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HISTORICITY, VALUE AND MATHEMATICS

ABSTRACT At the beginning of the present century a series of beautiful but threatening paradoxes were discovered within mathematics, paradoxes which suggested a fundamental unclarity in traditional mathematical methods. These methods rested on the assumption of a realm of mathematical idealities existing independently of our thinking activity, and in order to arrive at a firmly-grounded mathematics different attempts were made to formulate a conception of mathematical objects as purely human constructions. It was, however, realised that such formulations necessarily result in a mathematics which lacks the richness and power of the old "platonistic" methods, and the latter are still defended, in various modified forms, as embodying truths about self-existent mathematical entities. Thus there is an idealism-realism dispute in the philosophy of mathematics in some respects parallel to the controversy over the existence of the experiential world to the settlement of which Ingarden devoted his life. The present paper is an attempt to apply Ingarden's methods to the sphere of mathematical existence. This exercise will reveal new modes of being applicable to non-real objects, and we shall put forward arguments to suggest that these modes of being have an importance outside mathematics, especially in the areas of value theory and the ontology of Art.

1. TOWARDS A PHENOMENOLOGICAL ONTOLOGY OF MATHEMATICS

According to their reaction to the mathematical paradoxes philosophers of mathematics divided themselves into several conflicting 'schools'; each school based itself on one or other dogmatic notion of what is truly *given* in mathematics, and then attempted to derive the whole of pre-established mathematics from this given core, or to 'salvage' as much as was possible, dismissing what could not be absorbed as the result of unclear mathematical thinking. Some groups of philosophers even found themselves committed to a programme for the creation of a completely new mathematics, since they dismissed all that had gone before as through and through unacceptable; hence there arose radically alternative universes of mathematical objects, in conflict with the universe of 'standard' mathematics.

A phenomenological philosophy of mathematics would not, of course, seek to become just one further school with its own particular notion of what is given and its own particular methods of derivation. It is the es-

Tymieniecka (ed.), Analecta Husserliana, Vol. IV, 219–239.

sence of phenomenology to be open to all modes of givenness, and hence it must somehow enable the development of an all-embracing account of mathematics. With respect to this account each dogmatic school will 're-appear' as a special case resulting when we adopt a particular 'adumbrative' attitude with regard to the totality of mathematically given data. A phenomenological philosophy must recognise, in particular, that every species of mathematical object, whatever its mode of givenness in actual mathematical practice, has its own determinate mode of existence (this being true even in the case of objects postulated by mathematical works subsequently proved inconsistent). Hence the phenomenologist's universe of mathematical objects must be all-inclusive. With respect to this universe each particular school of mathematical activity is seen as having an effective commitment only to one or other possible 'sub-universe'.

In his masterpiece on *The Controversy about the Existence of the World*[1] Ingarden has provided us with a framework within which this all-embracing philosophy of mathematics can be developed at the level of a sophisticated ontology. Ingarden's phenomenological analyses of what is involved in the givenness to consciousness of objects possessing the various different modes of being, reveal the presence of 'existential moments'. In virtue of its possession of a particular combination of existential moments an object is distinguished as having real being, ideal being, intentional being, and so on. For example, if we confine ourselves to candidate real objects, we can say that what distinguishes a given real apple from a merely thought or intended apple is the moment of self-existence possessed by the former; the latter exists as a mere fiction, having no immanent qualities and no being-status of its own, for it is existentially 'dependent' upon the given intention, with the retraction of which it ceases to have any existence. What is involved here is a dichotomy between 'autonomous' and dependent or 'heteronomous' objects, and this is the first pair of existential moments distinguished by Ingarden.[2] The second pair concerns the existential 'source' of an object. For example, if we consider a block of stone out of which a sculptor intends to create a particular statue, the virgin block is given as having an existential source which is extrinsic to the sculptor, but this is not the case with regard to the finished statue. We say that the block has the moment of existential 'originality' relative to the sculptor whilst the statue has the opposite moment of existential 'derivation'. That we can distinguish

this notion of relative originality implies the possibility of an absolutely original being, i.e. a being which owed its existence to no other object or objects. Such a being would be primordial and permanent, containing within itself the guarantee of its own existence. Developed religions award this sort of absolutely original status to God, and Ingarden notes[3] that dialectical materialism attributes the same status to matter.

Ingarden's philosophy of the real world consists in the systematic investigation of all possible interrelations of these and other moments distinguished as applicable to reality as a whole and to the monad of pure consciousness in which this reality has its reflection, in such a way that we shall be able to determine once and for all and without prejudice the nature of the relation of dependence which holds between the two. Many emptily possible combinations of moments fall away as incoherent or in conflict with the results of 'formal' and 'material' ontology, and Ingarden's arguments demonstrate that only those combinations which survive exhaustive phenomenological analyses are possible 'solutions' to the problem of the existence of the world. But the 'impossible solutions' continue to have a role to play as indices of the different more or less dogmatic attitudes which it is possible to take up with regard to this controversy, amongst which are, for example, absolute realism and Husserlian subjective idealism. Each such attitude corresponds, on this level of generality, to one or other possible manner of reflectively inhabiting the real world.[4] We can now proceed to develop our 'all-embracing' philosophy of mathematics by carrying out for the universe of mathematical objects just that exposure of all the different possible combinations of relevant existential moments which Ingarden carried out for the real world in its relation to pure consciousness.

2. Mathematical Existence

There is an important difference between the real world and the universe of mathematical objects. The former is homogeneous, its objects all possess the same (real) mode of being, whilst the world of mathematical objects seems to admit certain modal partitions. The mode of existence appropriate to the natural numbers, for example, does not seem to be appropriate to transfinite (ordinal and cardinal) numbers. In general, as we leave the 'central core' of standard mathematics (finite number theory,

Euclidean geometry, etc.) we seem to encounter changes in the mode of givenness of the objects with which we deal, and these changes can have a correlate ontological significance. Hence an adequate phenomenology of mathematics must refrain from prejudgments to the effect that the mathematical world be considered as an ontologically homogeneous whole. At this stage, in fact, we must recognise the existence not only of objects (such as 1, 2, 3) which are directly given as possessing the moments of autonomy and originality, but also of objects (such as i, j, k within Hamilton's theory of quaternions) which are given as enjoying a merely heteronomous, derivative existence. This implies at least a dualist ontology, i.e. one which distinguishes two (possibly empty) object-regions as follows:

autonomy	heteronomy
originality	derivation.

However there seems to be no *a priori* reason why the limit of existential originality should coincide in the mathematical universe with the limit of heteronomy; thus the moment-combinations:

autonomy	and	heteronomy
derivation		originality

present themselves as corresponding to possible being-modes for mathematical objects. Plausible cases of the former would be arithmetical fractions ($\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, etc.) which are given as enjoying an autonomous existence alongside the natural numbers, but which are also given as non-original since they owe the source of their existence to the discovery of an adequate conception of them as ordered pairs of natural numbers, corresponding to the discovery of an adequate method of representation. The moment-combination

heteronomy
originality

is, however, dismissed by Ingarden as impossible.[5] This is because the dependence of heteronomous objects upon intentional acts implies that such objects cannot possess the moment of originality relative to consciousness. Nevertheless certain philosophers of mathematics have attempted to affirm that mathematical objects possess this sort of being-status. Such philosophers believe that mathematical objects enjoy a merely fictional existence, but they wish to stress a distinction between "natural" fictions and those which are in some sense arbitrary and contrived. Here they point to the difference in creative freedom which is experienced between e.g. writing a novel and developing an extension of standard mathematical set theory.[6] This doctrine of "objectivism without objects"[7] can be accounted for within our framework by exploiting a distinction, recognised by Ingarden[8], between the existential 'foundation', the existential 'source' and the existential 'basis' of an object. The autonomy/heteronomy opposition depends upon the existential foundation of an object, i.e. upon the manner in which it is maintained in existence, an autonomous object being one which 'founds' itself. Similarly the moment-pair originality/derivation is a matter of existential source. Existential basis is explained as follows:[9] "something which has its own existential basis in itself is as if it could *afford* to be founded in itself," i.e. the object is, although heteronomous, in some sense intrinsically stable and self-contained. Such an object is, we shall say, existentially 'self-basic', and we shall introduce the dichotomy: 'self-basis'/'artificiality' as a new pair of existential moments.[10]

This means that our ontology of mathematics now embraces a four-region universe of objects as follows:

autonomy / originality	autonomy / derivation	heteronomy / derivation / self-basis	heteronomy / derivation / artificiality.

A central core of ideal and eternal mathematical objects is surrounded by a sphere of autonomous derived objects brought into being from out of the central core, effectively by way of adequate definitions. More or less fragmentarily attached to these internal spheres are regions of heteronomous objects, some of which are given as intrinsically stable, whilst others are given as artificial inventions brought into being for particular mathematical purposes. The remotest of these heteronomous regions give way to an empty nothingness, i.e. to that area which has not yet been conquered by mathematicians in the sense of being made accessible to 'concretisation'[11] via chains of definitions and proofs.

3. Traditional philosophies of mathematics

Traditional schools of mathematical thought, platonism, intuitionism, formalism, psychologism, etc. are included as special cases within our phenomenological philosophy in the same way that traditional positions with respect to the idealism-realism dispute were included in the meta-philosophy which Ingarden puts forward in *The Controversy about the Existence of the World*. Each traditional conception of the universe of mathematical objects can be correlated with one of the (at this stage, fifteen) emptily possible specialisations of the four region ontology which we have outlined above: four one-region ontologies (monisms), six dualisms, four troilisms and a single four-region ontology.[12] We must also recall that within the different adumbrative attitudes which correspond to each of the traditional schools, the mathematical universe which results appears truncated when compared to the all-inclusive mathematical universe which is embraced on the level of phenomenological reflection. For example, 'platonists' do not recognise as existent what they regard as "unacceptable" intuitionistic objects, and the same holds in general for each school with regard to the objects of those schools with which it is in conflict. At this level of investigation the precise nature of each such 'truncation' can be only emptily indicated, sub-universes being 'small' or 'large' according to the relative extent of their commitment; further determinations must await the analyses of a 'material ontology' of mathematics.[13]

Monistic philosophies of mathematics (one-region ontologies) 'impose' a single mode of being throughout their mathematical universe. Such philosophies fall into four categories corresponding to the four possible moment-combinations so far distinguished.

(i) autonomy
originality.

Corresponding to *platonistic* philosophies of mathematics this category implies the affirmation that all mathematical objects are ideal and eternal, discovered and not invented, transcendent entities. Where our actual congress with mathematical objects leads us to question the ideal, eternal existence of mathematical objects which this doctrine implies – perhaps

because particular objects appear to have been *brought into being* by creative mathematical activity – platonists argue that this appearance is merely the result of an epistemological inadequacy on our part; this sort of difference in mode of givenness corresponds, they claim, to nothing on the side of the object itself. In general platonist philosophies allow for the existence of 'large' mathematical universes since the self-existence of mathematical objects implies that the mathematician has at his disposal powerful methods, such as the law of excluded middle and the method of impredicative definition, which are unavailable when objects are conceived as merely human creations. Platonism is not, however, incompatible with a restricted mathematical universe. *Pythagoreanism*, for example, holds that only finite numbers and possibly sets exist in the required platonistic sense and that other candidate mathematical objects do not exist in any sense.

(ii) autonomy
derivation.

Many philosophers of mathematics have held that mathematical objects are autonomous but that they owe the *source* of their existence to some realm of being extrinsic to mathematics. The *logicism* of Frege and Russell, for example, holds that mathematics is reducible to logic, and this seems to amount, in our terminology, to the characterization of mathematical objects as derived from purely logical objects. *Materialist* philosophy of mathematics, on the other hand, argues that material objects constitute the sphere of (relative) originality from which mathematical objects are derived.[14] Finally we can consider those varieties of *formalism* which hold, in effect, that the status of mathematical objects is derivative from that of concrete configurations of 'meaningless' mathematical symbols.

(iii) heteronomy
derivation
self-basis.

The *intuitionism* of Brouwer and the Dutch school requires mathematical objects to be constructible in an ideally conceived sequence of

mental acts or "intuitions" occurring at discrete quasi-temporal intervals by analogy with acts of counting out loud.[15] In comparison with platonist mathematics this results in a 'small' mathematical universe. Intuitionism amounts to a commitment to mathematical objects as heteronomous ('mental mathematical constructions') and as derivative, having the source of their existence in the ideally conceived mathematical intuition. The intuitionists also clearly believe that their constructions possess the moment of existential self-basis; they claim that the mathematics which results from their programme reveals 'mental architecture' with an intrinsic value of its own "which it is difficult to define beforehand, but which is clearly felt in dealing with the matter,"[16] i.e. in actually engaging in intuitionistic mathematics. This claim also finds some extrinsic support in the fact that intuitionistic logic (a logic within which it is impossible to derive, for example, the laws of excluded middle and of double negation) has been shown to have an elegant and 'natural' formalisation.[17]

(iv) heteronomy
derivation
artificiality.

The final type of monistic ontology corresponds to the commitment which is implicit in the statements of those who see mathematics as a matter of arbitrary human conventions, an 'anthropological phenomenon'[18] with regard to which we are free at every stage.[19]

Having considered each of the four possible species of mathematical monism we shall have space here, with regard to the six possible dualisms, to mention only two particular cases which have played an important role within the tradition.

(i) autonomy / heteronomy
originality / derivation
artificiality.

Weak realism consists in a commitment to the existence of only *'some'* mathematical objects as independent of our thought activity.[20] The remaining, derived objects in the mathematical universe exist as mere fictions. For example, Kronecker's statement "God made the natural numbers, all the rest is the work of man" can be interpreted as a statement of the (relative) originality of the natural numbers and of the heteronomous and artificial derivation of all other mathematical objects.

(ii) autonomy / heteronomy
derivation / derivation
self-basis.

The *dualist formalism* of Hilbert[21] consists in the argument that whilst all mathematical objects enjoy a merely derivative existence relative to certain configurations of mathematical symbolism (formal systems), nevertheless we must distinguish in our derived universe between *Realelemente* which are purely finitistic mathematical objects, and peripheral *Idealelemente*, the latter being purely instrumental aids to the smoother achievement of mathematical results, which have meaning only in so far as they concern *Realelemente*. Hilbert would admit only those *Idealelemente* which were 'self-basic' in the sense that he wished to restrict admissable 'instrumental aids' to those whose underlying formal system had been proved consistent using purely finitistic methods. Gödel's incompleteness theorem showed however that the carrying out of this programme was an impossible task, and this might be taken as suggesting that for a sufficiently powerful mathematics non-self-basic objects can never be excluded.[22]

4. Historicity

The existential moment of 'actuality' concerns the concrete temporal presence of an object in the full totality of its being, the dichotomy actuality/non-actuality is then conceived as embodying the phenomenological content of the scholastic opposition between objects which *exist* and those which merely *pertain* or *subsist*. We can now represent the four modes of being distinguished by Ingarden as follows:

A.	*Absolute (Timeless) Being* of, for example, God.	autonomy originality actuality
B.	*Ideal (Extratemporal) Being.*	autonomy originality non-actuality

C.	*Real (Temporal) Being* of objects in the present.[23]	autonomy derivation actuality fragility[24]
D.	*Purely Intentional Being.*	heteronomy derivation non-actuality.

In the present paper we are concerned with non-actual objects, i.e. with modes B and D; but our consideration of mathematical existence has revealed the importance of distinguishing modes of being 'between' the two extremes of ideal and purely intentional being which Ingarden, with some reservations,[25] put forward as together exhausting the sphere of the non-actual.

The intervening modes belong to *eidos* which do not have the existential permanence and primordiality of Ingarden's pure idealities; they possess a factor of historicity *in their being* and not merely at the level of our knowledge of them as is the case with true idealities. Arising at determinate points in time our 'historical' *eidos* can undergo specific types of evolutionary development and they can also, in some cases, be destroyed. We can represent the situation as follows:

B_1.	*Purely Ideal Being* (Extratemporal).	autonomy originality non-actuality
B_2.	*Derived Ideal Being* of autonomous historical *eidos*	autonomy derivation non-actuality
D_1.	*Stable Intentional Being* of heteronomous historical *eidos*	heteronomy derivation self-basis non-actuality
D_2.	*Unstable (Purely) Intentional Being*	heteronomy derivation artificiality non-actuality.

The recognition of dimensions of historical *eidos* alongside the realm of purely ideal being does not effect the validity of Ingarden's arguments[26] that only the existence of purely ideal concepts and essences can account for the intersubjective identity and individuality of intentional objectivities and in particular, for example, of linguistic meanings. Indeed, since the constitution of historical *eidos* is itself effected only through intentional objectivities of this kind it follows that the *eidos* themselves depend for their existence upon that of pure idealities. Thus commitment to historical *eidos* is not an 'alternative' to platonism, and this is the root of the failure of Husserl's attempts[27] to produce a constitutive phenomenology of *eidetic* experience without resort to platonic entities of any kind. The nature of Husserl's failure is interestingly revealed by a consideration of the philosophy of mathematics which is implicit in Husserl's idealism. The works of Becker[28] and Kaufmann[29] can be interpreted as demonstrating that a truly 'Husserlian' mathematics would be a counterpart of Brouwerian intuitionism; in particular it would have a 'small' universe within which uncountably infinite mathematical objects would be unattainable.[30] Gadamer[31] has expressed the Ingardenian, realist argument that "mathematical objects cannot be conceived as the result of human activity, that enumerating is not the origin of number. It seems impossible to admit that all logical and mathematical entities would depend upon the interrelations among meanings" as is implied by the idealism of Husserl for which "the transcendental ego has no ontological but only a *rational* claim."[32]

Husserlian mathematical philosophy is therefore seen to be inadequate,[33] however Ingarden's over-sharp formulation of a dichotomy between purely ideal and purely intentional being seems to commit him to an equally inadequate monistic platonism. In the next section an attempt is made to develop a *via media* between the two.

5. Contentual Mathematics

The bringing into being of a derived mathematical object is a double achievement of consciousness. The mathematician must first form a conception of the given object, and he must then find a way of communicating his conception. This communication can be achieved only through the medium of a pre-understood mathematical language whose 'nominal

meanings'[34] have determined references to pre-given mathematical objects (ultimately the latter must be autonomous individuals of the central directly-given mathematical core). The mathematician must contrive to order his symbols in such a way that the intention to these pre-given references can be effected only 'through' an intervening noematic layer of 'higher-order' objects (within which in particular his own newly-created object finds its place), just as the word and sentence-meanings are determinately ordered within a *literary* work in such a way that they can be properly understood only when intention of them results in the constitution of a (noematic) stratum of objects 'represented' within the work.[35] A 'concretisable' ordering of mathematical symbols constitutes a mathematical work; the publication of such a work endows the objects represented within it with a truly objective (intersubjectively identifiable and freely accessible) existence.[36]

A derived mathematical object, once brought into being by a mathematical work, is either autonomous or heteronomous depending upon the manner in which it is maintained in existence. In the former case the object is such that, to a consciousness which has adequately concretised the defining work, it is thereafter *given* as purely ideal, i.e. it comes to have the same direct mode of givenness as the central core mathematical objects. Therefore there is a sense in which, once concretised, derived autonomous objects do not require mathematicians' conscious *activity* to maintain them in existence: they fall into the passive, received dimension in such a way that their existence cannot be brought into question. (A possible *quasi*-platonist 'solution' to the problem of the existence of the mathematical world would consist in a demonstration that *all* mathematical objects can receive a representation in a mathematical work which is concretisable in such a way that its objects come to be endowed with this direct mode of givenness.)

In the case where a derived mathematical object is given as heteronomous, however, this is because the chain of definitions and proofs which must be followed if we are to achieve an adequate concretisation of the appropriate mathematical work and therefore also of its higher object stratum, does not have the required 'inexorability'. The created object, if concretised at all, does not in any case achieve direct givenness and thus it continues to depend for its existence upon the activity of mathematicians in bringing to life the noematic layer in which it rests.

Such an object is said to require a double existential foundation, (i) in the transcendent structure which is the mathematical work[37] and (ii) in the acts of consciousness of those who successfully concretise this work. In order to remain in existence, therefore, a created object relies upon there being some 'justification' for the continued concretisation of its defining work; otherwise it 'sinks and fades into oblivion'.[38] In the case where this justification is fully intrinsic to the work (is a matter of *aesthetics*) we say that the object is self-basic; otherwise, e.g. where there is a merely pragmatic justification for the concretisation of the work, the object has the moment of artificiality.

In the case of heteronomous derived objects we can never attain a direct mode of givenness of the object itself; all we can ever achieve is a 'frail' concretisation of the noematic layer of its defining work. In his investigation of the concretisation of literary works[39] Ingarden shows that there are manifold differences between concretisations effected by different individuals at different times, and that these differences can endow the work with a sort of 'life'. This applies also to mathematical works, in particular to 'creative' mathematical works, i.e. those which effect the ontogenesis of heteronomous mathematical objects.[40] As Ingarden points out, "one can speak of the 'life' of a work in a two-fold and, in both cases, figurative sense: (1) the work 'lives' while it is expressed in a manifold of concretisations; (2) the work 'lives' while it undergoes changes as a result of ever new concretisations formed by conscious subjects."[41] These latter changes in 'the work itself' need not involve actual alterations in the text of the work, (although important hidden features of a mathematical work, brought to the surface in concretisations, may be incorporated in successive *editions*); a creative mathematical work can evolve 'in itself' also imperceptibly, as the concretisation-possibility of the identical work changes from one generation of mathematicians to the next. A given mathematical community in its simple apprehension of the work is not conscious of the fortuitousness of an 'accepted' concretisation of the work, nor is it aware "of those points in which it materially and necessarily differs from the work, nor, finally, of the concretisation as something to be contrasted to the work itself."[42] As a result the mathematical community "absolutises the given concretisation, identifies it with the work, and in a naive way directs itself intentionally to the work thus intended. Everything that pertains

to the content of the given concretisation is then ascribed to the work"[43] and becomes further absolutised e.g. in the mathematical text-books produced by the given mathematical community. The 'material points of difference' between a concretisation and the transcendent identical structure which is the work itself hinge on the fact that the creative mathematical work is, like the work of literature, a 'schematic formation';[44] it possesses 'loci of indeterminacy' some of which must be 'filled in' with alien contents imported by consciousness in its efforts to concretise the given work.[45] Only in virtue of such an importation can the noematic layer 'held in readiness' within the work be brought to actualisation.[46] The mode of existence of created and heteronomous mathematical objects as schematic then has important mathematical consequences, for loci of indeterminacy conceal potential mathematical ambiguities which can be drawn out in subsequent mathematical research; indeterminacy can, most especially, conceal the fact that the work is inconsistent. This implies a certain 'provisionality' with regard to every created mathematical object as intended within a given absolutised concretisation (provisionality is in fact an existential moment of such objects, parallel to the moment of fragility which is possessed by real objects). The presence of this moment follows from the fact that candidate properties of heteronomous objects revealed in new concretisations are always dependent for their acceptance upon particular historical reactions of the mathematical community. Some of these different possible reactions lead to the effective destruction of one region of objects in the mathematical universe in favour of new regions embodying more subtle distinctions and having a more determinate concretisability. Lakatos[47] has ably characterised all the different types of such reaction and has emphasised that their co-existence makes of creative mathematics a 'dynamic development'. For him however this 'development' signifies the evolution of conflicting linguistic frameworks[48] having no ontological significance. Lakatos fails to recognise that language forms merely one existential foundation of mathematics, and therefore that adequate concretisation of mathematical language depends upon the constitution by consciousness of *sui generis* mathematical objects. The difficulty has been to reconcile the ontological aspect of mathematics with its 'heuristic' character as revealed by Lakatos. It is the argument of the present paper that such a reconciliation, leading to a truly contentual (*inhaltliche*) mathematics can be effected by exploiting the notion of historical *eidos*; but if historical *eidos* are objects which come into being merely as a result of our mathematical activity then such a conception might make of creative mathematics nothing more than a gratuitous game or ritual[49] with, at best, a merely pragmatic value. In particular we can ask how this 'activity' can be a matter of reason when, just as is the case with regard to literary creative activity, there is no suggestion of a pre-existent domain against which our 'results' can be tested. Its rationality, we suggest, consists in the fact that the intentional objects which it brings into existence must be 'stable' entities with respect to which thought as such finds itself particularly 'at home'. Such intentional *eidos* are not, like Ingarden's purely intentional objects, confined to the noematic layer of one particular work as this receives its manifold of concretisations,[50] for their being is such that they can float free and play a part in other areas of mathematics and within new creative mathematical works.

This mode of being is not confined to the objects of creative mathematics, we can distinguish other areas in which stable intentional *eidos* are brought into being and with regard to the totality of all such *eidos* it might be appropriate to develop a theory of 'intellectual aesthetics'[51] whose role would be as necessary supplement to direct *Wesensschau* in the laying bare of the eidetic universe. As applied, in particular, to mathematics such a theory would make explicit those evaluative criteria and objectives which are at present only unreflectively and unsystematically brought to bear in actual mathematical practice. It is these criteria and objectives which, stumblingly, and via the continual excision of false-starts under the pressure of criticism, lead to the constitution of stable intentional *eidos* and to the growth of the mathematical universe which they co-constitute.

6. Art, values, and ontology

Ingarden developed a full and adequate conception of the 'purely intentional object' through his investigation of the work of literature, and the question can be raised whether the conception thus developed can be applied indiscriminately to all areas of intentional being as Ingarden assumed. For him a purely intentional object is not a 'genuine creation' but rather a case of arbitrary properties being merely 'assigned' to an

illusory correlate of particular intentional acts.[52] This mode of existence is certainly appropriate to, say, the characters of a minor novel; for here we do have nothing other than illusory nuclei to which determinate properties are assigned by the given work and whose existence consists purely in their being intended within adequate concretisations of the work. But the arbitrariness of this kind of 'assignation' of properties does not seem to be present in all aesthetic objects.[53] There are different 'degrees' of arbitrariness, encapsulated in our notion of the (relative) existential self-basis of an object. As we distinguished stable created mathematical *eidos* which "float free from their defining work," so we must recognise the existence of self-basic literary *eidos*, particular characters, atmospheres, 'metaphysical qualities',[54] etc., which are so constituted that they survive, historically, to enjoy a stable (although heteronomous) existence independently of the work which brought them into being. Hamlet, for example, enjoys the moment of existential self-basis relative to what we might call the post-Shakespearean consciousness; he has become an eidetic component in our thinking.[55]

Art in general and literature in particular thus have the function of creating stable experiential *eidos*. A similar function is attributed, e.g. by Grabau,[56] to works of existential philosophy such as, for example, Heidegger's *Sein und Zeit*. Grabau criticises Heidegger's conception of this work as revealing purely ideal structures which are "conditions for the possibility of experience as such." Grabau argues, in effect, that all such 'experiential universals' are created intentional *eidos*. Thus he compares Heidegger's notion of *Sein zum Tode* with William James' conception of a 'moral equivalent of war', claiming that both are invented constructs "in terms of which energies which could find other discharges ... are organised and given direction."[57] If "to view one's life under the construct of Heidegger's theory of death perhaps lifts it from the level of the chaotic to that of the significant,"[58] then we would argue that such would be the case for those relative to whom the corresponding experiential *eidos* possessed the existential moment of self-basis.

Such 'experiential' historical *eidos* effectively 'mediate' between the realm of human experience and the realm of purely ideal concepts and essences in that it is the existence of pure idealities which give an intersubjective meaning to those e.g. aesthetic and philosophic works through which we can concretise particular historical *eidos*.

Commitment to a dimension of mediating historical *eidos* can also 'reconcile' a platonistic conception of the value-pantheon[59] with the dominion of different and sometimes conflicting systems of value-*eidos* over different civilisations. A derived, heteronomous system of value-*eidos* is adopted by a given civilisation should a stage arise in its moral development when the *eidos*-system has acquired the moment of existential self-basis relative to the civilisation as a whole. The given historical value-*eidos* then derive not only their intersubjective meaning but also their peculiar subjective imperativity from purely ideal and extra-temporal values-in-themselves. The value-*eidos* can become concretised by consciousness only when an intention of them is effected 'through the noematic layer' which they co-constitute and 'onto' self-existent values-in-themselves;[60] consciousness is thus able to adequately concretise historical value-*eidos* only to the extent that it understands such *eidos* in the guise of transcendent values-in-themselves.

Autonomous non-actual entities, both purely ideal concepts and essences and derived autonomous historical *eidos*, have a peculiar 'postulational' mode of givenness. This depends on the fact that stable intentional *eidos*, in particular linguistic meanings and value-*eidos*, etc., depend for their intersubjective existence upon an existential foundation in the realm of autonomous idealities. But pure idealities themselves "never achieve genuine realisation at all."[61] Consciousness "can produce only *actualisations* of ideal meaning contents of concepts and form them into new wholes,"[62] into, e.g., meaning-contents of sentences and into historical *eidos* generally. This negative non-immediate mode of givenness is clearly 'inferior' as compared with immediate givenness of objects of perception, memory, imagination, etc., and phenomenologists have justifiably wished to account for all experience in terms of the latter, to 'reduce' all ideal being to the level of the immediately given, i.e. to the level of the *Lebenswelt*.[63] We hope that our investigation of mathematical existence and our brief remarks on art and value theory have suggested the need for an alternative to reductive phenomenology of eidetic experience; we also hope that this suggestion is within the spirit of Ingarden's philosophy.

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NOTES

[1] *Der Streit um die Existenz der Welt*, 3 vols., Tübingen, 1964–5, hereafter referred to as SEW (original Polish edition, Cracow, 1947–8). Volume I, *Existentialontologie*, with which we shall here principally be concerned, has received a partial English translation, *Time and Modes of Being*, Illinois, 1964, referred to as TMB. Volumes II/1 and II/2 concern Ingarden's *Formalontologie*; the culminating *Materialontologie* he did not live to complete.

Throughout the present paper we use single quotation marks to introduce technical terms, especially as derived from Ingarden in SEW in the English forms used in TMB.

[2] SEW, I, 79–87; TMB, 43–51.

[3] SEW, I, 112; TMB, 79.

[4] Sixty-four empty possibilities are distinguished by Ingarden. This aspect of SEW is emphasised in G. Küng, 'The World as Noema and as Referent', *Journal of the British Society for Phenomenology* III (1972) 15–26.

[5] SEW, I, 114f.; TMB, 81f.

[6] Cf. H. Wang, *From Mathematics to Philosophy*, London, 1974, p. 25.

[7] For an account of this view and its relation to the philosophy of K. Gödel, see Wang, *op. cit.*, 53–6.

[8] SEW, I, 79; TMB, 43. (Ingarden here refers to a similar distinction made by H. Conrad-Martius, in her *'Zur Ontologie und Erscheinungslehre der realen Aussenwelt'*, *Jahrbuch für Philosophie und phänemonologische Forschung*, III, 1916.)

[9] SEW, I, 79.

[10] In section 6 below we shall make explicit the distinction between relative and absolute existential self-basis. In the context of mathematics self-basis or stability is a moment possessed by objects relative to particular 'mathematical communities'. This concept of mathematical community requires its own phenomenological explication for which Husserl's analyses of intersubjectivity in general provide a starting point.

[11] The notion of concretisation is developed by Ingarden in his *Das Literarische Kunstwerk*, Halle, 1931, referred to as LKW, English translation: *The Literary Work of Art*, Evanston, 1973, below: LWA. See also Section 5 of the present paper.

[12] A far greater number of possibilities and therefore also a more subtle framework result when we introduce the further moments of existential separateness/inseparateness, and existential self-dependence/contingency (see SEW, I, 115–23; TMB, 82–91). Thus although in the present paper we have used the word 'object' (*Gegenstand*) in the widest sense, when these moments are incorporated we shall find ourselves able to distinguish between modes of being of mathematical objects-proper, mathematical relations, and mathematical properties and states of affairs. Such differentiations are often crucial within the philosophy of mathematics, e.g. where we wish to assert that a mathematical property such as the property of being an inductive number is autonomous and original, whilst at the same time asserting that natural numbers *sui generis* are mere fictions.

[13] The differences between sub-universes are never simply a matter of magnitude, i.e. of the relative strength of axioms of infinity which hold within given sub-universes. Such differences also concern the *effectivity* of methods which can be used to derive consequences from such axioms. Cf. S. C. Kleene, *Introduction to Metamathematics*, Amsterdam, 1952, *passim*.

[14] *Psychologism*, the (ambiguous) doctrine that mathematical objects are 'mentalistic', seems most adequately to be conceived as a special case of materialism.

[15] Brouwer identifies his doctrine closely with that of Kant. See his paper 'Intuitionism and Formalism', *Bulletin of the American Mathematical Society* 20 (1913) 81–96.

[16] A. Heyting, *Intuitionism: An Introduction*, Amsterdam, 1971, p. 10.

[17] Heyting, *op. cit.*, 101–18.

[18] Cf. L. Wittgenstein, *Remarks on the Foundations of Mathematics*, Oxford, 1964, p. 160, *et passim*.

[19] Michael Dummett's paper on 'Truth', *Proc. Arist. Soc.* 59 (1958–9), contains a discussion of the relation between doctrines of type (iii) and those of type (iv).

[20] See J. M. B. Moss, 'Kreisel's Work on the Philosophy of Mathematics – Realism', in *Logic Colloquium '69*, Amsterdam (1969) 411–38.

[21] *'Über das Unendliche'*, *Mathematische Annalen* 95 (1926) 161–90, English translation in van Heijenoort (ed.), *From Frege to Gödel*, Harvard, 1967, pp. 367–92.

[22] For a discussion of 'Gödel's Theorem' and its implications see J. Myhill, 'Some Philosophical Implications of Mathematical Logic', *Review of Metaphysics* VI (1952) 165–95.

[23] Ingarden also distinguished modes of being of real objects of the past and of the future.

[24] This moment is defined in SEW, I, 113–14; TMB, 124–56.

[25] SEW, I, 259–60; TMB, 79–80: "It seems doubtful that we could succeed in proving that everything we encounter in ideal being exists in its essence of necessity, such that it could not *not* be. On the other hand it also does not seem probable that all ideal objects (especially mathematical objects, logical relations, ideals, ideal qualities) could be regarded as existentially derivative, as created by some original being. ... There is also the possibility of acknowledging *two* different varieties of an extratemporal mode of being: one, in which existential originality would appear and in which there would be primary elements of the realm of ideal objects of a certain type, and a second, which would contain existential derivation."

[26] Reinforced in all his major works, see especially LKW, 381–90; LWA, 356–64.

[27] In *Erfahrung und Urteil*, Hamburg, 1948, English translation, London, 1973; and in *Die Krisis der Europäischen Wissenschaften*, Den Haag, 1954, English translation, Evanston, 1970.

[28] O. Becker, *'Mathematische Existenz'*, *Jahrbuch für Philosophie und phänomenologische Forschung* VIII (1927) 441–809.

[29] Felix Kaufmann, *Das Unendliche in der Mathematik und seine Ausschaltung*, Leipzig and Vienna, 1930.

[30] A similar point is made by S. Bachelard in her study of Husserl's *Formale und Transzendentale Logik*, original French edition, Paris, 1957, English translation: *Husserl's Logic*, Evanston, 1968, p. 122f.

[31] In *Analecta Husserliana*, II, Dordrecht, 1972, p. 109.

[32] Gadamar, *op. cit.*, p. 111.

[33] But note the reservations of Bachelard, *Husserl's Logic*, p. 123.

[34] LKW, 62–71; LWA, 63–71.

[35] The 'noematic layer' of the creative mathematical work parallels, in many respects, the 'stratum of represented objects' which Ingarden distinguished in the structure of the literary work (LKW, LWA, ch. 7), but it cannot be *identified* with the latter. This is because the mathematical work has a *double* object-stratum. The 'higher' stratum has the character of a 'stratum of aspects' in which the 'lower' pre-given 'central core' mathematical objects are received into consciousness. But it cannot be identified, either, with the 'stratum of aspects' as distinguished by Ingarden in the literary work, since it has in turn its own 'higher' stratum of aspects. We might remark that so-called "higher mathematics" deals, in effect, with the objects of higher strata of creative mathematical works.

On the notion of intending 'through' a noema and 'onto' a referent, see Küng, *op. cit.*, p. 20f.

[36] The concept of the mathematical work as a 'borderline case' of the literary work of art is developed in B. Smith, 'The Ontogenesis of Mathematical Objects', *Journal of the British Society for Phenomenology* VI (1975).
[37] Smith, *op. cit.*, Section 2.
[38] Cf. LKW, 376 lines 10–11, LWA, 351 (line 15).
[39] LKW, LWA, ch 13.
[40] Such works, which will be at the centre of our attention in what follows, have a vital role in the advance of mathematics comparable to works of 'revolutionary' as opposed to 'normal' science as these are distinguished by T. S. Kuhn, *The Structure of Scientific Revolutions*, Chicago, 1970. Perhaps the best example of a 'creative' mathematical work would be that of G. Cantor, *Beiträge zur Begründung der Transfiniten Mengenlehre*, 1895–7, reprinted in his *Gesammelte Abhandlungen*, Berlin, 1932, English translation, *Contributions to the Founding of the Theory of Transfinite Numbers*, New York, 1915.
[41] LKW, 380–1; LWA, 346–7.
[42] Cf. LKW, 378; LWA, 353.
[43] *Ibid.*
[44] LKW, 278–93; LWA, 262–275.
[45] This is emphasised by Wittgenstein who, however, arbitrarily asserts that 'schematisation' pervades the whole of mathematics, even, for example, simple addition requires the introduction of what he calls a 'paradigm', cf. his *Remarks*, pp. 1–5, and especially p. 3, "We say, for instance, to someone who uses a sign [say x^2] unknown to us: 'If by "x^2" you mean x^2, then you get this value for y, if you mean $\sqrt{x}$, that one.' Now ask yourself: how does one *mean* the one thing or the other by 'x^2'?".
[46] Cf. Wittgenstein, *op. cit.*, on the need for 'pictures', 'patterns', 'conventions' in our understanding of mathematics, e.g. p. 60, ll. 11.
[47] 'Proofs and Refutations', *British Journal for the Philosophy of Science* 14 (1963) (in four parts).
[48] *Op. cit.*, pp. 296–342, see especially p. 324.
[49] "The comparison with alchemy suggests itself. We might speak of a kind of alchemy in mathematics. It is the earmark of this mathematical alchemy that mathematical propositions are regarded as statements about mathematical objects, – therefore mathematics becomes the *exploration* of these objects?" Wittgenstein, *op. cit.*, p. 142.
[50] Note that this includes concretisations 'absolutised', e.g., in mathematical textbooks, and then also concretisations of the original work as mediated through such textbooks.
[51] Cf. H. Osborne, 'Notes on the Aesthetics of Chess and the Concept of Intellectual Beauty', *British Journal of Aesthetics* 4 (1964) 160–3.
[52] LKW, 127; LWA, 122.
[53] This is argued by van Breda, in *Analecta Husserliana*, II, *op. cit.*, p. 112, "I think personally that the freedom of Othello is extremely relative. There are a lot of things stemming from the transcendental world in Hamlet. The question is the extent of the freedom of the one who is creating poetical works, the one who is making a statue, the one who is making Flemish paintings. In the end, you depend, for instance, on the tools you use."
[54] LKW, LWA, ch. 10.
[55] This status is not confined to literary objects. It can also come to be applied, by history, to individual real objects (compare the English adjective 'Churchillian'), and also to 'higher-order' realities such as democracy, war, the proletariat, etc., and even certain religious 'objects'.
[56] R. F. Grabau, 'Existential Universals', in Edie (ed.), *Invitation to Phenomenology*, Chicago, 1965, pp. 147–60.

[57] *Op. cit.*, p. 155.
[58] *Ibid.*
[59] Especially that of Nicolai Hartmann, *Ethik*, Berlin, 1925, English translation, London, 1932.
[60] Compare the first paragraph of Section 5 above.
[61] LKW, 387, LWA, 362.
[62] *Ibid.*
[63] See note 27 above.