Boundaries: An Essay in Mereotopology[1]

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Introduction

Of Chisholm's many signal contributions to analytic metaphysics, perhaps the most important is his treatment of boundaries, a category of entity that has been neglected, to say the least, in the history of ontology. We can gain some preliminary idea of the sorts of problems which the Chisholmian ontology of boundaries is designed to solve, if we consider the following Zeno-inspired thought-experiment.

We are to imagine ourselves proceeding along a line through the middle of a disk that is divided into two precisely symmetrical segments, one of which is red, the other green, and that we move continuously from the red to the green segment. What happens as we pass the boundary between the two? Do we pass through a last point p_1 that is red and a first point p_2 that is green? Clearly not, given the density of every continuum; for then we should have to admit an indefinite number of further points between p_1 and p_2 which would somehow have no color. To acknowledge the existence of just one of p_1 and p_2 but not the other, however, as is dictated by the standard mathematical treatment of the continuum, would be to countenance a peculiar privileging of one of the two segments over the other, and such an unmotivated asymmetry can surely be rejected as a contravention of the principle of sufficient reason.

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Perhaps, then, the line on which we move is colorless at the point where it crosses from one segment into the other. The two segments would then be analogous to open intervals in the set-theoretic sense. One might seek support for this idea by reflecting that points and lines are not in any case the sorts of things which can be colored, since color properly applies only to extended regions and not to the unextended boundaries thereof. Imagine, however, a perfectly homogeneous red surface. Is a point or a line within the interior of this surface not then also red? In the end, however, it does not much matter how we answer this question, since an argument exactly analogous to the one here presented can be formulated also in relation to a range of other sorts of cases (indeed to qualities in general), including cases of qualities for which it is not attractive to suppose that extendedness in space is a precondition of existence.

Thus the argument can be applied to purely temporal phenomena, such as the beginnings and endings of mental processes. In the work of Zeno and of Bolzano it has been applied to the phenomena of motion and bodily contact. Imagine a body which is for a certain period at rest and then begins to move. Is there a last point in time p_1 when the body is at rest and a first point p_2 when it is in motion? Clearly not, given the density of every continuum; for then we should have to admit an indefinite number of further points between p_1 and p_2 at which the body would somehow be neither at rest nor in motion. To acknowledge one of p_1 and p_2 but not the other would again be to countenance a peculiar privileging of one of the two temporal segments over the other. Perhaps, then, the body is neither in motion nor at rest at the point in time where it crosses the segmentary divide, so that the two temporal intervals would be analogous, once again, to open regions. But is it even coherent to suppose that a body might be neither in motion nor at rest at a certain point in time?

Imagine, to pursue an example from Bolzano, two perfect spheres at rest and in contact with each other. What happens at the point where they touch? Is there a last point p_1 that belongs to the first sphere and a first point p_2 that belongs to the second? Again: clearly not, for then we should have to admit an indefinite number of further points between p_1 and p_2 and this would imply that the two spheres were not in contact after all. To acknowledge one of p_1 and p_2 but not the other would be to countenance an asymmetry of a quite peculiarly unmotivated sort. And our third alternative seems here to be ruled out also. For to admit that the point where the two spheres touch belongs to neither of the two spheres seems to amount to the thesis that the two spheres do not touch at all.

The Brentano-Chisholm Theory of the Continuum

What, then, is to be done? As Chisholm has insisted, there is in fact an alternative account of the actual reality, as far as color is concerned, at the point on the line where the red and green segments meet, an account which can be smoothly and uniformly extended to the other cases mentioned. This affirms that there is but one (albeit complex) point of the line which lies precisely on the border between the two segments. This point is colored, but not in simple fashion, for it is in a certain sense *both red and green.* To put the matter in another way: it is at one and the same time a *ceasing to be red* and a *beginning to be green*. And in yet another way still: it is a point where a red point and a green point *coincide.* Similarly in the case of the particle that begins to move: here too there is a single point at which the body is *both at rest and moving* (or more precisely: it is at one and the same time *ceasing to be at rest* and *beginning to move)*. The terminal boundary of the initial interval coincides with the initial boundary of the subsequent interval. And the same account can be given also of what occurs when two perfect spheres touch: a point on the boundary of the one sphere coincides with a point on the boundary of the adjacent sphere. All bodies and all temporal intervals are on this account analogous to closed regions – or perhaps we should more properly say that there is no analogue in the world of spatial and temporal continua of the standard opposition between open and closed.

It is the theory of coincidence, and the account of boundaries and the continuum which this dictates, which will occupy us in what follows. We shall concentrate especially on four papers in which Chisholm treats the theory of coincidence of boundaries in space (1983, 1989, 1992/93 and 1994). As will already be clear, analogous reasoning can be applied also to the coincidence of boundaries in time (to beginnings and endings, for example: see Chisholm 1982, 1992), though we shall here leave these, temporal, matters out of account.[2]

Chisholm's theory of coincidence is drawn from the work on space, time and the continuum of Franz Brentano and above all from Brentano's idea that what is above all characteristic of a continuum is 'the possibility of a coincidence of boundaries'. (Brentano 1988, pp. 4f.) Brentano's thesis runs: if something continuous is a mere boundary then it can never exist except in connection with other boundaries and except as belonging to a continuum of higher dimension. This must be said of all boundaries, including those which possess no dimension at all such as spatial points and moments of time and movement: a cutting free from everything that is continuous and extended is for them, too, absolutely

[2] For further discussion of temporal analogues of the issues here discussed see Pianesi and Varzi 1996. See also, for a general formal treatment, Galton 1995.

impossible. Brentano's ideas are based in turn on the conception of boundaries and continua sketched by Aristotle in the *Physics*. As Brentano developed and elaborated Aristotle's sketchy remarks, so Chisholm sought to render Brentano's ideas in a formal manner, and to incorporate them into a general theory of kinds or categories of being. I shall attempt here to extend and to complete Chisholm's formalizations and to show how much further works need to be done. But I shall seek also to demonstrate that he has in a sense domesticated Brentano's ideas by dissociating them from a number of difficult and puzzling consequences which a more detailed analysis will show to be bound up inextricably with the notion of coincidence.

Set Theory

It is above all the predominance of set theory as an instrument of (or more precisely as a substitute for) ontological investigation that has dictated the neglect of the category of boundary on the part of those working in the field of analytic metaphysics.[3] Indeed already in the early years of this century Brentano had seen the need to criticize standard set-theoretic writings on the continuum where, as he points out, the idea of coincidence will be sought after entirely in vain'. (Brentano 1988, p. 5)

From the point of view of set theory, boundaries are logical constructions, or in other words talk of boundaries is seen as a mere *façon de parler* about other things (effectively limits of sequences, or other like abstracta). Such treatments are of unquestioned value for mathematical purposes, and we must stress that we are not here attempting an alternative foundation of the mathematics of the continuum of the sort which Lesniewski projected. Rather, we are concerned with boundary-continuum structures as these make themselves manifest concretely, in bodies, or in what Chisholm calls 'spatial individuals'. The set-theoretic account of the continuum proves to be inadequate as an account of such concrete continua for at least the following reasons.

1. The latter are *qualitative* structures. This is so not merely in the sense that they are (standardly) filled by qualities (of color, temperature, hardness, etc.), but also in the sense that standard mathematical oppositions, for example between countable and uncountable magnitudes or between dense and continuous series, seem here to gain no purchase. Nothing

[3] The popularity of set theory among contemporary philosophers has been further sustained by remnants of older corpuscularistic ideas to the effect that atomistic physics (or some similar deep-level theory) enjoys a privileged status over other, competing assays of reality.

like Cantor's continuum problem arises for the concrete continuum, and indeed the very existence of this problem testifies to a certain weakness in the set-theoretic approach to the problems at issue.

2. The set-theoretical construction of the continuum is predicated on the highly questionable thesis that out of unextended building blocks an extended whole can somehow be constructed. Yet however many entities of zero dimension are assembled together, it seems difficult to comprehend that a whole of higher dimension will somehow be formed.

3. The application of set theory to a subject-matter presupposes quite generally the isolation of some basic level of *Urelemente* in such a way as to make possible a simulation of the structures appearing on higher levels by means of sets of successively higher types. In the world of concrete continua, in contrast, there need be no relevant *Urelemente* which could serve as such a starting point of ontological construction. It is however unproblematic that concrete continua are organized in such a way that parts, including boundary-parts, are capable of being discriminated within them.

4. Set theory sees the continuum as homogeneous, as made up of only one sort of ultimate part (according to preference: the empty set, points, atoms, or real numbers). Concrete continua are in contrast made up of different sorts of parts; above all, they are made up of boundaries of different numbers of dimensions, on the one hand, and of extended bodies or regions which these boundaries are the boundaries of, on the other.

Mereology

As an alternative to set theory, Chisholm adopts as his framework for dealing with the kinds or categories of individual beings the theory of part and whole or *mereology*:[4] boundaries are *parts* of the things they bound. Mereology has the advantage that we can use it to study the ontological structures in a given domain even in the absence of any ultimate knowledge as to the atoms, if any, out of which the domain is constructed. For the axioms of mereology apply whether the world is an infinitely divisible fluid or an edifice constructed out of atoms – or indeed some combination of the two of the sort that is exemplified, for example, by Descartes' bicategorial ontology of *res extensa* and *res cogitans*.

Boundaries of bodies are actual parts of the bodies which they bound. But they are not

[4] See Simons 1987.

just any sort of part; rather they are parts which, as a matter of necessity, can exist only as proper parts of things of higher dimension which they are the boundaries of (where from the set-theoretic point of view, isolated extensionless points are presented as existing in complete independence of any larger wholes).

Boundaries cannot exist in isolation: there are, in reality, no isolated points, lines or surfaces. As Brentano himself would express it, our healthy common sense, which is here evidence itself, would raise its head in violent protest at the postulation of such entities.[5] Boundaries are in this respect comparable to universal forms or structures (for example the structure of a molecule as this is realized in a given concrete instance), as also to shadows and holes.[6] Entities in all of these categories are such that, while they require of necessity hosts which instantiate them, they can be instantiated by an indefinite variety of different hosts.[7]

Consider, for example, the surface of an apple. The whole apple can here serve as instantiating host, but so also can the apple minus core, which might have been eaten away to varying degrees from within.

In light of some of Chisholm's remarks on the nature of souls or minds,[8] we might point out that souls or minds, too, may have similar features. If there are souls, and if souls are hosted by bodies, then the same soul can be hosted by many bodies in the sense that a body may lose molecules, cells and even limbs and yet preserve its relation to the same identical soul. The soul might even *be* a boundary. (This would be the case, for example – though it seems that an option along these lines is not what Chisholm has in mind – if the soul's (or mind's) activity were essentially a matter of what happens where nerve-endings are in multivariously patterned contact with each other inside the brain.)[9]

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[5] Cf. Brentano 1988, p. 22.

[6] On holes and their hosts see Casati and Varzi 1994.

[7] Brentano even went so far as to identify boundaries as special sorts of universals:

> Because a boundary, even when itself continuous, can never exist except as belonging to something continuous of more dimensions (indeed receives its fully determinate and exactly specific character only through the manner of this belongingness), it is, considered for itself, nothing other than a universal, to which as to other universals more than one thing can correspond. (Brentano 1988, p. 12)

[8] See for example his 1989a, pp. 115, 157.

[9] An alternative view along these lines is defended by Brentano in the following passage:

> What is continuous can be sub-divided … into what is continuously many and what is continuously manifold. … As an example of the continuously many we can give a body, of the continuously manifold someone who sees something spatial precisely in so far as he sees it. The body is a unity which can be

All the mentioned types of entities share further the fact that they license certain sorts of ontological inference (*if* there is a boundary/structure/hole/soul having these and those properties, *then* there is a host having these and those properties). We cannot infer to any specific host, however. Thus it cannot be said of any definite continuum that a boundary is dependent on *it*: that which a boundary is dependent on can be designated rather only via a general term: what is required by a boundary is, Brentano says, 'not this or that particular continuum, but any continuum of the appropriate kind.'[10] For while no boundary can exist without being connected with a continuum, 'there is no specifiable part, however small, of the continuum, and no point, however near it may be to the boundary, which is such that we may say that it is the existence of *that* part or of *that* point which conditions the boundary.' (Brentano 1981, p. 56)

Plerosis

As will by now be clear, boundaries can be classified as *external*, for example a point on the surface of a sphere, and *internal*, for example a point or line or surface entirely within the interior of a sphere.[11] In contrast to standard set-theoretic treatments, the Brentano-Chisholm theory is now able to do justice to the fact that the boundaries given in experience are in many cases *asymmetrical* (so that we might in certain circumstances talk of 'oriented boundaries').

> decomposed into a plurality in such a way that if one of its parts is destroyed the remainder can continue to exist just as it was before. … Things are quite different in the case of someone who intuits something spatially extended. This someone is as such not something simple but something manifold, since he sees not merely one but many parts of a continuum and could go on to see one such part while he ceases to see the others. But in so far as he sees the one part he does not amount to something totally other than what he is in so far as he sees the other part. We have before us not a duality, as we would have in the case where it was one who sees this part and another who sees that. … I cannot speak of a plurality of unities here but only of the manifold character of something that is itself one. … This distinction between the continuously many and the continuously manifold was borne insufficiently in mind by Aristotle when he inferred a spatial extension of the sensing subject from the spatial extension of the objects of sense. The consideration of what is given when someone who momentarily presents to himself a temporal continuum could have kept him from this false conclusion. (Brentano 1988, pp. 32f.)

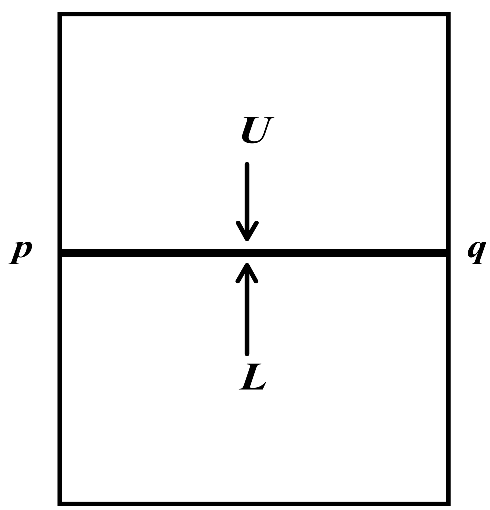
[10] Brentano 1981, p. 56: translation corrected. The continuum is specifically dependent on its boundary, but the boundary is not in this same sense dependent on its continuum; it is only generically so. See § 10 of Smith 1992 for further discussion of this generic dependence of boundaries upon their hosts.

[11] See Brentano 1988, pp. 10f. A line stretching from the surface of the sphere to its midpoint is partly internal, partly external. Note that a boundary on the surface of a cavity inside a sphere is an external boundary, in our present terminology.

This applies, for example, to the external boundaries of bodies and to the beginnings and endings of processes extended in time. Intuitively it seems not to be the case that the external boundary of a substance is in the same sense a boundary of the complement entity (i.e. of the entity which results when we imagine this substance as having been subtracted from the universe as a whole). (Even the thesis that there is such an entity is something which, from our present perspective, has to be taken with a pinch of salt.)

Boundaries, accordingly, may be boundaries only in certain directions and not in others. Imagine a line that is tangent to a circle, and meets the circle at a certain point. Strictly speaking we need here to recognize two points, a point on the line and a point on the circle, which *coincide*, the one with the other. The two points are not identical since they serve as boundaries in different directions. The point on the line is a boundary in two rectilinear directions, the point on the circle is a boundary in two directions of a certain determinate curvature.[12]

Every point, every line, every surface, must serve as a boundary in at least one direction. A point within the interior of a solid sphere is a boundary in all possible directions. The analogous point on the plane surface of a solid hemisphere is, in contrast, a boundary in exactly half this maximal number of directions. Brentano introduces at this point the notion of the *plerosis* or 'fullness' of a boundary. The mentioned interior boundary has full plerosis, the external hemispherical boundary has only half plerosis. Imagine that we have two cubes designed to fit exactly inside a container in such a way that the upper surface U of the lower cube coincides (in our technical sense) with the lower surface L of the upper cube. Consider two points *p* and *q* on opposite interior walls of the container and each located on the plane where the two cubes meet:



[12] Since boundaries in general exist always in consort with, and are determined in their nature by the things they bound, we might think of boundaries as being 'unsaturated' in something like Frege's sense, which is to say: they need to be completed in certain predetermined ways and in certain predetermined directions.

There are then *two* shortest lines connecting the given points: the line l_U in or on U and the coincident line l_L in or on L. This implies that 'the geometer's proposition that only one straight line is conceivable between two points, is strictly speaking false'. (Brentano 1988, p. 12)

The degree of plerosis of a boundary, together with the direction or directions in which it serves as boundary, yield further criteria in terms of which boundaries can be classified. As Brentano puts it in his *Theory of Categories*:

> a point differs spatially in its species [*spezifisch*] according to whether it serves as a boundary in all or only in some directions. Thus a point located inside a physical thing serves as a boundary in all directions, but a point on a surface or an edge or a vertex serves as a boundary only in some directions. And the point on a vertex will differ in its species in accordance with the shape and direction of the vertex. (Brentano 1981, p. 60)

In this geometrical sense, then, the boundary is determined in its nature by the continuum which it bounds.[13] Boundaries are determined by their hosts also qualitatively or materially:

> Imagine the mid-point of a blue circular surface. This appears as the boundary of numberless straight and crooked blue lines and of arbitrarily many blue sectors in which the circular area can be thought of as having been divided. If, however, the surface is made up of four quadrants, of which the first is white, the second blue, the third red, the fourth yellow, then we see the mid-point of the circle split apart in a certain way into a fourness of points. (Brentano 1988, p. 11)

Points, therefore, may have parts (called 'plerotic parts' in what follows). The parts of a point coincide with each other and with the point as a whole.

> Euclid's supposition that a point is that which has no parts was seen already by Galileo to be in error when he drew attention to the fact that the mid-point of a circle allows the

[13] Note that, if a boundary is determined essentially by the direction in which it bounds, then the following question presents itself: if a point is the interior boundary of a straight line and the line becomes bent into a semi-circle, does the point retain its identity?

distinction of just as many parts as there are points on the circumference, since it differs in a certain sense as starting point of the different individual radii. (Brentano 1988, p. 41)

Each point within the interior of a two- or three-dimensional continuum is in fact an infinite (and as it were maximally compressed) collection of distinct but coincident points: punctiform boundaries of straight and crooked lines, of two-dimensional segments of surfaces and of interior regular and irregular cone-shaped portions within three-dimensional continua, etc. (Not for nothing were the scholastic philosophers exercised by the question as to how many zero-dimensional beings might be fitted onto the head of a pin.) This ontological profligacy has its limits, however: the left punctiform boundary of a one-inch line is identical (and not merely coincident) with the left punctiform boundary of the corresponding initial half-inch segment. This is because it is only the immediate neighborhood of a boundary that is relevant to determining the nature of the boundary itself.

Why are plerotic parts important? One reason might be this: that everything (including you and me) is one. This is the case insofar as everything exists only in the present moment of time, which is itself a boundary of the past and future. Everything exists, as Brentano puts it, only *einer Grenze nach*, or in other words only according to the manner of existing of a boundary.

The Formalization of the Brentano-Chisholm Theory

We shall orient ourselves in what follows around Chisholm's formalization of Brentano's ideas in his rich and compressed paper "Spatial Continuity and the Theory of Part and Whole". Chisholm takes as primitives the concepts of individual thing, coincidence, and *de re* possibility. These notions, which will be introduced in succession in what follows, are employed within a mereological framework constructed around the primitive *is a proper part of,* which we symbolize here by means of: <. We use ≤ to represent the relation *part of*, which is defined in the usual way. Three further primitives will be introduced along the way: the notions of body, existence, and of sameness of dimension.

If we define *overlaps*:

DO. $xOy := \exists z(z \leq x \wedge z \leq y)$ (overlaps) then Chisholm's axioms for *part* can be formulated as follows (where initial universal quantifiers, here and in the sequel, are to be taken as understood):

A<1. $(x < y \wedge y < z) \rightarrow x < z$ (transitivity)

A<2. $x < y \rightarrow \neg\, y < x$ (asymmetry)

A<3. $x < y \rightarrow \exists z(z \leq y \wedge \neg zOx)$ (remainder)[14]

Here variables are to be conceived intuitively as ranging over individual spatial things, a primitive notion, which comprehends bodies and their boundaries, as well as continuous and non-continuous collectives comprised of bodies and boundaries. We are not, in this preliminary formal foray through the territory of coincidence, too concerned with the question whether this set of axioms for mereology is the most adequate set, though we note that we shall have need of a mereological summation or fusion principle (or more precisely, a principle-schema), which we hereby add to the set of axioms supplied by Chisholm himself:

A<4. $\exists y(y = \sigma x\phi x)$ (sum)

for each unary predicate ϕ which is satisfied (i.e. yields the value true for at least one argument).[15]

$\sigma\, x\phi x$ is then defined contextually as follows:

D σ. $\sigma x(\phi x) := \iota y\, (\forall w(wOy \rightarrow \exists v(\phi v \wedge wOv)))$

(the sum of ϕ–ers is the entity y which is such that w overlaps with y if and only if w overlaps

[14] This is in fact a slightly weakened version of Chisholm's remainder principle:

$y < x \rightarrow \exists z(z < x - y)$.

The latter would imply that atoms do not exist, an implication which, for the sake of neutrality and generality, we here wish to avoid.

[15] Varzi 1994 provides an extended treatment of the problems here at issue.

with something which ϕ□s).

In these terms we can define the *mereological difference* of two objects by:

D–. $x – y := \sigma z(z \leq x \wedge \neg zOy)$ (difference)

and similarly for the other standard mereological constants, above all *union* and *intersection*:

D∪. $x \cup y := \sigma z(z \leq x \vee z \leq y)$ (union)

D∩. $x \cap y := \sigma z(z \leq x \wedge z \leq y)$ (intersection)

A strengthened version of the remainder principle A<3 might now be formulated as a step towards ensuring the density of wholes along the lines presupposed by Chisholm in his comment on A<3 (1992/93, p. 14), for example a principle of the form:

T<1. $x < y \rightarrow \exists z(z = y – x)$ (exact remainder)

T<1 can be proved in the presence of A<4 by setting $\phi z := z < y – x$, a predicate we know is satisfied in virtue of A<3.

Note that T<1 is still rather weak. Thus it does not for example exclude a world containing only one single point that would be mereologically complex but only in the sense of containing Brentanian plerotic parts. (If a soul is zero-dimensional, and if solipsism is true, then the world would be precisely as thus described.)

Coincidence

Coincidence, as we shall here understand the notion, is exclusively the sort of thing that pertains to boundaries.[16] Bodies do not coincide (not even with themselves); nor do they coincide with the spatial regions they occupy. Other sorts of coincidence may be

[16] To aid his intuitions, the reader might provisionally view coincidence as a relation which obtains between two boundaries whenever they occupy exactly the same spatial location.

contemplated, thus for example of *the road from Athens to Thebes* with *the road from Thebes to Athens*, of *Bill Clinton* with *the President of the United States*, of the *mind* and its *brain,* of *this clumsily carved statue* and *this lump of bronze*. Here, however, potential generalizations of the theory of coincidence along these lines are left out of account.

Chisholm's axioms for *coincides* are (amended slightly):

A~1. $x \sim y \rightarrow y \sim x$ (symmetry)

A~2. $(x \sim y \wedge y \sim z) \rightarrow x \sim z$ (transitivity)

From which we can easily derive

T~1. $x \sim y \rightarrow x \sim x$ (all coincident entities are self-coincident)

and plerotic parts can be defined as those parts which self-coincide.

Chisholm adds a further axiom (1992/93, p. 15):

(*) $\Diamond_x \exists y(x \sim y) \rightarrow \Diamond_x \exists y(x \sim y \wedge x \neq y)$

(possibly coincident entities are possibly such that they coincide with something other than themselves).

Here '$\Diamond_x$' is an operator of *de re* possibility (read: 'x is possibly such that'). (Later we shall introduce the operator '$\Box_x$' for: 'x is necessarily such that'). Note that if, as it seems reasonable to suppose, all boundaries are self-coincident and all coincident entities are boundaries, then the first modal operator is redundant, since being possibly coincident is tantamount to being (self–)coincident.

We can now postulate as our equivalent of Chisholm's third axiom for coincidence an axiom asserting the possible non-self-coincidence of self-coincidents:

A~3. $x \sim x \rightarrow \Diamond_x \exists y(x \sim y \wedge x \neq y)$

(an entity which coincides with itself is of its nature an entity which can possibly coincide with something other than itself).

What Chisholm has in mind in inserting the second modal operator in (*) is the thesis that every external boundary is possibly such that, through *touching*, it can come to coincide with the external boundary of *some other thing*, and A~3, even in its revised form, is much too general as a rendering of this thesis. Indeed it seems on reflection that, in almost every type of case, the second modal operator is redundant also. This follows from the considerations on plerosis above. Suppose x is a point or line or internal surface, then in every case there is some larger continuum which it is a boundary in or of. x is then either complex, in which case it is actually (and not merely possibly) coincident with certain of its plerotic proper parts, or it is simple. In the latter case, however, it is difficult to conceive of examples which might fit the bill. Even setting x as identical with the point at the very tip of a cone, for example, would mean that x has parts, according to our present conception, corresponding to the indefinite number of points coincident at this tip which serve, respectively, as the punctiform boundaries of the indefinite number of straight and crooked lines which there converge.

An improvement on A~3 might accordingly assert simply, and non-modally:

A~3*. $x \sim x \rightarrow \exists y(x \sim y \wedge x \neq y)$.

In order to capture more closely what Chisholm has in mind in the case of touching bodies we might then countenance a further axiom to the effect that self-coincident entities are possibly such as to be coincident with entities with which they do not overlap:

A~3**. $x \sim x \rightarrow \Diamond_x \exists y(x \sim y \wedge \neg xOy)$.

We shall not seek to draw out the implications of these axioms here.We note only that it is primarily in the case of external surfaces where we would have need for Chisholm's second modal operator: for external surfaces, unlike boundaries of other sorts, can at any given time coincide at most with one other entity discrete from themselves, and they do not need to coincide with any other entity at all.

To do justice to the phenomena in hand we need to add to Chisholm's axioms a further summing principle to the effect that, if two entities coincide with two further entities, then the mereological sum of the first two coincides with the mereological sum of the second two:

A~4. $(x \sim y \wedge v \sim w) \rightarrow x \cup v \sim y \cup w$ (finite sum)

We will also need to add the following principle:

A~5. $[\exists y\phi y \wedge \forall y(\phi y \rightarrow x \sim y)] \rightarrow x \sim \sigma y\phi y$ (restricted sum)

(if something ϕ□s and if everything which ϕ□s coincides with x, then x coincides with the sum of ϕ-ers).

Thus in particular if x coincides with both y and z, then it coincides also with the sum of y and z. From A~5 we can prove also that, for satisfied predicates ϕ and ψ:

$$\forall xy[(\phi x \wedge \psi y) \rightarrow x \sim y] \rightarrow \sigma x\phi x \sim \sigma y\psi y.$$

A~4 and A~5 are modelled on standard axioms of general topology.[17] They will help us to move towards a position where we are able to define non-set-theoretic analogs of such topological notions as ‘connectedness’ and ‘dimension’ which are central to Chisholm’s ontological scheme but which cannot be defined on the basis of the axioms he supplies.

We shall also adopt an axiom to the effect that a boundary coincides with its plerotic parts (for example where the mereological sum of a coincident red and blue line coincides with the red line taken singly):

A~6. $x \sim x \rightarrow \forall y(y < x \rightarrow y \sim y)$ (parts of self-coincidents self-coincide)

Bodies

In the spirit of the definition of ‘spatial individual’ given by Chisholm on p. 15 of his

[17] See for example chapter 1 of Steen and Seebach 1978.

1992/93:[18]

> x is a *spatial individual* := (1) x is an individual thing; (2) x has a constituent that coincides with something; and (3) x is not possibly such that it coincides with anything,

we might seek to define what it is for an individual thing to be a body (*Körper*) as follows:

$$Kx := \neg\exists y(x \sim y) \wedge \exists y(y < x \wedge y \sim y)$$

(a body is an individual thing which does not coincide with anything and which has as proper part something which is self-coincident).

This definition, taken in conjunction with our schema A<4, does not impose very strong constraints on 'body'. Thus it allows as bodies not only scattered bodily collectives,[19] but also, and more worrisomely, randomly assorted collective wholes containing boundaries and non-boundaries as mutually unconnected parts. It would allow as bodies wholes consisting of bodies with isolated boundary-like outgrowths, thus for example an apple from which there protrudes an infinitely thin line. No formal means can be found to exclude these and other eldritch creatures from the realm of body in terms of the primitive notions thus far introduced. We shall accordingly embrace the concept of body as a further primitive, and seek to exclude these different sorts of counterexample by imposing constraints by means of axioms.

Our notion of body is to be conceived widely enough to allow as bodies both (scattered and non-scattered) collectives of bodies and (scattered and non-scattered) bodily parts of bodies (for example a sphere of 1-inch diameter that is abstractly discriminable within a concentric sphere of 2-inch diameter). Later we shall define the narrower concept of *substance* which will exclude these sorts of cases.

First we impose on bodies a principle to the effect that every body contains a self-coincident entity as proper part:

AK1. $Kx \rightarrow \exists y\,(y < x \wedge y \sim y)$ (bodies have plerotic parts)

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[18] This definition is formulated by Chisholm in terms of the notions: individual thing, part, coincidence and possibility. We note in passing that there is a question whether the concept of what is spatial can truly be defined exclusively by means of such purely formal, abstract notions.

[19] See Chisholm 1989a, pp. 90–99, Cartwright 1975.

Second we impose on bodies a constraint of *density*:

AK2. $Kx \rightarrow \exists yz\,(Ky \wedge Kz \wedge y \cup z = x \wedge \neg yOz)$ (density)

(each body can be divided without remainder into two further bodies which are discrete).

AK2 rules out the idea that there is a simplest body and it implies that all bodies are in the relevant sense bulky (are possessed of a certain material thickness).

Thirdly we assert:

AK3. $(Kx \wedge Ky \wedge xOy) \rightarrow K(y \cap x)$ (bodily intersection)

from which we can infer:

TK1. $(Kx \wedge Ky \wedge x < y) \rightarrow K(y–x)$ (subtraction)

(the mereological difference between two bodies is in every case a body).

AK3 rules out that bodies may manifest what we might refer to in standard topological terms as total or partial openness. Thus the interior of a body (the body minus its exterior boundary) is not a body, by AK3 and TK1, since the boundary itself is not a body. AK3 guarantees, too, that a hole or slit in a body always has a certain finite thickness.

Boundaries

We now have two (in the end equivalent) alternatives in regard to the definition of *boundary*: on the one hand we might exploit in Chisholmian fashion the use of *de re* modalities and define boundaries as entities that are necessarily such as to exist as parts of bodies; on the other hand we might exploit the notion of coincidence and define boundaries as coincident entities. Here we follow Chisholm in taking the former course. We shall then lay down the interrelation between boundaries and coincidence by means of an axiom.

Chisholm's D2 (1992/93, p. 16) would seem to amount, in our present context (where we are dealing exclusively with the spatial case) to the following definition of boundary:

$$Bx := \Diamond_x \exists y\,(Ky \wedge x < y)$$

(x is a boundary iff x is necessarily such that there is some body of which it is a proper part).

Boundaries are necessary proper parts of bodies. This will not do as it stands, however, first of all because there are other sorts of things which are necessary parts of this sort. Thus the definition is satisfied also by the interior of a body (i.e. by the result of deleting from a body its external boundary). It may be satisfied, too, by minds or souls (though then the proponent of the Brentano-Chisholm view of boundaries might argue in turn that minds or souls are themselves a species of zero-dimensional boundary). It may be that there are boundaries which are not parts of that which they bound: holes, for example, have boundaries of this sort on the view defended by Casati and Varzi (1994).

We can easily construct further counterexamples to the definition of boundary that is here formulated: take x to be the mereological sum of a banana together with a point on the surface of an apple. Then the point on the apple is a necessary proper part (it cannot exist without relevant larger apple-parts), and thus so also is the sum in question (it cannot exist without analogous larger parts of a banana-including sum); yet x is not itself a boundary. A more adequate definition of boundary along Brentanian-Chisholmian lines would therefore be (in keeping with Chisholm's own approach at p. 85 of his 1989):

DB. $$Bx := \Box_x \forall z(z \leq x \rightarrow \exists y(Ky \wedge z < y))$$

(a boundary is an entity which is as a matter of necessity such that it *and all its parts* are necessary proper parts of bodies).

In this way, too, we rule out interiors of bodies (though we seem not to rule out minds or souls, and other like examples).

Recall our principle to the effect that that which is above all else characteristic of a continuum is the possibility of a coincidence of boundaries. Chisholm gives as axiom (1992/93, p. 17):

$$Bx \leftrightarrow \Diamond_x \exists y\,(x \sim y \wedge \neg\, y < x)$$

A simpler axiom is:

AB1. $Bx \leftrightarrow x \sim x$ (boundaries are self-coincident)

Boundaries fall into two classes: punctiform boundaries, which have no extension, and boundaries of other sorts which share with bodies some of the extension which the latter in every case enjoy. Defining *point* as follows:

DPT. $PTx := Bx \wedge \forall y(y \leq x \rightarrow y \sim x)$ (point)

we can then assert by analogy with AK2:

AB2. $Bx \rightarrow PTx \vee \exists yz(By \wedge Bz \wedge y \cup z = x \wedge \neg yOz)$ (weak density)

(every non-punctiform boundary can be divided into two discrete boundary parts).

If There Be Monsters

Consider the following superficially attractive density principle:

(**) $\forall x \exists y(y \leq x \wedge y \sim y)$.

This asserts that boundaries are thick on the ground (that everything has or includes one: that boundaries and non-boundaries mutually penetrate). Unfortunately we can easily prove from this principle that the world is built up out of boundaries alone (or in other words that the sum total of everything is equal to the sum total of self-coinciders: $\sigma x(x = x) = \sigma x(x \sim x)$).

Let us suppose, for the sake of argument, that we are able to protect our theory from a conclusion of this sort. (Recall that one of our reasons for rejecting the set-theoretical conception of the continuum above was our rejection of the view that what is extended can somehow be built up by combining together a sufficiently large number of extensionless parts.) We are then able to countenance a thought-experiment along the following lines.[20] We define the *form* of a body as the sum total of all the boundaries within it; a body's *matter* (its unformed *materia prima*) can then be defined as what results when this form is taken away (the latter is of course merely the abstract result of a purely abstract subtraction, for form and matter are each as a matter of necessity such that they cannot exist without the other):

DF. form(x) $:= \sigma y(y < x \wedge y \sim y)$

DM. matter(x) $:= x - \text{form}(x)$.[21]

Note that both the form and the matter of a body are dense; but neither constitutes a continuum, since a continuum is essentially such as to contain both boundaries and the things they bound.

But now other such artefacts can be constructed (or defined within our theory). Given any point within the interior of a body, we can define its left- and right-directed point-counterparts, two different sorts of punctiform plerotic parts within the given point, as follows: take a spherical neighborhood around the given point as center, and imagine a vertical plane bisecting the neighborhood. Then the left-directed points (LDP's) are all those points coincident with the given point which serve as boundaries in directions properly included within the left hemisphere, the right-directed points (RDP's) are all those points coincident with the given point which serve as boundaries in directions properly included within the left hemisphere. We can now define two new sorts of bodily parts: the left- and right-point-surrogates of a body, respectively:

DLPS.LPS(x) $:= \sigma y(y < x \wedge \text{LDP}y)$

DLPR. RPS(x)$:= \sigma y(y < x \wedge \text{RDP}y)$.

The principle that from extensionless points extended wholes can never be constructed

[20] Compare Zimmerman 1996.

[21] These definitions may capture part of what is involved in the insight that bodies, in contrast to boundaries, are bulky: 'matter', on our present conception, is another term for bulk.

implies that these peculiar bodily surrogates are strictly distinct from the body itself with which we begin. They are analogues, within our mereotopological framework, of the space-filling curves, the Menger-Sierpinski sponges and the deleted Tychonoff planks of standard set theory.[22] Such monsters force us to be on our guard when formulating axioms and definitions, since our definitions may be satisfied by the monsters in question even though the latter correspond in no wise to the intuitive understanding which these definitions were designed to express. If, on the other hand, we accept the density principle (**) above, and swallow the consequences which we have otherwise seen reason to reject, then we thereby exclude these eldritch creatures by having them collapse onto each other and onto their respective bodies. This in turn, however, will carry a further price to the effect that our initial mereological axioms will have to be adjusted. Specifically, the standard ('additive') mereological summing principle will need to be replaced by a principle which allows that the sum of two objects may contain as parts objects which do not overlap with either of the two objects with which we begin.[23]

Varieties of Connectedness

A pair of spatial entities are in contact each other directly when their respective boundaries, in whole or in part, coincide. Chisholm defines *direct contact* as follows (1992/93, p. 16):

> x is in direct spatial contact with y := a constituent of x spatially coincides with a constituent of y. (1992/93, p. 16)

Unfortunately, however, this definition does not work in the general case. Thus for example it cannot capture the case in which a point-boundary x inside the interior of a body y is in direct contact with the punctured neighborhood $y - \sigma z(z \sim x)$. We can, however, define a relation of direct contact for bodies:

DDCOK. $xDCOKy := Kx \wedge Ky \wedge \exists vw(v < x \wedge w < y \wedge v \sim w \wedge v \neq w)$

(bodily direct contact)

[22] See, again, Steen and Steebach 1978.

[23] For a discussion of such non-additive summation principles see Smith 1991.

Note, however, that this definition imposes no constraint of connectedness on either x or y or their sum. Consider the case where x is the mereological sum of a banana and a point on an apple, y the mereological sum of some coincident point on the same apple together with a second banana some miles distant from the first. We then have xDCOKy even though the whole $x \cup y$ is not connected.

At this point Chisholm writes:

> A thing that turns back on itself (e.g. a tire, a hoop, or a doughnut) is in contact with itself. Lines and surfaces can turn back on themselves. (1992/93, p. 17)

A pliable rod, it seems clear, can be turned back upon itself in some special sense (the two ends can be brought into contact with each other). But surely the sense in which a doughnut is in contact with itself applies to all connected bodies (consider v in DDCOK as the right boundary of the left hemisphere of a sphere, w as the left boundary of the right hemisphere of the same sphere). Indeed it follows from our considerations on internal boundaries above that xDCOKx holds for every body x, since every body is large or thick enough to contain at least two coincident entities as parts.

Touching

In fact, to do justice to what Chisholm has in mind, we must therefore distinguish *touching* as a special case of direct contact which applies only to coincident parts of external boundaries of mutually discrete bodies: an entity x touches an entity y if each is such that it can exist without detriment even should the other be destroyed. We might, accordingly, introduce a new primitive 'exists' (symbolized by 'E!') and formulate a definition along the lines of:

DTO. $x \text{ TO } y := x \text{ DCOK } y \wedge \neg\, xOy \wedge \forall z(z \leq y \rightarrow \Diamond_x \neg E!z) \wedge \forall z(z \leq x \rightarrow \Diamond \neg E!z)$ (touching)

(x touches y iff x and y are discrete bodies in direct contact and, given any part z of y, x is possibly such that z does not exist and, given any part z of x, y is possibly such that z does not exist).

Thus imagine a pair of exactly similar hemispheres h_1 and h_2 which touch each other in such a way that the flat portions of each coincide in a horizontal plane. Imagine, on the other hand, a single sphere *s,* of identical proportions. *s* has running through its central horizontal plane a boundary in full plerosis that is in many respects similar to the sum of coincident boundaries, each in half plerosis, existing where h_1 and h_2 touch. The former differs from the latter, now, in that we can separate this sum of coincident boundaries without detriment to either half of the pair. The two correspondingly coincident boundaries in the solid sphere cannot, correspondingly, be cut apart: they belong together intrinsically (as a matter of necessity).

Contact

Bodies are *in contact* in the broader sense when they and all their parts are connected to one another, possibly via others, in such a way as to establish a seamless chain of direct contact. Chisholm seeks to define *contact* in this wider sense as the successor-relation of direct contact as follows:

> x is in *spatial contact* with y := x belongs to every class C which contains y and anything that is in direct spatial contact with any member of C (1992/93, p. 17)

Here we take a different tack, one surely more in keeping with the remainder of the present theory, and define first of all what it is for a body to be *connected*:

DCNK. $\mathrm{CNK}x := \mathrm{K}x \wedge \forall yz[(\mathrm{K}y \wedge \mathrm{K}z \wedge x = y \cup z) \rightarrow y\mathrm{DCOK}z]$

(connectedness for bodies)[24]

(x is a connected body iff all partitions of x into a pair of bodies y and z are such that y is in direct contact with z).

We go on to set as a definition of *bodily contact:*

DCOK. $x\mathrm{COK}y := \mathrm{K}x \wedge \mathrm{K}y \wedge \exists z\,(\mathrm{CNK}z \wedge x \cup y \leq z)$ (bodily contact)

[24] We might seek to define *connectedness for boundaries* analogously by means of:

$\mathrm{CNB}x := \mathrm{B}x \wedge \forall yz[(\mathrm{B}y \wedge \mathrm{B}z \wedge x = y \cup z) \rightarrow \exists vw(v \leq y \wedge w \leq z \wedge v \neq w \wedge v \sim w)]$

This will not work, however, as can be seen if we take a connected line x, and define y as the result of subtracting from x the sum z of all points in x coincident with a given interior point. We must return to this issue later, when we have a notion of *sameness of dimension* (and when we are in a position to eliminate punctured entities and other similar monsters from the class of boundaries).

(bodies x and y are in bodily contact iff their sum is part of some connected body).

My left hand and my right hand are in contact with each other in this sense as long as they remain attached to my body; they are in direct contact only if they touch each other.

A further sort of contact, illustrated by the case of two coincident surfaces, s_1 and s_2, where every part of s_1 is in contact with some part of s_2 and vice versa, is called by Chisholm *total contact*:

$$xTCOy := \forall z(z \leq y \rightarrow \exists w[w \leq x \wedge z \sim w]) \wedge \forall z(z \leq x \rightarrow \exists w[w \leq y \wedge z \sim w])$$

(x is in total contact with y iff x and y are in contact and all parts of x and y are in contact with corresponding parts of y and x).

Total contact is clearly impossible between entities that have any sort of thickness. (Such thickness would, as it were, shield certain interior parts from contact with the other entity.) Accordingly, Chisholm asserts as axiom:

$$xTCOy \rightarrow x \sim y.$$

Boundary Of

Rushing in where Chisholm fears to tread, we may now define a series of further mereotopological concepts on the basis of the notions defined thus far. Thus we may define the relational concept x *is a boundary of the body* y, where x is to be conceived as an exterior boundary – and thus as a boundary in the surface of y (which may mean: in the surface of an internal cavity of y):

DBK. $x\ BK\ y := \quad Bx \wedge x < y \wedge Ky \wedge \Diamond_y \exists z[y\ TO\ z \wedge \exists w(w < z \wedge x \sim w)]$

(boundary of body)

(x is a boundary of a body y iff x is a boundary and a part of y and y is possibly such as to touch some z with part of which x is coincident).

We may then define the notion of a maximal (exterior) boundary (complete boundary or ‘envelope’) of a body as follows:

DCBK. $x \text{ CBK } y := x \text{ BK } y \wedge \forall z(z \text{ BK } y \rightarrow z \leq x)$ (envelope)

Connectedness for Boundaries

We may define *connectedness for boundaries* as follows:

DCNB. $\text{CNB}x := \text{B}x \wedge \forall yz(y \cup z = x \wedge \forall uv[(\text{K}u \wedge \text{K}v \wedge y \text{ BK } u \wedge z \text{ BK } v) \rightarrow u \text{ DCOK } v])$ (connectedness for boundaries)[25]

(a boundary is connected iff any partition into y and z is such that if y is a boundary of body u and z is a boundary of body v then u and v are in direct contact)

We may then define *connectedness in general* as follows:

DCN. $\text{CN}x := \text{CNK}x \vee \text{CNB}x$ (connectedness)

In addition, and at the risk of some redundancy, we can assert the following principles for boundaries:

AB3. $(x \text{ B } y \wedge y \text{ B } z) \rightarrow x \text{ B } z$ (transitivity)

AB4. $(x \text{ B } z \wedge y \text{ B } z \wedge x \sim y) \rightarrow x = y$ (coincident boundaries of identicals are identical)

AB5. $(x \text{ B } z \wedge y \text{ B } z) \rightarrow x \cup y \text{ B } z$ (finite union)

We can prove:

[25] With the help of this concept of connectedness for boundaries, the definition of x BK y would enable us to formulate the equivalent of the “second Brentanian thesis” of Smith 1993, which affirms, for connected boundaries, the existence of connected bodies which they are the boundaries of:

$$(\text{B}x \wedge \text{CNB}x) \rightarrow \exists y(x \text{ BK } y \wedge \text{CNK}y).$$

TB1. $x \mathrm{B} y \rightarrow \neg y \mathrm{B} x$ (antisymmetry)

from which it follows trivially that it is never the case that xBx.

Substance

Our gloss on the primitive concept of *body* told us that bodies can fall short of being substances in two different ways: (1) in being too big, they contain two or more bodies as non-connected parts; (2) in being too small: they are parts of larger connected bodies (as one solid metal sphere may be discriminable inside a second, larger sphere).[26]

To exclude the first sort of counterexample we shall need to insist, following Chisholm,[27] that substances are connected bodies. To exclude counterexamples of the second sort we shall need to require in addition that substances are maximally connected bodies. This yields as candidate definition:

$$\mathrm{S}x := \mathrm{CNK}x \wedge \forall y(x < y \rightarrow \neg\, \mathrm{CNK}y)$$

We still, however, need to take account of the possibility that one substance might touch (be more or less momentarily in contact with) another. (The mereological sum of two such substances is connected, by our definition of CNK above.) Accordingly we set:

DS. $\mathrm{S}x := \mathrm{CNK}x \wedge \forall y[(x < y \wedge \mathrm{CNK}y) \rightarrow \exists t(x \cup t = y \wedge x\ \mathrm{TO}\ t)]$ (substance)

(a substance is a connected body which is such that if it serves as part of a larger connected body then this only because it touches some second body).

[26] Every substance contains *substantials* which are in this sense too small. Thus my arm is a substantial in relation to me as substance. See Smith 1997.

[27] Chisholm gives the definition (1992/93, p. 17):

$$\mathrm{S}x := \mathrm{K}x \wedge \forall y[y < x \wedge \mathrm{K}y \rightarrow \exists z(z < x \wedge z \neq y \wedge y\mathrm{CO}z)]$$

(a substance is a body all of whose proper bodily parts are in contact with some other proper parts).

This is too weak however. Even collectives made up of several separate bodies are such that all parts y are in contact either with their own respective proper parts or, in case y is a point which has no proper parts, with the surrounding portion of the relevant body.

Note that DS is consistent with the fact that substances may have holes of various shapes and sizes.

Dimensions

Boundaries can be classified according to the number of their dimensions. Thus we can distinguish one-, two- and three-dimensional continua and we can even contemplate continua of higher numbers of dimensions. From Brentano's point of view, a continuum

> is to be designated as one-dimensional if it has no other boundaries than such as are not themselves continuous. … The spatial line, too, has no boundaries other than non-extended ones, namely the spatial points, and it is for this reason that Euclid defined the point as that which has no parts. The surface, in contrast, belongs with the two-dimensional continua since its boundaries comprehend not only points but also lines. And a body is to be designated as a three-dimensional continuum since not only is the whole body bounded by a surface but so also each one of its parts is separated from the remainder by a surface that is a two-dimensional boundary. (1988, p. 10)

Chisholm's approach to the problem of dimension is to begin by defining *surface* as follows:

$$SFx := Bx \wedge \Diamond_x \neg \exists y\ (x < y \wedge By)$$

(a surface is a boundary – an envelope – which is possibly such that it is not a proper part of a boundary).

One problem with this definition, conceived as a definition of what we normally think of as two-dimensional boundaries, turns on the fact that there might be what Brentano calls 'topoids' of four or more dimensions, whose boundaries would satisfy the definition yet would not be surfaces in the intended (two-dimensional) sense. This we might solve by means of an axiom ruling out such cases, for example by setting as axiom:

$$\forall x \exists y (Ky \wedge x \leq y)$$

This will serve our purposes, however, only if we know in advance that bodies have always exactly three dimensions. This in its turn cannot be stipulated as an axiom unless we already have a concept of dimension at our disposal. No simple way out of this impasse presents itself, though for the moment we can follow Chisholm and impose (in effect) the requirement that the range of variables of our theory be restricted to spatial objects of three or less than three dimensions.

Another problem with Chisholm's proposed definition of surface is that it does not do justice to what we might call open surfaces, surfaces which are arbitrarily delineated sub-regions of other, larger surfaces.

A better definition can be achieved if we use an earlier proposal (advanced by Chisholm in his 1989, p. 88) and define surfaces by appealing to the fact that a surface, unlike other boundaries, can coincide at most with one other surface (where points and lines can coincide with an infinity of other points and lines). We then have (provisionally and tentatively):

DSF. $SFx := Bx \wedge \Box_x \forall yz[(x \sim y \wedge y \sim z) \rightarrow (z = x \vee z = y)]$

Points, Lines and Surfaces

We have already given the following definition of point:

DPT. $PTx := Bx \wedge \forall y(y \leq x \rightarrow y \sim x)$

Chisholm defines point as follows (1992/93, p. 19):

$$PTx := Bx \wedge \neg\Diamond_x \exists y(y < x)$$

The problem with a definition of this sort is that, as we have seen, the mereological sum of coincident points (the red, green, blue points at the center of a disk divided into three differently colored segments) is itself a point, and thus such as to have proper parts.

Chisholm seeks to define *line* as follows (1992/93, p. 19):

$$LNx := Bx \wedge \exists y(y < x) \wedge \neg SFx.$$

A line would then be a boundary which has parts but is not a surface. To rule out complex points from counting as lines, the middle conjunct needs to be adjusted to:

$$\exists y(y < x \wedge \neg x \sim y).$$

Sums of separate boundaries (pairs of separate points, sums of points and lines, etc.) would still provide counterexamples to the definition. To solve these difficult problems in the space available to us here we therefore introduce the notion of sameness of dimension ($=_{dim}$) as an additional primitive notion of our theory.[28]

We can now affirm a generalization of the subtraction theorem for bodies (TK1) above:

ADim1. $(x < y \wedge x =_{dim} y) \rightarrow x =_{dim} (x–y)$

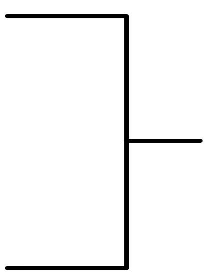
We can also define, by analogy with the case for bodies, what it is for two boundaries to be in direct contact:

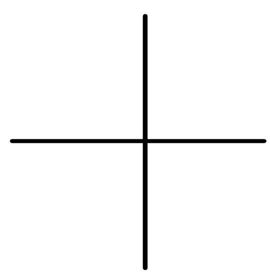
DDCOB. $xDCOBy := Bx \wedge By \wedge x =_{dim} y \wedge \exists vw(v < x \wedge w < y \wedge v \sim w \wedge v \neq w)$

and we can affirm an axiom of *density for non-punctiform boundaries:*

ADim2. $Bx \rightarrow PTx \vee \exists yz\,(By \wedge Bz \wedge y \cup z = x \wedge \neg yOz \wedge x =_{dim} y =_{dim} z)$

Can we now define a line as a suitably complex dense boundary that is not a surface and that is connected? To see why not, consider complexes of lines like these, which meet all of these conditions are yet are not *lines*:





[28] This notion can, incidentally, enable us to formulate an alternative definition of connectedness for boundaries along the following lines:

DCNB*. CNBx := $Bx \wedge \forall yz[(By \wedge Bz \wedge x =_{dim} y =_{dim} z \wedge x = y \cup z) \rightarrow \exists vw(v \leq y \wedge w \leq z \wedge v \neq w \wedge v \sim w)]$

Let us therefore define first of all the notion of a *line-complex* via:

DLC. $LCx := Bx \wedge CNBx \wedge \neg PTx \wedge \neg\exists y(SFy \wedge y \leq x)$ (line-complex)

We can eliminate those and other troublesome line-complexes by defining a relation of *contact for line-complexes* and restricting lines to those line complexes where at most two constituents are in contact at any given point:

DLCC. $xLCCy := LCx \wedge LCy \wedge CNBx \cup y \wedge \neg xOy$ (connectedness for line-complexes)

We then set:

DLN. $LNx := LCx \wedge \forall yz[(y < x \wedge z < x \wedge yLCCz) \rightarrow \forall w((w < x \wedge wLCCy) \rightarrow wOz)]$

(line)

(lines are line-complexes which, for every point of their extension, can be split into at most two mutually disjoint constituent line-complexes)

A similar operation will now have to be mounted in order to distinguish genuine surfaces from surface-complexes which arise where surfaces cross or split. (We shall then need to reexamine our definition of line-complex, which employs our earlier, provisional definition of surface.)

Even when this is done, much will have been left unsaid. Thus we have not specified that points are parts of lines, that lines are parts of surfaces. Thus *a fortiori* we have not said either that lines are not mere sums of points and that surfaces are not mere sums of lines. Nor have we said that lines and surfaces *have* boundaries. We have not defined the single dimensions, nor ruled out fractional dimensions, and nor have we said that points, lines and surfaces are entities of zero, one and two dimensions, respectively. All of these things need to be proved, or stipulated axiomatically on the basis of intuitively reasonable, sound and satisfactory mereotopological considerations. Only then will we have more than the beginnings of a theory of boundaries and coincidence.

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