



Resolving Frege's Other Puzzle

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ABSTRACT

Number words seemingly function both as adjectives attributing cardinality properties to collections, as in Frege's 'Jupiter has four moons', and as names referring to numbers, as in Frege's 'The number of Jupiter's moons is four'. This leads to what Thomas Hofweber calls Frege's Other Puzzle: How can number words function as modifiers and as singular terms if neither adjectives nor names can serve multiple semantic functions? Whereas most philosophers deny that one of these uses is genuine, we instead argue that number words, like many related expressions, are polymorphic, having multiple uses whose meanings are systematically related via type shifting.

1. FREGE'S OTHER PUZZLE

In the *Grundlagen*, Frege [1884, §57] observes that number words, such as 'four', appear to have two importantly different kinds of uses:

Since what concerns us here is to define a concept of number that is useful for science, we should not be put off by the attributive form in which number also appears in our everyday use of language. This can always be avoided. For example, the proposition 'Jupiter has four moons' can be converted into 'The number of Jupiter's moons is four'. Here the 'is' should not be taken as a mere copula ... Here 'is' has the sense of 'is equal to', 'is the same as' ... We thus have an equation that asserts

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that the expression ‘the number of Jupiter’s moons’ designates the same object as the word ‘four’.

Specifically, ‘four’ has an “attributive form” witnessed in (1b), and an apparently referential form witnessed in (1a).

- (1) a. The number of Jupiter’s moons is four.
- b. Jupiter has four moons.

On its face, the function of ‘four’ in (1b) is to count moons, by attributing a certain numerical property to them. In this respect, ‘four’ seemingly resembles the adjective ‘green’ in (2), which intuitively attributes a color property to those same entities:

- (2) Jupiter has green moons.

In contrast, (1a) looks like an identity statement, equating the referents of ‘the number of Jupiter’s moons’ and ‘four’. In this respect, ‘four’ in (1a) seemingly resembles the name ‘Wagner’ in (3), due to Hofweber [2007].

- (3) The composer of *Tannhäuser* is Wagner.

This parallel may suggest that ‘four’ in (1a) is also a name, *i.e.*, a prototypical singular term.

Frege’s primary purpose was not to provide an empirically plausible semantics for number words in natural language, but instead to develop an ideal language suitable for science (a *Begriffsschrift*). Nevertheless, his observation that such expressions appear to serve opposing referential and non-referential semantic functions in natural language is puzzling. That is because, to quote Thomas Hofweber [2016, pp. 115–116] at length:

On the one hand, “four” occurs as an adjective [in (1b)], which is to say that it occurs grammatically in sentences in a position that is commonly occupied by adjectives ... similar to “green” in [(2)] ... On the other hand, “four” occurs as a singular term [in (1a)], which is to say that it occurs in a position that is commonly occupied by paradigmatic cases of singular terms, ... [so that] “four” [in (1a)] seems to be just like “Wagner” in [(3)] ... and both of these statements seem to be identity statements, ones with which we claim that what two singular terms stand for is identical.

That number words can occur both as singular terms and as adjectives is puzzling. Usually adjectives cannot occur in a position occupied by a singular term, and the other way round, without resulting in ungrammaticality and nonsense. To give just one example, it would be ungrammatical to replace “four” with “the number of moons of Jupiter” in [(1b)]:

- (4) * Jupiter has the number of moons of Jupiter moons.

This ungrammaticality results even though “four” and “the number of moons of Jupiter” are both apparently singular terms standing for the same object in [(1a)]. So, how can it be that number words can occur both as singular terms and as adjectives, while other adjectives cannot occur as singular terms, and other singular terms cannot occur as adjectives?

Even though Frege raised this question more than 100 years ago, I dare say that no satisfactory answer has ever been given to it.

Hofweber calls this FREGE'S OTHER PUZZLE (henceforth FOP).¹

Hofweber's comments suggest characterizing FOP as an inconsistent tetrad:

Same Expression: Both occurrences of ‘four’ in (1a,b) are witness to the same expression.

Non-Referentiality: ‘Four’ in (1b) occurs as an adjective, acceptably functioning as a modifier.

Referentiality: ‘Four’ in (1a) occurs as a name, acceptably functioning as a singular term.

Single Function: Different occurrences of the same expression cannot acceptably serve different semantic functions.

Specifically, different occurrences of adjectives or names cannot acceptably function as both singular terms and modifiers. Thus, FOP concerns the semantic function of number words in natural language. In brief, how can different occurrences of ‘four’ acceptably serve different semantic functions if names only acceptably function referentially and adjectives only acceptably function non-referentially?

Since FOP consists of four claims, this suggests four main potential solutions, each corresponding to the denial of one of them. For example, one solution would be to deny Same Expression, thus leading to what we call THE HOMONYM STRATEGY. According to it, the different occurrences of ‘four’ in (1a,b) are homonyms, *i.e.*, different expressions which just happen to be spelled and pronounced alike. Specifically, whereas ‘four’ in (1a) is a name, ‘four’ in (1b) is a lexically distinct adjective. Thus, it is hardly surprising that these are both acceptable despite having different semantic functions.

Generally, we do not expect homonyms, such as the noun ‘fire’ and the verb ‘fire’, to be acceptably intersubstitutable. Consider (5a,b), for instance, paralleling (1a) and (4).

- (5) a. The rapid oxidation of combustible materials is fire.
 b. Let's {fire/??the rapid oxidation of combustible materials} John.

¹ “Other” since the label ‘Frege's Puzzle’ is already used to describe a different, well-known linguistic puzzle concerning identity statements, also due to Frege. See [Salmon, 1986].

Perhaps this would explain why Hofweber's (4) is similarly unacceptable. However, because homonyms are characteristically spelled and pronounced alike as a matter of historical accident, we do not expect their meanings to be related. For example, there appears to be no relation in meaning between the noun 'fire' and the verb 'fire'. Thus, the problem with the Homonym Strategy is that the meanings witnessed in (1a,b) clearly *are* related. In fact, (1a,b) both tell us something about the cardinality of Jupiter's moons.

Two further strategies are suggested by Dummett [1991, p. 99], in response to Frege's observation:

Number-words occur in two forms: as adjectives, as in ascriptions of number, and as nouns, as in most number-theoretic propositions. When they function as nouns, they are singular terms ... Frege tacitly assumes that any sentence in which they occur as adjectives may be transformed either into an ascription of number ... or into a more complex sentence containing an ascription of number as a constituent part. Plainly, any analysis must display the connection between these two uses ... Evidently, there are two strategies. We may first explain the adjectival use of number-words, and then explain the corresponding numerical terms by reference to it: this we may call *the adjectival strategy*. Or, conversely, we may explain the use of numerals as singular terms, and then explain the corresponding number-adjectives by reference to it; this we may call *the substantival strategy*.

According to THE SUBSTANTIVAL STRATEGY, or SUBSTANTIVALISM, both occurrences of 'four' in (1a,b) are in fact names, *contra* Non-Referentiality, and the apparently non-referential use witnessed in (1b) is to be explained in terms of the genuinely referential use witnessed in (1a). According to THE ADJECTIVAL STRATEGY, or ADJECTIVALISM, both occurrences of 'four' in (1a,b) are either adjectives or determiners,² *contra* Referentiality, and the apparently referential use witnessed in (1a) is to be explained in terms of the genuinely non-referential use witnessed in (1b).

Both Strategies enjoy a distinguished philosophical pedigree. For example, following Dummett, Frege's own analysis may be construed as a form of Substantivalism, at least with respect to the goal of developing a *Begriffsschrift*. Specifically, Frege analyzes (1a) as an identity statement, as suggested in (6), where '#' is a function mapping concepts to natural numbers representing how many objects fall under that concept.

$$(6) \quad \#[\lambda x. \text{moon-of-Jupiter}(x)] = 4.$$

²The label 'the Adjectival Strategy' is somewhat unfortunate, because it suggests that non-referential uses of, *e.g.*, 'four' must be adjectives. However, the intended view is that both occurrences of 'four' in (1a,b) are witness to the same non-referential expression, be it an adjective or determiner.

Accordingly, (1a) is true just in case the natural number referenced by 'the number of Jupiter's moons' is the same one referenced by the numeral 'four'. In the quote above, Frege suggests that "attributive" uses like (1b) can "always be avoided", namely by paraphrasing them in terms of equivalent identity statements like (1a). If so, then (1a,b) both have the truth conditions suggested in (6), so that 'four' is ultimately analyzed as a singular term in both cases.

However, there are good reasons to reject this as a piece of empirical linguistics. Generally, we expect coreferential singular terms to be acceptably intersubstitutable, and (1a) would, if true,³ establish that 'the number of Jupiter's moons' and 'four' are coreferential singular terms. Thus, as Hofweber's (4) suggests, it is rather mysterious why, on this proposed solution, 'the number of Jupiter's moons' is not acceptably substitutable for 'four' in (1b). This casts significant doubt on the claim that 'four' in (4) is a genuine singular term.

So, this would appear to leave us with two viable solutions to FOP: denying Referentiality, thus leading to a form of Adjectivalism, or else denying Single Function, thus leading to what we call THE POLYMORPHIC STRATEGY. As it turns out, the only extant, developed solution to FOP within the philosophical literature is a version of Adjectivalism, due to Hofweber [2005; 2007]. In contrast, the primary purpose of this paper is to defend the Polymorphic Strategy. In doing so, we argue for two claims. First, despite having the apparent theoretical advantage of vindicating NOMINALISM, *i.e.*, the view that numbers do not exist, Hofweber's Adjectivalism is empirically inadequate, on semantic grounds. An immediate result is that Hofweber's proposed solution to FOP fails, as it depends wholly on the empirical adequacy of his Adjectivalism.

Second, not only is the Polymorphic Strategy naturally suggested by our best extant analyses within linguistic semantics, it is the only available solution capable of explaining what is perhaps the most empirically significant semantic fact about number words, namely that they can take on a wide variety of different but related meanings, including but not limited to those witnessed in (1a,b).

The rest of the paper is laid out as follows. §2 sketches Hofweber's Adjectivalism and offers novel criticisms of it. §3 outlines the Polymorphic Strategy, its motivations, and how it naturally resolves FOP and its potential extensions. We conclude the paper in §4, by considering three potential objections to the Polymorphic Strategy.

2. HOFWEBER'S ADJECTIVALISM AND ITS PROBLEMS

As originally formulated, FOP specifically concerns Frege's [1884] (1a,b). However, (1a) is only one of many apparently referential occurrences of 'four'. Others include (7a,b).

- (7) a. Four is a number.
b. Two plus two equals four.

³ Actually, according to NASA (<https://solarsystem.nasa.gov/moons/jupiter-moons/overview/>), Jupiter has up to 79 moons.

With respect to FOP, (7a,b) are just as “puzzling” as (1a). Specifically, in both cases, ‘four’ appears to be a singular term, unlike ‘four’ in (1b). Thus, it is straightforward to formulate alternative versions of FOP using (7a,b), substituting them for (1a) in the original formulation. Furthermore, as with (1a), the truth of (7a,b) appears to support REALISM, *i.e.*, the view that numbers exist. For example, (7a) is true, intuitively, just in case ‘four’ refers to a number.

Thus, as a potential defense of nominalism, Adjectivalism must be extended so as to account for all apparently referential occurrences of ‘four’, not just (1a) or its kin. The version of Adjectivalism defended by Hofweber [2005; 2007; 2016] is designed to do just that. It is initially motivated by the success of GENERALIZED QUANTIFIER THEORY (GQT, [Barwise and Cooper, 1981]). According to GQT, ‘four’ in (1b) is a quantificational determiner denoting a relation between sets, namely those whose intersection has at least four members, as suggested in (8).

$$(8) \quad \llbracket \text{four} \rrbracket = \lambda P. \lambda Q. |P \cap Q| \geq 4.$$

As such, ‘four’ belongs to the same class of prototypical non-referential expressions as ‘no’, ‘some’, and ‘every’. Hofweber’s key contention is that, despite surface syntactic appearances, all apparently referential occurrences of ‘four’, including (1a) and (7a,b), are actually witness to the same determiner in (1b). In particular, they are what Hofweber calls a BARE DETERMINER, or a determiner occurring without an overt accompanying noun, similar to ‘some’ in (9).

(9) Several men came to the bar. Some stayed until sunrise.

Although ‘some’ occurs bare here, it is nevertheless interpreted as restricted by the noun in the preceding sentence: some men stayed until sunrise. In contrast, ‘some’ in Hofweber’s (10a,b) occurs bare but is not restricted by any overt or preceding noun.

- (10) a. Some are more than none.
b. Four are more than three.

Rather, (10a) is intended to be interpreted *generically*: no matter what we are talking about, some are more than none. Likewise for (10b): no matter what we are talking about, four are more than three. Accordingly, Hofweber calls these SEMANTICALLY BARE DETERMINERS.

Hofweber’s central semantic thesis is that all apparent numerals, such as ‘four’ in (7a,b), are actually semantically bare determiners, having something like the GQT meaning in (8). As such, (7a,b), as well as Frege’s (1a), are all seemingly consistent with nominalism: since no single occurrence of ‘four’ is a singular term referring to a number, explaining their truth does not require positing numbers as referents of those expressions.

If so, then why do these various occurrences of ‘four’ *appear* to function referentially, contrary to fact? Hofweber provides different explanations for different cases. With respect to (1a), ‘four’ is said to “move” out of its determiner position in (1b), thus resulting in (1a), via a syntactic operation Hofweber posits

called EXTRACTION. In contrast, arithmetic equations such as (7b) are likened to (11a), analyzed informally as (11b).

- (11) a. Two and two are four.
 b. Two things and two (more) things are four things.

According to Hofweber, the apparent numerals in (11a) are again semantically bare determiners, similar to (11b). Thus, (11a) does not involve identifying the referents of two singular terms. Rather, it involves *counting* things, though in a completely general manner.

To explain why (11a) appears to involve singular terms, Hofweber posits an operation within the language of thought he calls COGNITIVE TYPE COERCION. To quote Hofweber [2016, pp. 136–137]:

[R]easoning can be seen as operating on the syntax of the language of thought, and not directly on its semantic features ... These operations ... are well developed in creatures like us, but we can't use them to reason with thoughts that would be expressed with bare determiners and that involve higher types. We thus have a mismatch between the form of the representations that we want to reason with and the form of a representation that is required for our powerful reasoning mechanisms to be employed. But this mismatch can be overcome quite simply. We can force the representation to take on a form that fits our reasoning mechanism. The representation will have to change its syntactic form by systematically lowering the type of the operation on determiners and of the bare determiners to the lower type of operations on objects and objects, respectively. Once this is done the reasoning mechanisms we have can get a grip ... I will call the process of changing the type of the form of a representation to facilitate cognition *cognitive type coercion*.

According to Hofweber, we are better at reasoning about objects than about relations between sets. Consequently, (11a) results in a kind of cognitive “mismatch”: because the higher type of bare determiners does not match the form of representations we are better equipped to reason with, we must “coerce” the form of the original representation into having a more suitable form. Crucially, however, this has no effect on the semantic types of the expressions involved in (11a). In particular, the number words retain their non-referential semantic function as bare determiners. As a result, we are able to reason arithmetically as if numbers were objects without actually saying or thinking that numbers are objects.

Cognitive type coercion may also explain why ‘four’ in (7a) only appears to function referentially. To quote Hofweber [2016, p. 145]:

Here there is a natural way to extend our account so far to cover cases like [(7a)]. After cognitive type coercion happens for bare determiner statements number determiners become available to be subjects in singular subject-predicate sentences. They still are not referential expressions on

such uses, but determiners. Correspondingly, the predicate “is a number” in [(7a)] is not predicated of the referent of [‘four’], but forms a meaningful and true subject-predicate sentence nonetheless.

Let us call examples like (7a), where ‘four’ occurs as the subject of a simple subject-predicate statement, SIMPLE NUMERICAL STATEMENTS. Hofweber’s comments suggest that in such statements, subjects meaningfully combine with predicates so as to result in a truth value. As we will see below, there are different ways this could happen, compositionally. For now, the important contention is that ‘four’ in (7a), like the number words in (11a,b), is really a semantically bare determiner, and only appears to function referentially thanks to cognitive type coercion.

Ultimately, then, Hofweber’s analysis purports to substantiate the following thesis:

Non-Referentialism: No English expressions refer to numbers. Specifically, all apparent numerals are actually non-referential semantically bare determiners.

This in turn provides a uniform solution to Frege’s Other Puzzle: since all occurrences of ‘four’ noted above are witness to the same non-referential determiner, there is no puzzle as to how one and the same expression can serve *different* semantic functions. It also purports to make sense of the truth of our number discourse, but without needing to postulate numbers as objects. In fact, given Non-Referentialism, Hofweber [2016] argues that the natural language evidence is not merely consistent with nominalism, it actually *implies* nominalism.

Most extant critical discussion of Hofweber’s Adjectivalism focuses on the empirical viability of extraction and Hofweber’s evidence for it.⁴ We focus here instead on the comparatively neglected treatment of simple numerical statements. In what follows, we challenge the empirical viability of Non-Referentialism with respect to these statements, by challenging some of its more significant semantic predictions.

2.1. The Objection from Coordination

By Non-Referentialism, the apparent numeral ‘four’ in (7a) is not a genuine singular term. In contrast, proper names occurring in subject-predicate sentences are paradigmatic singular terms, according to Hofweber [2016, p. 205].

[W]e are entitled to take names to be at least broadly referential, certainly in many of their uses where they are in subject position. Reference is thus paradigmatically the relationship that holds between (certain uses of) proper names and objects.

⁴See, e.g., [Brogaard, 2007; Balcerak-Jackson, 2013; Moltmann, 2013; Snyder *et al.*, 2022].

Specifically, despite having similar grammatical structures, 'Fido' in Hofweber's (12) functions referentially, unlike 'four' in (11a).

(12) Fido is a dog.

Put differently, 'four' in (7a) and 'Fido' in (12) are predicted to have different semantic types: whereas the latter should be of type-*e*, the former should have the type of a determiner.⁵

Our first objection challenges this prediction. It proceeds from an observation concerning expressions coordinated under conjunction, such as those underlined in (13a–c).

- (13) a. Mary and John are two of my favorite people.
 b. Mary sang and danced.
 c. Mary danced furiously and with purpose.

Speaking of such examples, Hofweber [2016, p. 138] says:

Semantic type shifting has been proposed to solve problems in natural language semantics that arise from expressions apparently having different types on different occasions. The multiple uses of “and” are a good example of this. “And” can conjoin expressions of many different types: sentences, verbs, determiners, and others . . . But it would be a mistake to think that these cases involve different words “and” all pronounced the same way. A better way to go is to think of “and” as having variable type: it can take on different types on different occasions, but all the types it can take on are related in a certain way.

Here, Hofweber is referring to the theory of GENERALIZED CONJUNCTION due to Partee and Rooth [1983]. On this influential theory, grammatical conjunctions can take on a range of different semantic types, in different syntactic environments. The core underlying assumption is:

Coordination: Expressions can be coordinated under grammatical conjunction only if they have the same semantic type.

This not only explains why expressions of the same category can be coordinated under grammatical conjunction, *e.g.*, names and verbs, but also why expressions of different categories, such as the adverb 'furiously' and the prepositional phrase 'with purpose', can likewise be coordinated: they have the same semantic type.

With this in mind, consider (14), which is clearly true in the following context.

⁵ We assume familiarity with basic type-theory as employed within linguistic semantics. For an introduction, see [Dowty *et al.*, 1981].

Context: For homework, John is to list five of his favorite things. The next day in class, Mary asks John to give two items in his list. John responds: “Well, let’s see. Fido is my favorite dog. Four is my favorite number. So ...

- (14) Fido and four are two of my favorite things.

By Coordination, which Hofweber seemingly endorses, ‘Fido’ and ‘four’ must have the same semantic type. There are two options, apparently: both are of type-*e*, denoting entities, or else both are of type- $\langle\langle e, t \rangle, \langle\langle e, t \rangle, t \rangle\rangle$, denoting relations between sets.

Clearly, the first option is inconsistent with Non-Referentialism, as ‘four’ would then have the type of a singular term. On the other hand, there is little empirical plausibility to the second option. The problem is that no widely accepted set of type-shifting principles, such as those of Barbara Partee [1986], would permit names to shift so as take on meanings relevantly similar to those of uncontroversial determiners within GQT.⁶ And for good reasons. In combination with the rest of Partee’s type-shifters, the prediction would be that names and determiners can take on the same range of semantic functions. However, this appears to be mistaken. For example, in contrast to uncontroversial determiners like ‘no’ or ‘some’, names can function acceptably as predicates and modifiers, as suggested in (15b,c).

- (15) a. {Einstein/??No/??Some/??Most/??Every} was a genius.
 b. She’s a(n) {Einstein/genius/??no/??some/??most/??every}.
 c. The {Einstein/large/??no/??some/??most/??every} family came for dinner.
 d. {??Einstein/Einstein’s/No/Some/Every} family came for dinner.

Furthermore, whereas uncontroversial determiners cannot acceptably occur bare in argument positions (*cf.* (15a)), names cannot generally occupy determiner positions (*cf.* (15d)).⁷ As we will see below, number words are more like names in these respects. In fact, this constitutes the primary semantic argument *against* number words being determiners.⁸ The important point for now, however, is that ‘four’ in (14) is most plausibly type-*e*, contra Non-Referentialism.

⁶ An anonymous reviewer rightly observes that a slight extension of the system of type-shifting principles employed below could permit names to shift to type- $\langle\langle e, t \rangle, \langle\langle e, t \rangle, t \rangle\rangle$. However, the resulting meanings are not those typically attributed to names within GQT, and they would not be appropriate for examples such as (14) or (15a–d).

⁷ To be clear, names can shift types to that of expressions like ‘every girl’, ‘most boys’, or ‘nobody’ on Partee’s [1986] analysis. The latter are type- $\langle\langle e, t \rangle, t \rangle$ expressions, or “generalized quantifiers”, denoting sets of sets, not relations between sets. Thanks to an anonymous referee for raising this issue.

⁸ See [Partee, 1986; Landman, 2003; Rothstein, 2017].

2.2. A Dilemma for Non-Referentialism

According to Hofweber, recall, number words featuring as subjects of simple numerical statements meaningfully combine with predicates to produce truth values.

(7a) Four is a number.

There are two ways this can happen, compositionally speaking, given the semantic type assumed in (8). On the first, 'four' combines directly with the predicate. In particular, just as 'is a dog' in (12) takes 'Fido' as argument and returns a truth value, 'is a number' in (7a) takes 'four' as argument and returns a truth value.

(12) Fido is a dog.

Since 'four' has the type of a determiner, by hypothesis, the prediction is that 'is a number' should be of type- $\langle\langle\langle e, t \rangle, \langle\langle e, t \rangle, t \rangle\rangle, t \rangle$, denoting a set of relations between sets. (7a) will then be true if the relation between sets specified in (8) is a member of this set.

However, one significant empirical problem with this suggestion is that it renders intuitively valid arguments, such (16), invalid, or at best equivocal.

- (16) a. Every number is divisible by one.
 b. Four is a number.
 c. So, four is divisible by one.

Within GQT, determiners generally denote relation between sets, or predicate extensions. For example, 'every' denotes the subset relation.

- (17) a. $\llbracket \text{every} \rrbracket = \lambda P. \lambda Q. P \subseteq Q$;
 b. $\llbracket \text{some} \rrbracket = \lambda P. \lambda Q. P \cap Q \neq \emptyset$.

This requires 'number' in (7a) to have the usual type of a predicate, namely $\langle e, t \rangle$, denoting a set of entities. Accordingly, the inference from (16a,b) to (16c) is valid only if 'four' in (7a) denotes a member of this set. Yet 'four' does not denote an entity, and 'is a number' does not denote a set of entities, on the present hypothesis. For similar reasons, the inference from (18a) to (18b) is wrongly predicted to be invalid, or at best equivocal.

- (18) a. Four is divisible by two.
 b. Some number is divisible by two.

On the present suggestion, the predicate in (18a) would need to have the same semantic type as 'is a number' in (7a). Yet by (17b), (18b) is true only if 'is divisible by two' denotes a set of entities, namely those which overlap with the set of numbers. Simply put, the types assumed under the present hypothesis are inconsistent with those required by GQT, and thus the original motivation for Hofweber's Adjectivalism.

The second hypothesis, suggested by Ionin and Matushansky [2006], avoids this problem. According to it, the underlying syntax of (7a) is something

like (19a), where ‘ \emptyset ’ represents an unpronounced noun having the denotation in (19b).

- (19) a. $[_S [_{DP} [_{DET} \text{Four}] [_N \emptyset]]] [_{VP} \text{is a number}]]$;
 b. $[[_N \emptyset] = \lambda x. x = x$.

According to (8), ‘four’ takes two type- $\langle e, t \rangle$ expressions as arguments. However, only one such expression is overt in (7a), namely ‘is a number’. Thus, the unpronounced noun in (19a) provides the missing second argument, in the form of an identity predicate, thus allowing composition to proceed. This has two apparent advantages over the first hypothesis. First, ‘is a number’ has a type consistent with GQT. Second, it aligns Hofweber’s analysis of arithmetic equations like ‘Two and two are four’ with simple numerical statements. Specifically, both involve counting. Indeed, since everything is self-identical, (7a) will be true on the present suggestion just in case at least four things are numbers.

According to these truth conditions, the truth of (7a) is seemingly inconsistent with Hofweber’s nominalism, since at least one thing would be a number. However, Hofweber has the linguistic resources to avoid this conclusion. Hofweber [2016, Ch. 4] defends the following thesis:

Underspecification: All natural language determiners are semantically “underspecified” for two “readings”: an internal reading, and an external reading.

The distinction between INTERNAL and EXTERNAL READINGS corresponds to a familiar distinction between SUBSTITUTIONAL and OBJECTUAL QUANTIFICATION. Roughly, on the latter, ‘Something is F ’ is true in a model if the domain contains at least one object satisfying F , while on the former, it is true in a model if there is at least one term ‘ t ’ in the language such that ‘ $F(t)$ ’ is true in that model. Thus, according to Underspecification, determiners may generally be interpreted substitutionally or objectually. Crucially, however, Hofweber [2016, Ch. 5] insists that numerical determiners are generally interpreted substitutionally. The upshot is that (7a) need not entail objects which are numbers. Rather, it will be true if there are at least four terms substitutable for the formula ‘ $x = x$ and x is a number’. And this is true, according to Hofweber: there are infinitely many such terms.

Nevertheless, significant empirical problems remain. Clearly, if Non-Referentialism is to be maintained in full generality, then all apparent numerals must be analyzed as semantically bare determiners. Thus, the present suggestion predicts that the semantic function of all apparent numerals is to count. However, there are excellent linguistic reasons for thinking that this prediction is mistaken, regardless of whether we are counting objects or terms. Indeed, a wealth of contrasts due mostly to Susan Rothstein [2017] reveal that the present hypothesis makes numerous false semantic predictions.

First, Rothstein observes that the verb 'count' is ambiguous between two senses, corresponding to what Benacerraf [1965] calls INTRANSITIVE and TRANSITIVE COUNTING.⁹ These are witnessed respectively in Rothstein's (20a,b):

- (20) a. Mary counted to thirteen (??things).
 b. Mary counted thirteen (things).

Thus, as the labels suggest, transitive 'count' requires a direct object, whereas intransitive 'count' does not. Now consider Rothstein's (21a,b):

- (21) a. ?? Mary counted to thirteen. — Thirteen what?
 b. Mary counted thirteen. — Thirteen what?

It is hard to see how there could be any differences in acceptability here if the present hypothesis were correct. In that case, the semantic function of all apparent numerals would be to count transitively. Specifically, 'thirteen' would transitively count entities in (20a) and (21a), just as it does in (20b) and (21b). Yet (20a) and (21a) quite clearly do not involve transitive counting. Rather, they involve intransitive counting, where 'thirteen' supplies the numeral up to which Mary intransitively counted.

Secondly, numerals and bare determiners differ in their agreement features. Specifically, whereas the former require singular morphology, the latter require plural morphology.¹⁰

- (22) a. Which one of these three numbers is Mary's favorite? Four {is/??are}.
 b. How many people are coming to the party? Four {??is/are}.

And the same holds for numerals in comparative constructions, as Rothstein points out.

- (23) Four {is/??are} bigger than three.

Again, however, neither contrast is expected on the current hypothesis. According to it, the function of 'four' in (22a) and (23) should be to count (transitively), and so should always require plural morphology, contrary to fact.

Finally, and perhaps most significantly, the present analysis wrongly predicts that (24a,b) should be truth-conditionally equivalent.

- (24) a. Two is an even prime.
 b. Two things are even primes.

Again, regardless of whether 'two' in (24b) is interpreted objectually or substitutionally, the truth of (24a) clearly does not imply the truth of (24b). Yet that is precisely what the present analysis predicts.

⁹To a first approximation, intransitive counting consists in reciting the numerals in their canonical order: "1, 2, 3, ...". In contrast, transitive counting consists in the counting of things. That is, when transitively counting we use the numerals to answer how-many questions.

¹⁰See [Snyder, 2017].

To summarize, there are two potential analyses of simple numerical statements that preserve compositionality in a manner consistent with Non-Referentialism. However, both are plagued by substantial empirical problems. This suggests that numerals should not be analyzed uniformly as semantically bare determiners, in which case Hofweber's proposed solution to Frege's Other Puzzle fails to generalize. What is needed, apparently, is an altogether different solution, one which does not analyze all occurrences of 'four' as having the same (non-referential) semantic function. As we will see in §3, this is exactly what our best comprehensive analyses of number words within linguistic semantics recommend.

3. THE POLYMORPHIC STRATEGY

We have argued that the only extant, developed solution to FOP suffers from serious empirical difficulties. The purpose of the present section is to develop an alternative solution, the Polymorphic Strategy. According to this view, the same expression serves different referential and non-referential semantic functions in Frege's (1a,b), *contra* Single Function.

Indeed, one might reasonably reject Single Function on independent linguistic grounds. For example, consider color words like 'green'. While these are typically assumed to be adjectives,¹¹ 'green' functions acceptably as a singular term in (25).

(25) Green is a color.

Conversely, consider the name 'Trump', which functions acceptably as a modifier in (26).

(26) The country club has {Trump/wealthy} supporters.

Thus, (25) and (26) plausibly constitute counterexamples to Single Function.

However, this alone will not suffice to resolve FOP. After all, most adjectives cannot function acceptably as singular terms in a manner parallel to color words. For example, consider 'nice' and 'stupid', which cannot acceptably occupy a similar position.

- (27) a. {??Nice/Being nice/To be nice} is nice.
 b. {??Stupid/Stupidness/Stupidity} is annoying.

So why think that numerals are more like 'green' than 'nice' or 'stupid' in this respect? Conversely, names used as modifiers, such as 'Trump' in (26), typically have a distinctly relational character: intuitively, 'Trump supporter' is roughly synonymous with 'supporter of Trump'. In contrast, 'four' in (1b) is not roughly synonymous with 'moons of four'. So why think that 'four' in (1b) is like 'Trump' in this respect?

In short, even if Single Function is false, questions pertinent to adequately resolving FOP remain. These include:

¹¹ See, e.g., [Partee, 1992] and [McNally and de Swart, 2011].

Categorization: If the different occurrences of 'four' in (1a,b) are witness to the same expression, then it is presumably either a name or an adjective. If so, then which is it?

Opposing Functions: If 'four' functions referentially in (1a) but non-referentially in (1b), then why can the same expression serve different, and indeed opposing, functions?

Polysemy: Since expressions performing different semantic functions typically have different meanings, why are the different meanings of 'four' in (1a,b) related, both conveying something about cardinality?

We provide answers to these questions by highlighting some assumptions which are widespread within linguistic semantics. The first is that number expressions, such as 'four', can serve many semantic functions. Consequently, 'four' can take on correspondingly many meanings, including, but not limited to, those witnessed in (1a,b). Secondly, these meanings, despite being distinct, are *related*. In other words, number words are POLYSEMOUS. Third, the means by which these meanings are related is *systematic*: they are generated from a single lexical meaning via independently motivated and well-attested type-shifting principles. In short, number expressions are POLYMORPHIC, taking on different but related semantic types, and thus meanings, in different syntactic environments, thanks to type-shifting. Finally, number words are not the only polymorphic expressions. In particular, color words and measure phrases are also polymorphic, and pattern in remarkably similar ways to number words. Consequently, analogs of FOP are easily formulated using these other expressions, thus suggesting that a more general solution is not only desirable, but indeed required.

3.1. Generalizing Frege's Other Puzzle

A basic semantic fact about number words is that they have a wide variety of uses. But the same is true of color words and measure phrases. For example, 'four' has a nominal use, similar to 'green' in (28b) and 'four pounds' in (28c).

- (28) a. Four is a number.
 b. Green is a color.
 c. Four pounds is a weight.

All three sorts of expressions also have predicative uses.

- (29) a. Jupiter's moons are four (in number).
 b. Mary's socks are green (in hue).
 c. Mary's hammers are four pounds (in weight).

They can also be used as modifiers.

- (30) a. Those are (Jupiter's) four moons.
 b. Those are (Mary's) green socks.
 c. Those are (Mary's) four-pound hammers.

They also have “attributive” uses, to reuse Frege’s [1884] label.

- (31) a. Jupiter has four moons.
 b. Mary has green socks.
 c. Mary has four-pound hammers.

And they can occur in the post-copular position of so-called SPECIFICATIONAL SENTENCES.¹²

- (32) a. The number of Jupiter’s moons is four.
 b. The color of Mary’s socks is green.
 c. The weight of Mary’s hammers is four pounds.

Since color words and measure phrases pattern just like number words in these respects, it is straightforward to formulate a more general version of FOP in light of (28)–(32): How can number words, color words, and measure phrases function referentially, predicatively, attributively, *etc.* if other, similar expressions cannot function acceptably in these ways? Minimally, this suggests that, all else being equal, whichever solution one offers to FOP, that solution should also generalize to color words and measure phrases.

These examples highlight two important semantic facts. First, names, predicates, and modifiers are typically assumed to have different semantic types, and thus meanings. Thus, at least some of the different occurrences of ‘four’, ‘green’, and ‘four pounds’ in (28)–(32) plausibly have different meanings. Recall the quote from Dummett [1991, p. 99] above, according to whom “any analysis must display the connection between” Frege’s two uses. Now that we have distinguished yet further uses of ‘four’, we might formulate a more general condition of adequacy on analyses of number words, namely:

Dummett’s Condition: An analysis of number words ought to explain how the meanings of their various uses are related.

Substantivalism and Adjectivalism meet Dummett’s Condition in the case of Frege’s (1a,b): trivially, these meanings are related, because they are identical. However, neither view meets Dummett’s Condition with respect to *all* relevant cases. There is hardly any empirical plausibility to the claim that ‘four’ in (29a), for instance, has the same referential meaning witnessed in (28a) or non-referential meaning witnessed in (31a).

Likewise for color words and measure phrases: there is little empirical plausibility to the claim that the predicative uses in (29), for instance, have the

¹²The semantic motivations for distinguishing specificational sentences and equatives like ‘Cicero is Tully’ are well known (see, *e.g.*, [Mikkelsen, 2011]). For arguments that (1a) is in fact a specificational, as opposed to equative, sentence, see [Moltmann, 2013; Felka, 2014; Snyder, 2017].

same meanings as those witnessed in (28) and (31). Thus, the analogs of Substantivalism and Adjectivalism would be hardly plausible with respect to these other expressions, and thus to these potential extensions of FOP.

Secondly, it is highly implausible that the different meanings witnessed in (28)–(32) are completely unrelated in a manner resembling, *e.g.*, ‘bank’. After all, (28)–(32) all intuitively convey something about number, color, and weight, respectively. In other words, (28)–(32) plausibly reveal that number words, color words, and measure phrases are *polysemous*. Within linguistic semantics, this kind of polysemy is standardly explained via polymorphism. In the next section, we sketch how this is accomplished, using color words to illustrate. In §3.3, we then deploy similar resources to handle number words.

3.2. Illustrating Polymorphism: The Case of Color

Within linguistics, color words like ‘green’ are commonly assumed to be polymorphic, taking on different but related semantic types, and thus meanings, in different syntactic environments. In developing this view, semanticists commonly assume that each color word has a single lexical meaning — crudely, a “basic” meaning from which others are derivable. Further, they assume an inventory of independently motivated and well-attested type-shifting principles, which permit the derivation of other meanings from the basic, lexical one. Below, we consider how meanings appropriate for (28b)–(32b) might be derived from a single lexical meaning. In doing so, we adopt a standard assumption about the lexical meaning of color terms, and restrict ourselves to some type-shifting principles that have common currency in linguistic semantics: principles labelled ‘ADJUNCT’, ‘NOM’, and ‘A’.

As already mentioned, ‘green’ has both a predicative and attributive form, respectively witnessed in (29b) and (30b). Unlike, *e.g.*, ‘former’, ‘green’ is INTERSECTIVE, giving rise to the following characteristic entailment.

- (33) a. Those are green socks \models Those are socks and they are green (in hue).
 b. She is a former senator $\not\models$ She is a senator and she is former.

Thus, intersective adjectives are so-called because their attributive form intuitively denotes the intersection of two sets, namely one denoted by the accompanying noun and one denoted by the predicative form of the adjective.

Usually, this is explained by assuming that the attributive meaning is derived from a lexical predicate via type-shifting. Thus, assume that ‘green’ has the lexical meaning in (34).

$$(34) \llbracket \text{green} \rrbracket = \lambda x. \text{green}(x).$$

So, the lexical meaning of ‘green’ is that of a predicative adjective true of all and only green things. (34) provides a meaning appropriate for predicative uses like (29b), of course. Accordingly, (29b) is true just in case Mary’s socks are among the green things.

What of the other meanings that ‘green’ can take on? A meaning appropriate for (30b) is derivable from (34) via Landman’s [2004] type-shifting principle

‘ADJUNCT’, defined in (35a). This takes a predicate and returns an intersective modifier, as demonstrated in (35b).¹³

- (35) a. $\text{ADJUNCT} = \lambda P. \lambda Q. \lambda x. P(x) \wedge Q(x);$
 b. $\text{ADJUNCT}(\lambda x. \text{green}(x)) = \lambda Q. \lambda x. \text{green}(x) \wedge Q(x);$
 c. $\llbracket \text{green socks} \rrbracket = \lambda x. \text{green}(x) \wedge \text{socks}(x).$

Combining the result with a noun like ‘socks’ thus returns a predicate true of things which are both green and socks, *cf.* (35c). This in turn predicts the entailment in (33a): if those things are green socks, then those things must be socks which are green. Likewise, (30b) is just in case the things deictically referenced by ‘those’ are socks which are green.

Similar derivations are available for the other uses of ‘green’ above. Consider first nominal uses like (28b), where ‘green’ functions as a singular term. Such meanings are usually assumed to arise through NOMINALIZATION, and there are different ways of nominalizing adjectives in English. One is to form gerunds and infinitives like those in (27a). A second is to use nominalizing suffixes, as witnessed in (27b). A third, evidently witnessed by ‘green’ in (28b), is to form a name directly from its lexical root.¹⁴

Semantically speaking, to nominalize is to coin a name for a property expressed by a given predicate, and to use that name to refer to that property, as a thing. This is codified in Partee’s [1986] ‘NOM’ principle, defined in (36a), where ‘ \cap ’ is Chierchia’s [1984] nominalization operation.

- (36) a. $\text{NOM} = \lambda P. \cap[\lambda x. P(x)];$
 b. $\text{NOM}(\lambda x. \text{green}(x)) = \cap[\lambda x. \text{green}(x)].$

Properties play two roles in Chierchia’s PROPERTY THEORY, roughly corresponding to Frege’s distinction between concept and object. Specifically, properties may be predicated of an object, as witnessed in (29b), or they may be viewed as entities, as witnessed in (28b). In the latter case, ‘green’ references the INDIVIDUAL PROPERTY CORRELATE of being green, or roughly the property of being green viewed as an entity in its own right, given in (36b).¹⁵ Accordingly, (28b) is true just in case that entity is a color.

Next, consider (31b). Intuitively, this would not be false if Mary also happened to have red socks. Rather, all that is apparently required is that Mary has

¹³ There are a number of important theoretical questions concerning type-shifting we cannot answer here. For example, how exactly does type-shifting get triggered — only in the presence of type-mismatches, when interpretation requires it, given conversational context, or something else? How exactly are we supposed to understand type-shifting principles — as unpronounced morphemes, as general semantic principles available in the grammar, or something else? These are open questions within linguistic semantics (see, *e.g.*, [Partee, 1986; Barker, 1992; Chierchia, 1998] for discussion). For our purposes here, however, it suffices to note that any polymorphic account of (1a,b) will deny Single Function.

¹⁴ See [Alexiadou, 2017].

¹⁵ This threatens paradox if properties can be self-instantiating. See [Chierchia and Turner, 1988].

some green socks. This is captured via another of Partee's principles, namely 'A', defined in (37a).

- (37) a. $A = \lambda P. \lambda Q. \exists x. P(x) \wedge Q(x);$
 b. $A(\lambda x. \text{green}(x) \wedge \text{socks}(x)) = \lambda Q. \exists x. \text{green}(x) \wedge \text{socks}(x) \wedge Q(x).$

Combining this with 'green socks' in (35c) thus results in (37b). Consequently, (31b) is predicted to be true just in case Mary has some green socks, as desired.

Finally, consider specificational uses like (32b). These too are readily captured via type-shifting. Specifically, assume the meaning of the specificational copula in (38), due to Romero [2005], where ' y ' ranges over INDIVIDUAL CONCEPTS, or functions from worlds to entities.¹⁶

- (38) $[\text{be}] = \lambda x. \lambda w. \lambda y. y(w) = x.$

According to (38), post-copular 'green' in (32b) is a singular term, and we have already seen how to generate a meaning appropriate for such a term via nominalization. Hence, according to the resulting truth conditions, (32b) will be true just in case the actual color instantiated by Mary's socks is the same one named by 'green' in (28b).

In summary, meanings suitable for the different uses of 'green' in (28b)–(32b) are derivable from a single lexical meaning via independently motivated type-shifting principles. What does this show? First, it shows that a single expression — 'green' — can in fact serve different semantic functions in different syntactic contexts. Secondly, because the corresponding meanings are derived via type-shifting, they are systematically related, thus capturing the polysemy of color words. Thus, it is not generally puzzling how one and the same expression can serve different, but related, semantic functions. Furthermore, it turns out that these same type-shifting principles are applicable to measure phrases and number words, thus explaining why all three can take on similar semantic functions.¹⁷ Seen this way, FOP is puzzling only if we neglect the polymorphic nature of number words, and indeed a variety of other English expressions.

3.3. Resolving Frege's Other Puzzle

Recall the various uses of 'four' mentioned above, repeated here in (39).¹⁸

- (39) a. Four is a number.
 b. Jupiter's moons are four (in number).
 c. Those are (Jupiter's) four moons.
 d. Jupiter has four moons.
 e. The number of Jupiter's moons is four.

¹⁶ To be clear, this is not the only available analysis of specificational sentences. For alternative analyses, see, *e.g.*, [Moltmann, 2013; Felka, 2014].

¹⁷ See [Rothstein, 2017] and [Snyder, 2021a] for measure phrases.

¹⁸ To be sure, (39a–e) are not the only potential uses of 'four'. For a more comprehensive list and treatment of number word meanings, see [Snyder, 2021b].

There are, in fact, many proposed polymorphic analyses of number words, differing in various formal details. Fortunately, for our purposes they can be grouped into two kinds, without much loss in generality. According to the first, POLYMORPHIC SUBSTANTIVALISM (PS), the lexical meaning of ‘four’ is that of a numeral naming the number four, and all other meanings ‘four’ can take on are derivable from it via type-shifting. Versions of PS have been defended by Landman [2004], Scontras [2014], and Snyder [2017]. In contrast, according to POLYMORPHIC ADJECTIVALISM (PA), the lexical meaning of ‘four’ is that of a non-referential adjective or determiner, and all other meanings ‘four’ can take on are derivable from it via type-shifting. Versions of PA have been defended by Partee [1986], Geurts [2006], Rothstein [2013; 2017], and Kennedy [2015].

It is important to emphasize that for the purposes of resolving FOP, either kind of polymorphic analysis will suffice, as both ultimately recommend the Polymorphic Strategy. That is, independent of whether the lexical meaning of ‘four’ is referential (PS) or non-referential (PA) in character, it will take on the same range of potential meanings, using broadly similar semantic resources to do so. Thus, our purpose here is not to *adjudicate* between PS and PA, but rather to show how both meet Dummett’s Condition.

We use the analyses of Fred Landman [2004] and Susan Rothstein [2013; 2017] to represent PS and PA, respectively. These analyses are very similar in that they invoke the same repertoire of type-shifting principles mentioned in §3.2. The crucial difference between them lies in the lexical meanings assumed. These are given in (40a,b), where ‘4’ represents the number four, and ‘ $\mu_{\#}$ ’ is a measure function mapping pluralities to numbers representing how many countable individuals are part of that plurality.

$$(40) \quad \begin{array}{ll} \text{a. } \llbracket \text{four} \rrbracket = 4 & \text{(PS)} \\ \text{b. } \llbracket \text{four} \rrbracket = \lambda x. \mu_{\#}(x) = 4 & \text{(PA)} \end{array}$$

Thus, for PS, ‘four’ is lexically referential, and meanings appropriate for predicative uses are derived via a type-shifting principle we dub ‘NUM’.¹⁹

$$(41) \quad \begin{array}{ll} \text{a. } \text{NUM} = \lambda n. \lambda x. \mu_{\#}(x) = n \\ \text{b. } \text{NUM}(4) = \lambda x. \mu_{\#}(x) = 4. \end{array}$$

In contrast, according to PA, the lexical meaning of ‘four’ is that of a predicative adjective, and meanings appropriate for numerals are derived via nominalization. Specifically, applying Partee’s ‘NOM’ to the cardinality predicate in (40b) results in a nominalized cardinality property, namely the property of being four in number, viewed as an entity.

$$(42) \quad \text{NOM}(\lambda x. \mu_{\#}(x) = 4) = \cap [\lambda x. \mu_{\#}(x) = 4].$$

However, this alone will not guarantee that the referent of the numeral ‘four’ is a number, consistent with the truth of (39a). Thus, Rothstein additionally

¹⁹Note that NUM is a function from numbers to a function from pluralities to truth values. Thus, applying NUM to the number four, as in (41b), returns a function equivalent to the set of pluralities having exactly four atomic parts.

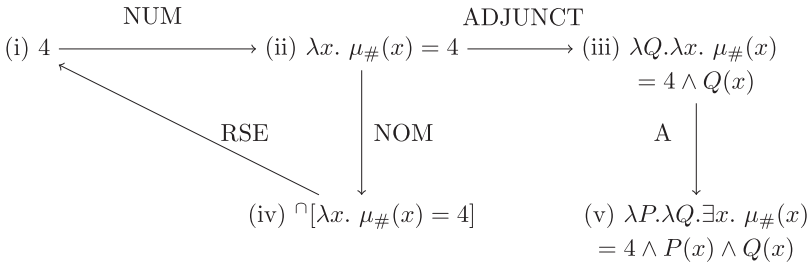


Fig. 1. Map of some potential meanings for ‘four’.

posits the schematic equation in (43), which we call ROTHSTEIN’S SCHEMATIC EQUATION (RSE).

$$(43) \quad n = \cap[\lambda x. \mu_{\#}(x) = n].$$

Here, ‘*n*’ ranges over numbers.²⁰ Thus, according to RSE, numbers are NOMINALIZED CARDINALITY PROPERTIES, *i.e.*, individual correlates of being *n* in number. Accordingly, (39a) will be true just in case the individual correlate of being four in number is among the set of numbers.

Despite this difference in lexical meanings assumed, PS and PA derive meanings appropriate for (39a–e) via broadly the same type-shifting principles. Moreover, *mutatis mutandis*, they do so in precisely the manner suggested for ‘green’ in §3.2. These derivations and resulting meanings are summarized by the map in Figure 1, where arrows indicate the outputs of type-shifting principles or the intended effect of RSE.²¹

Here, node (i) provides a meaning appropriate for (39a); (ii) for (39b); (iii) for (39c); (iv) for (39e); and (v) for (39d). Consequently, PS and PA both predict the following truth conditions:

- (44) a. (39a) is true iff the referent of ‘four’ is a number.
- b. (39b) is true iff the plurality referenced by ‘Jupiter’s moons’ has exactly four countable parts.

²⁰As Snyder [2021b] observes, RSE threatens a regress, assuming identities license substitution of like for like. In fact, as an anonymous reviewer points out, if each occurrence of *n* within RSE permits substitution of its corresponding \cap -term, then we get the following infinite, and possibly vicious, regress:

- (i) a. $n = \cap[\lambda x. \mu_{\#}(x) = n]$;
- b. $n = \cap[\lambda x. \mu_{\#}(x) = \cap[\lambda x. \mu_{\#}(x) = n]]$;
- c. $n = \cap[\lambda x. \mu_{\#}(x) = \cap[\lambda x. \mu_{\#}(x) = \cap[\lambda x. \mu_{\#}(x) = \dots]]]$.

Many thanks to the reviewer for this interesting and important observation.

²¹This slightly misrepresents the effect of Partee’s [1986] ‘A’, which takes a predicate, rather than a modifier, as argument. Thus, ‘four’ must first combine with a noun such as ‘moons’ before ‘A’ applies.

- c. (39c) is true iff the plurality deictically referenced by ‘those’ consists of exactly four countable parts, each of which is a moon (belonging to Jupiter).
- d. (39d) is true iff there is at least one plurality having exactly four countable parts, each of which is a moon belonging to Jupiter.
- e. (39e) is true iff the nominalized (quantitative) property instantiated by Jupiter’s moons is the property of being four in number.

The analyses just sketched highlight three important observations. First, as with color words, ‘four’ takes on different semantic types, and thus functions, in different syntactic environments. These include, but are not limited to, Frege’s examples. Secondly, while those types determine different meanings, there is an element shared by all of (i)–(v), namely a number. In a sense, it is the formal witness to the polysemy of ‘four’, tying its meanings together. Finally, PA and PS naturally recommend the Polymorphic Strategy as the correct solution to FOP. Indeed, on both kinds of analyses, ‘four’ in Frege’s original examples is witness to a single expression, one which takes on different, though systematically related, semantic functions.²²

How, then, do these analyses answer our three questions from above? First, consider Categorization: is ‘four’ a name or an adjective? This requires clarification. Within the context of polymorphic analyses, a distinction should be drawn between the lexical meaning of an expression and the meaning of one of its occurrences in a given syntactic environment. As we have seen, while PS and PA differ with respect to the lexical meaning of ‘four’, they agree about the meanings of its occurrences in (39). Categorization is thus rightly understood as a question about lexical meaning, and answering it ultimately requires taking a stand on which of these two kinds of polymorphic analyses is correct. However, this remains an open, substantive empirical issue within linguistics. Fortunately, since both kinds of analyses deny Single Function, adjudicating between them is not required to solve FOP.

Next, consider Opposing Functions: why can ‘four’ serve different, opposing semantic functions in (1a,b)? The answer relies on type-shifting, of course. On both PS and PA, ‘four’ comes to serve both referential and non-referential semantic functions, thanks to independently available, and general, type-shifting operations. Specifically, according to PS, ‘four’ is referential by default, and comes to function non-referentially in certain syntactic environments, thanks to type-shifting.²³ In contrast, according to PA, ‘four’ is non-referential by default, and comes to function referentially in certain syntactic environments, again thanks to type-shifting. In this regard, number words are no more puzzling than color words, measure phrases, or names.

²²This raises important theoretical questions regarding lexical individuation, of course. However, addressing them adequately is far beyond the scope of the present paper.

²³Admittedly, however, ‘NUM’ is not uncontroversial among semanticists, unlike the other type-shifting principles employed.

Finally, consider Polysemy: why are the occurrences of 'four' in (1a,b) related, both conveying something about cardinality? Again, the answer concerns type-shifting. The characteristic feature of polymorphic expressions is that they can take on different meanings which are either witness to, or else derivable from, a single lexical meaning, via type-shifting, thereby explaining their polysemy. In the case of (1a,b), the meanings of 'four' are related in virtue of both having the cardinality function ' $\mu_{\#}$ ' as a component, thus explaining why both intuitively convey something about how many moons belong to Jupiter.

We conclude that widespread assumptions within linguistic semantics naturally recommend the Polymorphic Strategy as the correct resolution to FOP and its potential extensions to other, similar polymorphic expressions. Ultimately, then, the lesson of FOP is that contrary to the predominant assumption within the philosophy of mathematics, there is no good empirical reason for thinking that number words serve the same semantic function in Frege's original examples, or indeed many others like them.

4. OBJECTIONS TO THE POLYMORPHIC STRATEGY

We have argued that Hofweber's Adjectivalism is empirically inadequate, and thus fails as a solution to FOP. Furthermore, the Polymorphic Strategy is superior in two respects. First, it accommodates the available semantic evidence regarding number words. Secondly, it provides a uniform solution to FOP and its potential extensions. We close by considering what we take to be the three most substantive potential objections.

4.1. The Objection from Faulty Predictions

Though it is largely uncontroversial within linguistics that number words are polymorphic, the same apparently cannot be said of philosophy. Indeed, according to Hofweber [2005, p. 205]:

I think that semantic type coercion [*i.e.*, type-shifting] is the second-best attempt to solve Frege's Other Puzzle, but once we look at the details we can see that it can't be right.

That is because, as Hofweber [2016, p. 139–140] later clarifies, it (allegedly) makes false predictions.

Note that the type of a determiner is higher than that of a noun phrase, and that by the standard interpretations strategy widely assumed in the literature on type-shifting principles and cited above, the lowest type should produce the standard reading when both types are available. But this just does not seem to be the case. When both types are available the determiner reading is more prominent, and the noun phrase reading is harder to get.

Specifically, Hofweber argues that if number words were polymorphic, then (45a) would not only be ambiguous between (45b,c), (45b) should be the *preferred* interpretation.

- (45) a. I want two or three beers.
 b. I want the number two or three beers.
 c. I want two beers or three beers.

The presumption is that given a choice between two potential interpretations corresponding to different available types of a polymorphic expression, the preferred interpretation should be the one corresponding to its simplest available type. So, if ‘two’ can be interpreted both referentially, referring to a number, and quantificationally, counting things like beers, then (45b) ought to be preferred over (45c), contrary to fact. Call this THE OBJECTION FROM FAULTY PREDICTIONS.

The problem with this objection is that polymorphic analyses make no such prediction. In fact, the objection appears to misconstrue a certain semantic principle governing polymorphic expressions, succinctly summarized by Partee [1986, p. 359] as: “Each basic expression is lexically assigned the simplest type adequate to capture its meaning”. Thus, we might call this THE LOWEST AVAILABLE TYPE PRINCIPLE (LATP):

LATP: For any polymorphic expression α and range of semantic types suitable for α , assign to α that lexical meaning corresponding to its simplest available type.

For example, LATP plausibly requires assigning to names, such as ‘Fido’, a lexical meaning corresponding to their simplest available type, namely type-*e*. For the same reason, LATP may also require assigning to ‘two’ this same type.

It does not follow from this, however, that (45b) should be preferable to (45c). For one thing, polymorphic analyses need not adopt LATP. Rather, this is a strictly additional assumption, one initially defended by Partee and Rooth [1983] on broadly cognitive grounds. However, polymorphic analyses may consistently maintain that ‘two’ takes on various semantic types, and yet its lexical type is different from the simplest type it can, in principle, take on. Indeed, we have already seen one such analysis, namely PA from §3.3.

Secondly, even if an advocate of polymorphism does accept LATP, it need not follow that interpretations resulting from the simplest available type should always be preferred. Clearly, if LATP is to confer any cognitive benefit, then the simplest available type of a polymorphic expression must be processed before other available types. For example, Partee and Rooth argue that given a correspondence between complexity in semantic type and facility in semantic processing, LATP would ease semantic processing if simpler types are processed prior to more complex types. However, priority in processing does not imply priority in interpretation preference. Rather, these are simply different kinds of priority: roughly, temporal precedence vs ranking a speaker’s likely intentions

given the conversational circumstances of an utterance. Thus, LATP does not imply that (45b) ought to be preferable to (45c), any more than, say, the fact that painting A was produced before painting B implies that A should more accurately represent a certain scene than B. In both cases, we simply have distinct kinds of priority.²⁴

4.2. The Objection from Uniformity

The second objection to the Polymorphic Strategy comes from a parallel debate concerning proper names. Specifically, as mentioned, these can plausibly function as both singular terms and as predicates, as witnessed respectively in (46a,b).

- (46) a. Mary danced.
b. Every Mary danced.

Two corresponding views have thus come to prominence. According to REFERENTIALISM, defended by, *e.g.*, Robin Jeshion [2015], names are exclusively singular terms, and the meaning of 'Mary' in (46a) is Mary herself. In contrast, according to PREDICATIVISM, defended by, *e.g.*, Delia Graff-Fara [2015], names are exclusively predicates, and the meaning of 'Mary' in (46a,b) is the property of being called by that name.

According to Jeshion, the primary argument for Predicativism over Referentialism appeals to the following principle.

Uniformity: A semantic theory which offers a unified semantic analysis for all occurrences of a given class of expressions is better than one that does not.

A "unified semantic analysis" is one which assigns a uniform *semantic type*, and thus meaning, to the expressions in question. The motivation for Uniformity is presumably GRICE'S RAZOR: Do not postulate ambiguities unnecessarily. Thus, the claim is that since Predicativism accounts for both (46a,b) with only a single meaning,²⁵ but Referentialism does not, the former is preferable to the latter. Clearly, an analogous argument could be made for Hofweber's Adjectivalism: since number words uniformly denote relations between sets, Hofweber's analysis is preferable to polymorphic analyses, thus undermining the Polymorphic Strategy. Call this THE ARGUMENT FROM UNIFORMITY.

The problem with this argument is the same for both number words and names: as stated, Uniformity begs the question against polymorphic analyses. In fact, Uniformity precludes the possibility of polymorphism, by fiat. While it is certainly true that, all else being equal, one should not posit more meanings than required, the matter of which meanings a given class of expressions have

²⁴We would like to thank an anonymous reviewer for valuable comments on this issue.

²⁵Thanks to an unpronounced determiner in the syntax of (46a). See [Fara, 2015].

is an empirical one, and according to extant polymorphic analyses, number words, as a matter of empirical fact, have multiple meanings.

This problem could be avoided by modifying Uniformity: *All else being equal*, we ought to prefer uniform analyses over non-uniform analyses. In other words, given a choice between two analyses which are otherwise equally empirically viable, we ought to prefer one which posits just a single meaning. This, we believe, is plausible. However, it should be evident why it cannot be used as an argument in favor of Hofweber's Adjectivalism over the Polymorphic Strategy: in this case, all things are *not* equal. First, we have seen that Hofweber's analysis suffers from serious empirical difficulties. Second, since the empirical evidence clearly supports number words taking on different meanings in different syntactic environments, there is every reason to prefer a polymorphic analysis over a uniform analysis in this case.

4.3. The Objection from Divergent Commitments

A final potential objection concerns another puzzle suggested by Hofweber [2007]. It comes in the form of an inconsistent triad, the first claim of which is:

Same Circumstances: (1a,b) are true in exactly the same circumstances.

The motivation for Same Circumstances should be clear: (1a,b) are arguably both true just in case Jupiter has exactly four moons.²⁶ The second thesis is:

Same Commitment: Statements which are true in exactly the same circumstances imply the same ontological commitments.

The motivation for Same Commitment should also be intuitively clear. If two statements are true in exactly the same circumstances, then whatever is required to make one true must also be what is required to make the second true. Specifically, it cannot be that whereas the truth of one requires a certain domain of entities, the truth of the other does not. The final thesis is:

Ontological Difference: (1a,b) entail different ontological commitments.

²⁶Of course, one might plausibly reject Same Circumstances based on contrasts like (ia,b).

- (i) a. Jupiter has four moons. In fact, Jupiter has at least seventy moons.
- b. ?? The number of Jupiter's moons is four. In fact, Jupiter has at least seventy moons.

Whereas (ia) is seemingly consistent, (ib) appears to be an explicit contradiction, thus suggesting (1a,b) are not equivalent [Horn, 1972]. However, this objection is easily avoided by adding 'exactly' to (1b).

Again, the intuitive rationale for Ontological Difference should be evident. As an apparent identity statement, (1a) seemingly requires reference to a number. In contrast, (1b) appears to involve counting moons, something which can be represented within predicate logic without numbers. Thus, (1a,b) appear to entail different ontological commitments.

Obviously, no puzzle arises if Hofweber's Adjectivalism is correct, in which case neither (1a) nor (1b) would involve reference to numbers. In contrast, the Polymorphic Strategy would appear to endorse Ontological Difference, as 'four' functions referentially in (1a) but non-referentially in (1b). Hence, it would appear that the Polymorphic Strategy requires rejecting Same Circumstances or Same Commitment, contrary to intuition or perhaps even empirical fact. Call this THE OBJECTION FROM DIVERGENT COMMITMENTS.

Clearly, this objection secures a theoretical advantage for Hofweber's Adjectivalism only if it supplies a more attractive solution to THE COMMITMENT PUZZLE than the Polymorphic Strategy. However, this is not the case. In fact, both strategies deny the same thesis, though for importantly different reasons.

As mentioned, polymorphic analyses explain the polysemy of number words by employing roughly the same stock of type-shifting principles. As a result, all potential meanings of 'four' share an element in common, namely a number. Apparently, then, (1b) could not be true if numbers did not exist, as evidenced by the predicted truth conditions for (1b).

$$(47) \quad \exists x. \mu_{\#}(x) = 4 \wedge \text{moons}(x) \wedge \text{has}(j, x).$$

Relatedly, note that (1b) seemingly entails (48a), which is plausibly analyzable on both PS and PA as (48b), where 'n' ranges over numbers.

- (48) a. Jupiter has a certain number of moons.
 b. $\exists n. \exists x. \mu_{\#}(x) = n \wedge \text{moons}(x) \wedge \text{has}(j, x).$

Minimally, this shows that just because (1b) can be regimented within predicate logic without commitment to numbers, it does not follow that the proposed regimentation has any plausibility as an empirical semantic hypothesis. In other words, the original motivation for Ontological Difference is especially weak. More substantially, (48a,b) suggest that the truth of (1b) may well imply the existence of a number.²⁷ If so, then Ontological Difference is simply false, despite 'four' functioning differently in (1a,b).

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²⁷ Cf. [Snyder, 2017].

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