STRUCTURAL OPTIMIZATION WITH RELIABILITY CONSTRAINTS

J. D. Sørensen & P. Thoft-Christensen
University of Aalborg
Sohngaardsholmsvej 57
DK-9000 Aalborg
Denmark

1. INTRODUCTION

During the last 25 years considerable progress has been made in the fields of structural optimization and structural reliability theory. In classical deterministic structural optimization all variables are assumed to be deterministic. Due to the unpredictability of loads and strengths of actual structures it is now widely accepted that structural problems are non-deterministic. Therefore, some of the variables have to be modelled as random variables/processes and a reliability-based design philosophy should be used, Cornell [1], Moses [2], Ditlevsen [3] and Thoft-Christensen & Baker [4].

In this paper we consider only structures which can be modelled as systems of elasto-plastic elements, e.g. frame and truss structures. In section 2 a method to evaluate the reliability of such structural systems is presented. Based on a probabilistic point of view a modern structural optimization problem is formulated in section 3. The formulation is a natural extension of the commonly used formulations in determinstic structural optimization. The mathematical form of the optimization problem is briefly discussed.

In section 4 two new optimization procedures especially designed for the reliability-based optimization problem are presented. In some examples in section 5 the optimization procedures are compared.

2. RELIABILITY OF STRUCTURAL SYSTEMS

The loads on the considered structures and the strengths of the structural elements are modelled as time-invariant stochastic variables. All other variables such as geometrical quantities are assumed to be deterministic.

Failure of the structural system can be defined in a number of ways. For a detailed description, see e.g. Thoft-Christensen [5]. The computationally simplest definition which in statically indeterminate structures can be taken as a serviceability limit state is to define failure of the system as failure of one of the structural elements. This is called a level 1 definition of failure. As a measure of the reliability of a structural element the reliability index β can be used, see Thoft-Christensen & Baker [4]. Let the N basic random variables be collected in the vector

$\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_N)$ with given density function $f_{\underset{\sim}{Y}}(\underset{\sim}{y})$ and the failure surface which defines the separation between the safe and failure areas in the sample space be given by the equation $g(\underset{\sim}{y}) = 0$. When $\underset{\sim}{Y}$ is non-normally distributed a transformation from $\underset{\sim}{Y}$ to the normally distributed standardized vector $\underset{\sim}{U} = \underset{\sim}{T}(\underset{\sim}{Y})$ is established (e.g. the Rosenblatt transformation can be used, [6]). The reliability index β is now defined as the shortest distance from the origin in the $\underset{\sim}{u}$-space to the failure surface:

$$\beta = \min_{g(\underset{\sim}{T}(\underset{\sim}{z}))=0} \left(\sum_{i=1}^{N} z_i^2 \right)^{\frac{1}{2}} \tag{1}$$

The reliability of the structural system can now be estimated by modelling each structural element as an element in a series system.

The above definition of failure can be generalized to a level m definition of failure. Failure of the structural system is then defined as the event that m structural elements have failed. Such a failure mode can be modelled as a parallel system with m elements. To estimate the reliability of the structural system each parallel system is modelled as an element in a series system.

Usually in elasto-plastic systems the ultimate limit state is defined by the formation of a mechanism (i.e. collapse). The number of possible mechanisms in a structural system is usually very large. For the types of structure considered here the so-called safety margins for the mechanisms can be written

$$M_i = \sum_{j=1}^{N_R} a_{ij} R_j - \sum_{j=1}^{N_P} b_{ij} P_j \quad , \quad i = 1, 2, \ldots, h \tag{2}$$

where $\underset{\sim}{R}$ and $\underset{\sim}{P}$ model the yield strength and load variables. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are matrices which contain coefficients of influence. $N = N_R + N_P$ and h is the number of mechanisms. Here $\underset{\sim}{R}$ and $\underset{\sim}{P}$ are assumed to be normally distributed with expected values $\underset{\sim}{\mu}_R$ and $\underset{\sim}{\mu}_P$ and covariance matrices $\underset{\sim}{C}_R$ and $\underset{\sim}{C}_P$. $\underset{\sim}{R}$ and $\underset{\sim}{P}$ are assumed independent.

The reliability index β_i for the i^{th} safety margin is then as follows

$$\beta_i = \frac{\mu_i}{\sigma_i} = \frac{\sum_{j=1}^{N_R} a_{ij}\mu_{R_j} - \sum_{j=1}^{N_P} b_{ij}\mu_{P_j}}{\left(\sum_{j=1}^{N_R}\sum_{k=1}^{N_R} a_{ij}a_{ik}C_{R_{jk}} + \sum_{j=1}^{N_P}\sum_{k=1}^{N_P} b_{ij}b_{ik}C_{P_{jk}}\right)^{\frac{1}{2}}} \tag{3}$$

where μ_i and σ_i are the expected value and the standard deviation of M_i. The coefficient of correlation between the i^{th} and j^{th} safety margin is

$$\rho_{ij} = \frac{\sum_{k=1}^{N_R}\sum_{\ell=1}^{N_R} a_{ik}a_{j\ell}C_{R_{k\ell}} + \sum_{k=1}^{N_P}\sum_{\ell=1}^{N_P} b_{ik}b_{j\ell}C_{P_{k\ell}}}{\sigma_i\sigma_j} \tag{4}$$

These failure modes are modelled as elements in a series system, and an upper-bound estimate of the reliability of the elasto-plastic structural system is given by

$$\beta_S = -\Phi^{-1}(1 - \Phi_h(\underset{\sim}{\beta}; \underset{\sim}{\rho}))\tag{5}$$

where Φ^{-1} is the inverse standard normal distribution function and $\Phi_h(\cdot\,; \underset{\sim}{\rho})$ is the standard distribution function for h normal variables with correlation coefficient matrix $\underset{\sim}{\rho}$.

In real structures (e.g. offshore steel jacket structures) the number of possible failure modes is generally very large. It is therefore important to be able to identify the most significant failure modes. For that purpose the so-called β-unzipping technique has been developed by the authors, Thoft-Christensen & Sørensen [7].

The basic idea in the β-unzipping technique is that a failure tree is successively formed. Each node signifies a modified structure where a number of elements have failed and each branch is an element. The critical elements are selected on the basis of the safety indices of the elements in the modified structures.

3. RELIABILITY-BASED STRUCTURAL OPTIMIZATION

In classical deterministic structural optimization for truss and frame structures the design variables are usually the cross-sectional areas x_i, $i = 1, 2, \ldots, n$, where n is the number of sets of different structural members. Each structural element is characterized by one number. This is fully satisfactory for truss structures where only tensile/compressive forces exist. However, when bending occurs in a structural member, the plastic section moduli w_i, $i = 1, 2, \ldots, n$ and the second moments of area I_i, $i = 1, 2, \ldots, n$ are significant. To maintain the great computational advantage of having only one design variable for each structural member it is often assumed that

$$w_i = k_1 x_i^{2/3}\tag{6}$$

$$I_i = k_2 x_i^2\tag{7}$$

where k_1 and k_2 are constants.

As objective function a natural choice would be the total cost of the structure. But due to the difficulties in assigning monetary values to failure consequences and to the initial cost we have in this paper used the structural weight as objective function. If the structure is made of only one type of material the weight is proportional to

$$W(\underset{\sim}{x}) = \sum_{i=1}^{n} \ell_i x_i\tag{8}$$

where ℓ_i is the total length of the elements having the area x_i.

In classical structural optimization the constraints usually signify that the stresses and/or displacements should be smaller than some prescribed values. In reliability-based structural optimization a choice for the constraints could be that the reliability index in all elements should be greater than some target value. However, based on the discussion in section 2 a more natural choice would be to use the system reliability index

$$\beta_S(\underset{\sim}{x}) - \beta_S^0 \geq 0\tag{9}$$

where β_S^0 is some target system reliability index and $\beta_S(\underset{\sim}{x})$ is given by (5). Because the areas of the structural elements have to be non-negative we also have the constraints

$$x_i \geqslant 0 \quad , \quad i = 1, 2, \dots, n \tag{10}$$

The optimization problem is seen to have a linear, objective function. Since an optimum point is a global optimal point if the optimization problem is convex it is important to investigate if the constraint (9) is concave. A precondition is that the Hessian matrix is negative semi-definite. The elements in the gradient vector and the Hessian matrix are

$$\frac{\partial \beta_S(\underset{\sim}{x})}{\partial x_i} = \frac{1}{\varphi(-\beta_S)}\Big[\sum_{k=1}^{n} \frac{\partial \Phi_h}{\partial \beta_k}\frac{\partial \beta_k}{\partial x_i} + \sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n} \frac{\partial \Phi_h}{\partial \rho_{k\ell}}\frac{\partial \rho_{k\ell}}{\partial x_i}\Big] \tag{11}$$

where $\underset{\sim}{\beta}$ and $\underset{\sim}{\rho}$ are given by (3) and (4). φ is the standard normal density function.

$$\frac{\partial^2 \beta_S(\underset{\sim}{x})}{\partial x_i \partial x_j} = \frac{1}{\varphi(-\beta_S)}\Big[\beta_S \varphi(-\beta_S)\frac{\partial \beta_S}{\partial x_i}\frac{\partial \beta_S}{\partial x_j}$$

$$+ \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 \Phi_h}{\partial \beta_k \partial \beta_\ell}\frac{\partial \beta_k}{\partial x_i}\frac{\partial \beta_\ell}{\partial x_j} + \sum_{k=1}^{n}\frac{\partial \Phi_h}{\partial \beta_k}\frac{\partial^2 \beta_k}{\partial x_i \partial x_j}$$

$$+ \sum_{k=1}^{n-1}\sum_{K=k+1}^{n}\sum_{\ell=1}^{n-1}\sum_{L=\ell+1}^{n} \frac{\partial^2 \Phi_h}{\partial \rho_{kK}\partial \rho_{\ell L}}\frac{\partial \rho_{kK}}{\partial x_i}\frac{\partial \rho_{\ell L}}{\partial x_j} + \sum_{k=1}^{n-1}\sum_{K=k+1}^{n}\frac{\partial \Phi_h}{\partial \rho_{kK}}\frac{\partial^2 \rho_{kK}}{\partial x_i \partial x_j}\Big] \tag{12}$$

The derivatives in (11) and (12) can be derived from the definition of Φ_h, (3) and (4). As seen from (12) it is very difficult to establish whether the constraint in a given problem is concave. To show that the fulfilment of the concavity condition depends on the parameters in the given problem consider the following simple example.

Example

In (2) we assume that $h = 1$, $N_R = 2$, $a_{11} = a_{12} = 1$, $\sum_{j=1}^{N_P} b_{ij}\mu_{P_j} = 2$, $\sum_{j=1}^{N_P}\sum_{k=1}^{N_P} b_{ij}b_{ik}C_{P_{jk}} = 0.05$,

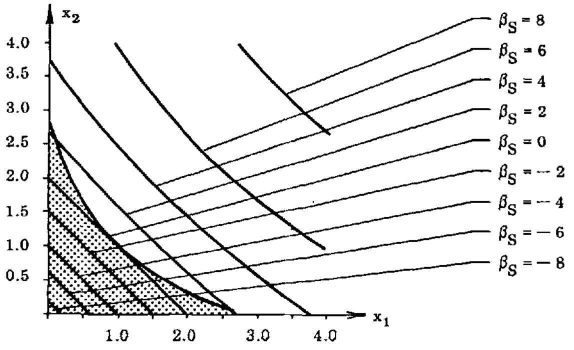


Figure 1.

$E[R_1] = x_1, E[R_2] = x_2, C_{R_{11}} = (0.1\ x_1)^2, C_{R_{22}} = (0.1\ x_2)^2, C_{R_{12}} = (0.1)^2 \rho x_1 x_2, \rho = 0.3.$
In figure 1 contours of $\beta_S(x_1, x_2)$ are shown. The Hessian matrix is determined according to (12). The hatched area in figure 1 shows the area where the Hessian is not negative semi-definite. It is seen that only for combinations of x_1 and x_2 where the reliability index is relatively small, the Hessian is not negative semi-definite.

The above example indicates that for a structure with a reasonable reliability index ($\beta_S \gtrsim 3$) the reliability-based optimization problem will in most cases be convex.

4. OPTIMIZATION PROCEDURES

In this section we describe two optimization procedures. The computational work involved in solving the optimization problem described above can be divided in three parts:

I Identification of critical failure modes by the β-unzipping method.

II Evaluation of the systems reliability index for a given set of critical failure modes.

III Optimization calculations.

Due to the great complexity of the constraint (9) the derivatives of (9), if needed, will be calculated by using finite differences. Each time calculation of the constraint (9) is requested by the optimization algorithm both part I and II have to be performed. But because part I is very time-consuming compared to part II and because the set of critical failure modes cannot be expected to change significantly due to small changes in the design vector $\underset{\sim}{x}$, the critical failure modes are only identified when one of the following conditions is fulfilled (the latest identification is performed at iteration step i with the design vector $\underset{\sim}{x}^i$):

1. The actual iteration step is equal to $i + I_c$.

2. $\sqrt{\sum_{j=1}^{n} (x_j - x_j^i)^2} \geqslant x_{max}$, where $\underset{\sim}{x}$ is the actual design vector.

Evaluation of the systems reliability index given a set of significant failure modes (part II) has to be performed many times. Generally, the evaluation has to be made approximately. In this paper we have used partly the so-called PNET method, Ang & Ma [8] and partly the average correlation coefficient method, Thoft-Christensen & Sørensen [9].

The special formulation of the optimization problem where the constraint (9) is very important has caused the following test for optimality to be used:

$$\sum_{j=1}^{n} c_1 \left| \frac{x_j^i - x_j^{i-1}}{x_j^i} \right| + c_2 \left| \beta_S^0(\underset{\sim}{x}^i) - \beta_S^0 \right| + c_3 \left| \frac{W(\underset{\sim}{x}^i) - W(\underset{\sim}{x}^{i-1})}{W(\underset{\sim}{x}^i)} \right| < \epsilon \tag{13}$$

where $\underset{\sim}{x}^i$ is the value of $\underset{\sim}{x}$ at iteration level i. This stopping criterion can only be used when the set of significant failure modes is updated. c_1, c_2, and c_3 are prescribed constants.

The two different optimization algorithms which have been used in the above optimization procedure are

a. The non-linear programming code NLPQL developed by Schittkowski [10]. This mathematical method is based on the successive solution of quadratic programming sub-problems and a subsequent one-dimensional line search with an augmented Lagrange function as merit function. The optimality test (13) has been added to the tests in NLPQL.

b. In structural optimization a sequential linear programming technique based on the inverse areas of the structural elements is often used, e.g. Fleury [11]. At the i^{th} iteration level the constraint (9) is linearized

$$\beta_S(\underset{\sim}{x}) - \beta_S^0 \cong \beta_S(\underset{\sim}{x}^i) + \sum_{j=1}^{n} \left(\frac{\partial \beta_S(\underset{\sim}{x})}{\partial(\frac{1}{x_j})} \right|_{\underset{\sim}{x}=\underset{\sim}{x}^i} \left(\frac{1}{x_j} - \frac{1}{x_j^i}\right)\Bigg) - \beta_S^0 = \sum_{j=1}^{n} \frac{d_j^i}{x_j} - \beta^i \tag{14}$$

At iteration level i the new design vector $\underset{\sim}{x}^{i+1}$ is then found by solving the linearized Problem ((8) + (14) + (10)):

$$x_j^{i+1} = \max\{0, \sqrt{\frac{D_j^i}{\ell_i}} \frac{\sum_{k=1}^{n} \sqrt{D_k^i \ell_k}}{\beta^i}\} \tag{15}$$

where $\underset{\sim}{\ell}$ is defined in (8) and

$$D_j^i = \frac{1}{i} \sum_{k=1}^{i} d_j^k \tag{16}$$

(16) is added to stabilize the iteration.

5. EXAMPLES

Consider the frame shown in figure 2. It has 4 different structural elements with areas x_1, x_2, x_3, and x_4. k_1 and k_2 in (1) - (2) are chosen as (Gorman [12]) $k_1 = 1.84$ and $k_2 = 3.20$.

The loading (5 concentrated loads) and the 19 failure elements (potential yield hinges) are modelled by 4 + 19 normal stochastic variables with constant coefficients of variation. The expected values of the failure elements are determined by

$$\mu_i = w_i \cdot 270 \cdot 10^3 \text{ kNm}^{-2} \quad , \quad i = 1, 2, 3, 4$$

where w_i is given by (6). Further details concerning the loading, the correlation structure, and the β-unzipping can be found in Thoft-Christensen & Sørensen [13]. The constants in (13) are chosen as $c_1 = c_2 = c_3 = 1$ and $\epsilon = 0.01$.

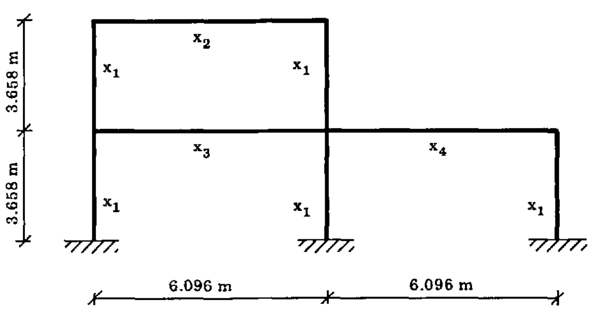


Figure 2. Geometry and optimization variables.

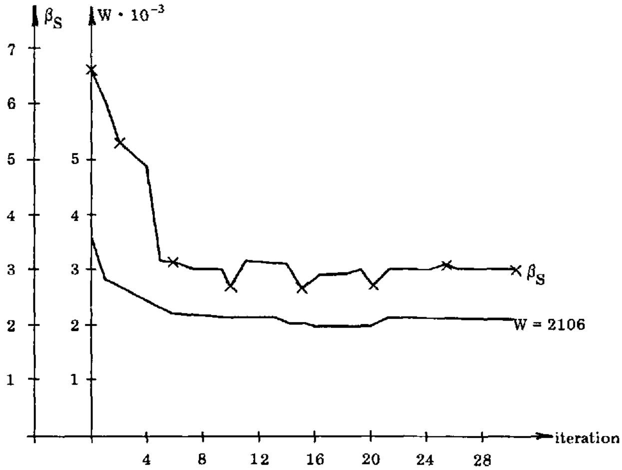


Figure 3. Iteration history for failure defined at level 2. X indicates identification of significant failure modes.

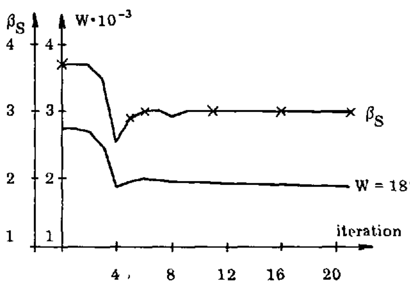


Figure 4. Iteration history for failure defined at mechanism level. X indicates identification of significant failure modes. The NLPQL algorithm is used.

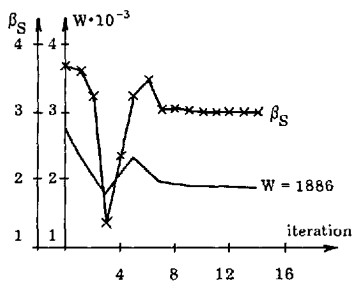
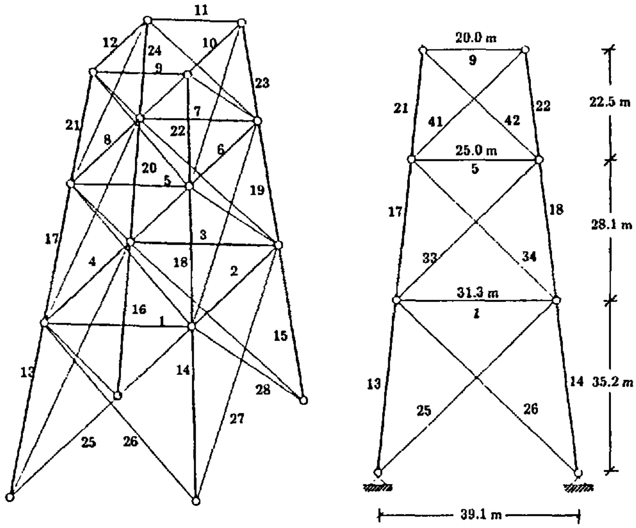


Figure 5. Iteration history for failure defined at mechanism level. X indicates identification of significant failure modes. The simple optimization algorithm in section 4 is used.

For failure defined at level 2, $I_c = 5$, $x_{max} = 20$, the PNET method used to evaluate the systems reliability index β_S approximately and the NLPQL algorithm used for the optimization the iteration history is shown in figure 3. The computer time is 1203 sec. (CDC Cyber 170-730). Some fluctuations are observed. These are mainly due to the stepwise updating of the significant failure modes. The optimal areas are $\underset{\sim}{x}$ = (52.0, 51.9, 82.6, 55.0).

With the same parameters the iteration history for failure defined at mechanism level is shown in figure 4. The computer time is 150 sec. Again it is seen that the process converges, although there is a great fluctuation at iteration no. 4. The optimal areas are $\underset{\sim}{x}$ = (42.9, 50.7, 70.3, 59.8).

In figure 5 the iteration history is shown for the same example as in figure 4. The only differences are that the simple optimization algorithm b in section 4 is used instead of the advanced NLPQL algorithm and $I_c = 1$. Also with this algorithm the process converges. The optimum



area	elements
x_1	1, . . . , 4
x_2	5, . . . , 8
x_3	9, . . . , 12
x_4	13, . . . , 24
x_5	25, . . . , 32
x_6	33, . . . , 40
x_7	41, . . . , 48

Figure 6. Spatial truss tower.

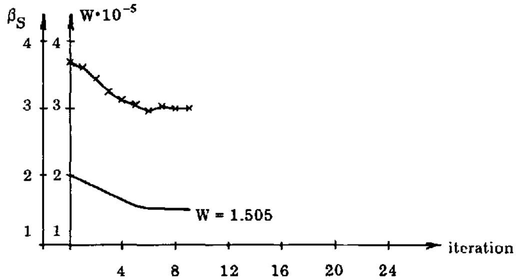


Figure 7. Iteration history for failure defined at mechanism level. X indicates identification of significant failure modes. The simple optimization algorithm in section 4 is used.

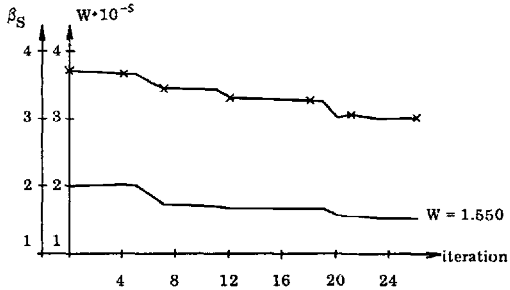


Figure 8. Iteration history for failure defined at mechanism level. X indicates identification of significant failure modes. The NLPQL algorithm is used.

point is the same and the same great fluctuation is found (here at iteration no. 3). The computer time is 124 sec.

In the following example consider the three-dimensional truss structure in figure 6. The structural system is a model of a steel jacket offshore platform and has 48 structural elements. But only 7 of them are chosen to be different in the optimization, see figure 6.

The loading (16 concentrated loads) and the 48 failure elements (potential axial yielding elements) are modelled by 2 + 48 normal stochastic variables with constant coefficients of variation. The expected values of the failure elements are determined by

$$\mu_i = x_i \cdot 270 \cdot 10^3 \text{ kNm}^{-2} \quad , \quad i = 1, 2, \ldots, 7$$

Further details concerning the loading, the correlation structure, and the β-unzipping can be found in Sørensen et al. [14]. The method of average correlation coefficients, see section 4, is used to evaluate β_S approximately.

With the same parameters as used in the first example and failure defined at mechanism level iteration histories are shown in figures 7 and 8. In figure 7 the result from using the simple optimization algorithm b in section 4 is shown. Convergence is obtained after 9 iterations. The total computer time is 8191 sec. (2805 sec. for identification of failure modes, 4625 sec. for evaluation of β_S, and 761 sec. for optimization calculations). The optimal areas are (0, 105, 127, 172, 0, 2.7, 271).

In figure 8 the result of a run with the NLPQL algorithm is shown. After 26 iterations convergence is not obtained. The run stopped because the algorithm could not find a better point. The reason is probably that the reliability constraint is very flat in the area about the minimum point. The object function value is 3% greater than the value which was found using the simple optimization algorithm and the areas at the point where the algorithm stopped are (0, 60, 88, 248, 0, 5.5, 212). The computer time is 11266 sec. It is seen that three of the areas are almost 0.

The results of the latter example therefore indicate that the simple optimization algorithm (see section 4) is better than the advanced NLPQL algorithm to find the optimal areas in a structural system with reliability constraint. Further investigations of the effect of the choice of the parameters in the algorithms are being performed.

6. CONCLUSION

The optimization problem which is considered in this paper is to find the minimum weight of a structural system subject to the constraint that the reliability of the structure exceeds a critical value.

In this paper it is shown that it is generally not possible to establish that the optimization problem is convex. A simple example demonstrates this.

To solve the optimization problem a new optimization procedure is developed. The procedure is composed of three main parts, namely identification of significant failure modes, evaluation of the systems reliability index, and calculation of the optimal point. Since the first two parts are the most computer time consuming special considerations are given to these parts in designing the procedure. Two different optimization algorithms are investigated, namely a simple procedure based on linearization of the constraint and the NLPQL algorithm which is an advanced procedure based on solving sequential quadratic subproblems.

In two examples the procedure is investigated. In the first example runs with both algorithms converge. In the second example only the run with the simple optimization algorithm converges. The run with the NLPQL algorithm stops at a point with a value of the weight which is 3% greater. This result indicates that for complex structural systems with reliability constraint it is better to use a simple optimization algorithm.

7. REFERENCES

[1] Cornell, C. A.: *A Probability-Based Structural Code.* ACI Journal, American Concrete Institute, Vol. 66, 1969, pp. 974-985.

[2] Moses, F.: *Design for Reliability-Concepts and Applications.* Optimum Structural Design (eds. Gallagher & Zienkiewics). John Wiley, N. Y., 1973, pp. 241-265.

[3] Ditlevsen, O.: *Basic Reliability Concepts.* Proceedings, NATO Advanced Study Institute (ed. P. Thoft-Christensen), Martinus Nijhoff Publishers, The Netherlands, 1983, pp. 1 - 56.

[4] Thoft-Christensen, P. & Baker, M. J.: *Structural Reliability Theory and Its Applications.* Springer-Verlag, Berlin, 1982.

[5] Thoft-Christensen, P.: *Reliability Analysis of Structural Systems by the β-Unzipping Method.* Structural Reliability Theory, Paper no. 3, Institute of Building Technology and Structural Engineering, The University of Aalborg, 1984.

[6] Lind, N. C. & Krenk, S. & Madsen, H. O.: *Safety of Structures.* Prentice Hall, 1985.

[7] Thoft-Christensen, P. & Sørensen, J. D.: *Calculation of Failure Probabilities of Ductile Structures by the β-Unzipping Method.* Institute of Building Technology and Structural Engineering. The University of Aalborg, 1982.

[8] Ang, A. H.-S. & Ma, H. F.: *On the Reliability Analysis of Framed Structures.* Proc. ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability, Tucson, 1979, pp. 106-111.

[9] Thoft-Christensen, P. & Sørensen, J. D.: *Reliability of Structural Systems with Correlated Elements.* Applied Mathematical Modelling, Vol. 6, 1982, pp. 171-178.

[10] Schittkowski, K.: *Theory, Implementation, and Test of a Nonlinear Programming Algorithm.* Proc. Euromech-Colloquium 164 on »Optimization Methods in Structural Design», 1982, Bibliographisches Institut, Mannheim, 1983, pp. 122-132.

[11] Fleury C.: *Structural Weight Optimization by Dual Methods of Convex Programming.* International Journal for Numerical Methods in Engineering, Vol. 14, 1979, pp. 1761-1783.

[12] Gorman, M. R.: *Reliability of Structural Systems.* Report No. 79-2, Case Western Reserve University, Ohio, Ph.D. Thesis, 1979.

[13] Thoft-Christensen, P. & Sørensen, J. D.: *Optimization of Structural Systems.* Structural Reliability Theory, Paper no. 6, Institute of Building Technology and Structural Engineering, The University of Aalborg, 1984

[14] Sørensen, J. D. & Thoft-Christensen, P. & Sigurdsson, G.: *Development of Applicable Methods for Evaluating the Safety of Offshore Structures, Part 2.* Structural Reliability Theory, Paper no. 11, Institute of Building Technology and Structural Engineering, The University of Aalborg, 1985.