The NILPOTENT Characterization of the finite neutrosophic p-groups

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Abstract

A well known and referenced global result is the nilpotent characterisation of the finite p-groups. This undoubtedly transends into neutrosophy. Hence, this fact of the neutrosophic nilpotent p-groups is worth critical studying and comprehensive analysis. The nilpotent characterisation depicts that there exists a derived series (Lower Central) which must terminate at $\{\epsilon\}$ (an identity) , after a finite number of steps. Now, Suppose that $G(I)$ is a neutrosophic p-group of class at least $m \geq 3$. We show in this paper that $L_{m-1}(G(I))$ is abelian and hence $G(I)$ possesses a characteristic abelian neutrosophic subgroup which is not supposed to be contained in $Z(G(I))$. Furthermore, If $L_3(G(I)) = 1$ such that p^m is the highest order of an element of $G(I)/L_2(G(I))$ (where $G(I)$ is any neutrosophic p-group) then no element of $L_2(G(I))$ has an order higher than p^m.

Keywords: Neutrosophic p-groups ; Nilpotency; central series, order; commutator; abelian
SUBJECT CLASSIFICATION MSC. 2020 Primary: 20D15; 20F18 Secondary 20F19

1 introduction

N.B : *Throughout this paper , please note that our BINARY OPERATION is strictly the usual ordinary addition (as the operation of multiplication may not be defined due to the fact that* I^{-1} *does not exist)*

The numbers of the form $a + bI$, are the basis of the Neutrosophic Algebraic Structures. Here, a and b are real or complex entities and I is a literal indeterminacy with $I^2 = I$. The generalization of the classical Algebraic Structures to NeutroAlgebraic Structures was formalized by Smarandache in 2019. The operations and axioms of such structures are partially true, partially indeterminate, and partially false. This actually is as extensions of Partial Algebra, and to AntiAlgebraic Structures otherwise called the AntiAlgebras . Since then, more ideas have so far being developed. Now, our real world is having more applications and developments from the NeutroAlgebras and AntiAlgebras. These actually pave ways for the formation, proposition and the developments on new field of research . (For detailed information on this, please, see [G1]) Furthermore, the introduction and research developments on the refined neutrosophic algebraic structures and studies on refined neutrosophic groups were carried out through the able efforts of Agboola Adesina (please, see [A1]) . After the successful feat, many other neutrosophic researchers have as well tried to establish and studied

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further more on the refined neutrosophic algebraic structures. (please, see [A5]). Further studies on refined neutrosophic rings and refined neutrosophic subrings, their presentations and fundamental were also worked upon. Also, Agboola, in his paper [Q] has examined and as well studied the refined neutrosophic quotient groups, where more properties of refined neutrosophic groups were presented and it was shown that the classical isomorphism theorems of groups do not hold in the refined neutrosophic groups. The existence of classical morphisms between refined neutrosophic groups $G(I_1; I_2)$ and neutrosophic groups $G(I)$ were established. The readers can as well consult [A2, A4, A7, A13] in order to have detailed knowledge concerning the refined neutrosophic logic, neutrosophic groups, refined neutrosophic groups and neutrosophy, in general.

Definition 1 (please, see [1]): Suppose that $(X(I_1; I_2); +; .)$ is any refined neutrosophic algebraic structure. Here, + and . are ordinary addition and multiplication respectively. Then I_1 and I_2 are the split components of the indeterminacy factor I that is $I = \alpha_1 I_1 + \alpha_2 I_2$ with $\alpha_i \in\ C; i = 1; 2$.

Definition 2 (please, see [1]): Suppose that $(G; *)$ is any group. Then, the couple $(G(I_1; I_2); *)$ can be referred to as the refined neutrosophic group. Furthermore, this group can be said to be generated by G, I_1 and I_2 and $(G(I_1; I_2); *)$ is said to be commutative if $\forall x; y \in G(I_1; I_2)$, we have $x * y = y * x$. Otherwise, $(G(I_1; I_2); *)$ can be referred to as a non-commutative refined neutrosophic group.

Theorem (please, see [1]): (1) Every refined neutrosophic group is a semigroup but not a group. (2) Every refined neutrosophic group contains a group.'

Corollary (please, see [1]): Every refined neutrosophic group $(G(I_1; I_2); +)$ is a group.

Definition (please, see [1]): Let $(G(I_1; I_2); *)$ be a refined neutrosophic group and let $A(I_1; I_2)$ be a nonempty subset of $G(I_1; I_2)$. $A(I_1; I_2)$ is called a refined neutrosophic subgroup of $G(I_1; I_2)$ if $(A(I_1; I_2); *)$ is a refined neutrosophic group. It is essential that $A(I_1; I_2)$ contains a proper subset which is a group. Otherwise, $A(I_1; I_2)$ will be called a pseudo refined neutrosophic subgroup of $G(I_1; I_2)$.

Definition (please, see [1]): Let $H(I_1; I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group$(G(I_1; I_2); .)$. Define $x = (a; bI_1; cI_2) \in G(I_1; I_2)$.

Theorem (please, see [1]): Let $(G(I_1; I_2); +)$ be a refined neutrosophic group and let $(G(I); +)$ be a neutrosophic group such that where $I = xI_1 + yI_2$ with $x; y \in\ C$. Let $\varphi : G(I_1; I_2) \to G(I)$ be a mapping defined by $\varphi((a; xI_1; yI_2)) = (a; (x + y)I)\forall(a; xI_1; yI_2) \in (G(I_1; I_2)$ with $a; x; y \in G :$ Then φ is a group homomorphism.

Sequel to the discovery of the existence of p-groups, various work have been done while researches are continuously being carried out from day to day by a number of eminent personalities.

Joseph L. Lagrange, in 1771 had a theorem accredited to him based on finite Group. Meanwhile, he did not prove this theorems all he did, essentially, was to discuss some special cases. The first complete proof of the theorem was provided by Abbati. In 1872, the Normegran mathematician L. Sylow had a collection of theorems on finite group named after him [7], [10], [11].

Moreover, the Sylow theorems have been proved in a number of ways, and the history of the proofs themselves are the subjects of many papers including (Waterhouse 1979), (Scharlau, 1988), (Casadia & Zappa 1990), (Gow 1994), and to some extent (Meo 2004). Wielandt (1959) used combinatorial arguments to show part of the Sylow theorems [9].

Frattini had his argument on Sylow subgroups of a normal subgroup which was slightly generalized by Burnside as Burnside's fusion theorem.

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Others are Brauer, Gorenstein and J.L. Alperin. [6], [8].

A neutrosophic subgroup $H(I)$ of a neutrosophic p-group $G(I)$ may be said to be characterisstic if $\alpha(H(I)) \leq H(I)$ for all $\alpha \in Aut(G(I))$.

Definition: A neutrosophic group $G(I)$ can be said to be nilpotent if it has a normal series of a finite length n. That is,

$$G(I) = G_0(I) \geq G_1(I) \geq G_2(I) \geq \cdots \geq G_n(I) = \{e\},$$

where

$$G_i(I)/G_{i+1}(I) \leq Z(G(I)/G_{i+1}(I)).$$

By this notion, every finite neutrosophic p-group $G(I)$ is nilpotent. The nilpotence property is an hereditary one and for any finite neutrosophic p-group, the product also forms t neutrosophic groups which are also supposed to be nilpotent. If $G(I)$ is nilpotent of a class c, then, every neutrosophic subgroup as well as the neutrosophic quotient group of $G(I)$ is nilpotent and of class $\leq c$. Here, I represents the indeterminacy factor such that $I = \alpha_1 I_1 + \alpha_2 I_2$ with $\alpha_i \in\ C; i = 1; 2$.

Definition: If the lower central series terminates after a finite number of steps such that $G_n(I) = \{e\}$ for some n), then $G(I)$ is said to be nilpotent.

Proposition : A neutrosophic p-group must be nilpotent.

Proof: Assume $G(I) > \{e\}$. Then

$$Z(G(I)) > \{e\}$$

Here, $Z(G(I))$ is the centre of the neutrosophic group $G(I)$ and $Z(G(I)) = \{(x_1; x_2 I_1; x_3 I_2) = x \in G(I) | g^{-1}xy = x,\ \ (a_1; a_2 I_1; a_3 I_2) = g \in G(I)\}$.
Define $G\bar{(I)} = G(I)/Z(G(I))$. Then we have the existence of the series given as follows :

$$G\bar{(I)} = G\bar{(I)}_0 \geq \cdots \geq G\bar{(I)}_{n+1} = Z(G(I))$$

show that $G\bar{(I)}$ is nilpotent with the identity $Z(G(I))$. And hence, $G(I)$ is nilpotent.

2 Second characterization of the Nilpotent Group

Theorem:
Suppose that $G(I)$ is a neutrosophic, noncyclic nilpotent group. Then, we would have that : (i) If $(a_1; a_2 I_1; a_3 I_2) = a \in G(I)$, then $\langle a^x | (x_1; x_2 I_1; x_3 I_2) = x \in G(I) \rangle\ \langle G(I) \rangle$. (ii) $G/Z(G(I))$ is noncyclic.

Proof:
Since the neutrosophic group $G(I)$ is nilpotent, there is a lower central series:

$$G(I) = G_0(I) \geq G_1(I) \geq \cdot \geq G_{n-1}(I) \geq G_n(I) = \{\epsilon\}$$

Here, it is very clear that $G(I) > \{\epsilon\}$.
Hence, if $a, x \in G(I)$. Then $(x_1^{-1}; x_2^{-1} I_1; x_3^{-1} I_2) = x^{-1} \in G(I)$ and $ax^{-1} \in G(I)$.
So, $xax^{-1} \in G(I)$ and $\langle a^x | (x_1; x_2 I_1; x_3 I_2) = x \in G(I) \rangle \langle G(I)$ where $a^x = xax^{-1}$.

$$Z(G(I)) = \{x \in G(I) | x = g^{-1}xg,\ \ (a_1; a_2 I_1; a_3 I_2) = g \in G(I), a_1, a_2, a_3, \in G\}$$

Suppose that $G(I)/Z(G(I))$ is cyclic, then $G(I)/Z(G(I)) = \{az | z \in G(I)\}$ is cyclic $\Longrightarrow \Longleftarrow$
Thus $G(I)/Z(G(I))$ is noncyclic.

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The idea of higher commutators can be used to define a sequence of a neutrosophic subgroup of a neutrosophic group $G(I)$, which is the lower central series of $G(I)$, by the rules given by :

$$\begin{aligned} L_1(G(I)) &= G(I), \;\; L_2(G(I)) = [G(I), G(I)] = G'(I), \ldots \\ L_i(G(I)) &= [L_{i-1}, G(I)] \;\text{ for }\; i > 2. \end{aligned}$$

Definition: The lower central series of a group $G(I)$ is given by

$$G(I) = G_0(I) \supseteq G_1(I) \supseteq G_2(I) \supseteq \cdots$$

where for $i > 0$, $G_i(I) = [G_{i-1}(I), G(I)]$.
A group $G(I)$ is called nilpotent if $L_m(G(I)) = 1$ for some m. If $n+1$ is the least value of m satisfying this condition, then n is called the class of $G(I)$, i.e., $cl(G(I)) = n$.

Proposition A_1 :

(i) $L_i(G(I))\ Char(G(I))$ for all i.

(ii) $L_{i+1}(G(I)) \subseteq L_i(G(I))$ and $L_i(G(I))/L_{i+1}(G(I)) \subseteq Z(G(I)/L_{i+1}(G(I)))$

Proposition A_2 : Let x, y, z be elements of $G(I)$ and $H(I), K(I)$ be neurosophic subgroups of the neurosophic group $G(I)$. Then,

$$[H(I), K(I)] = [K(I), H(I)] \quad [5]$$

Proposition (α): Let $G(I)$ be a neurosophic p-group of class at least $m \geq 3$. Then $L_{m-1}(G(I))$ is abelian and hence $G(I)$ possesses a characteristic abelian neurosophic subgroup which is not contained in the centre of the neurosophic group $G(I)$.

Proof: By (A_2), $L_{m-1}(G(I))$ is abelian since

$$L_{m-1}(G(I)) = [L_{m-2}(G(I)), G(I)] = [G(I), L_{m-2}(G(I))]$$

Now, by $(A_1(ii))$, $L_{m-1}(G)/L_m(G(I)) \subseteq Z(G(I)/L_m(G(I)))$
$\Rightarrow$ if $L_m(G(I)) = 1$, then, $L_{m-1}(G(I)) \subseteq Z(G(I))$. But $L_m(G(I)) \neq 1$.
$\Rightarrow L_{m-1}(G(I)) \supseteq Z(G(I))$. □

Proposition (β): Let $G(I)$ be a neutrosophic p-group with $L_3(G(I)) = 1$. If p^m is the highest order of an element of $G(I)/L_2(G(I))$, then no element of $L_2(G(I))$ has an order higher than p^m.[5]

Proof: By definition,

$$G = L_1(G(I)) \supset L_2(G(I)) \supset \cdots \supset L_i(G(I)) \supset L_{i+1}(G(I)) = 1$$

$$\Rightarrow \quad G(I) = L_1(G()I) \supset L_2(G(I)) \supset L_3(G(I)) = 1$$

$$\therefore \quad G(I)/L_2(G(I)) = \{a | o(a) \leq p^m\}$$

$$= \{gL_2(G(I)) | g \in G(I) \;\text{ and }\; |gL_2(G(I))| \leq p^m\}$$

$$(G(I) \supset L_2(G(I)) \supset 1).$$

$$\Rightarrow \quad |g||L_2(G(I))| \leq p^m$$

$$\Rightarrow \quad |L_2(G(I))| \leq \frac{p^m}{p^k}, \quad k \geq 1$$

$$\Rightarrow \quad |L_2(G(I))| \leq p^{m-k} < p^m$$

□

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3 Applications

The neutrosophic nilpotent p-groups possesses a wide range of several appllications and so it is worth comprehensive analysis and detailed investigations .

4 Conclusion

The neurosophic p-group of a given class gives very important and highly intresting results in the general concept of classical, logic fuzzy as well as the neutrosophy.

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