Minimal inconsistency-tolerant logics: a quantitative approach

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Abstract

In order to reason in a non-trivializing way with contradictions, paraconsistent logics reject some classically valid inferences. As a way of recovering some of these inferences, Graham Priest ([Priest, 1991]) proposed to nonmonotonically strengthen the Logic of Paradox by allowing the selection of "less inconsistent" models via a comparison of their respective inconsistent parts. This move recaptures a good portion of classical logic in that it does not block, e.g., disjunctive syllogism, unless it is applied to contradictory assumptions. In Priest's approach the inconsistent parts of models are compared in an extensional way by considering their inconsistent objects. This distinguishes his system from the standard format of (inconsistency-)adaptive logics pioneered by Diderik Batens, according to which (atomic) contradictions validated in models form the basis of their comparison. A well-known problem for Priest's extensional approach is its lack of the Strong Reassurance property, i.e., for specific settings there may be infinitely descending chains of less and less inconsistent models, thus never reaching a minimally inconsistent model.

In this paper, we demonstrate that Strong Reassurance holds for the extensional approach under a cardinality-based comparison of the inconsistent parts of models. Furthermore, we introduce and investigate the metatheory of the class of first-order nonmonotonic inconsistency-tolerant construct over the extensional or quantitative comparisons of their respective models. Core model-theoretic properties for these logics, such as the Löwenheim-Skolem theorems, along with other nonmonotonic properties, are further studied.

1 Introduction and outline of the paper

In this paper, we investigate a specific family of nonmonotonic logics that interpret a given set of premises as consistently as possible. While these logics,

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unlike classical logic, tolerate inconsistencies, they re-capture a good portion of the inferential power of classical logic. The basic idea is the following. We start with a fixed monotonic paraconsistent logic PL. Given a possibly inconsistent premise set Σ, PL offers a class of possible interpretations or models of Σ: $\mathcal{M}_{\mathsf{PL}}(\Sigma)$. Some of these interpretations will be less inconsistent, while others will be more so. The meaning of interpreting the premises "as consistently as possible" can be disambiguated in several ways. Throughout the paper we follow *quantitative* or *qualitative* considerations, coupled with considerations over *linguistic* or *extensional* aspects. Further details about these approaches will be provided in Section 2. What is common to both is that they equip us with an ordering $\prec$ on $\mathcal{M}_{\mathsf{PL}}(\Sigma)$, where $M \prec M'$ if M offers a less inconsistent interpretation of Σ than M'. Then, one can choose the minimally inconsistent models in $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ to define a consequence relation $\vDash_{\mathsf{nmPL}}$: $\Sigma \vDash_{\mathsf{nmPL}} A$ iff for all $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$, A holds in M.

In a nutshell, the linguistic approach considers (formulas expressing) contradictions validated in a model to measure how inconsistent is the interpretation offered by the model. On the other hand, the extensional approach considers inconsistent extensions of predicates in a model. When comparing different models M and M' through the ordering $\prec$ with respect to their inconsistent interpretations, be it linguistically or extensionally, one can either consider subset relations by demanding that every contradiction in M is also in M', or one can compare the cardinality of their respective set of contradictions.

The linguistic approach in both its *qualitative* and *quantitative* flavour, in the form of adaptive logics ([Batens, 2007, Straßer, 2014]), has been shown to be based on a robust meta-theory with many properties that are deemed to be at the core of defeasible inference (such as cumulativity [Gabbay, 1985, Kraus et al., 1990]). The situation for the extensional approach, on the other hand, is unflattering. One of the main reasons is that its qualitative version lacks the property of strong reassurance ([Batens, 2000]), and therefore also lacks central properties of nonmonotonic inference, such as cautious monotony and cumulativity. *Strong reassurance* states that for any model $M \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ either M is $\prec$-minimal or we can find a minimally inconsistent $M' \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ that is less inconsistent than M (so, $M' \prec M$).

As a central result, we show that if one adopts a quantitative extensional approach, strong reassurance and the core properties of nonmonotonic inference hold. Furthermore, we investigate under what circumstances classical logic (or another Tarskian deductive standard) is recaptured by the nonmonotonic logic, whether it makes a difference for the nonmonotonic entailment relation to interpret the equality symbol as identity or merely as a congruence relation, and whether the nonmonotonic logic also satisfies the Löwenheim-Skolem theorems.

We proceed as follows. In Section 2 we motivate the move from monotonic paraconsistent logics to nonmonotonic inconsistency-tolerant logics. We introduce the linguistic and the extensional approaches, as well as the qualitative and the quantitative model comparisons. After having demonstrated that the *qualitative* extensional approach does not satisfy strong reassurance and related properties, we move our focus in Section 3 to the *quantitative* extensional approach. There we introduce different ways to disambiguate the phrase "interpreting the premises as consistently as possible". In Section 4 we show that the latter approach is based on a robust meta-theory which satisfies strong reassurance and the core properties of nonmonotonic inference. In Section 5 we discuss whether classical logic or another deductive standard is recaptured when the nonmonotonic logic takes as an input a premise set that can be interpreted consistently. In Section 6 we recall and enhance some of the discussion concerning philosophical problems underlying the extensional or the linguistic approaches. Section 7 is concerned with the interpretation of the equality symbol as either identity or congruence. Finally, we investigate nonmonotonic and abnormality-aware versions of the Löwenheim-Skolem theorems in Section 8. In Section 9 we conclude our paper.

2 Reasoning with contradictions: from monotonic to nonmonotonic logics

It has been argued that, in specific reasoning tasks, humans and artificial agents benefit from adopting a tolerant attitude towards contradictions. For instance, scientific theories (especially in their earlier stages) have been argued to be inconsistent[1], semantic paradoxes such as the liar paradox give rise to contradictions, databases may contain contradictions, and some philosophical traditions embrace conceptual contradictions (see e.g., [Priest, 2002]). Classical logic's attitude towards contradictions is inconsistency-intolerant, as it does not allow for models in which contradictions are true and therefore validates the explosion principle, according to which anything follows from a contradiction.

As a remedy, several families of paraconsistent logics have been proposed in the literature (see [Priest et al., 2018] for an overview). A common feature of these logics is that they make use of a negation ($\neg$) which is inferentially weaker than its classical counterpart ($\sim$), thus allowing for both A and $\neg A$ to be true. The price to pay for inconsistency-tolerance is that some classical inferences such as disjunctive syllogism (if A and $\sim A \vee B$, then B) fail in paraconsistent logics. To address this issue, nonmonotonic systems , have been proposed that allow to interpret a given premise set as consistently as possible, and thereby to validate some instances of disjunctive syllogism.

2.1 Paraconsistent logics considered in this contribution

Before we equip paraconsistent logics with more strength, let us shortly shed some light on what kind of paraconsistent logics we consider in this paper. Since our results apply to a whole class of logics, we decided to keep most of the informal discussion on an abstract level. Nevertheless, some constraints are in place.

[1]Such as Bohr's atomic theory and Newton's cosmology. See [Vickers, 2013, Šešelja and Straßer, 2014].

Firstly, our perspective is a semantic one. We consider any Tarskian logic[2] PL in a first order language $\mathcal{L}$ with a countable set of predicate symbols $\{P_i \mid i \in I\}$ (with I being an initial subset of $\mathbb{N}$) with corresponding arities denoted simply in typewriter font by $\mathtt{i}$ (so $\mathtt{i} =_{\mathsf{df}} \mathsf{arity}(P_i)$), countably many sentential letters p_i and a set of countably many constants denoted by $\mathsf{Const}_{\mathcal{L}}$. We denote the set of $\mathcal{L}$-sentences [resp. wffs] by $\mathsf{sent}_{\mathcal{L}}$ [resp. by $\mathsf{wffs}_{\mathcal{L}}$]. We only consider logics with finitely many truth-values collected in $\mathbb{V}$ and truth-functional (deterministic) matrices. We assume there to be a non-empty, finite and strict subset $\mathbb{D}$ of $\mathbb{V}$ of designated truth-values, and a non-empty set of designated values $\mathbb{A} \subseteq \mathbb{D}$ that are considered "abnormal". Typically, these will be truth-values tracking inconsistencies. E.g., in three-valued logics with values $\mathbb{V} = \{0, \mathsf{i}, 1\}$ and $\mathbb{D} = \{\mathsf{i}, 1\}$, $\mathbb{A}$ will typically be the singleton $\{\mathsf{i}\}$. In what follows, we introduce nonmonotonic logics that "avoid" interpreting formulas with values in $\mathbb{A}$ as much as possible.[3]

As usual, we consider models of PL to be structures $M = \langle \mathcal{D}, v\rangle$, where $\mathcal{D}$ is a non-empty discourse domain and v interprets each constant, each sentential letter, and each predicate symbol. For this, each constant is associated with a member of $\mathcal{D}$ and each sentential letter is associated with a truth-value in $\mathbb{V}$. Furthermore, for each predicate symbol P_i and each tuple $(d_1, \ldots, d_{\mathtt{i}}) \in \mathcal{D}^{\mathtt{i}}$, v maps $(P_i, (d_1, \ldots, d_{\mathtt{i}}))$ to $\mathbb{V}$. Similarly for identity $=$, non-identity $\neq$ and congruence symbols $\approx$.[4] In that case v additionally satisfies the following constraints:[5]

Eq For all $d, d' \in \mathcal{D}$, $v(=, (d, d')) \in \mathbb{D}$ iff $d = d'$.

InEq For all $d, d' \in \mathcal{D}$, $v(\neq, (d, d')) \in \mathbb{D}$ iff $d \neq d'$.

Cong For all $d, d', d'' \in \mathcal{D}$,

(Ref) $v(\approx, (d, d)) \in \mathbb{D}$, and

(Str) For all P_i, for all $(d_1, \ldots, d_{\mathtt{i}-1}) \in \mathcal{D}^{\mathtt{i}-1}$, and for all $1 \leq j \leq \mathtt{i}$, if $v(\approx, (d, d')) \in \mathbb{D}$ then

$$v(P_i, (d_1, \ldots, d_{j-1}, d, d_{j+1}, \ldots, d_{\mathtt{i}-1})) = v(P_i, (d_1, \ldots, d_{j-1}, d', d_{j+1}, \ldots, d_{\mathtt{i}-1})).$$

The reader may note that from (Ref) and (Str) follow, e.g., symmetry (if $v(\approx, (d, d')) \in \mathbb{D}$ then $v(\approx, (d', d)) \in \mathbb{D}$) and transitivity (if $v(\approx, (d, d')), v(\approx, (d', d'')) \in \mathbb{D}$ then also $v(\approx, (d, d'')) \in \mathbb{D}$).

[2] A logic PL is *Tarskian* if its consequence relation $\vDash_{\mathsf{PL}}$ is reflexive ($\Sigma \vDash_{\mathsf{PL}} A$ for all $A \in \Sigma$), transitive (if $\Sigma \vDash_{\mathsf{PL}} A$ for all $A \in \Delta$ and $\Sigma \cup \Delta \vDash_{\mathsf{PL}} B$ then $\Sigma \vDash_{\mathsf{PL}} B$) and monotonic (if $\Sigma \vDash_{\mathsf{PL}} A$ then $\Sigma \cup \Delta \vDash_{\mathsf{PL}} A$).

[3] In order to reduce clutter –especially in the already spacious meta-theoretic proofs– we will not consider functional symbols in the first-order languages studied in this paper.

[4] Of course, whether a symbol $\circ \in \{=, \neq, \approx\}$ is interpreted as identity, non-identity or a congruence depends on constraints imposed on the given models. In this paper we will conventionally use the symbol "$=$" [resp. "$\neq$", "$\approx$"] only in the context of models that satisfy **Eq** [resp. **InEq**, **Cong**].

[5] We use lazy notation here by using also in the meta-language the symbols "$=$" and "$\neq$". Also, all expressions in the meta-language have their usual classical interpretation.

A	$\neg A$
$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{V}\setminus\mathbb{A}$
$\mathbb{A}$	$\mathbb{A}$
$\mathbb{D}\setminus\mathbb{A}$	$\mathbb{V}\setminus\mathbb{D}$

A	$\neg A$
0	1
i	i
1	0

A	$\neg A$
f	**t**
n	**n**
b	**b**
t	**f**

$\wedge$	$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{A}$	$\mathbb{D}\setminus\mathbb{A}$
$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{V}\setminus\mathbb{D}$
$\mathbb{A}$	$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{A}$	$\mathbb{A}$
$\mathbb{D}\setminus\mathbb{A}$	$\mathbb{V}\setminus\mathbb{D}$	$\mathbb{A}$	$\mathbb{D}\setminus\mathbb{A}$

$\wedge$	0	i	1
0	0	0	0
i	0	i	i
1	0	i	1

$\wedge$	**f**	**n**	**b**	**t**
f	**f**	**f**	**f**	**f**
n	**f**	**n**	**f**	**n**
b	**f**	**f**	**b**	**b**
t	**f**	**n**	**b**	**t**

Table 1: The schematic truth-tables for negation (left, row 1), conjunction (left, row 2) express our requirements (Neg) and (Con). E.g., the first line of the left table for negation reads: $f_\neg[\mathbb{V}\setminus\mathbb{D}] \subseteq \mathbb{V}\setminus\mathbb{A}$. These schemes are not to be misunderstood as a nondeterministic truth-tables (as in [Avron, 2005]). We also list the truth-tables for the paraconsistent negation and conjunction of LP (center) and of the four-valued characterization of FDE (where $\mathbb{A} = \{\mathbf{b}\}$ and $\mathbb{D} = \{\mathbf{t}, \mathbf{b}\}$, see [Omori and Wansing, 2017] for an introduction to FDE). We highlight the correspondences between the schemes on the left and the truth-tables of LP and FDE.

Formulas are associated with truth-values by means of a valuation function v_M which proceeds in a bottom-up way:

- for atomic formulas $A = P_i(c_1, \dots, c_{\mathtt{i}})$ we let

$$v_M(P_i(c_1, \dots, c_{\mathtt{i}})) = v(P_i, (v(c_1), \dots, v(c_{\mathtt{i}})))$$

- for atomic formulas $A = p_i$ we let $v_M(p_i) = v(p_i)$.

For complex formulas we assume each n-ary connective in the language to be truth-functional. That means, for each connective $\circ$ of arity n there is a function $f_\circ : \mathbb{V}^n \to \mathbb{V}$ such that $v_M(\circ(A_1, \dots, A_n)) = f_\circ(v_M(A_1), \dots, v_M(A_n))$.

A typical paraconsistent logic has a unary connective $\neg$ that allows for truth-value gluts, i.e., for which $f_\neg[\mathbb{D}] \cap \mathbb{D} \neq \emptyset$ (see Fig. 2.1 (left) for the schematic truth-table).[6] In what follows, let (Neg) and (Con) be the respective truth-conditions based on the truth-tables for negation and conjunction (Fig. 2.1).

Fact 2.1. *For any valuation function v_M that satisfies (Neg) for $\neg$ and (Con) for $\wedge$ (see Fig. 2.1)*[7]*, we have: $v_M(A) \in \mathbb{A}$ iff $v_M(A \wedge \neg A) \in \mathbb{A}$.*

[6] For functions $f : \mathcal{X} \to \mathcal{Y}$ and a non-empty $X \subseteq \mathcal{X}$ we will use the notation $f[X]$ for the set $\{f(x) \mid x \in X\}$.

[7] If $f_\wedge$ is a min-operator, this means that $\wedge$ respects the ordering $\mathbb{V}\setminus\mathbb{A} < \mathbb{A} < \mathbb{D}\setminus\mathbb{A}$ (in lazy notation, more precisely: for all $\mathsf{v} \in \mathbb{V}\setminus\mathbb{A}$, for all $\mathsf{v}' \in \mathbb{A}$ and for all $\mathsf{v}'' \in \mathbb{D}\setminus\mathbb{A}$ we have $\mathsf{v} < \mathsf{v}' < \mathsf{v}''$).

Proof. Suppose $v_M(A) \in \mathbb{A}$. By (Neg), $v_M(\neg A) \in \mathbb{A}$ and by (Con), $v_M(A \wedge \neg A) \in \mathbb{A}$.

Suppose $v_M(A \wedge \neg A) \in \mathbb{A}$. By (Con), we have the following cases: (1) $v_M(A) \in \mathbb{A}$ and $v_M(\neg A) \in \mathbb{A}$, (2) $v_M(A) \in \mathbb{A}$ and $v_M(\neg A) \in \mathbb{D} \setminus \mathbb{A}$, and (3) $v_M(A) \in \mathbb{D} \setminus \mathbb{A}$ and $v_M(\neg A) \in \mathbb{A}$. For cases (1) and (2), we're done. For case (3), by (Neg), $v_M(A) \in \mathbb{A}$ follows. □

The fact shows that in a typical paraconsistent logic, in order to interpret a premise set "as consistently as possible", sentences need to be interpreted in $\mathbb{V} \setminus \mathbb{A}$ as much as possible.

For technical reasons we will consider the following (rather harmless) constraints on quantifiers and the presence of an existential quantifier throughout the paper. Namely, for each quantifier μ we assume there to be a function $f_\mu : \wp(\mathbb{V}) \setminus \{\emptyset\} \to \mathbb{V}$ such that[8]

Q0. $v_M(\mu x A(x)) = \mathsf{v}$ iff $f_\mu(\{v_M(A(\overline{d})) \mid d \in \mathcal{D}\}) = \mathsf{v}$,

and for all $\mathcal{V}, \mathcal{V}' \in \wp(\mathbb{V}) \setminus \{\emptyset\}$,

Q1. $f_\mu(\mathcal{V}) \in \mathcal{V}$, **Q2.** if $f_\mu(\mathcal{V} \cup \mathcal{V}') \in \mathcal{V}$ then $f_\mu(\mathcal{V}) = f_\mu(\mathcal{V} \cup \mathcal{V}')$.

Finally, we assume the presence of an existential quantifier, i.e., a quantifier $\exists$ that fulfills (for all $\mathcal{V} \in \wp(\mathbb{V}) \setminus \{\emptyset\}$):

Q∃. $f_\exists(\mathcal{V}) \in \mathbb{D}$ iff $\mathcal{V} \cap \mathbb{D} \neq \emptyset$.

Fact 2.2. *Whenever $\mathbb{V}$ is totally ordered by some $\sqsubset$, the requirements **Q0**, **Q1**, and **Q2** are fulfilled for the standard definitions of $\exists$ and $\forall$ in terms of $f_\exists = \max_\sqsubset$ and $f_\forall = \min_\sqsubset$.*[9]

As usual, we define $M \models A$ iff $v_M(A) \in \mathbb{D}$, else $M \not\models A$. We shall write $\mathcal{M}_{\mathsf{PL}}$ for the class of all PL-models and $\mathcal{M}_{\mathsf{PL}}(\Sigma)$ for the class of PL-models of a given set of $\mathcal{L}$-sentences Σ.

Example 2.1 (The logic of paradox, LP.)**.** *The logic of paradox* LP *([Asenjo et al., 1966, Priest, 1979]) is one of the best-known and most simple paraconsistent logics. In the rest of the paper we will use it in our examples. It comes with three truth-values: $\mathbb{V} = \{0, \mathsf{i}, 1\}$, where $\mathbb{D} = \{\mathsf{i}, 1\}$ and $\mathbb{A} = \{\mathsf{i}\}$. The values 0 and 1 behave as their classical counter-parts, while* i *allows for negation-gluts. This is captured by the truth-table for negation in Table 2.1 (right). Disjunction and conjunction behave in the usual way in that $f_\wedge = \min_\sqsubset$ and $f_\vee = \max_\sqsubset$ where $0 \sqsubset \mathsf{i} \sqsubset 1$. In this way, both (Neg) and (Con) are satisfied in* LP *and so*

[8]Where d is an element of the given domain $\mathcal{D}$, we denote by $\overline{d}$ the pseudo/auxiliary-constant whose interpretation is d. Moreover, we use the notation $A(x_1, \ldots, x_n)$ to indicate that all free variables in A are among $x_1, \ldots, x_n$. In particular, $A(x)$ denotes that the only free variable in A is x or A is a sentence. Furthermore, $A(c)$ denotes the result of the uniform substitution of all free variables in A by c.

[9]Recall that $\mathbb{V}$ is finite. For infinite $\mathbb{V}$ with $f_\exists = \sup$ or $f_\forall = \inf$ Fact 2.2 does not in general hold.

Fact 2.1 applies. Similarly, the quantifiers ∃ and ∀ are defined by $f_\forall = \max_\sqsubset$ *and* $f_\exists = \min_\sqsubset$. *Identity '=' fulfills our requirement and identity-gluts of the type* $(c = c) \wedge \neg(c = c)$ *are allowed for the case in which* $v(=, (v(c), v(c))) = \mathsf{i}$.

Example 2.2. *Other paraconsistent logics that fall in our class are, e.g.,* CLuNs, FDE, J_3 *and* RM_3.

2.2 Inconsistency-tolerant nonmonotonic logics

The central question is how to tolerate contradictions without paying the full price in inferential power. The solution adopted here is to sacrifice another totem of classical logic: monotonicity ([Priest, 1991, Batens, 2000]). Inconsistency-tolerant nonmonotonic logics (in short, ITNMLs) interpret a given set of premises "as consistently as possible".[10] In other words, we take a given paraconsistent logic PL – let us call it the *base logic*–, but instead of considering all its models/interpretations, we only select those which are most consistent, i.e., the "less inconsistent" ones.

Example 2.3. *Suppose our premise set is* $\{\neg p, p \vee q\}$. *An ITNML, let us call it* nmPL, *will consider models of its base logic PL (e.g., LP from Ex. 2.1) in which* $\neg p$ *is true,* p *is false, and therefore* q *is true. This means, disjunctive syllogism will be validated:* $\{\neg p, p \vee q\} \vDash_{\mathsf{nmPL}} q$. *Nonmonotonicity strikes again if we add more information to our premises in the form of* p: $\{\neg p, p, p \vee q\}$. *In this case,* nmPL *will only consider models in which both* p *and* $\neg p$ *are true. And since* p *is true,* $p \vee q$ *will also be true.*[11] *Thus, there are models of* $\{\neg p, p, p \vee q\}$ *where* q *is false. Hence,* $\{\neg p, p, p \vee q\} \nvDash_{\mathsf{nmPL}} q$. *This is nonmonotonicity on the level of the entailment relation of an ITNML: adding new information may cause the loss of consequences.*

The underlying rationale of ITNMLs is the following assumption:

Consistency by Default (CbD) It is plausible to assume that (our description of) the world is consistent in a given aspect X unless we have reasons to believe the opposite.

There are several ways to interpret a set of premises "as consistently as possible", each of which we discuss in more detail below:

1. quantitative vs. qualitative
2. extensional vs. linguistic

Example 2.4. *We start off with an example in the propositional version of* LP *with the premises*

$$\Sigma = \{\neg p, \neg q, \neg s, p \vee q, p \vee s, p \vee t, q \vee t, s \vee t\}.$$

[10] As a clarification: one can, of course, turn any nonmonnotonic logic into a paraconsistent logic by merely changing its base logic. ITNMLs cover specifically those logics whose purpose is to interpret a possibly (classically) inconsistent set of premises "as consistently as possible".

[11] We suppose the disjunction ∨ behaves as expected. The reader may assume LP to be the underlying logic.

We distinguish the following models of our premises relative to the literals that are true in them:

model	*literal*	*contradictions*
M_1	$p, \neg p, \neg q, \neg s, t$	$p \wedge \neg p$
M_2	$\neg p, q, \neg q, s, \neg s, t$	$q \wedge \neg q, s \wedge \neg s$
M_3	$p, \neg p, q, \neg q, \neg s, t$	$p \wedge \neg p, q \wedge \neg q$
M_4	$p, \neg p, \neg q, s, \neg s, t$	$p \wedge \neg p, s \wedge \neg s$
M_5	$p, \neg p, q, \neg q, s, \neg s, \neg t$	$p \wedge \neg p, q \wedge \neg q, s \wedge \neg s$
M_5'	$p, \neg p, q, \neg q, s, \neg s, t$	$p \wedge \neg p, q \wedge \neg q, s \wedge \neg s$
M_6	$p, \neg p, q, \neg q, s, \neg s, t, \neg t$	$p \wedge \neg p, q \wedge \neg q, s \wedge \neg s, t \wedge \neg t$

In order to select those models of Σ in which the given information is interpreted as consistently as possible, it is useful to define the inconsistent or abnormal part of a model. In this section we let (see Section 3 for a more detailed discussion):

$$\mathsf{Ab}(M) =_{\mathsf{df}} \{A \in \mathsf{Atoms} \mid v_M(A) \in \mathbb{A}\}.$$

In view of Fact 2.1, this can equivalently be expressed by $\mathsf{Ab}(M) = \{A \in \mathsf{Atoms} \mid M \models A \wedge \neg A\}$. We can then compare models in terms of their respective abnormal parts.

According to the *qualitative comparison*, a model M of Σ is more consistent than another model M' of Σ iff $\mathsf{Ab}(M) \subsetneq \mathsf{Ab}(M')$. The most consistent models of Σ according to this approach are M_1 and M_2. In Figure 1 (right) we illustrate how the given models compare.

According to the *quantitative comparison*, M_1 is better than all other models since it validates the least number of contradictions (in atoms). According to this approach, a model M of Σ is more consistent than another model M' of Σ iff $\mathsf{card}(\mathsf{Ab}(M)) < \mathsf{card}(\mathsf{Ab}(M'))$.[12] In Figure 1 (left) we illustrate how the given models compare.

In order to discuss the last distinction, extensional vs. linguistic, we better move to the more expressive language of predicate logic. Suppose we have a unary predicate symbol P in our language. The *extensional* approach to the abnormal part of a model $M = \langle \mathcal{D}, v \rangle$ is defined by considering the inconsistent extension of P, i.e., all objects $d \in \mathcal{D}$ for which $v(P, \overline{d}) \in \mathbb{A}$.[13] For instance, in the case of LP, all objects $d \in \mathcal{D}$ for which $v(P, \overline{d}) = \mathsf{i}$. So the abnormal part consists of inconsistent objects relative to P. The *linguistic* approach, on the other hand, only considers those inconsistencies in atomic formulas such as Px and Pc that are expressible as sentences in the object language, such as $\exists x(Px \wedge \neg Px)$ or $Pc \wedge \neg Pc$, for a constant c.

[12] The quantitative comparison was first mentioned in [Priest, 1988] for propositional LP. It was not further studied in view of the fact that minimal and LP-models collapse in the context of premise sets that give rise to infinitely many contradictions in atoms, i.e., in case the inconsistent part of a model is taken to be the set of all contradictory atoms. See also Remark 3.1. In the present paper, we take a more general perspective in that we consider various ways of comparing models quantitatively and in that we consider a more expressive 1st order language.

[13] In [Batens, 1999] the extensional approach is called *ontological*.

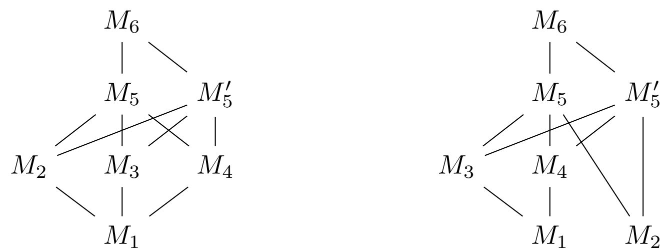

Figure 1: Orderings of models based on the quantitative [qualitative] (left) [right] approach.

Example 2.5. *To illustrate the difference between the two approaches consider the premise set*

$$\Sigma = \big\{(c_1 = c_2 \wedge Pc_1 \wedge \neg Pc_1) \vee (Pc_1 \wedge \neg Pc_1 \wedge c_2 = c_3 \wedge c_4 = c_5)\big\}.$$

We consider two types of models:

1. *M_1 for which the left disjunct of our premise holds and $M_1 \not\models Pc_i \wedge \neg Pc_i$ for any $3 \leq i \leq 5$.*
2. *M_2 for which the right disjunct of our premise holds and $M_2 \not\models Pc_i \wedge \neg Pc_i$ for any $2 \leq i \leq 5$.*

In the linguistic approach there is no need to further consider the extensions of our predicates to determine the abnormal parts of our models. We have: $\mathsf{Ab}(M_1) = \{\exists x(Px \wedge \neg Px), Pc_1 \wedge \neg Pc_1, Pc_2 \wedge \neg Pc_2\}$ and $\mathsf{Ab}(M_2) = \{\exists x(Px \wedge \neg Px), Pc_1 \wedge \neg Pc_1\}$. Hence, M_2 fares better in our comparison and models such as M_2 will be selected. Thus, for instance, $c_4 = c_5$ is entailed. Notice that in M_1, despite the fact that $M_1 \models c_1 = c_2$, we have two linguistic entities $Pc_1 \wedge \neg Pc_1$ and $Pc_2 \wedge \neg Pc_2$ that count as different abnormalities. In the extensional approach, with its focus on extensions, this linguistic difference will not matter and both models will turn out equally inconsistent.

To see this, we first have to consider the structures of our models in more detail. For simplicity, suppose that both models have the same domain consisting of the individuals d_1, d_2 and d_3. The following table gives an overview of the underlying interpretations.

$M = \langle \mathcal{D}, v \rangle$	$\mathcal{D}$	$\mathsf{Ab}_P(M)$	$v(c_1)$	$v(c_2)$	$v(c_3)$	$v(c_4)$	$v(c_5)$
M_1	$\{d_1, d_2, d_3\}$	$\{d_1\}$	d_1	d_1	d_2	d_2	d_3
M_2	$\{d_1, d_2, d_3\}$	$\{d_1\}$	d_1	d_2	d_2	d_3	d_3

In both models only object d_1 falls both in the extension and the "anti-extension" of P and therefore both models have the same abnormal parts. In this case, $c_4 = c_5$ is not derivable.

Let us sum up:

1. An ITNML is based on a paraconsistent base logic PL whose class of interpretations for a given premise set is further refined by implementing the rule of **Consistency by Default**.

2. For this purpose, models of PL are subjected to an ordering $\prec$ where $M \prec M'$ if the abnormal part $\mathsf{Ab}(M')$ of M' is to be considered less inconsistent than $\mathsf{Ab}(M)$.

3. The qualitative and the quantitative approach compare these abnormal parts according to different considerations: either by cardinality or via set-inclusion.

4. Given the resulting ordering $\prec$, models which are deemed sufficiently consistent are selected. In this paper we focus on the selection of $\prec$-minimal models, which means that the set of selected models is $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$.[14]

5. The induced nonmonotonic entailment relation of nmPL is defined by: $\Sigma \models_{\mathsf{nmPL}} A$ iff for all $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$, $M \models A$.

In view of this we can consider an ITNML a pair $\langle \mathsf{PL}, \prec \rangle$ consisting of a paraconsistent (monotonic) base logic and a relation that compares models relative to their consistency (by taking into account the set of abnormalities Ab in M).

2.3 (Strong) reassurance

An important meta-theoretic desideratum for a given ITNML $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ is that it does not trivialize premise sets which are not already trivialized by its corresponding paraconsistent base logic PL. So,

Reassurance. For all sets of $\mathcal{L}$-sentences Σ, $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma)) \neq \emptyset$ if $\mathcal{M}_{\mathsf{PL}}(\Sigma) \neq \emptyset$.

In case the trivial model either doesn't exist or is always strictly worse than any non-trivial model, we can characterize reassurance also in view of the following consequence relation: if there is a $\mathcal{L}$-sentence A for which $\Sigma \not\models_{\mathsf{PL}} A$ then there is an $\mathcal{L}$-sentence B for which $\Sigma \not\models_{\mathsf{nmPL}} B$.

A similar property is *strong reassurance*.

Strong Reassurance. For all sets of $\mathcal{L}$-sentences Σ and all M in $\mathcal{M}_{\mathsf{PL}}(\Sigma) \setminus \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$, there is an $M' \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M' \prec M$.[15]

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[14] In the context of adaptive logics many other types of selections were already presented (see [Straßer, 2014, Ch. 5]).

[15] This property is also known under the name *smoothness* ([Kraus et al., 1990]) or *stopperedness* ([Makinson, 2005]): an ordered structure $(X, \prec)$ is *smooth* iff for every $x \in X$, either $x \in \min_{\prec}(X)$ or there is an $x' \in \min_{\prec}(X)$ for which $x' \prec x$. So, strong reassurance means that for any premise set Σ, the structure $(\mathcal{M}_{\mathsf{PL}}(\Sigma), \prec)$ is smooth.

Strong reassurance has important merits when it comes to properties of nonmonotonic entailment. E.g., Cautious Monotony, which is considered one of the core properties of defeasible inference, is a corollary of strong reassurance and the monotonicity of the core logic PL.

Cautious Monotony For all sets of $\mathcal{L}$-sentences $\Sigma \cup \{A, B\}$, if $\Sigma \vDash_{\mathsf{nmPL}} A$ then

1. $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})) \subseteq \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ and
2. if $\Sigma \vDash_{\mathsf{nmPL}} B$ then $\Sigma \cup \{A\} \vDash_{\mathsf{nmPL}} B$.

Theorem 2.1. *If $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ satisfies Strong Reassurance, it also satisfies Cautious Monotony.*

Proof. Suppose $\Sigma \vDash_{\mathsf{nmPL}} A$. We show that $\min_{\prec}\big(\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})\big) \subseteq \min_{\prec}\big(\mathcal{M}_{\mathsf{PL}}(\Sigma)\big)$. From this it immediately follows that if $\Sigma \vDash_{\mathsf{nmPL}} B$ then also $\Sigma \cup \{A\} \vDash_{\mathsf{nmPL}} B$.

Suppose $M \in \min_{\prec}\big(\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})\big)$. Since $M \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$, by Strong Reassurance, either $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ or there is a $M' \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M' \prec M$. Assume the latter. Since $\Sigma \vDash_{\mathsf{nmPL}} A$, $M' \in \mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})$, in contradiction to the fact that M is $\prec$-minimal in $\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})$. So we are left with the former case which is what had to be shown. □

In view of these considerations we have good reasons to grant strong reassurance a special status in obtaining a well-behaved formal model of defeasible reasoning. Accordingly, we have good reasons to worry about violations of strong reassurance. The danger of violations of strong reassurance looms since it is by no means clear why the structure $(\mathcal{M}_{\mathsf{PL}}(\Sigma), \prec)$ should not give rise to infinitely $\prec$-descending chains of better and better models without ever reaching a best model in $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. And indeed, the extensional qualitative approach doesn't have the property of (strong) reassurance as the following example by Batens ([Batens, 2000]) shows[16]:

Example 2.6. *We consider the base logic LP. Let $\Sigma = \{A_n \mid n \geq 2\}$, where*

$$A_n = \exists x_1 \cdots \exists x_n \left(\bigwedge_{i=1}^{n} (Px_i \wedge \neg Px_i) \wedge \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j) \right).$$

We consider the model $M_0 = \langle \mathbb{N}, v_0 \rangle$ for which $v_0(P, n) = \mathsf{i}$ for all $n \in \mathbb{N}$ and

$$v_0(=, (n, n')) = \begin{cases} 0 & n \neq n' \\ 1 & \textit{else}. \end{cases}$$

We now show that there is an infinitely descending chain of $\prec$-better and better models in the extensional qualitative approach. For this let $\mathbb{N}_i =_{\mathsf{df}} \{0, \ldots, i\}$ and $M_i = \langle \mathbb{N}, v_i \rangle$ be just like M_0 except that the inconsistent extension of P is given by $v_i(P, n) = 0$ for all $n \in \mathbb{N}_i$ and $v_i(P, n) = \mathsf{i}$ for all $n \in \mathbb{N} \setminus \mathbb{N}_i$. Note that

[16] This is a slightly simplified version of his example. Other examples can be found in [Crabbé, 2011].

$M_{i+1} \prec M_i$ *in the extensional qualitative approach since* $\mathsf{Ab}(M_{i+1}) \subsetneq \mathsf{Ab}(M_i)$. *Therefore we have*

$$\ldots \prec M_{i+1} \prec M_i \prec \ldots \prec M_1 \prec M_0$$

and, moreover, there is no $\prec$*-minimal model* $M_\star$ *of* Σ *for which* $M_\star \prec M_i$ *for all* $i \geq 0$.

The good news is that strong reassurance has been established for the linguistic approach: for the qualitative comparison in [Batens, 2000] and for the quantitative comparison in [Straßer, 2014].

It is an open question whether the *quantitative* extensional approach is similarly doomed as its qualitative counterpart. In fact, the question has not been considered in the literature. In the remainder of the paper we will fill this lacuna by

1. proposing ways of comparing models under the extensional and the quantitative approach (Section 3),

2. demonstrating that strong reassurance holds for all of them (Section 4) and give some more meta-theoretic insights into them (Sections 5, 7 and 8).

3. Additionally, we introduce some linguistic properties as a way to highlight some differences between the linguistic and the extensional approaches (Section 6).

3 Counting inconsistencies

Having adopted an extensional and quantitative approach in the context of ITNMLs still allows for several different ways of comparing models relative to how inconsistent their underlying interpretations are. In this section, we present several such variants. We will proceed in increasing degrees of fine-grainedness for each comparison.

While in our examples, so far, we have considered only minimizing abnormalities in predicates, we now generalize this to an ordered set of arbitrary formulas $\Phi = \langle \alpha_i(x_1, \ldots, x_{a_i}) \rangle_{i \in \mathbb{P}}$, where $\mathbb{P}$ is a initial subset of $\mathbb{N}$. We then let the abnormal part of some model M in an $\alpha_i(x_1, \ldots, x_{a_i})$ be $\mathsf{Ab}_{\alpha_i(x_1,\ldots,x_{a_i})}(M) =_{\mathsf{df}} \{(d_1, \ldots, d_n) \in \mathcal{D}^n \mid v_M(\phi(\overline{d_1}, \ldots, \overline{d_n})) \in \mathbb{A}\}$, for which we shall simply write $\mathsf{Ab}_i(M)$.[17] Examples for a choice of Φ are $\Phi = \mathsf{Pred}$, where $\mathsf{Pred} =_{\mathsf{df}} \langle P_i(x_1, \ldots, x_{\mathtt{i}}) \rangle_{i \in I}$ is the set of all predicates. Another possibility is to consider gluts or abnormal behavior in other connectives, say some $\circ$ by letting $\alpha_i(x_1, x_2) = Px_1 \circ Qx_2$, etc. In the remainder of this section, we distinguish several ways of determining and comparing the abnormal parts of models relative to Φ: globally by simply

[17] In order to reduce clutter from hereafter we shall neglect abnormalities in sentential letters. To consider them, all we need to do is to add the set of abnormal sentential letters $\{p \mid v(p) \in \mathbb{A}\}$ to the set of abnormalities considered.

considering $\bigcup_{i \in \mathbb{P}} \mathsf{Ab}_i(M)$, by means of the product order, and by means of the lexicographic order. After that, we refine each ordering by taking into account the distinct comparison types.

3.1 Comparing abnormalities

The most coarse-grained comparison is to consider the contradictory extension of all formulas α_i in Φ at once by letting $\mathsf{Ab}_\Phi(M) = \bigcup_{i \in \mathbb{P}} \mathsf{Ab}_i(M)$. Proceeding quantitatively, we let:

Definition 1 (Global order)**.** *$M \prec^g_\Phi M'$ iff $\mathsf{card}(\mathsf{Ab}_\Phi(M)) < \mathsf{card}(\mathsf{Ab}_\Phi(M'))$.*[18]

As the reader may note, this global type of comparison may lead to undesired outcomes in specific scenarios.

Example 3.1. *Let $\Sigma = \{\forall x(P_1x \wedge \neg P_1x) \vee \forall x(P_2x \wedge \neg P_2x), P_3c, \neg P_3c \vee P_4c, P_1c', \neg P_1c' \vee P_4c'\}$ and $\Phi = \langle P_1(x), P_2(x), P_3(x), P_4(x)\rangle$. Consider the following models $M_i = \langle \mathbb{N}, v_i\rangle$ of Σ:*

M	$\mathsf{Ab}_1(M)$	$\mathsf{Ab}_2(M)$	$\mathsf{Ab}_3(M)$	$\mathsf{Ab}_4(M)$	$\mathsf{Ab}_\Phi(M)$	$M \models P_4c$	$M \models P_4c'$
M_1	$\mathbb{N}$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathbb{N}$	✓	
M_2	$\emptyset$	$\mathbb{N}$	$\emptyset$	$\emptyset$	$\mathbb{N}$	✓	✓
M_3	$\mathbb{N}$	$\emptyset$	$v_3(c)$	$\emptyset$	$\mathbb{N}$		

According to $\prec^g_\Phi$, M_1 and M_2 are not strictly better than M_3, as may be expected. In a sense, the contradictions in P_1 resp. P_2 contaminate the global abnormal part so that we are not anymore able to take into consideration that M_1 and M_2 fares better than M_3 relative to the predicate P_3.

Motivated by the previous example, we now consider abnormalities for each formula in Φ separately[19]. For this we use the *product order* relative to Φ:

Definition 2 (Product order)**.** *$M \prec^p_\Phi M'$ iff*

1. *for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M)) \leq \mathsf{card}(\mathsf{Ab}_i(M'))$, and*
2. *there is an $i \in \mathbb{P}$ for which $\mathsf{card}(\mathsf{Ab}_i(M)) < \mathsf{card}(\mathsf{Ab}_i(M'))$.*

Example 3.2 (Ex. 3.1 cont.)**.** *In our Example 3.1 we now get $M_1 \prec^p_\Phi M_2$ and so P_4c will be nonmonotonically entailed since it is true in all $\prec^p_\Phi$-minimal models.*

In some scenarios it may be intuitive to order the members of Φ according to how important it is to avoid inconsistencies in them. Suppose contradictions in α_i are worse than contradictions in α_j for any $i, j \in \mathbb{P}$ for which $i < j$. For this we use the lexicographic order relative to Φ:

[18] The ordered nature of Φ plays no role for this type of comparison of models, but will be relevant for the lexicographic comparison below. We chose Φ for the sake of a unified parametrized notation.

[19] This corresponds to the way models are compared in, e.g., [Crabbé, 2011], except for the fact that the qualitative approach is explored by the author.

Definition 3 (Lexicographic comparison)**.** *$M \prec^l_\Phi M'$ iff there is an $i \in \mathbb{P}$ such that*

1. *for all $1 \leq j < i$, $\mathsf{card}(\mathsf{Ab}_j(M)) = \mathsf{card}(\mathsf{Ab}_j(M'))$ and*
2. *$\mathsf{card}(\mathsf{Ab}_i(M)) < \mathsf{card}(\mathsf{Ab}_i(M'))$.*

Example 3.3 (Ex. 3.2 cont.)**.** *Now, we have also $M_2 \prec^l_\Phi M_1$ since abnormalities in P_1 are considered more severe than those in P_2. Now, also P_4c' will be nonmonontonically entailed since it is true in all $\prec^l_\Phi$-minimal models of Σ.*

Remark 3.1. *To keep things streamlined, we focused in this section on abnormal parts relative to the extension of predicates. Concerning propositional letters, similar distinctions can be made, of course. Take, e.g., the premise set $\Sigma = \{!p \vee !q_1, !p \vee !q_2\}$, where $!A \stackrel{def}{=} A \wedge \neg A$, and consider the following models:*

model M	$M \models !p$	$M \models !q_1$	$M \models !q_2$
M_1	✓		
M_2		✓	✓

If we consider $\mathsf{Prop} = \langle p, q_1, q_2 \rangle$, then $M_1 \prec^g_{\mathsf{Prop}} M_2$. If we compare them by $\prec^p_{\mathsf{Prop}}$, the two models are incomparable. Note also that in a purely propositional logic the quantitative approach in which propositional letters are compared by $\prec^p_{\mathsf{Prop}}$ coincides with the qualitative approach.

Example 3.4 (Ex. 3.3 cont.)**.** *Taking a closer look at Example 3.1, we can identify another possibly strange property of our comparisons. Consider for this an LP-model $M_0 = \langle \{d\}, v_0 \rangle$ of Σ with only one element in its domain and for which $\mathsf{Ab}_\Phi(M_0) = \mathsf{Ab}_2(M_0) = \{d\}$, while $\mathsf{Ab}_1(M_0) = \mathsf{Ab}_3(M_0) = \mathsf{Ab}_4(M_0) = \emptyset$. We can observe that any model with at least two elements in the domain will be strictly worse for any comparison $\prec^\star_\Phi$ where $\star \in \{g, l\}$. For instance, $M_0 \prec^\star_\Phi M_1$, $M_0 \prec^\star_\Phi M_2$, and $M_0 \prec^\star_\Phi M_3$. (Of course, a similar problem can be identified for $\star = p$.) One may, of course, critically ask whether we should derive anything about the size of the domain based on the premises in Σ.*

3.2 Additionally comparing the domains of models

The situation critically analyzed in Example 3.4 can be improved by imposing a necessary condition for $M \prec M'$ to hold in terms of the domains $\mathcal{D}_M$ and $\mathcal{D}_{M'}$ of M and M': they have to be in some sense comparable. Options are:

[C**f**]	no requirement (free)	[C$\supseteq$]	$\mathcal{D}_M \supseteq \mathcal{D}_{M'}$,
[C$=$]	$\mathcal{D}_M = \mathcal{D}_{M'}$,	[C$\geq$]	$\mathsf{card}(\mathcal{D}_M) \geq \mathsf{card}(\mathcal{D}_{M'})$.
[C$=_c$]	$\mathsf{card}(\mathcal{D}_M) = \mathsf{card}(\mathcal{D}_{M'})$,		

Definition 4. *Where $\pi \in \{\mathbf{f}, =, =_c, \supseteq, \geq\}$ and $\dagger \in \{g, p, l\}$, we define $M \prec^{\pi,\dagger}_\Phi M'$ iff [Cπ] holds and $M \prec^\dagger_\Phi M'$.*

Remark 3.2. *The [C=] restriction for the $\prec^{p}_{\mathsf{Pred}}$ ordering appears in Priest's original proposal ([Priest, 1991]). However, due to some shortcomings pointed out by [Batens, 1999, Batens, 2000], Priest decided to drop the conditon and accept [C**f**] [Priest, 2017]. Some of these problems, as shown in [Crabbé, 2011], can be avoided by adopting [C$\supseteq$]. The choice of [Cπ] is also consequential for properties such as Recapture (see Section 5).*

It is not our task in this paper to ultimately pick one comparison type over the others (this may very well depend on the given application context), but rather to study their meta-theoretic properties and in this way to provide further evidence and criteria for such a comparison.

Example 3.5 (Ex. 3.4 cont.)**.** *We have $M_0 \prec^{\mathbf{f},l}_{\Phi} M_2$ while for $\pi \in \{=, =_c, \supseteq, \geq\}$ the models M_0 and M_2 are incomparable (given their different "sized" domains) and only $M_2 \prec^{\pi,l}_{\Phi} M_1$ and $M_2 \prec^{\pi,l}_{\Phi} M_3$. So, both M_0 and M_2 are selected and we can conclude P_4c and P_4c', but nothing concerning the size of the discourse domain. This may be considered the intuitive outcome.*

Fact 3.1. *Where $\star \in \{g, p, l\}$ and $\pi \in \{\mathbf{f}, =, =_c, \supseteq, \geq\}$, $\prec^{\pi,\star}_{\Phi}$ is transitive and irreflexive.*

Proof. Irreflexivity follows directly from the definition of $\prec^{\pi,\star}_{\Phi}$. We paradigmatically give a proof of transitivity for $\prec \, = \, \prec^{\leq,p}_{\Phi}$. Assume $M_1 \prec M_2$ and $M_2 \prec M_3$. By definition of $\prec$, it follows that (1) for all $i \in I$, $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_2))$ and (2) there is a $i \in I$ for which $\mathsf{card}(\mathsf{Ab}_i(M_1)) < \mathsf{card}(\mathsf{Ab}_i(M_2))$. Similarly, (1′) for all $i \in I$, $\mathsf{card}(\mathsf{Ab}_i(M_2)) \leq \mathsf{card}(\mathsf{Ab}_i(M_3))$ and (2′) there is a $i \in I$ for which $\mathsf{card}(\mathsf{Ab}_i(M_2)) < \mathsf{card}(\mathsf{Ab}_i(M_3))$. Moreover, $\mathsf{card}(\mathcal{D}_1) \leq \mathsf{card}(\mathcal{D}_2) \leq \mathsf{card}(\mathcal{D}_3)$. Clearly, (1″) for all $i \in I$, $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_3))$ and (2′) there is a $i \in I$ for which $\mathsf{card}(\mathsf{Ab}_i(M_1)) < \mathsf{card}(\mathsf{Ab}_i(M_3))$. Hence, $M_1 \prec M_3$. □

4 Nonmonotonic Reasoning Properties

In this section we demonstrate that the quantitative extensional approach has a robust underlying metatheory. In particular, it is not haunted by the lack of strong reassurance (and therefore the lack of cumulativity) as its qualitative extensional relative. In the following, $\prec$ is any of the paradigmatic comparison types defined in the previous section, i.e., $\prec \, = \, \prec^{\pi,\star}_{\Phi}$, where $\pi \in \{\mathbf{f}, =, =_c, \supseteq, \geq\}$, $\star \in \{g, p, l\}$, $\Phi = \{\alpha_i(x_1, \ldots, x_{a_i}) \mid i \in \mathbb{P}\}$ and $\mathbb{P}$ is an initial subset of $\mathbb{N}$.

As pointed out in the introduction, strong reassurance is (a) a necessary requirement for core properties of nonmonotonic inference, such as cumulativity, and (b) it doesn't hold for the qualitative extensional approach. Hence, our first result is a key contribution of the paper paper since it shows that the meta-theory of the *quantitative* extensional approach can be built on a firm foundation.

Theorem 4.1 (Strong Reassurance)**.** nmPL *satisfies* Strong Reassurance.[20]

Cautious monotony states that if we add a nonmonotonic consequence of Σ to Σ, our nonmonotonic consequence set does not *decrease*.[21]

Theorem 4.2 (Cautious Monotony)**.** nmPL *satisfies* Cautious Monotony.[22]

Cautious transitivity (or cautious cut) is the inverse of cautious monotony: if we add nonmonotonic consequences of Σ to Σ, our nonmonotonic consequence set does not *increase*. Its proof does not rely on strong reassurance.

Theorem 4.3 (Cautious Transitivity)**.** nmPL *satisfies* Cautious Transitivity. *That is, for any set of $\mathcal{L}$-sentences Σ and any $\mathcal{L}$-sentences A and B, if $\Sigma \vDash_{\mathsf{nmPL}} A$ then*

1. $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma)) \subseteq \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\}))$, *and*
2. *if $\Sigma \cup \{A\} \vDash_{\mathsf{nmPL}} B$ then also $\Sigma \vDash_{\mathsf{nmPL}} B$.*

Proof. Suppose $\Sigma \vDash_{\mathsf{nmPL}} A$. We show Item 1 from which Item 2 follows immediately. If $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma)) = \emptyset$, the claim follows trivially. Otherwise, suppose $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Since $\Sigma \vDash_{\mathsf{PL}} A$, $M \in \mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})$. Assume for a contradiction that there is an $M' \in \mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\})$ for which $M' \prec M$. Since $M' \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$, this contradicts $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Thus, $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\}))$. □

As a corollary we get one of the core properties of nonmonotonic inference.

Corollary 1 (Cumulativity)**.** nmPL *satisfies* Cumulativity. *That is, for any set of $\mathcal{L}$-sentences Σ and any $\mathcal{L}$-sentences A and B, if $\Sigma \vDash_{\mathsf{nmPL}} A$ then*

1. $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma)) = \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma \cup \{A\}))$, *and*
2. $\Sigma \cup \{A\} \vDash_{\mathsf{nmPL}} B$ *iff* $\Sigma \vDash_{\mathsf{nmPL}} B$.

Another property often considered in the context of nonmonotonic inference is rational monotony. Similar to cautious monotony, it concerns the robustness of consequences under the addition of information to a given premise set Σ. In particular, it concerns the addition of information that is consistent with the nonmonotonic consequences of Σ.

Rational Monotony For all sets of $\mathcal{L}$-sentences Σ, if $\Sigma \vDash_{\mathsf{nmPL}} A$ and $\Sigma \nvDash_{\mathsf{nmPL}} \neg B$ then $\Sigma \cup \{B\} \vDash_{\mathsf{nmPL}} A$.

[20] Since the proof of this result is technically involved, we present it in the technical Appendix 9.

[21] Often nonmonotonic reasoning properties are only presented in terms of their characterization via the consequence relation. Due to its informativeness, we will in the following also list the semantic characterization in terms of set-theoretic relations among classes of models.

[22] The proof was presented in the introduction (see Theorem 2.1).

In the following example we show that Rational Monotony does not hold in general for nmPL. In that respect, the quantitative approach follows the footsteps of many central nonmonotonic logics, such as Default Logic ([Reiter, 1980]) and, in particular, of the qualitative approach to measuring inconsistency, for which Rational Monotony also fails ([Straßer, 2014]).

Example 4.1. *Let $\mathcal{L}$ be a language with four unary predicates P, S, Q, R, and a constant c. Let*

$$\Sigma = \{ \ !Qc \vee !Sc, \ \neg Pc \vee !Sc, \ !Pc \vee Rc \ \}$$

We consider $\prec = \prec^{\pi,p}_{\mathsf{Pred}}$, for any $\pi \in \{=, =_c, \supseteq, \geq, \mathsf{f}\}$. We have four types of models, M_1, M_2, and M_2' in $\min_\prec(\mathcal{M}(\Sigma))$, listed in the following table.[23]

M	$\#(\mathsf{Ab}_P(M))$	$\#(\mathsf{Ab}_Q(M))$	$\#(\mathsf{Ab}_S(M))$	$M \models Pc$	$M \models \neg Pc$	$M \models Rc$
M_1	*0*	*1*	*0*		✓	✓
M_2	*0*	*0*	*1*	✓		✓
M_2'	*0*	*0*	*1*		✓	✓
M_3	*1*	*1*	*0*	✓	✓	

Note that $\Sigma \not\models_{\mathsf{nmPL}} \neg Pc$ (in view of model M_2), while $\Sigma \models_{\mathsf{nmPL}} Rc$. Considering $\Sigma \cup \{Pc\}$, model M_1 and M_2' are not in $\mathcal{M}(\Sigma \cup \{Pc\})$, but we have minimal models of the type M_2 and M_3. In view of M_3, however, $\Sigma \cup \{Pc\} \not\models_{\mathsf{nmPL}} Rc$, the opposite to what rational monotony requires.[24]

A similar property to Rational Monotony is that of Disjunctive Rationality.

Disjunctive Rationality For all sets of $\mathcal{L}$-sentences Σ and all $\mathcal{L}$-sentences A, B and C:
if $\Sigma \cup \{A \vee B\} \models_{\mathsf{nmPL}} C$ then $(\Sigma \cup \{A\} \models_{\mathsf{nmPL}} C$ or $\Sigma \cup \{B\} \models_{\mathsf{nmPL}} C)$.

We also have to report negative results for this property in the general case.

Example 4.2. *Let $\mathcal{L}$ be a language with the unary predicates P, P', Q, Q', and S. Let*

$$\Sigma = \left\{ (!Qc \wedge !Q'c) \vee (!Pc \wedge !P'c), \ !Pc \vee Sc, \ !Qc \vee Sc \right\}.$$

We distinguish the models in the following table under the quantitative ordering $\prec = \prec^{\pi,p}_{\mathsf{Pred}}$, where $\pi \in \{=, =_c, \supseteq, \geq\}$.

M	$\#(\mathsf{Ab}_Q(M))$	$\#(\mathsf{Ab}_{Q'}(M))$	$\#(\mathsf{Ab}_P(M))$	$\#(\mathsf{Ab}_{P'}(M))$	$M \models Sc$
M_1	*1*	*1*	*0*	*0*	✓
M_1'	*1*	*1*	*1*	*0*	
M_2	*0*	*0*	*1*	*1*	✓
M_2'	*1*	*0*	*1*	*1*	

[23] In this and the following tables we often write # instead of card to save space.

[24] The example also shows that rational monotony does not hold for the qualitative comparison under $\prec^{\pi,p}_{\mathsf{Pred}}$.

All models in $\min_\prec(\mathcal{M}(\Sigma \cup \{!Pc \vee !Qc\}))$ *are of the types* M_1 *and* M_2 *relative to their abnormal parts. Models in* $\min_\prec(\mathcal{M}(\Sigma \cup \{!Pc\}))$ *are of types* M_2 *and* M_1'*, and models in* $\min_\prec(\mathcal{M}(\Sigma \cup \{!Qc\}))$ *are of types* M_1 *and* M_2'*. We observe that* $\Sigma \cup \{!Pc \vee !Qc\} \vDash_{\mathsf{nmPL}} Sc$*, while* $\Sigma \cup \{!Pc\} \nvDash_{\mathsf{nmPL}} Sc$ *and* $\Sigma \cup \{!Qc\} \nvDash_{\mathsf{nmPL}} Sc$*. This shows that Disjunctive Rationality does not hold in general for comparisons based on the product order.*

A slightly more complicated example demonstrates that also the global and lexicographic orderings do not satisfy Disjunctive Rationality, if not combined with **f** (see Ex. H.1 in Appendix 9). For orderings $\prec_\Phi^{\mathsf{f},\star}$ with $\star \in \{g, l\}$, we can report positive results.

Lemma 4.1. *Where* $\star \in \{g, l\}$ *and* $\Sigma \subseteq \mathsf{sent}_\mathcal{L}$*,* $\langle \mathcal{M}(\Sigma), \prec_\Phi^{\mathsf{f},\star} \rangle$ *is modular.*[25]

Proof. Let $\prec = \prec_\Phi^{\mathsf{f},\star}$. We consider $\star = l$, the other case is similar and left to the reader. Suppose $M_1, M_2, M_3 \in \mathcal{M}(\Sigma)$ and $M_1 \prec M_2$ while $M_3 \not\prec M_2$. We have to show that $M_1 \prec M_3$. As $M_1 \prec M_2$ there is a minimal $k \geq 1$ such that for all $1 \leq j < k$, $\mathsf{card}(\mathsf{Ab}_j(M_1)) = \mathsf{card}(\mathsf{Ab}_j(M_2))$ and $\mathsf{card}(\mathsf{Ab}_k(M_1)) < \mathsf{card}(\mathsf{Ab}_k(M_2))$. If for all $j \geq 1$, $\mathsf{card}(\mathsf{Ab}_j(M_1)) = \mathsf{card}(\mathsf{Ab}_j(M_3))$ then $M_3 \prec M_2$ which contradicts our assumption. So, there is a minimal $i \geq 1$ such that for all $1 \leq j < i$, $\mathsf{card}(\mathsf{Ab}_j(M_1)) = \mathsf{card}(\mathsf{Ab}_j(M_3))$ and $\mathsf{card}(\mathsf{Ab}_i(M_1)) \neq \mathsf{card}(\mathsf{Ab}_i(M_3))$. If $\mathsf{card}(\mathsf{Ab}_i(M_3)) < \mathsf{card}(\mathsf{Ab}_i(M_1))$, then $M_3 \prec M_1$ and by transitivity (Fact 3.1) $M_3 \prec M_2$ which contradicts our assumption. So, $\mathsf{card}(\mathsf{Ab}_i(M_1)) < \mathsf{card}(\mathsf{Ab}_i(M_3))$ and thus, $M_1 \prec M_3$. □

As in [Lehmann and Magidor, 1992], the proofs of the following two theorems make essential use of the modularity of our order (Lemma 4.1).[26]

Theorem 4.4. *Where* $\prec = \prec_\Phi^{\mathsf{f},\star}$*,* $\star \in \{g, l\}$*,* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *satisfies Disjunctive Rationality if* $f_\vee[(\mathbb{V} \setminus \mathbb{D}) \times (\mathbb{V} \setminus \mathbb{D})] \subseteq \mathbb{V} \setminus \mathbb{D}$.[27]

Proof. Suppose $\Sigma \cup \{A \vee B\} \vDash_{\mathsf{nmPL}} C$. Assume for a contradiction that $\Sigma \cup \{A\} \nvDash_{\mathsf{nmPL}} C$ and $\Sigma \cup \{B\} \nvDash_{\mathsf{nmPL}} C$. Thus, there are $M_A \in \min_\prec(\mathcal{M}(\Sigma \cup \{A\}))$ and $M_B \in \min_\prec(\mathcal{M}(\Sigma \cup \{B\}))$ for which $M_A \not\models C$ and $M_B \not\models C$. Since $\Sigma \cup \{A \vee B\} \vDash_{\mathsf{nmPL}} C$, by Strong Reassurance (Theorem 4.1), there are $M_B', M_A' \in \min_\prec(\mathcal{M}(\Sigma \cup \{A \vee B\}))$ for which $M_B' \prec M_A$ and $M_A' \prec M_B$. Since $M_A \in \min_\prec(\mathcal{M}(\Sigma \cup \{A\}))$, $M_B' \notin \mathcal{M}(\Sigma \cup \{A\})$. Since $M_B' \in \mathcal{M}(\Sigma \cup \{A \vee B\})$ and since $f_\vee[(\mathbb{V} \setminus \mathbb{D}) \times (\mathbb{V} \setminus \mathbb{D})] \subseteq \mathbb{V} \setminus \mathbb{D}$, $M_B' \in \mathcal{M}(\Sigma \cup \{B\})$. Similarly, $M_A' \in \mathcal{M}(\Sigma \cup \{A\})$. By the modularity of $\langle \mathcal{M}, \prec \rangle$ (Lemma 4.1), $M_A' \prec M_A$ or $M_B' \prec M_A'$. The former is impossible since $M_A \in \min_\prec(\mathcal{M}(\Sigma \cup \{A\}))$. The second is impossible since $M_A' \in \min_\prec(\mathcal{M}(\Sigma \cup \{A \vee B\}))$. Altogether this shows that our assumption is false and so $\Sigma \cup \{A\} \vDash_{\mathsf{nmPL}} C$ or $\Sigma \cup \{B\} \vDash_{\mathsf{nmPL}} C$. □

[25] An order $\langle X, \prec \rangle$ is *modular* if for all $x, y, z \in X$, $x \prec y$ implies $x \prec z$ or $z \prec y$ [Lehmann and Magidor, 1992].

[26] Since unlike [Lehmann and Magidor, 1992] our systems are not necessarily based on classical logic and since we don't operate with single-premise consequence relations we state the short proof.

[27] The requirement on the disjunction is fulfilled for instance in LP and FDE.

Theorem 4.5. *Where* $\prec = \prec_{\Phi}^{\mathsf{f},\star}$, $\star \in \{g, l\}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *satisfies Rational Monotony if* $f_{\neg}[\mathbb{V} \setminus \mathbb{D}] \subseteq \mathbb{D}$.

Proof. Suppose $\Sigma \vDash_{\mathsf{nmPL}} A$ and $\Sigma \nvDash_{\mathsf{nmPL}} \neg B$. Thus, there is a $M_B \in \min_{\prec}(\mathcal{M}(\Sigma))$ for which $M_B \not\models \neg B$. Since $f_{\neg}[\mathbb{V} \setminus \mathbb{D}] \subseteq \mathbb{D}$, $M_B \models B$. Suppose $M \in \min_{\prec}(\mathcal{M}(\Sigma \cup \{B\}))$. Assume for a contradiction that $M \not\models A$. Hence, $M \notin \min_{\prec}(\mathcal{M}(\Sigma))$. Thus, by Strong Reassurance (Theorem 4.1) there is a $M' \in \min_{\prec}(\mathcal{M}(\Sigma))$ such that $M' \prec M$. By the modularity of $\langle \mathcal{M}, \prec \rangle$ (Lemma 4.1), $M_B \prec M$ or $M' \prec M_B$. The latter is impossible since $M_B \in \min_{\prec}(\mathcal{M}(\Sigma))$ and the former is impossible since $M_B \in \mathcal{M}(\Sigma \cup \{B\})$ and $M \in \min_{\prec}(\mathcal{M}(\Sigma \cup \{B\}))$. Hence, our assumption is false and so $\Sigma \cup \{B\} \vDash_{\mathsf{nmPL}} A$. □

5 Recapture

In [Crabbé, 2011], Marcel Crabbé shows that for some choices of π under the extensional qualitative approach, the logic at hand is not able to agree with classical logic in consistent situations.

Example 5.1. *Consider the premise set* $\{\exists x Px, \exists x \neg Px\}$ *in a logic* PL *with an existential quantifier (satisfying **Q0-Q2** and **Q**∃). Note that there are no consistent PL-models with cardinality 1, however for each higher cardinality there are consistent models. Thus, where* $\prec = \prec_{\mathsf{Pred}}^{\pi,\star}$, $\pi \in \{=_c, =\}$, *and* $\star \in \{g, p, l\}$, $\min_{\prec}(\mathcal{M}(\Sigma))$ *contains models of size 1 that have an P-inconsistent element. In contrast, where* $\pi \in \{\supseteq, \geq, \mathbf{f}\}$, $\min_{\prec}(\mathcal{M}(\Sigma))$ *only contains models all of whose individuals are P-consistent.*

This gives rise to the following simple observation.

Definition 5. *We call* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ [upwards/downwards] cardinality-sensitive*, iff there is a cardinality* $\kappa > 0$ *for which there is a set of sentences* Σ *such that* $\mathcal{M}_{\mathsf{PL}}(\Sigma)$ *contains models* M *of some cardinality* κ *and there are no models* $M' \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ *of cardinality* κ *[but there are models of higher/of lower cardinality in* $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$*].* nmPL *is fully cardinality-sensitive iff there is a cardinality* κ *such that there are models in* $\mathcal{M}_{\mathsf{PL}}(\Sigma)$ *with cardinalities smaller and larger than* κ*, but* $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ *only contains models of cardinality* κ*.*

Fact 5.1. *Where* $\prec = \prec_{\mathsf{Pred}}^{\pi,\star}$, $\star \in \{g, p, l\}$, $\pi \in \{\mathsf{f}, \supseteq, \geq\}$*, the approach is linguistic or extensional, (i)* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *is cardinality-sensitive. (ii) Where* $\pi \in \{\supseteq, \geq\}$, nmPL *is upwards cardinality-sensitive. (iii) Where* $\pi = \mathsf{f}$ *and* $\mathcal{L}$ *contains an identity,* nmPL *is fully cardinality-sensitive.*

Proof. Items (i) and (ii) are shown in Example 5.1. For (iii) consider $\Sigma = \{\forall x(c = x \vee c' = x \vee !Px), \neg(c = c')\}$. The only models without inconsistencies are those of cardinality 2. □

Crabbé shows that, in the context of LP, one can also recover classical logic for premise sets that have a classical model for orderings based on $\pi = \supseteq$ and $\star = p$. The rationale is the following. Consider we have LP with $\mathbb{A} = \{\mathsf{i}\}$. Suppose we are dealing with a premise set that does not give rise to contradictions, such as $\{\forall x Px\}$. In that case we may expect that nmLP interpret the given premises only in terms of the classical truth-values 0 and 1, and so "recaptures" classical logic. Indeed, the result of restricting the truth-tables of LP to $\{0, 1\}$ is exactly the truth-tables of the classical connectives.

In our more abstract setting the desideratum can be informally phrased as follows. For a given logic PL, based on the truth-values $\mathbb{V}$, and given a set of abnormal truth-values $\mathbb{A} \subseteq \mathbb{V}$, let $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$ be the result of restricting PL-truth-assignments to $\mathbb{V} \setminus \mathbb{A}$ (we make this more precise below). Then, *recapture* expresses that if a premise set Σ has PL-models with assignments only based on the truth-values $\mathbb{V} \setminus \mathbb{A}$, then nmPL interprets Σ just like $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$.

Clearly, not every PL truth-assignment can be restricted to $\mathbb{V} \setminus \mathbb{A}$ in such a way that every formula obtains a value in $\mathbb{V} \setminus \mathbb{A}$. E.g., suppose there is a unary connective $\circ$ such that $f_\circ$ maps a value $\mathsf{v} \in \mathbb{V} \setminus \mathbb{A}$ to a value $\mathsf{v}_a \in \mathbb{A}$. Then in some model M, even if $v_M(A) = \mathsf{v}$, $v_M(\circ A) \notin \mathbb{V} \setminus \mathbb{A}$. Thus, we need our truth-tables to separate abnormal truth-values from "normal" truth-values.

Definition 6. *Where* PL *is a logic based on the language* $\mathcal{L}$ *and* $\mathcal{L}'$ *is a sub-language of* $\mathcal{L}$*,* PL *is* $\mathbb{A}$-separable *(on* $\mathcal{L}'$*) iff*

1. *for all* n*-ary connectives* π *(in* $\mathcal{L}'$*),* $f_\pi[(\mathbb{V} \setminus \mathbb{A})^n] \in \mathbb{V} \setminus \mathbb{A}$ *and* 2. $\mathbb{D} \setminus \mathbb{A} \neq \emptyset$.

Definition 7. *Where* PL *is* $\mathbb{A}$*-separable, let* $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$ *be the logic which is defined just like* PL*, but the set of truth-values is restricted to* $\mathbb{V} \setminus \mathbb{A}$*.*

Let us come back to our previous example.

Fact 5.2. LP *is* $\{\mathsf{i}\}$*-separable and* $\mathsf{LP}^{\{1,2\}} = \mathsf{CL}$*.*

We are now able to phrase our desideratum in a formally precise way:

Definition 8. *Where* PL *is a logic based on the language* $\mathcal{L}$ *and* $\mathcal{L}'$ *is a sub-language of* $\mathcal{L}$*,* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *satisfies* semantic recapture *(for* $\mathcal{L}'$*), iff,*

1. PL *is* $\mathbb{A}$*-separable (on* $\mathcal{L}'$*) and*
2. *for all* $\Sigma \subseteq \mathsf{sent}_{\mathcal{L}'}$ *that are* $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$*-satisfiable,* $M \in \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$ *iff* $M \in \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$*.*[28]

We additionally define:

[28] We slightly abuse notation here since $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-models are not, strictly speaking, PL-models, since their assignment functions have restricted domains. More precisely one can phrase item 2 as follows. We say that a PL-model $M = \langle \mathcal{D}, v \rangle$ has corresponding $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-model iff $v_M(A) \in \mathbb{V} \setminus \mathbb{A}$ for all $A \in \mathsf{sent}_{\mathcal{L}[\mathcal{D}]}$. In that case the corresponding $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$ model is $\langle \mathcal{D}, v' \rangle$ where v' assigns the same values as v (since M has a corresponding $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-model this doesn't violate the domain restrictions underlying v'). Given a $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-model $M = \langle \mathcal{D}, v \rangle$ we let its corresponding PL-model be $\langle \mathcal{D}, v' \rangle$ where v' assigns identical values to v. We can now rephrase item 2 as follows: for each $M \in \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$ there is a corresponding $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-model and for each $M \in \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$ there is a corresponding PL-model in $\mathcal{M}_{\mathsf{nmPL}}(\Sigma)$.

- nmPL *satisfies* entailment recapture *(for $\mathcal{L}'$) iff for all* $\Sigma \subseteq \mathsf{sent}_{\mathcal{L}'}$ *that are* $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$*-satisfiable and all* $A \in \mathsf{sent}_{\mathcal{L}'}$, $\Sigma \vDash_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}} A$ *iff* $\Sigma \vDash_{\mathsf{nmPL}} A$.
- nmPL *satisfies* recapture *(for $\mathcal{L}'$) iff it satisfies semantic and entailment recapture (for $\mathcal{L}'$).*
- nmPL *satisfies* classical recapture *iff* $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}} = \mathsf{CL}$ *and* nmPL *satisfies recapture.*

In the following, we will focus on semantic recapture which implies recapture.

Fact 5.3. *If* nmPL *satisfies semantic recapture it satisfies entailment recapture and therefore also recapture.*

Fact 5.4. *Where* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$*, if* PL *is* $\mathbb{A}$*-separable, then* $\mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma) \subseteq \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ *for any set of sentences* Σ.

This holds since for any $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$, $\mathsf{Ab}_i(M) = \emptyset$ for every P_i.

As a first positive result we notice that for any ordering without restrictions on the size of the domains of models we get recapture for $\mathbb{A}$-separable base logics.

Theorem 5.1. *Where* $\star \in \{g, p, l\}$, $\prec\, = \,\prec_{\Phi}^{\mathbf{f},\star}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$*, if* PL *is* $\mathbb{A}$*-separable then* nmPL *satisfies recapture.*

Proof. Let Σ be $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-satisfiable. Suppose $M \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Assume for a contradiction that $\mathsf{Ab}_i(M) \neq \emptyset$ for some $i \in \mathbb{P}$. Let $M' \in \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$. Thus, $\mathsf{Ab}_j(M) = \emptyset$ for every $j \in \mathbb{P}$. Thus, $M' \prec M$ which is a contradiction. The other direction is due to Fact 5.4. □

However, in the presence of an identity $=$ or a non-identity $\neq$, recapture does not hold for typical core logics for any ordering $\prec_{\Phi}^{\dagger,\star}$ where $\dagger \in \{=, \supseteq, \geq\}$. We illustrate this point with LP.

Fact 5.5. *Where* $\dagger \in \{=, \supseteq, \geq\}$ *and* $\star \in \{g, p, l\}$, $\mathsf{nmLP} = \langle \mathsf{LP}, \prec_{\mathsf{Pred}}^{\dagger,\star} \rangle$ *does not satisfy recapture.*

Proof. Take as an example $\Sigma = \{\exists x \forall y (y = x \vee !Py)\}$. It is easy to see that there is a normal model of cardinality 1 and every model of higher cardinality has abnormalities in P and is therefore not a $\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}$-model. In view of the selection type $\dagger$ these models of higher cardinality are selected.
For a counterexample with $\neq$, consider $\Sigma = \{\exists x \forall y (\neg(y \neq x) \vee !Py)\}$. Every model of cardinality 1 is consistent, but models of higher cardinality have abnormalities relative to either $\neq$ or P. □

The situation improves for logics without (non-)identity. Such a logic may still feature a congruence relation $\approx$. In fact, we show something stronger: given any $\mathbb{A}$-separable base logic PL and a premise set Σ without occurrences of (non-)identities, nmPL satisfies recapture for any ordering of the type $\prec_{\Phi}^{\pi,\star}$ where $\pi \in \{\geq, \supseteq\}$ and $\star \in \{g, l, p\}$.[29]

[29]The proof can be found in Appendix 9.

Theorem 5.2. *Where* PL *is* $\mathbb{A}$*-separable and based on* $\mathcal{L}$ *and* $\mathcal{L}'$ *is a sublanguage of* $\mathcal{L}$ *without identity and non-identity (but possibly with a congruence* $\approx$*),* $\star \in \{g, p, l\}$*, and* $\dagger \in \{\geq, \supseteq\}$*,* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec_{\Phi}^{\dagger,\star} \rangle$ *satisfies recapture for* $\mathcal{L}'$*.*

6 The Linguistic (In)Dependence of the Linguistic and the Extensional Approach

In this section we highlight two ways in which the extensional approach is less dependent on linguistic subtleties than the linguistic approach. We first consider the following example.

Example 6.1. *Let* $\Sigma = \{\exists x!Px\}$*. We first consider a language* $\mathcal{L}_0$ *without constants and only one unary predicate symbol* P *and then move to a language* $\mathcal{L}_c$ *that adds a constant* c *to* $\mathcal{L}_0$*. In the* linguistic approach *the minimal* $\mathcal{L}_0$*-models of* Σ *will be identical to the* PL*-models of* Σ *since all these models have the abnormal part* $\{\exists x!Px\}$*. The situation changes in the context of* $\mathcal{L}_c$*. All models of cardinality 1 will have the abnormal part* $\{\exists x!Px, !Pc\}$ *while minimal models of higher cardinalities will have the smaller abnormal part* $\{\exists x!Px\}$ *and be therefore preferable (if we compare with* $\prec_{\mathsf{Pred}}^{\pi,g}$ *where* $\pi \in \{\mathbf{f}, \supseteq, \geq\}$*). In terms of nonmonotonic entailment, in the context of the language* $\mathcal{L}_c$ *we get the nonmonotonic consequence* $\exists x \exists y \neg(x = y)$ *which is not entailed in the context of the language* $\mathcal{L}_0$*. In contrast, when considering the* extensional approach*, the class of minimal* $\mathcal{L}_0$ *models is identical to the class of minimal* $\mathcal{L}_c$ *models according to any comparison type* $\prec$*. In the context of both languages we have the same nonmonotonic entailments.*

We now show that the extensional approach (unlike the linguistic approach) is robust under extending the (non-logical) signature of the language. In the following we denote by $\mathcal{L}_{\mathsf{PL}}$ the class of languages that contain the logical symbols of PL but vary in their non-logical symbols.

Definition 9. *Where* $\mathcal{L}, \mathcal{L}' \in \mathcal{L}_{\mathsf{PL}}$*, let* $\mathcal{L}'$ *be an extension of* $\mathcal{L}$ *by non-logical symbols (i.e., predicate symbols and constants). Where* $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_{\mathsf{PL}}^{\mathcal{L}'}$ *let* $M{\downarrow}\mathcal{L} =_{\mathsf{df}} \langle \mathcal{D}, v{\downarrow}\mathcal{L} \rangle$ *where* $v{\downarrow}\mathcal{L}$ *is the result of restricting* v *to the language* $\mathcal{L}$*. Moreover, where* $\mathcal{M} \subseteq \mathcal{M}_{\mathsf{PL}}^{\mathcal{L}'}$ *is a set of* $\mathcal{L}'$*-models, we let* $\mathcal{M}^{\downarrow\mathcal{L}} =_{\mathsf{df}} \{M{\downarrow}\mathcal{L} \mid M \in \mathcal{M}\}$*.*

Definition 10. *Where* $\mathcal{L}, \mathcal{L}' \in \mathcal{L}_{\mathsf{PL}}$*,* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *is* linguistically robust relative to $\mathcal{L}$ *if for any set of* $\mathcal{L}$*-sentences* Σ *and any extension* $\mathcal{L}'$ *of* $\mathcal{L}$ *by non-logical symbols,* $\min_{\prec}(\mathcal{M}_{\mathsf{PL}}^{\mathcal{L}}(\Sigma)) = \min_{\prec}(\mathcal{M}_{\mathsf{PL}}^{\mathcal{L}'}(\Sigma)^{\downarrow\mathcal{L}})$*. We say* nmPL *is* linguistically robust *if for any* $\mathcal{L} \in \mathcal{L}_{\mathsf{PL}}$*,* nmPL *is linguistically robust relative to* $\mathcal{L}$*.*

As shown in Example 6.1, the linguistic approach is not linguistically robust. As the next proposition shows, the extensional approach is.

Proposition 6.1. *Where* $\prec = \prec_{\Phi}^{\pi,\star}$*,* $\star \in \{g, l, p\}$*,* $\pi \in \{\mathbf{f}, =, =_c, \supseteq, \geq\}$*,* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *is linguistically robust.*

Proof. Let $\mathcal{L}'$ be an extension of $\mathcal{L}$ by non-logical symbols.

($\subseteq$) Suppose $M = \langle \mathcal{D}, v \rangle \in \min_{\prec}(\mathcal{M}^{\mathcal{L}}_{\mathsf{PL}}(\Sigma))$. We let $M' = \langle \mathcal{D}, v' \rangle \in \mathcal{M}^{\mathcal{L}'}_{\mathsf{PL}}$ be such that v' extends v to $\mathcal{L}'$ in the following way: 1. for each constant c in $\mathcal{L}$, $v'(c) = v(c)$; 2. for each predicate P_i in $\mathcal{L}$ and each tuple $d_1, \ldots, d_{\mathtt{i}} \in \mathcal{D}^{\mathtt{i}}$ we let $v'(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v(P_i, (d_1, \ldots, d_{\mathtt{i}}))$; 3. for each constant c in $\mathcal{L}'$ that is not in $\mathcal{L}$, we let $v'(c) = d$ for an arbitrary $d \in \mathcal{D}$; 4. for each predicate P_i in $\mathcal{L}'$ that is not in $\mathcal{L}$ and each tuple $(d_1, \ldots, d_{\mathtt{i}}) \in \mathcal{D}^{\mathtt{i}}$ we let $v'(P_i, (d_1, \ldots, d_{\mathtt{i}})) = \mathsf{v}$ for an arbitrary $\mathsf{v} \in \mathbb{V} \setminus \mathbb{A}$.

It is easy to see that (a) $M' \in \mathcal{M}^{\mathcal{L}'}_{\mathsf{PL}}(\Sigma)$ since Σ is a set of sentences of $\mathcal{L}$ and (b) $\mathsf{Ab}_i(M) = \mathsf{Ab}_i(M')$ for all predicates $i \in \mathbb{P}$ (note that $\alpha_i \in \mathsf{wffs}_{\mathcal{L}}$). Assume for a contradiction that there is a $M'' \in \mathcal{M}^{\mathcal{L}'}_{\mathsf{PL}}(\Sigma)$ for which $M'' \prec M'$. In view of (b), also $M''{\downarrow}\mathcal{L} \prec M$. This is not possible since $M \in \min_{\prec}(\mathcal{M}^{\mathcal{L}}_{\mathsf{PL}}(\Sigma))$ and $M''{\downarrow}\mathcal{L} \in \mathcal{M}^{\mathcal{L}}_{\mathsf{PL}}(\Sigma)$.

($\supseteq$) Suppose now that $M'' \in \min_{\prec}(\mathcal{M}^{\mathcal{L}'}_{\mathsf{PL}}(\Sigma))$. Assume for a contradiction that there is a $M \in \mathcal{M}^{\mathcal{L}}_{\mathsf{PL}}(\Sigma)$ for which $M \prec M''{\downarrow}\mathcal{L}$. By the ($\subseteq$)-direction, there is a $M' \in \mathcal{M}^{\mathcal{L}'}_{\mathsf{PL}}(\Sigma)$ for which items (a) and (b) hold. Thus, also $M' \prec M''$ which is a contradiction. ☐

Another type of linguistic dependency has to do with naming conventions and the cardinality of the selected models. We take a look at another example.

Example 6.2. *Consider the premise set*

$$\Sigma = \big\{(c_1 = c_2 \wedge !Pc_1 \wedge \neg p) \vee (c_2 = c_3 \wedge !Pc_1 \wedge p)\big\}$$

and the following models of Σ (where $v[P, \mathsf{i}]$ denotes the set $\{d \in \mathcal{D} \mid v(P, d) = \mathsf{i}\}$):

$M = \langle \mathcal{D}, v \rangle$	$\mathcal{D}$	$v[P, \mathsf{i}]$	$v(c_1)$	$v(c_2)$	$v(c_3)$	$M \models p$	$M \models \neg p$
M^1_a	$\{d_1\}$	$\{d_1\}$	d_1	d_1	d_1		✓
M^1_b	$\{d_1\}$	$\{d_1\}$	d_1	d_1	d_1	✓	
M^2_a	$\{d_1, d_2\}$	$\{d_1\}$	d_1	d_1	d_2		✓
M^2_b	$\{d_1, d_2\}$	$\{d_1\}$	d_1	d_2	d_2	✓	

While for M^i_a (where $i \in \{1, 2\}$) we have $M^i_a \models c_1 = c_2 \wedge !Pc_1 \wedge \neg p$, for M^i_b we have $M^i_b \models c_2 = c_3 \wedge !Pc_1 \wedge p$. Now, according to the linguistic approach we have:

$$\mathsf{Ab}_P(M^2_b) = \{\exists x!Px, !Pc_1\} \subset \mathsf{Ab}_P(M^2_a) = \{\exists x!Px, !Pc_1, !Pc_2\} \subset$$
$$\mathsf{Ab}_P(M^1_a) = \mathsf{Ab}_P(M^1_b) = \{\exists x!Px, !Pc_1, !Pc_2, !Pc_3\}$$

Where $\prec = \prec^{\pi,\star}_{\{P\}}$ and $\pi \in \{\mathbf{f}, \geq, \supseteq\}$, $\star \in \{g, p, l\}$ we have therefore $M^2_b \prec M^2_a \prec M^1_a, M^1_b$ in the linguistic approach and we can conclude p. We note that

1. *the linguistic approach is sensitive to* naming conventions*: for instance, the main difference between M^2_b and M^2_a is that in the situation depicted*

by M_a^2 two names are used for the inconsistent object d_1 (namely, c_1 and c_2) while in M_b^2 only one name is used namely, c_1.[30]

2. *observation 1 also has repercussions on the cardinality of the selected models (for $\pi \in \{\mathbf{f}, \geq, \supseteq\}$): models M_a^1 and M_b^1 are not selected since (due to the size of their respective domains) there are more names for inconsistent objects than in their larger counterparts M_a^2 and M_b^2.*

The situation is different in the extensional approach, where all four models are on a par since P has in all of them the same abnormal extension. All depicted models will be selected and therefore p is not nonmonotonically entailed.

Altogether, we observe that the linguistic approach interprets the premise sets also by adhering to pragmatic considerations according to which naming conventions are such that they minimize the number of names for inconsistent objects. Let us make this more precise.

Definition 11. *We say that* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *is* non-pragmatic *iff for every $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_{\mathsf{PL}}$ and every $M' = \langle \mathcal{D}, v' \rangle$ for which v' agrees with v on all predicates (but differs possibly in the interpretation of the constants), $M' \not\prec M$ and $M \not\prec M'$. We say* nmPL *is* pragmatic *if it is not non-pragmatic.*

As we have seen in Example 6.2, the linguistic approach is pragmatic since M_a^2 and M_b^2 agree on all predicates and still $M_b^2 \prec M_a^2$.[31] The extensional approach, in contrast, is non-pragmatic:

Fact 6.1. *Where $\prec = \prec_{\Phi}^{\pi,\star}$, $\star \in \{g, p, l\}$, $\pi \in \{\mathbf{f}, =, =_c, \supseteq, \geq\}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ is non-pragmatic.*

In sum, we present an overview of our results in Table 2. We have shown that the linguistic and extensional perspectives are based on different intuitions as to what should count as an "abnormality" when measuring and comparing the inconsistency of two given models. Moreover, in the linguistic approach pragmatic considerations are taken into account, while the extensional approach is linguistically more robust. Depending on whether one leans more toward one or the other side, one will find certain logical behavior of the respective accounts

[30] In the context of some applications, a reader may critically ask as to why interpreting a premise set as consistent as possible should tell us anything about naming conventions. E.g., our model M_a^2 may depict a situation in which two groups of scientists discover an element d_1 which behaves abnormal w.r.t. property P. Each group gives d_1 a different name, say "strangelet" (c_1) and "weirdlet" (c_2). Why should interpreting the world as consistently as possible exclude such a scenario? Of course, in some situations there may be pragmatic arguments in favor of not multiplying names for the same object (e.g., to reduce linguistic "clutter" and this may be modeled by a nonmonotonic logic), but these considerations seem to be independent of an object's status of being inconsistent or abnormal. Similar critical thoughts can be raised in the context of observation 2. For a relevant defense of this behavior see [Batens, 2000].

[31] In a presentation of this paper it has been suggested to us that the linguistic approach can easily be adjusted to obtain a non-pragmatic variant. This seems not be the case, however. In Appendix 9, we demonstrate that some intuitive suggestions along these lines do not work as intended.

approach	π	recapture	card.-sensitive	pragmatic	ling. robust
extensional	$\pi \in \{=, =_c\}$				✓
linguistic	$\pi \in \{=, =_c\}$			✓	
extensional	$\pi \in \{\geq, \supseteq\}$		✓ (upwards)		✓
linguistic	$\pi \in \{\geq, \supseteq\}$		✓ (upwards)	✓	
extensional	$\pi = \mathbf{f}$	✓	✓ (fully with =)		✓
linguistic	$\pi = \mathbf{f}$	✓	✓ (fully with =)	✓	

Table 2: Overview results

more or less compelling. The fact that logical principles are connected to different styles of reasoning is well known in the context of nonmonotonic logic: see, for instance, the discussion on the so-called floating conclusions ([Horty, 1994]), on the credulous vs. the skeptical reasoning style ([Meheus et al., 2013]), and disagreements on the status of Rational Monotony ([Rott, 2017]).[32]

7 Identity, Congruence and their (Ir)Reducibility

In this section we investigate whether it matters how the equality symbol $\approx$ is interpreted in models for the resulting nonmonotonic entailment relation. More precisely, the question is whether we get different consequences if we select only minimal models that interpret $\approx$ as an identity (so satisfying the constraint **Eq**) or if we consider the class of minimal models that interpret $\approx$ as a congruence (so, satisfying the weaker constraint **Cong**).

In the following we consider a language $\mathcal{L}$ with a symbol $\approx$ and we suppose that all models $\mathcal{M}_{\mathsf{PL}}$ satisfy the constraint **Cong** for it. We call these models *general models*. We suppose the logic PL to be fixed in the background and will from now on skip the subscript. Let $\mathcal{M}^{\mathsf{id}}$ be the subclass of *id-normal models* that satisfy **Eq** for $\approx$. Similarly, where $\Sigma \subseteq \mathsf{sent}_{\mathcal{L}}$, we write $\mathcal{M}(\Sigma)$ resp. $\mathcal{M}^{\mathsf{id}}(\Sigma)$ for the class of models of Σ in $\mathcal{M}$ resp. in $\mathcal{M}^{\mathsf{id}}$. Let in the following $\mid\!\sim$ denote $\models_{\mathsf{nmPL}}$ where $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$. We define two variants $\mid\!\sim^{\mathsf{id},1}$ and $\mid\!\sim^{\mathsf{id},2}$ of $\mid\!\sim$ as follows:

Definition 12. *Where* $\Sigma \cup \{A\} \subseteq \mathsf{sent}_{\mathcal{L}}$*,*

1. $\Sigma \mid\!\sim^{\mathsf{id},1} A$ *iff for all* $M \in \min_{\prec}(\mathcal{M}(\Sigma)) \cap \mathcal{M}^{\mathsf{id}}$*,* $M \models A$*.*
2. $\Sigma \mid\!\sim^{\mathsf{id},2} A$ *iff for all* $M \in \min_{\prec}(\mathcal{M}^{\mathsf{id}}(\Sigma))$*,* $M \models A$*.*

Our initial question can then be phrased by: Do $(\star_1)$ resp. $(\star_2)$ hold?, where

[32] It is not our intention to give ultimate arguments in favor of one account (linguistic vs. extensional) over the other, especially since we consider this an issue depending on the context of application and so an issue which cannot be decided from the abstract meta-theoretical perspective of this paper. Rather, our aim is to provide precise formal criteria/properties that may assist a formal modeler in choosing an adequate formal system (besides issues concerning the content of the to be formalized reasoning).

property	qualitative	quantitative
$(\star_1, \Rightarrow)$	Fact 7.1	Fact 7.1
$(\star_1, \Leftarrow)$	Cor. 2	Ex. 7.2
$(\star_2, \Rightarrow)$	Ex. 7.1	Ex. 7.1
$(\star_2, \Leftarrow)$	Cor. 3	Ex. 7.2

Table 3: Comparison of quantitative and qualitative approaches. The results apply to any ordering $\prec$ considered in this paper. The gray background indicates that the property does not hold and a reference to a counter-example is given.

$$(\star_1) \quad \Sigma \mathrel{|\!\sim} A \text{ iff } \Sigma \mathrel{|\!\sim}^{\mathsf{id},1} A. \qquad\qquad (\star_2) \quad \Sigma \mathrel{|\!\sim} A \text{ iff } \Sigma \mathrel{|\!\sim}^{\mathsf{id},2} A.$$

Crabbé in [Crabbé, 2011, Remark 7.2] observed that $(\star_1)$ holds for $\langle \mathsf{LP}, \prec^{\supseteq,p}_{\mathsf{Pred}} \rangle$ for qualitative comparisons. In the following we will give some more positive as well as negative results. We provide an overview in Table 3. By the definition of $\mathrel{|\!\sim}^{\mathsf{id},1}_{\prec}$ we immediately get:

Fact 7.1. *If* $\Sigma \mathrel{|\!\sim} A$ *then* $\Sigma \mathrel{|\!\sim}^{\mathsf{id},1} A$.

Furthermore, we have:

Fact 7.2. $\min_\prec(\mathcal{M}(\Sigma)) \cap \mathcal{M}^{\mathsf{id}} \subseteq \min_\prec(\mathcal{M}^{\mathsf{id}}(\Sigma))$.

We now consider the $(\Rightarrow)$-direction of $(\star_2)$.

Example 7.1. *We consider the base logic* LP *and* $\prec \, = \, \prec^{\dagger,\star}_{\mathsf{Pred}}$ *where* $\dagger \in \{\supseteq, =, \geq, =_c\}$ *and* $\star \in \{g, p, l\}$. *For the example it will not matter whether we compare the abnormal parts of our models qualitatively or quantitatively. For simplicity we consider the language* $\mathcal{L}$ *consisting of only one unary predicate symbol,* P. *Let our premise set* Σ *be:*

$$\Sigma = \{\forall x(x \approx c) \vee !Pc\}$$

We consider the model $M = \langle \mathcal{D}, v \rangle$ *where* $\mathcal{D} = \{d_1, d_2\}$, $v(c) = d_1$, $[d_1]_\approx = \{d_1, d_2\}$ *(where* $[d]_\approx = \{d' \in \mathcal{D} \mid v(\approx, (d, d')) \in \mathbb{D}\}$*) and the extensions of our predicate as follows (where the* $v[P, \mathsf{v}]$ *denotes the set* $\{d \in \mathcal{D} \mid v(P, d) = \mathsf{v}\}$*):*

$v[P, 1]$	$v[P, 0]$	$v[P, \mathsf{i}]$
$\{d_1, d_2\}$	$\emptyset$	$\emptyset$

This model is minimal, i.e., $M \in \min_\prec(\mathcal{M}(\Sigma))$. *However, it is not id-normal, i.e.,* $M \notin \mathcal{M}^{\mathsf{id}}$. *Consider now* $M' = \langle \mathcal{D}, v' \rangle$ *where* $v(c) = d_1$, $v[P, \mathsf{i}] = d_1$, *and* $\approx$ *is interpreted as identity. Note that* $M' \models !Pc$ *since* $v_{M'}(\forall x(x \approx c)) = 0$ *(in view of* ***Eq*** *and since* $v'(\approx, (d_1, d_2)) \notin \mathbb{D} = \{1, \mathsf{i}\}$*). Clearly,* $M' \notin \min_\prec(\mathcal{M}(\Sigma))$ *since* $M \prec M'$. *Nevertheless,* $M' \in \min_\prec(\mathcal{M}^{\mathsf{id}}(\Sigma))$. *In fact, we have* $\Sigma \mathrel{|\!\sim} \forall x(c \approx x)$ *while* $\Sigma \not\mathrel{|\!\sim}^{\mathsf{id},2} \forall x(c \approx x)$. *This illustrates that for all our orders* $\prec$ *the left-to-right direction of* $(\star_2)$ *does not hold. As we have seen (Fact 7.1), the situation is different for* $(\star_1)$.

$\langle\mathcal{D}, v\rangle$	$\mathcal{D}$	$\mathsf{Ab}_R(M)$	$[c]_\approx$	$[c']_\approx$	$[c'']_\approx$	$M \models \forall x!R(x,x)$	$M \models B$
M_1	$\{1\}$	$\{(1,1))\}$	$\{1\}$	$\{1\}$	$\{1\}$	✓	✓
M_3^1	$\{1,2,3\}$	$\{(1,2),(1,3)\}$	$\{1\}$	$\{2\}$	$\{3\}$		✓
M_3^2	$\{1,2,3\}$	$\{(i,i)\}_{i=1}^3$	$\{1\}$	$\{2\}$	$\{3\}$	✓	
M_ω^1	$\mathbb{N}_1$	$\{(1,i)\}_{i\geq 2}$	$\{1\}$	$\{2\}$	$\mathbb{N}_3$		✓
M_ω^2	$\mathbb{N}_1$	$\{(i,i)\}_{i\geq 1}$	$\{1\}$	$\{2\}$	$\mathbb{N}_3$	✓	

Table 4: Models for Example 7.2, where $B = \forall x \exists y(!R(x,y) \vee !R(y,x))$ and $\mathbb{N}_i = \{j \in \mathbb{N} \mid j \geq i\}$.

We now show that for quantitative comparisons also the ($\Leftarrow$)-direction of ($\star_1$) and ($\star_2$) fail.

Example 7.2. *Let $\prec = \prec^{\dagger,\star}_{\mathsf{Pred}}$ where $\dagger \in \{=, =_c, \geq, \supseteq\}$ and $\pi \in \{g,p,l\}$ under the quantitative comparison. Let*

$$\Sigma = \{\forall x(x \approx c \vee x \approx c' \vee x \approx c''),\ \forall x!R(x,x) \vee B\}.$$

and

$$B = \forall x \exists y((!R(x,y) \wedge (x \approx c)) \vee (!R(y,x) \wedge (y \approx c))).$$

We consider the models in $\mathcal{M}(\Sigma)$ presented in Table 4. We also assume that M_1 and M_3^1 are in $\mathcal{M}^{\mathsf{id}}(\Sigma)$. Clearly, M_ω^1 and M_ω^2 are not id-normal since $\forall x(x \approx c \vee x \approx c')$ holds in them while their domains are infinite. In fact, there are no id-normal models of Σ with domains of cardinality greater than 3. Note that $M_\omega^1, M_\omega^2 \in \min_\prec(\mathcal{M}(\Sigma))$. We also observe that $M_3^1 \prec M_3^2$ and $\Sigma \mathrel{|\!\!\sim}^{\mathsf{id},k} B$ for $k \in \{1,2\}$, while $\Sigma \mathrel{|\!\!\not\sim} B$ in view of models like M_ω^2. Thus, the right-to-left directions of ($\star_1$) and ($\star_2$) fail for quantitative comparisons. We also notice that M_3^2 is a quotient model of M_ω^2 (see Def. 13 below) and, while the latter is $\prec$-minimal, the former is not.

Interestingly, the ($\Leftarrow$)-direction of ($\star_2$) and both directions of ($\star_1$) hold for the qualitative comparison. The reason is the following observation by Crabbé, here slightly generalized for different qualitative comparison types and different base logics. We first define quotient models.

Definition 13. *Where $M = \langle\mathcal{D}, v\rangle \in \mathcal{M}$ and $\approx$ satisfies **Cong**, we define the quotient model $M_\approx = \langle\mathcal{D}_\approx, v_\approx\rangle$ as follows:*

1. *$\mathcal{D}_\approx = \{[d]_\approx \mid d \in \mathcal{D}\}$, where $[d]_\approx =_{\mathsf{df}} \{d' \in \mathcal{D} \mid v(\approx, (d,d') \in \mathbb{D})\}$,*
2. *$v_\approx(P, ([d_1]_\approx, \ldots, [d_n]_\approx)) = v(P, (d_1, \ldots, d_n))$*
3. *$v_\approx(\approx, ([d]_\approx, [d']_\approx)) = v(\approx, (d,d'))$*
4. *$v_\approx(c) = [v(c)]_\approx$*

Lemma 7.1. *Let the underlying comparison be qualitative,* $\prec\, = \,\prec_{\Phi}^{\dagger,\star}$*,* $\dagger \in \{=, \supseteq, \geq, =_c, \mathbf{f}\}$*, and* $\star \in \{g, p, l\}$*: if* $M \in \min_{\prec}(\mathcal{M}(\Sigma))$ *then* $M_{\approx} \in \min_{\prec}(\mathcal{M}(\Sigma)) \cap \mathcal{M}^{\mathsf{id}}$*.*[33]

As we have seen in Example 7.2, this fails for quantitative comparisons.

Corollary 2. *Where* $\prec\, = \,\prec_{\Phi}^{\pi,\star}$*,* $\pi \in \{=, \supseteq, \geq, =_c, \mathsf{f}\}$*,* $\star \in \{g, p, l\}$ *and the underlying comparison is qualitative, if* $\Sigma \mid\!\sim^{\mathsf{id},1} A$ *then* $\Sigma \mid\!\sim A$*.*

Proof. Suppose $\Sigma \mid\!\sim^{\mathsf{id},1} A$ and suppose $M \in \min_{\prec}(\mathcal{M}(\Sigma))$. Thus, by Lemma 7.1, $M_{\approx} \in \min_{\prec}(\mathcal{M}(\Sigma)) \cap \mathcal{M}^{\mathsf{id}}$. By the first supposition, $M_{\approx} \models A$ and hence $M \models A$. Thus, $\Sigma \mid\!\sim A$. □

Corollary 3. *Where* $\prec\, = \,\prec_{\Phi}^{\pi,\star}$*,* $\pi \in \{=, \supseteq, \geq, =_c, \mathsf{f}\}$*,* $\star \in \{g, p, l\}$ *and the underlying comparison is qualitative, if* $\Sigma \mid\!\sim^{\mathsf{id},2} A$ *then* $\Sigma \mid\!\sim A$*.*

Proof. Suppose $\Sigma \mid\!\sim^{\mathsf{id},2} A$. Thus, for all $M \in \min_{\prec}(\mathcal{M}^{\mathsf{id}}(\Sigma))$, $M \models A$. Suppose $M \in \min_{\prec}(\mathcal{M}(\Sigma))$. By Lemma 7.1, $M_{\approx} \in \min_{\prec}(\mathcal{M}(\Sigma)) \cap \mathcal{M}^{\mathsf{id}}$ and thus by Fact 7.2, $M_{\approx} \in \min_{\prec}(\mathcal{M}^{\mathsf{id}}(\Sigma))$. Hence, $M_{\approx} \models A$ and so $M \models A$. □

8 Löwenheim-Skolem

The Löwenheim-Skolem theorems tell us, for instance, that for any model of a countable language $\mathcal{L}$ with an infinitely large domain, we find a model with the same theory whose domain has cardinality κ, where κ is an arbitrary infinite cardinal.[34] It is not obvious that this property also holds when considering semantic selections. The reason is that abnormal parts may grow in models with larger cardinality. While we expect this not to affect orderings $\prec_{\Phi}^{\dagger,\star}$ where $\dagger \in \{=, =_c, \supseteq, \geq\}$ and $\star \in \{g, p, l\}$, for $\pi = \mathsf{f}$ one can easily show that for specific premise sets, even if infinitely large models exist, the only selected ones have countable cardinality.

Example 8.1. *Consider* $\Sigma = \{\forall x!Px\} \cup \{\neg(c_i = c_j) \mid i \neq j \text{ and } i, j \geq 1\}$*. Models* M *without abnormalities in* $=$ *will have an infinitely large domain* $\mathcal{D}$ *for which* $\mathsf{Ab}_P(M) = \mathcal{D}$*. Two such models are* M_ω *and* M_κ *in Table 5 (where* $\kappa > \aleph_0$*). Some of these will be minimal according to* $\prec_{\mathsf{Pred}}^{\mathsf{f},p}$*, namely models* $M = \langle \mathcal{D}, v \rangle$ *for which* $\mathsf{card}(\mathcal{D}) = \aleph_0$*. E.g.,* M_ω *is minimal. Models with uncountable domains, such as* M_κ*, may have the same theory as* M_ω *but are not selected, since for these models* $\mathsf{Ab}_P(M_\kappa) > \aleph_0$*.*[35]

In view of this example we observe, that the upwards version of Löwenheim-Skolem does not hold for orderings $\prec_{\mathsf{Pred}}^{\mathsf{f},\star}$ where $\star \in \{p, l\}$.

[33] We prove this Lemma in Appendix G and refer the reader interested in the intricacies of quotient models in the context of many-valued logic to [Ferguson, 2020].

[34] In this section we focus on countable languages $\mathcal{L}$. The generalization to the uncountable case works as expected.

[35] A similar reasoning applies to $\prec_{\mathsf{Pred}}^{\mathsf{f},l}$ where abnormalities in $=$ are prioritized to abnormalities in P.

$M = \langle \mathcal{D}, v \rangle$	$\mathsf{card}(\mathcal{D})$	$\mathsf{card}(\mathsf{Ab}_P(M))$	$\mathsf{card}(\mathsf{Ab}_=(M))$
M_1	1	1	1
M_ω	$\aleph_0$	$\aleph_0$	0
M_κ	κ	κ	0

Table 5: Models of Example 8.1. We have: $M_\omega \prec^{\mathsf{f},p}_{\mathsf{Pred}} M_\kappa$ while M_1 and M_ω (resp. and M_κ) are $\prec^{\mathsf{f},p}_{\mathsf{Pred}}$-incomparable.

In the remainder of this section we give more results. We start with the monotonic base logic, for which we state "abnormality-aware" versions of the two Löwenheim-Skolem theorems. They illustrate that, given a model M with infinite domain, for all predicates whose abnormal parts do not have the same cardinality as M's domain, we find a just as good model with the same theory but arbitrarily larger domain. In fact, for all such predicates, the abnormal part will maximally be countable. We paradigmatically show this for orderings $\prec^{\dagger,\star}_{\Phi}$ where $\star \in \{p, l\}$ and any $\dagger$. We first define:

Definition 14. *Where $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}$ is a model, let*

1. *$\mathbb{P}_{\mathsf{coinf}}(M)$ be the set of all $i \in \mathbb{P}$ for which $\aleph_0 \leq \mathsf{card}(\mathsf{Ab}_i(M)) < \mathsf{card}(\mathcal{D})$;*
2. *$\mathbb{P}_{\mathsf{fin}}(M)$ be the set of all (i, l) for which $i \in \mathbb{P}$ and $\mathsf{card}(\mathsf{Ab}_i(M)) = l \in \mathbb{N}$;*
3. *$\mathbb{P}_{\mathsf{fin/coinf}}(M)$ be the set of all $i \in \mathbb{P}$ for which $\mathsf{card}(\mathsf{Ab}_i(M)) < \mathsf{card}(\mathcal{D})$.*

Definition 15. *For a logic L, a cardinal κ and a set of $\mathcal{L}$-sentences Σ let $\mathcal{M}^{=\kappa}_{\mathsf{L}}(\Sigma)$ [resp. $\mathcal{M}^{<\kappa}_{\mathsf{L}}(\Sigma)$, $\mathcal{M}^{\leq\kappa}_{\mathsf{L}}(\Sigma)$, $\mathcal{M}^{>\kappa}_{\mathsf{L}}(\Sigma)$, $\mathcal{M}^{\geq\kappa}_{\mathsf{L}}(\Sigma)$] be the sub-class of models in $\mathcal{M}_{\mathsf{L}}(\Sigma)$ with cardinality equal to [resp. smaller than, smaller or equal to, greater than, greater or equal to] κ. We abbreviate $\mathcal{M}^{\dagger}_{\mathsf{L}}(\emptyset)$ by $\mathcal{M}^{\dagger}_{\mathsf{L}}$ for $\dagger \in \{=\kappa, <\kappa, \leq\kappa, >\kappa, \geq\kappa\}$. Similarly, we define $\Sigma \Vdash^{\dagger}_{\mathsf{L}} A$ iff for all $M \in \mathcal{M}^{\dagger}_{\mathsf{L}}(\Sigma)$, $M \models A$.*

In addition to the property of preservation of validity (Item 4 in the following definition) that is expressed in the Löwenheim-Skolem theorems, we will also consider properties concerning preservation relative to the cardinality of abnormal parts (Items 1–3):

Definition 16 (LS-Ab). *Where $\aleph_0 \leq \kappa < \lambda$ (**upwards version**) resp. $\aleph_0 \leq \lambda < \kappa$ (**downwards version**), for all $\Sigma \subseteq \mathsf{sent}_{\mathcal{L}}$ and for all $M_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{L}}(\Sigma)$, there is a $M_\lambda \in \mathcal{M}^{=\lambda}_{\mathsf{L}}(\Sigma)$ for which*[36]

1. *$\mathbb{P}_{\mathsf{fin}}(M_\kappa) = \mathbb{P}_{\mathsf{fin}}(M_\lambda)$ (and so, for all $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M_\kappa)$, $\mathsf{card}(\mathsf{Ab}_i(M_\kappa)) = l = \mathsf{card}(\mathsf{Ab}_i(M_\lambda))$);*

[36] In the context of a monotonic logic L and in virtue of Item 4, stating "for all $\Sigma \subseteq \mathsf{sent}_{\mathcal{L}}$ and for all $M_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{L}}(\Sigma)$ there is a $M_\lambda \in \mathcal{M}^{=\lambda}_{\mathsf{L}}(\Sigma)$" is equivalent to "for all $M_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{L}}$ there is a $M_\lambda \in \mathcal{M}^{=\lambda}_{\mathsf{L}}$". Since we below also consider nonmonotonic logics we state it more generally.

2. *for all* $i \in \mathbb{P}_{\mathsf{coinf}}(M_\kappa)$, $\mathsf{card}(\mathsf{Ab}_i(M_\lambda)) = \aleph_0 \leq \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$*;*

3. *if* $\kappa, \lambda > \aleph_0$, $\mathbb{P}_{\mathsf{coinf}}(M_\kappa) \subseteq \mathbb{P}_{\mathsf{coinf}}(M_\lambda)$*;*

4. *for all* $A \in \mathsf{sent}_{\mathcal{L}}$, $v_{M_\kappa}(A) = v_{M_\lambda}(A)$*.*

In the following we write $M_\lambda \sqsubseteq M_\kappa$ *in case Items 1–4 hold for* M_κ *and* M_λ*.*

Lemma 8.1. *Let* $M_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{PL}}$ *and* $M_\lambda \in \mathcal{M}^{=\lambda}_{\mathsf{PL}}$ *such that* $M_\lambda \sqsubseteq M_\kappa$*. Then for all* $i \in \mathbb{P}_{\mathsf{fin/coinf}}(M_\kappa)$, $\mathsf{card}(\mathsf{Ab}_i(M_\lambda)) \leq \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$*. If* $\kappa \geq \lambda$*, then for all* $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M_\lambda)) \leq \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$*.*

Proof. Let $i \in \mathbb{P}$. If there is an l for which $(i,l) \in \mathbb{P}_{\mathsf{fin}}(M_\kappa)$ then $(i,l) \in \mathbb{P}_{\mathsf{fin}}(M_\lambda)$ and therefore $\mathsf{card}(\mathsf{Ab}_i(M_\kappa)) = \mathsf{card}(\mathsf{Ab}_i(M_\lambda))$. If $i \in \mathbb{P}_{\mathsf{cofin}}(M_\kappa)$ then $\mathsf{card}(\mathsf{Ab}_i(M_\kappa)) \geq \aleph_0 = \mathsf{card}(\mathsf{Ab}_i(M_\lambda))$. If $i \in \mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}(M)$ and $\kappa \geq \lambda$, then $\mathsf{card}(\mathsf{Ab}_i(M_\kappa)) = \kappa \geq \lambda \geq \mathsf{card}(\mathsf{Ab}_i(M_\lambda))$. □

We now state the abnormality-sensitive versions of the Löwenheim-Skolem theorems for the monotonic base logic.[37]

Theorem 8.1. ***LS-Ab (upwards)*** *holds for every logic* PL*.*

Theorem 8.2. ***LS-Ab (downwards)*** *holds for every logic* PL*.*

Corollary 4. *For all sets of* $\mathcal{L}$*-sentences* $\Sigma \cup \{A\}$, $\Sigma \Vdash_{\mathsf{PL}} A$ *iff* $\Sigma \Vdash^{\leq\aleph_0}_{\mathsf{PL}} A$*.*

The next question is whether we get **LS-Ab** also for our nonmonotonic logics $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$. Our previous theorem should make us optimistic since it shows that for many predicates the size of the abnormal part is robust under changes of the size of the domains of our models (given they are infinitely large). Nevertheless, our hopes are only partially fulfilled.

Example 8.2. *Let* $\Sigma = \{(\forall x!P_2x \wedge !P_1c_i) \vee \forall x!P_1x \mid i \geq 1\}$ *and* $\kappa > \aleph_0$*. Let, moreover,* $\prec = \prec^{\dagger,\star}_{\mathsf{Pred}}$ *where* $\dagger \in \{=, =_c, \geq, \supseteq\}$ *and* $\star \in \{p, l\}$*. Consider the models in the following table.*

$\langle \mathcal{D}, v\rangle$	$\#(\mathcal{D})$	$\#(\mathsf{Ab}_{P_1}(M))$	$\#(\mathsf{Ab}_{P_2}(M))$	$M \models \forall x!P_2x \wedge !P_1c_i$	$M \models \forall x!P_1x$
M_κ	κ	$\aleph_0$	κ	✓	
M'_κ	κ	κ	0		✓
M_ω	$\aleph_0$	$\aleph_0$	$\aleph_0$	✓	
M'_ω	$\aleph_0$	$\aleph_0$	0		✓

Note that $M_\kappa \in \min_\prec(\mathcal{M}(\Sigma))$*. However, there is no minimal model with the same theory as* M_κ *with cardinality* $\aleph_0$*. Note for this that* $M'_\omega \prec M_\omega$*. This shows that Theorem 8.2 and Corollary 4 do not generalize to the nonmonotonic setting.*

Example 8.3. *Let* $\Sigma = \{\forall x!Px \vee p\} \cup \{!Pc_i \mid i \geq 1\} \cup \{\neg(c_i = c_j) \mid i \neq j\}$ *and* $\prec = \prec^{\dagger,\star}_{\mathsf{Pred}}$ *(where* Pred *includes* $=$*), where* $\dagger \in \{=, =_c, \geq, \supseteq\}$ *and* $\star \in \{g, p, l\}$*. Consider the models in the following table, where* $\kappa > \aleph_0$*.*

[37] Proofs are provided in Appendix C.

$M = \langle \mathcal{D}, v\rangle$	$\#(\mathcal{D})$	$\#(\mathsf{Ab}_P(M))$	$\#(\mathsf{Ab}_=(M))$	$M \models \forall x!Px$	$M \models p$
M_κ	κ	κ	*0*	✓	
M'_κ	κ	$\aleph_0$	*0*		✓
M_ω	$\aleph_0$	$\aleph_0$	*0*	✓	
M'_ω	$\aleph_0$	$\aleph_0$	*0*		✓

Although $M_\omega \in \min_\prec(\mathcal{M}(\Sigma))$, there is no model in $\min_\prec(\mathcal{M}(\Sigma))$ with the theory of M_ω and cardinality κ. Note for this that $M'_\kappa \prec M_\kappa$.

Despite these examples one can establish two weaker results for nmPL.

Definition 17 (**LS-Ab**->ω)**.** *We define **LS-Ab**->ω just as **LS-Ab**, except that $\kappa, \lambda > \aleph_0$.*

Lemma 8.2. *Let $\star \in \{g, p, l\}$, $\dagger \in \{=, =_c, \geq, \supseteq\}$, $\prec = \prec_\Phi^{\dagger,\star}$. Moreover, let $M_1, M_4 \in \mathcal{M}_{\mathsf{PL}}^{=\kappa}$, $M_2 \in \mathcal{M}_{\mathsf{PL}}^{=\lambda'}$, $M_3 \in \mathcal{M}_{\mathsf{PL}}^{=\lambda}$ where $\kappa, \lambda, \lambda' > \aleph_0$. Then, $M_1 \sqsubseteq M_2$ and $M_2 \prec M_3$ and $M_3 \sqsubseteq M_4$ implies $M_1 \prec M_4$.*

Proof. We first prove the case for $\star = p$. Since $M_2 \prec M_3$, for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M_2)) \leq \mathsf{card}(\mathsf{Ab}_i(M_3))$ and there is a $k \in \mathbb{P}$ for which $\mathsf{card}(\mathsf{Ab}_k(M_2)) < \mathsf{card}(\mathsf{Ab}_k(M_3))$.

We show (1) that for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_4))$ and (2) $\mathsf{card}(\mathsf{Ab}_k(M_1)) < \mathsf{card}(\mathsf{Ab}_k(M_4))$. This suffices for $M_1 \prec M_4$.

Let $i \in \mathbb{P}$. Assume first that $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M_4)$ for some $l \geq 0$. Since $M_3 \sqsubseteq M_4$, $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M_3)$. Since $M_2 \prec M_3$, $(i, l') \in \mathbb{P}_{\mathsf{fin}}(M_2)$ for some $l' \leq l$. If $i = k$, $l' < l$. Since $M_1 \sqsubseteq M_2$, $(i, l') \in \mathbb{P}_{\mathsf{fin}}(M_1)$. So, $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_4))$ and, if $i = k$, $\mathsf{card}(\mathsf{Ab}_i(M_1)) < \mathsf{card}(\mathsf{Ab}_i(M_4))$.

Assume now that $i \in \mathbb{P}_{\mathsf{cofin}}(M_4)$. Since $M_3 \sqsubseteq M_4$, $\mathsf{card}(\mathsf{Ab}_i(M_3)) = \aleph_0$. Since $M_2 \prec M_3$, $\mathsf{card}(\mathsf{Ab}_i(M_2)) \leq \aleph_0$ and if $i = k$, there is an $l \geq 0$ for which $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M_2)$. Since $\lambda' > \aleph_0$, $i \in \mathbb{P}_{\mathsf{fin/coinf}}(M_2)$. Since $M_1 \sqsubseteq M_2$, $i \in \mathbb{P}_{\mathsf{fin/coinf}}(M_1)$ and $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \aleph_0$. So, $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_4))$. Moreover, if $i = k$, $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M_1)$ and therefore $\mathsf{card}(\mathsf{Ab}_i(M_1)) < \mathsf{card}(\mathsf{Ab}_i(M_4))$.

If $i \in \mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}(M_4)$, $\mathsf{card}(\mathsf{Ab}_i(M_4)) = \kappa$ and trivially $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_4))$, since $M_1 \in \mathcal{M}_{\mathsf{PL}}^{=\kappa}$. Suppose $i = k$. Since $M_3 \sqsubseteq M_4$, $i \in \mathbb{P} \setminus \mathbb{P}_{\mathsf{fin}}$. We have two cases, (i) $\mathsf{card}(\mathsf{Ab}_k(M_3)) = \lambda$ or (ii) $\aleph_0 \leq \mathsf{card}(\mathsf{Ab}_k(M_3)) < \lambda$. Moreover, since $M_2 \prec M_3$, $\lambda' \geq \lambda$. So, in both cases, $\mathsf{card}(\mathsf{Ab}_k(M_2)) < \lambda$ and therefore $i \in \mathbb{P}_{\mathsf{fin/coinf}}(M_2)$. Since $M_1 \sqsubseteq M_2$, $i \in \mathbb{P}_{\mathsf{fin/coinf}}(M_1)$. Thus, $\mathsf{card}(\mathsf{Ab}_k(M_1)) < \mathsf{card}(\mathsf{Ab}_k(M_4))$.

Let now $\star = l$. Since $M_2 \prec M_3$ there is a k such that for all $i < k$, $\mathsf{card}(\mathsf{Ab}_i(M_2)) = \mathsf{card}(\mathsf{Ab}_i(M_3))$ and $\mathsf{card}(\mathsf{Ab}_k(M_3)) < \mathsf{card}(\mathsf{Ab}_i(M_2))$.

We show (1) that for all $i < k$, $\mathsf{card}(\mathsf{Ab}_i(M_1)) \leq \mathsf{card}(\mathsf{Ab}_i(M_4))$ and (2) $\mathsf{card}(\mathsf{Ab}_k(M_1)) < \mathsf{card}(\mathsf{Ab}_k(M_4))$. This suffices to show that $M_1 \prec M_4$.

Let $i \leq k$. We can prove (1) and (2) analogously to how we proceeded for $\star = p$. The case $\star = g$ is similar and left to the reader. □

Theorem 8.3 (Löwenheim-Skolem, uncountable, nonmonotonic)**.** *Where $\star \in \{g, p, l\}$, $\dagger \in \{=, =_c, \geq, \supseteq\}$, $\prec = \prec_\Phi^{\dagger,\star}$, **LS-Ab**->$\omega$ (both upwards and downwards) hold for $\mathsf{nmPL} = \langle \mathsf{PL}, \prec\rangle$.*

Proof. Suppose $\aleph_0 < \lambda \leq \kappa$ resp. $\aleph_0 < \kappa \leq \lambda$ and $M_\kappa \in \min_\prec(\mathcal{M}_{\mathsf{PL}}(\Sigma) \cap \mathcal{M}^{=\kappa}_{\mathsf{PL}})$. Let $M_\lambda \sqsubseteq M_\kappa$ in $\mathcal{M}^{=\lambda}_{\mathsf{PL}}$ as in Theorem 8.1 resp. 8.2. Assume for a contradiction that there is an $M'_{\lambda'} \in \mathcal{M}^{=\lambda'}_{\mathsf{PL}}(\Sigma)$ such that $M'_{\lambda'} \prec M_\lambda$ (therefore $\lambda' \geq \lambda$). Then let $M'_\kappa \sqsubseteq M_\kappa$ in $\mathcal{M}^{=\kappa}_{\mathsf{PL}}$ as in Theorem 8.1 resp. 8.2. By Lemma 8.2 (with $M_1 = M'_\kappa$, $M_2 = M'_{\lambda'}$, $M_3 = M_\lambda$ and $M_4 = M_\kappa$) we have $M'_\kappa \prec M_\kappa$ which contradicts the $\prec$-minimality of M_κ. Thus, $M_\lambda \in \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$. □

Although we have no equivalent result to Corollary 4 for the nonmonotonic setting, we have the following weakened version:

Corollary 5. *Where* $\star \in \{g, p, l\}$, $\dagger \in \{=, =_c, \geq, \supseteq\}$, $\prec = \prec^{\dagger,\star}_\Phi$, *and* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec\rangle$, *for all sets of* $\mathcal{L}$*-sentences* $\Sigma \cup \{A\}$,

$$\Sigma \Vdash_{\mathsf{nmPL}} A \text{ iff } \Sigma \Vdash^{\leq\aleph_1}_{\mathsf{nmPL}} A.$$

Finally, for the global (Thm. 8.4) or f-based orderings (Thm. 8.5), Löwenheim-Skolem downwards holds for the standard version:

Theorem 8.4 (Löwenheim-Skolem, downwards, nonmonotonic, global)**.** *Where* $\dagger \in \{=, =_c, \geq, \supseteq\}$, $\prec = \prec^{\dagger,g}_\Phi$, ***LS-Ab*** *(downwards) holds for* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec\rangle$.

Proof. Let $M_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{nmPL}}(\Sigma)$ and $\aleph_0 \leq \lambda < \kappa$. By Theorem 8.2, there is an $M_\lambda \in \mathcal{M}^{=\lambda}_{\mathsf{PL}}(\Sigma)$ that satisfies Items 1–4 of Definition 16.

Consider first the case $\mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa)) < \kappa$. Then, $\mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}(M_\kappa) = \emptyset$. Also, by Items 1-3 of Definition 16, $\mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) = \mathsf{card}(\bigcup_{i \in \mathbb{P}_{\mathsf{fin}}(M_\kappa)} \mathsf{Ab}_i(M_\kappa)) + \mathsf{card}(\mathbb{P}_{\mathsf{coinf}}(M_\kappa) \times \aleph_0)$. Hence, $\mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) \leq \aleph_0$ and $\mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) \leq \mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa))$. Assume for a contradiction that $M_\lambda \notin \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$. Thus, there is a $M' \in \mathcal{M}^{\lambda'}_{\mathsf{PL}}(\Sigma)$ for which $\mathsf{card}(\mathsf{Ab}_\Phi(M')) < \mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda))$ and $\lambda' \geq \lambda$. Therefore, $\mathsf{card}(\mathsf{Ab}_\Phi(M')) = n$ for some $n \in \mathbb{N}$. Note that $\lambda' \neq \kappa$ since $n < \mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa))$ and by the $\prec$-minimality of M_κ. By Theorem 8.1 (if $\lambda' < \kappa$) resp. Theorem 8.1 (if $\lambda' > \kappa$), there is a model $M'_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{PL}}(\Sigma)$ for which $M'_\kappa \sqsubseteq M'$. Thus, $\mathsf{card}(\mathsf{Ab}_\Phi(M'_\kappa)) = n < \mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) \leq \mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa))$. This contradicts the $\prec$-minimality of M_κ. So, $M_\lambda \in \mathcal{M}^{=\lambda}_{\mathsf{nmPL}}(\Sigma)$.

Consider now the case in which $\mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa)) = \kappa$. Assume first for a contradiction that $\mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}(M_\kappa) = \emptyset$. Then, by Items 1-3 of Definition 16, $\mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) = \mathsf{card}(\bigcup_{i \in \mathbb{P}_{\mathsf{fin}}(M_\kappa)} \mathsf{Ab}_i(M_\kappa)) + \mathsf{card}(\mathbb{P}_{\mathsf{coinf}}(M_\kappa) \times \aleph_0) = \mathsf{card}(\bigcup_{i \in \mathbb{P}_{\mathsf{fin}}(M_\lambda)} \mathsf{Ab}_i(M_\lambda)) + \mathsf{card}(\mathbb{P}_{\mathsf{coinf}}(M_\kappa) \times \aleph_0)$. Note that $\mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) \leq \aleph_0$. By Theorem 8.1, there is a model $M'_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{PL}}(\Sigma)$ for which $M'_\kappa \sqsubseteq M_\lambda$. Hence, $\mathsf{card}(\mathsf{Ab}_\Phi(M'_\kappa)) = \mathsf{card}(\bigcup_{i \in \mathbb{P}_{\mathsf{fin}}(M_\lambda)} \mathsf{Ab}_i(M_\lambda)) + \mathsf{card}(\mathbb{P}_{\mathsf{coinf}}(M_\kappa) \times \aleph_0) = \mathsf{card}(\mathsf{Ab}_\Phi(M_\lambda)) \leq \aleph_0$. But then, $\mathsf{card}(\mathsf{Ab}_\Phi(M'_\kappa)) < \mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa))$, which contradicts the $\prec$-minimality of M_κ.

Thus, $\mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}(M_\kappa) \neq \emptyset$. Assume for a contradiction that $M_\lambda \notin \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$. Thus, there is a $\lambda' \geq \lambda$ and a $M'_{\lambda'} \in \mathcal{M}^{=\lambda'}_{\mathsf{PL}}(\Sigma)$ for which $M'_{\lambda'} \prec M_\lambda$. So, $\mathsf{card}(\mathsf{Ab}_\Phi(M'_{\lambda'})) < \lambda$.

In case $\lambda = \aleph_0$, $\mathsf{card}(\mathsf{Ab}_\Phi(M'_{\lambda'})) < \aleph_0$ and therefore $\mathbb{P} \setminus \mathbb{P}_{\mathsf{fin}}(M'_{\lambda'}) = \emptyset$. By Theorem 8.2 resp. 8.1 there is a $M'_\kappa \in \mathcal{M}^{=\kappa}_{\mathsf{PL}}(\Sigma)$ for which $M'_\kappa \sqsubseteq M_\kappa$.

logic L	LS	$\Vdash_{\mathsf{L}} = \Vdash_{\mathsf{L}}^{\geq\kappa}$
PL	LS-Ab	$\aleph_0$
$\langle \mathsf{PL}, \prec_\Phi^{\dagger,\star} \rangle$	LS-Ab->ω	$\aleph_1$
$\langle \mathsf{PL}, \prec_\Phi^{\ddagger,g} \rangle$	LS-Ab (down)	$\aleph_0$
$\langle \mathsf{PL}, \prec_\Phi^{\mathsf{f},\star} \rangle$	LS-Ab (down)	$\aleph_0$

Table 6: Results related to the Löwenheim-Skolem Theorems, where $\star \in \{g, p, l\}$, $\dagger \in \{=, =_c, \supseteq, \geq\}$ and $\ddagger \in \{\mathsf{f}, =, =_c, \supseteq, \geq\}$.

So, $\mathbb{P} \setminus \mathbb{P}_{\mathsf{fin}}(M'_\kappa) = \emptyset$ and $\mathsf{card}(\mathsf{Ab}_\Phi(M'_\kappa)) \leq \aleph_0$. But then $M'_\kappa \prec M_\kappa$ which contradicts the $\prec$-minimality of M_κ.

Suppose then that $\lambda > \aleph_0$. By Theorem 8.2 resp. 8.1 there is a $M'_\kappa \in \mathcal{M}_{\mathsf{PL}}^{=\kappa}(\Sigma)$ for which $M'_\kappa \sqsubseteq M_\kappa$. By Lemma 8.2, $M'_\kappa \prec M_\kappa$. □

Corollary 6. *Where* $\dagger \in \{=, =_c, \geq, \supseteq\}$, $\prec = \prec_\Phi^{\dagger,g}$, *and* $\Sigma \cup \{A\} \subseteq \mathsf{sent}_{\mathcal{L}}$, $\Sigma \Vdash_{\mathsf{nmPL}} A$ *iff* $\Sigma \Vdash_{\mathsf{nmPL}}^{\leq\aleph_0} A$.

Theorem 8.5 (Löwenheim-Skolem, downwards, nonmonotonic, f)**.** *Where* $\star \in \{g, p, l\}$ *and* $\prec = \prec_\Phi^{\mathsf{f},\star}$, ***LS-Ab*** *(downwards) holds for* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$.

Proof. Where $\mathcal{M}_{\mathsf{nmPL}}^{=\kappa}(\Sigma) = \min_\prec(\mathcal{M}(\Sigma)) \cap \mathcal{M}^{=\kappa}$, suppose $M_\kappa \in \mathcal{M}_{\mathsf{nmPL}}^{=\kappa}(\Sigma)$ and $\aleph_0 \leq \lambda < \kappa$. By Theorem 8.2, there is a $M_\lambda \in \mathcal{M}_{\mathsf{PL}}^{=\lambda}(\Sigma)$ for which $M_\lambda \sqsubseteq M_\kappa$. By Lemma 8.1, for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M_\lambda)) \leq \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$. Assume for a contradiction that there is a $M' \in \mathcal{M}_{\mathsf{PL}}^{=\lambda}(\Sigma)$ such that $M' \prec M_\lambda$. Clearly then also $M' \prec M_\kappa$. This contradicts the minimality of M_κ. So, $M_\lambda \in \mathcal{M}_{\mathsf{nmPL}}^{=\lambda}(\Sigma)$. □

Corollary 7. *Where* $\star \in \{g, p, l\}$, $\prec = \prec_\Phi^{\mathsf{f},\star}$, *and* $\Sigma \cup \{A\} \subseteq \mathsf{sent}_{\mathcal{L}}$, $\Sigma \Vdash_{\mathsf{nmPL}} A$ *iff* $\Sigma \Vdash_{\mathsf{nmPL}}^{\leq\aleph_0} A$.

9 Conclusion

In this paper we have presented a study of a class of nonmonotonic inconsistency-tolerant logics. The underlying rationale is to interpret a premise set as consistently as possible. For this we only considered models of the premise set in a given paraconsistent base logic PL that give rise to minimally many contradictions. Given a model M, we count for each of the given predicates how many inconsistent individuals they give rise to and prefer those models that give rise to less. Thus, we follow an *extensional approach* in that we count inconsistent individuals in the extension of predicates as opposed to linguistic representations of contradictions. Also, the approach is *quantitative* in that we consider the cardinality of the inconsistent extension of predicates. In contrast, in the qualitative approach that was originally proposed by Priest (following the extensional approach) and Batens (following the linguistic approach) a model is better than another one if its set of contradictions is a subset of the other sets of contradictions. By turning quantitative we solved the problem of a lack of strong reassurance that underlies the extensional approach and demonstrated that our approach is based on a robust meta-theory by, for instance, validating the cumulativity property and specific versions of the Löwenheim-Skolem theorems. Our approach is highly modular in that the resulting nonmonotonic logics can be based on any paraconsistent and finitely-valued Tarskian base logic with truth-functional connectives.

Some paths for future exploration are left open, such as to extend our study to infinitely-valued, non-deterministic, and non-truth-functional paraconsistent logics,[38] or, as suggested in [Priest, 2014], to a second-order language.

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References

[Asenjo et al., 1966] Asenjo, F. G. et al. (1966). A calculus of antinomies. *Notre Dame Journal of Formal Logic*, 7(1):103–105.

[Avron, 2005] Avron, A. (2005). A non-deterministic view on non-classical negations. *Studia Logica*, 80(2-3):159–194.

[Avron and Zamansky, 2005] Avron, A. and Zamansky, A. (2005). Quantification in non-deterministic multi-valued structures. In *35th International Symposium on Multiple-Valued Logic (ISMVL'05)*, pages 296–301. IEEE.

[Batens, 1999] Batens, D. (1999). Linguistic and ontological measures for comparing the inconsistent parts of models. *Logique et Analyse*, pages 5–33.

[Batens, 2000] Batens, D. (2000). Minimally abnormal models in some adaptive logics. *Synthese*, 125(1-2):5–18.

[Batens, 2007] Batens, D. (2007). A universal logic approach to adaptive logics. *Logica universalis*, 1(1):221–242.

[Crabbé, 2011] Crabbé, M. (2011). Reassurance for the logic of paradox. *The Review of Symbolic Logic*, 4(03):479–485.

[Ferguson, 2014] Ferguson, T. M. (2014). On non-deterministic quantification. *Logica Universalis*, 8(2):165–191.

[Ferguson, 2020] Ferguson, T. M. (2020). Variations on the collapsing lemma. In *Graham Priest on Dialetheism and Paraconsistency*, pages 249–270. Springer.

[38] First order accounts as in [Avron and Zamansky, 2005] and [Ferguson, 2014] will be useful for this.

[Gabbay, 1985] Gabbay, D. M. (1985). Theoretical foundations for non-monotonic reasoning in expert systems. In *Logics and models of concurrent systems*, pages 439–457. Springer.

[Horty, 1994] Horty, J. F. (1994). Some direct theories of nonmonotonic inheritance. *Handbook of logic in artificial intelligence and logic programming*, 3:111–187.

[Kraus et al., 1990] Kraus, S., Lehmann, D., and Magidor, M. (1990). Non-monotonic reasoning, preferential models and cumulative logics. *Artificial intelligence*, 44(1-2):167–207.

[Lehmann and Magidor, 1992] Lehmann, D. and Magidor, M. (1992). What does a conditional knowledge base entail? *Artificial intelligence*, 55(1):1–60.

[Makinson, 2005] Makinson, D. (2005). Bridges from classical to nonmonotonic logic, volume 5 of texts in computing. king's college publications. *London, UK.*

[Makinson and Van Der Torre, 2000] Makinson, D. and Van Der Torre, L. (2000). Input/output logics. *Journal of Philosophical Logic*, 29(4):383–408.

[Meheus et al., 2013] Meheus, J., Straßer, C., and Verdée, P. (2013). Which style of reasoning to choose in the face of conflicting information? *Journal of logic and Computation*, 26(1):361–380.

[Omori and Wansing, 2017] Omori, H. and Wansing, H. (2017). 40 years of fde: an introductory overview. *Studia Logica*, 105(6):1021–1049.

[Priest, 1979] Priest, G. (1979). The logic of paradox. *Journal of Philosophical logic*, 8(1):219–241.

[Priest, 1988] Priest, G. (1988). Consistency by default. *Technical Report Reasoning Project, Australian National University.*

[Priest, 1991] Priest, G. (1991). Minimally inconsistent LP. *Studia Logica*, 50(2):321–331.

[Priest, 2002] Priest, G. (2002). Paraconsistent logic. In *Handbook of philosophical logic*, pages 287–393. Springer.

[Priest, 2014] Priest, G. (2014). *One: Being an Investigation into the Unity of Reality and of its Parts, including the Singular Object which is Nothingness.* Oxford University Press.

[Priest, 2017] Priest, G. (2017). What if? the exploration of an idea. *The Australasian Journal of Logic*, 14(1).

[Priest et al., 2018] Priest, G., Tanaka, K., and Weber, Z. (2018). Paraconsistent Logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy.* Metaphysics Research Lab, Stanford University, summer 2018 edition.

[Reiter, 1980] Reiter, R. (1980). A logic for default reasoning. *Artificial intelligence*, 13(1-2):81–132.

[Rott, 2017] Rott, H. (2017). Stability and scepticism in the modelling of doxastic states: Probabilities and plain beliefs. *Minds and Machines*, 27(1):167–197.

[Šešelja and Straßer, 2014] Šešelja, D. and Straßer, C. (2014). Concerning peter vickers's recent treatment of 'paraconsistencitis'. *International Studies in the Philosophy of Science*, 28(3):325–340.

[Straßer, 2014] Straßer, C. (2014). *Adaptive logics for defeasible reasoning*. Springer.

[Vickers, 2013] Vickers, P. (2013). *Understanding inconsistent science*. OUP Oxford.

APPENDIX

A Outline of the proof of strong reassurance

Before we jump into the rather technical proofs (see Sections B–E of this Appendix), let us outline the basic underlying ideas. For this outline we consider the logic $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ where $\prec = \prec_\Phi^{=_c,l}$ for some set of $\mathcal{L}$-formulas $\Phi = \langle \alpha_i(x_1, \ldots, x_{a_i}) \mid i \in \mathbb{P} \rangle$ and PL satisfies the requirements from Section 2.1. We discuss other cases $\prec_\Phi^{\pi,\star}$ where $\star \in \{g, p, l\}$ and $\pi \in \{\mathbf{f}, =, =_c, \supseteq, \geq\}$ in Section E. Without loss of generality we will assume that $\mathbb{P} = \mathbb{N}$, but the proof generalizes to $\mathbb{P}$ being any initial sequence of $\mathbb{N}$. We write $\alpha_i(x_1, \ldots, x_{a_i})$ to signify that the (only) free variables in α_i are $x_1, \ldots, x_{a_i}$. We will also write $\alpha_i(\mathbf{x})$ for $\alpha_i(x_1, \ldots, x_{a_i})$ where $\mathbf{x} = x_1, \ldots, x_{a_i}$.

Given a model $M_\top = \langle \mathcal{D}_\top, v_\top \rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \backslash \min_\prec (\mathcal{M}_{\mathsf{PL}}(\Sigma))$, we have to show that there is a $M_\bot \in \min_\prec (\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M_\bot \prec M_\top$. Let $\kappa = \mathsf{card}(\mathcal{D}_\top)$. We will address the case where $\kappa \geq \aleph_0$. The other case $\kappa < \aleph_0$ will be treated in similar but simpler ways (see Remark D.1).

The following definitions will be useful for this.

Definition 18. $\mathcal{M}_\downarrow =_{\mathsf{df}} \big\{ M \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \mid M \prec M_\top \big\}$.

We have to show that $\min_\prec(\mathcal{M}_\downarrow) \neq \emptyset$. We first select models in $\mathcal{M}_\downarrow$ that minimize abnormalities in α_1, then out of these those that minimize abnormalities in α_2, and so on. Some definitions will help to make the idea precise.

Definition 19. *For any $i \in \mathbb{P}$ we define $\prec_i \subseteq \mathcal{M}_\downarrow \times \mathcal{M}_\downarrow$ and $\preceq_i \subseteq \mathcal{M}_\downarrow \times \mathcal{M}_\downarrow$, where*

$$M \prec_i M' \quad \text{iff} \quad \mathsf{card}(\mathsf{Ab}_i(M)) < \mathsf{card}(\mathsf{Ab}_i(M')),$$
$$M \preceq_i M' \quad \text{iff} \quad \mathsf{card}(\mathsf{Ab}_i(M)) \leq \mathsf{card}(\mathsf{Ab}_i(M')).$$

Furthermore, let $\mathcal{M}_0 =_{\mathsf{df}} \mathcal{M}_\downarrow$, $\mathcal{M}_1 =_{\mathsf{df}} \min_{\prec_1}(\mathcal{M}_\downarrow)$, and $\mathcal{M}_{i+1} =_{\mathsf{df}} \min_{\prec_{i+1}}(\mathcal{M}_i)$.

Three key insights will be used in our proof:

1. $\mathcal{M}_i \neq \emptyset$ for each $i \in \mathbb{P}$ (Lemma D.1),
2. $\bigcap_{i \geq 1} \mathcal{M}_i \subseteq \min_\prec(\mathcal{M}_\downarrow)$ (Lemma D.2), and
3. $\bigcap_{i \geq 1} \mathcal{M}_i \neq \emptyset$ (Corollary 10).

From this follows that $\min_\prec(\mathcal{M}_\downarrow) \neq \emptyset$, what is to be shown.

Step 3 will be the most demanding. We will construct a model $M \in \bigcap_{i \geq 1} \mathcal{M}_i$. For this we will enhance the premise set Σ to gain more control in the construction of M. In order to explain the nature of this extended premise set we first define:

Definition 20. *Let* α_i be l-finitary in $\mathcal{M}_\downarrow$ *iff for any/all*[39] $M' \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M')) = l$. *We let*

$$\mathbb{P}_{\mathsf{fin}} =_{\mathsf{df}} \big\{(i,l) \in (\mathbb{P} \times \mathbb{N}) \mid \alpha_i \textit{ is } l\textit{-finitary in } \mathcal{M}_\downarrow\big\}.$$

Let α_i be co-infinitary relative to $\mathcal{M}_\downarrow$ *iff for any/all* $M' \in \mathcal{M}_i$, $\aleph_0 \leq \mathsf{card}(\mathsf{Ab}_i(M')) < \kappa$. *Let*

$$\mathbb{P}_{\mathsf{coinf}} =_{\mathsf{df}} \big\{i \in \mathbb{P} \mid \alpha_i \textit{ is co-infinitary in } \mathcal{M}_\downarrow\big\}, \textit{ and}$$
$$\mathbb{P}_{\mathsf{fin/coinf}} =_{\mathsf{df}} \mathbb{P}_{\mathsf{coinf}} \cup \big\{i \in \mathbb{P} \mid \exists l((i,l) \in \mathbb{P}_{\mathsf{fin}})\big\}.$$

In view of our abnormality-sensitive version of Löwenheim-Skolem (**LS-Ab**, Theorems 8.1 and 8.2) we will observe in Lemma D.4 that

- for all $i \in \mathbb{P}_{\mathsf{coinf}}$ and all $M \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M)) = \aleph_0$.

Clearly, for all $(i,l) \in \mathbb{P}_{\mathsf{fin}}$ and all $M' \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M')) = l$. That means that the model $M_\perp$ we are searching for has to satisfy the following three demands:

1. for all $(i,l) \in \mathbb{P}_{\mathsf{fin}}$, $\mathsf{card}(\mathsf{Ab}_i(M_\perp)) = l$,
2. for all $i \in \mathbb{P}_{\mathsf{coinf}}$, $\mathsf{card}(\mathsf{Ab}_i(M_\perp)) = \aleph_0$, and
3. for all $i \in \mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}$, $\mathsf{card}(\mathsf{Ab}_i(M_\perp)) = \kappa$.

Enhancing our premise Σ to a set $\mathcal{T}_\star$ will help us expressing some of these conditions formally. We will proceed in a similar way as in the proof of the upwards version of the Löwenheim-Skolem Theorem. We enhance our language $\mathcal{L}$ with κ many new constants k_i, resulting in $\mathcal{L}_\kappa$, for which we require that they refer to different entities and that they are "normal" relative to all $i \in \mathbb{P}_{\mathsf{fin/coinf}}$. That is, the formula $\alpha_i(k_1, \ldots, k_{a_i})$ has to have a truth-value in $\mathbb{V} \setminus \mathbb{A}$.

In many paraconsistent logics this is not expressible since the available negation is not sufficiently expressive (LP is a case in point). Similarly, the non-identity of new constants may be inexpressible in our given language. In order to prove the satisfiability of $\mathcal{T}_\star$ in the enriched language we will utilize a "Henkin-extended" language in which witnesses are provided for quantified formulas. For this we need to express that, in every model, $A(c_\exists^A)$ (where $c_\exists^A$ is our "Henkin-witness" for $\exists x A(x)$) has the same truth-value as $\exists x A(x)$. Again, a strong bi-conditional that can express such dependencies is typically not available in a given paraconsistent logic with no detachable conditional. It will be our task to conservatively extend our given language adequately, i.e., in such a way that the generality of our results remains unrestricted.[40]

In sum, our first sub-task in Section B will be to develop a meta-theory in the conservatively extended language, including a proof of compactness which

[39] In view of the iterative construction of $\langle \mathcal{M}_i \rangle_{i \geq 1}$ it is easy to see that for all $M, M' \in \mathcal{M}_i$ and any $i \geq 1$, $\mathsf{card}(\mathsf{Ab}_i(M)) = \mathsf{card}(\mathsf{Ab}_i(M'))$.

[40] We will also add a conjunction and disjunction to the language in case they are not already available and an operator @ which expresses that a formula has an abnormal truth-value. We show that the extension of the language is conservative.

will help us (a) to express all necessary requirements in $\mathcal{T}_\star$ and (b) to prove the satisfiability of $\mathcal{T}_\star$.

Having a model $M_\star$ of $\mathcal{T}_\star$ is not the end of the story, however. Even if we can ensure that the domain of $M_\star$ has the size κ, and all newly added κ many constants are normal relative to all $i \in \mathbb{P}_{\mathsf{fin/coinf}}$ and refer to different κ many entities, there may still be some $i \in \mathbb{P}_{\mathsf{coinf}}$ for which $\mathsf{card}(\mathsf{Ab}_i(M_\star)) > \aleph_0$. The reason is that having κ many "normal" counter-instances does not warrant that there are not also more than $\aleph_0$ (possibly even κ many) abnormal elements in the domain of $M_\star$. How do we get rid of them? This is where things get a bit tricky. One idea could be to build a term-model for the theory of $M_\star$. But the problem is that with the Henkin enrichment of $\mathcal{L}_\kappa$ we introduce κ many new constants which can serve as witnesses for formulas such as $\exists x A(x) \in \mathcal{L}_\kappa$. So, whenever for a $i \in \mathbb{P}_{\mathsf{coinf}}$, $\mathsf{Ab}_i(M_\star) > \aleph_0$, we will get more than $\aleph_0$ many witnesses to this effect. Clearly, we would introduce too many witnesses were we to Henkin-enhance $\mathcal{L}_\kappa$.

To avoid this problem, we will only introduce Henkin-witnesses for our base language $\mathcal{L}$ (resulting in $\mathcal{L}_h$): this will result in $\aleph_0$ many new constants in $\mathcal{L}_h$ to serve as witnesses of quantified formulas in $\mathcal{L}_h$, but not for the full $\mathcal{L}_h \cup \mathcal{L}_\kappa$. Clearly, for all $i \in \mathbb{P}_{\mathsf{coinf}}$, there will be $\aleph_0$ many Henkin-witnesses which will be abnormal relative to i. The idea is then to build a term-model based on (the theory of) $M_\star$ and on the constants in $\mathcal{L}_h$ and $\mathcal{L}_\kappa$. Since all constants in $\mathcal{L}_\kappa \setminus \mathcal{L}_h$ are normal relative to all $i \in \mathbb{P}_{\mathsf{fin/coinf}}$, the only candidates to be abnormal are constants in $\mathcal{L}_h$, which –and this is the important bit– are "only" $\aleph_0$ many. This is exactly what we need to satisfy our requirement 2.

The resulting model $M_\perp$ will satisfy requirements 1–3 which suffices to show that $M_\perp \in \bigcap_{i\geq 1} \mathcal{M}_i$ (and so $\bigcap_{i\geq 1} \mathcal{M}_i \neq \emptyset$).

In the remainder of the appendix we will present the proof in all its details. In Appendix B we enrich the language of PL and prove compactness. In Appendix C we prove abnormality-aware versions of the Löwenheim-Skolem theorems. Appendix D provides the core of our proof of Strong Reassurance. We focus there on orderings of the type $\prec_\Phi^{=_c,l}$. Finally, in Appendix E we show Strong Reassurance for the remaining other ordering types from Section 3. Given the results established before this will not require much labor anymore. Finally, in Appendix G we prove a technical lemma of Section 7.

B Compactness for an enriched language

For the proof of the strong reassurance of nmPL in Section D it is key to show that PL is compact.[41] There it will be most useful to employ a more expressive object language. Note that we do not require that the base logic PL comes

[41] In [Ferguson, 2014] the reader finds compactness results for finitely valued non-deterministic logics. However, the proof there is (a) based on a different Fregean semantics (based on a set-theory without function extensionality) and (b) on a construction based on the ultraproduct. For what follows (such as the Löwenheim-Skolem theorems) it will be useful to construct a Henkin model as part of the compactness proof.

readily equipped with the demanded expressive power. If it is not, we can simply superimpose the needed connectives on the language $\mathcal{L}$ underlying PL, resulting in a richer language $\mathcal{L}'$. Super-imposing them is easy from a technical point of view. We will comment more on this after introducing the necessary ingredients. We will show compactness for this enriched language, but the result of course also applies to PL in its original form based on $\mathcal{L}$. $\mathcal{L}'$ is supposed to include the following ingredients:[42]

- an inequality relation $\neq$ for which **InEq** holds;
- an existential quantifier $\exists$ for which **Q0**–**Q2** and**Q**$\exists$ hold;
- a conjunction $\wedge$ for which (Con) holds; and
- a unary connective @ which tracks abnormalities and respects the following schematic truth-table:

$$\begin{array}{c|c} A & @\,A \\ \hline \mathbb{A} & \mathbb{D}\setminus\mathbb{A} \\ \mathbb{V}\setminus\mathbb{A} & \mathbb{V}\setminus\mathbb{D} \end{array} \tag{T@}$$

Furthermore, we require that all connectives are truth-functional, and that all quantifiers satisfy **Q0**–**Q2**.

We will use these ingredients, e.g., to express that there are at least n many elements (1) resp. abnormalities (2) for a given formula $\alpha_i(\mathbf{x})$. To avoid clutter, when using the notation "$\exists\mathbf{x}_1\ldots\exists\mathbf{x}_n\ldots\alpha_i(\mathbf{x}_1)\ldots\alpha_i(\mathbf{x}_n)$" below we suppose that each $\mathbf{x}_i$ has the form $\mathbf{x}_j = x_{j_1},\ldots,x_{j_{a_i}}$ where for all $1\leq j<k\leq n$ and all $1\leq j'<k'\leq a_i$, $x_{j_{j'}}\neq x_{k_{k'}}$. Also, $\exists\mathbf{x}_j$ abbreviates $\exists x_{j_1}\cdots\exists x_{j_{a_i}}$. Finally, we abbreviate $\bigvee_{1\leq l\leq a_i} x_{j_l}\neq x_{k_l}$ by $\mathbf{x}_j\neq\mathbf{x}_k$.

$$\exists x_1\ldots\exists x_n \bigwedge_{1\leq j<i\leq n} x_i\neq x_j \tag{1}$$

$$\exists\mathbf{x}_1\ldots\exists\mathbf{x}_n\left(\bigwedge_{l=1}^{n} @\,\alpha_i(\mathbf{x}_l)\wedge\bigwedge_{1\leq j<j'\leq n}\mathbf{x}_j\neq\mathbf{x}_{j'}\right) \tag{2}$$

The reader may wonder why we require a primitive inequality symbol instead of an equality symbol = that satisfies (**Eq**) and instead of expressing (1) simply by

$$\exists x_1\cdots\exists x_n \bigwedge_{1\leq j<i\leq n}\neg(x_i=x_j) \tag{3}$$

The reason is that in paraconsistent logics such as LP (3) may be true in models which have less than n elements simply because the value of both $x_i = x_j$ and

[42] See Section 2.1 for the definitions of the constraints.

$\neg(x_i = x_j)$ may be i where x_i and x_j refer to the same entity. In contrast, this is not possible with an inequality symbol that satisfies (**InEq**).

As mentioned above, if (some of) these ingredients are not part of PL already, we superimpose them on the language $\mathcal{L}$. Superimposing a set of connectives $\pi_1, \ldots, \pi_n$ of arities $\zeta_1, \ldots, \zeta_n$ on a language $\mathcal{L}$ (supposing no π_i is already in the logical signature of $\mathcal{L}$) is to close the set of well-formed formulas of $\mathcal{L}$ under these connectives. As a result, no π_i will occur in the scope of one of the connectives of $\mathcal{L}$.

By superimposing them we don't disturb the inner workings of PL and obtain, without much ado, a conservative extension PL' of PL. Given our finite set of truth-values $\mathbb{V}$ such connectives and quantifiers can easily be characterized. E.g., where $\mathsf{v}_1 \in \mathbb{D} \setminus \mathbb{A}$ and $\mathsf{v}_0 \in \mathbb{V} \setminus \mathbb{D}$ are fixed, let

$$v(\neq, (d, d')) =_{\mathsf{df}} \begin{cases} \mathsf{v}_1 & \text{if } d \neq d' \\ \mathsf{v}_0 & \text{else.} \end{cases} \tag{4}$$

Similarly, if there is no conjunction in the language that satisfies (Con), one may define $\wedge$ by imposing a total order $\sqsubset$ on $\mathbb{V}$ that separates $\mathbb{V} \setminus \mathbb{D}$ from $\mathbb{D}$ such that for all $\mathsf{v} \in \mathbb{V} \setminus \mathbb{D}$ and all $\mathsf{v}' \in \mathbb{D}$, $\mathsf{v} \sqsubset \mathsf{v}'$. From that just define

$$f_\wedge(\mathsf{v}, \mathsf{v}') =_{\mathsf{df}} \min_\sqsubset(\mathsf{v}, \mathsf{v}'). \tag{5}$$

The exact nature of $\sqsubset$ does not matter for our purposes.

Most probably there will be no operator @ which satisfies (T@) in our language. In that case we define it by

$$f_@(\mathsf{v}) =_{\mathsf{df}} \begin{cases} \mathsf{v}_1 & \mathsf{v} \in \mathbb{A} \\ \mathsf{v}_0 & \text{else.} \end{cases} \tag{6}$$

Finally, if there is no existential quantifier in the language that fulfills **Q0**–**Q2** and **Q**$\exists$ we may define $\exists$ via

$$f_\exists(\mathcal{V}) =_{\mathsf{df}} \max_\sqsubset(\mathcal{V}). \tag{7}$$

Let the enriched language be $\mathcal{L}'$ and the enriched logic based on $\mathcal{L}'$ be PL'. This enrichment of PL leads to a conservative extension.

Proposition B.1.

1. *For every $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_{\mathsf{PL}}$ there is an $M' = \langle \mathcal{D}, v' \rangle \in \mathcal{M}_{\mathsf{PL}'}$ such that (a) for all $A \in \mathsf{sent}_{\mathcal{L}[\mathcal{D}]}$, $v_M(A) = v_{M'}(A)$ (where $\mathcal{L}[\mathcal{D}]$ is the language $\mathcal{L}$ enriched with pseudo-constants $\overline{d}$ for every $d \in \mathcal{D}$ which are interpreted by $v(\overline{d}) = v'(\overline{d}) = d$), (b) for all predicates P_i (incl. identities, non-identities and congruences), $v(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v'(P_i, (d_1, \ldots, d_{\mathtt{i}}))$ and (c) for all $i \in \mathbb{P}$, $\mathsf{Ab}_i(M) = \mathsf{Ab}_i(M')$.*

2. *For every $M' \in \mathcal{M}_{\mathsf{PL}'}$ there is an $M \in \mathcal{M}_{\mathsf{PL}}$ such that (a) for all $A \in \mathsf{sent}_{\mathcal{L}[\mathcal{D}]}$, $v_M(A) = v_{M'}(A)$ and (b) for all predicates P_i (incl. identities, non-identities and congruences), $v(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v'(P_i, (d_1, \ldots, d_{\mathtt{i}}))$ and (c) for all $i \in \mathbb{P}$, $\mathsf{Ab}_i(M) = \mathsf{Ab}_i(M')$.*

Proof. Given $M = \langle \mathcal{D}, v\rangle \in \mathcal{M}_{\mathsf{PL}}$ we let $M' = \langle \mathcal{D}, v'\rangle$, where v' is defined just as v on the language $\mathcal{L}$ and atomic formulas that are inequalities are characterized as in (4). The claim that $v_M(A) = v_{M'}(A)$ is then shown inductively over the length of A. The proof is trivial in view of v' being identical to v on $\mathcal{L}$.

Given $M' = \langle \mathcal{D}, v'\rangle \in \mathcal{M}_{\mathsf{PL}'}$ we let $M = \langle \mathcal{D}, v\rangle$, where v is the restriction of v' to $\mathcal{L}$. Again, $v_M(A) = v_{M'}(A)$ is then shown inductively over the length of A. The proof is trivial in view of v' being identical to v on $\mathcal{L}$. ☐

Corollary 8. PL' *conservatively extends* PL*, i.e., for all set of* $\mathcal{L}$*-sentences* Σ *and every* $\mathcal{L}$*-sentence* A*,* $\Sigma \vDash_{\mathsf{PL}} A$ *iff* $\Sigma \vDash_{\mathsf{PL}'} A$*.*

In view of the conservativity of PL' relative to PL, we will henceforth show Strong Reassurance for the richer language $\mathcal{L}'$. The result then immediately applies to PL as well. To avoid clutter in the notation we will henceforth skip the prime notation for $\mathcal{L}'$ and PL'.

To express that, for instance, there are maximally n many abnormalities for a formula α, or to express that our Henkin witnesses have in every interpretation the same truth-value as the quantified formula for which they are witnesses, we go one step further in generalizing the underlying language $\mathcal{L}$ of PL by working with io-formulas ("io" for "input/output").[43]

Definition 21. *Where* $\mathcal{L}$ *is a 1st-order language, an* io-formula [io-sentence] *in* $\mathcal{L}$ *is of the form* (A, B) *or* $(\emptyset, B)$ *or* $(A, \emptyset)$ *where* A *and* B *are formulas [sentences] in* $\mathcal{L}$*. An* io-theory *is a set of io-sentences, which will be denoted by calligraphic letters* $\mathcal{S}, \mathcal{T}$*, etc. We write* $\mathsf{sent}^{\mathsf{io}}_{\mathcal{L}}$ *resp.* $\mathsf{wffs}^{\mathsf{io}}_{\mathcal{L}}$ *for the set of io-sentences resp. well-formed io-formulas in* $\mathcal{L}$*.*

The interpretation of (A, B) is that B holds or A doesn't hold. The interpretation of $(\emptyset, B)$ is that B holds, and that of $(A, \emptyset)$ is that A doesn't hold.

We define a generalized validity relation to capture this intuition:

Definition 22. *Where* M *is a* PL*-model,* $M \models^{\mathsf{io}} (A, B)$ *iff* $M \models B$ *or* $M \not\models A$*. And similarly,* $M \models^{\mathsf{io}} (\emptyset, B)$ *iff* $M \models B$*, and* $M \models^{\mathsf{io}} (A, \emptyset)$ *iff* $M \not\models A$*.*

For a given io-theory $\mathcal{T}$ *we write* $\mathcal{M}_{\mathsf{PL}}(\mathcal{T})$ *for the class of models* M *for which* $M \models^{\mathsf{io}} (A, B)$ *for all* $(A, B) \in \mathcal{T}$*.*

This way we can express that there are maximally n many abnormalities for $\alpha_i(\mathbf{x})$ (where $i \in \mathbb{P}$) by (recall the notation introduced in the context of Eq. (2)):

$$M \models^{\mathsf{io}} \left(\left(\exists \mathbf{x}_1 \cdots \exists \mathbf{x}_{n+1} \left(\bigwedge_{j=1}^{n+1} @\, \alpha_i(\mathbf{x}_j) \wedge \bigwedge_{1 \leq j < j' \leq n+1} \mathbf{x}_j \neq \mathbf{x}_{j'} \right) \right), \emptyset \right)$$

[43] One finds these kind of syntactic units with the same interpretation in Input/Output logics [Makinson and Van Der Torre, 2000]. One can also think of io-formulas as expressing a superimposed and detachable implication that cannot be nested. Note that some paraconsistent logics do not have a detachable implication (e.g., standard LP).

Indeed this expresses that

$$M \not\models \exists \mathbf{x}_1 \cdots \exists \mathbf{x}_{n+1} \left(\bigwedge_{j=1}^{n+1} @\, \alpha_i(\mathbf{x}_j) \wedge \bigwedge_{1 \leq j < j' \leq n+1} \mathbf{x}_j \neq \mathbf{x}_{j'} \right)$$

and therefore there can be maximally n different abnormal entities in the extension of $\alpha_i(\mathbf{x})$.

Remark B.1. *We proceed by utilizing io-formulas because –in the general case– connectives such as a classical negation, a detachable implication, etc., that could be used to express the same, may not be available in the language $\mathcal{L}$ of the underlying* PL. *We found working with io-formulas as less intrusive and more elegant to work with than super-imposing a classical negation, a classical co-implication, etc., on $\mathcal{L}$ (e.g., we don't need to bother with truth-tables over $\mathbb{V}$, questions of nesting, etc.). Since this choice has no impact on the studied system and its language $\mathcal{L}$, it is inconsequential for the main results of this paper. Note also that the presence of an inequality symbol that satisfies (**InEq**) is in general not sufficient to define a classical negation.*

For technical reasons that have to do with the construction of a term-model for the compactness proof (see, e.g., Definition 27), we will work with sentences that are annotated with truth-values.

Definition 23. *A $\mathbb{V}$-annotated* io-formula *[resp.* sentence*] (in $\mathcal{L}$) is given by $(A, B, \mathcal{V})$, where $A, B \in \mathsf{wffs}_{\mathcal{L}}$ [resp. $A, B \in \mathsf{sent}_{\mathcal{L}}$] and $\mathcal{V} \subseteq \mathbb{V}$ is non-empty. Where $\mathcal{V}$ is a singleton $\{\mathsf{v}\}$ we write (A, B, v). We write $\mathsf{sent}_{\mathcal{L}}^{\mathbb{V}}$ [resp. $\mathsf{wffs}_{\mathcal{L}}^{\mathbb{V}}$] for the set of $\mathbb{V}$-annotated io-sentences [resp. formulas].*

Fact B.1. *The set $\mathsf{sent}_{\mathcal{L}}^{\mathbb{V}}$ is countable.*

This follows by basic cardinality calculations in view of the fact that $\mathcal{L}$ is countable and $\mathbb{V}$ is finite.

We generalize $\models$ and $\models^{\mathsf{io}}$ for PL-models M as expected:

Definition 24. *Where $(A, B, \mathcal{V})$ is an $\mathbb{V}$-annotated io-formula, we define:*

$$M \models^{\mathbb{V}} (A, B, \mathcal{V}) \quad \textit{iff} \quad v_M(A) \notin \mathcal{V} \textit{ or } v_M(B) \in \mathcal{V}.$$

*A $\mathbb{V}$-*annotated theory *is a set of $\mathbb{V}$-annotated io-sentences. We again use calligraphic letter to denote such theories. For a given $\mathbb{V}$-annotated theory $\mathcal{T}$ we write $\mathcal{M}_{\mathsf{PL}}(\mathcal{T})$ for the class of models M for which $M \models^{\mathbb{V}} (A, B, \mathcal{V})$ for all $(A, B, \mathcal{V}) \in \mathcal{T}$.*

Fact B.2. *We have: $M \models^{\mathsf{io}} (A, B)$ iff $M \models^{\mathbb{V}} (A, B, \mathbb{D})$.*

Satisfiability and consistency are then defined in the standard way:

Definition 25. *A set of sentences Σ [io-theory $\mathcal{T}$, $\mathbb{V}$-annotated theory $\mathcal{T}$] is*

1. satisfiable *iff $\mathcal{M}_{\mathsf{PL}}(\Sigma) \neq \emptyset$ [$\mathcal{M}_{\mathsf{PL}}(\mathcal{T}) \neq \emptyset$],*

2. consistent *iff every finite subset of it is satisfiable.*

Below we show the following generalized version of compactness for $\mathbb{V}$-annotated theories, namely:

Proposition B.2 ($\mathbb{V}$-Compactness)**.** *Where $\mathcal{T}$ is a $\mathbb{V}$-annotated theory, $\mathcal{T}$ is satisfiable iff it is consistent.*

Before proving it, we point out that in view of it we immediately get compactness for io-theories (Proposition B.3) and for PL (Corollary 9):

Proposition B.3 (io-Compactness)**.** *Where $\mathcal{T}$ is an io-theory, $\mathcal{T}$ is satisfiable iff it is consistent.*

For the proof of Proposition B.2 we utilize the following lemma.

Lemma B.1. *Where $\mathcal{T}$ is an io-theory and $\mathcal{T}^v = \{(A, B, \mathbb{D}) \mid (A, B) \in \mathcal{T}\}$: $\mathcal{T}$ is satisfiable iff $\mathcal{T}^v$ is satisfiable.*

Proof. Suppose $\mathcal{T}$ is satisfiable. Let M be a model of $\mathcal{T}$. For every $(A, B) \in \mathcal{T}$, $M \models B$ or $M \not\models A$. Thus, $v_M(B) \in \mathbb{D}$ or $v_M(A) \notin \mathbb{D}$. Thus, M is a model of $\mathcal{T}^v$. The other direction is analogous. □

Proof of Proposition B.3. Suppose the io-theory $\mathcal{T}$ is not satisfiable. Let $\mathcal{T}^v = \{(A, B, \mathbb{D}) \mid (A, B) \in \mathcal{T}\}$. By Lemma B.1, $\mathcal{T}^v$ is not satisfiable. By $\mathbb{V}$-compactness (Proposition B.2), there is a finite $\mathcal{T}^v_f \subseteq \mathcal{T}^v$ that is not satisfiable. By Lemma B.1, also $\mathcal{T}_f = \{(A, B) \mid (A, B, \mathbb{D}) \in \mathcal{T}^v_f\}$ is not satisfiable.

If the io-theory $\mathcal{T}$ is satisfiable then it trivially is consistent as well. □

Since any set of $\mathcal{L}$-sentences Σ can be expressed by $\big\{(\emptyset, A) \mid A \in \Sigma\big\}$ as an io-theory, we immediately get compactness for PL.

Corollary 9 (Compactness)**.** *Where Σ is a set of $\mathcal{L}$-sentences, Σ is satisfiable iff it is consistent.*

For proving the right-to-left direction of Proposition B.2, we proceed as usual by building a term-model based on the Henkin-enrichment of our language $\mathcal{L}$. In particular, given a consistent $\mathbb{V}$-annotated theory $\mathcal{T}$, we show that its Henkin extension $\mathcal{T}_h$ (Definitions 26 and 27) is consistent as well (Lemma B.4). We then build a maximal consistent extension $\mathcal{T}_m$ of $\mathcal{T}_h$ by the usual Lindenbaum construction (Lemma B.5). The purpose of $\mathcal{T}_m$ is to build a term-model M of $\mathcal{T}$ to demonstrate its satisfiability. For this we first show that $\mathcal{T}_m$ is complete in several meanings of the term (Definition 28 and Lemmas B.6, B.7, B.8, B.10). This allows us to "read off" the term-model: $v_M(A) = \mathsf{v}$ iff $(\emptyset, A, \mathsf{v}) \in \mathcal{T}_m$.

Definition 26. *We define the Henkin extension of a language $\mathcal{L}$ in the usual way inductively, where $\mathcal{L}_{h,0} =_{\mathsf{df}} \mathcal{L}$:*

1. *where $A(x) \in \mathsf{wffs}_{\mathcal{L}_{h,i}}$ and μ is a quantifier, we add a new constants c^{μ}_{A} to the language $\mathcal{L}_{h,i}$. Let the resulting language be $\mathcal{L}_{h,i+1}$.*

2. $\mathcal{L}_h$ is $\mathcal{L}$ enriched with all new constants c^{μ}_{A} and $\mathsf{wffs}_{\mathcal{L}_h} = \bigcup_{i \geq 0} \mathsf{wffs}_{\mathcal{L}_{h,i}}$.

Fact B.3. *The set of sentences $\mathsf{sent}_{\mathcal{L}_h}$ in the enriched language is countable.*

This fact follows in the usual way by basic cardinality calculus in view of the fact that $\mathcal{L}$ is a countable language.

Definition 27. *Given a Henkin-enriched language $\mathcal{L}_h$, we define $\mathcal{H}$ as the $\subset$-smallest set containing for each quantifier μ, each truth-value $\mathsf{v} \in \mathbb{V}$ and each $\mu x A(x) \in \mathsf{Sent}(\mathcal{L}_h)$ both*

$$\text{1. } \big(\mu x A(x), A(c^{\mu}_{A}), \mathsf{v}\big) \qquad \text{and} \qquad \text{2. } \big(A(c^{\mu}_{A}), \mu x A(x), \mathsf{v}\big).$$

Given a $\mathbb{V}$-annotated theory $\mathcal{T}$ in $\mathcal{L}$, we define its Henkin-extension $\mathcal{T}_h$ by $\mathcal{T} \cup \mathcal{H}$,

Lemma B.2. *Where $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_{\mathsf{PL}}$ and μ is a quantifier, there is a $d \in \mathcal{D}$ for which $v_M(\mu x A) = v_M(A(\overline{d}))$ (where $\overline{d}$ is the pseudo-constant for which $v(\overline{d}) = d$).*

Proof. By **Q0**, $v_M(\mu x A) = f_\mu(\{v_M(A(\overline{d})) \mid d \in \mathcal{D}\})$. By **Q1**, $f_\mu(\{v_M(A(\overline{d})) \mid d \in \mathcal{D}\}) \in \{v_M(A(\overline{d})) \mid d \in \mathcal{D}\}$ and so there is a $d \in \mathcal{D}$ for which $f_\mu(\{v_M(A(\overline{d})) \mid d \in \mathcal{D}\}) = v_M(A(\overline{d}))$. □

Lemma B.3. *If the $\mathbb{V}$-annotated $\mathcal{L}$-theory $\mathcal{T}$ is satisfiable (by an $\mathcal{L}$-model $M = \langle \mathcal{D}, v \rangle$), then also $\mathcal{T} \cup \mathcal{H}$ is satisfiable (by an $\mathcal{L}_h$-model $M^h = \langle \mathcal{D}, v^h \rangle$ such that v^h conservatively extends v).*

Proof. Suppose $\mathcal{T}$ is satisfiable and let $M = \langle \mathcal{D}, v \rangle$ be a model of $\mathcal{T}$. We now conservatively extend the $\mathcal{L}$-assignment v to v^h defined over the language $\mathcal{L}_h$ to obtain $M^h = \langle \mathcal{D}, v^h \rangle$. Since we let v^h be identical to v on $\mathcal{L}$ we only need to interpret each new constant c^{μ}_{A} in $\mathcal{L}_h$ (where μ is a quantifier and $A(x)$ is in $\mathcal{L}_h$). We do so inductively starting with the constants in $\mathcal{L}_{h,1}$ (recall Def. 26).

- Let c^{μ}_{A} in $\mathcal{L}_{h,1}$. Then $\mu x A(x) \in \mathsf{sent}(\mathcal{L})$. By Lemma B.2 there is a $d \in \mathcal{D}$ for which $v_M(\mu x A(x)) = v_M(A(\overline{d}))$. Let $v^h(c^{\mu}_{A}) = d$.

- Let $c^{\mu}_{A} \in \mathcal{L}_{h,i+1}$. Then $\mu x A(x) \in \mathsf{sent}(\mathcal{L}_i)$. By Lemma B.2 there is a $d \in \mathcal{D}$ for which $v_M(\mu x A(x)) = v_M(A(\overline{d}))$. Let $v^h(c^{\mu}_{A}) = d$.

We note that for all new constants c^{μ}_{A} we have: (†) $v_{M^h}(\mu x A(x)) = v_{M^h}(A(c^{\mu}_{A}))$. We also note that (‡) M^h is a model of $\mathcal{T}$ since the interpretation of all formulas in $\mathcal{T}$ is identical to M (since these formulas do not contain any of the new constants). We need to still show that $M^h \in \mathcal{M}^{\mathcal{L}_h, \mathbb{V}}(\mathcal{H})$. All elements in $\mathcal{H}$ have the form: $(\mu x A(x), A(c^{\mu}_{A}), \mathsf{v})$ or $(A(c^{\mu}_{A}), \mu x A(x), \mathsf{v})$.

Consider the former. We have to show that $v_{M^h}(\mu x A(x)) \neq \mathsf{v}$ or $v_{M^h}(A(c^{\mu}_{A})) = \mathsf{v}$. Suppose $v_{M^h}(\mu x A(x)) = \mathsf{v}$. By (†), $v_{M^h}(A(c^{\mu}_{A})) = \mathsf{v}$. The other case is analogous. Thus, M^h is a model of $\mathcal{H}$ and in view of (‡) also of $\mathcal{T}_h$. □

Let in the following $\mathcal{T}$ be a $\mathbb{V}$-annotated theory in $\mathcal{L}$. We now show that moving from $\mathcal{T}$ to $\mathcal{T}_h$ preserves consistency.

Lemma B.4. *If $\mathcal{T}$ is consistent, so is $\mathcal{T}_h$.*

Proof. Suppose $\mathcal{T}$ is consistent and let $\mathcal{T}_f$ be a finite subset of $\mathcal{T}_h$. By the consistency of $\mathcal{T}$ there is a $\mathcal{L}$-model $M = \langle \mathcal{D}, v \rangle$ of $\mathcal{T}_f \cap \mathcal{T}$. By Lemma B.3, also $(\mathcal{T}_f \cap \mathcal{T}) \cup \mathcal{H}$ is satisfiable and so trivially also $\mathcal{T}_f$ since $\mathcal{T}_f \subseteq (\mathcal{T}_f \cap \mathcal{T}) \cup \mathcal{H}$. $\square$

As usual, we build a maximal consistent extension of $\mathcal{T}$ by the Lindenbaum construction.

Lemma B.5. *If a set of sentences $\mathcal{S}$ in $\mathcal{L}_h$ is consistent, there is a $\subset$-maximal consistent extension $\mathcal{S}_m$ of $\mathcal{S}$.*

Proof. We first note that $\mathsf{sent}^{\mathbb{V}}_{\mathcal{L}_h}$ is countable (see Facts B.1 and B.3). Let now $A_1, A_2, \ldots$ be a list of $\mathsf{sent}^{\mathbb{V}}_{\mathcal{L}_h}$. We build $\mathcal{S}_m$ in the usual way, where $\mathcal{T}_0 = \mathcal{S}$ and

$$\mathcal{T}_{i+1} = \begin{cases} \mathcal{T}_i \cup \{A_{i+1}\} & \text{if } \mathcal{T}_i \cup \{A_{i+1}\} \text{ is consistent} \\ \mathcal{T}_i & \text{else.} \end{cases}$$

we define $\mathcal{S}_m = \bigcup_{i \geq 0} \mathcal{T}_i$. It is easy to show that each $\mathcal{T}_i$ is consistent, and that $\mathcal{S}_m$ is $\subset$-maximal consistent. $\square$

Definition 28. *A $\mathbb{V}$-annotated theory $\mathcal{S}$ in $\mathcal{L}_h$ is*

1. $\mathbb{V}$-classical*, iff, for every sentence A there is a $\mathsf{v} \in \mathbb{V}$ such that*
   - $(\emptyset, A, \mathsf{v}) \in \mathcal{S}$,
   - $(A, \emptyset, \mathsf{v}) \notin \mathcal{S}$, *and*
   - *for all $\mathsf{v}' \in \mathbb{V} \setminus \{\mathsf{v}\}$ we have $(A, \emptyset, \mathsf{v}') \in \mathcal{S}$ and $(\emptyset, A, \mathsf{v}') \notin \mathcal{S}$.*
2. $\emptyset$-prime*, iff, $(A, B, \mathsf{v}) \in \mathcal{S}$ iff $(A, \emptyset, \mathsf{v}) \in \mathcal{S}$ or $(\emptyset, B, \mathsf{v}) \in \mathcal{S}$,*
3. ω-complete *(relative to $\mathcal{L}_h$, see Definition 26) iff, for all $\mathsf{v} \in \mathbb{V}$ and all quantifiers μ there is a constant c^{μ}_{A} such that, $(\emptyset, \mu x A(x), \mathsf{v}) \in \mathcal{S}$ iff $(\emptyset, A(c^{\mu}_{A}), \mathsf{v}) \in \mathcal{S}$,*
4. closed under connectives*, iff, for every n-ary connective π and all $\mathcal{L}$-sentences $A_1, \ldots, A_n$ we have:*
   (a) *for all $\mathsf{v} \in \mathbb{V}$, if $(\emptyset, \pi(A_1, \ldots, A_n), \mathsf{v}) \in \mathcal{S}$ then there are $\mathsf{v}_1, \ldots, \mathsf{v}_n \in \mathbb{V}$ for which $f_\pi(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{v}$ and $(\emptyset, A_i, \mathsf{v}_i) \in \mathcal{S}$ for every $1 \leq i \leq n$; and*
   (b) *for all $\mathsf{v}_1, \ldots, \mathsf{v}_n \in \mathbb{V}$, if $(\emptyset, A_i, \mathsf{v}_i) \in \mathcal{S}$ for every $1 \leq i \leq n$ then also*
   $$(\emptyset, \pi(A_1, \ldots, A_n), f_\pi(\mathsf{v}_1, \ldots, \mathsf{v}_n)) \in \mathcal{S}.$$
5. closed under substitution*, iff, whenever $(\emptyset, c_i \neq c'_i, \mathsf{v}_i)$ and $\mathsf{v}_i \in \mathbb{V} \setminus \mathbb{D}$ for each $1 \leq i \leq n$, then for all sentences A and B and every $\mathsf{v} \in \mathbb{V}$,*
   $$(A, B, \mathsf{v}) \in \mathcal{S} \text{ iff } (A[c_1/c'_1, \ldots, c_n/c'_n], B[c_1/c'_1, \ldots, c_n/c'_n], \mathsf{v}) \in \mathcal{S},$$

where $A[c_1/c'_1, \ldots, c_n/c'_n]$ and $B[c_1/c'_1, \ldots, c_n/c'_n]$ are the results of substituting each c_i for c'_i.

Lemma B.6. *A maximal consistent $\mathbb{V}$-annotated theory $\mathcal{S}$ in $\mathcal{L}_h$ is $\mathbb{V}$-classical.*

Proof. Suppose $\mathbb{V} = \{\mathsf{w}_1, \ldots, \mathsf{w}_n\}$. Assume for a contradiction that $(\emptyset, A, \mathsf{w}_i) \notin \mathcal{S}$ for all $1 \leq i \leq n$. Thus, $\mathcal{S} \cup \{(\emptyset, A, \mathsf{w}_i)\}$ is inconsistent for each $1 \leq i \leq n$ and so there is a finite $\mathcal{S}_i \subseteq \mathcal{S}$ such that $\mathcal{S}_i \cup \{(\emptyset, A, \mathsf{w}_i)\}$ is not satisfiable. Note that $\mathcal{S}_f = \bigcup_{j=1}^{n} \mathcal{S}_i$ is a finite subset of $\mathcal{S}$ and as such satisfiable by the consistency of $\mathcal{S}$. Let M be a model of it. Trivially $v_M(A) \in \mathbb{V}$ and so $M \models^{\mathbb{V}} (\emptyset, A, \mathsf{w}_i)$ for some $1 \leq i \leq n$. This is a contradiction. So, there is a $1 \leq i \leq n$ for which $(\emptyset, A, \mathsf{w}_i) \in \mathcal{S}$. Without loss of generality suppose $i = 1$.

For classicality we have to show two additional things: (1) $(A, \emptyset, \mathsf{w}_1) \notin \mathcal{S}$ and for all $2 \leq j \leq n$, (2) $(\emptyset, A, \mathsf{w}_j) \notin \mathcal{S}$ and (3) $(A, \emptyset, \mathsf{w}_j) \in \mathcal{S}$.

1. Assume for a contradiction that $(A, \emptyset, \mathsf{w}_1) \in \mathcal{S}$. By the consistency of $\mathcal{S}$, $\{(\emptyset, A, \mathsf{w}_1), (A, \emptyset, \mathsf{w}_1)\}$ is satisfiable which is impossible.
2. Let $2 \leq j \leq n$. Assume for a contradiction that $(\emptyset, A, \mathsf{w}_j) \in \mathcal{S}$. By the consistency of $\mathcal{S}$, $\{(\emptyset, A, \mathsf{w}_1), (\emptyset, A, \mathsf{w}_j)\}$ is satisfiable which is impossible.
3. Similar and left to the reader. □

Lemma B.7. *A maximally consistent $\mathbb{V}$-annotated theory $\mathcal{S}$ is $\emptyset$-prime.*

Proof. Suppose $(A, B, \mathsf{v}) \in \mathcal{S}$. Assume for a contradiction that $(\emptyset, B, \mathsf{v}) \notin \mathcal{S}$ and $(A, \emptyset, \mathsf{v}) \notin \mathcal{S}$. So, $\mathcal{S} \cup \{(\emptyset, B, \mathsf{v})\}$ and $\mathcal{S} \cup \{(A, \emptyset, \mathsf{v})\}$ are inconsistent. Thus, there are finite $\mathcal{S}_1 \subseteq \mathcal{S}$ and $\mathcal{S}_2 \subseteq \mathcal{S}$ for which (†) $\mathcal{S}_1 \cup \{(\emptyset, B, \mathsf{v})\}$ and $\mathcal{S}_2 \cup \{(A, \emptyset, \mathsf{v})\}$ are not satisfiable. By the consistency of $\mathcal{S}$, the finite $\mathcal{S}_f = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{(A, B, \mathsf{v})\}$ is satisfiable. So there is a model M of $\mathcal{S}_f$ for which $v_M(B) = \mathsf{v}$ or $v_M(A) \neq \mathsf{v}$. This is a contradiction to (†). □

Lemma B.8. *A maximally consistent $\mathbb{V}$-annotated theory $\mathcal{S}$ is closed under connectives.*

Proof. Let π be an n-ary connective. Suppose $(\emptyset, \pi(A_1, \ldots, A_n), \mathsf{v}) \in \mathcal{S}$. By the $\mathbb{V}$-classicality of $\mathcal{S}$ (Lemma B.6), for each $1 \leq i \leq n$ there is a $\mathsf{v}_i \in \mathbb{V}$ for which $(\emptyset, A_i, \mathsf{v}_i) \in \mathcal{S}$. Assume for a contradiction that $f_\pi(\mathsf{v}_1, \ldots, \mathsf{v}_n) \neq \mathsf{v}$. Since $\mathcal{S}$ is consistent,

$$\{(\emptyset, A_1, \mathsf{v}_1), \ldots, (\emptyset, A_n, \mathsf{v}_n), (\emptyset, \pi(A_1, \ldots, A_n), \mathsf{v})\}$$

is satisfiable by some model M. However, since

$$v_M(\pi(A_1, \ldots, A_n)) = f_\pi(v_M(A_1), \ldots, v_M(A_n)) \neq \mathsf{v},$$

this is a contradiction. The other direction is similar and left to the reader. □

Lemma B.9. *Any maximal consistent $\mathbb{V}$-annotated theory $\mathcal{S}$ is closed under substitution.*

Proof. Suppose $(\emptyset, c_1 \neq c_1', \mathsf{v}_1), \ldots, (\emptyset, c_n \neq c_n', \mathsf{v}_n) \in \mathcal{S}$ where $\mathsf{v}_1, \ldots, \mathsf{v}_n \in \mathbb{V} \setminus \mathbb{D}$. And suppose that $(A, B, \mathsf{v}) \in \mathcal{S}$. Assume for a contradiction that

$$(A[c_1/c_1', \ldots, c_n/c_n'], B[c_1/c_1', \ldots, c_n/c_n'], \mathsf{v}) \notin \mathcal{S}.$$

Thus, $\mathcal{S} \cup \{(A[c_1/c_1', \ldots, c_n/c_n'], B[c_1/c_1', \ldots, c_n/c_n'], \mathsf{v})\}$ is not consistent. Hence, there is a finite $\mathcal{S}_f \subseteq \mathcal{S}$ such that

$$\mathcal{S}_f \cup \{(A[c_1/c_1', \ldots, c_n/c_n'], B[c_1/c_1', \ldots, c_n/c_n'], \mathsf{v})\}$$

is not satisfiable. Since $\mathcal{S}$ is consistent,

$$\mathcal{S}_f' = \mathcal{S}_f \cup \{(A, B, \mathsf{v}), (\emptyset, c_1 \neq c_1', \mathsf{v}_1), \ldots, (\emptyset, c_n \neq c_n', \mathsf{v}_n)\}$$

is satisfiable. Let M be a model of $\mathcal{S}_f'$. Since, for all $1 \leq i \leq n$, $v_M(c_i \neq c_i') = \mathsf{v}_i \in \mathbb{V} \setminus \mathbb{D}$, by **InEq**, $v(c_i) = v(c_i')$. Thus, $v_M(B) = v_M(B[c_1/c_1', \ldots, c_n/c_n'])$ and $v_M(A) = v_M(A[c_1/c_1', \ldots, c_n/c_n'])$. Thus,

$$M \models^{\mathbb{V}} (A[c_1/c_1', \ldots, c_n/c_n'], B[c_1/c_1', \ldots, c_n/c_n'], \mathsf{v})$$

since $M \models^{\mathbb{V}} (A, B, \mathsf{v})$, which is a contradiction.

The other direction is similar and left to the reader. ☐

Lemma B.10. *Any maximal consistent* $\mathbb{V}$*-annotated theory* $\mathcal{S}$ *that contains its Henkin-extension* $\mathcal{S}_h$ *is* ω*-complete (relative to the Henkin enhancement* $\mathcal{L}_h$*).*

Proof. Suppose $(\emptyset, A(c_A^\mu), \mathsf{v}) \in \mathcal{S}$. Assume for a contradiction that $(\emptyset, \mu x A(x), \mathsf{v}) \notin \mathcal{S}$. Thus, $\mathcal{S} \cup \{(\emptyset, \mu x A(x), \mathsf{v})\}$ is inconsistent. Thus, there is a finite $\mathcal{S}' \subseteq \mathcal{S}$ for which $\mathcal{S}' \cup \{(\emptyset, \mu x A(x), \mathsf{v})\}$ is not satisfiable. By the consistency of $\mathcal{S}$, and since $(\emptyset, A(c_A^\mu), \mathsf{v}), (A(c_A^\mu), \mu x A(x), \mathsf{v}) \in \mathcal{S}$, $\mathcal{S}' \cup \{(\emptyset, A(c_A^\mu), \mathsf{v}), (A(c_A^\mu), \mu x A(x), \mathsf{v})\}$ is satisfiable. Let M be a model of $\mathcal{S}' \cup \{(\emptyset, A(c_A^\mu), \mathsf{v}), (A(c_A^\mu), \mu x A(x), \mathsf{v})\}$. Thus, $v_M(A(c_A^\mu)) = \mathsf{v}$ and since $M \models^{\mathbb{V}} (A(c_A^\mu), \mu x A(x), \mathsf{v})$, $v_M(\mu x A(x)) = \mathsf{v}$. But then $M \in \mathcal{M}_{\mathsf{PL}}(\mathcal{S}' \cup \{(\emptyset, \mu x A(x), \mathsf{v})\})$ which is a contradiction.

The other direction is similar and left to the reader. ☐

Lemma B.11. *Any maximal consistent* $\mathbb{V}$*-annotated theory* $\mathcal{S}$ *that contains its Henkin-extension* $\mathcal{S}_h$ *is satisfiable.*

Proof. We define a model $M = \langle \mathcal{D}, v \rangle$ based on $\mathcal{S}$. Let for this $c \sim c'$ iff $(\emptyset, c \neq c', \mathsf{v}) \in \mathcal{S}$ for some $\mathsf{v} \in \mathbb{V} \setminus \mathbb{D}$, and $[c] = \{c' \in \mathsf{Const}_{\mathcal{L}_h} \mid c \sim c'\}$. Let $\mathcal{D} = \mathsf{Const}_{/\sim} = \{[c] \mid c \in \mathsf{Const}_{\mathcal{L}_h}\}$. The interpretation v is defined as follows:

- We let $v(c) = [c]$ for all $c \in \mathsf{Const}_{\mathcal{L}_h}$.
- For all predicates P_i and all $c_1, \ldots, c_{\mathtt{i}} \in \mathsf{Const}_{\mathcal{L}_h}$, we let

$$v(P_i, ([c_1], \ldots, [c_{\mathtt{i}}])) = \mathsf{v} \text{ iff } (\emptyset, P_i(c_1, \ldots, c_{\mathtt{i}}), \mathsf{v}) \in \mathcal{S}.$$

- For all $c, c' \in \mathsf{Const}_{\mathcal{L}_h}$ we let $v(\neq, ([c], [c'])) = \mathsf{v}$ iff $(\emptyset, c \neq c', \mathsf{v}) \in \mathcal{S}$.

- If there is an identity $=$ in the language $\mathcal{L}$, for all $c, c' \in \mathsf{Const}_{\mathcal{L}_h}$ we let $v(=, ([c], [c'])) = \mathsf{v}$ iff $(\emptyset, c = c', \mathsf{v}) \in \mathcal{S}$.
- If there is a congruence $\approx$ in the language $\mathcal{L}$, for all $c, c' \in \mathsf{Const}_{\mathcal{L}_h}$ we let $v(\approx, ([c], [c'])) = \mathsf{v}$ iff $(\emptyset, c \approx c', \mathsf{v}) \in \mathcal{S}$.

We have to show that v is well-defined. Note for this that with Lemma B.6, for any sentence A, there is a unique $\mathsf{v} \in \mathbb{V}$ for which $(\emptyset, A, \mathsf{v}) \in \mathcal{S}$. We also have to show that the definitions are independent of the representatives of the equivalence classes $\sim$. This is a direct consequence of the closure under substitution (Lemma B.9).

Finally, we have to verify that our requirements (**Eq**) resp. (**InEq**) resp. (**Cong**) for $=$ (if there is an identity in $\mathcal{L}$) resp. for $\neq$ resp. for $\approx$ (if there is a congruence in $\mathcal{L}$) are met. We first discuss (**Eq**) (assuming there is a $=$ in our language).

Suppose $\mathsf{v} \in \mathbb{D}$ and $v(=, ([c], [c'])) = \mathsf{v}$. Thus, $(\emptyset, c = c', \mathsf{v}) \in \mathcal{S}$. We have to show that $[c] = [c']$ and thus that $c \sim c'$. For this we have to show that $(\emptyset, c \neq c', \mathsf{v}') \in \mathcal{S}$ for some $\mathsf{v}' \in \mathbb{V} \setminus \mathbb{D}$. Assume for a contradiction that there is no $\mathsf{v}' \in \mathbb{V} \setminus \mathbb{D}$ for which $(\emptyset, c \neq c', \mathsf{v}') \in \mathcal{S}$. By Lemma B.6, there is a $\mathsf{v}' \in \mathbb{D}$ for which $(\emptyset, c \neq c', \mathsf{v}') \in \mathcal{S}$. By the consistency of $\mathcal{S}$, $\{(\emptyset, c = c', \mathsf{v}), (\emptyset, c \neq c', \mathsf{v}')\}$ is satisfiable by some model $M' = \langle \mathcal{D}', v' \rangle$. Thus, $v_{M'}(c = c') = \mathsf{v}$ and $v_{M'}(c \neq c') = \mathsf{v}'$. By **Eq**, $v'(c) = v'(c')$ and by **InEq**, $v'(c) \neq v'(c')$. This is a contradiction. Thus, there is a $\mathsf{v}' \in \mathbb{V} \setminus \mathbb{D}$ for which $(\emptyset, c \neq c', \mathsf{v}') \in \mathcal{S}$ and so $c \sim c'$. The other direction is analogous and left to the reader.
We now discuss (**Cong**).

(Ref) Let $c \in \mathsf{Const}_{\mathcal{L}_h}$. By the $\mathbb{V}$-classicality of $\mathcal{S}$ (Lemma B.6) there is a $\mathsf{v} \in \mathbb{V}$ for which $(\emptyset, c \approx c, \mathsf{v}) \in \mathcal{S}$. Assume $\mathsf{v} \notin \mathbb{D}$. By the consistency of $\mathcal{S}$, $\{(\emptyset, c \approx c, \mathsf{v})\}$ is satisfiable by a model $M' = \langle \mathcal{D}', v' \rangle$. However, by (**Cong** (Ref)), $v'(\approx, (v(c), v(c))) \in \mathbb{D}$ which is a contradiction since $\mathsf{v} \notin \mathbb{D}$. So $\mathsf{v} \in \mathbb{D}$.

(Str) Suppose $v(\approx, ([c], [c'])) = \mathsf{v} \in \mathbb{D}$ and thus $\phi_0 = (\emptyset, c \approx c', \mathsf{v}) \in \mathcal{S}$. Assume for a contradiction that $v(P, ([c_1], \ldots, [c_{i-1}], [c], [c_{i+1}], \ldots, [c_n])) = \mathsf{v}_1 \neq \mathsf{v}_2 = v(P, [c_1], \ldots, [c_{i-1}], [c'], [c_{i+1}], \ldots, [c_n])$. So,

$$\phi_1 = (\emptyset, P(c_1, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_n), \mathsf{v}_1) \in \mathcal{S},$$
$$\text{and } \phi_2 = (\emptyset, P(c_1, \ldots, c_{i-1}, c', c_{i+1}, \ldots, c_n), \mathsf{v}_2) \in \mathcal{S}.$$

By the consistency of $\mathcal{S}$ there is a model $M' = \langle \mathcal{D}', v' \rangle$ of $\{\phi_0, \phi_1, \phi_2\}$. However, v' violates (**Cong**) (Str) and so we reached a contradiction.

The case for $\neq$ and (**InEq**) is analogous and left to the reader.

We now show that $v_M(A) = \mathsf{v}$ iff $(\emptyset, A, \mathsf{v}) \in \mathcal{S}$ inductively over the length of A.

For the *base case* this follows right from our definition.

- $v_M(P_i(c_1, \ldots, c_{\mathtt{i}})) = \mathsf{v}$ iff $v(P_i, ([c_1], \ldots, [c_{\mathtt{i}}])) = \mathsf{v}$ iff $(\emptyset, P_i(c_1, \ldots, c_{\mathtt{i}}), \mathsf{v}) \in \mathcal{S}$.

- If there is an identity in $\mathcal{L}$, $v_M(c = c') = \mathsf{v}$ iff $v(=, ([c], [c'])) = \mathsf{v}$ iff $(\emptyset, c = c', \mathsf{v}) \in \mathcal{S}$.

- $v_M(c \neq c') = \mathsf{v}$ iff $v(\neq, ([c], [c'])) = \mathsf{v}$ iff $(\emptyset, c \neq c', \mathsf{v}) \in \mathcal{S}$.

- If there is a congruence in $\mathcal{L}$, $v_M(c \approx c') = \mathsf{v}$ iff $v(\approx, ([c], [c'])) = \mathsf{v}$ iff $(\emptyset, c \approx c', \mathsf{v}) \in \mathcal{S}$.

We now move to the *inductive step*.

- We first look at connectives. Let π be an n-ary connective.

  – Suppose $v_M(\pi(A_1, \ldots, A_n)) = \mathsf{v}$. Therefore, $f_\pi(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{v}$, where $v_M(A_i) = \mathsf{v}_i$ for each $1 \leq i \leq n$. By the inductive hypothesis, $(\emptyset, A_i, \mathsf{v}_i) \in \mathcal{S}$. By Lemma B.8, $(\emptyset, \pi(A_1, \ldots, A_n), \mathsf{v}) \in \mathcal{S}$.

  – Let now $(\emptyset, \pi(A_1, \ldots, A_n), \mathsf{v}) \in \mathcal{S}$. By Lemma B.8, there are $\mathsf{v}_1, \ldots, \mathsf{v}_n \in \mathbb{V}$ for which $f_\pi(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{v}$ and $(\emptyset, A_i, \mathsf{v}_i) \in \mathcal{S}$. By the inductive hypothesis, $v_M(A_i) = \mathsf{v}_i$. Therefore $v_M(\pi(A_1, \ldots, A_n)) = f_\pi(\mathsf{v}_1, \ldots, \mathsf{v}_n) = \mathsf{v}$.

- We consider quantifiers μ. Suppose $v_M(\mu x A(x)) = \mathsf{v}$. Thus, by **Q0**, $f_\mu(\{A(c) \mid [c] \in \mathcal{D}\}) = \mathsf{v}$. Where $\{v_M(A(c)) \mid [c] \in \mathcal{D}\} = \{\mathsf{w}_1, \ldots, \mathsf{w}_m\}$ let $c_1, \ldots, c_m$ be such that $v_M(A(c_i)) = \mathsf{w}_i$. By Lemma B.6, there is a unique v' for which $(\emptyset, \mu x A(x), \mathsf{v}') \in \mathcal{S}$. We have to show that $\mathsf{v} = \mathsf{v}'$. By Lemma B.10, $(\emptyset, A(c^\mu_A), \mathsf{v}') \in \mathcal{S}$. By the inductive hypothesis, $v_M(A(c^\mu_A)) = \mathsf{v}'$ and $(\emptyset, A(c_i), \mathsf{w}_i) \in \mathcal{S}$ (for each $1 \leq i \leq m$). As $\mathcal{S}$ is consistent there is a model $M' = \langle \mathcal{D}', v' \rangle$ of

  $$\big\{(\emptyset, A(c_1), \mathsf{w}_1), \ldots, (\emptyset, A(c_m), \mathsf{w}_m), (\emptyset, A(c^\mu_A), \mathsf{v}'), \\ (\mu x A(x), A(c^\mu_A), \mathsf{v}'), (A(c^\mu_A), \mu x A(x), \mathsf{v}')\big\}.$$

  Note that in M' we have $v_{M'}(\mu x A(x)) = \mathsf{v}'$ since $v_{M'}(A(c^\mu_A)) = \mathsf{v}'$. Note also that $\{v_{M'}(A(\overline{d})) \mid d \in \mathcal{D}'\} \supseteq \{v_M(A(c)) \mid [c] \in \mathcal{D}\}$. Thus, by **Q2**, $f_\mu(\{v_{M'}(A(\overline{d})) \mid d \in \mathcal{D}_{M'}\}) = \mathsf{v}' = f_\mu(\{v_M(A(c)) \mid [c] \in \mathcal{D}\}) = \mathsf{v}$.

  For the other direction suppose $(\emptyset, \mu x A(x), \mathsf{v}) \in \mathcal{S}$. Let $\{v_M(A(c)) \mid [c] \in \mathcal{D}\} = \{\mathsf{w}_1, \ldots, \mathsf{w}_m\}$. Let $c_1, \ldots, c_m$ be such that $v_M(A(c_1)) = \mathsf{w}_1, \ldots, v_M(A(c_m)) = \mathsf{w}_m$. By the inductive hypothesis, $(\emptyset, A(c), \mathsf{w}_i) \in \mathcal{S}$. By the consistency of $\mathcal{S}$ there is a model M' of $\{(\emptyset, \mu x A(x), \mathsf{v}), (\emptyset, A(c_1), \mathsf{w}_1), \ldots, (\emptyset, A(c_m), \mathsf{w}_m)\}$. So, $f_\mu(\{\mathsf{w} \mid \exists d \in \mathcal{D}'(v_{M'}(A(\overline{d})) = \mathsf{w})\}) = \mathsf{v}$. Note that $\{\mathsf{w} \mid \exists d \in \mathcal{D}'(v_{M'}(A(\overline{d})) = \mathsf{w})\} \supseteq \{\mathsf{w}_1, \ldots, \mathsf{w}_m\}$ and $\mathsf{v} \in \{\mathsf{w}_1, \ldots, \mathsf{w}_m\}$. So, by **Q2**, $f_\mu(\{\mathsf{w}_1, \ldots, \mathsf{w}_\mathsf{m}\}) = \mathsf{v}$. Thus, $v_M(\mu x A(x)) = f_\mu(\{\mathsf{w}_1, \ldots, \mathsf{w}_m\}) = \mathsf{v}$.

We finally show that $M \in \mathcal{M}_{\mathsf{PL}}(\mathcal{S})$. Let for this $(A, B, \mathsf{v}) \in \mathcal{S}$. By Lemma B.7, $(\emptyset, B, \mathsf{v}) \in \mathcal{S}$ or $(A, \emptyset, \mathsf{v}) \in \mathcal{S}$. In the former case, as shown above $v_M(B) = \mathsf{v}$, and so $M \models^{\mathbb{V}} (A, B, \mathsf{v})$. Assume the latter. By Lemma B.6, there is a $\mathsf{v}' \in \mathbb{V} \setminus \{\mathsf{v}\}$ such that $(\emptyset, A, \mathsf{v}') \in \mathcal{S}$. As shown, $v_M(A) = \mathsf{v}'$. Therefore $v_M(A) \neq \mathsf{v}$ and so $M \models^{\mathbb{V}} (A, B, \mathsf{v})$. □

We are now ready to prove Proposition B.2 and the model existence lemma.

Lemma B.12 (Model Existence)**.** *Any consistent $\mathbb{V}$-theory $\mathcal{T}$ in a language $\mathcal{L}$ has a $\mathcal{L}$-model with maximal cardinality* $\mathsf{card}(\mathcal{L})$*.*

Proof. Suppose the $\mathbb{V}$-theory $\mathcal{T}$ in $\mathcal{L}$ is consistent. By Lemma B.4, also its Henkin-extension $\mathcal{T}_h$ is consistent. Let $\mathcal{T}_m$ be a maximal consistent extension of $\mathcal{T}_h$ in $\mathcal{L}_h$ (which exists with Lemma B.5). By Lemma B.11, $\mathcal{T}_m$ is satisfiable in $\mathcal{L}_h$ by a model $M = \langle \mathcal{D}, v \rangle$. Note that $\mathsf{card}(\mathcal{D}) = \mathsf{card}(\{[c] \mid c \in \mathsf{Const}_{\mathcal{L}_h}\}) \leq \mathsf{card}(\mathcal{L}_h) = \mathsf{card}(\mathcal{L})$. Clearly, then also $\mathcal{T}$ is satisfiable in $\mathcal{L}$ (e.g., just restrict the assignment v of M to $\mathcal{L}$). □

Proof of Proposition B.2. This follows directly from Lemma B.12. □

C Abnormality-aware versions of the Löwenheim-Skolem Theorems

We now prove Theorems 8.1 and 8.2 from Section 8. Recall for this

- Definition 14 of $\mathbb{P}_{\mathsf{x}}(\cdot)$ where $\mathsf{x} \in \{\mathsf{fin}, \mathsf{coinf}, \mathsf{fin/coinf}\}$,
- Definition 16 of our abnormality-aware versions of the Löwenheim-Skolem Theorems, and
- Definition 15 of $\mathcal{M}^{\leq\kappa}(\cdot)$ and $\Vdash^{\leq\kappa}_{\mathsf{L}}$.

For the proof we need some more definitions first.

Definition 29. *Where $i \in \mathbb{P}$ and $l \geq 1$, we introduce the following abbreviations:*

$$\exists_i^{\geq l} =_{\mathsf{df}} \left(\emptyset, \exists \mathbf{x}_1 \cdots \exists \mathbf{x}_l \left(\bigwedge_{j=1}^{l} @\,\alpha_i(\mathbf{x}_j) \wedge \bigwedge_{1 \leq j < j' \leq l} \mathbf{x}_j \neq \mathbf{x}_{j'} \right), \mathbb{D} \right)$$

$$\exists_i^{\leq l} =_{\mathsf{df}} \left(\exists \mathbf{x}_1 \cdots \exists \mathbf{x}_{l+1} \left(\bigwedge_{j=1}^{l+1} @\,\alpha_i(\mathbf{x}_j) \wedge \bigwedge_{1 \leq j < j' \leq l+1} \mathbf{x}_j \neq \mathbf{x}_{j'} \right), \emptyset, \mathbb{D} \right)$$

Lemma C.1. *Where $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_{\mathsf{PL}}$, $\mathsf{card}(\mathsf{Ab}_i(M)) = l$ iff ($M \models^{\mathbb{V}} \exists_i^{\geq l}$ and $M \models^{\mathbb{V}} \exists_i^{\leq l}$).*

Proof. Let $M = \langle \mathcal{D}, v\rangle$. Suppose that $\mathsf{card}(\mathsf{Ab}_i(M)) = l$. Thus, $\mathsf{Ab}_i(M) = \{\mathbf{d}_1, \ldots, \mathbf{d}_l\} \subseteq \mathcal{D}^{a_i}$. Where $\mathbf{d} = \langle d_1, \ldots, d_{a_i}\rangle$ we write $\overline{\mathbf{d}}$ for $\langle \overline{d_1}, \ldots, \overline{d_{a_i}}\rangle$. So, $M \models \bigwedge_{j=1}^{l} @\,\alpha_i(\overline{\mathbf{d}_j})$ and $M \models \overline{\mathbf{d}_j} \neq \overline{\mathbf{d}_{j'}}$ for all $1 < j \leq j' \leq l$ (compare the notation in the context of Eq. 2). Thus, $M \models^{\mathbb{V}} \exists_i^{\geq l}$. Assume now for a contradiction that $M \not\models^{\mathbb{V}} \exists_i^{\leq l}$. Thus,

$$M \models \exists \mathbf{x}_1 \cdots \exists \mathbf{x}_{l+1} \left(\bigwedge_{j=1}^{l+1} @\,\alpha_i(\mathbf{x}_j) \wedge \bigwedge_{1 \leq j < j' \leq l+1} \mathbf{x}_j \neq \mathbf{x}_{j'} \right)$$

Thus, we have:

- There are $\mathbf{d}_1, \ldots, \mathbf{d}_{l+1} \in \mathcal{D}^{a_i}$ for which
$$M \models \left(\bigwedge_{j=1}^{l+1} @\,\alpha_i(\overline{\mathbf{d}_j}) \right) \wedge \left(\bigwedge_{1 \leq j < j' \leq l+1} \overline{\mathbf{d}_j} \neq \overline{\mathbf{d}_{j'}} \right).$$
- Thus, (a) for each $1 \leq j \leq l+1$, $M \models @\,\alpha_i(\overline{\mathbf{d}_j})$ and (b) for each $1 \leq j < j' \leq l+1$, $M \models \overline{\mathbf{d}_j} \neq \overline{\mathbf{d}_{j'}}$.
- By (**InEq**) and (b), $\mathbf{d}_j \neq \mathbf{d}_{j'}$ for all $1 \leq j < j' \leq l+1$.
- By (a), $\{\mathbf{d}_1, \ldots, \mathbf{d}_{l+1}\} \in \mathsf{Ab}_i(M)$ and so $\mathsf{card}(\mathsf{Ab}_i(M)) \geq l+1$.

The other direction is similar and left to the reader. □

Definition 30. *As before, let $\mathcal{L}_h$ be the language $\mathcal{L}$ enriched by Henkin-constants (see Definition 26). Let $\mathcal{L}_\kappa$ be the language $\mathcal{L}$ enhanced with κ-many constants $\{k_i \mid i \in \mathcal{I}\}$.*

Finally, let $\mathcal{L}_{h,\kappa} = \mathcal{L}_h \cup \mathcal{L}_\kappa$ where we suppose that $\mathsf{Const}(\mathcal{L}_h) \cap \mathsf{Const}(\mathcal{L}_\kappa) = \mathsf{Const}(\mathcal{L})$ (so, the new κ-many constants in $\mathcal{L}_\kappa$ do not occur as Henkin-constants of $\mathcal{L}_h$).

We denote $\mathcal{L}$ $[\mathcal{L}_h, \mathcal{L}_\kappa, \mathcal{L}_{h,\kappa}]$-models of PL *by $\mathcal{M}^{\mathcal{L}}$ $[\mathcal{M}^{\mathcal{L}_h}, \mathcal{M}^{\mathcal{L}_\kappa}, \mathcal{M}^{\mathcal{L}_{h,\kappa}}]$.*

Definition 31. *Where $M = \langle \mathcal{D}, v\rangle \in \mathcal{M}^{\mathcal{L}}$ let $\mathcal{T}_M$ be the $\mathcal{L}_{h,\kappa}$-$\mathbb{V}$-annotated theory built step-wise as follows:*

1. *We start with $\mathcal{T} = \{(\emptyset, A, \mathsf{v}) \mid A \in \mathsf{sent}(\mathcal{L}), v_M(A) = \mathsf{v}\}$.*
2. *We then let $\mathcal{T}_h$ be the Henkin-extension of $\mathcal{T}$ in $\mathcal{L}_h$ as defined in Definition 27. We then extend $\mathcal{T}_h$ in $\mathcal{L}_{h,\kappa}$ by:*
3. *$(\emptyset, k_i \neq k_j, \mathbb{D})$ where $i \neq j$ and $k_i, k_j \in \mathsf{Const}(\mathcal{L}_\kappa) \setminus \mathsf{Const}(\mathcal{L})$*
4. *$\mathsf{Norm}_i(\mathbf{k})$ for all $i \in \mathbb{P}_{\mathsf{fin/coinf}}(M)$ and all $\mathbf{k} \in (\mathsf{Const}(\mathcal{L}_{h,\kappa}) \setminus \mathsf{Const}(\mathcal{L}_h))^{a_i}$, where $\mathsf{Norm}_i(\mathbf{k}) =_{\mathsf{df}} \left(@\alpha_i(\mathbf{k}), \emptyset, \mathbb{D}\right)$;*
5. *and both $\exists_i^{\geq l}$ and $\exists_i^{\leq l}$ for every $(i,l) \in \mathbb{P}_{\mathsf{fin}}(M)$.*

6. $\exists_i^{\geq l}$ *for all* (i, l) *such that* $i \in \mathbb{P} \setminus \mathbb{P}_{\mathsf{fin}}(M)$ *and* $l \geq 1$.

Fact C.1. *Where* $\mathsf{card}(\mathcal{D}) = \kappa \geq \aleph_0$ *and* $\mathsf{card}(\mathcal{D}') < \kappa$, $\mathsf{card}(\mathcal{D} \setminus \mathcal{D}') = \kappa$.

Proof. Note that (given the axiom of choice):

$$\mathsf{card}(\mathcal{D}) = \mathsf{card}(\mathcal{D} \setminus \mathcal{D}') + \mathsf{card}(\mathcal{D}') = \max(\mathsf{card}(\mathcal{D} \setminus \mathcal{D}'), \mathsf{card}(\mathcal{D}')) = \kappa.$$

So $\mathsf{card}(\mathcal{D} \setminus \mathcal{D}') = \kappa$ since $\mathsf{card}(\mathcal{D}') < \kappa$. □

Lemma C.2. *Where* $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}^{\mathcal{L}}$ *and* $\mathsf{card}(\mathcal{D}) \geq \aleph_0$, $\mathcal{M}^{\mathcal{L}_{h,\kappa}}(\mathcal{T}_M) \neq \emptyset$.

Proof. Let $M^h = \langle \mathcal{D}, v^h \rangle$ be the $\mathcal{L}_h$-model of $\mathcal{T}_h$ from Lemma B.3. By compactness (Proposition B.2), we have to show that any finite $\mathcal{T}_f \subseteq \mathcal{T}_M$ is satisfiable. We conservatively extend the $\mathcal{L}_h$-assignment v^h of M^h to an $\mathcal{L}_{h,\kappa}$-assignment v^f and show that the resulting model $M^f = \langle \mathcal{D}, v^f \rangle$ is a model of $\mathcal{T}_f$. Let $i \in \mathbb{P}$ be the maximal index for which some $\mathsf{Norm}_i(\mathbf{k})$ or $\exists_i^{\leq l}$ or $\exists_i^{\geq l}$ occurs in $\mathcal{T}_f$. We define the assignment v^f as follows:

- $v^f(c) =_{\mathsf{df}} v^h(c)$ for all $c \in \mathsf{Const}(\mathcal{L}_h)$.
- $v^f(P_i, (d_1, \ldots, d_{\mathtt{i}})) =_{\mathsf{df}} v^h(P_i, (d_1, \ldots, d_{\mathtt{i}}))$ for all $i \in \mathcal{I}$ and all $d_1, \ldots, d_{\mathtt{i}} \in \mathcal{D}^{\mathtt{i}}$.
- and analogous for $=, \approx$ and $\neq$.

We still have to fix the reference of the new constants in $\mathcal{L}_\kappa$. Where $\mathsf{Ab}_j^\star(M_h)$ contains all individuals d_l contained in some $\langle d_1, \ldots, d_{a_j} \rangle \in \mathsf{Ab}_j(M_h)$, let $\mathcal{A} =_{\mathsf{df}} \bigcup \{\mathsf{Ab}_j^\star(M_h) \mid j \leq i \text{ and } j \in \mathbb{P}_{\mathsf{fin/coinf}}(M)\}$.

Suppose first that $\mathsf{card}(\mathcal{D}) = \aleph_0$. In that case $\mathbb{P}_{\mathsf{coinf}}(M) = \emptyset$. Thus, $\mathsf{card}(\mathcal{A}) < \aleph_0 = \mathsf{card}(\mathcal{D})$.

Suppose now that $\mathsf{card}(\mathcal{D}) > \aleph_0$. In that case $\mathsf{card}(\mathcal{A}) = \max(\{\mathsf{card}(\mathsf{Ab}_j(M)) \mid j \in \mathbb{P}_{\mathsf{fin/coinf}}(M), j \leq i\}) < \mathsf{card}(\mathcal{D})$. In any case, by Fact C.1, there is a bijective function μ from $\mathsf{Const}_{\mathcal{L}_\kappa}$ to $\mathcal{D} \setminus \mathcal{A}$. We let: $v^f(k_i) = \mu(k_i)$ for each $1 \leq i \leq m'$.

We now show that M^f is a model of $\mathcal{T}_f$. Clearly, M^f is a model of $\mathcal{T}_h \cap \mathcal{T}_f$ since v^f is a conservative extension of v^h and $\mathcal{T}_h$ only contains elements of the language $\mathcal{L}_h$.

- Suppose $(\emptyset, k_i \neq k_j, \mathbb{D}) \in \mathcal{T}_f$. Since $v^f(k_i) \neq v^f(k_j)$, $v^f(\neq, (v^f(k_i), v^f(v_j))) = v^h(\neq, (v^f(k_i), v^f(k_j))) \in \mathbb{D}$ by **InEq**.
- Suppose $\mathsf{Norm}_j(\mathbf{k}) \in \mathcal{T}_f$, where $\mathbf{k} = \langle k_{i_1}, \ldots, k_{i_{a_j}} \rangle$. Since $v^f(k_{j_i}) \in \mathcal{D} \setminus \mathcal{A}$, $v_M(\alpha_j(\overline{v^f(k_{i_1})}, \ldots, \overline{v^f(k_{i_{a_j}})})) \notin \mathbb{A}$. Therefore, $v_{M^f}(\alpha_j(k_{i_1}, \ldots, k_{i_{a_j}})) \notin \mathbb{A}$ and so $v_{M^f}(\mathsf{Norm}_j(\mathbf{k})) \in \mathbb{D}$.
- The cases for $\exists_i^{\leq l}$ and $\exists_i^{\geq l}$ are shown analogously. □

Definition 32. *Where* $M_\star = \langle \mathcal{D}_\star, v_\star \rangle \in \mathcal{M}^{\mathcal{L}_{h,\kappa}}(\mathcal{H})$ *(recall Definition 27), we define* $M'_\star = \langle \mathcal{D}'_\star, v'_\star \rangle \in \mathcal{M}^{\mathcal{L}_{h,\kappa}}$ *as follows:*

1. *Where* $c, c' \in \mathsf{Const}(\mathcal{L}_{h,\kappa})$*, let* $c \sim c'$ *iff* $v_\star(c) = v_\star(c')$*. Let* $[c] = \{c' \in \mathsf{Const}(\mathcal{L}_{h,\kappa}) \mid c \sim c'\}$ *be the* $\sim$*-equivalence class of* c*.*

2. *Let* $\mathcal{D}'_\star = \{[c] \mid c \in \mathsf{Const}(\mathcal{L}_{h,\kappa})\}$*.*

3. *Let* $v'_\star(c) = [c]$ *for all* $c \in \mathsf{Const}(\mathcal{L}_{h,\kappa})$*.*

4. *If "=" is part of* $\mathcal{L}$*, let* $v'_\star(=, ([c], [c'])) = v_\star(=, (v_\star(c), v_\star(c')))$*.*

5. *If "≈" is part of* $\mathcal{L}$*, let* $v'_\star(\approx, ([c], [c'])) = v_\star(\approx, (v_\star(c), v_\star(c')))$*.*

6. *Let* $v'_\star(\neq, ([c], [c'])) = v_\star(\neq, (v_\star(c), v_\star(c')))$*.*

7. *Let* $v'_\star(P_i, ([c_1], \ldots, [c_{\mathtt{i}}])) = v_\star(P_i, (v_\star(c_1), \ldots, v_\star(c_{\mathtt{i}})))$*.*

The next two lemmas are relative to Definition 32

Lemma C.3. $v'_\star$ *is well-defined and* $M'_\star$ *is a model.*

Proof. We first show that $v'_\star$ is well-defined in the sense that its interpretation is independent of the choice of representatives of the equivalence classes of constant. I.e., if $v'_\star(=, ([c], [c'])) = \mathsf{v}$ (resp. $v'_\star(\neq, ([c], [c'])) = \mathsf{v}$ resp. $v'_\star(\approx, ([c], [c'])) = \mathsf{v}$) then for all $c_1 \in [c]$ and all $c_2 \in [c']$ also $v'_\star(=, ([c_1], [c_2])) = \mathsf{v}$ (resp. $v'_\star(\neq, ([c_1], [c_2])) = \mathsf{v}$ resp. $v'_\star(\approx, [c_1], [c_2]) = \mathsf{v}$).

So, assume $c_1 \sim c$ and $c_2 \sim c'$. Thus, by Item 1 (in Definition 32), (†) $v_\star(c_1) = v_\star(c)$ and $v_\star(c_2) = v_\star(c')$. Let $\circ \in \{=, \neq, \approx\}$. Suppose $v'_\star(\circ, ([c], [c'])) = \mathsf{v}$. Therefore, by Items 4–6, $v_\star(\circ, (v_\star(c), v_\star(c'))) = \mathsf{v}$. By (†), $v_\star(\circ, (v_\star(c_1), v_\star(c_2))) = \mathsf{v}$ and thus, by Items 4–6, $v'_\star(\circ, ([c_1], [c_2])) = \mathsf{v}$.

Similarly, suppose $v'_\star(P_i, ([c_1], \ldots, [c_{\mathtt{i}}])) = \mathsf{v}$ and $c'_i \in [c_i]$ (where $1 \leq i \leq \mathtt{i}$). Then $v_\star(P_i, (v_\star(c_1), \ldots, v_\star(c_{\mathtt{i}}))) = \mathsf{v}$ by Item 7. Since, by Item 1, $v_\star(c_i) = v_\star(c'_i)$, also $v_\star(P_i, (v_\star(c'_1), \ldots, v_\star(c'_{\mathtt{i}}))) = \mathsf{v}$ and so by Item 7, $v'_\star(P_i, ([c'_1], \ldots, [c'_{\mathtt{i}}])) = \mathsf{v}$.

We now show that **Eq** (if there is an identity), **InEq**, and **Cong** (if there is a congruence) are fulfilled in $M'_\star$.

Concerning **Eq** suppose $v'_\star(=, ([c], [c'])) = \mathsf{v} \in \mathbb{D}$.
Therefore, by Item 4, $v_\star(=, (v_\star(c), v_\star(c'))) = \mathsf{v}$ and by **Eq**, $v_\star(c) = v_\star(c')$. By Item 1, $[c] = [c']$.

Concerning **InEq** suppose that $v'_\star(\neq, ([c], [c'])) = \mathsf{v} \in \mathbb{V} \setminus \mathbb{D}$.
Therefore, $v_\star(\neq, (v_\star(c), v_\star(c'))) = \mathsf{v}$ and by **InEq**, $v_\star(c) \neq v_\star(c')$. By Item 1, $[c] \neq [c']$.

Cong is shown in a similar straight-forward way. This is left to the reader. □

Lemma C.4. *For every* $A \in \mathcal{L}_{h,\kappa}$*,* $v_{M'_\star}(A) = v_{M_\star}(A)$*.*

Proof. We show this inductively over the length of $A \in \mathsf{sent}(\mathcal{L}_{h,\kappa})$.
Base. (We refer to justifications of the items in the Definition 32 of $v'_\star$.)

- Let $c, c' \in \mathsf{Const}(\mathcal{L}_{h,\kappa})$ and let $\circ \in \{=, \neq, \approx\}$. In view of Items 4–6 we have: $v_{M'_\star}(c \circ c') = \mathsf{v}$ iff $v'_\star(\circ, ([c], [c'])) = \mathsf{v}$ iff $v_\star(\circ, (v_\star(c), v_\star(c'))) = \mathsf{v}$ iff $v_{M_\star}(c \circ c') = \mathsf{v}$.

- Let $c_1, \ldots, c_n \in \mathsf{Const}(\mathcal{L}_{h,\kappa})$. In view of Item 7 we have: $v_{M'_\star}(P_i(c_1, \ldots, c_{\mathtt{i}})) = \mathsf{v}$ iff [by Item 3] $v'_\star(P_i, ([c_1], \ldots, [c_{\mathtt{i}}])) = \mathsf{v}$ iff $v_\star(P_i, (v_\star(c_1), \ldots, v_\star(c_{\mathtt{i}}))) = \mathsf{v}$ iff $v_{M_\star}(P_i(c_1, \ldots, c_{\mathtt{i}})) = \mathsf{v}$.

Inductive step.

- Consider an n-ary connective π. We have:

$$v_{M'_\star}(\pi(A_1, \ldots, A_n)) = f_\pi(v_{M'_\star}(A_1), \ldots, v_{M'_\star}(A_n)) \text{ and}$$
$$v_{M_\star}(\pi(A_1, \ldots, A_n)) = f_\pi(v_{M_\star}(A_1), \ldots, v_{M_\star}(A_n)).$$

  By the inductive hypothesis, $v_{M'_\star}(A_i) = v_{M_\star}(A_i)$ for each $1 \leq i \leq n$. Thus, $v_{M'_\star}(\pi(A_1, \ldots, A_n)) = v_{M_\star}(\pi(A_1, \ldots, A_n))$.

- Consider the quantifier μ. We have, by Items 2 and 3,

$$v_{M'_\star}(\mu x A(x)) = f_\mu(\{v_{M'_\star}(A(\overline{d})) \mid d \in \mathcal{D}'_\star\}) = f_\mu(\{v_{M'_\star}(A(c)) \mid c \in \mathsf{Const}(\mathcal{L}_{h,\kappa})\})$$

  and $v_{M_\star}(\mu x A(x)) = v_{M_\star}(A(c^A_\mu))$ since $M_\star \in \mathcal{M}(\mathcal{H})$. By the inductive hypothesis, $\{v_{M'_\star}(A(c)) \mid c \in \mathsf{Const}(\mathcal{L}_{h,\kappa})\} = \{v_{M_\star}(A(c)) \mid c \in \mathsf{Const}(\mathcal{L}_{h,\kappa})\} \subseteq \{v_{M_\star}(A(\overline{d})) \mid d \in \mathcal{D}_\star\}$ and $v_{M'_\star}(A(c^A_\mu)) = v_{M_\star}(A(c^A_\mu))$. By **Q2**, $f_\mu(\{v_{M'_\star}(A(\overline{d})) \mid d \in \mathcal{D}'_\star\}) = v_{M'_\star}(A(c^A_\mu))$ and therefore, $v_{M'_\star}(\mu x A(x)) = v_{M_\star}(\mu x A(x))$. □

Lemma C.5. *Where $M \in \mathcal{M}^{\mathcal{L}}$ and $M_\star \in \mathcal{M}^{\mathcal{L}_{h,\kappa}}(\mathcal{T}_M)$ (see Lemma C.2) and $M'_\star \in \mathcal{M}^{\mathcal{L}_{h,\kappa}}$ is as defined in Definition 32,*

1. *$\mathbb{P}_{\mathsf{fin}}(M) = \mathbb{P}_{\mathsf{fin}}(M'_\star)$;*
2. *for all $i \in \mathbb{P}_{\mathsf{coinf}}(M)$, $\mathsf{card}(\mathsf{Ab}_i(M)) \geq \mathsf{card}(\mathsf{Ab}_i(M'_\star)) = \aleph_0$;*
3. *$\mathbb{P}_{\mathsf{coinf}}(M) \subseteq \mathbb{P}_{\mathsf{coinf}}(M'_\star)$ in case $\kappa > \aleph_0$,*
4. *$\mathsf{card}(\mathcal{D}'_\star) = \kappa$.*

Proof. Ad 1. Let $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M)$. By Lemma C.1, $M \models^{\mathbb{V}} \exists^{\geq l}_i$ and $M \models^{\mathbb{V}} \exists^{\leq l}_i$. Since $M'_\star \in \mathcal{M}(\mathcal{T}_M)$, also $M'_\star \models^{\mathbb{V}} \exists^{\geq l}_i$ and $M'_\star \models^{\mathbb{V}} \exists^{\leq l}_i$. Again by Lemma C.1, $\mathsf{card}(\mathsf{Ab}_i(M'_\star)) = l = \mathsf{card}(\mathsf{Ab}_i(M))$. Thus, $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M'_\star)$.

For the other direction suppose $(i, l) \in \mathbb{P}_{\mathsf{fin}}(M'_\star)$. By Lemma C.1, $M'_\star \models^{\mathbb{V}} \exists^{\geq l}_i$ and $M'_\star \models^{\mathbb{V}} \exists^{\leq l}_i$. In case there is a k for which $(i, k) \in \mathbb{P}_{\mathsf{fin}}(M)$, then $(i, k) \in \mathbb{P}_{\mathsf{fin}}(M'_\star)$ (as established in the previous paragraph). Again, by Lemma C.1, $M'_\star \models^{\mathbb{V}} \exists^{\geq k}_i$ and $M'_\star \models^{\mathbb{V}} \exists^{\leq k}_i$. But then it is easy to see that $l = k$. Suppose, then, that there is no $k \in \mathbb{N}$ for which $(i, k) \in \mathbb{P}_{\mathsf{fin}}(M)$. Thus, $\exists^{\geq k}_i \in \mathcal{T}_M$ for every $k \geq 1$. In particular, $\exists^{\geq l+1}_i$ which contradicts the fact that $M'_\star \models \exists^{\leq l}_i$.

Ad 2. Let $i \in \mathbb{P}_{\mathsf{coinf}}(M)$. So, $\mathsf{card}(\mathsf{Ab}_i(M)) \geq \aleph_0$. Since for all k in $(\mathsf{Const}(\mathcal{L}_{h,\kappa}) \setminus \mathsf{Const}(\mathcal{L}_h))^{a_i}$, $M'_\star \models^{\mathbb{V}} \mathsf{Norm}_i(\mathbf{k})$ and since $\mathsf{Const}(\mathcal{L}_h)$ is countable, $\mathsf{card}(\mathsf{Ab}_i(M'_\star)) = \aleph_0 \leq \mathsf{card}(\mathsf{Ab}_i(M))$.

Ad 3. This follows immediately with Item 2.

Ad 4. Note that $\mathsf{card}(\mathsf{Const}(\mathcal{L}_{h,\kappa})) = \kappa$ which implies $\mathsf{card}(\mathcal{D}'_\star) \leq \kappa$. Also, for each $i, j \in \mathcal{I}$ (see Definition 30), if $i \neq j$ then $(\emptyset, k_i \neq k_j, \mathbb{D}) \in \mathcal{T}_\star$. By **InEq**, $v_\star(c) \neq v_\star(c')$ and so $[c] \neq [c']$ (see Definition 32). Thus, $\mathsf{card}(\mathcal{D}'_\star) = \kappa$. □

We are now in a position to prove Theorems 8.1 and 8.2.

Theorem C.1. ***LS-Ab (upwards)*** *holds for every logic* PL.

Proof. We proceed in the following steps:

1. first we enrich the language $\mathcal{L}$ with κ many new constants obtaining $\mathcal{L}_{h,\kappa}$ (Definition 30);
2. then we enrich the premise set, to encode that the new constants refer to different entities and that they are, whenever possible, non-abnormal, obtaining $\mathcal{T}_M$ (Definition 31);
3. we show that the enriched premise set $\mathcal{T}_M$ is satisfiable by an $\mathcal{L}_{h,\kappa}$-model $M'_\star = \langle \mathcal{D}'_\star, v'_\star \rangle$ of cardinality κ that has the desired properties (Lemma C.5).
4. Finally, we let v' be the restriction of $v'_\star$ to $\mathcal{L}$ to obtain the $\mathcal{L}$-model $M' = \langle \mathcal{D}'_\star, v' \rangle$ that has all the desired properties. □

Theorem C.2. ***LS-Ab (downwards)*** *holds for every logic* PL.

Proof. Suppose $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}$ and $\mathsf{card}(\mathcal{D}) = \lambda > \aleph_0$. Let $\lambda > \kappa \geq \aleph_0$. We define $\mathcal{T}_M$ as in Def. 31 and $M'_\star$ as in Def. 32. In view of Lemma C.5, $M'_\star$ has all the desired properties. □

D Strong Reassurance for $\prec_\Phi^{=_c,l}$.

In the following we show that $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ has the property of Strong Reassurance for $\prec = \prec_\Phi^{=_c,l}$ following the path outlined in Section A. In Section E we discuss how the proof can be adjusted for other orderings.

Given a $M_\top = \langle \mathcal{D}_\top, v_\top \rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \setminus \min_\prec\big(\mathcal{M}_{\mathsf{PL}}(\Sigma)\big)$, we have to show that there is a $M^\star \in \min_\prec(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M^\star \prec M$. As in Section A we will first work under the assumption that, where $\kappa = \mathsf{card}(\mathcal{D}_\top) \geq \aleph_0$ and address the other simpler case in the end of this section (Remark D.1). Following the strategy outlined in Section A we show that

1. $\bigcap_{i \geq 1} \mathcal{M}_i \neq \emptyset$ and that 2. $\bigcap_{i \geq 1} \mathcal{M}_i \subseteq \min_\prec(\mathcal{M}_\downarrow)$.

For what follows recall Definition 19.

Lemma D.1. *For each* $i \geq 0$, $\langle \mathcal{M}_i, \prec_{i+1} \rangle$ *is well-founded and therefore* $\mathcal{M}_i \neq \emptyset$.

Proof. We show this by induction. Let $i = 0$. Assume for a contradiction that there is an infinitely $\prec_1$-descending sequence of models $M_\top = M_1, \ldots, M_n, \ldots$ in $\mathcal{M}_\downarrow$ (so $M_{i+1} \prec_1 M_i$ for all $i \geq 1$). Hence, $\mathsf{card}(\mathsf{Ab}_1(M_{i+1})) < \mathsf{card}(\mathsf{Ab}_1(M_i))$. But (given the axiom of choice) there is no infinitely decreasing sequence of cardinals. The inductive step is analogous and left to the reader. □

Lemma D.2. $\bigcap_{i\geq 1}\mathcal{M}_i \subseteq \min_{\prec}(\mathcal{M}_{\downarrow})$.

Proof. Assume for a contradiction that there are models $M \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ and $M^\star \in \bigcap_{i\geq 1}\mathcal{M}_i$ for which $M \prec M^\star$. Thus, there is a $j \in \mathbb{P}$ such that (1) for all $i \in \mathbb{P}$ smaller than j, $\mathsf{card}(\mathsf{Ab}_i(M)) = \mathsf{card}(\mathsf{Ab}_i(M^\star))$ while (2) $\mathsf{card}(\mathsf{Ab}_j(M)) < \mathsf{card}(\mathsf{Ab}_j(M^\star))$. Note that since $M^\star \in \mathcal{M}_{\downarrow}$, it follows that $M^\star \prec M_\top$. And since $M \prec M^\star$, by the transitivity of $\prec$ (Fact 3.1), we have $M \prec M_\top$, and so $M \in \mathcal{M}_{\downarrow}$. Moreover, by (1), $M \in \mathcal{M}_i$ for all i smaller than j. But then $M^\star \notin \mathcal{M}^j$ since $M \prec_j M^\star$. This is a contradiction with $M^\star \in \bigcap_{i\geq 1}\mathcal{M}_i$. □

In order to establish that $\bigcap_{i\geq 1}\mathcal{M}_i \neq \emptyset$ we will construct a model M_Φ in $\bigcap_{i\geq 1}\mathcal{M}_i$. Let, in the following, $\mathcal{T} =_{\mathsf{df}} \{(\emptyset, A, \mathbb{D}) \mid A \in \Sigma\}$.

As in Definition 30, we enrich our language $\mathcal{L}$ to a language $\mathcal{L}_\kappa$ with κ many new constants k_i. We also define the Henkin enrichment $\mathcal{L}_h$ of the base language $\mathcal{L}$ as in Definition 26. The joint language of $\mathcal{L}_h$ and $\mathcal{L}_\kappa$ is denoted by $\mathcal{L}_{h,\kappa}$. We denote sets of $\mathcal{L}$ $[\mathcal{L}_h, \mathcal{L}_\kappa, \mathcal{L}_{h,\kappa}]$-models of PL by $\mathcal{M}^{\mathcal{L}}$ $[\mathcal{M}^{\mathcal{L}_h}, \mathcal{M}^{\mathcal{L}_\kappa}, \mathcal{M}^{\mathcal{L}_{h,\kappa}}]$. Similarly, we denote the set of constants by $\mathsf{Const}(\mathcal{L})$ $[\mathsf{Const}(\mathcal{L}_h), \mathsf{Const}(\mathcal{L}_\kappa), \mathsf{Const}(\mathcal{L}_{h,\kappa})]$.

Definition 33. *We now enrich $\mathcal{T}$ to $\mathcal{T}_\star$ in two steps (recall the Definition 20 of $\mathbb{P}_{\mathsf{fin}}$, $\mathbb{P}_{\mathsf{coinf}}$ and $\mathbb{P}_{\mathsf{fin/coinf}}$, and Definition 29 of $\exists_i^{\geq l}$ and $\exists_i^{\leq l}$):*

1. *First we build its Henkin extension $\mathcal{T}_h$ in $\mathcal{L}_h$ as defined in Definition 27.*
2. *Then we enhance $\mathcal{T}_h$ to a set of $\mathcal{L}_{h,\kappa}$-$\mathbb{V}$-annotated sentences $\mathcal{T}_\star$ by adding:*
   - $(\emptyset, k_i \neq k_j, \mathbb{D})$ *where $i \neq j$ and k_i and k_j are newly added constants in $\mathsf{Const}(\mathcal{L}_\kappa)$;*
   - $\mathsf{Norm}_i(\mathbf{k})$ *for all $i \in \mathbb{P}_{\mathsf{fin/coinf}}$ and all $\mathbf{k} = \langle k_{j_1}, \ldots, k_{j_{a_i}}\rangle \in \mathsf{Const}_{\mathcal{L}_\kappa}{}^{a_i}$, where $\mathsf{Norm}_i(\mathbf{k}) =_{\mathsf{df}} \big(@\,\alpha_i(\mathbf{k}), \emptyset, \mathbb{D}\big)$;*
   - *and both $\exists_i^{\geq l}$ and $\exists_i^{\leq l}$ for every $(i,l) \in \mathbb{P}_{\mathsf{fin}}$.*

Definition 34. *Given a model $M = \langle \mathcal{D}, v\rangle$ in a language $\mathcal{L}$, we let in the following $\mathcal{L}[\mathcal{D}]$ be the language $\mathcal{L}$ enriched by pseudo-constants $\overline{d}$ for every $d \in \mathcal{D}$.*

Lemma D.3. *For every model $M = \langle \mathcal{D}, v\rangle \in \mathcal{M}^{\mathcal{L}}(\mathcal{T})$, there is a $M_h = \langle \mathcal{D}, v_h\rangle \in \mathcal{M}^{\mathcal{L}_h}$ for which:*

1. *$v_M(A) = v_{M_h}(A)$ for every wff A in $\mathcal{L}[\mathcal{D}]$,*
2. *$M_h \models^{\mathbb{V}} (A(c_\mu^A), \mu x A(x), \mathsf{v})$ for every $A(x)$ in $\mathcal{L}_h$ and every $\mathsf{v} \in \mathbb{V}$,*
3. *$M_h \models^{\mathbb{V}} (\mu x A(x), A(c_\mu^A), \mathsf{v})$ for every $A(x)$ in $\mathcal{L}_h$ and every $\mathsf{v} \in \mathbb{V}$,*
4. *$M_h \in \mathcal{M}^{\mathcal{L}_h}(\mathcal{T}_h)$, and*
5. *for every $i \in \mathbb{P}$, $\mathsf{Ab}_i(M) = \mathsf{Ab}_i(M_h)$.*

Proof. Where $M = \langle \mathcal{D}, v\rangle \in \mathcal{M}^{\mathcal{L}}$ is a model of $\mathcal{T}$, let the $\mathcal{L}_h$-model $M_h = \langle \mathcal{D}, v_h\rangle$ be defined as follows:

- Let $v_h(c) = v(c)$ for all $c \in \mathsf{Const}(\mathcal{L})$.
- For c_μ^A we proceed in the usual iterative way over the inductive structure of $\mathcal{L}_h = \bigcup_{i \geq 1} \mathcal{L}_{h,i}$ (see Definition 26).
  - Consider $\mathcal{L}_{h,1}$. In view of **Q0** and **Q1** we know that for each $A(x) \in \mathcal{L}$ there is a $d_\mu^A \in \mathcal{D}$ for which $v_M(A(\overline{d_\mu^A})) = v_M(\mu x A(x))$. Let $v_h(c_\mu^A) = d_\mu^A$.
  - Consider $\mathcal{L}_{h,i+1}$. Again, for each $A(x) \in \mathcal{L}_{h,i}$ there is a $d_\mu^A \in \mathcal{D}$ such that $v_M(A'(\overline{d_\mu^A})) = v_M(\mu x A'(x))$ where A' is the result of substituting each $c_\mu^B \in \mathcal{L}_{h,i}$ in A for $\overline{d_\mu^B}$. Let $v_h(c_\mu^A) = d_\mu^A$.
- For all P_i and all $d_1, \ldots, d_{\mathtt{i}} \in \mathcal{D}$, let $v_h(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v(P_i, (d_1, \ldots, d_{\mathtt{i}}))$.
- For all $d, d' \in \mathcal{D}$, (if there is an identity) let $v_h(=, (d, d')) = v(=, (d, d'))$, $v_h(\neq, (d, d')) = v(\neq, (d, d'))$, and (if there is a congruence) let $v_h(\approx, (d, d')) = v(\approx, (d, d'))$.

By the definition of v_h, $v_M(A) = v_{M_h}(A)$ for all $A \in \mathcal{L}[\mathcal{D}]$ (Item 1) and $\mathsf{Ab}_i(M) = \mathsf{Ab}_i(M_h)$ for all $i \in \mathbb{P}$ (Item 5). Items 2 and 3 follows immediately since by the definition of v_h, $v_M(A(\overline{d_\mu^A})) = v_M(\mu x A(x)) = v_{M_h}(A(\overline{d_\mu^A})) = v_{M_h}(A(c_\mu^A)) = v_{M_h}(\mu x A(x))$. Item 4 follows from Items 1–3 and the fact that $M \in \mathcal{M}^{\mathcal{L}}(\mathcal{T})$. □

The following fact is a direct consequence of the definition of $\mathbb{P}_{\mathsf{fin}}$.

Fact D.1. *Where $(i, l) \in \mathbb{P}_{\mathsf{fin}}$, for all $M \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M)) = l$.*

Lemma D.4. *Where $i \in \mathbb{P}_{\mathsf{coinf}}$, for all $M \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M)) = \aleph_0$.*

Proof. Let $M \in \mathcal{M}_i$. By Theorems 8.1 and 8.2 there is a M' that satisfies items 1–4 of Definition 16 relative to M whose domain has the same size as the domain of M. Thus, $M' \in \mathcal{M}(\Sigma)$ (by Item 4) and $M' \in \mathcal{M}_i$ by Items 1–3. Since $\mathsf{card}(\mathsf{Ab}_i(M')) = \aleph_0$ also $\mathsf{card}(\mathsf{Ab}_i(M)) = \aleph_0$. □

Lemma D.5. *$\mathcal{T}_\star$ is satisfiable. (See Definition 33)*

Proof. Consider a finite $\mathcal{T}_f \subseteq \mathcal{T}_\star$. We show that $\mathcal{T}_f$ is satisfiable and so, by compactness (Proposition B.2), is $\mathcal{T}_\star$.

Let $i \in \mathbb{P}$ be the maximal index such that some $\mathsf{Norm}_i(\mathbf{k})$ or $\exists_i^{\geq l}$ or $\exists_i^{\leq l}$ is contained in $\mathcal{T}_f$. Consider an $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}_i$ (recall that by Lemma D.1, $\mathcal{M}_i \neq \emptyset$). Thus, $\mathsf{card}(\mathcal{D}) = \kappa$. Let $M_h = \langle \mathcal{D}, v_h \rangle$ be the ω-complete enhancement of M based on the new constants in $\mathcal{L}_h$ as in Lemma D.3. We define $M' = \langle \mathcal{D}, v' \rangle \in \mathcal{M}^{\mathcal{L}_{h,\kappa}}$ as follows:

1. $v'(c) = v_h(c)$ for all $c \in \mathsf{Const}(\mathcal{L}_h)$.
2. Where $\mathsf{Ab}_j^\star(M_h)$ contains all individuals d_l contained in some $\langle d_1, \ldots, d_{a_j} \rangle \in \mathsf{Ab}_j(M_h)$, let $\mathcal{A} =_{\mathsf{df}} \bigcup \{\mathsf{Ab}_j^\star(M_h) \mid j \leq i, j \in \mathbb{P}_{\mathsf{fin/coinf}}(M)\}$.

3. Note that $\mathsf{card}(\mathcal{A}) \leq \aleph_0$ in case $\kappa > \aleph_0$ and $\mathsf{card}(\mathcal{A}) < \aleph_0$ in case $\kappa = \aleph_0$. The reason is that with Lemma D.3, Lemma D.4 and Fact D.1, for every $j \in \mathbb{P}^i_{\mathsf{fin/coinf}}$ we have $\mathsf{card}(\mathsf{Ab}_j(M)) = \mathsf{card}(\mathsf{Ab}_j(M_h)) \leq \aleph_0$ and so also $\mathsf{card}(\mathsf{Ab}^\star_j(M)) = \mathsf{card}(\mathsf{Ab}^\star_j(M_h)) \leq \aleph_0$. In case $\kappa = \aleph_0$, $\mathbb{P}_{\mathsf{coinf}} = \emptyset$ and so $\mathsf{card}(\mathsf{Ab}_j(M)) = \mathsf{card}(\mathsf{Ab}_j(M_h)) < \aleph_0$ whence $\mathsf{card}(\mathsf{Ab}^\star_j(M)) = \mathsf{card}(\mathsf{Ab}^\star_j(M_h)) < \aleph_0$.

4. By Fact C.1, there is a bijection $\eta : \mathsf{Const}(\mathcal{L}_{h,\kappa}) \setminus \mathsf{Const}(\mathcal{L}_h) \to \mathcal{D} \setminus \mathcal{A}$. Let $v'(c) = \eta(c)$ for all $c \in \mathsf{Const}(\mathcal{L}_{h,\kappa}) \setminus \mathsf{Const}(\mathcal{L}_h)$.

5. Let $v'(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v_h(P_i, (d_1, \ldots, d_{\mathtt{i}}))$ for all $d_1, \ldots, d_{\mathtt{i}} \in \mathcal{D}$ and all predicate symbols P_i.

6. Let $v'(\neq, (d, d')) = v_h(\neq, (d, d'))$ for all $d, d' \in \mathcal{D}$.

7. If there is an identity "$=$" in $\mathcal{L}$, let $v'(=, (d, d')) = v_h(=, (d, d'))$ for all $d, d' \in \mathcal{D}$.

8. If there is a congruence "$\approx$" in $\mathcal{L}$, let $v'(\approx, (d, d')) = v_h(\approx, (d, d'))$ for all $d, d' \in \mathcal{D}$.

Note that:

(†) $v_{M'}(A) = v_{M_h}(A)$ for every $A \in \mathcal{L}_h$ since v' conservatively extends v_h on $\mathcal{L}_h$ in view of Items 1, 5–9. Thus, since $M_h \in \mathcal{M}(\mathcal{T}_h)$, also $M' \in \mathcal{M}(\mathcal{T}_h)$.

(‡) For the same reason, $\mathsf{card}(\mathsf{Ab}_j(M')) = \mathsf{card}(\mathsf{Ab}_j(M_h)) = \mathsf{card}(\mathsf{Ab}_j(M))$ for every $j \in \mathbb{P}$.

We still have to show that M' verifies all members of $\mathcal{T}_f \cap (\mathcal{T}_\star \setminus \mathcal{T}_h)$.

- Consider $(\emptyset, k_i \neq k_j, \mathbb{D})$, where $i \neq j$ and $k_i, k_j \in \mathsf{Const}(\mathcal{L}_\kappa)$. We have $v_{M'}(k_i \neq k_j) = \mathsf{v}$ (for some $\mathsf{v} \in \mathbb{D}$), iff, $v'(\neq, (v'(k_i), v'(k_j))) = \mathsf{v}$, iff, [by Item 7] $v_h(\neq, (v'(k_i), v'(k_j)) = \mathsf{v}$, iff, [by **InEq**] $v'(k_i) \neq v'(k_j)$. Note that $v'(k_i) = \eta(k_i)$, $v'(k_j) = \eta(k_j)$, and since η is injective [by Item 5] and $i \neq j$, $\eta(k_i) \neq \eta(k_j)$.

- Consider $\mathsf{Norm}_j(\mathbf{k})$, where $\mathbf{k} = \langle k_{l_1}, \ldots, k_{l_{a_j}} \rangle \in \mathsf{Const}(\mathcal{L}_\kappa)^{a_j}$ and $j \leq i$. This holds in M' since $v'(k_{l_1}), \ldots, v'(k_{l_{a_j}}) \in \mathcal{D} \setminus \mathcal{A}$ by Items 2–5.

- Since $M \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_j(M)) = l$ for all $(j, l) \in \mathbb{P}_{\mathsf{fin}}$ for which $j \leq i$. Thus, by Item (‡) also $\mathsf{card}(\mathsf{Ab}_j(M')) = l$. By Lemma C.1, $M' \models^{\mathbb{V}} \exists^{\geq l}_j$ and $M' \models^{\mathbb{V}} \exists^{\leq l}_j$.

This concludes our proof. □

Lemma D.6. *Let $M'_\star = \langle \mathcal{D}'_\star, v'_\star \rangle \in \mathcal{M}^{\mathcal{L}_{h,\kappa}}$ be the term-model based on $M_\star$ defined in Definition 32 and $M_\perp = \langle \mathcal{D}'_\star, v_\perp \rangle \in \mathcal{M}^{\mathcal{L}}$ where $v_\perp$ is the restriction of $v'_\star$ to $\mathcal{L}$.*

1. $M'_\star \in \mathcal{M}(\mathcal{T}_\star)$ *and so* $M'_\star \in \mathcal{M}(\Sigma)$ *and* $M_\perp \in \mathcal{M}(\Sigma)$,
2. $\mathsf{card}(\mathcal{D}'_\star) = \kappa$,
3. $\mathbb{P}_{\mathsf{fin}}(M'_\star) = \mathbb{P}_{\mathsf{fin}}(M_\perp) = \mathbb{P}_{\mathsf{fin}}$,
4. $\mathbb{P}_{\mathsf{coinf}}(M'_\star) = \mathbb{P}_{\mathsf{coinf}}(M_\perp) = \mathbb{P}_{\mathsf{coinf}}$, *and*
5. $M_\perp \in \bigcap_{i\geq 1} \mathcal{M}_i$.

Proof. Ad 1. Since $M_\star \in \mathcal{M}(\mathcal{T}_\star)$ it is sufficient to show $v_{M'_\star}(A) = v_{M_\star}(A)$, for all $A \in \mathcal{L}_{h,\kappa}$. This is Lemma C.4.

Ad 2. Since $\mathsf{card}(\mathsf{Const}_{\mathcal{L}_{h,\kappa}}) = \kappa$, $\mathsf{card}(\mathcal{D}'_\star) \leq \kappa$. Also, $\mathsf{card}(\mathcal{D}'_\star) \geq \kappa$ since $M \models (\emptyset, k_i \neq k_j, \mathbb{D})$ for all $k_i, k_j \in \mathsf{Const}(\mathcal{L}_\kappa) \setminus \mathsf{Const}(\mathcal{L}_h)$ and by (**InEq**).

Ad 3. This follows by Item 1 and Lemma C.1.

We show Items 4 and 5 together. By Item 1, $M'_\star \models \mathsf{Norm}_i(\mathbf{k})$ for all $i \in \mathbb{P}_{\mathsf{coinf}}$ and all $\mathbf{k} \in {\mathsf{Const}_{\mathcal{L}_\kappa}}^{a_i}$. Thus, $\mathsf{Ab}_i(M'_\star) \subseteq {\mathsf{Const}_{\mathcal{L}_h/\sim}}^{a_i}$. Since $\mathcal{L}_h$ is countable, ${\mathsf{Const}_{\mathcal{L}_h/\sim}}^{a_i}$ is countable as well and so $\mathsf{card}(\mathsf{Ab}_i(M'_\star)) \leq \aleph_0$. Thus, (†), $\mathbb{P}_{\mathsf{coinf}} \subseteq \mathbb{P}_{\mathsf{fin/coinf}}(M'_\star)$. Thus, (‡), for any $i \in \mathbb{P}$ and any $M_i \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M_i)) \geq \mathsf{card}(\mathsf{Ab}_i(M'_\star))$. To see this we distinguish the three cases (a) $(i, l) \in \mathbb{P}_{\mathsf{fin}}$ for some $l \geq 0$, (b) $i \in \mathbb{P}_{\mathsf{coinf}}$ and (c) $i \in \mathbb{P} \setminus \mathbb{P}_{\mathsf{fin/coinf}}$. In case (a), $\mathsf{card}(\mathsf{Ab}_i(M_i)) = l = \mathsf{card}(\mathsf{Ab}_i(M'_\star))$ by Item 3 and Lemma C.1. In case (b), $\mathsf{card}(\mathsf{Ab}_i(M_i)) \geq \aleph_0$ and by (†), $\mathsf{card}(\mathsf{Ab}_i(M'_\star)) \leq \aleph_0$. In case (c), $\mathsf{card}(\mathsf{Ab}_i(M_i)) = \kappa \geq \mathsf{card}(\mathsf{Ab}_i(M'_\star))$. This is (‡).

Now assume for a contradiction that there is an $i \in \mathbb{P}$ and an $M_i \in \mathcal{M}_i$ for which $\mathsf{card}(\mathsf{Ab}_i(M_i)) > \mathsf{card}(\mathsf{Ab}_i(M'_\star))$. Let i be minimal with this property and therefore for all $j < i$ and all $M_j \in \mathcal{M}_j$, $\mathsf{card}(\mathsf{Ab}_j(M_j)) = \mathsf{card}(\mathsf{Ab}_j(M'_\star)) = \mathsf{card}(\mathsf{Ab}_j(M_\perp))$ and so $M_\perp \in \mathcal{M}_j$. But then also $\mathsf{card}(\mathsf{Ab}_i(M_i)) > \mathsf{card}(\mathsf{Ab}_i(M_\perp))$ and so $M_\perp \in \mathcal{M}_i$ and therefore $M_i \notin \mathcal{M}_i$ which is a contradiction. Hence, (‡′), for all $i \in \mathbb{P}$ and all $M_i \in \mathcal{M}_i$, $\mathsf{card}(\mathsf{Ab}_i(M_i)) = \mathsf{card}(\mathsf{Ab}_i(M'_\star)) = \mathsf{card}(\mathsf{Ab}_i(M_\perp))$. Thus, Items 4 and 5 hold. □

Corollary 10. $\bigcap_{i\geq 1} \mathcal{M}_i \neq \emptyset$.

This concludes our proof, we only have to put together the pieces as explained in Section A (see Proposition D.1 below). Recall, however, that our initial assumption was that $\kappa \geq \aleph_0$ (where $\kappa = \mathsf{card}(\mathcal{D}_\top)$). We will therefore now briefly discuss the remaining simpler case: $\kappa < \aleph_0$.

Remark D.1. *We proceed analogous to the proof above but we also encode in* $\mathcal{T}_\star$ *that the size of our domain is* κ *by adding*

$$\exists(\leq \kappa) =_{\mathsf{df}} \left(\exists x_1 \cdots \exists x_{l+1} \bigwedge\nolimits_{1\leq i<j\leq\kappa} x_i \neq x_j, \emptyset, \mathbb{D}\right) \text{ and}$$

$$\exists(\geq \kappa) =_{\mathsf{df}} \left(\emptyset, \exists x_1 \cdots \exists x_l \bigwedge\nolimits_{1\leq i<j\leq\kappa} x_i \neq x_j, \mathbb{D}\right)$$

to $\mathcal{T}_\star$. *In order to show that* $\mathcal{T}_\star$ *is satisfiable, we again have to find a model of a finite* $\mathcal{T}_f \subseteq \mathcal{T}_\star$ *which can be established just like in Lemma D.5. Finally, any model* $M_\perp$ *of* $\mathcal{T}_\star$ *will be in* $\bigcap_{i\geq 0} \mathcal{M}^{1+i}$ *and* $M_\perp \prec M_\top$ *(this time there is no need to construct a term model as in Lemma D.6 because any model of* $\mathcal{T}_\star$ *has the right size since the size of the domain is encoded in* $\mathcal{T}_\star$*).*

$\pi \in \dots$	$\star \in \dots$	Str. Reas. for $\prec_{\Phi}^{\pi,\star}$
$\{=_c\}$	$\{l\}$	Prop. D.1
$\{\geq\}$	$\{l\}$	Prop. E.1
$\{\mathsf{f}, =, =_c, \geq, \supseteq\}$	$\{g\}$	Prop. E.2
$\{=_c, \geq\}$	$\{p\}$	Prop. E.3
$\{\supseteq\}$	$\{p, l\}$	Prop. E.4
$\{=\}$	$\{p, l\}$	Prop. E.5
$\{\mathsf{f}\}$	$\{p, l\}$	Prop. E.6

Table 7: Overview: Results concerning Strong Reassurance

Proposition D.1. $\mathsf{nmPL} = \left\langle \mathsf{PL}, \prec_{\Phi}^{=_c,l} \right\rangle$ *satisfies Strong Reassurance.*

Proof. Suppose $M_{\top} = \langle \mathcal{D}_{\top}, v_{\top} \rangle \in \mathcal{M}(\Sigma) \setminus \min_{\prec}(\mathcal{M}(\Sigma))$, where $\prec = \prec_{\Phi}^{=_c,l}$. We consider the case that $\mathsf{card}(\mathcal{D}_{\top}) > \aleph_0$. By Corollary 10, there is a $M_{\bot} \in \bigcap_{i \geq 1} \mathcal{M}_i$ and by Lemma D.2, $M_{\bot} \in \min_{\prec}(\mathcal{M}_{\downarrow})$. Hence it follows that $M_{\bot} \in \min_{\prec}(\mathcal{M}(\Sigma))$. And since $M_{\bot} \in \mathcal{M}_{\downarrow}$, we have $M_{\bot} \prec M_{\top}$, what completes our proof. For the case in which $\mathsf{card}(\mathcal{D}_{\top}) < \aleph_0$ we use the simplified proofs outlined in Remark D.1. □

E Strong Reassurance for other orderings

So far we have discussed strong reassurance for the order $\prec_{\Phi}^{=_c,l}$ for some $\Phi = \{\alpha_i(x_1, \dots, x_{a_i}) \mid i \in \mathbb{P}\}$. We will now consider the other cases. In Table 7 we give an overview of the results. We first consider the case of $\prec_{\Phi}^{\geq,l}$.

Lemma E.1. *Let $M \in \mathcal{M}_{\mathsf{PL}}^{=\lambda}$, $M_\kappa, M' \in \mathcal{M}_{\mathsf{PL}}^{=\kappa}$ and $\lambda \geq \kappa$. Then $M' \sqsubseteq M$ and $M \prec M_\kappa$ implies $M' \prec M_\kappa$.*

Proof. Since $M' \sqsubseteq M$ and $\lambda \geq \kappa$, by Lemma 8.1, for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M')) \leq \mathsf{card}(\mathsf{Ab}_i(M))$.

Let first $\dagger = p$. Since $M \prec M_\kappa$, for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M)) \leq \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$ and there is a $k \in \mathbb{P}$ for which $\mathsf{card}(\mathsf{Ab}_k(M)) < \mathsf{card}(\mathsf{Ab}_k(M_\kappa))$. So, for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M')) \leq \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$ and where $i = k$, $\mathsf{card}(\mathsf{Ab}_i(M')) < \mathsf{card}(\mathsf{Ab}_i(M_\kappa))$. This suffices to show that $M' \prec M_\kappa$. The proof for $\dagger = l$ is analogous.

Let $\dagger = g$. Then, $\mathsf{card}(\mathsf{Ab}_\Phi(M)) < \mathsf{card}(\mathsf{Ab}_\Phi(M_\kappa))$ since $M \prec M_\kappa$. Also, $\mathsf{card}(\mathsf{Ab}_\Phi(M')) \leq \mathsf{card}(\mathsf{Ab}_\Phi(M))$ and so $\mathsf{card}(\mathsf{Ab}_\Phi(M')) < \mathsf{card}(\mathsf{Ab}_{Phi}(M_\kappa))$ which suffices to show that $M' \prec M_\kappa$. □

Proposition E.1. $\mathsf{nmPL} = \langle \mathsf{PL}, \prec_{\Phi}^{\geq,l} \rangle$ *satisfies Strong Reassurance.*

Proof. Let $\prec = \prec_{\Phi}^{\geq,l}$. Suppose $M_{\top} = \langle \mathcal{D}_{\top}, v_{\top} \rangle \in \mathcal{M}(\Sigma) \setminus \min_{\prec}(\mathcal{M}(\Sigma))$. We consider the case where $\kappa = \mathsf{card}(\mathcal{D}_{\top}) \geq \aleph_0$. The case $\kappa = \mathsf{card}(\mathcal{D}_{\top}) < \aleph_0$ is a simple variation of the proof below (see also Remark D.1). Where $\prec' = \prec_{\Phi}^{=_c,l}$, by Proposition D.1, there is a $M_\bot = \langle \mathcal{D}_\bot, v_\bot\rangle \in \min_{\prec'}(\mathcal{M}(\Sigma))$ for which $M_\bot \prec' M_\top$.

Assume for a contradiction that there is an $M' \in \mathcal{M}(\Sigma)$ for which $M' \prec M_\bot$. Thus, $\mathsf{card}(\mathcal{D}') > \mathsf{card}(\mathcal{D}_\bot)$. By Theorem 8.3 there is a $M'' = \langle \mathcal{D}'', v''\rangle$ with $\mathsf{card}(\mathcal{D}'') = \mathsf{card}(\mathcal{D}_\bot)$ for which $M'' \sqsubseteq M'$. By Lemma E.1, $M' \prec' M_\bot$ which is a contradiction since $M'' \in \mathcal{M}(\Sigma)$ and $M_\bot \in \min_{\prec'}(\mathcal{M}(\Sigma))$. Thus, $M_\bot \in \min_\prec(\Sigma)$. $\square$

We now consider the case of $\prec^{\pi,g}_{\Phi}$ where $\pi \in \{\mathbf{f}, =, =_c, \geq, \supseteq\}$.

Proposition E.2. *Where* $\pi \in \{\mathbf{f}, =, =_c, \geq, \supseteq\}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec^{\pi,g}_{\Phi}\rangle$ *satisfies Strong Reassurance.*

Proof. Let $\prec\, = \,\prec^{\pi,g}_{\Phi}$ and $M_\top \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \setminus \min_\prec(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Assume for a contradiction that there is no model $M_\bot \in \min_\prec(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M_\bot \prec M_\top$. Thus, there is an infinitely descending chain $M_\top = M_1 \succ M_2 \succ \ldots \succ M_n \succ \ldots$ of better and better models (so: $M_{i+1} \prec M_i$ for all $i \geq 1$). Thus, $\mathsf{card}(\mathsf{Ab}(M_{i+1})) < \mathsf{card}(\mathsf{Ab}(M_i))$. But then we get an infinitely descending sequence of smaller and smaller cardinals which is impossible (given the Axiom of Choice). $\square$

We now consider $\prec^{\pi,p}_{\star}$ where $\pi \in \{=_c, \geq\}$. The following lemma will be useful for several results below.

Lemma E.2. *Where* $\pi \in \{\mathbf{f}, =, =_c, \geq, \supseteq\}$, $\min_{\prec^{\pi,l}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma)) \subseteq \min_{\prec^{\pi,p}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$.

Proof. Suppose $M \in \min_{\prec^{\pi,l}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Assume for a contradiction that there is a $M' \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ for which $M' \prec^{\pi,p}_{\Phi} M$. Thus, for all $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M')) \leq \mathsf{card}(\mathsf{Ab}_i(M))$ and there is a $j \in \mathbb{P}$ for which $\mathsf{card}(\mathsf{Ab}_j(M')) < \mathsf{card}(\mathsf{Ab}_j(M))$. Thus, there is a minimal $i \in \mathbb{P}$ for which $\mathsf{card}(\mathsf{Ab}_{i'}(M')) = \mathsf{card}(\mathsf{Ab}_{i'}(M))$ for all $i' < i$ and $\mathsf{card}(\mathsf{Ab}_i(M')) < \mathsf{card}(\mathsf{Ab}_i(M))$. Thus, $M' \prec^{\pi,l}_{\Phi} M$ which is a contradiction. $\square$

Proposition E.3. *Where* $\pi \in \{=_c, \geq\}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec^{\pi,p}_{\Phi}\rangle$ *satisfies Strong Reassurance.*

Proof. This follows in view of Propositions D.1, E.1, and Lemma E.2. $\square$

We now move to the case $\prec\, = \,\prec^{\supseteq,\star}_{\Phi}$ where $\star \in \{p, l\}$.

Proposition E.4. *Where* $\star \in \{p, l\}$, $\prec\, = \,\prec^{\supseteq,\star}_{\Phi}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec\rangle$ *satisfies Strong Reassurance.*

Proof. Suppose (1) $M_\top = \langle \mathcal{D}_\top, v\rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \setminus \min_\prec(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Assume for a contradiction that (2) $M_\top \in \min_{\prec^{\geq,\star}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. By (1) there is a $M' = \langle \mathcal{D}', v'\rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ for which $M' \prec M_\top$. Thus, $\mathcal{D}' \supseteq \mathcal{D}_\top$. Without loss of generality we assume $\mathcal{D}' \cap \mathcal{D}_\top = \emptyset$. By (2), $\mathsf{card}(\mathcal{D}') < \mathsf{card}(\mathcal{D}_\top)$ which is a contradiction.

So, $M_\top \notin \min_{\prec^{\geq}_{\star}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. By Proposition E.1, there is an $M' = \langle \mathcal{D}', v' \rangle \in \min_{\prec^{\geq}_{\star}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M' \prec^{\geq}_{\star} M_\top$ and so $\mathsf{card}(\mathcal{D}') \geq \mathsf{card}(\mathcal{D}_\top)$. That means there is an injective function $g : \mathcal{D}_\top \to \mathcal{D}'$. Let $\eta : \mathcal{D}' \to \mathcal{D}_\top \cup (\mathcal{D}' \setminus \mathsf{Im}(g))$ be defined by

$$d \mapsto \begin{cases} d & d \in \mathcal{D}' \setminus \mathsf{Im}(g) \\ g^{-1}(d) & \text{else.} \end{cases}$$

Note that η is bijective. Let $M'' = \langle \mathcal{D}_\top \cup (\mathcal{D}' \setminus \mathsf{Im}(g)), v'' \rangle$ where v'' is defined as follows:

- $v''(c) = \eta(v'(c))$ and
- $v''\big(P_i, (d_1, \ldots, d_{\mathtt{i}})\big) = v'\big(P_i, (\eta^{-1}(d_1), \ldots, \eta^{-1}(d_{\mathtt{i}}))\big)$.

Clearly, $v_{M'}(A) = v_{M''}(A)$ for all sentences A in $\mathcal{L}$. Thus, $M'' \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ and $M'' \prec M_\top$ since $\mathcal{D}'' \supseteq \mathcal{D}_\top$. Since for all $i \in \mathbb{P}$, $\mathsf{card}(Ab_i(M')) = \mathsf{card}(\mathsf{Ab}_i(M''))$ and $\mathsf{card}(\mathcal{D}'') = \mathsf{card}(\mathcal{D}')$, $M'' \in \min_{\prec^{\geq,\star}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$.

We now show that $M'' \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Assume for a contradiction that there is a $M^\star = \langle \mathcal{D}^\star, v^\star \rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ for which $M^\star \prec M''$ and thus $\mathcal{D}^\star \supseteq \mathcal{D}''$. So, $\mathsf{card}(\mathcal{D}^\star) \geq \mathsf{card}(\mathcal{D}'') = \mathsf{card}(\mathcal{D}')$ and so $M^\star \prec^{\geq,\star}_{\Phi} M''$ which is a contradiction. This completes our proof. □

Proposition E.5. *Where* $\star \in \{p, l\}$, $\prec\, = \,\prec^{=,\star}_{\Phi}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *satisfies Strong Reassurance.*

Proof. Suppose $M_\top = \langle \mathcal{D}_\top, v_\top \rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \setminus \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. Thus, there is a $M' = \langle \mathcal{D}, v' \rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ such that $M' \prec M_\top$. Clearly, $M' \prec^{=_c,\star}_{\Phi} M_\top$ and so $M_\top \in \mathcal{M}_{\mathsf{PL}}(\Sigma) \setminus \min_{\prec^{=_c,\star}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$. By Proposition D.1, there is an $M'' \in \min_{\prec^{=_c,\star}_{\Phi}}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ for which $M'' = \langle \mathcal{D}'', v'' \rangle \prec^{=_c,\star}_{\Phi} M_\top$. Hence, $\mathsf{card}(\mathcal{D}'') = \mathsf{card}(\mathcal{D}_\top)$. Thus, there is a bijective function $g : \mathcal{D}_\top \to \mathcal{D}''$. Let $M^\star = \langle \mathcal{D}_\top, v^\star \rangle$ where $v^\star$ is defined as follows:

- $v^\star(c) = g^{-1}(v''(c))$ and
- $v^\star(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v''(P_i, (g(d_1), \ldots, g(d_{\mathtt{i}})))$.

So, for each $i \in \mathbb{P}$, $\mathsf{card}(\mathsf{Ab}_i(M^\star)) = \mathsf{card}(\mathsf{Ab}_i(M''))$ and hence $M^\star \prec M$.

To show that $M^\star \in \min_{\prec}(\mathcal{M}_{\mathsf{PL}}(\Sigma))$ assume for a contradiction that there is an $M^\dagger = \langle \mathcal{D}, v^\dagger \rangle \in \mathcal{M}_{\mathsf{PL}}(\Sigma)$ for which $M^\dagger \prec M^\star$. But then also $M^\dagger \prec^{=_c,\star}_{\Phi} M^\star$ which is a contradiction. This concludes our proof. □

Proposition E.6. *Where* $\star \in \{p, l\}$, $\prec\, = \,\prec^{\mathbf{f},\star}_{\Phi}$, $\mathsf{nmPL} = \langle \mathsf{PL}, \prec \rangle$ *satisfies Strong Reassurance.*

Proof. We consider the case $\star = l$. The case $\star = p$ then again follows in view of Lemma E.2.

We proceed in a similar way as in Remark D.1. Where $\mathcal{T} = \{(\emptyset, A, \mathbb{D}) \mid A \in \Sigma\}$ we let $\mathcal{T}_\star$ be $\mathcal{T}$ enriched by $\exists^{\leq l}_i$ and $\exists^{\geq l}_i$ for all $(i, l) \in \mathbb{P}_{\mathsf{fin}}$ (see Def. 29). We show that $\mathcal{T}_\star$ is satisfiable, and build a term-model $M_\perp$ of $\mathcal{T}_\star$ in $\mathcal{L}_h$. For this model we have:

- for all $(i,l) \in \mathbb{P}_{\mathsf{fin}}$, $\mathsf{card}(\mathsf{Ab}_i(M_\perp)) = l$, and
- for all $i \in \mathbb{P}$ for which there is no l such that $(i,l) \in \mathbb{P}_{\mathsf{fin}}$, $\mathsf{card}(\mathsf{Ab}_i(M_\perp)) = \aleph_\emptyset$.

As such we can show that $M_\perp \in \mathcal{M}_{1+i}$ for each $i \geq 0$ (similar as in Lemma D.6) and so $M_\perp \in \bigcap_{i\geq 1} \mathcal{M}_i$. By Lemma D.2, $M_\perp \in \min_\prec(\mathcal{M}_\downarrow)$ and so $M_\perp \in \min_\prec(\mathcal{M}(\Sigma))$. Since $M_\perp \prec M_\top$ this concludes our proof. □

F Proof of Theorem 5.2

Theorem 5.2. *Where* PL *is* $\mathbb{A}$*-separable and based on* $\mathcal{L}$ *and* $\mathcal{L}'$ *is a sublanguage of* $\mathcal{L}$ *without identity and non-identity (but possibly with a congruence* $\approx$*),* $\star \in \{g,p,l\}$*, and* $\dagger \in \{\geq, \supseteq\}$*,* $\mathsf{nmPL} = \langle \mathsf{PL}, \prec_\Phi^{\dagger,\star}\rangle$ *satisfies recapture for* $\mathcal{L}'$*.*

Proof. In the following we suppose that PL is $\mathbb{A}$-separable and $\mathcal{L}'$ is a fragment of $\mathcal{L}$ without identity and without non-identity. Let Σ be a set of $\mathcal{L}'$-sentences that is $\mathsf{PL}^{\mathbb{P}\setminus\mathbb{A}}$-satisfiable. We have to show that $\mathcal{M}_{\mathsf{nmPL}}(\Sigma) = \mathcal{M}_{\mathsf{PL}^{\mathbb{P}\setminus\mathbb{A}}}(\Sigma)$. We show the theorem for $\dagger = \geq$. The proof for $\supseteq$ is very similar and left to the reader.

($\supseteq$) This is Fact 5.4. ($\subseteq$) Suppose $M'' = \langle \mathcal{D}'', v''\rangle \in \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$. Assume for a contradiction that $M'' \notin \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$. Let $M = \langle \mathcal{D}, v\rangle \in \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$. Since $M'' \notin \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}$, $\mathsf{Ab}_i(M'') \neq \emptyset$ for some $i \in \mathbb{P}$. So, $\mathsf{card}(\mathcal{D}) < \mathsf{card}(\mathcal{D}'')$ since otherwise $M \prec M''$ which is impossible since $M'' \in \mathcal{M}_{\mathsf{nmPL}}(\Sigma)$.

Let $\mathcal{N}$ be a set of points for which $\mathcal{N} \cap \mathcal{D} = \emptyset$ and $\mathsf{card}(\mathcal{D} \cup \mathcal{N}) = \mathsf{card}(\mathcal{D}'')$. We now define a model $M' = \langle \mathcal{D} \cup \mathcal{N}, v'\rangle$ as follows: First we fix an arbitrary $\mathsf{d} \in \mathcal{D}$. We also fix an arbitrary $\mathsf{v}_+ \in \mathbb{D} \setminus \mathbb{A}$ and $\mathsf{v}_- \in \mathbb{V} \setminus \mathbb{D}$.

1. for all constants c we let $v'(c) = v(c)$;
2. for all $P = P_i$ we let $v'(P, (d_1, \ldots, d_{\mathtt{i}})) = v(P, (d_1, \ldots, d_{\mathtt{i}})[\mathcal{N}/\mathsf{d}])$, where $(d_1, \ldots, d_{\mathtt{i}})[\mathcal{N}/\mathsf{d}]$ denotes the result of replacing all elements $d_k \in \mathcal{N}$ by d;
3. if there is a non-identity $\neq$ in $\mathcal{L}$, we let
$$v'(\neq, (d_1,d_2)) = \begin{cases} v(\neq, (d_1,d_2)) & d_1, d_2 \in \mathcal{D} \\ \mathsf{v}_+, & \text{if } d_1 \neq d_2 \text{ and } \{d_1,d_2\} \not\subseteq \mathcal{D} \\ \mathsf{v}_-, & \text{if } d_1 = d_2 \text{ and } \{d_1,d_2\} \not\subseteq \mathcal{D} \end{cases};$$
4. if there is an identity $=$ in $\mathcal{L}$, we let
$$v'(=, (d_1,d_2)) = \begin{cases} v(=, (d_1,d_2)) & d_1, d_2 \in \mathcal{D} \\ \mathsf{v}_+, & \text{if } d_1 = d_2 \text{ and } \{d_1,d_2\} \not\subseteq \mathcal{D} \\ \mathsf{v}_-, & \text{if } d_1 \neq d_2 \text{ and } \{d_1,d_2\} \not\subseteq \mathcal{D} \end{cases}$$

5. if there is a congruence $\approx$ in $\mathcal{L}$, we let $v'(\approx, (d_1, d_2)) = v(\approx, (d_1, d_2)[\mathcal{N}/\mathsf{d}])$.

We now show the following:

Lemma F.1. *For all $A \in \mathcal{L}'[\mathcal{D} \cup \mathcal{N}]$, $v_{M'}(A) = v_M(A[\mathcal{N}/\mathsf{d}])$ where $A[\mathcal{N}/\mathsf{d}]$ is the result of replacing for all $d \in \mathcal{N}$ each $\overline{d}$ occurring in A with $\overline{\mathsf{d}}$.*

Proof. We show this by induction over the length of A.

Base. This follows directly by Items 1, 2 and 5 of the definition of v'. (Note that $=$ and $\neq$ are not part of $\mathcal{L}'$.)

Inductive Step.

- Let π be an n-ary connective. $v_{M'}(\pi(A_1, \ldots, A_n)) = f_\pi(v_{M'}(A_1), \ldots, v_{M'}(A_n))$ and $v_M(\pi(A_1, \ldots, A_n)[\mathcal{N}/\mathsf{d}]) = f_\pi(v_M(A_1[\mathcal{N}/\mathsf{d}]), \ldots, v_M(A_n[\mathcal{N}/\mathsf{d}]))$. By the inductive hypothesis, for each $1 \leq i \leq n$, $v_{M'}(A_i) = v_M(A_i[\mathcal{N}/\mathsf{d}])$. Thus, $v_{M'}(\pi(A_1, \ldots, A_n)) = v_M(\pi(A_1, \ldots, A_n)[\mathcal{N}/\mathsf{d}])$.
- Let μ be a quantifier. We have: $v_{M'}(\mu x A) = f_\mu(\{v_{M'}(A(\overline{d})) \mid d \in \mathcal{D} \cup \mathcal{N}\})$ and $v_M(\mu x A) = f_\mu(\{v_M(A(\overline{d})) \mid d \in \mathcal{D}\})$. Since by the inductive hypothesis, if $d \in \mathcal{N}$, $v_{M'}(A(\overline{d})) = v_M(A(\overline{\mathsf{d}}))$, $\{v_{M'}(A(\overline{d})) \mid d \in \mathcal{D} \cup \mathcal{N}\} = \{v_M(A(\overline{d})) \mid d \in \mathcal{D}\}$ and so $v_{M'}(\mu x A) = v_M(\mu x A)$. □

By Lemma F.1, $M' \in \mathcal{M}_{\mathsf{PL}^{\mathbb{V}\setminus\mathbb{A}}}(\Sigma)$. But then $M' \prec M''$ since for all P_i, which is a contradiction. □

G Proof of Lemma 7.1

We start with a fact about quotient models.

Fact G.1. *Where $M = \langle \mathcal{D}, v \rangle \in \mathcal{M}$,*

1. *$M_\approx$ is id-normal, i.e., $M_\approx \in \mathcal{M}^{\mathsf{id}}$,*
2. *$v_M(A(\overline{d_1}, \ldots, \overline{d_n})) = v_{M_\approx}(A(\overline{[d_1]_\approx}), \ldots, \overline{[d_n]_\approx})$ for all $A(\overline{d_1}, \ldots, \overline{d_n}) \in \mathcal{L}[\mathcal{D}]$,*
3. *$v_M(A) = v_{M_\approx}(A)$ for all $A \in \mathcal{L}$.*

Proof. Ad 1. Suppose $v_\approx(\approx, ([d]_\approx, [d']_\approx)) \in \mathbb{D}$. Thus, $v(\approx, (d, d')) \in \mathbb{D}$. Thus, $[d]_\approx = [d']_\approx$. The other direction of **Eq** follows with (Ref).

Ad 2. This is shown inductively. For atomic formulas the claim follows in view of Def. 13. Note that $v_\approx(P, ([d_1]_\approx, \ldots, [d_n]_\approx)) = v(P, (d_1, \ldots, d_n))$ is well-defined: it is independent of the representative of the respective $[\cdot]$-class due to requirement (Str) in (Cong). Similar for the other cases. Consider, for the inductive step, an n-ary connective $\circ$. Then, in view of the inductive hypothesis, $v_{M_\approx}(\circ(A_1, \ldots, A_n)) = f_\circ(v_{M_\approx}(A_1), \ldots, v_{M_\approx}(A_n)) = f_\circ(v_M(A_1), \ldots, v_M(A_n)) = v_M(\circ(A_1, \ldots, A_n))$.

Consider now a quantifier μ. Then, in view of the inductive hypothesis, $v_{M_\approx}(\mu x A(x)) = f_\mu(\{v_{M_\approx}(A(\overline{[d]})) \mid d \in \mathcal{D}\}) = f_\mu(\{v_M(A(\overline{d})) \mid d \in \mathcal{D}\}) = v_M(\mu x A(x))$.

Ad 3. This follows directly with Item 2. □

Lemma 7.1. *Where the underlying comparison is qualitative,* $\prec = \prec_{\Phi}^{\dagger,\pi}$*,* $\dagger \in \{=, \supseteq, \geq, =_c, \mathsf{f}\}$*, and* $\pi \in \{g, p, l\}$*, if* $M \in \min_{\prec}(\mathcal{M}(\Gamma))$ *then* $M_{\approx} \in \min_{\prec}(\mathcal{M}(\Gamma)) \cap \min_{\prec}^{\mathsf{id}}(\mathcal{M}(\Gamma))$*.*

Proof. We paradigmatically give the proof for $\dagger = \supseteq$ and $\pi \in \{p, l\}$.

Suppose $M = \langle \mathcal{D}, v \rangle \in \min_{\prec}(\mathcal{M}(\Sigma))$. Let $M_{\approx} = \langle \mathcal{D}_{\approx}, v_{\approx} \rangle$ be the quotient model of M. Assume for a contradiction that there is a $M' = \langle \mathcal{D}', v' \rangle \in \mathcal{M}(\Sigma)$ for which $M' \prec M_{\approx}$ (and so $\mathcal{D}' \supseteq \mathcal{D}_{\approx}$). We define a new model $M^{\star} = \langle \mathcal{D}^{\star}, v^{\star} \rangle$ where

- $\mathcal{D}^{\star} = (\mathcal{D}' \setminus \mathcal{D}_{\approx}) \cup \mathcal{D}$,[44]
- $v^{\star}(c) = \gamma(v'(c))$ for all constants $c \in \mathsf{Const}(\mathcal{L})$, where

$$\gamma : \mathcal{D}' \to \mathcal{D}^{\star}, d \mapsto \begin{cases} d & d \in \mathcal{D}' \setminus \mathcal{D}_{\approx} \\ d'' & d = [d']_{\approx} \in \mathcal{D}_{\approx} \text{ and } d'' \in [d']_{\approx} \text{ arbitrary.} \end{cases}$$

- $v^{\star}(P_i, (d_1, \ldots, d_{\mathtt{i}})) = v'(P_i, (\mu(d_1), \ldots, \mu(d_{\mathtt{i}})))$ where

$$\mu : \mathcal{D}^{\star} \to \mathcal{D}', d \mapsto \begin{cases} d & d \in \mathcal{D}' \setminus \mathcal{D}_{\approx} \\ [d] & d \in \mathcal{D} \end{cases},$$

- $v^{\star}(\approx, (d, d')) = v'(\approx, (\mu(d), \mu(d')))$.

We now show inductively that for all $(d_1, \ldots, d_n) \in \mathcal{D}^{\star n}$, and all $A(x_1, \ldots, x_n)$ in $\mathcal{L}$,

$$v_{M^{\star}}(A(\overline{d_1}, \ldots, \overline{d_n})) = v_{M'}(A(\overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})).$$

In view of this, $M^{\star} \in \mathcal{M}(\Sigma)$.

The base step follows directly with the definition of $v^{\star}$. For the inductive step we consider a quantifier δ and a formula $\delta x A(x, \overline{d_1}, \ldots, \overline{d_n})$. We have:

$$v_{M^{\star}}(\delta x A(x, \overline{d_1}, \ldots, \overline{d_n})) = f_{\delta}(\{v_{M^{\star}}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}^{\star}\})$$

and

$$\begin{aligned} &v_{M'}(\delta x A(x, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) = \\ &\quad f_{\delta}(\{v_{M'}(A, (\overline{d}, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) \mid d \in \mathcal{D}'\}) = \\ &\quad f_{\delta}(\{v_{M'}(A(\overline{d}, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) \mid d \in \mathcal{D}' \setminus \mathcal{D}_{\approx}\} \cup \\ &\qquad \{v_{M'}(A(\overline{[d]_{\approx}}, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) \mid [d]_{\approx} \in \mathcal{D}_{\approx}\}). \end{aligned}$$

By the inductive hypothesis,

$$\begin{aligned} &\{v_{M'}(A, (\overline{d}, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) \mid d \in \mathcal{D}' \setminus \mathcal{D}_{\approx}\} = \\ &\qquad \{v_{M^{\star}}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}' \setminus \mathcal{D}_{\approx}\} \end{aligned}$$

[44] We assume for simplicity that $\mathcal{D}' \cap \mathcal{D} = \emptyset$.

and (since for all $d \in \mathcal{D}$, $\mu(d) = [d]_\approx$)

$$\{v_{M'}(A(\overline{[d]_\approx}, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) \mid [d]_\approx \in \mathcal{D}_\approx\} = \{v_{M^\star}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}\}.$$

Since

$$\{v_{M^\star}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}' \setminus \mathcal{D}_\approx\} \cup \{v_{M^\star}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}\} = \{v_{M^\star}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}^\star\},$$

and

$$\{v_{M'}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}' \setminus \mathcal{D}_\approx\} \cup \{v_{M'}(A(\overline{[d]_\approx}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}\} = \{v_{M'}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}'\},$$

we have

$$f_\delta(\{v_{M^\star}(A(\overline{d}, \overline{d_1}, \ldots, \overline{d_n})) \mid d \in \mathcal{D}^\star\}) = f_\delta(\{v_{M'}(A, (\overline{d}, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)})) \mid d \in \mathcal{D}'\})$$

and so $v_{M^\star}(\delta x A(x, \overline{d_1}, \ldots, \overline{d_n})) = v_{M'}(\delta x A(x, \overline{\mu(d_1)}, \ldots, \overline{\mu(d_n)}))$. For our truth-functional connectives the case is shown similar and left to the reader.

We now show that $M^\star \prec M$ which is a contradiction to our main supposition. Since $M' \prec M_\approx$ there is an $i \in \mathbb{P}$ for which

1. $\mathsf{Ab}_i(M') \subset \mathsf{Ab}_i(M_\approx)$ and
2. for all $j \geq 1$, $\mathsf{Ab}_j(M') \subseteq \mathsf{Ab}_j(M_\approx)$ (for $\pi = p$), resp., for all $i' < i$, $\mathsf{Ab}_{i'}(M') = \mathsf{Ab}_{i'}(M_\approx)$ (for $\pi = l$).

We will now show that $M^\star \prec M$ by showing that:

1. $\mathsf{Ab}_i(M^\star) \subset \mathsf{Ab}_i(M)$ and
2. for all $j \geq 1$, $\mathsf{Ab}_j(M^\star) \subseteq \mathsf{Ab}_j(M)$ (for $\pi = p$), resp., for all $i' < i$, $\mathsf{Ab}_{i'}(M^\star) = \mathsf{Ab}_{i'}(M)$ (for $\pi = l$).

Let $\pi = p$. (The case for $\pi = l$ is analogous.) For Item 2, consider some $j \geq 1$. Suppose $v_{M^\star}(\alpha_j(d_1, \ldots, d_{a_j})) \in \mathbb{A}$. So, $v_{M'}(\alpha_j(\mu(d_1), \ldots, \mu(d_{a_j}))) \in \mathbb{A}$. Since $M' \prec M_\approx$, $v_{M_\approx}(\alpha_j(\mu(d_1), \ldots, \mu(d_{a_j}))) \in \mathbb{A}$ and therefore $\mu(d_1), \ldots, \mu(d_{a_j}) \in \mathcal{D}_\approx$. So, for each $1 \leq k \leq a_j$, $d_k \in \mu(d_k)$. So, $v_M(\alpha_j(d_1, \ldots, d_{a_j})) \in \mathbb{A}$. Thus, item 2 holds.

For Item 1 suppose that $v_{M'}(\alpha_i([d_1]_\approx, \ldots, [d_{a_i}]_\approx)) \notin \mathbb{A}$, while $v_{M_\approx}(\alpha_i([d_1]_\approx, \ldots, [d_{a_i}]_\approx)) \in \mathbb{A}$. By the latter $v_M(\alpha_i(d_1, \ldots, d_n)) \in \mathbb{A}$. By the former and since for each $1 \leq k \leq a_i$ we have $\mu(d_k) = [d_k]_\approx$, $v_{M^\star}(\alpha_i(d_1, \ldots, d_n)) \notin \mathbb{A}$. This suffices to establish Item 1. □

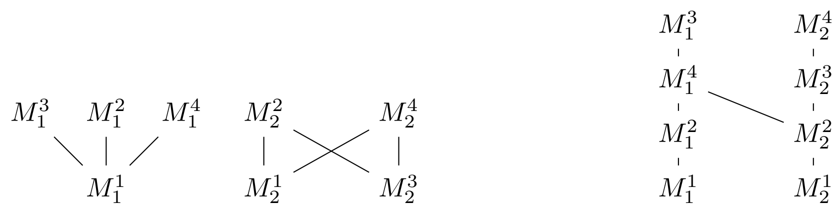

Figure 2: Orderings of models in Example H.1. To the left: $\prec^{\pi,g}_{\mathsf{Pred}}$-ordering (where $\pi \in \{=, =_c, \supseteq, \geq\}$), to the right: $\prec^{\geq,l}_{\mathsf{Pred}}$-ordering.

H Counterexample to Disjunctive Rationality for $\prec^{\pi,\star}_{\Phi}$, $\star \in \{g, l\}$, $\pi \neq \mathsf{f}$

Example H.1. *We consider a language with two unary predicate symbols Q and S, a binary predicate symbol P, a congruence $\approx$ and two constants c and c'. Let*

$$\Sigma = \big\{ \big(\forall x(c \approx x) \wedge !P(c,c)\big) \vee \big(\neg(c \approx c') \wedge !Qc \wedge !Qc'\big),\ !P(c,c) \vee Sc,\ !Qc \vee Sc \big\}$$

We consider the models in the following table.

$M = \langle \mathcal{D}, v\rangle$	$\#(\mathsf{Ab}_P(M))$	$\#(\mathsf{Ab}_Q(M))$	$\#(\mathsf{Ab}_\approx(M))$	$\#(\mathcal{D})$	$M \models Sc$
M_1^1	1	0	0	1	✓
M_1^2	1	1	0	1	
M_1^3	0	1	1	1	✓
M_1^4	1	0	1	1	✓
M_2^1	0	2	0	2	✓
M_2^2	1	2	0	2	
M_2^3	0	1	1	2	✓
M_2^4	1	1	1	2	

We discuss the quantitative approach. Given the order $\prec = \prec^{\pi,g}_{\mathsf{Pred}\cup\{\approx\}}$ where $\pi \in \{=, =_c, \supseteq, \geq\}$, we have the relations between the models depicted in Figure 2 (left). Similarly, for $\prec = \prec^{\geq,l}_{\mathsf{Pred}}$ we have the ordering depicted in Figure 2 (right). (where $\approx$ is listed before Q which is listed before P.).

Let us first consider $\prec^{\pi,g}_{\mathsf{Pred}}$. Note that models in $\min_\prec(\mathcal{M}(\Sigma \cup \{!P(c,c) \vee !Qc\}))$ with domains of sizes 1 and 2 are of the type M_1^1, M_2^1 and M_2^3. We have $\Sigma \cup \{!P(c,c) \vee !Qc\} \models_{\mathsf{nmLP}} Sc$. The models in $\min_\prec(\mathcal{M}(\Sigma \cup \{!P(c,c)\}))$ with domains of sizes 1 and 2 are of the type M_1^1, M_2^2 and M_2^4. Thus, $\Sigma \cup \{!P(c,c)\} \not\models_{\mathsf{nmLP}} Sc$. Similarly, the models in $\min_\prec(\mathcal{M}(\Sigma \cup \{!Qc\}))$ are of the type M_1^2, M_1^3, M_2^1, and M_2^3. Again, $\Sigma \cup \{!Qc\} \not\models_{\mathsf{nmLP}} Sc$.

The situation is similar for $\prec^{\pi,l}_{\mathsf{Pred}}$. While M_1^1 and M_2^1 are minimal models in $\mathcal{M}_{\mathsf{nmPL}}(\Sigma \cup \{!P(c,c) \vee !Qc\})$, M_1^1 and M_2^2 are minimal in $\mathcal{M}_{\mathsf{nmPL}}(\Sigma \cup \{!P(c,c)\})$, and M_1^2 and M_2^1 are minimal in $\mathcal{M}_{\mathsf{nmPL}}(\Sigma \cup \{!Qc\})$. Again, we have $\Sigma \cup \{!P(c,c) \vee !Qc\} \models_{\mathsf{nmPL}} Sc$ while $\Sigma \cup \{!P(c,c)\} \not\models_{\mathsf{nmPL}} Sc$ and $\Sigma \cup \{!Qc\} \not\models_{\mathsf{nmPL}} Sc$.

I Difficulties in turning the linguistic approach non-pragmatic

When pointing out the linguistic non-robustness and pragmatic nature of the linguistic approach in Section 6, we have only considered the standard way of defining abnormalities in the linguistic approach. An open question is whether some of these potentially counter-intuitive results can be avoided by defining the abnormal parts of models differently. In the following we shortly focus on two such variants:[45]

$$\mathsf{Ab}^1_{\mathsf{ling}}(M) = \left\{ \bigwedge_{i=1}^{n} !Pc_i \wedge \bigwedge_{1 \leq i < j \leq n} c_i \neq c_j \mid M \models \bigwedge_{i=1}^{n} !Pc_i \wedge \bigwedge_{1 \leq i < j \leq n} c_i \neq c_j \right\}$$

and $\mathsf{Ab}^2_{\mathsf{ling}}(M) = \{ [!Pc]_{\sim_M} \mid M \models !Pc \}$, where $!Pc_i \sim_M !Pc_j$ iff $M \models c_i = c_j$.

We illustrate the basic idea behind approach 1 with the following example. Let $M \prec^1_{\mathsf{ling}} M'$ iff $\mathsf{Ab}^1_{\mathsf{ling}}(M) \subseteq \mathsf{Ab}^1_{\mathsf{ling}}(M')$. Now consider $\Sigma_1 = \{!Pc_1, !Pc_2\}$ and the models $M_1 = \langle \mathcal{D}, v_1 \rangle$ and $M_2 = \langle \mathcal{D}, v_2 \rangle$ with the following abnormal parts:

1. $\mathsf{Ab}^1_{\mathsf{ling}}(M_1) = \{!Pc_1, !Pc_2\}$ and

2. $\mathsf{Ab}^1_{\mathsf{ling}}(M_2) = \{!Pc_1, !Pc_2, !Pc_1 \wedge !Pc_2 \wedge c_1 \neq c_2\}$.

According to approach 1 model M_1 is better than model M_2, as expected, since it identifies c_1 and c_2 (i.e., $v_1(c_1) = v_1(c_2)$ while $v_2(c_1) \neq v_2(c_2)$). In this example, approach 1 thus mirrors the extensional approach in that it considers models worse with more *(extensionally) different* abnormalities.

According to approach 2 we get:

1. $\mathsf{Ab}^2_{\mathsf{ling}}(M_1) = \left\{[c_1]_{\sim_{M_1}}\right\}$ and

2. $\mathsf{Ab}^2_{\mathsf{ling}}(M_2) = \left\{[c_1]_{\sim_{M_2}}, [c_2]_{\sim_{M_2}}\right\}$.

Note, however, that the two models are still incomparable if we naively compare them according to subset-comparisons, for $[c_1]_{\sim_{M_1}} = \{c_1, c_2\}$ and $[c_1]_{\sim_{M_2}} = \{c_1\}$. We can refine the comparison by defining $M \prec^2_{\mathsf{ling}} M'$ iff

1. for all $\Delta \in \mathsf{Ab}^2_{\mathsf{ling}}(M)$ there is a $\Delta' \in \mathsf{Ab}^2_{\mathsf{ling}}(M')$ such that $\Delta' \subseteq \Delta$, and

2. there is a $\Delta' \in \mathsf{Ab}^2_{\mathsf{ling}}(M')$ for which there is no $\Delta \in \mathsf{Ab}^2_{\mathsf{ling}}(M)$ such that $\Delta \subseteq \Delta'$.

Now we have: $M_1 \prec^2_{\mathsf{ling}} M_2$, as expected.

[45] We simplify things slightly by supposing (i) that all predicates are unary and (ii) that in case of $\mathsf{Ab}^1_{\mathsf{ling}}$ a non-identy $\neq$ is available. An analogous treatment can be achieved by using a classical negation and an identity.

However, both approaches are neither linguistically robust nor non-pragmatic. First, both approaches are not linguistically robust, as our Example 6.1 shows. Second, if we complicate the picture slightly we can also see that the two proposals don't deliver expected outcomes. For this we consider again our Example 6.2. According to the first approach we have $\mathsf{Ab}^1_{\mathsf{ling}}(M^2_a) = \{!Pc_1, !Pc_2\}$, while $\mathsf{Ab}^1_{\mathsf{ling}}(M^2_b) = \{!Pc_1\}$, and so still $M^2_b \prec^1_{\mathsf{ling}} M^2_a$. Similarly, according to the second approach, we have $\mathsf{Ab}^2_{\mathsf{ling}}(M^2_a) = \{\{c_1, c_2\}\}$ and $\mathsf{Ab}^2_{\mathsf{ling}}(M^2_b) = \{\{c_1\}\}$, thus $M^2_b \prec^2_{\mathsf{ling}} M^2_a$. Both approaches are thus pragmatic according to Def. 11. Altogether our discussion shows that it is difficult to alter the linguistic approach so that it doesn't run into the pragmatic oddities pointed out in our examples above.