THEOREM. *If f and g are integral forms of unit determinant, then f and g are integrally equivalent if and only if (1) they have the same number of variables, (2) they represent the same numbers modulo π^{2e+1}, (3) $S(f) = S(g)$, and (4) $d(f)/d(g)$ is a square.*

THE AXIOM OF CHOICE IN QUINE'S NEW FOUNDATIONS FOR MATHEMATICAL LOGIC

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The object of this note is to disprove the axiom of choice in Quine's "New Foundations."[1] As the axiom of choice is provable for finite sets, we obtain the axiom of infinity as a corollary.[2]

There will be references to Rosser's *Logic for Mathematicians*,[3] although the axiom of infinity is assumed there; but it is readily eliminated for our purposes if one replaces Quine's ordered pair by Kuratowski's.

The proof will be by *reductio ad absurdum.*

1.1 V is the universal set, Λ the null set (R 256).

1.2 $SC(a)$ is the set of subsets of a (R 255).

1.3 $SC(V) = V$ (R 256).

1.4 $USC(a)$ is the set of unit subsets of a, $USC^2(a) = USC(USC(a))$ (R 255).

1.5 *Fin* is the set of finite sets (R 417).

2.1 Cardinal numbers are construed as saturated sets of equivalent sets (R 371). $Nc(a)$ is the cardinal number of a; so $a \in Nc(a)$.

2.2 NC is the set of cardinal numbers; $\Lambda \notin NC$.

2.3 $Nc(SC(a)) = Nc(SC(b))$ if $Nc(a) = Nc(b)$ (R 369).

2.4 $Nc(USC(a)) = Nc(USC(b))$ if and only if $Nc(a) = Nc(b)$ (R 368).

2.5 $Nc(SC(V)) = Nc(V)$ (1.3).

2.6 $Nc(SC(USC(a))) = Nc(USC(SC(a)))$ (R 368).

2.7 $Nc(SC(USC(V))) = Nc(USC(V))$ (2.6 and 1.3).

2.8 A cardinal number is finite if it is a subset of *Fin*. FNC is the set of finite cardinal numbers.

2.9 1, 2, 3 are defined as cardinal numbers of sets with one, two, three elements; $1, 2, 3 \in FNC$.

3.1 Definition of the sum of two cardinal numbers m, n: $m + n = Nc(a \cup b)$ if $m = Nc(a)$, $n = Nc(b)$ and $a \cap b = \Lambda$; if there are no such a, b, then $m + n = \Lambda$ (R 373).

3.2 $1 + 1 = 2, 1 + 2 = 3$.

3.3 If n is a finite cardinal number and $n + 1 \neq \Lambda$, then $n + 1$ is a finite cardinal number.

3.4 If m is a finite cardinal number, then there are finite cardinal numbers n, p, q such that either $m = n + n + n$ or $m = p + p + p + 1$ or $m = q + q + q + 2$; either of these three cases excludes the two others.

3.5 The cardinal numbers are well-ordered by the relation "there are sets a, b such that $a \epsilon m, b \epsilon n$ and $a \subseteq b$" (axiom of choice).

3.6 If $n + 1$ is a finite cardinal number, then $n < n + 1$.

4.1 Definition of 2^m for cardinal numbers m: If $m = Nc(USC(a))$, then $2^m = Nc(SC(a))$ (cf. 2.3, 2.4; R 389); if there is no set a such that $USC(a) \epsilon m$, then $2^m = \Lambda$.

4.2 $2^{Nc(USC)(a))} = Nc(SC(a))$.

4.3 $2^{Nc(USC(V))} = Nc(V)$ (4.2 and 2.5).

4.4 $2^{Nc(USC^2(V))} = Nc(USC(V))$ (4.2 and 2.7).

4.5 $2^m = \Lambda$ if and only if $Nc(USC(V)) < m$.

4.6 If $2^m \neq \Lambda$, then $m < 2^m$ (R 390).

4.7 $Nc(USC(V)) < Nc(V)$ (4.3 and 4.6).

4.8 If $m \leq n$ and $2^n \neq \Lambda$, then $2^m \neq \Lambda$ and $2^m \leq 2^n$.

4.9 "$2^m = n$" is stratified if "m" and "n" have the same type.

5.1 Definition of $T(m)$ for cardinal numbers m: $T(Nc(a)) = Nc(USC(a))$ (cf. 2.4).

5.2 $T(1) = 1, T(2) = 2, T(Nc(V)) = Nc(USC(V)), T(Nc(USC(V))) = Nc(USC^2(V))$.

5.3 If $m, n, m + n$ are cardinal numbers, then $T(m + n) = T(m) + T(n)$.

5.4 If m is a finite cardinal number, then $m \neq T(m) + 1$ and $m \neq T(m) + 2$ (3.4, 5.2, 5.3).

5.5 If m, n are cardinal numbers, then $m \leq n$ if and only if $T(m) \leq T(n)$ (2.4).

5.6 If $m \leq T(n)$, then there is a p such that $m = T(p)$

5.7 If $m \leq Nc(USC(V))$, then there is a p such that $m = T(p)$.

5.8 $2^{T(m)} \neq \Lambda$ (4 5)

5.9 If $2^m \neq \Lambda$, then $T(2^m) = 2^{T(m)}$ (2.6 and 4.2).

6.1 If m is a cardinal number, then $\Phi(m)$ is the set of cardinal numbers $m, 2^m, 2^{2^m}, \ldots$. To formalize this, define $Q(m, n)$ for $m, n \epsilon NC$ and $2^m = n$; $\phi(m) = \text{Clos}(\{m\}, Q)$ (R 245; stratification 4.9).

6.2 If $2^m = \Lambda$, then $\Phi(m) = \{m\}$.

6.3 $\phi(Nc(V)) = \{Nc(V)\}, Nc(\phi(Nc(V))) = 1$.

6.4 If $n \epsilon \phi(m)$, then $m \leq n$.

6.5 If $2^m \neq \Lambda$, then $m \notin \phi(2^m)$ (4.6).

6.6 If $2^m \neq \Lambda$, then $\phi(m) = \{m\} \cup \Phi(2^m)$.

Proof: (1) $\phi(m) \subseteq \{m\} \cup \phi(2^m)$: $m \epsilon \{m\} \cup \phi(2^m)$; assume $n \epsilon \{m\} \cup \phi(2^m)$ and $2^n \neq \Lambda$; if $n = m$, then $2^n \epsilon \phi(2^m)$; if $n \epsilon \phi(2^m)$, then $2^n \epsilon \phi(2^m)$.

(2) $\phi(2^m) \subseteq \phi(m) - \{m\}$: by 4.6, $2^m \epsilon\, \phi(m) - \{m\}$; assume $n \epsilon\, \phi(m) - \{m\}$ and $2^n \neq \Lambda$. So $2^n \epsilon\, \phi(m)$; by 6.4, $m \leq n$; by 4.6 and 4.8, $m < 2^m \leq 2^n$, so $2^n \epsilon\, \phi(m) - \{m\}$.

6.7 If $2^m \neq \Lambda$, then $Nc(\phi(m)) = Nc(\phi(2^m)) + 1$ (6.5 and 6.6).

6.8 If $2^m = \Lambda$, then $Nc(\phi(T(m))) = 2$ or 3.

Proof: By the hypothesis and 4.5, $Nc(USC(V)) < m$; so by 5.2, 5.5, $T(m) \geq T(Nc(USC(V))) = Nc(USC^2(V))$. So by 4.8 and 4.4, $2^{T(m)} \geq 2^{Nc(USC^2(V))} = Nc(USC(V))$. If $2^{T(m)} > Nc(USC(V))$, then $\phi(T(m)) = \{T(m), 2^{T(m)}\}$. If $2^{T(m)} = Nc(USC(V))$, then by 4.3 $2^{Nc(USC(V))} = Nc(V)$ and $\phi(T(m)) = \{T(m), Nc(USC(V)), Nc(V)\}$.

7.1 If $\phi(T(m))$ is finite, so is $\phi(m)$.

Proof by induction on $Nc(\phi T(m))$: If $2^m = \Lambda$, then $\phi(m) = \{m\}$. If $2^m \neq \Lambda$, then by 5.8, 5.9, 6.7, $Nc(\phi(T(m))) = Nc(\phi(2^{T(m)})) + 1 = Nc(\phi(T(2^m))) + 1$; so by 3.6, $Nc(\phi(T(2^m))) < Nc(\phi(T(m)))$. If $\phi(2^m)$ is finite, so is $\phi(m)$ by 6.7 and 3.3.

7.2 If $\phi(m)$ is finite, so is $\phi(T(m))$ and $Nc(\phi(T(m))) = T(Nc(\phi(m))) + k$, where $k = 1$ or $k = 2$.

Proof by induction on $Nc(\phi(m))$: (We have achieved stratification by introducing a "T" on the right-hand side.) Assume $Nc(\phi(m)) = 1$; so $\phi(m) = \{m\}$, $2^m = \Lambda$; so by 6.8, $Nc(\phi(T(m))) = 2$ or 3, by 5.2, $2 = T(1) + 1$, $3 = T(1) + 2$. Assume $Nc(\phi(m)) > 1$; so $2^m \neq \Lambda$ and $Nc(\phi(m)) = Nc(\phi(2^m)) + 1$. By 3.6, $Nc(\phi(2^m)) < Nc(\phi(m))$; by 5.2, 5.3, and 6.6, $Nc(\phi(T(m))) = Nc(\phi(2^{T(m)})) + 1 = Nc(\phi(T(2^m))) + 1 = T(Nc(\phi(2^m))) + k + 1 = T(Nc(\phi(2^m)) + 1) + k = T(Nc(\phi(m))) + k$, where $k = 1$ or 2.

7.3 There is a cardinal number m such that $\phi(m)$ is finite and $T(m) = m$.

Proof: Let c be the set of cardinal numbers n such that $\phi(n)$ is finite. By 6.3, c is not the null set. Let m be the smallest cardinal number in c; so $\phi(m)$ is finite. By 7.2, $\phi(T(m))$ is finite, so $m \leq T(m)$. By 5.6, there is a cardinal number p such that $m = T(p)$; $T(p) \leq T(T(p))$ and by 5.5, $p \leq T(p)$. By 7.1, $\phi(p)$ is finite, so $p = T(p)$, $m = T(m)$.

7.4 There is a finite cardinal number n such that $n = T(n) + 1$ or $n = T(n) + 2$.

Proof: Choose m such that $\phi(m)$ is finite and $T(m) = m$ and let $n = Nc(\phi(m))$. By 7.2, $n = Nc(\phi(T(m))) = T(Nc(\phi(m))) + k = T(n) + k$, where $k = 1$ or 2.

7.5 Contradiction: 5.4 and 7.4.[4]

8.1 Generalized continuum hypothesis in "New Foundations": If m, 2^m, n are cardinal numbers, m not finite and $m \leq n \leq 2^m$, then either $m = n$ or $n = 2^m$. The generalized continuum hypothesis does not hold in "NF."[5] The proof is by proving the theorem of Lindenbaum and Tarski in "NF" according to which the axiom of choice is a consequence of the generalized continuum hypothesis.

[1] Quine, W. V., "New Foundations for Mathematical Logic," *Am. Math. Monthly*, **44**, 70–80 (1937).

[2] The axiom of infinity has been proved in a paper submitted to the *Journal of Symbolic Logic;* the constant use of cardinal number in the present note goes back to the referee of that paper.

[3] Rosser, J. B., *Logic for Mathematicians*, McGraw-Hill Book Company, Inc., New York, 1953. References to this book will be of the form "(R 256)," where the number indicates the page.

[4] By a slight modification, we can prove the following theorem (without the axiom of choice): If $m = Nc(a) = Nc(SC(a))$, then there is a cardinal number n such that neither $n \leq T(m)$ nor $T(m) \leq n$. A finite set is therefore not equivalent to its power set; this has been proved in the paper mentioned in reference 2.

[5] This has been proved in the paper mentioned in reference 2 with the axiom of choice.

NUCLEAR REACTIONS WITH ENERGETIC NITROGEN IONS

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1. *Introduction.*—The possibility of obtaining information concerning the nucleus with accelerated heavy ions was discussed recently by Breit, Hull, and Gluckstern.[1] Several sources of energetic heavy nuclei have been reported. Sextuply-charged carbon ions have been accelerated in the 60-inch cyclotron[2] at Berkeley and in the 170-inch synchrocyclotron[3] at the University of Chicago to energies of 120 Mev and 1000 Mev, respectively. The Berkeley group has reported several reactions produced by carbon ions, including some reactions resulting in the production of californium.[4] Recently, Miller[5] investigated nuclear reactions occurring in emulsions exposed to the external carbon beam of the 60-inch cyclotron. Heavy nuclei have been observed in nuclear emulsions exposed to cosmic rays at high altitudes, but, due to the high energies and low fluxes of these cosmic-ray particles,[6] they do not lend themselves conveniently to a study of nuclear properties at low energies.

The experiments reported here utilized triply-charged nitrogen ions accelerated in the ORNL 63-inch cyclotron to an $H\rho$ corresponding to an energy of approximately 25 Mev. They represent an extension of the activation experiments of Wyly and Zucker[7] in which radioactive products from nitrogen bombarded targets are detected and identified mainly from their half-lives. The choice of nitrogen as the accelerated particle was based on the following considerations: (1) the magnetic field available to the resources of the program was limited to an $H\rho$ of 3.75×10^5 oersted-inches; in order that the accelerated particles penetrate the coulomb