Preprint version of paper in Achille Varzi and Laure Vieu (eds.), *Formal Ontology in Information Systems. Proceedings of the Third International Conference* (FOIS 2004), Amsterdam: IOS Press, 2004, 37–48.

Individuals, Universals, Collections: On the Foundational Relations of Ontology

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Abstract. This paper provides an axiomatic formalization of a theory of foundational relations between three categories of entities: individuals, universals, and collections. We deal with a variety of relations between entities in these categories, including the is-a relation among universals and the part-of relation among individuals as well as cross-category relations such as instance-of, member-of, and partition-of. We show that an adequate understanding of the formal properties of such relations – in particular their behavior with respect to time – is critical for formal ontology. We provide examples to support this thesis from the domain of biomedicine.

1 Introduction

Biomedical ontologies such as the Gene Ontology [13] must be correlated with terminology systems developed for clinical medicine such as GALEN [9], the Foundational Model of Anatomy (FMA) [10], and other terminologies collected in the Unified Medical Language System (UMLS) [2].

A critical requirement for such correlations is the alignment of the fundamental ontological relations these different systems use [7]. These include above all the relations of: (1) subsumption or taxonomic inclusion (human being *is-a* mammal), (2) instantiation (I am an *instance of* human being), (3) the individual part-of relation (my heart is *part of* me), (4) the membership relation (this tooth is a *member of* the collection of teeth in my dental arcade), (5) partonomic inclusion between universals (every instance of the universal *human nervous system* is an individual part of some instance of the universal *human being*), (6) the partition-of (or subdivision-of) relation (the collection of my cells forms a *partition of* me).

While these relations are ubiquitous in bioinformatics ontologies and terminologies, they are unfortunately not always clearly distinguished. Moreover, as is pointed out in [11], the relations of class subsumption and partonomic inclusion are often treated in ways which are inconsistent or otherwise problematic. Several authors have discussed *part-of* and *is-a* relations in the context of biomedical ontologies, e.g., [5, 9, 4]. However, a unifying framework incorporating all the relations mentioned above has not yet been supplied. In this paper, we provide an axiomatic theory that is designed to fill this gap.

2 Methodology and basic categories

We here focus on independent endurants, entities such as molecules, cells, and organisms, which survive self-identically through time while undergoing changes of various sorts. Dependent endurants (qualities, functions, powers, attributes) and perdurants (events, actions, processes) are treated for example in [6].

We first distinguish three disjoint sorts of entities which are assumed in our treatment of endurants: (i) individual endurants (you, me, your heart, my left hand); (ii) endurant universals (*human being*, *heart*, *oxygen*); and (iii) collections of individual endurants (the collection of grocery items in my shopping bag, the collection of cells in your body, the collection of all human beings existing at a given time).

Note that individuals, universals, and collections have different temporal properties. Individuals can gain and lose parts. (For example, organisms gain and lose cells.) Universals gain and lose instances. (For example, the universal *human being* gains or loses instances every time a person is born or dies.) Collections, on the other hand, are identified through their members and thus cannot have different members at different times.

While individuals are tied to universals through the instantiation relation, certain collections are tied to universals through the extension-of relation. The *extension* of a universal at a given time is the collection of individuals which instantiate the universal at that time. Not every universal has an extension at every time – think of extinct species. Not every collection is the extension of a universal: think of the collection of the cells in your body or the collection of human beings in Saarbruecken.

Besides the extensions of universals, other important collections are those that consist of disjoint parts of an individual y which jointly sum up to y. We call such collections *partitions* of y. Partitions are critical for the representation of anatomical knowledge [7]. An example is a partition of a human body into its constituent cells. Many partitions consist of fiat parts [12] like the head, neck, torso, and limbs of a human body or the right and left hemispheres of a brain.

Given the three categories – individuals, universals, and collections – we can distinguish the following relations according to the kinds of entities they relate:

first term	second term	relation	symbol
Individual	Individual	individual-part-of	$\leq$
Individual	Universal	instance-of	*Inst*
Individual	Collection	member-of	$\in$
Universal	Universal	taxonomic inclusion (*is-a*)	$\sqsubseteq$
Universal	Universal	partonomic inclusion of universals	$\lhd$
Collection	Universal	extension-of	*Ext*
Collection	Collection	partonomic inclusion of collections	$\preceq$
Collection	Individual	partition-of	*Pt*

In the remainder of this paper, we give an axiomatic characterization of these relations and of the interrelationships between them.

3 Universals and individuals

We present the theory in a sorted first-order predicate logic with identity. All quantification is restricted to a single sort. Restrictions on quantification will be understood by conventions on variable usage. We use the letters t, t_1, t_2 as variables ranging over instants of time; w, x, y, z as variables ranging over individuals; c, d, e, g as variables ranging over universals; and p, q, r as variables ranging over collections. Leading universal quantifiers are omitted. Labels for axioms begin with 'A', labels for theorems begin with 'T', and labels for definitions begin with 'D'.

3.1 Temporal mereology of individuals

We here develop a temporal version of mereology based on the ternary primitive $\leq$, where $x \leq_t y$ is interpreted as: individual x is part of individual y at time-instant t. For example this blood cell was part of my body yesterday, but it is not a part of my body now. We will refer to $\leq$ as the *individual parthood* relation in order to distinguish it from the partonomic relations that we will introduce later for collections and universals. We define the relations of proper parthood and overlap among individuals in the usual way:

$$D_< \quad x <_t y \equiv x \leq_t y \land \neg(x = y) \qquad\qquad D_{O_t} \quad O_t\, xy \equiv (\exists z)(z \leq_t x \land z \leq_t y)$$

We use $\leq$ to distinguish the time-instants at which individuals exist. We will say that x exists at time t (symbolically $\boldsymbol{E}\, xt$) if and only if x is a part of itself at t. We then add an axiom ensuring that for every individual there is some time at which it exists.

$$D_E \quad \boldsymbol{E}\, xt \equiv x \leq_t x \qquad\qquad AM1 \quad (\exists t)\boldsymbol{E}\, xt$$

We add the following axioms. At each time instant, the individual parthood relation is anti-symmetric and transitive (AM2–3). If x is a part of y at t then both x and y exist at t (AM4). We can then prove that x is a part of itself whenever x is either itself a part or has a part (TM1).

$$AM2 \quad x \leq_t y \land y \leq_t x \to x = y \qquad\qquad AM4 \quad x \leq_t y \to \boldsymbol{E}\, xt \land \boldsymbol{E}\, yt$$
$$AM3 \quad x \leq_t y \land y \leq_t z \to x \leq_t z \qquad\qquad TM1 \quad (\exists y)(x \leq_t y \lor y \leq_t x) \to x \leq_t x$$

Finally we require that if everything that overlaps x at t also overlaps y at t then x is a part of y at t (AM5).

$$AM5 \quad (z)(O_t\, zx \to O_t\, zy) \to x \leq_t y,$$

Using AM5, we can prove that two individuals x and y are identical if and only if they overlap the same individuals at any instant t:

$$TM2 \quad x = y \leftrightarrow (z)(O_t\, zx \leftrightarrow O_t\, zy)$$

We can also derive the so-called weak supplementation principle (WSP). This tells us that if x is a proper part of y at t then there exists an individual z such that z is a proper part of y at t and x and z do not overlap at t:

$$TM3 \quad x <_t y \to (\exists z)(z <_t y \land \neg O_t\, zx)$$

Consequently there cannot be an individual with a single proper part.

We now define the sum of ϕ-ers at t, i.e., the sum of all individuals satisfying the formula ϕ at t, as the individual z such that w overlaps z at t if and only if there exists a ϕ-er x which overlaps w at t. We write $z\sigma_t\phi$ in order to signify that z is the sum of all ϕ-ers at t.

$$(D_{\sigma_t}) \quad z\sigma_t\phi \equiv (w)(O_t\, wz \leftrightarrow (\exists x)(\phi\, x \land O_t\, xw))$$

Using AM2 and AM4 we can prove that sums are unique (TM4).

$$TM4 \qquad (z_1\sigma_t\phi \land z_2\sigma_t\phi \to z_1 = z_2)$$

We call the theory formed by AM1–5 temporal extensional mereology TEM.

As pointed out for example in [1], the individual parthood relations employed in biomedical ontologies often satisfy not only axioms the AM1–5 of TEM but also the no-partial-overlap principle (NPO):

$$NPO \quad O_t\, xy \rightarrow x \leq_t y \vee y <_t x$$

NPO tells us that if two individuals overlap, then one is a part of the other. But NPO is too strong for many purposes. We normally assume that individuals such as the front half and the right half of my desk or my pelvis and my vertebral column can properly overlap. Though we are interested in the consequences of NPO, we want to keep our theory as general as possible and therefore we do not add NPO as an axiom to our theory. Rather, we will explicitly mark those consequences that follow from adding NPO to TEM. A model of TEM that does satisfy NPO has the structure of a tree.

3.2 The is-a relation

Recall that the variables c, d, e, g range over universals (*human being*, *cell*, *organ*, and so forth), which are here assumed to form hierarchies ordered by the *is-a*sub-universal relation. This relation holds between two universals when the first is subsumed by the second. For example, the universal *human being* is a sub-universal of (is subsumed by) the universal *mammal*. We use the symbol '$\sqsubseteq$' for this relation. Like the identity relation and unlike the individual part-of relation, $\leq$, the sub-universal relation, $\sqsubseteq$, is atemporal.

We define the relations proper sub-universal ($\sqsubset$) and taxonomic overlap ($O_{\sqsubseteq}$) (sharing a common sub-universal) in terms of $\sqsubseteq$. Consider, for example, the universals *protein* and *hormone*. Neither is a sub-universal of the other, but both have *insulin* as a (proper) sub-universal and therefore stand in the taxonomic overlap relation. We also introduce a predicate for the root universal (*Root*), a universal which subsumes all universals (D_{root}).

$$D_{\sqsubset} \quad c \sqsubset d \equiv c \sqsubseteq d \wedge \neg(c = d)$$
$$D_{O_{\sqsubseteq}} \quad O_{\sqsubseteq}\, cd \equiv (\exists e)(e \sqsubseteq c \wedge e \sqsubseteq d)$$
$$D_{root} \quad \mathit{Root}\, c \equiv (g)(g \sqsubseteq c)$$

The *is-a* relation $\sqsubseteq$ is governed by the axioms (AU1–4), which are the atemporal analogues of the axioms AM2–5 of TEM and also by AU5, which postulates the existence of a root-universal. In the most general case, this will be the universal *substance* (or *independent endurant*). Theorem TU1 tells us that the root universal is unique.

$$AU1 \quad c \sqsubseteq c$$
$$AU2 \quad (c \sqsubseteq d \wedge d \sqsubseteq c) \rightarrow c = d$$
$$AU3 \quad (c \sqsubseteq d \wedge d \sqsubseteq e) \rightarrow c \sqsubseteq e$$
$$AU4 \quad (e)(O_{\sqsubseteq}\, ec \rightarrow O_{\sqsubseteq}\, ed) \rightarrow c \sqsubseteq d$$
$$AU5 \quad (\exists c)\mathrm{Root}\, c$$
$$TU1 \quad \mathrm{Root}\, c \wedge \mathrm{Root}\, d \rightarrow c = d$$

We call the theory formed by the axioms AU1–5 extensional universal mereology, EUM.

As is seen in the the *protein–hormone* example above, universals can properly overlap (i.e., stand in the taxonomic overlap relation without either being a sub-universal of the other). However, as has been pointed out by authors such as [4] and [8] it is often desirable to isolate sub-ontologies which are limited to universals that form tree structures with respect to the subsumption relation. For example, [8] proposes to have two separate sub-ontologies: one of body-substance universals (such as *protein*) distinguished according to structure and a second of body-substance universals (such as *hormone*) distinguished according to function. Taken separately, each sub-ontology has a tree structure. The phenomenon of multiple inheritance arises only when they are combined.

These isolated tree-structured sub-ontologies satisfy in addition to AU1–5 the following no-partial-overlap principle:

$$NPO_U \quad O_{\sqsubseteq}\, cd \rightarrow (c \sqsubseteq d \vee d \sqsubset c)$$

Notice, again, that we do not add NPO$_U$ as an axiom to our theory. It can be regarded rather merely as a design principle ensuring that certain sub-ontologies are easy for human beings to maintain. We will later discuss some consequences of adding NPO$_U$ to EUM which reveal certain other benefits of tree-structured sub-ontologies for universals.

3.3 Instantiation

The relation of instantiation is time-dependent and it holds between individuals and universals (in that order). We write *Inst* xct to signify that the individual x instantiates the universal c at time-instant t. Since universals and individuals are disjoint sorts, instantiation is irreflexive and asymmetric.

Axioms (AI1–2) establish the relationship between instantiation and the *is-a* relation. AI1 tells us that if c is a sub-universal of d then the instances of c at any given time are also instances of d at that time. AI2 says that if two universals share an instance x at some time t then the universals taxonomically overlap. AI3 tells us that if x is an instance of a universal at t then x exists at t.

$$AI1 \quad c \sqsubseteq d \rightarrow (\textit{Inst}\, xct \rightarrow \textit{Inst}\, xdt)$$
$$AI2 \quad (\textit{Inst}\, xct \wedge \textit{Inst}\, xdt) \rightarrow O_{\sqsubseteq}\, cd$$
$$AI3 \quad \textit{Inst}\, xct \rightarrow \textit{E}\, xt$$

We then add axioms stating that every universal is instantiated at some time (AI4); and that at every time at which an individual exists it is an instance of some universal (AI5):

$$AI4 \quad (\exists t)(\exists x)(\textit{Inst}\, xct)$$
$$AI5 \quad \textit{E}\, xt \rightarrow (\exists c)\textit{Inst}\, xct$$

Finally we add an axiom stating that if two universals have the same instances at every time then they are identical (AI6):

$$AI6 \quad (t)(x)(\textit{Inst}\, xct \leftrightarrow \textit{Inst}\, xdt) \rightarrow c = d$$

In the presence of NPO$_U$, i.e., in sub-ontologies in which universals form a tree, we can prove that if two universals share an instance x at some time t then one is a sub-universal of the other:

$$TI1 \qquad NPO_U \vdash (\textit{Inst}\, xct \wedge \textit{Inst}\, xdt) \rightarrow (c \sqsubseteq d \vee d \sqsubset c)$$

Thus, there cannot be universals like *upper human limb* and *left human limb* in such a tree structure, since neither universal is a sub-universal of the other, though both have my left arm as an instance. We can however have the universals *upper human limb*, *left upper human limb*, and *right upper human limb* in a tree-structured sub-ontology (and these are in fact included in the FMA).

Notice that AI2 is consistent with the existence of individuals that instantiate disjoint universals at different times, as in the case of *child* and *adult*, or *larva* and *butterfly*. Standard biomedical ontologies, including the Gene Ontology (GO), because they have no facility for reasoning about time, fail to do justice to the existence of universals connected to each other by the fact that the instances of one develop from instances of the other.

In our framework, we can very easily define an *EvolvesFrom* relation among universals. Given a linear ordering $\ll$ on time instants (where $t_1 \ll t_2$ means: t_1 is earlier than t_2) we can define:

$$D_{\textit{EvolvesFrom}} \qquad \textit{EvolvesFrom}\, dc \equiv (t_1)(x)(\textit{Inst}\, xdt_1 \rightarrow (\exists t_2)(t_2 \ll t_1 \wedge \textit{Inst}\, xct_2))$$

This tells us that universal d evolves from universal c if and only if every instance of d was at some earlier time an instance of c. For example, *Adult* evolves from *child* and *Butterfly* evolves from *larva*.

We can also define the more general relation *GenFrom* between universals where *GenFrom* dc holds if and only if whenever x is an instance of d, x either currently is or was earlier a part of some instance of c:

$$D_{\textit{GenFrom}} \qquad \textit{GenFrom}\, dc \equiv (t_1)(x)(\textit{Inst}\, xdt_1 \rightarrow (\exists t_2)(\exists y)((t_2 \ll t_1 \vee t_2 = t_1) \wedge \textit{Inst}\, yct_2 \wedge x \leq_{t_2} y))$$

For example, *apple* is generated from *apple tree*, *ovum* is generated from *female organism*, *sperm* is generated from *male organism*, and *human skull* is generated from *human being*.

4 Collections

We are interested not only in individuals and universals but also in *collections* of individuals, for instance in collections of infected cells or in collections of infected persons. In particular, we need to consider two special sorts of collections: those that form partitions of individuals and those that are the extensions of universals at given times.

We use $\in$ to stand for the member-of relation between individuals and collections, and p, q, r for variables ranging over collections. Sometimes it is useful to refer to a finite collection p by listing its members, $x_1, \ldots, x_n$, thus: $\{x_1, \ldots, x_n\}$. Since collections and individuals are disjoint sorts, $\in$ is irreflexive and asymmetric.

Two collections are identical if and only if they have the same members (AC1). This makes explicit the extensional character of collections. We also require that all collections are non-empty (AC2):

$$AC1 \qquad p = q \leftrightarrow (x)(x \in p \leftrightarrow x \in q) \qquad\qquad AC2 \qquad (\exists x)(x \in p)$$

We do, however, allow singleton collections which have only one member. For example, we assume that there can be a collection p whose only member is the individual Fred. Notice that, in this case, p is not identical to Fred – Fred, unlike p, is a human being with parts like a head, arms, fingers, cells, etc. that can change over time. We say that the collection p is a sub-collection of the collection q, $p \subseteq q$, if and only if every member of p is also a member of q:

$$D_{\subseteq} \qquad p \subseteq q \equiv (x)(x \in p \rightarrow x \in q)$$

We can then prove that $\subseteq$ is reflexive, antisymmetric, and transitive. Thus, $\subseteq$ is a partial ordering.

We finally add an axiom schema which states that if some member of p satisfies the formula ϕ, then there is a sub-collection q of p whose members are those members of p that satisfy ϕ (AC3):

$$AC3 \quad (\exists x)(x \in p \wedge \phi\, x) \rightarrow (\exists q)(x)(x \in q \leftrightarrow (x \in p \wedge \phi\, x))$$

We define relations that allow us to distinguish between the times when all, some, or none of the members of a collection exist. We say that a collection p is *fully present* at t iff all its members exist at t (D_{FP}). p is *partly present* at t iff some of its members exist at t (D_{PtP}). Finally, p is *non-present* at t iff none of its members exist at t (D_{NP}).

$$D_{FP} \quad \textit{\textbf{FP}}\, pt \equiv (x)(x \in p \rightarrow \textit{\textbf{E}}\, xt) \qquad\qquad D_{NP} \quad \textit{\textbf{NP}}\, pt \equiv \neg \textit{\textbf{PtP}}\, pt$$

$$D_{PtP} \quad \textit{\textbf{PtP}}\, pt \equiv (\exists x)(x \in p \wedge \textit{\textbf{E}}\, xt)$$

Consider the collection p of cells which are in my body at this instant. p as an atemporal entity cannot cease to exist, but by the time you read this sentence, many of the cells that form p will no longer exist. Thus, p is now fully present but will be only partly present next week. In 500 years, p will be non-present.

Notice that, since every collection has at least one member (AC2), full presence is a special case of partial presence. In other words, if p is fully present at t, then p is also partly present at t.

4.1 Partitions

The individuals in a given collection may overlap. (Consider for example a collection which includes my body and my heart.) Some collections, however, are formed by individuals which are at a given time pair-wise disjoint. For example, all individuals in the collection of cells currently in my body are currently disjoint. All individuals in the current extension of the universal *planet* are currently disjoint. We call a collection p *discrete* at time t if and only if p is partly present at t and its members do not overlap at t:

$$D_D \qquad D\, pt \equiv \textit{\textbf{PtP}}\, pt \wedge (x)(y)(x \in p \wedge y \in p \wedge O_t\, xy \rightarrow x = y)$$

Notice, that the same collection can be fully present and non-discrete today but fully present and discrete tomorrow. Think, for example, of Siamese twins before and after separation. A collection can also be non-discrete today and discrete tomorrow if members that overlap today cease to exist tomorrow.

A collection p *partitions* the individual y at time t ($PT\ pyt$) if and only if (i) p is fully present at t, (ii) the members of p jointly sum up to y at t, and (iii) p is discrete at t:

$$D_{PT} \qquad PT\, pyt \equiv \textit{\textbf{FP}}\, pt \wedge y\sigma_t[x \in p] \wedge D\, pt$$

We can prove that if p partitions y at t and x is a member of p then x is a part of y at t (TPT1). We can also prove that all fully present singleton collections partition their only member (TPT2).

$$TPT1 \quad PT\, pyt \wedge x \in p \rightarrow x \leq_t y \qquad\qquad TPT2 \qquad \textit{\textbf{FP}}\, \{x\}t \rightarrow PT\, \{x\}xt$$

Examples of partitions are anatomic subdivisions such as: the subdivision of my body into my head, my neck, my trunk, and my limbs; the subdivision of my brain into its right and left hemispheres.

A single individual can be partitioned by multiple partitions. The following example (which we refer to later as (Ex1)) presents the collections $p_1, \ldots, p_4$, which partition the individual Fred in different ways at some given time:

$p_1 = \{$Fred$\}$
$p_2 = \{$Fred's head, Fred's neck, Fred's torso, (the mereolog. sum of) Fred's limbs$\}$
$p_3 = \{$Fred's head, Fred's neck, Fred's torso, Fred's left leg, Fred's right leg, Fred's left arm, Fred's right arm$\}$ (Ex1)
$p_4 = \{$Fred's head, Fred's neck, Fred's torso, Fred's left leg, Fred's right leg, Fred's left upper arm, Fred's left lower arm, Fred's left hand, Fred's right arm$\}$

4.2 Collections as Extensions of Universals

We now define the extension relation between universals and collections. The collection p is the *extension* of the universal c at time t, *Ext pct*, if and only if the members of p are those individuals that instantiate c at t:

$$D_{Ext} \quad \textit{Ext}\, pct \equiv (x)(x \in p \leftrightarrow \textit{Inst}\, xct)$$

We can prove: a universal has at most one extension at a time (TE1); if a universal has an extension at t, then it has an instance at t (TE2); and if p is the extension of a universal at time t, then p is fully present at t (TE3).

$$TE1 \quad \textit{Ext}\, pct \wedge \textit{Ext}\, qct \rightarrow p = q$$
$$TE2 \quad (\exists p)\textit{Ext}\, pct \rightarrow (\exists x)(\textit{Inst}\, xct)$$
$$TE3 \quad \textit{Ext}\, pct \rightarrow \textit{FP}\, pt$$

We continue by adding an axiom stating that if the universal c has an instance at t then there is some collection which is the extension of c at t:

$$AE1 \quad (\exists x)(\textit{Inst}\, xct) \rightarrow (\exists p)(\textit{Ext}\, pct)$$

From AI6 it follows that two universals are identical if and only if they have identical extensions at every time:

$$TE4 \quad c = d \leftrightarrow (t)(p)(\textit{Ext}\, pct \leftrightarrow \textit{Ext}\, pdt)$$

As we already noted, some collections are never the extensions of a universal. For example, the collection formed by you and me is not the extension of any universal at any time (though it is currently a proper sub-collection of the extension of the universal *human being*).

There is clearly a correspondence between the sub-universal structure of universals and the sub-collection structure of their extensions. We can indeed prove that if c is a sub-universal of d and c is instantiated at t, then the extension of c at t is a sub-collection of the extension of d at t:

$$TE5 \quad (\exists x)(\textit{Inst}\, xct) \rightarrow (c \sqsubseteq d \rightarrow (\exists p)(\exists q)(\textit{Ext}\, pct \wedge \textit{Ext}\, qdt \wedge p \subseteq q))$$

Notice however that there may be points in time where distinct universals have identical extensions. For example, if at some point in time all mammals except whales are extinct, then the extensions of *mammal* and *whale* at that time are identical even though the corresponding universals are distinct.

As was shown above, the sub-collection relation is a partial ordering. This leaves open the possibility that there are collections that partially overlap in the sense that they share members but neither is a sub-collection of the other. For those collections that are extensions of universals in tree-forming sub-ontologies (i.e., where the no-partial-overlap principle (NPO_U) holds), we can prove that if two such extensions share a member then one is a sub-collection of the other:

$$TE6 \quad NPO_U \vdash [(\exists x)(x \in p \wedge x \in q) \wedge (\exists c)\textit{Ext}\, pct \wedge (\exists d)\textit{Ext}\, qdt] \rightarrow (p \subseteq q \vee q \subseteq p)$$

We can also prove (even without NOP_U) that at every time t the extension of the root universal has the extensions of all other universals as sub-collections:

$$TE7 \quad (\textit{Root}\, c \wedge \textit{Ext}\, pct \wedge \textit{Ext}\, qdt) \rightarrow q \subseteq p$$

We now can see that the sub-collection relation, when restricted to extensions of tree-forming universals, also generates a tree-structure. This is because, on the domain of extensions of tree-forming universals, the sub-collection relation is at every time a partial ordering (i) for which the no-partial-overlap principle holds (TE6) and (ii) which has a unique root collection (TE7). Notice, however, that the two structures are not necessarily identical, since there may be universals with no extensions at t or distinct universals which share the same extension at t.

5 Partonomic inclusion

The domain of individuals is governed by temporal mereology, i.e., by the relation $\leq$. Partonomic inclusion is a relation between collections which is determined by the $\leq$ relations among the members of collections. The time-dependent character of the individual part-of relations implies that partonomic inclusion is time-dependent too. As (Ex2) consider the relation of partonomic inclusion between the collections p^* and q^*:

$$\begin{aligned} p^* &= \{\text{my left hand, my left arm, my right foot}\} \\ q^* &= \{\text{my left arm, my right leg}\} \end{aligned} \tag{Ex2}$$

Assuming that all my limbs, hands, etc. are parts of my body at t, every member of p^* is a part of some member of q^* at t and every member of q^* has some member of p^* as part at t. We then say that the relation of partonomic inclusion holds between p^* and q^* at time t.

To see the time-dependent character of partonomic inclusion, imagine that I have an accident at time t_1 and lose my right foot. At t_1, my right foot is no longer part of my right leg. Therefore, there is a member of p^* (my right foot) which is not a part of any member of q^* at t_1. Consequently, the relation of partonomic inclusion does not hold between p^* and q^* at time t_1.

Formally, we say that the collection p is *partonomically included* in the collection q at t, $p \preceq_t q$, if and only if (i) for every member x of p there is a member y of q such that x is part of y at t, and (ii) for every member y of q there is a member x of p such that x is a part of y at t (D$_{\preceq_t}$):

$$\begin{aligned} D_{\preceq_t} \qquad p \preceq_t q \equiv\ & (x)(x \in p \rightarrow (\exists y)(y \in q \wedge x \leq_t y)) \wedge \\ & (y)(y \in q \rightarrow (\exists x)(x \in p \wedge x \leq_t y)) \end{aligned}$$

We can then prove that at a fixed time, $\preceq$ is transitive (TPI1). We can also prove that if p is partonomically included in q at t then both p and q are fully present at t (TPI2) and that a collection is partonomically included in itself at t if and only if it is fully present at t (TPI3).

$$TPI1 \quad p \preceq_t q \wedge q \preceq_t r \rightarrow p \preceq_t r$$
$$TPI2 \quad p \preceq_t q \rightarrow (\textit{FP}\, pt \wedge \textit{FP}\, qt) \qquad\qquad TPI3 \quad p \preceq_t p \leftrightarrow \textit{FP}\, pt$$

Partonomic inclusion is not antisymmetric – even on the sub-domain of collections that are fully present at a given time – and therefore it is not a partial ordering relation. However, we can prove that at a fixed time $\preceq$ is antisymmetric on the sub-domain of discrete collections:

$$TPI4 \quad D\, pt \wedge D\, qt \wedge p \preceq_t q \wedge q \preceq_t p \rightarrow p = q$$

5.1 Partonomic inclusion among partitions

We define the relation of *strong* partonomic inclusion $\preccurlyeq$, a stronger version of $\preceq$, as follows. Collections p and q stand in relation $\preccurlyeq$ at t if and only if (i) p and q are discrete at t, (ii) for any member y of q there is a sub-collection r of p such that r partitions y at t, and (iii) for any member x of p there is a sub-collection r of p with $x \in r$ and an individual $y \in q$ such that r partitions y at t.

$$D_{\preccurlyeq_t} \quad p \preccurlyeq_t q \equiv D\,pt \wedge D\,qt \wedge (y)(y \in q \rightarrow (\exists r)(r \subseteq p \wedge Pt\,ryt)) \wedge$$
$$(x)(x \in p \rightarrow (\exists r)(\exists y)(x \in r \wedge r \subseteq p \wedge y \in q \wedge Pt\,ryt))$$

Example (Ex1) gives four different partitions of Fred. It is easy to verify that, for any time t at which these collections partition Fred, each collection is *strongly* partonomically included in those above it: $p_4 \preccurlyeq_t p_3 \preccurlyeq_t p_2 \preccurlyeq_t p_1$. But in Example 2, $p^* \preccurlyeq_t q^*$ does not hold even for times t at which both p^* and q^* are fully present. This is because: i) whenever my left hand exists p^* is not discrete, ii) there is no sub-collection of p^* that ever partitions my right leg (a member of q^*), and iii) no sub-collection of p^* that includes my left hand (a member of p^*) ever partitions any member of q^*.

We can prove that strong partonomic inclusion implies partonomic inclusion:

$$TPO4 \qquad p \preccurlyeq_t q \rightarrow p \preceq_t q$$

Example 2 shows that the implication in the other direction does not hold.

We can also derive the following theorems: If p is strongly partonomically included in q at t, then both p and q are fully present at t (TPO5). p is strongly partonomically included in itself at t if and only if p is discrete and fully present at t (TPO6). If p is strongly partonomically included in q at t and q is strongly partonomically included in r at t, then p is strongly partonomically included in r at t (TPO7). If p is strongly partonomically included in q at t and q is strongly partonomically included in p at t, then p and q are identical (TPO8). If p is strongly partonomically included in q at t, then p partitions x at t if and only if q partitions x at t (TPO9).

$$TPO5 \quad p \preccurlyeq_t q \rightarrow (\mathit{FP}\,pt \wedge \mathit{FP}\,qt)$$
$$TPO6 \quad p \preccurlyeq_t p \leftrightarrow D\,pt \wedge \mathit{FP}\,pt$$
$$TPO7 \quad p \preccurlyeq_t q \wedge q \preccurlyeq_t r \rightarrow p \preccurlyeq_t r$$
$$TPO8 \quad p \preccurlyeq_t q \wedge q \preccurlyeq_t p \rightarrow p = q$$
$$TPO9 \quad p \preccurlyeq_t q \rightarrow (x)(PT\,pxt \leftrightarrow PT\,qxt)$$

5.2 Universal parthood

We now introduce a partonomic inclusion relation for universals in order to do justice to the way partonomy relations are used, albeit under different names, in ontologies such as the FMA, GALEN, and GO. For example assertions like: *human head* part-of *human being*, *nucleus* part-of *cell*, *tooth* part-of *dental arcade*, *aortic bitucahein* part-of *abdominal aorta*, etc., all claim that some sort of partonomic relation holds between universals.

We define a partonomic inclusion relation $\lhd$ between *universals*, as follows. The universal c is partonomically included in the universal d if and only if c and d have extensions at the same times and, at those times, the extension of c is partonomically included in the extension of d:

$$D_{\lhd} \quad c \lhd d \equiv (t)[((\exists p)\mathit{Ext}\,pct \leftrightarrow (\exists q)\mathit{Ext}\,qdt) \wedge (p)(q)(\mathit{Ext}\,pct \wedge \mathit{Ext}\,qdt \rightarrow p \preceq_t q)]$$

We can prove that $\lhd$ is reflexive and transitive. We can also prove the following equivalence: c is partonomically included in d if and only if (i) any instance x of c at t is a part of some instance of d at t and (ii) any instance y of d at t has some instance of c as a part at t (TUP1).

$$TUP1 \quad c \lhd d \leftrightarrow (x)(t)(\mathit{Inst}\,xct \rightarrow (\exists y)(\mathit{Inst}\,ydt \wedge x \leq_t y) \wedge$$
$$(y)(t)(\mathit{Inst}\,ydt \rightarrow (\exists x)(\mathit{Inst}\,xct \wedge x \leq_t y)$$

Thus, for example, the universal *human nervous system* is partonomically included in the universal *human being* – at any given time, each human nervous system is part of some human being and every human being has a human nervous system as one of its parts.

However, as defined above, $\lhd$ does not capture many examples of partonomy listed in the FMA, Galen, GO, and other ontologies. A severed human head is an instance of *human head* that is no longer part of a human being. Also, not every tooth is part of a dental arcade and not every cell has a nucleus as one of its parts. Thus, the following do NOT hold:
human head $\lhd$ *human being*, *tooth* $\lhd$ *dental arcade*, *nucleus* $\lhd$ *cell*.

One strategy for extending our analysis to better fit these kinds of cases is to distinguish different varieties of universal partonomy in terms of the relations already available in our theory. Similar to [11] we could, for example, introduce the relations $\lhd_1$ and $\lhd_2$ as follows: $c \lhd_1 d$ means 'at any given time, every instance of universal c is included in some instance of universal d' ($D_{\lhd_1}$); and $c \lhd_2 d$ means 'at any given time, every instance of universal d has an instance of universal c as a part' ($D_{\lhd_2}$).

$$D_{\lhd_1} \quad c \lhd_1 d \equiv (x)(t)(\mathit{Inst}\, xct \rightarrow (\exists y)(\mathit{Inst}\, ydt \wedge x \leq_t y)$$
$$D_{\lhd_2} \quad c \lhd_2 d \equiv (y)(t)(\mathit{Inst}\, ydt \rightarrow (\exists x)(\mathit{Inst}\, xct \wedge x \leq_t y)$$

Thus, *human head* $\lhd_2$ *human being*, but not *human head* $\lhd_1$ *human being* or *human head* $\lhd$ *human being*.

It is easy to see that both $\lhd_1$ and $\lhd_2$ are reflexive and transitive and that the following equivalence holds.

$$TUP2 \quad c \lhd d \leftrightarrow (c \lhd_1 d \wedge c \lhd_2 d)$$

A second strategy, which may be used either alone or in combination with the first strategy, is to introduce a relation that distinguishes *normal* instances of a given universal. The idea is that, in addition to *Inst*, we have also the primitive relation Inst_N where $\mathit{Inst}_N\ xct$ means: x is a normal instance of universal c at instant t.

The question of exactly how Inst_N should be axiomatized and interpreted is a difficult one which goes beyond the scope of this paper. We will say only that we expect that the interpretation of Inst_N will depend in part on the research context at hand. When we are dealing with the anatomy and function of the human digestive system, only human beings whose digestive track is in proper working order will count as normal instances of *human being*. But whether or not these human beings are flat-footed, near-sighted, or missing a finger is irrelevant in this context. In another context, different criteria may be used to distinguish normal instances of *human being*.

Using Inst_N, we could define the following normalized version of our original universal partonomy relation. (If desired, normalized versions of $\lhd_1$ and $\lhd_2$ could also be defined.)

$$D_{\lhd_N} \quad c \lhd_N d \equiv (x)(t)(\mathit{Inst}_N\, xct \rightarrow (\exists y)(\mathit{Inst}\, ydt \wedge x \leq_t y) \wedge$$
$$(y)(t)(\mathit{Inst}_N\, ydt \rightarrow (\exists x)(\mathit{Inst}\, xct \wedge x \leq_t y)$$

Thus, although *human head* $\lhd$ *human being*, *tooth* $\lhd$ *dental arcade*, and *nucleus* $\lhd$ *cell* do not hold, given appropriate interpretations of Inst_N, the following should hold:
human head $\lhd_N$ *human being*, *tooth* $\lhd_N$ *dental arcade*, *nucleus* $\lhd_N$ *cell*.

6 Conclusions

We distinguished three categories of entities: individuals, universals, and collections, and provided an axiomatic theory that formalizes relations between the entities in these categories in such a way as to make explicit their different temporal behavior. This work is designed as a first step towards an ontological framework which can do justice to temporalized versions of taxonomic and partonomic inclusion, relations widely used in current bio-medical ontologies.

Further work on this topic will include a suitable axiomatization for the normal-instantiation relation $Inst_N$. Another important direction for further work is to define additional relations among universals in terms of containment, connection, and other spatial relations among individuals [3]. In ontologies such as the FMA, we find not only assertions that partonomic relations hold among certain universals, but also assertions such as *stomach* contained-in *abdominal cavity* or *stomach* continuous-with *esophagus*. An analysis of such assertions requires counterparts of $\lhd$, $\lhd_1$, $\lhd_2$, or $\lhd_N$ which are defined using individual containment or connection relations.

Acknowledgements

This work was written under the auspices of the Wolfgang Paul Program of the Alexander von Humboldt Foundation, the EU Network of Excellence in Semantic Datamining, and the Volkswagen Foundation Project "Forms of Life".

References

[1] T. Bittner. Axioms for parthood and containment relations in bio-ontologies. In O. Hahn, editor, *Proceedings of KR-MED 2004: First International Workshop on Formal Biomedical Knowledge Representation*, 2004.

[2] O. Bodenreider. The unified medical language system (UMLS): integrating biomedical terminology. *Nucleic Acids Res*, 2004.

[3] M. Donnelly. A formal theory for reasoning about parthood, connection, and location. *Journal of Artificial Intelligence*, to appear.

[4] N. Guarino and C. Welty. A formal ontology of properties. In R. Dieng and O. Corby, editors, *Proceedings of EKAW-2000: The 12th International Conference on Knowledge Engineering and Knowledge Management*, LNCS. Spring-Verlag, 2000.

[5] U. Hahn, S. Schulz, and M. Romacker. Partonomic reasoning as taxonomic reasoning in medicine. In *Proceedings of the 16th National Conference on Artificial Intelligence and 11th Innovative Applications of Artificial Intelligence Conference*, pages 271–276, 1998.

[6] M. Masolo, S. Borgo, A. Gangemini, N. Guarino, A. Oltramari, and A. Oltramari. WonderWeb deliverable D18 – ontology library (final). Technical report, ISTC-CNR, 2004.

[7] J. L. V. Mejino, A. V. Agoncillo, K. L. Rickard, and C. Rosse. Representing complexity in part-whole relationships within the Foundational Model of Anatomy. In *Proceedings of the American Medical Informatics Association Fall Symposium*, 2003.

[8] R. Rector. Modularization of domain ontologies implemented in description logics and related formalisms including OWL. In *Proceedings of the international conference on Knowledge capture*, pages 121–128, 2003.

[9] J. Rogers and A. Rector. GALEN's model of parts and wholes: experience and comparisons. In *Proceedings of the AMIA Symp 2000*, pages 714–8, 2000.

[10] C. Rosse and J. L. V. Mejino. A Reference Ontology for Bioinformatics: The Foundational Model of Anatomy. *Journal of Biomedical Informatics*, in press, 2003.

[11] B. Smith and C. Rosse. The role of foundational relations in the alignment of biomedical ontologies. In *Proceedings of MedInfo*, 2004.

[12] B. Smith and A.C. Varzi. Fiat and bona fide boundaries. *Philosophy and Phenomenological Research*, 60(2):401–420, 2000.

[13] The Gene Ontology Consortium. Creating the gene ontology resource: Design and implementation. *Genome Res*, 11:1425–1433, 2001.