means that mathematics does not exist in itself, independent of the doer, but also that it does not exist independent of the observer, who assesses these actions as mathematical. Hence, assessing students’ mathematics is as much assessing one’s own mathematics. The sentence could then read: “[…] investigating students’ mathematical knowledge, through investigating one’s own mathematical knowledge.”

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Methodological Issues of Second-order Model Building
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>Upshot  I argue that radical constructivism poses a series of deep methodological constraints on educational research. We focus on the work of Ulrich et al. to illustrate the practical implications of these constraints.

1 The target article “Constructivist Model Building: Empirical Examples from Mathematics Education” by Catherine Ulrich et al. is a fine piece of educational radical constructivist research. In particular, it covers several relevant aspects regarding the application of radical constructivism (RC) to an actual teaching-learning situation. On the other hand, due to its applied character, the authors do not pay attention to the theoretical assumptions underlying their approach nor to the far-reaching methodological implications that stem from it. In my commentary, I will focus on these basic issues. It must be stressed that while the target article deals with mathematics education, the aspects I am going to address can be extrapolated to any field of educational research – provided, of course, that the obvious differences are taken into account.

2 Ulrich et al. advocate the construction of second-order models of the mathematical thinking of students as an optimal way to bridge the gap between radical constructivist theory and the practical understanding of actual teaching-learning settings (§1). A second-order model (SOM) is an explanatory reconstruction of a student’s thought on the basis of his or her observed behaviour in the didactic interaction, together with a conceptual analysis of his or her mathematical thinking (§10). In this commentary, I focus on the implicit psychological theory that necessarily underlies the construction of a SOM.

3 In order to deal with these questions, let me introduce some concepts from analytical philosophy of mind, in particular the concepts of psychological state and narrow psychological state introduced by Hilary Putnam in his 1975 paper “The Meaning of ‘Meaning.’” According to Putnam, a psychological state is a two-place predicate of a psychological property whose arguments are an individual and a time (Putnam 1975: 136). Brushing the analytical jargon out of this definition, a psychological state can be defined as an array of three elements, namely a subject, a temporal specification, and, of course, a psychological property.

4 One kind of psychological states is those which are said to include a representation of the world. In common terms, the psychological property that typifies these states is being aware of something. These representations, the “something” that we are aware of, are usually called the content of a psychological state.

5 A psychological state is said to be narrow if it can be defined without any reference external to the subject. The necessary and jointly sufficient conditions for a psychological state to be narrow are immediate:

- The subject of the state must be an individual.
- The psychological property that is attributed must be definable without any reference external to this individual.

For obvious reasons, the content of a narrow psychological state is usually called narrow content.

6 In sum, in analytical terms an SOM is a model of a psychological state of a pupil whose content is related to a school topic. Hereafter we shall use the acronym SOM in this sense, and in particular we shall speak of the content of an SOM.

7 It is easy to see that within RC, psychological states must be narrow (see Glasersfeld 1995b: 1 for an absolutely clear, if synthetic, programmatic statement in this sense). This implies that the locus of educational radical constructivist research must be the individual student, not any form of collective subject, such as, for example, the classroom group. Besides, in order to set the content of an SOM, nothing external to the mental realm of the student to whom the state is attributed must be required. Since the work of Ulrich et al. explicitly complies with the requirement of individualism, we shall focus on the latter condition.

8 The question of whether there is such a thing as a narrow mental content is one of the open questions in analytical philosophy. Putnam (1975) rejected the possibility of a narrow content. Other philosophers, however, have defended that at least some mental contents are narrow. These authors have proposed several ways of understanding what narrow content is, and have proposed different strategies for gaining access to it (for a recent revision, see Brown 2011). For our purposes, we shall focus on a crucial methodological issue implicit in these strategies.

9 In order to determine the narrow content of the mental state of an individual, the researcher must gather signs of this content out of his or her behavior. In other words, and using the well known Saussurean distinction, some actions of the subjects must be interpreted as signifiers of a fully internal signified narrow content. In many cases, the action that is taken to be the via regia to these contents is the individual’s speech, but other ways are also possible, at least in some cases. In the work of Ulrich et al., several examples of these semiotic inferences can be found. In particular, the reader is presented with pieces of the behaviour of the subjects under study that are significant, as they convey information about some mental mathematical procedures. This, which is rather clear in §§13, 14, 21 and 22, becomes transparent in §32, where the
reader is even presented with a graphic sign (Figure 3) of the mathematical reasoning of an individual.

Assuming that language (or any other semiotic system) can express a narrow content has some severe semantic implications. In particular, it implies that for a piece of speech, or any other signifier, to be taken to be a sign of a narrow content, its meaning must be assumed to be fully determined in the inner sphere of the speaker. Or, using the Putnamian terminology, the determination of narrow content by semiotic means necessarily rests on an internalist semantics.

Internalism imposes a stark methodological restriction to the works that are useful from a radical constructivist point of view. Only transcriptions of students' speech, and other qualitative methods that present the ideas of individual pupils untouched, can be acceptable since it is assumed that they convey the narrow content of their mental states in their own terms. Thus, only qualitative studies of students' thought would be valid to make hypotheses about the content of an SOM of a pupil. I cannot see any way to escape this limitation. RC implies a hard-line qualitative educational research program when it comes to studying students' thought. The paper of Ulrich et al. is an example of such a qualitative methodology.

From this qualitative educational point of view, and in accordance with the tenets of RC, SOMs are idiiosyncratic. Each SOM is in principle distinct. While acknowledging this fact, the authors claim that substantial commonalities have been found between SOMs attributed to different students in equivalent educational settings (§27). That is, and this is a crucial point, the possibility of an extension2 of an SOM to new individuals poses to RC. and thus, the validity of the application of a given SOM to a new individual must be evaluated case by case.

Let us now think of an investigation of one student's thought on a particular school topic. In order to identify the narrow content of an SOM attributed to a pupil, it is necessary to know in advance the type of content we are interested in. At the very least, we must know the issue the content is about. In other words, even the purest qualitative study of the student's thought rests on a previous knowledge that cannot be reduced to particular specimens of the pupil's speech. Without this typical previous knowledge, the discourse of an individual student is a formless stream of information that cannot be linked to psychological states with a discrete content.

Giving up on any form of previous type-like knowledge in educational radical constructivist research implies rejecting a discretized model of the psyche of students. Students' thought should be re-

http://www.univie.ac.at/constructivism/journal/9/3/328.ulrich
garded as an unstructured whole, as a men-
tal continuum. While this perspective may
be acceptable from certain points of view, it
is definitely not the approach of the different
subject-specific didactics, such as those of
mathematics or the experimental sciences.
A holistic vision of the mental content of
the students is arguably a defendable posi-
tion, but it simply rules out any model of the
students’ thought based on school topics or
disciplinary contents. Educational research
would be reduced to a pure psychological
discipline, where curricular contents would
play little, if any, role.

It is through the type that the
wealth of disciplinary knowledge enters into
educational radical constructivist research.
The two-faced character, type and token,
of SOMs is what actually permits the re-
search. The two-faced character, type and token,
is contested, this delicate equilibrium fails.

To sum up, the work of Ulrich at
al. is a fine example of several traits that nec-
essarily accompany radical constructivist
educational research:
- **Individualism**: The locus of the research is a
  specific student.
- **Methodological solipsism**: The properties at-
  tributed to the student must be definable
  without any reference external to him or
  her.
- **Semantic internalism**: The meaning of a
  word, or any other sign, uttered by stu-
dents is fully determined in their inner
sphere. This, together with the previous
point, is the analytic philosophy counter-
part of the operational closure that
has been outlined as an attribute of RC
(Riegler 2001: 4).

Qualitative methodologies: Adapting the
well-known Foucaultian saying, stu-
dents’ documents must be treated as
monuments. In an ideal limit, the mate-
rials on which the investigation is based
should be presented untouched to the
 eventual reader. While this is obviously
a utopian limit, a hard-line qualitative
program is the methodological frame-
work of educational radical constructiv-
ist research.

Type-token dichotomy of SOMs: SOMs must
be regarded as tokens, since they refer to
individual students. On the other hand,
the only meaningful way to extend them
is by grouping them in types. The obvi-
ous way to define these types in educa-
tional research is by means of the school
contents.

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Second-order Models of Students’ Mathematics:
Delving into Possibilities

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> Upshot - I look at the different possibil-
ities offered by the trajectory of second-
order models in mathematics education.
It seems to me that although possibili-
ties are extended as models become
more elaborate, this is only the case if
teacher/researchers remain cognisant of
a radical constructivist perspective. I also
suggest that broad-ranging research on
the models affords insight into the “big”
mathematical ideas to which all stu-
dents should have access.

Before I begin this commentary,
I need to declare my own interests in con-
structivism. My doctoral research (Dooley
2010) concerned the construction of math-
ematical insight by individual children in
the context of whole-class conversation. My
theoretical stance was emergent, that is, an
approach in which an account of individual
students’ learning is coordinated with the
means by which the learning is supported.

Paul Cobb and Erna Yackel suggest that “(f)
from this vantage point, individual activity is
seen to be situated within the practices of a
local community such as that constituted by
the teacher and students in the classroom”
(Cobb & Yackel 1996: 188). In taking this
viewpoint, I did not dismiss the broader so-
cial and cultural frameworks of mathemat-
ics teaching and learning but, in terms of
the aims of my research, the emergent ap-
proach as a theoretical position afforded me
the requisite “grain size” (Cobb et al. 2001).
From a psychological perspective, the level
of analysis is that of the teacher and pupils
as individuals constructing knowledge as
they interact; from a social perspective, it
is the class as a local community establishing
communal norms and practices.

I conducted research in three dif-
ferent schools. All classroom interactions
were audio-taped; additional data, such as
children’s written artefacts, photographs of
materials, etc., were also gathered. For each
lesson, I identified “knowledge elements”
(Ron, Dreyfus & Hershkowitz 2010), that is,
the constructs that pupils might be expected
to develop; these informed a hypotheti-
cal learning trajectory. Initially I garnered
these knowledge elements from relevant
research but, as the project continued, I
incorporated constructs that emerged in
previous lessons. For example, I taught the
“Handshakes” problem (Slavitt 1999) in
two different classes – analysis of the first
of these lessons informed the hypothetical
trajectory that I formulated for the second.
As was the case with some of the teacher/
researchers in this article, I became more
aware of possible ways that children might
engage in tasks as I progressed through the
study.

A particular challenge that I en-
countered was attending simultaneously to
the personal nature of constructions and
the ways knowledge was distributed among
various individuals in the group. In order to
address this challenge, I first explored the
new mathematical ideas (with reference to
knowledge elements) that were created by
the class members over the course of a lec-
tion or series of lessons, e.g., development
of an explicit rule to develop the number of
handshakes for n (number of individuals)
in the “Handshakes” problem. Later, I iso-
lated individuals’ contributions and traced