The Conditional in Three-Valued Logic

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Abstract

By and large, the conditional connective in three-valued logic has two different functions. First, by means of a deduction theorem, it can express a specific relation of logical consequence in the logical language itself. Second, it can represent natural language structures such as "if/then" or "implies". This chapter surveys both approaches, shows why none of them will typically end up with a three-valued material conditional, and elaborates on connections to probabilistic reasoning.

1 Introduction

I open this chapter with a catchy quote by the relevance logicians Alan Ross Anderson and Nuel D. Belnap:

> Although there are many candidates for logical connectives, such as conjunction, disjunction, negation, quantifiers, and for some writers even identity of individuals, we take the heart of logic to lie in the notion "if ... then ... " [...]. (Anderson and Belnap 1975, p. 3)

In this quote, Anderson and Belnap reverse the traditional relationship between logical connectives. Most logic textbooks consider the Boolean connectives $\neg$, $\wedge$ and $\vee$ to be fundamental. The conditional connective $A \to B$ appears later and is typically identified with the logical disjunction $\neg A \vee B$, i.e., the material conditional. In other words, it has secondary importance. Anderson and Belnap, however, take the conditional connective to be the *primary* connective, thanks to its privileged relationship to logical consequence: it expresses what follows from what, and how we make suppositions and conditional inferences. This is indispensable for our understanding of valid reasoning.

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Let me give two examples. First, both Hilbert-style calculi and (Fitch-style) natural deduction are based on Modus Ponens as the main inference rule: $A \to B, A \vdash B$. If there is a specific type of connection between A and B, we can infer B from A. The conditional connective is essential for making these inferences work. Second, useful meta-inference rules such as Conditional Proof are based on the logical properties of the conditional connective. For example, to show that Transitivity holds in a given logic (i.e., $A \to B$, $B \to C \vdash A \to C$), we typically assume that A is true, and then derive from the premises that C must be true, too. But this suffices as a proof only if we have a rule for *introducing* the conditional. Conditional Proof, i.e., the meta-theorem that $\Gamma, A \vdash B$ implies $\Gamma \vdash A \to B$, does the job.

Both Modus Ponens and Conditional Proof enjoy strong support among logicians and they have an important role in our reasoning and proof practices. When we take them together, we obtain the

Deduction Theorem For any set of formulas Γ and formulas A, B: $\Gamma, A \models B$ if and only if $\Gamma \models A \to B$.

(I am switching back to the semantic formulation here.) When a conditional connective $\to$ satisfies the deduction theorem, the notion of valid inference is mirrored on the level of the logical connectives. That is, the connective $\to$ "internalizes" the relation of logical consequence $\models$ into the language itself.

These introductory paragraphs were supposed to counter a prejudice that philosophy freshmen (including my past self) may have after following Logic 101. In simple propositional and predicate logic, the standard conditional $A \to B$ is simply the disjunction $\neg A \vee B$. The truth conditions of the conditional connective therefore seem to be subordinate to the truth conditions of the Boolean connectives and to not have independent significance. This view is mistaken: logicians have, at all times, identified the behavior of the conditional connective as a central question of logic.

This chapter tries to explore the role of the conditional in three-valued logics, i.e., logics which are characterized by having a third semantic values in addition to the classical values of "true" and "false". The interpretation of this third value differs according to the intended application: it can be "still possible" (e.g., in quantum mechanics, or in evaluation of future contingent sentences), "unknown" (in epistemic interpretations), "undecidable" (in intuitionistic interpretations), "nonassertive" or "void" (when evaluating conditional assertions), "both true and false" (in dialethical interpretations), or "inconsistent" (in computer science applications). It can also be defined negatively as "neither true nor false" or as a "truth-value gap". In this chapter, I will denote the classical values with the numbers of 1 and 0 and the third value with $1/2$. The chapters by Graham Priest on interpretations of the third value, and by John Cantwell on metaphysical indeterminacy, survey these options in greater detail.

Simplifying a bit, the conditional connective can have two basic functions in the numerous applications of three-valued logic. First, there is the *external* function of modeling natural language discourse, or a specific structure in reasoning with a conditional component, such as "if A, then B", "A implies B", or restricted quantification. I call this the external function because the adequacy of the truth conditions of the conditional is mainly evaluated as a function of how well it describes the target system it is supposed to model.

Many three-valued logics that wish to have such a conditional connective without the paradoxes of material implication (especially without the inference $\neg A \models A \to B$) go paraconsistent. This means that they will typically not have a single designated value, i.e., $D \neq \{1\}$. Moreover, the conditional connective, denoted by '$\to$', will differ from $A \supset B := \neg A \vee B$. Specific three-valued systems aiming at the natural language indicative conditional are de Finetti (1936), Cooper (1968), Belnap (1970, 1973), McDermott (1996), Cantwell (2008), and Égré, Rossi, and Sprenger (2021a, 2023b, forthcoming). Some of these logics satisfy a deduction theorem, but not all of them do.

Second, there is the *internal* function of the conditional connective: its truth conditions should mirror the valid inferences of a particular (three-valued) logic by means of a deduction theorem. Often, the essential features of the logic are determined by considerations not pertaining to the conditional, but to the intended application. The question is then how we should choose a conditional connective that mirrors logical consequence, and for the rest validates desirable and blocks undesirable inference patterns involving the conditional (e.g., Modus Ponens: $A, A \to B \models B$, or the Law of Identity: $\models A \to A$). In classical logic with bivalent valuations, the conditional that internalizes truth preservation is the material conditional. This changes when we move to non-classical logic and three-valued valuations. A classical representative of this approach is Asenjo and Tamburino's (1975) "Logic of Antinomies", but we may also cite Łukasiewicz's logic L_3 (1920; 1930; 1951).

This second approach, which relates a conditional connective closely to logical consequence, is of particular interest for proof-theoretic systems. It is quite common in mathematics and computer science, while it is less popular among philosophers. In practice, both types of considerations can also be mixed: the adequacy of a conditional connective can be evaluated both by means of its faithfulness to the modeling target *and* by its formal properties related to the consequence relation. The three-valued relevance logic RM_3, for example, is motivated by explicating what we mean by the predicate "implies", but in doing so, it needs to be connected tightly to the notion of (relevant) logical consequence. In other words, it has both "internal" and "external" traits. I will therefore devote a separate section to this logic.

The chapter is structured as follows: In Section 2, I compare the conditional in classical and three-valued logic. Section 3 surveys the interaction between the conditional and logical consequence in some well-known three-valued logics. Section 4 describes three-valued semantics for the natural language indicative conditional. Section 5 elaborates on that theme and studies the relationship between the indicative and the material conditional in three-valued logic. Section 6 surveys three-valued semantics for probabilistic reasoning with a non-monotonic conditional, and Section 7 studies three-valued semantics for relevance logics. Section 8 concludes.

2 Classical and Three-Valued Logic

In the remainder, I work with a simple propositional language $\mathcal{L}$ whose sentential variables are denoted by uppercase Roman letters. Let us get back for a moment to classical logic: its conditional connective is the material conditional $\supset$ although it is not obvious that $A \supset B$ is a good explication of "A implies B". Indeed, the famous paradoxes of material implication, $\neg A \models A \supset B$ and $B \models A \supset B$, show that its truth conditions do not square well with our intuitive use of "A implies B" or "if A, then B". Indeed, why should the falsity of the antecedent suffice for the truth of "A implies B"? First-year logic instructors know how difficult it is to convince students that the material conditional should be the conditional connective that we use in ordinary reasoning! Psychological experiments such as the Wason Selection Task demonstrate that this is not only anecdotal evidence, but represents a persistent phenomenon in conditional reasoning.[1]

Unfortunately, freshmen have not followed enough logic classes in order to appreciate the many virtues of the material conditional: it satisfies Modus Ponens and is the only truth-functional connective that represents truth preservation in the propositional language of classical logic. That is, whenever $\Gamma, A \models B$, we also have $\Gamma \models A \supset B$ and vice versa. This is extremely useful since it allows us to reduce any proof of a logical implication to the proof of a logical theorem, and to "introduce" the conditional in proofs of conditionals. Such connectives are therefore called *proper implications* by Arieli, Avron, and Zamansky (2011a,b).

[1] In the Wason Selection Task, participants are faced with a set of cards with a number on one side, and a letter printed on the other side. Then they are asked (e.g.) "which cards do you have to turn in order to test the hypothesis that every card with a vowel has an odd number on the other side?" While participants usually respond correctly that cards with a vowel need to be turned, they rarely respond that also cards with an even number need to be turned in order to rule out counterexamples to the hypothesis.

In classical propositional logic, the choice of the consequence relation is simple since there is no ambiguity about valid inference: true premises must yield true conclusions, or in other words, the semantic value 1 should be preserved. Since both the consequence relation and the truth tables for the Boolean connectives are canonical, it is clear that the conditional must adapt itself to what the logic requires (i.e., an internalization of logical consequence into the language). The paradoxes of material implication in classical logic, such as $\neg A \models (A \supset B)$, $B \models (A \supset B)$ and $A \models (B \supset B)$ are, on that view, not really paradoxical: the only way $A \supset B$ ("if A, then B") can be false is if A is true and B is false. Hence the above inferences just assert that if A is false or B is true, $A \supset B$ cannot be false. Moreover, $B \supset B$ can never be false, regardless of the premises. Once we fully understand that valid inference is about truth preservation, the paradoxes cease to be a problem. This "official view" (Anderson and Belnap 1975) sounds eminently reasonable, even if it is not that successful at convincing people who want a conditional connective with more demanding truth conditions—such as most freshmen in logic courses.

Of course, there are alternatives to the material conditional in bivalent logic, for example the Stalnaker-Lewis conditionals with their modal semantics (Stalnaker 1968; Lewis 1973). But there is a crucial advantage in going three-valued: truth-functional conditional connectives are not exhausted by the material conditional. Specifically, we can define conditional connectives that are proper implications without validating the paradoxes of material implication. Moreover, defining a conditional connective in a three-valued setting offers more choices than generalizing the usual Boolean connectives to three truth values. Let me elaborate.

The truth tables of the Boolean connectives are often determined by adequacy conditions mirroring the truth conditions of "and", "not", and "false" in natural language. For example, for negation and conjunction it is natural to assume:

- $\neg A$ is true if and only if A is false; and it is false if and only if A is true;
- $A \wedge B$ is true if and only if A is true and B is true;
- $A \wedge B$ is false if and only if A is false or B is false.

For a rigorous definition of conjunction in the context of three-valued logic, see Ciucci and Dubois (2013). Analogously, we have for disjunction:

- $A \vee B$ is true if and only if A is true or B is true;
- $A \vee B$ is false if and only if A and B are false;

When we identify "true" with semantic value 1, and "false" with semantic value 0, these requirements imply in the context of three-valued logic the so-called Strong Kleene truth tables for the Boolean connectives:

	$\neg$
1	0
1/2	1/2
0	1

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\supset$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

Table 1: Strong Kleene truth tables for negation, conjunction, disjunction, and material implication.

Alternative proposals, such as the Weak Kleene truth tables (Bochvar 1937; Halldén 1949), which assign value 1/2 to any formula where a component is valued 1/2, violate the above constraints. For example, in the case of Weak Kleene, $0 \wedge 1/2 = 1/2$ and $1 \vee 1/2 = 1/2$. For this reason, the Strong Kleene truth tables are perhaps the most widespread truth tables for the Boolean connectives in three-valued logic. They can also be represented as the functions $A \wedge B = \min(A, B)$ and $A \vee B = \max(A, B)$ and therefore they naturally match the *meet* and *join* operators in algebraic representation of three-valued logics (i.e., as Kleene algebras, a subclass of de Morgan algebras).

There is much less agreement on the conditional connective $\rightarrow$. This is because its semantics can be determined on the basis of two types of criteria: (1) by adequacy conditions relating to our use of "if A, then B" or "A implies B"; (2) by motivating, on independent grounds, a logical consequence relation, and by choosing the semantics in a way that the connective *represents logical consequence* in the language itself. This means in particular that $A \rightarrow B$ should be a theorem if and only if A logically implies B, as mentioned in the introduction.

We now consider the behavior of the material conditional $\supset$ defined by the Strong Kleene tables according to criterion (2). (The relationship between material and natural language conditionals will be discussed in Section 5.)

Fact 1. *The material conditional $\supset$ of the Strong Kleene Truth tables is no proper implication for a relation of logical consequence defined as preservation of semantic values $D = \{1\}$ or $D = \{1, 1/2\}$. Specifically:*

- *For $D = \{1\}$, Modus Ponens holds, but Conditional Proof fails.*
- *For $D = \{1, 1/2\}$, Modus Ponens fails, but Conditional Proof holds.*

On the other hand, the Strong Kleene material conditional satisfies the deduction theorem for *mixed consequence relations* where we apply different standards of truth to the premises and the conclusions. One of them is the "strict-to-tolerant" (ST) consequence investigated in detail by Frankowski (2004) and Cobreros et al. (2012): $\Gamma \models_{ST} B$ if and only if for all Strong Kleene valuations such that $v(A) = 1\ \forall A \in \Gamma$, we have $v(B) \geq 1/2$. In other words, strictly true premises must yield at least tolerantly true conclusions. This consequence relation may seem unintuitive, but Cobreros et al. (2012) show that it is useful for analyzing paradoxes of semantic vagueness and soritical reasoning.

The counterpart of strict-to-tolerant implication is tolerant-to-strict (TS) implication, called q-consequence by Malinowski (1990). Here tolerantly true (=non-false) premises must yield a strictly true conclusion: if $v(A) \geq 1/2$ $\forall A \in \Gamma$, we have $v(B) = 1$. Both mixed consequence relations yield a deduction theorem for the Strong Kleene material conditional:

Fact 2. *The Strong Kleene material conditional $\supset$ is a proper implication for $\models_{ST}$ and $\models_{TS}$.*

Chemla and Égré (2021) show another interesting fact about these consequence relations. Consider the constraint

$$\Gamma, A \to B \models \Delta \qquad \text{if and only if} \qquad (\Gamma \models A, \Delta \text{ and } \Gamma, B \models \Delta)\,. \tag{G}$$

This can be read as expressing that $A \to B$ is not designated if the antecedent is designated and the consequent not designated. Both constraints, the deduction theorem and (G), have first been proposed by Gentzen (1935) and so, Chemla and Égré (2021) call a conditional that satisfies both of them *Gentzen-regular*. They then show that the Strong Kleene material conditional is the *only* Gentzen-regular conditional connective with respect to $\models_{ST}$ and $\models_{TS}$, while there are many more Gentzen-regular conditional operators for the strict-to-strict and tolerant-to-tolerant consequence relations $\models_{SS}$ ($D = \{1\}$) and $\models_{\mathsf{TT}}$ ($D = \{1, 1/2\}$). The available options are shown below in Table 3.

It should be remarked that the mixed consequence relations $\models_{ST}$ and $\models_{TS}$ are not Tarskian: they fail either reflexivity or transitivity. $\models_{TS}$ fails reflexivity ($A \models A$) while $\models_{ST}$ fails transitivity (from $\Gamma, A \models B$ and $\Gamma, B \models C$ it follows that $\Gamma, A \models C$). This is not necessarily an argument against them: for example, Cobreros et al. (2012) argue that the failure of transitivity is important for understanding what goes on in soritical reasoning. But since most logicians and philosophers are primarily interested in Tarskian consequence relations, this chapter will mainly focus on $\models_{SS}$, $\models_{\mathsf{TT}}$ and their intersection, which are all Tarskian. By Fact 1, we know that the Strong Kleene material conditional is not Gentzen-regular for these logics. Hence, logicians have looked for alternative conditional connectives that could internalize these consequence relations. The next section surveys some of these attempts.[2]

3 Some Three-Valued Logics and Their Conditionals

A classic three-valued logic is the Strong Kleene logic K_3, developed by Stephen Cole Kleene (1938) in a paper on computability theory. Kleene was interested in a logic for combining the truth values of relations and functions which are not defined everywhere. Take, for example, the relation "is a more authentic Neapolitan pizzeria than" in the domain of all New York restaurants. The third truth value is therefore interpreted as "undefined". When we combine partial relations by means of Boolean operators, plausibly they respect the Strong Kleene tables (e.g., the conjunction of a true sentence and "the *Bratwurststube* is a more authentic Neapolitan pizzeria than the *Dragon Palace*" is undefined, see also Spector, this volume). The natural generalization of valid deductive inference to such three-valued valuations is to preserve only the semantic value 1 (i.e., $D = \{1\}$), i.e., *strict* three-valued consequence, and this logic is called K_3. When interpreting the third value as "undefined", as Kleene does, there is indeed no reason why valid reasoning should preserve undefinedness. Kleene also takes the material conditional $A \supset B = \neg A \vee B$, defined according to the Strong Kleene truth tables, to be the intended conditional of his logic (Kleene 1952, §64).

Kleene's logic K_3 has a large number of valid inferences, and Modus Ponens is among them. However, unless one adds logical constants such as $\top$ and $\bot$, it has no theorems at all: when we assign value $1/2$ to all propositional atoms, *any* formula takes value $1/2$, too, and so it fails to be a theorem. In particular, the Law of Identity $\models A \to A$—usually seen as an adequacy criterion for a conditional—does not hold for the material conditional of Strong Kleene logic. For the same reason, the deduction theorem fails: we cannot derive $\Gamma \models_{K_3} A \supset B$ from $\Gamma, A \models_{K_3} B$, simply because K_3 has no theorems. (Assume $\Gamma = \emptyset$.) The conditional of K_3 is thus not a conditional which "internalizes" the logical consequence relation in any interesting sense.

Note that K_3 can also be applied to the logical analysis of liar sentences such as "This sentence is false" or "Every Cretan is a liar" (uttered by a Cretan). This is because the Law of Excluded Middle fails in K_3 and so we can consider the liar sentence and its negation—or more precisely, the truth predicate as applied to the liar sentence—as yielding the third truth value. This is the road taken by Kripke (1975) and numerous successor papers. But the best-known attempt to analyze liar sentences with three-valued

[2]For a more detailed characterization of logical consequence in three-valued logic, I recommend, in addition to the cited papers, Chemla, Égré, and Spector (2017).

logic does not give up on the Law of Excluded Middle, but on the Law of Non-Contradiction. This is the road taken by Priest (1979) and his Logic of Paradox *LP*, which was introduced some years earlier by Asenjo (1954, 1966) as a "logic of antinomies", but with less philosophical discussion and elaboration. Priest notes that when we assume that a liar sentence such as "This sentence is false" is true, we can establish by means of valid arguments that it is false, and vice versa. For Priest, such sentences are simply true and false at the same time. Indeed, "both true and false" appears to be a much more natural interpretation for liar sentences than "undefined", "unknown" or "neither true nor false". Giving up the Law of Non-Contradiction, advocated by Priest, is known as *dialetheism*. The position can be motivated by well-known problems of modern logic (e.g., how to model a naive truth predicate, Russell's antinomies which are the set-theoretic version of Liar sentences), but also by the occurrence of dialethical views in ancient and medieval Western philosophy (e.g., the pre-socratics, Chrysippus, and Jean Buridan) as well as in Chinese and Indian schools of philosophy. Specifically, *LP* tries to model how one can give up the Law of Non-Contradiction and still obtain an interesting and insightful theory of truth values and reasoning (for further information, see Priest, Berto, and Weber 2022).

Consequently, the third value, still denoted as $1/2$, is on this approach not a truth-value *gap*, like "neither true nor false", but a truth-value *glut*: "both true and false". This means that the concept of truth preservation is translated as preservation of two designated values $D = \{1, 1/2\}$. The Boolean connectives are interpreted according to the Strong Kleene tables since there is no reason to deviate from the reasoning that motivated them (e.g., the conjunction of a sentence that is true and a sentence that is both true and false should plausibly be both true and false). The logic generated from these premises is the paraconsistent logic *LP*, which satisfies Disjunction Introduction ($A \models A \vee B$), but not Disjunctive Syllogism ($\neg A, A \vee B \models B$).

L_3	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

$K_3^{\rightarrow}$	1	1/2	0
1	1	1/2	0
1/2	1	1	1
0	1	1	1

$LP^{\rightarrow}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1	1	1

Table 2: The truth tables for the conditionals of Łukasiewicz's logic L_3, and for the logics $K_3^{\rightarrow}$ and $LP^{\rightarrow}$ that satisfy a deduction theorem with respect to the consequence relations of K_3 and *LP*. See Asenjo and Tamburino (1975) and Middelburg (2020).

Priest then chooses the Strong Kleene material conditional from Table 1 as the conditional connective of his logic. This choice implies that *LP* fails Modus Ponens, but it satisfies Conditional Proof. Priest's choice may look surprising since there is an alternative that satisfies the full deduction theorem with respect to $D = \{1, 1/2\}$, given in Table 2 and discussed below. Such a conditional would internalize implication of *LP*.

However, Priest has specific reasons for *not* wanting a deduction theorem and for preferring the Strong Kleene material conditional. Suppose we have a conditional connective $\rightarrow$ that satisfies both Modus Ponens and Contraction (i.e., $A \rightarrow (A \rightarrow B) \models_{LP} A \rightarrow B$). For motivating *LP* as a logical analysis of Liar sentences, it is crucial that the language is semantically closed, i.e., that we have the expressive resources to make statements about sentences of the language within the language itself, in particular with the truh predicate. But in such a language, a conditional that satisfies Modus Ponens and Contraction leads to absurd results (Priest 1979, pp. 232-233). Take the Curry sentence

$$T(\text{Curry}) \rightarrow A \tag{Curry}$$

where T denotes the truth predicate. We can then derive from the Tarski truth scheme, Modus Ponens and Contraction:

1:	$T(\text{Curry}) \leftrightarrow (T(\text{Curry}) \rightarrow A)$	Tarski truth scheme for (Curry)
2:	$T(\text{Curry}) \rightarrow A$	Contraction (1)
3:	$T(\text{Curry})$	Modus Ponens (1, 2)
4:	A	Modus Ponens (2, 3)

and so any sentence A is a theorem of the language. Something has to go. For Priest, a restriction to the object language is not an option since a satisfactory analysis of self-referential sentences is a primary philosophical motivation for developing *LP*. Giving up Modus Ponens and the deduction theorem is therefore not unwelcome, but in line with the philosophical motivations of his logical project. And so Priest retains the material conditional of the Strong Kleene truth tables as the conditional of *LP*. While this conditional does not satisfy Modus Ponens, it satisfies Conditional Proof, i.e., the meta-theorem that allows us to pass from $A \models_{LP} B$ to $\models_{LP} A \rightarrow B$. That is, while *LP* does not have a full deduction theorem, it internalizes the logical consequence relation at least partially.

Regardless of the philosophical benefits and drawbacks, from a purely logical point of view it is interesting to ask the following question. Is it possible to add a conditional connective to K_3 and *LP* that satisfies the deduction theorem and internalizes the relation of logical consequence? The answer is yes. In the case of K_3 it is necessary that valuations of sentential variables with $1/2$ do not "propagate" to the implication relation, i.e., that $v(1/2 \rightarrow 1/2) = 1$ for any valuation v—and more specifically that one adopts the truth table in the middle of Table 2. The logic one obtains, let us call it $K_3^{\rightarrow}$ to point out the newly introduced conditional, is the propositional fragment of the Logic of Partial Functions studied by Barringer, Cheng, and Jones (1984). See also the survey articles of Middelburg (2020) and Chemla and Égré (2021).

The same move is possible for LP with a slightly modified conditional of $K_3^{\rightarrow}$, shown in the right part of Table 2: the truth table of the middle row is now aligned with the upper rather than the lower row. This conditional has been proposed by Asenjo and Tamburino (1975) as an enhancement of their "logic of antinomies". The resulting logic, denoted by $LP^{\rightarrow}$, satisfies a deduction theorem with respect to the conditional $\rightarrow$, and has thus like $K_3^{\rightarrow}$ a "proper" implication.

While these logics may, especially in the case of $LP^{\rightarrow}$, not be suitable for all of the original philosophical purposes, they have several desirable properties with an eye to algebraization and extended formal analysis:

(a) They are *contained in classical logic*; i.e., each inference is classically valid: $\models \subseteq \models_{\mathsf{CL}}$.

(b) They have *proper connectives*, i.e., conjunction, disjunction and implication behave as expected with respect to logical consequence:

  ($\wedge$) $\Gamma \models A \wedge B$ if and only if $\Gamma \models A$ and $\Gamma \models B$;

  ($\vee$) $\Gamma, A \vee B \models C$ if and only if $\Gamma, A \models C$ and $\Gamma, B \models C$;

  ($\rightarrow$) $\Gamma, A \models B$ if and only if $\Gamma \models A \rightarrow B$.

Specifically, for $LP^{\rightarrow}$:

(c) $LP^{\rightarrow}$ is a paraconsistent logic that is *weakly maximal relative to classical logic*: for any classical theorem A that is not a theorem of $LP^{\rightarrow}$, if there is a consequence relation $\models' \supseteq \models_{LP^{\rightarrow}}$ such that $\models' A$, then $\models'$ has the same theorems as classical logic.

(d) $LP^{\rightarrow}$ is a *strongly maximal absolute paraconsistent logic*, i.e., there is no paraconsistent propositional logic with the same logical constants and connectives as $LP^{\rightarrow}$ and a stronger notion of logical consequence than $LP^{\rightarrow}$ (in other words, if there were such a logic, it would cease to be paraconsistent).

Logics which satisfy (a), (b), (c) and (d) are defined as "ideal paraconsistent logics" by Arieli, Avron, and Zamansky (2011a). They also show that a paraconsistent logic L is ideal if and only if it has a proper implication and is *negation-contained* in classical logic, i.e., there is a two-valued interpretation $\mathbf{F}$ of the connectives of L such that $\mathbf{F}(\neg)$ is classical negation, and all valid inferences of L are also classically valid with respect to the models given by $\mathbf{F}$. An analogous characterization can be given for $K_3^{\rightarrow}$, when we replace "paraconsistent" by "paracomplete", i.e., the property that there is a formula A such that $\not\models A \vee \neg A$. However, it is not known whether $K_3^{\rightarrow}$ satisfies the analogue of condition (d) (Middelburg 2020).

We can systematize these observations beyond K_3, LP and their versions with a proper implication, and make some general observations on when a three-valued conditional connective internalizes a consequence relation. I focus on paraconsistent logics, i.e., logics which do not satisfy the explosion principle $A \wedge \neg A \models B$. Three-valued logics with two designated values are typically paraconsistent as long as $\neg 1/2 = 1/2$ and $1/2 \wedge 1/2 = 1/2$—natural conditions that most three-valued logics satisfy. The reason for focusing on paraconsistent logics is that with just one designated value, such as in paracomplete logics like K_3, the choice of a proper conditional is very restricted: the conditional must take a designated value (i.e., semantic value 1) almost everywhere apart from the combinations. See the left part of Table 3.

Fact 3. *If $D = \{1\}$, a three-valued conditional connective $f_{\rightarrow}$ validates the deduction theorem if and only if $f_{\rightarrow}(1, 1/2) = f_{\rightarrow}(1, 0) \leq 1/2$ and $f_{\rightarrow} \equiv 1$ for all other arguments.*

$\rightarrow_{\{1\}}$	1	1/2	0
1	1	$\leq 1/2$	$\leq 1/2$
1/2	1	1	1
0	1	1	1

$\rightarrow_{\{1,1/2\}}$	1	1/2	0
1	$\geq 1/2$	$\geq 1/2$	0
1/2	$\geq 1/2$	$\geq 1/2$	0
0	$\geq 1/2$	$\geq 1/2$	$\geq 1/2$

Table 3: Restrictions on truth tables for conditional connectives $\rightarrow$ that are proper implications (i.e., satisfy a deduction theorem) with respect to preservation of the designated values $D = \{1\}$, and with respect to preserving $D = \{1, 1/2\}$.

There are thus only four options for conditionals with a deduction theorem, limiting one's choice considerably. A famous logic which is *not* in that class is Łukasiewicz's logic L_3, intended as a logic for the evaluation of future contingent sentences, such as "Tomorrow there will be a sea-battle". The classic discussion of this sentence is found in Aristotle's *On Interpretation*: shall we evaluate it as true, as false, or as neither of them? After all, it seems that today, the truth value of this sentence is not yet settled. The question has generated a large amount of philosophical literature over the centuries, but Łukasiewicz (1920, 1930, 1951) was the first to formalize it rigorously using three-valued logic, interpreting the third value as "possibility" or "indeterminacy" (see also Malinowski 1993, ch. 2). He takes the conditional and the negation operator as primitive, uses Strong Kleene negation and adopts the truth table at the left of Table 2 for the conditional. The other

Boolean operators are then introduced as

$$A \vee B := (A \to B) \to B \qquad\qquad A \wedge B := \neg(\neg A \vee \neg B).$$

Łukasiewicz pairs these truth tables with a notion of logical consequence as preservation of the designated value $D = \{1\}$ and obtains Modus Ponens, but not Conditional Proof (and therefore, no deduction theorem).

The paraconsistent case, by contrast, is more interesting (see the right part of Table 3). In a short note, Jeffrey (1963) showed that if $D = \{1, 1/2\}$, any conditional with non-false antecedent and false consequent must be false if it is supposed to validate Modus Ponens.

Fact 4. *If $D = \{1, 1/2\}$, a three-valued conditional connective $f_\to$ validates Modus Ponens only if $f_\to(1, 0) = f_\to(1/2, 0) = 0$.*

More generally, the deduction theorem requires that the conditional takes designated value everywhere apart from $v(1 \to 0) = v(1/2 \to 0) = 0$, but it says nothing on *which* designated value it should take, thus leaving a large set of options to philosophical applications. In other words, there is a large number of proper conditional connectives, since the values 1 and 1/2 are interchangeable from the point of view of logical implication. See the right part of Table 3.

Fact 5. *If $D = \{1, 1/2\}$, a three-valued conditional connective $f_\to$ validates the deduction theorem if and only if $f_\to(1/2, 0) = f_\to(1, 0) \notin D$ and $f_\to \in D$ for all other arguments of $f_\to$.*

$\to_{\{1,1/2\}+\mathsf{CL}}$	1	1/2	0
1	1	$\geq 1/2$	0
1/2	$\geq 1/2$	$\geq 1/2$	0
0	1	$\geq 1/2$	1

Table 4: Restrictions on truth tables for a conditional connective $\to$ that (i) is a proper implication with respect to preservation of the designated values $D = \{1, 1/2\}$; (ii) generalizes the material conditional of classical logic.

Arieli, Avron, and Zamansky (2011a) show that if we want a *normal paraconsistent logic*, i.e., a logic with a proper implication that is contained in classical logic, *and* this conditional generalizes the classical material conditional to three-valued logic, we have not more than 16 possible truth tables, shown in Table 4. It is, however, questionable whether a three-valued conditional should by necessity generalize the material conditional. For example, we will see in Section 4 that all three-valued truth tables for a conditional connectives modeling natural language "if/then" consider conditional assertions with false antecedents as "void" and neither true nor false (compare also Belnap 1970). We will also see that some of these conditionals are logically stronger than the material conditional while others are weaker; also the deduction theorem may or may not be satisfied (Égré, Rossi, and Sprenger 2021a). Comprehensive surveys and analyses of three-valued logics with respect to the behavior of the conditional are given in Avron (1991), Middelburg (2020), and Chemla and Égré (2021), while Arieli, Avron, and Zamansky (2011a,b) investigate which paraconsistent logics are "ideal" and show that ideality and normality are co-extensive notions.

4 The Natural Language Conditional

One of the most important varieties of the natural language conditional is the indicative conditional, as in "if Mary passed the exam, she went for drinks". It is controversial whether such conditionals have factual truth conditions and can be treated as expressing propositions (e.g., see the dialogue in Jeffrey and Edgington 1991). Theorists such as Ernest W. Adams (1965, 1975), Dorothy Edgington (1986, 1995, 2009) and David Over and Baratgin (2017) claim that indicative conditionals do not express propositions; at most they have partial truth conditions.

> [...] the term 'true' has no clear ordinary sense as applied to conditionals, particularly to those whose antecedents prove to be false [...]. In view of the foregoing remarks, it seems to us to be a mistake to analyze the logical properties of conditional statements in terms of their truth conditions. (Adams 1965, pp. 169-170)

Zooming in on truth-functional truth conditions, the situation seems to be even worse. The only sensible candidate for a truth-functional analysis of an indicative conditional $A \to B$ in bivalent propositional logic is the material conditional $A \supset B$. This has the unwelcome consequence that the falsity of the antecedent suffices for the truth of the conditional: $\neg A \models A \supset B$ (=one of the paradoxes of material implication). But on which basis could our conditional "if Mary passed the exam, she went for drinks" be called true if Mary failed the exam? The inference from $\neg A$ to $A \to B$ seems plain invalid—not because the proposition expressed by Mary's failure at the exam is *irrelevant* for the conditional, but because it does not provide the right kind of justification for asserting or accepting the conditional.

On the other hand, giving up on truth conditions altogether throws out the baby with the bathwater. It seems that a conditional "if A, then B" has been *verified* if we observe both A and B, and *falsified* if we observe A and $\neg B$. If Mary passed the exam and went for drinks, the conditional seems true, whereas it seems false if she did not go for drinks after passing the exam. Indeed, what else could be required for determining the truth or falsity of the sentence?

If you want to treat conditionals as propositions and assign them truth conditions, there have three principled strategies:

(1) to retain truth-functionality and bivalent valuations, i.e., to defend the material conditional analysis against the above objections (e.g., Lewis 1976; Jackson 1979; Grice 1989);

(2) to give up truth-functionality while

  (2a) retaining bivalent valuations and classical consequence, e.g., in the possible world semantics by Stalnaker (1968) or Lewis (1973);

  (2b) giving up bivalent valuations and adopting a non-classical consequence relation, such as in informational state semantics (e.g., Gillies 2009; Ciardelli 2021; Santorio 2022)

(3) to retain truth-functionality, but to give up bivalent valuations, i.e., to develop truth conditions for the conditional in many-valued logic (typically three-valued logic).

A variant of (3) is to claim that the conditional has only *partial* truth conditions and no truth value, or an *ersatz* truth value such as the conditional probability, when the antecedent is false. See the chapter on conditional probability (Cruz and Over, this volume) for more details.

Strategy (1) is motivated by the claim that perceived differences between the indicative and the material conditional are due to pragmatic, not to semantic factors. However, it has never gained widespread acceptance, neither in the logic nor in the philosophy community—the shortcomings of the material conditional are simply too obvious to be accounted for by pragmatics alone. In addition to the aforementioned paradox of material implication, the *probability* of the material conditional exceeds the corresponding conditional probability: $p(C|A) \leq p(A \supset C)$, with equality only in very special cases. Since the conditional probability is a good measure of the plausibility of an indicative conditional (for theoretical and empirical arguments, see Adams 1965, 1975; Evans and Over 2004), this account does not explain why we find the indicative conditional often more demanding to assert than the material conditional. Indeed, "if Mary passed the exam, she went for drinks" seems less probable than "Mary did not pass the exam or she went for drinks". This is hard or impossible to explain away if their truth conditions are identical. Conversely, an account that links the probability of the conditional to the conditional probability of the consequent given the antecedent can explain this phenomenon.

Strategy (2) comes with a variety of logical tools, such as possible world semantics, premise semantics, information state semantics, dynamic semantics, and falls outside the scope of this volume. Strategy (3) is the research program that I will describe in this section. Note that I only cover *indicative* conditionals in this section since it is consensus that counterfactuals, with their irreducibly modal content, require a richer semantics than truth-functional accounts in three-valued logic can provide.

The main idea of (3) is to represent the truth value of conditionals with false antecedents by a third truth value. "If A, then B" is interpreted as a *conditional assertion*—i.e., as an assertion about B upon the supposition that A is true. Similar to the Adams quote shown above, this view interprets a conditional with a false antecedent as having a non-classical truth value. The first articulation of this view is from the 1930s, by Bruno de Finetti in his essay "La logique de la probabilité":

> C'est ici qu'il paraît indiqué d'introduire une logique spéciale à trois valeurs, comme nous l'avions déjà annoncé : B et A étant deux événements (propositions) quelconques, nous dirons *triévénement* B/A (B subordonné à A), l'entité logique qui est considérée
>
> 1. *vraie* si B et A sont vrais;
> 2. *fausse* si B est faux et A est vrai;
> 3. *nulle* si A est faux
>
> (on n'a pas de distinction entre "non A et B" et "non A et non B", le triévénement ne devant être fonction que de A et $C \wedge A$).[3] (de Finetti 1936, 568, emphasis in original)

The French "nulle" for the value 1/2 can be interpreted as "nonassertive", "void", or "indeterminate", depending on the reader's taste. De Finetti's proposal is represented graphically in Table 5, but it neither evaluates nested conditionals, nor does it specify how the conditional interacts with the Boolean connectives. A central question in three-valued analyses of conditionals is therefore how Table 5 should be completed, and how it should be paired with truth tables for the Boolean connectives, and a logical consequence relation. All proposals, in sharp contrast to relevance logics, assign

[3] In the English translation of R. Angell, the quote goes: "It is here that introduction of a special logic of three values seems indicated, as we have already announced: B and A being any two events (propositions) whatever, we will speak of the *tri-event* B/A (B given A), the logical entity which is considered:

1. *true* if B and A are true;
2. *false* if B is false and A true;
3. *null* if A is false

(one does not distinguish between 'not A and B' and 'not A and not B', the tri-event being only a function of A and $A \wedge B$)."

the third truth value to a conditional with false antecedent, and moreover, they treat a conditional with true antecedent as taking the value of the consequent. Opinions diverge, however, on "void" antecedents (e.g., antecedents containing a conditional). The two main options have been proposed by Bruno de Finetti (1936) and William Cooper (1968), and have been rediscovered several times later (e.g., Belnap 1973; Cantwell 2008). See Table 6.[4]

$A \to B$	1	0
1	1	0
0	1/2	1/2

Table 5: De Finetti's (restricted) truth table for a conditional $A \to B$, with truth values of A at the left and truth values of B on top of the table.

$\to_{DF}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	1/2
0	1/2	1/2	1/2

$\to_{CC}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

Table 6: Truth tables for the de Finetti conditional (left) and the Cooper(-Cantwell) conditional (right).

It is hard to determine on purely axiomatic grounds which of the two truth tables is a more adequate representation of the indicative conditional. It depends a lot on the interaction of the conditional with the other connectives, and the consequence relation we choose. We will be able to appreciate their respective advantages and drawbacks later on. Pairing the de Finetti and the Cooper conditional with Strong Kleene negation is uncontroversial because this choice yields for both conditionals the following schemes usually seen as desirable (Williams 2010; Santorio 2022):

Commutation with Negation $\neg(A \to B)$ and $A \to \neg B$ have the same truth and falsity conditions.

Conditional Excluded Middle (CEM) $(A \to B) \vee (A \to \neg B)$ cannot be false (in particular, it is a logical validity when $D = \{1, 1/2\}$).

Both principles square well with the use of the indicative conditional in hypothetical reasoning. Take Ramsey's famous observation about how we *argue* about the truth value of a conditional $A \to B$: both sides *presuppose* the antecedent A, but disagree on whether this implies commitment to B

[4]A third option, proposed by Farrell (1986), assumes that the middle row reads $(1/2, 1/2, 0)$. For discussion and evaluation of that option, I refer to Égré, Rossi, and Sprenger (2021a).

or rather to $\neg B$ (Ramsey 1929/1990, p. 247). To the degree that we agree with Ramsey's analysis, both Commutation with Negation and Conditional Excluded Middle (which is an immediate consequence) look eminently sensible.

The Strong Kleene connectives for disjunction and conjunction are more controversial. On an abstract level, when we interpret the third truth value as "void", it seems that a conjunction of assertive and void sentences may still assert something, and the Strong Kleene tables do not mirror this intuition (because $1 \wedge 1/2 = 1/2$). Specifically, "partitioning sentences" such as $(A \to B) \wedge (\neg A \to C)$ will always be void or false (Belnap 1973; Bradley 2002, pp. 368-370). However, a sentence such as:

> If the sun shines tomorrow, John goes to the beach; and if it rains,
> he goes to the museum.

seems to be true (with hindsight) if the sun shines tomorrow and John goes to the beach. This intuition is completely lost in Strong Kleene semantics, regardless of whether we use the de Finetti or the Cooper table for the conditional. Even worse, "obvious truths" such as $(A \to A) \wedge (\neg A \to \neg A)$ are always classified as void.

	$f_\neg$
1	0
1/2	1/2
0	1

$\wedge$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	0
0	1	0	0

Table 7: Truth tables for negation, quasi-conjunction and quasi-disjunction as defined by Cooper (1968).

For this reason, various theorists such as Cooper (1968), Belnap (1970), Dubois and Prade (1994), McDermott (1996) and Égré, Rossi, and Sprenger (forthcoming, 2023b) adopt an alternative to Strong Kleene connectives for conjunction and disjunction, represented in Table 7 and called *quasi-conjunction* and *quasi-disjunction*.[5] Indeed, the truth tables of Table 7 evaluate partitioning sentences according to our intuitions, i.e., as capable of being true. Moreover, quasi-disjunction avoids the Linearity principle that $(A \to B) \vee (B \to A)$ cannot be false. This schema was famously criticized by MacColl (1908), who pointed out that neither "if John is red-haired, then John is a doctor", nor "if John is a doctor, then he is red-haired", nor their disjunction seems acceptable in ordinary reasoning. Also Dubois and Prade (1994) and McDermott (1996) point out the advantages of this choice, but do

[5]The truth tables for the material conditional implied by quasi-disjunction has already been proposed by Sobociński (1952), but not in the context of the natural language conditional.

not commit to it without reservation (e.g., McDermott believes that we need both Strong Kleene conjunction and quasi-conjunction to account for the behavior of the natural language conditional). The quasi-connectives do not have the favorable algebraic properties of the Strong Kleene connectives (i.e., they do not behave like the meet and join operators in de Morgan algebras), but in the context of developing a three-valued logic for studying natural language conditionals, these considerations may be secondary to how well the connectives behave with respect to truth conditions and inference. All of the possible combinations of truth tables for the conditional and conjunction satisfy, however, the Law of Import Export in its strongest possible semantic form

Import-Export (semantic version) $A \to (B \to C)$ and $(A \wedge B) \to C$ take the same semantic values, i.e., they have the same truth conditions.

That is, the two formulas are not only logically or materially equivalent, but they really have the same meaning. Although the general validity of Import-Export is debated in the philosophical literature (e.g., Khoo and Mandelkern 2019), it enjoys substantial support both from an empirical and a theoretical perspective (e.g., McGee 1989, p. 489; van Wijnbergen-Huitink, Elqayam, and Over 2015).

The biggest choice regards the logical consequence relation. The two main options are preservation of "strict truth" (i.e., $D = \{1\}$) and preservation of "tolerant truth" (i.e., $D = \{1, 1/2\}$, Cobreros et al. 2012). As it is often the case in three-valued logic, this choice corresponds to a choice between a paracomplete logic (i.e., $\not\models A \vee \neg A$) and a paraconsistent logic (i.e., $A \vee \neg A \not\models B$). The other relevant choices concern the conditional connective (Cooper vs. de Finetti) and the truth tables for the Boolean operators (Strong Kleene vs. quasi-connectives). In total, we obtain eight options, which I write as (Q)DF/SS, (Q)DF/TT, (Q)CC/SS, and (Q)CC/TT (compare Égré, Rossi, and Sprenger 2023a). Q denotes quasi-connectives, DF or CC determines the conditional connective, and SS or TT fixes the logical consequence relation. For example, CC/TT denotes the paraconsistent logic where $D = \{1, 1/2\}$ is preserved, with the Cooper conditional and Strong Kleene Boolean connectives, whereas QCC/TT—the preferred logic of the author of this chapter—differs from CC/TT by interpreting $\wedge$ and $\vee$ as quasi-conjunction and -disjunction.

The choice between these options can be made in various ways. Égré, Rossi, and Sprenger (forthcoming) define a probability function based on three-valued valuations and show an equivalence between probabilisitic and three-valued semantics for logics that preserve certainties (i.e., sentences held with maximal probability). Taking a credence or weight function c on possible worlds that represent different states of affairs. We can then define the probability of a sentence A as the ratio of the weight of possible worlds where A is true, written $c(A_T)$, divided by the weight of possible worlds where A receives *classical* truth value, i.e., $c(A_T) + c(A_F)$:

$$p(A) := \frac{c(A_T)}{c(A_T) + c(A_F)} \qquad \text{if } \max(c(A_T), c(A_F)) > 0. \qquad \text{(Probability)}$$

This approach is the standard generalization of probability to semantics non-classical truth values (for a survey, see Williams 2016). Using the Cooper conditional and the quasi-connectives, Égré, Rossi, and Sprenger then consider the certainty-preserving logic C on such probability functions:

Logic C of Certain Inference $\Gamma \models_{\mathsf{C}} B$ if and only if for all probability functions $p : \mathcal{L}^{\to} \longmapsto [0,1]$ based on QCC-valuations: if $p(A) = 1$ for all $A \in \Gamma$, then also $p(B) = 1$.

They then show that C is equivalent to QCC/TT, or in other words, they give a three-valued semantics for C, greatly simplifying the analysis of valid and invalid inferences.

Trivalent Characterization of C $\Gamma \models_{\mathsf{C}} B$ if and only if $\Gamma \models_{\mathrm{QCC/TT}} B$, i.e., for all QCC-valuations $v : \mathcal{L} \longmapsto \{0, 1/2, 1\}$: $v(A) \geq 1/2 \ \forall \ A \in \Gamma \Rightarrow v(B) \geq 1/2$.

Similarly, if one adopted Strong Kleene connectives instead of the quasi-connectives, one would obtain CC/TT as the logic corresponding to certainty preservation, and if one adopted the De Finetti conditional, one would obtain DF/TT or QDF/TT as the logic of certainty-preserving inference.

These correspondence results suggest that the family of TT-logics has some substantial advantages over the family of SS-logics. Indeed, while the TT-logics preserve certainties, the SS-logics esssentially preserve possibilities (e.g., in single-premise inference, if a premise has probability greater than zero, so has the conclusion). Since preserving certainties is arguably more important in reasoning than preserving possibilities, we have a principled argument for working with TT-logics rather than SS-logics.

It is possible to make the same point in favor of TT-logics by means of considering adequacy conditions on the interaction of a conditional with a logical consequence relation. All of the three following conditions seem to be plausible constraints on how "if/then" should interact with logical consequence (compare Égré, Rossi, and Sprenger 2021a):

Law of Identity The conditional "if A, then A", should be a validity of the logic: $\models A \to A$.

	$\models A \to A$	Modus Ponens	$\to \not\models \leftarrow$
DF/SS and QDF/SS	×	✓	×
DF/TT and QDF/TT	✓	×	✓
CC/SS and QCC/SS	×	✓	✓
CC/TT and QCC/TT	✓	✓	✓
Ideal case	✓	✓	✓

Table 8: The behavior of the various three-valued logics for the indicative conditional with respect to the Law of Identity, Modus Ponens and implication to the converse conditional.

Modus Ponens From $A \to B$ and A we should be able to infer B, that is: $A \to B, A \models B$.

No Inference to the Converse From the conditional "if A, then B", we should *not* be able to infer the converse, that is: $A \to B \not\models B \to A$.

The Law of Identity is invalid in all SS-logics, while the TT-logics validate it. Modus Ponens is valid in DF/SS, CC/SS and CC/TT (and their variants with the quasi-connectives), but invalid in DF/TT. Finally, all SS-logics validate the (undesirable) implication to the converse conditional, while the TT-logics avoid it. See Table 8.

Taken together, these observations make a strong case that a three-valued connective aiming at the natural language indicative conditional should be paired with a TT- rather than an SS-consequence relation—also because the author is aware of no substantive drawbacks to this choice. (The mixed case of a SS∩TT-consequence relation will be discussed in Section 6.) If we accept this argument, then it is also compelling to prefer the Cooper truth table for the conditional to the de Finetti truth table, because TT-logics validate Modus Ponens only if paired with the Cooper conditional.[6]

In fact, CC/TT and QCC/TT are the only logics that satisfy all three conditions, i.e., they validate Modus Ponens and the Law of Identity without licensing the implication to the converse. In particular, the CC-logics satisfy the deduction theorem with respect to the Cooper conditional, or in other words, the Cooper conditional is a proper implication with respect to $\models_{\mathsf{TT}}$. I close this section by surveying some of its properties in the variant QCC/TT, i.e., the logic C that we have encountered above. C is a paraconsistent logic almost equivalent to Cooper's—his propositional logic of Ordinary

[6]Of course, since the paper by McGee (1985), there is a substantial debate about whether Modus Ponens *should* hold in conditional inference. But McGee's argument relies essentially on the uncertainty of the premise; there is as of now no argument why Modus Ponens should fail when we take the premises to be true. For a three-valued analysis of why Modus Ponens holds in certain and fails in uncertain inference, see Égré, Rossi, and Sprenger (forthcoming).

Discourse OL—except that Égré, Rossi, and Sprenger (forthcoming) do not restrict C to atom-classical valuations.

As we should expect from an adequate logic for the indicative conditional, the inference from $\neg A$ to $A \to B$ is blocked in C. However, the conditional behaves monotonically, i.e., both the meta-inference from $\Gamma \models A \to B$ to $\Gamma, C \models A \to B$ and the inference $B \models A \to B$ are valid. This is in agreement with the interpretation of C as a certainty-preserving logic. ("If Bob is coming to the party, then he will come (in particular) if Alice comes.") The laws of classical logic in the conditional-free language $\mathcal{L}$ (=the Boolean fragment of $\mathcal{L}^{\to}$) are also theorems of C, if we restrict ourselves to atom-classical valuations. It retains Disjunctive Syllogism ($A \vee B, \neg A \models B$), but gives up Disjunction Introduction ($A \models A \vee B$). However, the counterexample necessarily involves the semantic value 1/2: when we restrict ourselves to bivalent *valuations* of sentential variables, the only invalid instances of $A \models A \vee B$ occur when A is itself a conditional with a false antecedent. This shows that exceptions to the otherwise intuitive rule of Disjunction Introduction addition are quite modest.

Historically, it is interesting that such a well-behaved three-valued conditional logic has been developed in the 1960s, but has had little echo in the philosophical discussion on conditionals. Specifically, Cooper's logic OL anticipates important elements of Belnap's (1970; 1973) three-valued account of restricted quantification, but Belnap neither cites Cooper, nor seems to be aware of his work.[7] Neither did Adams, Cooper's PhD supervisor, develop the ideas of his student further. Only very recently, Égré, Rossi, and Sprenger (2021a,b, forthcoming, 2023b) have highlighted the groundbreaking nature of Cooper's ideas on analyzing conditionals with three-valued logic.

5 Relation to the Material Conditional

The previous section proposed to treat conditionals as conditional assertions, with a truth table that differed from the material conditional $A \supset B := \neg A \vee B$. This does not mean, however, that the material conditional has no role in conditional reasoning. Some three-valued logics we have seen before, such as $K_3^{\to}$, L_3, and $LP^{\to}$, dismiss the material conditional in favor of a proper implication, and basically "forget" about the former. However, an account

[7]It is worth mentioning Belnap's argument for quasi-conjunction as opposed to Strong Kleene conjunction: "a paragraph should not be thrown out as nonassertive on the basis of a single nonassertive sentence. [...] Nor should a book be consigned to the flames because containing a single "if-then" with a false antecedent" (Belnap 1973, p. 61). In general, the interaction of Belnap-style restricted quantification with three-valued conditionals needs to be studied more systematically.

of natural language conditionals needs to explain how the indicative relates to the material conditional. This is for two reasons. First, the paradoxes of material implication (in particular, $\neg A \models A \supset B$) require that an indicative conditional $\rightarrow$ be stronger, i.e., have more demanding truth conditions, than the material conditional $\supset$:

Stronger-than-Material $A \rightarrow B \models A \supset B$.

This condition is quoted as an adequacy condition in various philosophical analyses of indicative conditionals: it is shared by accounts as diverse as the possible world semantics of Stalnaker (1968), the probabilistic logics of Adams (1975) and McGee (1989), the dynamic semantics of Gillies (2009) and the restrictor semantics of Kratzer (2012), to name a few.

On the other hand, the scheme Or-to-If seems to be extremely compelling: if we know that $\neg A$ or B is the case, then we can infer to "if A, then B". Writing the disjunction as a material conditional, we obtain

Or-to-If $\neg A \vee B \models A \rightarrow B$

and thus, the indicative conditional seems to collapse to the material conditional since intuitions push us into accepting both implications in Stronger-Than-Material and in Or-To-If.[8] Any sensible account of conditionals that does not identify the material and the indicative conditional should therefore explain which of the two principles is invalid, and how the intuitions for its validity can be accounted for.

The orthodox solution is to give up Or-To-If and to retain Stronger-Than-Material. The idea is that Or-To-If is invalid as soon as we introduce *uncertainty*. A good illustration is given by Edgington (1986, p. 191): if I am 90% confident that it is 8 o'clock, then I am at least as confident that it is 8 or 11 o'clock, but that does not give me the same confidence that if it is not 8 then it is 11 o'clock.

However, this strategy comes at a price: one needs to protect such an analysis of conditionals against Gibbard's famous triviality result. Gibbard (1981) shows in his paper "Two recent theories of conditionals" that any binary conditional connective $\rightarrow$ collapses to the material conditional of classical logic $\supset$ if the following conditions hold: (i) the conditional connective satisfies Import-Export, (ii) it validates Stronger-Than-Material, (iii) it reproduces the valid inferences of classical logic as theorems of the logic of the conditional, i.e., $\models A \rightarrow B$ whenever $A \models_{\mathsf{CL}} B$. From (i)–(iii) and some natural background assumptions, Gibbard infers $A \supset B \models A \rightarrow B$.

[8] Or-to-If is in C equivalent to Disjunctive Syllogism: since $\rightarrow$ satisfies the deduction theorem, we can freely move from $A \supset B \models A \rightarrow B$ to $\neg A \vee B, A \models B$ and back.

The most common reaction is to give up (i), i.e., Import-Export, either by restricting the language to simple conditionals, as in Adams's logic of p-valid inference, or by invalidating it, as in Stalnaker-Lewis conditional logics. Paraconsistent logics of the conditional avoid the collapse result because a valid inference of classical logic such as $A \vee \neg A \models_{\mathsf{CL}} B$ is usually no theorem of the target logic, i.e., $\not\models (A \vee \neg A) \to B$ (take $v(A) = 1/2$, $v(B) = 0$). This is also why DF/TT and CC/TT do not fall prey to Gibbard's theorem, although they satisfy Import-Export and have a conditional connective that is logically stronger than the (Strong Kleene) material conditional of three-valued logic. For their paracomplete counterparts DF/SS and DF/CC, the same reasoning can be made using an example of the form $\models_{\mathsf{CL}} A \vee \neg A$ but $\not\models A \vee \neg A$. The other strategy, to retain Or-to-If and to give

$\supset$	1	1/2	0
1	1	0	0
1/2	1	1/2	0
0	1	1	1

$\to$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

Table 9: Truth tables of the material and the indicative conditional in the logic C = QCC/TT.

up Stronger-Than Material, is less explored.[9] It corresponds to the diagnosis given by C, i.e., QCC/TT. The material conditional of C is *logically stronger* than the indicative conditional since it is based on the truth tables for quasi-disjunction: see Table 9. This sounds paradoxical, but makes a lot of sense given that C is a theory of *certainty-preserving* inference. When there is no uncertainty about the premises, i.e., we know that $\neg A \vee B$, the inference to $A \to B$ is iron-cast. This means that we need a separate logic for conditional reasoning with *uncertain* premises—a topic that we will elaborate on in the next section—in order to account for the fact that the indicative conditional often *appears* to be stronger and more demanding than the material conditional. $p(B|A) \leq p(A \supset B)$ is a well-known theorem of probability theory, and it also holds in the three-valued generalization of probability (in the form $p(A \to B) \leq p(A \supset B)$, for conditional-free A and B). When we identify assertability and probability, this means that the indicative conditional is, in the fragment of $\mathcal{L}^{\to}$ containing at most simple conditionals, indeed less assertable than the material conditional. The next section studies these questions in more detail.

[9]Defenders of the material conditional analysis such as Lewis (1976), Jackson (1979), and Grice (1989) retain *both* principles and just accept Gibbard's collapse result: it is what they have argued for all the way.

6 Reasoning with Non-Monotonic Conditionals

There is a strong research tradition that models the behavior of conditionals as *non-monotonic*, invalidating the inference from $A \to C$ to $(A \wedge B) \to C$. "If Alice goes to the party, then Carol goes" does not imply that "If Alice and Bob go to the party, then Carol goes": imagine that Carol cannot stand Bob, even if she likes being in Alice's company (compare Adams 1965; Lewis 1973, for examples of this type).

The *suppositional account of conditionals* is especially natural for modeling this feature. The basic idea of that tradition goes back to Ramsey (1929/1990). but its modern form has mainly been developed by Ernest W. Adams (1965, 1975) and Dorothy Edgington (1986, 1995). On their view, simple conditionals $A \to B$ do not express standard propositions, but we can express their probability as the conditional probability $p(B|A)$:

$$p(A \to B) = p(B|A) \qquad \text{(Adams's Thesis)}$$

While Adams's Thesis is a *theorem* of the three-valued semantics of probability, Adams uses this equality as a *definition* of the probability of conditionals. Arguably, our intuitive judgments of the probability of conditionals agree at least in many paradigmatic cases with the conditional probability (for empirical defenses of Adams's Thesis, see Evans and Over 2004; Evans, Handley, et al. 2007; for critical discussions, see Douven and Verbrugge 2010; Skovgaard-Olsen, Singmann, and Klauer 2016).

The relation of logical consequence in this probabilistic semantics is given by the probability order of the sentences of the language. For single-premise inferences, $A \models B$ if and only if for all probability distributions, $p(A) \leq p(B)$. This criterion captures the above aspect of non-monotonicity of the conditional since $p(C|A, B)$ and $p(C|A)$ are not ordered in general. In particular, $B \not\models A \to B$ because it is not a theorem that $p(B) \leq p(B|A)$.

Let $U(A) := 1 - p(A)$ denote the *uncertainty* of a sentence A. Then the above criterion can be generalized as follows to multi-premise inference (Suppes 1966; Adams 1975):

Adams's Criterion for p-valid inference $\Gamma \models_p B$ if and only if for all probability functions $p : \mathcal{L}_1^{\to}[0, 1]$, the uncertainty of the conclusion does not exceed the cumulative uncertainty of the premises:

$$U(B) \leq \sum_{A_i \in \Gamma} U(A_i) \qquad \text{(p-valid inference)}$$

Here the language $\mathcal{L}_1^{\to}$ denotes the flat fragment of a propositional language $\mathcal{L}^{\to}$ with a conditional connective, i.e., conditionals have at most one level of embedding. For conditional-free sentences in $\mathcal{L}$, p-valid inference, preservation of certainty and truth preservation all agree. But when we move to $\mathcal{L}_1^{\to}$, they fall apart: "truth preservation" has no more a canonical definition since it depends on the truth conditions, while p-valid inference and certainty preservation single out different logics.

Specifically, compared to C, the logic of certainty preservation in $\mathcal{L}^{\to}$, p-valid inference takes a different stance on Stronger-Than-Material and Or-To-If. Stronger-Than-Material is a theorem for p-valid inference in $\mathcal{L}_1^{\to}$ since $p(B|A) \leq p(A \supset B)$ for conditional-free A and B. Or-To-If fails, on the other hand, because p-valid inference is essentially a theory of *uncertain reasoning*. Recall Edgington's example from the previous section: if I am 90% confident that it is 8 o'clock, then I am at least as confident that it is 8 or 11 o'clock, but that does not give me the same confidence that if it is not 8 then it is 11 o'clock. Since p-valid inference demands more than preservation of certainty in passing from premises to the conclusion, Adams and his followers can say that Or-To-If is invalid: we just mistake the logic of certainty preservation, where Or-To-If is valid, for the logic of uncertain inference with conditionals.

Recently, p-valid inference has been represented and generalized to arbitrary compounds and nestings of conditionals within a three-valued semantics by Égré, Rossi, and Sprenger (forthcoming, 2023b). They use the same semantics for the connectives as in C and define a the logic U as probability preservation according to the non-classical probability function presented in Section 4.

Logic U of Uncertain Inference $\Gamma \models_{\mathsf{U}} B$ if and only if:

there is a finite subset of the premises $\Delta \subseteq \Gamma$ such that for all probability functions $p : \mathcal{L}^{\to} \longmapsto [0,1]$ based on QCC-valuations, $p(\bigwedge_{A \in \Delta} A) \leq p(B)$.

(The quantification over subsets is required to preserve the structural monotonicity of the logic.) They then show that this logic does not only generalize Adams's logic to nested conditionals and their Boolean compounds, but has an equivalent three-valued characterization:

Three-Valued Characterization of U For a consistent set of formulas[10] $\Gamma \subseteq \mathcal{L}^{\to}$ and $B \in \mathcal{L}^{\to}$ with $\not\models_{\mathsf{C}} B$, $\Gamma \models_{\mathsf{U}} B$ holds if and only if there is a finite subset of premises $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathsf{QCC/SS\cap TT}} B$: for all QCC-valuations v, $v(\bigwedge_{A_i \in \Delta} A_i) \leq v(B)$.

In other words, they show that three-valued logic offers an efficient decision criterion for validating probability-preserving inferences in conditional logic.

[10]This means that there is no set $\Delta \subseteq \Gamma$ such that $p(\bigwedge_{A \in \Delta} A) = 0$ for all probability functions.

The reader may have noticed that this definition of three-valued consequence is very similar to Adams's *yielding condition*, developed in a framework where "neither true nor false" is no truth value in its own right (e.g., interpreted as "void"), but simply seen as a *truth-value gap*. Conditionals with a false antecedent, on this account, simply do not take a semantic value and valuations are partial. Adams (1966, Theorem 8) then shows that p-valid inference $\Gamma \models_p B$ coincides with the existence of a subset $\Delta \subset \Gamma$ such that Δ "strongly entails" B. This is defined as follows:

> Δ strongly entails B if and only if for all [classical] truth assignments f [to sentences] of $\mathcal{L}$:
>
> (i) if no $A \in S$ is falsified under f [=taking semantic value 0], then B is not falsified under f, and
>
> (ii) if no $A \in S$ is falsified under f, and at least one A is verified [=taking semantic value 1], then B is verified under f. (Adams 1966, 297, notation adapted)

It is easy to check that these conditions coincide with TT- and SS-validity using the quasi-connectives defined in Section 4. While the yielding condition generates the same decision procedures for checking the validity of inferences, it is arguably more limited because Adams does not provide a semantics for nestings and compounds of conditionals. Moreover, the definition in terms of three-valued semantics is much more elegant and accessible.

The above result by Égré, Rossi, and Sprenger (forthcoming) seems to be in tension with a well-known result by McGee (1981) that Adams's logic of p-valid inference cannot be given a full characterization by a finitely valued logical matrix, and thus, in particular, not by a three-valued logic. How is it then possible that U generalizes p-valid inference? Schulz (2009) generalizes McGee's impossibility result and shows that it depends on the assumption that the many-valued logic has a *proper conjunction connective*, i.e., that

$$\Gamma \models \phi \wedge \psi \qquad \text{if and only if} \qquad \Gamma \models \phi \text{ and } \Gamma \models \psi.$$

Since U is based on quasi-connectives for disjunction and conjunction, this equivalence fails (in particular the right-to-left direction). The McGee/Schulz result should therefore interpreted as saying that something has to go: we cannot have a proper conjunction connective *and* a three-valued logic that generalizes p-valid inference. In fact, neither the conjunction, nor disjunction nor the conditional are proper connectives in U: Modus Ponens fails because the conditions on logical implication $A \models B$ (i.e., combined SS- and TT-implication) are *stronger* than the conditions on the theoremhood of $A \to B$. While we have Conditional Proof, we have no full deduction theorem. The failure of Modus Ponens occurs, by the way, only for nested conditionals, in line with the counterexamples that McGee (1985) presented against its validity in uncertain reasoning with conditionals.

U is not the only three-valued logic that aims at modeling non-monotonic behavior of the conditional. The accounts developed by Dubois and Prade (1994) and McDermott (1996) are similar both in spirit and content. They propose the same relation of logical consequence (i.e., the combination of SS- and TT-validity), but stick to de Finetti's truth table for the conditional. Moreover, McDermott uses Strong Kleene truth tables for conjunction and disjunction in the definition of valid consequence, which is for the rest identical to the second part of the definition of U (i.e., SS∩TT-consequence). On the level of inferences, many features are similar, but McDermott's logic validates Transitivity ($A \to B$, $B \to C$, therefore $A \to C$). While this is acceptable and even desirable in the framework of *certain* inference, it is arguably problematic when reasoning from *uncertain* premises since the probability of $p(C|A)$ is in no way controlled by $p(C|B)$ and $p(B|A)$; in fact, it can be arbitrarily low. Suppose that you live in a very sunny, dry place. Consider the sentences A = "it will rain tomorrow", B = "I will work from home", C = "I will work on the balcony". Clearly, both $A \to B$ and $B \to C$ are highly plausible, but $A \to C$ isn't. Dubois and Prade avoid that feature because their definition of logical consequence is identical to Égré, Rossi and Sprenger's, but like Adams and Cooper, they restrict their account to the flat fragment of $\mathcal{L}^{\to}$, i.e., allowing only simple, non-nested conditionals. In any case, all these accounts show that uncertain reasoning with non-monotonic conditionals can be fruitfully captured in three-valued logic.

7 Relevant Implication and Three-Valued Logic

Can we also find a three-valued connective that describes the meaning of "A implies B" or "A entails B"? Clearly, an explication of "implies" should satisfy a deduction theorem, i.e., correspond to a notion of logical consequence: what else should "implies" mean, if not "assures the truth of" or "is deducible from"?

Before answering this question, let us have a short look at classical logic. Anderson and Belnap (1975, §1–3) argue that the material conditional of classical propositional logic, $A \supset B$, is no adequate explication of "A implies B". True, it satisfies the deduction theorem with respect to classical logical consequence. But on the other hand, it validates the inferences $A \models B \supset B$, that $A \models B \supset A$, and that $\neg A \models A \supset B$. Moreover, their conditional counterparts $A \supset (B \supset B)$, $A \supset (B \supset A)$ and $\neg A \supset (A \supset B)$ are theorems. All of these results are in blatant violation of what we mean by "imply" or "entail". While $B \supset B$ ought to be a theorem, but it is in no way *entailed* by A. The (contingent) truth of A should not be sufficient to make us infer that B implies A, and the (contingent) falsity of A should not be sufficient to make us infer that A implies B. All these results of classical logic do not take seriously that implication is a proper two-place relation, and that when we prove a conclusion entailed by the premises, we prove it *from* the premises.

If we buy these constraints on valid entailment, as relevance logicians do, we have to revise the rules of our syntactic calculus, e.g., Natural Deduction. Suppose we have a deduction theorem for a conditional connective $\rightarrow$ that moreover satisfies the axiom $A \rightarrow A$. Then we can reason as follows:

1:	A	hypothesis
2:	$B \rightarrow B$	axiom
3:	$A \rightarrow (B \rightarrow B)$	deduction theorem ($\Rightarrow$)

leading to one of the paradoxes of classical implication. This is, of course, also a problem for the three-valued explications of "if/then": C blocks the inference $\neg A \models A \rightarrow B$, but it validates both $A \models B \rightarrow B$ and $A \models B \rightarrow A$. While C satisfies the most important principles of connexive logic, i.e., Aristotle's Thesis $\neg(\neg A \rightarrow A)$ and Boethius's Thesis $(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$, it fails to model *relevant* implication, and derivation *from* a set of premises.

Anderson and Belnap (1975, p. 18) suggest to solve this problem by "keeping track of the steps used", and to allow the introduction of a conditional $A \rightarrow B$ only "when A is relevant to B in the sense that A is *used* in arriving at B." They restrict Modus Ponens—their main rule of inference—to cases where at least one of the premises is a hypothesis of the deduction, or has been derived using the hypothesis. Similarly, they require that $A \rightarrow B$ can only be introduced when both A and B have been proven *from* the hypothesis. Such a logic $\mathsf{R}_{\rightarrow}$, restricted to the pure implication fragment of the language, can be characterized by the following axioms:

$R1_{\rightarrow}$:	$A \rightarrow A$	identity
$R2_{\rightarrow}$:	$(A \rightarrow B) \rightarrow (C \rightarrow A \rightarrow (C \rightarrow B))$	transitivity
$R3_{\rightarrow}$:	$A \rightarrow (B \rightarrow C) \rightarrow (B \rightarrow (A \rightarrow C))$	permutation
$R4_{\rightarrow}$:	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	contraction

This logic, already developed by Church (1951), also satisfies a deduction theorem, which Anderson and Belnap give in its syntactic version: if $A_1, \ldots, A_n \vdash B$, then $A_1, \ldots, A_{n-1} \vdash A_n \rightarrow B$, i.e., there is a proof of the conditional $A_n \rightarrow B$ from the first $n-1$ premises. However, it just covers the fragment of the language with a conditional connective. Its classical ("canonical") extension to a language that also contains the Boolean connectives $\neg$, $\wedge$ and $\vee$ is known as R and was studied in great detail in Anderson and Belnap (1975). It rejects all of the "irrelevant" inferences listed at the beginning of the section, and is therefore a proper relevance logic.

It is characteristic of $\mathsf{R}_{\rightarrow}$ and R that they have the *variable-sharing property*, i.e., if $\models_{\mathsf{R}} A \rightarrow B$ or equivalently, $A \models_{\mathsf{R}} B$, then A and B share at least one propositional variable. This is not the case for RM—the logic one gets from R by adding the "mingle axiom" $A \rightarrow (A \rightarrow A)$. Actually, while RM still blocks the above paradoxes of implication, it has been dubbed at most a "semi-relevant" logic because it validates inferences such as $A \models (\neg A \rightarrow A)$ or $\neg(A \rightarrow A) \models (B \rightarrow B)$ (Anderson and Belnap 1975, §29.5). Still, it is advocated as a valid alternative to R in §29.3 of the same book because it simplifies the calculus of R considerably, for its mathematical and algebraic properties, and because it may be "good enough when *some* relevance is desirable". It has been studied extensively by Dunn (1976a,b) and Avron (1986).

$\rightarrow_{RM_3}$	1	1/2	0
1	1	0	0
1/2	1	1/2	0
0	1	1	1

Table 10: The three-valued truth tables for the conditional in the semi-relevant logic RM_3. The truth table corresponds to Sobocinki's (1952) conditional and the material conditional of C, i.e., Cooper's logic OL.

For RM, one can obtain a three-valued semantics as soon as one adds the further axiom $A \vee (A \rightarrow B)$. The simplest interpretation of the semantic values is quasi-intuitionistic: 1 and 0 mean that a sentence is derivable from the premises, or that its negation its derivable, while 1/2 means that neither the sentence nor its negation is derivable. While negation, conjunction and disjunction are defined using the ordinary Strong Kleene tables, the conditional follows the truth table of Table 10 where $v(0 \rightarrow 1/2) = v(1/2 \rightarrow 0) = 0$, like for quasi-disjunction in the previous version. This three-valued logic was introduced by Sobociński (1952) for the fragment containing only implication and negation as connectives; its extension to all Boolean connectives and its application to modeling relevant implication is due to Anderson and Belnap (1975) and figures under the name of RM_3. Note that the conditional of RM_3 has the same truth table as the material conditional in C, i.e., Cooper's logic OL.

Obviously, a relevance logic should be paraconsistent; explosion is a paradigmatic case of a non-relevant inference. So it is natural to define logical consequence as preservation of designated values $D = \{1, 1/2\}$. Then it is not hard to see that RM_3 satisfies Modus Ponens for the conditional in Table 10, but not Conditional Proof (counterexample: $1 \models_{\mathsf{RM}_3} 1/2$, but $\not\models_{\mathsf{RM}_3} 1 \rightarrow 1/2$). Hence it does not satisfy the deduction theorem. For further analysis of RM_3, see for example Robles (2016).[11]

While RM_3 is certainly not the most popular of all relevant logics, it is notable that a simple, truth-functional three-valued logic can validate a great number of principles of relevant logics. The semantics is remarkably simple compared to other examples of relevant logics. If one wants more—for example, if one wishes a full deduction theorem or to block further irrelevant inferences—three semantic values may not be enough. Adding a fourth value, one obtains a semantics for the flat (single-implication) fragment of the most famous of all relevance logics: Belnap and Dunn's logic of first-degree entailment E or FDE.

8 Conclusions

This article has surveyed the behavior of the conditional in three-valued logic from two perspectives: (i) how to align it with a definition of logical consequence; and (ii) how to explicate natural language structures such as "if/then" and "implies" using three-valued logic.

Question (i) poses non-trivial challenges since the material conditional of the Strong Kleene truth tables (which have independent plausibility as generalizations of "and" and "or" to three-valued logic) does not align with a standard definition of logical consequence. In particular, regardless of whether we work with just one designated value ("strict truth", $D = \{1\}$) or with two designated values ("tolerant truth" or non-falsity, $D = \{1, 1/2\}$), we will not have a material conditional that aligns with the logical consequence relation and yields a deduction theorem. What is more, even Modus Ponens fails. This behavior of the material conditional shows a fundamental difference between two-valued and three-valued logic.

Some three-valued logics based on the Strong Kleene tables, like Priest's logic of paradox LP, bite the bullet and argue on independent grounds that Modus Ponens should be invalid, and that a deduction theorem is not desirable at all. This is because the intended applications (e.g., in a semantically closed language with a naive truth predicate) are such that Modus Ponens would lead to disastrous consequences. Other theorists decide to save Modus Ponens by introducing a new conditional connective that "internalizes" logical implication, such as the logics $K_3^{\rightarrow}$ and $LP_3^{\rightarrow}$ introduced in Section 3.

In the second part of this chapter, I have focused on question (ii) and shown that three-valued logic is a fruitful framework for analyzing the se-

[11] The Sobociński conditional is exactly the material conditional of the logic C.

mantics and epistemology of the natural language indicative conditional. The truth tables for the indicative conditional are based on de Finetti's idea to interpret a conditional as a conditional assertion which is *void* or *nonassertive* when the antecedent is false. This basic idea has many advantages, as it leads almost automatically to the validation of desirable principles such as Import-Export and avoids Gibbard's and Lewis's triviality results.

Choosing a (full) truth table for the conditional, and pairing it with truth tables for the Boolean connectives and a relation of logical consequence, requires further choices. I have argued that if one is seriously interested in the behavior of the natural language conditional, one should (i) interpret conjunction and disjunction according to Cooper's quasi-connectives; and (ii) go for a logical consequence relation that is either preservation of non-falsity (for inference with certain premises) or preservation of strict truth *and* non-falsity. This approach can also be applied to restricted quantification, reasoning with non-monotonic conditionals, and generalizing Adams's theory of p-valid inference to arbitrary formulas of a propositional language $\mathcal{L}^{\rightarrow}$ with a conditional connective. The penultimate section briefly surveyed three-valued semantics for the relevance logic R-Mingle. All in all, three-valued logic emerges as a powerful framework for explicating natural language structures such as "if/then" and "implies".

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