PENULTIMATE DRAFT – PLEASE CITE THE PUBLISHED VERSION
Philosophia, DOI: 10.1007/s11406-018-9953-1.

Mathematical Knowledge and Naturalism

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ABSTRACT

How should one conceive of the method of mathematics, if one takes a naturalist stance? Mathematical knowledge is regarded as the paradigm of certain knowledge, since mathematics is based on the axiomatic method. Natural science is deeply mathematized, and science is crucial for any naturalist perspective. But mathematics seems to provide a counterexample both to methodological and ontological naturalism. To face this problem, some naturalists try to naturalize mathematics relying on Darwinism. But several difficulties arise when one tries to naturalize in this way the traditional view of mathematics, according to which mathematical knowledge is certain and the method of mathematics is the axiomatic method. This paper suggests that, in order to naturalize mathematics through Darwinism, it is better to take the method of mathematics not to be the axiomatic method.

Keywords: Axiomatic method; Darwinism; Mathematical knowledge; Mathematical Platonism; Naturalism.

1. Introduction

The scope of this article is quite limited, since it is mainly intended to point out how "the very commonplace view that mathematical knowledge is essentially obtained by deductive inference" (Prawitz 2014, p. 73) cannot easily be made compatible with a naturalist stance. As "Quine and any other naturalists would claim," from a naturalist point of view all "knowledge is part of scientific knowledge; natural science is the one and only source of reliable beliefs, including reliable beliefs about the nature of belief itself," and mathematical knowledge "is a part of this" (Brown

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Philosophia, DOI: 10.1007/s11406-018-9953-1.

2012, p. 117). But the idea that mathematical knowledge is part of our scientific knowledge is in contrast with the traditional view of mathematics, according to which mathematical knowledge has a special epistemic status with respect to knowledge provided by natural sciences (Paseau 2013). This article does not even try to provide a fully developed alternative view to the traditional view of mathematics, it just suggests that if one adopts a naturalist stance, one should at least carefully reflect before accepting the claim that the axiomatic method is the method of mathematics, and that it is likely that, upon reflections, a naturalist will dismiss such claim.

The article is organized as follows: in Section 2, the traditional view of mathematical knowledge is presented; in Section 3, it is presented the axiomatic view of the method of mathematics; in Section 4, the issue of a naturalist perspective on mathematics is discussed; in Section 5, some attempts aimed at naturalizing mathematics through Darwinism are discussed; Section 6 contains a brief digression on whether consistency is a sufficient condition for truth, and whether the idea that mathematical knowledge is acquired by deduction can account for the ampliation of mathematical knowledge; finally, in Section 7 some conclusions are drawn.

2. Mathematics and Knowledge

Although there are some exceptions,[1] mathematics is still regarded as “the paradigm of certain and final knowledge” (Feferman 1998, p. 77) by most mathematicians and philosophers. According to many authors, the degree of certainty that mathematics is able to provide is one of its qualifying features. For example, Byers states that the certainty of mathematics is “different from the certainty one finds in other fields [...]. Mathematical truth has [...] [the] quality of inexorability. This is its essence” (Byers 2007, p. 328). Mathematics is also usually thought to be objective, in the sense that it is regarded as mind-independent, and so independent from our biological constitution. For example, George and Velleman state that understanding the nature of mathematics does not require asking “such questions as ‘What brain, or neural activity, or cognitive architecture makes mathematical thought possible?’,” because

[1] See Kline 1980; Cellucci 2017, 2013; Bell, Hellman 2006; Clarke-Doane 2014.

Philosophia, DOI: 10.1007/s11406-018-9953-1.

"such studies focus on phenomena that are really extraneous to the nature of mathematical thought itself' (George, Velleman 2002, p. 2). Mathematics proved tremendously useful for dealing with the world. Indeed, current natural science is "mathematical through and through: it is impossible to do physics, chemistry, molecular biology, and so forth without a very thorough and quite extensive knowledge of modern mathematics" (Weir 2005, p. 461).

But, despite its being so pervasive in scientific knowledge, we do not have yet an uncontroversial and science-oriented account of what mathematics is. So, "in a reality [...] understood by the methods of science," we are unable to answer to the following question: "where does mathematical certainty come from?", even because most mathematicians and scientists "do not take seriously the problem of reconciling" the certainty of mathematical knowledge "with a scientific world-view" (Deutsch 1997, p. 240). Moreover, many authors are skeptical about the very possibility of developing a naturalist perspective on mathematics. They think that "mathematics is an enormous Trojan Horse sitting firmly in the center of the citadel of naturalism," because even if "natural science is mathematical through and through," mathematics seems to "provide a counterexample both to methodological and to ontological naturalism." Indeed, mathematics ultimately rests on axioms, which are "traditionally held to be known a priori, in some accounts by virtue of a form of intuitive awareness." The epistemic role of the axioms in mathematics seems "uncomfortably close to that played by the insights of a mystic. When we turn to ontology, matters are, if anything, worse: mathematical entities, as traditionally construed, do not even exist in time, never mind space" (Weir 2005, p. 461-462). In fact, the majority of mathematicians and philosophers of mathematics argues for some form of mathematical realism (Balaguer 2009). Thus, it is very difficult even to envisage how it could be possible to naturalistically account for what mathematics is and how we acquire mathematical knowledge.[2]

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[2] In this article 'naturalism' is understood as it is usually understood in the philosophy of science (see below, Sect. 4). In order to avoid misunderstanding, it is important to note that 'naturalism' is used in a quite different sense in the philosophy of mathematics proper, where it indicates a philosophical position, according to which, roughly, the only authoritative standards in the philosophy of mathematics are those of mathematics itself (Paseau 2013).

Philosophia, DOI: 10.1007/s11406-018-9953-1.

A clarification is in order here. There is a huge amount of work in cognitive science devoted to study numerical capacities in human and non-human animals (see e.g. Cohen Kadosh, Dowker 2015; Dehaene, Brannon 2011), but we will not be primarily concerned with those works here. Indeed, these researches may well shed light on how to naturalistically conceive of mathematics. But they have so far investigated the origin and functioning of just some very basic numerical abilities. These basic capacities are thought to have evolved because they allowed our ancestors to approximately deal with numerosities sufficiently well to ensure their survival. But this ability seems insufficient to justify the claim that mathematical knowledge is knowledge of the most certain kind. And no adequate scientific account of how we develop advanced mathematics starting from those basic numerical abilities has been provided yet (see e.g. Spelke 2011). Thus, even if *prima facie* the study of such basic cognitive abilities does not support the traditional view of mathematics, it seems at the moment even unable to definitely confute that view. Indeed, according to many authors that support the traditional view of mathematics, showing the evolutionary roots of these numerical capacities is insufficient to naturalistically explain the degree of certainty and effectiveness that our advanced mathematics displays. For example, Polkinghorne states that while it is easily conceivable that "some very modest degree of elementary mathematical understanding [...] would have provided our ancestors with valuable evolutionary advantage," it is, on the contrary, very difficult to evolutionarily explain "the human capacity [...] to attain the ability to conjecture and eventually prove Fermat's Last Theorem, or to discover non-commutative geometry." Indeed, that ability appears not only to convey no direct survival advantage, but it also seems "vastly to exceed anything that might plausibly be considered a fortunate spin-off from such mundane necessity" (Polkinghorne 2011, p. 31-32). Since we are dealing here with the issue of whether the traditional view of mathematics is compatible with a naturalist stance, we will not dwell on those attempts that (1) try to naturalize mathematics by focusing on discoveries related to our basic numerical abilities, but (2) do not address the issue of whether or not the traditional view of mathematics should be maintained in the light of our scientific understanding of those basic abilities.

Turning to the issue at stake, the difficulty of accommodating mathematical knowledge within a coherent scientific world-view is what Mary Leng called 'the problem of mathematical knowledge'. According to her, "the most obvious answers

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to the two questions 'What is a human?' and 'What is mathematics?' together seem to conspire to make human mathematical knowledge impossible" (Leng 2007, p. 1).

This article aims to suggest that a promising step towards the elaboration of an adequate naturalist account of mathematics and mathematical knowledge based on Darwinism, may be to take the method of mathematics *not* to be the axiomatic method. It will be argued that it is impossible to naturalize mathematics relying on Darwinism without challenging at least some crucial aspects of the traditional view of mathematics, according to which mathematical knowledge is certain and the method of mathematics is the axiomatic method. Nor does it seem possible to keep maintaining that mathematical knowledge is the paradigm of certain knowledge, if we dismiss the claim that the method of mathematics is the axiomatic method.[3]

3. Mathematics and Method

The certainty of mathematical knowledge is usually supposed to be due to the method of mathematics, which is commonly taken to be the axiomatic method.[4] In this view, the method of mathematics differs from the method of investigation in the natural sciences: whereas "the latter acquire general knowledge using inductive methods, mathematical knowledge appears to be acquired [...] by deduction from basic principles" (Horsten 2015). According to Frege, when we do mathematics we form chains of deductive "inferences starting from known theorems, axioms, postulates or definitions and terminating with the theorem in question" (Frege 1984, p. 204). In the same vein, Gowers states that what mathematicians do is that they "start by writing down some axioms and deduce from them a theorem" (Gowers 2006, p. 183). So, it is the deductive character of mathematical demonstrations that

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[3] For a positive alternative view on the method of mathematics, see e.g. Cellucci (2013; 2017), who takes the method of mathematics to be the analytic method. Such proposal cannot be illustrated here for reason of space. See also Sterpetti 2018; Bertolaso, Sterpetti 2017.

[4] Cf. e.g. Baker 2016, Sect. 1: "there is a philosophically established received view of the basic methodology of mathematics. Roughly, it is that mathematicians aim to prove mathematical claims [...], and that proof consists of the logical derivation of a given claim from axioms."

Philosophia, DOI: 10.1007/s11406-018-9953-1.

confers its characteristic certainty to mathematical knowledge, since demonstrative "reasoning is safe, beyond controversy, and final" (Pólya 1954, I, p. v), precisely because it is deductive in character. In this view, "deductive proof is almost the defining feature of mathematics" (Auslander 2008, p. 62).

If the method of mathematics is the axiomatic method, mathematics mainly consists in deductive chains from given axioms.[5] So, in order to claim that mathematical knowledge is certain, we have to know that those axioms are 'true', where 'true' is usually understood as 'consistent with each other'. As well as the consistency of axioms, the problem of justifying our reliability about mathematics is also related to the problem of justifying our reliability about logic. Indeed, if we think that the method of mathematics is the axiomatic method, proving the reliability of deductive inferences is essential for claiming for the certainty of mathematical knowledge.

Thus, there are two statements that one should be able to prove in order to safely claim that mathematical knowledge is certain: (α) axioms are consistent; (β) deduction is truth-preserving. Indeed, a deductive proof "yields categorical knowledge [i.e. knowledge which is independent of any particular assumptions] only if it proceeds from a secure starting point and if the rules of inference are truth-preserving" (Baker 2016).

Now, while whether it is possible to deductively prove (β) is at least controversial (see e.g. Haack 1976; Cellucci 2006), it is almost uncontroversial that it is generally impossible to mathematically prove (α), i.e. that axioms are consistent, because of Gödel's results.[6] By Gödel's second incompleteness theorem, for any consistent, sufficiently strong deductive theory *T*, the sentence expressing the consistency of *T* is undemonstrable in *T*. Nevertheless, despite Gödel's incompleteness theorems seem to refute the view that the method of mathematics is the axiomatic method, this view is still in fact the most widespread view among mathematicians (see Cellucci

[5] Cf. Prawitz 2014, p. 78: "mathematics, after its deductive turn in ancient Greek, is essentially a deductive science, which is to say that it is by deductive proofs that mathematical knowledge is obtained."

[6] Cf. Baker 2016, Sect. 2.3: "Although these results apply only to mathematical theories strong enough to embed arithmetic, the centrality of the natural numbers (and their extensions into the rationals, reals, complexes, etc.) as a focus of mathematical activity means that the implications are widespread."

Philosophia, DOI: 10.1007/s11406-018-9953-1.

2017, Sect. 20.12). Usually, those authors that despite these results maintain that mathematical knowledge is certain, make reference to a sort of faculty that we are supposed to possess, a faculty that would allow us to 'see' that axioms are consistent. For example, Brown states that we "can intuit mathematical objects and grasp mathematical truths. Mathematical entities can be 'perceived' or 'grasped' with the mind's eye" (Brown 2012, p. 45).

This view has been advocated by many great mathematicians and philosophers. Detlefsen describes the two main claims of this view as follows: (1) "mathematicians are commonly convinced that their reasoning is part of a process of discovery, and not mere invention;" (2) "mathematical entities exist in a noetic realm to which the human mind has access" (Detlefsen 2011, p. 73). With respect to the ability of grasping mathematical truths, i.e. accessing the mathematical realm, this view traditionally assumes "a type of apprehension, *noēsis*, which is characterized by its distinctly 'intellectual' nature. This has generally been contrasted to forms of *aisthēsis*, which is broadly sensuous or 'experiential' cognition […]." (Ibidem, p. 73). For example, Gödel states that "despite their remoteness from sense experience we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true" (Gödel 1947: 1990, p. 268).

The problem is that this view is commonly supported by authors that are anti-naturalists.[7] More precisely, many of them are explicitly anti-Darwinist, in the sense that they overtly deny that our intellectual ability to grasp mathematics can be made compatible with the claim that all our cognitive abilities have evolutionary roots. For example, Gödel claims that mathematical intuition is a superior faculty which is not "derived from subconscious induction or Darwinian adaptation" (Gödel 1953: 1995, p. 354). Hence, those anti-naturalist authors do not take care of articulating a scientifically plausible account of how such 'intuition' may work or may have evolved.

Obviously, adopting the same attitude is less easy for the naturalists. Evolution is central to naturalism. For example, Giere states that if "evolutionary naturalism is

[7] On the anti-naturalism of many supporters of this view, cf. e.g. Gödel 1947: 1990, p. 323: "There exists [...] an entire world consisting of the totality of mathematical truths, which is accessible to us only through our intelligence, just as there exists the world of physical realities; each one is independent of us, both of them divinely created."

understood to be a general naturalism informed by the facts of evolution and by evolutionary theory, then no responsible contemporary naturalist could fail to be an evolutionary naturalist in this modest sense" (Giere 2006, p. 53). As modest this commitment can be, if one commits oneself to naturalism, one would find difficult to claim that some cognitive ability cannot be explained in the light of evolutionary theory. The following question then arises: Is it possible to naturalize the human ability to grasp mathematics and logic, and keep maintaining the traditional view of mathematical knowledge, i.e. that mathematical knowledge is certain, and the method of mathematics is the axiomatic method? In other words, how can we account for the reliability of mathematics and logic, if we accept the idea that they are both produced by humans and humans are evolved organisms?

There is no clear answer to this question. Some authors tried in the last decade to naturalize mathematics and logic by relying on Darwinism (see e.g. De Cruz 2006, 2004; Krebs 2011; Núñez 2009; Schechter 2013; Smith 2012; Woleński 2012; Ye 2011).[8] The main difficulties afflicting those approaches derive from the fact that they try both to (1) naturalize mathematics through Darwinism and (2) avoid the risk of being excessively revisionary on what we take mathematical knowledge to be.[9] In other words, they try to show that mathematics rests on some evolved cognitive abilities, and that this evolutionary ground confers a degree of epistemic justification to how we actually do mathematics which is able to secure our traditional convictions on what mathematical knowledge is. The fact is that it is not easy to defend the claim that evolution can provide the degree of justification needed to maintain the traditional view of mathematics as the paradigm of certain knowledge. Briefly, in order to claim that natural selection gave us the ability at attaining the truth with regard to mathematics, we should demonstrate that natural selection is an aimed-at-truth process in the first place. For example, Wilkins and Griffiths state that to "defeat evolutionary skepticism, true belief must be linked to evolutionary success

[8] Those interesting proposals cannot be individually discussed here for reason of space.

[9] Cf. Paseau 2013, Sect. 4.1: "Scientific naturalism about mathematics proper is thus a philosophically revolutionary view, since it advocates a different set of standards with which to judge mathematics […] from the traditional ones […]. It is also potentially revolutionary about mathematics itself, as it might lead to a revision of mathematics […]. Having said that, recent scientific naturalists have tended to be mathematically conservative in temperament and have advocated little or no revision of mathematics."

in such a way that selection will favour organisms which have true beliefs" (Wilkins, Griffiths 2013, p. 134). The problem is exactly *how* to justify such a link, and the issue is at least very controversial (see e.g. Vlerick, Broadbent 2015; Sage 2004).

Consider, for instance, our confidence in the fact that deduction is truth-preserving, while non-deductive inference rules are not. Since there is no non-circular justification of the validity of deductive inferences rules (Cellucci 2006), nor there is an uncontroversial justification of the claim that circular justifications are acceptable, non-deductive rules and deductive rules seem to be on a par with respect to the issue of the formal justification of their validity (see also Carroll 1895; Haack 1976). Thus, what we take to be the distinctive feature of deductive rules, i.e. truth-preservation, has to be grounded in some different way. According to many authors, the justification of the claim that deduction is truth-preserving is grounded in our intuition. For example, Kyburg states that "our justification of deductive rules must ultimately rest, in part, on an element of deductive intuition: we see that M[odus] P[onens] is truth-preserving − this is simply the same as to reflect on it and fail to see how it can lead us astray" (Kyburg 1965, p. 276). The problem is that our failing in conceiving an alternative to some issue we reflect on could justify the reliability of the deductive rules only if our ability in conceiving alternatives to some issue we reflect on could be shown to be able to reliably exhaust the space of all the possible alternatives to such issue. The fact is that there is not such a demonstration of the reliability of our ability in conceiving alternatives, so that in order to ground our confidence in such ability we can only rely on our 'intuition'. So, in the ultimate analysis, our confidence in the truth-preservation of deductions relevantly rests on (1) our failing in finding any counterexample able to convince ourselves that MP can lead us astray, and on (2) the fact that the 'intuition' that no possible alternative has been overlooked in the search for a counterexample appears self-evident to us. But the fact that some statements appear to us as self-evidently true it is not by itself a guarantee of their truth, if our ability in evaluating the self-evident truth of a statement is an evolved capacity. Our 'sense' of the self-evident may be not only oriented towards contingent connections which were useful in the past and that do not reflect necessary and eternal truths, but given that we are not able to demonstrate that only true beliefs can permit us to successfully deal with the world,[10] we cannot even eliminate the possibility that an ability in perceiving as self-evident some falsities has been selected because perceiving such falsities as self-evident truths was adaptively useful (Nozick 2001; see also Vaidya 2016).

It is worth noting that, if we want to maintain that mathematical knowledge is certain, and we want to naturalize mathematical knowledge, the evolved cognitive ability to grasp whether axioms are consistent cannot be supposed to be fallible. Indeed, if this faculty is fallible, and we are not able to correctly determine whether axioms are true in all the cases we examine, then we will generally be unable to claim that our mathematical knowledge is certain in any particular case. Indeed, as we have seen, a mathematical result is true and certain if the axioms from which it is derived are true (at least in the sense of 'being consistent'), and deduction is truth-preserving. If naturalizing mathematics implies that our evolved ability in assessing the truth of the axioms is fallible, and we have no other way to verify our verdict, we find ourselves in a situation in which we may have erred in assessing the consistency of axioms, and we are unable to detect whether or not we made an error. Thus, we would never be able to safely claim that we judged correctly, and so that our mathematical knowledge is actually certain. So, if the justification of our mathematical knowledge rests on some fallible faculty, then the attempt to naturalize mathematics cannot maintain the traditional view of mathematics.

Contrary to this perspective, McEvoy (2004) argues for the compatibility of reliabilism and mathematical realism. According to him, our mathematical intuition may be at the same time an *a priori*, reliable, and fallible faculty of reason. In a similar vein, Brown (2012) maintains that platonism and fallibilism can be combined. But, even if we concede that fallibilism in epistemology is compatible with platonism in ontology, this view seems not compatible with a *naturalist* stance, since it is not able to give a naturalist account of *how* we "can intuit mathematical objects and grasp mathematical truths", given that in this perspective mathematics is *a priori*, and mathematical truths are necessary truths. This view has to face the same difficulties discussed above with regard to the justification of the claim that

[10] Cf. e.g. McKay, Dennett 2009, p. 507: "In many cases [...], beliefs will be adaptive by virtue of their veridicality. The adaptiveness of such beliefs is not independent of their truth or falsity. On the other hand, the adaptiveness [...] of some beliefs is quite independent of their truth or falsity."

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deduction is truth-preserving: when evolution enters the picture, it is not easy to justify the claim that we are able to correctly assess what is possible or impossible through reasoning alone. This impinges on the possibility of claiming that our mathematical beliefs *are* certainly true because they *cannot* be otherwise. So, any kind of evolutionary reliabilism seems not really able to provide a naturalist way of supporting the traditional view of mathematics, since it is not able to secure the certainty of our knowledge (Sage 2004). That a belief is reliably produced may be insufficient for conferring to mathematics the degree of certainty that many platonists are looking for. If instead a platonist accepts the idea that mathematical knowledge can be fallible, i.e. she claims that, although mathematics is an *a priori* discipline, mathematical knowledge need not be certain (see e.g. Brown 2012), and so she rejects (at least to some extent) the traditional view of mathematics, she has now to face the problem of justifying her own view of mathematics: If mathematical knowledge is fallible, how can the platonist justify the claim that the platonist view of mathematics is the *true* one? If mathematical knowledge is fallible, what we think about mathematics may be incorrect or even false. If the adoption of a platonist attitude depends on (or it is part of) our mathematical knowledge, platonism itself may be incorrect or even false. Many platonists would be unwilling to accept this claim, so rejecting the traditional view of mathematics is not an easy way for the platonist to take.

4. Mathematics and Naturalism

With regard to the issue of how to understand the term 'naturalism' in the context we are dealing with, we will not be concerned here with any specific view of naturalism, nor we will survey the many criticisms that have been so far moved to this (family of) view(s) (for a survey on naturalism, see Clark 2016; Papineau 2016; see Paseau 2013 for a survey on naturalism in the philosophy of mathematics). For the purpose of this article, 'naturalism' can just be understood as the claim that we should refute accounts or explanations that appeal to non-natural entities, faculties or events, where 'non-natural' has to be understood as indicating that those entities, faculties or events cannot *in principle* be investigated, tested and accounted for in the way we usually do in science.[11] In other words, non-natural entities, faculties or events are those that are characterized and defined precisely by their *inaccessibility*, by the impossibility of being assessed, empirically confirmed, or even made compatible with what we consider genuine knowledge in the same or in some close domain of investigation. In all those cases, we have to face a problem of accessibility[12] and a claim of exceptionality that usually lacks sufficiently strong reasons to be conceded.

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[11] Cf. e.g. Lacey 2005, p. 604: "[Naturalism is] the view that everything is natural, i.e., that everything there is belongs to the world of nature, and so can be studied by the methods appropriate for studying that world."

[12] The access problem, first raised in the philosophy of mathematics by Benacerraf (1973), is now thought to arise in many other domains. It is the problem of justifying the claim that our D-beliefs align with the D-truths of a given domain D, if D is regarded as an *a priori* domain, i.e. a domain whose objects cannot *in principle* be empirically investigated (see Clarke-Doane 2016). Benacerraf's epistemological challenge to mathematical platonism has been criticized because it assumes the causal theory of knowledge, which nowadays is discredited among epistemologists. But Benacerraf's argument may be raised against platonism without assuming the causal theory of knowledge, as Field (1989) maintains. On this issue, cf. Baron 2015, p. 152: "Field's version of the access problem focuses on mathematicians' mathematical beliefs. The mathematical propositions that mathematicians believe tend to be true. If platonism is correct, however, then these propositions are about mathematical objects. So, the mathematical beliefs held by mathematicians [...] are reliably correlated with facts about such objects. The challenge facing the platonist, then, is to provide an account of this reliable correlation." It may be objected that this formulation implicitly assumes a sort of correspondence view of truth, and that this view of truth has not to be necessarily held by platonists. But, even if accepting the correspondence view of truth is not strictly mandatory for a realist, the correspondence view is in fact the view of truth usually adopted by realists of all stripes. And according to many authors, the "major argument for mathematical realism appeals to a desire for a uniform semantics for all discourse: mathematical and non-mathematical alike [...]. Mathematical realism, of course, meets this challenge easily, since it explains the truth of mathematical statements in exactly the same way as in other domains" (Colyvan 2015, Sect. 5), i.e. by assuming that there is a correspondence between the realm of mathematical objects and our mathematical knowledge. So, if platonists try to avoid Benacerraf's challenge by rejecting the correspondence view of truth, they risk dismissing one of the most convincing reasons for adopting platonism in the first place.

Although such a characterization of naturalism is quite broad, it nevertheless retains the idea that every naturalist view requires both (1) an *ontological* and (2) an *epistemological* commitment. This means that, in order to naturalize a domain D, it is insufficient to merely specify what kind of entities we can admit in our ontology of D. We have also to provide a naturalistic (i.e. a scientific adequate and reliable) account of *how* we can acquire knowledge of those D-entities.

This point is relevant also for those attempts aimed at naturalizing mathematics by relying on Darwinism. To see this, it is important to distinguish (a_1) the fact that a subject S has some D-beliefs about a domain D, and (a_2) the ability to deal with the world that those D-beliefs confer to S, from (b_1) the beliefs that S has about the nature of such D-beliefs, and (b_2) the beliefs that S has about the reasons why those D-beliefs give her such an ability to deal with the world.

In this regard, consider sight, i.e. the ability to see. Sight gives us an ability to deal with the world (a_2) and allows us to form beliefs related to what we see and about the possibility of seeing that may well be regarded as reliable to some extent (a_1). Nevertheless, for many millennia humans have known very little about how sight was possible (b_1), and many ideas we humans have proposed to explain this phenomenon proved untenable in the light of successive scientific inquiry (b_2). We can conceive of our ability to see as evolutionarily rooted. Thus, natural selection may well have equipped us with the ability to form reliable (at least to some extent) beliefs through our innate ability to see, and these beliefs proved very useful for dealing with the world. Nevertheless, we would not draw the conclusion that our innate ability to see allowed us to form reliable beliefs about how it is that we can see, or about how human sight works. On the contrary, it is starting from our current scientific knowledge that we can assess whether (and to what extent) our innate ability to see allows us to form reliable beliefs.

Consider now mathematics. Mathematicians are supposed to have highly justified beliefs about mathematics (a_1), and mathematics certainly helps us to deal with the world (a_2). But this does not mean that the beliefs that philosophers of mathematics or mathematicians have about what mathematics is (b_1), or about the reasons why mathematics proves so helpful to deal with the world (b_2), are justified or reliable. In other words, even if natural selection gave us the ability to produce some reliable D-beliefs in some domain D, because these beliefs were useful to deal with our ancestors' environment, this does not mean that any belief we may produce about D

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or about how D-beliefs are produced is reliable or justified. So, even if mathematics is useful to deal with the world and mathematicians are reliable when doing mathematics, this may be insufficient to take for granted what mathematicians think about what mathematics is and the way we acquire mathematical knowledge. In a naturalist perspective, such claims should be supported by adequate scientific findings, or at least be compatible with our scientific knowledge on similar issues (e.g. what we know about how we acquire different kinds of knowledge).

This is the reason why naturalists should resist those attempts aimed at 'platonizing naturalism' (see e.g. Linsky, Zalta 1995), which are mainly devoted to defending the claim that an ontology which comprehends abstract objects is not incompatible with a science-oriented world-view. Those accounts support what Brown (2012) calls *semi-naturalism*, in the sense that they aim at supporting a platonist ontology in some domain D, while they reject the classical platonist account of how we come to know D-entities. For example, in the case of mathematics, Linsky and Zalta (1995) wish both to (1) maintain a platonist perspective on mathematical objects, according to which mathematical entities are abstract objects (i.e. non-spatiotemporally located), and (2) reject the platonist view according to which we come to know about such abstract objects through a sort of intuition either of those objects, or of the truth of the axioms from which we can derive them.

According to Linsky and Zalta, we have not to conceive of abstract objects on the model of physical objects. Indeed, unlike "ordinary objects, for which reference proceeds by some combination of causal processes, referential intentions, and [...] descriptive properties, reference to abstract objects is ultimately based on descriptions alone" (Linsky, Zalta 1995, p. 546). In this view, mathematical objects are described through a comprehension principle for individual abstract objects.[13] This principle says that for "every condition on properties, there is an abstract individual that encodes exactly the properties satisfying the condition" (Ibidem, p. 536). This means that "if a mathematical entity is logically possible, then it is actual" (Brown 2012, p. 122). In this way, every consistent mathematical structure can be said to exist, and no epistemic contact with any mathematical object is needed in order to say that it exists. In this view, "our cognitive faculty for acquiring

[13] Comprehension principles are general existence claims stating which conditions specify an object of a certain sort.

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knowledge of abstracta is simply the one we use to understand the comprehension principle" (Linsky, Zalta 1995, p. 547).

The main problem with this perspective, as it has been correctly pointed out by Brown (2012), is that it does not provide any naturalist account of *how* we come to know the comprehension principle.[14] Moreover, Linsky's and Zalta's idea that there are as many abstract objects as there could possibly be seems to commit them to possible-worlds modal realism. For example, they claim that every "consistent mathematical theory describes an abstract mathematical realm that, however bizarre or convoluted, might be needed to characterize some portion of the physical reality of some metaphysically possible world" (Linsky, Zalta 1995, p. 550). Thus, the argument goes, the existence of abstract objects can be derived by their indispensability in constructing possible worlds. Since platonism supports realism with respect to mathematics, if Linsky and Zalta derive the existence of abstract objects by relying on their indispensability in constructing possible worlds, it is fair to suppose that Linsky and Zalta embrace some sort of possible-worlds modal realism. Adopting a realist attitude on possible worlds and modality amounts to claim that we are able to know what is necessary and what is possible. Obviously, if one adopts such a stance, one immediately has to face the problem of justifying the claim that one knows what is necessary and what is possible, i.e. one has to explain *how* one comes to know what is necessary and what is possible.

Now, possible-worlds modal realism is usually deemed to be incompatible with a naturalist stance. The reason for this incompatibility is analogous to the one seen above with regard to the incompatibility between platonism and naturalism, i.e. possible-worlds modal realism implies the existence of *empirically inaccessible* domains. If some worlds are empirically inaccessible, and nevertheless we do know what is necessary or possible in such worlds, and we are realist about the existence of such worlds, this means that we can have *knowledge* of some inaccessible domain D which is independent from any empirical confirmation of our D-beliefs. This knowledge is a sort of *a priori* knowledge, in the broad sense that it is knowledge

[14] Cf. Linsky, Zalta 1995, p. 547: "The comprehension principle […] is known a priori. The reason is that it is not subject to confirmation or refutation on the basis of empirical evidence." But many naturalists are unwilling to concede that a priori knowledge is possible (see Devitt 1998).

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reached by virtue of reasoning alone.[15] Indeed, modal realists claim that, for every way the world could be, there is a world that is that way (Lewis 1986). This means to assume that if something is impossible in our world, but it is conceivable, it is true in some other possible world *causally isolated* from ours. So, the adoption of possible-worlds modal realism amounts to assuming that there is something "like a realm of metaphysical possibility and necessity that outstrips the possibility and necessity that science deals with, but this is exactly what naturalists should not be willing to concede" (Morganti 2016, p. 87).[16]

It is important to stress that possible-worlds modal realism rests on an analogy between modal knowledge and mathematical knowledge developed by Lewis (1986): "the key idea is that we have mathematical knowledge by drawing (truth-preserving) consequences from (true) mathematical principles. And we have modal knowledge by drawing (truth-preserving) consequences from (true) modal principles" (Bueno, Shalkowski 2004, p. 97). This means that possible-worlds modal realism rests on the traditional view of mathematics, according to which axioms are known to be true, and mathematical knowledge is ampliated by deducing theorems from those axioms. To the extent that Linsky's and Zalta's attempt can be regarded as an attempt aimed at securing the traditional view of mathematical knowledge by grounding it on possible-worlds modal realism, if possible-worlds modal realism rests in its turn on the traditional view of mathematical knowledge, Linsky's and Zalta's move displays a sort of circularity. But what is more interesting, is that Lewis' analogy between modality and mathematics can be developed in exactly the opposite direction, if one wishes to adopt a naturalist stance. Let's unpack this claim a bit. Mathematics is regarded by some authors as able to provide support for anti-

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[15] On the realist attitude on coexisting parallel worlds that possible-worlds modal realism implies, cf. e.g. Norris 2000, p. 109: "Lewis himself arrives at this conclusion by way of modal logic and the argument that necessary truths are those that hold good across all possible worlds rather than obtaining only in a certain limited subset of worlds which happen to resemble our own in respect of various contingent features. In this form the theory goes back to Leibniz and involves the essentially rationalist belief that thinking can indeed deliver such real-world applicable truths through a priori reflection on the scope and limits of human knowledge in general."

[16] For a survey of the problems afflicting possible-worlds modal realism, see Vaidya 2016, Bueno, Shalkowski 2004.

naturalism. Indeed, mathematical knowledge is usually regarded as an instance of genuine knowledge despite we do not possess any naturalist account of mathematics. So, the argument goes, if we do not possess any naturalist account of how we form beliefs about a given domain D, our knowledge of D may well be regarded as an instance of genuine *a priori* knowledge in the same way mathematics is regarded as an instance of genuine *a priori* knowledge. This means that the burden of proof is on the naturalist. D can be safely regarded as an *a priori* domain and our knowledge of D can be safely regarded as an instance of genuine *a priori* knowledge at least until an adequate account of how we form D-beliefs will be provided by the naturalists.

But this view rests on the simplistic idea that mathematical statements can *really* be proved to be true by reasoning alone. And so that also D-statements can be regarded as true, if D is an *a priori* domain in the same sense in which mathematics is an *a priori* domain. In fact, things are more complicated. According to Bueno and Shalkowski (2004), for instance, as in mathematics, due to Gödel's results, we are generally unable to prove with certainty that the axioms of the theory we are dealing with are true, and thus that the theorems that we derive from such axioms are actually true, when dealing with modality our modal knowledge may be of the same kind, i.e. knowledge whose truth depends on whether the metaphysical assumptions from which we start are true, but we are unable to prove whether such assumptions are actually true. Indeed, when dealing with non-actual cases, the possibility of determining whether something is possible or not will depend on controversial assumptions. There are several incompatible and competing assumptions available to be taken as the starting point from which we derive our target conclusions on what is possible, and there is not a way of proving that such 'first assumptions' are in their turn 'true' without ending in an infinite regress or committing a *petitio principii*.[17]

[17] Cf. Bueno, Shalkowski 2004, p. 97-98: "If the analogy with mathematics is taken seriously, it may actually provide a reason to *doubt* that we have any knowledge of modality. One of the main challenges for platonism about mathematics comes from the epistemological front, given that we have no access to mathematical entities – and so it's difficult to explain the reliability of our mathematical beliefs. The same difficulty emerges for modal realism, of course. After all, […] we have *no access* to […] [possible worlds]. Reasons to be skeptical about a priori knowledge regarding mathematics can be easily 'transferred' to the modal case, in the sense that difficulties we may have to

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It seems fair to conclude that Linsky's and Zalta's view is mainly concerned with the idea of facing the challenge raised by Benacerraf with regard to the access problem for some kind of objects, but it completely misses the other requirement that is implied by any naturalist perspective, i.e. that it should be possible (at least in principle) to account in naturalistic terms for the means by which we develop our scientific knowledge. Thus, naturalists should resist those attempts aimed at platonizing naturalism, because they are not really compatible with a naturalist stance.

5. Mathematics and Darwinism

From the previous sections, it should be clear that in this article we are exclusively concerned with those strategies aimed at naturalizing a given domain D, which has traditionally been regarded as affected by an 'access problem' (e.g. mathematics, morality, modality, etc.), by providing a plausible evolutionary account of some cognitive abilities that would make our knowledge of some aspects of some D-objects a natural fact. As an example, Timothy Williamson's approach to modality can be regarded as a way to naturalize modality, by firstly reducing the problem of explaining our modal knowledge to the problem of explaining our capacity to correctly perform counterfactual reasoning, and then by giving some reasons to think that an evolutionary account of the emergence of this capacity may be plausible (Williamson 2000).[18] This example might give rise to some misunderstanding, since

establish a given mathematical statement may have a counterpart in establishing certain modal claims. For example, how can we know that a mathematical theory, say ZFC, *is consistent*? Well, we can't know that in general; we have, at best, relative consistency proofs. And the consistency of the set theories in which such proofs are carried out is far more controversial than the consistency of ZCF itself, given that such theories need to postulate the existence of inaccessible cardinals and other objects of this sort."

[18] Cf. also Kitcher 1988, fn. 10, p. 322-323: "it seems to me to be possible that the roots of primitive mathematical knowledge may lie so deep in prehistory that our first mathematical knowledge may be coeval with our first prepositional knowledge of any kind. Thus, as we envision the evolution of human thought (or of hominid thought, or of primate thought) from a state in which there is no prepositional knowledge to a state in

Williamson justifies our ability to deal with metaphysical modality by relying on a sort of evolutionary argument, i.e. an argument which justifies some kind of beliefs by appealing to their evolutionary roots. Above such kind of arguments has been defined 'very controversial' (Sect. 3). In Williamson's view, our evolved ability to perform counterfactual reasoning justifies our reliability to deal with metaphysical modality, because if an ability has been selected for by natural selection, this means that it is reliable in tracking some aspect of the world. It is precisely this inference from 'having been selected for' to 'being reliable in tracking some truths' that is very controversial. But it is not because of this aspect that Williamson's move may be suitable for the naturalists. Rather, it is the general structure of Williamson's strategy that can be of interest if one tries to naturalize a given domain. As already said, Williamson's strategy consists in reducing a controversial domain C, for which we do not possess any naturalist explanation of our ability to deal with C-objects, to a more familiar domain F, for which a plausible evolutionary explanation *f* of our ability to deal with F-objects is available. In this way, by means of *f* we can now explain our ability to deal with C-objects in naturalistic terms. It is important to stress that this strategy is neutral with respect to the issue of whether or not evolved abilities are truth-tracking. In order to explain some aspect of domain C, *f* need not be true, since an ability may have been selected for despite it does not track any truth. So, Williamson's move does not imply, by itself, a commitment to a realist or an antirealist perspective on a given domain. As well as Williamson supports a non-sceptical attitude toward metaphysical modality, someone else can support modal scepticism by performing a similar reasoning. One can indeed firstly show that our ability to deal with modality can be reduced to our ability to perform counterfactual reasoning. Then, by noting that counterfactual reasoning is an evolved ability, and that natural selection does not guarantee the reliability of evolved abilities in tracking truths, one can conclude that we should be skeptical about the reliability of our ability to deal with metaphysical modality.[19]

Turning to the issue of the naturalization of mathematics, a naturalistic account of mathematics has to assume the hypothesis that the human mathematician is "a

which some of our ancestors know some propositions, elements of mathematical knowledge may emerge with the first elements of the system of representation." For several criticisms of Kitcher's mathematical naturalism, see Brown 2012.

[19] I wish to thank an anonymous reviewer for urging me to clarify this point.

thoroughly natural being situated in the physical universe," and that therefore "any faculty that the knower has and can invoke in pursuit of knowledge must involve only natural processes amenable to ordinary scientific scrutiny" (Shapiro 1997, p. 110). Is this assumption compatible with the traditional view of mathematics?

Recently, Helen De Cruz argued that an evolutionary account of mathematics may well be compatible with a realist view of mathematics. According to her, "animals make representations of magnitude in the way they do because they are tracking structural (or other realist) properties of numbers" (De Cruz 2016, p. 7). In this view, "realism about numbers could be true, given what we know about evolved numerical cognition" (Ibidem, p. 2). Indeed, "it seems plausible that numerical cognition has an evolved, adaptive function," and it has been demonstrated that numerical cognition "plays a critical role in our ability to engage in formal arithmetic" (Ibidem, p. 4). According to De Cruz, "the brain has as proper function the production of beliefs that are fitness-enhancing," and it is possible to develop an evolutionary argument, according to which "natural selection will form animal brains that tend to produce true beliefs, because true beliefs are essential for adaptive decision making" (De Cruz, De Smedt 2012, p. 416-417). With regard to mathematics, De Cruz states that since mathematics is the product of evolution by natural selection, it "must somehow have promoted the survival and reproductive success of the ancestors of those organisms" (De Cruz 2004, p. 80). If the beliefs produced by humans need to be true in order to be fitness-enhancing, and mathematics is produced by humans because it has been fitness-enhancing, we can conclude that in this line of reasoning mathematical beliefs are true. This means that mathematical beliefs need to be derivable from axioms which are true, at least in the sense that they are consistent.

The main problem with this view is that if one tries to (1) naturalize mathematics and (2) maintain the traditional view, i.e. the view according to which (a) the method of mathematics is the axiomatic method and (b) mathematical knowledge is certain, then our naturalized account of mathematics risks being incompatible with Gödel's results. Indeed, in the traditional view, as we have already noted, in order to justify mathematical knowledge, at least two requirements have to be fulfilled: (α) axioms have to be consistent, and (β) deduction has to be truth-preserving.

We have already mentioned (Sect. 3) the difficulties that arise when one tries to justify the claim that deduction is truth-preserving (β), if one takes a naturalist stance.

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Let’s now focus on the first requirement (α), i.e. axioms have to be consistent. If we maintain that evolution is able to justify the traditional view of mathematics, this amounts to claim that evolution, in some way, gave us an ability to know with certainty whether a set of axioms are true, at least in the sense that they are consistent. Let’s name T the ‘result’ that we can obtain thanks to such evolved ability. Consider, for example, that T expresses the following content: ‘the axioms of the axiomatic system we are considering are consistent’.

The problem is that, by Gödel’s second incompleteness theorem, it is impossible to demonstrate in any sufficiently powerful axiomatic system that the axioms of such system are consistent. Let’s name this result G.

Now, if the method of mathematics is really the axiomatic method, how could we accept that T holds? Or, to put it slightly differently, should we consider T be part of our mathematical knowledge?

If T is part of our mathematical knowledge, then the axiomatic method is not really the unique method of mathematics, since a crucial result as T is not obtained by this method, and so the method by which T has been obtained should be added to the list of the legitimate methods of mathematics. This would render G almost irrelevant. Indeed, if the axiomatic method is not the only acceptable method in mathematics, and we can know that a set of axioms is consistent thanks to some evolved faculty, then that in some axiomatic systems we cannot prove whether or not a set of axioms is consistent is irrelevant to us. We could safely take as established that those axioms are consistent by our evolved faculty and go on.

But the majority of mathematicians, even of platonist mathematicians, would be unwilling to regard Gödel’s contributions as irrelevant, and the consistency of axioms establishable by merely relying on an evolved sort of intuition. Precisely because they *do* believe the axiomatic method to be the method of mathematics, they tend to confer a higher degree of certainty to Gödel’s results, which have been established according to such method, than to the intuitions of an evolved faculty, the reliability of which cannot be proved by the same method. Indeed, in a naturalist framework, our evolved intuitions can be shown to be reliable only through some inductive method, which is peculiar of natural science. If we concede that the method of mathematics is distinct from the method of natural science, as the traditional view holds, and that the method of mathematics is the axiomatic method, then we will be unable to sufficiently justify the belief that our evolved intuition is reliable up to a

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degree which is comparable with the confidence that the axiomatic method is supposed to confer to mathematical results. Thus, even if our evolved intuition were in fact reliable and infallible, we would be unable to scientifically demonstrate its infallibility with the same degree of certainty with which Gödel's results can be proven, given that they are *mathematical* results.

If, on the other hand, we take T *not* to be part of our mathematical knowledge, and protest that T is not really a 'mathematical result', we nevertheless find ourselves in an uncomfortable position: we should maintain that we possess some knowledge about some mathematical issue, and that this knowledge is not part of our mathematical knowledge. It is not easy to accommodate this claim by the usual epistemological standards. Since knowledge requires (at least) truth and justification, if we take T to be knowledge, T is true and justified. If T expresses something true about some mathematical issue, then we can affirm that T expresses a mathematical truth. But if we refuse to regard T as a part of our mathematical knowledge, and we are not able to express the same mathematical truth that T expresses by the means of what we take to constitute our current mathematical knowledge, then T would be able to express a mathematical truth that cannot be derived in our mathematical knowledge.

It may be objected that T is a mathematical truth that cannot be derived on the basis of our mathematical knowledge, because the justification requirement that a true belief needs to fulfill in order to become *mathematical* knowledge is stricter than the justification requirement that has to be fulfilled in other domains. Let's concede, for argument's sake, such claim on the justification requirement for mathematics.[20] If this is the case, T could well able to express a truth about some mathematical issue, but this truth may nevertheless be insufficiently justified in order to become part of our mathematical knowledge. And this would explain the fact that T is able to express a truth on a mathematical issue, and that this truth does not figure among our known mathematical truths. But this would amount to saying that our mathematical knowledge is a kind of knowledge with a higher degree of certainty than T, since the

[20] Cf. e.g. Kitcher 1988, p. 297: "The obvious way to distinguish mathematical knowledge from mere true belief is to suggest that a person only knows a mathematical statement when that person has evidence for the truth of the statement—typically, though not invariably, what mathematicians count as a proof."

true beliefs that constitute our mathematical knowledge are supposed to display a higher degree of justification than T.

But if we try to naturalize mathematics in the way here we are dealing with, things should go the other way around. Since T, i.e. the claim that the axioms of the theory we are dealing with are consistent, in order to be able to justify the traditional view of mathematics has to be certain, T has to be knowledge with the *highest* degree of certainty. Thus, the degree of certainty that our mathematical knowledge may display is in some sense subordinated to the degree of certainty that T displays, since the certainty of our mathematical knowledge is dependent on the certainty of T. Mathematical knowledge would be in this way a kind of knowledge with a *less* high degree of certainty than T. Thus, it cannot be the case that T is a mathematical truth which is not an instance of mathematical knowledge because it is insufficiently justified. So, this objection is inadequate.

In both the cases we analyzed, i.e. either we take T to be part of our mathematical knowledge or not, we end with implausible scenarios. So, the supporters of the traditional view seem unable to really find an adequate way to justify the claim that axioms are consistent with each other in naturalistic terms. Since, as we tried to show, if one adopts a naturalist stance, both the requirements that should be fulfilled in order to maintain the traditional view of mathematics (i.e.: to show that axioms are consistent, and deduction is truth-preserving) cannot be fulfilled, it seems fair to conclude that if one adopts a naturalist stance, one cannot maintain the traditional view of mathematics.

6. A Brief Digression on Consistency and Ampliativity

Throughout this article we accepted as undisputed that there are two statements that one should be able to prove in order to safely claim that mathematical knowledge is certain, and so maintain the traditional view of mathematical knowledge, namely: (α) axioms are consistent; and (β) deduction is truth-preserving. But, in fact, the idea that these two statements adequately describe the necessary requirements to account for mathematical knowledge has been disputed (see e.g. Cellucci 2017). Indeed, both (α) and (β) are inadequate to account for mathematical knowledge. Although what follows does not impinge on the argumentation we developed above, which assumes,

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for the sake of the argument, that (α) and (β) are adequate requirements to account for mathematical knowledge, in this section we will sketch some of the main problems that afflict (α) and (β), because we think that the naturalists should keep in mind these concerns on (α) and (β) when developing their view on mathematical knowledge.

As regard to (α), the problem is that consistency is not a sufficient condition for truth. So, even if it were possible to prove the consistency of the axioms of a given theory, as it is required by the traditional view, this would not be sufficient to justify the traditional claim that such axioms are true. For example, Hilbert states that, "if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true" (Hilbert 1980, p. 39). But, as Cellucci (2017) clearly points out, the concept of truth as consistency is inadequate, because by a corollary of Gödel's first incompleteness theorem, for any consistent, sufficiently strong, formal system S, there is a consistent extension T of S in which some false sentence is demonstrable. From the corollary it follows that the "axioms of T, though consistent, cannot be said to be true – if they could be said to be true, only true sentences would be demonstrable in T" (Cellucci 2017, p. 103). Thus, contrary to Hilbert's claim, consistency is not a sufficient condition for truth.

As regard to (β), according to the traditional view mathematical knowledge is acquired by deductive proofs from previously acquired mathematical truths, and so on. This gives raise to two main problems: (β_1) the problem of accounting for how we acquired the initial body of mathematical truths from which mathematics originated; (β_2) the problem of accounting for how mathematical knowledge can be ampliated, given that in some cases the advancement of mathematics cannot be reduced to the derivation of consequences from already established mathematics.

As regard to (β_1), even if ultimately all mathematical knowledge were acquired by deductive proofs from mathematical truths, there would remain "the challenge to explain in what way the ultimate starting points for mathematical proofs are obvious truths" (Prawitz 2014, p. 90).[21] According to Prawitz, explaining this is still "an open question" (Ibidem). And certainly, it is not an easy issue to deal with. As noted

[21] Cf. Kitcher 1988, p. 299: "Our present body of mathematical beliefs is justified in virtue of its relation to a prior body of beliefs; that prior body of beliefs is justified in virtue of its relation to a yet earlier corpus; and so it goes. Somewhere, of course, the chain must be grounded."

above, postulating the consistency of the axioms would not solve the problem, given that consistency is insufficient as a condition for truth. Nor resorting to some evolutionary account of some cognitive capacity to grasp fundamental mathematical truths would solve the problem, because, as we noted, if one wishes to support the traditional view, natural selection cannot guarantee the degree of certainty required by such view.

As regard the difficulty of reaching a consensus on what are the 'obvious' truths (where 'obvious' can be understood as 'easy to agree on' and 'undisputed') which we should regard as the ultimate starting points for mathematical proofs, consider set theory. Even if one concedes, for argument's sake, that the axioms of Zermelo-Fraenkel set theory, ZF, are generally accepted and undisputed, and so can be regarded as 'obvious truths' from which mathematical proofs are derived, those axioms proved insufficient for deriving many relevant mathematical results, as for instance, that every vector space has a basis in algebra. So, some extra axiom has to be added to ZF in order to derive those results. A good candidate is the axiom of choice, C, which asserts that given a collection of sets, it's possible to choose a single element from each set. But C, although it proved very useful, has been disputed because, among other things, it leads to some results which are unpalatable to many. For instance, in ZF + C the Banach-Tarski Paradox can be proved: a solid sphere can be cut into finitely many pieces, which can then be reassembled to form two solid spheres of the same size as the original. Those who reject C in order to avoid weird results such as the Banach-Tarski Paradox, have to propose some other axiom to be added to ZF in order to derive the important results that cannot be derived in ZF. For instance, some prefer axiom LM, according to which every set of reals is Lebesgue measurable. In ZF + LM the Banach-Tarski Paradox becomes untenable. But, as Stewart (2017) clearly points out, this does not eliminate the possibility of deriving weird results, because ZF + LM implies "an arguably worse paradox, the Division Paradox. Consider the additive group of reals **R** and its subgroup of rationals **Q**. Then in ZF + LM, the cardinality of **R** is *less* than that of the quotient group **R**/**Q**" (Stewart 2017, p. 78). In other words, in ZF + LM the set **R** "can be partitioned into disjoint non-empty subsets, in such a way that the number of subsets is greater than the number of points" (Ibidem). If one finds this result disturbing, in order to show that it is untenable, one should be allowed to rely precisely… on C, i.e. the axiom of choice. Indeed, in ZF + C the Division Paradox becomes untenable. Now, consider two subjects, one who argues for ZF + C and one who argues for ZF + LM. It seems fair to say that such a case is a good example of 'faultless' disagreement (Kölbel 2004; Clarke-Doane 2014), i.e. disagreement which is not originated by some incorrectness in one's starting points, rather disagreement derives from the differences between one's starting points and the starting points of one's opponent. Such differences lead to different and incompatible conclusions, hence disagreement. If disagreement is 'faultless', different starting points have to be regarded as equally legitimate. And, indeed, there is nothing blatantly incorrect in both ZF + C and ZF + LM, nor in the weird results that one can derive in them. Both choices are equally legitimate. Supporters of the opposing views give arguments for and against each possible choice and argue for its plausibility, and there is disagreement precisely because assessing the plausibility of each choice cannot be reduced to a mere matter of immediately giving one's assent to some 'obvious' truths.[22] In fact, one chooses between ZF + C and ZF + LM according to one's ideas on the plausibility of the results that can be derived in ZF + C and ZF + LM. It is difficult to say whether C and LM have to be regarded as 'truths'. Certainly, both C and LM are disputed, so they cannot be said to be 'obvious' truths. Nevertheless, they are undisputedly regarded as genuine axioms by many mathematicians, i.e. possible starting points of mathematical proofs according to the axiomatic view. And they lead to two incompatible results. There is clearly a tension between the disagreement on C and LM, and the idea that mathematical knowledge is acquired

[22] As Hellman and Bell writes, contrary to the "popular (mis)conception of mathematics as a cut-and-dried body of universally agreed-on truths and methods, as soon as one examines the foundations of mathematics, one encounters divergences of viewpoint […] that can easily remind one of religious, schismatic controversy" (Bell, Hellman 2006, p. 64). Cf. Clarke-Doane 2014, p. 243: "consider […] the Axiom of Foundation (which states that every set occurs at some level of the cumulative hierarchy). The key questions here are whether it is plausible that there are sets that contain themselves or whether it is plausible that there are sets with infinitely descending chains of membership. Some seem to think that it is—to banish such sets would be unnaturally restrictive. But many others seem to think that it is not—such sets are pathological. This seems to be a straightforward case of people disagreeing as to the *plausibility*—not just the truth—of epistemically basic mathematical propositions." For a survey of the arguments for and against some crucial axioms given by great mathematicians at the beginning of the 20th century, see Kline 1980, especially Chap. IX.

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by deductive proofs from ‘obvious’ truths. If all results were really deductively derivable from the same set of ‘obvious’ truths, results should not be incompatible. They are incompatible because they are derived from different axioms. But if the truths from which we derive mathematical results were ‘obvious’, they should have been shared, so it should have not been possible to have different starting points and reach incompatible results, and so disagreement should have been impossible.[23]

As regard to (β_2), i.e. the problem of accounting for how mathematical knowledge can be ampliated, the point is that deduction is regarded as truth-preserving, but it is also usually regarded as non-ampliative. This implies that the traditional view of mathematical knowledge is not able to account for all those cases in which new mathematics has been introduced in order to solve some mathematical problem. For example, when Cantor demonstrated that to every transfinite cardinal there exist still greater cardinals, “he did not deduce this result from truths already known […] because it could not be demonstrated within the bounds of traditional mathematics. Demonstrating it required formulating new concepts and new hypotheses about them” (Cellucci 2017, p. 310). So, not all mathematical knowledge is acquired by deductive proofs from already established mathematical results.

Contrary to the view that deduction is non-ampliative, it may be objected that deductive arguments are at least ampliative in an *epistemic* sense, because otherwise we should say that we learn nothing in mathematics beyond what we already knew

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[23] It may be objected that according to Prawitz the problems afflicting the axiomatic method do not affect the view that mathematical knowledge is acquired by deductive proofs from obvious truths, because “this view is not tied to the idea that one can specify once and for all a set of axioms from which all deductive proofs are to start” (Prawitz 2014, p. 90). Even granting this point, there remains the incompatibility between the idea that there are ‘obvious’ truths from which we can derive mathematical proofs and the *possibility* of disagreement on which starting points we should choose. Indeed, mathematical “disagreement raises doubts about the supposed self-evidence of the relevant propositions” (Clarke-Doane 2014, p. 243). And in mathematics disagreement is widespread and persistent: “for core claims in every area of mathematics—from set theory to analysis to arithmetic—*there are some* […] who deny those claims” (Ibidem, p. 241). If disagreement persists, this means that either we have so far failed to identify the *right* ‘obvious’ truths, or that mathematics cannot be reduced to deductions from ‘obvious’ truths. But, why should an ‘obvious’ truth be so hard to find out? Shouldn’t be more ‘obvious’?

by knowing the premises of a proof. For instance, Dummett famously objects that, if deductive rules were non-ampliative, then, "as soon as we had acknowledged the truth of the axioms of a mathematical theory, we should thereby know all the theorems. Obviously, this is nonsense" (Dummett 1991, p. 195). It can be conceded that mathematical proofs can be regarded as ampliative in this epistemic sense, since we are not deductively omniscient, and so it is impossible for us to know all the derivable theorems by simply acknowledging the truth of the axioms of a mathematical theory. But epistemic ampliation is not equivalent to knowledge ampliation (Cellucci 2017, Sect. 12.7). When dealing with deduction, irrespective of how much we are surprised by a given conclusion, nothing objectively new (i.e. independent from the agent's epistemic condition) can be found in such conclusion with respect to the premises. If mathematical proofs rest exclusively on deductions, a mathematical theorem "asserts nothing that is *objectively* or *theoretically new* as compared with the postulates from which it is derived, although its content may well be *psychologically new* in the sense that we were not aware of its being implicitly contained in the postulates" (Hempel 1945, p. 9). Deduction is extremely useful because it "discloses what assertions are concealed in a given set of premises, and it makes us realize to what we committed ourselves in accepting those premises." Nevertheless, "none of the results obtained by this technique ever goes by one iota beyond the information already contained in the initial assumptions" (Ibidem). So, if deduction is ampliative in a merely epistemic sense, this means that it is unable to account for how we reach mathematical knowledge which cannot *even in principle* be deductively derived from what we already know.

Both the challenges to the traditional view illustrated above show that assuming that the axiomatic method is the method of mathematics is not compatible with the adoption of a naturalist stance. On the contrary, it is often claimed that if one dismisses naturalism, one can address these challenges and secure the traditional view of mathematics. As Brown claims, if we adopt naturalism, we "may have a question-begging circle: How do we know mathematical result R is correct? Because it was produced by method M," namely by the axiomatic method, i.e. by deductive proofs from given axioms, but: "How do we know method M is reliable? Because it produced result R" (Brown 2012, p. 54). Indeed, there is no other way for the naturalist to justify our confidence in our intuition that, for instance, MP cannot lead us astray: we have to know that the result we arrived at by means of MP is *the correct*

Philosophia, DOI: 10.1007/s11406-018-9953-1.

one in order to confirm our intuition that MP *cannot* lead us astray. This kind of naturalistic confirmation is obviously circular, and this makes, in Brown's view, the naturalist position weaker than the anti-naturalist one. According to Brown, such circle "is not 'vicious', since it does not lead to contradiction, but neither is it 'virtuous'. Platonists can break into the circle by means of intuitions, but […] naturalists are trapped" (Ibidem). This means that the axiomatic method is not really able to describe by itself how we acquire mathematical knowledge. It must be complemented with some sort of 'intuition' in order to escape circularity.[24] Obviously, the platonist can do better than the naturalist only if we are willing to admit in our view the existence of a faculty such as 'platonist intuition' for which we have no plausible scientific account, nor even a clue that it exists. If instead one does not wish to commit oneself to such an esoteric faculty, one should dismiss the claim that the method of mathematics is the axiomatic method.

7. Conclusion

[24] This does not mean that by adding intuition to the axiomatic method one is *really* able to secure the platonist view. On the difficulty of securing platonism by relying on intuition, cf. e.g. Cellucci 2017, p. 255: "Gödel claims that we can extend our knowledge of the abstract concepts of transfinite set theory by focusing more sharply on the concepts concerned. So we will arrive at an intuitive grasping of ever newer axioms, which is necessary for the solvability of all problems. This, however, is problematic. Suppose that, by focusing more sharply on the concept of set Σ, we get an intuition of that concept. Let S be a formal system for set theory, whose axioms this intuition ensures us to be true of Σ. So Σ is a model of S, hence S is consistent. Then, by Gödel's first incompleteness theorem, there is a sentence A of S which is true of Σ but is unprovable in S. Since A is unprovable in S, the formal system $S' = S \cup \{\neg A\}$ is consistent, and hence has a model, say Σ′. Then ¬A is true of Σ′ and hence A is false of Σ′. Now, Σ and Σ′ are both models of S, but A is true of Σ and false of Σ′, so Σ and Σ′ are not equivalent. Suppose next that, by focusing more sharply on the concept of set Σ′, we get an intuition of this concept. Then we have two different intuitions, one ensuring us that Σ is the concept of set, and the other ensuring us that Σ′ is the concept of set, where the sentence A is true of Σ and false of Σ′. This raises the question: Which of Σ and Σ′ is the genuine concept of set? Gödel's procedure gives no answer to this question."

Philosophia, DOI: 10.1007/s11406-018-9953-1.

This article tried to suggest that, since it seems that the traditional view of mathematics cannot be naturalized, if one wishes to maintain a naturalist stance, a promising way to start developing a naturalist account of mathematics and mathematical knowledge may be to take the method of mathematics not to be the axiomatic method.

It has been argued that it is impossible to naturalize mathematics without challenging at least some crucial aspects of the traditional view. Indeed, in order to justify the traditional view of mathematical knowledge, at least two requirements have to be fulfilled, i.e. axioms have to be consistent, and deduction has to be truth-preserving. But, if one adopts a naturalist stance, it seems that there is no way to show that these requirements can be fulfilled. If these requirements cannot be fulfilled, mathematical knowledge cannot be said to be certain in the same way it can be said to be certain in the traditional perspective. Since the certainty of mathematical knowledge is supposed to be due (at least in part) to the method of mathematics, which the traditional view takes to be the axiomatic method, and since the adoption of a naturalist stance seems to imply that mathematical knowledge cannot be said to be certain, if one wishes to develop a naturalist account of mathematics, one should deem inadequate the claim that the method of mathematics is the axiomatic method.

Certainly, the rejection of the claim that the axiomatic method is the method of mathematics comes with a cost: it forces us to rethink the whole traditional image of mathematics. Indeed, if one takes the method of mathematics not to be the axiomatic method, mathematical knowledge cannot be said to be certain, and the only kind of mathematical knowledge that one can have is knowledge which is plausible (Cellucci 2013; 2017). But even the alternative option, i.e. maintaining the traditional view, comes with a cost for the naturalist: she would be unable to scientifically account for mathematics, while she maintains the primacy of a thorough mathematized science in her world-view. So, dismissing the claim that the axiomatic method is the method of mathematics may represent a promising route to take for the naturalist.

References

Auslander, J. (2008). On the Roles of Proof in Mathematics. In: Gold, B., Simons, R. (eds.), Proof and Other Dilemmas: Mathematics and Philosophy. Washington: The Mathematical Association of America: 62-77.

Baker, A. (2016). Non-Deductive Methods in Mathematics. In: Zalta, E.N. (ed.), The Stanford Encyclopedia of Philosophy, URL = <https://plato.stanford.edu/archives/win2016/entries/mathematics-nondeductive/>.

Balaguer, M. (2009). Realism and Anti-Realism in Mathematics. In: Gabbay, D., Thagard, P., Woods, J. (eds.), Handbook of the Philosophy of Science. Volume 4. Philosophy of Mathematics. Amsterdam: Elsevier: 117-151.

Baron, S. (2015). Mathematical Explanation and Epistemology: Please Mind the Gap. Ratio, 29: 14-167.

Bell, J., Hellman, G. (2006). Pluralism and the Foundations of Mathematics. In: Kellert, S.H., Longino, H.E., Waters, C.K., Minneapolis: University of Minnesota Press: 64-79.

Benacerraf, P. (1973). Mathematical Truth. The Journal of Philosophy, 70: 661-679.

Bertolaso, M., Sterpetti, F. (2017). Evidence Amalgamation, Plausibility, and Cancer Research. *Synthese*, DOI: 10.1007/s11229-017-1591-9.

Brown, J.R. (2012). Platonism, Naturalism, and Mathematical Knowledge. New York: Routledge.

Bueno, O., Shalkowski, S. (2004). Modal Realism and Modal Epistemology: A Huge Gap. In: Weber, E., De Mey, T. (eds.), Modal Epistemology. Brussels: Koninklijke Vlaamse Academie van Belgie: 93-106.

Byers, W. (2007). How Mathematicians Think. Princeton: Princeton University Press.

Carroll, L. (1895). What the Tortoise Said to Achilles. Mind, 4: 278-280.

Cellucci, C. (2006). The Question Hume Didn't Ask: Why Should We Accept Deductive Inferences? In: Cellucci, C., Pecere, P. (eds.), Demonstrative and Non-Demonstrative Reasoning. Cassino: Edizioni dell'Università degli Studi di Cassino: 207-235.

Cellucci, C. (2013). Rethinking Logic. Dordrecht: Springer.

Cellucci, C. (2017). Rethinking Knowledge. Dordrecht: Springer.

Clark, K.J. (2016). Naturalism and its Discontents. In: Clark, K.J. (ed.), The Blackwell Companion to Naturalism. Oxford: Blackwell: 1-15.

Clarke-Doane, J. (2014). Moral Epistemology: The Mathematics Analogy. Noûs, 48: 238-255.

Clarke-Doane, J. (2016). What Is the Benacerraf Problem? In: Pataut, F. (ed.), Truth, Objects, Infinity. New Perspectives on the Philosophy of Paul Benacerraf. Cham: Springer: 17-43.

Cohen Kadosh, R., Dowker, A. (eds.) (2015). The Oxford Handbook of Numerical Cognition. Oxford: Oxford University Press.

Colyvan, M. (2015). Indispensability Arguments in the Philosophy of Mathematics. In: Zalta, E.N. (ed.), The Stanford Encyclopedia of Philosophy, URL = <https://plato.stanford.edu/archives/spr2015/entries/mathphil-indis/>.

De Cruz, H. (2004). Why Humans Can Count Large Quantities Accurately. Philosophica, 74: 63-83.

De Cruz, H. (2006). Towards a Darwinian Approach to Mathematics. Foundations of Science, 11: 157-196.

De Cruz, H. (2016). Numerical Cognition and Mathematical Realism. Philosophers' Imprint,16: 1-13.

De Cruz, H., De Smedt, J. (2012). Evolved Cognitive Biases and the Epistemic Status of Scientific Beliefs. Philosophical Studies, 157: 411-429.

Dehaene, S., Brannon, E.M. (eds.) (2011). Space, Time and Number in the Brain. Searching for the Foundations of Mathematical Thought. Amsterdam: Elsevier.

Detlefsen, M. (2011). Discovery, Invention and Realism: Gödel and Others on the Reality of Concepts. In: Polkinghorne, J. (ed.), Meaning in Mathematics. Oxford: Oxford University Press: 73-94.

Deutsch, D. (1998). The Fabric of Reality. New York: Penguin Books.

Devitt, M. (1998). Naturalism and the A Priori. Philosophical Studies, 92: 45-65.

Dummett, M. (1991). The Logical Basis of Metaphysics. Cambridge (MA): Harvard University Press.

Feferman, S. (1998). In the Light of Logic. Oxford: Oxford University Press.

Field, H. (1989). Realism, Mathematics and Modality. Oxford: Blackwell.

Frege, G. (1984). Collected Papers on Mathematics, Logic, and Philosophy. Oxford: Blackwell.

George, A., Velleman, D.J. (2002). Philosophies of Mathematics. Malden: Blackwell.

Giere, R.N. (2006). Modest Evolutionary Naturalism. Biological Theory, 1: 52-60.

Gödel, K. (1947). What is Cantor's Continuum Problem? In: Idem (1990), Kurt Gödel. Collected Works. Volume II. Oxford: Oxford University Press: 176-187.

Gödel, K. (1951). Some Basic Theorems on the Foundations of Mathematics and their Implications. In: Idem (1995), Kurt Gödel. Collected Works. Volume III. Oxford: Oxford University Press: 304-323.

Gödel, K. (1953). Is Mathematics Syntax of Language? In: Idem (1995), Kurt Gödel. Collected Works. Volume III. Oxford: Oxford University Press: 324-355.

Gowers, T. (2006). Does Mathematics Need a Philosophy? In: Hersh, R. (ed.), 18 Unconventional Essays on the Nature of Mathematics. New York: Springer: 182-200.

Haack, S. (1976). The Justification of Deduction. Mind, 85: 112-119.

Hempel, C.G. (1945). Geometry and Empirical Science. The American Mathematical Monthly, 52: 7-17.

Hilbert, D. (1980). Letter to Frege, 29 December 1899. In: Frege, G., Philosophical and Mathematical Correspondence. Oxford: Blackwell: 38-41.

Horsten, L. (2015). Philosophy of Mathematics. In: Zalta, E.N. (ed.), The Stanford Encyclopedia of Philosophy, URL = <http://plato.stanford.edu/archives/spr2015/entries/philosophy-mathematics/>.

Kitcher, P. (1988). Mathematical Naturalism. In: Aspray, W., Kitcher, P. (eds.), Minnesota Studies in the Philosophy of Science. Volume XI. History and Philosophy of Modern Mathematics. Minneapolis: University of Minnesota Press: 293-325.

Kline, M. (1980). The Loss of Certainty. New York: Oxford University Press.

Kölbel, M. (2004). Faultless Disagreement. Proceedings of the Aristotelian Society, 104: 53-73.

Krebs, N. (2011). Our Best Shot at Truth: Why Humans Evolved Mathematical Abilities. In: Frey, U.J., Störmer, C., Willführ, K.P. (eds.), Essential Building Blocks of Human Nature. Dordrecht: Springer: 123-141.

Kyburg, H. (1965). Comments on Salmon's 'Inductive Evidence'. American Philosophical Quarterly. 2: 274-276.

Lacey, A. (2005). Naturalism. In: Honderich, T. (ed.), The Oxford Companion to Philosophy. Oxford: Oxford University Press: 640-642.

Leng, M. (2007). Introduction. In: Leng, M., Paseau, A., Potter, M. (eds.), Mathematical Knowledge. Oxford: Oxford University Press: 1-15.

Lewis, D. (1986). On the Plurality of Worlds. Oxford: Blackwell.

Linsky, B., Zalta, E.N. (1995). Naturalized Platonism Versus Platonized Naturalism. The Journal of Philosophy, 92: 525-555.

McEvoy, M. (2004). Is Reliabilism Compatible with Mathematical Knowledge? The Philosophical Forum, 35: 423-437.

McKay, R.T., Dennett, D.C. (2009). The Evolution of Misbelief. Behavioral and Brain Sciences, 32: 493-510.

Morganti, M. (2016). Naturalism and Realism in the Philosophy Science. In: Clark, K.J. (ed.), The Blackwell Companion to Naturalism. Oxford: Blackwell: 75-90.

Norris, C. (2000). Quantum Theory and the Flight from Realism. London: Routledge.

Nozick, R. (2001). Invariances. Cambridge (MA): Harvard University Press.

Núñez, R. (2009). Numbers and Arithmetic: Neither Hardwired nor Out There. Biological Theory, 4: 68-83.

Papineau, D., (2016). Naturalism. In: Zalta, E.N. (ed.), The Stanford Encyclopedia of Philosophy, URL: <https://plato.stanford.edu/archives/win2016/entries/naturalism/>.

Paseau, A. (2013). Naturalism in the Philosophy of Mathematics. In: Zalta, E.N. (ed.), The Stanford Encyclopedia of Philosophy, URL: <https://plato.stanford.edu/archives/win2016/entries/naturalism-mathematics/>.

Polkinghorne, J. (2011). Mathematical Reality. In: Polkinghorne, J. (ed.), Meaning in Mathematics. Oxford: Oxford University Press: 27-34.

Pólya, G. (1954). Mathematics and Plausible Reasoning. Princeton: Princeton University Press.

Prawitz, D. (2014). The Status of Mathematical Knowledge. In: Ippoliti, E., Cozzo, C., (eds.), From a Heuristic Point of View: Essays in Honour of Carlo Cellucci. Newcastle upon Tyne: Cambridge Scholars Publishing: 73-90.

Sage, J. (2004). Truth-Reliability and the Evolution of Human Cognitive Faculties. Philosophical Studies, 117: 95-106.

Schechter, J. (2013). Could Evolution Explain Our Reliability about Logic. In: Gendler, T.S., Hawthorne, J. (eds.). Oxford Studies in Epistemology. Volume 4. Oxford: Oxford University Press: 214-239.

Shapiro, S. (1997). Philosophy of Mathematics: Structure and Ontology. Oxford: Oxford University Press.

Smith, J.M. (2012). Evolution and Logic. In: Dybjer, P., Lindström, S., Palmgren, E., Sundholm, G. (eds.) Epistemology versus Ontology. Dordrecht: Springer: 129-138.

Spelke, E.S. (2011). Natural Number and Natural Geometry. In: Dehaene, S., Brannon, E.M. (eds.), Space, Time and Number in the Brain. Searching for the Foundations of Mathematical Thought. Amsterdam: Elsevier: 287-317.

Sterpetti, F. (2018). Mathematical Knowledge, the Analytic Method, and Naturalism. In S. Bangu (ed.), *Naturalizing Logico-Mathematical Knowledge. Approaches from Psychology and Cognitive Science*. New York: Routledge: 268-293.

Stewart, I. (2017). Xenomath! In: Sriraman, B. (ed.), Humanizing Mathematics and its Philosophy. Essays Celebrating the 90th Birthday of Reuben Hersh. Cham: Birkhäuser: 69-83.

Vaidya, A. (2016). The Epistemology of Modality. In: Zalta, E.N. (ed.), The Stanford Encyclopedia of Philosophy, URL: <https://plato.stanford.edu/archives/win2016/entries/modality-epistemology/>.

Vlerick, M., Broadbent, A. (2015). Evolution and Epistemic Justification. Dialectica, 69: 185-203.

Weir, A. (2005). Naturalism Reconsidered. In: Shapiro, S. (ed.), The Oxford Handbook of Philosophy of Mathematics and Logic. Oxford: Oxford University Press: 460-482.

Wilkins, J.S., Griffiths, P.E. (2013). Evolutionary Debunking Arguments in Three Domains: Fact, Value, and Religion. In: Maclaurin, J., Dawes, G. (eds.), A New Science of Religion. New York: Routledge: 133-146.

Williamson, T. (2000). Knowledge and Its Limits. Oxford: Oxford University Press.

Woleński, J. (2012). Naturalism and Genesis of Logic. Studies in Logic, Grammar and Rhetoric, 27: 223-240.

Ye, F. (2011). Strict Finitism and the Logic of Mathematical Applications. Springer: Dordrecht.