On geometric nature of numbers and the non-empirical scientific method

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Abstract

We give a brief overview of the evolution of mathematics, starting from antiquity, through Renaissance, to the 19th century, and the culmination of the train of thought of history's greatest thinkers that lead to the grand unification of geometry and algebra. The goal of this paper is not a complete formal description of any particular theoretical framework, but to show how extremisation of mathematical rigor in requiring everything be drivable directly from first principles without any arbitrary assumptions actually leads to relaxing the computational difficulty along with maximal conceptual clarity. With this, we consider a revision of the foundations of elementary geometry and algebra based on the work of Grassmann and Clifford and apply it to conceptual and practical problems of past and present modern mathematics and mathematical physics.

$a \cdot (b \wedge c) = a \cdot bc - a \cdot cb$

"For geometry, you know, is the gate of science, and the gate is so low and small that we can only enter it as a little child. "

-William Kingdon Clifford

'Now it came to me:...the independence of the gravitational acceleration from the nature of the falling substance, may be expressed as follows: In a gravitational field (of small spatial extension) things behave as they do in space free of gravitation,... This happened in 1908. Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning."

Albert Einstein

0.1 Introduction

First principle thinking is often dismissed as philosophy and cast to the margins of science, yet it could not be more essential for practical efficiency.
Especially in the long run, as often the hardest thing in science isn't learning new ideas and concepts but recognizing and unlearning the progress impeding ones.

Einstein had gone through great trouble to learn the mathematics (algebra and geometry) of relativity, and then through even more trouble to unlearn its flawed assumptions. If he had a deeper understanding of the first principles(of the underlying mathematics) his work would have been even more efficient.
Here we take a deep dive into the foundations of elementary geometry and algebra, and the deeper we probe in their fundamental nature, the more hints and clues appear that they are but the two sides of the same mathematical coin. While pure (axiomatic) geometry is superior conceptually, algebraic methods on the other hand offer more computational power. The goal here is to use a precise theoretical method to set strong mathematical foundations that can, together with calculus, form a geometro-algebraic unified theory, which lets us understand geometry from algebra, and algebra from geometry.

We show that this geometric calculus works without any basis and without any axes needed to facilitate the integration of algebraic and geometric methods, because fundamentally ;
Geometry is intrinsically algebraic and algebra intrinsically geometric.

0.2 A brief history

Mathematics in general is the foundation of human civilization and is fundamental to all science, yet it is not considered science by itself. It began with algebra and geometry which first arose as practical observational sciences. Algebra started as the science of counting and quantity while geometry dealt with form and quality (properties of space and objects therein). We will see that they are two aspects of a deeper reality, and glimpses of that realization have appeared throughout history.

Investigations into (platonic) reality naturally started with geometry, which being explicitly visual was always more intuitive to the human mind. Particularly perhaps to a civilization such as ancient Greece, with strong emphasis on clear conceptual understanding and artistic expression. Ancient Greeks defined multiplication, division and even square roots of geometric quantities by geometric (constructive) methods, which in turn invariably followed the laws of algebra.
Euclid for example, in his Elements, described geometrically the multiplication of parallelogram's base and height and its distributivity with respect to area addition.

With the passing of the Dark ages and the period of apparent stagnation, the evolution of mathematics began picking up speed during the time of Renaissance Italy. It was the era of mathematical duels, and the solution to the cubic was a secret weapon. It was also a period with strong affiliations for arts and science drawing its cultural and scientific inspirations from ancient Greece. Though the main problems were of algebraic nature, geometric reasoning and intuition were as strong as ever with even the solutions to numerical equations being expressed as geometric poems about completing particular geometric objects (squares and cubics), hence the names for operations of multiplying numbers by themselves.
But as the geometric intuition failed the Greeks when they came upon the perplexing irrational numbers as certian square roots (diagonals), so were the great Italian duelists stumped with the realization that some solutions and algorithms for solving cubics involve numbers with negative squares. Despite their great power and utility, the square roots of negatives were dubbed imaginary, akin to the square root of 2 and its dismissive nomenclature.
Eventually, irrationals were recognized as an integral part of a more complete number system of real numbers. Further algebraic completion of reals was continued by including imaginary numbers to form the algebraic field of complex numbers. Quite surprisingly, along with immense algebraic utilities, these number systems also turned out to be deeply connected with the geometry of Euclidean space.

Algebra evolved fully from arithmetic with the introduction of a more suitable algebraic language that could express its meaning more efficiently than even the poetic words of Tartaglia. Thus operations on particular numericals were abstracted to general algebraic relations on abstract symbols (Vieta), allowing a more powerful understanding of numbers based on their abstract algebraic properties and definitions.
With the number concept rooted in a stronger foundation, the elusive complex numbers could finally be integrated into standard mathematical practices, and though at first dismissed based on geometric intuition, they proved to be indispensable geometric tool, as they were a " 2-dimesninal" set of numbers that spanned the 2-d (complex) plane.
Their addition follows the same rules as parallelogram addition of directed line segments, a well-known geometric law whose descriptions date back to ancient Greek thinkers like Aristotle and Heron.
That was also the first corollary in Newton's Principia (1687.), which was founded upon such fundamentally "vectorial" entities (force, velocity).

Systematic studies of vectors started with the dawn of the 19th century, but despite great progress, nobody was able to extend the primary idea behind vectors as geometric representations of 2-d numbers to 3 dimensions, while preserving the basic algebraic properties of real and complex numbers.
After a long period of trial and error, in 1843. Hamilton finally succeeded in this, expanding the complex numbers to the hypercomplex Quaternions. However, as their name suggests, they were not of the of the next higher dimension (3), but were a 4-d number system that when interpreted geometrically described the rotational properties of the 3-d Euclidean space.
At about the same time Hermann Grassmann was working on his algebra of extension, a purely geometric calculus developed as part of his study of the theory of tides. His techniques had great success in simplifying and improving on the great work of Lagrange and Laplace in celestial and analytical mechanics, and even more remarkably, Grassman's calculus generalized the algebra of space to an arbitrary number of dimensions.
The only issue was that the formalism was highly abstract and written in archaic language making it almost impossible to understand and integrate into existing mathematical disciplines, even for the likes of Mobius who despite investing a great deal of effort was unable to comprehend it fully.

Nobody could until a brilliant English mathematician, William Clifford.
Clifford managed to understand the great insights of Grassman and he even expanded further on his work by giving an expression for a product of two quaternions by vector products, the scalar product and a vectorial outer product. The (inner) scalar product of vectors gave the "spatial algebra" its metrical properties needed for distance and angle relations between lines, planes and volumes, while the outer product ensured that 2 and more dimensional objects(planes, volumes) had vector-like properties of direction and orientation.
Combining the two products brought the algebraic operation of division into play which greatly increased the qualitative and conceptual understanding of the spatial interaction of geometric objects. It also provided the efficient practical means of quantitively calculating the spatial consequences of those interactions. This was the long-sought algebraic theory of geometry which directly emulated the movements and projections of objects in 3- and higher-dimensional space.

Clifford's work showed enormous promise, and he was also the first to suggest that gravitation could be a manifestation of the underlying geometry with curvature determined by the presence of matter and fields.
Unfortunately, he couldn't pursue his ideas further because his life ended tragically in 1879. at the young age of 34. This was sadly also a pivotal moment that would change the course of history of mathematical science.

The algebra of space (algebra of geometric vectors) as we know it today took shape at the end of the 20th century as Gibbs and Heaviside developed and pushed into the mainstream thier 3-d vector calculus devoid of any advances made by Clifford and Grassman. This was the outcome of an intellectual war between the proponents of Quaternion and vector methods where the latter prevailed and would eventually shape the foundations of all geometry-related mathematics and mathematical physics of the 20th century. Almost every major mathematical discipline that later emerged had been greatly influenced by the school of thought that arose with the understanding of mathematical space/time based on the vector calculus, and an entire plethora of related formalisms was created which extended the idea of 3-vectors to more general algebraic objects like tensors and 4-vectors, and also to more general geometric (curved) spaces through tangent vectors and their duals, differential forms.

0.3 Is mathematics invented or discovered?

And it is a noteworthy fact that ignorant men have long been in advance of the learned about vectors. Ignorant people, like Faraday, naturally think in vectors. They know nothing of their formal manipulation, but if they think about vectors, they think of them as vectors, that is, directed magnitudes. No ignorant man could or would think about the three components of a vector separately, and disconnected from one another. That is a device of learned mathematicians, to enable them to evade vectors. The device is often useful, especially for calculating purposes, but for general purposes of reasoning the manipulation of scalar components instead of the vector itself is entirely wrong.

-Oliver Heaviside

Formalism was a stance invented to avoid this question, but it was to no avail.
Curt Godel, a declared platonist, crushed the formalists' hopes of proving the consistency of mathematics from itself, while greatly attributing his discoveries to his metamathematical viewpoint.
Through his remarkable theorems, Truth was revealed to be more fundamental than provability and not entirely within reach of bare formal deduction.

But how can it be known, without formal logical proof, that the platonic world of mathematics truly is universal and fundamental, and not mere product of the human mind?
In other words, is mathematics invented, discovered , or something in between?
Considering the semantics more precisely, the word invented here pertains to that which has no non-arbitrary relation to something fundamental.
Discovered being its opposite, is used in regards to something with its own intrinsic reality devoid of any arbitrariness, subjectivity or particulars, at least with respect to a given set of axioms (first principles).
The answer to the great question then, is very clear and concrete ;

Mathematics is fundamental, real and not invented ;

if it's not invented.
The idea that coordinates have an immediate metrical meaning is clearly an example of invented mathematics. The general method to discover mathematics rather to invent is rather simple :

Given first principles, not to force upon them any further(non-canonical) structure or arbitrary assumptions. All theorems should be derivable directly and continuously from first principles without breaking the conceptually clear rigorous flow of logical deduction. The method is to, instead of an introduction of arbitrary particulars in proofs and computations, work with the intrinsic data already available in the first principles(axioms). Provided this can't be done, instead of adding elements imbued with arbitrary choice, the method suggests improving or expanding the foundation (axioms) along the lines of existing intrinsic natural structure to supplement it and work with it.

This is the only way to ensure that nothing is invented or imaginary, simply because at no point is anything invented or added based on arbitrary and subjective imagination.

To discover s a theorem is to discover a proof. If the proof is not discovered but invented(based upon a choice of arbitrary objects) then the theorem cannot be deemed a discovery.

As no theory containing basic arithmetic can even prove its own consistency, a theory as a whole can not be known absolutely to be discovered, only its particular theorems.
We cannot know absolutely whether a set of axioms is invented, but if a theorem comes up that can be proven within the theory but can't be proven without arbitrary structure, then the theory is not sufficiently expressive.
Retroactively then based on their arbitrary structure-free expressive power, axioms themselves are re-evaluated.
The key is not only not to introduce any arbitrary structure, but to not disregard any available intrinsic structure, to avoid redundancy and achieve efficiency.
A great hint that one has discovered a theory tends to be that the theorems flow effortlessly without the need for forcing and patching any additional structure, with more and more surprising things discovered. Almost as if there is intelligence behind the axioms beyond the original conception.

Geometric theories are mostly affected by an introduction of arbitrary choices as they obscure the invariant geometrical nature and complicate calculations as a result of various redundancies and appearances of pseudogeometric objects.
Geometry was the first theory of the physical world, and has always been the foundation of scientific theories, both theoretical and experimental.
Even more abstract theories like topology and cohomology have their roots in geometry, and are ultimately theories of invariants. They should be treated as scientific theories studying those invariants directly, without any baggage of non-fundamental structure.
Algebra at first glance seems to be the complete opposite of geometry, but history and modern developments alike say otherwise. Vectors are the most obvious bridge between the two fundamental mathematical disciplines, and on closer inspection it becomes even more unclear whether they are in fact geometric or algebraic objects, or whether the physical reality, fundamentally described by vectors, is algebraic or geometric.

The goal here is to give a technical and natural account of the concept of vectors as directed magnitudes, one that is fully computationally operational as well as conceptually true to its intuitive geometric nature. Let V be an n-dimensional linear space over the reals. Just from the axioms and by basic definitions, we can see it has $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ linear subspaces. In this set of subspaces lies latent computational power, one that does mitigate geometric intuition, but enhances it.

0.4 Geometry of elementary algebra

Quando chel cubo con le cose appresso
Se aqquaglia ´a qualche numero discreto $x^3 + ax = b$
Trouan duo altri differenti in esso $u - v = b$

Dapoi terrai questo per consueto
Che"llor productto sempre sia equale $uv = (a/3)^3$
Alterzo cubo delle cose neto,

El residuo poi suo generale
Delli lor lati cubi ben sottrati $x = u^{\frac{1}{3}} + v^{\frac{1}{3}}$
Varra la tua cosa principale.

The first half of Tartaglia's algebraic poem

Where Euclid's geometry lacks in computational efficiency, basic algebra lacks in geometric intuition. We will re-discover a stronger elementary number system, with enough expressive power to allow geometric and algebraic calculations to be done in an invariant and elementary way, such that they don't require any further structure patches.
The algebraic concept of an n-dimensional linear space has a clear geometrical interpretation, but as it is, it is not by itself fully operational, for it cannot utilize its own intrinsic structure (subspaces) in computation. Rather it has to rely on patches of additional data and operators to prove its theorems.

By applying the theoretical scientific method, we will see how to, purely from algebraic considerations, (re)discover a true geometric theory of vector spaces that completely characterizes the original concept of vectors as directed magnitudes or directed numbers.

Starting from the natural numbers, expanding to whole and rational numbers, we complete our number system by extending it to include the upper bounds of its subsets, thus getting the real number system. To get an algebraically complete field, it must be closed under the square root operation. Historically this was done through the introduction of imaginary numbers, which together with reals form a linear space (over the reals).
However, the standard algebraic theory of (finite dimensional) linear spaces and their morphisms (linear functions) is not expressive enough computationally without inventing arbitrary structure as general linear operators need to be encoded in lists of numbers which are neither intrinsic algebraic nor geometric objects. The theory is therefore invented and not fundamentally sound.
Also, vectors as elements of bare linear spaces, or even inner product spaces, are quite incompatible with complex numbers, despite both being deeply connected to Euclidean geometry. And the reason lies fundamentally in what is considered basic or elementary algebra.

Complex numbers are the most general number system based on standard elementary algebra but are defined solely through a non-algebraically fundamental operation of square rooting.

Considering the proposed scientific method, if the theory is not intrinsically expressive enough (without additional arbitrary structure), the fundament axioms should be made more powerful. In this way, theory becomes akin to experimental science, where one can perform trial and error. With a more general fundamental axiom of multiplication, which allows for generally non-commutative multiplication, we can once again try using that field expanding operation (the square root) to get a new set of numbers.

Generalizing the theory in this way we get a new kind of numbers defined directly through the fundamental axioms. These noncommuting numbers form a linear space over reals but their dimensionality is not limited to 2 and they also come equipped with more a powerful product promising greater potential for a naturally richer (linear)algebra.
We call these numbers* vectors. With this we also immediately get the square roots of negative reals as vectors with negative real magnitudes which form a linear space of a negative signature, but for now we will focus on exploring the algebra of linear(vector) spaces with a positive signature ; $A(R^{n,0})$.
Vectors are thus simply square roots of (positive)reals that behave non-commutatively under multiplication.
It is this non-commutativity that encodes the intuitive geometric notion of a directed magnitude or an "arrow in space".

(* A concept of a number is not defined in general. Here a number simply means an element of a number system, which is taken to be a set closed under addition and multiplication, with its every element being either contained in a well-ordered subset or intrinsically mapped to an element of a well-ordered subset(to its magnitude) of the number system.)

(Geometric) Multiplication of two linearly independent vectors immediately gives rise to two canonical products; commutative and anticommutative, whose (symmetric and antisymmetric) properties follow directly from fundamental algebraic properties, the symmetry and antisymmetry,
of addition and subtraction respectively :

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba)$$

$$a \cdot b \equiv \frac{1}{2}(ab + ba) = b \cdot a = \frac{1}{2}(ba + ab)$$

$$a \wedge b \equiv \frac{1}{2}(ab - ba) = -b \wedge a = \frac{1}{2}(ba - ab)$$

From distributivity and the fact that the square of any vector is a real it follows that the symmetric(commutative) product of two vectors is also a scalar(real).

$$a \cdot b \equiv \frac{1}{2}(ab + ba) = \frac{1}{2}((a + b)^2 - a^2 - b^2))$$

This is equivalent to the standard inner product.
Thus such noncomuttative numbers are indeed Euclidean vectors as they form an inner product space over the reals, which in turn defines the geometry of Euclidean space through free and transitive actions of the (free) vectors on the abstract affine set of points ;
as per the standard Kleininian definition of Euclidean space $E(n) \equiv (\{p\}, \vec{\mathbf{E}}(n))$, b+v=c , v=c-p , d(c,b)=$|v| = \sqrt{v^2} = \sqrt{v \cdot v}$, where b,c $\in \{p\}$ with vectors acting as geometric transformations i.e. group actions (translations) on the set of points $\{p\}$.

The inner product has a direct geometrical interpretation as orthogonality is implied for vectors whose inner product is zero. It can in turn also be seen as the algebraic realization of the (geometric) perpendicular projection of line segments as it designates algebraically that the two lines are orthogonal if the inner product of the vectors representing them is zero.

The other part of the geometric product of two vectors is called the outer or exterior product. It is axiomatically irreducible to being a vector or a scalar.
It can be interpreted geometrically as directed area because two noncolinear vectors span an area. Dually it is the algebraic expression of the geometric relations between parallelogram area and its base and height.
If the product of two vectors is to represent the spanned area, it has to be zero for all colinear vectors implies antisymmetry of the outer product for all vectors a, b,c $\in \mathbf{E(n)}$:

$$a \wedge d = a \wedge (\alpha a) = \alpha a \wedge a = 0 \text{ iff } a \wedge a = 0 = (b + c) \wedge (b + c) = b \wedge b + c \wedge c + b \wedge c + c \wedge d = 0$$

Hence if the outer product of 2 vectors is zero they are collinear.
Viewed from the side of geometry, if the outer product of two vectors, each representing a line, vanishes, then the two lines are parallel.
In this the geometric concept of vectors as directed magnitudes is naturally realized algebraically, as their geometric product is a measure of their relative direction. This on the other is just a natural completion of the(multi-linear) algebra of vectors by the general noncommutative product that has a direct geometric interpretation; if vectors commute they are collinear, if they anticommute they are orthogonal.

Thus we have found the very core of Euclidean geometry (its translation vector inner product space and its exterior algebra, by applying the theoretical method to elementary algebra, in this far surpassing the computational and even conceptual potential of classical geometry. In this algebra of nonabelian numbers, the whole algebra of geometry can be located, the algebra of subspaces. Thereby directly algebraically relating points lines, planes, lengths, areas, volumes, etc.., and utilizing fully the latent power of the subspace structure in linear spaces. Subspaces of different grade, corresponding to the dimension of the spanned geometric space, are generated by linear combinations of algebraic elements called n-blades enumerated iteratively by natural numbers ;

Starting from vectors labeled as first-grade elements or 1-blades, and considering two parts of multiplication with a vector:

$$aa_n = \frac{1}{2}(aa_n - (-1)^n a_n a) + \frac{1}{2}(aa_n + (-1)^n a_n a) \qquad (0)$$

$$a \wedge a_n \equiv \frac{1}{2}(aa_n + (-1)^n a_n a) \qquad (1)$$

$$a \cdot a_n \equiv \frac{1}{2}(aa_n - (-1)^n a_n a) \qquad (2)$$

with (1) giving a n+1 grade element or (n+1) blade.

It follows then that the interior product (2) is a grade lowering operation. It is more general than a metric and it can have non-metrical, signature-independent interpretations, but unlike exterior and geometric products, it is not associative.
To reduce the number of braces, as a convention the interior and exterior products are given priority over the geometric product.

The above also implies that n-blades are totally antisymmetric parts of products of more than two vectors, with a direct geometric interpretation of oriented n-volume elements. Orthogonality and colinearity relations between higher grade blades are also determined by inner and outer products as was the case for vectors (1-blades).
General elements (of mixed grade) are called (geometric) numbers or multivectors. Multivectors that are linear combinations of n-blades are general elements of the n^{th} grade called n-vectors. A vector space is n-dimensional iff n-blades are the nonvanishing elements of the highest grade. Such blades are called pseudoscalars as they are equal up to a real scale factor and are thus 1-dimensional numbers as are reals (scalars).

In this algebra of subspaces, we can represent geometry directly by algebraic operations gaining even more geometric intuition and conceptual clarity than by using just "geometry", and remarkably even more efficient computational efficiency than using the "standard elementary algebra" applied to geometry or even applied to algebra itself.
The inner and outer products are hereditarily bi-linear and multilinear, respectively, in this immediately yielding canonical algebraic structure of linear and multilinear operators. But this hides even more intrinsic, non-trivial complexity, as the inner and outer are just two sides of the ununifying fundamental (geometric)multiplication.
The symmetric part of the product of two vectors (the inner product) is a scalar. It is a bilinear operator equivalent to scaling of the geometric projection. The outer product is also bilinear operator, producing a bi-vector in an equivalently linear way to the parallelogram base and height multiplication.
The geometric (fundamentally multilinear) product of two vectors, however, is neither a scalar nor a bi-vector, or even a vector, but a fundamental geometro-algebraic object called spinor, which is a key for the complete invariant geometric representation of (multilinear)algebra.

0.5 Algebra of elementary geometry

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."
- Michael Francis Atiyah

"An algebra that is so close to geometry that every expression in it has clear geometric meaning, that the algebraic manipulations of the expressions correspond to geometric constructions. Such an algebra, if exists, is rightly called geometric algebra.
-Leibniz's Dream of "Geometric Algebra"

This paper is not about any particular algebra but an algebra free from particulars. The algebra that uses its intrinsic structure (of subspaces) for computation and can operate on geometric objects and relations(directions ,projections ,..)directly through its fundamental axioms (commutativity, distributivity ,..)
By the theoretical scientific method, the notion of a linear space is expanded into a a structure that can intrinsically describe itself, called a universal vector or geometric algebra. Thus a vector becomes not just an element of a linear space but an element of a universal vector algebra here taken to be the fundamental (division) algebra behind the "new" basic number system.

Traditional elementary algebra (of the field of reals or complex numbers) seems simpler as the multiplication is always commutative, however, it is not intrinsically equipped to handle algebraic structures that arise within it (groups, linear spaces) because the algebra of just real or complex numbers is not abstract and powerful enough. Then one has to introduce non-fundamental algebraic structures, composed of lists of numbers and not numbers of the algebra themselves, to apply general axioms and definitions in practice. It can't either be applied to geometry without the introduction of arbitrary structure that then introduces further complications.
Starting from the basic algebra we arrived at a more fundamental theory of algebra which is also the fundamental foundation of Euclidean geometry in its modern transformational (Kleinian) formulation. This is simply a more condensed and efficient version of the original Euclid's theory, directly expressing its already existing, intrinsic algebraic aspects, particularly in regards to the transformational properties of geometric spaces.
Aside from the trivial identity maps, there exist transformations that act on spaces and objects in non-trivial ways yet still leave certain properties invariant. Such transformations are the symmetries or symmetry transformations.

The basic symmetries of Euclidean space are reflections, rotations and translations.
The set of all translations of points in a space naturally forms (or equivalently is a basic consequence of) the standard Euclidean geometry defining Euclidean vector space with the inner product giving translations (vectors) their magnitude. This naturally gives rise to real numbers as the only set of numbers complete enough to cover all possible geometric lengths, including irrational and non algebraic, transcendental magnitudes.
However, the symmetric (inner) product alone does not tell us what (Euclidean) vectors really are, particularly not what they are concretely. It does imbue vectors with magnitudes but it does not completely describe their directional nature. Furthermore, it is not enough to invariantly express their full (linear) algebra nor to achieve its complete geometric realization, not without the addition of non-fundamental structure ; particular basis vectors, vector products based on arbitrary orientation (right-hand rule), special origin,.. etc, none of which exist in real Euclidean geometry.
So in most instances, one is left only to work with arbitrary representations of vectors instead of vectors themselves, which are then in most applications, along with other geometric objects, encoded in lists of numbers which renders any connection to their true (geometric) nature essentially non-existent.
But merely removing the non-canonical structure is not enough as also "removing the removal " of the canonical structure in the axioms is needed to discover a fully expressive invariant theory.

On a 2-d plane the fundamental (linear) symmetries are reflections and rotations which are distance preserving transformations with a fixed line and a fixed point respectively.
We can, working in just a special case of an inner product space, find a canonical (basis free) reflection (linear) operator : $R_b(a) = a - 2a \cdot \hat{b}\hat{b}$
Although the operator itself is constructed in an invariant way, the composition of such simple operators gets very complicated very quickly.
The theory of laws of the composition of linear operators is called linear algebra. It would be great if the (linear) algebra of geometric transformations was as simple and as natural as possible, both conceptually and computationally.
Aside from the multiplication with -1(180° rotation) the simplest nontrivial rotation operator is a 90° rotation which produces a vector orthogonal to the original vector. Just with the inner product, we can also construct a canonical antisymmetric product which is also an operator $(a \circ b)x = a \cdot xb - b \cdot xa$ that produces a vector in the plane spanned by a and b, orthogonal vector to the input vector x.

$$x \cdot (a \cdot xb - b \cdot xa) = a \cdot xb \cdot x - b \cdot xa \cdot x = 0$$

Thus if we normalize the operator to output an orthogonal vector with the same magnitude as the input vector we get a 90° rotation. However, although we used only canonical structure, considerable complexity and computational inefficiency is introduced, almost as much as in using arbitrary structure. The reason for this lies in the incomplete utilization of the natural structure resulting in the introduction of canonical but redundant objects and concepts, which ultimately also leads to the inability to invariantly express all objects and concepts of the theory.

As an example, a complete fundamental characterization of linear operators needs to include an invariant way of calculating the effect a particular operator has on the oriented volumes(subspaces) of a given space, to see also how they affect the orientations of geometric objects.
This is captured by the determinant of a transformation, which is just a scalar by which oriented n-volumes of n-spaces are multiplied (scaled up) when the operator acts on a given space. But to define this precisely we need to know the exact algebra of n-volumes (volume forms).

Operators affect the oriented volumes through their action on vectors which produce a definite oriented volume if they are multiplied in an antisymmetric and multilinear way to express its magnitude as well as orientation. A determinant of a linear operator is then precisely the scalar which multiplies elements of the n^{th} exterior power as the induced transformation on the algebra generated by the exterior (antisymmetric) product on the n-dimensional linear space, where the k^{th} exterior power is the $\binom{n}{k}$-dimensional vector subspace of the algebra. Thus we need to introduce another product on the algebra in addition to the inner product. However, calculating the determinant in this way is also very cumbersome when at all possible, because the inner and exterior products defined separately in this way are not related algebraically(or geometrically in any natural way).

The inner product alone reveals that algebra can be seen as being simply a natural language of geometry. Instead of the complex grammar and syntax of the synthetic language with straight lines and parallelism as primitives resulting in more involved and inefficient proofs (sentences), geometric relations can instead naturally be expressed as basic algebraic identities. The most fundamental theorem in Euclidean geometry, the Pythagorean theorem for example, $|\overline{AC}|^2 = |\overline{AB}|^2 + |\overline{BC}|^2 \Leftrightarrow \overline{AB} \perp \overline{BC}$ can derived as simply as :
$|\overline{AC}|^2 = \overrightarrow{AC}\cdot\overrightarrow{AC} = (\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{AB}+\overrightarrow{BC}) = \overrightarrow{AB}\cdot\overrightarrow{AB}+\overrightarrow{BC}\cdot\overrightarrow{BC}+2\overrightarrow{AB}\cdot\overrightarrow{BC} = |\overline{AB}|^2+|\overline{BC}|^2 \Leftrightarrow \overrightarrow{AB}\cdot\overrightarrow{BC} = 0$

Immediately also generaling to the "cosine rule" if the sides are not exactly orthogonal, which is given by the explicit trigonometric relations in ch.6,eq.(9).
Geometrically the reasoning behind the proof lies in the equality of square areas corresponding to each side of the triangle.
So even without any preconceptions of linear algebra, naturally linear translations of Euclidean points are geometrically equivalent to directed line segments which in turn correspond to squares with the sides of the same magnitude.
That the square of a directed line (translation vector) as its side has an area A, is from the geometric point of view more fundamental than the algebraic square of real number.(Historically also , the square and the square root of a number were first conceived as purely geometric objects.)

The area of a square is a scalar indifferent to the orientation (direction) of the vector a as one of its sides, as a vector a has no absolute direction ($aa = |a||a|$) -The square of a side $a = \overrightarrow{AB}$ and the square of a magnitude $|a|$ represent the same geometric area.
This area is just a scalar multiple of the rectangle area ab with sides of magnitudes $|a|, |b|$ if the two vectors in th is equal in area to the square with the side of magnitude $\alpha|a|$ if : $ab = a\alpha a = \alpha|a||a| = ba$.
In this way the "algebra" of collinear translations follows the same rules as the" algebra" of distance relations (real numbers). Thus at the very least algebra is a language directly and naturally arising from geometry, but even this does not do it justice, al the very word language is generally understood as being a human invention.
But ultimately algebra is not just a language of geometry,
algebra is geometry. And from another point of view, geometry is algebra. This refers particularly to the complete theory of elementary algebra (fundamental number systems), and the full invariant theory of basic geometry.
Just as magnetic and electric fields are but one unified electromagnetic field viewed from different frames of reference, so are geometry and algebra just one mathematical field, which is also (non-) coincidentally the ideal system for physical equations like the Maxwell equations, unlike the vector calculus that was originally marketed as the best tool for vector field equations.
Addition and multiplication are just operational aspects of the geometry of lengths, areas and directions of geometric objects symbolically written in the standard algebraic notation.
If $a \neq \alpha b$ then $ab \neq ba = \alpha aa = \alpha|a||a|$
breaking the symmetry of multiplication of the scalar algebra, which unfolds a larger multi-dimensional algebra.
If ab is a geometric object(number), it can be scaled up in a distributive way with regards to associative addition of geometric objects(directions, areas, magnitudes etc..) and thus can then be written as :

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba)$$

As the square of any (translation) vector is a scalar, the inner and the outer products of 2 vectors are again revealed to be just two parts of one fundamental associative and distributive geometric product.

$$a \cdot b \equiv \frac{1}{2}(ab + ba) = b \cdot a = \frac{1}{2}(ba + ab)$$

$$a \wedge b \equiv \frac{1}{2}(ab - ba) = -b \wedge a = \frac{1}{2}(ba - ab)$$

This is the Grassmann's exterior product which allows for the anticommutative multiplication needed to construct volume forms which can now also be used in a operational manner with its laws of geometric interaction with other geometric objects now fully described.
By takeing the exterior product of the triangle equation $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ with $\overrightarrow{AC} = c, \overrightarrow{AB} = a, \overrightarrow{BC} = b$ in succession we have :

$$a \wedge c = a \wedge b = c \wedge b \Rightarrow |a \wedge c| = |a \wedge b| = |c \wedge b|$$

Which with dividing by abc and by eq.(9) is exactly the sine rule :

$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ where $\alpha.\beta, \gamma$ are the corrsponding angles of the triangle.

Thus ab is indeed a geometric object consistent with the laws of the standard intuitive axiomatic geometry of Euclidean space. But the visual spatial experience and intuition are not the ultimate validation and a fundamental theoretical guiding principle, as any scientific theory must ultimately be logically consistent and operational without invented structure. The missing object in the n-dimensional inner product and synthetic geometries is the oriented n-volume form, and this is how algebra naturally comes up out of geometry as its transformational and orientational (directional) aspect. An oriented n-volume form is then embodied in the totally antisymmetric part of a generally non-symmetric product of n vectors :

$$V(n) = \langle a_1 a_2 ... a_n \rangle = a_1 \wedge a_2 \wedge ... a_n$$

For an n-dimensional vector space, these are exactly the pseudoscalars contained in its geometric algebra, which is the algebra of its subspaces encoded in the structure generated by the iterative application of the geometric (noncommutative, associative and distributive) product. In this we have an intrinsic way to utilize the canonical structure of subspaces for computation, a key thing missing in the theory of (symmetric) inner product spaces.
To invariantly describe the transformational properties of linear subspaces we also need the (linear) algebra of operators acting on such geometric subspaces and not just on vectors. Remarkably the algebra of operators and the algebra of subspaces are part of the very same unified (geometric) algebra, the elementary non-abelian (division)algebra which was discovered as the fundamental number system just from purely algebraic considerations. In this generalized algebra(geometry) of geometry(algebra) which includes both linear subspaces and operators as its elements, unit-oriented area turns out to be a canonical (90°) rotation operator, and an oriented volume is the highest grade element of the included exterior algebra .

The abstract noncommutative multiplication of vectors produces a spinor, an element of the even algebra subalgebra which is the simplest canonical algebraic representation of rotations and dilations, directly showing how a composition of an even number reflections produces a rotation(ch. 8.).

Thus by applying the theoretical method either to the foundation of elementary algebra and removing the assumption(axiom) of a generally commutative multiplication of numbers, or by the same method applied to geometry resulting in losing the assumption that the product of geometric translations (vectors) is symmetric in general we arrive at one and the same unified theory which can also take us far beyond elementary algebra and geometry.

0.6 Geometric algebra

The geometric algebra is the tool which allows to study and solve geometric problems through a simpler and more direct way than a purely geometric reasoning, that is, by means of the algebra of geometric quantities instead of synthetic geometry. In fact, the geometric algebra is the Clifford algebra generated by the Grassmann's outer product in a vector space, although for me, the geometric algebra is also the art of stating and solving geometric equations, which correspond to geometric problems, by isolating the unknown geometric quantity using the algebraic rules of the vectors operations , such as the associative, distributive and permutative properties.
-Giuseppe Peano

From (0),(1),(2) it follows that :

$a \wedge b + a \cdot b = ab$

which relates the interior and exterior products of 2 vectors additively, and from (2) their multiplicative relation is derived as follows ;

$abc = -bac + 2a \cdot bc$

$= -b(-ca + 2c \cdot a) + 2a \cdot bc$

$\Rightarrow a \cdot bc - a \cdot cb = \frac{1}{2}(abc - bca)$

$= \frac{1}{2}(ab \wedge c - b \wedge ca) = a \cdot (b \wedge c)$ from (1),(2)

$\Rightarrow a \cdot (b \wedge c) = a \cdot bc - a \cdot cb$ (3)

Thus the interior product of a 2-blade and a vector produces a vector.

The complement of the projection operator P(a, b)=$a \cdot bb^{-1} = a \cdot \hat{b}|b|\hat{b}|b|^{-1} = a \cdot \hat{b}\hat{b}$ -(3a) ;

is the perpendicular rejection operator R(a,b) which outputs the orthogonal component of a vector to a particular direction. This follows from (3) :

R(a,b)=$a \wedge bb^{-1} = a - a \cdot bb^{-1} = a - a \cdot \hat{b}\hat{b}$ (3b)

General projection and rejection operators giving parallel and orthogonal components to higher grade elements representing n-dimensional spaces(planes, volumes etc) follow from (0),(1),(2) :

$a = aa_n{a_n}^{-1} = a \cdot a_n{a_n}^{-1} + a \wedge a_n{a_n}^{-1} = P(a, a_n) + R(a, a_n)$

$P(a, a_n) = a \cdot a_n{a_n}^{-1} = a_{\parallel}$ (3c)

$R(a, a_n) = a \wedge a_n{a_n}^{-1} = a_{\perp}$ (3d)

With the inner and outer products combined multiplicatively in (3) another, operational, interpretation of the outer products of 2 vectors can unfold; it is the simplest canonical generator of rotations in a plane ;

Taking another contraction :
$a \cdot (a \cdot (b \wedge c)) = a \cdot ((a \cdot b)c - (a \cdot c)b) = (a \cdot b)(a \cdot c) - (a \cdot c)(a \cdot b) = 0$ (4)

This proves that the inner product of a 2-blade with a vector produces a vector orthogonal to the original vector, in the plane represented by the 2-blade (spanned by vectors whose outer products are blades collinear with the 2-blade) which is also rotated by 90°. If the outer product of a vector and a 2-blade is zero they are collinear and their inner product is equal to the associative geometric product. Thus we get a rotational operator in a plane acting and composing simply and efficiently through the fundamental product of the algebra.
In this way, unit blades rotate vectors in their planes by $\pi/2$ radians, and we will see this in ch.8. in more detail and in the 3-d case.
The composition of two successive rotations in a plane by $\pi/2$ is equal to multiplication by -1. This can be proven by aain taking the inner product of the plane rotation operator (unit 2-blade) and the rotated vector, but we can also see this as the direct composition of ($\pi/2$) plane rotation operators,

the 2-blades, as they compose through the basic(general or elementary) multiplication, in this case squaring to -1, thus in turn also revealing their "imaginary" nature.

$a^2b^2 = abba = (ab)(ba) = (a\cdot b + a\wedge b)(b\cdot a + b\wedge a) = (a\cdot b)^2 + (a\wedge b - a\wedge b)(a\cdot b) - (a\wedge b)^2 = (a\cdot b)^2 - (a\wedge b)^2$

$\Rightarrow (a\wedge b)^2 = (a\cdot b)^2 - a^2b^2 \qquad (5)$

Also :

$a\cdot(b\cdot(a\wedge b)) = (a\cdot b)^2 - a^2b^2 = (a\wedge b)^2 = (a\wedge b)\cdot(a\wedge b)$ from (3) and (1)

$\Rightarrow (a\wedge b)^2 = a\cdot(b\cdot(a\wedge b)) \qquad (6)$

A more general contraction rule for n-blades with m-blades can also be proven from (1),(2).

To get a normalized (unit) 2-blade the Cauchy-Schwarz rule is needed and it is proven as follows:

$0 < (|a|b - a\cdot b\hat{a})^2 = a^2b^2 - |a|b(a\cdot b)\hat{a} - \hat{a}|a|b(a\cdot b) + (a\cdot b)^2$

$= a^2b^2 - 2|a|\hat{a}\cdot b(a\cdot b) + (a\cdot b)^2 = a^2b^2 - (a\cdot b)^2$

Thus we have : $|a\wedge b| = (a\wedge b)(b\wedge a)$ and $a\,\hat{\wedge}\,b = \dfrac{a\wedge b}{|a\wedge b|} = R_{\pi/2} \qquad (7)$

Since R_θ is a rotation in a plane, that is a rotation operator on a 2-d Euclidean space , rotation angles will add simply in compositions.

$\Rightarrow R_\pi = R_{\pi/2+\pi/2} = R_{\pi/2}\circ R_{\pi/2} = R^2{}_{\pi/2}$

$= \dfrac{a\wedge b}{|a\wedge b|}^2 = \dfrac{(a\wedge b)(a\wedge b)}{(b\wedge a)(a\wedge b)} = \dfrac{a^2b^2 - (a\cdot b)^2}{(a\cdot b)^2 - a^2b^2} = -1 \qquad (8)$

$\Rightarrow \sqrt{-1} = \pm i = \pm\dfrac{a\wedge b}{|a\wedge b|}$

$i(a) = ia$ is indeed a rotation in the plane its representing , the 2-d Euclidean space E(2) =({p},$\vec{\mathbf{E}}(2)$) with $a\wedge b\wedge c = 0$ $a, b \in \vec{\mathbf{E}}$ and $\pm i = a\,\hat{\wedge}\,b \in \mathbf{A}(\vec{\mathbf{E}}(2))$ are unit pseudoscalars in the geometric algebra each determining rotation directions opposite to one another.

Rotations are the fundamental symmetries of Euclidean spaces as they preserve the shapes and sizes of all embedded objects including their orientations (unlike reflections which reverse them). To see this it will suffice to prove that reflections operate in a covariant manner in the sense that they preserve the fundamental product of the algebra of the 2-d Euclidean (vector)space.

$i(x)i(y) = ixiy = ix\cdot iy = -ii\cdot xy = -iixy = xy = x\wedge y + x\cdot y$

Thus (90°)rotations preserve the geometric product and with that both the outer and inner products, unlike reflections which only preserve orthogonality and inner products (ch. 8.)

As the normalized outer product of two vectors $\dfrac{a\wedge b}{|a\wedge b|}$ is a 90° plane rotation operator (generator) i, we can expect that the normalized geometric product $\dfrac{a\ b}{|a\ b|} = \dfrac{a\ b}{\sqrt{a^2b^2}} = \hat{a}\hat{b}$ is a general plane rotation operator with the outer product $\hat{a}\wedge\hat{b}$ being only its a special case when the vectors in the product are exactly orthogonal.
The geometric product of two vectors, however, is neither a scalar, a vector nor a 2-blade but the object with most general rotational (transformational) properties called spinor.

A unit spinor $\hat{S} = \dfrac{ab}{|ab|} = \dfrac{ab}{\sqrt{a^2b^2}} = \hat{a}\hat{b}$ acts on vectors by geometric multiplication and reduces to a unit 2-blade $i = \dfrac{a\wedge b}{|a\wedge b|} = \hat{a}\wedge\hat{b}$ (a 90° rotation) when $\hat{a}$ and $\hat{b}$ are orthognal.
It can be confirmed simply that unit spinor multiplication preserves distances between points in E(2), which is represented by vector magnitudes ;

$|\hat{S}a| = |\hat{a}\hat{b}a| = \sqrt{\hat{a}\hat{b}aa\hat{b}\hat{a}} = |a|$

Every vector in $\vec{\mathbf{E}}(2)$ a is transformed this way, with the only vector being unmoved (transformed to

itself) is the zero vector. Thus unit spinor multiplication in a 2-plane has a single fixed point(vector), or equivalently the eigenvector of this canonical rotation operator is the zero vector, proving that it is in fact a rotation.
To express the rotation operator more explicitly geometric product can be put into the exponential form :

$$ab = a \wedge b + a \cdot b = |a||b|\sin(\theta)\frac{a \wedge b}{|a \wedge b|} + |a||b|\cos(\theta) = |a||b|(\sin(\theta)i + \cos(\theta)) = |a||b|e^{i\theta} \qquad \text{-(9)}$$

Which determines the angle relation between vectors as $\theta(a,b) = \arccos(\frac{a \cdot b}{|a||b|})$ giving more explicit expressions also for the inner (scalar) and outer (bi-vector) parts of the product of two vectors in terms of the radian angle measure, by equating the scalar and bi-vector parts of (9) ;

$$a \cdot b = |a||b|\cos(\theta)$$

$$a \wedge b = |a||b|\sin(\theta)\frac{a \wedge b}{|a \wedge b|}$$

We can confirm this by substitution into (5):

$$abba = (|a||b|)^2 = (a \cdot b)^2 - (a \wedge b)^2 = (|a||b|)^2(\sin^2(\theta) + \cos^2(\theta) = (|a||b|)^2$$

Thus a rotation through an angle θ $R_\theta(a)$ can be put in the (spinor) operator form :
$R_\theta(a) = \hat{a}\hat{b}a = (\sin(\theta)i + \cos(\theta))a = e^{i\theta}a$, for all vectors a=P(a,i) or equivalently for all vectors in $\in \vec{\mathbf{E}}(2)$

$e^{i\theta}a$ is indeed a rotation angle θ by the definition of the angle relation of two vectors:

$$\arccos(a \cdot e^{i\theta}a = \cos(\theta)a + i\sin(\theta)a = \arccos(\cos(\theta)) = \theta, \qquad \text{as } a \cdot ia = 0$$

Also, $re^{i\theta}$ describes a circle of radius $|r|$ in the complex plane - the circle group S(1) which is just a set of complex numbers of magnitude $|r|$ which in turn also form a group U(1) under complex(geometric multiplication). The circle (lie) group S(1) is diffeomorphic to Spin(2) the group of unit spinors in 2-d represented by oriented unit circle arcs composing additively :

$$e^{i\theta}e^{i\alpha}e^{i\beta}\hat{r} = e^{i(\theta+\alpha+\beta)}\hat{r}$$

The circle group S(1)≅U(1) can be smoothly parametrised by a real number t : $t \to r(t) = r(t_0)e^{i\theta(t)}$, determining a circular trajectory with (a tangent) velocity $v = \dot{r}(t) = r(t_0)i\dot{\theta}(t)e^{i\theta(t)}$ -where i produces a tangent to the trajectory (circle) by rotating r throught 90°,
and acceleration a=$\ddot{r} = -r(t_0)\theta(\dot{t})^2 e^{i\theta(t)} + r(t_0)i\ddot{\theta}(t)e^{i\theta(t)}$
which if $\theta(t)$ is linear simply reduces to centripetal acceleration :
a=$\ddot{r} = -w^2 r(t_0)e^{iwt}$

This can also be applied the other way around starting from a dynamical equation $\ddot{r}(t) = -|f(r)|\hat{r}(t)$ and deriving the kinematical trajectory which is exactly circular for the following inital conditions :
$|\dot{r}(0)| = |\dot{r}_0| = |v_0| = \sqrt{|r_0 f(r)|} \Leftrightarrow w^2|r| = |f(r)|, v_0 \cdot r_0 = 0$;

$$r(t) = r_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t}$$

$$\dot{r}(t) = v(t) = \sqrt{|f(r)/r_0|}r_0\hat{r}_0 \wedge \hat{v}_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t} = \sqrt{|f(r)/r_0|}\hat{v}_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t}$$

$$\ddot{r}(t) = \dot{v}(t) = a(t) = \hat{r}_0 \wedge \hat{v}_0{}^2 \sqrt{|f(r)/r_0|}^2 r_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t} = -|r_0|/|r_0||f(r)|\hat{r}_0 e^{\hat{r}_0 \wedge \hat{v}_0 \sqrt{|f(r)/r_0|}t}$$

$$= -|f(r)|\hat{r}(t)$$

With that, one of the most important equations in all of mathematics and physics;

$$e^{i\pi} + 1 = 0$$

can now be seen in a whole new light, directly through geometry and uncrucified from the axes.
Thus we discover numbers that are square roots of negative reals without only asserting their existence or that of any objects with negative squares.
And there is nothing imaginary about these numbers. There is even no imaginary axis needed to realize their extraordinary geometric interpretation. The fractal Mandelbrot set for example is the set of all

complex numbers c for which the function $f_c(z) = c + z^2$ converges when iterated from z=0. To realize its geometry one has to imagine two perpendicular axes in the complex plane, as is the case in any geometric application of complex numbers. But we have seen how real imaginary numbers are directly and intrinsically part of (the real algebra of) Euclidean geometry, without any artificial structure. The Mandelbrot set then takes its real manifestly geometric form $r_c(z) = (c + z^2)\hat{r}$, where $\hat{r}$ is the direction of the set along its axis of symmetry directed from the origin of the set, and z is a quaternion in the i plane composed of a real and a real imaginary number proportional (collinear)to the unit bivector i representing the plane.
Note that the real algebra of 3-d Euclidean vector space contains another set of imaginary numbers of a higher grade ; $\pm\sqrt{-1} = \frac{a \wedge b \wedge c}{|a \wedge b \wedge c|} = I$ which can be proven analogously to (5). I is the unit 'pseudoscalar' because tri-vectors are 1-dimensional as are scalars(0-vectors), in the case of 3-d generating vector space. It is particularly important for projective(non metrical) incidence relations, with duality expressed as simply as $a^* = aI^{-1}$.
This reveals what square roots of negative numbers really look like, and that they are simply a natural consequence of the real algebra of physical (Euclidean) space.

Here we have adopted the standard definition of Euclidean space, as an affine space with the Euclidean vector space acting on it freely and transitively. Thus vectors and their fundamental algebra give geometric (algebraic)relations between abstract elements of the space called points. We will not elaborate further on at these points are, though in various interpretations they can be directly modeled with vectors and thus also be found in the algebra.

Imaginary numbers do make a comeback when it comes to fundamental algebraic models of Euclidean space, though not in the form of imaginary scalars, but imaginary vectors. This further completes the algebra with all reals having vector square roots. (Here we are only considering the square roots of positive reals ie Euclidean signature $A(R^{n,0})$. Geometric points can then also be interpreted as elements of the algebra located in $A(R^{4,1})$, the 5-d conformal model with "imaginary" vectors. The conformal geometric algebra model of Euclidean geometry is a particularly interesting discovery as it unifies the orthogonal group with translations into a linearised Euclidean group quite elegantly, essentially by considering translations as rotations at infinity.[8]

Vectors can in general be imbued with various interpretations depending on the model used. They can be interpreted as points, line segments, relations between points, directions, magnitudes, and even planes or spheres (conformally). They are an example of how a theory can express itself intrinsically and directly from fundamental axioms. Thus one can compute straight from first principles and require no non fundamental choice of particulars. This also solves the problem of the weakness of the axioms of linear space in regards to expressing their full linear algebra which can now be expressed as geometric algebra, with the basic product (geometric or noncommutative product) being naturally (multi)linear due to distributivity.

This powerful system of geometric elementary algebra is ultimately equivalent to the notions of geometry both in terms of Riemann and Klein programs, as well as in the synthetic case, hence the name universal geometric algebra[2]. More on that in the last section. Unlike algebraic geometry where one interprets equations as geometry with the help of pseudo geometric structure, and unlike analytic geometry where geometry is encoded through algebra with non-algebraic structure, in geometric algebra, geometry (including nonmetrical, and non-Euclidean) is located directly in the algebra itself, and the algebra with its relations is in turn itself just an expression of geometric relations between geometric objects.

The empowered elementary algebra lets us tap into that old intelligence fully capable of operating purely with geometric laws, drawing its origins to the time of Euclid but evolved into a modern form, yet still staying true to the ways of reasoning behind the work which marked the beginning of theoretical science, the Mathematical Principles of Natural Philosophy.

0.7 Theory in practice

The ponderous instrument of synthesis, so effective in Newton's hands, has never since been grasped by anyone who could use it for such purpose; and we gaze at it with admiring curiosity, as some gigantic implement of war, which stands idle among the memorials of ancient days, and makes us wonder what manner of man he was who could wield as a weapon what we can hardly lift as a burden"

William Whewell on Newton's geometric proofs, 1847

"The geometric calculus differs from the Cartesian geometry in that whereas the latter operates analytically with coordinates, the former operates directly on the geometrical entities".
-Giuseppe Peano

Here we will solve a problem from Principia of the translation of the kinematics of celestial objects into a fundamental equation of their dynamical interaction described by equations of the form : $\ddot{r} = f(r, \dot{r})$ where f is a vector-valued function (force divided by a scalar mass constant m) and r is a Euclidean vector with $\dot{r}$ being its tangent vector, as generally in classical mechanics forces can be functions of both position and velocity, as are magnetic and drag forces for instance.
The goal is to derive the function $f(r, \dot{r})$ from the kinematics of a given trajectory, and in expressing geometry through algebra directly, which will allow effortless calculations using both the pure conceptual clarity of geometry coupled with the computational power of algebra, together also with calculus, in what Feynman called an "elementary derivation", that, as he defines it, "requires no heavy machinery of multivariable calculus, only infinite intelligence". Luckily we have in the discovered elementary geometric calculus the computational tecquiniqes that contain greater intelligence than that of theories which are inventions of the human mind. As the universe and the human body are infinitely more complex than any machine or computer simulation, so are theoretically discovered structures in that sense real and "alive", seamlessly providing ever unveiling new interesting structures and objects, as is in every case of an invention as opposed to a fundamental discovery.

We will derive the law governing the gravitational interaction (gravitational force inverse square law) of celestial bodies up to a constant from 2 (empirical) kinematical assumptions :

(1.) Orbits of celestial objects are conic sections, where the Sun is a focus.

(2.) The "angular momentum" with respect to that focus point is time-symmetric. The second assumption is equivalent to that of the area swept out by the line joining the Sun and a celestial object being constant in time.

We will need to use the fact that the trajectory is a conic as per assumption (1.). It should be a function of position r and velocity $\dot{r}$, and also of area so we can use the area sweeping time symmetry. The basic combination of position and velocity that produces the area up to a scale factor is their outer product $r \wedge \dot{r}$. If we divide it by its magnitude (scale) we get the unit area bi-vector $i = r \hat{\wedge} \dot{r}$ whose square will give us a form of a conic with a constant real in the numerator and a vector dot product in the denominator ; $\frac{c}{e \cdot \hat{d} + 1} = \frac{c}{e \cos(\theta) + 1} = |d|$, provided the eccentricity vector (Lagrange vector) e is constant in time.

$$i^2 = -1 = \frac{r \wedge \dot{r}}{|\dot{r} \wedge r|}^2 = -\frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{(r \wedge \dot{r})(\dot{r} \wedge r)} \qquad \text{from (7),(8)}$$

$$= -\frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{(r \cdot (\dot{r} \cdot (\dot{r} \wedge r))} = -\frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{|r|\hat{r} \cdot (\dot{r}\dot{r} \wedge r)} \qquad \text{from (2),(6)}$$

$$\Rightarrow |r| = \frac{(r \wedge \dot{r})(\dot{r} \wedge r)}{\hat{r} \cdot (\dot{r}\dot{r} \wedge r + k\hat{r} - k\hat{r})} \qquad \text{-Assumtion (1.) - this curve has has to be a conic.}$$

$$= \frac{(r \wedge \dot{r})(\dot{r} \wedge r)/k}{\hat{r} \cdot e + 1} \qquad \text{- Thus } \dot{r}\dot{r} \wedge r - k\hat{r} \equiv e \text{ must be a constant.}$$

$$\Rightarrow \frac{d}{dt}(\dot{r}\dot{r} \wedge r - k\hat{r}) = 0$$

$$= r \wedge \dot{r}\ddot{r} - k\dot{\hat{r}} \qquad \text{-Assumtion (2.) - Conservation of angular momentum.}$$

$$= |r|^2 \dot{\hat{r}} \wedge \hat{r}\ddot{r} - k\dot{\hat{r}} = 0 \qquad \text{- As } \dot{\hat{r}} \dot{\cdot} \hat{\boldsymbol{r}} = \frac{1}{2}\hat{r} \dot{\cdot} \hat{\boldsymbol{r}} = 0.$$

$$- |r|^2 \dot{\hat{r}}\hat{r}\ddot{r} = k\dot{\hat{r}}$$

$$\Rightarrow \ddot{r} = -k\frac{\hat{r}}{|r|^2}$$

Additionally, with the knowledge of the orbital period of closed orbits (Kepler's 3rd law), the exact value of the constant k (mass m of the orbiting object is also suppressed in k here)can be determined. Note that the number of dimensions here is irrelevant, as opposed to the Gibbs vector theory where the dimension is fixed at 3, coupled with the fact that one has to invent an arbitrary structure to compute the antisymmetric vector product. Also, there are no dimension axes or a particular set of basis vectors. Nature doesn't pick a basis so why should we.

Thus we have realized vectors as direct concrete manifestations of the fundamental abstract laws of algebra, without losing any abstraction, while simultaneously achieving concreteness of realization, due to working with the canonical structure in the axioms and not adding any particular arbitrary (subjective)information. In this, we achieve an almost impossible and paradoxical union of the "painful abstraction" of algebraic identities and their concrete operational representation. Thus vectors and their algebra are described as simply as possible containing no superficial information unrelated to objective mathematical reality. Vectors being defined through the core algebraic operation of multiplication are numbers that are the fundamental manifestations of elementary algebra itself as they are directly connected to its first principles, making them a powerful representation tool that does not give up the abstraction of algebra's axioms and definitions even when used in concrete applications.

0.8 Symmetry of Euclidean space

No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the 'square root' of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors.

-Michael Atiyah

The normalization requires the possibility of extracting square roots. The constructions in Euclidean geometry with ruler and compass are algebraically equivalent to the four species and the extraction of square roots. A field in which every quadratic equation $x^2 - p = 0$ is solvable may therefore be called a Euclidean field. Our result is then that in every Euclidean field we can construct the spin representation; the Euclidean nature of the field is essential. The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures. In some way Euclid's geometry must be deeply connected with the existence of the spin representation.

-Hermann Weyl

Real numbers show obvious geometric properties, from constituting a real line, to completing the set of distance relations between points in space. We have seen they are a subsystem of a more general system of numbers with even stronger geometric properties.
The (geometric) algebra of 3-d Euclidean vector space is 8-dimensional and contains the vector space as its first-grade subset. The 2-vectors and 3-vectors are the algebras 2^{nd}-grade 3-dimensional and 3^{rd}-grade 1-dimensional subsets respectively. They both are real imaginary numbers leaving vectors as the only real Euclidean algebra subspace whose elements have positive squares. This explains the appearance of imaginary numbers in so many areas of mathematics and physics as they are revealed to be a natural structure of Euclidean geometry. Still, it is quite interesting that simple Euclidean space contains this "complex "structure. We will see in the following that this is a special case of an even more mysterious spin structure of Euclidean space and that the imaginary number i is just a special kind of spinor or quaternion.
As complex numbers form a 2-dimensional vector space, so do real imaginary numbers represented by 2-vectors together with reals form a 4-d space of real quaternions, hence the name.
Freedman Dyson wondered why quaternions are incompatible with vectors while they both describe the geometry of Euclidean 3-space.
The reality is that quaternions could not be more compatible with vectors as they are fundamentally simply products of pairs of vectors and act and compose the by same product as vectors (acting as reflections), also directly reflecting the fact(geometric law) that the composition of two reflections is a rotation through twice the angle between the normals to the reflecting planes :

A reflection operator U(x,y) is a linear operator in its first slot of which the component parallel to the second slot input is reversed in the output. If we fix the second slot to be a constant vector u we get a linear operator U(x) reflecting x in the plane normal to a vector u :

U(x)=$-u^{-1}xu = (-u^{-1}\cdot x - u^{-1}\wedge x)u = -x\cdot -u^{-1}u - (u^{-1}\wedge x)\cdot u = x - 2x\cdot u^{-1}u = x - 2x\cdot \hat{u}\hat{u}$ -From (3b)

$= x - 2x_{\parallel} = x_{\perp} = x_{\parallel}$

This can also be seen as :

$-u^{-1}xu = -u(x_{\parallel} + x_{\perp})u = -u(x\cdot uu + x\wedge uu)u$ - from (3a),(3b)

$= -x\cdot uu + x\wedge uu$ -From (1), (2)

$= -x_{\parallel} + x_{\perp}$

-The scalar part of the induced transformation of a reflection operator U(x)on the geometric product computes the effect a reflection operator U(x) has on the inner product :

$(Ux)(Uy) = (Ux)\cdot(Uy) + (Ux)\wedge(Uy)$

$= (-uxu)(-uyu) = uxyu = ux\cdot yu + ux\wedge yu$

$(Ux)\cdot(Uy) = x\cdot y$

-Thus reflections are inner product preserving operators called orthogonal transformations.

The determinant (action on the volume form) of reflections is the scalar which multiplies pseudoscalars as the induced transformation on the geometric product :

$(Ux)(Uy)(Uz) = (-uxu)(-uyu)(-uzu) = -u(xyz)u$

$-ux \wedge y \wedge zu = -x \wedge y \wedge z$ - Taking the trivector part , using (1).

Thus reflections aren't exterior product preserving (outer morphisms) as they don't preserve oriented n-volumes of n-spaces, which get multiplied by - 1 designating the value of their determinant.

Compositions of reflection operators determine their (linear) algebra :
$Vx = -vxv$

$VUx = vuxuv$
Reflection operators compose(multiply) through the fundamental multiplication of algebra, inherited from their direct vector (sandwich)representations.

ψ=VU - The composition of two reflections is a linear operator ψ ;

$\psi x = \tilde{\hat{S}}x\hat{S}$

-Where $\tilde{x}$ is the reverse operation on multivectors which reverses the order of multiplication, here equivalent to (quaternion) conjugation, $\hat{S}$ is a unit quaternion also called rotor (unitary spinor in 3-d)

$\hat{S} = uv = u \cdot v + u \wedge v = e^{(1/2)i\theta}$ -From (9).

-Parametrisation by the multivector exponential gives explicit (bi-vector) angle relations(parametrization).

$\tilde{\hat{S}} = \hat{S}^{-1} = uv = u \cdot v - u \wedge v = e^{(-1/2)i\theta}$

$\hat{S}\tilde{\hat{S}} = 1$ -Here we use unitary spinors to get (special) orthogonal transformations and not dilations.

$i = \dfrac{u \wedge v}{|u \wedge v|}$ -The unit 2-blade i determines the rotation plane and is itself a$(\pi/2)$rotation operator in its plane (ch. 6.)eq.(7).

$x_{\|}\theta = x \cdot \theta\theta^{-1}\theta = x \cdot \theta = -\theta \cdot x = -\theta x_{\|}$ From (3c),(2).

$x_{\perp}\theta = x \wedge \theta\theta^{-1}\theta = x \wedge \theta = \theta \wedge x = \theta x_{\|}$ From (3d),(1).

$\tilde{\hat{S}}x_{\|} = x_{\|}S$ -As in addition to the 2-blade θ , $\hat{S}$ has only a genrally commuting (sclalar)part which doesn't affect the sign.

$\tilde{\hat{S}}x_{\perp} = x_{\perp}\tilde{\hat{S}}$

$\tilde{\hat{S}}x\hat{S} = \tilde{\hat{S}}(x_{\perp} + x_{\|})\hat{S}$

$= \tilde{\hat{S}}\hat{S}x_{\perp} + \hat{S}^2 x_{\|} = x_{\perp} + e^{i\theta}x_{\|}$

Reflections act on the parallel component ($x_{\|}$) (to the operation generator) by reversing its sign, while rotations rotate it (by a radian angle parameter).
Rotations are thus shown to be special(oriented volume-preserving) orthogonal transformations made of atomic orthogonal transformations called reflections directly represented by vectors.

The orthogonal group O(3) composed of inner product preserving linear transformations(rotations and reflections) is the fundamental, canonical algebraic structure of Euclidean space and is a subgroup of the group of general linear transformations GL(n) (of on the Euclidean vector space). O(n) together with translations (which do become linear themselves in conformal models) form the group of isometries(distance preserving transformations) of Euclidean space called the Euclidean group E(n). Through elementary (geometric) algebra which contains not only numbers that represent only quantities, but also numbers expressing qualities (directions, symmetries), the nature of the particular geometry can truly be understood while also gaining immense computational power.
In this one can see the full richness of all its forms of symmetry, of the formless emptiness of space.

From (1) and (2) we see that a real quaternion can be expressed sum of a scalar (real) and a 2-vector (real Euclidean imaginary number).
Quaternions are numbers that dilate and rotate 3-Euclidean space. Unitary quaternions rotate 3-space. They are numbers expressing (rotational) symmetry of Euclidean space.

Quaternions are a special case of more general (transformational) numbers called spinors. Euclidean spinors are elements of the even subalgebra of geometric algebra. They are the numbers that dilate and rotate n-dimensional Euclidean spaces. Unitary spinors are (bi)rotations of Euclidean space(sandwich operators) forming the Spin (3) group with their composition (group product) simply being the fundamental (geometric) multiplication. Spin(3) determines a double cover of the special orthogonal group SO(3), (because opposite by sign spinors are mapped to the same rotation due to two sandwiching minus signs).

Generally, spinors appear as certain square roots of differential forms. Hence the ”square root of geometry”, as forms represent oriented volumes on general spaces.
In the sense of Cartan, a differential form is an element of the exterior algebra of a module* dual to a module of derivations (linear functions with a Leibniz rule) on the algebra of smooth functions on a smooth manifold. (*a module is a generalized linear space with ”scalars” not necceserely being an algebraic field like reals)
In the elementary (geometric) algebra interpretations of geometric spaces, points themselves can be modeled by algebraic elements and thus have intrinsic algebraic structure, without the need of imposing the algebra of functions to gain algebraic structure. Even if this is done, the algebra of multivector (general number) functions compared to scalar functions is much more powerful. In this inventing maps from points of space to n-tuples of reals and considering such strings of reals(which themselves are not numbers) becomes generally operationally redundant, to say the least. The only reason such maps are introduced is that real and complex numbers are not powerful enough as elementary algebraic (number) systems.
Thus despite its great power and utility, the theory of classical Cartan’s forms is not powerful (expressive) enough for differential geometry without introducing (differential) geography (charts, atlases). But geography is only partly natural and partly humanitarian. Differential geometry with Cartan’s forms can’t prove its flagship theorem, the general stokes, without invented mathematical structure. The reason again fundamentally lies in the elementary algebra upon which the theory is based.
Behind Stokes theorem, lies the algebra of scalar-valued forms, ultimately built on too weak elementary algebra limited to real numbers.

Algebra is the fundamental language upon which all equations are built.
Behind any fundamental equation lie the elementary equations of algebraic identities, therefore one better understand the foundations of algebra, for unlike set theory, it is a foundational theory with not only conceptual but immense computational implications
(there are subtle operational issues dealing directly with set theory and real numbers, for example when it comes to integration and measure theory, but this is avoided with suitable measure definition, for which in GA there is a particularly powerful way [2] as naturally contains oriented n-volumes).
Thus to even begin to understand general (spin) geometries the foundational understanding of the Euclidean case upon which they are based must be foolproof.
Unitary Spinors of the Euclidean 3-space(real unit quaternions) form a lie group (Spin(3))as they are naturally diffeomorphic to the 3-sphere. This unveils a very interesting connection (Hopf map) between the 2- and 3-dimensional spheres and Euclidean geometry.

General spaces, not just multivector lie groups, can also be treated as sets of points that are elements of geometric algebra. (Nash) Guarantees that n-dimensional manifolds can be embedded in a Euclidean space of sufficiently higher dimension. The arbitrariness of embedding can be handled and removed subtly and elegantly since any space can be seen as embedded in an infinite-dimensional geometric algebra[2].
General spinors are then just elements of even subalgebras of tangent algebras on abstract multivector manifolds.

0.9 The path forward

"I learned to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting interesting mathematics." ...

"Just by studying mathematics we can hope to make a guess at the kind of mathematics that will come into the physics of the future. A good many people are working on the mathematical basis of quantum theory, trying to understand the theory better and to make it more powerful and more beautiful. If someone can hit on the right lines along which to make this development, it may lead to a future advance in which people will first discover the equations and then, after examining them, gradually learn how to apply them."
— Paul Dirac

Beauty and simplicity were the basic guiding principles behind many great theoretical discoveries, but are often misunderstood and overlooked in modern research. Mathematical simplicity(beauty), if well defined (ch. (3.)) can be used as the principal scientific guideline, not only for "testing" and optimizing present theories but also as a scientific method for discovering new theories from scratch. This doesn't in any way diminish the importance of the experimental method, only supplements it and invites a healthy caution against hasty interpretations of data.

It is said only God and Dirac understood spinors, however, Dirac was never able to express them as simply as possible.
Towards the end of his life, he stood firmly against further building upon his QFT, saying it wasn't simple (beautiful) enough.
This was so for many reasons, none more obvious perhaps than neglecting the infinite (instead of infinitesimal) quantities. A more subtle reason is the fact that his whole theory is founded upon arbitrary representations of spinors, not on actual, real spinors, that are themselves the fundamental algebraic representation of the (special) orthogonal group, which in turn is the basic canonical algebraic structure of Euclidean geometry.

Einstein equated truly understanding something to the ability to express it simply. And to express something fully and maximally simply, that is understanding it completely, is to express it fully using nothing but the core fundamentals. This is exactly the way to do fundamental theoretical science.
He arrived at General relativity from purely theoretical considerations and mental experiments, as a means of mathematical (geometrical) extension and simplification of the classical Newtonian theory.
This is just one of many examples showing how the deeper the theory in mathematical physics, the more simple it is when expressed in invariant geometric terms [4].
GR has a remarkably simple invariant geometric foundation, unlike Newtonian theory(with its "zeroth law" of absolute for space-time) which becomes very complicated when expressed in a manifestly invariant (differential) geometric manner. On the other hand, Newton's axioms take a very simple form when expressed through Cartesian geometry in which the Einstein field equation $\mathbf{G} = 8\pi\mathbf{T}$ for the GR metric becomes a complete mess. Non-invariant formulations can be a practical shortcut in certain engineering applications, but they have no place in fundamental research, for both practical and conceptual reasons. A common misconception is that the field equation $\mathbf{G} = 8\pi\mathbf{T}$ is just a compact representation of the 10 differential equations for the metric coefficients, or that the position vector $\vec{\mathbf{r}}$ actually just compactly represents 3 spatial coordinates, and not the other way around (even claimed by Feynman). The reason for such misunderstandings is that there is no common knowledge and understanding of how to practically and concretely manipulate abstract vector (field)equations and geometric objects in general.
A fundamental reason why gravity (in the standard GR formulation) can't be quantized is that the Einstein field equation can't be applied in a way that is without any kind of coordinate dependence. (Non-smooth quantum spaces remind us more explicitly that nature doesn't pick coordinate systems). Thus GR is generally covariant only in theory, not in practice. Thus it is either not the fundamental mathematical interpretation (theory) of gravity, or only its underlying mathematics is flawed and not fundamental. To discover a fully covariant unified theory, one must first discover a completely invariant unified theory of (the underlying) mathematics.
While (Clifford) Algebra of spinors and differential geometry are on an abstract level very well understood, at least algebraically and in the formal sense, the universal geometric algebra is an even more abstract and unified theory [2] (it is not just a particular algebra, nor a purely metrical theory) that remarkably is also simultaneously intrinsically equipped with efficient geometric (including projective) realizations and powerful invariant ways to do concrete, specific calculations directly from abstract

axioms.

Quantum theory can also largely be demystified and conceptually simplified by its explicit deduction from elementary algebraic, geometric and statistical considerations[7]. However, to the knowledge of the author, no formulation of general relativity or quantum field (Dirac) theory has been discovered that is in accordance with the proposed scientific method, (without employing arbitrary structure). This is either indicative of the fundamental incompleteness of these theories or the shortcoming of the sophistication of understanding, interpretation, and application of fundamental elementary algebra.

It would be interesting to see what do further scientific investigations in mathematical reality say about the physical world. In other words, what kind of a theory could be constructed directly from fundamental mathematical invariants in a way of working with them without any arbitrariness. Like the spin representation, there are many incredible mathematical structures,(Hopf fibrations, exceptional lie groups, symplectic forms,..etc), with many other fundamental objects in topology and cohomology, which are all more and more revealed to be deeply interconnected. They would be interesting to study in their own right, in an invariant scientific way used to an extent in all history's great theoretical discoveries.

The literature, however, is full of redundancy and is in desperate need of standardization. We can see that just by considering a relatively sophisticated theory of forms which reduces Maxwell's equations to two along with many additional interesting and useful revelations [3]. Interior and exterior products are the two basic antiderivations of the algebra of forms, recovered directly in the (tangent)geometric algebra which reduces the set of equations to just one, with considerable operational simplifications [5]. Thus even the powerful theory of forms (including also the more general vector-valued forms [3]) is not as efficient and powerful as the more fundamental algebra that contains them, regardless if it is done in as invariant and fundamental way as possible for that theory, which in most instances isn't the case anyway.
This is just a mild example of an enormous problem in mathematics, drawing its origin all the way back to the 19th century and the calculus of Gibbs, with most formalisms particularly connected to geometry containing only fragments of mathematical reality wrapped in redundant structural baggage. Even theories with an accent on invariance, like differential geometry are founded on manifolds without intrinsic fundamental geometric/algebraic structure, and instead use invented structure like charts and atlases, with conceptual understanding only clouded by the lack of fundamentality of a particular formalism, instead of being broth more to the light.
With this also comes practical inefficiency and an ominous warning that the apparent development and an explosion of the now individually inaccessible large body of mathematics with its ever more branching fields and increasingly stricter specialization, might in many ways not be much more than just smoke and mirrors, a large wall built upon the foundations of salt and sand, sealing the truth ever further away from understanding of men. This especially is true for the theories of physics, and the promises of a unified theory (string theories) that now almost appear as an ingenious disinformation ploy to lead the entire community into fooling itself in conflating the exploration of hypothetical landscapes of multiverses with actual scientific progress, all the while completely disregarding the already very mysterious "nuts and bolts' aspects (spinors, exceptional groups, etc.) of our physical and mathematical reality. Such approaches were always destined to fail miserably, especially without the right invariant geometrical tools needed to discern at least mathematical fact from fiction.
If one does the mathematical science right, he is on a certain line of progress, both in understanding and application as even the most distant and abstract mathematical advances have also always been shown to be inseparable from a real-world application.

Conclusions

Spinors are nature's way of getting structure and complexity out of "nothing". A simple bare Euclidean space contains this hidden structure, as also its core symmetry space SO(3), is no longer a simple, (-y connected)space, of which the spin(3) group of unitary spinors(quaternions) is a simply connected double cover and the fundamental manifestly invariant algebraic expression.
Euclidean spinors arise naturally as elements of the even subalgebra of the fundamental (geometric) algebra of the Euclidean vector space which defines the affine Euclidean space through group actions (geometric transformations) on its points. Euclidean vectors naturally generate their intrinsic algebra if the more general non-symmetric vector product is adopted instead of the special case of the symmetric inner product. This allows a completely expressive, invariant unification of multilinear algebra and calculus in their full geometric glory allowing for efficient calculations with fundamental geometric objects instead of their arbitrary representations.

Spinors turn out to also be the fundamental reason why any manifest (physical) structure exists instead of a vacuum of nothingness, giving rise to matter with spin-1/2 wavefunctions and interactions described by integer spin wavefunctions. Spin 1/2 essentially just implies orientation entanglement with all other spatial points, thus a 720° symmetry and 360° antisymmetry, which is possible due to the natural (spin) structure of ordinary Euclidean space, which also naturally includes complex numbers and quaternions in their true geometric form)

Ultimately the magic and the fundamental mystery of quantum theory might lie in the spin "1/2" representation itself, which in turn can be clearly understood as the fundamental representation of elementary(geometric) algebra. Understanding this does not eliminate the wonder in the discovery, the wonder which is the geometric algebra itself, just pushes it deeper to the very foundation.
This is where the intellectual mind and Platonic reductionism meet their fundamental limit. By the scientific method, the clutter of imaginary structure is removed and the understanding becomes as clear as it can be, but the fundamental mystery shines through ever more brightly, revealing ultimately that it will never be fully tamed by the intellectual reductionistic mind. There will always be (algebraic/geometric) mathematical primitives that are by definition(of being primitives, and Godel) not completely within intellectual grasp. But this is not just a matter of logical incompleteness.
If no arbitrariness is introduced one can follow the conceptual understanding arbitrarily deep, and follow a train of thought as far as it can go, all the way to the very core of mathematical reality. Yet still, there it stands strong, the basic wonder at the very foundation of (mathematical) creation itself, and of all the forms of beauty that are to a clear mind revealed to arise so wondrously out of seeming nothingness.
As any scientific theory is ultimately rooted in this, science and mystery can be reconciled. Fundamentally, one does not eliminate the other.

Geometric algebra and its contributions are often dismissed asserting they present no new discoveries. On the contrary, the beauty and truth of geometric forms are through it for the first time are seen as they truly are, uncluttered from the baggage of arbitrary structure.
As is with any other theoretical work, when done using the scientific method, nothing imaginary is created. So is also true for GA, when done right, everything is discovered.
And there lies ahead a whole universe of real mathematics left to discover.

For I remain completely confident that the labour which I have expanded on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remained unused for another seventeen years or even longer, without entering into the actual development of science, still the time will come when it will be brought forth from the dust of oblivion, and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me in a position (which I have up to now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich further these ideas, nevertheless there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into living communication with contemporary developments. For truth is eternal and divine, and no phase in the development of truth, however small may be the region encompassed, can pass on without leaving a trace; truth remains, even though the garment in which poor mortals clothe it may fall to dust." Hermann Grassmann

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