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Hale's argument from transitive counting

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Abstract
A core commitment of Bob Hale and Crispin Wright's neologicism is their invocation of Frege's Constraint—roughly, the requirement that the core empirical applications for a class of numbers be "built directly into" their formal characterization. According to these neologicists, if legitimate, Frege's Constraint adjudicates in favor of their preferred foundation—Hume's Principle—and against alternatives, such as the Dedekind–Peano axioms. In this paper, we consider a recent argument for legitimating Frege's Constraint due to Hale, according to which the primary empirical application of the naturals is transitive counting, or answering 'how many'-questions using numerals. We make two claims regarding Hale's argument. First, it fails to legitimate Frege's Constraint in virtue of resting on unsupported and highly contentious assumptions. Secondly, even if sound, Hale's argument would vindicate a version of Frege's Constraint which fails to adjudicate in favor of Hume's Principle over alternative characterizations of the naturals.

Keywords Neologicism · Frege's Constraint · Hume's Principle · Structuralism · Counting

1 Introduction

This paper is concerned with Frege's so-called *application constraint*, a core methodological commitment of a popular brand of neologicism. Roughly, the requirement is that the primary empirical applications of a class of mathematical entities be "built directly into" their characterization.[1] In particular, we are concerned with a recent and intriguing argument, due to Bob Hale (2016), for Frege's Constraint

[1] On Wright (2000)'s gloss, as well as Hale (2016)'s, Frege's constraint is limited specifically to empirical applications. Of course, one could reformulate Frege's constraint so as to include non-empirical applications as well. However, because our concern here is with Hale's argument, we will ignore this latter possibility.

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as it applies to the natural numbers, which we call *the Argument from Transitive Counting*.

Although Hale's argument possesses considerable initial plausibility, we show that it is unsound. We thereby take ourselves to establish the untenability of one of the very few *prima facie* plausible arguments for Frege's Constraint. Though establishing this negative result does not constitute a rejection of Frege's Constraint itself, our discussion is instructive in ways that are of broader significance to debates over the nature of natural numbers and our concepts thereof. In particular, it helps clarify the notion of counting, the relationship between the possession of natural number concepts and counting, and the prospects of vindicating the neologicst program for arithmetic by invoking Frege's Constraint. It also differs from extant discussions of Frege's Constraint in that it draws on related empirical research –especially regarding concept possession, the semantics of number words, and the acquisition of number concepts.

2 Background: formal characterizations of natural number, and Frege's Constraint

How ought we to formally characterize the natural numbers? Consider the following pair of characterizations. The first, the familiar Dedekind–Peano (DP) axioms, characterizes the natural numbers relationally. Informally:

(DPA1) 0 is a natural number.
(DPA2) Every natural number has a unique successor.
(DPA3) The successor relation is one-to-one.
(DPA4) 0 is the successor of no natural number.
(DPA5) For every concept F, if F holds of 0 and F's holding of n implies F holds of n's successor, then F holds of every natural number.

These axioms specify an abstract structure—an ω-sequence—which is isomorphic to the natural number system. Accordingly, structuralists, such as Michael Resnik (1997) and Stewart Shapiro (1997), identify the naturals with "places" or "positions" within an ω-sequence.

According to a second characterization, defended by Bob Hale (2016) and his collaborator Crispin Wright, whom we collectively refer to as "the abstractionist neologicists", or "the abstractionists",[2] the naturals are instead characterized by *Hume's Principle* (HP).

(HP) $\forall F, G.\ \#F = \#G \leftrightarrow F \approx G$

Here, "#" is a cardinality-operator mapping a concept φ to a cardinal number representing the number of objects falling under φ, and "≈" is the relation of equinumerosity holding between two concepts just in case every object falling under F can be

[2] We use this label to distinguish the view from other versions of neologicism, such as Tennant (1997)'s Constructive Logicism.

mapped to a unique object falling under G, and vice versa. Thus, two cardinal numbers will be identical just in case they number equinumerous concepts.

HP can be used to derive cardinal numbers. For example, cardinal zero can be identified with the the number of the concept BEING NON-SELF-IDENTICAL[3]; cardinal one with the number of the concept of BEING IDENTICAL TO ZERO; cardinal two with the number of the concept of EITHER BEING IDENTICAL TO ZERO OR BEING IDENTICAL TO ONE, etc.

$$\begin{aligned}
&0 = \#[\lambda x.\ x \neq x]\\
&1 = \#[\lambda y.\ y = \#[\lambda x.\ x \neq x]]\\
&2 = \#[\lambda z.\ z = \#[\lambda y.\ y = \#[\lambda x. \neg x = x]] \vee z = \#[\lambda x. \neg x = x]]\\
&\ldots
\end{aligned}$$

The result is a system of cardinal numbers isomorphic to the naturals. Accordingly, abstractionist neologicists identify the naturals with these finite cardinals.

So we have two different, seemingly legitimate, formal characterizations of the natural numbers.[4] Moreover, it is well known that the DP axioms are derivable from HP, once suitable definitions are added,[5] and that a finite version of HP is derivable from the DP axioms.[6] So, there is a sense in which if we accept the truth of one, then we must also accept the truth of the other. Nevertheless, one might reasonably think that even if the DP axioms and (finite) HP are both true, their status with respect to the natural numbers are quite different. Specifically, one of them might be, in some appropriate sense, more *basic* than the other. That is, one of them might characterize what the natural numbers *are* –their *essence*, if you will—whereas the other might instead be a (mere) necessary consequence of this (correct) account of what the naturals are.

Suppose one accepts this picture of the contrast. (We don't.) How would one determine which putative characterization of the natural numbers is more basic than the other? The question, it would seem, cannot be resolved on purely formal grounds. So, recourse to extra-formal considerations is required. But what might these be? Here is a tempting thought: One might determine which of these two formal characterizations of the natural numbers is more basic by determining which

[3] For the purposes of this paper, we will adopt the convention of naming concepts in CAPITALS.

[4] It has been suggested to us that 'natural number' is "systematically ambiguous", in that the DP axioms and HP characterize "different things". In particular, the DP axioms characterize the "natural numbers in sequence", presumably the natural number structure, while HP characterizes "individual natural numbers"; and this is somehow reflected in 'natural number' or the corresponding concept(s). However, the structuralist *does* define the "individual natural numbers"—they are places in the natural number structure—and the abstractionist neologicist *does* characterize the "natural numbers in sequence"—they are the finite cardinal numbers ordered in the usual way. The present paper concerns the priority between these two accounts, an issue posed by the abstractionists themselves. Of course, the DP axioms and HP are not the only formal characterizations of the natural numbers (either "individually" or "in sequence"). For example, a third characterization is to be found in Tennant's Constructive Logicism; and a fourth, ordinal-based characterization, is due to Linnebo (2009). We will return to both these proposals in subsequent work.

[5] See Heck (2011).

[6] Details for this can be found in Dedekind (1888); see Sect. 5 below.

(if either) best captures *our actual natural number concepts*—the ones we routinely employ in our mundane numerical activities, such as counting cats or performing simple arithmetic calculations. If one could garner evidence that a specific characterization of the naturals captured *these* concepts, then (so this line of thought goes) we would have reason to think this characterization is more basic than the alternative. But how could we determine that one of these characterizes our actual natural number concepts?

Here's where Frege's Constraint comes in. Roughly put, for a foundation for a mathematical theory—in this case arithmetic—to satisfy Frege's Constraint it "must somehow build its applications, actual and potential, into its core—into the content it ascribes to the statements of the theory—rather than merely 'patch them on from the outside'" (Wright (2000, p. 324)). According to Wright, and Hale as well, the primary empirical application of the naturals is answering 'how many'-questions such as 'How many Elmos are on the table?'. Yet the DP axioms provide no means for answering such questions. So, if HP can explain how the naturals can be used in this manner, then we would appear to have a good reason for preferring the abstractionist characterization of the naturals over the structuralist alternative—so long, of course, as this version of Frege's Constraint is independently justified.

3 The argument from transitive counting

After rejecting a well-known argument from Wright (2000), Hale (2016) offers an alternative argument for Frege's Constraint, which turns crucially on the idea that *counting* is the primary means by which we determine cardinality (at least for small, but not too small, collections). The argument depends crucially on Paul Benacerraf's (1965) distinction between *intransitive counting* and *transitive counting*. To a first approximation, the former consists in reciting the numerals in their canonical order starting with one–"1, 2, 3,…"—whereas the latter consists in using the numerals to answer 'how many'-questions, roughly by establishing a one-to-one correspondence between an initial segment of those numerals and a collection of objects being counted. With this distinction in hand, Hale's argument proceeds from three claims:

TCP1 One who has learned to count both intransitively and, crucially, transitively, but not yet to add, multiply, etc., has at least a basic grasp of (the concepts of) the natural numbers

TCP2 It is clearly a possibility that a trainee should by-pass the second stage—that is, the trainee should learn to count intransitively, but not transitively, and then proceed directly to learn to do arithmetic. She could be introduced to the successor operation as one which takes one from any given number to the next (i.e. to the number she has learned to count after the given number), and be taught to add, multiply, etc., perhaps by being given the usual recursive definitions of + and x, or perhaps by means of tables

TCP3 A trainee who realized the possibility just described would not yet have a basic grasp of (the concepts of) the natural numbers

It is important to stress two points. First, when speaking of *concepts* of natural numbers, Hale has in mind concepts such as TWO, THREE, and so on. He is not concerned with the concept of NATURAL NUMBER per se. That is, he is concerned with concepts for specific natural numbers, and not the system of natural numbers as such. Second, Hale provides no arguments in support of these premises. Instead, their assertion appears to rest entirely on the presumption that they are intuitively plausible. In Sect. 4, the status and plausibility of these claims will be discussed in greater detail. For the present, however, we focus on spelling out their implications.

According to TCP1, the ability to count both intransitively and transitively is sufficient for possessing at least a basic grasp of natural number concepts. However, according to the conjunction of TCP2 and TCP3, the abilities to count intransitively and perform arithmetic operations are not jointly sufficient for possessing such concepts. Suppose, for example, there was a trainee—*the DP Novice*—who, by virtue of grasping the DP axioms (along with second order logic), was able to count intransitively, and to perform basic arithmetic operations. Such an agent might nonetheless be unable to count transitively since the DP axioms (plus second order logic) do not ensure that our trainee will be able to use numerals to assign cardinalities to collections.

Thus, suppose we grant TCP1–TCP3. Together they appear to entail TCC1:[7]

TCC1 Transitive counting is essential to the possession of a basic grasp of (the concepts of) the natural numbers.[8]

In which case, it follows that our DP Novice does not possess a basic grasp of natural number concepts. Now, TCC1 does not by itself constitute an argument for Frege's Constraint. Rather, we need to add some further premises, also endorsed by Hale.[9]

TCP4 If transitive counting is essential to the possession of the concepts of the natural numbers, then the fact that the natural numbers can be used to transitively count is essential to the natural numbers themselves

TCP5 A philosophically adequate characterization of what natural numbers are ought to directly reflect what is essential to them

[7] In fact, one must further suppose that the conditions described by TCP1 and TCP2 respectively exhaust the relevant alternatives for concept possession. However, since this assumption will play no role in our discussion, we set it to one side.

[8] As one anonymous reviewer notes, one might think it is more felicitous to express claims about essentiality as involving a relationship between kinds or nominalized properties –e.g. "Being striped is essential to being a zebra." That said, as our presentation of the argument makes clear, Hale uses other constructions to express essentialist claims. For present purposes, however, nothing of philosophical significance turns on this. For example, the claim in TCC1 can be readily reformulated as follows: "One's being able to transitively count is essential to one's possessing natural number concepts." *Mutatis mutandis* for other claims in the argument.

[9] Hale (2016, p. 340): "If [TCC1] is right, then the fact that the natural numbers can be used to count collections of things is no mere accidental feature, but is essential to them. And if that is so, then a satisfactory definition of the natural numbers–a characterisation of what they essentially are–should reflect or incorporate that fact."

In which case, according to Hale, TCC2 follows:

TCC2 A philosophically adequate characterization of the natural numbers ought to build transitive counting directly into their characterization

And this, it would seem, is just a restatement of Frege's Constraint, albeit one which takes transitive counting to be the relevant empirical application of the natural numbers.

One intended implication of the argument is that a structuralist characterization of the naturals is philosophically inadequate. More specifically, if sound, the argument would show that the DP axioms fail to provide an adequate philosophical account of the natural numbers, because they fail to capture one of their essential features: their role in transitive counting. But notice that this conclusion is a purely negative one. If sound, the argument would rule out the structuralist characterization of natural numbers. Yet it remains entirely silent regarding the adequacy of the abstractionist's proposed alternative, i.e. HP. We return to this issue in the final section of this paper, where we argue that in fact *neither* the DP axioms *nor* HP satisfy the condition imposed by TCC2. If so, then since TCC2 just is Hale's preferred version of Frege's Constraint, so construed, Frege's Constraint fails to do the philosophical work required of it by abstractionists—to justify HP as a satisfactory foundation for arithmetic as opposed to the structuralist account. Before we turn to this issue, however, we develop a pair of objections to the argument from transitive counting itself. In Sect. 3, we focus on the inference from TCC1 to TCC2; and in Sect. 4, we challenge the argument for TCC1.

4 Concepts, possession conditions, and reference

For the sake of argument, suppose TCC1 is true—that transitive counting is essential to the possession of natural number concepts. Hale maintains that, given minimal additional assumptions, this provides good reason to endorse a version of Frege's Constraint, i.e. TCC2.

This conclusion builds on Wright's original gloss of Frege's Constraint in two respects. First, it specifies that the application relevant to meeting Frege's Constraint is transitive counting. Second, it makes explicit that respecting Frege's Constraint demands that the relevant application be built into the *referents* of the corresponding numerals, i.e. into our characterization of the natural numbers themselves.

Although we do not purport to establish the *impossibility* of inferring TCC2 from TCC1, we propose to argue that this is an instance of a kind of inference, what we call a *Possession-to-Object inference,* which in the *general case* is illegitimate. Further, we argue that the task of justifying this specific instance of the inference poses a serious challenge to Hale's argument since it is far from obvious how to justify it in a plausible, non-question-begging fashion.

4.1 Possession-to-object inferences

Let us start by clarifying the sort of inference Hale is relying upon. TCC1 and TCC2 are claims regarding quite different sorts of entities. TCC1 is a claim about the *possession conditions* for number *concepts*, whereas TCC2 is a claim regarding the adequacy conditions on an account of the entities that *fall under* such concepts. More precisely, TCC1 concerns what's required for the possession of a concept, whereas TCC2 concerns the properties that an object must have in order to fall under the concept possessed. Hale's argumentative burden is to infer a claim of the latter kind from a claim of the former kind, and this is accomplished via TCP4:

TCP4 If transitive counting is essential to the possession of the concepts of the natural numbers, then the fact that the natural numbers can be used to transitively count is essential to the natural numbers themselves

TCP4 encodes what we will call a *Possession-to-Object* (P-to-O) *inference*, and is a particular instance of what we call *the Possession-to-Object Principle* (POP):

POP If an entity O falls under a concept C, and some ability or capacity E is essential to the possession of C, then E is essential to O

With TCP3, the relevant objects are natural numbers, the relevant concepts are natural number concepts, and the relevant ability or capacity is transitive counting.

Now it is quite clear that, in the general case, instances of POP are not all legitimate. By way of illustration, suppose it is essential to possessing our concept WATER that one has the ability to recognize certain things as samples of water. Plausibly, an account of the *possession* conditions for WATER should reflect this fact—we come to possess WATER in virtue of recognizing stuff as water. In contrast, it is highly implausible that an account of *what water is* should reflect this ability. Indeed, it is no part of the chemical identity of water that it can be recognized as such. Moreover, there is nothing unusual about the case of WATER. Though there is not the space here to argue the point in detail, much the same is plausibly true, *mutatis mutandis*, for a great many concepts. For example, it is plausibly true of chemical kind concepts quite broadly—gold, aluminum, helium, and so on. Similarly, we maintain that it is plausibly true of core concepts in biology—such as concepts for species—as well as concepts in physics, and perhaps even concepts in psychology. But if this is so, then P-to-O inferences are clearly not generally legitimate. Why, then, should it be otherwise for natural number concepts, such as TEN?

Our initial complaint, then, is that Hale's argument relies on the unjustified assumption that, seemingly in contrast to many other concepts,[10] number concepts conform to POP. Further, and in view of this, we maintain that if the Argument from

[10] It is possible, we suppose, that someone might subscribe to POP for *all* concepts, not just natural number concepts. Since this view is both implausible and unnecessary for Hale's argument to go through, we don't discuss it here.

Transitive Counting is to have any probative force, a proponent of the argument owes a response to the following challenge:

> *The Justificatory Challenge:* A proponent of the Argument from Transitive Counting must justify the assumption that natural number concepts conform to POP.

In what follows, we explore what we take to be the most natural and plausible candidate responses to this challenge, and argue that none of them are suitable for the abstractionist's purposes. Unless a better response can be found, TCP4 will remain unwarranted.

4.1.1 Conceptualist metaphysics of number

Let us start with a possible attempt to meet the Justificatory Challenge which is easily dismissed. An obvious reason why someone might endorse TCP4 is that they advocate a *conceptualist* metaphysics on which the existence and nature of number is *grounded* in—or metaphysically dependent upon—our psychology. In particular, suppose they think the essential properties of numbers are grounded in those properties essential to the possession of number concepts. If we further assume that the capacity to transitively count is essential to the possession of such concepts, then it would be plausible to suppose that it is essential to natural numbers themselves that they figure in the process of transitive counting.

Such a line of reasoning might be attractive to some, but it is not available to the abstractionists. Indeed, they are *platonists* about natural numbers, whereby the naturals exist independently of our ability to cognize or form concepts about them. Thus, TCP4 cannot be a defended on the general grounds that the essential properties of natural numbers are grounded in facts about our concepts or their possession.

4.1.2 Abstractionism

A different possible attempt to meet the Justificatory Challenge appeals to how the natural numbers are identified. More specifically, Hale could observe that the naturals are identified in a way that is interestingly different than water: whereas water is identified directly with a certain chemical compound (H_20), the naturals are instead identified indirectly via an abstraction principle (HP). The suggestion we would like to entertain here, on Hale's behalf, is that this difference explains why POP holds with respect to the naturals but not with respect to water and other so-called natural kind concepts.

Generally speaking, abstraction principles take the form of AP, where '*a*' and '*b*' are variables of a given type (typically first-order, ranging over individual objects, or second-order, ranging over concepts or properties), 'Σ' is a higher-order operator, denoting a function from items of the given type to objects in the range of the first-order variables, and '$\approx$' is an equivalence relation over items of the given type.

(AP) $\forall a \forall b.\ \Sigma(a) = \Sigma(b) \leftrightarrow a \approx b$

Abstraction principles perform at least two tasks.[11] The first is *semantic*: they allow us to introduce new singular terms in virtue of introducing identity statements whose truth-conditions are fixed by already familiar sentences of the form on the right-hand side of AP. To illustrate, consider the following abstraction principle originally suggested by Frege (1884, §66):

(1) $\forall l \forall l'. \mathrm{D}(l) = \mathrm{D}(l') \leftrightarrow l$ is parallel to l'

Here, 'D' is a direction-operator mapping lines to directions. We may assume that we already know when sentences of the form 'l is parallel to l'' are true. On the other hand, we may suppose that we have no prior terms for referring to directions. What (1) does, in effect, is establish truth-conditions for identity statements involving direction terms, i.e. statements of form '$\mathrm{D}(l) = \mathrm{D}(l')$'. Since identity statements are statements involving singular terms, we now have the means for referring to objects which, by assumption, we could not refer to before.

The second role of abstraction principles is *epistemic:* they grant us knowledge of *abstracta* on the basis of facts concerning *concreta*. Directions are apparently abstract, not existing in physical space. How, then, are we able to have knowledge about them? (1) offers an answer. Thanks to the equivalence established in (1), our knowledge of concrete lines, and a certain already familiar relation between them, we can come to have knowledge about the identity of abstract directions. What's more, because this equivalence establishes the *identity* of directions, (1) plausibly encodes one of their essential features, namely that they can be individuated according to whether the lines they are associated with are parallel. So, if we assume that grasping the concept of direction requires grasping all features essential to directions, then it will follow that someone fails to grasp the concept of direction if they fail to grasp this feature encoded by (1).

Similarly, if we assume that the identity of an object is tied directly to its essential features, so that a putative object of a certain sort F fails to be an F if it does not have all essential features of Fs, then it will also follow that it is essential to directions themselves that they can be so individuated. Thus, we would appear to have a valid instance of POP, as (2a) would appear to entail (2b):

(2) a. It is essential to possessing the concept of direction that one grasps that directions can be individuated according to whether the lines they are associated with are parallel

b. It is essential to directions themselves that they can be individuated according to whether the lines they are associated with are parallel

If so, then it might be thought that abstraction principles provide the resources necessary for distinguishing valid from invalid instances of POP. Moreover,

[11] See MacBride (2003) and Ebert and Rossberg (forthcoming).

because an exactly similar sort of argument can be given for HP with respect to the natural numbers, it might seem equally reasonable to think that this is the strategy Hale should pursue in closing the gap between TCC1 and TCC2.

This suggestion ultimately fails, however, for following reason: Even if we grant that grasping a concept requires grasping all features essential to the objects falling under the concept, and that HP encodes a feature essential to natural numbers, it still will not follow that an ability to *transitively count* is essential to those numbers. To see why, consider again directions as characterized in (1). One clear, arguably "primary", empirical application of directed lines is predicting the trajectory of an object, e.g. the path of a ball thrown with a certain velocity. Obviously, this application is not "built directly into" (1), nor was it intended to be. Rather, the only potential empirical application (1) encodes is distinguishing directions on the basis of whether the lines they are associated with are parallel. A parallel point holds for HP: the only potential empirical application it encodes is distinguishing cardinal numbers on the basis of whether the concepts they are properties of are equinumerous, and as we will see in §5, this alone is not sufficient to ensure an ability to transitively count, contrary to what TCC2 requires.

We can summarize the problem as follows. In order for Hale's argument to meet the Justificatory Challenge, it needs to be that HP, qua abstraction principle, licenses the inference from (3a) to (3b).

(3) a. It is essential to the possession of natural number concepts that natural numbers feature in transitive counting

b. It is essential to the natural numbers themselves that they feature in transitive counting.

However, without further argument, HP would at most license the inference from (4a) to (4b).

(4) a. It is essential to the possession of natural number concepts that natural numbers can be individuated according to whether the concepts they are properties of are equinumerous

b. It is essential to the natural numbers themselves that they can be individuated according to whether the concepts they are properties of are equinumerous.

Since an ability to individuate concepts in this manner is not the same thing as to being able to transitively count, appealing to the fact that HP is an abstraction principle will not suffice to meet the Justificatory Challenge.

4.1.3 The nature of transitive counting

Here is a third response to the Justificatory Challenge, and the one most apparent from Hale's own discussion: P-to-O inferences are acceptable for number, but not in cases like water and gold, due to the nature of transitive counting itself. When introducing his argument for Frege's Constraint, Hale observes that when counting transitively we use "the *numbers* themselves *in counting*" (Hale 2016, p. 338, italics in the original). This suggests a way of trying to render TCP4 plausible. If transitive counting is essential to the possession of natural number concepts, and transitive counting involves the natural numbers themselves, then it is thereby *essential* to them that they can be so used. So we have reason to accept TCP4 and, hence, to accept the soundness of the inference from TCC1 to TCC2.

However, we think this line of inference is problematic for at least two reasons.

First, there are different kinds of transitive counting, each plausibly corresponding to a different stage in the acquisition of number concepts. On at least one legitimate construal, a person might count without *using* the natural numbers at all. We presume that any sensible construal of transitive counting characterizes it as a procedure for satisfactorily answering 'how many'-questions. Further, employing such a procedure requires the use of some appropriately organized system of labels to "tag" items when counting collections. But this generic construal of transitive counting can be elaborated in quite different ways, some of which tells us nothing whatsoever about the natural numbers as such.

To develop this point, we start with Benacerraf's (1965, p. 275) original characterization of transitive counting:

> To count the members of a set is to determine the cardinality of the set. It is to establish that a particular relation C obtains between the set and one of the numbers–that is, one of the elements of $\mathbb{N}$... Practically speaking, and in simple *cases*, one determines that a set has k elements by taking (sometimes metaphorically) its elements one by one as we say the numbers one by one (starting with i and in order of magnitude, the last number we say being k). To count the elements of some k-membered set b is to establish a one-to-one correspondence between the elements of b and the elements of $\mathbb{N}$ less than or equal to k. The relation "pointing-to-each-member-of-b- in-turn-while-saying-the-numbers-up-to-and-including-k" establishes such a correspondence.

Notice that what Benacerraf is describing here is a *procedure*: transitive counting involves reciting numerals in their canonical order (starting with "one", or the equivalent numeral) while "tagging" the objects being enumerated, thus establishing a one-to-one correspondence between the numerals and those objects; if 'k' is the terminal numeral used, then 'k' is correctly answers the salient "How many?"-question.

If this is how we are to construe transitive counting, then one might think that there is a sense in which we use the natural numbers themselves in transitive counting. We use them, because we *say* them. But this way of putting things is misleading, if not outright false. Numbers are not the sorts of things that are *said*—any more than *dogs* are said when we use the word 'dog'. Rather, when counting, we deploy semantically interpretable items—number words—that (we suppose) denote natural numbers. In view of this, natural numbers ought not to be thought of as directly *used* when counting. They are not the vehicles deployed in such a process, in the way that, say, knives are used when cutting. Rather, numbers are the referents of semantically evaluable vehicles—concepts, words, or numerals—that we deploy in such processes.

Thus, we distinguish between two sorts of counting that are locally behaviorally indistinguishable. Both involve using numerals to answer 'how many'-questions. In the first case, the counter grasps concepts of natural number when doing so. Call this *numerical transitive counting*. In the second case, the counter does not grasp natural number concepts when deploying numerals to answer 'how many'-questions—even though the numerals they use may express *our* concepts of natural number, or denote natural numbers in the public language. Call this *nominal transitive counting*. The crucial difference between these two kinds of counting is that whereas numerical transitive counters will deploy natural number concepts when performing the transitive counting procedure, nominal transitive counters will not. Rather, nominal transitive counters have merely memorized a routine, without recognizing that the result of performing that routine designates the cardinality of the collection being counted.

We will clarify this distinction further in §5.[12] For now, the important point is that the inference from TCC1 to TCC2 goes through only if "transitive counting" is construed as *numerical* transitive counting. Otherwise, TCP4 could be false. That is, it could be that transitive counting is essential to the possession of natural number concepts, and yet transitive counters do not use the natural numbers at all.

Thus, for the purposes of Hale's argument, TCP4 should be re-construed as TCP4*:

TCP4* If *numerical* transitive counting is essential to the possession of the concepts of the natural numbers, then the fact that the natural numbers can be used to numerically transitively count is essential to the natural numbers themselves

Yet this is no better for Hale's purposes than the more inclusive reading of TCP4. When unpacked, TCP4* asserts the following sufficiency claim:

(N1) Numerical counting – a procedure which essentially involves number concepts – is essential to the possession of natural number concepts.
is a sufficient condition for
(N2) It is essential to the natural numbers themselves that they can be used in numerical counting.

[12] There is philosophical precedent for the distinction, however. Indeed, Heck (2000) speculates that children, relatively early in their cognitive development, might be what we are calling nominal transitive counters: "Such children may well understand the numerals as mere tags, having no independent significance. For them, 'There are four hats on the table' *really does* mean something like: I ended with 'four' when I counted the hats. But they seem to have no grasp at all of the *point* of such 'ascriptions' of number."

But the sufficiency claim is far from obvious. To appreciate the point, consider the following (partially) analogous sufficiency claim:

(W1) Knowingly drinking water – which essentially involves the water concept – is essential to the possession of the concept WATER.
is a sufficient condition for
(W2) It is essential to water *itself* that it can be knowingly drunk.

This claim is false. Water would still be what it is—H_2O—even if no one could drink it, knowingly or otherwise. Indeed, even if W1 were true, and it was a *necessary* truth that water can be knowingly drunk, it would still be unobvious that the possibility of being drinkable would be *essential* to water itself. This is because not every necessary truth about an entity is essential to its being the (sort of) thing it is.[13] Of course, there is an assumption that, if accepted, would make the sufficiency of W1 for W2 rather more plausible:

(The Use Principle) The core uses of an entity are *essential* to what that entity is.

If this principle were true, and we further assume that being knowingly drunk is a core use of water, then it follows that this use is essential to what water is. In which case, we would be in a position to criticize the chemists for failing to "directly reflect" this fact in their account of water.

As far as we call tell, the situation is exactly similar for TCP4*. Just as water is essential to the activity of drinking water, the natural numbers are essential to the activity of numerical transitive counting. But it is the converse claim that Hale needs to establish, i.e. N2.

(N2) It is essential to the natural numbers themselves that they can be used in numerical counting.

And, for reasons analogous to the case of W2, it is far from obvious that this is true. Indeed, even if N1 were true, and it was a *necessary* truth that numbers can be used to numerically transitively count, it would *still* be far from obvious that their role in counting would be essential to them. This is because, as we discuss in §5, one can *derive* the role of numbers in numerical transitive counting, from a broadly structuralist conception of the naturals—the DP axioms—which does not even pretend to specify transitive counting as essential to those numbers.

Of course, as in the case of W2, there is an assumption, which if accepted, would make the sufficiency of N1 for N2 rather more plausible:

(The Number Use Principle) The core uses of a kind of number are *essential* to what that entity is.

If this principle were true, and if we further assume that numerical transitive counting is a core use of natural numbers—as it plausibly is amongst humans—then it

[13] This is, of course, a point on which Hale, qua modal essentialist, is wholly in agreement. See Hale (2013), especially Chapter 6.

follows that numerical transitive counting is essential to the naturals. In which case, we would be in a position to criticize structuralists, such as Dedekind, for failing to "directly reflect" this fact in their account of the natural numbers.

But now the problem with the present attempt to meet the Justificatory Challenge becomes entirely apparent. Hale's aim was to *argue* for Frege's Constraint. Yet the Number Use Principle *just is* a version of Frege's Constraint. More precisely, it's a generic version of the Constraint that omits reference to any specific application. To the extent, then, that Hale aims to argue against the structuralist—who explicitly disavows the need to treat core applications as essential to the natural numbers—the present proposal is question-begging.

To summarize, we have argued that there is a problematic gap in the inference from TCC1 to TCC2. First, this inference presupposes, without justification, the acceptability of P-to-O inferences in the case of natural number. Second, in view of this, a proponent of the Argument from Transitive Counting must meet the challenge of justifying this assumption, if the argument is to have any probative force. Third, we have considered and rejected what we take to be the three most plausible responses to this challenge. In all three cases, the abstractionist is forced to draw on objectionable assumptions. Finally, we take it to be far from obvious that the Justificatory Challenge can be satisfactorily met. Consequently, even if transitive counting were essential to possessing natural number concepts, it is far from obvious that a philosophically adequate characterization of the naturals ought to build that into their characterization.

5 Counting, arithmetic, and two notions of number

We now turn our attention to Hale's case for TCC1. For the sake of argument, we grant Hale's first two premises—that the ability to intransitively and transitively count jointly imply at least a basic grasp of natural number concepts, and that someone equipped with knowledge of just the DP axioms and second-order logic can do basic arithmetic. In this section, we focus on the third premise:

TCP3 A trainee who realized the possibility just described [i.e. the DP Novice who does basic arithmetic] would not yet have a basic grasp of (the concepts of) the natural numbers

The idea is that the DP Novice—despite her grasp of the DP Axioms and second-order logic, and despite her ability to intransitively count and do basic arithmetic—*still* fails to have even a basic grasp of natural number concepts, because she is incapable of numerical transitive counting. No argument is presented for this claim, but Hale takes it to be an intuitively plausible one and, moreover, one he expects his reader will also find intuitive. Unless the intuition is widely shared, it is hard to see how the argument could be of any probative value.

However, it is far from obvious to us that TCP3 is true—not, at any rate, on an interpretation that will allow it to do the work required of it by Hale's argument. To be sure, the DP novice cannot transitively count. But she does possess other core numerical capacities. In particular, by assumption, she has the ability to do basic arithmetic—surely a central numerical competence and, moreover, one that is typically assumed to be *more sophisticated* than transitive counting.[14] So, why exactly should transitive counting suffice for the possession of natural number concepts, where the capacity for basic arithmetic does not? And if basic arithmetic doesn't suffice, then why should transitive counting? If it was just *obvious* that the ability to do basic arithmetic did not suffice for a grasp of number concepts, then perhaps such questions could be sidestepped. But we think that they should not be sidestepped.

The purpose of this section is to explain why. We will present two considerations against TCP3. The first engages in a parallel kind of intuition mongering Hale adopts when defending this premise, via a character we call "the DP Expert". In effect, we find it implausible to insist that an idealized individual who has knowledge of the DP axioms and can prove various theorems in number theory, but who also cannot count, fails to have at least a basic grasp of natural number concepts. We then strengthen this contention by providing linguistic evidence suggesting that the predicate 'natural number' more plausibly denotes objects of the sort characterized by the DP axioms than HP, thus casting further doubt on Hale's assumption that the DP Novice fails to grasp basic natural number concepts.

5.1 The DP expert

Recall the DP Novice. She has access to the DP axioms, second-order logic, and can do very elementary arithmetic. Suppose she goes on to become adept in number theory. She learns a full range of theorems in arithmetic, say by studying a textbook on arithmetic such as the classic Hardy and Wright (1938). To get fanciful, suppose that she even contributes to the field, publishing articles on number theory in mainstream mathematics journals, all without knowing how to numerically transitively count. Call this character *the DP Expert.*

According to Hale (2016, p. 339), even our DP Expert "would not yet have a basic grasp of (the concepts of) the natural numbers", since she does not have what is putatively necessary for this grasping—the ability to numerically transitively count.[15] But this strains credulity. By hypothesis, she is an established authority on *number theory*. She has read and digested standard texts on the subject and even contributes to the field in professional journals. The coherence of this scenario is surely *prima facie* reason to suppose that she possesses at least a basic grasp of natural number concepts.

Admittedly, she lacks the notion of a *cardinal* number—or at least she does not tie her concept of natural number to that of cardinal number. As a consequence, there will be certain number theoretic propositions she does not grasp, e.g. the prime

[14] At least in terms of the development of number concept acquisition, which we return to in §5.

[15] Wright (2000, p. 327) is committed to a similar conclusion. He notes that the structuralist, who bases her account of the natural numbers on the DP axioms, will "be open to the charge of changing the subject: whatever the detail of her epistemological story about the simplest truths of arithmetic, the content of the knowledge thereby explained will not be that of the knowledge we actually have."

number theorem. Yet this alone is not a reason to deny her possession of natural number concepts. After all, Hale's transitive counter will fail to grasp *many* number theoretic propositions, even very simple ones such as $4+3=7$.

Nor will it do merely to insist that natural number concepts *just are* concepts of cardinal numbers or, perhaps, finite cardinal numbers. For this is, in large measure, the point at issue between abstractionists, such as Hale and Wright, and those of more structuralist inclination. Without such an assumption, however, we see no reason to suppose that the DP Expert lacks basic natural number concepts.

Of course, the abstractionist might seek to explain away the DP Expert's seeming grasp of natural number concepts. For they might claim that she merely grasps a structure isomorphic to the natural numbers, and that, for this reason, she is able to contribute to number theory—for isomorphic structures share their structural properties. But we can see no non-circular reason for insisting on this interpretation of the case, i.e. not without appealing directly to TCP3.

We might develop the point further by exploiting Heck (2011)'s distinction between being *expressible* within a certain formal theory and being *interpretable* within that theory.[16] Thanks to her knowledge of the DP Axioms, the DP Expert is an expert on a theory which overlaps with, but does not entirely exhaust, number theory. As a result, she cannot express all number theoretic propositions. Nevertheless, she can *interpret* those propositions within her theory, and this gives us a seemingly strong, a priori reason for thinking that she grasps concepts corresponding to those objects which number theory aims to describe.

To draw an analogy, consider the concept WINNING A BATTLE, and consider two potential formal theories. The first is stocked with concepts like SOLDIER, GENERAL, TANK, etc., and is capable of describing what these things are, where they are located on the battlefield, how they interact, etc. The second, more impoverished theory, does not have these concepts, though it is capable of describing which things belong to which sides, as well as which things are located on the battlefield and which are not. Intuitively, both theories are capable of describing when a battle is won: when there are no more things on the battlefield belonging to one side. It seems highly implausible, therefore, that only someone grasping the first theory possesses the concept WINNING A BATTLE. Yet the DP Expert would appear to be a similar situation with respect to natural number concepts.

To draw a different analogy, consider the *HP Novice*, i.e. someone who has epistemic access to just HP and second-order logic. Thanks to his knowledge of HP, he can tell when collections have the same cardinality. However, because he does not yet have access to Frege's Theorem, he cannot do basic arithmetic. Now consider a parallel claim to Hale's TCP3.

> (5) A trainee who can tell whether collections are equinumerous but cannot yet do basic arithmetic would not yet have a basic grasp of (the concepts of) the natural numbers.

[16] Many thanks to an anonymous referee for this suggestion.

If true, (5) would show that knowledge of HP is not sufficient for possessing even a basic grasp of genuine natural number concepts. Obviously, abstractionists will reject the intuition underlying (5). They might suggest that because the DP Axioms are derivable from HP along with some suitable definitions, the HP Novice at least has the conceptual ingredients sufficient to *eventually* do basic arithmetic, and that this is enough to grasp basic natural number concepts.

However, as we will see in §5, something similar can be said for the DP Novice with respect to transitive counting. Though she does not have the conceptual ingredients required to transitively count, the additional resources required can be supplied, in which case a parallel claim can be made on her behalf: she at least has the conceptual resources sufficient to *eventually* transitively count. So why think that the HP Novice is any better off than the DP Novice with respect to possessing basic natural number concepts? Again, the only apparent reason is that possessing basic natural number concepts requires an ability to transitively count, and this precisely the assumption our DP Expert calls into question.

5.2 Two notions of number in natural language

So far, our conclusion is that because the DP Expert not only has knowledge of, but can work extensively with, the axioms typically taken to characterize the natural numbers, there is at least some intuitive plausibility to the claim that she possesses basic natural number concepts. In this section, we strengthen this contention by drawing on natural language evidence. Following Moltmann (2013) and Snyder (2017), we will argue that natural language distinguishes between two kinds of number—what we call *arithmetic number* and *cardinal number*. Moreover, we claim that the English expression ‘natural number’ more plausibly denotes the former sorts of objects, and that these are more plausibly viewed as characterized by the DP axioms. Thus, of our two Novices, the DP Novice is more plausibly viewed as having a basic grasp of corresponding natural number concepts, despite her inability to numerically transitively count.

We begin with a number of semantic contrasts observed originally by Moltmann and Snyder. What they purport to reveal is that the noun ‘number’ is ambiguous between a monadic predicate true of individual arithmetic objects, and a relational predicate—‘number of’—true of pairs of collections and cardinalities representing how many objects belong to that collection. The former is witnessed in (6a,b), the latter in (7a,b).

(6) a. Four is an even <u>number</u>
b. The <u>number</u> {four/Mary is thinking about} is even.

(7) a. Mary saw a <u>number of</u> ducks on the bridge
b. The <u>number of</u> ducks Mary saw is four.

To see that these in fact have different meanings, it suffices to note that they are not acceptably intersubstitutable in a variety of different environments. For example, Moltmann points out that whereas cardinal ‘number’ is acceptable with predicates like ‘notice’, ‘count’, and ‘compare’, arithmetic ‘number’ is not.[17]

(8) a. Mary noticed the number {of ducks/??four}.
b. Mary counted the number {of ducks/??four}.
c. Mary compared the number {of ducks/??four} to the number of geese.
d. The number {of ducks/??four} is larger than the number of geese.

Even if Mary happened to notice, count, or compare exactly four ducks, there is a clear contrast in acceptability between ‘the number of ducks’ and ‘the number four’ in these examples, thus suggesting that they cannot refer to the same sort of object.

Now consider the contrast in (9):

(9) a.?? The number of ducks Mary noticed is the number four
b. The number Mary is thinking about is the number four.

Assuming ‘number of’ in (9a) relates a collection of ducks to their cardinality, the fact that ‘the number four’ is unacceptable suggests that it must refer to something other than a cardinality. On the other hand, assuming ‘the number Mary is thinking about’ in (9b) denotes a number, the acceptability of ‘the number four’ there suggests that it does refer to such an object. We see an exactly similar contrast in (10):

(10) a.?? How many ducks did Mary notice? The number four
b. Which number is Mary thinking about? The number four.

Again, this makes sense if (10a) asks about a cardinality, i.e. the sort of thing answering a ‘how many’-question, if (10b) asks about a particular number, and if ‘the number four’ refers to the latter rather than the former.

Finally, Moltmann notes numerous differences between arithmetic ‘number’ and cardinal ‘number’ with respect to various mathematical modifiers. For example, whereas arithmetic ‘number’ acceptably combines with modifiers like ‘natural’ and ‘prime’, cardinal ‘number’ does not.[18]

(11) a. the {natural/prime/rational} number Mary is thinking about
b.?? the {natural/prime/rational} number of ducks

[17] Some may find certain of these examples more acceptable than others. To be clear, the judgments reported here are those of Moltmann (2013) and Snyder (2017), and they appear to be shared by many native English speakers, though this is ultimately an empirical question. What’s important for our purposes is that there is a contrast, witnessed in a variety of contexts, between *the number of*-terms and terms like *the number four*.

[18] To be clear, not all mathematical modifiers are unacceptable with *the number of*-terms. For example, as Moltmann (2013) observes, there is no difference in acceptability between ‘the even number {of ducks/four}’.

Also, whereas arithmetic 'number' acceptably combines with various functional expressions like 'successor', 'square', and 'square root', cardinal 'number' does not.

(12) a. the {successor/square/square root} of the number four
b.?? the {successor/square/square root} of the number of ducks

Based on these and further similar contrasts, Moltmann and Snyder come to the same conclusion: natural language distinguishes between numbers *qua* arithmetic objects and cardinalities *qua* representations of the cardinal size of collections.[19]

Assuming this is correct, two questions naturally arise. First, which of these two notions is relevant to TCP3? Secondly, which of our two candidate characterizations—HP or the DP axioms—more plausibly capture this notion? As for the first question, it seems quite clear that notion relevant to TCP3 is *arithmetic* number. After all, as TCP2 makes evident, TCP3 invites us to imagine someone who not only possesses the definition of zero and successor, but also has been taught to do basic addition and multiplication. Clearly, such an individual is being taught how to do basic *arithmetic*, despite an inability to transitively count. Moreover, examples like (11) and (13) strongly suggest that 'natural number' is most plausibly viewed as a predicate true of arithmetic objects,

(13) a. Four is a (natural) number
b. Some (natural) numbers are even, such as the (natural) number four.
c. Mary's favorite (natural) number is the (natural) number four.

and (12) suggests that these are the sorts of things which can bear various arithmetic relations to each other, including successor.

This in turn suggests an answer to the second question: the DP axioms are a more plausible candidate for characterizing arithmetic number, whereas HP is a more plausible candidate for characterizing cardinal number. After all, HP identifies #-terms based on whether concepts are equinumerous, and '#' codifies 'number of'. For example, consider the fact that while (14a) is faithfully rendered as (14b), (15a) cannot be faithfully rendered as (15b), since Mary can think about the number four without thinking about four things.

(14) a. The number of ducks Mary noticed is four
b. $\#[\lambda x.\ \text{duck}(x)\ \&\ \text{Mary-noticed}(x)] = 4$

(15) a. The number Mary is thinking about is four
b. $\#[\lambda x.\ \text{Mary-is-thinking-about}(x)] = 4$

[19] Balcerak-Jackson and Penka (2017) come a similar conclusion, though based on different considerations.

On the other hand, while the DP axioms tell us nothing about how to enumerate collections or answer 'how many'-questions, they are usually assumed to form the basis of number theory, i.e. the theory of the natural numbers.

All of this suggests that someone equipped with just the DP axioms and the ability to do elementary arithmetic is a more plausible candidate for grasping the extension of 'natural number' than someone equipped with just HP. In other words, the DP Novice is a more plausible candidate for having a basic grasp of the concept NATURAL NUMBER, along with (presumably) concepts corresponding to the members of its extension, i.e. the natural numbers themselves. Thus, the semantic evidence not only casts doubt on the intuitive plausibility of TCP3, but also Hale's first conclusion—that possessing a basic grasp of natural number concepts requires an ability to numerically transitively count. After all, since the HP Novice is a more plausible candidate for grasping cardinal 'number', he would appear to be a better candidate for being a genuine numerical transitive counter, at least initially (stay tuned). But since the DP Novice is the more plausible candidate for grasping basic natural number concepts, it thus seems even less plausible that having a basic grasp of natural number concepts should require being able to numerically transitively count.

To be clear, as with the intuitions regarding the DP Expert, we take the linguistic evidence to be suggestive, not definitive. Obviously, it remains open to abstractionists to dismiss this evidence as irrelevant or somehow misleading. Nevertheless, we believe that when taken along with the DP Expert, these considerations provide strong reasons for questioning the intuitive plausibility of Hale's third premise. It simply is not obvious that possessing basic natural number concepts requires an ability to numerically transitively count.

6 A final neologicist dilemma

We have argued that Hale's first conclusion—that numerical transitive counting is essential to possessing natural number concepts—is unwarranted because his third premise—that someone who could not count but could do basic arithmetic fails to have even a basic grasp of natural number concepts—is at best unobvious. Further, we have argued that even if we were to grant Hale's first conclusion, his second conclusion—that a philosophically adequate account of the naturals ought to observe Frege's Constraint—rests on an unsupported principle—the Possession-to-Object Principle—which we do not see how to justify in a non-circular manner.

In this final section, we sketch a different sort problem: Even if one grants Hale's second conclusion, it will not do what it is intended to do, namely adjudicate in favor of HP over the DP Axioms as uniquely characterizing our actual natural number concepts. That's because *neither* candidate set of characterizing principles captures numerical transitive counting, at least not without some additional cognitive resources at hand. Thus, if numerical transitive counting is the application relevant to satisfying Frege's Constraint, then neither the DP axioms nor HP satisfies Frege's Constraint. On the other hand, if additional cognitive resources are allowed, then *both* sets of characterizing principles will satisfy Frege's Constraint, using similar resources to do so.

6.1 Three forms of counting

Above, we distinguished three forms of counting. First, there was *intransitive* counting: reciting the numerals in their usual order, starting with 'one'. Next, there was *nominal* transitive counting: successfully performing the transitive counting procedure without grasping and deploying cardinality concepts. Finally, there was *numerical* transitive counting: successfully performing the transitive counting procedure while grasping and deploying cardinality concepts. In this section, we clarify the distinction between nominal and numerical transitive counting. We will see that while the DP Novice has the resources necessary for intransitive counting, the HP Novice does not. Nevertheless, even if we grant him those resources, both Novices would be capable of nominal transitive counting. Consequently, if Frege's Constraint is to adjudicate between these different characterizations of arithmetic, then it had better turn out that HP but not the DP Axioms captures *numerical* transitive counting. The ultimate purpose of this section is to show that, in fact, *neither* set of characterizing principles accomplishes this, at least not by itself.

Let's begin with intransitive counting. Because she has access to the DP Axioms and second-order logic, the DP Novice can derive a potentially infinite list of numerals in their appropriate order. These are given in (16), or what we call *DP numerals*:

(16) $<\mathrm{s(o), s(s(o)), \ldots}>$

As a result, the DP Novice is capable of intransitive counting: she simply recites the DP numerals in (16). On the other hand, because the HP Novice does not yet have access to Frege's Theorem, he cannot generate numerals in this manner. Nevertheless, thanks to his access to second-order logic, he can *form* the numerals given in (17), which we call *HP numerals*, repeated from above (but starting with the numeral for one).

(17) $<\#[\lambda y.\ y=\#[\lambda x.\ x\neq x]],\ \#[\lambda z.\ z=\#[\lambda y.\ y=\#[\lambda x.\ x\neq x]]\ \vee\ z=\#[\lambda x.\ x\neq x]]], \ldots>$

Supposing we supply the HP Novice with this list, he too is able to intransitively count: he simply recites the HP numerals in the order suggested in (16).

It is worth noting that while both Novices can intransitively count if supplied the numerals in (16) and (17), respectively, neither is actually capable of counting *in*transitively, at least if additional deductive reasoning is prohibited. One way to see this is through principles psychologists commonly use to characterize counting. Specifically, consider Gelman and Gallistel (1986)'s Stable Ordering Principle (SOP):

(SOP) The numerals employed in counting must occur in a stable, and thus repeatable, order

SOP is a minimal condition on successful intransitive counting. It requires that the count sequence contains a fixed first element, followed by a fixed sequence of successive elements. This is, of course, precisely what the DP Axioms provide, given some additional deductive reasoning: the axiom for zero provides a stable first element, while the axioms characterizing successor provide a sequence of stable successive elements. Similarly, HP numerals can be generated in their appropriate order within the abstractionist setting given additional deductive reasoning.[20] We begin with HP, and derive zero by considering the empty concept, i.e. $\lambda x.\ x \neq x$. We then derive one by taking the number of concepts equinumerous with zero, two by taking the number of concepts equinumerous with one, etc. Thus, both sets of base principles—HP and the DP Axioms—can be seen as meeting SOP, given some additional deductive resources.

That neither Novice can intransitively count without additional deductive resources at hand is significant because the ability to transitively count *presupposes* a capacity to count intransitively. If one cannot generate a stable ordering on the numerals, one cannot generate such an ordering while labelling objects.[21] In view of this, if neither Novice can produce numerals in their appropriate order without further deduction, then *a fortiori* they cannot use such numerals to perform the transitive counting procedure.

Our ultimate question here is whether an ability to transitively count adjudicates in favor of the HP Novice over the DP Novice. We have already seen that the answer is "No", simply because an ability to transitively count presupposes an ability to intransitively count, and neither Novice has this ability, assuming additional deduction is prohibited. Nevertheless, suppose for the sake of argument that both Novices are provided with their respective list of numerals from (16) and (17). Would they then be capable of transitively counting? Let's begin by first considering nominal transitive counting. Again, this involves performing the transitive counting procedure, which can be roughly characterized procedurally as follows:

(18) If asked "How many *F*s are there?",
i. Isolate the *F*s from the non-*F*s.
ii. Establish a mapping between the *F*s and an initial segment of the numerals in the count list $<$'n_1',…,'n_n'$>$ by reciting the numerals in order, starting with 'n_1', and correlating each F with a unique numeral in that list.
iii. If 'n_k' is the final numeral resulting from ii), then answer "There are n_k *F*s".

By stipulation, a *nominal* transitive counter is someone who can perform this procedure correctly without grasping that the answer delivered designates a cardinality. Indeed, such a counter could, in principle, perform this procedure successfully without attaching any significant *meaning* to 'n_k', let alone a *cardinal* meaning. Rather, performing the transitive counting procedure would merely amount to associating a collection with a provided label, reminiscent of Searle (1980)'s famous Chinese Room scenario. Since both Novices could do this when provided with (18) and their

[20] Many thanks to an anonymous referee for this observation.

[21] In view of this, it is unsurprising that ability to intransitively count comes prior to an ability to transitively count. That is, children learn to count intransitively well before learning how to count transitively. See Carey (2009), especially Chapter 4.

respective count lists, both would be *nominal* transitive counters. The only significant difference between the two performances would be the numerals employed: whereas the DP Novice would employ the count list in (16), the HP Novice would employ the count list in (17).

As nominal transitive counters, both Novices would plausibly grasp another commonly cited counting principle, namely Fuson (1988)'s Last Word Rule (LWR):

(LWR) The last numeral used in the transitive counting procedure answers the relevant 'how many'-question posed when performing that procedure

LWR simply encodes clause iii) in (18). Crucially, however, a grasp of LWR alone does not yet imply recognition of cardinality. Psychologically speaking, what separates nominal from numerical transitive counters is that only the latter grasp what Gelman and Gallistel call the Cardinal Principle (CP):

(CP) The last numeral used in the transitive counting procedure designates the cardinality of the collection being counted

As Gelman and Gallistel explain, grasping CP is a prerequisite for grasping cardinality concepts:

> The cardinal principle says that the final tag in the series has a special significance. This tag, unlike any of the preceding tags, represents a property of the set as a whole. The formal name for this property is the cardinal number of the set. Put more informally, the tag applied to the final item in the set represents the number of items in the set.[22]

Emphasis here is on *number of*: What CP-knowers grasp, and what mere LWR-knowers fail to grasp, is that successfully performing the transitive counting procedure results in tagging the collection with a numeral which designates the *cardinality* of the collection being counted. Thus, the distinction between CP-knowers and mere LWR-knowers mirrors our distinction between numerical and merely nominal transitive counters.

If grasping CP is necessary for numerical transitive counting, then what in addition to LWR must one grasp in order to be a CP-knower? According to the influential accounts like Sarnecka and Carey (2008, p. 665), the extra ingredient required is something like the successor relation:

> The cardinal principle is often informally described as stating that the last numeral used in counting tells how many things are in the whole set. If we interpret this literally, then the cardinal principle is a procedural rule about counting and answering the question 'how many'… Alternatively, the cardinal principle can be viewed as something more profound – a principle stating that a numeral's cardinal meaning is determined by its ordinal position in the list… If so, then knowing the cardinal principle means having some implicit

[22] Gelman and Gallistel (1986, p. 79–80).

knowledge of the successor function – some understanding that the cardinality for each numeral is generated by adding one to the cardinality for the previous numeral.

Two points are worth emphasizing here. First, insofar as CP is meant to separate nominal from numerical transitive counters, it must be *distinguished* from LWR. Secondly, insofar as a grasp of successor implies a grasp of CP, the relevant notion of successor involved had better be tied to *cardinality* in some way. After all, the DP Novice already has a notion of successor and can successfully perform the transitive counting procedure without having any obvious grasp of cardinal number, i.e. *number of*.

Thus, we distinguish here between two notions of successor. The first is *structural successor*, as defined by the DP axioms. As we have seen, grasping structural successor suffices to generate a stable order of numerals, but is insufficient to link numerals to cardinalities. The second is *cardinal successor*, as defined by Frege (1884, 1893). In modern notation, n' is the cardinal successor of n just in case:

(19) $\exists F.\exists G.\ n = \#F \ \&\ n' = \#G \ \&\ \exists x.\ Gx \ \&\ \forall y\ [Fy \equiv (y \neq x \ \&\ Gy)]$

In English, n' is the cardinal successor of n if n' enumerates a concept G, n enumerates a concept F, and exactly one more object falls under G than F. Unlike structural successor, cardinal successor ties successor directly to cardinalities, thanks to #. Thus, it represents a plausible candidate for linking the result of performing the transitive counting procedure to the cardinality of the collection being counted.

With this distinction in place, we lay down the following two conditions on numerical transitive counting.

(NuTC1) The counter must recognize that the first numeral in the count list designates the cardinal number one

(NuTC2) The counter must recognize that each numeral in the count list following the first designates a cardinal number which is the cardinal successor of the cardinal number designated by the immediately prior numeral

Plausibly, genuine numerical transitive counting requires recognizing that *all* numerals in one's count list potentially designate cardinalities. For instance, it seems that someone who counts four Elmos on the table, but fails to recognize that had the counting procedure ended on the previous numeral instead, then the cardinality of the Elmos would have been three, has yet to fully appreciate the purpose of performing that procedure, namely designating cardinalities. Together, NuTC1 and NuTC2 prevent this possibility. Specifically, NuTC1 guarantees that the first numeral on the count list designates a cardinality, while NuTC2 does likewise for all subsequent numerals in the count list. Consequently, anyone grasping NuTC1 and NuTC2 would recognize that all of their numerals potentially designate cardinalities, and so grasping NuT1 and NuTC2 is plausibly what separates merely nominal from genuine numerical transitive counters.

Given these conditions, it is easy to see that *neither* Novice is a genuine numerical transitive counter, even if they are provided with their respective count lists.

Even though the DP Novice can form a stable count list thanks to her knowledge of the DP axioms, at least given a small amount of additional deduction, she does not yet have a notion of *cardinal number*, as provided by HP or some other such principle.[23] In other words, she has no way of linking her DP numerals to #-terms, and thus the result of performing the transitive counting procedure to cardinalities. Hence, she fails to grasp *both* NuTC1 and NuTC2. In contrast, the HP Novice he can immediately see that the first HP numeral refers to a singleton set, and so can infer via HP that all one-membered classes are enumerated by it. Thus, he arguably grasps NuTC1. However, because he does not yet have access to Frege's definition of cardinal successor, he does not grasp NuTC2.[24]

We can summarize our results as follows: while the DP Novice has a general grasp of what we might call *structural number* and structural successor, the HP Novice has a general grasp of *cardinal number* but not cardinal successor. As we saw in §4, something like this distinction is plausibly reflected in natural language, specifically in the ambiguous noun 'number'. There we argued that because the DP Expert has knowledge of and is capable of working extensively with the axioms characterizing number theory, there is at least some intuitive plausibility to the claim that she possesses basic natural number concepts. Moreover, from the present perspective, to insist that she does not possess such concepts amounts to insisting without argument that natural number concepts are cardinal number concepts.

However, what our characters here reveal is that even if one insists on this identification, Hale's formulation of Frege's Constraint will not adjudicate in favor of HP over the DP axioms, since neither set of principles suffices to capture that proposed application. Consequently, knowledge of either set of principles will not suffice to possess cardinality concepts. Something more is needed in each case. In the next section, we spell out what exactly these additional resources are.

6.2 Recovering numerical transitive counting

We have argued that since neither Novice possesses the resources necessary for numerical transitive counting, neither HP nor the DP axioms satisfies Frege's Constraint on Hale's construal. More specifically, whereas the DP Novice fails to grasp both NuTC1 and NuTC2, the HP Novice arguably only fails to grasp the latter. This makes it fairly easy to see what additional resources each Novice would need to numerically transitively count, and thus what else needs to be added to each set of characterizing principles to satisfy Frege's Constraint on Hale's construal.

[23] For example Tennant (1997)'s Schema N.

[24] To illustrate, consider again the second HP numeral, and consider a class of two objects, say the moons belonging to Mars. In order to infer from HP that the number of Martian moons is the number referenced by the second HP numeral, the HP Novice needs to make an additional inference, given the numeral's disjunctive character. Namely, he needs to infer that the number of the concept BEING IDENTICAL TO THE NUMBER ZERO (i.e., THE CONCEPT BEING NON-SELF-IDENTICAL) and the number of the concept BEING IDENTICAL TO EITHER THE NUMBER ZERO OR THE NUMBER ONE are distinct. This is *not* something he can know from HP alone, and the point generalizes to all HP numerals beyond the first.

Let's begin with the HP Novice. Again, though he arguably recognizes that his first numeral designates the cardinal number one, he fails to have a general notion of cardinal successor, or *next number of*. What he needs, of course, is Frege's definition of successor, repeated here in (19).

(19) $\exists F.\exists G.\ n = \#F\ \&\ n' = \#G\ \&\ \exists x.\ Gx\ \&\ \forall y\ [Fy \equiv (y \neq x\ \&\ Gy)]$

Following Frege, the HP Novice could then go on to prove the existence of zero, that the successor relation is a one-to-one function on the finite cardinal numbers, and the induction principle. That is, he could establish the DP axioms for the notion of finite cardinal number and cardinal successor via Frege's Theorem. This, in turn, would provide him with a way of generating stable numerals, and thus a repeatable count list. And this, combined with his knowledge of HP and cardinal successor, provides the resources necessary for numerical transitive counting.

On the other hand, what the DP Novice is missing is, of course, a general notion of cardinal number. This would be remedied if she had access to what we call *Dedekind's Theorem*, stated in the following passage from Dedekind (1888):

> 161. Definition. If Σ is a finite system, then by (60) there exists one and by (120), (33) only one single number n to which a system Σ_n similar to the system Σ corresponds; this number n is called the number [*Anzahl*] of elements contained in Σ (or also the degree of the system Σ) and we say Σ consists of or is a system of n elements, or the number n shows how many elements are contained in Σ. If the numbers are used to express accurately this determinate property of finite systems they are called cardinal numbers.

In contemporary terms, "finite system" translates as "Dedekind finite set", and "similar" as "equinumerous". In effect, Dedekind's Theorem combines previous results—results obtained from the DP Axioms, suitable definitions, and second-order logic—to establish the sorts of one-to-one correspondences characteristic of transitive counting, and then defines "cardinal number of the Fs" as the terminal number resulting from performing that procedure on some finite collection of Fs.

With Dedekind's Theorem in hand, the DP Novice has, in effect, a finite version of HP. This is given in (20), where '$\sim$' is an equivalence relation holding between two concepts F and G just in case the result of performing the transitive counting procedure on the Fs terminates in the same DP numeral as performing that procedure on the Gs.

(20) $\forall F, G.\ \#F = \#G \leftrightarrow F \sim G$

This, in turn, will fix the meanings of #-terms for finite concepts, thus affording a finite notion of *number of*. She can also go on to prove that, in general, if $\#F = n$ for some DP numeral n and exactly one more object falls under G than F, then $\#G = \mathrm{s}(n)$, thus connecting her notion of structural successor to that of cardinal successor. Hence, she too would have the resources necessary for numerical transitive counting.

In sum, with some additional resources at hand, both Novices would be capable of numerical transitive counting. The problem, however, is that these additional resources come too late in the explanation by Neologicists' lights. They are "tacked on externally" to quote Frege (1903).[25] They are not "absolutely on the surface" to quote Wright (2000). More specifically, the problem is that they are *derived* from the original characterizing principles, as the following quote from Dummett (1991, p. 60) makes clear.

> Any specific type of application will involve empirical, or at least non-logical, concepts alien to arithmetic;… To make such applications intrinsic to the sense of arithmetical propositions is therefore to import into their content something foreign to it,… What is intrinsic to their sense, however, is the general principle governing all possible applications. That must accordingly be incorporated into the definitions of the fundamental arithmetical notions. It is not enough that they be defined in such a way that the possibility of these applications is subsequently provable; since their capacity to be applied in these ways is of their essence, the definitions must be so framed as to display that capacity explicitly.

Emphasis here is on *subsequent provability*. Because the empirical applications of arithmetic are essential to the naturals, those applications must be directly reflected in the principles characterizing those numbers, not subsequently derivable from those principles. But this is precisely what the cognitive situations of our two Novices reveal: the resources required for explaining numerical transitive counting are available only if subsequent derivation is allowed in the explanation.

Ultimately, then, the abstractionist faces a dilemma. Either derivation is allowed in the explanation of numerical transitive counting or it's not. If it is allowed, then HP is sufficient to explain transitive counting since it is possible to derive the additional resources required in the manner just indicated. But then so are the DP axioms, since a similar derivation is possible for them, using broadly similar resources. On the other hand, if derivation is not allowed in the explanation, as Dummett and others insist, then neither set of principles will satisfy Frege's Constraint. In either case, Frege's Constraint will not adjudicate in favor of HP over the DP axioms as the uniquely correct formal characterization of the natural numbers.

7 Conclusions

We have argued that the Argument from Transitive Counting ultimately fails to establish its intended conclusion: that because Frege's Constraint is justified, and since HP but not the DP Axioms satisfy that Constraint, only HP correctly characterizes of the natural numbers.

There are at least three problems. First, Hale's argument makes an unjustified, and indeed highly dubious, assumption, namely that because transitive counting is essential to possessing natural number concepts, an ability to transitively count is

[25] This translation is from Ebert & Rossberg (forthcoming), and differs from Wright (2000, p. 324)'s translation ("patch them on from the outside").

essential to the natural numbers themselves. We have seen that inferences of this form—possession-to-object inferences—are generally unwarranted, and that none of the most obvious and natural responses available to abstractionists succeed in explaining why natural number concepts but not a great many other kinds of concepts prohibit such inferences.

Secondly, Hale's argument relies on another unwarranted, and indeed highly dubious, assumption, namely that someone who had access to the DP Axioms but could not transitively count would fail to a genuine grasp of natural number concepts. We have argued that this is implausible, for at least two reasons. First, it suggests that someone who has mastered the DP Axioms and can derive many interesting number theoretic results on that basis nevertheless fails to have even a basic grasp of natural number concepts, which strains credulity. Secondly, English and other natural languages appear to draw an important semantic distinction arithmetic number and cardinal number, and the DP Axioms are arguably a better candidate for characterizing natural number concepts than HP. If so, then someone who possessed only the DP Axioms is arguably in a better position to possess natural number concepts than someone who only possessed HP.

Finally, even if we grant that Frege's Constraint is justified, and that the application relevant to satisfying that constraint is transitive counting, it does not follow that HP correctly characterizes the natural numbers. That's because neither HP nor the DP Axioms alone are sufficient to guarantee an ability to transitively count, at least not without further deductive resources available, which Frege's Constraint prohibits. Moreover, since the resources required are relevantly similar in both cases, Frege's Constraint would not plausibly adjudicate between these different candidate formal characterizations even if additional deductions were allowed.

The significance of this result extends beyond a mere interest in abstractionist neologicism. There are *many* formal characterizations of the natural numbers available, including e.g. the contructivist logicism of Tennant (1987,1997), modal accounts like that of Hellman (1989) and Zalta (1999), and ordinal characterizations like that of Linnebo (2009). Given the availability of these accounts, it is natural to want to identify one of them as somehow "correct", or at least "better" in some sense than the alternatives. Ultimately, the purpose of Frege's Constraint is to do just that—to adjudicate among the competing alternatives.

There are at least two challenges to this suggestion, however. First, we need a persuasive argument for *accepting* Frege's Constraint, despite a great many, seemingly legitimate, formal characterizations of various classes of entities failing to respect such a constraint. Secondly, we need to be able to identify what *the* "primary" (empirical or otherwise) application of the natural numbers is, despite the fact that the naturals appear to serve a number of legitimate functions, including counting, doing arithmetic, and ordering collections of objects.[26] The alternative, which we endorse, is to take a broadly "structuralist" attitude towards these various characterizations. Since they characterize the same *structure*, they are all "legitimate", and there is no obvious need for or benefit to legislating among them.

[26] See Snyder et al. (2018) for relevant discussion.

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