A formal introduction to Kolmogorov complexity is given, along with its fundamental theorems. Most importantly the theorem of undecidability of a random string and the information-theoretic reformulation of Gödel’s first theorem of incompleteness, stated by Chaitin. Then, the discussion moves on to inquire about some philosophical implications the concept randomness has in the fields of physics and mathematics. Starting from the notion of “understanding as compression” of information, as it is illuminated by algorithmic information theory, it is investigated (1) what K-randomness has to say about the concept of natural law, (2) what is the role of incompleteness in physics and (3) how K-randomness is related to unpredictability, chance and determinism. Regarding mathematics, the discourse starts with a general exposition of the relationship between proving and programming, to propose then some ideas on the nature of mathematics itself; namely, that it is not as unworldly as it is often regarded: indeed, it should be considered a quasi-empirical science (the terminology is from Lakatos, the metamathematical argument by Chaitin) and, more interestingly, random at its core. Finally, a further proposal about the relationship between physics and mathematics is made: the boundary between the two disciplines is blurred; no conceptual separation is possible. Mathematics can be seen as a computational activity (i.e. a physical process), the structure of such a type of process is analyzed.
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1. Introduction

“Yo sé de un laberinto griego que es una línea única, recta. En esa línea se han perdido tantos filósofos que bien puede perderse un mero detective”

Jorge Luis Borges

Randomness is a deep concept. When formally defined, not only does it play a major role in its proper field, algorithmic information theory (AIT), but it reveals profound interdisciplinary connections between fundamental concepts pertaining to different disciplines – such as logic, theoretical computer science, mathematics, evolutionary biology –, with far reaching implications in physics and philosophy. Since the sixties, when a new rigorous formulation of randomness was given – independently by Kolmogorov (Kolmogorov, 1965), Chaitin (Chaitin 1966) and Solomonoff (Solomonoff 1964) –, its impact has been immense, redefining and blurring both the internal and the external borders delimiting most of the logical disciplines.

The aim of the present work is to give a basic formal characterization of randomness, following the Kolmogorov-Chaitin definition, then justify it as a plausible notion, able to capture important aspects of the commonsensical idea of randomness. Some theorems are proven, to show some of the progress that can be made in the conceptual analysis of randomness when this is logically defined. The discussion moves on to investigate what the given definition can tell us about physics, when the latter is considered as a formal system. This inquiry raises more than one fundamental philosophical question about the nature of physical laws. Emphasis is given to the connection between incompleteness and the discipline of physics, as it is addressed by the Brazilian mathematician and logician Newton da Costa. The fourth and last chapter deals with metamathematics (the reflection on mathematics conducted using logical or mathematical tools). It presents some ideas of Chaitin on the implications the notion of randomness has on how mathematics must be looked at, especially while thinking about its limitations, but also when considering its own intrinsic nature. Section 4.4 investigates the mind-boggling question whether mathematics is structurally random deep down. Finally, the concluding section proposes some ideas about the relationship between physics and mathematics, when the latter is seen as a computational physical activity.
According to Kolmogorov, randomness can be defined informally as follows: a given string of characters is said to be random if and only if the shortest description we can give of the string is the string itself (a character by character copy of it). Another way to put it is: the information a random string conveys is not compressible; in order to communicate the string, through whatever process of transmission of information, we must exhibit the whole string, uncompressed. A third equivalent way of defining randomness is to say that a string is random if and only if it is patternless, namely that no kind of order or recurrence can be found in its sequence of characters. If a pattern may be found, we can compress the string, thanks to the pattern, which is informationally less cumbersome than generating the string in its entire length. We would be able, then, to communicate everything there is to say about the string without having to exhibit it completely, just providing an algorithm (a computer program) that yields the complete string as output. Consider the following examples:

10101010101010101010…
0010010011110110…

These are both infinite binary non-random strings. The first is an obvious alternation of ones and zeros, which can be compressed by means of the following algorithm: print 1, if the last printed digit is 0, then print 1, else 0. This algorithm, suitable encoded in any programming language, will provide the means of generating the whole string. However, the program itself, regarded as a string of symbols of the chosen programming language, is decisively shorter than the whole string. The second string is more interesting; it is a binary translation of the digits of \(\pi\). Hence, even if it may look random at first sight, it can be easily compressed using one of the many well-known algorithms to calculate \(\pi\).

The underlying concept of the three given equivalent definitions of randomness is that of complexity. A random string is a string whose complexity is roughly equal to the length of the string itself. The meaning of this will be soon explained (at the beginning of the next section).

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1 For a binary representation of \(\pi\) (http://www.befria.nu/elen/pi/binpi.html).

2 For example the Gregory-Leibniz series: \(\pi = 4 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \cdots \right)\). Please refer to Weisstein, Eric W. "Pi Formulas." From MathWorld--A Wolfram Web Resource. (http://mathworld.wolfram.com/PiFormulas.html), if interested in a review of the main formulas to calculate \(\pi\).

3 The complexity cannot be much greater, since in the worst case the shortest program which generates the string is long as the string itself plus the instruction to print it.
the moment let us just say that complexity is another way to talk about the possibility to shorten strings through compression.

It is also important to note that one must not confuse the concept of algorithmic randomness with that of statistical randomness. The former is concerned with the possibility of compression, and ultimately with the notion of order, while the latter is a probabilistic notion, it deals with the frequency of occurrence of the characters. There are several statistical tests for randomness; some of them search for *normality*, which is to say equal occurrence of every character (or substring of characters) present in the string. The probability of finding a certain character in a given position $i$ must be $\frac{1}{n}$, where $n$ is the number of different characters occurring in the string. Consider the binary translation of $\pi$ given above. Such a string is statistically random, yet $\pi$ is ordered, compressible, since it may be generated by algorithms (some of them quite short)$^4$.

Finally, another important distinction to make in order to avoid confusion is that between randomness and chance. They are two different concepts. Consider the procedure of determining a string by flipping a fair coin. If the outcome is head we write 1, if tail 0. There is obviously a non-zero probability that after $n$ flips the resulting string consists in $n$ equal characters, say 1. In fact, it has the same probability of any other string. A succession of ones is not a very complex string; in fact, its complexity is as low as possible, nevertheless one may be tempted to consider it random, being it the result of a random process. I will not tackle here the matter of the connection of Kolmogorov randomness with random processes, there will be more to say later in section 3.6, when the conceptual bases will be laid. What is relevant in the present discourse is that we can take coin flipping as a process whose outputs are chancy, capable of generating different strings, from highly ordered to random ones. Moral of the story: random processes do not necessarily generate random strings in the Kolmogorov sense. The property of a string of being Kolmogorov random does not depend on the physical processes generating the string; strings should be considered abstract objects. To avoid confusion, we will call *chancy* a string being the output of a random process$^5$. Furthermore, we should also keep the notion of random process separated from that of random object. The former notion has to do with indeterminism, while the latter with the

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$^4$ See note n. 2.

$^5$ For an in-depth introduction to these topics see (Eagle, 2005).
notion of ordered structure. There are of course interesting relationships, but they will be considered later.

2. Kolmogorov Complexity and Randomness

2.1. Formal Definitions and Introductory Theorems

We are ready now to present the notation necessary to give a more precise formulation of the above mentioned definition of Kolmogorov randomness.

- If \( \sigma \) is a string of characters, \(|\sigma|\) represents its length (in number of occurrences of characters).
- \( \Sigma = \{0, 1\} \) is the binary alphabet.
- \( \Sigma^* = \{0, 1\}^* \) is the set of strings over the binary alphabet (i.e. the set of binary strings).
- Let \( \varphi : \{0, 1\}^* \rightarrow \{0, 1\}^* \) be a partial recursive function. \( \tau \) is called a description of \( \sigma \), relative to \( \varphi \), if \( \varphi(\tau) = \sigma \). \( \tau \) may be seen as the program whose execution gives \( \sigma \) as output, while \( \varphi \) as a suitable encoding of the machine executing it.\(^6\)

Subtle paradoxes arise if \( \varphi \) is not subject to further restrictions. Consider for instance a description \( \tau_1 \) so defined: \textit{the smallest number not describable in fewer than} \( n \) \textit{characters} (leave apart for the moment the fact that the length of the description is language dependent; one can always add “in a given language \( L \)” at the end of the description). If \(|\tau_1| < n\) then there is a paradox. Take for example \( n=100 \). There must be, at some point, a number so big that is not possible to describe it in fewer than 100 characters, but in this case, if \( \tau_1 \) is true for some number \( m \), then there is a contradiction, since \(|\tau_1| < 100\). This is the so called Berry paradox. In order to avoid it, \( \varphi \) must be a \textit{computable function} (a function is \textit{computable} if there is a Turing machine that yields its result as output when given its argument as input)\(^7\).

\(^6\) Sometimes \( \varphi \) is referred to as the language in which \( \tau \) is expressed, see for example (Koucký, 2006). This is not entirely precise, but it can make sense if one notes that to execute \( \tau \) by a particular machine is implicitly the selection of a language.

\(^7\) Refer to (D’Agostino & Mondadori, 1997) for an introductory discussion on Turing machines.
In my opinion there is another way to dissolve the paradox. It can be negated that there is in fact such number satisfying description $\tau_1$. Consider a number $m$ such that it is the smallest number for which every non-self-referential (unlike $\tau_1$, which is self-referential) describing it must not be smaller than $n$ characters. Let us suppose that $m$ does not satisfy $\tau_1$ (if it really does, then the paradox does not hold), it cannot be said, therefore, that $m$ is the smallest number generally describable (using also self-referential descriptions) in fewer than $n$ characters. The same holds a fortiori for every number $p > m$, since if the self-referential description $\tau_1$ does not apply to $m$, then the same is true for every $p > m$, because $\tau_1$ does not increase in size when referring to $p$ bigger than $m$. Now, $n$ cannot be $p < m$ either, because $m$ is by definition the smallest number for which every non-self-referential description applying to it must be longer than $n$, there exist hence a possible non-self-referential description $\tau_2$ of $p$ that is smaller than $n$, making of $n$ not a possible candidate for $m$. This is to say that the requirements a number wanting to satisfy $\tau_1$ are too strict for such a number to exist; but this is not something paradoxical! It is just a matter of fact.

**Definition 1. (Kolmogorov Complexity)** for a string $\sigma \in \{0, 1\}$ the Kolmogorov complexity of $\sigma$, with respect to $\varphi$, is $C_{\varphi}(\sigma) = \min \{|\tau|, \tau \in \{0, 1\} \& \varphi(\tau) = \sigma\}$.

The Kolmogorov complexity of a string $\sigma$ is defined as the length of the shortest binary program $\tau$ which, executed by $\varphi$, outputs the string $\sigma$.

Consider the following particular case: if $\varphi$ is the identity function, $\varphi(x) = x$, then the Kolmogorov Complexity of any given string $x$ is the length of the string itself, $C_{\varphi} = |x|$. It can be easily seen that, were the chosen function different, so would be the related measure of Kolmogorov Complexity (also indicated as K-complexity). Without further specifications, hence, K-Complexity would be a context-dependent notion, being it relative to the language of description.

The necessary step to avoid such context-dependency is to introduce the following theorem:

If $U$ and $\varphi$ are partial recursive functions, and $c$ is a constant,

**Theorem 1. (Optimality)** $\exists U$ so that $\forall \varphi, \sigma, \exists c > 0$ such that:

$$C_U(\sigma) \leq C_{\varphi}(\sigma) + c$$
A partial recursive function \( U \), satisfying Theorem 1 is otherwise called a *Universal Turing Machine*.

**Proof:** enumerate all the partial recursive functions, \( \varphi_1, \varphi_2, \varphi_3, \ldots \), they all can be associated with a program, which can be uniquely mapped to an \( n \in \mathbb{Z} \). Define then \( U: \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^* \) as follows: \( U(w) = \varphi_i(\tau) \), where \( w \) must be decoded into \( i \) and \( \tau \), such that \( w = \langle i, \tau \rangle \) (note that there is the possibility that \( \varphi_i \) does not stop). An example may be: \( U(1^i0\tau) = \varphi_i(\tau) \), where \( 1^i \) represents a string consisting in \( i \) ones. \( U \) is a *partial recursive function* satisfying Theorem 1. □

We can now refer to the Kolmogorov complexity as \( C(\sigma) = C_U(\sigma) \), to simplify the notation.

**Definition 2. (Kolmogorov Randomness)** A string \( \sigma \) is Kolmogorov random if and only if

\[
C(\sigma) \geq |\sigma|.
\]

**Theorem 2.** \( \exists \sigma_n \forall n \) such that \( C(\sigma_n) \geq n \), for \( n \in \mathbb{Z} \).

There is a random string of every length.

**Proof:** There are \( 2^n \) strings of length \( n \), while the strings of length less than \( n \) are \( 2^n - 1 \). This can be calculated summing of the infinite series \( \sum_{i=0}^{n-1} 2^i = 2 \left( \frac{1 - 2^{n-1}}{1 - 2} \right) \), and then adding the string of length 0. Therefore, being the strings of length \( n \) more than those of length less then \( n \), there must always be some string \( \sigma_r \) such that their descriptions \( \tau_r \) are equal in length to the strings themselves. This means that there is a random string of every length. □

A string \( \sigma_1 \) consisting of \( n \) zeros has complexity \( C(\sigma_1) = \log_2 n + c \), where \( c \) is a constant, since to describe it only two things are needed: 1) the number of zeros \( \sigma_1 \) consists in (information which can be given in \( \log_2 n \) characters) and 2) a short program (of length \( c \)) capable of reconstructing the list of zeros from \( n \).

**Theorem 3.** Given any string \( \sigma_n \) of length \( |\sigma_n| = n \), \( C(\sigma_n) \leq n + c \), where \( c \) is a constant.

**Proof:** This is easily seen if one considers that in the worst of the cases a description of whatever string cannot be longer than the string itself plus the program necessary to give the string as output. □
**Theorem 4. (Undecidability)** It is uncomputable (undecidable)\(^8\) whether a string is K-random\(^9\).

**Proof:** Non-random strings are recursively enumerable. To do so one can follow this procedure: given a string \(\sigma\), run all programs shorter than \(\sigma\) in parallel. If anyone of them ever outputs \(\sigma\), then \(\sigma\) is not random. Assume now we dispose of a program \(p\), able to tell whether a string is random. We could then specify the lexicographically first Kolmogorov random string of length \(n\) using \(\log_2 n + c\) bits. To do so run \(p\) on all strings of length \(n\) in the lexicographical order until you find a string that is K-random; then print out such a string (how to print it is determined by the lexicographical order). This only requires to specify the program \(p\) and \(n\), a specification which requires \(|p| + \log_2 n + c\) bits, consisting in a compression of the string; this is impossible, since \(\sigma\) was supposed random. Hence, no such \(p\) can exist; therefore, it is undecidable whether a string is K-random. \(\blacksquare\)

Theorem 4 is an implication of the halting problem (the problem of determining whether a given program halts or not). Since the halting problem is undecidable, also Kolmogorov randomness is so. This is the reason why it is not so easy to exhibit a random string as an example. One can never be sure that the best (the shortest) description available is the best of every possible description. More generally stated Theorem 4 affirms that there cannot be a function which takes a string as an input and outputs its K-complexity.

**Theorem 5. (Incompleteness)** \(\exists\ \sigma\) such that it is random and unprovable to be random within a given consistent system.

**Proof:** Let \(T\) be an axiomatized theory which 1) contains arithmetic, 2) is consistent, 3) is describable in \(k\) bits (i.e. \(C(T) \leq k\)) and 4) is complete (i.e. all the true formula in \(T\) can be proved in \(T\)). Let \(P_n(x)\) means “\(x\) is the lexicographically least binary string of length \(n\) with \(C(x) \geq n\)”. It follows that \(C(P_n(x)) \leq \log_2 n + c\). Moreover, Theorem 2 shows that there is a K-random string of any length, so the existence of \(P_n(x)\) is guaranteed; it is also unique. For any given string \(\sigma\) of length \(n\) we can decide which between the following statements is true: \(P_n(\sigma) = True\) or \(\neg P_n(\sigma) = True\), simply enumerating the proofs of \(T\). It is possible to derive a description of \(x\) combining

\(^8\) A decision problem is a question formulated as to have only “yes” or “no” answer.

\(^9\) Cf. (Koucký, 2006, page 4).
the descriptions of $T$ and $P_n$ so that $C(x) \leq 2k + \log_2 n + c$, which is in contradiction with $C(x) \geq n$, for $n$ large enough.\textsuperscript{10}

Theorem 5 is a reformulation of G"odel’s first incompleteness theorem. In its original formulation the incompleteness theorem informally states that “in any consistent formal system $F$ within which a certain amount of arithmetic [namely, Peano Arithmetic] can be carried out, there are statements of the language of $F$ which can neither be proved nor disproved in $F$” (Raatikainen, 2015). G"odel’s proof uses a formalized version of the proposition “This statement is unprovable in $F$”. Such a proposition turns out to be true if and only if unprovable. Theorem 5 is based on the concept of Kolmogorov complexity, which is extraneous to G"odel’s formulation, yet it finds incompleteness in logical systems as the latter does. It is possible to prove that if a theorem contains more information than a given set of axioms, then it is impossible for the theorem to be derived from the axioms. Chaitin’s version is as follows:

Here is the first information-theoretic incompleteness theorem. Consider an N-bit formal axiomatic system. There is a computer program of size N which does not halt, but one cannot prove this within the formal axiomatic system. On the other hand, N bits of axioms can permit one to deduce precisely which programs of size less than N halt and which ones do not. Here are two different N-bit axioms which do this. If God tells one how many different programs of size less than N halt, this can be expressed as an N-bit base-two numeral, and from it one could eventually deduce which of these programs halt and which do not. An alternative divine revelation would be knowing that program of size less than N which takes longest to halt. (In the current context, programs have all input contained within them. (Chaitin, 1982, page 6)

\textbf{2.2. Plausibility of the Definitions}

It is legitimate to ask why a string satisfying definition 1 is called random. What has randomness to do with that definition?

Certainly one can define terms \textit{ad libitum}; definitions are just definitions, they are arbitrary in some sense; it is what derives from them which is not. Nevertheless, there are appropriate

\textsuperscript{10} Cf. (Bortolussi, online publication).
definitions, and that of Kolmogorov randomness is one of them. It captures some features of the commonsensical concept of randomness.

In common sense “random” is a synonym of “unpredictable”. Things are predictable if they show some kind of pattern, underlying structure. We already distinguished between random processes and random objects; these two notions must not be confused; however, they have something in common: they lack order. If something lacks any kind of order it cannot be schematized, summarized, compressed, grasped unless it is given as a whole, holistically. One is tempted to say that something totally disordered cannot be comprehended; and such an incomprehensibility is a property of K-randomness. Yet “randomness” is a better word than “disorder”, since, as Floridi puts it: “the former is a syntactical concept whereas the latter has a strongly semantic value, that is, it is easily associated to interpretations, (as I used to try to explain to my parents when I was young)” (Floridi, 2016, page 23).

3. Randomness and the Laws of Nature

3.1. Understanding as Compression of Information

Understanding is compression of information\(^\text{11}\). This is the intuition enabling us to export Kolmogorov Complexity, with everything it has to say about randomness, in a broader ambit of discourse. I will not try to propose here a formal argument in favor of such an intuition, since it will require a divagation into cognitive science and epistemology, which is beyond the scope of this work; yet I want to stress informally that the idea is quite reasonable, letting aside the illuminating capability of its far reaching implications on a plethora of disciplines (from philosophy of science to machine learning, to pattern individuation, passing through the mentioned broader fields of epistemology and cognitive science, to statistics, psychology, and finally but not exhaustively physics, which will be our focus).

The proposed idea, so, to put it in a slogan, is: comprehension is compression\(^\text{12}\) (expression which take advantage of the Latin etymology of “comprehension”, namely cumprehendere, take

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\(^{11}\) Cf. (Adriaans, 2007).

\(^{12}\) In Italian it sounds even better: comprensione è compressione.
together, contain inside. Comprehending as embracing. A beautiful interpretation of the possibility of encoding information by algorithms)\textsuperscript{13}. One may try to justify such statement, without claiming to argue rigorously about it, suggesting that understating is a glance at the underlying structure of things; it is a work of conceptual breaking down their apparent complexity into more fundamental bricks; reducing them, hence, to something simpler, graspable, something easier to operate with, in order to have a notion of the whole starting from its components and the relationships between them\textsuperscript{14}.

There are, of course, different competing conceptions of understanding, but what is quite interesting is that the understanding-as-compression idea captures many interesting aspects of them. Consider for instance two commonsensical – and not necessarily mutually exclusive – widespread conceptions of understanding: understanding as categorization/classification and understanding as seeing connections/relations between things. Both, categorization and individuation of connections, can be seen as procedures of compression. In the former case, categorizing would be the individuation of similarities between things (finding in them the same property, the same scheme) remarkable enough to enable us to put those different things in the same set (category), which is another way to compress information: make two distinct things the same, abstractly, under some particular given aspect. In the latter case the individuation of relations would be another process of simplification (compression). It is possible to acquire new knowledge, according to this view, when finding known relationships between the new unknown objects, situations, processes we meet, and what we already are familiar with.\textsuperscript{15} The matter is to identify the basic components of what we want to understand, to put them in relation with known old ones. Such a procedure demands the individuation of the fundamental (or at least a more fundamental) structure of things; such individuation is obtained by reduction.

What we call the logical structure of things (from natural laws in physics to universal grammar in linguistics, to give just a couple of examples) is a compression.

\textsuperscript{13} Have a look at (http://www.etimo.it/?term=comprendere\&find=Cerca), in Italian, or at (http://www.etymonline.com/index.php?term=comprehend), in English.
\textsuperscript{14} I don’t think one must necessarily embrace a reductionist view in order to agree with the exposed position, although reductionism is compatible with it. One may state, in fact, that the most fundamental bricks we are aiming to, when trying to comprehend something, are only conceptually fundamental, not ontologically. The present discourse encourages only epistemological reductionism, not reductionism \textit{tout court}.
\textsuperscript{15} The similarity between the two conceptions of understanding (categorization and relationship individuation) is not so striking if one considers that relationships, like properties, may be conceived as functions, with the difference of being binary and not unary.
It is worth noting that the idea of comprehension as compression is closely related to the famous principle of epistemological parsimony known as Occam’s razor, stating that other things being equal, among competing hypotheses, the one with the fewest assumptions should be selected. It should not be said that the Occam’s principle is the same as the comprehension as compression theory. The first is a norm of methodology, guiding research, both individual and collective, while the second is a definition of understanding, it may even be an empirical theory—although we are not dealing with such a version of it—of how intellect works. There is a subtle point here, which is where their similarity lies. If comprehension is compression, a more efficient compression is a better comprehension; in this sense Occam’s principle may be thought as a heuristic to achieve, if not always a more insightful comprehension, a better reorganization of the comprehension of what we already have, which is to say a better understanding. What is meant may be made clear by an example. Consider a totally accurate yet non-economic description of the universe, call it \( T_1 \); it is a mere enumeration of brute facts (let us not be bothered by the question whether a thing such a brute fact exists or not. Moreover, it does not matter whether the number of facts of the universe is finite or infinite).\(^{16} \) No pattern or underlying principle connects them. A description of the universe of this kind may be seen as the limiting case of a theory. Consider now a second description \( T_2 \), which is a proper theory, it makes connections, it compresses the data, but, we may assume, it is not exhaustive of everything there is to say about the universe. \( T_1 \) is complete but redundant (and in practice totally useless), \( T_2 \) is of some use but incomplete. At first sight one may think that Occam’s razor would suggest that \( T_2 \) is a better theory, but, at a more considerate inspection, it would be fair to admit that we did not specify how much \( T_2 \) was incomplete; maybe it is just a little compression of some secondary data; if this were the case, it would be hard to say which of the two theories is better. The concluding consideration is that to prefer a theory to another one must not only consider which one is compressing the most but also what is the span of the compressed information. A proposal on this theme can be found in the appendix.

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\(^{16} \) L. Floridi is proposes an interesting definition of datum, from which can be constructed a working definition of brute fact. The definition goes as follows: “a datum is a putative fact regarding some difference or lack of uniformity within some context. Nothing seems to be a datum per se. Rather, being a datum is an external property” (Floridi, 2016, online publication).
3.2. AIT and the Notion of Theory in Physics

A quite legitimate question to ask when considering the possibility of exporting Kolmogorov Complexity to the context of physics is whether it makes sense to do so or not. Complexity, as we defined it, is a technical term referring to a property of strings, pertinent to the particular scenario of algorithmic information theory; is it considerate to talk about the complexity of theories as we talk about the complexity of strings?

To answer to this question we must go back to definition 1. Rigorously defined complexity is not applicable to theories as we normally encounter them (mainly written in mathematical equations, like general relativity, if not in a natural language, as usually is natural selection). We may take two different roads, to search for a solution. The first is to loosen the definition of complexity, to make it applicable to formal theories, the other (which is the one here considered) is to show that formal theories can be encoded into binary strings.

We shall not embark in such an ambitious task as that of the explanation of how the binary translation of physical theories may be done. What is important here is to note that it can be done. Laws are sets of sentences indicating procedures, something having more than one affinity with algorithms, both being ordered set of instructions. If theories are something more than that, an implicit ontology for instance, namely the ideal substrate upon which the structure captured in the models operates, there is no reason to suppose the same does not hold for algorithms. Since algorithms are easily encoded into binary strings – informatics being the best example of this – it can be easily seen that theories, when formally and systemically defined, are translatable into binary strings.

Let us return to the notion of physical theory. Coherently with the idea of understanding as compression of information, one can say that the understanding of nature is identifying patterns and regularities in the physical quantities describing it – similar ideas outside of the AIT literature may be found for example in (Feynman, 1965) --. We can see theories as compressions of the packages of data we get from measurements and experiments, and, more interestingly, as a tool to calculate the future. According to this view, physical models do not only schematize nature, catching its most relevant aspects, they are better than schemes, they do not leave secondary features behind, since even these are captured through compression.
Pondering the question whether reality is ultimately ordered or not calls us back to the problem of K-randomness. What is the best possible description of reality? Is any aspect of reality non-reducible to laws? If reality were not reducible to generalizing principles, it would be K-random. The best possible theory would then be a simple exhibition of everything occurring in the universe. The same applies to some parts of a universe which is only ordered only to some degree: pertaining the components of such a universe which are random, only mere enumeration of facts is possible. A possibility that must be considered is that of quasiperiodicity. Quasiperiodicity is the property of those systems which display irregular periodicity. Where a periodic system displays the occurrence of something (say \(x\)) with a constant interval, a quasiperiodic system is so that the interval between the occurrences of \(x\) is variable. Can it be that reality is quasiperiodic? Can quasiperiodicity be viewed as a degree of complexity in the middle between high compressibility and total incompressibility (i.e. randomness)? Is a system of that kind ultimately random, due to the propagation of disorder into the system as time passes? The answer is negative, because saying that reality is ordered (or not) means that there are regularities (or that there are not), laws, and these regularities are patterns of the temporal development, they stretch across time: we are not considering order as something given in a precise moment; order is not a property of some particular situation situated in time (a conception related to entropy), so that order may decrease or increase in time, we apply the notion of order to the temporal development, which may display patterns or not.

A colorful expression of Chaitin may clarify further: “In medieval terms, \(H(X)\) [Chaitin’s notation for complexity] is the minimum number of yes/no decisions that God would have to make to create \(X\)” (Chaitin, 2014, page 5). We will have to say more regarding this formulation.

Physical theories are given in the form of mathematical equations, which individuate relations between mathematical objects; in the physical models those relations stand for data, or relations between set of data, if not relations between other relations. It is thanks to these structures that nature can be conceptually compactified. What is not modeled in our theories we cannot comprehend, since big amount of data are not manageable.

It is pertinent to note that we can never be sure that some particular aspect of reality which seems random to us is really random. Recall Theorem 4, it is not decidable whether a string is random. It is all a matter of how Kolmogorov complex the universe is. Moreover, since there is no
function which takes a string as input and outputs its K-complexity, we can never be sure that some theory is the most economical description of nature possible.

We have seen that the limiting case of a theory is plane description, no conceptual shortcut, no compactifying principle. The sunrise of a science starts from something approximated to those descriptions. The renaissance anatomical tables are sometimes just a figurative description of the body, accompanied by an enumeration of its organic functions. Those functions are not reduced to general biophysical principles; they are given as they appear: for example, the eye sees; vision is not explained in term of physiology and electromagnetism. The same holds for many ancient astronomic tables: they consist in lists of values (determined empirically), specifying the position of some celestial bodies in the various periods of the year, without an understanding of the geometrical curves describing the trajectories the celestial bodies follow. In the history of science famous is the case of Kepler using Tycho Brahe’s astronomic observations to derive its elliptical orbits; it is an example of compression\textsuperscript{17}.

Another wonderful example, which may be taken as an analogy for the role of theory in physics, comes from informatics. To store images of fractals pixel by pixel, say Mandelbrot set, would require (if you count that every pixel is a 24-bit color) millions bits for quite small images. Computers can reproduce images of the Mandelbrot sets simply deriving its from its defining equation:

$$Z_{n+1} = Z_n + C$$

With $Z_0 = C$, where points C in the complex plane for which the orbit of $Z_0$ does not tend to infinity are in the set\textsuperscript{18}.

\textsuperscript{17} The matter is not so trivial because there is nestled here the phantom of the problem of induction. Tycho Brahe did not give of course every single point of the curve, so that the single positions were compressed in the equation of an ellipsis by Kepler. Kepler’s elliptical orbits pass with a good degree of approximation through the positions specified by Brahe, they filled the holes, specified the unspecified traits. The generalization from the single discrete positions to the curves involved a reasoning by induction; for this reason, talking about compression is not unproblematic, yet sensible. It gives the idea.

3.3. Physics as a Formal System

We already mentioned in the previous section the fact that in order to apply AIT to physics we must work on bodies of theories “formally and systemically defined”; this actually means we need theories to be axiomatically organized. Let us make a little digression then to give a general idea of how the axiomatization of physics may be accomplished.

To axiomatize is no trivial task, especially in the natural sciences. We are talking about nothing less than Hilbert’s sixth problem:

Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: to treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. (Hilbert, 1902, page 12)

When talking about systematic and complete formalization of physics, the German structuralist program carried forward by Joseph Sneed, later popularized by W. Stegmüller, must be mentioned19. In order to solve problems concerning the definition of theoretical terms in scientific theories a set-theoretic approach to theory formalization was proposed. We deal here with the idea as later developed by da Costa and Doria, based on P. Suppes’ predicates.

According to da Costa and Doria (da Costa & Doria, 2007) to axiomatize a theory (in mathematics as in mathematical based sciences) is to define a set-theoretic predicate, that is a species of structures. More precisely: it is to exhibit a species of structures so that:

1. The primitive terms of the theory are the basic sets and the primitive relations of the species of structures.

2. The theorems of the theory are the logical consequences of the species of structures, whose primitive elements are replaced by the corresponding primitive terms of the theory. Proofs are made within set theory.

They also suggest that:

we may… identify theories with sets of sentences, to which we add an empirical counterpart, that is to say, observational terms, protocol sentences, and the like. Also, we

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19 The seminal work is (Sneed, 1971).
may assert that a theory is a family of structures (models). Finally, according to our view point a theory may also be identified to a triple $<S; D; R>$ where $S$ is a species of structures (Suppes predicates)\(^{20}\), $D$ the set of domains of applications, and $R$ the rules that relate $S$ to $R$. (da Costa & Doria, 2007)

“However – they worn next – we must distinguish the two steps required in the axiomatization of an empirical theory”:

1. The construction of the theory's *Suppes predicate*.
2. The characterization of $D$ and $R$, a procedure that depends on the science where we find that theory.\(^{21}\)

This being said, it must not be forgotten that in physics there are several important cases of useful procedures (sometimes essential for their own fields) which are mathematically uncertain, in the sense that they are not bridled in a nexus of theorems deriving from axioms; they do not consist in a completely formalized set of propositions, being rather useful procedures, heuristics to obtain the desired category of results while solving problems. An example da Costa and Doria give is Feynman integration (da Costa & Doria, 2007, page 34). These techniques, they say, are not likely to be absorbed – as they are now – into the axiomatization of their field.

Gregory Chaitin, upon whose work da Costa and Doria based their speculations about the notion of incompleteness in physics, a theme which will interest us in the next section, uses the Kolmogorov complexity machinery to draw keen conclusions about axiomatization:

The general flavor of my work is like this. You compare the complexity of the axioms with the complexity of the result you're trying to derive, and if the result is more complex than the axioms, then you can't get it from those axioms. (Chaitin, 1999)\(^{22}\)

We can now discuss the role of incompleteness in formalized physics.

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\(^{20}\) They call *Suppes predicates* those set-theoretic predicates they use in the semantical approach to axiomatization, as opposed to a syntactic approach à la Bourbaki, which however they do not disdain.

\(^{21}\) There is some interest in saying that da Costa and Doria consider this method extendable to mathematized economical science and mathematical biology.

\(^{22}\) Online publication.
3.4. Incompleteness in Physics

If physics is formalizable, then the physical interpretation of Kolmogorov complexity is possible. It could be applied to physics as if this were a set of strings (we can in fact translate the formal system of physics into programs, which are indeed binary strings).

Maybe when we prove that some aspect of the natural world is intrinsically random, as it seems to happen for the “no hidden variables” theorems in quantum mechanics\(^{23}\), we are just saying that some aspect of nature in uncompressible. We will have more to say about this theme in section 3.6. For the moment let us focus on the impact that Theorem 5 – which is, as we have already seen, a reformulation of the first incompleteness theorem – has on formalized natural science.

Gödel’s theorem applies to formal systems as expressive as Peano Arithmetic\(^{24}\); Formalized physics certainly is. It means that there must be true physical statements which are not provable in the theory of physics; not at least if extra axioms are not added the formal system. This fact maybe uncomfortable to the physicist who wish that the mathematical language may be able to reflect – capture – the structure of the world. No matter how adequate the models are, a formal systematization of them will always leave some feature of nature behind; unless one is willing to accept that there are natural phenomena appearing random, given a chosen formal system; phenomena whose explanation must coincide with their plane description (uncompressed exhibition). The dream of a formalized Theory of Everything, courted by many physicists, is a mirage; it must deal with incompleteness (Chaitin, 2003).

In the mentioned work, da Costa and Doria try to “obtain examples of Gödel sentences – undecidable sentences – within the axiomatic versions of those theories” (da Costa & Doria, 2007, page 47). We try to give the general flavor of the conclusions derived in their work, without presenting the details, since such a presentation would require a technical digression outside of the purposes of the present study.

They explain that the motivation at the origin of their research program was to identify some feature to distinguish chaotic systems from non-chaotic ones. A fertile intuition was to ask whether

\(^{23}\) See for example (Shimony, 2016).
\(^{24}\) For an explanation of this see (Raatikainen, 2016).
such a problem would not be in fact algorithmically undecidable. However, their discovery was that undecidability (and hence incompleteness) came from an unexpected origin: not from some property of the systems themselves, undecidability is rather what they call a “linguistic phenomenon”, which is to say something that “depends on the tools that we have within the formal systems to handle expressions for the objects in those systems” (da Costa & Doria, 2007, page 62).

They sketched a general method to find undecidable statements in formalized physical theories, to use then such a method to derive the undecidability and incompleteness of chaos theory.

3.5. Computational Universe (Laws as Software)

If the whole universe is conceived as an immense computer, then the natural laws, its internal processes, are the software. Physical states at some time correspond to outputs. We can imagine the entire history of the universe as a run of the universal software. This scenario, call it computational universe, enables us to investigate some interesting thoughts about K-complexity.

The computational universe hypothesis is part of that philosophical doctrine which may be called pancomputationalism, namely the conception of everything as a (physical) process computational in its nature. From quantum particle interaction to orbiting celestial bodies, to physiological and neurological processes, from fluid dynamics to chemistry (biochemical or not), everything seen as information processing, life and consciousness being not exceptions. For a semi-formalization of this ideas see (Schmidhuber, 1997, 2002).

Together with M. Davis one may ask: “what is the hardware on which the cosmic software is being run?” (Davies, 2007, page 72); question to which he answers:

The universe itself. And by this I mean the real, physical universe. I am not referring to some imaginary cosmic hardware in a Platonic heaven, but the real universe we observe… What matters for computational purposes is not the spatial extent of the universe, but the number of physical degrees of freedom located in a causally connected region. Information processed in causally disconnected parts of space cannot be considered as belonging to the same program. (Davies, 2007, page 72)
Scientific theories can be thought then as nested computer programs predicting our observations, inside the universal computer. They would be elegant compressions of data, basically algorithms to calculate the future. The point is that, in order to be able to completely predict a future (or a past) state of the universal computer, the universal computer itself must compute its next state; “as Chaitin already demonstrated there are incompressible truths which means truths that cannot be computed by any other computer but the universe itself”. (Dodig-Crnkovic, 2007, page 263)

Strictly connected to this view is the question of how ontologically fundamental information is. If we are living in a software, what is reality made of? How can information be the fundamental brick everything ultimately consists of? The idea that information may be so fundamental comes from theoretical physics. It started having important implication in cosmology when Berkstein (Berkstein, 1973) and Hawking (Hawking, 1975) applied quantum mechanics to black holes, discovering how to quantify the entropy of a non-rotating uncharged black hole. The formula they derived relates the total information content of a region of space to the area of the surface encompassing that volume. This principle was exported from black holes’ thermodynamics to other fields, giving birth to the variants of the holographic principle, being ultimately applied to the cosmological event horizon. It was find out that the universal information bound is $10^{122}$.

As Dodig-Crnkovic put it, “the universe is viewed as a structure (information) in a permanent process of change (computation)” (Dodig-Crnkovic, 2007, page 267). Such a worldview, according to Davies, can overturn our conception of the relationship between the laws and physical reality.

If traditionally natural laws are seen as the rules of the game, independent from matter, constraining and determining its behavior, and information is seen as arising from states of matter, dispositions, he has a different proposal. Consider now the traditional view:

(A) Laws of physics → Matter → Information

“There is thus a fundamental asymmetry: the states of the world are affected by the laws, but the laws are completely unaffected by the states.” (Davies, 2007, page 70)

An alternative view may be:

(B) Laws of physics → Information → Matter
This is the computational universe view. Information is something upon which matter depend.

Nature is regarded as a vast information-processing system, and particles of matter are treated as special states which, when interrogated by, say, a particle detector, extract or process the underlying quantum state information so as to yield particle-like results. It is an inversion famously encapsulated by Wheeler's pithy phrase “It from bit.” (Davies, 2007, page 75)

This view can be connected well with processual philosophy, that class of doctrines considering processes as something more fundamental than the “things” participating in those processes – for a general overview of processual philosophy see (Seibt, 2016) -. The idea that the execution of programs is the most fundamental thing (note that programs can be interpreted as processes).

Davies considers finally a third option:

(C) Information → Laws of physics → Matter

What he finds attracting in this view is that the laws of physics are informational statements. It is true that for the majority of practical cases the logical hierarchy does not matter very much; there are anyway some relevant situations: if (A) is the case, and “information is simply a description of what we know about the physical world… there is no reason why Mother Nature should care about the [computational] limit… But if information underpins physical reality – if, so to speak, it occupies the ontological basement – (as is implied in Scheme (C) and perhaps (B)) then the bound on \( I_{universe} \) represents a fundamental limitation on all reality, not merely on states of the world that humans perceive” (Dodig-Crnkovic, 2007, page 76). If the universe is really subject to an information bound, operating in limited resources and time, many concepts

like real numbers, infinitely precise parameter values, differentiable functions, the unitary evolution of a wave function, are a fiction: a useful fiction to be sure, but fiction nevertheless, and with the potential to mislead. It then follows that the laws of physics, cast as idealized

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\[ ^{25} \] Floridi defends a form of informational structural realism in (Floridi, 2008). The idea can interestingly be connected also with the late German idealism: the philosopher Herman Lotze discusses the doctrine of things as laws in his *Metaphysics* (Lotze, 1879, book 1, chapter 3, paragraph 32). In the presented informational view laws ontologically depend on informational states, it can be said that they exist as part of the inventory of the world. Lotze discusses the possibility that things in their changing as time passes must be considered as particular instances of applications of laws. The informational view reaches the conclusion that laws are things bottom-up (from information to laws), Lotze that things are laws top-down (from matter to laws).
infinitely precise mathematical relationships inhabiting a Platonic heaven, are also fictions when it comes to applications to the real universe. (Dodig-Crnkovic, 2007, page 76)

The connection between this story of the computational universe, the information bound and complexity are many. Consider life as an example. If everything in the universe is a computational process, so are the organism and their evolution. They are self-replicating algorithms, informational structures, being capable of achieving a momentary stability inside the ongoing computational process of the history of the universe (note how the boundaries between organisms and the environment get blurred), enough to replicate themselves. An interesting possibility is that when life is so defined, it may be possible to apply AIT to evolutionary biology, mathematizing it (Chaitin, 2011). Chaitin laid the foundations for a research of this kind: evolution may be modeled, he maintained, using an attractive expression, “as a hill-climbing random walk in software space” (Chaitin, 2014, page 9).

Stephen Wolfram is another computer scientist who did much to develop such an algorithmic view of life. A main idea of his immense opus, A New Kind of Science (Wolfram, 2002), is that if it were recognized that the universe is digital in nature, running on laws describable as simple programs, the impact of this on the natural sciences would be somewhat fertile (hence the title).

3.6. Unpredictability, Chance and Randomness

We already made a distinction between random processes and random objects (strings), such a distinction was important – it was said – to avoid confusion. Now, after a more precise characterization of Kolmogorov complexity, it is allowable to investigate the relationship between the two concepts.

We mentioned the fact that the processual notion of randomness has to do with unpredictability. Which is to say that the upshot of a random process is predictable only with a certain degree of uncertainty; if a probability is to be given to the output of the process, it must be different from one. On the other hand, random strings are those whose shortest description is long at least as the string itself (we saw that it cannot increase very much more than the length of the string). An important result obtained by Martin-Löf is that the collection of c-incompressible strings (where a
A string is a string such that $C(\sigma) \geq |\sigma| - c$ coincides with the collection of strings that pass all computably enumerable statistical tests for randomness. This discovery establishes a remarkable link whose investigation is not carried further here. Nonetheless, the aim of the present chapter is to inquiry about some other link worth considering, namely, what happens when we apply the notion of K-randomness to processes.

While outputs of random processes are by no means necessarily Kolmogorov random (recall the string consisting in a succession of ones, obtained flipping a coin: it is a possible non K-random outcome, in fact), one may wonder whether there is a relationship between random processes and K-randomness. To formulate a precise question on the matter a comment is necessary.

We already saw pancomputationalism, the view that everything occurring in the universe (actually the universe itself) is a computational process; in this worldview natural laws are the algorithms characterizing the dynamics of the physical processes. Processes are integrally captured in their structure by programs (these programs may be more or less compressing). We can hence associate a string (i.e. a program) to every process, so that the execution of such a program is an occurrence of the process.

A pertinent question would then be: are random processes (i.e. chancy events) random objects in the sense of Kolmogorov? To reformulate this question with more precision some notation is needed.

I call \textit{chancy} those processes having statistically random results, not to confuse the two meaning of randomness. Let us now say that $x$ is a process\footnote{One may think processes as physical events, but the specification is not needed, not to lose generality.} and $\sigma_x$ is the set of the possible strings a program associated to $x$ may consist in (in some given language). $\overline{\sigma_x}$ is the string – among those of $\sigma_x$ – with the lowest Kolmogorov complexity\footnote{It is important to note that this a necessary step to make K-randomness cease to be a property of the descriptions of nature (i.e. models) to become a property of the natural phenomena themselves.}. We say that a process $x$ is Kolmogorov random if $\overline{\sigma_x}$ is Kolmogorov random. Now the question is: is $\overline{\sigma_x}$ K-random for every chancy process $x$? If not, what is the relationship between the complexity of $\overline{\sigma_x}$ and $x$, if $x$ is chancy?
To answer these question some considerations must be made. If the process $x$ is not K-random, a compression of $\bar{\sigma}_x$ is possible. This is to say that there is an identifiable and reproducible (at least in principle) procedure going on in the process, which can be compressed into some general principles. Moreover, to answer is fundamental to have a precise idea of the relationship occurring between Kolmogorov randomness, chance and the following oppositions of concepts: determinism/indeterminism and predictability/unpredictability in principle.

Consider a widespread definition of determinism:

*A system of events (processes) is deterministic if and only if every event is necessitated by antecedent events and conditions*\(^{28}\)

This definition captures the idea that in deterministic systems things could not go differently as they do, given the same initial conditions. *Indeterministic processes* are those which are *non-deterministic*.

The conceptual map – in favor of which we will argue, and which will give us the coordinates to answer the posed questions – of the relationships occurring between the mentioned concepts is as indicated in figure 1.

\(^{28}\) Cf. (Hoefer 2016), online publication.
We indicate a deterministic process with $D(x)$, while an indeterministic one with $\neg D(x)$. If $x$ is a process whose output is predictable (at least in principle\textsuperscript{29}), we indicate it with $P(x)$, otherwise $\neg P(x)$, if $x$ is unpredictable (even in principle). Finally, let a Kolmogorov random process be indicated with $K(x)$ and with $\neg K(x)$ a process whose complexity is lower than that of a random one. $C(x)$ is a chancy process.

First of all, some clarity on the notion of chance. The notion of chance may be used in two senses: one epistemological (where the fact that something is chancy or not depends on our knowledge), and one real, where chance is the property of those processes having more than one possible output; in this latter sense chance coincides with indeterminism.

Now consider the argument in favor of the following argument: the whole discourse about complexity (and in particular K-randomness) must be applicable only to deterministic processes. If a process has more than one possible output – being hence an indeterministic one –, there is no way to translate it into a string, since the string cannot contain all the information about how to choose between the possible alternative outcomes; if this were the case, so that the information

\textsuperscript{29} With this I mean that the prediction may be computationally very hard, but still possible.
about how the choice can be made would be conveyable by the string, that would mean that there is a precise necessitating process to choose among the possible outcomes, given some initial condition, making so the process deterministic. The possibility of translation applies completely only to deterministic processes. Determinism is in fact the possibility to algorithmatize the process (algorithmatize at least in principle). Further attention is here necessary. There are of course the so called nondeterministic programs; those programs have non-necessitating if-then structures: they include choice points, where the program flow can take different directions. The matter here is that the source of the choice is external to the program, and as such it may be based on some external process deterministic (e.g. noise; or something connected to current time) as well as an indeterministic one (e.g. a quantum random source). “Nondeterministic” programs may well be deterministic if they depend on deterministic processes; it all depends on the nature of the source that determines the decisions in the choice points. If nondeterministic programs really rely on some indeterministic source of decision, then the whole process consisting in the source plus the program is indeterministic, hence non completely formalizable. To be completely algorithmatizable a process must be determinable in every detail, even in the mechanism of decision at the choice points. In fact, if the mechanism of choice is itself a nondeterministic program a regressus ad infinitum occurs. So, again, more clearly:

A process is completely algorithmatizable if and only if it is deterministic\(^\text{30}\).

An indeterministic process can be schematically characterized in the form of an algorithm, but a characterization of the source of choice at the branching points is possible, hence the algorithmicization is not complete\(^\text{31}\).

A natural law is said probabilistic when the process it describes has no single output, given the same initial conditions (again, nondeterministic programs); the different output alternatives may have different probabilities. But what is the nature of probabilistic laws? What kind of process is the choice of the outcome among the range of possibilities? It cannot be based on necessitation from previous conditions, otherwise a single probabilistic law would be a set of different deterministic laws (each one governing the process given some specific initial condition). Were

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\(^\text{30}\) We are here giving a tentative answer to one of a series of question Chaitin left open at the end of one of its articles: “Is there a physical phenomenon that computes something noncomputable?” (Raatikainen, 2015), online publication.

\(^\text{31}\) This is the reason non-deterministic programs are not simulable without an external indeterministic source.
probabilistic laws determining the output of some process behaving according to some higher level law, they would be another instance of determinism, but this idea is worth exploring. The notion of chance is clearly non-utilizable, since it is what we are trying to explain (through the definition of probabilistic behavior); we must avoid infinite regression. Probabilistic laws could be seen as *laws of laws* then: laws determining the outcome of the process according to the circumstances, so that also the law change according to the circumstances, but again they would reduce to determinism. The necessitating initial conditions would here include also the law that should be applied in any given moment. I suspect in fact that if the notion of probabilistic law is to be understood, a final reduction to determinism must be made. There is something dogmatic in the notion of a law describing a process whose choosing mechanism to determine the output remains non-defined.

Going back to the main discourse, even if Kolmogorov random events are deterministic (they must be so in order to be able to apply the notion of complexity to them), they are unpredictable. To be able to predict a process means being capable of telling *in advance* what the outcome will be. K-random processes are uncompressible, no shortcut to the outcome can be made. Since no exhaustive scheme or law applies, the only way to know the upshot is to run the process, to know *a posteriori* what the outcome was. While K-random processes imply unpredictability, there are also some deterministic unpredictable non-random processes. In fact, compressibility is no guarantee that a prediction is possible, even if we mean *predictable in principle*. Some process could be deterministically law-driven, but being so that the computation needed to extract (predict) their outcome may require more resources than actually running the process to find out the solution. This type of events is represented in figure 1 as the region superiorly limited by the dotted line$^{32}$.

The white region is the set of the *understandable* processes; they must be non-K-random; however – as we have said – a lower than K-randomness level complexity is not a sufficient condition for predictability.

To resume the argument, combining formal and unformal reasoning:

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$^{32}$ Chaotic systems are situated in this non-random unpredictable region. They are deterministic system whose possibility of prevision lies on the accuracy of how the initial conditions are given. They are uncertain, a feature that makes them unpredictable, yet they are deterministic and compressible. There is a connection with the work of M. D’Agostino in depth-bounded logic. See for example (D’Agostino & Finger & Gabbay, 2013).
∀x, D(x) ∨ ¬D(x) (either a process is deterministic or indeterministic)

∀x, P(x) ∨ ¬P(x) (the same holds for predictability)

∀x, K(x) ∨ ¬K(x) (and K-randomness)

∀x, P(x) → D(x) (we argued that if a process has more than one possible output, then it is not predictable with certainty)

∀x, K(x) → ¬P(x) (we have seen that K-random processes are unpredictable)

∀x, P(x) → ¬K(x) (contraposition, C2)

∃x, ¬K(x) ∧ ¬P(x) (this is reasonable, since if a process is compressible, it is then still possible that running the algorithm to predict is more expensive in term of resources then running the process itself)

The given conceptual map is justified.

Now we can go back to the questions we moved from. Are random processes (i.e. chancy events) Kolmogorov random objects? A question which we rephrased more precisely in: (1) Is \( \overline{\sigma_x} \) K-random for every chancy process \( x \)? To which we added: (2) If not, what is the relationship between the complexity of \( \overline{\sigma_x} \) and \( x \)? The answer to question (1) is no; as we have seen, there is no complete string (a string capturing everything there is to say about the process) associable to an indeterministic process. Question (2) is more interesting. We have seen that if \( \overline{\sigma_x} \) is K-random, then \( x \) is unpredictable. It is not true instead the inverse: unpredictability does not imply K-randomness; but, if a process is unpredictable and translatable into \( \overline{\sigma_x} \) (i.e. it is deterministic), then it is K-random.

When systems get complex in their behavior (mean complex in the technical Kolmogorovian sense), as in biology or in the social sciences, the possibility of describing them with deterministic simple laws decreases, reaching soon incompressibility – or structural disorder –. Nevertheless, disorder is only a particular case of unpredictability.

Another interesting question is: can a deterministic processes generate a K-random string? The answer is of course positive, since programs are deterministic processes, but a further specification is necessary. The processes having the string as output cannot be compressible in their behavior. A process \( x \) giving \( \sigma_1 \) as an output string must be so that its string-translation \( \overline{\sigma_x} \) is K-random;
otherwise there would be the possibility of compressing the string while compressing the string associated to the process which generates it\textsuperscript{33}.

The exposed reflections help to contextualize and understand an opinion of Wolfram, connected with the theme of the previous chapter, the computational universe: “Maybe the universe contains no randomness, maybe everything is actually deterministic, maybe it's only pseudo-randomness!”

Pseudo-randomness is the property of those systems or processes appearing statistically random, being hence able to generate strings passing statistical random tests, but non-random themselves (in the Kolmogorovian sense)\textsuperscript{34}. What Wolfram is saying is that even if some natural phenomena appear random (take quantum mechanics as the principal example), they may ultimately be deterministic, if not ordered in some sense. The interesting thing is that whether a phenomenon – say quantum mechanics – is K-random or probabilistic makes not a practical difference\textsuperscript{35}: they are both unpredictable. If a process is disordered, uncompressible, it appears random to us. The substantial difference is that K-random processes can be simulated, at least in principle (they can in fact be reproduced, if it is not left any detail behind, since every information is here relevant; no schematization is possible), while probabilistic processes are not simulable, even in principle – a simulation of a quantum event cannot be probabilistic unless the source of randomness is external to the simulating system (i.e. unless it is a quantum event itself): no complete “algorithmization” is possible for probabilistic phenomena\textsuperscript{36}.

\textsuperscript{33} Consider also this passage of A. Eagle: “So unpredictability of the generating process is a necessary condition on KML-randomness [Kolmogorov-Martin-Löf randomness] of any concrete outcome sequence. But unpredictability is not sufficient, for it may be that we cannot know everything true of the future outcomes of the process, and those truths might preclude a KML-random sequence. One way to see this draws on our discussion of chaotic dynamics. Let's say that a system exhibits apparent dependence on initial conditions if states indiscriminable to us may end up arbitrarily far apart after some relatively short period of time. (Another definition, if knowledge obeys a margin for error principle, would be: a system exhibits apparent dependence on initial conditions if, for all we know, it is sensitively dependent on initial conditions.)” (Eagle, 2016).

\textsuperscript{34} Pseudo-randomness is defined in cryptography in more precise terms: something (usually a function) is pseudorandom if it can’t be distinguished from something random in polynomial time.

\textsuperscript{35} Every physicist believe that coin flipping is a deterministic process (driven by the laws of mechanics), yet, being unable to compress the information relative to the context where the single flip occur (i.e. all the forces active on the coin every other time) makes such a process undistinguishable from a probabilistic one; even if the single flips are considered deterministic processes, it is as if it had multiple outcomes (two, in fact); the category of processes “coin flipping”, due to the unknown case by case initial conditions, is as if it were probabilistic.

\textsuperscript{36} Eagle (Eagle, 2005) argues later that the unpredictably generated sequences are a better fit to the theoretical role of randomness, and claims on that basis that randomness is unpredictability.
A final remark by Chaitin may close this section, connecting the exposed reflections about unpredictability with the concept of entropy, a must when thinking about predictability. The link is only suggested, as a call for future research:

Let's compare randomness and entropy in physics with lack of structure as defined via program-size complexity. It's just individuals versus ensembles! In statistical physics you have Boltzmann entropy which measures how well probability is distributed over an ensemble of possibilities. It's an ensemble notion. In effect, in AIT I look at the entropy/program-size of individual microstates, not at the ensemble of all possible microstates and the distribution of probability across the phase space. (Chaitin, 1999)\(^{37}\)

4. From Randomness to Metamathematics

4.1. What is Metamathematics?

There are at least three ways in which mathematics can enter into relation with philosophy: (1) by becoming its object of interest; this is the case of the philosophy of mathematics. (2) By being an argumentative or modeling tool for the philosophical discourse (e.g. the notion of function used to express propositions in the philosophy of language). (3) By having a mathematical approach to philosophical questions\(^ {38}\). This third relation is by far the most unexplored of the three.

Metamathematics is an instance of this third type, more precisely the inquiry of mathematics about its own nature. The questions posed by the discipline coincide with those of the philosophy of mathematics, which – also – is interested about the nature of mathematics, but the methodology of inquiry is different: it is mathematical in nature.

Kolmogorov randomness, an expression of algorithmic information theory, a branch at the interception of computer sciences and mathematics, has a lot to say about the structure of

\(^{37}\) Online publication.

\(^{38}\) One may wonder whether philosophy is defined by its method (or methods) and/or by the questions it tries to answer; we do not address this topic. What we want to say here is that there is a mathematical approach to questions traditionally pertinent to the philosophical research and debate, but not confined to it necessarily. Such an approach is something more than a contribution to philosophy by lending a formalized language, notations or concepts, it is philosophical mathematics. Cf. (Chaitin, 2014).
mathematics itself. An informal analysis of this issue will be the topic of the concluding sections, especially 4.4, with emphasis on the question whether mathematics is fundamentally random.

4.2. Proving and Programming

There are so many striking analogies between proving and programming that one may suspect these two are the same activity, or at least that one is a particular case of the other. If the similarities between the two concepts are enough, it may be possible to argue that some property of one, say programming, can be attributed to the proving too. Such a possibility interests us here because it may be possible to apply the concept of complexity to the notion of proof, as we do with programs; hence enabling us to say something about the role of randomness in mathematics.

So, is proving the same as programming? A first observation is that they are both procedures starting from some preliminary conditions (the input for programs, the axioms for proofs) from which, through some precise operations (computation for programs, logical reasoning for proofs) a result is reached (output for programs, theorems for proofs). The structure looks the same, yet a precaution is required. Proofs are less detailed than programs. For a proof to be valid means being understandable by the mathematical community, step by step. Although precision is compulsory – possibility of comprehension being otherwise undermined –, it is not necessary to specify every single logical operation as if the proof were to be understood by a machine. Human comprehension can be even hampered by such an exhaustivity. On the other hand, programs must be read by machines, they must leave no gaps. Mathematicians can intuitively understand the general thread of discourse, filling in the missing information themselves, machines cannot. This being said, one can still think of proofs as potentially specifiable in their details, so that they may be understood by a computer. A general tacit assumption of the mathematical community is in fact that such a pedantic work can always in principle be carried out, for every correct proof.

In the history of mathematics, the emphasis given to conceptual rigor – as opposed to an operationalist view – varied greatly (Mancosu, 2010). The higher the rigor, the more proofs approximate to programs. Operational mathematics prefers working heuristics and practical methods to obtain results to polished theory. We must hence not forget that mathematics is not
always about full formalization, even if most of the operational techniques of the past are now incorporated into systematized theory. Moreover, there are hybrid types of proofs that are increasingly used today; they combine theoretical arguments with calculations. Probabilistic proofs\textsuperscript{39} are an example, experimental proof are one other.

It must also be clarified that correctness in proofs and in programs is not quite the same. As S. Calude, E. Calude and S. Marcus argue:

It makes sense to prove the correctness of an algorithm, but not the correctness of a program… Programs are analogues of mathematical models; they may be more or less adequate to code algorithms. In the analogy between proving and programming, theorems correspond to algorithms not programs; programs correspond to mathematical models. The role of proof in mathematical modelling is very small: adequacy is the main issue! One can re-phrase the arguments against the idea of proof of correctness of programs… as arguments against the idea of proof of correctness of mathematical models.

For example, engineers use theorems by “plugging in” values and relying on some (physical) interpretations of the conclusion. This is what makes planes and bridges stand. (Calude & Calude & Marcus, 2007, page 331)

Nevertheless, mathematicians maintaining that the future of mathematical proving is programming are not missing:

The real work of us mathematicians, from now until, roughly, fifty years from now, when computers won't need us anymore, is to make the transition from human-centric math to machine-centric math as smooth and efficient as possible. (Zeilberger, 1999)\textsuperscript{40}

With the exposed considerations in mind, we can still say that Kolmogorov complexity is applicable to mathematics, making possible a particularly interesting evaluation of the discipline. Going back to the “understanding as compression idea”, if proofs consist in exhaustive (and non-redundant) step by step understandable descriptions of how the conclusions are connected to the premises, a humorous – yet true, I believe – sentence of Chaitin is pertinent: “To me, you understand something only if you can program it. (You, not someone else!) … Programming

\textsuperscript{39}“Probabilistically checkable proofs are mathematical arguments that can be checked probabilistically by reading just a few of their bits. In the early 1990's it was proved that every proof can be actively transformed into a probabilistically checkable proof with only a modest increase in the original proof length”, (Calude & Calude & Marcus, 2007, page 312).

\textsuperscript{40}Online publication.
something forces you to understand it better, it forces you to really understand it, since you are explaining it to a machine”. Where I take ‘can’ to mean ‘being able in principle’, even if you do not know any programming language: being able to draw up an algorithm of what is going on in what you pretend to understand (again, the shorter the algorithm, the deeper the comprehension)” (Chaitin, 2005, page XII).

Before moving to discuss what AIT has to say about mathematics, some further consideration on the conception of the mathematical practice seen as a computational process may be made. Computation is a physical process. Even if computations can be represented as abstract operations, they are embedded in space-time, or somehow depend on the notion of it: even if computation is conceived as an idealized process, it cannot be meaningfully thought of as completely free from spatiotemporal constraints. Computations consist in ordered succession of states, a succession which is intrinsically temporal. Of course there are non-temporal successions, numerical ones are an example; in their case, even if subsequent numbers depend on previous ones, so that a computation is needed to derive a successor from its antecedents, one can imagine the digits as occurring all together, abstractly, with no time development, even if they are recursively defined. Computation is of a different kind; it does not make sense to think of the ordered operations the computation consists in as happening all together. If an order of succession is to be conceived, it must imply time; this holds even if computational processes are abstract objects, a kind of entity usually conceived as timeless. We are not claiming of course that computational objects are affected by time, but rather that they imply the notion of time. Abstract objects are in their case not so abstract, sharing some intrinsic properties with the physical reality.

Letting aside the debate on whether the idealized notion of computation implies time, real computation is a physical process; must we then conclude that if mathematics is computational, then it is bounded to physics? There is an odd circularity going on. Mathematics describes physics while depending on it. Nonetheless, mathematics is a thinking practice, and thought is a kind of computation. A known comment by Rota does not solve the problem:

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41 Is there non computational thinking? Are there non computational physical processes?
One must guard, however, against confusing the presentation of mathematics with the content of mathematics. Proofs have to be written on paper, which means proofs are physical. From this perspective, proofs depend upon the physical universe. (Rota, 1997, page 96)

Yet, when we say that mathematics is based on thought we mean something more fundamental than the substrate (e.g. paper) through which it is represented. Of course even thinking may be conceived as a representational substrate, but the situation is subtler here. If one is not inclined to accept a Platonic realm of ideas, existing beyond the computational process of thought – a view with all its needs for ontological and epistemological justifications – mathematics should be seen as a computational practice occurring in the computational processes our mind (or calculators) consist of. The problem remains: how can mathematics depend on physics, when physics is applied mathematical.

If one puts mathematics first, so that there is mathematics and then, beside, its applicability to the natural world, the fact that the world is mathematically ordered – at least in some of its aspects – appears a miracle (Wigner, 1967). If otherwise physics is put first, so that it describes also the way our mind processes behave, physics determines our notion mathematics, and its application to the physical world itself. Both the conception may coexist: a mathematically ordered reality, determining the way we think about it: mathematically. We would partly comprehend reality, in its computational processes, through the computational process of thinking, itself part of the computational process of reality. No total comprehension of reality can be possible, since no model inside reality can picture it completely. Schematization cannot be avoided; since the model is a part of reality, no bi-univocal connection is possible (unless reality is a fractal). We will address less superficially the relationship between physics and mathematics in the closing section.

4.3. Math as a Quasi-Empirical Science

The use of the expression “quasi-empirical” in relation to mathematics comes from the philosophy of science of Hungarian thinker Imre Lakatos. He used it mean that the practice of mathematics has the methodological structure of an empirical science.
Not because it revives Mill’s idea that the truths of arithmetic are empirical generalizations, but because it ascribes to mathematics the same kind of hypothetico-deductive structure that the empirical sciences supposedly display, with axioms playing the part of theories and their mathematical consequences playing the part of observation-statements (or in Lakatos’s terminology, “potential falsifiers”).

The difference between science and mathematics consists in the differences between the potential falsifiers. (Musgrave & Pigden, 2016)42

Such an idea is not extraneous to Kolmogorov complexity. We can trace the outlines of an AI-theoretic argument for the quasi-empirical nature of mathematics, following some ideas of Chaitin; we will do it in section 4.4. The matter is strictly linked to that of randomness in mathematics. For the moment we want to give some examples of mathematical practices sharing the same structure of the empirical sciences.

First of all, axiomatization. Even in pure mathematics the choice of the axioms is guided by pragmatical reasons. Quite often self-evidence is subordinated to usefulness. Consider the following examples. Cryptosystems, where more often than not the proof of the security of a system is based on unproved statements (i.e. hypotheses) (Borwein & Bailey, 2003). \( P \neq NP \), which is sometimes taken as a working assumption (Chaitin, 2003). There is then highly computational mathematics, as the determination of the digits of \( \pi \). The use of computer-based tools is getting pervasive in mathematics. The tendency to regard the outcomes of computations as important mathematical results is expressed properly be the widely used expression experimental mathematics. J. M. Borwein and D. H. Bailey (J. M. Borwein & D. H. Bailey, 2003) talk about computers as the mathematician’s “laboratory” in which he or she can perform experiments: analyzing examples, testing out new ideas, or searching for patterns”. They identify eight ways in which mathematics can be computationally experimental:

1. gaining insight and intuition; 2. discovering new patterns and relationships; 3. using graphical displays to suggest underlying mathematical principles; 4. testing and especially falsifying conjectures; 5. exploring a possible result to see if it is worth formal proof; 6. suggesting approaches for formal proof; 7. replacing lengthy hand derivations with computer based derivations; 8. confirming analytically derived results. Note that the above activities are,

42 Online publication.
for the most part, quite similar to the role of laboratory experimentation in the physical and biological sciences. (J. M. Borwein & D. H. Bailey, 2003, pages 2-3)

What is interesting about this view of the discipline of mathematics is that it takes a clear position about the question whether mathematics is invented or discovered. It suggests that new mathematical results are discovered, as in any other natural science. What is humorous about such a position is that a Platonist would agree, despite being as far as possible in the belief of how close is mathematics to physical reality. In fact, a mathematical Platonist too would think that new results are discovered and not invented, in virtue of the existence of a region of abstract mathematical entities which is objectively explorable through thought. Ironically enough, we can read a passage written by Gödel (a Platonist!) from an experimental point of view: “If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics” (Gödel, 1995, 313).

4.4. Is Mathematics Fundamentally Random?

There are further reasons to conceive mathematics as a quasi-empirical science, they are connected to the question of the role of randomness in mathematics. Let us see them while pondering the question on the role of randomness in mathematics. They show that it is not a matter of how pervasive the computational tools are in the mathematical research; the reason is even deeper; it has to do with the nature of mathematics itself, randomness in fact.

To understand what we are talking about, a digression is needed.

The halting problem is a problem in computability theory which tries to determine when a given program $p$, with $i$ as an input, stops running. Chaitin uses the notion of halting problem to define a number, which he calls omega, or halting probability, as follow:

$$\Omega = \sum_{p \text{ that halts}} 2^{-|p|}$$

43 Posthumously published.
It is the sum of the ratios between one and two elevated to the length of the binary program \( p \), where \( p \) varies across the halting programs. Expressed simpler, it is the probability that a randomly chosen binary program will halt. It is necessary that \( p \) is a program such that no extension of it is a valid program, if this is the case omega converges to less than one.

Omega has many interesting features, most importantly it illuminates the primary role that complexity plays in mathematics. Imagine to define the numerical value of \( \Omega \), such a value may depend on the particular Turing machine you use, but as Chaitin says “its paradoxical properties remain the same” (Chaitin, 2016)\(^{44}\). You approximate it from the lower bound.

\[
\Omega = .01101\ldots
\]

Omega turns out to be a K-random string, as far as we can tell. The bits constituting omega look as independent flips of a coin. Moreover, it can be proved that it is a transcendental real (Chaitin, 2016)\(^{45}\). This tell us something about the very structure of mathematics: it is random too; even what is called pure mathematics. To quote Chaitin again:

The simplest reason that these bits [those of omega] are what they are is the bits themselves. There is no more concise axiom from which you can deduce this. It is a perfect simulation, in pure mathematics, of contingency. (Chaitin, 2016)\(^{46}\)

And the matter goes even further: omega is uncomputable in the worst way.

The smallest program that calculates the first \( N \) of bits of the base-two expansion of \( \Omega \) will also have \( N \) bits. That is precisely what it means to say \( \Omega \) is irreducible… And the smallest axioms from which you can deduce what those bits are instead of computing them will also have to have \( N \) bits.” (Chaitin, 2016)\(^{47}\)

Omega is both computationally and logically irreducible. This is the fatal blow to Hilbert’s idea of mathematics. Let us remind it. The German mathematician David Hilbert thought that mathematics could be completely reduced to simple basic axioms from which the whole edifice depended. Every true theorem would be deducible from the axiomatic foundation, at least in principle, no matter how complicated it was. This idea was very influential in the mathematical

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community, starting to be around at the turn of the XX century in relation to consistency proof in analysis, to be stated in a programmatic form in 1921 (Zach, 2016), yet already in the thirties it encountered a refutation: the two famous theorems of incompleteness found by the Austrian logician Kurt Gödel showed that there “are limits of provability in formal axiomatized theories” (Raatikainen, 2015)\(^48\). The first theorem of incompleteness – we know it from the first section – states that “in any consistent formal system \(F\) within which a certain amount of arithmetic can be carried out, there are statements of the language of \(F\) which can neither be proved nor disproved in \(F\)” (Raatikainen, 2015)\(^49\). Gödel’s work undermines Hilbert’s program, nevertheless the mathematical community did not embrace the implications completely, Chaitin complains (Chaitin, 2016)\(^50\). Theorem 5 is an AIT version of the first incompleteness theorem, and definitely a simpler version of it. It shows with even more clarity that mathematics is not completely reducible to a set of axioms. Sometimes – as in the case of omega – mathematical truths are not compressible; their complexity makes them non deducible from simpler propositions: they are Kolmogorov random. Gödel’s formulation didn’t make clear how pervasive incompleteness is in mathematics. The original proof is based on a type of uncommon self-referential statement coming from the liar paradox. Chaitin suggests that this may be the reason why the mathematical community, despite the notoriety of the incompleteness theorem, and its impact on the philosophy of mathematics, didn’t change the everyday conception the mathematics have of their activity, continuing to see it in a “Hilbertian spirit or following the Bourbaki tradition” (Chaitin, 2016)\(^51\). The self-referential statements that Gödel uses in his proof may well have been often seen as a logical oddity, making incompleteness a bizarre marginal fact we meet when dealing with those statements, unused in the daily practice of mathematics. The matter changes with the AI-theoretic version of the theorem: you can still continue to carry on your mathematical practice unaffected by Gödel’s self-referential statements, but it is not the same with complexity. As Chaitin puts it ironically: “Logicians hate randomness. That is precisely why they became logician.”, but AIT shows how pervasive incompleteness is in the mathematical activity. It hints at incompleteness as a limit to mathematics, as a limit to thought.

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We are back here to the quasi-empirical nature of mathematics. Let us tell it with Chaitin’s words:

My feeling about all of this is that it is not a sharp divide; it is a continuum. Absolute truth you can only approach asymptotically in the limit from below.

You graph it and there is a beautiful curve, and it is fit beautifully by a very simple equation. What if you cannot prove it? A physicist would publish anyway. But a pure mathematician does not care how much empirical evidence there is, or how accurately this simple formula fits the curve. You need a proof! (Chaitin, 2016)\(^{52}\)

We have mentioned contingency, before. Omega as a simulation of contingency in pure mathematics. This can be connected to Leibniz’s thoughts on the matter.

A contingent truth cannot be proved logically or mathematically; it is accidental, or historical. (Actually, according to Leibniz, the chain of reasons for a contingent truth is infinite and can only be perceived by God). (Chaitin, 2016)\(^{53}\)

Leibniz’s *principle of sufficient reason* states that if something is true, then there must be a reason for that. In mathematics reasons should be proofs; here is the problem. There is no proof for random statements, they are contingent. Only Leibniz’s God is capable of seeing the proofs for contingent mathematical truths.

A main consequence of the present discourse is that no unified theory of mathematics is possible; as in physics the dream of great unification result being a chimera. This bring us to a suggestion: “When mathematicians can't understand something they usually assume that it is their fault, but it may just be that there is no pattern or law to be discovered!” (Chaitin, 1987, page X). We should consider mathematics is an open texture.

Chaitin’s discourse is not exempt from severe criticism. Fallis (Fallis, 1996) dismisses the claim that the AI-theoretic proof the incompleteness theorem adds anything new to the original version, regarding it just as a clever reformulation of the theorem. The philosophical reflection on incompleteness does not gain new profound insights from the reformulation, he claims. Moreover, it does not change the daily practice more than what Gödel’s formulation does, as instead Chaitin maintains. The Finnish logician P. Raatikainen goes even further in

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the criticism. He exhibits counterexamples to the received view on the theorem (Raatikainen, 1998) – namely that the theorem shows that in a formalized theory one cannot prove an object to be more complex than the complexity of the theory itself –, showing that the constant associated to any formal system (the constant representing the complexity limits of what can be proven from the axioms) provided by the theorem depends on the choice of the coding of the Turing machines. Raatikainen’s review (Raatikainen, 2001) of the two books by Chaitin *Exploring Randomness* (Chaitin, 2000) and *The Unknowable* (Chaitin, 1999) insists on the matter.

Consider, for example, an enormously complex finite collection of axioms with the form \( n < n + 1 \); even the simple theory consisting of the single generalization “for all \( x \), \( x < x + 1 \)” can prove more. On the other hand, there exist very simple and compact axiom systems that are sufficient for the development of all known mathematics (e.g., the axioms of set theory) and that can in particular decide many more cases of program-size complexity than some extremely complex but weak axiom systems (such as the one above). (Raatikainen, 2001, page 4)

The view that mathematics is intrinsically random is no spared either:

> The individual bits of \( \Omega \) are 0 or 1 depending on whether certain Turing machines halt or not — that is the reason. It is an objective matter of fact; the truth here is completely determined, and Chaitin’s interpretation of the situation is quite misleading. (Raatikainen, 2001, page 4)

In this review Chaitin is accused of continuing to work on his track, ignoring the criticism from the literature of the field, celebrating AIT as a major paradigm shift, while sweeping “all such problems under the carpet with rather cheap rhetoric” (Raatikainen, 2001, page 4). The way Chaitin in his writings returns over and over on the significance of his formulation of the incompleteness theorem, as the last stage of an intellectual journey which, expanding the discoveries of Gödel and Turing, casts new light on the nature of mathematics, is in fact criticizable. P. Gacs comments on the matter complaining that the argentine thinker is even “rewriting the history of the field”, “presenting himself as the sole inventor of its main concepts and results” (Gacs, 1989, page 1), an opinion which does not appear excessive familiarizing with the texts of Chaitin.
A sensate balancing counterattack comes from Porter (Porter, 2013), which, recognizing the validity of the mentioned objections, proposes to see Chaitin’s theorem as a trade of for the definition of randomness:

On the one hand, if we require a sufficiently strong definition of randomness, then certain desiderata for our definition may have to be sacrificed. On the other hand, if we don't want to sacrifice these desiderata, then we must be willing to accept a weaker definition of randomness, one that counts more strings as random than stronger definitions do.

4.5. Physical Mathematics

In this section it will be argued that the dependency of mathematics on physics is even deeper than what was proposed so far. The boundary between the two disciplines is not well-defined. The conception of physics considered simply as the application of mathematics to the natural world is an idea that requires clarification. The proposed view suggests a logical scheme to conceptualize the physical structure of mathematics; moreover, it has implications on how we should look at the informational content of mathematics. A discourse on the presupposed circularity of the proposed view is also necessary.

Mathematical practice depends on the laws of physics: both computer-aided activities and thought are computational physical processes. The matter is not only ontological here: the fact that mathematics is physical imposes to look at the discipline in a certain perspective, which affects the mathematical practice as well as the philosophical discourse. The claims of Chaitin we presented take some precautions: they affirm that mathematics is quasi-empirical; it is said that mathematics has the same structure of the empirical sciences, not that it is really empirical itself, being really as physics. We want to go forward; suggest that it is fertile to consider mathematics as physical in nature.

We have shown that proving can be seen as a physical process, as a computational activity, but there is something more that can be said about this. Consider for example one of the many proofs of the Pythagorean theorem; they generally rely on some reasoning upon a geometrical
construction. This geometrical thinking is physically grounded in a non-trivial way. I do not mean to say that it is empirical in its origin, as we would say when affirming that ancient geometry originates from the necessity to divide agricultural fields. I maintain that the single logical steps a proof consists of have a truth-preserving function, and that such a function is possible only if those logical steps are themselves truth-preserving physical operations for some set of assertions. What I mean will be clear in a moment.

Consider the act of measuring a length with a meter. One puts the measuring length (the meter) near to the measured length, so that they are close enough and orientated in the same direction; a mark corresponding to the length of the measured object is then made on the meter (sometimes only ideally). This is trivial, but crucial. The length identified by the mark on the meter is as long as the measured length, and it is so in virtue of a physical procedure (putting the meter near and parallel to the object). Such a physical procedure makes it possible to apply the true statements about the length of the object to the length marked on the meter. Such a procedure preserves the truth of some set of sentences, so that if they are true for the measured object they are also true for the measuring one (more precisely: for the part identified by the mark on the meter).

Geometrical constructions consist in similar truth-preserving operations. Note that we are not saying that Pythagoras’s theorem is true in virtue of some measurement; the point is subtler: it is true due to some operations (being them spatial or simply logical) preserving the truth of some statements, so that we can say that something true about the geometrical construction is equivalent to the Pythagorean theorem. Such truth-preservation ultimately depends on physical processes, through the rules of logic, no matter how abstract the reasoning is. The applications of the rules of logic is an occurrence of truth-preserving processes. Let us clarify the terminology.

What does it mean to say that an operation is truth-preserving? How is it that some true statement about the length of the measured length is also true for the length indicated by the mark on the meter? It is so because they are of the same length, of course. But why are they of the same length? Or, better, what is it to say that the operation of measurement preserves the length?

Objects are specified by a set of questions; in the case of physical objects those questions correspond to the specification of their physical quantities: position, velocity, momentum, etc. It does not matter to the present discourse whether there is an ultimate limit to specification (as Heisenberg’s Uncertainty Principle affirms) or not; if such a limit exists it sets the level of a perfect
specification of an object – remember Chaitin: “In medieval terms, H(X) [Chaitin’s notation for complexity] is the minimum number of yes/no decisions that God would have to make to create X” (Chaitin, 2014, page 5) –. Information-theoretically objects may be seen as strings whose bits are “yes” or “no” answers to specifying questions (Rovelli, 1996). Those answers may be expressed in bits, so that strings are associated to objects. The questions are not decision problems stricto sensu, they can be physical interactions with a system in order to acquire information about it (i.e. measurements). For example, to determine the position of an electron we cast a beam of light on it; the specification of the physical quantities related the position are derived through the physical process of interaction between the ray and the particle. If the assertion $p$ is true for the object $x$, it means that a particular physical interaction with $x$ results in knowing the fact that $p$ ($p$ is the answer to a question)$^{54}$.

If the meter preserves the length it is because a set of the bits describing the object whose length is near the meter also apply to the length marked on the meter. If one keeps in mind that the specifying questions are – for objects of the natural world – measurements, hence physical interactions, truth-preservation may be seen under a new light: the physical interactions consisting in some specification of a physical quantity connected with the length marked on the meter are the same (physically the same) of those physical interactions intended to specify the corresponding physical quantities of the object to which we put the meter near to.

The same discourse holds for the abstract rules of natural deduction. If the premises are true, the conclusions are also true. As it is often stressed, it is as if the conclusion were included in the premises. This is so in virtue of the fact that the deducing rules are truth-preserving. Deduction here is comparable to the act of putting the meter near to the measured object; they both are physical processes: computation in the former case, spatial translation in the second (another form of computation). They are both so that a set of questions maintains its answers unvaried, under some physical transformation. Physical transformations identify equivalence classes whose

$^{54}$ In this view it is excluded the possibility that an objects has a certain property regardless of the fact that it is possible to know that the object has that that property. The nature of the objects is defined by their possibility of interactions, so that their properties are defined by what results from the interactions. It is meaningless to talk about a property which is beyond the possibility of interaction, which does not result in some physical aspect of the interaction. The epistemological horizon (i.e. what the possible physical interactions set as a limit of what is possible to detect) does not depend on technology (measuring tools), the possibilities of interaction are defined by the physical nature of reality. The epistemological horizon is not an anthropocentric notion; it depends on what physical systems can register of the other physical systems.
elements give the same answers to the same questions; they can be seen as a kind of salva veritate for sets of physical sentences.

Now, we can “ask” information-theoretical questions to abstract objects as well as to physical ones. If we interact with the physical objects physically (through measurement), we interact with abstract objects computationally (which is again to say physically). We specify the properties they have (which is compiling the strings describing them) making deductions about them; and those deductions are computational operations: physical processes. Take for example the task of sketching the graph of a function; we input specific crucial values and make computations to have an idea of the general trend. The matter is not different for objects more abstract than functions, the interaction is computational in nature. Note that it is not always possible to have mechanical methods to define some features of an abstract object; consider for example integration. Integrals are often solved using intuition, i.e. “seeing” a shape we already know how to solve. But the fact that we do not always dispose of an (efficient) mechanical method to obtain a result does not mean that our trial and error approach, based on experience, is not a computational interaction. The crucial point is this: saying that if something is true for the premises then it is also true for the conclusions is to say that some particular way in which we can interact causally with the premises through computation is the same causal interaction (same in the sense that gives the same results) we have with the conclusions, under some particular process of deduction (also a computational physical process). Deduction is a computational process whose truth-preservation, even in pure mathematics, is not different from that of putting a meter near to an object and asking to the measured object the same questions (related to the length, but not necessarily strictly about the length!) we ask to the part of the meter defined by the mark, and receive the same answers.

How can it be possible to interact physically with an abstract object? Abstract objects are captured through computations; they are patterns in computational processes. Their definition is formal, but such a form is grasped in a scheme of responses to computational questions. The matter here is not what one thinks the ontology of abstract objects is; of course the direction we hint at is that of making the existence of an abstract realm of ideas an unnecessary surplus, yet the possibility to acquire information about abstract entities through computation does not reveal much about their nature; it does not rule out Platonism. It is however just a matter of fact that to know the slope of a particular curve we do computations, applying a formula. Undecidable questions count as non-well posed ones; I mean count, not that they are the same. They have no answer: undecidable
questions do not leave an information hole; everything that there is to specify about an object can be conveyed in the decidable questions askable about the object.

A schematizing formalization of these ideas is as follow:

- \( S \) is some physical structure (It may be thought as an object, an event, a fact, a relationship, etc.; \( S \) is associable to a string, conveying binarily all the answers to the question specifying \( S \)).
- \( Q_i(S) = \sigma_s \) is the set of questions which is possible to ask to \( S \) in order to define its features, which are encoded in the string \( \sigma_s \) (\( i \) varies from 1 to \( n \) -we are assuming here that the possible questions are enumerable).
- \( T(S) = S^1 \) is some transformation of \( S \) into \( S^1 \) (such a transformation is interpreted here always as a physical process).
- The truth-preserving processes (as measurement for length and deduction for truth in general) are those \( T(S) = S^1 \) such that \( \exists k \in i \), \( Q_k(S) = Q_k(S^1) \).

Another equivalent formalization is:

- If \( \gamma_x = \{ P \mid P(x) = \text{true} \} \) is the set of true sentences about \( x \), the truth-preserving processes are those \( T(S) = S^1 \) such that \( \exists t, t^1, t \subseteq S \) (where \( t \) is a substructure of \( S \) - it may be well said that \( s \) is a substring of \( S \) -; it may be some isolated feature of \( S \), as “the length” or “everything correlated with the length”), \( t^1 \subseteq S^1 \), \( \gamma_s \supseteq \gamma_{s^1} \).

Both \( T(S) \) and \( Q_i(S) \) are physical processes. The first is a transformation (it is a process which keeps part of the structure the same – it may preserve or reproduce the structure –) the second is a physical interaction with \( S \), whose function is to acquire some particular information about it. \( T \) preserves – or reproduce – part of the structure of \( S \), so that the physical interactions \( Q_i(S) \) and \( Q_i(S^1) \) (i.e before and after \( T \)) may be similar enough to have the same answer to some set of questions \( k \).

\[
\begin{align*}
\text{S} & \quad \xrightarrow{T} \quad S^1 \\
Q_k(S) & \quad = \quad Q_k(S^1)
\end{align*}
\]
The claim is that the sketched logical schema – physically interpreted – applies to abstract logical reasoning. It is the computational universe view followed to the bottom.

In this view even the syntactical approach à la Bourbaki, which proudly declare itself meaningless, has a meaning: a physical meaning in fact. Those purely syntactical manipulations of symbols, which – they claim, (Bourbaki, 1986) – mathematics consists in, are some physical process of truth-preservation $T$.

It should be clear by now why the mathematical activity cannot be conceptually separated from the physical practice; the boundary between the two disciplines is blurred. Mathematics is a form of very sophisticated physical measurement, and proofs are physical process of a bizarre type. To have a concrete hint on what kind of physical process proofs are let us give an example.

Consider the following Chinese version of the proof of the Pythagorean theorem (figure 3):

![Figure 2.](image1)

*The usual visualization of the squares constructed on the sides of a rectangle.*

![Figure 3.](image2)

*The geometric construction used in a Chinese version of the Pythagorean theorem.*
The construction of the square over the hypotenuse is made so that the triangle lies inside it (figure 3). Such a construction makes the relationship \( c^2 = a^2 + b^2 \) visible. The area of the square over the hypotenuse is equal to the four rectangle triangles plus the little square in the middle.

\[
c^2 = 4 \left( \frac{ab}{2} \right) + (b - a)^2 = a^2 + b^2
\]

Seeing this relationship in the geometrical construction of figure 2 is not possible, yet the relationship is obviously valid also for it. The point is that when one says that the relationships found in figure 3 are general, it is so because they do not depend on that particular geometrical construction, and hence are valid for figure 2 too. It is so because figure 2 and figure 3 are, under some transformation \( T \) (we grasp just intuitively in the daily practice) which keeps many internal relationships – among which the Pythagorean theorem – unchanged, the same.

The proposed view – as we have already commented – must deal with the accusation of stumbling on a circularity: mathematics being at the same time the language of physics and an instance of it. Which come first, physics or mathematics? The point is that, since the practice of mathematics is a physical activity, we understand the physical world through mathematical models, and those models are physical representations of the physical world in our brains (or other substrates such as computers), we physically reproduce the universe through our models as a little diorama reproduces its object. This does not mean that if we think about black holes in scientific terms we have little black holes in our heads, but that our brain captures (if our knowledge is correct) some structural aspect of them, physically reproducing in our mental computational models the net of relationships the physics of a black hole consist in. Hence we understand physics through physics (physical simulations in our brains). Physical models are distinguished from mathematical concepts because we believe there is something “out-there” having the structure of our model-simulations, while mathematical concepts are just some kind of physical occurrence in our brain. There is no circularity: brains represent a portion of the world being themselves a portion of it. Again, no total representation is possible (unless the brain stands to the world as the part of a fractal to a fractal).

There are many implications of this proposed view. Let us just concentrate on a main one; it overturns the widespread idea that mathematics and logic are uninformative. If something is necessary (i.e. true in every possible world) it is by no means informative: it does not decrease the
uncertainty about which possible world we are in. So, since logical conclusions are necessary – it is maintained – they are uninformative. However, if mathematics is physical, in the explained sense, then its theorems are informative indeed. The possibility of deriving them depends on the laws of physics, which are contingent: the theorems do tell us something about the world; they rule out all the possible worlds where they could not be derived, namely those based on different laws of physics determining the derivation procedure. But what is more interesting is that the outlined logical schema can be the starting point to prove the first theorems of the information-theoretical discipline of physical mathematics.

5. Concluding Remarks

The observations of the previous section, “Physical Mathematics”, are quite far from where we started. If the reader has followed the line that the present work tries to trace up to here, he or she would certainly have realized, even if not agreeing with the particular arguments exposed, perhaps, what a knot of meaning Kolmogorov randomness is. It lies at a multidisciplinary intersection, being able to raise the hotly debated question whether mathematics is random at its core and, at the same time, to propose a model for understanding, impacting on physics, illuminating the notion of natural law; and there is more that was left outside the perimeter of the present study.

Consider for example the relation between randomness and consciousness (how reducible is consciousness may be a question addressable in AIT terms), or between randomness and creativity (is creativity so information-theoretically complex to be randomic?), a theme addressed in (Dodig-Crnkovic, 2007), randomness and conceptualization (why is that something which exists made of parts rather than in one single piece?) (Chaitin, 2014). Links with the traditional philosophical debates are not missing, with the notion of freewill/destiny (Meyerstein, 2007), for example, or, as we have seen, with Occam’s razor. Many are the studies addressing and exploring these relationships; more is left to inquire and understand.
6. Appendix

A proposal for further investigations related to section 3.1 is to combine linearly completeness (exhaustivity) and compression, so that the explanatory value of a theory is a function of both, as follows:

\[ f(T) = \alpha E + \beta \frac{1}{C_U(T)} \]

Where \( E \) is a measure of the exhaustivity of \( T \), \( C_U(\sigma) \) a measure of its achieved compression and \( \alpha \) and \( \beta \) are constants chosen by the working scientist, according to how much he is willing to put emphasis on exhaustivity rather than compression, or vice versa, in his evaluation of competing theories. How to measure completeness then?

It is possible to use the Kolmogorov complexity machinery to give a quantitative account of the compression the theory is capable of. Higher the compressibility, lower the complexity (and the randomness). To compute \( C_U(T) \) it is required to translate \( T \) into a string. This is no trivial step; it is topic which must be analyzed carefully in future investigations.

Finally, \( E \), a measure of the exhaustivity of \( T \), is another aspect to be careful about. To measure how wide is the explanatory span of \( T \) (how much wide is the portion of reality it is affirming something about) one cannot compare it with reality, to say that the theory is grasping only a particular aspect of it; the measure of exhaustivity must be a relative notion, when comparing \( T \) with the other competing theories.

References


31. (Hawking, 1975), Hawking, S.W., “Particle Creation by Black Holes”, *Communications in Mathematical Physics*, Vol. 43, 1975, 199-220.


