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Scotus Geometres
The longevity of Duns Scotus's geometric arguments against indivisibilism

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Two types of theses are in competition regarding the nature of extension. Matter, as well as pure space, are either indefinitely divisible, as Aristotle maintains, or they are made up of elementary parts that cannot be divided, whatever the nature of these parts is (as we shall see, several variants are possible). Some of the best arguments raised against indivisibilism are mathematical. Indivisibilism, as a matter of fact, is absolutely incompatible with Euclidian geometry. Such mathematical objections were proposed by the pseudo-Aristotelian treatise *On indivisible lines* (969b29-970a19), then by Avicenna.[1] John Duns Scotus adopted some of them but he also framed new ones. I intend, first, to explain and underscore the novelty of two arguments he has set forth, then to call attention to the persistence of these arguments in 17th century debates.

I Scotus's demonstrations

In the course of a discussion on the angels' mode of presence in the world, where he maintains that angels are able to move in space with continuity, Scotus has to face the objection that space is composed of indivisibles – which would entail that motion is discontinuous.[2]

Whom does Scotus mean to refute? Who defended such indivisibilism before him? It is hard to tell. Scotus had only indirect knowledge of Motazillite indivisibilism through Al-Ghazzâlî, and possibly also through Maimonides, although no explicit reference is made to the latter.[3] However, a Latin theory of indivisibles emerged in the 12th century, found, for instance, in Abelard and William of Conches. One should also mention Robert Grosseteste, who based his cosmogony on the model of light propagating from point to point, with the result that space is made up of indivisible points. One may imagine that Grosseteste's theories were preserved among some Franciscans, but so far

1 Vd. Avicenna, *Le livre de science*, 141-146. Cf. Dhanani, *Physical Theory*, 176.

2 Ioannes Duns Scotus, *Ord.* II, d. 2, pars 2a, q. 5, nn. 286-289, 279-280.

3 Shortage of room prevents me to give here detailed references. For a survey of the question of indivisibles in the Middle Ages and bibliographies, vd. Murdoch, „Infinity and continuity", and more recently Grellard and Robert, eds., *Atomism*. For the wider picture, vd. Pyle, *Atomism*, and Papst *Atomtheorien*.

we know nothing about them (they do not include [pseudo-]Rufus, Bacon, or Olivi).

On the other hand, Scotus might just deal with various objections against infinite divisibility, without these objections necessarily constituting a consistent theory. Some discrepancies among the assumptions of the opponent hint at that possibility. If such is the case, we would not have to worry too much about identifying a definite position in effect defended by some author.

Let us try, however, to outline the patterns of the arguments for indivisibilism that Scotus takes to task. First, although some of Scotus's responses are geometrical, the position he repels is not purely mathematical but lays claim to say something about the real composition of things – recall that the initial question is about angels in space. The first set *("via")* of arguments proposed by the opponent may apply to geometrical space, but more obviously bears on physical space. In particular, the second argument analyzes the phenomenon of succession. The claim is that a "*successivum*", an element in a succession, cannot be actual without being indivisible.[4] Furthermore, the second *via* of the opponent appeals to the Aristotelian notion of "*minimum naturale*".[5] This notion applies to physical realities such as the minimal size under which flesh cannot remain flesh or fire cannot remain fire, or to the first stage of a motion.[6] Scotus accepts to follow his opponent on that field and answers at great length.[7] The least one can say is that there is no clear-cut distinction between the mathematical and the physical levels.

Second, although there is talk of *minima naturalia*, the opponent explicitly considers them, in the secunda *via*, as devoid of magnitude: each quantity has by definition parts which are smaller than it is; now, by definition also, a minimum has no parts since these parts would be smaller than it is; therefore, a minimum is in no respect a quantity ("*omnino non-quantum*").[8] The *minima* that are hypothesized are then point-like, sizeless.[9] One may think that *a fortiori* it is also the case for the indivisibles defined by the *prima via*. I will call "pointillism" that particular sort of indivisibilism, in order to distinguish it from atomism. Thanks to Aristotle, mediaeval thinkers were well aware that Democritean atoms were supposed to have magnitude and theoretical parts,

4 Ioannes Duns Scotus, *Ord.* II d. 2 p. 2a q. 5 n. 289, 280. Granted that this *successivum* be wholly given in actuality and be divisible, all its parts would be given in actuality and therefore would co-exist, which means that within the succession there would be a type of reality called *permanens*. As a consequence, a *successivum* must be indivisible and fleeting ("*raptim transeunte*"). Paradigmatically, such a *successivum* is an instant in time.

5 *Ibid.* nn. 291-294, 280-281.

6 *Ibid.* nn. 295-300, 282-284.

7 *Ibid.* nn. 332-353, 298-311; nn. 376-411, 321-338.

8 *Ibid.* n. 291, 280-281.

9 That such physical minima be dimensionless derives from the fact that they are not indivisible according to matter, but according to the form, as the objector insists (*Ibid.* nn. 334-338, 300-302). Cf. Cross, *Physics*, 127-138.

which is to say, to be indivisible only physically, but it is not that kind of indivisibilism which is here supported.

Thirdly, the opponent maintains that the indivisibles succeed immediately to each other.[10] The notion of succession could be understood in the technical sense that Aristotle gives to it in *Physics* V.3 (226b34-227a1): two indivisibles are successive when there is no other indivisible between them. This would not preclude that there is between them something of a different nature, such as a gap for instance. However, as we shall see later, Scotus's geometrical arguments would not work if gaps were interspersed between the points. Scotus, then, cannot have in view here that kind of position. Rather, one must emphasize the adverb *immediate* in the description of the targeted thesis: the points succeed each other without any kind of intermediary between them. Does that mean that they are what Aristotle calls contiguous, namely, that their extremities touch each other, or are "together", that is, in the same place (227a10-17)? No, for the points here considered by Scotus cannot be in contact. In the parallel text of the *Lectura*, the opponent says that the indivisibles have no extremities ("*indivisibilia non habent ultima*"), that is, one cannot distinguish in them a middle and extremities.[11] They are therefore indivisible in the strictest sense: not only physically (as Democritean atoms) but also conceptually (as Epicurean "minimal parts" of atoms). Consequently, the points cannot be said to touch each other by their edge or any other kind of part. Theories in which the indivisibles are immediately adjacent to each other without being contiguous in the Aristotelian sense (so as to avoid Aristotle's objections) have been proposed by Henry of Harclay and Walter Chatton. For the latter, although they are sizeless the points occupy different *situs* and can be added to each other. However, both Harclay and Chatton wrote after Scotus. Again, we do not know whom Scotus may have in view.

Finally, such points could be (and have been historically thought of) as being, within a finite magnitude, either in a finite or in an infinite number. But as we shall see later,[12] Scotus's arguments work only against a finite number of indivisibles, and such must be, therefore, the assumption of his opponent(s) here.

I shall concentrate on the geometrical refutations Scotus opposes to the *prima via* of the adversary, since they are the ones which will be still discussed in the early modern era. One might have recourse to Aristotle's objections in *Physics* VI, such as: a point added to a point does not result in anything larger than a point, but, Scotus declares, geometry is here more efficient than physical arguments.[13]

[10] "Si aliquod indivisibile succedit sibi in continuo, *habetur propositum, quod indivisibile sit immediatum indivisibili* (my italics)" (*ibid.* n. 289, 280). In his reply Scotus again specifies this constraint when constructing the hypothesis he is going to prove false by contradiction: "*duo puncta sibi immediate signetur*" (*ibid.* n. 320, 292).

[11] Ioannes Duns Scotus, *Lect.* II d. 2 p. 2 a q. 5 n. 262, 179.

[12] Vd. infra pp. 145 and 147.

[13] Ioannes Duns Scotus, *Ord.* II d. 2 p. 2a q. 5 n. 320, 292. Scotus's geometrical arguments

These geometrical arguments share a common basis on projections and bi-unique mapping. They also all proceed by *reductio ad impossibilem.* Let us suppose, then, that every magnitude *is* composed of a finite number of extensionless points not separated by gaps.

Scotus's first objection is quite original. At least, I have not encountered any earlier instance of it.[14] Suppose two circles with the same center *A*.[15] Let us pick two adjacent points (*B* and *C*) on the outer circumference (for the sake of visibility, they are represented, in the figures below, distant from each other, but there should not be any gap between them). Let us now draw two radii from each point to the center. Do these lines intersect the inner circumference at one and the same point, or at two distinct points?

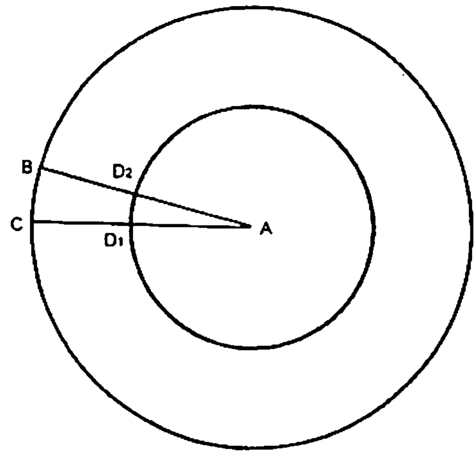
If at two distinct points, then every radius drawn from a point on the outer circle will define a corresponding point on the inner circle. In other words, we will have a one-to-one correspondence between the points of the outer circumference and those of the inner circumference. We will thus be forced to conclude that the two circumferences have the same number of points, that is to say, the same magnitude.[16]

have already been presented in various studies, but I have not found a detailed and satisfying account of the counter-objection in the second part of the first argument (that is, the case where the two radii intersect in one point of the small circle), nor of the second part of the second argument (which is Scotus's original input regarding this argument). That is why I am analyzing them again.

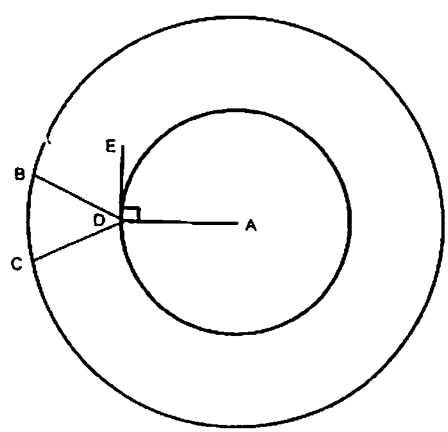
14 Avicenna has an anti-indivisibilist argument relying on the properties of the circle (quoted by Dhanani, *Physical Theory*, 172). But, first, he appeals to only one circle. Second, Avicenna's objection is not applicable to sizeless points. It addresses later Motazillite atomism, which supposes that the circumference has a certain thickness and that one can distinguish an outer and an inner side of the same circumference. Hence, Avicenna argues: if that was right, the outer perimeter would be longer than the inner one, whereas, since these perimeters would be completely in contact with each other, they should be equal because what is completely in contact must be equal to that with which it is in contact. Regarding Scotus's possible sources, one might think also to the mill stone objection used by al-Nazzam, Avicenna, Al-Ghazzâlî and Maimonides (vd. Dhanani, *Physical Theory*, 177). But that objection is more physical than mathematical since it relies on differences of speed.

15 Ioannes Duns Scotus, *Ord.* II d. 2 p. 2a q. 5 n. 320, 292.

16 *Ibid.* n. 321, 292-293.



If at one and the same point, let us call that point D.[17] Let us now draw DE, the tangent to the inner circle at D.



Euclid has demonstrated (III.17) that the tangent is perpendicular to the radius at D. Consequently (in virtue of Euclid I.13), ADE and EDC are two right angles. By the same token, ADE and EDB are two right angles as well, at least

[17] *Ibid.* Scotus points out that this second hypothesis must be addressed only because his opponent is a relentless one, a *protervus* (*ibid.* n. 326, 295). In Euclidian geometry, two non-parallel straight lines cannot have in common more than one point. But the opponent does not feel constrained by Euclidian geometry. His view would be better represented if, in the figure, the inner circle was extremely small, very close to the center, so that the segments AD, DB and DC would appear more as a 'V' than a 'Y'. In other words, the radii would merge insensibly, in less shocking a way. As we shall see later, such is Arriaga's and Hume's belief.

according to the hypothesis (which Scotus is fighting) that *AB* and *AC* are two distinct straight lines which nevertheless both contain *D* and are therefore both perpendicular to *DE*. (Again for the sake of visibility, *EDC* and *EDB* cannot be represented as two right angles, but that is what they are according to the pointillist hypothesis).

Then, if we subtract the angle that is common to both sets of angles, namely, *ADE*, we find that *EDB* = *EDC*. However (and this is the thrust of the argument), this would imply that the part is equal to the whole, since *EDB* must be included in *EDC* if *C* is "after" *B* on the circumference (turning counterclockwise).

Yet, Scotus realizes that his opponent could claim that *DB* and *DC* do not form an angle.[18] As a matter of fact, should they form an angle, the triangle *CDB* would have a base (opposing *D*), which means that there would be a line segment between *B* and *C*. But this cannot be, since, by hypothesis, *B* and *C* are immediately adjacent. Thus there is no angle *CDB* to be added to *EDB*. Therefore, angle *EDC* does not exceed *EDB*. It is not the case, then, that the part is equal to the whole.

Scotus retorts, first, that this reply entails new absurdities from the view point of Euclidian geometry.[19] Indeed it amounts to affirming that two distinct straight lines that intersect do not form an angle, contrary to Euclid's definition of an angle. But pointillists are again willing to endorse such absurdities, which are actually conform with their principles. Their geometry is a non-Euclidian one. Accordingly, in their theory two straight lines may be drawn from two adjacent points to a same point and yet not form an angle. As a consequence, they must be refuted in some other way. It is remarkable that Scotus takes the trouble to follow his opponents on their ground and still manages to find a geometrical refutation.

Although they deny that there is an angular difference between *EDC* and *EDB*, the pointillists nonetheless have to admit that there is a "difference of a point", Scotus says, between the angles *EDB* and *EDC*. Now, according to the pointillists, a point is an actual part of the extended magnitude (its most basic part). Thus the angle *EDC* is made up of the angle *EDB* plus a part. There consequently is a part-to-whole ratio between *EDB* and *EDC*.[20]

What does Scotus mean by "difference of a point"? He himself explains that two hypotheses must be considered.[21] An angle is defined as the space between two intersecting lines, but these lines can be considered either as extrinsic limits, or as intrinsic limits of the angle.

18 *Ibid.* n. 322, 294.

19 *Ibid.* n. 323, 294.

20 My two figures are inspired by Hugues, "Franciscans and mathematics", 118, and Podkonski, "Al-Ghazali's metaphysics", 623, but their figures are wrong regarding the position of *B* and *C*, since in Scotus's text *EDB* is smaller than *EDC*. The figure in Grant, *A source book in medieval science*, 317, is correct but must be read "clockwise".

21 *Ibid.* n. 324, 295.

In the first case, let us take on line *DB* the "first point that lies outside the inner circumference," which is to say, the point immediately after *D* (made possible only by the pointillist hypothesis). This point does not belong to angle *EDB* since the intersecting lines *DE* and *DB* are not part of this angle. It belongs, however, to angle *EDC* since it is included in the space between *ED* and *DC*. There is thus something more in *EDC* than in *EDB*.

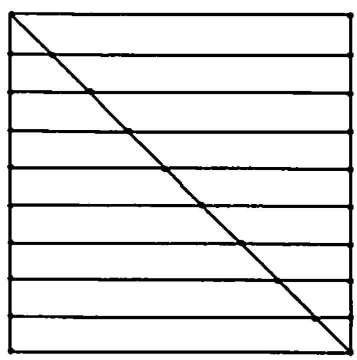
In the second case, let us take on line *DC* the first point after *D*. This point belongs to angle *EDC*, since *DC* is an intrinsic limit of this angle, but does not belong to *EDB*, since it lies outside of *DB*. Again, there is something more in *EDC* than in *EDB*.

Therefore, in no case can the two angles *EDB* and *EDC* be equal, which contradicts the conclusion that we must draw based on constructing the tangent *DE*. Thus the two radii *AC* and *AB* cannot intersect the inner circle at point *D*.

Nor can they intersect the inner circle at two distinct points, as we saw earlier. Conclusion of the whole first demonstration: the hypothesis that circumferences are made up of adjacent points is ruled out.

The second demonstration, although directly inspired from Roger Bacon, is improved and made more complex by Scotus.

Bacon himself probably borrowed his argument from Avicenna, through the presentation given by Al-Ghazzali.[22] But Bacon fine tunes the elementary argument of sixteen atoms assembled into a square by proceeding as follows.[23] Let us suppose a square with sides composed of ten adjacent points each (depicted with gaps between them only for the sake of illustration). From each point on one of the sides, draw perpendicular lines to each corresponding point on the opposite side. By definition, there cannot be more than ten lines.



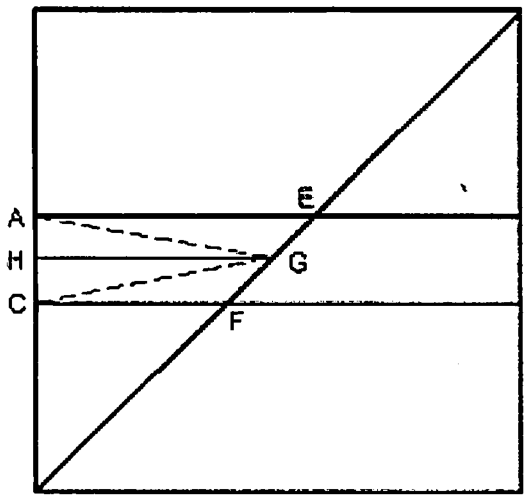
Yet these lines intersect the diagonals. Now, since the horizontal lines are immediately adjacent to each other and fill up, so to speak, the whole surface of the square, we are forced to conclude that each diagonal is exactly made up of

[22] Al-Ghazzâlî, *Metaphysics*, 12. Vd. Thijssen, "Roger Bacon", especially 29-30.
[23] Bacon, *Opus majus*, t. I, 151-152.

the points of intersection with the horizontal lines, numbering no more than ten. It follows that the diagonals and the sides have the same length – which is patently false.

Scotus adopts the same principle, but improves the demonstration.[24] He introduces a dilemma. Either the horizontal lines intersect the diagonal at each of its points (this is Bacon's construction, except not limited to ten points), and there will be the same number of points on the diagonal as on the side; or there are points on the diagonal that do not belong to any of the horizontal lines. The opponent might well be tempted to use the second hypothesis as an escape (so that the side is not as long as the diagonal), and Bacon had not addressed this possibility.

In that case, Scotus argues, let us take on the diagonal a point *(G)* which by hypothesis is situated between, and adjacent to, two points (*E* and *F*) through which two of the original horizontal lines pass. Let us draw from *G* a new line, parallel to those two lines.



Either this new line will intersect the opposite side at a point *(H)* that is situated between the two points *A* and *C*. But this cannot be, since *A* and *C* were earlier supposed to be immediately adjacent. Or the new line drawn from *G* intersects the opposite side at *A* or at *C*. But then it is not parallel to *AE* or *CF*, contrary to hypothesis. Therefore, there cannot be more points on the diagonal than on the side.

To close on Scotus, we may conclude that the *Doctor Subtilis* had a certain competence in geometry. His first argument seems to be new and his second argument is more sophisticated than Bacon's. As John Murdoch has emphasized, Scotus's treatment of the question of indivisibles will exert a widespread

[24] Ioannes Duns Scotus, *Ord.* II d. 2 p. 2a q. 5 n. 330, 297.

influence in the Middle Ages. The *rationes Scoti* will become standard arguments and will consecrate the importance of the mathematical approach.

However, Scotus's geometrical demonstrations do not refute two variants of pointillism. First, if the points were separated by more or less extended gaps, then, in the case of the square, we would be able to embrace the second limb of Scotus's dilemma: we would be able to find a point on the diagonal that has no corresponding point on the side. Put simply, we would be able to draw from *G* a line that passes through an empty gap between *A* and *C*. We could thus have many more points on the diagonal than on the side. The same reasoning would hold for the case of the two concentric circles.

Scotus's arguments do not allow, either, to refute the hypothesis of an infinite number of points. Admittedly, that solution, in turn, raises new questions. Typically, in the Middle Ages one would object: is the actual infinity of points on the outer circle larger than the infinity of points on the inner circle? The idea of infinites of different sizes remained a stumbling block for many medieval thinkers. Nevertheless, Robert Grosseteste endorsed, in his indivisibilist theory, the existence of different infinites.[25] Moreover, Scotus knew that position, since he notes that it can be used by those who want to argue for the eternity of the world.[26]

Now, these two last variants of pointillism will be formulated in the 17th century, when the debate over indivisibles will once again flare up.

II The rationes Scoti in the early modern era

As a matter of fact, pointillism and atomism enjoyed renewed favor in the 16th-17th centuries. The 1630's were particularly pivotal, marked by the publication of several important books on that topic.

In his *Labyrinthus sive de compositione continui* (1631), Froidmont thunders against those whom he calls Epicureans and denounces the "danger that comes from Spain", by which he means new Jesuit theories that resurrect Wycliff's and other heretics' error. The Louvain professor reminds the reader that the Council of Constance anathematized Wycliff's indivisibilist doctrine.[27] But, as he then stresses, Euclid is the atomists' fiercest enemy, and that is why he gives a prominent place to Scotus's (whom he cites by name) argument of the two circles.[28]

Froidmont, however, had not seen the worst of it, so to speak, since the following year Arriaga defended pointillism. Galileo, in turn, for very different reasons, launched a scientific and infinist indivisibilism, soon to be followed by Gassendi's rehabilitation of Epicurus.

25 Vd. Murdoch, "Beyond Aristotle", 21-24.
26 Ioannes Duns Scotus, *Ord.* II d. 1 q. 3 n. 171, 87; referred to by Cross, *Physics*, 125.
27 Froidmont, *Labyrinthus*, cap. IV.
28 *Ibid.*, cap. VII, 29-31.

The hierarchy of the Jesuit order actually shared Froidmont's concern. The General Chapters repeatedly prohibited teaching that *continua* are composed of indivisibles (in 1606, 1613, 1615, 1641, 1650-1).[29] The reason for these censorships was most likely (as it was for Froidmont's) the issue of the transubstantiation. As a matter of fact, one of the propositions banned in 1608 affirms that "Christ multiple times exists in the host, namely, as many times as there are indivisibles in the extended sacramental species, the extension of which is made up of indivisibles."

Nonetheless, not all Jesuits followed their superiors' admonitions, as the multiple reiteration of the ban shows. The "danger that comes from Spain" loomed again with Rodrigo Arriaga, whose *Cursus Philosophicus* (1632) granted the pointillist thesis (which he calls "Zenonist") a success that the Society was powerless to contain.[30] Admittedly, Arriaga acknowledges that there are "very grave objections" against the pointillist position.[31] The geometric arguments, however, do not strike him as decisive. Arriaga faithfully reports the argument of the two concentric circles, but gives the following reply. First, the natural light of reason makes us know evidently that we cannot draw as many lines from the head of a pin as from the whole circumference of the sky. We must, then, simply deny that the same number of lines may pass through the outer and inner circles. Consequently, whatever the nature of the continuum may be, some radii drawn from the circumference of the outer circle must eventually fuse into one another before reaching the center, and there simply are fewer radii issued from the circumference of the inner circle. Another Jesuit, Oviedo, adopts the same position: he resigns himself to admit the merging of radii. He acknowledges that this flies in the face of Euclid. However, Oviedo explains:

> "I have consulted expert mathematicians who have admitted that Euclid's assertion is not proved. It is not so self-evident as to be beyond all doubt. Maybe Euclid listed it as an axiom precisely because he could not establish it demonstratively."[32]

Similarly, Arriaga and Oviedo reject on the same grounds the argument of the square.

Next, Gassendi, joining the debate about the famous problem put by Poysson,[33] criticized invoking pure mathematics in physical questions. In a letter to Mersenne dated 13 December 1635, he writes, alluding to the two arguments coming from Scotus:

29 Palmerino, "Two Jesuit responses", 187.
30 Feingold, "Jesuits: savants", 27-29.
31 Arriaga, *Cursus*, Physica, sectio VIII, subsectio 4, 240.
32 Oviedo, *Cursus*, Controversia XVII (liber VI *Physicorum*), punctum X, 359, § 5.
33 Vd. Joy, *Gassendi*, 89 sqq.

"You will thus see what weight to give to the *subtle* objections that are raised against those who think that the continuum is composed of indivisibles – objections such as: *the diagonal of the square will be no greater in length than the side, concentric circles will be equal*, etc."[34]

Gassendi does not try to refute the geometric arguments. He simply attempts to persuade himself that his theory of indivisibles is beyond their reach. Gassendi later reiterates this stance in his *Syntagma Philosophicum*. He just dismisses the geometric objections (once again, Scotus's square and concentric circles) that are directed against atomism. Gassendi's view is that geometry reigns in its own ideal domain, separated from matter and the real world.[35] Ironically, the author of *Against Aristotelians* (1624) thus has recourse to a distinctly Peripatetic argument. It seems to me that Gassendi's defense is weak. True, as Gassendi insists, his physical atoms are not mathematical points: they have size, extension. Accordingly, if we suppose real, material concentric circles, with circumferences composed of atoms, Scotus's arguments become irrelevant. If we imagine radii to be drawn, for example, from the center of each material atom of the outer circle, Gassendi claims that there is no difficulty imagining that they traverse one and the same atom in the inner circle through different parts, for example through its center and through its edge. The problem, however, is that Gassendi remains silent about these theoretical "parts" of atoms. His atoms are not conceptually indivisible since it is possible to distinguish in them a center and a periphery. The issue of the possibility of indivisibility is just transferred to these parts and their relative spatial position. It is difficult to see how Gassendi can justify indivisibles of space and time (which he holds) without establishing entities that are indivisible both physically *and* conceptually.

The most scientific rehabilitation of indivisibilism, however, came from Galileo, under the form of a new "pointillism". As we saw, Scotus's arguments prove that the existence of a finite number of adjacent indivisibles is incompatible with Euclidian geometry. Nonetheless, they leave open the possibility of an *infinite* number of separated indivisibles. It is just such a theory that Galileo provides in *Two New Sciences* (1638). Salviati hypothesizes that the cohesion of solid bodies is tied to the intrinsic presence of empty interstices. As the dialogue unfolds, Sagredo asks what might be the number of these empty interstices. Salviati answers that there is an infinite number of them, just as there is an infinity of particles in the same body. But both these particles and the empty interstices between them must be extensionless. They are like mathematical points, not like atoms. Or else, their accumulation would result in an infinite extension.[36]

As Sagredo balks at the idea of an actual infinite, Salviati undertakes to prove that "it is not impossible to find an infinity of empty interstices in an

[34] Mersenne, *Correspondance*, vol. 5, 533-534. Emphasis is mine.
[35] Gassendi, *Syntagma (Physica)*, sectio I, liber III, 264.
[36] Galilei, *Discorsi*, 67 and 72.

extended continuum". He appeals to the famous paradox of "Aristotle's wheel"– which is not Scotus's concentric circles argument, but derives from the Aristotelian school's *Problemata*. The line that is traced by the inner circle, Salviati argues, appears to be continuous, but in reality is composed "of an infinity of points, some full, the others empty." He then applies the same principle to bodies.[37]

And yet, how will the infinity of points in a large body be greater than the infinity of points in a small body? Galileo answers that infinities cannot be compared: the relation "greater than" and "less than" and "equal to" simply do not apply.[38] Although this reply is far from providing a final answer to the problem of mathematical infinity, Galileo thus managed to remove an important epistemological obstacle.

Furthermore, the fruitfulness of indivisibles in mathematics will be illustrated in the 1630's by one of Galileo's disciples: Bonaventura Cavalieri. The basic idea of his method is the following: a solid may be decomposed into planes, or a plane into lines, and it makes sense to speak of "all the lines" of a given plane or of "all the planes" of a given solid, even if the lines or the planes are infinite in number. Cavalieri goes on to posit that the ratio of the areas of two figures is equal to the ratio of their sets of respective lines.

At first, Cavalieri's approach encountered fierce resistance. In order for there to be a ratio between the two sets of lines, those sets must be magnitudes, which indeed Cavalieri assumed. Jesuits such as Guldin or Tacquet, as well as Galileo himself, judged Cavalieri's attempt to determine a ratio between infinite sets to be doomed to failure. Against Cavalieri, Galileo invoked his famous paradox of the bowl, but also Scotus's concentric circles![39]

However, Evangelista Torricelli was inspired by Cavalieri's method and was soon followed by the majority of mathematicians of the middle of the 17th century. Did indivisibles thus win a decisive victory over Scotus's arguments? In fact, no, since Torricelli and others modified Cavalieri's idea on a crucial point. They consider that the so-called "indivisibles" have a size, albeit an extremely small one (as small as is wished, or less than any given quantity). This marks the birth of the infinitesimal calculus. After John Wallis, Leibniz, in his *Quadratura Arithmetica* (1676), replaces the indivisible lines with rectangles. The long side of each rectangle corresponds to one of Cavalieri's lines, but the short side is a segment of "indefinitely small" size, which means that, far from being indivisible, it may be decreased without limit. This is why there can be continuity among all of the rectangles. The upshot, then, is a sort of compromise: an actual infinite enters into the composition of the continuum, but the infinitesimals are not indivisible. They are quantities that are homogenous with the quantities of which they are the infinitesimals: they are not like lines

[37] *Ibid.*, 68-69.
[38] *Ibid.*, 77-79.
[39] Vd. Cavalieri, *Geometria*, 756-757.

as compared to surfaces, or like points as compared to lines, but like infinitely small surfaces as compared to the surfaces of which they are parts.

In an earlier work, the *De minimo et maximo* (Nov. 1672-Jan. 1673), Leibniz had invoked against indivisibilism the argument of the square. His presentation of that objection is so close to Scotus's that it is tempting to speculate that Leibniz drew it directly from a Scotist text. Exactly as Scotus, Leibniz deploys his proof in two stages.[40] First, parallels are constructed from one side of the square, cutting the diagonal, which serves to establish that there should be as many indivisibles on the side as on the diagonal. Next, in case someone wants to object that there may be more points on the diagonal, an intermediary point between two of the earlier points on the diagonal is hypothesized with the same resulting impossibility that Scotus pointed out.

Similarly, in his 1665 *Lectiones mathematicas*, Isaac Barrow, Newton's teacher and a likely precursor of infinitesimal calculus, rejected indivisibles on the ground that they contradict the basic principles of Euclidian geometry. Notably, he appeals to the argument of the two concentric circles–without naming Scotus, though.[41] A few decades later, the same argument is once more taken up (again without crediting Scotus with it) in the third lecture of John Keill's *Introduction to Natural Philosophy*.[42] Keill, one of Newton's disciples, taught natural philosophy at Oxford and published his lectures in Latin in 1702. The resulting *Introductio ad veram physicam* was a success, judging from the fact that the Latin text went through many reprints and was eventually translated into English in 1720 with the title above mentioned, and thenceforth republished many times. Keill's lectures even were adopted as a textbook both in Cambridge and Oxford. Thus at the start of the 18th century, Scotus's arguments against indivisibles had not lost currency, compatible as they were with the emergence of infinitesimals.

In addition, Scotus's arguments did not survive among mathematicians alone. Thanks to Pierre Bayle, they were disseminated to a wider philosophical audience. Long before he wrote his *opus magnum*, the *Dictionary*, Bayle issued the following warning in the philosophy lectures he gave from 1675 to 1680:

> "We have reached the most difficult question that there is in physics, namely the question of whether or not continua are composed of parts that are further divisible without limit, or of mathematical points, or of extended corpuscles that are indivisible because of their material solidity. Whatever sect we choose, we are faced with insoluble and incomprehensible difficulties. The extreme weakness of the human mind prevents us from discovering what we must think."[43]

In effect, Bayle does not rally to a particular position. He prefers to entertain his skepticism with the mutual refutations heaped on all sides. Both the Ari-

[40] Leibniz, "De minimo et maximo", 8-11.
[41] Barrow, *Lectiones*, 17.
[42] Keill, *Introduction*, 29-30.
[43] Bayle, *Institutio*, 295-296.

stotelian doctrine and atomism succumb, each to its own proper difficulties. Against the third thesis, that of the "mathematical points," Bayle appeals to the square and the concentric circles arguments, to which he adds the pyramid argument coming from Gregory of Rimini. Bayle's sources, as he himself indicates, are his Jesuit contemporaries: Oviedo and Arriaga, above mentioned, and Hurtado de Mendoza.

Bayle recycles the same *aporia* in the article "Zeno of Elea" of his *Historical and Critical Dictionary*. There, Bayle alludes only indirectly to Scotus's arguments, but the success of the *Dictionary* spread the issue into the 18th century. It is, for instance, probably in Bayle (perhaps also in Keill) that Hume found elements for his own discussion of the problem. His *Treatise on Human Nature* attacks the possibility of dividing space infinitely and defends indivisibilism. He rejects the geometrical arguments as "*scholastic* chicanery, undeserving of attention."[44] While he acknowledges the medieval origin of these arguments, he does not even bother to present them, assuming that they are familiar by now. And as all indivisibilists, he casually dismisses them rather than refutes them. A little farther, however, he feels compelled to overthrow the very foundations of Euclidian geometry. Hume's contention recalls Arriaga's and Oviedo's solutions and would indeed dissolve (only at a high price) the paradox of the two concentric circles. How might the mathematician "prove to me, for instance, that two right lines cannot have one common segment? Or that 'tis impossible to draw more than one right line betwixt any two points?"[45] If we suppose, he goes on, two straight lines getting closer by an inch every 20 leagues, "I perceive no absurdity in asserting that upon their contact they become one", that is to say, eventually share a common segment.

It is thus clear that while Scotus's own arguments are presented and discussed in less and less detail, they remain in the background and frame the discussion as late as the 18th century, which is an outstanding achievement.[46]

44 Hume, *Treatise*, book I, part II, section II, n. 9, 27.
45 *Ibid.*, section IV, n. 30, 38.
46 I would like to express my many thanks to Anne A. Davenport for her help.

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