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Phoronomy: space, construction, and mathematizing motion

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With his chapter, *Phoronomy*, Kant defies even the seasoned interpreter of his philosophy of physics.[1] Exegetes have given it little attention, and understandably so: his aims are opaque, his turns in argument little motivated, and his context mysterious, which makes his project there look alienating. I seek to illuminate here some of the darker corners in that chapter. Specifically, I aim to clarify three notions in it: his concepts of *velocity*, of *composite* motion, and of the *construction* required to compose motions.

I defend three theses about Kant. 1) his choice of velocity concept is ultimately insufficient. 2) he sided with the rationalist faction in the early-modern debate on directed quantities. 3) it remains an open question if his algebra of motion is a priori, though he believed it was.

I begin in § 1 by explaining Kant's notion of phoronomy and its argument structure in his chapter. In § 2, I present four pictures of velocity current in Kant's century, and I assess the one he chose. My § 3 is in three parts: a historical account of why algebra of motion became a topic of early modern debate; a synopsis of the two sides that emerged then; and a brief account of his contribution to the debate. Finally, § 4 assesses how general his account of composite motion is, and if it counts as a priori knowledge.

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[1] Hereafter, by '*Phoronomy*' I mean his chapter, "Metaphysical foundations of phoronomy," and by 'phoronomy' the discipline that Kant so denoted. The same goes for his other chapters (*Dynamics*, *Mechanics*, etc.). Throughout, I use '*Foundations*' as a convenient name for his book at issue in this volume, viz. *Metaphysical Foundations of Natural Science*. Unless noted otherwise, all translations are mine.

To achieve my aims, I use two methods. One is to translate his key terms into the language of modern kinematics. This approach lets us disambiguate his notions, clarify their explanatory connections, and better grasp the scope and limits of his phoronomic foundation. The other method is to read the problems and theses of *Phoronomy* in the context of early-modern efforts to mathematize motion. I show that Kant was part of a long effort to secure algebraic structure for directed quantities—which the new science of motion needed, but classical mathematics could not provide.[2] This casts fresh light on Kant's phoronomy, and on his place in the long history of foundational debates in mechanics.

More broadly, grappling with *Phoronomy* is good training for a much-needed examination of Kant's notion of proof in mathematical physics: its epistemology, sources of evidence, and reliance on central distinctions in his thought, such as a priori/a posteriori, pure/empirical, and the like.

1. The subject-matter of phoronomy

By his account, phoronomy studies matter regarded just as "the movable in space." It "abstracts," or leaves out, all internal structure in matter, and so it considers just "motion, and its magnitude." Specifically, its speed and direction (480). The challenge is to explain these ideas without paraphrase, in clear concepts—preferably, *our* concepts. That task faces several obstacles, so I try here to remove them first. In particular, there are some red herrings that can lead astray even the wary reader.

One is Kant's very term. 'Phoronomy' had been a coinage used just twice before *Foundations*, and in contexts quite unrelated to Kant's usage. Leibniz invented the word in the 1680s to denote the doctrine of the "laws of nature," whereby he meant the dynamical principles of *collision* theory.[3] Then his disciple, Jakob Hermann, adopted it for his own project, a 1716

[2] By 'algebraic structure' I mean the concepts and principles that legitimize addition, subtraction, and related operations. In this regard, I also use 'algebra of motions' as an auxiliary synonym for the idea that motion-quantities can be added and subtracted.

[3] The term came from the Greek for local motion (*phora*) and law (*nomos*). For a sense of Leibniz's meaning, cf. his dialogue *Phoranomus, sive de Potentia et legibus naturae.*

comprehensive treatise in particle dynamics.[4] Kant does not mean his term in these senses. For the Leibnizians, phoronomic doctrine dealt indispensably with the causes of motion processes, whereas Kant is clear that his phoronomy is a *pre*-causal treatment.

Another red herring is the modern notion, 'kinematics.' Just decades after Kant, Ampère and Poncelet invented it to denote the purely geometric, descriptive and *non-causal* account of motion. That sounds much like Kant's objective in *Phoronomy*, and so we might think that his term is an exact synonym for 'kinematics.' But that would be a mistake. There is *no* single, all-encompassing discipline that covers descriptively *all* species of motion; there are just local theories, fit to describe some species but not others. Namely, each branch of classical physics has its own kinematics, with just enough mathematical structure for the needs of that branch, not generally.[5] Hence saying that phoronomy is a kinematics leaves things fundamentally incomplete—we ought to also add, what his kinematics is *for*.

Finally, the third red herring is Kant's announcing that phoronomy is a general account, qua part of the "general doctrine" of *body*. Recall his claim to have "*completely* exhausted this metaphysical doctrine of body, so far as it *may* extend" (473, my italics). That is unhelpful in two respects. Qua descriptive theory of motion quantities, phoronomy is *not* general; far from it, in fact. There are many motion species that his chapter does not cover and could not possibly cover, because the conceptual basis he offers is too weak for them. Moreover, his phoronomy does not really describe the motion of *bodies*. Namely, it is too weak to describe their motion as *extended* volumes of matter, which require stronger concepts (and mathematics, too) well beyond the basis of his chapter. Insofar as it applies to bodies, phoronomy is valid of them only under very narrow, restrictive assumptions that Kant unfortunately mischaracterized. I detail these charges in Section 4.

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[4] The book's full title was *Phoronomy: the forces and motions of bodies, solid and fluid*. Kant had it in his library (Warda 1922: 34).

[5] That is, there is a kinematics of wave motion; one for free-particle orbits; one for rigid-body motion; yet another one for continuous deformations; a kinematics for constrained-motion systems; one for continua with microstructure, e.g. liquid crystals; and so on. No single branch of mathematics has the conceptual resources to treat all of these motions within one framework. E.g., for free particles and deformable continua, the most comprehensive kinematics requires as mathematics the differential geometry of skew curves and surfaces; whereas for rigid bodies we need to add tensor algebra, so as to describe the kind of rigid motion known as 'change of attitude.'

Then what *is* phoronomy? I propose here a construal of his movability doctrine in modern terms. To avoid the threat of incompleteness I signaled above, I specify these terms as best I can. I claim that Kant's phoronomy is a *kinematics for particle collision in a force-free vacuum.* Just what grounds I have for my construal will become more clear in Section 4.

2. Kant's concept of speed

The keystone of Kant's account of how we mathematize motion is a notion of speed at an instant. He was right to give it such attention, because it was a thorny problem. Between Galileo and Lagrange, mechanics relied on mathematical descriptions of motion that were often implicit, and dependent on foundations shifting rapidly. Kant found himself amidst this long age of drastic change. In particular, around the time of *Foundations* there were four notions of (instantaneous) speed available. Here I present those concepts, I use them to elucidate Kant's preferred concept, and then I assess his choice, as befits this guide. To keep them apart, I give them below custom names; in every case, let *P* be a particle moving in some curve.

Pre-classical. Speed is the 'intension' of a 'form.' Namely, it assumes that *P* has a 'form of motion' at every point of its trajectory. Imagine P to travel for some finite time T over some distance S *while having the same 'form' at every instant*, viz. 'uni-formly.' Instantaneous speed is the numeric 'intension' C of the form, and it equals S/T defined as above.

Condensive. Speed at a location *X* is the 'condensation point' of a series of decreasing values. Namely, let *XA*, *XB*, *XD*, etc. be segments standing for future paths of the particle *P* currently passing through *X*. And, let their lengths increase serially, i.e. $XA<XB<XD$, etc. The speed C is the value toward which these lengths tend, if taken smaller and smaller. In our terms, C is their 'limit from above.'

Differential. Speed is a ratio of two infinitesimals. Let *ds* be the infinitely small path that *P* crosses in an instant, i.e. an infinitely small time *dt.* P's speed is the ratio *ds*/*dt.*

Analytic. Speed is a part of an algebraic object. Let *P*'s motion be representable by an analytic function of time $\mathbf{f}(x, y, z, t)$. At any location *X* on its path, *P*'s speed equals the coefficient of the second term in the Taylor-series representation of the function **f** at *X*. Alternatively, at any instant, the speed of *P* is f's derivative with respect to time.

Now I give evidence for the concepts above. I called the first notion 'pre-classical' because it it shows up early, well before the law of inertia and the classical mechanics that it engendered. The notion received clear expression already in the 14th century from figures who reflected on mathematizing motion. At Paris in the 1340s, Nicole Oresme explained how speed links up with extended distance: "We *imagine* punctual instantaneous speed by means of a straight line" (1968: 292, my italics). The verb, 'to imagine,' was his way of conveying that in considering instantaneous speed we do not grasp a stretch of space actually crossed, but rather one that we may imagine the mobile to traverse [*spatium quod ymaginatur pertransiri*] counterfactually, if it crossed it at the same punctual speed (Oresme 2013: 686). This need (to rely counterfactually on stretches not actually traversed) is implicit in another, equivalent definition by Roger Swineshead at Oxford: "of local motions, one is swifter [*velocior*] when, by the intension of the former motion, one *could* cross [*poterit pertransiri*] a greater space, during some time, than by the latter motion's intension."[6] The thought that we may use line segments to quantify punctual speeds is even older; we see it voiced by 1260: "the proportion of the motions of points is as the proportion of straight lines described in the same time."[7]

And, the pre-classical notion of speed survived well into the early modernity. John Wallis relied on it: "Speed [*celeritas*] is an affection of motion, and it results from comparing distance [*longitudo*] and time; that is, from determining how much distance *is crossed* in how much time" (1695: 576). But Wallis and his age made a mistake while adopting the medieval concept. Note that, as Oresme and Swineshead above knew, to define punctual speed we must resort to *counterfactual* distances; but the early moderns omit to specify this crucial point. In so doing, their failure (to mention that S and T denote *non*-actual stretches) becomes a crippling defect. In sum, it replaces the desired concept (namely, instantaneous speed) with the wrong one, viz. *average* speed, which concept is not at issue here.

The condensive notion of speed is not explicit in 18th-century works. Only its underlying concept is, viz. of limit of a sequence, can be found. D'Alembert defined it as the fixed value toward which a convergent se-

[6] Translated from Roger's unpublished Erfurt Manuscript, page 43 retro, column *a*; excerpted in Sylla 1973: 245.
[7] Translated from Gerard of Bruxelles, *Liber de motu*, excerpted in Clagett (1984: 64).

quence tends: "a magnitude [*grandeur*] is the limit of another one when the latter can approach the former magnitude by more than any given, arbitrarily small magnitude, such that the difference (of the approaching magnitude) to its limit is unaccountably small" (1780: 52). Elsewhere, he singled out certain sequences—for instance, a sequence of ever-decreasing distances, as Kant's diagram (for his Parallelogram Rule) allows us to visualize—as especially easy to test for convergence: "for a series to be as perfect as possible, it must be the case that 1) its terms consecutively decrease, after the first one; and 2) all of its terms have the same sign" (d'Alembert: 1768: 177). But, I cannot find direct evidence for the condensive speed concept in the 1700s. Sutherland attributes it to Kant (2014: 703ff). But, we must count it merely as a possible interpretation—viz. just compatible with the textual evidence—because Kant himself did not declare it overtly.

The differential concept of speed is on display in Euler's *Theoria motus*, the first significant tract in rigid-body dynamics; Kant appears to have seen it, though perhaps too late.[8] There Euler introduces uniform and non-uniform motion, declares that "they differ in their essence," and goes on to treat non-uniform motion [*inaequabilis*] in a straight line first:

> [In problems of particle mechanics], the entire business reduces to finding the place where the moving point shall be at any arbitrary given time. Thus let AB be the straight line in which the point moves, starting from A. After a time $=t$, the point will be at place S, and let AS $=s$, the distance crossed in time t. [...] By differentiation, we obtain the element of distance ds that the particle crosses in an element of time dt. And the fraction ds/dt expresses the moving point's *speed at the place* S. Evidently, this fraction is a finite quantity. Hence, if we let v denote the speed *at* S, we have $v = ds/dt$. Consequently, for particle motion we can assign a *speed at any place or also at any instant*. (Euler 1765: 16; my italics)

Thus Euler taught lucidly the notion of instantaneous speed qua local magnitude defined at a point. And, he warned that C=S/T, the definition favored by Kant and others, is valid only in a very narrow context, namely when a particle moves in *uniform* translation.

Lastly, the analytic notion of speed was Lagrange's singlehanded creation. He began by taking for granted that we may represent particle position by some coordinate function; indeed, that was established practice by then. Lagrange refined this assumption in two respects. First, he imposed the

[8] Euler's book was reissued in 1790, and Kant in the *Opus postumum*, written largely in the 1790s, once refers to "Euler's *materia rigida*" (1993: 32; 22: 213).

condition that any such coordinate mapping must be an analytic function.[9] Second, he assumed that, for any such function, its value at a point $(x+i)$ can be represented by a Taylor-series expansion (1797: 8):

$$\mathbf{f}(x+i) = \mathbf{f}(x) + \mathbf{p}(i) + \mathbf{q}(i)^2 + \mathbf{r}(\mathrm{i})^3 + \ldots.$$

Against this backdrop, for any particle motion given by a function **f** of coordinates and time, its speed at point *W* is the second term in the Taylor-series representation of the value **f**(w) that the function takes at *W*:

> The functions [that we use in mechanics] inevitably relate to time, which I will always designate by *t*. And so—since a point's position in space depends on three rectangular coordinates *x*, *y*, and *z*—in all mechanical problems I will take these coordinates to be function of *t*. ...
>
> Generally then, in any rectilinear motion where the distance crossed is a given function of the time passed, the *first function of this function represents the speed*, and the second represents the accelerating force at some instant. (Lagrange [1797]: 223, 228; my italics)

That was his way of saying that speed is the first derivative of the particle's change of coordinates with respect to time.[10]

With this synopsis of kinematic concepts behind us, three questions need our attention now. Which concept of speed really was Kant's notion? We have no direct evidence for an answer—he remained oracular about it. Exegetes have ascribed him two such concepts. Sutherland says Kant used

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[9] Start with the notions of a *variable* quantity *x*, *y*, etc.; of *value* of a variable, viz. a particular number; and of a *constant* quantity *a*, *b*, *c*, etc. (The post-Leibnizians called them 'symbols,' as did Kant.) In the 18th century, an analytic function was any 'expression,' or syntactic string composed from variables, constants, and the common algebraic symbols +, –, ·, ÷, plus exponentiation. See the definition in Lagrange 1797: 7. In the formula above, **p**, **q**, **r** are component functions of Lagrange's "derived function." See below.

[10] Lagrange's phrasing is opaque, but easy to explain. For him, the Taylor expansion represents a function **f** by means of a "derived function" **g**, which is an (infinite) sum of *other* functions **f'**, **f''**, **f'''**, etc. Let *u*, *v*, *w* be consecutive values of some real variable, such that they differ from the next one by an infinitesimal amount *du*. (E.g. they could be points in space lying next to each other; or consecutive instants of time.) To represent **f**(v), i.e. **f**'s value at *v*, we use **g**(v), built as follows: $\mathbf{g}(v) = \mathbf{f}(u) + a\mathbf{f}'(v+du) + b\mathbf{f}''(v+du) + c\mathbf{f}'''(v+du)$, etc. In this infinite sum, **f**(u) is a *fixed* value, not a variable. Therefore, **f'** counts as Lagrange's "first function of the function" **g**. As is well known, **f'** is the first derivative of **f**, the represented function.

the condensive notion, but Friedman gave a strong argument that it would not be the right account to attribute to Kant.[11]

Why did Kant choose his particular speed-concept? There is an externalist explanation for his choice, but it is philosophically unsatisfying.[12] Fortunately, Kant had an internal reason to prefer the pre-classical concept. Namely, the concept integrates very naturally with his theory of mathematical cognition in two respects. One, it allows us to show that speed can be constructed geometrically, viz. represented by the picturable singulars that he considered essential to construction. Two, Kant thought that pre-classical speeds being constructable in *pure* intuition lets him argue that operations on speeds count as synthetic a priori knowledge. I turn to this point below, in Sec. 4.

Finally, was it a wise choice? On this point, I am not sanguine. Kant's velocity concept has an unobvious but real shortcoming. Namely, *it is not general*. More exactly put, the defect is this. Systematic reasons drove him to adopt geometric representings of the quantities crucial to a modern theory of motion—position, velocity, and acceleration. But, qua descriptive language for those parameters, the geometric framework is too weak for modern theory as it had grown by his age. First, it lacks a *general* way to express accelerations, though it admittedly allows us local, configuration-specific ways of representing velocity-difference, or acceleration.[13] Second, it cannot

[11] See Friedman (2013: 62-7) and Sutherland (2014: 703f). I am not sure what Sutherland's evidence is, for attributing the condensive speed-concept to Kant.

[12] Apart from brief interludes around mid-century, the German lands in the 1700s were a backwater of mathematics; what cutting-edge research there was (by Euler, and then Lagrange) was confined to the Berlin Academy, from which Kant and his peers—campus metaphysicians, with few exceptions—were cut off. To compound his predicament, Kant's mathematical training was always deficient, and his lack of French (the lingua franca for new science then, with Latin a distant second) prevented him from trying to keep up with the work of Euler and Lagrange. Cf. also Rusnock 2004, Sec. 2.

[13] One local approach to acceleration was by means of 'natural' coordinates. Specifically, at a given point C of the particle's motion (in a plane), one represented the induced acceleration as two speed increments, along two lines: one tangent to the trajectory *at C*—hence, locally—and one normal to the trajectory, likewise at C. The task was to infer the magnitude of these speed increments, from the known forces acting on the particle at that location. Once known, these increments would be 'composed'—via the Parallelogram Rule—with the particle's (already known) velocity at C. The result of this composition was knowledge of the location D where the particle would be at the *next* instant, once it moved past C. At that stage, the task above had to be reprised: a new 'composition of motions' was needed, to find the location E at the following instant; and so on. A vivid illustration of

be used to *write differential equations*. And yet, mechanics after 1760 had evolved into a stage that required the two capabilities above as sine qua non features of any mathematical language aspiring to be a representational vehicle for kinematics.

In particular, consider how Euler stated his "new principle of mechanics," i.e. the law of motion that governed *all* the mechanical processes solved mathematically by then, including some that neither Newton nor Kant had analyzed. Euler wrote his law in component form, viz. stated relative to each of three orthogonal axes of an inertial frame external to the system to be described. And, for each motion-component, he used a coordinate function to represent it:[14]

$$\text{I. } 2M\,ddx = P\,dt^2; \quad \text{II. } 2M\,ddy = Q\,dt^2; \quad \text{III. } 2M\,ddz = R\,dt^2.$$

Thus Euler makes clear that, in regard to mathematical form, all the laws of motion are differential equations. For the scientific elite then, successfully mathematizing matter in motion amounted to deriving equations of kinematic change at a point, over an instant, for that particular type of matter.

The import of these facts is: in Kant's time consensus among theorists had coalesced around three convictions. First, the laws of physics had to be stated in coordinate *functions*. Second, quantifying the motions of particular bodies amounted to finding the derivatives of these functions. Three, inferring to those derivatives was through deductive reasoning from differential equations, not geometric construction—because those equations *cannot* be represented geometrically. To be sure, some role remained for the geometric approach that Kant favored: it was useful for motions tractable in 'natural coordinates.' The fact is, however, that such geometric approaches were *not general.*

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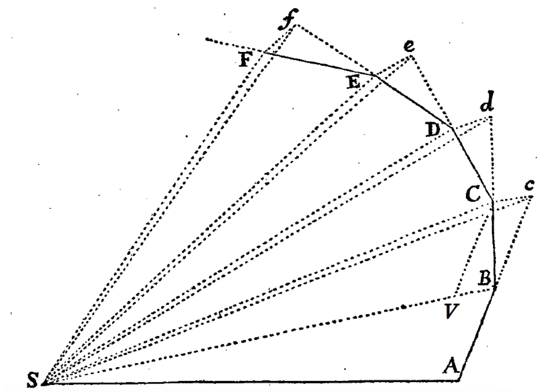
this natural-coordinates approach is Newton's proof of the Area Law in *Principia*, Book One, Proposition VI.

[14] P, Q, and R are the components of the total impressed force (on a point mass) along three orthogonal axes Ox, Oy, and Oz supposed immobile; x, y and z are coordinate functions, while d^2x/dt^2, d^2y/dt^2, and d^2z/dt^2 are the components of the resulting acceleration (induced by the total force). The 2 factor is needed because M is not really a mass; it is a *weight*, with the value of g, the acceleration of gravity, set by convention at ½.



[Fig. 2. Newton's diagram for his proof of Kepler's Area Law. *Bc* is a counterfactual path that the planet-qua-particle *would* cross in an instant, if it moved inertially with the velocity it had at location *B*. And, *BV* is another counterfactual path, which the planet *would* cross in an instant *if* it were sitting stationary at *B* (instead of moving, as it in fact does) while a gravity-like force (emanating from S) acted on it. Finally, *BC* is the effective path, which the particle *actually* crosses in an instant—as a result of arriving at B, where the force S acts on it. Note that *BC* results from applying the Parallelogram Rule to the two counterfactual motions above. Mutatis mutandis for the planet's motion at the subsequent locations *C, D, E*, etc.]

They worked just for particle motions under gravity-like forces as Newton had geometrized, with great success, in the previous century (see Fig. 2).

In sum, Kant's concept of velocity appears seriously hampered, if the foundational intent was to ground a sufficiently general theory of kinematic experience. If I am right, then we should ask again, just how much explanatory scope does his metaphysics of phoronomy have?

3. Kant on composite motion

The key result in *Phoronomy* is the Parallelogram Rule, a composition theorem for adding velocities. I begin with a baffling fact: Kant constructs the Rule—but, why? Why is there a geometric figure (and diagrammatic reasoning on it) in a treatise of *metaphysics*? After all, Kant is famous for drawing a sharp line between the evidence-gathering methods of metaphysics and mathematics.

One reason is that velocity crosses architectonic boundaries, so to speak. It counts as an intensive magnitude, but it is associated with extensive ones as well (the size of spaces crossed at various speeds). Thus treating degrees of speed with the mathematics of extension seems to rest on certain grounding assumptions that Kant wishes to uncover and defend: "it is not clear by itself that a given speed consists of smaller speeds (and a rapidity consists of slownesses) in the way that a space consists of smaller ones" (493).

Another rationale for him to discuss Composition is for the sake of a philosophical explanation. In particular, Sutherland has argued that Kant needs his diagram so as to explain how we *represent an identity*: of parts and the whole they make up. The compounding motions are the parts, and the composite motion is the whole (Sutherland 2014).

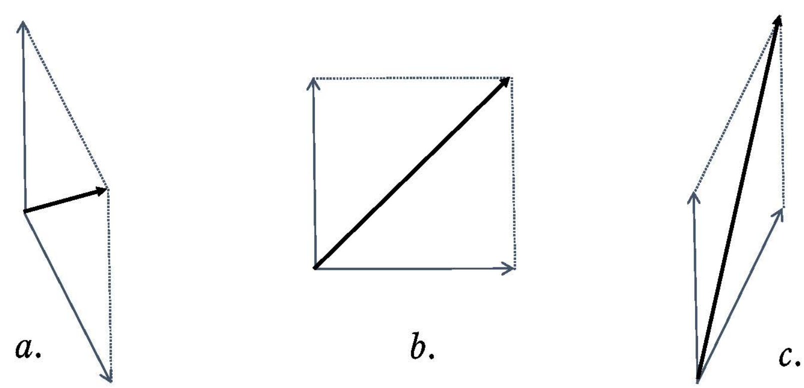
I suggest that Kant would have had yet another, third philosophical aim with his diagrammatic construction and philosophical explication of it. Specifically, I read Kant's diagram as an episode in the early-modern efforts to show that directed quantities have an algebra—they add and subtract—and to clarify the epistemology of their mathematization.

Adding velocities (by the Parallelogram Rule) was the gateway insight of early-modern science, on a par with the Law of Inertia but far more useful than it.[15] And yet, crucial and indispensable as it was, velocity addition was a foundational enigma for early-modern theorists. There were two sources of difficulty, and they required much skill and insight to navigate safely past them. Briefly, the difficulties were:

- velocities have directions too, not just sizes. And, *direction* makes a difference to the *size* of the sum, or resultant, of their addition.
- but the algebraic framework of classical mathematics had *no* way to determine the result of adding velocities. It lacked rules for *adding directions.*

[15] Like any inertial dynamics, early-modern theory grappled with the generic task of mathematizing *deflections* from inertial paths. (Inert translation, if it ever occurs, is trivial to quantify.) But that process always requires adding velocities as the sine qua non operation. It is because, from one instant to the next, forced particles move along the resultant, or 'composite' motion, of two velocities: one that the particle has and keeps (by the Law of Inertia), and an acquired velocity increment, due to the net impressed force (acting on it at that point and instant). See, again, Newton's above derivation of the Area Law.

To grasp the first difficulty above, consider the two combinations below. Each involves the addition of two velocities pairwise *equal in size*: their speeds are the same. But, one velocity differs in direction from its size-twin in the other pair. That difference alone is enough to make a difference to their addition—by affecting the size of the resultant.



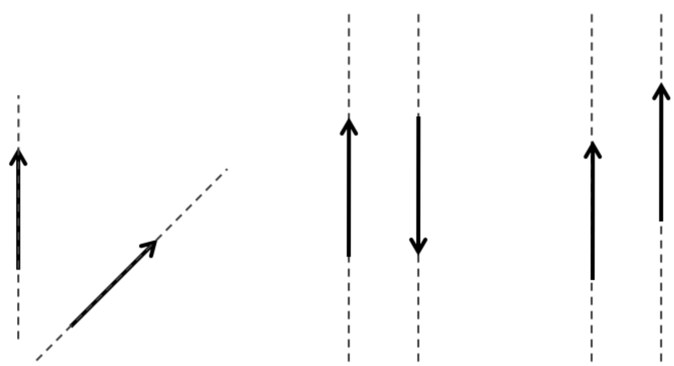
[Fig. 3. Pairs of velocities respectively equal in size. In each case, the heavy arrow represents their resultant, obtained in accordance with the Parallelogram Rule]

And so, a difficulty of composite motion was that, when it comes to adding velocities, knowing the size of the parts is *not* enough to infer the size of the whole. A confounding predicament, clearly.

To grasp the second difficulty, recall a key fact. Before the 1800s reforms in mathematics, the canon for adding magnitudes—more generally, the algebra of quantities, as we put it—was recorded halfway through Euclid's *Elements*. It is the theory of proportions historically credited to Eudoxus of Knidos. Like everyone then, Kant accepted it as *the* algebraic framework for mathematics qua general science of magnitudes.[16]

[16] See the conclusive case for that in Sutherland 2006.

But that framework had a debilitating gap. Born well before early modern kinematics, Eudoxean proportion theory was not designed to handle oriented magnitudes, viz. objects having a size and also a direction. The reason is that Euclid, Eudoxus and their pre-modern descendants counted as magnitudes just two basic species of object: discretes, viz. integers and their fractions; and continuous magnitudes, i.e. Euclidean multiples of straight-line segments.[17] In consequence, this algebra had no explicit rules for taking direction into account when adding motion-magnitudes. As illustration, consider the question (Fig 4): *what counts as two equal motions*?



[Fig. 4. Pairs of equal speeds. Which pair represents two equal motions, and *why*—on what grounds?]

This question is not well-posed in Eudoxean proportion theory—it has no rule or algorithm whereby to answer it.[18] But then, without an answer to

[17] Suppose a line segment AB to be given, and let it be unit length, by convention. Starting with AB and using *nothing but straightedge and compass*, construct a arbitrary segment CD. A Euclidean multiple is any magnitude *m* such that *m* is to 1 as CD is to AB.

[18] To be sure, embryonic forms of the Parallelogram Rule were *known* in Antiquity; for instance, in the Alexandrian tradition of statics. My point is that knowledge (of what results by applying the Rule) could not be fit into the official algebra, viz. Eudoxean proportion

this question there is no way to decide whether one motion counts as twice as much as another—it is a "doubled celerity," as Kant has it—or not; and so on, for every multiple. In sum: the classical theory of magnitude had no official rule for adding *oriented* magnitudes.

The lacuna was felt painfully when Descartes and others began to theorize in optics, collision theory, and dynamics—the 'mixed-mathematics' areas in which resolving and adding motions is the key device. Lacking any guidance from the Eudoxus and Aristotle, it took the 17th century many decades to understand how, and accept that, size is linked to direction inextricably, such that treating them separately would give the wrong result.[19] Then the early moderns faced up to another conundrum. Knowing that the Parallelogram Rule was true—but not in virtue of Eudoxean algebra—they had to answer the epistemological question, why is the Rule true? That is, what is the *evidence* for its truth, and how strong is it?

By the 1720s, two approaches to answering this question had emerged, and they would compete into Kant's age. Each approach had several species, but I can only very briefly mention them here; a real discussion of their subtleties belongs elsewhere. One approach was broadly empiricist, and indirect. That is, the approach rested on empirical facts about forces; and on conclusions about the motions—and *their* composition—associated with these forces. An example is Newton, who showed that his first two laws jointly entail that forces add by the Parallelogram Rule. In turn, single impressed forces generate accelerations, or velocity increments. It follows that a composite motion is proportional to (and inferable by the same Rule as) the composite force associated with it.[20] Later, 's Gravesande devised another proof procedure, even more decidedly empiricist. It relied on experiments with static forces on weights (1742: 94f). They showed that any two forces jointly acting on a body have a resultant equal to the diagonal of the parallelogram they form with each other (see Fig. 5, up).

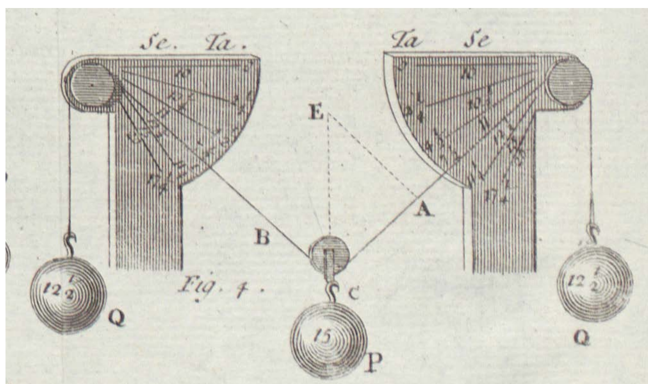
The other approach was broadly rationalist. It relied on premises made true by some broad principle of rationality; or on evidence accessible to

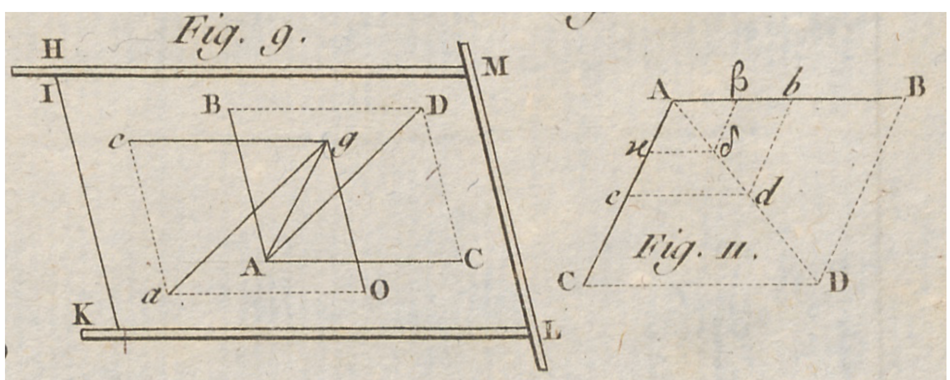
theory. As a result, knowers of the Rule would not have been able to explain why applying the Rule counted as an operation on *magnitudes*.

[19] These struggle are lucidly and rigorously explained in Miller 2017.

[20] See Corollary II (on composition of forces) and I (on composite motions) to the section 'Axioms, or Laws of Motion,' in Newton, *Principia*, Book I (1687: 13f).

rational intuition. The former was Daniel Bernoulli's way, who relied on the Principle of Sufficient Reason: he started with three static forces **f**, **g**, **h** that add up to zero (by the PSR, allegedly). Then he *argued* that any other three forces **a**, **b**, **c** obeying the Parallelogram Rule will be dynamically equivalent to **f**, **g**, **h** in that they too will result in equilibrium.[21] The latter was d'Alembert's procedure, who relied on a *Gedankenexperiment* with two parallel planes in uniform translation and a point moving relative to them—such that three relative velocities at issue (d'Alembert 1743: 35-7). From this thought scenario, he concluded we can grasp rationally that his three velocities above obey the Parallelogram Rule (see Fig. 5, down).





[21] Bernoulli's starting premise posits three forces **f**, **g**, and **h** to be equal, at 120 degrees with one another; he claims that PSR entails they are in equilibrium; cf. Bernoulli 1728.

[Fig. 5] [Up: 's Gravesande's experimental apparatus for confirming the Parallelogram Rule. The two outer weights a net resultant force R on the the middle body. R is equal and opposite to the body's weight (because the body remains at rest). The apparatus yields evidence that R is always equal and collinear with the diagonal of the parallelogram formed by the two outer-weight forces. Down: d'Alembert's a priori proof of the Rule. A is a point particle, mobile in a plane BDCA that sits, and slides without friction across, plane HMLK underneath it; *cg*O*a* represents where plane BDCA arrives, relative to absolute space, after a finite time.]

We can put the early-moderns' problem above in Kant's own terms, to make vivid his predicament. Let **a**, **b**, **c** be 'motions,' i.e. instantaneous velocities qua directed speeds, and let '⊕' denote addition with direction factored in. For him, statements like '**a** ⊕ **b** = **c**' have truth values, and they are knowable a priori. But Eudoxean algebra had no way to secure non-empirical evidence for their truth or falsity. Hence Kant had to expand that algebra, so as to make '**a** ⊕ **b** = **c**' into meaningful, determinate judgments.

In effect, he argued that the Parallelogram Rule (for inferring the size and direction of **c** in '**a** ⊕ **b** = **c**') is synthetic a priori. In *Phoronomy*, I suggest, he aimed to show that the Rule is true on non-empirical grounds, so it is a priori; and it is synthetic, not analytic, as the evidence for it comes from a *synthesis*, not from discursive reasoning with concepts. I assess his dual claim below.

Hopefully, knowing the early-modern context I uncovered above might make it easier to understand why Kant decided to insert a diagrammatic proof in a chapter ostensibly on *metaphysical* foundations.

4. Scope and warrant

This being a critical guide, I end my contribution by assessing phoronomy in regard to its scope, or descriptive reach; and in regard to Kant's warrant for his Composition Rule.

Generality. So far, I have hinted at how phoronomy lacks generality when examined from outside, as it were: in light of mechanics' then-growing need for a descriptive vocabulary (of motion quantities) that far exceeds geometry in representational content.

But the question of generality arises internally as well, from within his doctrine. In particular, his Phoronomy has *two* concepts of motion: 'change of outer relations in space' and rectilinear point-motion, respectively. Kant show that the latter motion is mathematizable: its size is C=S/T, and its directions are additive. Still, this concept is just a species of motion (and a narrow one, at that). It is the *former* motion-concept that is general. However, Kant left it fallow: he did not analyze the notion of outer relations in space, did not survey its scope, and did not try to show that it is mathematizable. He gave us no argument that "change of outer relations in space" belongs in the "pure mathematics of motion." To restate my worry above, then, *Phoronomy* has shown just that there is a Kantian-proper science of linear velocity, nothing more.

And yet. Phoronomy may not be general, but it is consilient—with his main agenda in *Foundations*. In *Mechanics* Kant's epitome of interaction is the collision of bodies, which he treats under very restrictive conditions: he reduces each body to a 'representative point' and treats *their* collision. In effect, Kant disregards the bodies' extension and the very complicated motions that bodies undergo *because* they are extended. He just singles out their mass-centers and analyzes their impulsive motion (in impact) as mere points. Given that restrictive treatment, to represent their motion quantitatively he needs just a *kinematics for particle collision in a force-free vacuum*. Which is exactly what I claimed (in Section I) that his phoronomy amounts to. Specifically, it is a meager kinematics that includes a notion of velocity of translation; and the Galilean transformation—that is what really Kant's 'composite motion' amounts to—which he needs so as to describe the collision from two different frames: the observer's perspective, and the 'absolute space' in which the two bodies collide with equal forces of motion.[22]

Apriority. Now I examine the claim that his proof (of the Parallelogram Rule) is a priori. That he thinks so emerges from his objection to empiricist attempts to prove it: the Rule "must be constituted *wholly a priori*, and indeed intuitively, on behalf of applied mathematics" (486; my italics). But, is he right to think that his own proof counts as synthetic a priori?

The Rule is synthetic in the strongest sense: he infers it by diagrammatic reasoning on the output of a *synthesis*, viz. a parallelogram constructed in

[22] Again, see Stan 2013 for the full details.

intuition. Still, is it a priori? That is not clear yet. There is a chance that, in the final analysis, the Rule might count as a posteriori. I give here two signposts to guide future discussion on the real epistemic status of his Rule, because I consider it an open question.

Recall that apriority is a dual notion in Kant: semantic and epistemological. On semantic criteria, the Rule would count as a priori if it constructs *pure* concepts in pure intuition. Epistemologically, the Rule counts as a priori if all the *evidence* for it comes from *non-empirical* sources. This clarification should help us see why we cannot yet grant that his Rule is apodictically true. Consider.

Semantically, the Rule relies on the concept of a relative space: some of the motions it constructs are motions of relative spaces relative to one another. But, it is unclear if ⟨relative space⟩ counts as an a priori concept. Kant own words suggest the opposite, when read naturally:

> In any experience, something must be sensed—that is the real of the sensible intuition. Hence, the space too (in which we set up our experience of motion) must be *perceptible*: we must designate it through what can be *sensed*. (481; my italics)

That is, ⟨relative space⟩ is an acquired representation—by Lockean abstraction, he thinks—so it must count as an *a posteriori concept*. Also, 'relative space' is not a term in the inventory of basic concepts of the mathematical disciplines that Kant regards as pure, viz. elementary geometry and arithmetic. In sum, if future exegesis wishes to count ⟨relative space⟩ as a pure concept, it must make a case for it. More broadly, Kant admits that ⟨motion⟩ is an empirical concept. Then rules for composing motion would count as operations on a posteriori inputs.

Epistemologically, things are just as ambiguous. Kant wishes to restrict the scope of his Rule to rectilinear motions alone. Namely, he infers the Rule from a 'phoronomic principle,' namely, that rest and straight-line motion are equivalent: "I assume here that all motions [subject to the Phoronomic Principle] are rectilinear." But, what justifies these restrictive premises? Why not let *all* motions be phoronomic-relative, i.e. equivalent to rest? He gives us a hint: "for, in the case of curvilinear motion, it is *not the same in all respects* whether I regard the body as moving and the relative space as resting," or vice versa (488). In plain English, he means to say that rest and straight-line motion are dynamically indistinguishable—they count as the *same state* of motion—because of Galilean relativity.[23] (Or, even more strongly, because of Newton's Corollary VI.)[24] However, things are not the same—and so the Phoronomic Principle is *not* true—in the case of rest vs circular motion.

But, Galilean relativity is an *empirical* fact. It falls out of Newton's Third Law, viz. that impressed forces come in pairs and balance each other. Newton regarded the law as true a posteriori.[25] In sum, a key part of Kant's warrant for his Rule is empirical. Then how can it count as a priori true?

Objection: Kant has his own version of the Third Law; he argues that *it* is a priori; and it too entails Galilean relativity. Ergo, all of the evidence for his Rule amounts to a priori warrant, ultimately. Answer: the law is established late in *Foundations*, as a foundation for mechanics. So, it is not available as a premise in Phoronomy, where he would need it for the Parallelogram Rule. Plus, he derives his law of action-reaction from premises about force (mechanical and dynamical) and about 'active relations of matters in space.'[26] But *those* notions do not count as pure concepts. This attempted defense thus must face up to Kant's own injunction against 'impure' constructions and inference from them:

> For the construction of concepts, we require that the condition for presenting them not be borrowed from experience. So, their construction must not presuppose certain *forces*, whose existence can be inferred only from experience. Put more generally: the condition for constructing must not be itself a concept that *cannot at all be given a priori in intuition.* Such are the concepts of cause and *effect*, *action* and resistance, etc. (486f.; my emphasis)

[23] In this context, Galilean relativity means two things. 1) Rest and uniform straight-lien motion are equivalent states: neither requires any forces, nor produces any effects, that the other one does not. 2) The laws of motion work just as well in a relative space at rest and one in uniform rectilinear motion. In contrast, *circular* motion is different: neither (1) nor (2) hold of it, as Kant knows well (see below). Further, in Kant's doctrine—and also in Newton's science—Galilean relativity obtains *only* because of their respective Third Law, not unconditionally.

[24] Newton's Corollary VI is a stronger version of Galilean relativity. It asserts that rest and all rectilinear motion (uniform *and* accelerated) count as equivalent states. Friedman 2013 argues that Kant meant his Phoronomic Principle to be compatible with Corollary VI.

[25] Based on evidence from collision experiments. After him, in the 18th century, new evidence for the Third Law came from the mutual perturbations that Saturn and Jupiter induce on each other gravitationally. Cf. Wilson 1985.

[26] I have spelled out these conceptual assumptions and premises in Stan 2013.

If some of the concepts required for a construction are empirical; and if some of its premises are justified empirically: does the purported conclusion still count as a priori knowledge? McNulty 2014 likewise notes these delicate aspects of constructing motion. His efforts to elucidate these aspects, combined with my worries above, support the notion that we need more scholarly work to untangle this knot in Kant.

Conclusions

On the reading I have proposed here, phoronomy appears to be Kant's cautious, conservative attempt to articulate a geometry of motion compatible with his overall framework, methodology, and foundational agenda for the science of nature. Read this way, his project succeeds. Phoronomy, I argued, is consilient with his broader picture of how we use geometry to describe nature; with his constructive methods for mathematics; and with the centrality of collision in his philosophy of mechanics.

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