap B)$.

Then we define appropriate operations in the framework of weak rough sets.

Definition 5.5. Let $X \neq \emptyset$ and assume that $\mathfrak{A} = (A_1, A_2)$ and $\mathfrak{B} = (B_1, B_2)$ are two weak rough sets. Then we define:

(1) $\mathfrak{A} \vee \mathfrak{B} = (A_1 \cap B_1, A_2 \cup B_2)$.
(2) $\mathfrak{A} \wedge \mathfrak{B} = ((A_1 \cup B_1) \cap (A_2 \cap B_2), A_2 \cap B_2)$.

Theorem 5.6. *Let $X \neq \emptyset$ and assume that $\mathcal{A}$, $\mathcal{B}$ are two possibly paraconsistent sets and $\mathbf{g}$ is a wrs-transforming function. Then the following properties are true:*

(1) $\mathbf{g}(\mathcal{A} \vee \mathcal{B}) = \mathbf{g}(\mathcal{A}) \vee \mathbf{g}(\mathcal{B})$.
(2) $\mathbf{g}(\mathcal{A} \wedge \mathcal{B}) = \mathbf{g}(\mathcal{A}) \wedge \mathbf{g}(\mathcal{B})$.

Proof:

(1) We have $\mathbf{g}(\widehat{\mathcal{A} \vee \mathcal{B}}) = (\widehat{\mathcal{A} \vee \mathcal{B}}, A \cup B) = (\widehat{\mathcal{A}} \cap \widehat{\mathcal{B}}, A \cup B) = (\widehat{\mathcal{A}}, A) \vee (\widehat{\mathcal{B}}, B) = \mathbf{g}(\mathcal{A}) \vee \mathbf{g}(\mathcal{B})$.
(2) We have: $\mathbf{g}(\widehat{\mathcal{A} \wedge \mathcal{B}}) = (\widehat{\mathcal{A} \wedge \mathcal{B}}, A \cap B) = ((\widehat{\mathcal{A}} \cup \widehat{\mathcal{B}}) \cap A \cap B, A \cap B) = (\widehat{\mathcal{A}}, A) \wedge (\widehat{\mathcal{B}}, B) = \mathbf{g}(\mathcal{A}) \wedge \mathbf{g}(\mathcal{B})$.

As for the behaviour of distinguished sets, it has been analyzed below (we omit the proof because it is simple):

Lemma 5.7. *Let $X \neq \emptyset$ and $\mathcal{A}$ be a possibly paraconsistent set on X. Then the following properties hold:*

(1) $\mathcal{A} \wedge \tilde{\emptyset} = \tilde{\emptyset}$, $\mathcal{A} \vee \tilde{\emptyset} = (A, X)$.
(2) $\mathcal{A} \wedge \tilde{X} = (A, A^c)$, $\mathcal{A} \vee \tilde{X} = (X, \sim A)$.
(3) $\mathcal{A} \wedge \overline{X} = (A, X)$, $\mathcal{A} \vee \overline{X} = \overline{X}$.

5.2. *Interpretation in terms of negotiations*

Our operations $\vee$ and $\wedge$ have somewhat interesting interpretation when they are interpreted in the framework of weak rough sets (as in Def. 5.5). In fact, we have already recognized these operations in our unpublished paper [17].

Imagine that there are two weak rough sets on some universe X. They are called $\mathfrak{A}$ and $\mathfrak{B}$. As for the operation $\vee$, it returns the intersection of necessity ranges and the union of possibility ranges. This can be interpreted as a hypothetical solution of some discussion between two decision makers (who evaluated the elements of X using weak rough sets). They compare and combine their sets to produce a new evaluation. The idea is that now *more* objects are acceptable (possible) and fewer are necessary. It is a kind of compromise. As for the necessity ranges, the approach of our decision makers is strict: they want to limit themselves only to those objects that are necessary in the eyes of both of them. Contrary to this, in case of possible objects they are ready to sum up their ranges.

The second operation, namely $\wedge$, can be described in a similar way. Clearly, the same can be said about the standard operations $\cup$ and $\cap$ but $\vee$ and $\wedge$ are less known and less typical. This makes them interesting (e.g. in the context of bisemilattices).

In [17] we proved that this bisemilattice is non-distributive. We gave appropriate counter-examples (in terms of weak rough sets). Clearly, this property is true in the setting of possibly paraconsistent sets too (by isomorphism). However, the reader is encouraged to find counter-examples that would be formulated exactly in the latter language.

6. **Conclusion and final remarks**

In this paper we introduced possibly paraconsistent sets. They can be analyzed in the context of three-valued logic. We proved that their algebra is isomorphic with the algebra of Çoker's intuitionistic sets (orthopairs) and thus with the algebra of weak rough (that is, double or. equivalently, flou) sets.

We have analyzed less typical operators that form bisemilattice with only one absorption law. We have shown that we can find corresponding operations in the setting of weak rough sets. One can think about finding appropriate operations in the setting of intuitionistic sets.

Regardless of isomorphisms, we gave exact proofs of some essential properties of possibly paraconsistent sets with all the operations introduced. We think that this is valuable from the practical point of view: even if the algebraic structure of different frameworks is the same, their meaning, interpretation and semantics can be different.

Another thing that is important, is the possible correspondence of our framework with the one that is generated by neutrosophic crisp sets of type 2. They are triples of the form (A_1, A_2, A_3) with the assumption that $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = X$. Clearly, this is just like our $A \cup (\mathcal{A} \setminus A^c) \cup A^c$ (in the possibly paraconsistent environment).

As for the neutrosophic crisp sets of type 2, they are a subclass of the wider class of neutrosophic crisp sets. The latter are defined just as triple (A_1, A_2, A_3) without any additional suppositions. Basically, their union and intersection are defined as:

(1) *Union*: $\mathfrak{A} \cup \mathfrak{B} =$
  (a) $(A_1 \cup B_1, A_2 \cap B_2, A_3 \cup B_3)$
  (b) or $(A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3)$.
(2) *Intersection*: $\mathfrak{A} \cap \mathfrak{B} =$
  (a) $(A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3)$
  (b) or $(A_1 \cap B_1, A_2 \cup B_2, A_3 \cap B_3)$.

Of course the class of *all* neutrosophic crisp sets is closed under all these operations. But the class of neutrosophic crisp sets of type 2 is not. This is because we do not have any guarantee that the components of the resulting set will sum up to X. For example, take $X = \{a, b, c, d, e\}$,

$\mathfrak{A} = (\{a, c\}, \{b\}, \{d, e\})$ and $\mathfrak{B} = (\{b, c\}, \{d\}, \{a, e\})$. Both these forms are neutrosophic crisp sets of type 2. Now use the first intersection: $\mathfrak{A} \cap \mathfrak{B} = (\{c\}, \emptyset, \{e\})$. Clearly, $\{c\} \cup \emptyset \cup \{e\} \neq X$.

Remark 6.1. Besides, there are some ambiguities in the theory of neutrosophic crisp sets. For example, in [14] we have Def. 3.1. where neutrosophic crisp sets *as such* are defined with the assumption that all three components are mutually exclusive. The same is repeated e.g. in [15]. But there is no such requirement in the book [13] that has been already mentioned, nor in [5] or [9] and many other papers. Thus we use the most general definition of neutrosophic crisp sets.

Hence, the first task is to introduce refined definitions of union and intersection (to make the class of NCS of type 2 closed under these operations). This will require some purely technical (but maybe a bit complicated) work. The next step would refer to the construction of an appropriate isomorphism. The whole program should be realized also with respect to the operations $\wedge$ and $\vee$.

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Key performance indicators in technology firms using generalized fermatean neutrosophic competition graph

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Abstract. This study investigates fermatean neutrosophic digraphs, generalized fermatean neutrosophic digraphs, and the out-neighborhood of vertices inside generalized fermatean neutrosophic digraphs. It looks at the qualities and characteristics of generalized fermatean neutrosophic competition graphs and their matrix representations. It also establishes the minimal graph, competition number for generalized fermatean neutrosophic competition graphs, and relevant features. Finally, the paper addresses a practical implementation of these ideas.

Keywords: Neutrosophicgraph, fermatean neutrosophic graph, fermatean neutrosophic digraphs, generalized fermatean neutrosophic digraphs.

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1. **Introduction**

An important area of applied mathematics is graph theory, which is utilized to address a wide range of issues in computer science, geometry, algebra, social networks, optimization, and other fields [1]. Cohen [2] introduced the competition graph and its use in ecosystems, focusing on species competition within food webs. When two species have at least one common prey, they are considered to be in competition in this context. Roberts et al. [3, 4] investigated the possibility of representing all networks with isolated vertices as competition graphs. The competition number represents the least number of such vertices. Opsut [5] studied how to calculate a graph's competition number. Kim and colleagues [6, 7] provided the p-competition number and graph. Brigham et al. [8] expanded the p-competition graph to incorporate the ∅-tolerance graph, enhancing its generality.

Cho and Kim [9] investigated the competition number of a graph with one hole. Li and Chang [10] investigated competition graphs with h holes. Factor and Merz initially proposed the (1,2)-step competition graph for tournaments [11], and then expanded the concept to the (1,2)-step competition graph. Kaufman [12] created the fuzzy graph, where each vertex and edge has various degrees of membership, to account for erroneous data in real life. Numerous scientific investigations have been conducted on fuzzy graphs [13]. Parvathi and Karunambigal introduced intuitionistic fuzzy graphs in [14]. It is a graph composed of vertices and edges with variable degrees of membership and non-membership. Akram and Dubek [15] introduced interval-valued fuzzy graphs, where vertices and edges' membership values are represented as intervals. However, even when competition is portrayed using competition graphs, these features are not fully realized.

Samanta and Pal [16] represented competition in a fuzzy environment more realistically, taking into account the ambiguity of prey and species in a food chain. Samanta and Sarkar [17, 18] proposed the generalized fuzzy competition graph and generalized fuzzy graph, where the vertices' membership values dictate edge membership values. Pramanik et al. [19] merged fuzzy tolerance graphs with fuzzy φ tolerance competition graphs.

Smarandache [20] developed neutrosophic logic, a cohesive framework for dealing with indeterminate and inconsistent information that extends classical and fuzzy logics. This fundamental discovery laid the groundwork for later discoveries in a wide range of fields, including decision-making and graph theory. Ye [21] expanded neutrosophic logic by developing a multi-criteria decision-making technique that leverages the correlation coefficient in a single-valued neutrosophic environment, proving its practical applicability in challenging decision circumstances.Akram, Siddique, and Davvaz [22] presented new concepts in neutrosophic graphs and studied their applications, proving the flexibility of neutrosophic logic in mathematical modeling. Quek et al. [23] made additional contributions to the topic by investigating graph theory in the context of complicated neutrosophic sets, revealing neutrosophic sets' ability to handle increasingly complex relational data. ahin [24] presented a practical approach to neutrosophic graph theory, highlighting its usefulness in tackling real-world situations. Huang et al. [25] investigated regular and irregular neutrosophic graphs, applying these principles to real-world circumstances and emphasizing their practical relevance. Mohanta et al. [26] investigated m-polar neutrosophic graphs, expanding the use of neutrosophic graph theory in intelligent and fuzzy systems and enlarging the scope and usefulness of neutrosophic logic in contemporary computing issues. Recent advances in neutrosophic logic and graph theory have greatly broadened the scope of decision-making and problem-solving in complicated and ambiguous situations. Mohanta, Dey, and Pal [27] investigated several neutrosophic graph products, stressing their potential for managing complex interactions in intelligent systems. Broumi et al. [28, 29] proposed interval-valued Fermatean neutrosophic graphs as a complete framework for dealing with data with a wider range of uncertainty and indeterminacy. Their contributions to "Collected Papers" and "Neutrosophic Sets and Systems" highlighted the theoretical underpinnings and practical uses of these graphs, particularly in scenarios needing improved decision-making capacities. Broumi et al. [30] extended the use of neutrosophic graph theory by developing complicated Fermatean neutrosophic graphs that were used in decision-making processes in management and engineering. This novel technique revealed the usefulness of neutrosophic graphs in solving diverse choice issues. Dhouib et al. [31] solved the Minimum Spanning Tree Problem with interval-valued Fermatean neutrosophic domains, demonstrating the usefulness of these graphs in optimizing network-related tasks. Additionally, Saeed and Shafique [32] examined the relationship of Fermatean neutrosophic soft sets with applications to sustainable agriculture, their findings showed that neutrosophic sets can help improve decision-making processes in agricultural sustainability. AL-Omeri et al. [[33]- [38]] discussed identify internet streaming services using max product of complement in neutrosophic graphs and give some real time applications.

1.1. *Motivation*

(1) To broaden graph theory by including fermatean neutrosophic graph, which covers membership, indeterminacy, and non membership.
(2) To develop powerful tools for modeling complicated real-world issues that go beyond the capability of classical binary logic.
(3) The inspiration for this study derives from the desire to better capture and evaluate dynamic interactions in such systems, where classical graph theory falls short.

1.2. *Novelty*

(1) The paper extends the notion of fermatean neutrosophic graphs to extended fermatean neutrosophic competition graphs, increasing the scope of fermatean neutrosophic graph theory.
(2) It defines and examines the minimal graph and competition number for generalized fermatean neutrosophic competition graphs, yielding novel theoretical insights and characteristics.
(3) The research presents a matrix form of generalized fermatean neutrosophic competition graphs that makes them easier to compute and see.
(4) The paper describes a practical application of generalized fermatean neutrosophic competition graphs in the context of technology firms, illustrating the relevance of the suggested ideas in capturing real-world contests and interactions.

1.3. *Structure of the article*

The research begins with an overview of fermatean neutrosophic digraphs(FNDG) and the reason behind expanding graph theory to include fermatean neutrosophic graph(FNG).
Section 2 covers the fundamental terminology and preliminary information required. Section 3 present generalized fermatean neutrosophic graph(GFNG), fermatean neutrosophic digraphs, generalized fermatean neutrosophic digraphs, fermatean neutrosophic competition graph(FNCG), generalized fermatean neutrosophic competition graph(GFNCG), minimal graph and competition number for generalized fermatean neutrosophic competition graphs, along with their features.
Section 4 presents the matrix form of GNCGs, followed by an appropriate example to demonstrate its use. section 5 discusses a practical application of the explored theoretical principles, demonstrating the importance and value of generalized fermatean neutrosophic digraphs(GFNDG) in solving real-world problems.
Summarizes the findings and makes recommendations for future study in fermatean neutrosophic graph(FNG) theory.

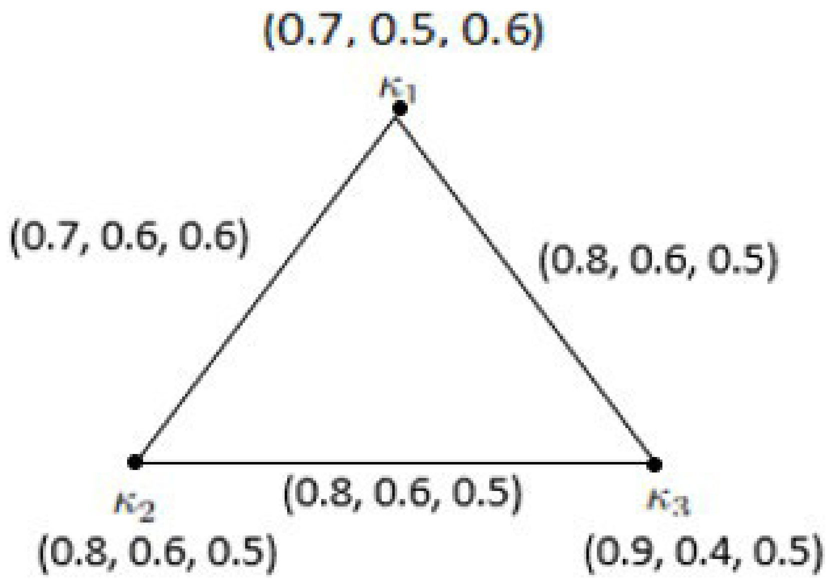
2. **Basic Definitions**

This part provides the essential components required for understanding the article.

Definition 2.1. Let X represent a universal set. A Fermatean Neutrosophic relation on X is a mapping $g = (\chi_\upsilon, \chi_\xi, \chi_\varsigma) : X \times X \to [0, 1]$ where $\chi_\upsilon(\kappa_r, \kappa_s), \chi_\xi(\kappa_r, \kappa_s), \chi_\varsigma(\kappa_r, \kappa_s) \in [0, 1]$.

Definition 2.2. Let X be a universal set. Let $G = (g, \vartheta)$ be FNG, where g is a fermatean neutrosophic set on X and ϑ is a fermatean neutrosophic relation on X. The pair fulfills the following requirements
$\delta_\upsilon(\kappa_r, \kappa_s) \leq min\{\chi_\upsilon(\kappa_r), \chi_\upsilon(\kappa_s)\}$
$\delta_\xi(\kappa_r, \kappa_s) \geq max\{\chi_\xi(\kappa_r), \chi_\xi(\kappa_s)\}$
$\delta_\varsigma(\kappa_r, \kappa_s) \geq max\{\chi_\varsigma(\kappa_r), \chi_\varsigma(\kappa_s)\}$
$0 \leq \delta^3_\upsilon(\kappa_r, \kappa_s) + \delta^3_\xi(\kappa_r, \kappa_s) + \delta^3_\varsigma(\kappa_r, \kappa_s) \leq 2$ for all $\kappa_r, \kappa_s \in X$ where $\delta_\upsilon : X \times X \to [0, 1], \delta_\xi : X \times X \to [0, 1]$ $and \delta_\varsigma : X \times X \to [0, 1]$ represents the degree of membership, indeterminacy-membership, and non-membership of ϑ, respectively. Here, the Fermatean Neutrosophic edge set of G is represented by ϑ, while the Fermatean Neutrosophic vertex set of G is represented by g.

Example 2.3. Consider the FNG $G = (g, \vartheta)$ where edge set of G is represented by ϑ, and vertex set of G is represented by g defined by $g = \{\kappa_1, \kappa_2, \kappa_3\}$ and edges $\vartheta = \{(\kappa_1, \kappa_2), (\kappa_1, \kappa_3), (\kappa_2, \kappa_3)\}$ as in figure 1.



FNG.PNG

Figure 1. FNG .

Definition 2.4. The cardinality of FNS χ is denoted as $|\chi| = (|\chi|_\upsilon, |\chi|_\xi, |\chi|_\varsigma)$. The total membership values are represented by $|\chi|_\upsilon$, indeterminacy values are represented by $|\chi|_\xi$ and non membership values are represented by $|\chi|_\varsigma$.

Definition 2.5. The height of an FNS $\chi = (X, \chi_\upsilon, \chi_\xi, \chi_\varsigma)$ is defined as $h(\chi) = (sup_{x\in X}\chi_\upsilon(x), inf_{x\in X}\chi_\xi(x), inf_{x\in X}\chi_\varsigma(x)) = (h_1(\chi), h_2(\chi))$

Definition 2.6. If $\vec{G} = (g, \vec{\vartheta})$ is defined as FNDG if

(i) $\chi_\upsilon : g \to [0, 1]$, $\chi_\xi : g \to [0, 1]$ and $\chi_\varsigma : g \to [0, 1]$ it denotes the degree of membership, indeterminacy and non membership respectively, such that $0 \le \chi_\upsilon^3 + \chi_\xi^3 + \chi_\varsigma^3 \le 2\ \forall \kappa_r \in g$.

(ii) $\delta_\upsilon : \vartheta \to [0, 1]$, $\delta_\xi : \vartheta \to [0, 1]$ and $\delta_\varsigma : \vartheta \to [0, 1]$ it denotes the degree of membership, indeterminacy and non membership of edge respectively.

$\delta_\upsilon(\overrightarrow{\kappa_r, \kappa_s}) \le min\{\chi_\upsilon(\kappa_r), \chi_\upsilon(\kappa_s)\}$

$\delta_\xi(\overrightarrow{\kappa_r, \kappa_s}) \ge max\{\chi_\xi(\kappa_r), \chi_\xi(\kappa_s)\}$

$\delta_\varsigma(\overrightarrow{\kappa_r, \kappa_s}) \ge max\{\chi_\varsigma(\kappa_r), \chi_\varsigma(\kappa_s)\}$

$0 \le \delta_\upsilon^3(\overrightarrow{\kappa_r, \kappa_s}) + \delta_\xi^3(\overrightarrow{\kappa_r, \kappa_s}) + \delta_\varsigma^3(\overrightarrow{\kappa_r, \kappa_s}) \le 2$

Example 2.7. The graph in Figure 2 is represented by the notation $\vec{G} = (g, \vec{\vartheta})$, with vertices $g = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and edges $\vartheta = \{(\overrightarrow{\kappa_1, \kappa_2}), (\overrightarrow{\kappa_1, \kappa_3}), (\overrightarrow{\kappa_2, \kappa_3}), (\overrightarrow{\kappa_3, \kappa_4})\}$

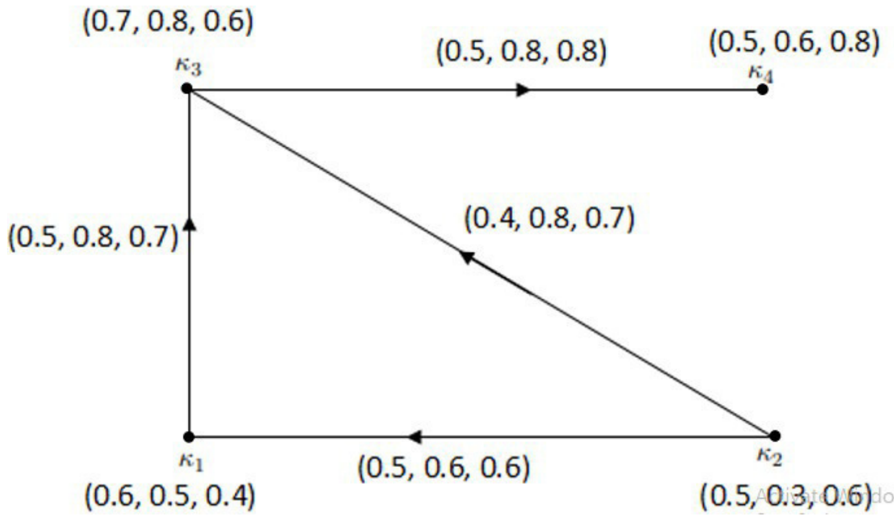


FIG1.PNG

Figure 2. FNDG .

Definition 2.8. The Fermatean Neutrosophic out-neighborhood of a vertex κ_r in a directed Fermatean Neutrosophic graph $\vec{G} = (g, \chi, \delta)$ is a FNS $F^{+}(\kappa_r) = \left(X^{+}_{\kappa_r}, \chi^{+}_{\upsilon_{\kappa_r}}, \chi^{+}_{\xi_{\kappa_r}}, \chi^{+}_{\varsigma_{\kappa_r}}\right)$, where $X^{+}_{\kappa_r} = \{\kappa_s | \delta_\upsilon(\overrightarrow{\kappa_r, \kappa_s}) > 0, \delta_\xi(\overrightarrow{\kappa_r, \kappa_s}) > 0, \delta_\varsigma(\overrightarrow{\kappa_r, \kappa_s}) > 0\}$ and $\chi^{+}_{\upsilon_{\kappa_r}} : X^{+}_{\kappa_r} \to [0, 1]$ defined by $\chi^{+}_{\upsilon_{\kappa_r}}(\kappa_s) = \delta_\upsilon(\overrightarrow{\kappa_r, \kappa_s})$,
$\chi^{+}_{\xi_{\kappa_r}} : X^{+}_{\kappa_r} \to [0, 1]$ defined by $\chi^{+}_{\xi_{\kappa_r}}(\kappa_s) = \delta_\xi(\overrightarrow{\kappa_r, \kappa_s})$ and $\chi^{+}_{\varsigma_{\kappa_r}} : X^{+}_{\kappa_r} \to [0, 1]$ defined by $\chi^{+}_{\varsigma_{\kappa_r}}(\kappa_s) = \delta_\varsigma(\overrightarrow{\kappa_r, \kappa_s})$.

Definition 2.9. The Fermatean Neutrosophic in-neighborhood of a vertex κ_r in a directed Fermatean Neutrosophic graph $\vec{G} = (g, \chi, \delta)$ is a FNS $F^{-}(\kappa_r) = \left(X^{-}_{\kappa_r}, \chi^{-}_{\upsilon_{\kappa_r}}, \chi^{-}_{\xi_{\kappa_r}}, \chi^{-}_{\varsigma_{\kappa_r}}\right)$, where $X^{-}_{\kappa_r} = \{\kappa_s | \delta_\upsilon(\overrightarrow{\kappa_s, \kappa_r}) > 0, \delta_\xi(\overrightarrow{\kappa_s, \kappa_r}) > 0, \delta_\varsigma(\overrightarrow{\kappa_s, \kappa_r}) > 0\}$ and $\chi^{-}_{\upsilon_{\kappa_r}} : X^{+}_{\kappa_r} \to [0, 1]$ defined by $\chi^{-}_{\upsilon_{\kappa_r}}(\kappa_s) = \delta_\upsilon(\overrightarrow{\kappa_s, \kappa_r})$, $\chi^{-}_{\xi_{\kappa_r}} : X^{+}_{\kappa_r} \to [0, 1]$ defined by $\chi^{-}_{\xi_{\kappa_r}}(\kappa_s) = \delta_\xi(\overrightarrow{\kappa_s, \kappa_r})$ and $\chi^{-}_{\varsigma_{\kappa_r}} : X^{+}_{\kappa_r} \to [0, 1]$ defined by $\chi^{-}_{\varsigma_{\kappa_r}}(\kappa_s) = \delta_\varsigma(\overrightarrow{\kappa_s, \kappa_r})$.

3. Generalized fermatean neutrosophic competition graph

Definition 3.1. A GFNG $G = (g, \vartheta)$ where $\vartheta \subseteq g \times g$ is examined if certain functions exist

$\chi_\upsilon : g \to [0, 1]$, $\chi_\xi : g \to [0, 1]$ and $\chi_\varsigma : g \to [0, 1]$.

$\delta_\upsilon : \vartheta \to [0, 1]$, $\delta_\xi : \vartheta \to [0, 1]$ and $\delta_\varsigma : \vartheta \to [0, 1]$.

$E_\upsilon : \vartheta_\upsilon \to [0, 1]$, $E_\xi : \vartheta_\xi \to [0, 1]$ and $E_\varsigma : \vartheta_\varsigma \to [0, 1]$,
such that $0 \leq \chi^3_\upsilon(\kappa_r) + \chi^3_\xi(\kappa_r) + \chi^3_\varsigma(\kappa_r) \leq 2 \ \forall \kappa_r \in g \ (r = 1, 2, ..., n)$ and

$\delta_\upsilon(\kappa_r, \kappa_s) = E_\upsilon(\chi_\upsilon(\kappa_r), \chi_\upsilon(\kappa_s))$

$\delta_\xi(\kappa_r, \kappa_s) = E_\xi(\chi_\xi(\kappa_r), \chi_\xi(\kappa_s))$

$\delta_\varsigma(\kappa_r, \kappa_s) = E_\varsigma(\chi_\varsigma(\kappa_r), \chi_\varsigma(\kappa_s))$

where

$\vartheta_\upsilon = \{(\chi_\upsilon(\kappa_r), \chi_\upsilon(\kappa_s)) : \delta_\upsilon(\kappa_r, \kappa_s) \geq 0\}$

$\vartheta_{\xi} = \{(\chi_{\xi}(\kappa_r), \chi_{\xi}(\kappa_s)) : \delta_{\xi}(\kappa_r, \kappa_s) \geq 0\}$

$\vartheta_{\varsigma} = \{(\chi_{\varsigma}(\kappa_r), \chi_{\varsigma}(\kappa_s)) : \delta_{\varsigma}(\kappa_r, \kappa_s) \geq 0\}$

and $\chi_{\upsilon}(\kappa_r)$, $\chi_{\xi}(\kappa_r)$ and $\chi_{\varsigma}(\kappa_r)$ denotes the degree of membership, indeterminacy and non membership of vertex respectively and $\delta_{\upsilon}(\kappa_r, \kappa_s)$, $\delta_{\xi}(\kappa_r, \kappa_s)$ and $\delta_{\varsigma}(\kappa_r, \kappa_s)$ denotes the degree of membership, indeterminacy and non membership of edges respectively.

Definition 3.2. A GFNG $\vec{Ģ} = (g, \vec{\vartheta})$ where $\vec{\vartheta} \subseteq g \times g$ is examined if certain functions exist

$\chi_{\upsilon} : g \rightarrow [0,1]$, $\chi_{\xi} : g \rightarrow [0,1]$ and $\chi_{\varsigma} : g \rightarrow [0,1]$.

$\delta_{\upsilon} : \vec{\vartheta} \rightarrow [0,1]$, $\delta_{\xi} : \vec{\vartheta} \rightarrow [0,1]$ and $\delta_{\varsigma} : \vec{\vartheta} \rightarrow [0,1]$.

$E_{\upsilon} : \vartheta_{\upsilon} \rightarrow [0,1]$, $E_{\xi} : \vartheta_{\xi} \rightarrow [0,1]$ and $E_{\varsigma} : \vartheta_{\varsigma} \rightarrow [0,1]$,

such that $0 \leq \chi^3_{\upsilon}(\kappa_r) + \chi^3_{\xi}(\kappa_r) + \chi^3_{\varsigma}(\kappa_r) \leq 2 \; \forall \kappa_r \in g \; (r = 1, 2, ..., n)$ and

$\delta_{\upsilon}(\overrightarrow{\kappa_r, \kappa_s}) = E_{\upsilon}(\chi_{\upsilon}(\kappa_r), \chi_{\upsilon}(\kappa_s))$

$\delta_{\xi}(\overrightarrow{\kappa_r, \kappa_s}) = E_{\xi}(\chi_{\xi}(\kappa_r), \chi_{\xi}(\kappa_s))$

$\delta_{\varsigma}(\overrightarrow{\kappa_r, \kappa_s}) = E_{\varsigma}(\chi_{\varsigma}(\kappa_r), \chi_{\varsigma}(\kappa_s))$

where

$\vartheta_{\upsilon} = \{(\chi_{\upsilon}(\kappa_r), \chi_{\upsilon}(\kappa_s)) : \delta_{\upsilon}(\kappa_r, \kappa_s) \geq 0\}$

$\vartheta_{\xi} = \{(\chi_{\xi}(\kappa_r), \chi_{\xi}(\kappa_s)) : \delta_{\xi}(\kappa_r, \kappa_s) \geq 0\}$

$\vartheta_{\varsigma} = \{(\chi_{\varsigma}(\kappa_r), \chi_{\varsigma}(\kappa_s)) : \delta_{\varsigma}(\kappa_r, \kappa_s) \geq 0\}$

and $\chi_{\upsilon}(\kappa_r)$, $\chi_{\xi}(\kappa_r)$ and $\chi_{\varsigma}(\kappa_r)$ denotes the degree of membership, indeterminacy and non membership of vertex respectively and $\delta_{\upsilon}(\overrightarrow{\kappa_r, \kappa_s})$, $\delta_{\xi}(\overrightarrow{\kappa_r, \kappa_s})$ and $\delta_{\varsigma}(\overrightarrow{\kappa_r, \kappa_s})$ denotes the degree of membership, indeterminacy and non membership of edges respectively.

Example 3.3. The graph in Figure 3 is represented by the notation $\vec{Ģ} = (g, \vec{\vartheta})$, with vertices $g = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and edges $\vartheta = \{(\overrightarrow{\kappa_1, \kappa_2}), (\overrightarrow{\kappa_1, \kappa_3}), (\overrightarrow{\kappa_4, \kappa_1}), (\overrightarrow{\kappa_3, \kappa_2})\}$

Definition 3.4. A GFNG $\vec{Ģ} = (g, \vec{\vartheta})$ is defined as GFNDG. The out-neighbourhood $\mathsf{F}(\kappa_r)$ of a vertex $\kappa_r \in g$ is denoted as $\mathsf{F}(\kappa_r) = \{\kappa_s, (\delta_{\upsilon}(\overrightarrow{\kappa_r, \kappa_s}), \delta_{\xi}(\overrightarrow{\kappa_r, \kappa_s}), \delta_{\varsigma}(\overrightarrow{\kappa_r, \kappa_s})) / (\overrightarrow{\kappa_r, \kappa_s}) \in \vartheta\}$.

Example 3.5. The graph in Figure 4 is represented by the notation $\vec{Ģ} = (g, \vec{\vartheta})$, with vertices $g = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and edges $\vartheta = \{(\overrightarrow{\kappa_1, \kappa_2}), (\overrightarrow{\kappa_1, \kappa_3}), (\overrightarrow{\kappa_1, \kappa_4}), (\overrightarrow{\kappa_3 \kappa_4}), (\overrightarrow{\kappa_2 \kappa_3})\}$

$\mathsf{F}(\kappa_1) = \{(\kappa_2, (0.7, 0.8, 0.6)), (\kappa_3, (0.9, 0.5, 0.6)), (\kappa_4, (0.6, 0.6, 0.7))\}$

$\mathsf{F}(\kappa_2) = \{(\kappa_3, (0.9, 0.8, 0.7))\}$ $\mathsf{F}(\kappa_3) = \{(\kappa_4, (0.9, 0.6, 0.7))\}$ $\mathsf{F}(\kappa_4) = \emptyset$

Definition 3.6. If $\vec{Ģ} = (g, \vec{\vartheta})$ is defined as GFNDG. The GFNCG $C(\vec{Ģ})$ of $\vec{Ģ} = (g, \vec{\vartheta})$ is GFNG that has the same vertex set g and contains a fermatean neutrosophic edge between κ_1 and κ_2 iff $\mathsf{F}(\kappa_1) \cap \mathsf{F}(\kappa_2) \neq \emptyset$. Furthermore, there exist sets

$\lambda_1 = \{\aleph^{\upsilon}_{\kappa_1} \kappa_1 \in g\}$, $\lambda_2 = \{\aleph^{\xi}_{\kappa_1} \kappa_1 \in g\}$, $\lambda_3 = \{\aleph^{\varsigma}_{\kappa_1} \kappa_1 \in g\}$ and functions $E_1 : \lambda_1 \times \lambda_1 \rightarrow [0,1]$, $E_2 : \lambda_2 \times \lambda_2 \rightarrow [0,1]$, $E_3 : \lambda_3 \times \lambda_3 \rightarrow [0,1]$ for each $(\kappa_1, \kappa_2) \in \vartheta$ where

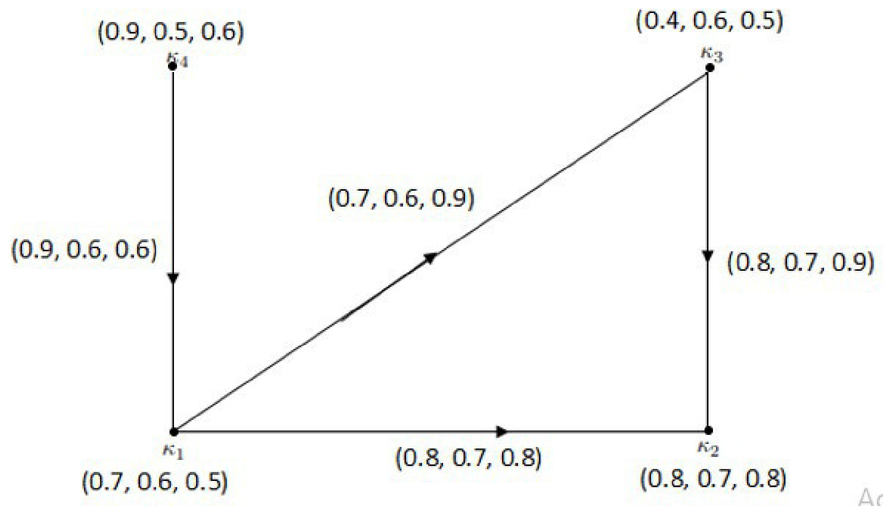


FIG2.PNG

Figure 3. GFNDG .

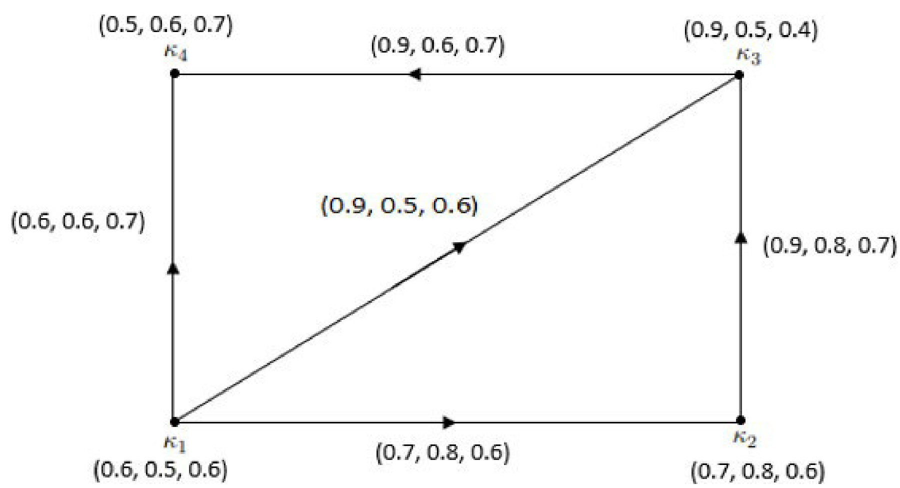


FIG3.PNG

Figure 4. FNDG.

$\delta_\upsilon(\kappa_1, \kappa_2) = E\left(\aleph^{\upsilon}_{\kappa_1}, \aleph^{\upsilon}_{\kappa_2}\right)$

$\delta_\xi(\kappa_1, \kappa_2) = E\left(\aleph^{\xi}_{\kappa_1}, \aleph^{\xi}_{\kappa_2}\right) -$

$\delta_\varsigma(\kappa_1, \kappa_2) = E\left(\aleph^{\varsigma}_{\kappa_1}, \aleph^{\varsigma}_{\kappa_2}\right)$

$\aleph^{\upsilon}_{\kappa_1} = min\{\delta_\upsilon(\overrightarrow{\kappa_1, \kappa_3}), \forall \kappa_3 \in F(\kappa_1) \cap F(\kappa_2)\}$

$\aleph^{\upsilon}_{\kappa_2} = min\{\delta_\upsilon(\overrightarrow{\kappa_2, \kappa_3}), \forall \kappa_3 \in F(\kappa_1) \cap F(\kappa_2)\}$

$\aleph^{\xi}_{\kappa_1} = max\{\delta_\xi(\overrightarrow{\kappa_1, \kappa_3}), \forall \kappa_3 \in F(\kappa_1) \cap F(\kappa_2)\}$

$\aleph^{\xi}_{\kappa_2} = max\{\delta_\xi(\overrightarrow{\kappa_2, \kappa_3}), \forall \kappa_3 \in F(\kappa_1) \cap F(\kappa_2)\}$

$\aleph^{\varsigma}_{\kappa_1} = max\{\delta_\varsigma(\overrightarrow{\kappa_1, \kappa_3}), \forall \kappa_3 \in F(\kappa_1) \cap F(\kappa_2)\}$

$\aleph^{\varsigma}_{\kappa_2} = max\{\delta_\varsigma(\overrightarrow{\kappa_2, \kappa_3}), \forall \kappa_3 \in F(\kappa_1) \cap F(\kappa_2)\}$

Example 3.7. The graph in Figure 4 is represented GFNDG $\vec{Ģ} = (g, \vec{\vartheta})$, with vertices $g = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and edges $\vartheta = \{(\overrightarrow{\kappa_1, \kappa_2}), (\overrightarrow{\kappa_1, \kappa_3}), (\overrightarrow{\kappa_1, \kappa_4}), (\overrightarrow{\kappa_3 \kappa_4}), (\overrightarrow{\kappa_2 \kappa_3})\}$
. The consequent competition graph (Figure 5)

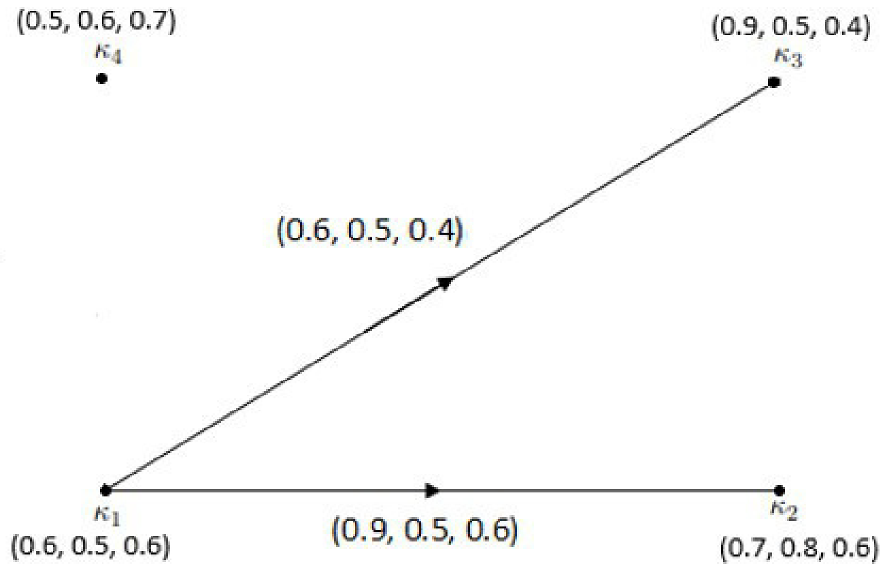


FIG4.PNG

Figure 5. GFNCG of graph (Figure 4).

Theorem 3.8. *If Ģ represent a GFNG, then there exist a GFNDG $\vec{Ģ}$ such that $C(\vec{Ģ}) = Ģ$*
proof Given a GFNG $Ģ = (g, \vartheta)$. (u_1, u_2) indicates an edge in Ģ, the goal is to create a GFNDG $\vec{Ģ}$, the competition graph $C(\vec{Ģ})$ is equal to Ģ. Let $\check{u}_1$ and $\check{u}_2$ be the corresponding vertices of u_1 and u_2 in Ģ. Then, from vertices $\check{u}_1$, $\check{u}_2$ we may construct two directed edges to a vertex $\check{u}_3 \in \vec{Ģ}$ such that $\check{u}_3 \in F(\check{u}_1) \cap F(\check{u}_2)$. In a similar manner, we may do this for each vertex and edge of Ģ, and as a result, $C(\vec{Ģ}) = Ģ$

Definition 3.9. Let Ģ represent a GFNG. The minimal graph $\vec{\check{Ģ}}$ of Ģ is a GFNDG with $C(\vec{\check{Ģ}}) = Ģ$ and $\vec{\check{Ģ}}$ has the minimum number of edges, i.e, if another graph, $\vec{\check{Ģ}}$ exists and $C(\vec{\check{Ģ}}^{1}) = Ģ$, then number of edges of $(\vec{\check{Ģ}})$ ≤ number of edges of $(\vec{\check{Ģ}}^{1})$.

Given a GFNCG, we may create a directed variant(a GFNDG) that emphasizes these competitive interactions. However, for a single GFNCG there may be many comparable digraphs with various amount of edges. Our objective is to identify the most compact digraph-one with the minimum number of edges- that appropriately represents the competition.

Theorem 3.10. *In a generalized fermatean neutrosophic connected graph $\vec{\check{Ģ}}$ which has an underlying complete graph with vertex n. The minimal graph of $\vec{\check{Ģ}}$ has 2n edges where $n \geq 2$.*

proof

Let $\overrightarrow{\check{G}}$ *be the connected GFNG which has an underlying complete graph with vertex n this means that each vertex is linked to every other vertex. Let k and r be two neighboring vertices in* $\overrightarrow{\check{G}}$ *and* k_1, r_1 *be the corresponding vertices in the minimal graph* $\overrightarrow{\check{G}}$. *Let* $\overrightarrow{\check{G}}$ *is a GFNDG every vertex except* k_1 *has only out neighbourhood as* k_1. *Hence* $\overrightarrow{\check{G}_1}$ *has* $n-1$ *edges. In a similar way, a GFNDG* $\overrightarrow{\check{G}_2}$ *is taken into consideration* r_1 *and consequently* $\overrightarrow{\check{G}_2}$. *There are* $n-1$ *edges in* $\overrightarrow{\check{G}_2}$. *Let us now examine GFNDG* $\overrightarrow{\check{G}_3}$ *including only the edges* $\overrightarrow{(k_1, u_1)}\overrightarrow{(r_1, u_1)}$. *Consequently, the combined graph* $\overrightarrow{\check{G}} = \overrightarrow{\check{G}_1} \cup \overrightarrow{\check{G}_2} \cup \overrightarrow{\check{G}_3}$ *has a total* $(n-1)+(n-1)+2=2n$ *edges.*

Definition 3.11. In a GFNG, the score of an edge (κ_1, κ_2) connecting the vertices is denoted by $S(\kappa_1, \kappa_2) = \dfrac{2\delta_\upsilon + \delta_\xi - 2\delta_\upsilon\delta_\varsigma}{3}$

Example 3.12. The GFNG in Figure 6 is represented by the notation $Ģ = (g, \vartheta)$, with vertices $g = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and edges $\vartheta = \{\overrightarrow{(\kappa_1, \kappa_2)}, \overrightarrow{(\kappa_1, \kappa_4)}, \overrightarrow{(\kappa_2, \kappa_3)}, \overrightarrow{(\kappa_3, \kappa_4)}, \overrightarrow{(\kappa_2, \kappa_4)}\}$

.

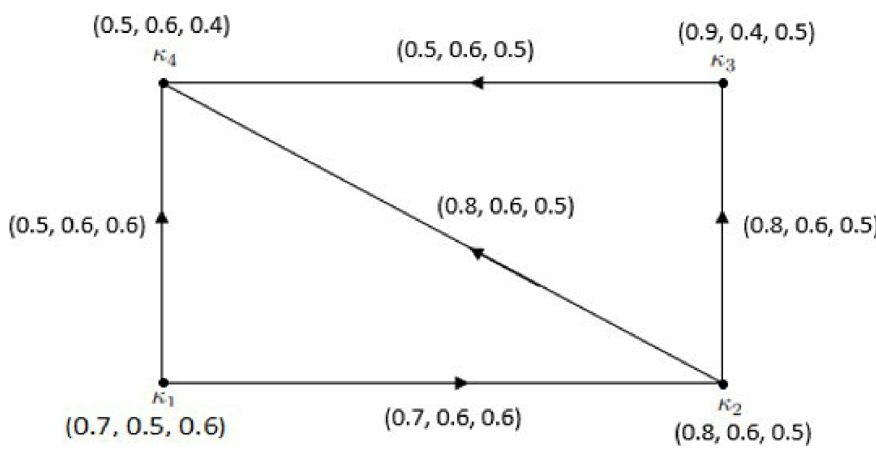


FIG5.PNG

Figure 6. GFNG .

Definition 3.13. The vertex κ_1 with neighbouring vertices $r_1, r_2, \ldots, r_h$ is considered isolated in GFNG if $S(\kappa_1, r_l)$ if $l = 1, 2, \ldots, h$.

Example 3.14. The GFNG in Figure 7 is represented by the notation $Ģ = (g, \vartheta)$, with vertices $g = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and edges $\vartheta = \{\overrightarrow{(\kappa_1, \kappa_2)}, \overrightarrow{(\kappa_1, \kappa_3)}, \overrightarrow{(\kappa_2, \kappa_3)}, \overrightarrow{(\kappa_2, \kappa_4)}\}$

. The neighboring vertex of κ_4 is κ_2, with the edge score (κ_2, κ_4) is 0, indicating the κ_4 is an isolated vertex.

Table 1. Edge score values.

Edges	Score Value
$\kappa_1\kappa_2$	0.386
$\kappa_2\kappa_4$	0.466
$\kappa_2\kappa_3$	0.466
$\kappa_3\kappa_4$	0.366
$\kappa_4\kappa_1$	0.33

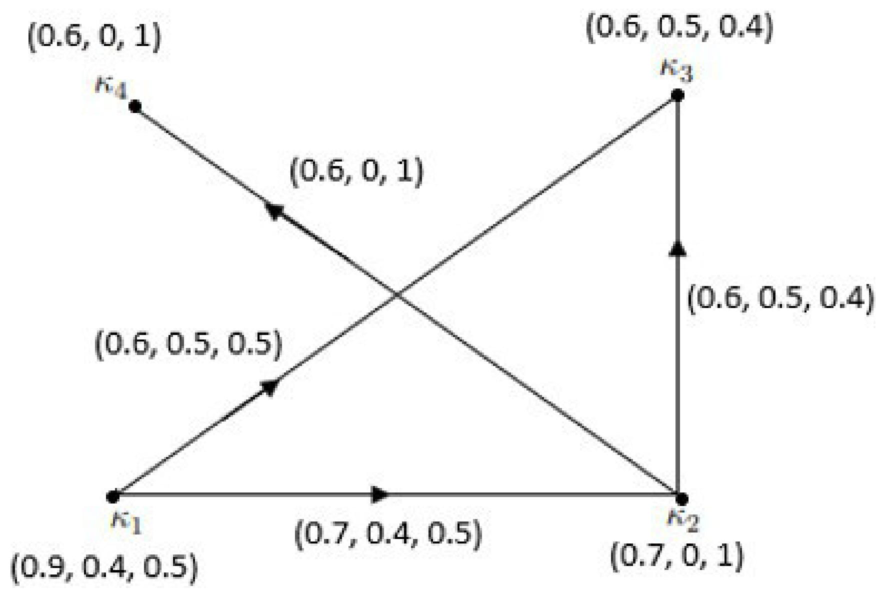


FIG6.PNG

Figure 7. GFNG with isolated vertex .

Definition 3.15. In a GFNG, a cycle with a length of more than 4 is referred to as a hole if each edge in the cycle has a score that is not zero.

Example 3.16. In Example 5, the graph $\kappa_1 - \kappa_2 - \kappa_3 - \kappa_4 - \kappa_1$ shows a 4-cycle with non-zero scores, indicating a hole.

Definition 3.17. In a generalized neighborhood graph, the competition number refers to the smallest isolated vertex, denoted by $C_F(Ģ)$.

Lemma 3.18. *A crisp graph with a single hole has a maximum completion number of two. A GFNG with a single hole may have a competition number larger than two.*

Consider a graph (Figure 8) with a single hole with competition number two. Edge Scores $(\overrightarrow{\kappa_1, \kappa_2})$, $(\overrightarrow{\kappa_2, \kappa_3})$, $(\overrightarrow{\kappa_3, \kappa_4})$ *and* $(\overrightarrow{\kappa_4, \kappa_1})$ *are non- zero by definition. However, the score of*

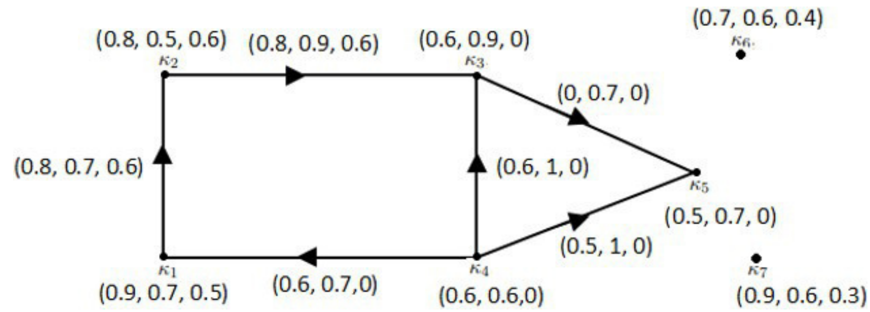


FIG7.PNG

Figure 8. GFNG with competition number 2 .

(κ_4, κ_5) *and* (κ_3, κ_5) *may be zero. Hence,* κ_5 *is an isolated vertex. The competition number is three.*

Definition 3.19. A fermatean neutrosophic chordal graph(FNCG) is one in which every hole has a chord with a score than zero.

Example 3.20. In Example 5, the graph if FNCG if the edges (κ_2, κ_4) are chords with non-zero scores and κ_1 is a hole.

Lemma 3.21. *A FNCG with a pendent vertex must have a competition number larger than one. The isolation of vertex* κ_5 *in the FNCG (Figure 9) results in a competition number larger than two.*

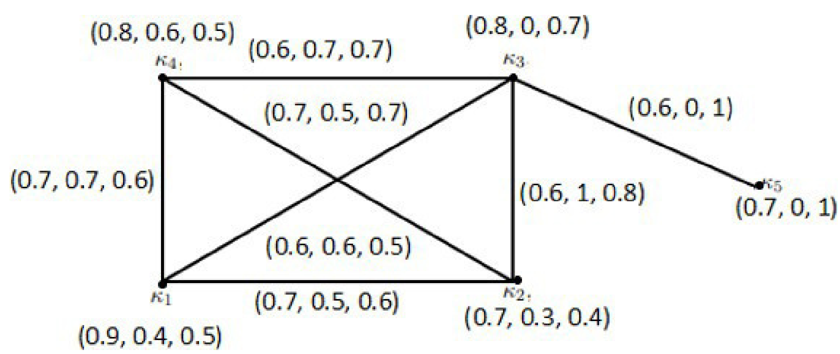


FIG8.PNG

Figure 9. FNCG .

4. GFNCG represented as a matrix

The following procedure calculates the elements of the adjacency matrix of a GFNCG.

(1) Consider GFNDG.

(2) Identify the vertices k_l and r_l for $l = 1, 2, ..., n$ such that their exist an edge $(k_l, u_p)(r_l, u_p)$ for $p = 1, 2, ..., m$ with $\digamma(k_l)$ and $\digamma(r_l)$

(3) Determine the set $\digamma(k_l) \cap \digamma(r_l) = \{u_q, q = 1, 2, ..., n\}$

(4) Compute

$$\aleph_k^{\upsilon} = min\left\{\delta_\upsilon\left(\overrightarrow{k_l, u_1}\right), \delta_\upsilon\left(\overrightarrow{k_l, u_2}\right), ..., \delta_\upsilon\left(\overrightarrow{k_l, u_n}\right)\right\}$$
$$\aleph_r^{\upsilon} = min\left\{\delta_\upsilon\left(\overrightarrow{r_l, u_1}\right), \delta_\upsilon\left(\overrightarrow{r_l, u_2}\right), ..., \delta_\upsilon\left(\overrightarrow{r_l, u_n}\right)\right\}$$
$$\aleph_k^{\xi} = max\left\{\delta_\xi\left(\overrightarrow{k_l, u_1}\right), \delta_\xi\left(\overrightarrow{k_l, u_2}\right), ..., \delta_\xi\left(\overrightarrow{k_l, u_n}\right)\right\}$$
$$\aleph_r^{\xi} = max\left\{\delta_\xi\left(\overrightarrow{r_l, u_1}\right), \delta_\xi\left(\overrightarrow{r_l, u_2}\right), ..., \delta_\xi\left(\overrightarrow{r_l, u_n}\right)\right\}$$
$$\aleph_k^{\varsigma} = max\left\{\delta_\varsigma\left(\overrightarrow{k_l, u_1}\right), \delta_\varsigma\left(\overrightarrow{k_l, u_2}\right), ..., \delta_\varsigma\left(\overrightarrow{k_l, u_n}\right)\right\}$$
$$\aleph_r^{\varsigma} = max\left\{\delta_\varsigma\left(\overrightarrow{r_l, u_1}\right), \delta_\varsigma\left(\overrightarrow{r_l, u_2}\right), ..., \delta_\varsigma\left(\overrightarrow{r_l, u_n}\right)\right\}$$

(5) For the pair of vertices k, r use functions E_1, E_2 and E_3 to get the combined membership degrees.

$$\delta_\upsilon(k, r) = E_1\left(\aleph_k^{\upsilon}, \aleph_r^{\upsilon}\right)$$
$$\delta_\xi(k, r) = E_1\left(\aleph_k^{\xi}, \aleph_r^{\xi}\right)$$
$$\delta_\varsigma(k, r) = E_1\left(\aleph_k^{\varsigma}, \aleph_r^{\varsigma}\right)$$

For the sake of simplicity, the functions E_1, E_2 and E_3 may be replaced by a single function E.

(6) A competition matrix is a square matrix. The number of vertices is equal to its order. The entries are as follows

$$\alpha_l = \begin{cases} \left(E_1\left(\aleph_l^{\upsilon}, \aleph_p^{\upsilon}\right), E_2\left(\aleph_l^{\xi}, \aleph_p^{\xi}\right), E_3\left(\aleph_l^{\varsigma}, \aleph_p^{\varsigma}\right)\right) & \text{if there exists an edge between l and p.} \\ (0, 0, 0) & \text{if there is no edge between l and p.} \end{cases}$$

Example 4.1. A matrix representation example is provided, complete with all phases.

Step 1: Consider GFNDG.

Step 2: $\digamma(k_1) = \{k_2\}$, $\digamma(k_2) = \{k_5\}$, $\digamma(k_3) = \{k_1, k_2\}$, $\digamma(k_4) = \{k_1, k_2, k_3\}$, $\digamma(k_5) = \{k_3\}$, $\digamma(k_6) = \{k_5\}$, $\digamma(k_7) = \{k_5\}$

Step 3: $\digamma(k_1) \cap \digamma(k_2) = \digamma(k_1) \cap \digamma(k_5) = \digamma(k_1) \cap \digamma(k_6) = \digamma(k_1) \cap \digamma(k_7) = \emptyset$, $\digamma(k_1) \cap \digamma(k_3) = \{k_2\}$, $\digamma(k_1) \cap \digamma(k_4) = \{k_2\}$, $\digamma(k_2) \cap \digamma(k_3) = \digamma(k_2) \cap \digamma(k_4) = \digamma(k_2) \cap \digamma(k_5) = \emptyset$, $\digamma(k_2) \cap \digamma(k_6) = \{k_5\}$, $\digamma(k_2) \cap \digamma(k_7) = \{k_5\}$, $\digamma(k_3) \cap \digamma(k_4) = \{k_1\}$, $\digamma(k_3) \cap \digamma(k_5) = \digamma(k_3) \cap \digamma(k_6) = \digamma(k_3) \cap \digamma(k_7) = \emptyset$, $\digamma(k_4) \cap \digamma(k_5) = \{k_3\}$, $\digamma(k_4) \cap \digamma(k_6) = \digamma(k_4) \cap \digamma(k_7) = \digamma(k_5) \cap \digamma(k_6) = \digamma(k_5) \cap \digamma(k_7) = \digamma(k_6) \cap \digamma(k_7) = \emptyset$

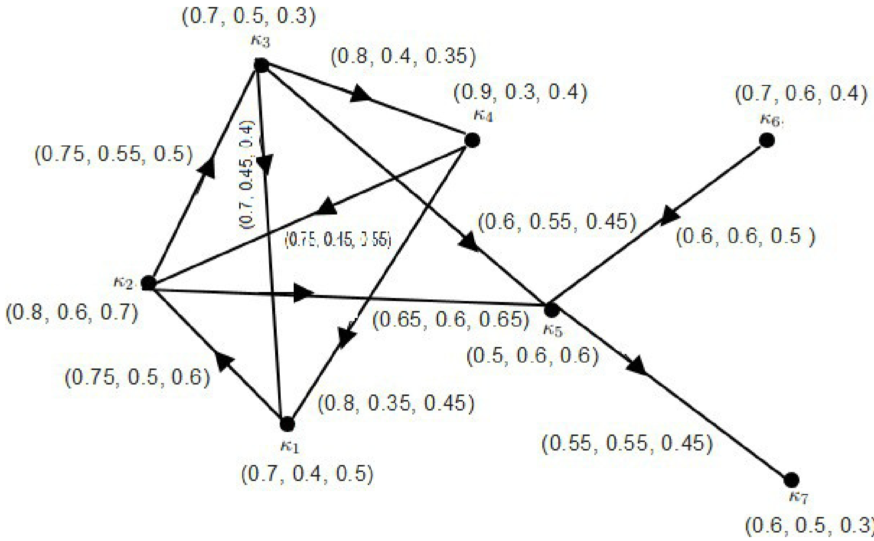


FIG9.PNG

Figure 10. GFNG with 7 vertices .

Step 4:

$\aleph_1^{\upsilon} = 0.75$	$\aleph_1^{\xi} = 0.5$	$\aleph_1^{\varsigma} = 0.6$
$\aleph_3^{\upsilon} = 0.75$	$\aleph_3^{\xi} = 0.55$	$\aleph_3^{\varsigma} = 0.5$
$\aleph_4^{\upsilon} = 0.8$	$\aleph_4^{\xi} = 0.4$	$\aleph_4^{\varsigma} = 0.35$
$\aleph_4^{\upsilon} = 0.8$	$\aleph_4^{\xi} = 0.35$	$\aleph_4^{\varsigma} = 0.45$
$\aleph_{31}^{\upsilon} = 0.7$	$\aleph_{31}^{\xi} = 0.45$	$\aleph_{31}^{\varsigma} = 0.4$
$\aleph_4^{\upsilon} = 0.75$	$\aleph_4^{\xi} = 0.45$	$\aleph_4^{\varsigma} = 0.55$
$\aleph_{53}^{\upsilon} = 0.6$	$\aleph_{53}^{\xi} = 0.55$	$\aleph_{53}^{\varsigma} = 0.45$
$\aleph_7^{\upsilon} = 0.55$	$\aleph_7^{\xi} = 0.55$	$\aleph_7^{\varsigma} = 0.45$
$\aleph_{25}^{\upsilon} = 0.65$	$\aleph_{25}^{\xi} = 0.6$	$\aleph_{25}^{\varsigma} = 0.65$
$\aleph_6^{\upsilon} = 0.6$	$\aleph_6^{\xi} = 0.6$	$\aleph_6^{\varsigma} = 0.5$

Step 5:

$\delta^{\upsilon}_{1} = 0$	$\delta^{\xi}_{1} = 0.1$	$\delta^{\varsigma}_{1} = 0.25$
$\delta^{\upsilon}_{1} = 0.3$	$\delta^{\xi}_{1} = 0.25$	$\delta^{\varsigma}_{1} = 0.25$
$\delta^{\upsilon}_{34} = 0.3$	$\delta^{\xi}_{34} = 0.25$	$\delta^{\varsigma}_{34} = 0.3$
$\delta^{\upsilon}_{45} = 0.4$	$\delta^{\xi}_{45} = 0.3$	$\delta^{\varsigma}_{45} = 0.35$
$\delta^{\upsilon}_{2} = 0.25$	$\delta^{\xi}_{2} = 0.35$	$\delta^{\varsigma}_{2} = 0$
$\delta^{\upsilon}_{2} = 0.3$	$\delta^{\xi}_{2} = 0.4$	$\delta^{\varsigma}_{2} = 0.25$
$\delta^{\upsilon}_{6} = 0.25$	$\delta^{\xi}_{6} = 0.25$	$\delta^{\varsigma}_{6} = 0.25$

Step 6:

–	(0,0,0)	(0,0.25,0.1)	(0.3,0.25,0.25)	(0,0,0)	(0,0,0)	(0,0,0)
(0,0,0)	–	(0,0,0)	(0,0,0)	(0,0,0)	(0.25,0,0.35)	(0.3,0.25,0.4)
(0,0.25,0.1)	(0,0,0)	–	(0.3,0.3,0.25)	(0,0,0)	(0,0,0)	(0,0,0)
(0.3,0.25,0.25)	(0,0,0)	(0.3,0.3,0.25)	–	(0.4,0.35,0.3)	(0,0,0)	(0,0,0)
(0,0,0)	(0,0,0)	(0,0,0)	(0.4,0.35,0.3)	–	(0,0,0)	(0,0,0)
(0,0,0)	(0.25,0,0.35)	(0,0,0)	(0,0,0)	(0,0,0)	–	(0.25,0.25,0.25)
(0,0,0)	(0.3,0.25,0.4)	(0,0,0)	(0,0,0)	(0,0,0)	(0.25,0.25,0.25)	–

5. **Application of key performance indicators in technology firms**

Many competitions exist in various aspects of everyday life, comparable to those seen in ecosystems. This study investigates the competition for technological innovation among major technology businesses in a fermatean neutrosophic environment. We take two things into consideration: Market share and R&D expenses. A company's market share is the total amount of sales it controls in its industry during a certain time period. The R&D Investment refers to the cash allocated for the company's research and innovation initiatives.

Market share increases represent true membership, whereas R&D spending measures non-membership. Uncertainty factors including market volatility, regulatory changes, and economic crises are measured against the level of indeterminacy membership. Data on market share and R&D investment are sourced from industry journals and business financial filings.

Leading technology companies such as Apple, Google, Microsoft, and Amazon are vying for technological superiority. Because all enterprises compete, the competition graph is complete. The membership values of the businesses (nodes) are displayed in tabular form (Table2 and Table3), whilst the membership values of edges are calculated using the following formula and provided in matrix style.

The matrix structure above depicts the competitiveness of technology businesses.

Table 2. Market share and R&D expenditures of technology firms.

Tech corporations	Market share	R&D expenditures
Apple	10	6.5
Microsoft	8.8	5.4
Google	7.2	7.5
Amazon	4	10
Meta	3.2	6.6
Samsung	2.8	4.3
Intel	2	3.6

Table 3. Normalized value of Market share and R&D expenditures of technology firms.

Tech corporations	NMS	N R&D	NMS ~NR&D
Apple	1	0.65	0.35
Microsoft	0.88	0. 54	0.34
Google	0.72	0.75	0.03
Amazon	0.4	0.10	0.6
Meta	0.32	0.66	0.34
Samsung	0.28	0.43	0.15
Intel	0.2	0.36	0.16

(1,0,1)	(0.06,0,0.055)	(0.14,0,0.05)	(0.3,0,0.175)	(0.34,0,0.005)	(0.36,0,0.11)	(0.4,0,0.145)
(0.06,0,0.055)	(1,0,1)	(0.08,0,0.105)	(0.24,0,0.23)	(0.28,0,0.06)	(0.3,0,0.055)	(0.34,0,0.09)
(0.14,0,0.05)	(0.08,0,0.105)	(1,0,1)	(0.16,0,0.125)	(0.2,0,0.045)	(0.02,0,0.085)	(0.26,0,0.195)
(0.3,0,0.175)	(0.24,0,0.23)	(0.16,0,0.125)	(1,0,1)	(0.04,0,0.17)	(0.06,0,0.285)	(0.1,0,0.32)
(0.34,0,0.005)	(0.28,0,0.06)	(0.2,0,0.045)	(0.4,0.35,0.3)	(1,0,1)	(0.02,0,0.115)	(.060,0,0.15)
(0.36,0,0.11)	(0.3,0,0.055)	(0.02,0,0.085)	(0.06,0,0.285)	(0.02,0,0.115)	(1,0,1)	(0.04,0,0.035)
(0.4,0,0.145)	(0.34,0,0.09)	(0.26,0,0.195)	(0.1,0,0.32)	(0.06,0,0.15)	(0.04,0,0.035)	(1,0,1)

The matrix structure above represents the competitiveness among tech corporations.

6. **Comparative Analysis**

The competitive environment in the technology industry is sophisticated and dynamic, demanding the adoption of current analytical tools to understand the intricate interrelationships between leading businesses. Such competitive settings have been described using Generalized Neutrosophic Competition Graphs (GNCGs), which provide a complex representation using truth, falsity, and indeterminacy memberships. GNCGs, on the other hand, have the potential to oversimplify competition through their linear combination strategy. Key Performance Indicators (KPIs) such as market share and R&D expenditures are also used to evaluate competitive performance; however, due to their quantitative nature, they frequently fail to capture the full range of dynamics, excluding indeterminate factors such as market volatility and regulatory changes. The introduction of Generalized Fermatean Neutrosophic Competition Graphs (GFNCGs) represents a substantial advance. GFNCGs use a complex algorithm to score edges between vertices that better captures the reality of competitive dynamics by allowing for uncertainty. This method provides more detailed insights for strategic planning and decision-making. Compared to GNCGs and KPIs, GFNCGs give a more comprehensive view of the competitive environment by combining quantitative and qualitative metrics. This is vital in the rapidly changing technology industry, where understanding the link between market share growth, innovation investment, and external uncertainty is essential for maintaining a competitive edge. Thus, GFNCGs are superior tools for modeling competitive scenarios in the technology sector because they provide a balanced analytical framework that captures the complexities and unpredictable nature of industrial rivalry.

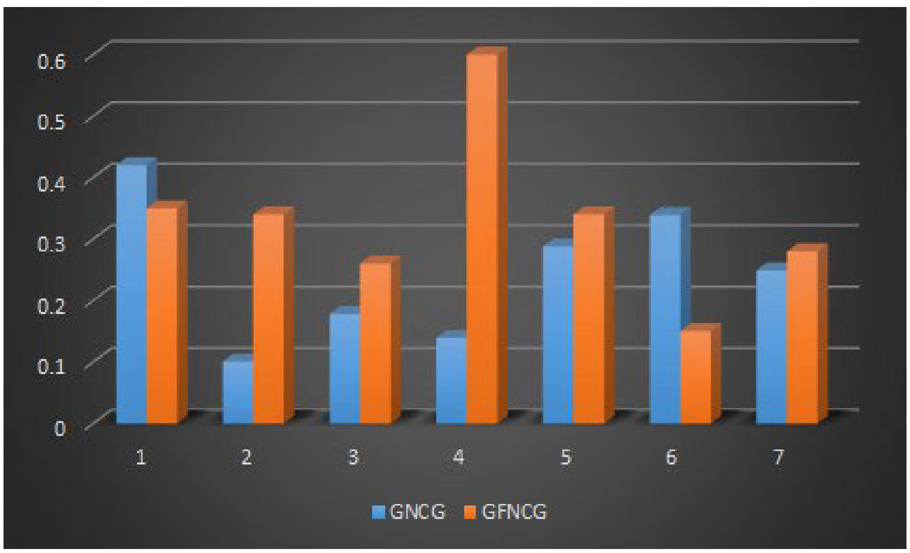


Figure 11. Comparision between GFNCG AND GNCG .

Conclusion

This work proposes a GFNCG that overcomes edge constraints. It depicts the GFNCG using a square matrix and explores concepts such as the minimal graph and competition number. In addition, the GFNCG framework is used to define a real-world application. In this application, nations' actual membership value is represented by their market share, whereas non-membership value is represented by the complement of their R&D spending. These criteria may be adjusted to capture different aspects of international competition, providing a useful perspective for studying real-world competitions. The research focuses on one-step competition, with plans to examine n-step fermatean neutrosophic competition graphs and other related concepts in the future. This study will serve as the foundation for subsequent research.

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Plithogenic Forest Hypersoft Sets in Plithogenic Contradiction Based Multi-Criteria Decision Making

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Abstract: Plithogenic based decision making approaches are more versatile and accommodative with regard to attribute handling This paper introduces the concepts of Plithogenic Forest hypersoft set (PFHS) and develops a Plithogenic centered decision-making model with PFHS representations. The Plithogenic method of devising decisions based on contradictions is integrated with the newly introduced representations of PFHS to develop a robust decision-making technique to deal with attributes and sub-attributive values at a larger scale. The integrated method proposed in this work is applied to a decision-making problem of site selection for establishing manufacturing plants. The core attributes are identified and the respective Plithogenic forest hypersoft sets are constructed with the possible sub-attributes. In this case, each of the core attributes itself forms a Plithogenic tree hypersoft set representations with several sub-attribute values, the alternatives are subjected to each of the criteria to determine the optimal ranking in specific to the criteria. Also, aggregate score values are determined to obtain a more comprehensive ranking. The concept of PFHS shall be integrated with other decision-making methods to evolve novel methods of decision-making.

Keywords: Plithogenic Forest Hypersoft sets, attributes, sub-attribute values, decision-making, site selection.

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1. Introduction

Smarandache [1] introduced the concept of hypersoft sets as an extension of soft sets to deal with attributes values subjected to each of the attributes say A1, A2,..An. The hypersoft set representations of the form Ai1, Aj2,..Akn are more compatible and comprehensive as they deal with various attribute values at a time whereas the soft sets deal with only a single attribute value at an instance.

The hypersoft sets shall be primarily classified into fuzzy, intuitionistic, neutrosophic based on the nature of the values indicating the magnitude of association between the attribute values and the elements of universe of discourse. The hypersoft sets are applied in several decision-making circumstances especially in ranking kind of multi-criteria decision-making problems. Researchers prefer these hypersoft sets of different types to handle the scenario involving several criteria or the attribute in designing optimal solution. Researchers have also discussed different forms of hypersoft sets such as Single-valued, multi-valued hypersoft sets[2], Bipolar Hypersoft sets[3], Picture Hypersoft sets[4], convex and concave hypersoft sets[5], N-hypersoft sets[6], bijective hypersoft sets[7] and many other as a means of extending the efficacy of hypersoft sets in data handling. Smarandache[8] extended hypersoft sets to Plithogenic hypersoft sets (PHS) in which the degree of appurtenance is presented to each of the attribute values with respect to the elements of the universe of discourse. Smarandache[20] has sketched out a vivid picture between Plithogenic soft sets and Plithogenic hypersoft sets. Plithogenic hypersoft sets are also applied in decision making to develop a more comprehensive solution to the decision-making problems. Researchers have developed PHS based decision models in different domains, To mention a few significant works, Martin and Smarandache[9] presented the applications of combined plithogenic hypersoft sets. Martin et al [10, 13] explored the applications of extended plithogenic hypersoft sets in Covid-19 decision making. Priya et al[11,12] induced the plithogenic cognitive analysis with combined connection. Martin and Smarandache[14] leveraged the notion of concentric plithogenic hypergraph embedded with Plithogenic hypersoft sets in decision making. Ahmad et al[15,17] formulated a multi-criteria decision-making model using plithogenic hypersoft sets. Rana et al[16,18] introduced plithogenic fuzzy whole hypersoft sets and generalized plithogenic whole hypersoft sets and applied multi-attribute decision making diagnostic models. Majid et al[19] formulated a decision model for site selection using plithogenic multipolar fuzzy hypersoft sets. These recent contributions substantiate the proficiency of plithogenic hypersoft sets in making optimal decisions.

Smarandache[21] also introduced few types of soft sets such as indeterm soft sets, indeterm hypersoft sets and tree soft sets. In the indeterm hypersoft sets, either any of the attributes deal with indeterminate values. In case of tree soft sets, the attributes that are considered form a tree structure with root attribute at level 0 and branches indicating the attribute values and attribute sub-values in the subsequent levels. However, in considering the attribute values, the tree soft set representations reflect tree hypersoft sets. This uniqueness of tree soft set has motivated the authors to evolve the concept of Plithogenic forest hypersoft sets (PFHS) which is a union of several Plithogenic tree soft sets (PTSS). The proposed notion of PFHS based representations is employed in making decisions on site selection on integrating with contradiction based Plithogenic decision method.

The remaining contents of the paper are structured into different sections as follows. Section 2 presents the conceptualization of Plithogenic Forest Hypersoft sets. Section 3 describes the decision-making model framework applied in this research work. Section 4 applies the model to the decision-making problem of site selection of manufacturing plants. Section 5 discusses the results and the last section concludes the work.

2. Theoretical Development of Plithogenic Forest Hypersoft Sets (PFHS)

This section presents the conceptualization of Forest Hypersoft sets and then describes the extension of the same to PFHS.

Let U be the universe of discourse, H be the non-empty subset of U, A be the set of attributes

Each of the attributes has different levels

Level 1 be the sub attribute values

Level 2 be the sub-sub attribute values:

Level n be the n-sub attribute values

Each of the attributes forms a Tree soft sets and all these tree soft sets together form a forest hypersoft sets.

The Forest Hypersoft Set shall be defined as G : P(Forest (A)) → P(H)

Where Forest (A) = {Tree (A)} and Tree (A) = {Ai1 | i1= 1, 2, ... }

Let us first discuss the construction of forest hypersoft sets with a simple example. Let us consider a decision-making problem on supplier selection based on different attributes say A1,A2,A3. In this case, the attributes A1, A2 and A3 forms the root level. Each of the attributes has different levels where each level indicates the sub attribute values. Thus, each attribute with its respective sets of sub attribute values together forms a tree soft set. In this case three tree soft sets are obtained for three different attributes. The union of these three tree soft sets together form a forest hypersoft sets.

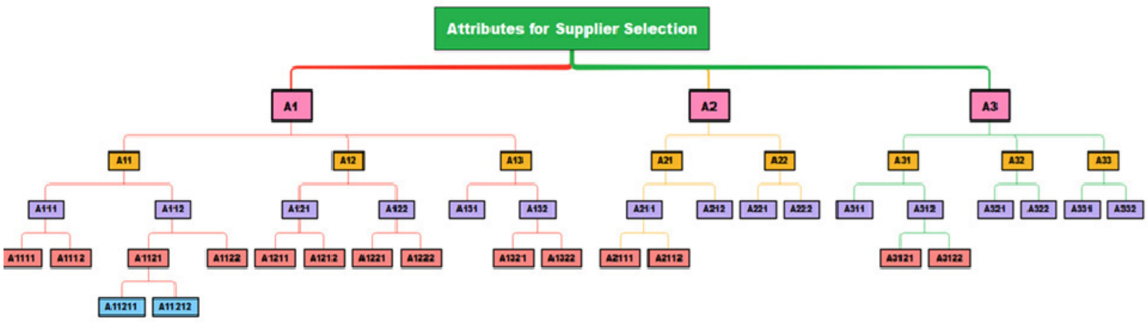

Fig.1. Plithogenic Forest Hypersoft sets

From the above fig.1., the attributes A1,A2 and A3 form the root level say L0. The next level L1 is subjected to each of the attributes which consists of the attribute values say for $A1$, the attribute sub values are A11, A12, A13 and for A2 it is, A21, A22 are the attribute values and then for A3 it is A31, A32 and A33. The level L2, consists of sub - sub – attribute values (sub_2 – attribute values) say for A11, the A111 and A1112 are the respective sub-attribute values however they are the sub_2 – attribute values of A1. Similarly, the attribute values A1111 and A1112 are the sub attribute values of A111 and sub_2 – attribute values of A11 and sub_3 – attribute values of A1. Also, A11211 and A11212 are the sub-attribute values of A1121, sub_2 – attribute values of A112, sub_3 – attribute values of A11 and sub_4 – attribute values of A1. A similar kind of discussion shall be made for the attributes A2 and A3.

If suppose the attribute A1 is considered for ranking the suppliers then, the attribute values from each of the sub_n attribute values has to be chosen and this will be of the form
F (A1112 × A11212 × A1211 × A1221 × A131 × A1322), this expression is derived from the representations of tree soft sets. If the decision maker considers the attribute A1, then the attribute values say A11, A12 and A13 have to be considered. In this case, each of the attribute values form a tree structure and hence ultimately at the end only the attribute values represented in last levels shall be taken for decision-making.

3. Classification of Forest Hypersoft sets

The forest hypersoft sets shall be classified generally as fuzzy forest hypersoft sets, intuitionistic forest hypersoft sets and neutrosophic hypersoft sets based on the membership values representing the degree of satisfaction, the alternatives make with the attribute values taken for consideration.

Example

Let the set of suppliers be {S1,S2,..Sn} and let us consider a set say H = {S1,S2,S3,S4}. Then let us consider the attribute values A1112 ,A11212 , A1211, A1221, A131 and A1322. Then Forest Hypersoft set is of the form F (A1112 × A11212 × A1211 × A1221 × A131 × A1322) = { S1 (0.2),S2 (0.5), S3 (0.5), S4 (0.7)} is termed as fuzzy forest hypersoft set.

F (A1112 × A11212 × A1211 × A1221 × A131 × A1322) = { S1 (0.2,0.7),S2 (0.5,0.4), S3 (0.5,0.3), S4 (0.7,0.2)} is termed as intuitionistic forest hypersoft set.

F (A1112 × A11212 × A1211 × A1221 × A131 × A1322) = { S1 (0.2,0.1,0.6),S2 (0.5,,0.2,0.4), S3 (0.5,0.3,0.4), S4 (0.7,0.2,0.1)} is termed as neutrosophic forest hypersoft set.

However, in case of Plithogenic forest hypersoft sets, the dominant attribute values are initially assigned and the contradiction degrees between the attribute values are determined. Then the degree of appurtenance of the alternative with respect to the dominant attribute values are determined and the contradiction degree are also considered in making decisions. For simplification, the plithogenic forest hypersoft sets are represented in a tabular form. Also, the Plithogenic forest hypersoft sets shall be classified based on their degree of appurtenance.

Table 1. Classification of Plithogenic forest hypersoft sets

Crisp	**A1112**	**A11212**	**A1211**	**A1221**	**A131**	**A1322**
S1	1	1	1	1	1	1
S2	1	1	1	1	1	1
S3	1	1	1	1	1	1
S4	1	1	1	1	1	1
S5	1	1	1	1	1	1
Fuzzy	**A1112**	**A11212**	**A1211**	**A1221**	**A131**	**A1322**
S1	0.2	0.4	0.5	0.3	0.7	0.8
S2	0.1	0.7	0.9	0.4	0.6	0.4
S3	0.5	0.6	0.3	0.2	0.7	0.1

S4	0.4	0.6	0.7	0.6	0.4	0.5
S5	0.6	0.4	0.2	0.7	0.5	0.8
Intuitionistic	**A1112**	**A11212**	**A1211**	**A1221**	**A131**	**A1322**
S1	(0.2, 0.8)	(0.4, 0.6)	(0.5, 0.5)	(0.3, 0.7)	(0.7, 0.3)	(0.8, 0.2)
S2	(0.1, 0.9)	(0.7, 0.3)	(0.9, 0.1)	(0.4, 0.6)	(0.6, 0.4)	(0.4, 0.6)
S3	(0.1, 0.9)	(0.7, 0.3)	(0.9, 0.1)	(0.4, 0.6)	(0.6, 0.4)	(0.4, 0.6)
S4	(0.4, 0.6)	(0.6, 0.4)	(0.7, 0.3)	(0.6, 0.4)	(0.4, 0.6)	(0.5, 0.5)
S5	(0.6, 0.4)	(0.4, 0.6)	(0.2, 0.8)	(0.7, 0.3)	(0.5, 0.5)	(0.8, 0.2)
Neutrosophic	**A1112**	**A11212**	**A1211**	**A1221**	**A131**	**A1322**
S1	(0.2,0.2,0.7)	(0.4,0.1,0.5)	(0.5,0.1,0.4)	(0.3,0.2,0.6)	(0.7,0.1,0.2)	(0.8,0.01,0.1)
S2	(0.1,0.15,0.8)	(0.7,0.20,0.2)	(0.9,0.1,0.0)	(0.4,0.1,0.5)	(0.6,0.2,0.3)	(0.4, 0.1, 0.5)
S3	(0.5, 0.1, 0.4)	(0.6, 0.2, 0.3)	(0.3,0.1,0.6)	(0.2,0.1,0.7)	(0.7,0.1,0.2)	(0.1, 0.1, 0.8)
S4	(0.4, 0.1, 0.5)	(0.6, 0.1, 0.3)	(0.7,0.2,0.2)	(0.6,0.1,0.3)	(0.4,0.1,0.5)	(0.5, 0.1, 0.4)
S5	(0.6, 0.1, 0.3)	(0.4, 0.1, 0.5)	(0.2,0.1,0.7)	(0.7,0.1,0.2)	(0.5,0.1,0.4)	(0.8, 0.1, 0.1)

In the above table 1., the values in each cell indicate the degrees of appurtenance of the suppliers with each of the dominant attribute values. The above table demonstrates the classifications of Plithogenic forest hypersoft sets into crisp, fuzzy, intuitionistic and neutrosophic based on the appurtenance values. In other representations of Forest Hypersoft sets the appurtenance degree reflect all the attribute values whereas in case of Plithogenic Forest Hypersoft sets, the appurtenance degree is considered for each of the attribute values.

Modelling Framework of Plithogenic Forest Hypersoft sets based Decision Making [22]

This section outlines the working procedure of the Plithogenic contradiction-based decision method with the representations of Plithogenic Forest Hypersoft sets.

Step 1: The decision making ecosystem is well defined by determining the alternatives say S1, S2,..Sn and attributes with sub_n attribute values. The dominant attribute values are identified and classified as benefit and non-benefit.

Step 2: The initial contradiction matrix considering the alternatives and the dominant attribute values is constructed from the attribute matrix with each cell representing the attribute value of the alternatives. The qualitative attribute matrix is then converted to the contradiction matrix by calculating the degree of contradiction existing between the attribute value of the alternative and the dominant attribute value.

	A_{1i}	A_{2j}	…..	…..	A_{nh}
S_1	A_{11}	A_{23}	…..	…..	A_{n4}
S_2	…..	…..	…..	…..	…..
:	…..	…..	…..	…..	…..
:	…..	…..	…..	…..	…..
S_n	A_{14}	A_{21}	…..	…..	A_{n2}

The above matrix representation is the attribute matrix which is constructed by considering the attribute values possessed by the alternatives in par with the dominant attribute values. The respective contradiction matrix is drawn by considering the degree of contradiction between the attribute values in the matrix and the dominant attribute values.

	A_{1i}	A_{2j}	…..	…..	A_{nh}
S_1	$C(A_{11}, A_{1i})$	$C(A_{23}, A_{2j})$	…..	…..	$C(A_{n4}, A_{nh})$
S_2	…..	…..	…..	…..	…..
:	…..	…..	…..	…..	…..
:	…..	…..	…..	…..	…..
S_n	$C(A_{14}, A_{1i})$	$C(A_{21}, A_{2j})$	…..	…..	$C(A_{n2}, A_{nh})$

The above is the contradiction matrix obtained from the above attribute matrix, where each of the cell values represents the contradiction degrees between the dominant attribute values and the actual attribute value of the alternatives

Step 3: The weighted contradiction matrix is calculated by multiplying the attribute value weights with the contradiction degrees.

	A_{1i}	A_{2j}	…..	…..	A_{nh}
S_1	$w_1C(A_{11},A_{1i})$	$w_2C(A_{23}, A_{2j})$	…..	…..	$w_nC(A_{n4},A_{nh})$
S_2	…..	…..	…..	…..	…..
:	…..	…..	…..	…..	…..
:	…..	…..	…..	…..	…..
S_n	$w_1C(A_{14},A_{1i})$	$w_2C(A_{21}, A_{2j})$	…..	…..	$w_nC(A_{n2},A_{nh})$

Step 4: The cumulative score values of the alternatives with respect to the benefit nature of attribute values and the cost nature of attribute values are determined. The cumulative score values of the benefit nature of attribute values are denoted by B_q and the cumulative score values of the cost nature of attribute values are denoted by C_k.

Step 5: The alternatives are ranked based on the differences between these benefit and non-benefit attribute values i.e B_q - $C_k$and the highest rankings are assigned to the alternatives with maximum differences of values.

The above working procedure shall be characterized into three major phased process. The first phase comprises defining decision making ecosystem, construction of attribute matrix and contradiction matrix. The second phase consists of steps 3 and 4 and the last phase consists of step 5.

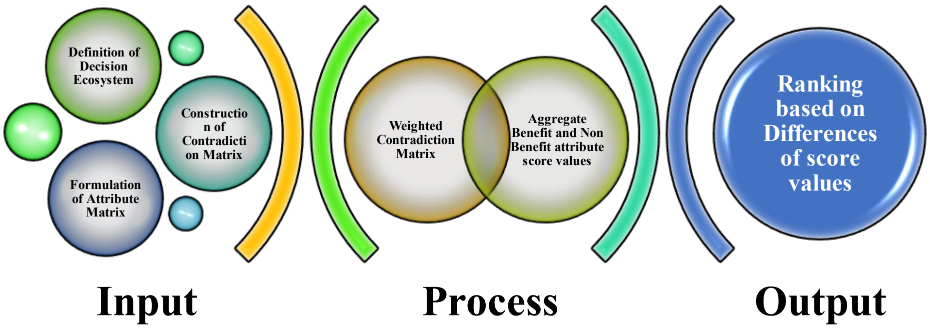

Fig.2. Process of three major phased

4. Application of the Proposed Modeling Framework in site selection

This section presents the modality of applying Plithogenic contradictions-based decision making with Plithogenic hypersoft sets in site selection for establishing manufacturing plants say P1,P2,P3,P4,P5. The core attributes considered in general are presented in Fig.3.

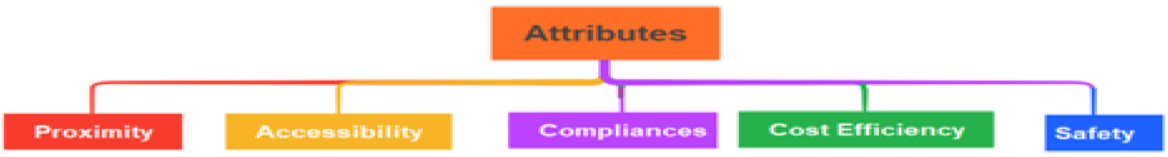

Fig.3. Classification of Attributes

The core attributes occupy the root level of the forest and each of the attributes takes a form of tree hypersoft sets and it is presented as follows.

Firstly, the attribute Proximity is considered. The following figure 4, represents the tree representations of the attribute Proximity and its sub attribute values at different levels.

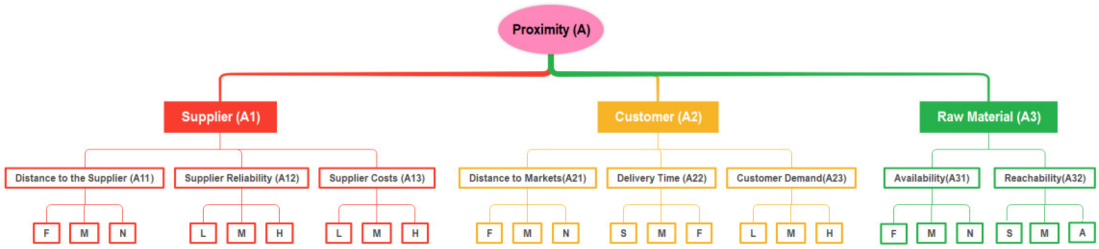

Fig.4.Proximity

The description of the attributes in the above fig.4. is sketched out in the following table 2.

Table 2. Description of the attributes

Attribute	**Sub-Attribute values**	**Sub₂ Attribute Values (Sub-sub Attribute Values)**	**Sub₃Attribute Values (Sub-sub – sub–Attribute Values)**
Proximity (A)	Supplier (A1)	Distance to the Supplier (A11)	Far (A111) Moderate (A112) **Near (A113)**
		Supplier Reliability (A12)	Low (A121) Moderate (A122) **High (A123)**
		Supplier Costs (A13)	**Low (A131)** Moderate (A132) High (A133)
	Customer (A2)	Distance to Markets (A21)	Far (A211) Moderate (A212) **Near (A213)**
		Delivery Time (A22)	Slow (A221) Moderate (A222) **Fast (A223)**
		Customer Demand (A23)	Low (A231) Moderate (A232) **High (A233)**
	Raw Material (A3)	Proximity to Sources (A31)	Far (A311) Moderate (A312) **Near (A313)**
		Raw material Availability (A32)	Scare (A321) Moderate (A322) **Abundant (A323)**

If the criteria proximity is alone considered the dominant sub-sub-sub attribute values have to be considered which are {Near, High, Low, Near, Fast, High, Near, Abundant}
Let us label it as P(A1111× A1123×A1131×A1213×A1223×A1233×A1313× A1323)

Table 3. Attribute matrix

Attributive Values	**P1**	**P2**	**P3**	**P4**	**P5**
A1133	Far	Far	Moderate	Near	Near
A1233	Low	Moderate	High	Low	Low
A1311	Moderate	Low	Moderate	High	Low
A2133	Near	Moderate	Fast	Moderate	Moderate
A2233	Moderate	Fast	Fast	Slow	Slow
A2333	High	Low	Low	Moderate	High
A3133	Scarce	Abundant	Moderate	Abundant	Scarce
A3233	Moderate	Near	Far	Far	Far

The above table 3, represents the attribute matrix.
The contradiction degrees with respect to the dominant attribute values are presented in Table 4.

Table 4. Contradiction value

Dominant Attribute Value	**Contradiction Degree**
A113	C(A111,A113) =2/3, C(A112,A113) =1/3
A123	C(A121,A123) =2/3, C(A122,A123) =1/3
A131	C(A132,A131) =1/3, C(A133,A131) =2/3
A213	C(A211,A213) =2/3, C(A212, A213)= 1/3
A223	C(A221,A223) =2/3, C(A222,A223) =1/3
A233	C(A231,A233) =2/3, C(A232, A233) =1/3
A313	C(A311, A313)=2/3, C(A312,A313)=1/3
A323	C(A321,A323) =2/3, C(A322,A323)= 1/3

Based on the contradiction values presented in the above Table 4, the respective contradiction matrix is as follows.

Table 5. Contradiction matrix

Attributive Values	**P1**	**P2**	**P3**	**P4**	**P5**
A1133 (C)	2/3	2/3	1/3	0	0

Attributive Values	P1	P2	P3	P4	P5
A1233 (B)	2/3	1/3	0	2/3	2/3
A1311 (C)	1/3	0	1/3	2/3	0
A2133 (C)	0	1/3	2/3	1/3	1/3
A2233 (C)	1/3	0	0	2/3	2/3
A2333 (B)	0	2/3	2/3	1/3	0
A3133 (C)	2/3	0	1/3	0	2/3
A3233 (B)	1/3	0	2/3	2/3	2/3

In the above matrix, the contradiction values of the attribute values with respect to the dominant attribute values are presented in the contradiction matrix. The weighted contradiction matrix is determined by considering equal weightage to all the dominant attribute values.

Table 6. Weighted contradiction matrix

Alternatives	A1133 (C)	A1233 (B)	A1311 (C)	A2133 (C)	A2233 (C)	A2333 (B)	A3133 (C)	A3233 (B)
P1	0.08333	0.08333	0.041667	0	0.041667	0	0.08333	0.041667
P2	0.08333	0.041667	0	0.041667	0	0.08333	0	0
P3	0.041667	0	0.041667	0.08333	0	0.08333	0.04167	0.08333
P4	0	0.08333	0.08333	0.041667	0.08333	0.04167	0	0.08333
P5	0	0.08333	0	0.041667	0.08333	0	0.08333	0.08333

By using step 4 and step 5, the score values of the alternatives are determined and presented in Table 7.

Table 7. Differences in Score values

Alternatives	B_A	C_A	$B_A - C_A$
P1	0.124997	0.249994	-0.124997
P2	0.125	0.124997	3E-06
P3	0.16666	0.208337	-0.04168
P4	0.20833	0.208327	3E-06
P5	0.16666	0.208327	-0.04167

Secondly, the attribute Accessibility is considered. The following figure 5. represents the tree representations of the attribute Accessibility and its sub attribute values at different levels.

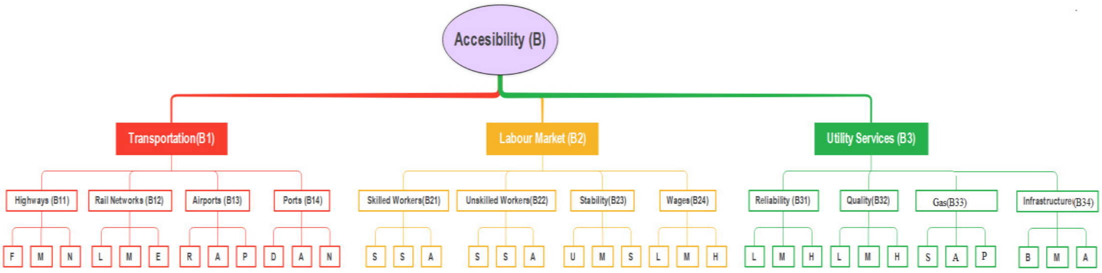

Fig.5. Accessibility

The description of the attributes in the above fig.5. is sketched out in the following table 8.

Table 8. Description of the attributes

Attribute	**Sub-Attribute values**	**Sub₂ Attribute Values (Sub-sub–Attribute Values)**	**Sub₃Attribute Values (Sub-sub – sub–Attribute Values)**
Accessibility (B)	Transportation (B1)	Proximity to Highways (B11)	Far (B111) Moderate (B112) **Near (B113)**
		Access to Rail Networks (B12)	Limited (B121) Moderate (B122) **Extensive (B123)**
		Access to Airports (B13)	Remote (B131) Accessible (B132) **Proximate (B133)**
		Access to Ports (B14)	Distant (B141) Accessible (B142) **Nearby (B143)**
	Labour Market (B2)	Availability to Skilled Workers (B21)	Scare (B211) Sufficient (B212) **Abundant (B213)**
		Availability to Unskilled Workers (B22)	Scare (B221) Sufficient (B222) **Abundant (B223)**
		Labor Market Stability (B23)	Unstable (B231) Moderate (B232) **Stable (B233)**
		Wage Labour (B24)	**Low (B241)** Moderate (B242) High (B243)

	Utility Services (B3)	Reliability of Electricity (B31)	Unreliable (B311) Moderate (B312) **Reliable (B313)**
		Availability and Quality of Water (B32)	Limited (B321) Adequate (B322) **Excellent (B323)**
		Availability of Gas (B33)	Scare(B331) Available (B332) **Plentiful (B333)**
		Telecommunications Infrastructure (B34)	Basic (B341) Moderate (B342) **Advanced(B343)**

If the criteria Accessibility is alone considered the dominant (sub_3) sub-sub-sub attribute values to be considered are {Near, Extensive, Proximate, Nearby, Abundant, Stable, Low, Reliable, Excellent, Plentiful, Advanced}

Let us consider the representation of the form

P(B1133xB1233xB1333xB1433xB2133xB2233xB233xB2411xB313xB3233xB3333xB3433)

The attribute matrix with respect to the attribute Accessibility is presented in Table 9.

Table 9. Attribute matrix

Attributive Values	**P1**	**P2**	**P3**	**P4**	**P5**
B1133	Near	Far	Moderate	Near	Far
B1233	Limited	Moderate	Limited	Extensive	Moderate
B1333	Accessible	Proximate	Proximate	Remote	Accessible
B1433	Nearby	Accessible	Distant	Nearby	Accessible
B2133	Scare	Abundant	Sufficient	Abundant	Scare
B2233	Sufficient	Scare	Scare	Sufficient	Abundant
B2333	Stable	Unstable	Stable	Unstable	Moderate
B2411	Moderate	Low	High	Moderate	High
B3133	Reliable	Moderate	Unreliable	Moderate	Unreliable
B3233	Adequate	Excellent	Limited	Adequate	Excellent
B3333	Plentiful	Available	Scare	Plentiful	Scare
B3433	Basic	Advanced	Basic	Moderate	Advanced

The contradiction degrees with respect to the dominant attribute values are presented in Table 10.

Table 10. Contradiction value

Dominant Attribute Value	Contradiction Degree
B113	C(B111, B113) = 2/3, C(B112, B113) =1/3
B123	C(B121, B123) = 2/3, C(B122, B123) =1/3
B133	C(B131, B133) =2/3, C(B132, B133) =1/3
B143	C(B141, B143) = 2/3, C(B142, B143) =1/3
B213	C(B211, B213) = 2/3, C(B212, B213) =1/3
B223	C(B221, B223) = 2/3, C(B222, B223) =1/3
B233	C(B231, B233) = 2/3, C(B232, B233) =1/3
B241	C(B242, B241) =1/3, C(B243, B241) =2/3
B313	C(B311, B313) =2/3, C(B312, B313) =1/3
B323	C(B321, B323) =2/3, C(B322, B323) =1/3
B333	C(B331, B333) = 2/3, C(B332, B333) =1/3
B343	C(B341, B343) =2/3, C(B412, B343) =1/3

The respective contradiction matrix is as follows and presened in Table 11.

Table 11. Contradiction matrix

Attributive Values	P1	P2	P3	P4	P5
B1133 (C)	0	2/3	1/3	0	2/3
B1233(C)	2/3	1/3	2/3	0	1/3
B1333 (C)	1/3	0	0	2/3	1/3
B1433 (C)	0	1/3	2/3	0	1/3
B2133(B)	2/3	0	1/3	0	2/3
B2233(B)	1/3	2/3	2/3	1/3	0
B2333(B)	0	2/3	0	2/3	1/3
B2411(C)	1/3	0	2/3	1/3	2/3
B3133(B)	0	1/3	2/3	1/3	2/3
B3233(B)	1/3	0	2/3	1/3	0
B3333(B)	0	1/3	2/3	0	2/3
B3433(B)	2/3	0	2/3	1/3	0

The weighted contradiction matrix is as follows

Table 12. Weighted contradiction matrix

Alternatives	B113 (C)	B123 (C)	B133 (C)	B143 (C)	B213 (B)	B223 (B)	B233 (B)	B241 (C)	B313 (B)	B323 (B)	B333 (B)	B343 (B)
P1	0	0.05556	0.027778	0	0.05556	0.027778	0	0.027778	0	0.027778	0	0.05556
P2	0.05556	0.027778	0	0.027778	0	0.05556	0.05556	0	0.027778	0	0.027778	0
P3	0.027778	0.05556	0	0.05556	0.027778	0.05556	0	0.05556	0.05556	0.05556	0.05556	0.05556
P4	0	0	0.05556	0	0	0.027778	0.05556	0.027778	0.027778	0.027778	0	0.027778
P5	0.05556	0.027778	0.027778	0.027778	0.05556	0	0.027778	0.05556	0.05556	0	0.05556	0

By following the similar procedure, the score values are presented in Table 13

Table 13. Difference in Score values

Alternatives	B_B	C_B	B_B- C_B
P1	0.111116	0.166676	0.05556
P2	0.111116	0.166676	0.05556
P3	0.194458	0.305578	0.11112
P4	0.083338	0.166672	0.083334
P5	0.194454	0.194458	4E-06

Thirdly the attribute Compliances is taken into account. Fig.6., represents the tree representations of the attribute

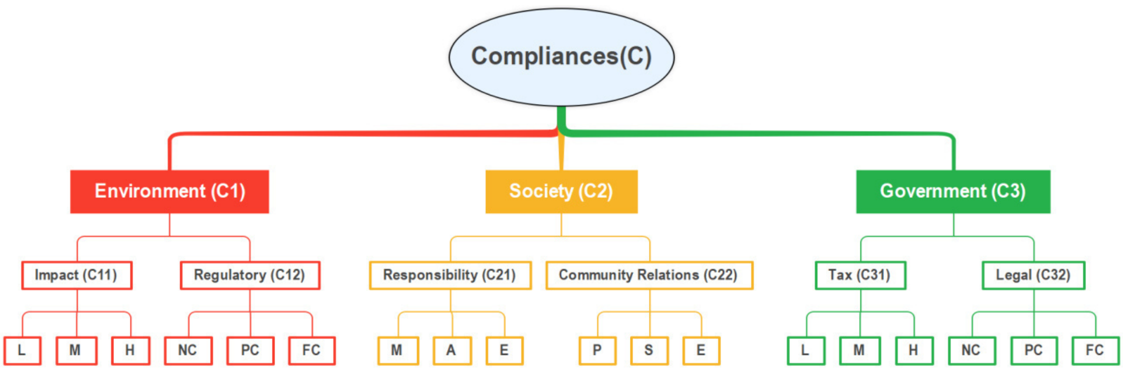

Fig.6. Compliances

The description of the attributes in the above fig.6. is sketched out in the following table 14.

Table 14. Description of attributes

Attribute	Sub-Attribute values	Sub2 Attribute Values (Sub-sub Attribute Values)	Sub3Attribute Values (Sub-sub – sub–Attribute Values)
Compliances (C)	Environment (C1)	Environmental Impact (C11)	**Low (C111)** Moderate (C112) High (C113)
		Regulatory Compliance (C12)	Non-Compliant (C121) Partially Compliant (C122) **Fully Compliant (C123)**
	Society (C2)	Social Responsibility (C21)	Minimal (C211) Adequate (C212) **Extensive(C213)**
		Community Relations (C22)	Poor (C221) Satisfactory (C222) **Excellent (C223)**
	Government (C3)	Tax Compliance (C31)	Low(C311) Moderate (C312), **High (C313)**
		Legal Compliance (C32)	Non- Compliant (C321) Partially Compliant (C322) **Fully Compliant (C323)**

If the criteria Compliances is alone considered the dominant sub-sub-sub attribute values to be considered are {Low, Fully Compliant, Extensive, Excellent, High, Fully Compliant}

Let us consider the representation of the form P(C1111xC1233xC2133xC2233xC3133xC3233)

The attribute matrix with respect to the attribute Compliances is presented in Table 15.

Table 15. Attribute matrix

Attributive Values	P1	P2	P3	P4	P5
C1111	High	Low	Moderate	High	Moderate
C1233	Partially Compliant	Non-Compliant	Non-Compliant	Partially Compliant	Fully Compliant
C2133	Extensive	Adequate	Minimal	Adequate	Minimal
C2233	Poor	Satisfactory	Excellent	Satisfactory	Excellent

Attributive Values	P1	P2	P3	P4	P5
C3133	Moderate	High	Low	Moderate	High
C3233	Non-Compliant	Partially Compliant	Fully Compliant	Non-Compliant	Fully Compliant

The contradiction degrees with respect to the dominant attribute values are presented in Table 16.

Table 16. Contradiction value

Dominant Attribute Value	Contradiction Degree
C111	C(C112, C111) = 1/3, C(C113, C111) = 2/3
C123	C(C121, C123) = 2/3, C(C122, C123) = 1/3
C213	C(C211, C213) = 2/3, C(C212 ,C213) = 1/3
C223	C(C221, C223)= 2/3, C(C222, C223) = 1/3
C313	C(C311,C313) = 2/3, C(C312, C313) = 1/3
C323	C(C321, C323) = 2/3 , C(C322, C323) = 1/3

The respective contradiction matrix is as follows and presented in Table 17.

Table 17. Contradiction matrix

Attributive Values	P1	P2	P3	P4	P5
C1111(C)	2/3	0	1/3	2/3	1/3
C1233(B)	1/3	2/3	2/3	1/3	0
C2133(B)	0	1/3	2/3	1/3	2/3
C2233(B)	2/3	1/3	0	1/3	0
C3133(C)	1/3	0	2/3	1/3	0
C3233(C)	2/3	1/3	0	2/3	0

The weighted contradiction matrix is as follows,

Table 18. Weighted contradiction matrix

Alternatives	C1111(C)	C1233(B)	C2133(B)	C2233(B)	C3133(C)	C3233(C)
P1	0.11111	0.055556	0	0.11111	0.055556	0.11111
P2	0	0.11111	0.055556	0.055556	0	0.055556

Alternatives	C1111(C)	C1233(B)	C2133(B)	C2233(B)	C3133(C)	C3233(C)
P3	0.055556	0.11111	0.11111	0	0.11111	0
P4	0.11111	0.055556	0.055556	0.055556	0.055556	0.11111
P5	0.055556	0	0.11111	0	0	0

By following the similar procedure, the score values are obtained and presented in Table 19.

Table 19. Difference in Score values

Alternatives	B_C	C_C	B_C- C_C
P1	0.166666	0.277776	-0.11111
P2	0.222222	0.055556	0.166666
P3	0.22222	0.166666	0.055554
P4	0.166668	0.277776	-0.11111
P5	0.11111	0.055556	0.055554

Fourthly the attribute Cost Efficiency is taken into account. Fig.7. represents the tree representations of the attribute

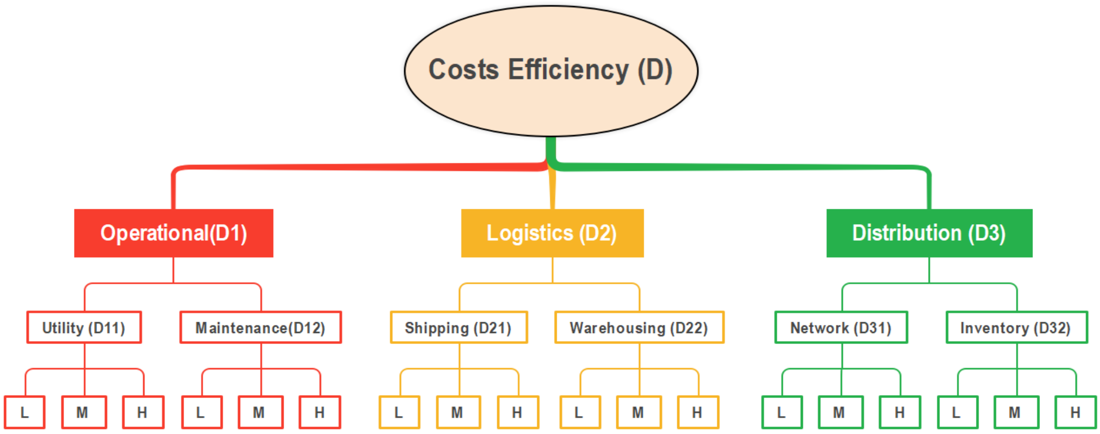

Fig.7. Cost Efficiency

The description of the attributes in the above fig.7. is sketched out in the following table 20.

Table 20. Description of Attributes

Attribute	Sub-Attribute values	Sub_2 Attribute Values (Sub-sub Attribute Values)	Sub_3Attribute Values (Sub-sub – sub–Attribute Values)
Costs Efficiency (D)	Operation (D1)	Utility Costs(D11)	**Affordable (D111)** Moderate (D112) Expensive (D113)
		Maintenance Costs (D12)	**Low (D121)** Moderate (D122) High (D123)
	Logistics (D2)	Shipping and Freight Costs (D21)	**Economical (D211)** Moderate (D212) Expensive (D213)
		Ware housing Costs (D22)	**Low (D221)** Moderate (D222) High (D223)
	Distribution (D3)	Distribution Network Costs (D31)	**Low-Cost (D311)** Moderate (D312) High-cost (D313)
		Inventory Holding Costs (D32)	**Low (D321)** Moderate (D322) High (D323)

If the criteria Cost Efficiency is alone considered the dominant sub-sub-sub attribute values to be considered are {Affordable, Low, Economical, Low, Low- Cost, Low}

Let us consider the representation of the form P(D1111xD1211xD2111xD2211xD3111xD3211)

The attribute matrix with respect to the attribute Cost Efficiency is presented in Table 21.

Table 21. Attribute matrix

Attributive Values	P1	P2	P3	P4	P5
D1111	Affordable	Moderate	Expensive	Affordable	Moderate
D1211	Moderate	High	Low	High	Low
D2111	Expensive	Economical	Moderate	Moderate	Expensive
D2211	Low	Moderate	High	Moderate	High
D3111	Moderate	Low-cost	Moderate	High-cost	Low-cost
D3211	High	Moderate	Low	Low	High

The contradiction degrees with respect to the dominant attribute values are presented in Table 22.

Table 22. Contradiction value

Dominant Attribute Value	Contradiction Degree
D111	C(D112, D111) = 1/3, C(D113, D111) = 2/3
D121	C(D122,D121) = 1/3 , C(D123, D121) =2/3
D211	C(D212, D211) = 1/3, C(D213, D211) =2/3
D221	C(D222, D221) = 1/3 , C(D223, D221) =2/3
D311	C(D312, D311) = 1/3 , C(D313, D311) = 2/3
D321	C(D322, D321) = 1/3 , C(D323, D321) = 2/3

The respective contradiction matrix obtained as follows

Table 23. Contradiction matrix

Attributive Values	P1	P2	P3	P4	P5
D1111(C)	0	1/3	2/3	0	1/3
D1211(B)	1/3	2/3	0	2/3	0
D2111(C)	2/3	0	1/3	1/3	2/3
D2211(C)	0	1/3	2/3	1/3	2/3
D3111(C)	1/3	0	1/3	2/3	0
D3211(B)	2/3	1/3	0	0	2/3

The weighted contradiction matrix is

Table 24. Weighted contradiction matrix

Alternatives	D1111(C)	D1211(B)	D2111(C)	D2211(C)	D3111(C)	D3211(B)
P1	0	0.05556	0.11111	0	0.05556	0.11111
P2	0.05556	0.11111	0	0.05556	0	0.05556
P3	0.11111	0	0.05556	0.11111	0.05556	0
P4	0	0.11111	0.05556	0.05556	0.11111	0
P5	0.05556	0	0.11111	0.11111	0	0.11111

By following the similar procedure, the score values are obtained and presented in Table 25.

Table 25. Difference in Score values

Alternatives	B_D	C_D	B_D- C_D
P1	0.16667	0.16667	0
P2	0.16667	0.11112	0.05555
P3	0	0.33334	-0.33334
P4	0.11111	0.22223	-0.11112
P5	0.11111	0.27778	-0.16667

Finally, the attribute Safety is considered. The diagrammatic representation is presented in Fig.8.

Fig.8. Safety

The description of the attributes in the above fig.8. is sketched out in the following table 26.

Table 26. Description of Attributes

Attribute	Sub-Attribute values	Sub$_2$ Attribute Values (Sub-sub Attribute Values)	Sub$_3$ Attribute Values (Sub-sub – sub–Attribute Values)
Safety (E)	Community (E1)	Community Safety (E11)	Low(E111) Moderate (E112) **High (E113)**
		Natural Disaster Risk (E12)	**Low (E121)** Moderate (E122) High (E123)
	Risk (E2)	Risk Assessment (E21)	**Low (E211)** Moderate (E212)

			High (E213)
		Emergency Response (E22)	Inadequate (E221) Adequate (E222) **Robust (E223)**
	Quality (E3)	Work Environment quality (E31)	Poor (E311) Acceptable (E312) **Excellent (E313)**
		Health and Safety (E32)	Basic (E321) Safety (E322) **Comprehensive (E323)**

If the criteria Compliances is alone considered the, respective dominant sub-sub-sub attribute values to be considered are {Affordable, Low, Economical, Low, Low- Cost, Low}

Let us consider a representation of the form P(E1133xE1211xE2111xE2233xE3133xE3233)

Table 27. Attribute matrix

Attributive Values	**P1**	**P2**	**P3**	**P4**	**P5**
E1133	Moderate	High	Low	High	Low
E1211	Low	Moderate	High	Low	Moderate
E2111	High	High	Low	Moderate	Moderate
E2233	Adequate	Robust	Adequate	Inadequate	Robust
E3133	Excellent	Acceptable	Excellent	Poor	Acceptable
E3233	Basic	Safety	Basic	Safety	Comprehensive

The contradiction degrees with respect to the dominant attribute values are presented in Table 28.

Table 28. Contradiction value

Dominant Attribute Value	**Contradiction Degree**
E113	C(E111, E113) = 2/3, C(E112, E113) = 1/3
E121	C(E122, E121) = 1/3, C(E123, E121) = 2/3
E211	C(E212,E211) = 1/3 , C(E213, E211) = 2/3
E223	C(E221, E223) = 2/3, C(E222, E223) = 1/3
E313	C(E311, E313) = 2/3, C(E312, E313) = 1/3
E323	C(E321, E323) = 2/3, C(E322, E323) = 1/3

By using the contradiction values, the contradiction matrix is as follows and presented in Table 29.

Table 29. Contradiction matrix

Attributive Values	P1	P2	P3	P4	P5
E1133(B)	1/3	0	2/3	0	2/3
E1211(C)	0	1/3	2/3	0	1/3
E2111(C)	2/3	2/3	0	1/3	1/3
E2233(C)	1/3	0	1/3	2/3	0
E3133(C)	0	1/3	0	2/3	1/3
E3233(B)	2/3	1/3	2/3	1/3	0

The respective weighted contradiction matrix is as follows,

Table 30. Weighted contradiction matrix

Alternatives	E1133(B)	E1211(C)	E2111(C)	E2233(C)	E3133(C)	E3233(B)
P1	0.05556	0	0.1111	0.05556	0	0.1111
P2	0	0.05556	0.1111	0	0.05556	0.05556
P3	0.1111	0.1111	0	0.05556	0	0.1111
P4	0	0	0.05556	0.1111	0.1111	0.05556
P5	0.1111	0.05556	0.05556	0	0.05556	0

By following the similar procedure, the score values are obtained and presented in Table 31.

Table 31. Difference in Score values

Alternatives	B_E	C_E	B_E- C_E
P1	0.16666	0.16666	0
P2	0.05556	0.22222	-0.16666
P3	0.2222	0.16666	0.05554
P4	0.05556	0.27776	-0.2222
P5	0.1111	0.16668	-0.05558

From the Table [7,13,19,25,31] the overall score values shall be determined and presented in Table 32.

Table 32. Overall score value

Alternatives	Cumulative Score Values
P1	-0.18057

Alternatives	Cumulative Score Values
P2	0.111119
P3	-0.1528
P4	-0.36109
P5	-0.20836

5. Results and Discussion

The final score values of the alternatives with respect to each of the core attributes and the aggregate score values of the alternatives are determined from the Tables respectively.

The ranks of the alternatives are presented in the Table 33.

Table 33. Ranks of the alternatives

Alternatives	Ranking Results Based on the Core Attributes and Aggregate Score Values					
	Proximity	Accessibility	Compliance	Cost Efficiency	Safety	Aggregate
P1	5	4	5	3	3	5
P2	1	4	1	1	5	1
P3	4	1	3	5	2	2
P4	1	2	5	4	4	4
P5	3	3	3	2	1	3

The graphical representation of the scores of the alternatives with respect to each of the attributes and the aggregate measures is presented as follows

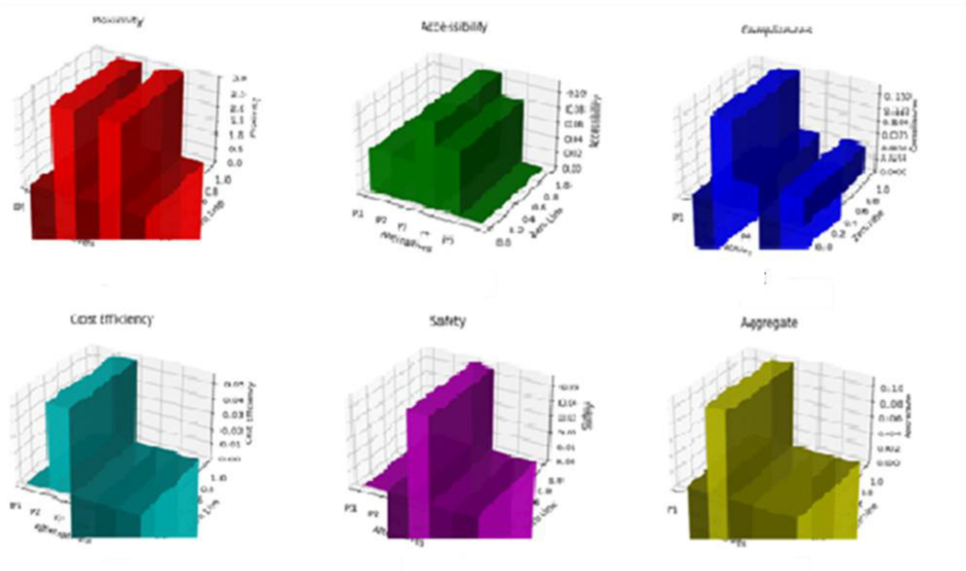

Fig.9. Graphical representation of Scores

From the ranking results presented in the table 33, the following inferences are obtained,

- P2 consistently performs well across most attributes, especially in compliance, cost efficiency, and overall aggregate score.
- P1 consistently ranks low in most attributes, particularly in proximity and aggregate score.
- P3 shows a strong performance in accessibility but is the worst in cost efficiency.
- P4 ranks well in proximity and accessibility but poorly in compliance and aggregate score.
- P5 excels in safety and has a balanced performance across other attributes, making it a solid all-around option.

Thus Plithogenic Forest Hypersoft sets are effectively applied in ranking the alternatives and these sets facilitates in choosing the alternatives based on attributes and sub_3 attribute values. The intervention of PFHS assists in making intense decisions by laying deep examination of the attributes. In other decision methods, the attributes are considered in shallow sense, however in this research work, the attributes are considered in a deeper manner.

Conclusion

This research work introduces the notion of Forest Hypersoft sets and Plithogenic Forest Hypersoft sets. The conceptualizations and classifications are well presented with suitable illustrations. The applications of Plithogenic Forest Hypersoft sets are sketched out with special reference to site selection of the manufacturing plants. The results are comprehensive in nature with the implications of Plithogenic based Forest Hypersoft sets. This decision approach shall be extended to different decision circumstances by augmenting with other decision methods.

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Integrated MEREC-CoCoSo approach using score function ofNeutrosophic for location selection of special economic zone

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Abstract. Special economic zones (SEZs) are essential if a developing country is to boost economic growth. The selection of an SEZ site might be viewed as a multi-criteria decision-making (MCDM) difficulty because of the many conflicting aspects connected to different locations. This work presents a flexible neutrosophic MCDM approach for optimal SEZ site search. The single-valued neutrosophic sets transform the qualitative evaluation of the criteria for SEZ alternatives into a quantitative evaluation. The method based on removal effect on criteria (MEREC) approach is improved in a neutrosophic environment to ascertain the relevance of every characteristic of SEZ alternatives. We suggested a three-tuple neutrosophic combined compromised solution (N-CoCoSo) to evaluate the choice of expert to solve the SEZ site ranking problem. According to the planned N-CoCoSo, coastal forest regions are the worst location for SEZ sites; agricultural land near coastal areas is ideal. The sensitivity and comparability study helps establish the suggested method's consistency and resilience.

Keywords: Special economic zone; MCDM; Single-valued neutrosophic set; CoCoSo; MEREC.)

1. Introduction

A Special Economic Zone (SEZ) refers to a specifically designated geographic region inside a nation subjected to distinct economic rules compared to the remaining areas of the country [1]. These policies frequently adopt a more permissive approach, intending to entice foreign investment, stimulate commerce, increase exports, create jobs, develop regional and national infrastructure, and foster economic expansion [2]. The SEZ commonly provides various advantages, including tax exemptions, exemptions from customs duties, and lenient labor regulations, to promote the establishment of firms within the designated zone [3]. The implementation of SEZ has emerged as a prominent issue in pursuing higher economic objectives at the governmental and policymaker levels, intending to attain sustainable development goals shortly [4]. Numerous nations across the globe have implemented SEZs as integral components of their economic growth strategies [5–7].

The crucial points of SEZ location selection depend on qualitative considerations like community backing and environmental effects against quantitative ones like land cost and transportation accessibility [8]. The complexity of decision-making in this situation can be addressed using multi-criteria decision-making (MCDM) methodologies enabling a structured study of numerous aspects pertinent to SEZ location selection [9]. Establishing meaningful criteria is an essential first step in applying MCDM to selecting SEZ locations [10]. These criteria are typically established by researchers drawing on prior research to reflect the unique requirements and goals associated with the SEZ's intended purpose. The selection of SEZ locations is being carried out using Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) [10] and Analytic Hierarchy Process (AHP) [11, 12]. Fuzzy MCDM's aggregation method provides a more thorough understanding of these trade-offs, allowing for more educated decision-making [13]. Crisp approaches can drastically change the ranking of alternatives since they are susceptible to even little changes in the criteria weights. This sensitivity can pose a concern in decisions like SEZ selection, where criterion weights are frequently subject to expert opinion. Because stakeholder input and new information might cause criterion weights to vary in SEZ selection, fuzzy MCDM offers a more flexible decision-making framework that can adjust to these changes without significantly impacting the results.

Considering these objectives, this article contributes the following advantages:

- ☐ A framework for location selection for SEZs is formulated, identifying different geographic locations with their crucial criteria.
- ☐ Criteria like public perception, environmental restoration, climate regulation, transport proximity, and risk assessment are quantified using single-valued neutrosophic sets (SVNSs).

- ☐ The method based on removal effect on criteria (MEREC) [14] is applied to determine the criteria weight to avoid subjective assessment.
- ☐ An integrated MCDM methodology is developed using neutrosophic rating of the criteria, MEREC, and the neutrosophic extension of combined compromise solution (CoCoSo) [15] approach to counter hesitant and uncertain MCDM issues.
- ☐ The proposed combined approach is applied to identify a suitable prospective SEZ location.
- ☐ The reliability and consistency of the proposed approach are evaluated by comparing it to several prominent MCDM approaches.
- ☐ The sensitivity analysis evaluates the influence of internal parameter modification and criteria weight variation on the ranking of the choices.

The description of the remaining portion of the paper is as follows: The relevant literature review is located in Section 2. The preliminary mathematics of SVNSs is the subject of Section 3. According to Section 4, the geographic context serves as the foundation for the SEZ's location. The proposed integrated neutrosophic methodology is described in Section 5. Figure 6 illustrates the proposed methodology through numerical examples. Section 7 comprises the findings of this article.

2. **Literature review**

Yazdani et al. [15] presented the CoCoSo method as a novel MCDM strategy for determining the best option by merging exponentially weighted product models with simple additive weighting. The benefits of the CoCoSo strategy are as follows: (i) it uses three separate compromise aggregation techniques [16, 17]; (ii) it offers optimum choices devoid of paradoxical events [18, 19], and (iii) it is mostly unaffected by the addition or deletion of options [20]. The CoCoSo method, in contrast to other MCDM strategies, improves the precision of the deciding mechanism and has a greater degree of resolution in discriminating the choices that are being considered [21–23]. Scholars were inspired by the advantages of the CoCoSo method and applied it to their research for various applications. Yazdani et al. [24] developed a grey interval-based CoCoSo technique to assess the efficiency of construction vendors. Ecer et al. [25] suggested an integrated BWM-CoCoSo framework for supplier selection. Peng and Huang [26] extended the CoCoSo approach in the q-rung ortho-pair fuzzy set and combined it with CRITIC for financial risk assessment. Peng and Smarandache [27] formulated a combined CRITIC-CoCoSo approach for the security evaluation of the earth industry. Liao et al. [28] suggested a pythagorean expansion of the CoCoSo technique to determine the optimal logistic distribution centre. Peng et al. [29] proposed a Pythagorean CRITIC-CoCoSo framework for evaluating 5G industries. Ulutacs et al. [30] developed a GIS-based SWARA-CoCoSo technique for selecting logistic center locations. Rani and Mishra [31] developed a similarity measure-based CoCoSo approach to determine the best way to recycle electrical and electronic equipment waste. Yazdani et al. [32] developed a neutrosophic CRITIC-CoCoSo method for supplier selection. Mishra and Rani [33] extended the CoCoSo approach in a neutrosophic context for performance evaluation of logistic suppliers. Peng and Luo [16] assessed the bubble in the China stock market through a picture fuzzy extension of the CoCoSo approach. Rani et al. [34] developed a combined SWARA-CoCoSo methodology to determine the optimal source of renewable energies. Alrasheedi et al. [19] suggested an interval-valued intuitionistic CoCoSo framework to identify factors in green growth in manufacturing sectors. Peng et al. [23] improved the CRITIC-CoCoSo technique in an interval-valued fuzzy soft context for assessing healthcare management through AI. Torkayesh et al. [20] developed a hybrid weighting method using BWM and LBWA with CoCoSo to rank the healthcare facilities in European countries. Demir et al. [35] extended FUCOM and CoCoSo in fuzzy environments to choose the best plan for sustainable urban transportation. A decision-making model using picture fuzzy-based CoCoSo was developed by Qiyas et al. [36] for choosing the best drug. Chen et al. [37] developed a fermatean CoCoSo method for risk evaluation in professional hazards related to health issues. Pamucar and Gorcun [38] proposed a combined fuzzy LBWA-CoCoSo approach to reduce transportation costs in shipping industries. Wei [39] assessed the quality of English courses for college standards through a neutrosophic CoCoSo approach. Peng et al. [40] proposed a neutrosophic CRITIC-CoCoSo model for assessing project-driven immersion teaching. Wei and Pan [41] assessed the teaching quality in sport management through the neutrosophic extension of the CoCoSo approach. Li and Qin [42] suggested a triangular neutrosophic CoCoSo method for college content assurance of innovation and entrepreneurship projects. Mohamed et al. [43] developed a triangular neutrosophic expansion of the AHP-CoCoSo method to identify the risk in the food supply chain. Nabeeh and Sallam [44] proposed a neutrosophic CoCoSo approach to choosing the best bearing ring in the medical field. Aytekin et al. [45] identified the barriers to blockchain technology implementation through the neutrosophic CoCoSo approach.

2.1. *MCDM approaches for location selection of SEZ*

SEZ location selection is based on several objectives: transportation proximity, land cost, local govt.'s rules and regulations, climate, environment, availability of labor, foreign investment, public perception, and many more. Several case studies are conducted in different countries or states using MCDM approaches [46]. Table 1 provides case studies conducted through MCDM for location selection of SEZ-related issues.

Table 1. Case studies of location selection issues using MCDM.

Contributor	Benchmark	Country	Application
Mohon and Naseer [47]	AHP	India	Port locations
Ahmed et al. [10]	AHP-VIKOR	Pakistan	SEZ location
Komchornrit [48]	AHP-PROMETHEE	Thailand	SEZ location
Asaad M. A. et al. [49]	AHP-RSW-SRS	Egypt	Landfill site selection
Tercan et al. [50]	AHP-WLC	Turkey	Solar power system location
Mulusew et al. [12]	AHP	Ethiopia	Waste disposal site selection
Saatsaz et al. [51]	AHP-GIS	Iran	Landfill selection
Zak et al. [46]	ELECTRE	Poland	Logistic center location selection
Nguyen et al. [52]	BWM-ELECTR	Vietnam	Dry port location selection
Rajput et al. [53]	AHP-LAM	India	Business climate

Table 1 shows that the AHP method has been applied primarily to conduct the case studies on location selection. The locations considered for prospective sites for constructing SEZ include only one location. On these points, the prospective research areas can include:

- There is a need for a more fuzzy MCDM approach in location selection problems. However, describing the nature of the essential factors of location selection issues through fuzzy sets is more acceptable.
- Regional geographic advantage is a crucial factor in SEZ location selection. The demographic characteristics, local weather, food habits, and available facilities are critical factors in searching for a suitable SEZ location. These crucial issues still need to be addressed in the literature.
- Determining the criteria weight is always an important factor in ranking the alternatives. Researchers often need to pay more attention since either they use subjective assessment or minimal objective procedure to determine the same.
- There is a need for more MCDM methods that consider uncertainty and hesitation in the computation process. The computation process should include a complete absorption of uncertainty to reflect the decision expert's opinions.

3. Preliminaries on neutrosophic set

Definition 3.1. An SVNS [54] on a fixed set X is defined by $A = (x,\ \varphi_A(x),\ \varpi_A(x),\ \varrho_A(x) | x \in X)$, where $\varphi_A(x),\ \varpi_A(x),\ \varrho_A(x) \in [0,\ 1]$ are sets of membership, non-membership, and indeterminacy degrees of the element $x \in X$ to the set A, respectively, with $0 \leq \varphi_A(x) + \varpi_A(x) + \varrho_A(x) \leq 3$. In general, $0 \leq (\varphi_A(x))^q + (\varpi_A(x))^q + (\varrho_A(x))^q \leq 3,\ \forall x \in X$. For simplicity, we use $A = (\varphi_A(x),\ \varpi_A(x),\ \varrho_A(x))$ as a single valued neutrosophic number (SVNE).

Definition 3.2. [55] Let $A_1 = (\varphi_1,\ \varpi_1,\ \varrho_1)$ and $A_2 = (\varphi_2,\ \varpi_2,\ \varrho_2)$ be two SVNEs and $\lambda \geq 0$ then the basic arithmetic operations are defined as follows:

$$A_1 \bigoplus A_2 = (\varphi_1 + \varphi_2 - \varphi_1.\varphi_2,\ \varpi_1\varpi_2,\ \varrho_1\varrho_1)$$
$$A_1 \bigotimes A_2 = (\varphi_1.\varphi_2,\ \varpi_1 + \varpi_2 - \varpi_1\varpi_2,\ \varrho_1\varrho_1)$$
$$\lambda A = (1 - (1 - \varphi)^\lambda,\ \varpi^\lambda,\ \varrho^\lambda)$$

Definition 3.3. Score function of SVNS: [56] Let $A = (\varphi_A(x),\ \varpi_A(x),\ \varrho_A(x)|x \in X)$ be a SVNE and $\varphi_A(x),\ \varpi_A(x),\ \varrho_A(x) \in [0,\ 1]$ are sets of membership, non-membership degree, and indeterminacy of the element $x \in X$ to the set A, respectively. Then the score function of A is defined as:

$$\mathrm{SC}(A) = \frac{3 + 3 * (\varrho_A(x))^q - 2 * (\varpi_A(x))^q - (\varphi_A(x))^q}{6} \tag{1}$$

Where $q \geq 1$ is the exponents and if $q = 1$, the score function represents for SVNE.

4. An MCDM framework of SEZ's location selection

The first step in formulating an MCDM issue is identifying available alternatives and their critical criteria. In this section, we have identified five prospective areas for SEZ and nine criteria that significantly impact SEZ location selection.

4.1. *Identification of SEZ location*

Identifying the appropriate site for a SEZ entails thoroughly assessing infrastructure, workforce availability, governmental regulations, market entry opportunities, and environmental and social considerations. A thorough evaluation should be conducted to choose sustainable solutions for long-term viability. Taking this into consideration, we have examined the following geographical locations as alternatives to encompass all potential areas:

Forestry coastal (O_1)**:** The coastal forestry region is valuable because it has raw materials that reduce logistics challenges, transit infrastructure that saves time, and precise regulations and environmental standards for managing the ecosystem. However, most coastal settlements are near cities, making marketplaces easier to reach.

Forestry hill (O_2)**:** Establishing a SEZ in a hilly region is feasible; however, it necessitates a meticulous assessment of the distinctive opportunities and challenges presented by the local communities, environment, and terrain. Strategic planning, targeted economic incentives, and environmental safeguards will be essential for the sustainability and viability of this endeavor.

Barren and fallow plane (O_3)**:** Developing supporting infrastructure and constructing networks that facilitate access to local and international markets may use the extensive land in Barren and fallow plains. The interconnections of these locations enable the conveyance of products and services between domestic and international markets without jeopardizing any environmental strategies. Consequently, it is imperative to engage in meticulous preparation to minimize adverse effects and optimize potential economic benefits.

Agricultural land coastal (O_4)**:** Coastal agricultural fields generally provide expansive areas of level or slightly inclined ground well-suited for industrial and commercial development. The presence of available land facilitates the establishment of manufacturing facilities, warehouses, residential complexes, and the essential supporting infrastructure for the activities of SEZs. In addition to this, it possesses the preexisting infrastructure for a comprehensive transportation network encompassing roadways, trains, and ports. The area would have an ample local workforce, decreasing recruiting expenses and promoting long-term job prospects for neighboring areas.

Agricultural land plane (O_5)**:** Agricultural plains are ideal for establishing large-scale industries because of the abundance of necessary resources such as infrastructure, transportation, labor, and easy access to raw materials. The industry also took advantage of local agricultural resources in many aspects. They are assessing the condition of natural resources, identifying threats, and assessing the efficacy of conservation strategies through scientific research and monitoring programs.

4.2. *Defining Criteria*

The literature assessment and experts' opinions in the relevant field are the sources from which the requirements for developing a special economic zone (SEZ) are gathered [10], [**?**] toward. The following are the primary factors to consider when surveying a location for a SEZ building:

Public Perception (Y_1)**:** Considering the local people's willingness to allow the establishment of industry on agricultural land, the impact on public health caused by pollution, and the long-term sustainability of the social and economic equilibrium defines this criterion.

Environmental restoration (Y_2)**:** Continuous monitoring is necessary to restore the environment damaged by deforestation, pollution, and carbon sequestration to promote the development of a healthy society organized by the region's policies.

Climate regulative (Y_3)**:** Ensuring environmental resilience that promotes human well-being requires understanding and protecting the climate regulator. The function of natural ecosystems in controlling the pattern at regional and international levels is being discussed.

Presence of labour and their settlement (Y_4)**:** To satisfy the operational requirement of industrialization, which drives economic success, this specifies the education and training for developing skilled laborers.

Foreign resources (Y_5)**:** This criteria investigates the knowledge, skills, intellectual property, technological advancement, transfer of consumer products, and collaboration that enhance local capital about global resources.

Transport facility (Y_6)**:** The success of a SEZ is contingent upon the presence of an efficient transportation infrastructure. It must be meticulously planned and integrated with the SEZ's overall infrastructure to ensure seamless connectivity to national and international markets, reduce costs, and support efficient logistics.

Installation cost (Y_7)**:** These cost parameters pertain to the system's installation, machinery relocation, and transportation, all of which benefit the organization.

Risk and uncertainty (Y_8)**:** The remainder of this piece elucidates the strategies for attaining enduring objectives about societal and economic advancement through adopting secure and salubrious regulations and using more environmentally friendly industrial technology.

The aforementioned criteria for identifying SEZ locations are the most effective for selecting alternatives. The beneficiary criteria are $Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5$, and Y_6, while the cost criteria are Y_7 to Y_8.

5. The proposed Methodology: Neutrosophic MEREC-CoCoSo Approach

The primary aim of this section is to create a decision-making method that employs the MEREC and CoCoSo approaches to address ambiguous MCDM problems. The proposed method is detailed below, with each phase delineated.

Step 1: Formulation of MCDM problem: Encourage a decision-maker (DM) to be accessible to evaluate the criteria of each option using the SVNE rating. Create a MCDM challenge considering 'p' alternatives $O = \{O_1,\ O_2,\ ...,\ O_p\}$ and 'q' criteria $Y = \{Y_1,\ Y_2,\ ...,\ Y_q\}$, where beneficiary and non-beneficiary criteria are denoted by Y_b and Y_{nb} respectively, such that $Y_b \cup Y_{nb} = Y$ and $Y_b \cap Y_{nb} = \emptyset$.

Step 2: Neutrosophic decision matrix: Gather DM's opinions to form a neutrosophic decision matrix $\beth = (a_{ij})_{p\times q}$, such that $a_{ij} = (\varphi_{ij},\ \varpi_{ij},\ \varrho_{ij}),\ i = 1(1)p;\ j = 1(1)q$. The rating of the i^{th} alternative and j^{th} criterion is a_{ij}, determined by the neutrosophic linguistic scale provided in Table 2.

Table 2. Criteria assessment scale.

Linguistic terms	SVNE (φ, ϖ, ϱ)
Worst Case Acceptable (EL)	(0.261, 0.813, 0.701)
Very Poorly Acceptable (TL)	(0.346, 0.728, 0.643)
Poorly Acceptable(AL)	(0.463, 0.651, 0.586)
Acceptable (N)	(0.558, 0.533, 0.551)
Fairly Acceptable (AH)	(0.646, 0.422, 0.386)
Moderately Acceptable (TH)	(0.753, 0.317, 0.215)
Highly Acceptable (EH)	(0.831, 0.262, 0.113)
Exceptionally Acceptable (NH)	(0.982, 0.142, 0.097)

Step 3: Preference of the alternative: Determine the primary preference based on the alternate choice $\mathrm{P}_{\mathrm{O}_i}$. Since the DM views the options as having an equal chance of occurring:

$$\mathrm{P}_{\mathrm{O}_i} = \frac{1}{p} \text{ such that } \sum_{i=1}^{p} \mathrm{P}_{\mathrm{O_i}} = 1.$$

Step 4: Neutrosophic MEREC approach to determine the criteria weight: Follow these procedures for establishing the criteria weight using the MEREC method:

Step 4.1: Use the decision matrix $\beth$ from Table 2 to evaluate the neutrosophic criteria weights.

Step 4.2: Compute the normalized neutrosophic decision matrix $\beth_N = (\eta_{ij})_{p\times q}$ using equation (2).

$$\eta_{ij} = \begin{cases} \frac{\min_k x_{kj}}{x_{ij}} & \text{for } j \in \mathrm{Y}_b \\ \frac{x_{ij}}{\max_k x_{kj}} & \text{for } j \in \mathrm{Y}_{nb} \end{cases} \tag{2}$$

Step 4.3: Compute the overall performance S_i of the options O_i using equation (3).

$$S_i = \ln(1 + (\frac{1}{p}\sum_j |\ln(\eta_{ij})|)) \tag{3}$$

Step 4.4: Determine the performance of each alternative, irrespective of the criteria S_{ij} using equation (4).

$$S_{ij} = \ln(1 + (\frac{1}{p}\sum_{k\neq j} |\ln(\eta_{ij})|)) \tag{4}$$

Step 4.5: To determine the removal effect of the j^{th} criteria, compute the sum of absolute deviation using equation (5).

$$E_j = \sum_j |S_{ij} - S_i| \;\; j = 1(1)q \tag{5}$$

Step 4.6: Establish the neutrosophic weights of the criteria using equation (6).

$$\omega_j = \frac{E_j}{\sum_j E_j}, \ j = 1(1)q \tag{6}$$

Step 5: Normalization of neutrosophic decision matrix: Normalize the neutrosophic decision matrix ($\beth$) using equation (7).

$$r_{ij} = \begin{cases} \frac{x_{ij} - \min_j x_{ij}}{\max_j x_{ij} - \min_j x_{ij}} \text{ for } j \in \mathrm{Y}_b \\ \frac{\max_j x_{ij} - x_{ij}}{\max_j x_{ij} - \min_j x_{ij}} \text{ for } j \in \mathrm{Y}_{nb} \end{cases} \tag{7}$$

Step 6: Real evaluation of the alternatives: Determine the real evaluation of neutrosophic weighted sum measure (WSM) (L_i) and weighted product measure (WPM) (M_i) corresponding performance indexes using equation (8).

$$\mathrm{L}_i = \sum_{j=1}^{q} \omega_j \bigotimes r_{ij} \text{ and } \mathrm{M}_i = \sum_{j=1}^{q} {r_{ij}}^{\omega_j} \tag{8}$$

Step 7: Neutrosophic appraisal score: Calculate the neutrosophic appraisal scores for the alternatives using equation (9).

$$\begin{cases} \kappa_{i\alpha} = \frac{\mathrm{L}_i + \mathrm{M}_i}{\sum_{i=1}^{m} (\mathrm{L}_i + \mathrm{M}_i)} \\ \kappa_{i\beta} = \frac{\mathrm{L}_i}{\min_i(\mathrm{L}_i)} + \frac{\mathrm{M}_i}{\min_i(\mathrm{M}_i)} \\ \kappa_{i\gamma} = \frac{\lambda \mathrm{L}_i + (1-\lambda)\mathrm{M}_i}{\lambda \max_i(\mathrm{L}_i) + (1-\lambda)\max_i(\mathrm{M}_i)}, \ \lambda \in (0, \ 1) \end{cases} \tag{9}$$

Step 8: Overall appraisal score: Compute overall appraisal score κ_i of each alternatives using equation (10)

$$\kappa_i = (\kappa_{i\alpha}\kappa_{i\beta}\kappa_{i\gamma})^{1/3} + \frac{1}{3}(\kappa_{i\alpha} + \kappa_{i\beta} + \kappa_{i\gamma}), \ i = 1(1)p \tag{10}$$

Step 9: Ranking order: Determine the ranking of each choice in decreasing order depending on the κ_i values.

Figure 1 shows the description of the SEZ selection problem by a sequential structure.

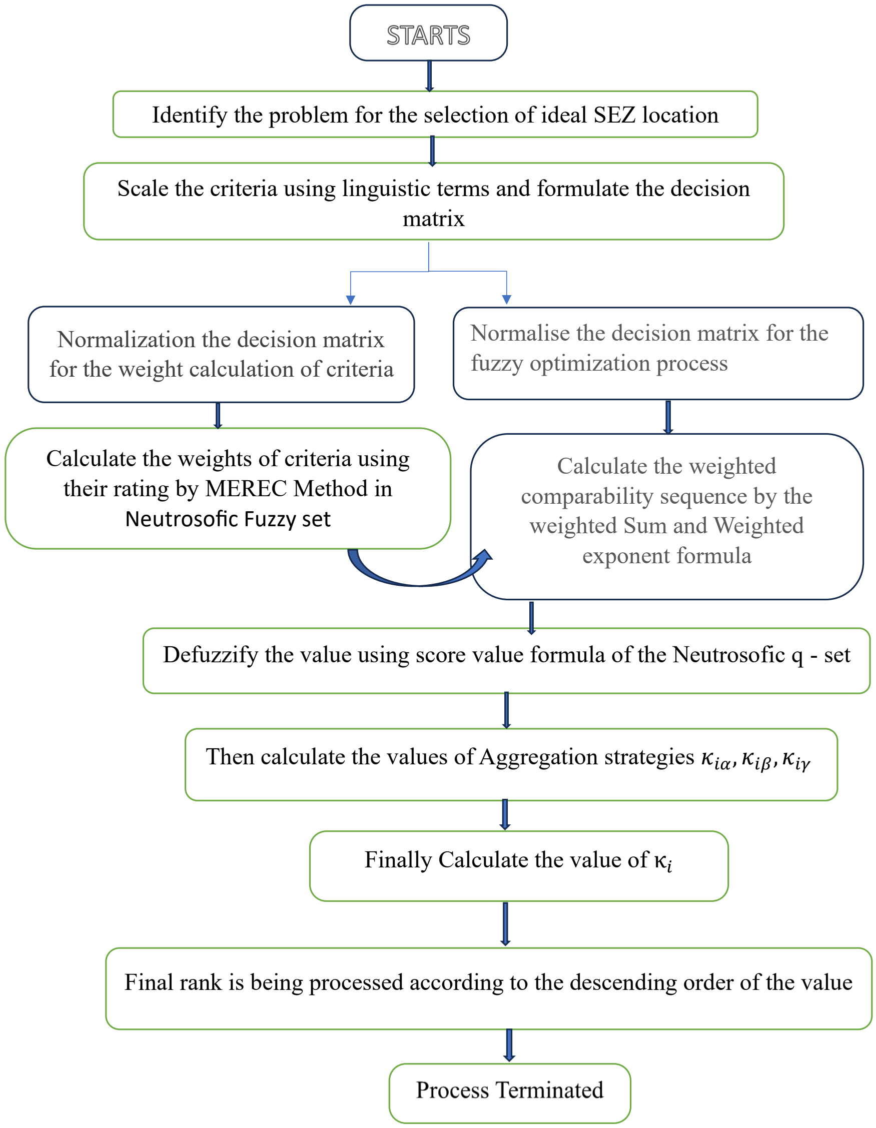

FIGURE 1. Description of SEZ selection problem by a sequential three-level structure.

6. Numerical illustration of proposed neutrosophic MEREC-CoCoSo approach

Step 1: The MCDM framework of the SEZ location selection has five options $O = \{O_1, O_2, O_3, O_4, O_5\}$ and eight criteria $Y = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8\}$ such that $Y_b = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\}$ and $Y_{nb} = \{Y_7, Y_8\}$.

Step 2: Table 3 provides the construction of the neutrosophic decision matrix $(\beth)_{5\times 8}$ in accordance with the SVN rating from Table 2.

TABLE 3. Linguistic neutrosophic decision matrix.

Alt./Cr.	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
O_1	EH	AN	AL	TL	TH	AN	AN	TL
O_2	AL	AH	EH	AL	AN	TL	AL	EL
O_3	TH	EH	AN	AN	AL	AH	EL	TH
O_4	AN	NH	TH	AH	TH	EH	TL	AL
O_5	NH	TH	AN	EH	AH	TH	AH	AH

Step 3: Since this MCDM issue of location selection for SEZ have five alternatives, $\mathrm{P}_{O_i} = \frac{1}{5}$ such that $sum_{i=1}^{5}\mathrm{P}_{\mathrm{O_i}} = 1$.

Step 4: Proposed neutrosophic MEREC method for criteria weight computation:

Step 4.1: The neutrosophic decision matrix of Table 3 is generated using the criteria rating from Table 2.

Step 4.2: The normalized decision matrix $\beth_N$ is provided in Table 4 using equation (2).

TABLE 4. Normalised neutrosophic decision matrix

Zone	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
O_1	(0.557,0.542,0.858)	(1,0.266,0.176)	(1, 0.402,0.193)	(1,0.360,0.176)	(0.615,1,1)	(0.620,0.492,0.205)	(0.864,0.656,0.786)	(0.459,0.895,0.917)
O_2	(1,0.218,0.165)	(0.864,0.336,0.251)	(0.557,1,1)	(0.747,0.402,0.193)	(0.829,0.595,0.390)	(1,0.359,0.176)	(0.717,0.801,0.836)	(0.858,0.519,0.551)
O_3	(0.615,0.448,0.451)	(0.672,0.542,0.858)	(0.829,0.492,0.205)	(0.620,0.492,0.205)	(1,0.487,0.367)	(0.536,0.621,0.293)	(0.404,1,1)	(1,0.390,0.307)
O_4	(0.829,0.266,0.176)	(0.568,1,1)	(0.615, 0.826,0.526)	(0.536,0.621, 0.293)	(0.615,1,1)	(0.416,1,1)	0.536,0.895,0.917)	(0.615,0.801,0.836)
O_5	(0.472,1,1)	(0.741,0.448,0.451)	(0.829,0.492,0.205)	(0.416,1,1)	(0.717,0.751,0.557)	(0.459,0.826,0.526)	(1.000,0.519,0.551)	(0.347,1,1)

Step 4.3: The equation (4) is employed to ascertain the aggregate performance of the alternatives S_i, which is detailed in Table 5.

TABLE 5. Overall performance of the alternatives.

Cr./Alt.	O_1	O_2	O_3	O_4	O_5
$(S_i, SN(i), SH(i))$	(0.269, 0.494, 0.641)	(0.192, 0.555, 0.710)	(0.326, 0.484, 0.663)	(0.433, 0.255, 0.395)	(0.429, 0.286, 0.422)

Step 4.4: Table 6 provides the efficacy of each alternative, which is determined by equation (4).

Step 4.5: The sum of the absolute deviations is calculated using equation (9) and is presented in Table 6.

Step 4.6: Using equation (6), the neutrosophic criteria weights are computed and are displayed in Table 6.

TABLE 6. Absolute deviation and neutrosophic criteria weights.

Alt./Cr.	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
O_1	(0.057,0.048,0.010)	(0,0.106,0.121)	(0,0.072,0.115)	(0,0.081,0.122)	(0.048,0,0)	(0.047,0.168,0.110)	(0.014,0.107,0.016)	(0.077,0.207,0.128)
O_2	(0.000,0.116,0.117)	(0.015,0.081,0.089)	(0.062,0,0)	(0.031,0.068,0.107)	(0.019,0.038,0.060)	(0.000,0.152,0.113)	(0.035,0.026,0.011)	(0.031,0.186,0.130)
O_3	(0.045,0.064,0.053)	(0.037,0.048,0.010)	(0.017,0.056,0.108)	(0.044,0.056,0.108)	(0.000,0.057,0.067)	(0.058,0.055,0.082)	(0.085,0.041,0.000)	(0.037,0.148,0.090)
O_4	(0.015,0.137,0.158)	(0.047,0,0)	(0.040,0.019,0.056)	(0.052,0.047,0.109)	(0.040,0,0)	(0.074,0.053,0)	(0.052,0.018,0.007)	(0.089,0.007,0.015)
O_5	(0.063,0,0)	(0.025,0.078,0.067)	(0.015,0.069,0.139)	(0.074,0,0)	(0.027,0.027,0.049)	(0.065,0.067,0.054)	(0.000,0.117,0.050)	(0.117,0.007,0.067)
SUM	(0.181,0.365,0.338)	(0.123,0.314,0.287)	(0.135,0.216,0.417)	(0.201,0.252,0.445)	(0.135,0.122,0.175)	(0.244,0.496,0.360)	(0.186,0.309,0.084)	(0.352,0.555,0.430)
W_i	(0.116,0.139,0.133)	(0.079,0.120,0.113)	(0.087,0.082,0.164)	(0.129,0.096,0.175)	(0.087,0.046,0.069)	(0.157,0.189,0.142)	(0.120,0.118,0.033)	(0.226,0.211,0.169)

Step 5: The normalized decision matrix of the neutrosophic decision matrix (Table 3 using equation (7) is displayed in Table 7.

TABLE 7. Normalized decision matrix according to proposed N-CoCoSo method

Alt./Cr.	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
O_1	(0.317,0.105,0.015)	(0.000,0.447,0.447)	(0.000,0.447,0.447)	(0.000,0.447,0.447)	(0.447,0.000,0.000)	(0.195,0.260,0.370)	(0.213,0.680,0.102)	(0.053,0.923,0.370)
O_2	(0.000,0.447,0.447)	(0.093,0.320,0.285)	(0.447,0.000,0.000)	(0.108,0.373,0.399)	(0.147,0.289,0.405)	(0.000,0.447,0.447)	(0.163,0.815,0.213)	(0.290,0.647,0.097)
O_3	(0.250,0.154,0.108)	(0.288,0.137,0.016)	(0.115,0.312,0.414)	(0.195,0.260,0.370)	(0.000,0.447,0.447)	(0.277,0.154,0.230)	(0.000,1.000,0.447)	(0.447,0.553,0.000)
O_4	(0.082,0.344,0.415)	(0.447,0.000,0.000)	(0.352,0.063,0.096)	(0.277,0.154,0.230)	(0.447,0.000,0.000)	(0.447,0.000,0.000)	(0.082,0.903,0.348)	(0.106,0.854,0.264)
O_5	(0.447,0.000,0.000)	(0.206,0.200,0.116)	(0.115,0.312,0.414)	(0.447,0.000,0.000)	(0.282,0.141,0.206)	(0.375,0.053,0.086)	(0.447,0.553,0.000)	(0.000,1.000,0.447)

Step 6: The neutrosophic WSM and WPM are computed using equation (8). Table 8 provides neutrosophic WSM, and Table 9 provides neutrosophic WPM.

TABLE 8. Neutrosophic weighted sum measure.

Alt./Cr.	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
O_1	(0.037,0.230,0.146)	(0,0.513,0.510)	(0,0.493,0.538)	(0,0.501,0.544)	(0.039,0.046,0.069)	(0.031,0.400,0.459)	(0.026,0.717,0.132)	(0.012,0.940,0.477)
O_2	(0,0.524,0.521)	(0.007,0.402,0.366)	(0.039,0.082,0.164)	(0.014,0.433,0.504)	(0.013,0.322,0.446)	(0,0.551,0.526)	(0.020,0.836,0.239)	(0.066,0.722,0.251)
O_3	(0.029,0.271,0.227)	(0.023, 0.240,0.127)	(0.010,0.368,0.510)	(0.025,0.331,0.480)	(0,0.472,0.485)	(0.043,0.313,0.340)	(0.0000,1.0000,0.466)	(0.101,0.647,0.170)
O_4	(0.010,0.434,0.493)	(0.036,0.120,0.113)	(0.030,0.140,0.245)	(0.036,0.235,0.365)	(0.039,0.046,0.069)	(0.070,0.188, 0.142)	(0.010,0.914,0.370)	(0.024,0.885,0.388)
O_5	(0.052,0.139,0.133)	(0.016,0.296,0.216)	(0.010,0.368,0.510)	(0.058,0.096,0.175)	(0.024,0.181,0.261)	(0.059,0.231,0.216)	(0.054,0.605,0.033)	(0,1,0.541)

TABLE 9. Neutrosophic weighted product measures

Cr./Alt.	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
C_1	(0.869,0.014,0.002)	(0,0.069,0.070)	(0,0.070,0.070)	(0,0.072,0.070)	(0.901,0,0)	(0.818,0.036,0.055)	(0.819,0.136,0.013)	(0.686,0.281,0.055)
C_2	(0,0.070,0.070)	(0.751,0.045,0.040)	(0.906,0.000,0)	(0.756,0.057,0.060)	(0.780,0.043,0.062)	(0,0.070,0.070)	(0.792,0.195,0.029)	(0.853,0.125,0.012)
C_3	(0.844,0.020,0.014)	(0.861,0.018,0.002)	(0.768,0.045,0.063)	(0.814,0.037,0.055)	(0,0.074,0.070)	(0.854,0.020,0.032)	(0,1,0.070)	(0.902,0.098,0)
C_4	(0.736,0.050,0.064)	(0.908,0,0)	(0.880,0.008,0.012)	(0.851,0.021,0.032)	(0.901,0,0)	(0.906,0,0)	(0.725,0.259,0.051)	(0.749,0.219,0.037)
C_5	(0.906,0,0)	(0.827,0.026,0.015)	(0.768,0.045,0.063)	(0.904,0,0)	(0.849,0.019,0.028)	(0.886,0.007,0.011)	(0.902,0.098,0)	(0,1,0.070)

Step 7: The relative significance of the alternatives is determined by calculating the appraisal score using equation (1), which is presented in Table 10.

Step 8: The overall appraisal scores of the options are computed using equation (9) and is provided in Table 10.

TABLE 10. Overall appraisal scores of the alternatives.

Alt.	WS_i	WP_i	$\kappa_{i\alpha}$	$\kappa_{i\beta}$	$\kappa_{i\gamma}(\lambda = 0.5)$	κ_i
O_1	-1.187	2.265	0.116	1.977	-0.250	1.260
O_2	-1.215	2.660	0.156	2.174	-0.087	1.492
O_3	-1.066	2.533	0.158	1.996	0.022	1.426
O_4	-0.725	3.610	0.311	2.191	0.952	2.038
O_5	-0.683	3.092	0.260	1.927	0.743	1.747

For standardization, the coefficient of the compromise decision system indicated by $\lambda \in (0,\ 1)$ and is set to 0.5 to assign identical preference.

Step 9: The decreasing order of the overall appraisal scores determines the ranking order of the SEZ locations, which is $\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$.

The computation of the proposed N-CoCoSo method is carried out for $q = 2$ in equation (1) and the outcome is displayed in Table 11.

TABLE 11. Overall appraisal scores of the SEZ locations for $q = 2$ in equation (1).

Alt.	WS_i	WP_i	$\kappa_{i\alpha}$	$\kappa_{i\beta}$	$\kappa_{i\gamma}(\lambda = 0.5)$	κ_i
O_1	-4.664	8.180	0.066	1.972	0.177	1.022
O_2	-4.797	11.548	0.126	2.412	0.340	1.428
O_3	-4.020	12.068	0.150	2.313	0.405	1.477
O_4	-2.625	22.284	0.367	3.272	0.990	2.602
O_5	-2.418	17.978	0.291	2.702	0.783	2.109

Hence the ranking order of the alternatives is $\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_3 \succ \mathscr{O}_2 \succ \mathscr{O}_1$. The parameter q is not significantly affected by the ranking of the SEZ locations, despite the fact that the overall performance scores are sensitive to variations in q.

6.1. *Sensitivity Analysis*

To evaluate the influence of internal parameter variation and criteria weight variation on the sorting order of the proposed approach, this section implements two distinct categories of sensitivity analysis.

6.1.1. *Internal parameter variation*

The alteration in internal parameters has a major impact on the sorting sequence of the options. This impact plays an important role in checking the robustness of the proposed approach. The impact of variation in parameter λ to determine the neutrosophic appraisal score at Step 7 is depicted in Table 12.

TABLE 12. Impact of λ variation on ranking orders of SEZ locations.

Alt.	Ranking order
$\lambda = 0.1$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$
$\lambda = 0.2$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$
$\lambda = 0.3$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$
$\lambda = 0.4$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$
$\lambda = 0.5$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$
$\lambda = 0.6$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$
$\lambda = 0.7$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_3 \succ \mathscr{O}_2 \succ \mathscr{O}_1$
$\lambda = 0.8$	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_3 \succ \mathscr{O}_2 \succ \mathscr{O}_1$
$\lambda = 0.9$	$\mathscr{O}_2 \succ \mathscr{O}_1 \succ \mathscr{O}_3 \succ \mathscr{O}_4 \succ \mathscr{O}_5$

Table 12 shows that the λ alteration has marginally modified the ranking order of the choices. The rankings obtained for $\lambda = 0.1$ to $\lambda = 0.6$ are identical, whereas those for $\lambda = 0.7$ and $\lambda = 0.8$ deviate a bit. However, the ranking order produced for $\lambda = 0.9$ differs entirely from the others. In conclusion, ranking orders have little impact on λ variation since they only alter the value of a single component in the performance score. Figure 2 illustrates the influence of λ change on ranking order.

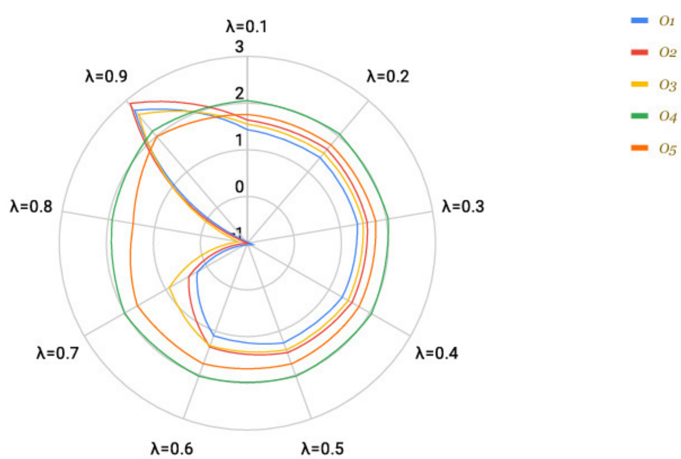

FIGURE 2. Impact of λ variation on performance scores of SEZ locations.

The performance scores of SEZ locations $\mathscr{O}_4$ and $\mathscr{O}_5$ are not affected by the λ variation, as illustrated in figure 2. Nevertheless, the performance values of $\mathscr{O}_1$, $\mathscr{O}_2$, and $\mathscr{O}_3$ have been substantial affected for $\lambda \in (0.7,\ 0.9)$. In summary, the performance values of the alternatives exhibit minimal fluctuation in response to variations in λ. This is because it can modify only one of the three parameters comprising the performance score.

6.1.2. *Criteria weight variation*

The relevance of the criterion may vary depending on the viewpoint of the decision expert. In this context, it is significant to examine the influence of criterion variation on the ordering of the choices. The rank-exponent approach [57] for criteria weight determination is described as follows:

Suppose that k_j and w_j represent the rank and preference degree of the j^{th} criterion, respectively, then $w_j = \frac{(n-k_j+1)^p}{\sum_{j=1}^{n}(n-k_j+1)^p}$, $p \geq 0$. When $p = 1$, it is called the rank-sum approach and assigns equal weights (holistic) when $p = 0$.

This article uses entropy, holistic, rank-sum, and rank-exponent ($p = 2$) methods to determine the criteria weight. To apply these approaches, the neutrosophic decision matrix (Table 6) is transformed into a crisp decision matrix using equation (1), regardless of the proposed N-CoCoSo approach in determining the ranking orders. Table 13 provides the computed criteria weights using these approaches.

Table 13. Weights of each criteria determined by various methods.

Methods	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Entropy [58]	0.138	0.131	0.124	0.123	0.119	0.126	0.116	0.123
Holistic	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
Rank-Sum	0.167	0.111	0.056	0.139	0.028	0.194	0.083	0.222
Rank-Exponent ($p = 2$)	0.078	0.044	0.240	0.314	0.123	0.176	0.020	0.005
MEREC [14]	0.122	0.120	0.122	0.126	0.130	0.123	0.129	0.129

Table 13 shows that the entropy and MEREC approaches assign almost identical weights to the criteria, consistent with the holistic preference. However, rank-sum and rank-exponent for $p = 2$ approaches assign specific preferences. According to rank-sum approach, criteria Y_1, Y_6, and Y_8 have much higher importance than the criteria average weighted criteria. The rank-sum approach assign significantly lower importance to the criteria Y_3, Y_5, and Y_7. The rank-exponent approach for ($p = 2$) assigns too high preference to the criteria Y_3, Y_4, and Y_6 than the remaining criteria. The reason for the difference in weight allocation in rank-sum and rank-exponent approaches may lie in the subjective assessment compared to objective assessment in entropy and MEREC approaches. Figure 3 shows the variation in criteria weight allocation through these approaches.

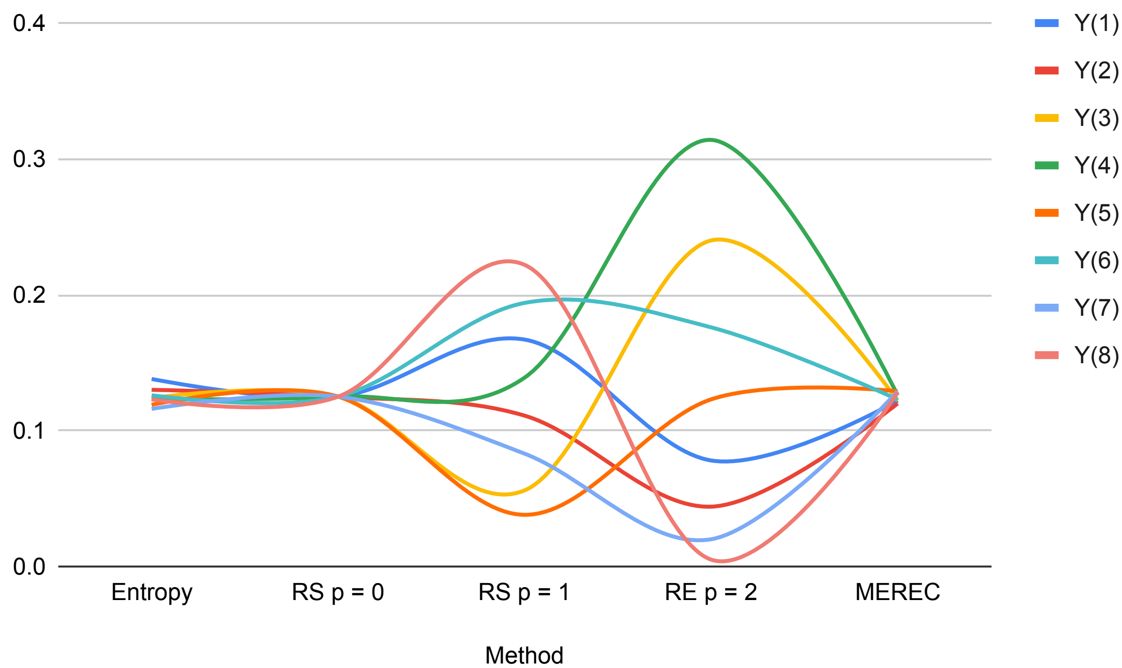

FIGURE 3. Criteria weight variation in different approaches.

The beginning and ending portions of the curves in figure 3 show that the criteria weight allotted to all the criteria is almost equal in entropy, rank-sum, and MEREC approaches. The maximum variation is seen in the weight allocation of the rank-exponent approach for $p = 2$. The most crucial criterion is Y_8 in the rank-sum approach, while Y_4 receives the highest preference in the rank-exponent approach for $p = 2$. The variation is observed for almost all criteria in their weight allocation in rank-sum and rank-exponent approaches; still, this variation is eye-catching in the criteria weight of Y_3 and Y_8. Table 14 illustrates the influence of criteria weight scheduling on the ranking sequence of the options.

TABLE 14. Ranking orders for criteria weight alteration.

Methods	Ranking order
Entropy	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_3 \succ \mathscr{O}_1 \succ \mathscr{O}_2$
Holistic	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_3 \succ \mathscr{O}_1 \succ \mathscr{O}_2$
Rank-Sum	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_3 \succ \mathscr{O}_1 \succ \mathscr{O}_2$
Rank-exponent ($p = 2$)	$\mathscr{O}_3 \succ \mathscr{O}_4 \succ \mathscr{O}_2 \succ \mathscr{O}_1 \succ \mathscr{O}_5$
MEREC	$\mathscr{O}_4 \succ \mathscr{O}_5 \succ \mathscr{O}_2 \succ \mathscr{O}_3 \succ \mathscr{O}_1$

The ranking sequence of the options in entropy and holistic criteria weight determination are identical, as expected from their criteria weight allocation, as depicted in Table 14. Surprisingly, the ranking sequence obtained from the rank-sum approach also coincides with the entropy and holistic approaches, though their criteria weight allocations are entirely different. The ranking order obtained in the MEREC approach has a slight variation between $\mathscr{O}_4$ and $\mathscr{O}_5$ compared to the ranking orders obtained from entropy and holistic approaches. The ranking order obtained through criteria weight determined by the rank-exponent approach for $p = 0$ is entirely different from those of the remaining approaches. However, the alternative $\mathscr{O}_4$ scores second in the ranking order, proving its superiority among the remaining alternatives. The influence of criteria weight alteration on the performance of the alternatives is depicted in figure 4.

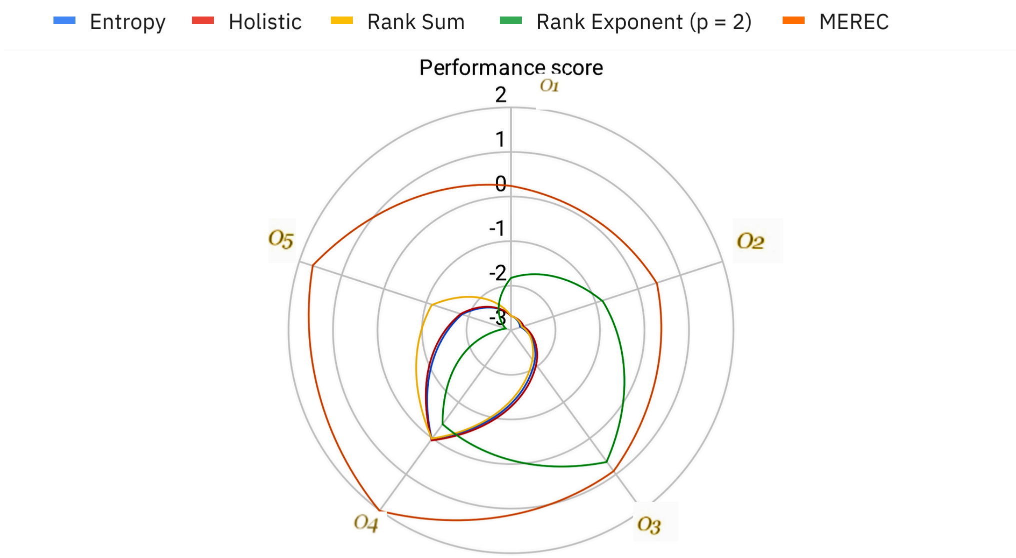

FIGURE 4. Variation of alternative's performance scores due to criteria weight alteration.

The performance scores of the locations for SEZ are significantly influenced by the criteria weight variation, as illustrated in Figure 4. Compared to the remaining locations, the SEZ location $\mathscr{O}_1$ only outperforms them regarding holistic criteria weight. The $\mathscr{O}_2$ and $\mathscr{O}_3$ options exhibit appalling performance in the entropy, rank-sum, and MEREC approaches, while they perform exceptionally well in the remaining two approaches. The $\mathscr{O}_4$ and $\mathscr{O}_5$ options exhibit comparable performance across all criteria weight determination procedures. Consequently, it is verifiable that these two alternatives are viable to select the location of a SEZ.

6.2. *Comparison analysis*

To establish the proposed neutrosophic MEREC-CoCoSo approach, it is imperative to check the outcomes of the suggested technique with some popular existing MCDM techniques like

MAIRCA [59], EDAS [60], CODAS [61], and MABAC [62] approaches. Here, we have determined the criteria weights through the MEREC approach to maintain logical similarity, and these approaches are applied to a crisp decision matrix obtained from neutrosophic decision matrix (Table 3) using equation (1). Table 15 displays the obtained outcomes of this comparison analysis.

TABLE 15. Comparison analysis of proposed approach.

Approach	MAIRCA		EDAS		CODAS		MABAC		SVN-CoCoSo	
Options	Score	Rank	Score	Rank	Score	Rank	Score	Rank	Score	Rank
$\mathscr{O}_1$	0.567	5	0.630	3	14.898	3	0.0356	3	1.259	5
$\mathscr{O}_2$	0.531	4	0.846	1	26.637	1	0.046	2	1.492	3
$\mathscr{O}_3$	0.827	3	0.672	2	11.492	4	0.0730	1	1.425	4
$\mathscr{O}_4$	0.857	2	0.138	5	15.457	2	-0.061	4	2.037	1
$\mathscr{O}_5$	1.057	1	0.199	4	10.120	5	-0.078	5	1.746	2

The proposed neutrosophic MEREC-CoCoSo approach shares significant similarities with the MAIRCA approach, while the outcomes in the EDAS, CODAS, and MABAC approaches are significantly different. The first two positions obtained by the alternatives $\mathscr{O}_4$ and $\mathscr{O}_5$ in SVN-CoCoSo and MAIRCA approaches are identical, while the remaining positions are identical. Hence, the alternative $\mathscr{O}_4$ has significant credibility than the remaining alternatives. The alternative $\mathscr{O}_1$ fails to prove its credibility as it is almost at the end of the ranking orders in all approaches. The performance score comparison of the alternatives is shown in figure 5.

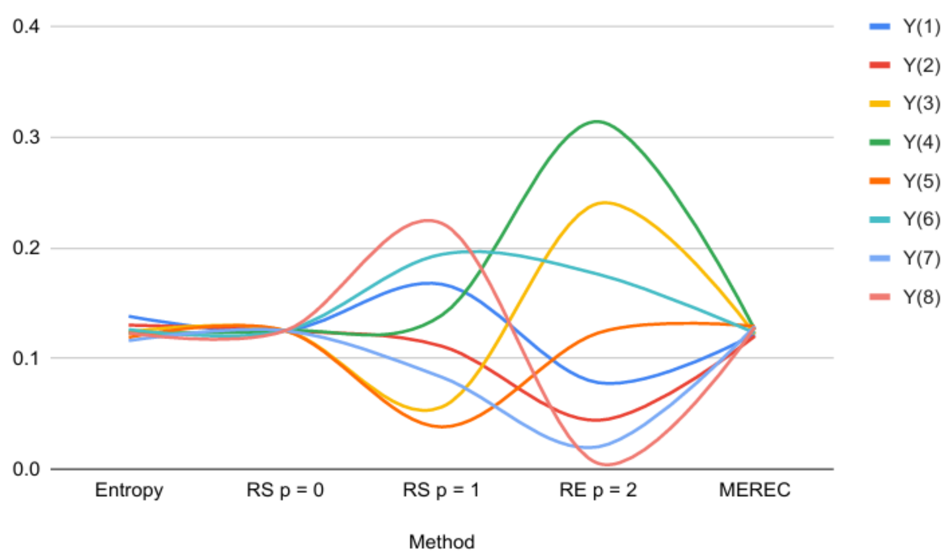

FIGURE 5. Comparison of performance scores of the options.

The performance scores of MAIRCA, EDAS, MABAC, and the proposed SVN-CoCoSo methods are within a very narrow range, as illustrated in figure 5. Conversely, the CODAS method exhibits a significantly higher performance score. Nevertheless, this observation does not significantly influence the ranking sequence of the options in various approaches. The alternative $\mathscr{O}_2$ ranks first in CODAS with a significant difference, whereas the same incident occurred in EDAS with a shallow margin. The reliability of $\mathscr{O}_2$ is uncertain due to its inconsistent performance across several methodologies. In MAIRCA, CODAS, and the proposed SVN-CoCoSo approaches, the performance of $\mathscr{O}_4$ is evident. However, in EDAS and MABAC, the performance is not as good, although the difference is minimal compared to the first. Therefore, $\mathscr{O}_4$ is a superior choice for a SEZ site. In conclusion, the proposed SVN-CoCoSo method exhibits a certain degree of similarity to the crisp approaches, with the distinction being the criteria analysis through neutrosophic information.

Conclusion

The advantages of carrying out the recommended N-CoCoSo method are as follows: (i) This strategy is significantly less complex compared to other fuzzy CoCoSo approaches, (ii) the suggested strategy evaluates the criteria using SVNS, which enhances its generality and adaptability, (iii) it employs the Entropy approach to determine the objective weights of the relevant criteria. Adopting and utilizing such emerging technology enhances the precision of the decision-making system, supports business policies, validates worldwide objectives, and yields advantageous outcomes for management control.

We have found that the land near the littoral area is significantly more advantageous than the plains far from the sea by utilizing the proposed N-CoCoSo approach for SEZ location selection. This may be because of proximity to the port, which is a critical requirement for SEZs. In the littoral region, agricultural land is more advantageous than forest land near the sea. The rationale is the straightforward and appropriate evaluation of the availability of large tracts of land and the environmental concerns. Mangroves frequently encircle the coastal forest areas, which substantially affects the preservation of biodiversity.

However, despite its numerous advantages, the proposed N-CoCoSo decision-making model has certain limitations. The limitations of the proposed approach are (i) The subjective bias and personal favoritism of the DM towards any specific criteria, and (ii) the fact that the criteria information needs a hesitancy component makes it impracticable to implement. Future research may include (i) the development of the suggested technique to accommodate more complex analyses in the context of Z-number, D-number, type II fuzzy, and hesitant fuzzy environments, (ii) the implementation of an appropriate blend of subjective and objective criteria preferences by the decision expert's qualifications; and (iii) the proposed N-CoCoSo method can be implemented to address decision-making challenges, including the evaluation of financial firms' performance and the selection of suppliers, ERP selection, and analysis of renewable energy sources.

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Strategy Selection for Natural Resource Conservation CapturingVagueness through Integrated Neutrosophic CRITIC-MAIRCA Approach

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Abstract. Conservation of natural resources like water, soil, and biodiversity is necessary for ecological balance and human survival. This article aims to identify potential ways to conserve natural resources while maintaining a balance between ecology and human survival. An integrated neutrosophic CRiteria Importance Through Intercriteria (CRITIC)-Multi-Atributive Ideal-Real Comparative Analysis (MAIRCA) approach is developed to handle uncertain and inconsistent information. The linguistic assessments of the criteria are quantified through single-valued neutrosophic sets (SVNSs). The CRITIC method evaluates the criteria weights to avoid subjective and biased assessments. The MAIRCA approach is improved in a neutrosophic context to identify the ranking of the alternatives. The combined neutrosophic CRITIC-MAIRCA approach determines the best way to preserve natural resources. The proposed method reveals that the most favourable ways to conserve natural resources are through research and monitoring, while the last option is sustainable practices. Comparison and sensitivity discussions of the proposed approach are conducted to check its consistency and robustness.

Keywords: Single-valued neutrosophic set; MAIRCA; CRITIC; MCDM; Objective weights; Natural resource conservation

1. **Introduction**

Managing and preserving water, soil, forests, and wildlife to guarantee their continued availability for future generations is known as natural resource conservation [1]. Protecting areas, restoring damaged ecosystems, and implementing sustainable practices are all part of it. Protecting natural resources aims to satisfy present-day human needs while preserving ecological balance by adopting laws and increasing public awareness [1]. Over-exploitation can result in species extinction, environmental damage, and resource depletion without conservation. Pollination, clean air, and water filtration are examples of ecosystem services maintained when these resources are protected [2]. Rich nations consume six times more resources and cause ten times more climate damage than lower-income countries. According to the survey, worldwide use of natural resources is expected to increase by 60 percent by 2060 compared to 2020 levels [3]. Therefore, it is necessary to address this issue with the utmost priority of conserving the elements of natural resources.

A multifaceted strategy is needed to address these issues, including increasing public awareness, creating sustainable practices, enforcing laws, and encouraging international cooperation. In this situation, decisions are influenced by various factors that cannot be easily quantified or compared directly. Various Multi-criteria decision-making (MCDM) methods, including outranking approaches, reference point approaches, and utility and value theory, are implemented to promote the conservation and management of natural resources [4, 5]. However, these crisp MCDM approaches need to deal with a qualitative assessment of the associated factors in natural resource conservation. In this regard, it becomes necessary to formulate an improved MCDM methodology to deal with vague and indeterminate descriptions of the criteria [6, 7]. Therefore, this article aims to introduce a completely flexible MCDM method for analyzing the available strategies for natural resource conservation and, if necessary, generating a new strategy. In conclusion, the objectives of this article include:

- ☐ To formulate an integrated MCDM method that can deal with indeterminate and incomplete information.
- ☐ To determine the objective criteria weight for avoiding subjective and biased assessment.
- ☐ To formulate an MCDM framework on natural resource conservation by identifying strategies and their related goals.
- ☐ To identify the executive strategy through the implementation of proposed MCDM methods

To reach the mentioned objectives, this article contributes as follows:

- Proposed an integrated neutrosophic Multi-Atributive Ideal-Real Comparative Analysis (MAIRCA) approach to address uncertain and indeterminate MCDM issues.
- Implemented the CRiteria Importance Through Intercriteria (CRITIC) approach to determine the objective criteria weights.
- Formulate an MCDM framework with six strategies and eleven criteria on natural resource conservation.
- Illustrate the proposed method numerically to select the best strategy for natural resource conservation.
- Established the proposed neutrosophic CRITIC-MAIRCA approach through comparison and sensitivity analysis.

To track the remaining article, the following section provides a brief literature study on the fuzzy MAIRCA approach and MCDM application on natural resource conservation. The proposed methodology with preliminary discussion on single valued neutrosophic sets (SVNSs) is demonstrated in Section 3. A framework on strategy selection to conserve natural resources is formulated in Section 4. Section 5 contains a comparison and sensitivity analysis-based numerical demonstration of the proposed approach. Section 6 accomplishes the article with some insights and findings.

2. Literature review

MAIRCA [8] simultaneously analyzes several competing criteria to provide an organized and methodical approach to decision-making. MAIRCA offers a fair evaluation of different options and their associated trade-offs by quantifying the criteria and lowering subjective biases through objective assessment. Though MAIRCA is helpful in handling multi-criteria, it cannot quantify qualitative assessment. Therefore, introducing fuzzy sets into the MCDM problem is a significant development to counter the uncertain decision-making process. It improves the capacity to handle ambiguity, imprecision, and qualitative aspects of decision criteria, resulting in decision-making processes that are more adaptable, practical, and focused on people. Chatterjee et al. [9] study the MAIRCA approach to assess the efficiency of environmentally friendly suppliers in the electronics sector. Trung et al. [10] and Nguyen et al. [11] compare the effectiveness of the MAIRCA approach with other MCDM methods. Hadian et al. [12] conducted a study that specifically examined flood susceptibility assessment using the MAIRCA approach. Maruf and Özdemir [13] work on tourism websites ranking by the MAIRCA method.

2.1. *Fuzzy MAIRCA approach*

The introduction of two membership degrees of intuitionistic fuzzy sets (IFSs) [14], Pythagorean fuzzy sets (PFSs) [15], and fermatean fuzzy sets (FFSs) [16] are the primary extensions of fuzzy sets that represent the acceptance and rejection of ambiguous information. However, these extensions fail to represent the indeterminacy of DM in quantifying inconsistent and hesitant data. The neutrosophic sets [17] are defined by membership, non-membership, and indeterminacy, which independently lies in $(0,\ 1)$ and can represent the indeterminacy in decision-making. The significance of neutrosophic fuzzy logic in decision-making has led to considerable advancements in SVNSs [18, 19]. Researchers developed their idea of neutrosophic fuzzy to analyze more conveniently to handle fuzziness [20, 21]. The fuzzy MAIRCA approach is utilized in several applications like defense system strategy selection [22], analysis of sustainable methods for treating wastewater [23], choosing a COVID-19 vaccination during the coronavirus pandemic [24], occupational health and environmental risk assessment [25], assessment of occupational risks [26]. Haq et al. [27] use the MAIRCA technique throughout an interval neutrosophic framework to determine the most suitable sustainable material for Human-Powered Aircraft.

The CRITIC [28] method establishes the objective weights of the criteria through the inter-criteria relationship established by statistical measures. The objectivity and certainty of selecting the best alternatives can be organized by combining MAIRCA and CRITIC approaches.

2.2. *MCDM in natural resource conservation*

The effective management and protection of natural resources have emerged as significant problems in biodiversity and ecological systems. Academic researchers are demonstrating commendable achievement in the domain of natural resource conservation. The literature has a large number of works about the same. The research by Regan et al. [29] examines the complete criteria for assessing biodiversity in forest conservation planning. Mendoza and Martins [30] examine the application of MCDM for natural resource management, with a specific focus on forest ecosystems. Hassangavyar et al. [31] investigate the methods used to mitigate soil erosion using a comparison of Vise Kriterijumska Optimizajica (VIKOR) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) models. Yang et al. [32] examine the connection between environmental preservation and managing natural assets through the VIKOR approach. Fuzzy MCDM techniques make a significant contribution to the preservation of natural resources. Researchers used several different applications of fuzzy MCDM for various reasons relating to the conservation of resources. Chen et al. [33] use the fuzzy Analytic Hierarchy Process (AHP) methodology to determine the optimal environment-watershed design. Narayanamoorthy et al. [34] conduct a thorough analysis of the selection of appropriate biomass conservation strategies using the fuzzy Decision-Making Trial and Evaluation Laboratory (DEMATEL) method. Table 1 summarizes the case studies conducted on natural resource conservation.

TABLE 1. Natural resource conservation: Fuzzy decision-making.

Benchmark	Fuzziness	Application	Contributor
AHP, WASPAS		Energy, deforestation, biodiversity	[35]
WASPAS	TFN	Climate change mitigation	[36]
AHP	TFN	Watersheds for conservation measures	[37]
AHP	TFN	Soil erosion conservation	[38]
VIKOR	TFN	Coastal areas conservation	[39]
TOPSIS	TFN	Forest conservation	[40]
TOPSIS	Trapezoidal fuzzy	Water resources	[41]
CoCoSo	Fermatean Fuzzy	Water save	[42]
MARCOS		Solar cite location	[43]
TOPSIS	Hesitant fuzzy set	Energy policy	[44]

It is evident from Table 1 that the researchers primarily used TFN for uncertainty representation while AHP, WASPAS, and TOPSIS dominate in the MCDM approaches. The applications of these fuzzy approaches mainly cover specific conservation, such as energy, deforestation, watersheds, soil, and forests.

2.3. *Research gap*

Through the pertinent literature review on the MAIRCA approach in fuzzy environments and MCDM application for natural resource conservation, we have identified the following as the progressive research area:

- ◇ To conserve natural resources, it is more meaningful to consider all aspects rather than a specific goal. Specific conservation may not be affected due to the interlinked properties of the elements related to natural resources.
- ◇ There needs to be more fuzzy decision-making approaches in natural resource conservation. The existing approaches only consider basic fuzzy sets for representing uncertainty, which may miss the hesitancy component of a decision expert.

- ◇ The researchers often need to pay more attention to the criteria weight determination procedure and deploy subjective assessment in this case. Hence, employing an objective assessment of the criteria is necessary to avoid subjective assessments and biases.
- ◇ The neutrosophic extension of the MAIRCA approach is available in the literature, but the neutrosophic MAIRCA was not combined with the CRITIC approach to produce a combined MCDM method.

3. **Material and method**

This section provides preliminary information on SVNSs and the proposed methodology.

3.1. *Preliminaries*

Definition 3.1. [17] An SVNS is represented by $A = (x,\ \varphi_A(x),\ \varpi_A(x),\ \varrho_A(x)|x \in X)$ on a fixed set X, where $\varphi_A(x),\ \varpi_A(x),\ \varrho_A(x) \in [0,\ 1]$ are degrees of membership, non-membership, and indeterminacy of the element $x \in X$ to the set A, respectively. For simplicity, $A = (\varphi_A,\ \varpi_A,\ \varrho_A)$ is used as single-valued neutrosophic number (SVNE).

Definition 3.2. [45] Let $A_1 = (x,\ \varphi_1,\ \varpi_1,\ \varrho_1)$ and $A_2 = (x,\ \varphi_2,\ \varpi_2,\ \varrho_2)$ be two SVNEs and $\lambda \geq 0$, the arithmetic operations are defined as follows:

$$A_1 \bigoplus A_2 = (\varphi_1 + \varphi_2 - \varphi_1.\varphi_2, \varpi_1\varpi_2, \varrho_1\varrho_1)$$
$$A_1 \bigotimes A_2 = (\varphi_1.\varphi_2,\ \varpi_1 + \varpi_2 - \varpi_1\varpi_2,\ \varpi_1 + \varpi_2 - \varpi_1\varpi_2,\ \varrho_1\varrho_1)$$
$$\lambda A_1 = (1 - (1 - \varphi_1)^\lambda,\ \varpi_1^\lambda,\ \varrho_1^\lambda)$$

Definition 3.3. [46] **Score function of SVNS:** Let $A = (x,\ \varphi_A(x),\ \varpi_A(x),\ \varrho_A(x)|x \in X)$ be a SVNS, then the score function of A is

$$\mathcal{SC}(A) = \frac{3 + \varphi_A(x) - 2.\varpi_A(x) - \varrho_A(x)}{4} \tag{1}$$

Let $\mathcal{SC}(A_1)$ and $\mathcal{SC}(A_2)$ be the score functions of two SVNEs A_1 and A_2 respectively, then

(i) If $\mathcal{SC}(A_1) \geq \mathcal{SC}(A_2)$ then $A_1 \geq A_2$
(ii) If $\mathcal{SC}(A_1) \leq \mathcal{SC}(A_2)$ then $A_1 \leq A_2$

3.2. *Proposed methodology*

In a neutrosophic context, this section develops a decision-making technique that employs the CRITIC and MAIRCA approaches to address imprecise MCDM problems. The suggested technique includes the following steps:

Step 1:: Formulate an MCDM issue with 'm' options $\mathcal{N} = \{\mathcal{N}_1,\ \mathcal{N}_2,\ ...,\ \mathcal{N}_m\}$ and 'n' criteria $\mathcal{R} = \{\mathcal{R}_1,\ \mathcal{R}_2,\ ...,\ \mathcal{R}_n\}$, where beneficiary and non-beneficiary criteria are denoted by $\mathcal{R}_{\mathcal{B}}$ and $\mathcal{R}_{\mathcal{NB}}$, respectively such that $\mathcal{R}_{\mathcal{B}} \cup \mathcal{R}_{\mathcal{NB}} = \mathcal{R}$ and $\mathcal{R}_{\mathcal{B}} \cap \mathcal{R}_{\mathcal{NB}} = \emptyset$.

Step 2:: Gather opinions of the DM using the seven-point linguistic scale provided in Table 2 to formulate a neutrosophic decision matrix D $= (\S_{ij})_{m\times n}$, where $\S_{ij} = (\Upsilon_{ij}, \aleph_{ij}, \daleth_{ij})$ such that Υ_{ij} is membership degree, $\aleph_{ij}$ is non-membership degree, and $\daleth_{ij}$ is indeterminacy degree.

Table 2. Scale for criteria assessment.

Linguistic terms	SVNS ($\Upsilon,\ \aleph,\ \daleth$)
Highly Oppose (HO)	(0.01, 0.75, 0.35)
Oppose (O)	(0.25, 0.55, 0.30)
Slightly Oppose (SO)	(0.30, 0.45, 0.25)
Neutral (N)	(0.50, 0.35, 0.20)
Slightly Favour (SF)	(0.75, 0.25, 0.15)
Favour (F)	(0.90, 0.15, 0.10)
Highly Favour (HF)	(0.99, 0.01, 0.01)

Step 3:: Determine the preference degree ($\mathcal{P}_{A_i}$) of the alternatives according to DM. Since the primary presumption of DM is unbiased about the alternatives, hence

$$\mathcal{P}_{A_i} = \frac{1}{m},\ \sum_{i=1}^{m} \mathcal{P}_{A_i} = 1$$

Step 4:: Criteria weight determination using the CRITIC method.

Step 4.1:: Evaluate crisp decision matrix from neutrosophic decision matrix D using equation (1).

Step 4.2:: Compute the criteria' respective standard deviations as follows: $\sigma_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\S_{ij} - \overline{\S_i}\right)^2},\ j = 1(1)n.$

Step 4.3:: Calculate the linear correlation coefficient of every element of crisp decision matrix as $\rho_{ij} = \frac{\sum_{j=1}^{n}(\S_{ij}-\overline{\S_i})((\S_{kj}-\overline{\S_k})}{\sqrt{\sum_{j=1}^{n}(\S_{ij}-\overline{\S_i})^2((\S_{kj}-\overline{\S_k})^2}}$

Step 4.4:: Determine key indicators of each criterion as:

$$\text{II}_j = \sigma_j . \sum_{i=1}^{m}(1 - \rho_{ik}) \tag{2}$$

and weights as

$$\omega_j = \frac{\text{II}_j}{\sum_{j=1}^{n} \text{II}_j}. \tag{3}$$

Step 5:: Compute the theoretical evaluation matrix

$\mathcal{T}_P = (\text{t}_{p_{ij}})_{m\times n}$ such that $\text{t}_{p_{ij}} = \omega_j \mathcal{P}_{A_i}$.

Step 6:: Determination of real evaluation matrix $\mathcal{T}_r = (\text{t}_{r_{ij}})_{m\times n}$ from equations (4) and (5).

(i) For beneficiary criteria

$$\text{t}_{r_{ij}} = \text{t}_{p_{ij}} \times (\frac{\S_{ij} - \S_i^-}{\S_i^+ - \S_i^-}) \tag{4}$$

(ii) For non-beneficiary criteria

$$\text{t}_{r_{ij}} = \text{t}_{p_{ij}} \times (\frac{\S_{ij} - \S_i^+}{\S_i^- - \S_i^+}) \tag{5}$$

where $\S_i^+ = \max_i(\S_{ij})$ and $\S_i^- = \min_i(\S_{ij}),\ i = 1(1)m$.

Step 7:: Compute the overall gap matrix $\mathcal{G} = (\text{g}_{ij})_{m\times n}$, where $\text{g}_{ij} = \text{t}_{p_{ij}} - \text{t}_{r_{ij}}$.

Step 8:: Determine the total of the criterion function $(mathcalQ_i)$ for every option derived from equation (6).

$$\mathcal{Q}_i = \sum_{j=1}^{n} \text{g}_{ij},\ i = 1(1)n. \tag{6}$$

Step 9:: Determine alternate ranking based on the decreasing order of the criterion function sum $(\mathcal{Q}_i)$.

4. Natural resource conservation via MCDM problem

The criteria of natural resource conservation and the strategies to conserve are identified in this section.

4.1. *Identification of strategies alternatives*

In order to maintain sustainable use for present and future generations, it is necessary to put strategies in place to protect ecosystems, biodiversity, and natural processes. The following are some crucial tactics that are frequently used to protect natural resources:

Protected areas and conservation reserves [47, 48] ($\mathcal{N}_1$):: Establishing designated areas where human activities are limited to protect biodiversity, habitats, and ecosystem functions includes biosphere reserves, marine protected areas, national parks, and wildlife sanctuaries.

Sustainable resource management [49] ($\mathcal{N}_2$)**::** Preserving natural resources while satisfying present demands without jeopardizing the capability of upcoming generations to satisfy their own. Sustainable practices in agriculture, fisheries management, and forestry, such as organic farming and agroforestry, are exemplified by measures like protected breeding areas, quotas, and selective logging and reforestation.

Biodiversity conservation [50] ($\mathcal{N}_3$)**::** Safeguarding an ecosystem's species diversity and genetic diversity, among other aspects of its diversity and variability. A few instances include conservation breeding initiatives for endangered species, species reintroduction initiatives, and habitat restoration and protection initiatives.

Ecosystem restoration [51] ($\mathcal{N}_4$)**::** Revitalization is the process of restoring the biodiversity and ecological functionality of harmed or destroyed ecosystems. Reforestation of degraded land, rehabilitation of coral reefs, and restoration of rivers, as well as wetland habitats, are a few examples.

Climate change adaptation and mitigation [52] ($\mathcal{N}_5$)**::** Employing adaptation of climate change mitigation techniques on ecosystems and natural resources while lowering greenhouse gas emissions. A few examples are implementing carbon sequestration projects (e.g., afforestation and mangrove restoration) and creating climate-resilient ecosystems.

Research and monitoring [53] ($\mathcal{N}_6$)**::** Assessing the condition of natural resources, identifying threats, and assessing the efficacy of conservation strategies through scientific research and monitoring programs. Studies on the ecological impacts caused by people, biodiversity surveys, and ecological monitoring programs are a few examples.

4.2. *Defining criteria*

Several variables influence the conservation and preservation of resources. These features above are considered to be essential factors for the conservation of natural resources. However, further study focuses on the specific criteria used to assess the rehabilitation and preservation of natural resources. The present study has effectively identified an array of eleven significant elements that are described in Table 3.

Table 3. Criteria description.

Criteria	Description	Symbol
Ecological Impact [54]	Consider biodiversity, ecosystem services, habitat quality, and climate change resilience when evaluating each strategy	$\mathcal{R}_1$
Economic Costs [55, 56]	Examine the financial effects of every approach, taking into account startup costs, ongoing expenses, and possible sources of income (such as eco-tourism or the sale of timber)	$\mathcal{R}_2$
Social Acceptance [57]	Various stakeholder groups, such as local communities, indigenous peoples, and future generations, should have their benefits and burdens distributed accordingly	$\mathcal{R}_3$
Regulatory Compliance	Ensure adherence to local, national, and international laws and regulations governing natural resource management	$\mathcal{R}_4$
Regulatory Effectiveness [58]	Think about the method's effectiveness in utilizing energy, land, and water as natural resources	$\mathcal{R}_5$
Technological Feasibility [59, 60]	Examine whether the technologies needed to implement the strategy are dependable and readily available	$\mathcal{R}_6$
Public Health and Safety	Determine the possible effects on people's health and safety, taking into account any exposure to risks or pollutants	$\mathcal{R}_7$
Long-Term Sustainability [61]	Evaluate the strategy's capacity to sustain social justice, economic feasibility, and ecological balance over time	$\mathcal{R}_8$
Risk and Uncertainty [62]	Analyze the strategy's degree of risk while taking the economy, society, and environment into consideration	$\mathcal{R}_9$
Scalability and Replicability	Examine the possibility of scaling up or replicating the strategy in different settings or areas.	$\mathcal{R}_{10}$
Stakeholder Acceptance [63]	Assess the degree of acceptance and support from important parties such as businesses, governmental organizations, environmental non-governmental organizations, and communities	$\mathcal{R}_{11}$

Among these criteria $\mathcal{R}_2$ and $\mathcal{R}_9$ are cost-base whereas remaining are benefit-base criteria.

5. Numerical analysis of proposed method on natural resource conservation

Step 1::

Create an MCDM problem with six alternatives $\mathcal{N} = \{\mathcal{N}_1,\ \mathcal{N}_2,\ \mathcal{N}_3,\ \mathcal{N}_4,\ \mathcal{N}_5,\ \mathcal{N}_6\}$ and eleven criteria $\mathcal{R} = \{\mathcal{R}_1,\ \mathcal{R}_2,\ \mathcal{R}_3,\ \mathcal{R}_4,\ \mathcal{R}_5, \mathcal{R}_6,\ \mathcal{R}_7,\ \mathcal{R}_8,\ \mathcal{R}_9,\ \mathcal{R}_{10},\ \mathcal{R}_{11}\}$.

Step 2:: The neutrosophic decision matrix D is formulated in Table 4, using criteria rating from Table 2.

Table 4. Linguistic neutrosophic decision matrix.

Alt./Cr.	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$	$\mathcal{R}_7$	$\mathcal{R}_8$	$\mathcal{R}_9$	$\mathcal{R}_{10}$	$\mathcal{R}_{11}$
$\mathcal{N}_1$	SF	F	O	HF	HF	HF	SF	HF	HF	F	HF
$\mathcal{N}_2$	F	F	O	HF	HF	HF	F	HF	HF	HF	F
$\mathcal{N}_3$	F	HF	SO	HF	HF	HF	SF	HF	HF	F	F
$\mathcal{N}_4$	HF	HF	SO	HF	HF	HF	SO	F	F	F	F
$\mathcal{N}_5$	HF	N	HO	N	SF	O	SO	F	HF	F	F
$\mathcal{N}_6$	HF	N	HO	F	HF	HF	F	F	HF	HF	HF

Step 3:: Since the problem regarding natural resource conservation contains six alternatives, the preference degree $(\mathcal{P}_{A_i})$ of each alternative is $\mathcal{P}_{A_i} = \frac{1}{6};\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$

Step 4:: Application of the CRITIC approach:

Step 4.1:: Crisp decision-matrix is determined from the linguistic neutrosophic decision matrix of Table 4 using equation (1).

Step 4.2: The standard deviations $\sigma_j,\ j = 1(1)11$ are computed for each criteria. The outcomes of these two steps are demonstrated in Table 5.

Table 5. Crisp decision matrix and standard deviations.

Alt./Cr.	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$	$\mathcal{R}_7$	$\mathcal{R}_8$	$\mathcal{R}_9$	$\mathcal{R}_{10}$	$\mathcal{R}_{11}$
$\mathcal{N}_1$	0.85	0.925	0.612	0.995	0.995	0.995	0.85	0.995	0.995	0.9	0.995
$\mathcal{N}_2$	0.925	0.925	0.612	0.995	0.995	0.995	0.925	0.995	0.995	0.992	0.925
$\mathcal{N}_3$	0.925	0.995	0.662	0.995	0.995	0.995	0.85	0.995	0.995	0.9	0.925
$\mathcal{N}_4$	0.995	0.995	0.662	0.995	0.995	0.995	0.6625	0.925	0.925	0.9	0.925
$\mathcal{N}_5$	0.995	0.75	0.465	0.75	0.85	0.662	0.662	0.925	0.995	0.9	0.925
$\mathcal{N}_6$	0.995	0.75	0.465	0.925	0.995	0.995	0.925	0.925	0.995	0.992	0.995
σ_j	0.059	0.113	0.092	0.098	0.059	0.136	0.121	0.038	0.029	0.048	0.036

Step 4.3:: Derive the linear correlation relationship of every element of the crisp decision matrix

Table 6. Correlation coefficient of each criterion.

Criteria	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$	$\mathcal{R}_7$	$\mathcal{R}_8$	$\mathcal{R}_9$	$\mathcal{R}_{10}$	$\mathcal{R}_{11}$
$\mathcal{R}_1$	1.000	-0.448	-0.473	-0.517	-0.396	-0.396	-0.448	-0.885	-0.396	0.165	-0.329
$\mathcal{R}_2$	-0.448	1.000	0.999	0.794	0.608	0.608	-0.019	0.566	-0.456	-0.360	-0.360
$\mathcal{R}_3$	-0.473	0.999	1.000	0.561	0.613	0.613	-0.002	0.586	-0.440	-0.348	-0.348
$\mathcal{R}_4$	-0.517	0.794	0.794	1.000	0.959	0.959	0.485	0.585	-0.261	0.138	0.138
$\mathcal{R}_5$	-0.396	0.608	0.613	0.959	1.000	1.000	0.608	0.447	0.447	-0.200	0.316
$\mathcal{R}_6$	-0.396	0.608	-0.002	0.959	1.000	1.000	0.608	0.447	-0.200	0.316	0.316
$\mathcal{R}_7$	-0.448	-0.019	-0.002	0.485	0.608	0.608	1.000	0.566	0.608	0.721	0.480
$\mathcal{R}_8$	-0.885	0.566	0.586	0.585	0.447	0.447	0.566	1.000	0.447	0.000	0.000
$\mathcal{R}_9$	-0.396	-0.456	-0.440	-0.261	0.447	-0.200	0.608	0.447	1.000	0.316	0.316
$\mathcal{R}_{10}$	0.165	-0.360	-0.348	0.138	-0.200	0.316	0.721	0.000	0.316	1.000	0.250
$\mathcal{R}_{11}$	-0.329	-0.360	-0.348	0.138	0.316	0.316	0.480	0.000	0.316	0.250	1.000

Step 4.4:: Determine key indicators of the criteria using equation 2 and corresponding weights using equation 3. Table 7 demonstrates the computational outcomes of criteria weight determination.

Table 7. Key indicator and criteria weight by CRITIC approach.

Criteria	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$	$\mathcal{R}_7$	$\mathcal{R}_8$	$\mathcal{R}_9$	$\mathcal{R}_{10}$	$\mathcal{R}_{11}$
$\mathcal{R}_1$	0.000	1.448	1.473	1.517	1.396	1.396	1.448	1.885	1.396	0.835	1.329
$\mathcal{R}_2$	1.448	0.000	0.001	0.206	0.392	0.392	1.019	0.434	1.456	1.360	1.360
$\mathcal{R}_3$	1.473	0.001	0.000	0.439	0.387	0.387	1.002	0.414	1.440	1.348	1.348
$\mathcal{R}_4$	1.517	0.206	0.206	0.000	0.041	0.041	0.515	0.415	1.261	0.862	0.862
$\mathcal{R}_5$	1.396	0.392	0.387	0.041	0.000	0.000	0.392	0.553	0.553	1.200	0.684
$\mathcal{R}_6$	1.396	0.392	1.002	0.041	0.000	0.000	0.392	0.553	1.200	0.684	0.684
$\mathcal{R}_7$	1.448	1.019	1.002	0.515	0.392	0.392	0.000	0.434	0.392	0.279	0.520
$\mathcal{R}_8$	1.885	0.434	0.414	0.415	0.553	0.553	0.434	0.000	0.553	1.000	1.000
$\mathcal{R}_9$	1.396	1.456	1.440	1.261	0.553	1.200	0.392	0.553	0.000	0.684	0.684
$\mathcal{R}_{10}$	0.835	1.360	1.348	0.862	1.200	0.684	0.279	1.000	0.684	0.000	0.750
$\mathcal{R}_{11}$	1.329	1.360	1.348	0.862	0.684	0.684	0.520	1.000	0.684	0.750	0.000
Sum	11.834	8.047	8.066	8.204	8.216	8.216	10.376	9.701	12.106	11.606	11.584
σ_j	0.059	0.113	0.092	0.098	0.059	0.136	0.121	0.038	0.029	0.048	0.036
II_j	0.696	0.908	0.741	0.807	0.486	1.115	1.255	0.372	0.346	0.554	0.419
ω_j	0.090	0.118	0.096	0.105	0.063	0.145	0.163	0.048	0.045	0.072	0.054

Step 5:: Calculate the theoretical evaluation matrix $\mathcal{T}_P$ and we get

Table 8. Theoretical evaluation matrix.

Criteria	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$	$\mathcal{R}_7$	$\mathcal{R}_8$	$\mathcal{R}_9$	$\mathcal{R}_{10}$	$\mathcal{R}_{11}$
$\mathcal{N}_1$	0.020	0.009	0.009	0.010	0.010	0.010	0.017	0.015	0.023	0.021	0.126
$\mathcal{N}_2$	0.020	0.009	0.009	0.010	0.010	0.010	0.017	0.015	0.023	0.021	0.126
$\mathcal{N}_3$	0.020	0.009	0.009	0.010	0.010	0.010	0.017	0.015	0.023	0.021	0.126
$\mathcal{N}_4$	0.020	0.009	0.009	0.010	0.010	0.010	0.017	0.015	0.023	0.021	0.126
$\mathcal{N}_5$	0.020	0.009	0.009	0.010	0.010	0.010	0.017	0.015	0.023	0.021	0.126
$\mathcal{N}_6$	0.020	0.009	0.009	0.010	0.010	0.010	0.017	0.015	0.023	0.021	0.126

Step 6:: Determination of real evaluation Equation $\mathcal{T}_r$ and we get

Table 9. Real evaluation matrix.

Cr./Alt.	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$
$\mathcal{R}_1$	0.000	0.011	0.011	0.020	0.020	0.000
$\mathcal{R}_2$	0.004	0.004	0.000	0.000	0.016	0.016
$\mathcal{R}_3$	-0.015	-0.015	-0.012	-0.012	-0.025	-0.025
$\mathcal{R}_4$	0.010	0.010	0.010	0.010	-0.007	-0.007
$\mathcal{R}_5$	0.010	0.010	0.010	0.010	0.000	0.000
$\mathcal{R}_6$	0.010	0.010	0.010	0.010	-0.017	-0.017
$\mathcal{R}_7$	0.000	0.009	0.000	-0.022	-0.022	-0.022
$\mathcal{R}_8$	0.015	0.015	0.015	0.008	0.008	0.008
$\mathcal{R}_9$	0.000	0.000	0.000	0.011	0.000	0.011
$\mathcal{R}_{10}$	0.011	0.021	0.011	0.011	0.011	0.011
$\mathcal{R}_{11}$	0.000	0.061	0.061	0.061	0.061	0.061

Step 7:: Obtain the gap matrix $\mathcal{G}$ shown in table 10

Step 8:: Calculate values of criteria-function $\mathcal{Q}_i$ shown in table 10

Step 9:: The ranking order of the alternatives are evaluated and listed Table 10.

Table 10. Gap matrix and ranking order of the alternatives.

Cr./Alt.	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$
$\mathcal{R}_1$	0.015	0.007	0.007	0.000	0.000	0.015
$\mathcal{R}_2$	0.010	0.010	0.020	0.020	-0.014	-0.014
$\mathcal{R}_3$	0.042	0.042	0.037	0.037	0.059	0.059
$\mathcal{R}_4$	0.000	0.000	0.000	0.000	0.030	0.030
$\mathcal{R}_5$	0.000	0.000	0.000	0.000	0.011	0.011
$\mathcal{R}_6$	0.000	0.000	0.000	0.000	0.064	0.064
$\mathcal{R}_7$	0.027	0.013	0.027	0.062	0.062	0.062
$\mathcal{R}_8$	0.000	0.000	0.000	0.004	0.004	0.004
$\mathcal{R}_9$	0.007	0.007	0.007	0.004	0.007	0.004
$\mathcal{R}_{10}$	0.006	0.000	0.006	0.006	0.006	0.006
$\mathcal{R}_{11}$	0.126	0.065	0.065	0.065	0.065	0.065
$\sum_{i=1}^{m} \mathcal{Q}_i$	0.234	0.146	0.169	0.197	0.293	0.305
Rank	3	6	5	4	2	1

5.1. *Sensitivity analysis*

A critical factor in determining the classification order of the alternatives is the influence of criteria weight variations. It is imperative to evaluate the effect on the robustness of the proposed neutrosophic MAIRCA approach. We employ the MEREC [64], Rank-Sum [65], Entropy [66], and FUCOM [67] methods to determine the weight, as illustrated in Table 11. To ensure that the proposed approach remains logically equivalent, the remaining procedures are maintained identically.

Table 11. Criteria preferences determined by various methods.

Methods	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$	$\mathcal{R}_5$	$\mathcal{R}_6$	$\mathcal{R}_7$	$\mathcal{R}_8$	$\mathcal{R}_9$	$\mathcal{R}_{10}$	$\mathcal{R}_{11}$
CRITIC	0.090	0.118	0.096	0.105	0.063	0.145	0.163	0.048	0.045	0.072	0.054
MEREC	0.166	0.025	0.061	0.027	0.198	0.026	0.095	0.026	0.200	0.024	0.151
RANK-SUM	0.167	0.076	0.061	0.030	0.045	0.152	0.106	0.136	0.121	0.091	0.015
ENTROPY	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091	0.091
FUCOM	0.0001	0.0002	0.0007	0.0001	0.003	0.007	0.018	0.034	0.088	0.245	0.596

The CRITIC approach assigns higher preference to $\mathcal{R}_2$, $\mathcal{R}_6$, and $\mathcal{R}_7$ while sets lower preferences to $\mathcal{R}_8$ and $\mathcal{R}_9$. The MEREC method allocates comparatively higher weights to the criteria $\mathcal{R}_1$, $\mathcal{R}_5$, and $\mathcal{R}_9$ while sets lower wights to the criteria $\mathcal{R}_2$, $\mathcal{R}_4$, $\mathcal{R}_8$, and $\mathcal{R}_{10}$. Surprisingly, the entropy approach assigns identical preferences to the criteria, which results in equal treatment. The FUCOM approach assigns significantly higher preferences to the criteria $\mathcal{R}_{10}$ and $\mathcal{R}_{11}$ compared to the remaining criteria. In conclusion, the cost-based criteria $\mathcal{R}_2$ and

$\mathcal{R}_9$ received higher preference than the remaining criteria except in the FUCOM approach. Figure 1 is drawn to understand the allocation of criteria weights in these approaches.

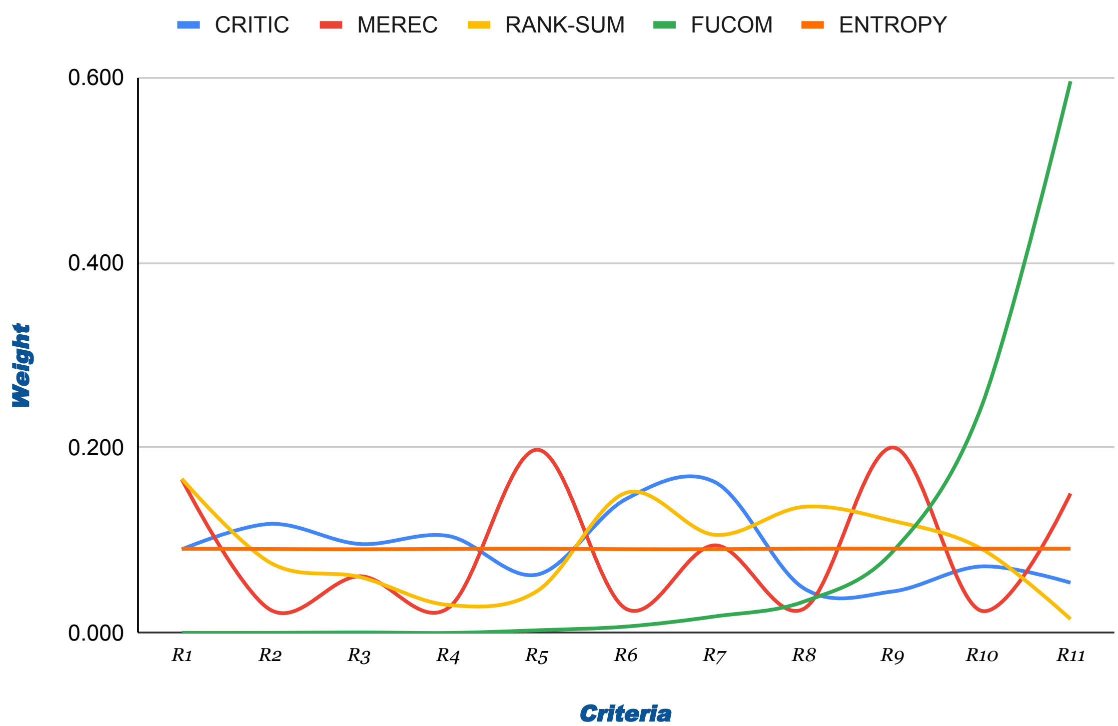

FIGURE 1. Criteria weight allocation in different approaches.

Figure 1 shows that the criteria weights of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ have low variation, $\mathcal{R}_5$, $\mathcal{R}_6$, $\mathcal{R}_7$, $\mathcal{R}_8$ and $\mathcal{R}_9$ have moderate variation, and $\mathcal{R}_{10}$ and $\mathcal{R}_{11}$ have high variation. The FUCOM approach has the highest degree of variation in assigning criteria weights as it gradually increases from the $\mathcal{R}_1$ criterion weight to $\mathcal{R}_8$ criterion weight. However, the curve instantly grows high for the criteria weights from $\mathcal{R}_9$ to $\mathcal{R}_{11}$. The impact of the criteria weights on the ranking sequence of the options is depicted in Table 12.

TABLE 12. Ranking of alternatives based on various methods.

Methods	Performance score						Rank
——	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\mathcal{N}_5$	$\mathcal{N}_6$	
CRITIC	0.234	0.146	0.169	0.197	0.293	0.305	$\mathcal{N}_6 \succ \mathcal{N}_5 \succ \mathcal{N}_1 \succ \mathcal{N}_4 \succ \mathcal{N}_3 \succ \mathcal{N}_2$
MEREC	0.234	0.149	0.157	0.150	0.226	0.237	$\mathcal{N}_6 \succ \mathcal{N}_1 \succ \mathcal{N}_5 \succ \mathcal{N}_3 \succ \mathcal{N}_4 \succ \mathcal{N}_2$
RANK-SUM	0.232	0.141	0.160	0.170	0.255	0.273	$\mathcal{N}_6 \succ \mathcal{N}_5 \succ \mathcal{N}_1 \succ \mathcal{N}_4 \succ \mathcal{N}_3 \succ \mathcal{N}_2$
ENTROPY	0.226	0.143	0.160	0.172	0.255	0.263	$\mathcal{N}_6 \succ \mathcal{N}_5 \succ \mathcal{N}_1 \succ \mathcal{N}_4 \succ \mathcal{N}_3 \succ \mathcal{N}_2$
FUCOM	0.164	0.082	0.103	0.102	0.113	0.106	$\mathcal{N}_1 \succ \mathcal{N}_5 \succ \mathcal{N}_6 \succ \mathcal{N}_3 \succ \mathcal{N}_4 \succ \mathcal{N}_2$

The alternative $\mathcal{N}_6$ ranks first in all weight determination techniques except FUCOM. In contrast, each technique allocates the alternative $\mathcal{N}_2$ to the last position. The alternative $\mathcal{N}_5$ always occupies the second position, except for the MEREC approach. In all methods, the fourth and fifth places are consistently occupied by either $\mathcal{N}_3$ or $\mathcal{N}_4$. Hence, the alternative $\mathcal{N}_6$ suppresses the remaining alternatives in terms of their performance through almost all weight determination procedures while $\mathcal{N}_2$'s performance is the weakest. This phenomenon can be conveniently viewed from figure 2.

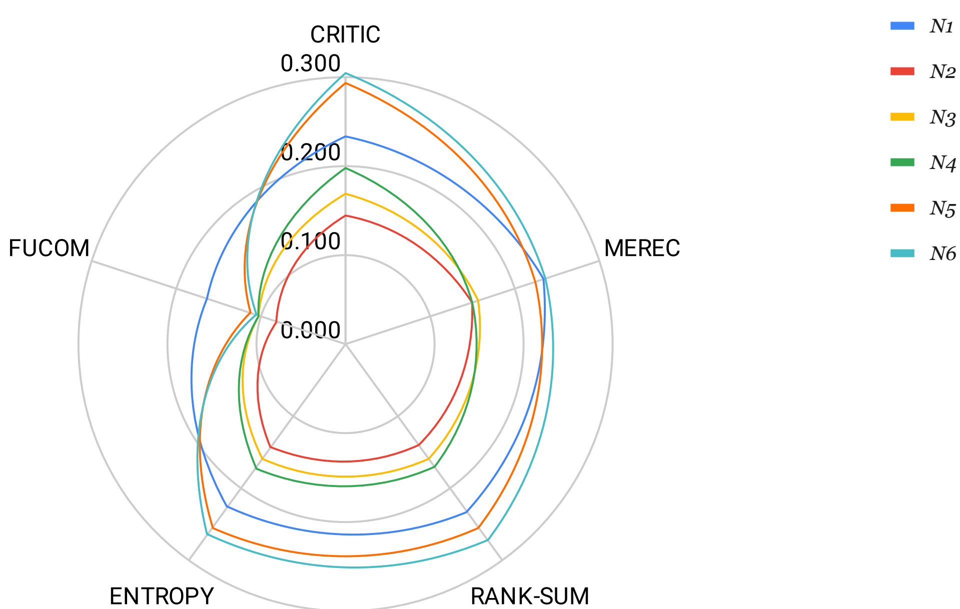

FIGURE 2. Performance score of each alternative by using various methods of weight calculation.

Figure 2 shows the performance score variation of alternatives for several weight methods. A marginal influence on the performance scores of the alternatives for the FUCOM approach is seen in figure 2. Based on the CRITIC approach, a significant disparity in performance scores is seen by which one can easily rank the options, with $\mathcal{N}_6$ occupying the top position and $\mathcal{N}_2$ ranking last. In the context of performance scores for the CRITIC, RANK-SUM, and ENTROPY methods, the figure illustrates how to rank the options easily. The MEREC technique offers an alternative approach that assigns greater performance priority to $\mathcal{N}_6$, $\mathcal{N}_5$, and $\mathcal{N}_1$ while assigning lesser preference to $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$.

5.2. *Comparison analysis*

To establish it, it is imperative to compare the outcomes of the suggested SVN-MAIRCA with those of the current MCDM techniques. We have considered popular MCDM approaches like TOPSIS, MARCOS, and MABAC to compare the outcome of the proposed SVN-MAIRCA approach. Evaluating alternative rankings using the CRITIC criterion weight maintains logical similarity in computation. The crisp MCDM approaches are applied to the crisp decision matrix from Table 7. Table 13 compares the proposed and existing techniques' ranking orders.

TABLE 13. Alternatives' ranking in suggested and existing methods.

Approach	SVN-MAIRCA		TOPSIS		MARCOS		MABAC	
Alternatives	Score	Rank	Score	Rank	Score	Rank	Score	Rank
$\mathcal{N}_1$	0.226	3	0.632	3	0.011	3	1.637	3
$\mathcal{N}_2$	0.136	6	0.634	1	0.011	1	1.748	2
$\mathcal{N}_3$	0.156	5	0.633	2	0.011	2	1.620	4
$\mathcal{N}_4$	0.164	4	0.630	5	0.011	4	1.544	5
$\mathcal{N}_5$	0.227	2	0.617	6	0.009	6	1.208	6
$\mathcal{N}_6$	0.236	1	0.631	4	0.010	5	1.780	1

Table 13 demonstrates that the ranking order derived from the SVN-MAIRCA method exhibits a substantial disparity compared to the rankings generated from other MCDM techniques. The first ranked alternative $\mathcal{N}_6$ in the SVN-MAIRCA model is only equivalent to MABAC. The alternative $\mathcal{N}_5$ obtained the lowest score in all crisp approaches, except for SVN-MAIRCA, which achieved the second highest ranking. The alternative $\mathcal{N}_2$ exhibits significant rank fluctuations when comparing the neutrosophic and crisp MCDM techniques. The remaining options exhibit a moderate degree of difference in ranking between the proposed and current techniques. Figure 3 compares the alternatives' ranking among SVN-MAIRCA and existing approaches.

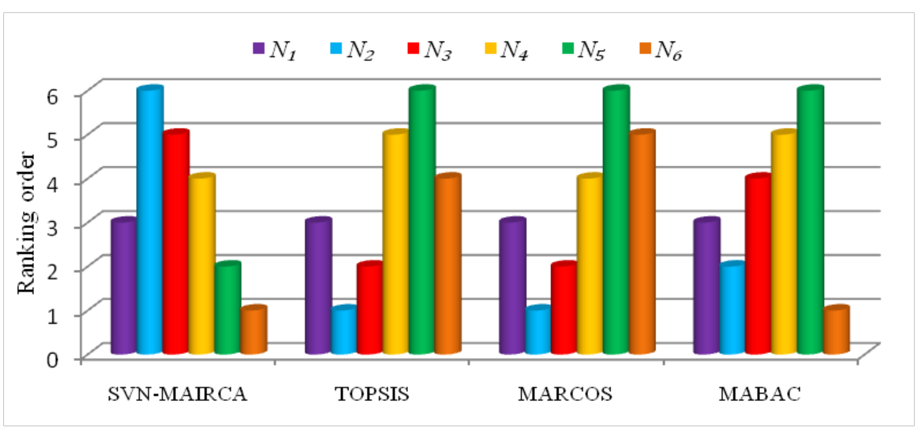

FIGURE 3. Ranking comparison of natural resource conservation strategy

The diagram 3 makes clear that the option $\mathcal{N}_1$ is consistently assigned third place by both the suggested SVN-MAIRCA technique and other approaches. This phenomenon indicates that the suggested technique is consistent. Every approach assigns $\mathcal{N}_4$ to either the third or fourth position. Figure 3 also shows that, in contrast to the suggested SVN-MAIRCA technique, which allocates $\mathcal{N}_5$ at the second place, all considered methods allocate it at the last position.

Conclusion

This paper presented an integrated neutrosophic MCDM methodology, that successfully determined the criteria weight and the ranking of the alternatives. The suggested method incorporates hesitation in decision-making through SVNS-rating of the criteria. The criteria weight variation shows that the proposed method is robust in decision-making. The comparison analysis reveals that the proposed neutrosophic methodology is inconsistent with the outcome of crisp MCDM approaches. The reason behind this inconsistency is the incorporation of uncertainty and hesitation in the computation procedures of the recommended approach. Based on the presented methodology, the alternative "research and monitoring" is the most optimal strategy, while "sustainable resource management" is deemed the least desirable. The critical criteria for ranking preference are ecological impact, economic costs, and stakeholder acceptance. Since the preference of the criterion is high, the financial cost is much higher than the remaining critical criteria; hence, it significantly impacts strategy selection for natural resource conservation.

Although the suggested method has several benefits, it has some constraints, as follows: (i) single DM may be subjective to some particular criterion, so producing a skewed assessment; (ii) the inclusion of uncertainty and hesitation in computation procedures is unavailable in criteria weight determination; and (iii) the availability of hesitant information about the criteria descriptions.

Recently developed fuzzy sets such as the Z-number, D-number, type-2 fuzzy set, and hesitant bi-fuzzy set may be used to update or enhance the criterion rating. The CRITIC approach may be expanded to include uncertainty in the criterion weight calculation in a Pythagorean, fermatean fuzzy environment. Introducing a panel of available DMs, a group decision-making methodology can be developed using the proposed N-MAIRCA approach for a more compact assessment of the criteria.

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On the basics of neutrosophic homotopy theory

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Abstract. In this paper, we introduce and study the concept of neutrosophic homotopic functions using topological properties. Also, we obtain some properties of neutrosophic homotopic functions and concept of neutrosophic homotopy.

Keywords: Neutrosophic sets, neutrosophic homotopic function, equivalence relation.

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1. Introduction

Zadeh [4] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophic" [French neutre, Latin neuter, neutral, and Greek sophia, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy / intuitionistic" logic/set and "neutrosophic" logic/set, that is, the included middle component, that is, the neutral/indeterminate/unknown part (besides the truth"/membership" and falsehood"/nonmembership" components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). The concept of neutrosophic set developed by Smarandache [2, 3] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm). In this paper, we introduced and studied the concept of neutrosophic homotopic functions using topological properties. Also, we obtained some properties of neutrosophic homotopic functions and concept of neutrosophic homotopy.

2. Preliminaries

Definition 2.1. Let A be a nonempty set. The neutrosophic set [2] on A is defined to be a structure

$$A := \{\langle x, \mu(x), \gamma(x), \psi(x)\rangle \mid x \in A\}, \tag{1}$$

where $\mu : A \to [0,1]$ is a truth membership function, $\gamma : A \to [0,1]$ is an indeterminate membership function, and $\psi : A \to [0,1]$ is a false membership function. The neutrosophic fuzzy set in (1) is simply denoted by $A = (\mu_A, \gamma_A, \psi_A)$.

For any neutrosophic set A of X and any $t, s, r \in [0,1]$, the (t, s, p)-cut of A is defined as $A_{t,r,s} = \{x \in X : \mu(x) \geq t \wedge \gamma(x) \leq s \wedge \psi(x) \leq p\}$.

Lemma 2.2. *A neutrosophic equivalence relation R on a set X is a relation that is reflexive, symmetric, and transitive:*

1. *Reflexive: $R_T(x,x) = 1$, $R_I(x,x) = 0$ and $R_F(x,x) = 0$ for all $x \in X$*
2. *Symmetric: $R(x,y) = R(y,x)$ for all $x, y \in X$*
3. *Transitive: $R \circ R \subseteq R$.*

Let R be an equivalence relation on A and $\langle x_{t,s,p}, y_{m,n,q}\rangle \in R$, $x, y \in X$, then we say $x_{t,s,p}$ equivalent to $y_{m,n,q}$ or $x_{t,s,p}$ and $y_{m,n,q}$ are equivalent. For any $x_{t,s,p} \in A$, using $(x_{t,s,p})_R = \cup\{y_{m,n,q} : \langle x_{t,s,p}, y_{m,n,q}\rangle \in R\}$. This is called R-equivalence class of $x_{t,s,p}$ and simply denoted by $(x_{t,s,p})$. Now, we put $A/B = \{(x_{t,s,p}) : x_{t,s,p} \in A\}$. We denoted, $R^{-1} = \cup\{\lambda R_\lambda^{-1} : t, s, p \in [0,1]\}$, it is inverse relation of R. If, $R = \cup\{\langle x_{t,s,p}, y_{m,n,q}\rangle : x, y \in X\}$, then $R^{-1} = \cup\{\langle y_{m,n,q}, x_{t,s,p}\rangle : x, y \in X\}$ and $(R^{-1})^{-1} = R$.

3. Basics of neutrosophic homotopy theory

Definition 3.1. Let (A, τ_1) and (B, τ_2) be neutrosophic topological spaces and $f, g : A \to B$ are neutrosophic continuous functions. Then f is neutrosophic homotope to g if there exists a neutrosophic continuous function $F : A \times I \to B$ such that for every $t \in I$, $F_{t,s,p}(x, 0) = f_{t,s,p}(x)$, $F_{t,s,p}(x, 1) = g_{t,s,p}(x)$, $F_{t,s,p}(x, (m,n,q)) = f^{m,n,q}_{t,s,p}(x)$. If f and g are neutrosophic homotopic functions, we write $f \sim g$.

Proposition 3.2. *It is clear that every continuous function homotopic to itself.*

Definition 3.3. Let (A, τ_1) and (B, τ_2) be neutrosophic topological spaces. Let $X_0 \subset X$ and there exists functions $f, g : X \to Y$ such that the condition for every $x_0 \in X_0$, $f(x_0) = g(x_0)$ is satisfied. Then f is neutrosophic homotopic to g relative to x_0 and written by $f \sim g/X_0$, if there exists a function $F : A \times I \to B$ such that the following conditions hold:

(1) $F_{t,s,p}(x, 0) = f(x)$, $F_{t,s,p}(x, 1) = g(x)$, for every $x \in X$,
(2) $F_{t,s,p}(x_0, (m, n, q)) = f(x_0) = g(x_0)$, for every $x_0 \in X_0$.

Theorem 3.4. *Neutrosophic homotopy relation is neutrosophic equivaliance relation.*

Proof. Now that for every continuous function f, $f \sim f$. Assume that $f \sim g$, then there exists a function F such that the conditions of the Definition are satisfied. If we rewrite the function F as $F'(x, (t, s, p)) = F(x, 1 - t)$, then we get $g \sim f$. The distributive condition is clear. □

Definition 3.5. Let X and Y be topological spaces and A and B are neutrosophic topological spaces on X and Y, respectively. Let $f, g \subset A \times B$ neutrosophic continuous and $f \sim g$. If $|img| = 1$, then it is called that f is homotopic to arbitrary.

Definition 3.6. X is contractibility or X may deformated to a point if the identity definition on X is homotopic to arbitrary.

Theorem 3.7. *Let X be a neutrosophic topological space and Y can be deformated to a point then all of the neutrosophic continuous function $f : A \to B$ is homotopic to arbitrary.*

Proof. B may deformated neutrosophic topological space then for every $t, s, p \in I$, $B_{t,s,p}$ may deformated topological subspace. Therefore there exists a function $g : B \to B$ such that for $y_0 \in B_{t,s,p}$ arbitrary, $g_{t,s,p}(y) = y_0$ such that $1_{B_{t,s,p}} : B_{t,s,p} \to B_{t,s,p}$ is homotopic to $g_{t,s,p}$, that is, $1_{B_{t,s,p}} \sim g_{t,s,p}$. Therefore, there exists a continuous function $f_{t,s,p} : B_{t,s,p} \times I^2 \to B_{t,s,p}$ such that for every $y \in Y$, $F_{t,s,p}(y, 0) = 1_{B_{t,s,p}}(y, 0) = 1_{B_{t,s,p}}(y), F_{t,s,p}(y, 1) = g_{t,s,p}(y)$. We assume that, $f : A_{t,s,p} \to B_{t,s,p}$ neutrosophic continuous function. We define that neutrosophic function $G_{t,s,p} : A_{t,s,p} \times I^2 \to B_{t,s,p}$ as $G_{t,s,p}(x, (m, n, q)) = F_{t,s,p}(f_{t,s,p}(x), (m, n, q))$. It is clear that $G_{t,s,p}$ is continuous function for every $t, s, p \in I$ and $G_{t,s}(x, 0) = F_{t,s,p}(f_{t,s,p}(x), 0) = f_{t,s,p}(x)$, $G_{t,s,p}(x, 1) = F_{t,s,p}(f_{t,s,p}(x), 1) = y_0$. Thus f is homotopic to arbitrary for every $t, s, p \in I$. Therefore f is neutrosophic homotopic to arbitrary. □

Theorem 3.8. *Let $A \in NS(X)$, $B \in NS(Y)$, $C \in NS(Z)$ be neutrosophic topological spaces and $f \subset A \times B$, $g \subset B \times C$ be neutrosophic continuous functions. If $g \sim h$, then $g \circ f$ and $h \circ f$ are neutrosophic continuous and $g \circ f \sim h \circ f$.*

Proof. If $g \sim h$, then there exists a neutrosophic continuous function F such that $F_{t,s,p}(y,0) = g_{t,s,p}$ and $F_{t,s,p}(y,1) = f_{t,s,p}$ for every $t,s,p \in I$. Let us define a function G with respect to F such that $G_{t,s,p}(x,(m,n,q)) = F_{t,s,p}(f(x),(m,n,q))$ for every $t,s,p \in I$. It is clear that G is a neutrosophic continuous function. However, $G_{t,s,p}(x,0) = F_{t,s,p}(f_{t,s,p}(x),0) = g_{t,s,p}f_{t,s,p}$ and $G_{t,s,p}(x,1) = F_{t,s,p}(f_{t,s,p}(x),1) = h_{t,s,p}f_{t,s,p}$ for every $t,s,p \in I$, too. Therefore, $g_{t,s,p}f_{t,s,p} \sim h_{t,s,p}f_{t,s,p}$, for every $t,s,p \in I$ thus $gf \sim hf$. □

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Neutrosophic Rhotrices for Improved Diagnostic Accuracythrough Score Rhotrix Computation

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Abstract. We introduce neutrosophic rhotrices, which serve as a novel extension of neutrosophic matrices. The primary objective is to establish the foundational structure for neutrosophic rhotrices and to define crucial operations that can be performed on them. We explore the concept of neutrosophic rhotrices in depth, outlining the fundamental operations necessary for their effective manipulation. Furthermore, we investigate the essential properties of neutrosophic rhotrices utilizing these newly established operations. In addition, we provide an algorithm designed to enhance decision-making processes in medical diagnostics, supported by an illustrative example to clarify its application.

Keywords: Neutrosophic Rhotrix, Rhotrix, Heart based Multiplication, Trace

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In 2003, Ajibade [1] introduced a method for representing arrays of numbers in a rhomboidal shape, which he named rhotrix and developed the structure of an n-dimensional rhotrix to fall between the $(n-1) \times (n-1)$ - dimensional matrix and the $n \times n$ - dimensional matrix. Ajibade [10] also proposed the first multiplication method for rhotrices, called heart-based multiplication. In 2004, B. Sani [2] introduced another multiplication method for rhotrices, known as row-column multiplication, which is similar to traditional matrix multiplication. These two methods are the primary techniques for rhotrix multiplication, though various other multiplications can be defined, none of which generate algebraic structures.

Sani [3] also introduced a technique for transforming a rhotrix into a matrix form, referred to as a coupled matrix. In 2008, A. O. Isere [4] expanded on Ajibade's work by introducing the concept of even-dimensional rhotrix, since Ajibade's rhotrix is always of odd dimension. Additionally, numerous authors have contributed to the development of rhotrix theory, exploring concepts such as rhotrix groups [7, 11], rhotrix rings [5], rhotrix vector spaces [6], and the application of rhotrices in various fields, including cryptography [12–14].

The concept of neutrosophy was introduced by Florentin Smarandache [8, 16] in the 1990s. Neutrosophic logic extends classical and fuzzy logic by considering three components: truth (T), indeterminacy (I), and falsity (F). This triad is utilized to handle real-world problems where information is incomplete, inconsistent, or uncertain.

Kandasamy and Smarandache [17–19] expanded on this concept by introducing neutrosophic algebraic structures, including neutrosophic fields, vector spaces, groups, and rings. In linear algebra, matrices are essential for understanding vector spaces and linear transformations, prompting the creation of neutrosophic matrices. A neutrosophic matrix is an extension of the classical matrix concept, incorporating neutrosophic logic, which deals with indeterminacy. Recently, Mohammad Abobala and et al. [15] investigated the algebraic properties of these matrices, such as diagonalization, invertibility, determinants, and their algebraic representations through linear transformations. Neutrosophic matrices have found applications in various fields, particularly in dealing with uncertain, inconsistent, and incomplete information. Mamoni Dhar [9], introduced neutrosophic soft matrices and also a score matrix addressing patient-symptoms and symptoms-disease neutrosophic soft matrices is also proposed to aid in decision-making. Recently, the authors [20] introduced fuzzy rhotrices and its application in decision making of medical diagnostics has been studied.

We seek to introduce the concept of neutrosophic rhotrices and establish the fundamental operations required for their manipulation. We will define the basic operations and furthermore, conduct an in-depth examination of the fundamental properties of neutrosophic rhotrices, utilizing the newly defined operations to explore their characteristics. Additionally, an algorithm is proposed for medical diagnosis using neutrosophic rhotrices accompanied with an illustrative example.

Let n be positive odd integer. A rhotrix of size n is represented as

$$\left\langle \begin{array}{ccccccccc} & & & & a_1 & & & & \\ & & & a_2 & a_3 & a_4 & & & \\ & & .. & .. & .. & .. & .. & & \\ & .. & .. & .. & .. & .. & .. & .. & \\ a_{\{\frac{t+1}{2}\}-\frac{n}{2}} & .. & .. & .. & a_{\frac{t+1}{2}} & .. & .. & .. & a_{\{\frac{t+1}{2}\}+\frac{n}{2}} \\ & .. & .. & .. & .. & .. & .. & .. & \\ & & .. & .. & .. & .. & .. & & \\ & & & a_{t-3} & a_{t-2} & a_{t-1} & & & \\ & & & & a_t & & & & \end{array} \right\rangle,$$

where $t = \frac{n^2+1}{2}$. In th above representation, the entries in red represents the first row, entries in blue represents first column, and entries in purple represents the diagonal. Here the major horizontal axis is $a_{\{\frac{t+1}{2}\}-\frac{n}{2}} \cdots a_{\frac{t+1}{2}} \cdots a_{\{\frac{t+1}{2}\}+\frac{n}{2}}$ and the major vertical axis is $a_1\ a_3 \cdots a_{\frac{t+1}{2}} \cdots a_{t-2}\ a_t$. Heart of rhotrix is the entry intersecting both major horizontal axis and major vertical axis of the rhotrix and that is $a_{\frac{t+1}{2}}$.

Heart-based Multiplication, as introduced by A. O. Ajibade, involves multiplying each entry of the first rhotrix by the heart of the second rhotrix, and then adding the product of the corresponding entry of the second rhotrix with the heart of the first rhotrix. In addition, the process of adding two rhotrices is done by summing each corresponding entry. To transpose a rhotrix, one flips its entries over the major vertical axis, keeping the entries on this axis unchanged.

1. Neutrosophic Rhotrix

Within this section, we lay the groundwork by presenting neutrosophic rhotrices and elucidating fundamental operations, namely neutrosophic rhotrix addition and heart-based multiplication. Subsequently, we analyze the algebraic attributes of neutrosophic rhotrices under these operations, with a specific emphasis on exploring their trace properties.

Definition 1.1. A ***neutrosophic rhotrix*** is defined as a rhotrix with each of the entries are from a neutrosophic set $R = [(r_i^T, r_i^I, r_i^F)]$, where r_i^T, r_i^I, r_i^F characterized by a membership function $\mu_R : F \to [0,1]$, indeterminate function $\mathcal{I}_R : F \to [0,1]$, falsity function $\mathcal{F}_R : F \to [0,1]$ respectively and F is any field.

Suppose $R_1 = \left\langle \begin{array}{ccc} & r_1 & \\ r_2 & r_3 & r_4 \\ & r_5 & \end{array} \right\rangle$ is a rhotrix in $R_n(F)$, then the neutrosophic rhotrix is represented as $(R_1)_N = \left\langle \begin{matrix} & (\mu_R(r_1), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & \\ (\mu_R(r_2), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & (\mu_R(r_3), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & (\mu_R(r_4), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) \\ & (\mu_R(r_5), \mathcal{I}_R(r_1), \mathcal{F}_R(r_1)) & \end{matrix} \right\rangle$

In simpler terms, it can be written as,

$$(R_1)_N = \left\langle \begin{matrix} & (r_1^T, r_1^I, r_1^F) & \\ (r_2^T, r_2^I, r_2^F) & (r_3^T, r_3^I, r_3^F) & (r_4^T, r_4^I, r_4^F) \\ & (r_5^T, r_5^I, r_5^F) & \end{matrix} \right\rangle$$

Definition 1.2. We define the **operations on neutrosophic rhotrices** as follows:

(i) The addition of two neutrosophic rhotrices is possible only if they are of the same size. Addition of two neutrosophic rhotrices $'A'_N$ and $'B'_N$ is defined as:

$$\text{The } i^{th} \text{ entry of } A_N + B_N = (max\{a_i^T, b_i^T\}, max\{a_i^I, b_i^I\}, min\{a_i^F, b_i^F\}), \text{ for all } i.$$

That is, for rhotrices $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$ of size 3, and $B_N = \left\langle \begin{matrix} & (b_1^T, b_1^I, b_1^F) & \\ (b_2^T, b_2^I, b_2^F) & (b_3^T, b_3^I, b_3^F) & (b_4^T, b_4^I, b_4^F) \\ & (b_5^T, b_5^I, b_5^F) & \end{matrix} \right\rangle$ of size 3,

$$A_N + B_N = \left\langle \begin{matrix} & (max\{a_1^T, b_1^T\}, max\{a_1^I, b_1^I\}, \\ & min\{a_1^F, b_1^F\}) & \\ (max\{a_2^T, b_2^T\}, max\{a_2^I, b_2^I\} & (max\{a_3^T, b_3^T\}, max\{a_3^I, b_3^I\} & (max\{a_4^T, b_4^T\}, max\{a_4^I, b_4^I\} \\ , min\{a_2^F, b_2^F\}) & , min\{a_3^F, b_3^F\}) & , min\{a_4^F, b_4^F\}) \\ & (max\{a_5^T, b_5^T\}, max\{a_5^I, b_5^I\}, & \\ & min\{a_5^F, b_5^F\}) & \end{matrix} \right\rangle$$

(ii) The heart based neutrosophic multiplication of two neutrosophic rhotrices is possible only if they are of the same size. Multiplication of two neutrosophic rhotrices 'A' and 'B' is defined as:

$$\begin{aligned} \text{The } i^{th} \text{ entry of } A_N \circ B_N = (&max\{min(a_i^T, e_b^T), min(b_i^T, e_a^T)\}, max\{min(a_i^I, e_b^I), min(b_i^I, e_a^I)\}, \\ &min\{max(a_i^F, e_b^F), max(b_i^F, e_a^F)\}), \text{ for all } i, \text{ except the } heart \end{aligned}$$

$$Heart \text{ of } A_N \circ B_N = (min\{e_a, e_b\}, min\{e_a, e_b\}, max\{e_a, e_b\})$$

Definition 1.3 (Trace of Neutrosophic Rhotrix)**.** The trace of a neutrosophic rhotrix of size 'n' is adding the entries in major vertical axis and it is denoted as $Tr(.)$.

That is, for a rhotrix $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$ of size 3,

$Tr(A_N) = (max\{a_1^T, a_3^T, a_5^T\}, max\{a_1^I, a_3^I, a_5^I\}, min\{a_1^F, a_3^F, a_5^F\})$

Theorem 1.4. *For any neutrosophic rhotrices with the neutrosophic addition operation, the following axioms holds:*

- $(A_N + B_N) + C_N = A_N + (B_N + C_N)$ *(Associativity),*
- $A_N + O = A_N$, *where* $O = \left\langle \begin{matrix} & (0,0,1) & \\ (0,0,1) & (0,0,1) & (0,0,1) \\ & (0,0,1) & \end{matrix} \right\rangle$ *(Additive Identity),*
- $A_N + B_N = B_N + A_N$ *(Commutativity), where* A_N, B_N, *and* C_N *are any neutrosophic rhotrices*

Proof. Let $A_N, B_N,$ and C_N of size 3 be any three neutrosophic rhotrices . Then,
$A_N + (B_N + C_N) = (A_N + B_N) + C_N =$

$$\left\langle \begin{matrix} & \begin{matrix}(max\{a_1^T, b_1^T, c_1^T\}, max\{a_1^I, b_1^I, c_1^I\}, \\ min\{a_1^F, b_1^F, c_1^F\})\end{matrix} & \\ \begin{matrix}(max\{a_2^T, b_2^T, c_2^T\}, max\{a_2^I, b_2^I, c_2^I\}, \\ min\{a_2^F, b_2^F, c_2^F\})\end{matrix} & \begin{matrix}(max\{a_3^T, b_3^T, c_3^T\}, max\{a_3^I, b_3^I, c_3^I\}, \\ min\{a_3^F, b_3^F, c_3^F\})\end{matrix} & \begin{matrix}(max\{a_4^T, b_4^T, c_4^T\}, max\{a_4^I, b_4^I, c_4^I\}, \\ min\{a_4^F, b_4^F, c_4^F\})\end{matrix} \\ & \begin{matrix}(max\{a_5^T, b_5^T, c_5^T\}, max\{a_5^I, b_5^I, c_5^I\}, \\ min\{a_5^F, b_5^F, c_5^F\})\end{matrix} & \end{matrix} \right\rangle$$

Suppose $O = \left\langle \begin{matrix} & (x_1^T, x_1^I, x_1^F) & \\ (x_2^T, x_2^I, x_2^F) & (x_3^T, x_3^I, x_3^F) & (x_4^T, x_4^I, x_4^F) \\ & (x_5^T, x_5^I, x_5^F) & \end{matrix} \right\rangle$. For $A_N + O = A_N$,

$$\left\langle \begin{matrix} & \begin{matrix}(max\{a_1^T, x_1^T\}, max\{a_1^I, x_1^I\}, \\ min\{a_1^F, x_1^F\})\end{matrix} & \\ \begin{matrix}(max\{a_2^T, x_2^T\}, max\{a_2^I, x_2^I\}, \\ min\{a_2^F, x_2^F\})\end{matrix} & \begin{matrix}(max\{a_3^T, x_3^T\}, max\{a_3^I, x_3^I\}, \\ min\{a_3^F, x_3^F\})\end{matrix} & \begin{matrix}(max\{a_4^T, x_4^T\}, max\{a_4^I, x_4^I\}, \\ min\{a_4^F, x_4^F\})\end{matrix} \\ & \begin{matrix}(max\{a_5^T, x_5^T\}, max\{a_5^I, x_5^I\}, \\ min\{a_5^F, x_5^F\})\end{matrix} & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$$

Then, $max\{a_i^T, x_i^T\} = a_i^T, max\{a_i^I, x_i^I\} = a_i^I and min\{a_i^F, x_i^F\} = a_i^F$, for all i, which implies $x_i^T = x_i^I = 0, x_i^F = 1$ for all i. Therefore, the additive identity for rhotrix size 3 is $\left\langle \begin{matrix} & (0,0,1) & \\ (0,0,1) & (0,0,1) & (0,0,1) \\ & (0,0,1) & \end{matrix} \right\rangle$. In general, the additive identity for rhotrix size 'n' is a rhotrix with all its entries as $(0,0,1)$.

Since each entry of $A_N + B_N = (max\{a_i^T, b_i^T\}, max\{a_i^I, b_i^I\}, min\{a_i^F, b_i^F\})$ and $max\{a_i, b_i\} =$

$max\{b_i, a_i\}$, and $min\{a_i, b_i\} = min\{b_i, a_i\}$ it is obvious that the commutative axiom holds for neutrosophic addition. □

Theorem 1.5. *The heart based multiplication operation among any neutrosophic rhotrices holds the following axioms:*

- $(A_N \circ B_N) \circ C_N = A_N \circ (B_N \circ C_N)$ *(Associativity),*
- $A_N \circ I = A_N$, *where* $I = \left\langle \begin{matrix} & (0,0,1) & \\ (0,0,1) & (1,1,0) & (0,0,1) \\ & (0,0,1) & \end{matrix} \right\rangle$ *(Multiplicative Identity),*
- $A_N \circ B_N = B_N \circ A_N$ *(Commutativity), where* A_N, B_N, *and* C_N *are any neutrosophic rhotrices*

Proof. Let us consider any three neutrosophic rhotrices $A_N, B_N,$ and C_N of size 3, and let $e_a, e_b,$ and e_c be the entries of heart of the rhotrices $A, B,$ and C, respectively. Then, the i^{th} entry of $(A_N \circ B_N) \circ C_N$

$$= \begin{cases} \Big(max\{min(a_i^T, e_b^T, e_c^T), min(b_i^T, e_b^T, e_c^T), min(c_i^T, e_b^T, e_c^T)\}, \\ max\{min(a_i^I, e_b^I, e_c^I), min(b_i^I, e_b^I, e_c^I), min(c_i^I, e_b^I, e_c^I)\}, \\ min\{max(a_i^F, e_b^F, e_c^F), max(b_i^F, e_b^F, e_c^F), max(c_i^F, e_b^F, e_c^F)\}\Big) & , \forall i, \text{ except } heart \\ \Big(min\{e_a^T, e_b^T, e_c^T\}, min\{e_a^I, e_b^I, e_c^I\}, max\{e_a^F, e_b^F, e_c^F\}\Big) & , \textit{for heart} \end{cases}$$

$= i^{th}$ entry of $A_N \circ (B_N \circ C_N)$

Suppose $I_N = \left\langle \begin{matrix} & (x_1^T, x_1^I, x_1^F) & \\ (x_2^T, x_2^I, x_2^F) & (x_3^T, x_3^I, x_3^F) & (x_4^T, x_4^I, x_4^F) \\ & (x_5^T, x_5^I, x_5^F) & \end{matrix} \right\rangle$

For $A_N \circ I_N = A_N$,

$$\text{The } i^{th} \text{ entry of } A_N \circ I_N = \begin{cases} \Big(max\{min(a_i^T, x_3^T), min(x_i^T, e_a^T)\}, \\ max\{min(a_i^I, x_3^I), min(x_i^I, e_a^I)\}, \\ min\{max(a_i^F, x_3^F), max(x_i^F, e_a^F)\}\Big) & , \forall i, \text{ except the } heart \\ (min\{e_a^T, x_3^T\}, min\{e_a^I, x_3^I\}, max\{e_a^F, x_3^F\}) & , \textit{for heart} \end{cases}$$

Then, $min\{e_a^T, x_3^T\} = e_a^T$, implies $x_3^T = 1$ and similarly $x_3^I = 1$. Also, $max\{e_a^F, x_3^F\} = e_a^F$, implies $x_3^F = 0$. Since $x_3^T = 1$ and $max\{min(a_i^T, x_3^T), min(x_i^T, e_a^T)\} = a_i^T$, $x_i^T = 0$. Similarly, $x_i^I = 0$ and $x_i^F = 1$

Therefore, the additive identity for rhotrix size 3 is $I = \left\langle \begin{matrix} & (0,0,1) & \\ (0,0,1) & (1,1,0) & (0,0,1) \\ & (0,0,1) & \end{matrix} \right\rangle$. In general, the multiplicative identity for rhotrix size 'n' is a rhotrix of size 'n'with all entries (0,0,1) and heart of the rhotrix as (1,1,0).

Since the heart based rhotrix multiplication is commutative, it is obvious that the commutative axiom holds for heart based neutrosophic multiplication. □

Theorem 1.6. *Heart based multiplication of neutrosophic rhotrices is distributive with respect to addition of neutrosophic rhotrices. That is,* $A_N \circ (B_N + C_N) = A_N \circ B_N + A_N \circ C_N$*, where* A_N, B_N, C_N *are neutrosophic rhotrices.*

Proof. Let $A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (e_a^T, e_a^I, e_a^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$,

$B_N = \left\langle \begin{matrix} & (b_1^T, b_1^I, b_1^F) & \\ (b_2^T, b_2^I, b_2^F) & (e_b^T, e_b^I, e_b^F) & (b_4^T, b_4^I, b_4^F) \\ & (b_5^T, b_5^I, b_5^F) & \end{matrix} \right\rangle$ be neutrosophic rhotrices.

Then, the i^{th} entry of

$$\begin{aligned} A_N \circ (B_N + C_N) = &\Big(max\big\{ min(a_i^T, max\{e_b^T, e_c^T\}), min(e_a^T, max\{b_i^T, c_i^T\})\big\}, max\big\{min(a_i^I, max\{e_b^I, e_c^I\}), \\ & min(e_a^I, max\{b_i^I, c_i^I\})\big\}, min\big\{max(a_i^F, min\{e_b^F, e_c^F\}), max(a_i^F, max\{e_b^F, e_c^F\})\big\}\Big) \\ = &\Big(max\{min(a_i^T, e_b^T, e_c^T), min(b_i^T, e_a^T, e_c^T), min(c_i^T, e_a^T, e_b^T)\}, max\{min(a_i^I, e_b^I, e_c^I), \\ & min(b_i^I, e_a^I, e_c^I), min(c_i^I, e_a^I, e_b^I)\}, min\{min(a_i^F, e_b^F, e_c^F), min(b_i^F, e_a^F, e_c^F), \\ & min(c_i^F, e_a^F, e_b^F)\}\Big) \\ = & i^{th} \text{ entry of } (A_N \circ B_N) + (A_N \circ C_N) \end{aligned}$$

Therefore, $A_N \circ (B_N + C_N) = (A_N \circ B_N) + (A_N \circ C_N)$. This means that, we have proved, for any rhotrices of size 3, distributive property holds. In similar manner, we also can prove, distributive axiom holds for any rhotrices of size 'n'. □

Theorem 1.7. *For two neutrosophic rhotrices* A_N *and* B_N *and any scalar* λ *such that* $0 \leq \lambda \leq 1$*, the following properties hold:*

(i) $Tr(A_N + B_N) = Tr(A_N) + Tr(B_N)$
(ii) $Tr(\lambda A_N) = \lambda Tr_f(A_N)$
(iii) $Tr(A_N) = Tr(A_N^T)$

Proof. Consider two neutrosophic rhotrices of size 3,
$A_N = \left\langle \begin{matrix} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{matrix} \right\rangle$ and
$B_N = \left\langle \begin{matrix} & (b_1^T, b_1^I, b_1^F) & \\ (b_2^T, b_2^I, b_2^F) & (b_3^T, b_3^I, e_b^F) & (b_4^T, b_4^I, b_4^F) \\ & (b_5^T, b_5^I, b_5^F) & \end{matrix} \right\rangle$.

(i) Since the maximum operation is associative, we have the following:

$$\begin{aligned} Tr(A_N + B_N) &= (max\{max(a_1^T, b_1^T), max(a_3^T, b_3^T), max(a_5^T, b_5^T)\}, max\{max(a_1^I, b_1^I), max(a_3^I, b_3^I), \\ & max(a_5^I, b_5^I)\}, min\{min(a_1^F, b_1^F), min(a_3^F, b_3^F), min(a_5^F, b_5^F)\}) \\ &= (max\{max(a_1^T, a_3^T, a_5^T), max(b_1^T, b_3^T, b_5^T)\}, max\{max(a_1^I, a_3^I, a_5^I), \\ & max(b_1^I, b_3^I, b_5^I)\}, min\{min(a_1^F, a_3^F, a_5^F), min(b_1^F, b_3^F, b_5^F)\}) \\ &= Tr_f(A_N) + Tr_f(B_N) \end{aligned}$$

Therefore, $Tr(A_N + B_N) = Tr(A_N) + Tr(B_N)$, for any neutrosophic rhotrices of size 'n'.

(ii) Now, $Tr(\lambda A_N) = (max\{min(\lambda, a_1^T), min(\lambda, a_3^T), min(\lambda, a_5^T)\}, max\{min(\lambda, a_1^I), min(\lambda, a_3^I), min(\lambda, a_5^I)\}, min\{max(\lambda, a_1^F), max(\lambda, a_3^F), max(\lambda, a_5^F)\}) = (min\{\lambda, max(a_1^T, a_3^T, a_5^T)\}), min\{\lambda, max(a_1^I, a_3^I, a_5^I)\}, max\{\lambda, min(a_1^F, a_3^F, a_5^F)\}$ ($\because$ (A∧B)∨(A∧C)∨(A∧D)=A∧(B∨C∨D), where ∧ and ∨ represents *min* and *max* operation resp.) and $\lambda Tr_f(A_f) = (min\{\lambda, max(a_1^T, a_3^T, a_5^T)\}), min\{\lambda, max(a_1^I, a_3^I, a_5^I)\}, max\{\lambda, min(a_1^F, a_3^F, a_5^F)\}$. Therefore, $Tr(\lambda A_N) = \lambda Tr_f(A_N)$.

(iii) Since there is no changes in the major horizontal axis in A_N^T and A_N, $Tr(A_N) = Tr(A_N^T)$.

□

2. Elevating Diagnostic Precision with Neutrosophic Rhotrices and Fuzzy rhotrices

In the context of decision-making, when using matrices for medical diagnosis, if an individual does not exhibit a particular symptom, the value corresponding to that symptom is set to zero, and computations proceed accordingly, accounting for the absence of that symptom. However, when applying rhotrices, individuals can be categorized based on the number of symptoms they exhibit. For example, patients with three symptoms can be grouped together, as can those with two symptoms, which simplifies the overall diagnostic process by creating more manageable groups.

Method 1 (Using Neutrosophic Rhotrices)

In this method, we incorporate neutrosophic rhotrices into medical diagnosis, this framework advances healthcare by effectively managing uncertainty, indeterminacy, and imprecision.

Let S represent the set of symptoms associated with certain diseases, D denote the set of diseases, and P refer to the set of patients.

Step 1: Construct a symptom-disease neutrosophic rhotrix $A = [a_ij]$ of size n (=number of symptoms and diseases) in the following manner.
Here, $s_i d_j^T$ represents the membership value indicating how much symptom s_i contributes to the occurrence of disease d_j, $s_i d_j^I$ represents the membership value indicating how much symptom s_i contributes to the indeterminacy of disease d_j, and $s_i d_j^F$ represents the membership value indicating how much symptom s_i contributes to the falsity of disease d_j.Thus, the symptom-disease neutrosophic rhotrix A can be formed as

$$\left\langle \begin{array}{ccccc} & & (s_1d_1^T, s_1d_1^I, s_1d_1^F) & & \\ & (s_3d_1^T, s_3d_1^I, s_3d_1^F) & (s_2d_2^T, s_2d_2^I, s_2d_2^F) & (s_1d_3^T, s_1d_3^I, s_1d_3^F) & \\ (s_5d_1^T, s_5d_1^I, s_5d_1^F) & (s_4d_2^T, s_4d_2^I, s_4d_2^F) & (s_3d_3^T, s_3d_3^I, s_3d_3^F) & (s_2d_4^T, s_2d_4^I, s_2d_4^F) & (s_1d_5^T, s_1d_5^I, s_1d_5^F) \\ & (s_5d_3^T, s_5d_3^I, s_5d_3^F) & (s_4d_4^T, s_4d_4^I, s_4d_4^F) & (s_3d_5^T, s_3d_5^I, s_3d_5^F) & \\ & & (s_5d_5^T, s_5d_5^I, s_5d_5^F) & & \end{array} \right\rangle$$

Step 2: Construct a patient-symptom neutrosophic matrix $B = [b_ij]$ of size n (=number of symptoms and patients) in the following manner.

Here, $p_i s_j^T$ represents the membership value indicating how much the patient p_i suffers from the symptom s_j, $p_i s_j^I$ represents the membership value indicating how much the patient p_i contributes to the indeterminacy of symptom s_j, and $p_i s_j^F$ represents the membership value indicating how much the patient p_i contributes to the falsity of the symptom s_j.Thus, the symptom-disease neutrosophic rhotrix B can be formed as

$$\left\langle \begin{array}{ccccc} & & (p_1s_1^T, p_1s_1^T, p_1s_1^T) & & \\ & (p_3s_1^T, p_3s_1^T, p_3s_1^T) & (p_2s_2^T, p_2s_2^I, p_2s_2^F) & (p_1s_3^T, p_1s_3^I, p_1s_3^F) & \\ (p_5s_1^T, p_5s_1^I, p_5s_1^F) & (p_4s_2^T, p_4s_2^I, p_4s_2^F) & (p_3s_3^T, p_3s_3^I, p_3s_3^F) & (p_2s_4^T, p_2s_4^I, p_2s_4^F) & (p_1s_5^T, p_1s_5^I, p_1s_5^F) \\ & (p_5s_3^T, p_5s_3^I, p_5s_3^F) & (p_4s_4^T, p_4s_4^I, p_4s_4^F) & (p_3s_5^T, p_3s_5^I, p_3s_5^F) & \\ & & (p_5s_5^T, p_5s_5^I, p_5s_5^F) & & \end{array} \right\rangle$$

Step 3: Compute the rhotrix C by performing heart-based neutrosophic multiplication between rhotrices A and B, where the operation $\circ$ denotes the heart based neutrosophic multiplication.

Step 4: Construct the complement rhotrices A^c and B^c for rhotrices A and B respectively, both having the same dimensions. Afterward, calculate the composition rhotrix D by applying the heart-based neutrosophic multiplication $\circ$ to the complement matrices A^c and B^c.

Step 5: Transform the rhotrix A and B to A_S and B_S by adding the truth and indeterminacy values, then subtracting the falsity value for each neutrosophic number in the rhotrix A and

B.

Step 6: Derive the score rhotrix M by applying the minimum operator, denoted as $(-)$, to the rhotrices A_S and B_S. This results in $M = A_S(-)B_S$.
This will be used to analyze and interpret the relationships or differences between the elements under study.

Illustrative Analysis: Assume there are five symptoms and five diseases. Out of these, three symptoms (s_1, s_3, s_5) are associated with three specific diseases (d_1, d_3, d_5), while the remaining two symptoms (s_2, s_4) correspond to two other diseases (d_2, d_4).

Let us, for illustration, take symptom disease neutrosophic rhotrix as

$$A = \left\langle \begin{array}{ccccc} & & (0.2,0.4,0.6) & & \\ & (0.5,0.1,0.3) & (0.2,0.5,0.2) & (0.4,0.4,0.1) & \\ (0.9,0.6,0.3) & (0.2,0.9,0.1) & (0.1,0.7,0.4) & (0.2,0.6,0.3) & (0.1,0.5,0.4) \\ & (0.3,0.4,0.8) & (0.7,0.9,0.1) & (0.5,0.5,0.5) & \\ & & (0.1,0.4,0.1) & & \end{array} \right\rangle$$

Consider a scenario with five symptoms and five diseases. Three patients (p_1, p_3, p_5) experience three symptoms (s_1, s_3, s_5), while the other two patients (p_2, p_4) exhibit the remaining two symptoms (s_2, s_4).

Let us, for illustration, take the patient symptom neutrosophic rhotrix as

$$B = \left\langle \begin{array}{ccccc} & & (0.3,0.4,0.1) & & \\ & (0.2,0.1,0.2) & (0.1,0.9,0.4) & (0.7,0.3,0.1) & \\ (0.4,0.3,0.3) & (0.7,0.1,0.3) & (0.4,0.1,0.3) & (0.7,0.7,0.7) & (0.4,0.2,0.7) \\ & (0.4,0.4,0.1) & (0.3,0.4,0.7) & (0.9,0.9,0.9) & \\ & & (0.4,0.4,0.2) & & \end{array} \right\rangle$$

For the above illustration, we heart based multiply the two rhotrices A and B and get

$$C = \left\langle \begin{array}{ccccc} & & (0.2,0.4,0.4) & & \\ & (0.4,0.1,0.3) & (0.2,0.7,0.3) & (0.4,0.3,0.3) & \\ (0.4,0.3,0.3) & (0.2,0.1,0.3) & (0.1,0.1,0.3) & (0.2,0.7,0.3) & (0.1,0.2,0.4) \\ & (0.3,0.4,0.4) & (0.4,0.4,0.3) & (0.4,0.7,0.5) & \\ & & (0.1,0.4,0.3) & & \end{array} \right\rangle$$

By using,

$$A^c = \left\langle \begin{array}{ccccc} & & (0.8, 0.6, 0.4) & & \\ & (0.5, 0.9, 0.7) & (0.8, 0.5, 0.8) & (0.6, 0.6, 0.9) & \\ (0.1, 0.4, 0.7) & (0.8, 0.1, 0.9) & (0.9, 0.3, 0.6) & (0.8, 0.4, 0.7) & (0.9, 0.5, 0.6) \\ & (0.7, 0.6, 0.2) & (0.3, 0.1, 0.9) & (0.5, 0.5, 0.5) & \\ & & (0.9, 0.6, 0.9) & & \end{array} \right\rangle \text{ and}$$

$$B^c = \left\langle \begin{array}{ccccc} & & (0.7, 0.6, 0.9) & & \\ & (0.8, 0.9, 0.8) & (0.9, 0.1, 0.6) & (0.3, 0.7, 0.9) & \\ (0.6, 0.7, 0.7) & (0.3, 0.9, 0.7) & (0.6, 0.9, 0.7) & (0.3, 0.3, 0.3) & (0.6, 0.8, 0.3) \\ & (0.6, 0.6, 0.9) & (0.7, 0.6, 0.3) & (0.1, 0.1, 0.1) & \\ & & (0.6, 0.6, 0.8) & & \end{array} \right\rangle, \text{ we get } D =$$

$$\left\langle \begin{array}{ccccc} & & (0.7, 0.6, 0.7) & & \\ & (0.8, 0.9, 0.7) & (0.9, 0.5, 0.6) & (0.6, 0.6, 0.9) & \\ (0.6, 0.4, 0.7) & (0.6, 0.3, 0.7) & (0.6, 0.3, 0.7) & (0.6, 0.4, 0.6) & (0.6, 0.5, 0.6) \\ & (0.6, 0.6, 0.7) & (0.7, 0.3, 0.6) & (0.5, 0.5, 0.6) & \\ & & (0.6, 0.6, 0.8) & & \end{array} \right\rangle$$

Then, by the aforementioned algorithm, we transform A to

$$A_S = \left\langle \begin{array}{ccccc} & & 0.2 & & \\ & 0.2 & 0.6 & 0.4 & \\ 0.4 & 0 & -0.1 & 0.6 & -0.1 \\ & 0.3 & 0.5 & 0.6 & \\ & & 0.2 & & \end{array} \right\rangle$$

and B to $B_S = \left\langle \begin{array}{ccccc} & & 0.6 & & \\ & 1 & 0.8 & 0.3 & \\ 0.3 & 0.2 & 0.2 & 0.4 & 0.5 \\ & 0.5 & 0.4 & 0.4 & \\ & & 0.4 & & \end{array} \right\rangle$.

Then, the score rhotrix thus obtained is, $\left\langle \begin{array}{ccccc} & & -0.4 & & \\ & -0.8 & -0.2 & 0.1 & \\ 0.1 & -0.2 & -0.3 & 0.2 & -0.6 \\ & -0.8 & 0.1 & 0.2 & \\ & & -0.2 & & \end{array} \right\rangle$

We can observe that, p_1 is affected by d_3, p_2 is affected by d_4, p_3 is affected by d_5, p_4 is affected by d_4 and p_5 is affected by d_1, by finding maximum of each row.

Method 2 (Using Fuzzy Rhotrices)

In optimizing diagnostics with fuzzy rhotrices, let S represent the set of symptoms associated with certain diseases, D denote the set of diseases, and P refer to the set of patients.

Step 1: Construct a symptom-disease fuzzy rhotrix $A = [a_ij]$ of size n (=number of symptoms and diseases) in the following manner, based on the following example. Consider there are 5 symptoms and 5 diseases, in which 3 symptoms(s_1, s_3, s_5) can indicate 3 diseases (d_1, d_3, d_5) and another 2 symptoms (s_2, s_4) indicate another 2 diseases (d_2, d_4).Here, s_id_j represents the membership value indicating how much symptom s_i contributes to the occurrence of disease d_j.

Then the symptom-disease fuzzy rhotrix A can be formed as

$$A = \left\langle \begin{array}{ccccc} & & s_1d_1 & & \\ & s_3d_1 & s_2d_2 & s_1d_3 & \\ s_5d_1 & s_4d_2 & s_3d_3 & s_2d_4 & s_1d_5 \\ & s_5d_3 & s_4d_4 & s_3d_5 & \\ & & s_5d_5 & & \end{array} \right\rangle$$

Suppose, for illustration,

$$A = \left\langle \begin{array}{ccccc} & & 0.2 & & \\ & 0.5 & 0.7 & 0.4 & \\ 0.6 & 0.1 & 0.9 & 0.6 & 0.4 \\ & 0.2 & 0.1 & 0.4 & \\ & & 0.5 & & \end{array} \right\rangle$$

Step 2: Construct a patient-symptom fuzzy matrix $B = [b_ij]$ of size n (=number of symptoms and patients) in the following manner, based on the following example. Consider there are 5 symptoms and 5 diseases, in which 3 patients (p_1, p_3, p_5) has 3 symptoms(s_1, s_3, s_5) and another 2 patients (p_2, p_4) indicate another 2 symptoms (s_2, s_4).

Here, p_is_j represents the membership value indicating how much a patient p_i have affected by the symptom s_j.

Then

the patient-symptom fuzzy rhotrix can be formed as

$$B = \left\langle \begin{array}{ccccc} & & p_1s_1 & & \\ & p_3s_1 & p_2s_2 & p_1s_3 & \\ p_5s_1 & p_4s_2 & p_3s_3 & p_2s_4 & p_1s_5 \\ & p_5s_3 & p_4s_4 & p_3s_5 & \\ & & p_5s_5 & & \end{array} \right\rangle$$

Suppose, for illustration,

$$B = \left\langle \begin{array}{ccccc} & & 0.5 & & \\ & 0.7 & 0.9 & 0.8 & \\ 0.2 & 0.5 & 0.9 & 0.2 & 0.3 \\ & 0.4 & 0.5 & 0.1 & \\ & & 0.2 & & \end{array} \right\rangle$$

Step 3: Evaluate $C = A \circ B$, where $\circ$ is heart based fuzzy multiplication

For the above illustration, $$C = \left\langle \begin{array}{ccccc} & & 0.5 & & \\ & 0.7 & 0.9 & 0.8 & \\ 0.6 & 0.5 & 0.9 & 0.6 & 0.4 \\ & 0.4 & 0.5 & 0.4 & \\ & & 0.5 & & \end{array} \right\rangle$$

Step 4: Build the complement matrix A^c and B^c of A and B of size n and form the composition matrix D by computing $A^c \circ B^c$.

Then, $$D = \left\langle \begin{array}{ccccc} & & 0.1 & & \\ & 0.1 & 0.1 & 0.1 & \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ & 0.1 & 0.1 & 0.1 & \\ & & 0.1 & & \end{array} \right\rangle$$

Step 5: Compute $M = C(-)D$, where (-) denotes min operator Then, M for the above illustration is $$\left\langle \begin{array}{ccccc} & & 0.4 & & \\ & 0.6 & 0.8 & 0.7 & \\ 0.5 & 0.4 & 0.8 & 0.5 & 0.3 \\ & 0.3 & 0.4 & 0.3 & \\ & & 0.4 & & \end{array} \right\rangle$$

Step 6: Calculating the relativity values and form the comparison matrix

Then, the comparison matrix for the above illustration is $$\left\langle \begin{array}{ccccc} & & 0 & & \\ & -0.143 & 0 & 0.143 & \\ 0.4 & -0.2 & 0 & 0.2 & -0.4 \\ & 0 & 0 & 0 & \\ & & 0 & & \end{array} \right\rangle,$$

We can observe that, p_1 is affected by d_3, p_2 is affected by d_4, p_3 is affected by d_3 and d_5, p_4 is affected by d_4 and p_5 is affected by d_1, by finding maximum of each row.

Comparative Analysis: The method applied for medical diagnostics using fuzzy rhotrices, observe that it only deals with truthfulness and do not explicitly address indeterminacy, which limits their ability to manage situations where medical data is incomplete or contradictory. But, our current algorithm uses neutrosophic rhotrices which are designed to handle three types of uncertainties: truth, indeterminacy, and falsity. In medical diagnosis, this means that a symptom can be partially present (truth), uncertain (indeterminacy), or absent (falsity), offering a richer framework for capturing ambiguity in symptoms.

Neutrosophic rhotrices, due to their sophisticated structure, provide significant advantages in decision-making processes, especially in situations where data can be organized into distinct categories or groups. For instance, in the field of medical diagnostics, neutrosophic rhotrices can facilitate the classification of patients based on their symptoms. This grouping allows healthcare professionals to make more precise and personalized decisions regarding diagnosis and treatment.

Compared to neutrosophic matrices, computations in neutrosophic rhotrices are much simpler. Rhotrices reduce computational complexity by handling multi-dimensional data in a more streamlined manner, enabling faster execution of operations like matrix multiplication and inversion. This efficiency is particularly valuable when dealing with large datasets in medical diagnosis.

In summary, the use of neutrosophic rhotrices in medical diagnostics not only streamlines the decision-making process but also promotes a more informed and personalized approach to patient care.

Conclusion

We introduced neutrosophic rhotrices, a novel extension of neutrosophic matrices, designed to effectively manage the complexities of uncertainty, indeterminacy, and falsity in decision-making processes, particularly in the context of medical diagnostics. By extending the traditional matrix structure with an additional dimension, neutrosophic rhotrices provide a more flexible and comprehensive framework for representing and processing medical data where ambiguity is inherent. Further, we explored the basic properties of neutrosophic rhotrices, such as their algebraic properties and their computational simplicity compared to neutrosophic matrices. Additionally, we compared the medical diagnostic method using neutrosophic rhotrices with that of fuzzy rhotrices and found that neutrosophic rhotrices provide significantly better results in the decision-making process.

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