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Doctrines of force in the Enlightenment

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The Enlightenment came on the heels of a great transformation: the birth of early modern science. Indeed, it was Enlightenment figures who first began to call it a scientific *revolution*.[1] At birth, that mode of knowledge was advertised as a science of matter in motion. However, it soon turned out that the key notion in their novel cognitive enterprise was really *force:* the new science was a mechanics of forces.

The Enlightenment contributed to it on two fronts. Mathematicians expanded its descriptive scope enormously, by applying or discovering laws of force in domains or to processes well beyond what pre-Enlightenment science had conquered. And, they complicated the semantics of the term 'force,' by giving it new meanings that have survived into our times. Natural philosophers, in turn, debated the ontology and epistemology of force, seeking to understand what made it foundational for the new science.

In consequence, this chapter is structured around the twofold contribution above. I distinguish in Enlightenment science two species of force: impressed and live. For each species, I explain to what novel phenomena it was applied; what semantic transformations in underwent; and the foundational debates it generated. Accordingly, I begin with impressed-force mechanics (sec. I) and some foundational issues associated with it (sec. II), then I move on to live-force dynamics (sec. III) and the philosophical questions it raised (sec. IV).

I aim to convey three lessons here. One, we must resist the temptation to call 'Newtonian' the exact science of nature created in the Enlightenment. Two, in the 1700s a framework for mechanics emerged that often succeeded as an alternative to Newton's theory; this framework was built on concepts inherited

[1] Natural philosophers and writers in the French Enlightenment—Clairaut, d'Alembert, du Châtelet, J.S. Bailly, and Condorcet—first began to call the period from Copernicus and Bacon to Descartes and Newton a revolution in science (Cohen 1985, ch. 13).

from Huygens and Leibniz. Finally, during the Enlightenment the project of mathematizing nature really emerged out of infancy. Next to it, all 17th-century achievements look rather diminutive. I rehash these lessons, with more specifics, in the concluding section.

Impressed force: theory development

The notion of impressed force came from Newton's treatise, the *Principia:* in essence, his book is a mathematized theory of two impressed forces. Newton there introduced the concept in two ways. First, he defined it: "an impressed force is an action to change the body's state of rest or uniform motion in a straight line" (Newton 1687, 2, Definition IV). Second, he stated certain laws that describe its chief attributes, three of which were specifically 'laws of motion.' On the face of it, they concern 'force of inertia,' 'impressed force,' and 'action and reaction,' respectively. And yet, each law contains insights about the main notion above, as follows:[2]

1. All impressed forces are outward-directed: by a mass on other masses, never on itself.
2. If the net impressed force on a mass is zero, the mass stays at rest, if it was at rest; and it moves with constant velocity, if it was moving.
3. An impressed force induces an acceleration, proportional to, and collinear with, the force.
4. An impressed force induces accelerations inversely proportional to the masses on which it acts.
5. Impressed forces are additive. To compute their sum, use the Parallelogram Rule.[3]
6. The effects of impressed force are independent of one another. A force will induce in a body the same acceleration, no matter whether it alone acts on it or there are other impressed forces at work.

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[2] Very often, modern interpreters think it is just Newton's *Second* Law (the famous equality, $\mathbf{f} = \mathrm{m}\mathbf{a}$, in our terms) that describe impressed force. I submit that it is possible to read all three laws as jointly constraining what impressed force is. To that end, it suffices to reflect on what the Second Law tacitly assumes; and on what Newton means by 'action' and 'reaction' in his Third Law. I am indebted to George E. Smith for this idea. (Note: hereafter, by lowercase boldface letters I denote vector, or oriented magnitudes; lowercase italic letters are for scalar variables, and roman letters for constants.)

[3] Briefly, the Parallelogram Rule says that, if two impressed forces **j** and **k** act on a body at the same time, the net action on it—the resultant force—is equal to, and in in the direction of, the long diagonal of the parallelogram formed from **j** and **k**.

7. Impressed forces act along the straight line between two particles. In extended bodies, the action is along the line between their respective mass centers.
8. Impressed forces come in pairs. If a mass *A* exerts a force on another, *B*, then *B* simultaneously exerts an impressed force on *A*. These two forces are always equal and opposite.
9. Impressed forces are seated in bodies or particles. They are never free-floating, unconnected to mass-carrying entities.
10. Any species of impressed force is individuated by a law.[4]

I spelled out these insights for two reasons: to help the reader understand how Newton's concept differs from the other species of force (in particular, *vis viva*) analyzed below; and how Enlightenment figures generalized or modified some of his insights above.

New territory. One type of generalization sought to extend the descriptive reach of Newton's theory. The *Principia* is a treatise that studies two species of impressed force—gravity, and a medium's resistance to motion through it—acting on 'particles,' or very small masses at large distances from other masses. After many years of great effort, Enlightenment mathematicians managed to quantify how impressed force acts on *extended* bodies, not just particles. In particular, on three classes of bodies: elastic, rigid, and fluid.

The first class was the hardest to treat. Much effort was spent on one-dimensional bodies: elastic solids that are much longer than they are thick or wide; for instance, beams, columns, elastic strings, and flexible laminas, or thin bands. In regard to such objects, applied mathematicians grappled with three generic questions. (i) How do they move when an impressed force acts on them—how will every point in that body change place in an instant? (ii) What shape will they take when several impressed forces in equilibrium act on them? (iii) What is the maximum impressed force they can carry, e.g. before a column buckles or a beam fractures? The figures who did the important work on this area were d'Alembert, Euler, and Lagrange. Among the key results they obtained were the equations of motion for a vibrating string and a circular thin membrane; the distinction between principal (normal) and mixed modes, in vibrating elastic bodies; and Euler's Buckling Formula, so called.[5] Behind these

[4] Some well-known examples are Hooke's Law for restoring forces in elastic solids, and Coulomb's Law for the force between electric charges. Newton in *Principia* discovered another species—the law that universal gravity obeys—and he explored, in Book II, various laws of drag force (exerted on a solid moving in a fluid).

[5] The vibrating-string equation was first derived in 1748 by d'Alembert, who built on previous research by Brook Taylor and Johann Bernoulli. Mathematical work on the equilibrium shape

breakthrough results was always the concept of impressed force; and when the processes studied involved actual motion (for instance, the vibration of elastic strings), the common premise was:

> the total impressed force $\propto$ (is proportional to) the acceleration at every point.

In essence, it is a generalization of the insight behind the Second Law in Newton's *Principia.*[6]

The second class, rigid bodies, likewise took much work to mathematize. The impetus behind it largely came from astronomy.[7] The general question they grappled with was: if an impressed force acts on a rigid body, and its line of action does *not* pass through the center of mass, how will the body move? D'Alembert first answered this question, in 1749. Then Euler took a decade to work out a general theory of rigid-body motion under impressed forces. Finally, Lagrange and Euler in the 1770s refined the mathematics behind this theory, so as to make it more tractable. For this particular process (rigid rotation), it is useful to refer the body's motion to a fixed point, say *Q*. Then we can put d'Alembert and Euler's key result in this area as follows:

$$\text{net impressed force} \times \text{distance}_{\textit{relative to } Q} =$$
$$= \text{change in angular momentum}_{\textit{relative to } Q}.$$

It was yet another great step forward in the quest to broaden the descriptive reach of the notion, *vis impressa.*[8]

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of thin rods and elastic strings was done between 1744 and 1768, mostly by Euler, d'Alembert, and Daniel Bernoulli. Euler derived his formula—which determines the maximum compressive force that a column will bear without buckling—in 1744. For details and analysis, see Truesdell 1960, Fraser 1991, and Brading & Stan 2023.

[6] An excellent, up-to-date overview of mechanics in the 1700s is Caparrini & Fraser 2013.

[7] Specifically, they had to represent the Earth as a rigid sphere (rotating around a fixed point inside it) for the sake of deriving from theory two patterns of terrestrial motion: the precession of the equinoxes (due to the earth's rotation axis slowly describing a large circle in the northern sky), and the nutation (regular wobbling) of that axis.

[8] Synoptic summaries of d'Alembert's and Euler's work in this area are Wilson 1987 and Stan 2017b. For a thorough analysis, heavily centered on Euler, see Verdun (forthcoming). So far, we lack a study of Lagrange's work in rigid-body dynamics.

The third class, fluids, was also difficult to treat. Success in this area was partial, much like the case of elastic solids. Theoreticians had to overcome two major hurdles, conceptual and mathematical. The conceptual obstacle was, understanding what agencies are at work when a particle in a fluid gets moved. One was known: gravity, acting from outside the fluid to move every particle downward. The other is pressure—or rather, pressure difference between different regions in a fluid—which acts *within* the fluid, to move particles toward regions of lower pressure. However, grasping what pressure *is*, and how it acts in a fluid, was very elusive. It took the greatest minds of the century to understand it, after many decades of slow and gradual progress.[9] The mathematical obstacle was this. When impressed forces act on a fluid particle, their strength is generally different in the three directions of space; and so is the velocity change they induce in the particle. To describe such differences exactly, we need partial derivatives. These representational devices (for capturing change in different directions) first became available in the 1740s.

Euler was best placed to make the most of these two advances, conceptual and mathematical, when they occurred. In an epoch-making paper from 1757, he described exactly the strength of impressed forces on a fluid particle, and how it moves in response to them, from one instant to the next. In words, he showed that

external impressed force + gradient of internal force =
= particle acceleration.

Expressed in mathematical terms, this result is known as Euler's Equation for an ideal fluid.[10]

In sum, during the Enlightenment mathematized mechanics built on the notion of impressed force made significant advances on the path to descriptive

[9] Before the 1720s, they conceived of pressure as the force (weight) exerted by the *atmosphere* on the free surface of a fluid. By the 1730s, the Bernoullis had reconceived pressure as the force exerted by the fluid on the *walls* of its container. At last, in the 1740s Johann Bernoulli seems to have finally grasped that pressure is the force exerted by fluid particles on *other* fluid particles, in contact with them. For a terse, but illuminating account, see Darrigol and Frisch 2008.

[10] In an 'ideal' fluid, there is no viscosity: no internal friction between its layers as they slide past each other. Informally, this sort of stuff is known as 'dry water.' For a survey of the progress in fluid dynamics during the 18th century, see Darrigol 2005, and the older but still classic Truesdell 1954.

generality. And, there was significant progress in elucidating the conceptual foundations of that theory.

Impressed force: foundations

The emergence and steady growth of rational mechanics based on impressed force was accompanied by a cluster of foundational issues—questions about the ontology and epistemology of such forces. I survey here three such issues.

Deviant forces. In the 1780s, Lagrange set out to articulate a unified theory, by showing that all the known results in statics and dynamics can be deduced from a single premise, viz. a principle of virtual work.[11] His success came at a price, however; it required him to admit into mechanics two types of force that lack one or more of the ten canonical features (of impressed force) that I listed in Section I. Namely, he had to allow forces that are *not seated in bodies*, and forces whose *specific law is unknown*. That makes them non-standard, deviant entities in classical mechanics. I explain them in turn.

Lagrange's approach to unification was to 'reduce' dynamics to statics. To that aim, he drew on an insight d'Alembert had had before him: there is a way to represent a system *in motion* as a system *at rest*. We may do so without paradox or contradiction if we imagine, or suppose counterfactually, that additional forces act on the system. Namely, forces individually equal to the target body's mass times actual (effective) acceleration, but in the opposite direction. If such forces *were* present, they would exactly balance the impressed forces that *are* present in the system. Thereby, the component bodies *would* come to rest: the system would be in equilibrium, and then we can treat it mathematically with the methods of statics. This enabled Lagrange to unify theory by reconceiving mechanics as a kind of generalized statics, so to say:

> Geometers have long accepted the principle [of virtual work] as the fundamental principle of equilibrium. ... This principle of *statics*, if we combine it with the principle of *dynamics* given by Mr. d'Alembert, results in a general formula that contains the solution to all the problems on the motion of bodies. (Lagrange 1873 [1764], 10, 12; my italics)

[11] The usual opposition is between statics qua theory of masses at mutual *rest* (in equilibrium) vs dynamics qua theory of masses in actual motion.

However, the move requires him to suppose the existence of certain forces *in addition* to the ones that are given, or known to exist:

> If we *imagine* impressing in each body the motion it is to have, but in the opposite direction, clearly the system will be reduced to rest. ... Thereby we can reduce the entire Dynamics to a single general formula. For, to apply the formula of equilibrium to the motion of a system, it is enough to *introduce* forces that come from the [actual] change in the motion of each body, which motions must be *destroyed*. (Lagrange 1811–5, 240; my italics)

Some modern authors call these forces 'kinetic reactions.'[12] They are impressed forces, but quite peculiar in two respects. One, they are fictitious, not real; the theorist must put them in by hand, as it were. Two, they are *not* seated in bodies. They do act on bodies, to be sure—the ones that make up the target system—but these bodies cannot act back on the sources of kinetic reactions, because they are *sourceless*. That makes them non-standard, or deviant: they violate Newton's conditions [8] and [9] on impressed forces listed above.

The other deviant type likewise arose in the context of Lagrange's theory. Among the types of behaviors he treated were systems with constraints: bodies whose motion is impeded or restricted in some respect.[13] These constraints have kinematic effects; they change the speed, direction, and relative distances of the bodies they constrain. Within the conceptual framework presented in this section, these changes must be due to impressed forces. Now in general, the specific laws that govern constraint forces—the actions that secure rigidity, solid and fluid incompressibility, flexible bending, and the like—are not given, or known in advance. Thus, constrained systems seem prima facie intractable. Lagrange devised an ingenious way around this obstacle: he *bracketed the physics* of these forces. Namely, he left out the question of the specific laws they obey, and found instead a quantitative fact about their effect as a whole (not individually). From this global fact and the geometry of the particular

[12] For discussion, see Brading & Stan (2023, ch. 11), who call them 'reverse effective forces.'

[13] Here are some examples: the inclined plane (a body is constrained to roll down on it, even though gravity, an impressed force, urges it to move *through* the plane, straight down); a rigid body (particles in it are constrained to keep the same relative distance over time), fluid that is incompressible (its motion is constrained to maintain the same volume, no matter where the fluid goes); a set of masses connected by flexible strings (hence constrained to never exceed the maximum length of their connecting strings), etc.

constraints, he derived the individual quantities of their respective actions in the system.[14]

> I must note that, in the general equation of equilibrium, we may *regard* the [Lagrange multipliers] as designating the moments of various *forces* applied to the system. It follows that every *equation* of constraint is *equivalent* to one or more *forces* applied to the system, in the given directions. Ergo, the system's state of equilibrium will be the *same*, whether we consider it as induced by these forces [equal to the Lagrange multipliers] or we suppose it to ensue from the *equation* of constraint. (Lagrange 1788, 47, 49; my italics)

Nonetheless, in Lagrange's mechanics—the zenith of natural-philosophical theory building in the Enlightenment—constraint forces violate the above condition [10] that canonical, well-behaved impressed forces obey. Namely, they are not given by, or allowed into theory based on, a specific *law of force*.

Agnosticism. In the early Enlightenment, a few natural philosophers moved to argue that impressed forces are epistemically mysterious—they are unknowable, or at least not yet known clearly and intelligibly. Some, like Boerhaave and 's Gravesande, were agnostic about particular species, e.g. gravity. Others, like d'Alembert, doubted knowledge of force in general.

Before I delve into their reasons, it helps to disentangle things. We may distinguish three aspects of a mechanical process:

I. The acceleration: the fact of a body being deflected from its inertial path; and the oriented quantity associated with that fact.

II. The impressed force: the action that generates that acceleration, or velocity increment. "If a force *generates* a given motion, then twice that force will *generate* twice the motion; three times that force will *generate* thrice the motion, and so on." (Newton 1687, 12; my italics)

III. The cause of an impressed force: the entity, corporeal property, or event causally responsible for an impressed force obtaining.[15]

[14] This is the method of Lagrange Multipliers, as we call it nowadays. The general fact above—he stipulated it, really—is that the net virtual work of all the constraints in the system is null; see Lagrange 1788, 23, 439; for discussion, cf. Brading & Stan 2023, ch. 11.

[15] Note that these entities are fully distinct. For instance, an acceleration can occur in response to a change or transfer of energy, not just impressed force. A force need not generate an actual (effective) acceleration; for instance, forces in static systems induce relative rest, not motion. And, in the Enlightenment many kinds of things counted as 'causes' of impressed force; e.g. impenetrability, a property of body; or impact, a mechanical process. Further, nowadays we know that some impressed forces can have *other* impressed forces as causes: e.g. electric forces

Now I can better explain the agnostics' reactions.

A few figures in the Low Countries, though generally favorable to Newton and broadly influenced by him, ended up agnostic about one species of force, viz. gravity. After the *Principia* came out, some objected that Newton in his book had not established what the 'cause of gravity' was. This resulted in some passive-aggressive exchanges on the matter, and led some natural philosophers to strike an agnostic note:[16]

> 'Attraction' denotes nothing but an *unknown cause* that produces motions whereby certain bodies are brought into mutual contact. It does not explain what that cause is, nor does it show intelligibly how these motions are induced. (Boerhaave 1715, 18; my italics)
>
> Still, I do not deny that heaviness [*gravitas*] might come from some contact action. I just assert that it demonstrably does not come impact that follows the known laws of contact action. So, I declare that the cause of heaviness is *wholly hidden* from us. ...
>
> I call 'attraction' that force whereby two bodies tend toward each other, though this might be the result of impact action. By this term I denote the *phenomenon* [of mutual approach], not its cause. ('s Gravesande 1742, II.997, I.17; my italics)

Translated in terms of my tripartite distinction above, their claim is: Hereafter, we use 'gravity' and 'attraction' as just another name for a class of Type I entities: accelerations in conic orbits, and downward motion above ground. *Not* for impressed forces.

While their individual doctrines were complex, these Dutch figures counted among Newton's few philosophical friends in Europe before the High Enlightenment. For that reason, their agnostic turn is both disappointing and excessive. It is disappointing, because they tacitly grant the anti-Newtonians two framing assumptions to which they were not entitled. Namely, that all action is action by contact or reducible to it; and that gravity could not be a fundamental impressed force. Elsewhere, I showed how little right they had to these assumptions.[17] And, it is excessive, because the Dutch did not need to go to such an agnostic extreme. Prima facie, they did not *have* to say, as they did,

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(at molecular levels) are the cause of friction forces (at mesoscopic scales). And, that some impressed forces have *no* causes—they count as fundamental; e.g. gravity in classical physics.

[16] For additional discussion, see Ducheyne & van Besouw 2022, and Ducheyne 2017.

[17] See Stan 2017, Section III, for the case of post-Leibnizian Germany.

that gravitation theory is *just* a descriptive account of the 'phenomena of gravitation,' i.e. of Type I entities alone. They had two other options available.

One was to accept that some impressed forces are fundamental—not reducible to other impressed forces—and that gravity is one. To deny this option a priori is to assume *dogmatically* that only action-by-contact forces can be fundamental, or irreducible. But, what could be the evidence for this claim, and how good is it? Two, they could have asserted:

> Gravitation theory is about a species of *impressed force*, or Type II entity. Specifically, a central force whose strength at a point equals $M{\cdot}m/r^2$.
>
> The cause of that force, *if any*, is so far unknown.

This was in fact the position of Newton, who, as always, had seen farther than his opponents. He declared it in the General Scholium, added to the *Principia* in the second edition:

> I have not as yet been able to deduce from phenomena the reason for these properties of gravity.... And it is enough that *gravity really exists* and acts *according to the laws* that we have set forth. (Newton 2016 [1726], 589; my italics)

The Dutch seem to have missed this, however, and so they hurried to declare themselves agnostics about gravity.

Another expression of agnosticism about force came from d'Alembert. Unlike the Dutch Newtonians above, his doubts reached well beyond gravity. D'Alembert thought there are three species: action-at-a-distance force, the 'force of motion,' and the force of impact. He dismissed knowledge claims about all three species. In regard to the first (distance actions), he denied that we know their nature*:* "all other causes [of motion change, other than impact] are known to us only through their effects, and we are completely *ignorant of their nature*." Concerning the second species (the force that a moving body has just because it moves), he thought our knowledge of it is mediated by a certain principle, and *it* is doubtful: "the *vague and obscure* axiom that the effect is proportional to the cause." Lastly, he thought that claims about impact forces rest on a confusion: motion changes in collision are due to the bodies' impenetrability, not any force of impact. These changes have "their source in the sensible and mutual action of bodies, *resulting from their impenetrability*."[18]

Their doubts about forces led these agnostics to repurpose the term 'force' as a mere linguistic label for purely kinematic, Type I entities, e.g. mass times acceleration or just the acceleration:

> I will take the statement—that the accelerative force times the element of time equals the element of speed—for a *definition*. Thus, by the term 'accelerative force' I understand *merely the quantity* to which the speed increment is proportional. ... And so, in general by the term 'motive force,' I understand the product of the moving mass and the element of its speed; ... and by 'accelerative force' I mean *merely* the element of speed. (d'Alembert 1758, 25f.; my italics)

On this count, d'Alembert too disappoints the modern reader; his agnosticism about *impressed* force looks glib and perfunctory. Newton had defined impressed force, had explained how they work, and had devised a new way of gaining knowledge of them, viz. 'deduction from phenomena.'[19] The true measure of his conceptual innovation was the enormous wealth of results it produced, in *Principia*. Anyone who wished their agnosticism to be taken seriously should have made an honest effort to engage thoughtfully with Newton's realism about impressed force. Regrettably, neither d'Alembert nor the Dutch Newtonians did that.

Realism. In the Enlightenment, there were realists about force too, not just agnostics. The German states were home to most of them. They shared the premise that bodies, or even matter in general, *have* forces, understood as dispositional properties. This joint commitment puts them closer to Leibniz, but at some remove from Newton.[20] Christian Wolff argued that bodies by their

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[18] See, respectively, d'Alembert 1758, xi, xii, x; my italics.

[19] Recall, in *Principia*, Definition IV states what an impressed force is. The Second Law explains how it works: by generating motion,' viz. momentum increments, in the body on which it acts. 'Deduction from phenomena' is a deductive inference (really, an instance of *modus ponens*) from observed motion facts—an orbit shape, periodic time, rotation of the line of apses, etc.—to the impressed forces that induce them. In these inferences, the key premise is always a theorem that Newton proved in Book I: an if-then conditional whose generic structure is, 'If motion fact *M* obtains, then impressed force *F* obtains.' Some scholars have called them 'inference tickets,' because they license our inferential passage from observed motions to their generating forces. For details, see Ducheyne 2012.

[20] In Newton, bodies do not *have* forces (except for his *vis inertiae*, which is not an impressed force). They merely exert them in the appropriate circumstances; Newtonian forces are episodic exercises, not enduring properties. "Impressed force consists *solely* in the action, and

nature have two forces: active, or motive; and passive, or resistive. The former inheres in bodies that move—they have it *because* they move—and the latter is a force whereby they resist changes of state:

> A body in motion is endowed with a force of acting. This active force of bodies is the principle of all changes. ... Motive force [*vis motrix*], on which a body's action depends, consists in a continual endeavor to change place, or in a continual striving.
>
> Every body resists motion.... In bodies, the principle of resistance to motion is called Force of inertia, or passive Force. (Wolff 1731, §§ 236–7, 129–30)

Wolff denied action at a distance, so his two forces count as contact actions, paradigmatically exerted in collision, or impact. He conceives of it as a mechanical process in which (two) colliding bodies play distinct causal roles. One is the agent, and the other counts as a passive subject, or patient; the former exerts active force onto the latter, which puts up a force of resistance.[21]

In the High Enlightenment, Kant defended a more ambitious version of realism. His natural philosophy endows material objects with two kinds of force, 'dynamical' and 'mechanical.' Any piece of matter has the former forces merely because it is *matter;* they secure—by underwriting causally—universal properties like impenetrability, cohesion, and having a stable configuration. The latter are forces whereby a body acts on another, by contact or at a distance. There are two dynamical forces, attraction and repulsion. Mechanical force comes in just one kind, namely, 'moving force,' which is his term of art for linear momentum:

> Matter fills a space, not through its mere *existence*, but through a *particular moving force*.... Matter fills its space through the repulsive forces of all of its parts, that is, through an expansive force of its own, having a determinate degree.
>
> The possibility of matter requires an *attractive force* as the second essential fundamental force of matter. The *attraction essential to all matter* is an immediate action of matter on other matter through empty space. ...
>
> Explication 1: Matter is the movable insofar as it, as such a thing, has moving force. ... In mechanics, the force of a matter set in motion is considered as *communicating* this motion to another. (Kant 2004 [1786], 34ff., 46ff., 75)

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does not subsist in bodies *after* the action has ended. The body persists in its new state by way of its *force of inertia alone*."—Newton 1687, 2f.; my italics.

[21] For further explanations, see Stan 2017.

Kant's dynamical forces are not really kinds of *impressed* force. His 'original attraction' is really a species of conservative potential U given by some (unknown) function of the relative distances between particles. However, its actual exercise counts as an impressed force. His 'original repulsion' resembles the notion of compressive stress in modern continuum mechanics. So, it responds like an impressed force to any relevant strain or deformation.[22]

Having questioned the agnostics' reasons for their doubts, it is only fair to press the realists for their epistemic confidence. On this count, things are somewhat murky. Wolff does not explain what evidence he has for his claims about active and passive forces. There are some indications that he thought it is a priori knowledge, obtained by conceptual analysis.[23] Kant is less opaque on this count. His claim that any moving body has 'mechanical' force seems inferred by conceptual analysis—of the notion of body presupposed in mechanical theory. There is no effort to show, however, that the analysis is *correct*. For instance, three of the four greatest theorists of that age—Newton, d'Alembert, and Euler—would have disagreed with this particular outcome of Wolff's and Kant's analyses. Regrettably, neither philosopher stopped to address this difficult point.[24]

As warrant for his other species, viz. 'dynamical' force, Kant devised a sui generis type of inference called a Balance Argument. In essence, it supposes that all matter has one such dynamical force (repulsion), and it argues that it must have the other type (attraction) as well, so as to balance the repulsion into matter configurations stable over time, which are known to exist. Here too, the question of evidence rears its head. Kant does not just assume without proof that all matter is endowed with repulsion force. Rather, he gives a reason for it: an argument in which the key premise is, "the cause of a motion is *called* a moving force."[25] But the evidential status of *this* claim is uncertain. For one, not everyone then—certainly not the leading theorists of mechanics in the Enlightenment—thought his claim was true. For another, in a treatise of

[22] Lucid accounts of Kant's 'dynamical' forces are Smith 2013 and Friedman 2013, 170–80.
[23] Wolff 1731b explains that, in *Cosmologia rationalis*, he took the 'analytic approach' to the concept of body.
[24] For Newton, a body in motion has a 'force of inertia' (whereby it continues in that state, and resists changes to it), not a 'force of motion.' Likewise for d'Alembert: "force of inertia: the bodies' *property* of remaining in their state" (1758, 3; my italics). For Euler, motion-changes in a body—the action of impressed forces in general—are the outcome of corporeal *impenetrability*, not Kant's 'moving force'; see Veldman (ms.).
[25] See Kant 2004, 34 (4: 497); emphasis added.

metaphysics (which his book is) it is not enough to report on linguistic usage—on what some people *call* things; at least not until that usage is fully entrenched. A metaphysician ought to show that we have a *right* to call things by that name. Kant does not appear to have examined the case for that right, if there is one.

In general, the epistemology of Enlightenment pictures of force, be they agnostic or realist, is an area where much work remains to be done. My sketch above is really just that—a forward-looking outline, not a summary of completed scholarship.

Live force: theory development

As the Enlightenment began, at least two force concepts were available for theory building. One was the notion above, inherited from Newton. The other came from Leibniz, who had called it *vis viva*, and also *force vive*. Thanks to his skillful maneuvering, live force became the core notion in a new agenda for theory building in the Enlightenment. Indeed, for some five decades, live-force dynamics seemed strong enough to compete with Newton's legacy on equal terms. Though this hope ultimately did not come to pass, by 1800 results and methods based in *vis viva* found a permanent home in modern mechanics.

Leibniz introduced his notion as part of a duality, viz. 'live' and 'dead' force, that he claimed was sufficient foundation for the new science of motion:

> Likewise, force is of two kinds. One is elementary, which I also call 'dead force,' because in it there is *no motion yet*, but only an urge [*solicitatio*] to move. ... The other kind is ordinary force, which accompanies actual motion, and I call it 'live force.' (Leibniz 1695, 49)

While the Leibnizian picture of dead force is murky and equivocal, his other notion is clear enough: if a body is in actual motion—if it changes place from one instant to the next, for a finite duration—it *has* live force: it is a property of the body during that time interval. Leibniz, who had glimpsed the notion in Huygens' work on elastic collision, borrowed from him its measure as well: a body's live force equals its weight times the (instantaneous) speed squared.[26] His mathematical disciples subtly corrected this idea, by substituting mass for weight, and so they took live force to be equal to ms^2.

[26] Lacking the modern notion of mass, Huygens and Leibniz used weight as a proxy for it.

Enlightenment figures had various motivations for adopting the notion of live force. Some, like Johann Bernoulli and Christian Wolff, were moved by loyalty to Leibniz or antipathy toward Newton.[27] Others, like Emilie du Châtelet, did so for independent reasons; or because they endorsed deeper Leibnizian tenets, such as the Principle of Sufficient Reason and his Law of Continuity. Yet others, like Daniel Bernoulli and Euler, adopted it because they saw genuine scientific merit in his concept.

In their eyes, the chief merit of live force was that it could further the mathematization of nature: by extending the descriptive reach of mechanics, and by unifying local descriptions under a principle of broader scope. To that aim, however, Enlightenment theorists needed to alter Leibniz's legacy in two ways: they changed the meaning of his fundamental principle; and they gave 'live force' a novel sense that departed from Leibniz. I examine these two innovations next.

When he introduced live force, Leibniz attached to it, qua governing law, a conservation principle:

> The effect contains neither more nor less power than its cause. ... As this law is not derived from the notion of bulk, it must follow from some other thing in bodies—namely, from force itself, *whose quantity remains the same*, even though it may be exerted by different bodies. (Leibniz 1695, 152; my italics)

Stated more precisely, his law says that, when several bodies interact,

$$\text{Sum of live forces}_{\textit{before interaction}} = \text{Sum of live forces}_{\textit{after}}\,. \tag{3a}$$

Equivalently, in a *finite* system, the total quantity of actual *vis viva* remains unchanged (if the system is closed off from causal actions originating outside it). After 1710, however, some followers, especially the Bernoullis, gave the principle—that live force is conserved—a new sense:

$$\text{Change in height}_{\textit{under gravity}} = \text{Change in live force}_{\textit{during the process}}\,. \tag{3b}$$

[27] Wolff to Manteuffel, 6 January 1741: "The Newtonians are arrogant creatures that despise anyone who does not sing their tune. And yet, no one who really understands philosophy could grant that their so-called Newtonian philosophy is one" (Ostertag 1910, 62). For a sample of Bernoulli's feelings about Newton, see his *De vera notione virium vivarum* (1735).

Thus stated, it becomes clear that, by 'conservation of live force,' the Bernoullis meant a species of the Work-Energy Theorem, so-called, which is quite different from a true conservation principle; see below.[28]

The new sense of 'live force' was due to Daniel Bernoulli's research in the theory of elasticity. In that context, it was known that an elastic lamina (a very thin, very narrow blade) has a certain power to move an object in its path while it unbends, or reverts to its 'natural' configuration. Mathematicians then grappled with the question, how strong is that power to move—on what quantities or parameters does it depend? It was widely suspected that, inter alia, it is a function of the lamina's curvature, or amount of bending.[29] However, a proof was needed. As he grappled with one, Daniel Bernoulli introduced, in a letter to Euler, the term 'potential live force' [*vis viva potentialis*] to denote the moving power stored, as it were, in the bent elastic blade. In a 1742 letter to Euler, he explained:

> Let a lamina ABC be fixed at the endpoints A, C, and the tangents at A and C be given. For this generic boundary condition I have not yet found a solution except by the isoperimetric method [viz. the calculus of variations]. Using this approach, I assume that the *potential live force* inherent in the lamina must be a minimum, as I once told you. (Bernoulli 1843 [1742], 505–6; my italics)

Euler adopted the new term without hesitation, apparently:

> Let AB be an elastic lamina, arbitrarily curved, $AM = s$ an arc length on it, and MR = R be the radius of the osculating circle at M. Then the *potential force* [*vis potentialis*] contained in the lamina is given by the formula $\int ds/R^2$. (Euler 1744, 247; my italics)

Bernoulli's terminological innovation was prescient; his term denotes what we call nowadays elastic-strain energy, which is a species of *potential* energy. Still, his notion is a radical departure from Leibniz (see below). And yet, at the time no one objected to Daniel Bernoulli's semantic heresy. Perhaps it was because,

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[28] The theorem says, the mechanical work done by a system against a force equals the change (the loss) in its kinetic energy; for instance, a body moving upward, under terrestrial gravity. In terms familiar to them, the system's weight times the height it crosses (in a given time, by rising or descending) equals the change in the system's *vis viva*.

[29] Specifically, that it is proportional to the square of R, the radius of curvature at every point.

with his terminological innovation, he was walking down a path that his father, Johann, had broken. A personal friend and acolyte of Leibniz, Johann Bernoulli had reconceptualized live force significantly. He thought that, properly understood, even certain bodies *at rest* can have live force—not just masses in actual motion:

> Live force does not consist in an actual exercise, but in the power to act [*facultas agendi*]. It subsists even when it does not act—and even when it does not have a thing to act on. Thus, for instance, a tensed spring ... has in itself a power to act even when there is nothing external to it on which it could exert its power....
>
> Hence it is clear that live force (for which a better name would be the power to act; in French, *le pouvoir*) is something real and substantial, which subsists by itself and, insofar as in it lies [*quantum in se est*], does not depend on another. (Bernoulli 1735, 210, 211)

Thus, in calling it potential live force, Daniel signaled that *vis viva* is a genus concept with two main species: latent (in bodies at rest) and actual (in bodies that move). We saw Euler above take up both Daniel's initial questions (about the mechanical power of a tensed lamina) and also his terminology: Euler called it 'potential force.' In the 1840s, Gauss, generally a careful reader of Euler's work, introduced the phrase 'potential function' to denote the energy stored in mechanical systems.[30]

It is easy to miss how radical the Bernoullis' semantic change was. From Leibniz to the mid-Enlightenment, the official definition of *vis viva* was, a force found in bodies that *move actually*, or change place at every instant: live force "accompanies actual motion."[31] And yet, Johann and Daniel Bernoulli above redefined live force such that it can inhere in bodies *at rest*. Not only was that a sharp departure from Leibniz, but it actually went against his framework; what they called *vis viva potentialis* counted for Leibniz as a species of *dead* force, not live: "Another example of dead force is that whereby a tensed spring begins to restore itself" to its equilibrium shape (Leibniz 1695, 49). More seriously, the Bernoullis did not show that *their* distinction (live vs dead force) was principled, or based in a clear, univocal criterion. They certainly could no

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[30] Euler 1744; for the creation of the term 'potential function,' see Gauss 1840.

[31] See Leibniz 1695, 49. We find the same idea in Poleni 1718, 46; Wolff 1731, §357; Hanov 1762, 158; Brisson 1781, and many others.

longer avail themselves of Leibniz's, original criterion for that distinction.[32] This difficulty notwithstanding, without the Bernoullis' reconceptualization it is anyone's guess how much *new* mechanics would have emerged from Leibniz's canonical, unmodified notion of live force.

But thanks to the two Swiss mathematicians, new mechanics did in fact emerge. Johann Bernoulli took a live-force approach so as to derive two results about the motion of oscillating systems: the frequency of an elastic string vibrating in the fundamental mode, and the period of a column of fluid oscillating in a U-shaped tube. Likewise, he derived the speed of two masses (connected by a flexible string) as they move on curved surfaces under gravity; and the speed of a constrained rigid body falling under gravity.[33]

Then his son, Daniel, went on to broaden the descriptive scope of live force, in the process obtaining some novel, key results. One was in fluid mechanics, to which he contributed path-breaking results; I single out the most important one. In a 1738 treatise, *Hydrodynamica*, Bernoulli sought inter alia to infer a formula for the "pressure" of a fluid moving through a pipe.[34] To that end, he reaches for a key premise:

> Now I must give an account of the principles I employ in this book. Chief among them is the Conservation of Live Forces. I would rather call it the principle of the *Equality of Actual Descent and Potential Ascent.* ...
>
> From this axiom [of equality], the conservation of live forces follows directly, as Huygens showed. To wit: if a number of weights begin to move under gravity, the sum of their individual masses times the speed squared is proportional to the height (from which their common gravity center descended) times the mass of the whole system. (Bernoulli 738, 11, 12)

[32] Recall, Leibniz in *Specimen dynamicum* had explained that dead force is associated with bodies that *do not* move, but have an "urge to move," whereas live force inheres in bodies "in actual motion" (1695, 49).

[33] See Bernoulli 1729 and 1735. For some glosses, see Brading & Stan 2023, ch. 7. Daniel's result below in celestial mechanics is derived in Bernoulli 1750.

[34] By 'pressure' in that context, Bernoulli meant the impressed force that the fluid exerts *on the walls* of the pipe as it moves through it. Not what *we* mean by pressure, viz. the force on a particle by the surrounding particles in a fluid.

Thus, in *Hydrodynamica* the key premise, talk of live force notwithstanding, is really a principle that comes from *Huygens*, not Leibniz.[35] Stated more precisely, the premise amounts to the claim that:

$$\textit{Increase in}\ \text{vis viva},\ v^2{}_{\textit{over an instant}} = \\ = \text{Maximum height}{}_{\textit{of travel upwards at initial speed}\ \mathrm{v}}\ . \qquad (3c)$$

With these preliminaries in place, Bernoulli's inference proceeds as follow. From his premise (3c) above, he derived an expression *E* relating changes in fluid velocity at two locations in the pipe. Then he offers a counterfactual scenario: to find the force ***f*** on the pipe at a location *c*, imagine that the (very small) pipe section at *c* were instantaneously removed. Then the fluid there—no longer contained by the pipe section—would undergo an acceleration ***a*** proportional to the force ***f***. Lastly, he finds a way to connect ***a*** to quantities in the expression *E*, which are known, or given. In essence, that gives him a way to calculate the sought pressure ***f*** on the container walls.[36]

Another set of results were in celestial dynamics, and motion under central forces more generally. Daniel Bernoulli's motivation in this area was the need to 'generalize' Conservation of Live Force, by showing that it holds in cases beyond what his predecessors had envisaged. For Leibniz and his age, a body's *vis viva* at some instant presupposed the relation:

$$\text{Quantity of live force},\ v^2{}_{\textit{at some instant}} \propto \\ \propto \text{Height of fall from rest, under gravity}{}_{\textit{until the speed}\ \mathrm{v}\ \textit{is reached}} \qquad (3d)$$

This relation tacitly assumes certain facts about gravity—its strength, direction, and distance to its source—that hold only approximately, in narrow situations.[37] Bernoulli wished to show what becomes of Leibniz's relation in

[35] Huygens had stated it in his 1673 *Horologium Oscillatorium*, and used it to solve a number of problems, e.g. elastic collision and the 'center of oscillation.' Before Daniel, Johann Bernoulli likewise acknowledged Huygens' authorship when he used the principle to derive new results based on *vis viva* approaches; cf. Bernoulli 1735, 221.
[36] For additional explanation, I recommend Mikhailov 2005 and Darrigol 2005, 6ff.
[37] Namely, that gravity—its acceleration—is uniform (has the same value everywhere) and parallel, viz. points in the same direction. This assumption is empirically accurate only for motion above the ground, not too far from it.

the general case of 'fall' under a variable central force from a nearby source, not just terrestrial gravity:

> The principle of live forces commonly assumes that gravity is always uniform and parallel at every location. However, when—in consequence of a system of bodies changing position—gravity changes too, be it in strength or in direction, we can no longer measure their live force as we ordinarily do, viz. from their masses multiplied by the height of fall of the system's gravity center. Still, conservation of live forces obtains in this case too, provided we understand it correctly. ... Hence, it might not be useless to examine here how we can employ this principle in cases other than parallel, uniform gravity. (Bernoulli 1750, 356)

He showed that, in the case of bodies orbiting around a center of force:

$$\text{Quantity of live force, } v^2_{\textit{at some instant}} \propto$$
$$\propto \text{ Path integral of the central force function}_{\textit{until speed } \mathrm{v} \textit{ is reached}} \qquad (3e)$$

Put in modern terms, Daniel Bernoulli proved the Work-Energy Theorem for particles moving under any *conservative* force (given by some power of the distance to the source), not just Newtonian gravity. From this relation, Bernoulli then derived two particular results: the speed of a comet in a near-parabolic orbit around the sun; and an 'inequality' in lunar theory, i.e. a difference in the moon's speed around the earth during a full cycle (Bernoulli 1747 and 1750).

Some of the results above had been known already; the Bernoullis merely re-derived them from different premises. In doing so, they had two aims. One was to show that live-force approaches can be more economical—inferentially shorter—than derivations based on impressed force.[38] Their second aim was to exhibit the heuristic power of a *vis viva*-based approach, which they called the 'indirect method.'[39] By the mid-Enlightenment, they had amassed enough

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[38] Euler to Daniel Bernoulli, September 1738: "I very much liked Your Lordship's paper on the principle of live forces. Not just for the insight it contains, viz. how to apply the principle to your topic—but especially for the utility and *uncommon advantage* it has for knowing the motion of the moon. Without your principle, that knowledge can only be gotten by means of the *most complicated* equations" (Euler 2016, 272; my italics).

[39] The relevant contrast class then was the 'direct method.' For both approaches, the objective was to quantify the motion of a single particle or body. The direct method identified the various impressed forces acting on that *individual* particle, and added them up; the resultant force is then proportional to the ensuing acceleration, which solves the problem. The 'indirect' method

successes to show that mechanics based on live force was a real competitor to the impressed-force approach.

However, after 1750, the notion of live force becomes confined to just two areas of mechanics. One was the domain of phenomena where it was known that some *vis viva* goes lost during a motion process. For instance, as water flows through pipes, it loses some speed, hence also some actual live force; as do solid bodies moving on rough surfaces, where friction slows them down. In such cases, theorists—rather than trying to explain away the loss of *vis viva* as merely apparent, which Leibniz and Wolff had done—sought to *quantify* the loss, so as to better predict the bodies' effective speed when friction or viscosity impede them.[40] This area of research was long-lived; applied mathematicians and engineers contributed to it equally. For a sample of the results they pursued, see Navier 1818.

The other domain was the motion of free particles, rigid bodies, and vibrating solids.[41] Lagrange in his synthetic treatise of 1788, *Mechanique analitique*, devised a uniform heuristic for treating their varied motions. For such systems, he found that it is useful to study them by means of a quantity which he later labeled Z, defined as follows:

$$Z = \text{Sum of actual live force} - \text{Sum of potential live force} \qquad \text{(3f)}$$

Specifically, while the system moves it satisfies certain relations between the changes (the derivatives) of Z with respect to time, space, and speed. Lagrange discovered a differential formula that relates these changes, and used it to quantify the mechanical behavior of those systems. In our times, that formula has come to be known as the Euler-Lagrange Equation.[42]

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would start with a fact (e.g., a conservation law) about the *whole system*, or set of bodies, and seek to infer from it an expression for the motion of every individual mass. Recurrent mentions of these two methods are in Daniel Bernoulli's correspondence with Euler (Fuss 1843). In modern terms, it is the contrast between differential and integral approaches to deriving the equations of motion.

[40] In this chapter I chose to focus on the loss of *vis viva* merely in *mechanical* contexts. After Lazare Carnot, the recognition grows that some lost *vis viva* has been converted to heat, and can be studied separately, qua object of 'heat science' and, subsequently, thermodynamics.

[41] In modern terms: *unconstrained* masses with finite degrees of freedom; and *undamped* oscillators. These restrictions are significant—no one in the Enlightenment (and beyond) knew whether the Euler-Lagrange Equation applies beyond these classes of motion.

[42] Lagrange introduced his formula (in a slightly different from) without any fanfare, in his 1788, 212–6; he created the label and the function Z in the second edition of his tract (1811–

Thus, in the Enlightenment, we see two groups emerge. One took *vis viva* to be a local concept: useful for describing the behavior of a few classes of body (e.g., elastic solids), though not much else.[43] The other thought that live force is really *the* fundamental concept for the entire theory of mathematical dynamics. As I explained above, ultimately their hope did not come to pass. But, it drove them to produce much novel, path-breaking work in rational mechanics based on Huygens' and Leibniz's ideas. Thanks to their efforts, as the Enlightenment wore on, *vis viva* became an entrenched concept of mechanics, but relegated to a subservient position, unequal (in descriptive power) to the notion of impressed force, which gradually turned out to be truly fundamental concept.

Live force: foundations

In parallel with theory development, certain foundational aspects of live force were debated at the time. Admittedly, these debates petered out by the mid-Enlightenment, without any clear resolution. I rehash below three such contested aspects: the measure of live force; whether live force is *the* basic notion for all mechanics; and whether it is conserved.

The first controversy was framed by an assumption that many in Europe (though much less so in Britain) shared: if a body actually moves, it has a 'force of motion':

> Definition 4. I call *force* that which, in a body in motion, transports it from one place to another. ... A body's force changes when its speed changes. The force inheres in the body, and can only be changed by the action of an external cause. ('s Gravesande 1722, 4–5)

Because the science of motion had to be mathematical, the question naturally arose: what is the quantity of this force? Two camps arose.[44] One, inspired by

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15). For context and explanation, see Brading & Stan 2023, ch. 11. Nowadays, the function Z is called the Lagrangian, often relabeled as L.

[43] This group was largely based in the British Isles; for a brief survey of their views on the place of *vis viva* in mechanics, see McCormmach 2004, 91–3.

[44] In scholarship, this controversy is sometimes described as Newtonians vs Leibnizians. But that's just because a few participants were British, thus presumably 'Newtonian' in some sociological sense. Caution: in *Newton's* mechanics a moving body does not *have* a 'force of motion.' It merely has a 'force of inertia,' whose quantity is the body's *mass*. No aspect of speed

Descartes and Malebranche, claimed that its measure was the product of quantity of matter times speed, *qs*. The other group thought it was qs^2, viz. proportional to the speed squared. Heavily influenced by Leibniz, they endorsed as well the notion of *vis viva*. Hence, they thought that live force is the true measure of a body's force of motion.

This latter group generally defended their position by two patterns of argument, a priori and a posteriori. Illustrative of the first was an inference by Christian Wolff: "in line with Leibniz's designs for dynamics, we should derive all that pertains to the forces and actions of bodies ... from concepts alone." In a nutshell, his supposed proof went as follows. A body's force of motion equals the 'action' it effects by means of its motion. That action is equal to the quantity of motion, *qs*, multiplied by the distance, *s*. But, that makes qs^2, which equals the body's live force. Ergo, the correct measure of 'force of motion' is *vis viva*.[45]

There were many a posteriori 'proofs' for the conclusion above; one achieved notable fame. In 1722, 's Gravesande tried to resolve the matter by experiment. He dropped spheres of different weight on a bed of wet clay, and assumed that the force of motion equals the effect it induces, which he took to be proportional to the indentation caused in the clay by the landing sphere. He found the depth proportional to s^2, the body's speed squared:

> The experiments on impact that I performed led me to see indubitably that Mr. Leibniz's position is correct, namely, the forces of two different bodies are in proportion to their masses times their speeds squared. ('s Gravesande 1722, 2)

Ergo, he concluded, live force is the true measure. To be sure, the other side had their own arguments; and neither group could produce conclusive evidence. As a result, by mid-century some thought the Vis Viva Controversy had run out of steam.[46]

The second debated aspect was mostly tacit, barely ever thematized. We recognize it as controversial in hindsight. That aspect was the place of live force in the foundations of mechanics. Leibniz took the notion of live force to be

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is relevant to it. The only genuine force is impressed force. Thus, no one who endorsed Newton's mechanics could be part of the Vis Viva Controversy *qua Newtonian*.

[45] See Wolff 1728; the citation above is from p. 218.

[46] For lucid commentary, see Ducheyne and van Besouw 2022. Before 's Gravesande, Giovanni Poleni at Padua had performed a similar experiment, with balls dropped on a bed of tallow (Poleni 1718, 56–8). Synoptic accounts and evaluations of the Vis Viva Controversy are Iltis 1973 and Smith 2006, which I have found particularly helpful. Brading 2019 analyzes developments in 1740s France, a stage often left out of standard histories of the Controversy.

indispensable for treating *any* temporally-extended motion process; and he thought that Conservation of Vis Viva was *the* law of dynamics, the allegedly new science he had created. However, in the Enlightenment many prominent figures demurred from this view. Christian Wolff demoted live force to the periphery of his natural philosophy: he used it just to explain elastic collision, Leibniz's old paradigm of live force at work. For Wolff, the truly general concepts were 'active-' and 'passive force,' with *vis viva* merely a species of the former. In *Foundations of Physics*, Emilie du Châtelet gave new arguments that *vis viva* was the true measure of 'force of motion,' but while her tract contains three laws of motion, presumably general, Conservation of Live Force is not among them. And, d'Alembert in his *Treatise of Dynamics* prided himself to have shown this conservation law to be a theorem—not a fundamental principle—inferable from *his* new laws of motion: "based on the principle whereby I solved the problems in this book ... I conclude with a *demonstration* of the principle commonly called the 'conservation of live forces'" (d'Alembert 1758, xxxiv; my italics). Those laws, in fact, did not even contain a force concept, let alone Leibniz's canonical notion.

The third aspect, likewise debatable in retrospect, was whether live force is truly conserved in all mechanical processes. The two conceptual innovations above—the idea of apparent loss (in inelastic impact), and the notion of potential live force—had an unintended effect: in the Enlightenment, Conservation of Vis Viva becomes *deeply equivocal.* In addition to the two senses (3a) and (3b), the conservation law also denoted two more, different ideas. Here is one:

$$\text{Live force}_{\textit{end of process}} = \text{Actual live force}_{\textit{beginning of process}} + \\ + \text{Apparent loss}_{\textit{transferred to the parts}}. \qquad (4a)$$

This is what Christian Wolff—and others like him (including Leibniz) meant by their conservation principle; see below. And, here is the other sense:

$$\text{Total live force, actual and potential}_{\textit{beginning of process}} = \\ = \text{Total live force, actual and potential}_{\textit{end of process}}. \qquad (4b)$$

This is what Daniel Bernoulli meant by *his* conservation principle—and others like him (e.g., Euler) who thought that mechanical processes amount to transfer of live force between bodies, and to its conversion, from potential to actual or vice versa.

Here is what led to the emergence of the novel sense (4a) above. Later in his career, Leibniz felt he had to say something about the fact, obvious to the senses, that when inelastic bodies collide, their actual live force after impact is *less* than before: some *vis viva* appears to have been lost or destroyed. Leibniz predictably claimed that it was not *really* lost; rather, the impact had transferred it to the 'smallest parts' of soft bodies. From this, he abruptly concluded that live force may not be conserved in every *finite system*, but it is conserved in the *universe as a whole:*

> Now when bodies' parts absorb the force of impact, be it entirely or in part, ... the third equation [viz. the conservation of actual *vis viva*] does not hold. ... And yet, this failure does not detract from the conservation of the same quantity of force *in the world.* For what is absorbed by the small parts is not absolutely lost *for the universe*, even though it is lost by the total force of the colliding bodies. (Leibniz 1860 [1700], 230–1; my italics)

Though Leibniz did not stop to comment, this change of mind marks a drastic shift in the status of his conservation law. In modern terms, he changed it from a *local* to a *global* conservation principle. Wolff, in his influential compendium of natural philosophy, endorsed this picture, and the resulting claim:

> The vis viva that goes missing in impact is not really destroyed; it is conserved. ... In the universe as a whole, the same quantity of vis viva is always conserved. ... Even though, when inelastic bodies collide, some live force goes missing, it does not perish; rather, it is conserved in another matter. (Wolff 1731, §§486–7; his italics)

As a result of these developments—discounting loss of live force as merely apparent, and the revisions made to its semantics—by 1750 Conservation of Vis Viva came to denote four distinct claims:

1. Actual *vis viva* is conserved. A local principle. (Leibniz, D. Bernoulli)
2. *Vis viva* is conserved in the universe as a whole (though not in every finite system). A global principle. (The later Leibniz, Wolff)
3. In closed systems, the actual- plus the apparently lost live force equals the initial quantity of actual *vis viva.* (Leibniz, Wolff)

4. In *some* mechanical processes, the sum of potential and actual live force is conserved throughout the interaction. (D. Bernoulli)

Thus disambiguating the exact content of the claim that live force is conserved helps us better keep track of the various commitments of Enlightenment figures who endorsed that claim; and it helps us diagnose when their agreement (that it is conserved) was genuine, or it rested on mutual incomprehension.

Conclusions

I take this synopsis of developments in Enlightenment science and philosophy to suggest three lessons.

In the century after *Principia*, an extensive brand of mechanics emerges, built on a concept central to Newton's book, viz. impressed force. And yet, it would be a mistake to view these developments as the rise of 'Newtonian' mechanics. Associating it with his name does little to explain, and does much to misattribute credit. After 1700, it was not clear to *anyone*, not even to Euler, that by 1780 the most powerful formulation of mechanics would be a theory of impressed forces. That it ended up being so was the result of a half-century of fortuitous insight, good luck—and stumbles by its competitor, live-force mechanics. If the theory of impressed force *must* be named, then we ought to call it 'Newton-Euler mechanics.' That points clearly to it being a child of the Enlightenment, in addition to the name giving credit where due.

Moreover, in the Enlightenment *another* brand of dynamics—based in *vis viva*, a notion and approach due to Huygens and Leibniz—emerges and grows to the point where, in the 1740s, it seemed a plausible alternative to impressed-force theory. While that prospect ultimately did not come to pass, live-force dynamics produced a body of results that lived on long after that. In effect, to Enlightenment figures from Johann Bernoulli to Lagrange we owe the creation and early application of what we call energy methods in mechanics: approaches based on the idea that a body or system always moves so as to either minimize its potential *vis viva* or to conserve its total live force. It is a cluster of very powerful heuristics that, for many classes of systems, works as a full alternative to the Newton-Euler approach based on impressed force. These developments add to our understanding of Leibniz's legacy for exact science in the Enlightenment, which now we know to have been quite significant (McDonough 2022).

And, with the benefit of hindsight we can see that the Enlightenment is when the mathematization of nature began to turn from a program into a fact. Next to them, the work of Huygens and Newton in the preceding century looks like a promissory note. A strong, ambitious and significant promise, to be sure, but relatively embryonic when compared to the collective output of the Bernoullis, d'Alembert, Euler, and Lagrange.[47] This fact alone is reason to regard Enlightenment mechanics as a self-standing achievement, worthy of consideration on its own rather than a coda—'Newtonian science,' as the old cliché had it—to the *Principia* of 1687.

Finally, thinking about force and its instantiations was the work of many traditions and communities of inquiry; some emerged in the 1600s, and others in the following century. Often, they grappled with this concept without an eye to terminological consensus. As a result, by the Late Enlightenment natural philosophy had made room for many kinds of force: absolute-, expansive-, accelerative-, driving-, dead-, live-, centrifugal-, elastic-, relative-, retarding-, variable-, constant-, and projectile force; and even force of inertia.[48] But, these species did not amount to a complete partition of the class of all forces; and some overlapped, whereas other subclasses eventually turned out to be empty. The underside of this linguistic exuberance, however, was that by the late 19th century the term 'force' had come to seem rife with confusion and equivocation. Tidying up its semantics, it appears, was one of the aims behind Heinrich Hertz's foundational project in mechanics (Eisenthal 2021).

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[47] Decades ago, some historians pointed out just how much remained to be done in mechanics as of 1720; see Truesdell 1960b.
[48] See the explications in Brisson 1781 and Gehler 1798.

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