Unity for Kant's Natural Philosophy

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I uncover here a conflict in Kant's natural philosophy. His matter theory and laws of mechanics are in tension. Kant's laws are fit for particles but are too narrow to handle continuous bodies, which his doctrine of matter demands. To fix this defect, Kant ultimately must ground the Torque Law; that is, the impressed torque equals the change in angular momentum. But that grounding requires a premise—the symmetry of the stress tensor—that Kant denies himself. I argue that his problem would not arise if he had kept his early theory of matter as made of mass points, or "physical monads."

1. Introduction. Several times in *Metaphysical Foundations of Natural Science*, Kant claims to offer the "laws of general mechanics" (1911, 551; cf. also 548 and 539).[1] It is a bold claim, undeniably, but it is rather mysterious. Attention to Kant's historical context reveals that we may take his phrase in two senses. One is the way of Descartes, Leibniz, and Christian Wolff: general laws are statements holding of any body qua body, matter or corporeal substance. The other is the way of Newton, Euler, and Lagrange: general laws are dynamical laws that yield equations of motion for all possible bodies. As Lagrange put it in 1763, "the Statical principle [of Virtual Work] I just explained, combined with the Dynamical principle of Mr. d'Alembert, establishes a sort of *general* formula"—to wit, the Euler-Lagrange equation—"containing the solution to *all the problems of the motion* of bodies" (1763/1878, 12; my emphasis). Kant gives strong indications that he means gen-

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1. Hereafter, I refer to this work simply as *Foundations* and cite it by volume and page number in the Academy Edition (Kant 1911), e.g., 4:551.

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erality in the second sense. In chapter 3, where he alleges to have general laws, he outlines a geometric approach to deriving equations of motion for direct particle collision. Then he claims that his procedure "brings with it, as its necessary condition" a *synthetic a priori* law, that is, the Equality of Action and Reaction (4:549). However, that law by itself is not general, as Kant wants. For true generality, it needs supplementation by another principle, which I call the Torque Law, namely, the impressed torque equals the rate of change in angular momentum.

In fact, the Torque Law is general in the first sense too, so it ought to concern Kant all the more. For him, mechanics is the theory of "communication of motion," or momentum transfer. That task requires matter to be essentially the "movable insofar as it, as mobile, has moving force" (4:536). That is, mechanics is possible just in case all matter is *movent*, or able to move things in its path as it moves—causally efficient in virtue of having momentum. And, Kant declares, the "necessary condition" for mechanics is synthetic a priori laws of matter as such (i.e., as movent). However, note that matter is movent as it spins too, not just as it travels in a straight line. And, it is movent by making bodies spin too, not just translate. This basic fact obtains at all scales—molecular, mesoscopic, and celestial.[2] Therefore, as matter by its essence is movent through spinning, Kant must supply laws of matter qua movent by spinning. But that is just the Torque Law and its zero case, Conservation of Angular Momentum (CAM). Hence, by Kant's own standard, the Torque Law cannot be empirical. It counts as synthetic a priori, so Kant must ground it. Incidentally, this should obviate a basic serious objection to my project, that is, that the law is empirical, and thus Kant has no obligation to prove it.[3] The Torque Law is mandatory, not optional, for him.

But could he have known it? After all, the law is an advanced principle, and classical mechanics had a long infancy. It turns out that the makers of Enlightenment dynamics were quite sophisticated. Daniel Bernoulli, Patrick d'Arcy, Euler, and Lagrange knew about CAM and argued for it from various vantage points. Euler found and advertised the Torque Law in Kant's metonymic backyard, at the Royal Academy in Berlin, where Lagrange then used it for his theory of lunar motion. So Kant could, and should, have known and grounded the basic law of rotation.

And yet he did not. Can we, perhaps, do so, on his behalf and with his resources? It turns out that there are two ways to achieve that. The first relies on a strong version of Newton's Third Law. The second needs the symmetry of the stress tensor as a premise. Neither strategy is too anachronistic

2. Consider two spinning galaxies in eccentric collision, a tennis ball hitting a racquet off center, two grains of sand rubbing against each other.

3. I thank an anonymous referee for pressing me on this point.

to be interesting: both emerge after Kant yet in the classical age, before relativistic and quantum mechanics.

But there are repercussions for Kant in pursuing either strategy. The first demands that he go back to his early theory of matter, the "physical monadology." The second requires some a priori argument that certain kinds of shear forces are equal, or at least that the Law of Inertia holds for rotations. Alas, Kant leaves himself little room for doing that.

Failure to carry out the latter strategy comes at a price. Kant's natural philosophy ends up too narrow: although he offers it as a basis for *allgemeine Mechanik*, it is not truly general: it cannot ground celestial mechanics for extended bodies—the full theory of the "starry sky above" him—or much mechanics of continua. Another heavy price is loss of unity: his theory of matter as a physical continuum is too strong for his mechanics. Kant, I will show, would have been better off retaining the core of his early view that bodies are lattices of mass points.[4]

Below, I explicate or argue for these points, each in its section. Kant needs the Torque Law in his foundations for mechanics (sec. 2). He was in a position to know about it, and so to think about how he might anchor it in his system (sec. 3). After Kant, two ways of grounding the Torque Law emerge. One rests on Newton's Third Law (sec. 4). The other requires the symmetry of the Cauchy stress tensor (sec. 5). Prima facie, Kant cannot implement the latter (sec. 6). His theory of matter ultimately prevents him from grounding the Torque Law in his system (sec. 7). Changing that theory, by reverting to mass points, would give his foundations the unity and generality he seeks (sec. 8).

In conclusion, Kant's natural philosophy in *Foundations* lacks internal unity at a critical joint. Namely, his theory of matter does not fit his mechanics: the former needs stronger laws of motion than the latter has to offer. To be sure, I do not deny that Kant's doctrine does have 'external' unity, imposed on it from the outside, as it were, by the transcendental categories of the First Critique (for that, see Friedman 2013).

2. The Torque Law in Kant's System. I have argued, in the introduction, that Kant needs the Torque Law indispensably. More specifically, he must have that law for three reasons.

He calls mechanics the theory of "communication of motion," or momentum transfer—by contact and at a distance. His paradigm of the former is impact, and he tasks mechanics with "constructing" momentum transfer in intuition. To wit, it must represent the result of an arbitrary collision by

4. If that was indeed his considered view. It seems that, by the end of his *Physical Monadology* (1756/1992), Kant had quietly switched to continuous matter. See the lucid Smith (2013).

geometric means. But, he asserts, mechanics needs philosophy to supply it with "principles for the construction" of contact interaction. Among these principles are a priori dynamical laws to determine the outcome of any impact, given initial conditions. To illustrate his point, Kant constructs a special case, the direct collision of two symmetric bodies in pure translation.[5] His principle relevant to this case is the Equality of Action and Reaction: the law is the "necessary condition" for the "construction of the communication of motion" (4:549). However, that law is insufficient for the general task of constructing velocity exchanges in impact. Colliding bodies may rotate too, thereby exchanging angular momentum, and so a new principle is needed to construct the outcome: the Torque Law.[6]

Further, the Torque Law is crucial to Kant's very project of grounding philosophically Newtonian science. Unhappy with the empiricist justification of the three laws of motion in Newton's *Principia*, Kant replaces them with equivalents he thinks are demonstrably necessary. Yet that gives him at most principles to treat planets as particles. However, actual planets and stars are extended bodies. As such, they exhibit a host of behaviors—precession, nutation, libration, tidal friction, deformation—that neither Newton's nor Kant's laws can handle unless we add the Torque Law to them. That is because Newton's $\mathbf{F} = m\mathbf{a}$ really applies just to causes of linear, not angular, acceleration. But extended bodies also exert gravitational torques on each other.

More generally, and crucially, the Torque Law is necessary to underwrite equations of motion for all continuous bodies, deformable and rigid, and bodies are continuous in Kant's doctrine. Now classical continuum mechanics is built on two dynamical principles: the Force Law, a generalization of Newton's *Lex Secunda*, and the Torque Law.[7] Kant has an analogue of the Force Law; see below. But he still needs the latter.

5. That is, the bodies are nonrotating, mass density within each is constant, and they collide along a common axis of geometric symmetry (see 4:545–50).

6. In fact, initial spin is not even needed. Consider the ex-centric collision of two extended bodies in pure translation. They will exchange linear momentum and angular momentum. (I thank Sheldon Smith for drawing my attention to this case.)

7. Compare Gurtin (1981) and Truesdell (1991). A caveat: the Torque Law as I give it here holds only in standard continuum mechanics. Some modern theories relax it. On the left side of the law, they introduce sources of torque in addition to the impressed forces of classical theory, e.g., body couples and couple stresses. On the right side, they posit kinematic effects in addition to change in angular momentum, e.g., increments of spin angular momentum. This nonstandard, 'micropolar' theory was devised to deal with special anisotropic materials: 'oriented media', or 'Cosserat continua', e.g., liquid crystals in nematic phases. Micropolar theory endows the basic unit—the material point—with extra kinematic structure, e.g., a director field. Compare Jaunzemis (1967) and Murdoch (2012). This structure underlies the extra torques above. I thank Sheldon Smith for pressing me on this point.

In conclusion, Kant's mature natural philosophy needs to incorporate and ground the Torque Law. Without it, his foundation of mechanics is just a feeble shadow of what it aims to be.

3. The Torque Law in Enlightenment Dynamics. The Torque Law was not unknown in Kant's age. In fact, that is just when it was found. So, it is not unreasonable to expect that Kant should have it in his foundations, seeing as he needs it.

Enlightenment theorists first discovered the null case of that principle (i.e., CAM). Three figures in the 1740s state it independently. Daniel Bernoulli in "Nouveau problème de Mécanique" asserts that, if a body slides outward in a rotating massless tube, the system's angular momentum "will always be the same" (1746, sec. 8). And, he notes that Euler knew it too. In "Problème de Dynamique," Patrick d'Arcy stated a "general principle of dynamics": in a system of interacting bodies in motion, the product of their respective masses and areas swept relative to a fixed point "is always a quantity proportional to the times" (1752, 348). Then in 1760, the young Lagrange obtained CAM as a theorem from his then-basic law, the Principle of Least Action. He applied the latter to a system of free bodies driven "to move around some fixed point" by forces, integrated twice, and found that the area swept by each body around the fixed point times its mass "is always proportional to the time." Later, in *Méchanique analitique*, he derived CAM from a different basis: the Principle of Virtual Work (Lagrange 1762, 212; 1788, 202ff.).

While many Enlightenment figures found that CAM holds in isolated systems, Euler saw further than all. He alone grasped the Torque Law, or that the moment of the external forces equals the increment of angular momentum. Still, even Euler took decades to grasp it with full clarity. In 1736, he discovered that a rotating body in impact acquires two independent kinds of motion: a translation and a spin (Euler 1744, sec. 16). Then, in *Scientia navalis*, he found that if a force turns a ship around an axis of symmetry, the total spin equals the torque divided by the ship's moment of inertia around that axis (Euler 1749, sec. 458).

However, in the 1740s Euler's insight was limited—he knew how to apply it only to a special case, that is, rigid spin around a fixed axis. His real pursuit was rotation around a variable axis. During the subsequent decade, he showed twice how to obtain differential expressions relating componentwise the impressed torque and the angular acceleration of a rigid body in arbitrary motion (Euler 1752, sec. 48; 1765, sec. 24). Nowadays we call his threefold expression 'Euler's dynamical equations'. He also codified another insight: that mass opposes rotation unlike how it resists translation. He named the former "moment of inertia," as do we, in his wake. Euler systematized all these results, including his key equations, in *Theoria mo-*

tus corporum solidorum seu rigidorum, a tract published twice in Germany and easily accessible to Kant (Euler 1765/1790).

One might retort that this is irrelevant to Kant, who denies rigid matter. That would be hasty. Euler's results were indispensable for mechanical knowledge of actual objects. For instance, he used the Torque Law to model the vibrating string as a set of tiny rigid bodies connected by flexible lines, in "De motu corporum flexibilium" (Euler 1751). More importantly, he used rigid dynamics to handle torques on planets, in "Recherches sur le mouvement de rotation des corps celestes" (Euler 1766) and elsewhere. Lagrange too used 'Euler's equations' to explain the libration of the moon, which he modeled as a rigid ellipsoid (Lagrange 1763/1878).

Finally, in the 1770s, Euler's research began to make clear that Newton's Second Law by itself is not general and needs the Torque Law to supplement it. Together—and only together—these two principles yield equations of motion for a vast range of mechanical systems. Thus, in "Genuina principia doctrinae de statu aequilibrii et motu corporum tam perfecte flexibilium quam elasticorum," Euler showed that the external actions on an element in a plane elastic continuum come in two kinds: forces and "moments of forces" (i.e., torques; Euler 1771, 384–85). Hence, we need two laws to handle them. (In turn, he explained, the first kind produces two sorts of effects: normal and shear stresses.) The same insight underlies his mature work on vibrating systems (Euler 1776a). And, so it does in *Nova methodus motum corporum rigidorum determinandi*, his last word on rigid dynamics. There, he states the Torque Law as the second fundamental principle governing the action of forces on rigid bodies (Euler 1776b, sec. 29):[8]

$$\mathbf{H} = \mathbf{L}'. \qquad (1)$$

To conclude: by 1776 Kant was in a position to know that the Newtonian tradition of mechanics, to which he is committed philosophically, requires the Torque Law to be a fully general theory. There is, however, no clear trace of the law in his works, let alone a robust account.[9] Then a friendly interpreter may be allowed to try and derive the law from resources Kant already

8. Euler's other principle is his generalization of Newton's *Lex Secunda*, often called 'Euler's First Law' in analytic mechanics. He wrote the Torque Law in differential form. I give here the integral form, to make his insight lucid. 'H' is torque; 'L' is angular momentum. The prime symbol denotes time differentiation.

9. It is hard to know how much of this literature Kant had studied. Warda's catalog of Kant's personal library lists none of the titles above. But that is no proof that he had read none. They were all published in proceedings of the academies of Berlin, Paris, and St. Petersburg and thus not beyond Kant's reach. For instance, in his tract on *vis viva*, written as an impecunious youth in 1747, Kant cites volume 1 (1728) of the proceedings of the Imperial Academy of St. Petersburg. My point is that he could have read Euler's papers, not that he did so.

has, so as to broaden the scope of his natural philosophy as needed. Next I present one way to do just that and the historical context of its birth.

Before I move on, I must remove an obstacle. In both proofs of the Torque Law, Newton's Second Law is the indispensable premise. Kant's official laws of mechanics are not identical to Newton's, but they overlap greatly with the first and third laws in the *Principia*. However, the Second Law is visibly absent from Kant's *Foundations*. Then what chance does he have to even embark on the proofs I propose?

That is no real impediment, I submit. Kant does have the premises for a full equivalent of *Lex Secunda*, namely, the Parallelogram of Forces. Newton had proved, in Corollary II to his laws of motion, that these two principles are dynamically equivalent. Kant proves first the Parallelogram of Velocities in his Phoronomy (4:487). Then, in his Dynamics, he argues that forces act in the direction of the accelerations, or velocity increments, they produce. That entails that forces add like vectors, that is, according to the parallelogram rule. It is Kant's declared aim to make the Parallelogram of Forces obtainable a priori, by proving first "the principles of the composition [of motions] in general" (i.e., the Parallelogram of Velocities; 4:498; cf. also Friedman 2013, 373–75).

Newton's Second Law is not an official tenet in Kant's doctrine, for two reasons. (i) In the Enlightenment, a sizable group aiming to secure apodictic foundations for mechanics—Varignon, Daniel Bernoulli, d'Alembert, de Foncenex, Kant—replaced the Second Law with the Parallelogram of Forces. They thought the latter was confirmable by a priori methods, thus shown to be necessary, whereas *Lex Secunda* seemed undeniably contingent. (ii) Among these figures, two factions emerged. Varignon, Bernoulli, and Lagrange's student Daviet de Foncenex tried to deduce directly the Parallelogram of Forces, from allegedly indubitable axioms, for example, the Principle of Sufficient Reason and the Law of the Lever.[10] D'Alembert and Kant sought to prove—by the same approach, that is, the method of moving frames—that the Parallelogram of Velocities was apodictic. That was enough for d'Alembert, a neo-Cartesian who grounded his entire mechanics in "motion," not force. Kant, who was a realist about force, added a second premise—the colinearity of force and its resultant acceleration—to infer that forces add like vectors. This explains the otherwise baffling absence of the Second Law from Kant's *Foundations*.[11]

Note that thereby the Second Law—as equivalent to the Parallelogram of Forces—counts as synthetic a priori for Kant. So, he does have a key element to show that the Torque Law is likewise a priori.

10. I am not here concerned with the cogency of these proofs. In their time, they were found compelling. By the way, this story has a fascinating sequel in the nineteenth century; see Lange (2011). For some eighteenth-century authors, cf. Radelet de Grave (1992).

11. This point deserves its own paper, in due time.

4. Generality for Kant's Mechanics: The First Way. Kant has a choice between two strategies to secure the Torque Law. They rest on the same idea, that is, proving the law as a theorem from other more basic dynamical principles. The two strategies share a premise, namely, the Force Law **F** = m**a**. But they diverge as to the other key premise they employ. That is because these strategies rely on different theories of matter. Let us examine the first (I follow here Joos 1934).

The given is a system of interacting particles, subject in addition to a net external force. Apply the Force Law above to each component particle. The equation of motion for the *j*th particle becomes

$$^{\mathbf{E}}\mathbf{F}_j + \sum_k \mathbf{F}_{jk} = m_j \mathbf{a}_j. \tag{2}$$

The left side is the total force on the particle. It comes from two sources: the external force $^{\mathbf{E}}\mathbf{F}$ originating outside the system, and the sum of all the forces individually exerted inside the system by every *k*th particle on *j*. Now recall that a torque is really the moment of a force around a point (i.e., the center of moment). Its measure is the cross-product of the force vector and its 'arm', or distance vector **r**, to that center. Then cross-multiply by **r** each member in the equality (2) above:

$$\sum_j \mathbf{r}_j \times {}^{\mathbf{E}}\mathbf{F}_j + \sum_j \sum_k \mathbf{r} \times \mathbf{F}_{jk} = \sum_j \mathbf{r}_j \times m_j \mathbf{a}_j. \tag{3}$$

In (3) above, the right-side term is the time rate of change in angular momentum, and the first term on the left side is the external torque. For these two quantities, we use conventionally the labels 'L' and 'H'. Then we may rewrite (3) as the claim that the external and the internal torques on a particle jointly equal its rate of change in angular momentum:

$$\mathbf{H} + \sum_j \sum_k \mathbf{r} \times \mathbf{F}_{jk} = \mathbf{L}'. \tag{4}$$

We are now at a crucial step in the proof. Let us invoke a fact about the dynamics of particles. The forces between them obey the strong version of Newton's Third Law: they are pairwise equilibrated and central. That is, any two interparticle forces are equal, opposite, and directed along the straight line between them.[12] In precise terms, the Strong Third Law says

$$\mathbf{F}_{jk} = -\mathbf{F}_{kj}. \tag{5}$$

12. The centrality condition distinguishes the 'strong' Third Law from its 'weak' counterpart, on which the forces are just equal and opposite. The Weak Third Law holds, e.g., between any two contiguous volume elements in a physical continuum.

That is, any two interparticle forces are equal and opposite, and

$$(\mathbf{r}_i - \mathbf{r}_k) \times \mathbf{F}_{jk} = 0. \tag{6}$$

That is, the forces are central.[13] Together, expressions (5) and (6), namely, the Strong Third Law, entail that

$$\sum_j \sum_k \mathbf{r} \times \mathbf{F}_{jk} = 0. \tag{7}$$

Which is to say, the internal torques mutually induced in each other by the particles within the system vanish. Finally, the two results under (4) and (7) jointly entail that

$$\mathbf{H} = \mathbf{L}'. \tag{8}$$

That is the Torque Law for a system of particles. It says that the external torque equals the system's rate of change in angular momentum.

The first explicit proof along these lines comes from Poisson. Near the end of the second edition of his international best seller *Traité de mécanique*, Poisson had declared that, for a system of free mass points, "the forces arising from their mutual actions not only do not enter into the equations of their translation—they also vanish from the equations of their rotation around the origin of coordinates" (1833, sec. 554). Poisson had written the latter in component form. Specifically, he wrote the Torque Law for any two particles *j* and *k* that are internal to the body or system and invoked the Strong Third Law to prove that their mutual torques vanish: "Hence, the terms arising from the mutual action of the points in the system destroy each other pairwise" (sec. 554). In other words, the self-torque on a body or system is zero. Thereby, he concluded that only external torques change its angular momentum. But his result is indemonstrable without the Strong Third Law. This proof differs radically from the one I present next.

5. Another Way: The Boltzmann-Hamel Axiom. Kant's program has another path, better suited to his theory of matter in the Critical period. Like the first, it starts with the Force Law $\mathbf{F} = m\mathbf{a}$ and turns the Torque Law into a theorem, by an additional premise—which, however, is quite unlike the Strong Third Law above. The reason is that the second strategy starts with matter as physical continuum, not discrete particles. In this picture, force takes on new guises.

The external forces acting on continuous bodies come in two kinds. One is *body* force, acting directly on every volume inside the body, such as gravity

13. To see that, consider that $\mathbf{r}_i - \mathbf{r}_k$ is really the vector between particles *j* and *k*. To say that an interparticle force is central is to say that its cross-product with the interparticle, relative-distance vector is zero; i.e., they are collinear.

and electromagnetism. The other is *contact* force, or traction, which deforms the bounding surface of the body.[14] Kant too has this duality: in his language, forces are either "penetrating" or "surface" actions (4:516). Now, to write equations of motion, and so predict how the body will respond to contact actions, we need to know how surface tractions will affect its interior. To describe the state of contact-force transmission at a point inside the body, continuum mechanics uses the concept of *stress*. The stress measures how that point is driven to move, by the total traction on the body's surface. Cauchy discovered that the stress at a point is always the sum of traction forces on three mutually normal planes intersecting at that point and that the sum is independent of the planes' orientation. In turn, the stress vector on each plane may be resolved into three components: one normal to the plane and two tangential, also called *shear* stresses, each tending to turn its plane toward one of the other two planes (Hjelmstad 2005, 103–18).

The stress being equivalent to three traction vectors, each with its own three components, entails that, to represent the stress at a point mathematically, we need a second-rank tensor—really, a 3 × 3 matrix. In the stress tensor, usually labeled 'T', each row stands for the stress vector on one of the three arbitrary, mutually normal planes at that point. In turn, each row has three entries, one normal and two shear stresses:[15]

$$\begin{array}{llll} \mathbf{T_x}: & \mathbf{X_x} & \mathbf{Y_x} & \mathbf{Z_x} \\ \mathbf{T_y}: & \mathbf{X_y} & \mathbf{Y_y} & \mathbf{Z_y}\,. \\ \mathbf{T_z}: & \mathbf{X_z} & \mathbf{Y_z} & \mathbf{Z_z} \end{array} \tag{9}$$

Let $\mathbf{T}_{ij}$ be the *j*th entry in the *i*th row of a tensor. Then the tensor is symmetric if $\mathbf{T}_{ij} = \mathbf{T}_{ji}$. If $\mathbf{T}_{ij} = -\mathbf{T}_{ji}$, the tensor is skew. Finally but crucially, *conjugate shear stresses* are those tending to turn any two normal planes toward each other (Malvern 1969, 77–78).[16]

Georg Hamel first grounded rigorously the Torque Law for continua—as a theorem, in 1909. His dynamical premises were the Force Law and the axiom that the stress tensor is symmetric. I reconstruct below a proof, from Hamel's terse sketch (Hamel 1909, 360ff.).

14. Strictly speaking, both are force densities over continuous volumes or surfaces. The force at a point is the limit of such densities, as the volume or surface, respectively, shrinks to that point. We must assume, of course, that such limits exist at every point and that their upper bound is always finite.

15. For instance, $\mathbf{X_x}$ is the component that presses against or pulls normal to the plane YZ, $\mathbf{Y_x}$ is a shear stress that turns the plane toward the *y*-axis, and the shear stress $\mathbf{Z_x}$ turns it toward the *z*-axis.

16. In the above, $\mathbf{X_y}$ and $\mathbf{Y_x}$ are conjugate shear stresses; so are $\mathbf{X_z}$ and $\mathbf{Z_x}$ and $\mathbf{Y_z}$ and $\mathbf{Z_y}$.

Start with the key premise, the Force Law for a continuous medium, or Cauchy's First Law of motion. It says that the rate of change in linear momentum equals the contact forces plus the body forces:[17]

$$\mathrm{d}/\mathrm{d}t \int_V \rho \mathbf{v} \, \mathrm{d}V = \int_S \mathbf{Tn} \, \mathrm{d}S + \int_v \rho \mathbf{b} \, \mathrm{d}V. \tag{10}$$

With the distinctions above as a backdrop, the Torque Law for continua says that a body's rate of change in angular momentum equals the moment of the contact forces plus the moment of the body forces, both external, acting on it. (Recall that the first kind acts on surfaces, the second on volumes.) More precisely, cross-multiplying each term in the Force Law above by its distance **r** to the center of moment, we get[18]

$$\mathrm{d}/\mathrm{d}t \int_V (\mathbf{r} \times \rho \mathbf{v}) \, \mathrm{d}V = \int_S (\mathbf{r} \times \mathbf{Tn}) \, \mathrm{d}S + \int_V (\mathbf{r} \times \rho \mathbf{b}) \, \mathrm{d}V. \tag{11}$$

The rate of change above provably equals the moment of the acceleration of mass, for any volume element (Gurtin 1981, 92):

$$\mathrm{d}/\mathrm{d}t \int_V (\mathbf{r} \times \rho \mathbf{v}) \, \mathrm{d}V = \int_V (\mathbf{r} \times \rho \mathbf{a}) \, \mathrm{d}V. \tag{12}$$

Apply the divergence theorem, once called Gauss's Principle, and transform the surface integral of the tractions into a volume integral:

$$\int_S (\mathbf{r} \times \mathbf{Tn}) \, \mathrm{d}S = \int_V \nabla(\mathbf{r} \times \mathbf{T}) \, \mathrm{d}V. \tag{13}$$

It can be proven (Bowen 2010, secs. 3.3.12–13) that the integrand on the right side can be rewritten as

$$\nabla(\mathbf{r} \times \mathbf{T}) = (\mathbf{r} \times \nabla\mathbf{T}) + (\mathbf{e}_j \times \mathbf{T}_j), \tag{14}$$

where j is a dummy index, $\mathbf{e}_j$ are basis vectors, and $\mathbf{e}_j \times \mathbf{T}_j$ is the Einstein summation convention. Substitute all these equivalences into (11) above, and the Torque Law becomes

17. Cauchy's First Law is Newton's Second Law $\mathbf{F} = \mathrm{m}\mathbf{a}$ for continuous bodies. A limpid account of this principle, for philosophers, is Smith (2007), sec. 4.

18. Here, ρ is the mass density, **b** the body force, **T** the Cauchy stress tensor, and **r** is the distance vector $(\mathbf{x} - \mathbf{x}_O)$ from a point inside the volume element to O, the center of moment. The vector **a** is the acceleration, and **n** is the unit vector normal to a surface element dS pointing outward.

$$\int_V (\mathbf{r} \times \rho\mathbf{a})\, \mathrm{d}V = \int_V (\mathbf{r} \times \nabla\mathbf{T})\, \mathrm{d}V + \int_V (\mathbf{r} \times \rho\mathbf{b})\, \mathrm{d}V + \int_V (\mathbf{e}_j \times \mathbf{T}_j)\, \mathrm{d}V. \tag{15}$$

Because all the functions in the integrand are assumed to be continuous and the integral holds for any arbitrary volume V inside the body, we can rewrite it in vector form:

$$\mathbf{r} \times \rho\mathbf{a} = \mathbf{r} \times \nabla\mathbf{T} + \mathbf{r} \times \rho\mathbf{b} + \mathbf{e}_j \times \mathbf{T}_j. \tag{16}$$

We can prove (Truesdell 1991, 363–64) that

$$\mathbf{e}_j \times \mathbf{T}_j = \mathbf{T} - \mathbf{T}^{\mathrm{T}} = 2\mathbf{D}. \tag{17}$$

Substituting (17) in expression (16) above gives

$$\mathbf{r} \times \rho\mathbf{a} = \mathbf{r} \times \nabla\mathbf{T} + \mathbf{r} \times \rho\mathbf{b} + 2\mathbf{D}. \tag{18}$$

Note some facts: **D** above is a vector. It behaves like a torque: when included in (18), the body still obeys the Second Law (Hamel 1909, 366). Critically, **D** results from the action of contiguous volume elements *within* the body. Thus, **D** counts as an internal torque, generated by stresses in the interior of the body. Hence, the Torque Law—namely, that the external torque equals the rate of change in angular momentum—is true if and only if

$$\mathbf{D} = 0. \tag{19}$$

But that just means that the skew part of the stress tensor **T** is nil; hence, **T** itself is symmetric. To conclude: the Torque Law holds for continuous bodies just in case the stress tensor is symmetric. And that entails trivially the equality of conjugate shear stresses.

Hamel made the symmetry of the stress tensor an axiom in his formalization and credited Boltzmann with the insight.[19] The latter in 1905 had explained that, to ground continuum mechanics, we need new assumptions that go beyond what particle dynamics requires. Boltzmann singled out an assumption he called "Principle X": "If we now apply the principle of static moments to a volume element, we will find that, in the case of equilibrium, the force acting in the y-direction on a surface element normal to the x-axis must be equal to the force acting in the x-direction on an equal surface element normal to the y-axis" (1905, 298). In retrospect, we recognize Prin-

19. It is his Axiom VII. His Axiom VI is Cauchy's First Law. Axioms I–V introduce inertial-kinematic structure.

ciple X as the assumption that the conjugate shear stresses at a point are equal. Hamel then baptized it "Boltzmann's Axiom" and generalized it as his Axiom VII: "For any volume element, the moment of all external forces, divided by ΔV, relative to the point X inside the volume element toward which ΔV converges, vanishes in the limit" (Hamel 1909, 358).[20] This too entails the symmetry of the stress tensor.

6. Kant at an Impasse. The strong presumption—it may be resisted, although I explain below that it is costly to do so—is that Kant has to choose Hamel's strategy above. If one starts with continuous matter, as he does, to obtain the Torque Law a priori one must ground the symmetry of the stress tensor. Clearly, neither Kant nor anyone before Cauchy in the 1820s had a concept of a stress tensor (Truesdell 1968), let alone Hamel's insight that it must be symmetric. But Enlightenment figures, and so Kant too, could have had an equivalent idea and tried to ground it philosophically. Such an idea is Boltzmann's principle that conjugate shear forces must be equal.[21] Another equivalent, inchoate but correct, was well within Kant's reach: the Law of Inertia for rotation, that is, the principle that, in the absence of external unbalanced forces, a continuous body does not change its own angular momentum.[22]

I say that rotational inertia was within Kant's reach because he does have a proof for inertia of *translation*. For him, the latter was a synthetic a priori law, which he sought to derive from the Second Analogy. Then if we show he could have proved a rotational analogue of that law, we would defuse the challenge that the Torque Law poses to his system.

Unfortunately, the prospect is bleak for the interpreter. Any attempt to prove Inertia of Rotation for Kant seems doomed, for he denies himself the very possibility of doing so. For that result, he ought to establish that the internal contact forces—between any two parts created by an arbitrary Euler cut—produce no net self-torque. That is a direct consequence of Boltzmann's Principle X: conjugate shear forces are equal everywhere. However, this avenue is closed to Kant in principle, for he denies that there

20. More precisely, if **z** is the distance vector to a point X inside a volume ΔV, and **F** is the total external force on the volume element, $\lim_{\Delta V \to 0} 1/\Delta V \int (\mathbf{z} \times d\mathbf{F}) = 0$. Regrettably, Hamel is somewhat confusing here. Context alone makes it clear that, by "external," he means external to the volume yet internal to the body. His point is really that the internal forces in a continuous body generate no net self-torque.

21. The Torque Law and the Symmetry of the Stress Tensor are equivalent: either follows from the other, with the Force Law as an additional premise. In turn, the Symmetry of the Stress Tensor and the Equality of Conjugate Shear Stresses entail each other.

22. Remember what the (integral) Torque Law really says: that the total change in angular momentum is equal to the net external torque; hence, the self-torque on a body is always zero.

can be any shear forces at all. Kant in the Critical years declares that the only conceivable forces are central, along the line between two "matters," as forces of mutual approach or receding: "only these two moving forces [of matter] can be thought. For all motion that one matter can impress on another, since in this regard each of them is considered only as a point, must always be viewed as imparted in the straight line between the two points" (4:498). But that rules out all shear forces inexorably. Recall that only the normal components of the mutual tractions by two volume elements are normal to their common deformed surface and thus—loosely and improperly speaking—along the line "between" elements. Shear forces are tangential to that surface, thus lateral between two elements, and so by Kant's lights are inconceivable and hence nonexistent.[23] Then we cannot appeal to them so as to ground for him inertia of rotation, Principle X, or any equivalent premise that yields the Torque Law as a theorem. Kant will not allow it. His drastic conceivability claim above makes it very hard for him to have continuous matter and fully general laws of mechanics.

Note that Kant's centrality claim above is the same thought that lets him infer that force is collinear with its acceleration, and thereby prove the Parallelogram of Forces, his alternative to *Lex Secunda*. Thus, to ground Newton's Second Law, Kant commits to a view of force—as central interaction—that leaves him eventually unable to ground contact stresses in a true continuum, which has noncentral, shear components.[24] In light of these difficulties, I argue below that he would be better off changing his matter theory—to something he used to have.

Objection: why not just conclude that Kant's counterfactual attempt to prove the Torque Law fails and end my inquiry here? I have two responses. First, I take it to be sound hermeneutics that we must not give up on a major thinker unless we have exhausted all reasonable, well-grounded attempts to make her look coherent. But there is a good way to salvage Kant's important generality claim. Second, I think a negative verdict on Kant is less interesting than the alternative I have to offer. His presumptive inability to pursue the proof strategy most natural to his foundations is symptomatic of larger problems in them. More exactly, his theory of matter is too strong: it requires his kinematics, dynamical laws, and concept of mechanical objectivity to bear a load they cannot support. Thus, I will argue, deemphasizing one aspect of Kant's theory of matter—to wit, the continuity of mass distribution—will yield vast benefits for his overall natural philosophy.

23. Mark Wilson puts it delicately: Kant "cannot find a way to incorporate [some source of shear] into his framework" (2013, 89 n. 39).

24. I thank Michael Friedman for drawing my attention to this point.

7. Must Kant Follow Hamel? If Kant were to pursue Hamel's proof strategy, he evidently has conceptual problems—but must he do so? Perhaps I was too hasty to conclude that he must. There are two reasons to doubt my inference; I mark them A and B and answer them below.

> **A.** If pressed, Kant could reply that, as all fundamental forces are central, all shear forces are zero everywhere; hence, conjugate shears are trivially equal. Ergo, the Torque Law follows as a matter of course.

This answer is unsatisfactory, for two reasons. Systematic: the proposal would make the foundation coherent and general, but explanatorily inadequate, as it fails to ground a general fact about the empirical behavior of continuous matter, which does exert actual nonzero shear forces (except in inviscid fluids). Historical: Kant's foundation thereby would be inadequate even for the mechanics of his time. By then, Euler had already computed the shear force (*vis tangentialis*) at a point on a continuous elastic line and a lamina (Euler 1771, 1776a). His contemporaneous result would contradict directly Kant's claim that all shear forces vanish.

> **B.** Take visible bodies to be lattices of 'squishy atoms', or microscopic but continuous and deformable particles, separated by empty space but kept in equilibrium configurations by action-at-a-distance forces. This retains Kant's essential tenets—that matter is continuous, compressible, and 'dynamical', or endowed with forces—but restricts them to the fundamental level.[25] Construing macroscopic bodies that way would let Kant take Poisson's route to the Torque Law, not Hamel's. Then the symmetry of the stress tensor is no problem for Kant, because he needs not invoke it. That is presumably because the forces between 'squishy atoms'—being all gravity-like distance interactions—obey the Strong Third Law.

However, I do not think this interpretive strategy is very wise.

'Squishy atoms' are foreign to the spirit of Kant's mature doctrine. Unlike modern approaches, for which continuity is scale dependent, Kant takes continuous mass distribution to hold at all scales, not just microscopic. Witness his strenuous efforts, in *Foundations*, to argue against empty space. Further, it would be hard to motivate, on Kant's behalf, restricting continuity to 'atoms', given that visible bodies *appear* continuous. (That phenomenological fact is likewise the chief motivation for contemporary mechanics to model matter as a continuum.)

25. More advantageously, this view is latent in Kant's *Physical Monadology* (1756/1992) and thus not too alien to his doctrine; see Smith (2013).

Kant is deeply interested in collision mechanics, although scholars have so far neglected it, unduly. His program is to show that the mechanism of velocity exchanges in impact is metaphysically intelligible, in the sense of not amounting to the actual transfer of properties like ‘motion’, ‘moving force’, or momentum between corporeal substances.[26] Ultimately, he will have to describe what happens on the boundary when two ‘squishy atoms’ collide and how that results in changes to the velocity of their mass centers—or else his self-imposed obligation remains unfulfilled. Now he has two basic choices. Either he will admit that there is genuine kinematic contact between ‘atoms’. Then talking about ‘atomic’ torques and shear may be inevitable, and lending him the Poisson strategy merely delays the inexorable. (In modern terms, if Kant posits his ‘atoms’ as true continua, he must endow them with microstrains.) Or he will claim that impact is really scattering caused by short-range, action-at-a-distance repulsive force. Then all collision—a fortiori all “communication of motion”—ends up mediated by body forces, not any genuine contact force.[27] In that case, the continuity of matter becomes explanatorily idle for his mechanics—all interactions are really between entities modeled as point masses—and also unmoored from his kinematics (see below). Then why not do away with it, as I suggest below, and thereby make Kant’s natural philosophy wholly coherent?

Kant’s kinematics likewise militates against ‘squishy atoms’. His Phoronomy allegedly spells out the a priori laws of “matter as the movable in space.” However, all we get from him under that heading is the Parallelogram of (Instantaneous Linear) Velocities. Combined with ‘squishy atoms’, this generates tension. (i) Either, by ‘movability’, Kant means the possible motions of an atom’s mass center. Then in effect he treats the ‘atom’ as a mass point, and so basic ‘atomic’ extension, deformability, and continuous mass density have no kinematic or dynamical import. Ockham’s Razor demands that we might as well excise them and give Kant the mass points I advocate for below. (ii) Or, by ‘atomic’ movability, he really means the full range of its possible kinematic behaviors. Then, for ‘squishy atoms’, ‘motion’ means

26. Kant excoriates some predecessors who explained impact as literal transfer of momentum. That is metaphysically absurd, he objects correctly, since properties do not migrate across substances. So, the natural philosopher must “explain the communication of motion itself with regard to its *possibility*” (4:550). All he offers, however, is that the possibility of momentum transfer rests on the repulsive force essential to matter. That is a contact force, in *Foundations*, and so our task as interpreters is to reconstruct a plausibly Kantian account of collision that escapes his own censure above: i.e., we must offer a description of the mechanism whereby two bodies suffer changes in momentum as a result of action by contact.

27. Anyway, that would be against his explicit assertion that repulsion is a genuine contact force.

a translation, rotation, or deformation. But Kant's kinematic foundation is too weak for that: his Parallelogram of Velocities holds for 'atomic' translations alone.[28] Ergo, Kant has failed to give us all the laws of "matter as the movable in space"—an undesirable outcome, completely avoidable if we have him go back to mass points. In conclusion, if Kant must insist that matter is continuous, he ought to follow Hamel in the attempt to derive the Torque Law.

8. Reclaiming Mass Points for Kant. It turns out that Kant at some point had the conceptual resources to replicate Poisson's crucial result. That put him in a position to obtain the Torque Law, and so to ensure that his philosophical basis is wide enough for a truly general theory of classical mechanics, as he requires and is desirable. Next, I present these resources and explain how they entail the Strong Third Law.

For decades in his youth, Kant had embraced a theory of matter he called "physical monadology." In his official view, the unit of matter, or physical monad, was an entity endowed with two forces—of repulsion and attraction. Both were actions at a distance, "exerted from a given point," and thus were central forces (1992, 1:483). Kant made repulsive force act radially outward "from the *central point* of the space occupied" by a monad and speculated that the force would be an inverse-cube function of the "distance from the *center* of its presence." As a result, this repulsion will be "infinite at the central *point* itself" (484, 487; my italics). Finally, Kant's physical monad also had a "force of inertia . . . called its mass" (485). Crucially, the mass of a Kant monad is located at a point, not distributed over a volume. Thus, in some essential respects his monads are just like our modern mass points.

Unfortunately, the young Kant did not hold on steadily to this picture. Rather, he also bestowed on his monads a property that puts them closer to deformable continua. Kant posited repulsive force so as to secure impenetrability. But he conceived of that force—and thus monadic impenetrability—in two very different ways. Early in his *Physical Monadology* (1756/1992), he thinks of repulsion as a distance force. This makes a monad impenetrable in the sense that no two monads can be superimposed: the mechanical work required would be infinite. By the end of the essay, though, Kant came to speak of repulsion as a contact force. That changes the meaning of impenetrability: now no monad can be compressed to zero volume. Crucially, in this conception monads are deformable and can act by kine-

28. What is missing: a composition theorem for (infinitesimal) rotations, i.e., the Parallelogram of Angular Velocities, and basic kinematics for strain, e.g., the Polar Decomposition Theorem.

matic contact.[29] In that respect, they resemble elastic volume elements, not mass points.

Officially, however, Kant had monads be endowed with point-sized mass and purely action-at-a-distance forces. Admittedly, he did not adhere consistently to this view. Had he done so, his monads would have had crucial commonalities with modern mass points, and so would have allowed him to take the route Poisson opened to the Torque Law.

But do Kant's physical monads, in his *official* view, obey the Strong Third Law, as they must if Poisson's proof is to hold for them? Yes. Purely Kantian premises yield a twofold case that they do.

A. One is an argument for the Weak Third Law: intermonadic forces of the same kind are pairwise equilibrated. Kant sketched it twice, in the 1780s, and it presupposes Transcendental Idealism—specifically, the claim that Newton's absolute space, metaphysically distinct from body, is "no object of experience." That is, we cannot possibly know absolute space, whether directly by acquaintance or indirectly, from its inertial effects on corporeal motions (4:481). For that reason, we cannot possibly know accelerations relative to absolute space: they are (epistemically) impossible, Kant claims.

With this premise in place, Kant's case for the Weak Third Law is a proof by reductio. In a nutshell, it goes as follows. Assume that interbody forces are not equal and opposite. Then interactions will result in a net increase of linear momentum for the entire physical universe. That would displace the mass center of the "world whole" in absolute space. But that would be a "motion" empirically inaccessible to us and so "simply impossible" for us as knowers. Then its entailing premise—that interactions are not equal and opposite—is apodictically false. Ergo, the Weak Third Law must be true. That is Kant's meaning, I submit, behind these declarations:

> Absolute motion would be simply the motion a body had without relation to any other matter. Of this kind would be just the rectilinear motion of the *whole world*, or of the system of all matter. . . . Because of that fact, every proof for a law of motion which argues that the law's opposite would entail the rectilinear motion of the entire world-edifice is an apodictic proof of that law. That is simply because, from its opposite, absolute motion would follow, which is simply impossible. Such is the law of *antagonism* [i.e., of action and reaction] in all community of matter through motion. (4:562–63; cf. also Kant 1925, 14:166)

If this is too vague, there is a rigorous way to spell out Kant's idea. Some decades ago, Walter Noll gave a strict deductive proof for the Weak Third Law,

29. Smith (2013) argues for these findings conclusively. I thank Michael Friedman for pressing me on this point.

in one of his axiomatizations of mechanics (Noll 1973).[30] His key relevant premise is that interactions between bodies are balanced, in the sense that, for every body A, the interaction between itself and its exterior A^e is null: $\mathrm{I}(A, A^e) = 0$. Charitably read, Kant's ruling out cosmic self-acceleration in absolute space is philosophical support avant la lettre for a special version of Noll's key premise: namely, all interactions by forces are balanced, because the "universal body," or Kant's *Weltganze*, is in static equilibrium of force.[31] With Noll's proof as a backdrop, Kant turns out to have had the right insight, although not the means to give it rigorous expression. Then an interpreter may reconstruct, just from Kantian resources, an argument that the Weak Third Law holds for physical monads.[32]

B. Kant also has a way to show that the basic forces between his monads are central. Although not explicit in his work, the idea is naturally at home in his physical monadology. It is this: the very mathematics of monadic forces entails that they are central. Kant thinks of those basic forces—repulsive and attractive—as subject to power laws, on the model of gravity.[33] As such, any monad induces irrotational scalar potentials around its point source. Then, for any particular location outside the monad, the 'Newtonian' force—really, the action and reaction in the Third Law—is given in strength and direction by the negative gradient of the potential at that location. In particular, the direction is always normal to the equipotential surface passing through that point. For Kant's monadic forces, those surfaces are all concentric spherical shells of a different radius. It follows that the accelerations induced by those forces, and so the action, always point radially to the center, where the monad sits—mutatis mutandis for the second monad, in an interaction. Ergo, monadic forces are central.

The Weak Third Law A plus the centrality condition B entail the Strong Third Law for physical monads. Then, if Kant can have it, and also an equivalent of Newton's Second Law, he can avail himself of the Torque Law.

30. But, unlike Kant's proof, Noll's is direct, and he did not set out to vindicate Kant.

31. Noll's "interaction" is a genus concept meant to cover four species pertinent to continuum mechanics: forces, torques, body couples, and distributed couple stresses. Kant's argument from absolute motion applies only to the first species. For instance, he admittedly cannot prove that torque interactions are balanced, as their resultant might well be nonzero; see 4:563.

32. Of course, Noll's proof has other substantive premises, i.e., his six axioms of body. But he admits that they "reflect our common experiences with physical bodies" and are so general as to hold for continuous bodies and discrete objects like Kant's physical monads (Noll 1973, 91).

33. Although that is clear, it is quite obscure why Kant thinks he is allowed to do that.

9. Conclusion. I have uncovered a potential conflict in Kant's mature philosophy of physics. For him, matter is a continuum, but his laws of motion work for particles, not true continua. (And so does his kinematics.) To ground mechanical phenomena in general, and angular momentum exchange in particular, Kant needs the Torque Law $\mathbf{H} = \mathrm{d}\mathbf{L}/\mathrm{d}t$. The best hope for Kant would be to obtain the law as a theorem, from premises he already has. I examined above two strategies for doing just that. But either strategy demands an additional premise beyond what Kant has to offer. The first requires the Strong Third Law. The second needs the Symmetry of the Stress Tensor. Kant banishes shear forces from his system, and so the second strategy comes at a prohibitive price for him. To pursue the first, Kant ought to bring back the physical monadology, his old theory of matter. Physical monads, or mass points, would be fully cohesive with his kinematics and dynamical laws. They alone can give Kant the generality and unity he seeks for his foundations of mechanics.

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