The Material and the Suppositional Conditional

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Abstract

The material conditional and the suppositional analysis of the indicative conditional are based on different philosophical foundations and they leave important successes of their competitor unexplained. This paper unifies both accounts within a truth-functional, trivalent model of the suppositional analysis. In this model, we observe that the material and the suppositional conditional exhibit the same logical behavior while they have different truth conditions and different probabilities. The result is a unified semantic analysis that closes an important gap in the suppositional story and explains the persistent appeal of the material conditional analysis for philosophers and psychologists of reasoning.

1 Introduction

The material conditional analysis (MCA) is among the oldest and most venerable analyses of indicative conditionals (henceforth simply "conditionals"). According to MCA, the truth conditions of "if A, then B" are given by the disjunction $A \supset B := \neg A \vee B$. The analysis is simple, truth-functional and therefore fully compositional. Philosophers such as David Lewis, Frank Jackson, H.P. Grice and Timothy Williamson have endorsed MCA despite some well-known problems such as the paradoxes of material implication.

By contrast, suppositional analyses are based on the *Ramsey test* (Ramsey 1929/1990): we evaluate a conditional by supposing the antecedent and evaluating the consequent under this assumption. Typically, suppositionalists endorse Adams's Thesis (Adams 1965, 1975), i.e., the probability of a simple conditional is given by the corresponding conditional probability:

$$p(\text{"if } A, \text{ then } B\text{"}) = p(B|A) \qquad \text{(Adams's Thesis)}$$

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Adams's Thesis squares well with intuitive judgments on the probability of conditionals. Suppose we toss a fair die. Then the probability of "if an even number occurs, it will be six" should be 1/3, and so on. MCA does not capture this intuition because in general $p(A \supset B) \geq p(B|A)$, with equality only in degenerate cases. Defenders of MCA react by appealing to pragmatic norms governing the assertability of conditionals (Lewis 1976; Jackson 1979; Grice 1989), or by asserting that suppositional reasoning is just a fallible *heuristic* for evaluating conditionals and their probability (Williamson 2020).

In addition to the disagreement about the probability of conditionals, MCA analyzes conditionals as standard propositions. Suppositionalists cannot do this due to pressure from Lewis's (1976) triviality results. Indeed, the classical suppositionalist theory by Adams (1975) and Edgington (1986) gives up on the idea that conditionals have truth conditions. Hence, the two accounts do not only make different predictions: they disagree about the very nature of conditionals.

This paper claims that both analyses can be unified within trivalent semantics. The basic idea of the latter is that conditionals with false antecedents are assigned a third semantic value ("void"). Moreover, logical consequence is explicated as *acceptance preservation* (an idea originally proposed by Stalnaker 1975). By giving up on bivalence and driving a wedge between the truth conditions of a conditional (i.e., the assignment of semantic value) and the notion of logical consequence, we can preserve the best of both worlds: that is, we can explain the successes of MCA and the Adams-style suppositional analysis without buying into their drawbacks.

Like MCA, the trivalent model analyzes conditionals as propositions, in agreement with their surface structure, and gives them a fully compositional, truth-functional semantics. Moreover, MCA captures the (trivalent) logic of acceptance preservation, explaining why the indicative conditional behaves so often like the material one. But like in the suppositional analysis, we obtain Adams's Thesis for the probability of conditionals as well as an attractive theory of uncertain reasoning.

The paper is structured as follows. To motivate our project, Section 2 refers some key successes of MCA. Section 3 relates MCA to the suppositional analysis. Section 4 introduces the trivalent analysis as a truth-functional semantic model of the suppositional account. Section 5 analyzes how the material and the suppositional conditional interact in the trivalent setting, and Section 6 studies their interaction in context-sensitive suppositional accounts. Section 7 draws the balance.

Finally, some notational conventions: $\mathcal{L}$ denotes a simple propositional language with Boolean connectives ($\neg$, $\wedge$, $\vee$). $\mathcal{L}^{\rightarrow}$ denotes the same language with an added conditional connective $\rightarrow$, and $\mathcal{L}^{\rightarrow}_1$ the restriction of $\mathcal{L}^{\rightarrow}$ to the fragment involving at most simple conditionals. Formulae of these languages are denoted in uppercase Roman letters (A, B, C, $\dots$) and sets of formulae in uppercase Greek letters (Γ, Δ).

2 The Material Conditional Analysis

The material conditional $A \supset B$ is the simplest possible logical model of the indicative conditional "if A, then B". It evaluates the conditional as false when A is true and B is false, and true in all other cases. See Table 1. Due to its truth-functional, fully compositional nature, MCA is a very simple and elegant theory. It also makes several notable and desirable predictions.

Truth value of $A \supset B$	B true	B false
A true	true	false
A false	true	true

Table 1: Truth table for a the material conditional $A \supset B$.

First, MCA validates the Import-Export scheme, i.e., $A \supset (B \supset C) = (A \wedge B) \supset C$, in agreement with stable intuitions and natural language data on nested conditionals (McGee 1989, p. 489; see also van Wijnbergen-Huitink, Elqayam, and Over 2015). Moreover, no account of conditionals with stronger truth conditions that MCA can validate Import-Export and classical laws without collapsing into the material conditional (Gibbard 1981, pp. 234-235; see also Égré, Rossi, and Sprenger 2023). The argument runs as follows, with conditional-free $A, B \in \mathcal{L}$:

1. Consider $(A \supset B) \to (A \to B)$. By Import-Export, this is equivalent to $((A \supset B) \wedge A) \to B$, which is the same as $(A \wedge B) \to B$.

2. The conditional connective $\to$ validates laws of classical logic. Therefore $(A \wedge B) \to B$ must be true.

3. Combining both observations, we infer that $(A \supset B) \to (A \to B)$ must be true, too.

4. Since $\to$ is stronger than $\supset$, we can infer that $(A \supset B) \supset (A \to B)$ is true, too. Therefore $\to$ and $\supset$ are logically equivalent.

In the light of this result, any truth-conditional analysis of conditionals that deviates from MCA must explain why Import-Export is invalid, make radical semantic changes (e.g., evaluate conditionals as context-sensitive), or bite the bullet and explain why the collapse is not harmful.

Second, MCA validates widely endorsed inference schemes, such as Modus Ponens and Modus Tollens. Specifically, it explains why the Or-to-If inference rings correct: "Either the butler or the gardener did it. Therefore, if the butler didn't do it, the gardener did it." For MCA, the conclusion is just a linguistic restatement of the premise: both sentences express the same proposition.

More generally, $A \supset B$ is true in models of classical logic if and only if the inference from A to B is valid. MCA therefore aligns well with the intuition that true conditionals express valid inferences (see also Iacona 2023). It is exactly this intuition that makes the above Or-to-If inference so plausible.

Third, MCA performs well on so-called "conditional standoffs". For simplicity, I will not deal with the famous Sly Pete case (Gibbard 1981), but with Bennett's (2003) more linear examples. A system regulating water flow consists of Top Gate, East Gate and West Gate. Top Gate is on top of the other two gates and distributes water to East Gate and West Gate. It is impossible that all three gates are open at the same time. An observer sees that East Gate is open and utters

(1) If Top Gate is open, all water will flow through East Gate.

while another observer sees that West Gate is open and utters

(2) If Top Gate is open, all water will flow through West Gate.

None of the observers seems to assert anything false, or to have false beliefs. Such cases are apparently grist on the mill of accounts that evaluate conditionals as context-sensitive: in this case (1) and (2) could both be true. But MCA accounts for the same intuition in a much simpler way: the logical form of (1) and (2) is (approximately) $A \supset B$ and $A \supset \neg B$. Their truth entails that A must be false. In other words, Top Gate must be closed. This is indeed what we would infer from the observers' reports.

Fourth—and this is fact is rarely observed—MCA squares well with Bayesian learning. Bayesian conditionalization on proposition E, i.e., the rule $p'(X) := p(X|E)$, can be characterized equivalently as finding the posterior distribution p' that minimizes the (Kullback-Leibler) divergence to the prior distribution conditional on the constraint $p'(E) = 1$ (Diaconis and Zabell 1982). MCA generalizes this rationale to a language with a conditional: conditionalizing on the material conditional i.e, $p'(X) := p(X|A \supset B)$ is the only updating procedure which minimizes the prior-posterior divergence conditional on the constraint $p'(B|A) = 1$.[1] Hence, MCA delivers sensible rules for dynamic reasoning with conditionals, and for updating our beliefs.

These are important *prima facie* successes of MCA, achieved within the bounds of a very simple and linear theory. They are characteristic of deduc-

[1] The statement is formalized and proved in Sprenger and Hartmann 2019, Theorem 4.3.

tive reasoning with conditionals; in fact the standing of MCA is much less clear when probability enters the picture, due to the failure of Adams's Thesis.[2] Moreover, one has to respond to the paradoxes of material implication ($\neg A \models A \supset B$ and $B \models A \supset B$). But the successes of MCA are notable enough that any rivaling theory of conditionals should explain them.

3 MCA and the Suppositional Analysis

The suppositional analysis of indicative conditionals goes back to Ramsey (1929/1990) and contends that the evaluation of "if A, then B", written as $A \to B$, requires us to suppose that A, and to make a hypothetical judgement about B under that supposition.

The most well-known and elaborate version of the suppositionalist analysis gives up on truth conditions for conditionals and focuses on their probability as given by Adams's Thesis (Adams 1965, 1966, 1975; Edgington 1986, 1995). Valid reasoning with conditionals is analyzed in terms of the probabilistic properties of premises and conclusions.

Two probability-based conditional logics surveyed by Adams (1975, 1996) deserve special mention. The first is *probability-preserving inference* or p-valid inference. Without going into details, the idea is that valid inferences do not increase uncertainty. For single-premise inference $A \models B$, this means that for any probability distribution p, $p(A) \leq p(B)$. This conception of valid inference is especially useful when we are reasoning under conditions of genuine uncertainty.

The second logic of interest is *certainty-preserving inference*. On this view, a valid inference requires that whenever we assign probability 1 to all premises of an argument (i.e., we are certain of them), the conclusion must have probability 1, too. This conception of valid inference can be motivated by the idea that probability 1 is a good proxy for full *acceptance* of a proposition, and that acceptance of propositions should be preserved in valid reasoning. Stalnaker (1975, p. 271) proposes a similar notion:

> an inference from a sequence of assertions or suppositions (the premises) to an assertion or hypothetical assertion (the conclusion) is *reasonable* just in case, in every context in which the premises could appropriately be asserted or supposed, it is impossible for anyone to accept the premises without committing himself to the conclusion

Stalnalker's idea of reasonable inference has been developed over the years. For example, Bledin (2014, p. 277) rejects truth preservation as a default no-

[2] For a recent book-length defense of MCA, see Williamson (2020). A critical discussion of how Williamson explains the failure of Adams's Thesis within MCA is given by Krzyżanowska and Douven (2022) and van Rooij, Krzyżanowska, and Douven (2023).

tion for logical consequence: he considers logic "a descriptive science that is fundamentally concerned not with the preservation of truth, but with the preservation of structural features of information". This view of logical consequence is now rather popular among semanticists and philosophers of language working on conditionals, especially in accounts that evaluate conditionals relative to information states (for example Yalcin 2007; Gillies 2009; Bledin 2015; Punčochár and Gauker 2020). Santorio (2022, p. 81) even claims that defining logical consequence along these lines is "the obvious notion of consequence for assessing consistency and validity for asserted claims in natural language". Certainty-preserving inferences also coincide with classically valid inferences in the Boolean fragment $\mathcal{L}$ (e.g., Leblanc 1979). This justifies our focus on certainty-preserving inference in the remainder of the paper.

The challenge of connecting MCA and the suppositional analysis concerns both probability and logic. On the side of logic, we note, generalizing some remarks by Edgington (1995, Section 6):

Observation 1: Adams's logic of certainty-preserving inference and MCA have the same theorems and valid inferences in the fragment $\mathcal{L}_1^{\rightarrow}$.

The observation will be stated more precisely in Section 5: for the moment the informal version suffices. Relatedly, we noted in Section 2 that

Observation 2: Conditionalizing on the material conditional $A \supset B$ and prior-posterior minimization relative to the constraint $p'(B|A) = 1$ generate the same posterior distribution.

To explain these agreements, note that $p(A \rightarrow B) = p(B|A) = 1$ will be the case (assuming $p(A) > 0$) if and only if $p(A \wedge \neg B) = 0$. This is equivalent to $p(A \supset B) = 1$. Thus, the suppositional conditional is a certainty whenever the material one is, and this explains why they generate the same logic. On the other hand, the agreement on logic co-exists with a blatant disagreement on probability since in general, $p(A \supset B) \geq p(B|A)$.

This divergence may be surprising, but it has an interesting application to the paradoxes of material implication. If we are *certain* that John is not in the office, it seems reasonable to infer to "(hence, in particular) if the bar is open, John won't be in the office". Similarly, there is nothing paradoxical in inferring that conditional from the premise that the bar is closed: this is just a variation of *ex falso quodlibet*. Indeed, both MCA and Adams's logic of certainty preservation validate the inferences $\neg A \models A \rightarrow B$ and $B \models A \rightarrow B$.

The above rationalization strategy does not work in contexts where we are *uncertain* about the premises. In those cases, we do not recognize the above inferences as valid, unless we know more about John's drinking habits. Indeed, for the suppositional analysis the inequalities $p(\neg A) \leq p(B|A)$ and $p(B) \leq p(B|A)$ will *not* hold in general. On the other hand, the analogous equalities $p(\neg A) \leq p(A \supset B)$ and $p(B) \leq p(A \supset B)$ hold for MCA, casting doubt on its adequacy as a theory of conditionals for uncertain inference.

The following section develops a semantic framework where we can explain these divergences from a semantic perspective and sketch a more precise picture of how the material and the suppositional conditional interact. It also addressess two principal shortcomings of the suppositional analysis: its limited scope, i.e., the missing coverage of nested conditionals and compounds of conditionals (see also McGee 1989, p. 485), and the lack of a truth-conditional model that explains how probability judgments are grounded in the attribution of truth values.

4 The Trivalent Analysis

Trivalent semantics of conditionals, an idea going back to Reichenbach (1935) and de Finetti (1936), is a peculiar semantic implementation of the suppositional analysis. It deviates from the Adams-Edgington view that conditionals do not express propositions and analyzes them as *partial* propositions: they have classical truth value at some, but not at all worlds.

The idea is to treat "if A, then B" as a *conditional assertion*—i.e., as an assertion about B upon the supposition that A is true (see Quine 1950). Whereas, when the antecedent is false, the assertion is "void": the speaker is committed to neither truth nor falsity of the consequent. This view takes into account Adams's (1965, p. 169) observation that "true" has no clear sense when applied to indicative conditionals, in particular to those with false antecedents.[3] Table 2 visualizes this basic idea. The third semantic value, "void", is represented by the symbol 1/2.

Truth value of $A \to B$	B true	B false
A true	true	false
A false	void	void

Table 2: Partial truth table for a conditional $A \to B$ analyzed as a conditional assertion.

To deal with nested conditionals, this truth table has to be extended to a full 3x3 table. Moreover, we have to define the meaning of the Boolean con-

[3]Another motivation for the trivalent view is that we tend to think that an assertion "if A, then B" has been *verified* if we observe both A and B, and *falsified* if we observe A and $\neg B$. But if the antecedent turns out to be false, there is no factual basis for evaluating the assertion (see also Cooper 1968; Belnap 1970, 1973).

nectives. We follow Cooper (1968) in his analysis of the logical connectives and truth tables and adopt the connectives as defined in Table 3.

	$\neg$
1	0
1/2	1/2
0	1

$\wedge$	1	1/2	0
1	1	1	0
1/2	1	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	0
0	1	0	0

$\to$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

Table 3: Cooper's (1968) truth tables for the Boolean connectives and the indicative conditional.

The non-standard treatment of conjunction and disjunction deserves a brief justification. Cooper's conjunction and disjunction reduce to their classical counterparts when the arguments take classical values. However, when just one of the conjuncts/disjuncts takes a classical value, it determines the value of the conjunction/disjunction—the non-classical value "void" is simply ignored. Unlike the Strong or Weak Kleene truth tables, this choice allows us to evaluate sentences such as

(3) If the sun shines tomorrow, Mary will go to the beach; and if it rains tomorrow, she will go to the museum.

as true (e.g., when the sun shines tomorrow and Mary goes indeed to the beach), and it allows us to evaluate sentences such as

(4) If Alice is red-haired, then Bob is tall; or if Bob is tall, then Alice is red-haired.

as false (e.g., when Bob is tall and Alice is black-haired: compare Bradley 2002).[4] This motivates the following definition:

Definition 1 (Cooper valuations)**.** *A* Cooper valuation *is a valution function* $v : \mathcal{L}^{\to} \longmapsto \{0, 1/2, 1\}$ *that respects the truth tables from Table 3 and assigns classical truth values from the set* $\{0, 1\}$ *to all atomic formulae of* $\mathcal{L}$ *(and thus, their Boolean compounds).*

We now define a (non-classical) probability function $p : \mathcal{L}^{\to} \longmapsto [0, 1]$, taking into account that sentences of $\mathcal{L}^{\to}$ can receive three values: true, false, or void.[5] For convenience, define

$$A_T = \{w \in W \mid v(A, w) = 1\}$$
$$A_F = \{w \in W \mid v(A, w) = 0\}$$

[4] For a more detailed defense of these features of trivalent conjunction and disjunction, we refer to Égré, Rossi, and Sprenger (2021, 2024, forthcoming).

[5] For other occurrences of that definition, see de Finetti (1936), who pioneered it, Cantwell (2006), Rothschild (2014), and Lassiter (2020).

$$A_V = \{w \in W \mid v(A,w) = 1/2\}$$

as the sets of possible worlds where A is valued as true, false or void, relative to a (Cooper) valuation function $v : \mathcal{L}^{\rightarrow} \times W \longmapsto \{0, 1/2, 1\}$. In analogy to bivalent probability, we derive the probability of a (conditional) sentence A from the (conditional) betting odds on A: how much more likely is a bet on A to be won than to be lost? For this comparison, two quantities are relevant: (1) the cumulative weight of the worlds where A is true (i.e., $c(A_T)$), and (2) the cumulative weight of the worlds where A is false (i.e., $c(A_F)$). The *decimal odds* on A are $O(A) = (c(A_T) + c(A_F))/c(A_T)$, indicating the factor by which the bettor's stake is multiplied in case A occurs and she wins the bet. Then we calculate the probability of A from the decimal odds on A by the familiar formula $p(A) = 1/O(A)$, yielding

$$p(A) := \frac{c(A_T)}{c(A_T) + c(A_F)} \qquad \text{if } \max(c(A_T), c(A_F)) > 0. \qquad \text{(Probability)}$$

Hence, the probability of a sentence corresponds to its expected semantic value, *restricted to the worlds where the sentence takes classical truth value.*[6]

On this definition of probability, we obtain Adams's Thesis as a simple corollary (for conditional-free $A, B \in \mathcal{L}$):

$$p(A \rightarrow B) = \frac{c(A \rightarrow B)_T}{c(A \rightarrow B)_T + c(A \rightarrow B)_F} = \frac{c(A_T \cap B_T)}{c(A_T)} = \frac{p(A \wedge B)}{p(A)} = p(B|A) \qquad \text{(Adams's Thesis)}$$

Regarding logical consequence, what should be preserved in the trivalent setting? Truth, non-falsity or both? Rather than making an ad-hoc choice about which values should count as designated, we follow the Stalnaker-Bledin line and preserve acceptance of a proposition in valid reasoning (see page 5 for motivation). Similar to Adams, we model acceptance preservation as certainty-preserving inference, i.e., preservation of probability 1:[7]

Definition 2 (Certainty Preservation or C-validity)**.** *For $\Gamma \subset \mathcal{L}^{\rightarrow}$ and $B \in \mathcal{L}^{\rightarrow}$, the inference from Γ to B is* C*-valid, in symbols $\Gamma \models_{\mathsf{C}} B$, if and only if for all probability functions $p : \mathcal{L}^{\rightarrow} \longmapsto [0,1]$:*

If $p(A) = 1$ for all $A \in \Gamma$, then also $p(B) = 1$.

[6] Additionally, we stipulate that $p(A) = 1$ whenever $c(A_T) + c(A_F) = 0$, i.e., if it is certain that A takes the value $1/2$.

[7] It could be objected that acceptance of a proposition requires a lower probabilistic threshold than 1, but our condition is analogous to the following common condition from information state semantics: a state s accepts a proposition when it is true in *every* world w in the state. Provided that $p(w) > 0$ for all $w \in s$, this is just the same as the probability 1 requirement. See Section 6 for further details.

Since the trivalent analysis satisfies Adams's Thesis, C agrees with Adams's logic of certainty-preserving inference in the fragment $\mathcal{L}_1^{\rightarrow}$ involving Boolean propositions and simple conditionals.

Crucially, C has an equivalent characterization in trivalent logic: an inference is C-valid if and only if non-falsity is preserved in passing from Γ to *B* (see Égré, Rossi, and Sprenger forthcoming):

Proposition 1 (Trivalent Characterization of C)**.** *For a set of formulae* $\Gamma \subset \mathcal{L}^{\rightarrow}$ *and a formula* $B \in \mathcal{L}^{\rightarrow}$*, the following are equivalent:*

(1) $\Gamma \models_{\mathsf{C}} B$.

(2) For all Cooper valuations $v : \mathcal{L}^{\rightarrow} \longmapsto \{0, 1/2, 1\}$*: if* $v(A) \geq 1/2$ *for all* $A \in \Gamma$*, then also* $v(B) \geq 1/2$.

In other words, preserving non-falsity in the trivalent semantic model assures that we will reason from fully accepted premises to fully accepted conclusions, and vice versa.[8] This means that 1 and 1/2 are both designated values. In the next section, we apply this analysis of conditionals to the semantic phenomena from the previous two sections and relate it to MCA.

5 Toward a Unified Analysis

The trivalent model of the suppositionalist analysis is particularly fruitful for explaining the virtues and limitations of MCA. The material conditional $A \supset B$ is interpreted as the disjunction $\neg A \vee B$, and its trivalent truth conditions are given by Table 4, in agreement with the truth conditions for disjunction in Table 3.

Crucially, on the trivalent account, *the suppositional conditional* $A \rightarrow B$ *is logically weaker than the material conditional* $A \supset B$:

Proposition 2. *For* $A, B \in \mathcal{L}^{\rightarrow}$*:* $A \supset B \models_{\mathsf{C}} A \rightarrow B$*, and for* $A, B \in \mathcal{L}$ *(i.e., conditional-free A and B):* $A \supset B \;=\!\mid\models_{\mathsf{C}} A \rightarrow B$.

$\rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

$\supset$	1	1/2	0
1	1	0	0
1/2	1	1/2	0
0	1	1	1

Table 4: Truth tables for the suppositional and the material conditional in trivalent semantics.

[8] Instead, preserving semantic value 1 ("strict truth") would preserve strictly positive probability (Égré, Rossi, and Sprenger forthcoming).

The proof is immediate from Table 4: $A \supset B$ has more demanding non-falsity conditions than $A \to B$, even if it has more lenient truth conditions. Therefore $A \supset B \models_{\mathsf{C}} A \to B$, and for simple conditionals, the two expressions are logically equivalent.

From this proposition, Observation 1 from Section 3 follows as an immediate corollary. The material conditional describes the logic of deductive reasoning in the fragment $\mathcal{L}_1^{\to}$ involving at most simple conditionals. But can we we generalize this observation? That is, is logical equivalence preserved when we substitute the material conditional for the suppositional conditional in more complex formulae?

To answer this question in a rigorous way, we first define the *substitution function* $\tau : \mathcal{L}^{\to} \mapsto \mathcal{L}$, which replaces every occurrence of $\to$ in a $\mathcal{L}^{\to}$-formula with the material conditional $\supset$. We then note that τ does not generally preserve logical equivalence: for example, $\neg(A \to B)$ is *not* C-equivalent to $\neg(A \supset B) = A \wedge \neg B$ and $(A \to B) \vee (C \to D)$ is *not* C-equivalent to $(A \supset B) \vee (C \supset D)$. Nonetheless τ preserves logical equivalence in a rather large fragment of $\mathcal{L}^{\to}$.

Definition 3. *$\mathcal{L}_{1+}^{\to}$ denotes the fragment of $\mathcal{L}^{\to}$ that is generated by combining arbitrary Boolean formulae with the connectives $\wedge$ and $\to$. Specifically, negation and disjunction are allowed for combining Boolean formulae, but not between conditional expressions.*

For example, $(A \to B) \to (A \vee C)$ and $(\neg A \to B) \wedge (C \to \neg D)$ are formulae of $\mathcal{L}_{1+}^{\to}$, but $(A \to B) \vee C$ or $\neg(A \to B)$ are not. We can now show that substituting the suppositional with the material conditional in formulae of $\mathcal{L}_{1+}^{\to}$ preserves logical equivalence (proof in the appendix):

Theorem 1 (Substitution of $\to$ by $\supset$ in formulae of $\mathcal{L}_{1+}^{\to}$)**.** *As before, the function $\tau : \mathcal{L}_{1+}^{\to} \to \mathcal{L}; A \mapsto \tau(A)$ substitutes all occurrences of $\to$ in a $\mathcal{L}_{1+}^{\to}$-formula A by $\supset$. Then, for any $\Gamma \subset \mathcal{L}_{1+}^{\to}$ and $X \in \mathcal{L}_{1+}^{\to}$:*

- *$X \mathrel{=\!\!|\!\models_{\mathsf{C}}} \tau(X)$;*
- *$\Gamma \models_{\mathsf{C}} X$ if and only if $\tau(\Gamma) \models_{\mathsf{CL}} \tau(X)$.*

This means that key successes of MCA listed in Section 2—especially Or-to-If and conditional standoffs—are predicted by the suppositional analysis: none of the target phenomena involves an element of uncertainty. *The suppositional conditional should behave exactly as predicted by the material conditional in deductive reasoning*: they stand and fall together. This observation also accounts immediately for Gibbard's (1981) otherwise puzzling result that any conditional satisfying Import-Export, classical inference patterns and $A \to B \models A \supset B$ is equivalent to the material conditional.

The failure of preserving logical equivalence for negations and disjunctions of conditionals should not worry us too much.[9] Plausibly, conjunctions of conditionals like (3) ("if A, then B, *and* if C, then D") occur more frequently in natural language than disjunctions like (4) ("if A, then B, *or* if C, then D"). Similarly, negations of conditionals such as "it is not the case that if A then B" sound rather convoluted and unnatural compared to "if A, then $\neg B$" or "if A, then possibly B".

All in all, the logic of many natural language expressions with suppositional conditionals is correctly described by MCA. It is more than just a good logical theory for deductive reasoning with simple conditionals: it extends to nested conditionals and conjunctive compounds of conditionals of arbitary complexity.

Still, the result stated by Proposition 2 remains puzzling: if the suppositional conditional is not logically stronger than the material conditional, how can it be the case that for all conditional-free A and B, $p(A \supset B) \geq p(B|A) = p(A \to B)$? Another look at Table 4 dissolves the tension. Simple material conditionals $A \supset B$ have a higher probability than $p(B|A)$: since they are true when A is false, $c(A \supset B)_T$ will exceed $c(A \to B)_T$ in general, while $c(A \to B)_F = c(A \supset B)_F$. Therefore

$$p(A \to B) = \frac{c(A \to B)_T}{c(A \to B)_T + c(A \to B)_F} \leq \frac{c(A \supset B)_T}{c(A \supset B)_T + c(A \supset B)_F} = p(A \supset B).$$

We can summarize all these observations by saying that on the trivalent analysis, (i) the material conditional and the suppositional conditional have different truth conditions; (ii) therefore, they can have different probabilities, with the material conditional always having a higher probability than the suppositional conditional; (iii) nonetheless, simple material and suppositional conditionals are logically equivalent in deductive reasoning; and (iv) in a large fragment of $\mathcal{L}^{\to}$, we can substitute the suppositional for the material conditional and preserve logical equivalence. Note that (iii) and (iv) do not create any form of collapse or trivialization. Moreover, when generalizing Bayesian conditionalization to the trivalent setting, learning the suppositional and the material conditional produce the same posterior probability distribution (Égré, Rossi, and Sprenger 2024, Proposition 6; see also Santorio 2022, on the "update equivalence" of $\to$ and $\subset$). The material and the suppositional conditional are therefore aligned in various dimensions of inference and learning.

[9] Indeed, C does not satisfy the principle of Substitution of Logical Equivalents because valuations of propositional atoms are assumed to be classical.

The combination of (ii) and (iii) explains why we often accept Or-to-If inferences when the premise looks certain, but not when it looks uncertain. Consider the following inference:

(5) The butler or the gardener did it.

(6) Therefore, if the butler did not do it, the gardener did it.

Here, the premise is presented in a form that invites us to take it for granted that all other suspects can be excluded. In fact, Edgington (1995, p. 242) explicitly presents the Or-to-If argument as a certainty-preserving inference:

> *having eliminated all but two suspects*, I'm sure that either the gardener or the butler did it. So, if the gardener didn't do it, the butler did. (my emphasis)

In such a situation, the Or-to-If inference from the material conditional (5) to its indicative counterpart (6) looks sound, in agreement with what is predicted by C. But in contexts of genuine uncertainty, Or-to-If fails. Consider the following judgments of the police inspector:

(7) With 75% probability the butler or the gardener did it.

(8) ??Therefore, with 75% probability, if the butler did not do it, the gardener did it.

This inference looks invalid: the speaker's probabilities may support the premises (say 50% butler, 25% gardener, 25% cook), but if the butler did not do it, the cook is just as likely as the gardener to be the culprit.[10] Without additional information, we should not infer that the gardener did it with 75% probability if the butler didn't do it. This feature is correctly predicted by our analysis: from $p(A \supset B) \geq 0.75$ it does not follow that $p(A \rightarrow B) \geq 0.75$.

We can draw a general lesson. The classical picture with only two semantic values (true and false) and logical consequence as truth preservation forces truth conditions, logical consequence and probability to go together. When two propositions are logically equivalent, they must have the same truth conditions, and therefore the same probability. But the logical and probabilistic behavior of conditionals diverges. By giving up bivalence and describing conditional logic in a more coarse-grained way than what truth conditions tell us, we can understand this difference: the material and the suppositional conditional have the same falsity conditions, but different truth conditions. This allows for a split between logical and probabilistic behavior. In particular, since valid deductive inference preserves non-falsity

[10]This inference assumes that the elimination of the butler as a suspect does not provide reasons to favor one of the two remaining ones.

(by Proposition 1), MCA is a successful logic of conditionals while it is bound to struggle with the probability of conditionals.

6 Suppositionalist Alternatives to Trivalence

The trivalent account is not the only analysis of indicative conditionals motivated by the Ramsey test. Can we obtain a similarly illuminating diagnosis using one of its suppositionalist cousins?

As argued above, the original Adams-Edgington view, where conditionals do not have truth conditions, does not cover nested conditionals and compounds of conditionals. Thereby, it rules out a *general* comparison of $A \rightarrow B$ and $A \supset B$, including cases where A and B themselves involve conditionals.

Contextualist analyses of conditionals may be more promising. They are suppositional in spirit, based on the Ramsey test, but add a semantic parameter for evaluating conditionals: a *context* or *informational state*. These states are simply sets of possible worlds compatible with the speaker's knowledge, or the common ground of the conversation. Can such an analysis say anything illuminating about the relation between the material and the suppositional conditional?

The answer depends on the details of the specific account. I will survey some recent proposals, without claiming that the analysis is exhaustive. For the sake of presentation, I will simplify the details.

Yalcin (2007) and Gillies (2009), and essentially also Starr (2014), propose a picture where the conditional $A \rightarrow B$ is treated similarly to a strict conditional $\Box(A \supset B)$. Roughly, a conditional $A \rightarrow B$ is true at a world w in an informational state s if and only if *updating* the state with A, i.e., moving from s to $s[A]$, makes B true in $s[A]$. Put simply: $w, s \models A \rightarrow B$ if and only if $s[A] \models B$ (i.e., B is true at all worlds in $s[A]$). The details of the updating procedure need not worry us, but with Gillies, we can simply take intersect s with the A-worlds (i.e., $s[A] := \{w \in s \mid w, s \models A\}$).

On this picture, a simple conditional $A \rightarrow B$ has stronger truth conditions than the corresponding material conditional $A \supset B$: for the material conditional to be true at w, either $\neg A$ or B must be true at w. But this is compatible with $w' \models A, \neg B$ for some $w' \in s$. In this case, $s[A] \not\models B$ and $A \rightarrow B$ fails at w. Conversely, when the material conditional $A \supset B$ is false at $w \in s$, it follows that $w \models A, \neg B$. Therefore $s[A] \not\models B$ and $w, s \not\models A \rightarrow B$. The indicative conditional is logically stronger than the material conditional, and the context-sensitivity of $\rightarrow$ blocks Gibbard-style collapse results.

This reassuring fact comes at a price. Since the conditional $\rightarrow$ behaves similarly to a strict conditional, it will *not* satisfy Adams's Thesis. Such accounts therefore struggle to account for intuitive judgments about the probability of conditionals. Specifically, if a speaker at world w evaluates $A \to B$ with respect to information state s, then quite reasonably, she will assign zero probability to worlds outside s. This means that the probability of $A \to B$ must be zero or one (because the conditional is uniformly true or false at s), trivializing a theory of the probability of conditionals.

Thus, strict conditional accounts obtain an intuitively correct account of the logical relationship between the material and the indicative conditional, but at the price of giving up on a promising theory of the probability of conditionals. Moreover, since truth conditions and logic are context-dependent, this analysis struggles to explain the attractive features of MCA.

Another contextualist research program substitutes "classical" possible worlds by *paths* or *sequences* of possible worlds as semantic building blocks (Goldstein and Santorio 2021; Khoo 2022; Santorio 2022; see also Kaufmann 2009; Bacon 2015). Any path corresponds to a sequence of classical possible worlds (i.e., maximally consistent valuations of all formulae of $\mathcal{L}$). The first world in a path determines the truth values of all Boolean formulae. For example, at the path $p = (w_2, w_1, w_3, \ldots)$, the material conditional $A \supset B$ (for $A, B \in \mathcal{L}$) is true if and only if $\neg A$ or B are true at w_2. For determining the truth value of a conditional formula $A \to B$, we look instead at the first world in the path where A is true, similar to Stalnaker's semantics for conditionals. So if A is false at w_2 but true at w_1, the truth value of $A \to B$ at p corresponds to the truth value of B at w_1. In general, we evaluate $A \to B$ at any path p by evaluating B at the updated path $p + A$ where the $\neg A$-worlds have been eliminated.

Based on this semantics, Santorio (2022) obtains that the truth of the indicative conditional at a path p guarantees the truth of the material conditional.[11] Goldstein and Santorio (2021) also derive Adams's Thesis $p(A \to B) = p(B|A)$ for conditional-free A and B.

For reasons discussed in Section 3, Santorio works with an analogue of certainty preservation as definition of logical consequence: $\Gamma \models A$ whenever any information state s that accepts all elements of Γ also accepts A (i.e., A is true at all elements of s). As before, information states are sets of classical possible worlds. The above semantics then implies that whenever an information state s accepts $A \to B$, it will accept $A \supset B$, too. This is unsurprising since at any path, the truth conditions of $A \to B$ are more demanding than the truth conditions of $A \supset B$. What is more surprising also the contrary is valid: any information state s that accepts $A \supset B$ will

[11] The only way the material conditional $A \supset B$ can fail is that the first world in p is a $(A, \neg B)$-world. But in this case, B would be false at the updated path $p + A$ (since the first element of p will be the first element of $p + A$), and so also $A \to B$ must be false at p. Moreover, $A \to B$ can fail at some paths where $A \supset B$ is true.

accept $A \rightarrow B$, too. The material and the indicative conditional are thus logically equivalent for informational consequence, like in our Observation 1. This allows Santorio to validate the Or-to-If inference without assigning identical truth conditions to the material and the suppositional conditional.

Similarly, the material conditional $A \supset B$ and the indicative conditional $A \rightarrow B$ are *update-equivalent*, i.e., any information state where we learn $A \supset B$ (by updating the individual paths in that state) equals the same information state after learning $A \rightarrow B$. This is the analogue of our Observation 2. Santorio (2022, p. 85) concludes that

> [l]earning a material conditional is equivalent to learning the corresponding indicative, even though the two are not treated as equivalent by the semantics

and this is a surprising and desirable feature, in line with our diagnosis.[12]

In a nutshell, Santorio's path semantics assigns more demanding truth conditions to the indicative conditional (at a path) than to the material conditional, while reproducing their equivalence in logic and updating. At the same time, probability can be defined rather straightforwardly and a limited version of Adams's Thesis holds. Santorio's semantics is thus a serious alternative to the trivalent account when it comes to forging systematic connections between the material and the suppositional conditional.

On the other hand, the use of *sequences* of possible worlds as basic semantic building blocks is a substantial departure from the traditional semantic picture that MCA proponents want to maintain. Of course, also the trivalent account is non-classical because it gives up bivalence, but in its structure, it is very similar to MCA: the semantic building blocks are standard possible worlds. Even more importantly, valuations are truth-functional, reducing the truth conditions of conditionals to the truth conditions of non-conditional, factual sentences. Moreover, logical consequence can be defined in terms of preservation of designated value at a world, without reference to contexts (see Proposition 1). Compared to contextualist accounts, the trivalent account is closer to MCA in its conceptual repertoire and its take on logic and reasoning. And for this reason, it is a more promising candidate for a unified theory of the material and the suppositional analysis.

7 Conclusions

The material and the suppositional analysis of the indicative conditional have different strengths and weaknesses. The classical suppositionalist anal-

[12] Khoo (2022, p. 75) rejects this update equivalence in favor of a slightly different principle. Apart from the brevity of the presentation, this is another reason why I focus on Santorio's proposal.

ysis delivers a powerful theory probabilistic reasoning, but it lacks a truth-conditional model that explains the semantic grounds for probability judgments. Moreover, it does not apply to nested conditionals and compounds of conditionals and has therefore limited scope. MCA does not have this limitation and adequately captures deductive reasoning with conditionals, but it struggles to account for intuitive judgments on the probability of conditionals, and in particular for Adams's Thesis—the core of the suppositionalist account. How can we integrate all this into a unified account?

This paper has argued that trivalent semantics explains the attractive features of either account, without sharing their limitations. MCA emerges as the correct analysis of deductive reasoning with conditionals in a large fragment of the language, including nested conditionals and conjunctions of conditionals. In this case, suppositional reasoning will be exactly mirrored by classical truth-preserving inferences with the corresponding material conditionals (Theorem 1). In other words, MCA is a reasonable approximation of suppositional reasoning that allows us to preserve familiar assumptions: truth-functional semantics, bivalence of semantic value, and logical consequence as truth preservation. (The latter two are given up by trivalent semantics.) This may also explain the appeal of MCA for psychologists of reasoning (Johnson-Laird and Byrne 1991, 2002).

In other words, we invert Williamson's claim that suppositional reasoning is a heuristic for MCA: we have defined a precise sense in which MCA is a heuristic for suppositional reasoning with conditionals, namely when no uncertainty is at stake. The performance of MCA is much less convincing when probability and uncertain reasoning are involved.

Notably, the divergence of the material and suppositional conditional on the level of probability remains a purely semantic feature, firmly grounded in truth conditions. This agrees with the idea that probability should exclusively depend on the relative weight of possible worlds, and on the truth values of propositions at different worlds. No appeal to pragmatics is required to connect logic, truth conditions and probability. Similar results can be obtained in Santorio's (2022) update-based path semantics, but the trivalent construction is simpler and structurally more similar to MCA.

These findings also suggest that the Adams-Edgington claim that conditionals do not express propositions may be unnecessary baggage for the suppositional account. It is not required for obtaining Adams's predictions about probabilistic inference with conditionals (see also Égré, Rossi, and Sprenger 2024). Accepting that conditionals are propositions, as their surface form suggests, does not come with drawbacks, but only with benefits.

Finally, a note on wider implications. The present proposal seems to introduce a certain amount of logical pluralism: there is no one true logic of conditionals, but there are two of them: certainty-preserving and probability-preserving inference. This distinction goes back to Adams (1975, 1996) and Edgington (1995). While looking strange at first, it makes a lot of sense. First, for the conditional-free fragment $\mathcal{L}$, both C and its probability-preserving counterpart collapse to classical logic. Second, the variation in validity judgments for the Or-to-If inference in examples (5) to (8), dependent on the degree of uncertainty, suggests that two different notions of logical consequence may be at play (compare also the discussion about the validity of Modus Ponens in McGee 1985). Distinguishing between certain and uncertain inference simply widens our repertoire for analyzing the semantic and pragmatic phenomena surrounding indicative conditionals.

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Proofs

Lemma 1. *For $A, B, C, D \in \mathcal{L}^{\rightarrow}$: If $A \dashv\models_{\mathsf{C}} C$ and $B \dashv\models_{\mathsf{C}} D$, then also $A \wedge B \dashv\models_{\mathsf{C}} C \wedge D$.*

Proof. Assume that at a world w, $v(A \wedge B, w) \geq 1/2$: this is the case if and only if $v(A, w) \geq 1/2$ *and* $v(B, w) \geq 1/2$. Since C preserves designated values $\geq 1/2$, $A \dashv\models_{\mathsf{C}} C$ and $B \dashv\models_{\mathsf{C}} D$ implies that these conditions are equivalent to $v(C, w) \geq 1/2$ *and* $v(D, w) \geq 1/2$—which is the same as $v(C \wedge D, w) \geq 1/2$. □

Lemma 2. *For $A, B \in \mathcal{L}_{1+}^{\rightarrow}$ and Boolean formulae $C, D \in \mathcal{L}$: if $A \mathrel{=\!\mid\!\models}_{\mathsf{C}} C$ and $B \mathrel{=\!\mid\!\models}_{\mathsf{C}} D$, then $A \supset B \models_{\mathsf{C}} C \supset D$.*

Proof. At any world w, $v((A \supset B) \rightarrow (C \supset D), w) \geq 1/2$. Constructing a countermodel requires that at a world w', $v(C, w') = 1$ and $v(D, w') = 0$ (remember that C and D are Boolean formulae). The C-equivalences $A \mathrel{=\!\mid\!\models}_{\mathsf{C}} C$ and $B \mathrel{=\!\mid\!\models}_{\mathsf{C}} D$ imply $v(A, w') \geq 1/2$ and $v(B, w') = 0$, and so $v(A \supset B, w') = 0$. But then $v((A \supset B) \rightarrow (C \supset D), w) = 1/2$. Hence no such countermodel exists. It follows by the Deduction Theorem for C that $A \supset B \models_{\mathsf{C}} \tau(A) \supset \tau(B)$. □

Proof of Theorem 1. The proof of the first claim ($X \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(X)$) proceeds by induction on the complexity of the $\mathcal{L}_{1+}^{\rightarrow}$-formula with the connectives $\rightarrow$ and $\wedge$. The base case is given by Proposition 2.

For the inductive step, suppose the main connective is $\wedge$ and that $X = A \wedge B \in \mathcal{L}_{1+}^{\rightarrow}$. By the inductive hypothesis, we know that $A \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(A)$ and $B \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(B)$. Lemma 1 yields that $A \wedge B \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(A) \wedge \tau(B)$. Finally we note that $\tau(A \wedge B) = \tau(A) \wedge \tau(B)$, and therefore $A \wedge B \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(A \wedge B)$, as desired.

The proof for the main connective $\rightarrow$ is slightly more complex. Assume that $X = A \rightarrow B \in \mathcal{L}_{1+}^{\rightarrow}$. We need to show that this is C-equivalent to $\tau(A \rightarrow B) = \tau(A) \supset \tau(B)$. By the inductive hypothesis, we know that $A \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(A)$ and $B \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(B)$. For the direction $\tau(A) \supset \tau(B) \models_{\mathsf{C}} A \rightarrow B$, we make use of some properties of $\models_C$, shown in Égré, Rossi, and Sprenger (forthcoming):

(1)	$\tau(A), \tau(A) \supset \tau(B) \models_{\mathsf{C}} \tau(B)$	C generalizes classical logic
(2)	$\tau(A), \tau(A) \supset \tau(B) \models_{\mathsf{C}} B$	(1), $\tau(B) \models_{\mathsf{C}} B$ and Transitivity
(3)	$A, \tau(A) \supset \tau(B) \models_{\mathsf{C}} B$	(2) and Left Logical Equivalence
(4)	$\tau(A) \supset \tau(B) \models_{\mathsf{C}} A \rightarrow B$	(3) and Conditional Proof

For the direction $A \rightarrow B \models_{\mathsf{C}} \tau(A) \supset \tau(B)$, we reason by cases, showing that no valuation of A and B gives rise to a countermodel to the postulated implication. Since C preserves designated value $\geq 1/2$, such a model would require that at some world w and for some valuation function v, $v(A \rightarrow B, w) \geq 1/2$ and $v(\tau(A) \supset \tau(B)) = 0$.

- Consider $v(A, w) \geq 1/2$ and $v(B, w) = 0$. Then also $v(A \rightarrow B, w) = 0$ and no countermodel is possible.

- Consider $v(A, w) = 1$ and $v(B, w) = 1/2$. By the inductive hypothesis, $A \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(A)$ and $B \mathrel{=\!\mid\!\models}_{\mathsf{C}} \tau(B)$, and $\tau(A)$ and $\tau(B)$ are Boolean formulae with classical truth values. Therefore $v(\tau(A), w) = v(\tau(B), w) = 1$ and consequently, $v(\tau(A) \supset \tau(B), w) = 1$.

- For all other valuations of A and B, we have $v(A \supset B, w) \geq 1/2$. But we know from Lemma 2 that in this case, also $v(\tau(A) \supset \tau(B), w) \geq 1/2$.

Thus, there is no world w and valuation v where $v(A \to B, w) \geq 1/2$ but $v(\tau(A) \supset \tau(B), w) = 0$, concluding the proof.

For the second part of the theorem, note that C satisfies Left and Right Logical Equivalence. Thus, $\Gamma \models_{\mathsf{C}} X$ if and only if $\tau(\Gamma) \models_{\mathsf{C}} \tau(X)$. Since both $\tau(\Gamma)$ and $\tau(X)$ are conditional-free and since C generalizes classical logic, we obtain $\tau(\Gamma) \models_{\mathsf{CL}} \tau(X)$. The converse direction runs analogously: from $\tau(\Gamma) \models_{\mathsf{CL}} \tau(X)$ we infer $\tau(\Gamma) \models_{\mathsf{C}} \tau(X)$ and by Left and Right Logical Equivalence $\Gamma \models_{\mathsf{C}} X$. ($\tau$ has a unique inverse function τ^{-1}, which substitutes every occurrence of $\supset$ with $\to$.) $\square$