

THE ONTOGENESIS OF MATHEMATICAL OBJECTS

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The mathematician has the peculiar freedom of being able to bring into being at will whole universes of objects, and the peculiar enjoyment of being able to sit back and toy with his newly created universes, confident that they will remain in being in such a way as to be accessible to future mathematicians.

A parallel "ontogenesis" takes place with respect to literary creative activity, where the central importance and indeed indispensability of the *work* is taken for granted. We wish to argue that all ontogenesis involves the ontological dependence of the created objects upon some type of symbolic structure analogous to and on the same ontological level as the literary work, and that this dependence has important consequences for the general nature and specific properties of created objects, consequences which have not been taken into account by traditional philosophies of mathematics.¹

1. *Natural Objects and Created Objects*

Consideration of the way in which mathematical objects are given to consciousness reveals that such objects fall into two groups: (i) "natural" mathematical objects (such as 1, 2, 3, etc.) which are given as existing "in themselves" such that we cannot imagine their being otherwise; and (ii) "unnatural" mathematical objects (e.g. large infinite sets) given as having been "invented" in a series of creative conscious acts. It seems clear that a group (ii) object becomes public property, i.e. becomes freely accessible and intersubjectively identifiable only when the appropriate creative acts have become embodied in some type of

mathematical work. To the extent, therefore, that mathematics is concerned with group (ii) objects, philosophical reflection upon it must concern itself with works.²

Unfortunately it has always been possible to deny, monistically, that *any* objects fall into group (ii); platonism,³ for example, claims that all objects "in reality" belong to group (i) but that mortals find it difficult to recognise that this is so. The inadequacy of platonism is immediately brought to light if we attempt to affirm its analogue within literary theory by claiming that, e.g., a pre-existent Ophelia was "discovered" by Shakespeare. The alternative means of denying the existence of group (ii) objects is to restrict the domain of "mathematics" (e.g., to finitistic or predicative reasoning), castigating work outside one's chosen boundary as "mere ornamentation"; this has parallels with restricting the term "literature" to apply only to historical novels, or to novels which one thinks of as being "true to life". Among the traditional philosophies of mathematics only formalism⁴ comes near to a recognition of the freely-creative, proliferating nature of mathematics and of the indispensable role played by symbolism once we leave the central core of group (i) objects; but for positivistic reasons formalists

1. Nor, indeed, by the philosophy of literature, where they have been systematically investigated only by R. Ingarden, especially in his *The Literary Work of Art*, Eng. trans. of 3rd German edn., Evanston: Northwestern, 1973, hereafter referred to as LWA. Single quotation marks will be used to indicate the use of terms derived from LWA; a more broadly-based account of some of these is given in my review of this work on pp. 42-45 below.
2. In particular with all mathematical works (from which we exclude merely pedagogical textbooks, tables of logarithms, applied mathematical works, etc.) which effect their own ontogeneses: e.g., Cantor's works on set theory in which he put forward a theory of transfinite numbers. Despite the central importance of such works they will inevitably be rare as compared to "normal" mathematics whose works deal either with group (i) objects (especially with an eye to extramathematical applications) or with proving new results about objects which have been brought into being by previous ontogenetic works. Hence we can understand why Ingarden, in his consideration of the ontological status of "borderline cases of the literary work of art" (LWA, p. 328-30) should have overlooked the peculiar ontological and aesthetic properties possessed by works like those of Cantor, by treating all mathematical works as if they were in these respects no different from other kinds of "scientific" work.
3. See P. Bernays, "On Platonism in Mathematics" (1935) Eng. trans. in Benacerraf and Putnam, eds., *Philosophy of Mathematics*, Oxford: Blackwell, 1964.
4. See H. B. Curry, *Outlines of a Formalist Philosophy of Mathematics*, Amsterdam: North-Holland, 1951.

have turned away from the rich domain of creative mathematical activity as it receives concrete expression in actual books, journals, lectures, etc., turning instead to the meagre world of formal systems, a world which has never played a significant part in actual mathematical practice and which, as is shown by Gödel's Incompleteness Theorem, has essential limitations upon its power relative to that of informal mathematics.

Our adoption of the dichotomy between (in Ingarden's terms) 'autonomous' and 'heteronomous' mathematical objects⁵ does not imply that we can draw a sharp line between the two groups. We are reminded of the distinction between analytic and synthetic statements, the validity of which depends only on our being able to recognise clear cases of either category. Nor do we wish to rule out the possibility that there is no corresponding dichotomy on the side of the objects themselves,⁶ for such a possibility could do nothing to detract from the validity of our distinction on its own (phenomenological) terms.⁷

The existence of an autonomous core and of an ever-growing periphery of heteronomous objects is something which is found in every case of a non-experiential domain becoming the subject of theoretical investigations. For example, the autonomously existing subject-matter of theology consists of God Himself (if He exists); all other gods, etc., brought into being by false theological works exist merely heteronomously.

When we move into the sphere of objects of

experience this opposition corresponds broadly to that which holds between theorizing about the autonomously existing real world (e.g. in science and in history), and the creative constitution of heteronomous non-actual alternatives to that world, which takes effect, characteristically through literature. Reflection upon the real world therefore takes two modes: the theoretical mode, which concerns itself with pre-existent objects; and the aesthetic mode, which strives to create universes alternative to that which pre-exists. We shall argue that these two modes of reflection can also be distinguished in mathematics.

2. *Ingarden and the structure of the literary work*

In our first attempts to come to a conception of the nature of heteronomous objects we are fortunate that this task has already been carried out for one species of such objects by Ingarden in his brilliant analysis⁸ of the ontological structure of the literary work, and in what follows we shall attempt to duplicate this analysis in so far as it can be applied within the sphere of mathematics.

Ingarden offers an account of the phenomenologically given ways in which a literary work makes it possible for us to 'project' its characters, etc., into existence in as-if-real settings. A reading of a work results in what Ingarden calls a 'concretisation' of its characters, plots, rhythms, meanings, word-sounds, etc., on the part of the reader. Each one of these concretised components made actual in a faithful reading is correlated with a non-actual 'derived' component on the side of the work. The latter can be thought of as idealised objects precisely determined by the text

5. Autonomous or 'self-existent' objects are fully determined in themselves; they include all real and all ideal objects (such as blue, good, circle, etc.). Heteronomous objects are objects whose existence is dependent upon other objects, in particular upon minds; this category therefore includes all objects of thought, dream-objects, all ideas and images (considered not as psychic disturbances but as intentional contents), and all meanings. See LWA, pp. 117-125.
6. This would be the case given either the autonomous existence of something like Plato's heaven containing even those mathematical objects which are phenomenologically given as heteronomous, or the mind-dependence of seemingly autonomous mathematical objects.
7. The dualist philosophy of mathematics which is implied finds some support in certain views of Leibniz (e.g. "On Some Philosophical Axioms and Mathematical Fictions" (1692), Eng. trans. in P. P. Wiener, ed., *Leibniz: Selections*, New York: Scribner's 1951, pp. 70-73), of Hilbert (in his doctrine of *Idealelemente* thought of as supplementing the "real" objects at the core of mathematics, see "On the Infinite" (1923), Eng. trans. in van Heijenoort, ed., *From Frege to Gödel*, Harvard, 1967, pp. 367-392) and of Cassirer ("The Object of Mathematics", in vol. III of *The Philosophy of Symbolic Forms* (1929), Eng. trans., Yale, 1957, pp. 357-405). All of these thinkers, however, regard the commitment to heteronomous objects as something which in each case must have a pragmatic justification.
8. LWA (see note 1 above).

of the work, and as forming, when united together, what Ingarden calls the ontological 'structure' of the work. This structure is a transcendent higher-order object independent of any given concretisations and mental acts on the part of readers and also — once the work has been created — independent of the mental acts of the author who effected its creation.

This structure has a 'stratified' character which can be specified as follows: the work is conceived as a polyphony of four different voices, each of which lends its own aesthetically and ontologically valent contribution to the whole. The lowest stratum consists of an appropriate species of *symbolism*; this makes accessible the second stratum, that of *meaning-units*; above this is the stratum of *objects* 'represented' in the work. We cannot experience the objects given by a work "from all sides" as we can real objects, but only within or from prescribed vantage points, i.e. in 'aspects' determined by the work: hence it is necessary to distinguish the separate stratum of *sequences of aspects*.

The units which make up each stratum have determinate relations with the other strata in virtue of their being *intentional*. Symbols possess intentionality in being transparent to the meanings which have been bestowed upon them within the given language, and these meanings in turn point us out toward the objects to which the symbols refer. It follows that the symbol itself must be distinct from and transcendent to mere marks on paper or concrete sound-material, since the latter possess no intentionality; they are, so to speak, "inert". And the work as a whole possesses its own characteristic intentionality in referring its world of objects outward, setting it, illusorily, as if existing in some corner of the real world. It is consequently to be distinguished from any actual books, papers, or readings; for until the latter become constituted as intentional, i.e. become *works* — and not mere physical objects as they would be to a tribe which knew nothing of literature — they can have none of the effects which works have on our cognitive and spiritual lives.

It is clear that what has been said can be applied to mathematical (and in fact also to scientific and historical) works. Only truly ontogenetic works, however, share with literary works the bringing into being of their own object-strata; recognising that some mathematical works fall into this category allows us for the first time to take full cognisance of the fact that all mathematical statements are about mathematical objects *sui generis*. Positivism and psychologism deny this fact, claiming that in some "round-about way" they are "about" physical objects (such as marks on paper), or that they "express" psychological data about "the most general ways in which human beings think". To bring out clearly the distinction between such doctrines and our own, we must emphasise what Ingarden has shown, namely that works are intentional structures, transcendent to everything on the level of the concrete, including all mental acts and images. In particular, their heteronomous objects are in no way mentalistic: having been created by consciousness they have, so to speak, broken free to live a life of their own.

3. *Mathematical Works and Mathematical Theories*

It is the existence of mathematical *theories* which accounts for the constraints upon our freedom of ontogenesis in the mathematical domain. A mathematical theory exerts a systematic pressure upon the creator of mathematical works in such a way as to make him feel a continually diminishing degree of arbitrariness in his object-constituting axioms and definitions. This has led many philosophers of mathematics to assert the ontological primacy of theories, as opposed to what they would regard as merely contingently existing mathematical works, arguing that cognitive "pressure" can be exerted only by ideal theories which exist autonomously. We shall argue however that at least those theories which concern heteronomous objects are themselves heteronomous; and that the constraint which the mathematician feels derives from internal structural consequences of the manner in which the theory is brought into being by his object-creating acts. It will follow from this that a creative mathematical theory does not exist in a fulfilled

way until these acts have become embodied in appropriate works.

Real objects are merely contingently in existence, having no *a priori* relations with each other; but ideal objects do possess such relations, as also do the as-if-ideal objects brought into being by creative mathematical works. It follows that sentences of mathematical works in general possess essential connections (e.g. logical connections) with each other. Such connections can exist between the sentences of different works, and in so far as the mathematician is aware of the connections of his work with future possible works he will be aware also of a network of sentence-intentions emanating from his work, capable of being fulfilled in future "derived" works. These in turn will be capable of generating "derived" works of their own, and thus we can think of mathematical works as falling into sequences. This suggests that we define a theory as a structure coincident with the set of sentences of a given sequence of works ideally conceived as having been brought to its completion.⁹

The intentional self-transcendence of mathematical works applies also to works-in-progress: even though the network of sentence-intentions and also, therefore, the correlated theory are then only projectedly in existence, the mathematician is constrained by his unfulfilled awareness of this network and of the fact that his created sentences must be logically and aesthetically consistent with it.

Having been brought up on a diet of autonomous mathematical objects we have acquired the prejudice that mathematics is a science with its own pre-existent domain alongside other sciences. This prejudice is nurtured by the sequential aspect of works, which gives mathematics a cumulative quasi-scientific character even where it ceases to have any autonomous subject-matter. But in mathematics we have the possibility of several essentially different and conflicting

sequences of works being derived from the same initial work, or in other words of a "branching out", a multifurcation of mutually conflicting extensions of the one initially projected theory: this is quite impossible for a science, where there is always the necessity of discovering which such extension is "true" of the pre-existent domain, and of then eliminating those extensions which remain.

4. *Heteronomy and Multifurcation*

A heteronomous domain will clearly possess unusual ontological properties from the point of view of inhabitants of autonomous domains. Its objects, need not, for example, satisfy the law of non-contradiction; where we have an x for which Px and not- Px are both true (in different works) we shall have to conceive of a ramification in the domain. Along one "path" we find the object x given as having the property P , along another path the same object has not- P .

Those of us who have a taste for desert landscapes are right to demand to know what purpose is served by committing ourselves to such a strange domain in our philosophy of mathematics. Our answer is that it is only by a commitment of this sort that we can take into account the existence of thriving schools of mathematics, radically alternative to each other in the sense that their works *conflict ontologically*.¹⁰ Even the adoption of one privileged school of mathematical activity, accompanied by a dictatorial dismissal of non-absorbable results achieved by other schools as "not mathematics" would not, it seems, exclude heteronomy and multifurcation from the resultant universe. This follows from a series of post-Gödelian results which show that, for a sufficiently rich mathematics, a stage is always reached where the decision between two or more conflicting possible determinations or extensions of the object-domain cannot be made without going beyond the intuitive means which have hitherto

9. The theory is not however *identical* with this set of sentences for it seems that we must place it on a higher ontological level such that it holds, perhaps, the same position in mathematics as is occupied by the *genre* in the domain of literature.

informed that mathematics.¹¹ If we are not of the disposition which is prepared to accept this sort of “multifurcation” then we are required to creatively develop further, extra-intuitive machinery; but where the continued validity of that machinery is crucially important we have no pre-determinate object-domain against which it can be tested, hence its acceptance depends merely upon considerations like the “elegance” of the resulting theorems and proofs and the “smoothness” of the resulting ontology. Once accepted, however, such machinery can do no more than transport us to a further and more remote branching-point in mathematics, and aesthetically grounded decisions come once more to be required as to which of the conflicting newly-alternative branches is the one which we are to follow. Myhill¹² points to this open-endedness of mathematics (itself a consequence of Gödel’s Incompleteness Theorem) as evidence of the essential and permanent dependence of mathematics on creative insight.

5. *Multifurcating Ontologies.*

In order to develop a theory of mathematical multifurcation we shall first consider the parallel multifurcation which is clearly endemic in the universe of literary objects. Although the latter have no *a priori* relations between each other, we can distinguish, e.g., geographical and temporal relations between such objects. This suggests the possibility of the identity of two objects which are in conflict with regard to their properties. This (counter-example of Leibniz’ Law) is merely a

trivial consequence of our creative freedom in constituting objects in a literary work. I can, for example, write a novel in which a recognisable *Café X* is represented as being empty at time *t* and a second novel in which it is represented as being full. We can account for this conflict without relinquishing the ontological status and identity of the object which is *Café X*, by recalling Ingarden’s distinction in the structure of the literary work between the stratum of objects and the independent stratum of aspects in which those objects are given. We can accede to the given heteronomous object in either alternative aspect, of fulness and emptiness, by concretising the appropriate novel.

The universe of all mathematical objects, whose extent depends upon the currently existing library of creative mathematics, is similarly capable of being inhabited in different ways, depending upon which works from that library we have chosen to concretise. It is “ramified” in the regions which consist of objects for which we have conflicting characterisations in different works.¹³ At its core we find all the simple mathematical objects as they are given to consciousness. More or less fragmentarily “attached” to this core will be the clusters of created mathematical objects. It is this attachment to what is familiar, effected via definitions and proofs which guide us further and further outward into regions ever more ethereal, which explains the accessibility to consciousness of the objects in those regions. We can say that consciousness requires the aid of such an attachment in order to arrive by stages at objects which

10. Two works conflict ontologically when there is a domain of objects which they have in common and which they give in two mutually inconsistent systems of aspects. Works of classicism (i.e. standard mathematics) and of intuitionism (see section 6 below) can “conflict” in this sense, as can works of Euclidean and non-Euclidean geometry. This concept is discussed by S. Körner in the section “On the Philosophy of Competitive Mathematical Theories” of his paper “On the Relevance of Post-Gödelian Mathematics to Philosophy”, (Lakatos, ed., *Problems in the Philosophy of Mathematics*, Amsterdam: North-Holland, 1967, pp. 118-137).
11. The two classic results in this respect are given in K. Gödel, “The Consistency of the Continuum Hypothesis”, *Ann. Math. Stud.*, 3, Princeton, 1940, and P. J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, 1966; these together show that an infinite number of possible values for one particularly important mathematical object (the number of points on a geometrical line) are consistent with the accepted axioms of set theory. This gives rise to a multifurcation in the path of set theory (see A. Robinson, “Formalism ’64”, in *Logic, Methodology and Philosophy of Science*, Amsterdam: North-Holland, 1965, especially p. 233).
12. See his “Some Philosophical Implications of Mathematical Logic”, *Rev. Metaph.*, VI, 1952, pp. 165-195, and the references there given.
13. Where we talk of a ramified universe of mathematical objects, Körner, *op. cit.*, talks of different “possible worlds” to which conflicting works could be *applied*.

it could otherwise only emptily intend, the symbols providing a “hinge” for thought about objects far too sylph-like and volatile to be directly apprehended.¹⁴

It is only by following through one particular path of attachments through the universe as determined by a sequence of mathematical works that the mathematician can come to inhabit a peripheral region of created objects, and he will then experience those objects only from within the framework of aspects imposed by the given sequence of works. Hence ramification in the regions of created objects can have no mathematical consequences, since it is never possible to accede on the mathematical level to a given x represented as being such that both Px and *not- Px* , even though this is emptily possible on the level of philosophical reflection upon the entire domain of created objects.

Having arrived at the periphery of the mathematical universe the creative mathematician finds a number of possibilities are open to him. He can attempt to further extend the universe by effecting the ontogenesis of even more remote objects, which he does by making definitions and proving results about the defined objects in such a way as to attach them to the already established regions. By reflection on objects already concretised the mathematician can come to conceive other creative ways of effecting changes in the universe. His knowledge of the objects created by two disparate works might lead him, for example, to a more or less determinate intuition of a relation (e.g., the relation of identity of objects), between the two works. He would then have the task of developing a third work which would allow the re-achieving of the two systems of

objects in such a way as to present them in an aspect in which they were given as one and the same.¹⁵ It is the possibility of this sort of reflection bringing about imaginative changes in the fabric of connections between different regions of the universe which accounts for the ways in which creative activity at the periphery of that universe can fruitfully affect our conception of objects at the centre, as e.g., when we find high-power axioms of infinity can have number-theoretic consequences in the finite domain.

6. *The Law of Excluded Middle.*

Heteronomous objects need not satisfy the law of non-contradiction, and nor need they satisfy the law of excluded middle¹⁶: we cannot assign a role to a statement like: “either Ophelia was right-handed, or she was not”. The ‘spots of indeterminacy’¹⁷ possessed by literary objects follow from the fact that they are given only in a finite number of symbol-determined aspects, where real objects are given in an ever-changing sequence of concrete, intuitive aspects. The light thrown upon creative mathematics by considering it as a literature of the ideal suggests that such indeterminacy occurs also with respect to mathematical objects. The platonists’ doctrine implies that this could be no more than a provisional indeterminacy, and that it is always “in principle possible” to determine for any property P which of the two halves of Px or *not- Px* was true of a given x ¹⁸. They thus justify the explicit use of this law to obtain mathematical results. Certainly it is true that we can for various (e.g., applied mathematical) reasons eliminate by decree the indeterminacy from a domain of created mathematical objects; but this can also be achieved in literature. For example, by staging a dramatisation of Flaubert’s novel we can “find

14. For this attachment to be effective mere symbols, even when ordered into definitions, proofs, etc., are not sufficient; creative mathematics, like every other form of art, takes pains in ensuring that its creations are made as accessible as possible, consistent with the subtle complexity of each particular case. Definitions and proofs will therefore be arranged as coherently as possible within works, and moreover within works whose concretisation involves the most immediate and perspicuous giving of their objects.
15. This has obvious relevance to the problems Frege was trying to solve in his “On Sense and Reference” (1892), Eng. trans. in *Philosophical Writings of Gottlob Frege*, Oxford: Blackwell, 1960.
16. “What we mean when we say that an object does not *really* exist, is that it does not satisfy the laws of two-valued logic.” Michael Dummett (in lectures, Oxford, 1973).
17. LWA, pp. 246-287.
18. This is argued by Gödel, e.g. in his “What is Cantor’s Continuum Problem?” repr. in Benacerraf and Putnam, *op. cit.*, pp. 258-273.

out" how tall Madame Bovary is by measuring the principal actress. Brouwer and his school have shown that it is possible to develop a rich mathematics without making this sort of arbitrary imposition of determinacy upon created mathematical objects, despite the loss of power consequent on the refusal to use the law of excluded middle as a mathematical tool.¹⁹

The failure of the excluded middle law for heteronomous objects does not imply that intuitionism has a higher 'truth-value' than platonistic mathematics. On the domain where "truth" is an evaluative criterion relevant to mathematics, namely the central autonomous domain, the two schools effectively coincide. Further, any value which could be imputed to intuitionism on the ground that it was more adequate to the meta-mathematical properties of heteronomous objects could also be imputed to mathematics which disregarded the law of non-contradiction. No one²⁰ is moved by the possibility of such a mathematics, since it would have none of the aesthetic justification seen to be possessed by platonism and intuitionism conceived as alternative extensions of the central autonomous core.

7. Formalisations, Generalisations and Original Creations.

All judgements possess an intention to 'match' an appropriate autonomously existing state of affairs. This is the case even for ontogenetic quasi-judgements where no such state of affairs actually exists; the matching-intention will then not possess the 'seriousness'²¹ of, e.g., historical judgments, but to the extent that quasi-judgments are organised into the coherent framework of a work, their matching-intention has none of the total unseriousness possessed by isolated quasi-judgments (e.g.,

in idle imagination). In the case of a historical novel the matching-intention is toward actual states of affairs in the real world with the aim of making possible a vivid re-experiencing of the relevant events by presenting them in an original sequence of aspects. In period novels the intention is merely toward types of such states, with which we are then allowed free play of imaginative rearrangement. In the case of purely imaginary literary works (symbolistic poetry) the matching-intention is merely a totally bare and illusory "setting" of objects into some unspecified and indeterminate corner of the real world. Ingarden's 'degrees of seriousness' of the matching-intention of quasi-judgments can also be distinguished in mathematical judgments and quasi-judgments, and they lead to the following broad classification of mathematical works:

(i) works (e.g. Euclid's *Elements*) whose subject-matter is pre-given as existing autonomously or as established in a pre-existent mathematical work, and which attempt to be as faithful as possible to this subject-matter itself, whilst making it possible for us to re-achieve it within a new and perhaps richer system of aspects.²² We can think of category (i) mathematical works as re-expressions and in particular *formalisations* of pre-given theories, the motive behind such re-expression perhaps being the need to remove some inconsistency or inelegance in the initial theory.

(ii) works which *generalise* from pre-given subject-matter. Here the matching-intention is to types of mathematical states of affairs rather than a complete faithfulness to the individual objects themselves. Generalising mathematical works therefore effect their own ontogeneses but still within the constraints imposed by the pre-existence of the initial object-domain.

19. This refusal is of course only one element in the intuitionist programme; see A. Heyting, *Intuitionism: An Introduction*, Amsterdam: North-Holland, 3rd edn., 1971.
20. (Except possibly Wittgenstein; see his *Remarks on the Foundations of Mathematics*, Oxford: Blackwell, 1956, especially II 77-88, III 56-7, and V 28).
21. See LWA, pp. 160-173, especially p. 168.
22. For example in the finite arithmetical parts of different formal set theories we find the natural numbers represented within a highly original system of aspects, since: they are given as if they were equal to particular sets. From within such a theory (in particular when we are engaged in the concretisation of an appropriate set-theoretical work) it will be as though we are looking at the natural numbers through the wrong end of a telescope: only certain of their properties will survive.

(iii) truly creative mathematical works. Here there is absolutely no intention of an exact or of a generalised matching with any pre-given objects, rather do we have an imaginative *extension* of the universe of such objects further into the emptiness of the ideal. Such an extension might issue merely out of a heuristic delight in pathological cases for their own sake, or out of an arbitrary cancellation of axioms within familiar theories; interestingly original ontogeneses can also be effected, however, by the making explicit of tendencies already contained in established mathematics.

8. *Historicity in mathematics.*

We have so far refrained from drawing one important consequence of the ontic dependency of created mathematical objects on the acts of mathematicians as embodied in works, a consequence of the fact that such acts take place in time. Aleph seventeen, we can say, had no pre-Cantorian ontological status, just as Madame Bovary had no pre-Flaubertian ontological status. Only at times subsequent to the existence of literary or creative mathematical works will the objects which they bring into being be accessible to consciousnesses, and phenomenological ontology grants ontological status to objects if and only if there is given a determinate means of accession to them.²³

Peripheral regions of the mathematical universe are therefore to be conceived, along with man's other cultural creations (gods, institutions, ideals, etc.) as burgeoning up at determinate points in time.

Ingarden's discussion²⁴ of the relations between the literary work and the cultural experience of

human beings shows that the characteristic ways in which these relations change over time endows the work with a sort of 'life'. Subsequent to the first appearance of a novel, say, there is an accumulation of concretisations making possible a gradually richer appreciation of the total structure of the work, bringing forward hitherto hidden meaning-levels on which the work can be read, finally perhaps leading to the "death" of the work as it ceases to be the subject of aesthetically serious concretisation. Concretisation can thus be said to undergo different possible species of evolution, and the transcendent structure of the given work, derived from its concretisation-possibility, will in each case suffer a parallel derived evolution.

Mathematical works as transcendent structures also undergo different species of evolution (marked e.g., by the appearance of successive editions of the work). As an example of one possible evolution "schema" we can consider a creative mathematical work of the type which results from more or less arbitrary symbolic manipulation. Such a work begins as "dubious"; its objects, being given as shamefacedly heteronomous, will be regarded by the mathematical establishment as baroque illusions, tricks of symbolism of the given work. But should this work prove mathematically fertile by engendering new works, by making possible the solution of important central mathematical problems, or by becoming applied in areas external to mathematics — all of which are typical "events" in the life of a mathematical work — it will eventually become itself a part of established mathematics, when the objects it had brought into being will be regarded as having been "in themselves" ontologically autonomous from the beginning.²⁵

23. This "weak" existence-criterion is fundamental to the Husserlian tradition to which Ingarden, of course, belongs. Cf. e.g. Oskar Becker: "The absolutely universal claim of transcendental phenomenology: that all being is synonymous with being constituted", *Mathematische Existenz*, Halle a.S., 1927, repr. from Husserl's *Jahrbuch*, Bd. VIII, p. 502n. Note that this criterion is yet not so weak as that of Meinong, for whom an appropriate intention is sufficient (see his "Principle of the Unlimited Freedom of Assumption", in *Über Möglichkeit und Wahrscheinlichkeit*, Leipzig, 1915, p. 282). Meinong's criterion "has all the virtues of theft over honest toil", whereas our criterion requires that for an object to exist a certain amount of honest toil must first have been undertaken by the creative mathematician, to whom we are merely extending the hand of ontological credit.

24. LWA, p. 331-355.

25. Compare the initially hostile reception and the final acceptance of irrational numbers by the Greeks and of Cantor's infinite sets by 19th century German mathematicians. More complicated case-studies are provided by set theory in the twentieth century, complex numbers, the infinitesimal calculus (up to and including the invention of Non-Standard Analysis), and Hamilton's theory of quaternions.

Every event in the life of a mathematical work causes characteristic changes in the ways in which it can be concretised. Perhaps the most important such event would be the "death" of the work, which comes about when faithful animating concretisations are no longer possible. This can either be because logical or aesthetic aspersions have been cast against the work, or it can be a matter of accident. We consider in turn each of these three cases:

(i) Following the appearance of, for example, an inconsistency proof for a given mathematical work, it would no longer be possible to effect a mathematically serious concretisation; concretisation could be achieved only emptily on the level of philosophical reflection. It could be said therefore that for all mathematical purposes its objects had been "destroyed", and that in general there was an intermittent *disappearance* of whole regions of objects from the mathematical universe. It might be thought that the indestructibility of mathematical objects could be salvaged by decreeing that to be ontogenetically effective a work must from the start be provably consistent: this has the effect of strengthening our initially weak existence-criterion. Unfortunately we should then have to exclude many fundamental objects from our mathematical universe; in particular we should have to exclude sets, since we have no consistency proof for any of the standard set theories.

(ii) There is a reluctance to say that mathematical objects can be destroyed for aesthetic reasons. But if, for example, we consider the effect on our future concretisation of a given work following the appearance of a more powerful or more elegant work which incorporated the content of the original work within a new framework of aspects, we can see that the original object-domain would have "collapsed upon itself", in the sense that henceforth in the mathematical

universe it will occupy marginal land, which is never visited except by philosophers and historians of mathematics. The aesthetic *destruction* of mathematical objects can, however, take place at the stage prior to their embodiment and objectification within a work; for in the creation of free heuristic mathematics, objects continually arise from the mathematician's own personal whims, and those which, on reflection, he decides are insufficiently interesting, or are such that they cannot be smoothly attached to pre-established regions of the mathematical universe, he will destroy. This can be compared with the way in which a novelist toys with possible characters for a given work, destroying those with which he eventually becomes dissatisfied by disincluding them from that work.

(iii) Interesting problems are raised by the ontological status of accidentally destroyed objects. For example, several hundred pages of Lesniewski's work on logical systems were destroyed when a Warsaw printing house was burned down in the siege of 1939, the year in which Lesniewski himself died. Since that time his disciples have been attempting to re-create his work: can we say that the objects which he created are maintained in a "programmatically" existence by these efforts?

The historicist view of the universe of mathematical objects shares with intuitionism the notion of regions of that universe "coming into being as we probe".²⁶ For the intuitionist "*to exist* is synonymous with *to be constructed*"²⁷ in a series of acts which are assumed to take place at discrete intervals of time, on analogy with acts of counting. Concretely interpreted such a doctrine would yield a universe which was over-small,²⁸ hence the doctrine is idealised so that we need only be satisfied that a given construction is "in principle possible" for its objects to exist. Here intuitionism parts company with historicism, since *time* in

26. See Michael Dummett, "Truth", repr. in Strawson, ed., *Philosophical Logic*, Oxford, 1967, p. 68.

27. Heyting, *op. cit.*, p. 2.

28. (Nevertheless it need not be without aesthetic interest, see D. van Dantzig, "Is 10 to the 10 to the 10 a finite number?" *Dialectica*, IX, 1956, pp. 273-7. and A. S. Yessenin-Volpin, "The Ultra-Intuitionist Criticism and the Anti-traditional Program for Foundations of Mathematics", in *Intuitionism and Proof Theory*, Amsterdam, North Holland, 1970.)

mathematical experience has now been turned by the intuitionists into something purely formal. A second idealising departure from the actuality of mathematical experience is made by the intuitionists when they interpret “acts” as simple and purely introspective mental contents performed without having recourse to anything “external” such as language or logic.²⁹ It follows from this that intuitionism is limited in its power as compared to mathematics based on higher-order acts of consciousness dependent upon the essential mediation of symbolism, such as those involved, e.g., in Cantor’s diagonal argument for the existence of uncountably infinite sets. Therefore, concretising intuitionist mathematical works we can accede to only a limited region of the mathematical universe; what ensures such works a permanent place in the mathematical library is that the framework of aspects within which the objects of that region are given to us reveals subtle distinctions³⁰ which are quite indiscernible when those objects are looked at through the cruder telescopes of high-powered mathematics.

9. *Aesthetics and the Foundations of Mathematics.*

The different conflicting schools, whose co-existence points to the radical multifurcation of mathematics, arose in the face of the discovery around 1900 of the logical paradoxes which seemed to suggest a fundamental incoherence in the intuitive notions at the basis of our mathematical thinking. Philosophically-minded mathematicians felt the hard ground of the most certain of all disciplines begin to sway under their feet. They began to search for new and firmer “foundations” for mathematics in order to ensure that all traces of paradox (and of the inconsistency consequent upon it) had been eradicated from mathematics.

In the light of the present paper we can see why this “search” was misconceived. For in the case of central core mathematics we do not require “foundations”: its objects are phenomenologically given as autonomous, and this is enough to ensure freedom from paradox and inconsistency. When we move into the richer and seemingly more “dangerous” sphere of creative mathematics then, so long as we recognise that mathematics has ceased to be a science and has become a historically evolving form of art, we no longer see anything to fear in the “immanent frailty”³¹ of the objects which it brings into being.³²

The cry for “foundations” came in fact to be used merely as an excuse by each philosophical school to enforce the de-ramification of the heteronomous domain by making the objects there conform to the standards of autonomous objects as it conceived them. We hope that our distinction between autonomous and heteronomous mathematical objects will offer an alternative to this enforced ossification of mathematics, by developing a philosophical conception more closely in line with actual mathematical practice in its totality. This is not to say that mathematics is to be left as it was before we arrived on the scene; for once “truth” is seen to be an evaluative criterion appropriate only to the limited autonomous core of mathematics, then this implies the indispensability for the remainder of mathematics of alternative criteria and objectives which, we suggest, need to be grounded in aesthetics. Such criteria and objectives are only unreflectively and unsystematically brought to bear in current mathematical activity, and this means that two clearly philosophical tasks remain, namely (i) to make these criteria explicit and to indicate how they ought most methodically to be used in critical reflection upon pre-established mathematics, e.g.,

29. See Brouwer, “Consciousness, Philosophy and Mathematics” (1940), partly repr. in Benacerraf and Putnam, *op. cit.*, p. 78.

30. Heyting, *op. cit.*, p. 11.

31. This is a synonym for “heteronomy”, see LWA p. 122n.

32. Nor would “foundations” be needed here to prevent some deviant mathematical work being *applied*, e.g. to the building of bridges, since engineers, etc., are rightly encouraged to believe that only central-core mathematics is used in practical applications to the physical world. (We wish to leave open here the question as to whether *theoretical* physics might properly exploit richer mathematical models, and the question as to the existence of heteronomous objects within theoretical sciences in general.)

in deciding which mathematical results should be accepted and which regions of the mathematical universe should be refined or truncated in virtue of their “ugliness”³³; and (ii) to come to a conception of the aesthetic objectives of creative mathematics and of the ways in which these are fulfilled in the continual extension and enrichment of the universe of mathematical objects.

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33. Note in particular the acute ugliness of objects brought into being by inconsistent works.