

Variable-sharing as relevance^{*}

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Abstract

A challenge for relevant logicians is to delimit their area of study. I propose and explore the definition of a relevant logic as a logic satisfying a variable-sharing property and closed under detachment and adjunction. This definition is, I argue, a good definition that captures many familiar logics and raises interesting new questions concerning relevant logics.

As is familiar to readers of *Entailment or Relevant Logics and Their Rivals*, the motivations for relevant logics have a strong intuitive pull. The philosophical picture put forward by Anderson and Belnap (1975), for example, is compelling and has led to many fruitful developments. With some practice, one can develop a feel for what sorts of axioms or rules lead to violations of relevance in standard relevant logics. These sorts of intuitions only go so far, as some principles that lead to violations of relevance in stronger logics are compatible with it in weaker logics. There is a large number of relevant logics, but there is not much discussion of precise characterizations of the class of relevant logics.¹

It is well known that the standard relevant logics avoid C. I. Lewis's paradoxes of implication, such as $A \rightarrow (B \rightarrow B)$ and $A \rightarrow (B \rightarrow A)$, but avoiding the paradoxes does not provide an adequate characterization for two reasons. First, it is a negative characterization that is also open-ended, insofar as the list of paradoxes

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¹Avron (2014) expresses this concern and provides a characterization, but it is less inclusive than the one adopted here.

of implication is open-ended. Second, it is not clearly based on a formal principle. Lewis's paradoxes of implication present some counter-intuitive features of the classical material implication, but it is not clear what formal principles should be extracted from the intuitions motivating their rejection.

The lack of a general, abstract characterization of the class of relevant logics is unfortunate, because investigations of relevant logics typically proceed either by focusing on a particular logic or by presenting a more or less arbitrary selection of axioms and rules combination of which determine the logics under consideration. Further, while the motivations mentioned above are often compelling, it is not clear that they succeed in isolating only the proponents' preferred logics.² Different logicians have defended different views about relevant logics and these views may not lead to the same groups of logics. There is, then, a need to reconsider foundational aspects of relevant logics, one aspect of which is a principled delimitation of the area of study.

In contrast to the situation with relevant logics, the (classical) modal logician has a ready answer to the analogous question "what is a modal logic?" The modal logician can say that a (normal) modal logic is any logic extending classical logic with the (K) axiom, $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$, and closed under the rule (Nec), $A \Rightarrow \Box A$, to be understood as saying that if A is a logical truth, then so is $\Box A$.³ Similarly, the paraconsistent logician can delimit their area of study as those consequence relations where $A, \sim A \not\vdash B$, for some formulas A and B and a negation operator ' \sim '. These abstract characterizations allow the modal or paraconsistent logician to take a broader view of their areas, shifting the focus from the more or less standard logics, such as K or S4, to more general classes of logics. So, it seems like an abstract characterization of relevant logics would be useful.

The question is how to formulate an abstract characterization of relevant logics. There are prominent groups of relevant logics, such as the preferred logics of Anderson and Belnap, but those are not the only relevant logics. One could try to give a model-based characterization, such as using the ternary relational models of Sylvan and Meyer, but the models for relevant logics can also be used to adequately model logics containing $A \rightarrow (B \rightarrow B)$ and $A \rightarrow (B \rightarrow A)$. Additionally, it would be good to have a model-independent characterization, and

²As an example, see Meyer (1985) or Øgaard (2021b, 2020, 2023), for example.

³This may not be adequate as a definition once other intensional operators are permitted, such as the actuality operator. If one has contingent logical truths with actuality, as defended by Nelson and Zalta (2012), the rule of Necessitation can fail in what is intuitively a modal logic. I thank Ben Blumson for pointing this out.

this points us to the basis of the proposal.

A key feature of relevant logics, one of the few that is generally agreed upon, is that they satisfy Belnap's *variable-sharing criterion*:

Definition 1 (Variable-sharing criterion). *A logic L satisfies the variable-sharing criterion iff whenever $A \rightarrow B$ is a logical truth, then A and B share a propositional variable.*

It is often said that the variable-sharing criterion is a necessary but not sufficient condition on being a relevant logic. Indeed, as Anderson and Belnap say,

A formal condition for “common meaning content” becomes almost obvious once we note that commonality of meaning in propositional logic is carried by commonality of propositional variables. So we propose as a necessary, but by no means sufficient, condition for the relevance of A to B in the pure calculus of entailment, that A and B must share a variable.⁴

This view is fairly standard, as illustrated by Mares, who says,

The variable sharing principle is only a necessary condition that a logic must have to count as a relevance logic. It is not sufficient. Moreover, this principle does not give us a criterion that eliminates all of the paradoxes and fallacies. Some remain paradoxical or fallacious even though they satisfy variable sharing.⁵

Further examples are not hard to find. The variable-sharing criterion presents a *minimal* requirement of formal relevance for implicational logical truths. There are natural things one can add to it and ways to strengthen it, some of which I will return to in the final section of this paper, but the basic variable-sharing criterion provides a succinct specification of a minimal level of formal relevance. While it is commonplace to say that variable-sharing is not sufficient for being a relevant logic, it is worth considering how things look if we take it to be sufficient, with another minimal condition, seeing what follows from that and what requires more. The proposal, then, is to say that relevant logics are those that satisfy the variable-sharing criterion, along with two other conditions. I will explore this proposal, prove some initial results concerning this class of logics, and then discuss how more familiar families of logics fit into this class. In section 1, I will present the relevant background and the proposal for a definition of the class of

⁴Anderson and Belnap (1975, 33).

⁵Mares (2022).

relevant logics. In section 2, I explore the proposal, identify some consequences of the definition, and identify further conditions one might use to identify subclasses of logics. In section 3, I relate the proposal to some extant views about relevant logics and respond to some potential criticisms, closing with a brief discussion of other forms of variable-sharing that one can consider.

1 The proposal

Let the language under consideration be made from countably many propositional variables, p, q, r, \dots , and the connectives $\{\wedge, \vee, \sim, \rightarrow\}$, with $A \leftrightarrow B$ defined, as usual, as $(A \rightarrow B) \wedge (B \rightarrow A)$.⁶ To facilitate comparisons, we will take classical logic to be formulated in this vocabulary and to include $(A \rightarrow B) \leftrightarrow (\sim A \vee B)$ as a theorem. The language can be pared back, but the key thing is that there is an implication connective, \rightarrow , in the language. The other connectives are optional. We will, for this paper, focus just on logical truths, as relevant logics are most often studied under that guise. Indeed, the logical truths seem to be the key to distinguishing relevant logics.⁷ We will assume some familiarity with relevant logics, but only the better known ones, such as R, T, and B.⁸

We will treat logics as sets of formulas closed under uniform substitution.⁹ A substitution is a function σ from propositional variable to formulas that is extended to the a function σ^+ on the whole language such that

- $\sigma^+(p) = \sigma(p)$, for p a propositional variable,
- $\sigma^+(\sim A) = \sim\sigma^+(A)$, and
- $\sigma^+(A \star B) = \sigma^+(A) \star \sigma^+(B)$, for $\star \in \{\wedge, \vee, \rightarrow\}$.

⁶A comment on the language is in order. The Ackermann and Church truth constants, t and \top , are excluded from the language. The primary reason is that for formulas that contain them, the variable-sharing criterion does not apply. See Yang (2013) or Øgaard (2021a) for discussion of variable-sharing in the presence of truth constants. The formulation of a general definition of relevant logics that includes the truth constants, or at least the Ackermann constant, is left as an open problem.

We have omitted fusion from the language, but we can add it, making some minor changes below.

⁷Avron (1992, 2014) looks at consequence relations and contains an alternative, less inclusive, characterization of relevant logics.

⁸For axiomatic presentations of these logics, see Brady (1984b). For a general overview of relevant logics, see Read (1988), Dunn and Restall (2002), Bimbó (2007), or Mares (2022).

⁹This is the framework FMLA of Humberstone (2011, ch. 1.2).

With that definition in hand, we can define logics.

Definition 2 (Logics). *A set of formulas X is a logic iff for any substitution σ and formula A , if $A \in X$, then $\sigma^+(A) \in X$*

A logic L is a sublogic of another logic M iff $L \subseteq M$.

Throughout the paper, I will use formula schemes to present various principles, where the instances of the scheme are obtained by replacing the displayed letters with arbitrary formulas. When presenting an instance of a scheme, I will use a specific formula displaying the propositional variables. A logic *includes* a formula scheme iff it includes every instance of that scheme. A logic *excludes* a scheme iff there is an instance of the scheme it does not contain.

Before I get to the main definition, I will present two preliminary definitions of classes logics.

Definition 3 (Proto-relevant logics). *A logic L is a proto-relevant logic iff L satisfies the variable-sharing criterion.*

The class of proto-relevant logics captures an important feature of relevant logics, but below we will argue that a little bit more should be added. For the next definition, it will be useful to introduce the notation for rules: A logic L is closed under a rule, $X \Rightarrow A$, iff for all substitutions σ , if $\sigma^+(B) \in L$, for all $B \in X$, then $\sigma^+(A) \in L$.

Definition 4 (DA-logics). *A logic L is a DA-logic iff both*

- (1) *it is closed under detachment, also known as modus ponens, $A, A \rightarrow B \Rightarrow B$, and*
- (2) *it is closed under adjunction, $A, B \Rightarrow A \wedge B$.*

If one wants to consider only the \rightarrow -fragment of logics, then condition (2) can be dropped. DA-logics include many familiar logics, such as classical logic, intuitionistic logic, and R. As should be clear from the examples, some DA-logics are not proto-relevant logics.

The two classes of logics, proto-relevant logics and DA-logics, are brought together in our main definition.

Definition 5 (Relevant logics). *A logic L is a relevant logic iff L is a proto-relevant logic and is a DA-logic.*

I claim that this definition provides a good delimitation of the area of relevant logics, and I will explore and defend this definition in the remaining sections of this paper. I will begin by commenting on the parts of the definition 5.

First, being a proto-relevant logic immediately guarantees that every relevant logic will satisfy Belnap’s variable-sharing criterion. It immediately rules out logics that include some of the implicational paradoxes, such as $A \rightarrow (B \rightarrow B)$. These are, on their face, incompatible with being a relevant logic.¹⁰

The proto-relevant logics include many logics that are non-transitive, in the sense that $A \rightarrow B$ and $B \rightarrow C$ are logical truths but $A \rightarrow C$ is not.¹¹ The implicational form of the steps of C. I. Lewis’s argument for explosion will do.

- $(A \wedge \sim A) \rightarrow (\sim A \wedge (A \vee B))$
- $(\sim A \wedge (A \vee B)) \rightarrow B$
- $(A \wedge \sim A) \rightarrow B$

The first two are cleared by the variable-sharing criterion but not the third. While non-transitive logics are not unheard of, they are also typically excluded from the family of relevant logics.¹² As Anderson and Belnap (1975, 154) say,

And what this shows is that connection of meaning, though necessary, is not a sufficient condition for entailment, since the latter relation is transitive. Any criterion according to which entailment is non-transitive, is *ipso facto* wrong. It seems in fact incredible that anyone should admit that B follows from A, and that C follows from B, but feel that some further argument was required to establish that A entails C. What better evidence for $A \rightarrow C$ could one want?

The proto-relevant logics include many logics that are not transitive, and so, arguably, are not logics of *entailment*. Even the logics taken to be central relevant logics include some that are not logics of entailment, such as R. So, there does not seem to be an essential conflict between being a relevant logic and being non-transitive. This is a point of agreement with the criticisms of Copeland (1980, 331),

¹⁰Even the well known relevant logics, such as R and B, contain some instances of the scheme $A \rightarrow (B \rightarrow B)$, e.g. $(p \rightarrow p) \rightarrow (p \rightarrow p)$. The important thing is that they exclude the scheme by omitting some instances, such as $p \rightarrow (q \rightarrow q)$. I would like to thank Lloyd Humberstone for highlighting this point.

¹¹Szmuc (2021).

¹²See Smiley (1959), Ripley (2013), or Weir (2013), for examples of non-transitive logics.

which doubts that non-transitive logics should be excluded from consideration as relevant logics.

Requiring closure under detachment seems like a reasonable condition for a connective being an *implication* connective. Together with satisfying variable-sharing, it excludes many of the remaining paradoxes of implication, in particular $A \rightarrow (B \rightarrow A)$.

Theorem 6. *No relevant logic contains $A \rightarrow (B \rightarrow A)$.*

Proof. To see this, let C be $p \rightarrow (q \rightarrow p)$ and let D be $r \rightarrow (s \rightarrow r)$. Suppose that $A \rightarrow (B \rightarrow A)$ is a logical truth. Then $C \rightarrow (D \rightarrow C)$ and C will be logical truths, as they are instances of the scheme and so $D \rightarrow C$ will be as well, but that violates variable-sharing. \square

Another paradox that is excluded is $A \rightarrow (\sim A \rightarrow B)$. Let C be $p \rightarrow (\sim p \rightarrow q)$ and let D be $r \rightarrow (\sim r \rightarrow s)$. Then $C \rightarrow (\sim C \rightarrow D)$ and C are both in the logic, so by detachment, $\sim C \rightarrow D$ is as well, but $\sim C$ and D do not share a variable. Indeed, a reason not to claim that the proto-relevant logics should be the target class of logics for the relevant logician is that the logic consisting of every instance of $A \rightarrow (B \rightarrow A)$ satisfies variable-sharing, while violating the condition of closure under detachment.

Being a DA-logic ensures that all relevant logics are theories.

Definition 7 (Theory). *A theory is a set X of formulas such that (i) if $A \rightarrow B \in X$ and $A \in X$, then $B \in X$ and (ii) if $A \in X$ and $B \in X$, then $A \wedge B \in X$.*

Theories are typically defined with respect to a particular background logic under whose implications the theory is supposed to be closed, but since all the sets of formulas we are considering are supposed to be logics, the definition requires closure under their own implications. Theories are important in the development of models for relevant logics, and being a theory is a natural constraint on being a logic. Nonetheless, it is not clear that being a theory is an essential feature of being a relevant logic, so one could adopt a broader definition of relevant logics as those proto-relevant logics closed under detachment but not necessarily closed under adjunction.

2 Exploring the proposal

Let us explore the proposed definition of relevant logics, definition 5, in order to help justify the claim that this is a good definition. At the outset, I noted that

modal logicians have a snappy definition of a normal modal logic. That definition sets a baseline, minimal condition. There is a minimal (normal) modal logic, namely the logic K . Any extension of K is a modal logic. On definition 5, there is a unique, minimal relevant logic, namely the empty logic, which will appear again later. Unlike the situation with modal logics, there are extensions of the minimal logic that are not relevant logics. With the relevant logics, the interest is more on the *upper bounds* for relevant logics. The two parts of the definition do different jobs. Being a DA-logic forces certain formulas to be in the logic, given that others are. Being a proto-relevant logic, by contrast, does not require the inclusion of anything, but rather states that certain formulas cannot be in the logic.

Given that the upper bounds of the class of relevant logics is the point of interest, the natural question is whether there is a unique, strongest relevant logic. There is not, but before getting to this theorem and its proof, I need to briefly explain a connection between matrices and logics. A matrix is a non-empty set of values, V , a set of operations on V under which V is closed, and a non-empty set $D \subseteq V$, which is set of designated values.¹³ An assignment on a matrix assigns values from V to atoms and it is extended to the whole language using the set of operations. Matrices are often used to show that a logic has the variable-sharing property. For example, Belnap's matrix M_0 and Meyer's crystal lattice can be used to demonstrate variable-sharing.¹⁴ A matrix can be used to define a logic as the set of formulas that take designated values on all assignments based on that matrix. We will call the logics of the crystal lattice and M_0 , respectively, CL and M_0 . Swirydowicz (1999) has shown that there are two maximal extensions of R that satisfy variable-sharing. These are, in fact, CL and M_0 , the logics of the crystal lattice and M_0 . With that in mind, we turn to the theorem.

Theorem 8. *There is not a strongest relevant logic, ordered by the sublogic relation*

Proof. CL and M_0 both satisfy conditions on being a DA-logic. Their union, when closed under detachment and adjunction, does not satisfy the variable-sharing criterion. \square

A natural question that arises at this point is how to characterize the maximal relevant logics. As shown by Robles and Méndez (2011, 2012), there is a general sufficient condition on a matrix that can be used to show variable-sharing. The key feature, for our purposes, is that there are two disjoint sets of values, X and Y such that each is closed under the operations of the matrix and for $x \in X$ and

¹³See Dunn and Hardegree (2001, ch. 7) for more on matrices.

¹⁴See, respectively, Anderson and Belnap (1975, 252ff.) and Routley et al. (1982, §3.6).

$y \in Y$, $x \rightarrow y$ is not a designated value. There are some matrices that can be used to show variable-sharing that do not yield maximal relevant logics. As an example, the product matrix of M_0 with the crystal lattice yields a relevant logic, which is to say a DA-logic that satisfies variable-sharing, but is not maximal.

Before proceeding to further investigate the proposed definition of relevant logic, I want to pause to consider an alternative proposed demarcation that is sharp, albeit less snappy. Since R is often viewed as the strongest standard relevant logic, one might be tempted to define *relevant logics* as logics that are sublogics, or rather sublogics that are also DA-logics, of CL and M_0 . While this class would not have a greatest logic, it would have exactly two maximal logics. It has, however, a major problem: It would exclude some logics that have a strong claim to being in the relevant logic family. One example is TM, the logic T with the addition of the mingle axiom, $A \rightarrow (A \rightarrow A)$. As is well known, RM, the result of adding mingle to R, violates variable-sharing, as $\sim(A \rightarrow A) \rightarrow (B \rightarrow B)$ is a logical truth of RM. TM, however, maintains variable-sharing.¹⁵ TM is not, however, a sublogic of R or its two maximal extensions with variable-sharing. The proposed definition in terms of CL and M_0 is, then, inadequate, so I will return to the main proposal.

A virtue of the proposal of definition 5 is that it is *presentation-independent*. It does not involve presenting a proof system. Nor does it rely on having a class of models or frames. Rather, it isolates some properties of a logic, and uses those to characterize the broad area. This approach to logic, while perhaps not the most common, is in fact well-known and useful. It lets us ask general and abstract questions about logics and relations between logics, such as whether a given logic is identical to the intersection of two others, as Anderson and Belnap asked about their logic E in relation to R and S4.¹⁶ We can ask about operations under which classes of logics are closed.

Lemma 9. *The class of DA-logics is closed under intersection.*

Proof. Let L and K be DA-logics. Suppose A and $A \rightarrow B$ are in both. By condition (1) of definition 4, both are closed under detachment, so $L \cap K$ contains B. Next, suppose that A and B are in both. By condition (2) of definition 4, then, $A \wedge B \in L \cap K$. \square

Lemma 10. *Let L be a proto-relevant logic. Let K be a logic. Then, $L \cap K$ satisfies variable-sharing, which is to say that it is a proto-relevant logic.*

¹⁵Méndez et al. (2012).

¹⁶For the interested reader, see Anderson and Belnap (1975, §8.10).

Proof. Suppose L be a proto-relevant logic. Let A and B be arbitrary formulas such that $A \rightarrow B \in L \cap K$. It follows that $A \rightarrow B \in L$, so $A \rightarrow B$ share a propositional variable. Since A and B were arbitrary, it follows that $L \cap K$ satisfies variable-sharing. \square

The following theorem is then a corollary of the previous two lemmas.

Theorem 11. *The class of relevant logics is closed under intersection.*

Proof. Let L and K be relevant logics. By lemma 9, $L \cap K$ is a DA-logic. By lemma 10, it is also a proto-relevant logic. \square

We can also ask about operations under which classes of logics are not closed.

Lemma 12. *The class of DA-logics is not closed under union.*

Proof. Let L be the least DA-logic containing all substitution instances of excluded middle, $A \vee \sim A$. Let K be the least DA-logic containing $A \rightarrow A$. $L \cup K$ is not a DA-logic. To see this, note that $(p \vee \sim p) \wedge (q \rightarrow q) \notin L \cup K$. This conjunction is neither in L , as L contains no formulas whose main connective is \rightarrow , nor in K , as K contains no formulas whose main connective is disjunction. \square

Rather than simple union, the DA-logics are closed under the operation $\bar{\cup}$, where $L \bar{\cup} K$ is defined as the least DA-logic containing $L \cup K$. The DA-logics form a lattice with \cap as meet and $\bar{\cup}$ as join.

In contrast to the DA-logics, the proto-relevant logics are closed under \cup .

Lemma 13. *The proto-relevant logics are closed under union.*

Proof. Let L and K be proto-relevant logics. Let $A \rightarrow B \in L \cup K$. Then either $A \rightarrow B \in L$ or $A \rightarrow B \in K$. In both cases, A and B share a propositional variable. \square

This lemma, combined with lemma 10, tells us that that the proto-relevant logics form a lattice with \cap and \cup as meet and join, respectively. The proto-relevant logics, however, are not closed under the $\bar{\cup}$ operation that acts as join for the lattice of DA-logics.

Lemma 14. *The proto-relevant logics are not closed under $\bar{\cup}$.*

Proof. The logic $CL \bar{\cup} M_0$ violates variable-sharing. \square

Many examples provide alternative proofs of the preceding lemma, such as $\text{TM}\bar{\cup}\text{R}$. Lemma 14 and theorem 11 have the following corollary, given that the empty logic is a relevant logic.

Corollary 15. *The class of relevant logics is not a sublattice of the lattice of DA-logics, with \cap as meet and $\bar{\cup}$ as join.*

Corollary 16. *The class of relevant logics is a meet semi-lattice with a bottom element, where \cap is the meet and \emptyset is the bottom element.*

While the class of proto-relevant logics is closed downwards under the sublogic relation, the class of DA-logics, and so the class of relevant logics, is not. The reason is that closure under adjunction, as well as closure under detachment, need not be preserved by sublogics. The appropriate ordering for considering relevant logics is, rather, the ordering defined by the meet operation of the semi-lattice, $L \leq K$ iff $L \cap K = L$. Lemma 14 has the following as a corollary.

Corollary 17. *There is no greatest relevant logic according to the semi-lattice ordering, \leq .*

Next, I will comment upon some logics that are, on this definition, relevant logics. First, all the standard relevant logics in the literature, roughly those contained between R and B, are relevant logics, in the sense of definition 5. They are all theories and they all satisfy variable-sharing.

Second, lattice R, linear logic, and Brady’s logic MC count, even though they fail to extend B, seeing as they lack the distribution axiom, $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$.¹⁷ Since they fail to extend B, the standard ternary relational models cannot model them adequately. Despite this, these logics have many affinities with the more familiar relevant logics, and they are naturally included with them.

Third, many logics of analytic containment, studied by Parry and others, will be relevant.¹⁸ As with the non-distributive logics, the analytic containment logics do not extend B. Nonetheless, they have many affinities with relevant logics, and they are often presented with similar ends.

Fourth, the set of classical tautologies in the vocabulary $\{\wedge, \vee, \sim\}$, closed under substitutions, is a relevant logic, as it contains no implications as theorems.¹⁹

¹⁷See Thistlewaite et al. (1988), Restall (2000), and Brady and Meinander (2013), respectively. Note that by ‘linear logic’, I here mean MALL, multiplicative-additive linear logic. The exponentials are not included. The reason is that the exponentials bring with them violations of variable-sharing.

¹⁸See Ferguson (2017, 7–8). See also Szmuc and Rubin (2022).

¹⁹To be more precise, one takes the set of formulas in the vocabulary $\{\wedge, \vee, \sim\}$ that are classical tautologies and closes that under substitutions from the vocabulary $\{\wedge, \vee, \sim, \rightarrow\}$, so that the result is a logic in the full vocabulary.

This is a degenerate case of the definition, much like the empty logic, as it contains no implications. In fact, any logic lacking implications will qualify as a relevant logic under definition 5. While there may be much about these logics to be dissatisfied with, the dissatisfaction would have to be based on something apart from considerations of relevance, as expressed by the implication.²⁰ For example, one might object to such logics on the basis of having bad principles for the truth-functional connectives, but this is not a matter solely of relevance. In practice, one typically wants to consider a relevant logic that contains at least some implications, so these edge cases do not seem to tell against the definition.

It is worth pausing for a moment to consider the much-loved system, FDE, as it is not a logic in our sense.²¹ As defined by Anderson and Belnap (1975), the axiomatic form of FDE is a set of formulas $A \rightarrow B$, where neither A nor B contain any implications. This set of formulas is not a logic, in our sense, as it isn't closed under substitutions. We can obtain a logic from FDE by taking the set of formulas and closing it under substitutions. Let us call this system FDE^+ . FDE^+ contains theorems, such as $(p \wedge (p \rightarrow q)) \rightarrow p$, which FDE does not, however it treats the embedded implications the same as atoms. FDE^+ is a logic, and in fact it is a sublogic of E and B, since FDE is contained in B and B is closed under substitutions.²²

Some salient subclasses of relevant logics are easy to specify. This is parallel to the situation in modal logics. As noted above, the modal logician has an easy answer to the question "What is a modal logic?"²³ A simple answer is that a modal logic is a logic containing all classical tautologies, the (K) axiom, and closed under detachment and the rule (Nec). This identifies a large class of logics, but the necessity operators of those logics may, in some cases, be thought to be too strong. The modal logician can broaden the definition, following Segerberg (1971), and others. The definition just given is for a *normal* modal logic. A *monotonic* modal logic is a logic containing all classical tautologies, closed under detachment and the rule (Mono), $A \rightarrow B \Rightarrow \Box A \rightarrow \Box B$. A *congruential* modal logic is a logic containing all classical tautologies, closed under detachment and the rule (Cong), $A \leftrightarrow B \Rightarrow \Box A \leftrightarrow \Box B$. This gives the modal logician clear areas of study, at least when considering singulary necessity operators. Further refinements of these classes can be obtained by placing some conditions on the classes.

²⁰I would like to thank an anonymous referee for pressing this issue.

²¹See Omori and Wansing (2017) for a nice overview of this system.

²²I would like to thank an anonymous referee for raising questions about the status of FDE.

²³Segerberg (1982) notes that this question presents some difficulties, but we will set those aside for the time being.

Much as the modal logician can broaden the characterization of a modal logic to permit weaker and weaker modal logics, the relevant logician can add some conditions to constrain the upper bounds of their areas of interest. For example, one way to avoid the degenerate logic of all the classical tautologies in the vocabulary $\{\wedge, \vee, \sim\}$ is to require that *some* implication $A \rightarrow B$ be in the logic. Relevant logics containing an implicational logical truth we can call the *implicational logics*.²⁴

We might be interested in the *lattice relevant logics*, those that contain the following logical truths.

- (L1) $A \wedge B \rightarrow A, A \wedge B \rightarrow B$
- (L2) $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
- (L3) $A \rightarrow A \vee B, B \rightarrow A \vee B$
- (L4) $((B \rightarrow A) \wedge (C \rightarrow A)) \rightarrow ((B \vee C) \rightarrow A)$

These are the logics in which conjunction and disjunction behave in the more or less familiar lattice-connective fashion. The *distributive relevant logics* are the lattice relevant logics that contain

- (L5) $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$.

While lattice R, linear logic, and MC are all lattice relevant logics, they fail to be distributive. Most of the standard relevant logics fall into the class of distributive relevant logics, but distribution seems like an optional extra.

Above I noted that the class of relevant logics permits there to be non-transitive relevant logics in the sense that a logic could contain $A \rightarrow B$ and $B \rightarrow C$ but lack $A \rightarrow C$. The implication of such logics does not express a sufficiency relation or entailment, whereas expressing entailment was central to the project of Anderson and Belnap. To that end, it is natural to consider the class of *transitive logics*, those closed under the rule $A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$. Alternatively, one might consider the class of *affixing logics*, those closed under the rules $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$ and $A \rightarrow B \Rightarrow (C \rightarrow A) \rightarrow (C \rightarrow B)$.

Let us say that the γ -*relevant logics* are those that are closed under the rule γ , also known as disjunctive syllogism, $A, \sim A \vee B \Rightarrow B$. The question of whether a logic is a member of this class has played an important role in the development of

²⁴The implicational logics would include the logic S of Martin and Meyer (1982), modifying the definition of DA-logic in the absence of conjunction.

the technical apparatus of relevant logics. Many of the standard relevant logics are γ -relevant logics, the verification of which fact was much celebrated.²⁵

Anderson and Belnap were interested in logics that contained all the tautologies of classical logic in the vocabulary $\{\wedge, \vee, \sim\}$. As we have set things up, their logics are sublogics of classical logic, although this does not seem to be essential to the study of relevant logics. One might want to consider relevant logics that are *contra-classical*, in the sense that they have logical truths that classical logic lacks. One example is (the \rightarrow -fragment of) Abelian logic, which contains $((A \rightarrow B) \rightarrow B) \rightarrow A$ as a distinguishing principle.²⁶ Another example is the family of connexive logics, which contain principles such as $\sim(A \rightarrow \sim A)$ and $\sim(\sim A \rightarrow A)$.²⁷ There are relevant logics that contain the connexive principles, although those logics cannot be as strong as R.²⁸ Including some contra-classical logics in the family of relevant logics seems like another virtue of the proposal.

One further thing that speaks in favor of the utility of a presentation-independent definition of relevant logic is that it combines neatly with additional criteria used for extensions of the language. For example, at the outset we provided a characterization of normal modal logics for classical logic. We can specify that a normal modal relevant logic is a relevant logic that contains the (K) axiom, is closed under the rule (Nec), and also contains the $(\Box\wedge)$ axiom, $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$. A more appropriate definition for the relevant logic context is that a modal relevant logic is a relevant logic closed under the rule (Mono), $A \rightarrow B \Rightarrow \Box A \rightarrow \Box B$, and containing the $(\Box\wedge)$ axiom. It is, I think, a virtue of the proposal that a specification of relevant modal logics can be provided using it, in a manner similar to that of normal modal logics in the case of classical logic.

The definition excludes many logics, as it should. Classical logic (with ‘ \rightarrow ’ in the vocabulary), intuitionistic logic,²⁹ and RM all fail to be relevant logics.³⁰ Further, the addition of Boolean negation to some familiar relevant logics result in a non-relevant logic.³¹

²⁵Meyer and Dunn (1969).

²⁶See Meyer and Slaney (1989), Butchart and Rogerson (2014), and Paoli et al. (2008).

²⁷Omori and Wansing (2019), Francez (2019).

²⁸Routley et al. (1982, 343).

²⁹Since the logical truths of Tennant’s (2017) core logic are those of intuitionistic logic, that will fail to be a relevant logic on this definition.

³⁰It is worth noting that the addition of the mingle axiom, $A \rightarrow (A \rightarrow A)$, to the implicational fragment of R, results in an implicational logic, RM0, that has variable-sharing. See Anderson and Belnap (1975, 148) or Humberstone (2011, 333,363) for details and discussion. I would like to thank Lloyd Humberstone for pointing this out to me.

³¹See Meyer and Routley (1973, 1974). I’d like to thank Tore Fjeltand Øgaard for this suggestion.

Smiley (1959) defines an entailment relation via substitutions. We can adapt this definition for a kind of implication connective, by saying $A \rightarrow B$ is valid iff for some substitution σ and formulas C, D , $\sigma^+(C \rightarrow D) = A \rightarrow B$, where $C \rightarrow D$ is a classical tautology but neither $\sim C$ nor D is a classical tautology. Smiley presents this as an alternative account of entailment to E. Smiley's logic, however, fails to be a relevant logic, on the present criterion. The reason is that it fails to be closed under detachment. To see this, note that $(r \rightarrow r) \rightarrow (s \rightarrow (r \rightarrow r))$ is a substitution of $p \rightarrow (q \rightarrow p)$ and neither $\sim p$ nor $q \rightarrow p$ is a tautology. Further, $r \rightarrow r$ is a substitution of $p \rightarrow p$, and neither $\sim p$ nor p is a tautology. To see that $s \rightarrow (r \rightarrow r)$ is not valid on Smiley's criterion, we consider the options for substitution: $p, p \rightarrow q, p \rightarrow (q \rightarrow q)$, and $p \rightarrow (q \rightarrow r)$. Of these, only the third is a tautology, but its consequent is itself a tautology.³²

As noted in this section, definition 5 includes many logics that it should and it excludes many as well. It includes many relevant logics beyond the usual suspects. In the next section, I will discuss this feature, along with other consequences of the definition and some objections.

3 Discussion

I will begin with a comment on the proposal and its relation to extant work on relevant logics. The proposed definition is not an attempt to provide a formal criterion that captures Anderson and Belnap's philosophical motivations for relevant logics. Nor is it meant to capture the philosophical views of other relevant logicians, such as Sylvan, Meyer, or Brady. The proposal, instead, offers a formal criterion that includes all the standard relevant logics, as well as many of their close neighbors, and that uses only a concept that is widely agreed to be central to the study of relevant logics, namely variable-sharing. The philosophical reasons behind the importance of variable sharing, whether they rely on use, suppression-avoidance, meaning containment, or something else, may vary, but relevant logicians, by and large, have taken variable-sharing to be a feature of relevant logics. The suggestion is to take variable-sharing, along with being a DA-logic, as sufficient as well as necessary for being a relevant logic, with additional

³²An alternative way of adapting Smiley's definition for an implication connective is to focus on the implications between formulas in the $\{\sim, \wedge, \vee\}$ vocabulary and substitution instances of those. Adding to this all the substitution instances of classical tautologies in the vocabulary $\{\sim, \wedge, \vee\}$ results in a relevant logic, as defined above.

For further discussion of these logics defined by substitution in this manner, see Dunn (1980).

conditions imposed to reach one's preferred logic or class of logics.

It is plausible that the sentiment that variable-sharing is only necessary, not sufficient, for pinning down the relevant logics remains when closure under detachment and adjunction is added. One reason for this sentiment is, I think, that variable-sharing is not sufficient for pinning down R and its major sublogics, roughly the standardly studied relevant logics. When Anderson and Belnap were developing relevant logics, they had identified E and R, along with some of their neighbors. These logics are, in many ways, special: They are all transitive and they have important connections to combinatory logic and the structural rules of sequent systems.³³ These logics can be adequately modeled using classes of Routley-Meyer ternary relational frames.³⁴ Variable-sharing is not sufficient to isolate these logics.

The class of relevant logics, in the sense of definition 5, includes logics stronger than R and some incomparable with R. From our perspective, standing on the shoulders of giants, we can see that the class of relevant logics naturally extends beyond and around R. Yet, one might ask how the current definition relates to what Anderson and Belnap were doing. One approach to this question is to provide an additional condition to isolate a subclass of logics of R. It seems like something like this is operative in the Use Criterion of Anderson and Belnap, which highlights R, E, and some other logics as salient relevant logics. If this is right, then it is natural to view those logics as combining two features, relevance as variable-sharing and use, specified in a proof-theoretic way.³⁵ Similarly, the logics motivated by the sufficiency view of Routley (2019) or the meaning containment view of Brady (2006) are special in many ways, but they can plausibly be isolated by imposing additional constraints on being a relevant logic.³⁶

It is a perhaps surprising fact about the proposal that none of the standard relevant logics, namely those discussed in depth in any of the surveys of the area, really stand out as distinctive. For example, neither B nor R is natural stopping points according to the definition.³⁷ They may feature prominently in certain

³³For more on combinatory logic, see Bimbó (2009, 2011). For more on structural rules, see Restall (2000) and Bimbó (2014), among others.

³⁴See Routley et al. (1982, ch. 4).

³⁵That the Use Criterion and variable-sharing were distinct, separable aspects of relevant logics was raised, as a criticism, by Copeland (1980).

³⁶Philosophically, it isn't correct to view either of their views as combining relevance with another element, sufficiency or meaning containment, as they see relevance as arising from the other element. Formally, there is not a problem with proceeding in this way.

³⁷Avron (2014) highlights R as distinctive of relevant logics.

subclasses of logics, e.g. those that are complete with respect to a class of Routley-Meyer frames, but there are many subclasses of logics in which they do not mark a minimal or maximal logic. From an abstract point of view, they are not particularly special. This is not to say that they are not special. Indeed, the attention that they, and their neighbors, have received is because they have many qualities, formal and philosophical, that mark them off as distinctive.

The fact that the proposal does not highlight or privilege the standard relevant logics should be viewed as a virtue. The reason is that it underlines a sense in which the net of relevant logics has been cast too narrowly. This shift of focus will, I think, help to isolate the aspects of relevant logics that are philosophically important, as proponents of a particular logic or family of logics will have the burden of articulating the additional philosophical features and their formal counterparts that exclude the undesired relevant logics. This process of making explicit intuitive criteria may help to clarify the philosophical positions that support various relevant logics, such as R and B.³⁸

The next topic to discuss concerns principles that are, or are not, ruled out by the proposal. On that score, it is worth emphasizing an important point, namely that whether a particular axiom can be in a relevant logic is something that cannot, in general, be read off the axiom. An example is the mingle axiom, $A \rightarrow (A \rightarrow A)$. Its antecedent and consequent share a variable. Adding it to R results in a violation of variable-sharing. On the other hand, adding it to T allows one to stay within the class of relevant logics. The axiom form of γ , $(A \wedge (\sim A \vee B)) \rightarrow B$, has a variable shared between its antecedent and consequent, but it cannot be added to any transitive, distributive relevant logic.³⁹ Even the weakening axiom, $A \rightarrow (B \rightarrow A)$, shares a variable between antecedent and consequent while leading to immediate failures of variable-sharing.

A natural question is whether the proposal picks out only logics that exclude all the standard paradoxes of implication. As we saw above, two of the positive Lewis paradoxes, $A \rightarrow (B \rightarrow A)$ and $A \rightarrow (B \rightarrow B)$, are both ruled out. There are some paradoxes involving other connectives that will be excluded from the class of relevant logics, just in virtue of their form, such as $(A \wedge \sim A) \rightarrow B$ and $A \rightarrow (B \vee \sim B)$. It makes no difference what principles govern \vee , \wedge , and \sim in these cases, as those violate variable-sharing on their face.

Other putative paradoxes will depend on the principles of the logic. Two ex-

³⁸Brandom (1994)

³⁹See Øgaard (2021b,c) for discussion of γ , in rule and axiom form, in the context of relevant logics.

amples of such are

- Dummett’s axiom, $(A \rightarrow B) \vee (B \rightarrow A)$, and
- $A \vee (A \rightarrow B)$.

Robles and Méndez (2012) have provided a matrix whose logic satisfies variable-sharing while also containing Dummett’s axiom. In a similar vein, the logic CL contains the second formula, $A \vee (A \rightarrow B)$. While neither of these principles is, I think, particularly *compelling* from a relevant logical point of view, their unattractiveness is somewhat lessened by keeping in mind that logics containing them will, typically, not be *prime*, where a logic is prime iff whenever it contains a disjunction, it contains at least one disjunct. Typically when either of these axioms is in a logic, the logic will fail to be prime, and so containing one does not entail that either of its disjuncts is valid. Further, neither of the particular logics containing the above axioms is closed under γ , so the consequences the disjunctive principles above are more restricted than one might expect.⁴⁰ The logic containing Dummett’s axiom has fairly strong principles, and the second formula above is valid in an extension of R. So, worries about their potential for undermining variable-sharing should be somewhat tempered by the fact that they can be combined with comparatively strong logical principles without resulting in violations of variable-sharing. One might have philosophical reasons for wanting to exclude the above paradoxes from consideration, but they will go beyond the minimal sense of relevance of definition 5.

Another example of an axiom that can be added to some, but not all, relevant logics is $\sim A \rightarrow (A \rightarrow B)$. In these logics, one may not have any logical truths whose main connective is a negation. One such example is the least DA-logic containing $\sim A \rightarrow (A \rightarrow B)$. In such a logic, every implicational logical truth has a variable shared between antecedent and consequent. If in the logic, negation obeys the plausible principle, $A \rightarrow \sim\sim A$, or even $A \rightarrow \sim^k A$, where $k \geq 1$ and ‘ \sim^k ’ is a k -long sequence of ‘ \sim ’s, then $\sim A \rightarrow (A \rightarrow B)$ will be excluded. As proof, let us suppose $A \rightarrow \sim^k A$ is in the logic, $k = m + 1$, and let C be $\sim p \rightarrow (p \rightarrow q)$. Then C is a theorem, as is $\sim\sim^m C \rightarrow (\sim^m C \rightarrow B)$. From the assumptions, $\sim\sim^m C$ is

⁴⁰ As a witness to the failure of γ in the Robles and Mendez’s logic with Dummett’s axiom, $\sim((p \rightarrow p) \rightarrow \sim(q \rightarrow q))$ and $((p \rightarrow p) \rightarrow \sim(q \rightarrow q)) \vee (\sim(q \rightarrow q) \rightarrow (p \rightarrow p))$ are theorems, but $\sim(q \rightarrow q) \rightarrow (p \rightarrow p)$ is not, on pain of violating variable-sharing. As a witness to the failure of γ in the logic of the crystal lattice, $\sim\sim(p \rightarrow p)$ and $\sim(p \rightarrow p) \vee (\sim(p \rightarrow p) \rightarrow q)$ are both theorems, but $\sim(p \rightarrow p) \rightarrow q$ is not, on pain of violating variable-sharing. I would like to thank Tore Fjetland Øgaard for raising this issue with me.

a theorem, so then $\sim^m C \rightarrow B$ is as well. As B can be any formula, we can suppose that it contains only the atom r , so we have a violation of variable sharing.

Reflecting on two of the negative Lewis paradoxes, namely $A \rightarrow (\sim A \rightarrow B)$ and $\sim A \rightarrow (A \rightarrow B)$, we can pull out a general lesson about variable-sharing and singularly connectives.

Theorem 18. *Let $\#$ be a singularly connective. Then no relevant logic can include the theorem $A \rightarrow (\#A \rightarrow B)$. Further, no relevant logic can have as theorems, for all formulas A and B and formula contexts C , both $A \rightarrow \#C(A)$, and $\#A \rightarrow (A \rightarrow B)$.*

Proof. For the first part, let D be $p \rightarrow (\#p \rightarrow q)$. Then, D is a theorem and so is $D \rightarrow (\#D \rightarrow r)$. Thus, $\#D \rightarrow r$ is a theorem as well, which is a violation of variable sharing. For the second, let A be $\#p \rightarrow (p \rightarrow q)$ and let B contain no atoms in $C(A)$. It follows from the assumptions that $\#C(A)$ is a theorem. As $\#C(A) \rightarrow (C(A) \rightarrow B)$ is a theorem, it then follows by detachment that $C(A) \rightarrow B$ is a theorem, which is a violation of variable sharing. \square

Although the negative Lewis paradoxes feature negation, their relevance-violating features do not depend on features specific to negation at all. Rather, they provide a lesson about connectives obeying certain principles being compatible with relevance.⁴¹

Definition 5 casts a wide net, and some may worry that the net is cast *too* widely. As noted, some of the paradoxes of implication are excluded from the class of relevant logics at the outset and some are excluded from subclasses that contain plausible principles governing certain connectives. Some relevant logics, on our definition, can contain some odd or surprising principles, such as the following.⁴²

- (i) $(A \rightarrow B) \rightarrow (B \rightarrow A)$
- (ii) $\sim(A \rightarrow A) \rightarrow (A \rightarrow B)$
- (iii) $(A \rightarrow B) \rightarrow (B \wedge C)$
- (iv) $(A \vee \sim A) \rightarrow A$

⁴¹See Standefer (2022) for a general discussion of relevant connectives.

⁴²I thank an anonymous referee for raising this issue and supplying some example axioms to consider.

One obtains a relevant logic by taking exactly one of (i)–(iv) as the sole axiom and closing under detachment and adjunction. Those four logics, however, have very few principles governing their connectives, and so their logical behavior may diverge from the usual understanding of the connectives. It should be no surprise that in extremely weak logics, some odd looking axioms are compatible with variable sharing. When some plausible principles are added, variable sharing may be violated. For example, axiom (i) and (iii) lead to violations of variable sharing in the presence of the axiom (L1) and transitivity rules, axiom (iv) immediately yields violations of variable sharing in any logic with excluded middle, and axiom (ii) violates variable sharing in logics with permutation, $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$.⁴³ For principles that are compatible with variable-sharing and many plausible principles, it is up to their critics to formulate the reasons, formal and philosophical, that they should be excluded.

I will close with a short discussion of variable-sharing. The form of variable-sharing used in definition 3 can be called *simple variable-sharing*. There are stronger forms of variable-sharing that one might consider for isolating classes of logics.

One of the stronger, more refined forms of variable-sharing is known as *strong variable-sharing*.⁴⁴ For strong variable-sharing, each formula is assigned a polarity, positive or negative, based on its place a formula.⁴⁵ In $A \rightarrow B$, A is negative and B is positive. Being in consequent position, being a conjunct, and being a disjunct all maintain polarity, whereas being in antecedent position or being negated changes polarity. Strong variable-sharing says that if $A \rightarrow B$ is a logical truth, then A and B share a variable with the same polarity. Belnap’s original proof of variable-sharing actually demonstrated strong variable-sharing. Strong variable-sharing would be sufficient to rule out $\sim A \rightarrow (A \rightarrow B)$, which as we saw above was not ruled out by simple variable-sharing.

Another stronger form of variable-sharing is Avron’s strong variable-sharing property, which I will call *basic variable-sharing*, as the property of the previous paragraph is well established with the name ‘strong variable-sharing’.⁴⁶ The basic

⁴³Judgments of plausibility vary, and some have found excluded middle and permutation to be implausibly strong. See Slaney (1989), for example.

⁴⁴See Anderson and Belnap (1975), where the variable-sharing theorem establishes strong variable-sharing. Similarly, Robles and Méndez (2011, 2012) also discuss strong variable-sharing.

⁴⁵The terminology ‘consequent part’ and ‘antecedent part’, or ‘cp’ and ‘ap’, is commonly used in this context.

⁴⁶Avron (2022) presents his basic relevance property for consequence relations, and he proposes an implicational form, what is here being called the ‘basic variable-sharing property’. Under fairly minimal conditions, the consequence relation and implicational properties are equivalent.

variable-sharing property says that if (i) $\vdash_{\perp} (A \wedge B) \rightarrow C$ and (ii) A and $B \rightarrow C$ do not share a propositional variable, then $\vdash_{\perp} B \rightarrow C$. Avron shows that, under fairly minimal conditions, having the basic variable-sharing property entails having the simple variable-sharing property. This is a compelling principle, but it requires conjunction as well as implication to state, so we have not adopted it for the main definition of variable-sharing.

Another form of variable-sharing, proposed by Brady (1984a), is known as *depth relevance*.⁴⁷ The *depth* of an occurrence of a propositional variable in a formula is, roughly, the number of implications it is under in the formula's parse tree. For example, in $p \rightarrow (p \rightarrow q)$, the leftmost p has depth 1, whereas the rightmost p has depth 2. The depth relevance criterion requires that if $A \rightarrow B$ is a logical truth, then A and B share a variable at the same depth. Clearly $\sim A \rightarrow (A \rightarrow B)$ is ruled out by the depth relevance criterion.

Finally, strong variable-sharing and depth relevance can be combined into the *strong depth relevance criterion*, as studied by Logan (2021, 2022). The strong depth relevance criterion says that if $A \rightarrow B$ is a logical truth, then A and B share a propositional variable at the same depth with the same polarity.

These stronger variable-sharing criteria isolate what seem to be interesting and natural classes of relevant logics.⁴⁸ They capture stronger senses of connection than is required by (simple) variable-sharing. For purposes of identifying the area of relevant logics, taking the broader definition as provided in section 1, seems the most natural. The stronger criteria all require additional specification that goes beyond the core of the simple connection idea found in the variable-sharing criterion.

To conclude I will take stock of what has been done. I have motivated and presented a definition of the class of relevant logics as those logics (i) all of whose implicational logical truths share a variable between antecedent and consequent, and (ii) that are closed under detachment and adjunction. Some notable consequences of this definition is that there is no greatest relevant logic and many of Lewis's paradoxes of implication are excluded from relevant logics. All of the standard relevant logics and many of their close neighbors are included in the class of relevant logics, in the sense of definition 5. I noted that we can see the work of Anderson and Belnap as isolating a subclass of relevant logics, implicitly using some additional conditions for being a relevant logic, and I responded to

⁴⁷See also Robles and Méndez (2014a,b) and Salto et al. (2018).

⁴⁸There are further properties one might want to consider as well. One example is the "no loose pieces" principle discussed by Robles and Méndez (2012). Another example is the Ackermann property, discussed by Anderson and Belnap (1975, 243).

the objection that the definition is too inclusive. Finally, I noted some alternative forms of variable-sharing that could be used to identify interesting subclasses of logics. As noted, the definition of relevant logics given by definition 5 generalizes neatly to include relevant modal logics, and, as demonstrated by theorem 18, it can be useful in proving limitative results about what sorts of connectives can or cannot be included in relevant logics. Further investigation along these lines will be pursued in future work, and it will, I hope, prove useful in clarifying the foundations of relevant logics.

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