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Beyond Newton, Leibniz and Kant: insufficient foundations, 1687 to 1786

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My aim below is to examine critically whether Newton's, Leibniz's, and Kant's respective doctrines of body and motion are sufficient foundations for classical mechanics. I argue that they are not. Each comes up short in various respects and to various extents. In consequence, I urge that, to discover the true foundations of mechanics, we must look beyond these three figures. And also that, because their respective foundations are insufficient, we ought to debate exactly what we historians and philosophers may hope to get from looking at them.

To make my case, I begin with an analytic preamble that elucidates the various senses of sufficiency and mechanical foundations I assume in this chapter (section 1). Next, I dispel some old prejudices and misunderstandings about mechanics after Newton (section 2). And, I give a synopsis of its structure and scope, based on newer research (section 3). Then I check whether the foundations above are sufficient for *that* mechanics. In particular, I show that Newton's, Leibniz's, and Kant's *laws of motion* are too narrow for it (section 4). And, that their respective *pictures of matter* are likewise insufficient foundations (section 5). I end with some morals and suggestions for future research.[1]

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[1] For Newton, I use his *Principia.* For Leibniz, I use his *Specimen dynamicum* and related texts (see nos. 14, 16, 22-3, 31, A3-4 and E3-4 in Leibniz, *Essays*). For Kant, I rely on his 1786 tract, *Metaphysical Foundations of Natural Science* (henceforth *MAN*).

This paper is summative: it relies on, and connects, work I have done elsewhere. Thus, it really ought to be read in conjunction with that work, for my point to sink in.

1. BACKGROUND DISTINCTIONS

To preempt confusion, and to help the reader see the force of my concerns, I must explain my charge of insufficiency above. A mechanical foundation *F* can be sufficient in two senses.[2]

Weak *F* entails a theory of mechanics *M*, and all of *physics* can be reduced to *M*.[3]

Strong *F* is enough to represent all mechanical phenomena. Namely, it entails equilibrium conditions and equations of motion for all possible bodies.

I will explain these matters further as the need arises. Next, I must elucidate the particular foundations I have in mind for my case in this paper. Labeled for efficient use, they are as follows:

N Newton's Definitions I through VII, his three laws of motion (including **f** = m**a**), and their six corollaries in *Principia.* The law of

[2] Broadly speaking, *F* is a set that contains concepts, laws, mathematical theorems, and perhaps heuristics for problem-solving and theory buildup.

[3] This sense might seem idle, but it was long influential. Descartes' 1644 *Principles of Philosophy* advocated for reducing physics (optics, magnetism, heat flow, even physiology and earth science) to a mechanics of matter in motion (specifically, of action by contact via collision and ether pressure). And so did Hobbes, in *De Corpore* of 1655. Centuries later, Hertz urged: "All physicists agree that the problem of physics consists in tracing the phenomena of nature back to the simple laws of mechanics" (Hertz, *Mechanics*, xxi). Halfway between these termini was Fischer's program of a *mechanische Physik*, whose influence extended to France (see his *Physique mécanique*).

universal gravitation. Optional: the matter theory in Query 31 of his *Opticks*.

L Leibniz's taxonomy of forces in *Specimen dynamicum*. Conservation of Vis Viva. Matter regarded as a deformable continuum.

K Kant's primitive concepts, their explications, his derived theorems, and the three laws of mechanics in his *MAN*.

These preliminaries now enable me to state my thesis more precisely:

> *Neither N, nor L, nor K are strongly sufficient foundations. None has the resources to represent the mechanical behavior of all bodies.*

Most of my case below marshals evidence for the thesis above. Before I do so, however, I need to cast more light on two aspects above, viz. 'representing' and 'mechanical behavior.'

2. THE SHAPE OF MECHANICS AFTER 1730

It is seductive, and has long been entrenched, to use Newton as the best vantage point for grasping the structure and foundations of mechanics after him (up to 1905-18, when Einstein supposedly displaced Newtonian theory). Here I will explain briefly why that is wrong.

The *Principia* contains a rational mechanics of 'centripetal' forces—the mathematics of particle orbits in fields of central acceleration—and then an application of this apparatus to *one* particular species of centripetal force in nature, viz. gravity. Knowing that he had discovered just one species of impressed force, Newton in a famous exordium to his *Principia* urged future generations to continue his program of discovery:

> If only we could derive the other phenomena of nature from mechanical principles by the *same* kind of reasoning! For many things lead me to have a suspicion that *all* phenomena may depend on certain *forces* by

> which the particles of bodies, by causes not yet known, either are impelled towards one another and cohere in regular figures, or are repelled from one another and recede. As these forces are *unknown*, philosophers have hitherto made trial of nature in vain. But I hope that the *principles* set down here will shed some light on either this mode of philosophizing or some truer one. (Newton, *Philosophical Writings*, 60f.; my italics)

Now combine his exhortation with two pieces of received wisdom. One is Mach's old chestnut that Newton's *laws* of motion are a sufficient basis for *all* classical mechanics. The other is Kuhn's old image of mechanics after Newton being 'normal science.'[4]

Together, these elements can strongly seduce the reader into thinking that, from 1687 to Einstein, mechanics was in the business of discovering more forces and their specific laws by following Newton's *recipe* and by building on his *foundation N* above.[5]

However, the Mach-Kuhn framing is wrong, on conceptual and empirical grounds; hence so is the historical picture that falls out of it. It is *not* true that mechanics after Newton was in the business of discovering more forces and their laws, by emulating his success with gravity. And, it is not true that mechanics post *Principia* built on his foundation *N*. In fact, mechanics then had a different agenda, pursued with very different tools, and evaluated from very different criteria of success.

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[4] "Newton's principles suffice for solving *every* mechanical problem we encounter in practice, whether in statics or dynamics. We need not appeal to any *new principle* for that. If we run into obstacles, they are always just mathematical. Not difficulties with the *principles*" (Mach, *Mechanik*, 239; my emphasis). Kuhn counted Newton's *Principia* as a paradigm—the exemplary achievement of classical mechanics—and claimed that it "served for a time implicitly to define the *legitimate problems and methods* of a research field for *succeeding generations* of practitioners" (Kuhn, *Structure*, 10; my emphasis).

[5] Note that, if these framing assumptions were true, they would make short work of Leibniz, Kant, and anyone who diverged from the Newtonian program above. If mechanics post 1700 was in fact as the Mach-Kuhn has it, then attempts to supplant it (as Leibniz tried, with *vis viva)* or to correct it (as Kant tried, with foundation *K*) must appear as doomed to fail, or at least seriously misguided. I thank Katherine Brading for enlightening discussion of these broader points.

Here is why and how. To treat behaviors from theory—to incorporate them into mechanics—requires an indispensable thing, viz. obtaining the equation of motion for that particular type of behavior. For instance, the wave equation: the differential formula that quantifies how every point in a flexible string (more generally, in any harmonic oscillator) moves in an instant once the string is made to vibrate. By the way, that formula was discovered in 1747, by d'Alembert.

This requirement is crucial for understanding my claim that post-Newtonian mechanics is drastically different from Newton's approach and results in *Principia.* First, there are *no* equations of motion in his book.[6] Pursuing them became the chief priority after 1730, at first with the Bernoullis and their associates, then collectively in continental Europe. Second, equations of motion must be derived from *dynamical laws*: from principles that relate mechanical agency (be it force, power, work, energy, or action) to kinematic change in space and time. Third, post-Newtonian theorists seek dynamical laws shown to be *general.* That is, some one or two principles that entail equations of motion for *all* species of extended body.

In sum: the old prejudice was that post-Newtonian mechanics aimed to discover more forces and their laws, by emulating Newton's heuristics, and by starting from his principles. In contrast, recent research entails the key objective of mechanics was different—namely, to derive equations of motion for a very broad spectrum of bodies, from dynamical laws thereby shown to be general.

This radically novel and extremely demanding objective is sine qua non for understanding the true shape and growth of mechanics after 1700. It has no precedent in the theories and research programs of the 1600s; and it is easy to miss if one looks at mechanics with Newton's achievement as our lens for history.

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[6] Decades ago Truesdell had pointed out that differential equations—the key representational device of modern mechanics—are absent from Newton's tract (Truesdell, *Essays*, 90). We may wonder if Newton would have even recognized the need for them in mechanics. I thank George E. Smith for stimulating discussion on this topic.

3. SUFFICIENT FOUNDATIONS, 1760 TO 1830

For a century after 1730, mechanics sought to mathematize two very broad classes of kinematic behavior: the motion of *extended* bodies; and motion with external *constraints*, viz. obstacles to free translation. It was all slow work—a testimony to how difficult these classes were.[7]

Mechanics came close enough to conquering these two areas in the 1820s. By the end of the Old Regime, it had already made enormous advances at the hands of Euler and Lagrange; later progress was due to Navier and Cauchy. Thanks to them and a few others, toward the mid-19th century it became clear that only two candidates had any realistic chance to be foundations in the strong sense. I present them here.

Euler-Cauchy laws. The motion of extended bodies was conquered stepwise, one species at a time. First they mathematized low-dimensional elastics, then rigid bodies, then ideal fluids, then elastic solids, and then at last viscous fluids.[8] After a century of effort, theorists learned that one dynamical law is indispensable for all these species. Namely, it is a required premise for deriving the equations of motion for every species. That law is:[9]

$$\mathbf{b} + (-\nabla\mathbf{T}) = \rho\ddot{\mathbf{x}} \qquad (1a)$$

Expressed in words, it says: at every point of an extended object in motion, the net body force plus the gradient of the local contact forces (i.e. stresses) equals the point's change of linear momentum in an instant.

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[7] Claims in this section depend on research carried out in Brading & Stan, *Philosophical Mechanics*, chapters 8-12. For elaboration, the reader is invited to consult it.

[8] Other types of extended-body motion (e.g. plasticity, fracture, hysteresis, creep, and brittleness) had to wait until the 20th century for their mathematization.

[9] $\mathbf{b}$ is the net body force, $\mathbf{T}$ the stress, or internal force, ρ the mass density, and the acceleration (the second derivative of the position vector $\mathbf{X}$). An early version of this law is in Cauchy, "Sur les equations."

Bur for some motions (namely, for rigid bodies and elastic solids) the expression is not quite enough to represent their every quantitative aspect. Rather, it must be combined with *another* law; only together do they completely represent the change at a point. That second law is:[10]

$$\mathbf{H} = d\mathbf{L}/dt \tag{1b}$$

Again in words: the net torque on a body forces equals the change of angular momentum (at a point, in an instant).

The figures who did most to showcase the descriptive power of these laws—their vast descriptive reach—were Euler and Cauchy. Thus, following recent tradition, I named the laws after them.[11]

Lagrange's law. Another area of intense research after Newton was constrained motion.[12] It was an arduous domain in which the 1600s had bequeathed no useful principle or heuristic for problem solving. In particular, Newton's second law was of no help.[13] And so, theorists in effect had to create this area of mechanics from the ground up. Clairaut and d'Alembert made very important advances, but only Lagrange in 1788 would obtain a general solution. Namely, a dynamical law that governs all species of motion, free or constrained; plus a method for quantifying

[10] **H** is the net impressed torque, and **L** the angular momentum. The earliest expression of this law is Euler, "De motu in superficiebus," § 48.

[11] See especially Truesdell, *Rational Continuum Mechanics*, 64ff.

[12] Generally, a constraint is a limit on how a particle or a body is allowed to move. Some constraints are external to the body. E.g. an inclined plane, which prevents the body from moving straight down (under the force of gravity). Other constraints are internal to the body. E.g. rigidity, which prevents the body's component points from changing their relative distances.

[13] The reason is, the physical basis that secures the constraints—e.g., forces (if they are forces), their specific laws, and mechanisms of action—is not known in advance. It is not given at the outset of building the theory of mechanics. But, to apply Newton's second law, that required knowledge *must* be available at the outset. The law really says that $\sum\mathbf{f} = m\mathbf{a}$, viz. the actual acceleration is the result of *all* the forces acting at that point. Absent knowledge of some forces, the law becomes inapplicable. The "most widespread mistake about Newton's three laws of motion is that they alone sufficed for all problems in classical mechanics."—Smith, "Newton's *Principia*," § 5.

the action—the motion changes it induces—of any constraint, no matter its particular makeup.

I call his principle 'Lagrange's law,' in line with some latter-day authors who have explained the merits of his demarche.[14] The law is a statement about the virtual work done in a mechanical system. It reads:[15]

$$\sum \mathrm{F}\,\delta f \;+\; \sum (-m\ddot{\mathbf{x}})\,\delta x \;+\; \sum \boldsymbol{\lambda}_i\,\delta L = 0 \qquad (2)$$

It says that when a set of bodies move, we may regard it as the target of three mechanical agencies: actual forces applied to it; certain fictitious forces; and the action of constraints on its motion. Each agency does virtual work, viz. it *could* displace its target mass by an amount δr. The law says that the net virtual work of all these agencies (forces and constraints) vanishes across the system as a whole.[16]

Incidentally, the law includes Lagrange's great breakthrough—his general treatment of constraints, which Newton's laws cannot handle. Lagrange reasoned as follows. Let the action of any constraint in the system—the amount whereby it changes the motion of the mass it prevents from moving—be some amount $\boldsymbol{\lambda}$.[17] And, let δL be a virtual displacement (of the target mass) compatible with that constraint. Lagrange's insight was that, across the system as a whole,

$$\sum \boldsymbol{\lambda}_i\,\delta L = 0$$

[14] For the origin of the idea that this is really Lagrange's law, together with a lucid explanation of its role in solving constraints, see Papastavridis, *Analytic Mechanics.*

[15] See, for instance, Lagrange, *Mechanique*, 53ff. **F** is any actual, impressed force on a mass i in the system; δf is a virtual displacement that **F** would cause in i. And, $-mi\ddot{\mathbf{x}}i$ are so-called 'reverse effective forces' (or also 'kinetic reactions'), viz. fictitious forces supposed equal and opposite to the particle's effective acceleration $\ddot{\mathbf{x}}i$; and δx a virtual displacement in their direction. Finally, $\boldsymbol{\lambda} i$ is a Lagrange multiplier, and δL a virtual displacement compatible with the constraint given by $\boldsymbol{\lambda} i$.

[16] By a mechanical system I mean one or more masses, point sized or extended.

[17] The modern name for this quantity is a 'Lagrange multiplier.'

namely, the virtual work of all the constraints, together, cancels out.[18]

I end with an important note. Of the two laws above, Lagrange's is the most powerful. Namely, it can be used to quantify as well the kinds of motion that Euler-Cauchy principles can represent. The converse does not hold: Euler-Cauchy laws are not strong enough to describe the motion of masses with external constraints (which Lagrange's law was designed to treat). Recall, that was a blind spot of Newton's second law as well. In effect, Lagrange's is really the only principle sufficient in the strong sense.[19]

4. INSUFFICIENT FOUNDATIONS: LAWS

Central to the respective foundations of Newton, Leibniz, and Kant are their laws of motion. Below, I explain why those laws are insufficient. There are two ways to do that, long and short.

Here is the short one. Neither *N* nor *L* nor *K* are the same (in terms of physical content and mathematical structure) as the Euler-Cauchy laws or Lagrange's law. But, only these laws have been shown—they alone have the track record—to be sufficient foundations, or very nearly sufficient. Hence, Newton, Leibniz and Kant did not give us enough basis for mechanics. So, we must move beyond them.

This explanation is brief and hard to impeach, but is not as illuminating as it gets. Accordingly, next I review the three candidates, and point out precisely what they lack.

Newton. The foundation *N* is deficient in three respects: it lacks enough concepts of force, enough laws, and enough expressive power. From the Bernoullis to Cauchy, it took a century of struggles to gain the

[18] This capsule of Lagrange's result is perforce terse, hence hard to follow, understandably. For a longer, more lucid explanation, see Brading & Stan, *Philosophical Mechanics*, chapter 11

[19] The evidence for my claim is Hamel's extensive treatise *Theoretische Mechanik*, which, from Lagrange's law above, derives equilibrium conditions and equations of motion for all the species of body treated by then (viz. rigid, flexible, fluid, and elastic).

hard-won insight that, in order to mathematize the motion of extended bodies—fluids, elastics, plastic solids, and the like—we must distinguish between *internal* and *external* forces.[20] To describe exactly how matter moves at any point in an extended body, we must add up the actions of both kinds of force. However, these two kinds are *mathematically unlike*: external forces are vectors, internal forces are tensors.[21] To find out how the net internal force at a point (in an extended body) contributes to moving that point over an instant, we must use the *gradient* of that force. Not the quantity of the force itself.[22] That gradient then combines with the net external force to cause in that point a momentum increment $\rho\ddot{\mathbf{x}}$, just as the Euler-Cauchy law has it.

Against this background, here is how N is insufficient. First, Newton did not have the distinction internal vs external force. It is neither explicit in his definitions, nor implicit in his practice of building the rational mechanics of *Principia.* Second, because he lacked that distinction, his most important principle—the second law above—is too weak to apply to most bodies. It is really quite narrow: the law can describe just two species of object, namely, a free mass point and one special point in a rigid body, in very special situations.[23] This fact was known at the time:

> In a vacuum, a material *point* in projectile motion describes a parabola. From that, we can understand why a *body* too will cross a parabola, if

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[20] External forces originate—they are exerted by sources—outside the bounding surface S of an extended body. Examples: gravity and magnetic forces. Internal forces are exerted below S (inside the extended body), due to the body's parts acting on one another. Examples: pressure in a fluid, and stress in an elastic solid.

[21] Interpreted geometrically, a vector is an arrow-like object with a length (size) and a direction. It was used to represent the action of an impressed force on a mass point; e.g. the velocity increment (acceleration) in $\mathbf{f} = m\mathbf{a}$. A tensor is analogous to a bundle of 9 arrows, or vectors; see next footnote.

[22] A tensor-like force acts on a small volume of matter to compress, stretch, or twist it. Cauchy called it 'pressure or tension,' to indicate that it does more than just translate a point over a small distance (which vectorial forces do). We call is 'stress.'

[23] Namely, only when the net resultant (of all the external forces) pass through the body's mass center. If it does not, the resultant induces motion effects (e.g. precession) that Newton's second law cannot predict.

> we throw it. But, that point motion alone will *not* teach us the laws governing the motion of *individual parts* in a finite body. ... What Newton has proved about motion under centripetal forces applies just to a *single point.* (Euler, *Mechanica*, v-vi; added emphasis)

They just did not know how to overcome that major limitation of Newton's second law. It took a century to overcome it.[24] Third, the foundation *N* lacks the mathematical resources to even *represent* the action of internal forces. That task requires the calculus of partial derivatives. Newton did not have it, and the concepts he did have are too weak to make up for that lacuna.[25]

Leibniz. The chief law of *L* is not sufficient for the strong task either. My evidence for this comes from the growth of mechanics after 1700. Historically, Conservation of Vis Viva played two roles, and neither role counts as a fundamental law for all mechanics.

Some used it as a *premise* for deriving a narrow range of results, but none of these results was an equation of motion. Rather, each counts as an effective parameter, namely, a quantity specific to the whole motion (the path crossed in a finite time) of a *special point* in an extended body; or by a small part of a mechanical system. For instance, the maximum height to which a body's center of mass can rise under gravity. For instance, the characteristic frequency for the oscillating motion of the midpoint in a string made to vibrate. For instance, the sum of speed and pressure for a thin slice of fluid moving in a tube of variable width.[26]

And, some proved that, in certain cases, Conservation of Vis Viva was a *consequence* of the equations of motion. In particular, that when certain mechanical systems are left alone—if no exogenous force acts on them—they will move such that their total *vis viva* remains constant

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[24] Again, for details and history, cf. Brading & Stan, *Philosophical Mechanics*, ch. 10.

[25] Newton did not have the term 'derivative.' He just had a proto-version that he called a 'fluxion.' That Newtonian concept overlaps with our modern notion of rate of change (of a variable quantity in respect to another, e.g. dx/dt or even dr/dx). But, it cannot capture the idea of a partial rate of change, which, say, $\partial f/\partial x$ expresses.

[26] For the first example, cf. Propositions 39-41 of Newton's *Principia.* For the second, see the final section of Daniel Bernoulli's *Hydrodynamica* (1738).

over time. That is the class of systems made of masses interacting by 'conservative' forces, viz. derivable from a function V that depends only on the relative distances between these masses. Clairaut in the1740s and Lagrange in 1788 were the chief figures for this line of thought.[27]

In sum, Conservation of Vis Viva sometimes entailed some special quantitative aspect of a whole motion, or path integral; and sometimes it was a corollary of the equations of motion. Not a premise for them, which a fundamental law of mechanics must be.

Kant. The set K is even less sufficient as a descriptive basis for a broad theory of mechanics. Two laws in it (conservation of mass, and the equality of action and reaction) do useful work, but for a limited range of motions, and only as auxiliary premises. By themselves, neither suffices to determine *any* motion.

Specifically, Conservation of Mass is a co-premise in fluid dynamics. Euler derived a version of it—he called it the 'continuity equation,' as do we. From a species of the Euler-Cauchy law (1a) above, in 1755 the Swiss mathematician derived the equation of motion for the instantaneous change at any point in a frictionless fluid.[28] With his formula in place, Euler then explained that, to determine the fluid's motion *completely*, his formula is not quite enough. Rather, it must be supplemented with Conservation of Mass (which Kant has), but stated in the exact form that is the Continuity Equation above.[29] Then Navier in 1821 extended Euler's success, but for the more complicated case of viscous fluids. Navier first inferred the strength of the friction-like effect (the viscosity tensor) that

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[27] In our terms, they showed that, if interaction forces in a system are given by (monogenic) potentials, then Conservation of Vis Viva is a 'first integral of motion,' i.e. a quantity conserved over a finite stretch of time. For additional discussion and historical details, see Brading & Stan, *Philosophical Mechanics*, chs 8 and 11.

[28] That formula is known as the Euler Equation for an ideal fluid. For its history, see Darrigol, *Worlds of Flow*, ch. 1; Brading & Stan, *Philosophical Mechanics*, ch. 10.

[29] Why it is not enough: Euler's Equation determines just the change of *velocity* at a point; but when a fluid moves, there is *mass flow* as well—the density at that point changes over time. This latter change is what the Continuity Equation (or Conservation of Mass) describes exactly: $\partial\rho/\partial \mathrm{t} + \partial\rho\mathbf{v}/\partial x_i = 0$. In words, in a volume element, the mass density at an instant equals the mass in it at the previous instant, plus the rate of mass flow across the volume's bounding surface.

two adjacent volume elements in a fluid exert on each other. Which he then added to the dynamical part—the left-side half that sums the local actions at a point—in Euler's Equation for a perfect fluid. Thereby, he obtained the famous Navier-Stokes equation, a more realistic description of how fluids in our world move. Navier too knows that his equation alone is not enough to describe the fluid's behavior (at a point) completely. It needs supplementation: "we must add the equation of continuity."[30]

In sum, Kant's law (conservation of mass) works as an auxiliary premise, for the case of fluid motion alone, provided we state it carefully and exactly, viz. as the Continuity Equation.

What about Kant's law of action and reaction, the other non-trivial principle in K? It is even weaker than his first law above, I submit. It too does work merely as a *co-premise* for a narrow class of motions, viz. the exit speed of a special point (the mass center) in extended bodies undergoing impact, or collision. That is the only quantitative use to which Kant ever put it, even though that law spans three decades in his career as a natural philosopher.[31] Even so, that law by itself is unable to entail any determinate content about motion changes (in impact). It too needs supplementation with other premises, or laws, depending on the type of collision at issue. Only in conjunction with them does it yield a description of motion changes for that process.[32]

Even beyond Kant's narrow focus on impact—for instance, in the course of the gravitation theory articulated in *Principia*—the law of action and reaction remains insufficient. Newton there used it in two contexts. In one, the law allows Newton to redescribe a particle's motion

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[30] Navier, "Lois des fluides," 252.

[31] In Kant, the Equality of Action and Reaction first shows up in a 1758 paper on collision theory. In *MAN*, he again applies it to impact, and extends its range (without explanation) to action-at-a-distance forces too; cf. Stan, "Kant's third law" and Friedman, *Kant's Construction.*

[32] Outcomes of collision range between two limit cases. One is inelastic impact, the other is elastic collision. To infer the outcome of each case, another premise (beside the law of action-reaction) is needed. For inelastic collision, that premise is Zero Relative Speed (viz. that the two bodies move together after impact). For elastic collision, it is the conservation of kinetic energy, or *vis viva.*

(e.g. the Kepler orbit it crosses under a centripetal force) from one reference point to another.[33] But that presupposes the motion has been determined *already*, before he gets to appeal to the third law. In the other context, Newton uses the law of action-reaction so as to infer—from the existence of force (gravity) on a body—to the existence of another force, on a different body. No equation of motion is at issue there.

To sum up, the other law in *K* is likewise not enough. It does some regional work—it shows up in the inference to the mathematical description of two particular types of motion—but just as an *auxiliary* premise. By itself it is insufficient, even for that narrow domain.

I end with the most serious issue. There is a lacuna in Kant's foundation that makes it quite weak—really, the least sufficient among the three candidates *N*, *L*, and *K*. It is this: Kant lacks any principle that allows us to infer how the motion changes if a material point is disturbed: if an exogenous mechanical agency causes that point to change its state. There is nothing in *K* to let us infer any determinate answer to that generic question. But that is exactly what dynamical laws are for. It is the chief virtue of the laws of Euler-Cauchy and Lagrange. That is what makes them *necessary* foundations for all mechanics, not just sufficient. Kant lacks even the weakest species of this sort of indispensable foundation.[34]

5. INSUFFICIENT FOUNDATIONS: MATTER

Early modern doctrines contained another thing (beside laws and principles) designed to work as a foundation for mechanics. That thing was a picture of matter—an account of bodies qua material objects. Rational

[33] Newton models first the orbit that results if the force on a particle *P* is directed to a point *fixed* in space. Then he supposes that force to emanate from another *particle* that is itself in motion. He proves that, relative to the *mass center* of these two particles, then the orbit of *P* is likewise elliptical.

[34] Separately, Watkins, *Kant on Laws*, and Stan, "Evidence and explanation," have noted that not even Newton's second law—the basis for the equation of motion of all free mass points, though not extended bodies—is to be found in Kant's foundation. Here I am just explaining the force of that alarming conclusion.

mechanics was the study of *that* generic object: the motion of bodies. *N*, *L*, and *K* contain each a picture of material objects. In this section, I argue that these pictures are also insufficient for a general mechanics. Just like the laws I examined above. That completes my overall case that *N*, *L*, and *K* are not complete mechanical foundations.

Here too, some analytic clarification at the outset is in order. 'Matter theory,' or 'picture of body,' can mean one of two things:

> **Content** A list of *attributes* (viz. properties and causal powers) that all bodies have universally.
>
> **Architecture** An account of the *geometry of mass distribution* at basic scales. Examples: the mass point; the rigid body; the deformable continuum.

With this distinction in hand, my claim in this section is that—whether we read them as Content or as Architecture—the matter theories in *N*, *L*, and *K* are each insufficient in the strong sense.

For three reasons, in this section I discuss Kant at length, with little attention to the other two figures. For one, the picture of matter in *N* is easy to dispatch as insufficient; and it was not really part of Newton's considered view.[35] For another, the picture *L* is a rudimentary version of *K*, and so my verdict about Kant will carry over to Leibniz, *mutatis mutandis.* Last but most important, the matter theory in *K* is the most explicit and detailed—it takes up a good deal of Kant's *MAN*—and so it repays sustained discussion. Accordingly, I move on to examine *K* from the point of view of content.

Content. To decide if a matter theory is sufficient, let us begin by asking: given a proposed matter theory, what is it for—what does Content do for rational mechanics? At least among Kant scholars, a frequent answer is that Content is explanatory: it explains why bodies obey the

[35] In natural philosophy, Newton's standard of evidence was 'deduction from phenomena.' The above theory of matter did not clear his standard, and so he offered it (in Query 31 of his *Opticks*) not as considered doctrine, but as an (initially plausible) proposal for further research.

fundamental laws of that science.[36] Then let us assess how well *K* discharges this task. Assembled from his *MAN*, the Content view of a Kant body is:

> *Matter is mobile, impenetrable, and carries momentum. A body is a finite volume of matter.*

For it to be a sufficient explanatory basis, two things are needed. First, Kant's matter theory must help us understand how or why bodies (as he defines them) move as dictated by the *truly* general laws: the Euler-Cauchy or Lagrange laws. Not by the laws he gave in *MAN*, because those are not sufficiently general. Second, his Content must explain every feature of the general laws—every quantitative aspect of the motion behavior it represents—or else it is explanatorily insufficient.

But here is a reason for concern: *K* does not even explain the common minimum of the general laws, i.e. the mechanical behavior that the two laws above, otherwise distinct as they are, have in common. (Each law—because it is more determinate and specific than the common minimum—then places further explanatory demands on Content.) Now that common minimum is: *matter exerts, and responds to, impressed forces.* Both laws above explicitly contain the notion of impressed force; and neither can be stated without it.[37] If *K* has no matter-based explanation for it, then *K* is insufficient in that respect.

Architecture. The packages *N*, *L*, and *K* include as well a view about the distribution of mass at basic scales. Is that a sufficient foundation? I doubt it. To see why, here too I begin by asking what Architecture is for—what work does it do, within rational mechanics?

It plays two related roles. One is *metaphysical:* it is a sharper picture (a more precise, determinate description) of the fundamental object that

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[36] Often, they couch this answer in terms of grounding as explanation. For examples and critical discussion, see Stan, "Evidence and explanation."

[37] Stresses are a more general species of impressed force, and their gradients (as in the first Euler-Cauchy law) are impressed forces, like gravity. In Lagrange's law, both the applied and also the reverse effective forces are kinds of impressed force.

mechanics is a theory of: rational mechanics quantifies the motions of *those* objects.[38] The other role is *semantic:* an architecture models the reference of the fundamental laws. It pictures the objects to which the laws (qua the most general equations of mechanics) refer.

At this point, it becomes easier to see why the material foundations of *N*, *L*, and *K* are insufficient. I begin with Newton. His architecture of matter was the rigid body—tiny but finite, inflexible volumes of mass:

> All these things being considered, it seems *probable* to me, that God in the beginning formed matter in *solid*, massy, *hard*, impenetrable, moveable particles, of such sizes and *figures*, and with such other properties, and in such proportion to space, as most conduced to the end for which he formed them; and that these primitive particles being solids, are *incomparably harder* than any porous bodies compounded of them; even so very hard, as *never* to wear or break in pieces: no ordinary power being able to divide what God himself made one in the first creation. (Newton, *Philosophical Writings*, 184; my italics)

Here is why this picture falls short. Newton's rigid atoms are discrete: there are small yet finite distances between them. But there are vast classes of extended-body motion that presuppose matter to be *continuous.* Paradigmatically, that is the motion of fluids, elastic solids, and plastic bodies. The equations of motion for these body types suppose that matter is continuous, not discrete.[39]

Now let us examine *K*. For the domains above where *N* fails, *K* is just right. It is precisely the architecture of matter that those parts of mechanics presuppose. Kant's picture fails elsewhere, however. That picture models matter as deformable, but some areas of mechanics require us to model it as *rigid.* For instance, the statics and dynamics of bodies with

[38] Kant in fact called his treatise a "metaphysics of corporeal nature" (*MAN*, 13). A study of early-modern matter theories from this vantage point is Holden, *Architecture.*

[39] Look again at the Euler-Cauchy (1a) law above. On the kinematic side, it relies on the quantity ρ, viz. mass density. That property obtains only in continuous matter. Discrete particles do not *have* mass density; they have just mass, m. But m does not show up in the for fluids and elastics (e.g., it is not in the Navier-Stokes equation).

constraints. Or the domain then called 'the mechanics of machines,' nowadays known as engineering dynamics and the statics of rigid structures.[40] To insist, as Kant and Leibniz did, that *all* matter is deformable, at *all* scales, is to deprive the subfields above of an object domain. It turns their equations of motion into illicit fictions devoid of reference. Alternatively, we may conclude (as I am inclined to do) that *L* and *K* are insufficient foundations for a truly general mechanics.

Objection: mechanics does not *have* a single ontology, or architecture—after all, that is an upshot of my discussion above, and of recent work as well.[41] So, it is unfair to ask that Kant should provide what mechanics lacks to this day, viz. a preferred ontology.

I respond: this objection really boomerangs back to hit Kant's doctrine, for two reasons. First, *he* insisted that his picture of matter in *MAN* amounts to a "general doctrine of body."[42] A natural reading of that phrase is that his matter theory there supports a general mechanics: the mathematical description of *any* body's motion. Either that, or a key aspect of his doctrine becomes mysterious: what does he mean by a 'general doctrine of body,' and what makes it a foundation? Second, if Kant's architecture of matter was not meant as an ontology for *all* mechanics, we ought to ask, what is it for? What *partial* role was it designed to play, and why did he think that a partial role was so philosophically important? Thus, I do not think Kant can escape the charge of insufficiency unscathed—and neither do Leibniz and Newton, for that matter.

SOME MORALS

[40] Already ancient statics—the science of the five 'simple machines,' later with the inclined plane as a sixth—was a theory of rigid bodies: those 'machines' were all supposed undeformable. On the 'science of machines' in the 18th century, especially in France, see Chatzis, "Mécanique rationnelle."

[41] See, for instance, Wilson, *Physics Avoidance.*

[42] Kant, *MAN*, 13 (4: 478).

With Mach and the Marburg neo-Kantians, we have long been tempted to think that, between Newton, Leibniz, and Kant, enough foundations for all mechanics were given in the century after 1687. That thought is wrong, I have argued above. It *seems* plausible, but only because of some tacit beliefs that, as I explain next, are themselves wrong.

First, the temptation ignores that classical mechanics is old and *ongoing*. Along the way, its logical structure, descriptive scope, and representational frameworks have changed dramatically. But so have its foundations. Then it is ill-advised to expect that complete foundations for science that is 400 years old were discovered and expressed in the first century of its long life.

Second, it ignores that, between its Galilean birth and the death of Leibniz, mechanics was able to handle just the *simplest* kind of body—free particles, viz. unconstrained masses the size of a point—for that was all it had the resources to treat. Qua objects of rational mechanics, extended bodies that plausibly behave like things in our common experience (water flowing, trees swaying in the wind, fabric stretching, soil shifting under foot, etc.) were too hard for the 17th century. So, theorists avoided them until much later. But then it is unrealistic to expect that nomic and material foundations designed for the easiest, most rudimentary parts of mechanics will survive unscathed—with no need to massively change them—when mechanics has matured enough to handle real bodies in our world.

Lessons. Then what benefit may we expect from engaging philosophically with Newton's, Leibniz's, or Kant's foundations? I suggest that uncovering how they miss the mark can teach us three lessons.

First, we often underestimate just how elusive and difficult classical mechanics is, for the philosopher of science. Seeing Newton, Leibniz, and Kant come up short (in regard to its true basis) is a sobering experience. At the very least, it ought to cure us of the stubborn prejudice that classical mechanics is easy to figure out philosophically, and that it ended when Einstein allegedly supplanted Newton.[43]

[43] My message here dovetails with lessons that Mark Wilson has long tried to teach us, e.g. in the perceptive and rewarding studies assembled in his *Physics Avoidance.*

Second, their problems remain our problems. We should not presume that our age has solved their two questions above, i.e. what laws of motion and picture of body suffice to ground a general theory. The several axiomatic presentations of mechanics that we have are partial (none have been shown to entail all of mechanics). And, it is not yet clear that classical mechanics can sufficiently rest on a single theory of matter.[44] In fact, because the problem of a complete foundation has proven so hard—Newton, Leibniz, and Kant, three of the greatest early modern minds, could not solve it—perhaps we ought to be prepared for the prospect that the problem may be intractable.

Third, a lesson for historians. For a century now, much scholarship on Newton, Leibniz and Kant has focused on their inertial-kinematic foundations. Namely, on which quantities of motion they counted as 'absolute,' or objective; and whether the true carriers of those quantities were material, mental, divine, or otherwise.[45] Above I made an incipient case that proper foundations for mechanics require more than just space- and time structures. In particular, they require nomic foundations, and also matter-theoretic ones. It is high time that we look at these three major thinkers from this vantage point as well. Only then, I suggest, can we really hope to judge correctly the relative weight of their philosophical insight into early modern science.

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[44] On this point, see again Wilson, *Physics Avoidance.*

[45] For Newton and Leibniz, the floodgates of research on that topic opened after Reichenbach, "Bewegungslehre." For Kant, it began with Cohen, *Kants Theorie*, and Cassirer, *Erkenntnisproblem*; then it continued through Friedman, *Kant's Construction.*

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