

The Caesar problem – a piecemeal solution

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Abstract

The Caesar problem arises for abstractionist views, which seek to secure reference for terms such as ‘the number of X s’ or $\#X$ by stipulating the content of ‘unmixed’ identity contexts like ‘ $\#X = \#Y$ ’. Frege objects that this stipulation says nothing about ‘mixed’ contexts such as ‘ $\#X = \text{Julius Caesar}$ ’. This article defends a neglected response to the Caesar problem: the content of mixed contexts is just as open to stipulation as that of unmixed contexts.

1 The Caesar problem

In his *Grundlagen der Arithmetik*,¹ Frege confronts a strange question:

- (1) Is Julius Caesar a cardinal number?

Might the last Roman dictator moonlight as a mathematical object? Frege apologises for asking a question that ‘looks nonsensical’ (§65). All the same, Caesar has given his name to a long-standing problem facing Fregean approaches to abstract objects.

The *Caesar problem* arises in Frege’s discussion of how numbers are ‘given to us’ (§62). The answer he floats contains the kernel of what is now called abstractionism. Central to this view is a ‘top-down’ approach to the metasemantics of abstract terms.² Even if a community lack an ostensive or descriptive means to baptise numbers, they may accord reference to a term such as ‘the number of X s’ (in symbols: $\#X$)³ by determining the content of a suitable range of whole sentences involving it. Frege, and most abstractionists after him, focus on identity contexts of the form ‘ $\#X = \#Y$ ’. According to what I will call his *abstractionist proposal*, moreover, their content may be stipulated with a ‘definition’ (§63) that has come to be known as HP:

¹Frege (1980). Section references refer to this work unless indicated otherwise.

²Warren (2017) uses the label ‘top-down metasemantics’ in a similar way. Williams (2007) dubs this a ‘two-step’ approach to metasemantics.

³When there is no risk of confusion, quotes are sometimes omitted from formal expressions.

HP For any classes X and Y , the following are equivalent (TFAE):

$$\#X = \#Y; \quad X \text{ and } Y \text{ are equinumerous}$$

Construed in this way, HP *stipulates* that the identity context on its left-hand side ‘is to mean the same as’ (§65) the sentence on its right-hand side.

Frege’s interest in HP stems from his desire to give a purely logical foundation for arithmetic. As is now well known, the system that results from adding HP, construed as an axiom, to a standard formulation of second-order logic *interprets* Peano Arithmetic. In other words, Frege’s definitions of arithmetical expressions permit us to rewrite the theorems of arithmetic using no non-logical terms save for # so that their #-based translations emerge as theorems of the HP-based system.⁴

Does this provide a purely logic- and definition-based foundation of arithmetic? Frege objects that his abstractionist proposal faces the Caesar problem:

... our proposed definition ... does not provide for all cases. It will not, for instance, decide for us whether [Julius Caesar] is the same as [the number of Xs]... Naturally no one is going to confuse [Caesar] with [the number of Xs]; but that is no thanks to our definition of [number]. That says nothing as to whether the proposition [$\#X = q$] should be affirmed or denied, except for the one case where q is given in the form of [$\#Y$]. (§66, example changed.⁵)

The problem needs some unpicking. But abstractionists generally follow Frege in maintaining that a stipulation does not succeed in conferring reference on terms such as $\#X$ unless, at least in some cases, it does ‘decide’ questions such as (1). Frege rapidly concludes that HP’s stipulation does not settle *Caesar questions*, as I will call questions of this kind, and abandons his abstractionist proposal. He opts instead for his well-known extension-based account of number, leading him, in *Grundgesetze der Arithmetik*, to Basic Law V and Russell’s paradox.

Frege’s abrupt change of tack may seem like an overreaction. Might his abstractionist proposal, or something like it, instead solve the Caesar problem? Nowadays, the dominant abstractionist response contends, contrary to Frege, that HP *does decide* Caesar questions. According to what I will call *wholesale responses*, number–Roman identity conditions come bundled wholesale with number–number identity conditions. Less metaphorically, although HP only overtly stipulates the content of ‘unmixed’ identity contexts—those of the form $\#X = \#Y$ —this stipulation also serves to covertly determine the content of a range of other contexts, including ‘mixed’ identity contexts such as $\#X = q$, in which q is not a #-term. Consequently—given how the world is⁶—HP decides the corresponding Caesar questions.

⁴This result (Frege’s Theorem) is noted in Parsons (1964) and proved in Wright (1983).

⁵§66 considers ‘England is the same as the direction of the Earth’s axis’. The now-standard example (Caesar) appears in an objection to a competing account of number terms in §56.

⁶A sentence’s content determines its truth-conditions (in a context); its truth-value also depends on the world. Henceforth I sometimes follows Frege in leaving the world’s contribution tacit.

This has been the guiding approach behind the various iterations of Bob Hale and Crispin Wright’s influential response to the Caesar problem. According to Hale and Wright, the stipulation of $\#P$ fixes the meaning of the $\#$ -operator so as to introduce a sortal concept: *cardinal number*. Moreover, the concept’s application-condition, which settles whether something (e.g. Caesar) is a cardinal number is, as they put it, ‘extractable from’ the identity conditions for cardinal numbers, directly stipulated by $\#P$ (2001b, 369). Hale and Wright’s most recent proposal appeals to a category-based ontology to effect this extraction. Competing wholesale responses have deployed different assumptions about the ambient metaphysics or metasemantics to bridge the gap between the content $\#P$ explicitly stipulates and the content that, on a wholesale view, $\#P$ covertly determines.⁷

In my view, wholesale responses grossly overestimate what is achieved just by laying down $\#P$. On the face of it, however, there is a much easier way to solve the Caesar problem. According to my favoured response, the content of a mixed context such as ‘ $\#X = \text{Caesar}$ ’ is just as open to stipulation as the content of an unmixed context such as ‘ $\#X = \#Y$ ’. On this view, then, Caesar questions may be decided piecemeal, via further stipulations governing unmixed contexts laid down alongside $\#P$. This is the main idea behind the *piecemeal response* to the Caesar problem.

Suggestions along these lines occasionally surface in the extensive debate surrounding this issue. Frege, in *Grundgesetze*, makes short work of the Caesar questions that arise in the context of his formal system by stipulating, in effect, that each truth-value is identical to an extension, namely its singleton (2013, §10). In a similar spirit, Michael Dummett notes that it would be ‘straightforward’ to attain negative answers to the Caesar questions that arise in the *Grundlagen* by ‘direct stipulation’ (1978, 111). More recently, Øystein Linnebo suggests that answers to Caesar questions often turn on ‘conceptual decisions’ and not just ‘factual discoveries’ (2018, 160).⁸

However, the piecemeal response is seldom defended and remains underdeveloped. One reason for this, I suspect, is that a would-be stipulative solution to the Caesar problem is widely taken to be hopeless. Richard Kimberley Heck, for example, offers this verdict on the stipulative identifications in *Grundgesetze*:

Plainly, Frege is not here offering a solution to the Caesar problem: A piecemeal ‘solution’ is not a solution to the problem but a recipe for side-stepping it. (2005, n. 17)

Heck does not elaborate on why, in the general case, a piecemeal response falls short of a fully fledged solution. Fortunately, others have been more explicit about their misgivings. Fraser MacBride objects that piecemeal stipulations may conflict with ‘antecedent facts’ (2006, 193):

⁷For the category-based approach, see Wright (2020, 309–315) and Hale and Wright (2001b, 369). Earlier neologist responses are offered in Wright (1983, ch. 3) and Hale (1987, ch. 8). For a competing approach, based on ‘real definitions’, see Rosen and Yablo (2020).

⁸A view in a similar spirit is sometimes defended in the context of mathematical structuralism. Shapiro (1997, 81) maintains that cross-structural identifications are ‘matters of *decision*, based on convenience, not matters of discovery’.

Suppose that Caesar leads a double life. Suppose that in addition to leading his material existence Caesar is also a number. In that case the stipulation that sentences that say Caesar is a number are all false cannot succeed. For some of these sentences will be true and true sentences cannot be stipulated to be false. . . . Stipulation cannot suffice as a basis for determining that Caesar is no number. (2006, 192)

Another worry stems from Frege’s observation that his *Grundgesetze* stipulation cannot be coherently generalized to identify each item—and, in particular, each extension—with its singleton, on pain of conflict with the identity-conditions laid down for extensions (2013, §10, n. 1).⁹ Hale and Wright object that piecemeal stipulations risk incoherence:

... before we can safely stipulate that some object ... is a certain extension, we need an assurance that it is not (behind our back, as it were) some other extension—else our new stipulation might conflict with the original stipulation of identity-conditions ... A solution to the Caesar Problem is thus presupposed, and cannot be provided, by generalizing the kind of stipulation Frege envisages for truth-values. (2001b, n. 8)

Both worries merit closer examination. In the end, though, I will argue that neither poses a serious threat to the response I defend. Moreover, when properly developed, the piecemeal response not only withstands these objections but, in important respects, improves on its wholesale competitors. First, we need a better grasp of the underlying problem. The remainder of this section elaborates on what I take to be the metase-mantic core of abstractionism (1.1) and the challenge to this metasemantics posed by Caesar (1.2). Section 2 then argues that the problems facing the wholesale view are worse than has so far been recognized. Most damagingly, a wide range of wholesale views fall foul of Benacerraf’s multiple reduction problem and unduly constrain mathematical freedom. Section 3 then develops a piecemeal version of abstractionism that avoids these problems and has a robust reply to the MacBride and Hale–Wright worries.

1.1

Frege’s abstractionist proposal may be developed in various ways. The best known stops just short of full-blown logicism. According to Hale and Wright’s neologicist version of abstractionism, HP does not provide a foundation for arithmetic that is based purely on logic and definitions, strictly conceived; but HP is a definition-like truth, in a more liberal sense.¹⁰

To extend their programme to other branches of mathematics, neologicists have sought other axioms that enjoy a similar status. An *abstraction principle* is standardly taken to have an HP -like form:¹¹

⁹The incoherence here is separable from Basic Law V’s inconsistency. The piecemeal stipulation also conflicts with consistent versions of this axiom, such as New V (stated in 2.2).

¹⁰See, for instance, Hale and Wright (2001a, 4).

¹¹Formally, an instance of the AP -schema is canonically taken to be the universal closure of a

AP For any x and y , TFAE: $\sigma x = \sigma y$; $x \sim_{\sigma:\sigma} y$

In the axiom, x and y are first- or higher-order variables that range over entities of the corresponding type (individuals, classes of individuals, etc.); $x \sim_{\sigma:\sigma} y$ expresses an equivalence relation on these entities; and σ is a singular-term-forming operator. Let's call a term of the form σx an *abstract*- or σ -*term*, and its referent (if any) for a suitable value of x a σ -*abstract*, or simply, an *abstract*. The axiom AP thus states identity conditions for σ -abstracts in terms of a *unity relation* $\sim_{\sigma:\sigma}$ holding between their *specifications*. I also speak of #-abstracts as #-*cardinals*, and so on, when an abstraction principle seeks to introduce mathematical objects of a familiar kind.¹²

Two comments are in order. First, in cases where x and y are higher-order variables, the usual range of options for interpreting them is available. For concreteness, I call the value of a first-order variable— x, y , etc.—an individual (object, item) and the value of a monadic second-order variable— X, Y , etc.—a class of individuals. But an abstractionist may equally adopt a Fregean or plural interpretation of second-order variables. In this case, my 'class'-talk below should be seen as elliptical for discourse about Fregean concepts or pluralities. In each case, the class-, concept-, or plurality-interpretation may be extended to polyadic and higher-order variables in the standard way.¹³

Second, although official characterizations of abstraction principles in the AP-mould are widespread, abstractionists often rely on a more liberal conception of the distinguished class of axioms. For example, Hale (2000) proposes to obtain positive reals from a suitable domain of quantities, and then to identify reals, positive or non-positive, with *differences* of positive reals.¹⁴

D For any reals $x, y, z, w > 0$, TFAE: $\text{diff}(x, y) = \text{diff}(z, w)$; $x + w = y + z$

Hale's axiom fails to fit the AP-mould unless it is liberalized in two ways: (i) to permit polyadic σ -terms of the form σx for $x = x_1, \dots, x_n$ and (ii) to allow for the specification variables to be restricted to a domain, \mathcal{D}_σ .¹⁵ Since Hale intends D to 'work in essentially the same way as paradigm abstractions' (107),¹⁶ I can see no principled reason for abstractionists not to liberalize their characterization, as per (i) and (ii). In deference to tradition, however, I use the term 'abstraction principle' in its standard sense and use *unmixed postulate* for both strict instances of AP and their liberalized counterparts.

biconditional: $\sigma x = \sigma y \leftrightarrow x \sim_{\sigma:\sigma} y$. See, for instance, Hale (2000, 100), Hale and Wright (2001a, 16), and Ebert and Rossberg (2016, 3).

¹²My terminology is intended to be neutral on whether #-cardinals (items introduced by HP) are cardinals (familiar mathematical objects). I return to this question in 1.2.

¹³For the plural case, see, for instance, Linnebo and Rayo (2012, apps. A, B.2).

¹⁴Strictly, Hale abstracts D-reals from the abstracts he identifies with positive reals (namely, ratios of quantities). Similar remarks apply to CP and CP* below.

¹⁵Boldface x and y are henceforth used in this way for a sequence of one or more variables, which may differ in type. Officially, the domain \mathcal{D}_σ is specified by a condition $\delta_\sigma(x)$ associated with σ , and comprises the entities that satisfy this condition under the pre-abstraction interpretation.

¹⁶Shapiro (2000, 338) makes similar remarks about a polyadic postulate.

Terminology aside, a central neologicist contention is that, in a wide range of cases, we may secure reference to σ -abstracts, and gain apriori knowledge of their identity conditions, by laying down the relevant postulate as an ‘implicit definition’.¹⁷ Here I propose to bracket much-debated questions surrounding the epistemic status of abstraction principles and focus on what I take to be the *metasemantic core* of Frege’s abstractionist proposal, shared in its essentials by neologicism and other variants of abstractionism.

For simplicity, let’s focus on an idealized linguistic community who initially speak a higher-order language. The language need not contain abstract-term-forming operators; but it’s convenient to suppose that it does contain a singular term ‘Caesar’ that refers to Caesar.¹⁸ Suppose further that the language’s initial interpretation is encoded as a model-theoretic interpretation of the standard kind—an *MT-interpretation*—which assigns semantic values to its expressions and determines truth-values for its sentences in a broadly Tarskian way.¹⁹ In outline, an attempt to confer reference on σ -terms via abstraction takes place in three stages:²⁰

- First, if necessary, the community may expand their lexicon with one or more abstract-term-forming operators. This purely syntactic addition leaves them with an initial interpretation of the expanded lexicon which accords no reference to the newly available σ -terms.²¹
- Second, the community stipulate the content of a range of sentences framed in the expanded lexicon. To begin with, let’s assume, in line with the abstractionist mainstream, that an abstraction attempt is based on one or more unmixed postulates that each stipulate the content of the relevant identity context in terms of an antecedently expressible unity relation.²²
- Third, semantic values for expressions in the expanded lexicon (and thus referents for σ -terms) are selected subject, at least, to the constraint that these compositionally determine the stipulated sentential content. If the abstraction attempt is successful, the expanded lexicon is furnished with a new interpretation that extends the initial one in line with this constraint (leaving the referent of ‘Caesar’

¹⁷See, for instance, Wright (1997, 278–80).

¹⁸This account of abstraction is readily refined to permit the initial language to include higher-order predicates and function symbols other than σ -operators, which I omit for ease of exposition.

¹⁹Officially, an MT-interpretation is a pair $\langle M, I \rangle$ comprising a non-empty set M as its domain and a function I that assigns a suitable referent/extension to each expression in the lexicon. I say ‘broadly Tarskian’ since the semantics allows for empty abstract terms. For an n -ary operator σ , $I(\sigma)$ is a *partial function* from n -tuples of appropriately typed entities based on M (either members of M or classes built from members of M) to members of M . To assign truth-values to atomic contexts which contain undefined σ -terms, I adopt a ‘negative free’ semantics that rules them false, and otherwise assigns truth-values in the usual Tarskian way.

²⁰ Compare Hale (1997, 98), Wright (1997, 276–9), Heck (2011a, 51–2), Studd (2016, 583–90, 597–9) and Linnebo (2018, ch. 8).

²¹A community may make successive abstraction attempts. In this case, an attempt’s initial interpretation may accord reference to σ -terms introduced by earlier attempts.

²²The notion of sentential content here must be unstructured, permitting the same content to be attached to sentences with different syntactic structures.

unchanged). In this case, let's say that the post-abstraction interpretation *extends the initial interpretation according to the stipulated postulates*.²³

Abstractionists may flesh out this bare-bones sketch depending on their preferred view of stipulation, sentential content, semantic values, and so on. But here, I propose to stay as neutral as possible on the details and focus on the core account of how—to indulge in some suggestive metaphors— σ -abstracts are 'abstracted from their specifications', 'introduced' or 'given to us'.

1.2

Frege's complaint that HP fails to decide Caesar questions poses a significant challenge to the proposed metasemantics.²⁴ Consider the following question facing a community who make an abstraction attempt based solely on HP (writing Λ for the empty class):

- (2) Is $\#\Lambda = \text{Caesar}$?

The Caesar problem then emerges in the following inconsistent triad:

- C1 The HP -abstraction attempt determines a unique referent for $\#\Lambda$ (and leaves the referent of 'Caesar' unchanged): post-abstraction, under the new interpretation, 'Caesar' is a singular term that refers to Caesar and $\#\Lambda$ is a singular term that refers to a unique item b .
- C2 The HP -abstraction attempt confers the standard syntax and semantics on the identity predicate: post-abstraction, if s and t are singular terms that refer to a and b , then ' $s = t$ ' is a well-formed identity context that is true if a is identical to b and false if a is not identical to b .
- C3 The HP -abstraction attempt settles no determinate—yes/no—answer for (2): post-abstraction, ' $\#\Lambda = \text{Caesar}$ ' is neither true nor false.

A solution to this version of the Caesar problem calls for a well-motivated denial of one of C1–C3. How, then, should the abstractionist respond?

Before I come to the wholesale response, two less popular responses merit attention. One repudiates C1. This thesis *holds* if the HP -abstraction

²³Formally, when the initial and post-abstraction interpretations— \mathcal{I} and \mathcal{J} —are MT-interpretations (of the expanded lexicon), $\langle M, I \rangle$ and $\langle N, J \rangle$, then \mathcal{J} extends \mathcal{I} according to some stipulated unmixed postulates if two conditions are met: (i) \mathcal{J} is an extension of \mathcal{I} , in the model-theoretic sense that $M \subseteq N$, $I(R) = J(R) \cap M^n$, $I(\sigma) = J(\sigma) \upharpoonright \text{dom}(I(\sigma))$, and $I(c) = J(c)$, for each n -ary predicate R , operator σ , and constant c in the expanded lexicon; (ii) for each stipulated postulate, when x and y are assigned to members of \mathcal{D}_σ , the terms $\sigma(x)$ and $\sigma(y)$ are defined under \mathcal{J} , and the truth-value of the postulate's left-hand side under \mathcal{J} coincides with the truth-value of its right-hand side under \mathcal{I} .

²⁴Compare, for instance, Heck (1997, 277–8), MacBride (2006, 186–9), and Hale and Wright (2001b, 341–2). Heck and MacBride also emphasize an epistemological aspect to the problem. The main text continues to focus on the metasemantic issue. But let me add that, on the piecemeal view I defend, answers to Caesar questions are settled by postulates that plausibly enjoy a similar epistemic status to abstraction principles.

attempt singles out a unique model-theoretic interpretation as the new interpretation. But, on reflection, how could it do so? If there are any MT-interpretations that extend the initial one according to \mathbb{HP} , there are infinitely many isomorphic MT-interpretations that extend it according to \mathbb{HP} . Some take $\#A$ to refer to a non-Roman; others to a Roman. How, then, could the \mathbb{HP} -abstraction attempt determine a unique, and presumably non-Roman, referent for $\#A$?

Rather than attempt to answer this question, advocates of the first response reject C1 and maintain that the \mathbb{HP} -abstraction attempt, even if successful, leaves the reference of $\#$ -terms indeterminate. The usual range of options is available to account for indeterminacy. One straightforward way is to deploy a supervaluationist semantics.²⁵ On this view, the post-abstraction interpretation corresponds to a class of MT-interpretations, encoding admissible ways of selecting determinate semantic values for the extended lexicon. A sentence is then true (false) under the post-abstraction interpretation if it is true (false) under each admissible MT-interpretation; otherwise, it lacks a determinate truth-value. In the absence of further constraints on the reference of $\#$ -terms, the abstractionist might adopt a radical version of the indeterminacy response that deems admissible every MT-interpretation that extends the initial one according to \mathbb{HP} . According to the radical indeterminacy response, $\#A$ may be admissibly interpreted to refer to anything whatsoever, with the result that ' $\#A = \text{Caesar}$ ' lacks a determinate truth-value.

A second response shares something of the quasi-structuralist feel of the indeterminacy view. This response maintains that (2) embodies a category mistake.²⁶ Consequently, C2, which accords the standard syntax and semantics to the problematic identity context, is denied on the grounds that ' $\#A = \text{Caesar}$ ' is either syntactically or semantically defective. This response also admits of a radical version, which takes every Caesar question to be defective.

Might the indeterminacy or category mistake response be developed into a satisfying solution to the Caesar problem? Not, I think, in their radical versions. Caesar questions are *much* more commonplace than 'nonsensical'-looking examples like (1) and (2) may suggest. Mixed identity contexts also give rise to questions like the following:

(3) Is $\#A = 0_{\mathbb{N}}$?

Here, ' $0_{\mathbb{N}}$ ' is not presumed to be a $\#$ -term: instead it is a disambiguated version of the ordinary English numeral that refers to a familiar natural number.²⁷ (I adopt a similar convention with subscripts below, reserving ' \mathbb{N} ', ' \mathbb{Q} ', and ' \mathbb{R} ' for the *familiar* natural, rational, and real numbers.) More generally, the abstractionist also needs to account for atomic contexts of the form $R(t_1, \dots, t_n)$ where one or more of t_1, \dots, t_n are σ -terms. These give rise to a further stock of Caesar questions:²⁸

²⁵Compare Boccuni and Woods's (2020) supervaluationist treatment of what they take to be the arbitrary reference of $\#$ -terms. MacBride (2006, 190–3) critically discusses this option.

²⁶Compare Benacerraf (1965, 64–7) and Heck (1997, 280–1; 2011b, 17–9).

²⁷More cautiously, ' $0_{\mathbb{N}}$ ' purports to refer to a number. Since ontology is not at issue here, I omit this caveat, and take for granted the existence of familiar mathematical entities.

²⁸Questions such as (3) and (4) respectively exemplify what have been dubbed *the Counter-Caesar*

- (4) Is $\#A$ Roman?
- (5) Is $\#A$ non-concrete?
- (6) Is $\#A \in \mathbb{N}$?

As in the case of (2), there is a straightforward sense in which \mathbb{HP} ‘says nothing’ to answer questions such as (3)–(6): provided the abstract terms are undefined pre-abstraction, if any MT-interpretations extend the initial interpretation according to \mathbb{HP} , some of them render the embedded context true and others render it false.²⁹ According to the radical indeterminacy response, therefore, the Caesar questions (2)–(6) all lack determinate answers, as is also the case according to the radical version of the category mistake response.

Abstractionists, however, cannot leave (3) and (6) unanswered—at least, not if they wish to explain how *the natural numbers* are given to us. For, unless these numbers, themselves, are found among the $\#$ -abstracts introduced by \mathbb{HP} , abstraction at best introduces ersatz-copies of the natural numbers. If, on the other hand, each natural number is identical to the corresponding $\#$ -abstract, this requires an affirmative answer to (3) and (6).³⁰ A similar point can be made about (5). This time, an affirmative answer is required by the broader metaphysical outlook adopted by neologicist abstractionists like Hale and Wright.³¹

Abstractionists, then, have independent reasons to think that (3), (5), and (6) have determinate answers. Semantically ascending, each of the embedded contexts has a determinate truth-value and, therefore, is not semantically or syntactically defective. This rules out the radical versions of

problem (MacBride 2003) and *the Roman problem* (Fine 2002). In my view, the various issues are really different aspects of a single problem.

²⁹The ‘say nothing’ point may be established with a permutation argument. More generally, suppose that \mathcal{J} is an MT-interpretation $\langle N, \mathcal{J} \rangle$ that extends \mathcal{I} according to one or more unmixed postulates. Consider a Caesar question that embeds a context of the form $R(c_1, \dots, c_m, \sigma_1(x_1), \dots, \sigma_n(x_n))$ whose σ_i -terms, for $i = 1, \dots, n$, are formed using different operators and undefined under \mathcal{I} for all assignments. Then, except in a trivial case, whatever truth-value the R -context receives under \mathcal{J} (when its free variables are respectively assigned to a_1, \dots, a_n), there are also MT-interpretations \mathcal{J}_1 and \mathcal{J}_2 that extend \mathcal{I} according to the same unmixed postulates, and differ from \mathcal{J} at most on the reference of $\sigma_1, \dots, \sigma_n$, that respectively render the R -context true and false (under this assignment). The trivial case is when the n -ary relation on N defined by $R(c_1, \dots, c_m, x_1, \dots, x_n)$ under \mathcal{J} is permutation invariant. In this case, the Caesar question is insensitive to which members of N are the referents of σ_i -terms. Otherwise, there is a permutation π on N such that $\langle a_1, \dots, a_n \rangle$ stands in the defined relation and $\langle \pi a_1, \dots, \pi a_n \rangle$ does not. In this non-trivial case, for $i = 1, \dots, n$, let π_i transpose $(J(\sigma_i))(a_i)$ and a_i . Then \mathcal{J}_1 and \mathcal{J}_2 may be defined to be just like \mathcal{J} , except that $J_1(\sigma_i) = \pi_i \circ (J(\sigma_i))$ and $J_2(\sigma_i) = \pi \circ (J_1(\sigma_i))$. By construction, \mathcal{J}_1 and \mathcal{J}_2 differ on the Caesar question, and adapting the reasoning of Fine’s switching lemma (2002, 110) both \mathcal{J}_1 and \mathcal{J}_2 extend \mathcal{I} in accordance with the relevant postulates. The same argument may be applied, *mutatis mutandis*, when the embedded context has the form $\sigma x = c$. In this case, the non-triviality condition requires only that N contain more than one item. For discussion of whether Frege anticipates permutation arguments of this kind see Wehmeier and Schroeder-Heister (2005).

³⁰MacBride (2006, 134) emphasizes this point. How far ersatz copies would serve Frege’s logicist aims is open to question: see Benacerraf (1981), Weiner (1984), and Blanchette (2012, ch. 4). When it comes to abstractionism, however, the interest of the view is significantly diminished if it cannot explain reference to familiar mathematical objects. Wright (1999) is clear that his target is ‘genuine arithmetic’ (322).

³¹See, for instance, Hale and Wright (2001a, 7).

the indeterminacy and category mistake responses. But it is compatible with moderate versions of these responses that accord a determinate answer to (3), (5), and (6) and accord no determinate answer to other Caesar questions, perhaps including (2) and (4). Developed in this way, however, the indeterminacy and category mistake responses at best provide a partial solution to the Caesar problem. For no progress has yet been made in explaining how abstraction settles answers in the determinate, non-defective cases.

This brings me to the wholesale response. On this view, at least in some cases, abstraction does decide Caesar questions. In the case of Caesar himself, it's open to a wholesale abstractionist to follow Hale and Wright in maintaining that HP does decide (2), contrary to C3.

2 Wholesale responses

According to wholesale versions of abstractionism, an abstraction attempt not only stipulates the content of unmixed identity contexts, it also determines the content of mixed identity contexts or other atomic contexts and thereby—given how the world is—settles answers for the corresponding Caesar questions. The view encompasses a wide range of responses to the Caesar problem that differ on their accounts of *how* abstraction decides Caesar questions.³² Rather than attempting to pick these off one by one, my aim in this section is to raise three objections against this general style of response: (i) a broad range of wholesale responses conflict with various intuitive identity judgements; (ii) they give rise to a version of Benacerraf's multiple reduction problem; and (iii) they unduly constrain mathematical freedom.

2.1

The first objection arises in connection with Caesar questions concerning the identity of abstracts. Suppose that σ - and ρ -abstracts are introduced by different unmixed postulates. Under what conditions are they identical?

Two wholesale answers have dominated the discussion. According to the first, σ - and ρ -abstracts are distinguished in a fine-grained way:³³

FG For any $x \in \mathcal{D}_\sigma$ and $y \in \mathcal{D}_\rho$, TFAE:

$$\sigma x = \rho y; \quad \sim_{\sigma:\sigma} \text{ and } \sim_{\rho:\rho} \text{ express the same relation and } x \sim_{\sigma:\sigma} y$$

Despite its superficial similarity with an abstraction principle, FG is no stipulation. Instead, the wholesale proposal is intended as a substantive metaphysical thesis.³⁴ This particular proposal—which automatically distinguishes abstracts associated with different unity relations—may be mo-

³²Some of the options are listed in n. 7

³³Fine (2002, 48) outlines this fine-grained response.

³⁴Compare, for instance, Hale and Wright (2001b, 370).

tivated by Hale and Wright’s suggestion that ontology divides into disjoint categories:³⁵

Within a category, all distinctions between objects are accountable by reference to the criterion of identity distinctive of it, while across categories, objects are distinguished by just that—the fact that they belong to different categories. (2001b, 389)

Whatever the merits of a category-based ontology, Roy Cook and Philip Ebert (2005, 125) maintain that $\mathbb{F}\mathbb{G}$ is subject to a counterexample, deriving from Fine (2002).³⁶ Consider a variant of $\mathbb{H}\mathbb{P}$ they dub $\mathbb{F}\mathbb{H}\mathbb{P}$:

$\mathbb{F}\mathbb{H}\mathbb{P}$ For any classes X and Y , TFAE:

$$\aleph X = \aleph Y; \quad X \text{ and } Y \text{ are equinumerous or both infinite}$$

Although they come apart in infinite cases, the two postulates agree on the identity conditions for the finite $\#$ - and \aleph -cardinals which provide candidate natural numbers according to accounts of arithmetic based on $\mathbb{H}\mathbb{P}$ or $\mathbb{F}\mathbb{H}\mathbb{P}$.³⁷

$$\begin{array}{ll} 0_{\#} =_{\text{df}} \#\Lambda \text{ (i.e. } \#\{x : x \neq x\}) & 0_{\aleph} =_{\text{df}} \aleph\Lambda \\ (n+1)_{\#} =_{\text{df}} \#\{m_{\#} : m_{\#} \leq n_{\#}\} & (n+1)_{\aleph} =_{\text{df}} \aleph\{m_{\aleph} : m_{\aleph} \leq n_{\aleph}\} \end{array}$$

Cook and Ebert intuit that corresponding finite $\#$ - and \aleph -cardinals are identical: in general, $n_{\#} = n_{\aleph}$. According to $\mathbb{F}\mathbb{G}$, however, there is no overlap between $\#$ - and \aleph -cardinals (since $\mathbb{H}\mathbb{P}$ and $\mathbb{F}\mathbb{H}\mathbb{P}$ deploy different unity relations). If Cook and Ebert are right, the wholesale proposal incorrectly distinguishes identical abstracts: $\mathbb{F}\mathbb{G}$ is sometimes too fine-grained.

2.2

Cook and Ebert’s intuition is not beyond question; but let it stand for now.³⁸ Might a wholesale abstractionist accommodate this intuition? Fine proposes a more coarse-grained view (2002, 47–9):

cg For any $x \in \mathcal{D}_{\sigma}$ and $y \in \mathcal{D}_{\rho}$, TFAE:

$$\sigma x = \rho y; \quad \{z : z \sim_{\sigma:\sigma} x\} = \{z : z \sim_{\rho:\rho} y\}$$

On this view—which, again, is not intended as a stipulation— σ - and ρ -abstracts are identical when their specifications determine the same equivalence class under their respective unity relations; equivalently—dispensing with the ‘class’-talk—when the same specifications stand in the σ -abstract’s unity relation to its specification as stand in the ρ -abstract’s unity relation to its specification.³⁹

³⁵Compare Linnebo (2005; 2018, 167–9).

³⁶Fine offers a more cautious verdict on the cases discussed in 2.1–2.2. See 2.4.

³⁷Officially, I take ‘ $n_{\#}$ ’ and ‘ n_{\aleph} ’ as shorthand for the corresponding $\#$ - and \aleph -terms. Similar remarks apply to ‘ $0_{\#}$ ’ and ‘ 0_{sup} ’.

³⁸I return to the various intuitions in play in 2.4.

³⁹The right-hand side of cg may be formalized $\forall z(z \sim_{\sigma:\sigma} x \leftrightarrow z \sim_{\rho:\rho} y)$, following Cook and Ebert (2005, 136). Compare Fine (2002, 167).

It's not so clear what underlying metaphysics might be used to motivate CG. In its favour, it does deliver Cook and Ebert's intuitive judgment that $n_{\#} = n_{\mathbb{N}}$. Putative counterexamples to CG, however, have also been forthcoming. Here is another case deriving from Fine (2002). Neologicist attempts to recover set theory often deploy a variant of Basic Law V to introduce what I will call †-sets:⁴⁰

NEW V For any X and Y , TFAE:

$$\dagger X = \dagger Y; \quad X \text{ and } Y \text{ are coextensive or both universe-sized}$$

The second disjunct in NEW V's right-hand side implements a 'limitation-of-size' fix to Russell's paradox, which would emerge without it. This time the difficulty centres on the empty †-set: $\emptyset_{\dagger} =_{\text{df}} \dagger \Lambda$. According to CG, $0_{\#} = \emptyset_{\dagger}$ (since they are both associated with the same equivalence class). My intuitive judgment, however, is that there is no overlap between natural numbers and sets. Assuming that #-naturals and †-sets are identical with their familiar counterparts, it follows that $0_{\#} \neq \emptyset_{\dagger}$. If this is right, Fine's proposal incorrectly identifies distinct abstracts: CG is sometimes too coarse-grained.

2.3

The counterexamples levelled against the FG- and CG-proposals may prompt the wholesale abstractionist to seek an intuitive Goldilocks zone somewhere between the two—thus a medium-grained proposal:⁴¹

$$\text{MG} \quad \text{For any } x \in \mathcal{D}_{\sigma} \text{ and } y \in \mathcal{D}_{\rho}, \text{ TFAE: } \quad \sigma x = \rho y; \quad x \sim_{\sigma;\rho} y$$

Really, this is a proposal-schema. The relation $\sim_{\sigma;\rho}$ may be any 'medium-grained' unity relation, no finer than FG's and no coarser than CG's. The MG-proposal consequently shares two assumptions with FG and CG:

$$\text{NFF} \quad \text{If } \sim_{\sigma;\sigma} \text{ and } \sim_{\rho;\rho} \text{ express the same relation and } x \sim_{\sigma;\sigma} y, \text{ then } \sigma x = \rho y.$$

$$\text{NCC} \quad \text{If } \sigma x = \rho y, \text{ then } \{z : z \sim_{\sigma;\sigma} x\} = \{z : z \sim_{\rho;\rho} y\}.$$

The no-finer-than-FG assumption—NFF—is hard to resist if the wholesale abstractionist follows Hale and Wright in maintaining that the meaning of a σ -operator is fixed solely by the identity conditions laid down in the relevant unmixed postulate.⁴² On this view, successful attempts to introduce σ - and ρ -abstract using notational variants of the same unmixed postulate, each deploying the same unity relation, accord the same meaning to σ and ρ . Consequently, given a modest amount of compositionality,

⁴⁰See, for instance, Wright (1997) and Boolos (1998), who dubs the axiom 'NEW V'. It's important to note the type distinction between a †-set (a value of a *first-order* variable) and the corresponding class (a value of a *second-order* variable). I write \emptyset for the empty set and Λ for the empty class.

⁴¹Compare Fine (2002, 54). The 'mixed' unity relation $\sim_{\sigma;\rho}$ giving identity conditions for σ - and ρ -abstracts may depend on the 'unmixed' unity relations that govern these abstracts (as in the cases of FG and CG).

⁴²See, for instance, Wright (2020, 304).

' $\sigma x = \rho y$ ' is equivalent to ' $\sigma x = \sigma y$ '. And this is enough to secure NFF on the basis of the unmixed postulates.⁴³

The no-coarser-than-CG assumption— NCC —is less central to the wholesale view and potentially more controversial.⁴⁴ As Fine observes, however, if we deny NCC , 'it is hard to see what reasonably systematic view could associate the same abstract with two different equivalence classes' (2002, 47).

What I sometimes loosely call 'the' MG -proposal consequently encompasses a broad class of wholesale responses, including, as limiting cases, the FG - and CG -proposal. Different wholesale abstractionists may choose an instance appropriate to their preferred account of how abstraction decides Caesar questions. Once again, however, *any* MG -proposal must contend with *prima facie* counterexamples. This time, take Shapiro's (2000) proposal to abstract reals from classes of rationals:⁴⁵

CP For any $X, Y \subseteq \mathbb{Q}$, TFAE:

$\sup X = \sup Y$; X and Y have the same rational upper bounds

Now consider $0_{\text{sup}} =_{\text{df}} \sup\{r \in \mathbb{Q} : r < 0_{\mathbb{Q}}\}$. According to NCC , $0_{\text{sup}} \neq 0_{\#}$ (since the abstracts are associated with different equivalence classes). Yet, however naive it may be to think that the natural embedding from \mathbb{N} into \mathbb{R} is the identity function, I am unable to shake the belief that there is just one zero here, both a natural and a real (and a rational, etc.). Assuming that this familiar zero is identical with its $\#$ - and sup -counterparts, it follows that $0_{\text{sup}} = 0_{\#}$. If this is right, any instance of MG (including FG and CG) incorrectly distinguishes identical abstracts.

2.4

Let's no longer delay confronting the obvious rejoinder to the supposed counterexamples. The argument against FG , CG , and MG relies on two groups of assumptions. First, there are the various intuitions:

$$(i1) \quad n_{\#} = n_{\mathbb{N}} \quad (i2) \quad 0_{\mathbb{N}} \neq \emptyset \quad (i3) \quad 0_{\mathbb{N}} = 0_{\mathbb{R}}$$

Cook and Ebert's intuition, (i1), concerning abstracts is in immediate conflict with FG . My intuitions, (i2) and (i3), concerning familiar sets and

⁴³ Assuming the antecedent of NFF , ' $x \sim_{\sigma, \rho} y$ ' holds and, by the unmixed postulates, is equivalent to ' $\sigma x = \sigma y$ '; by the argument in the text, this is equivalent to ' $\sigma x = \rho y$ ', which consequently also holds, as per the consequent of NFF .

⁴⁴ Linnebo and Uzquiano (2009, 248) note that NCC 'figures as an uncontroversial minimal assumption' in all the 'technically precise' discussions they were then aware of. In the case of abstracts associated with 'coarsenings of HP ', such as FHP , however, an identity axiom since put forward by Ebels-Duggan (2021) runs contrary to this assumption: his structural identity principle identifies $\mathbb{N}(\mathbb{N})$ and $\#(\mathbb{N})$ even in cases when these abstracts are associated with different equivalence classes. Without restriction, however, Ebels-Duggan's proposal also runs contrary to NFF , by giving a negative answer to Caesar questions of the form 'Is $\#^1 X = \#^2 X$?' in cases where both operators are introduced by duplicates of HP . To avoid this, he restricts the structural identity principle to render it silent on such questions. Whether this proposal may be developed into a 'reasonably systematic' view, then, depends on whether there's a non *ad hoc* way to lift this restriction and to extend the account beyond coarsenings of HP .

⁴⁵ Shapiro calls the relevant abstracts *cuts*.

numbers conflict with CG and MG if it is assumed that these mathematical objects—the natural $0_{\mathbb{N}}$, real $0_{\mathbb{R}}$, and set \emptyset —are identical to the corresponding abstracts:

$$(r1) \quad 0_{\#} = 0_{\mathbb{N}} \qquad (r2) \quad 0_{\text{sup}} = 0_{\mathbb{R}} \qquad (r3) \quad \emptyset_{\dagger} = \emptyset$$

The importance of reduction theses such as (r1)–(r3) has already been noted. The abstractionist’s interpretation of arithmetic aspires to provide a *faithful reduction* in the sense that the intended meaning of arithmetical expressions is captured by their $\#$ -based translations. This requires, in particular, that ‘ $0_{\#}$ ’ is accorded the same meaning as ‘ $0_{\mathbb{N}}$ ’ and thus that (r1) is true.⁴⁶ Similarly, a faithful reduction of real analysis or set theory via the standard CP- or NEW v-based interpretation requires the truth of (r2) or (r3).

It’s all too easy, however, for a proponent of FG, CG, or MG to simply reject intuitions like (i1)–(i3). According to Fine (2002, 73), for example, it ‘is not clearly incorrect’ to reject (i1). There’s also room to question my intuitions, (i2) and (i3). Fine maintains that if we ‘conceive of’ sets as abstracts, it’s again ‘not clear’ that the number 0 is to be distinguished from the empty set (54).⁴⁷ In the case of (i3), prior to his more recent stipulative proposal, Linnebo (2005, 219) contends that intuitions of this kind present a ‘mis-diagnosis’ of the fact that numerals like ‘0’ are ambiguous, denoting natural numbers on some disambiguations, and distinct reals on others.⁴⁸

What becomes of the case against FG, CG, and MG if a wholesale abstractionist bites the bullet? Well, I stand by my intuitive judgements about numbers and sets (although, unlike Cook and Ebert, I do not have any direct intuitions about $\#$ - and \mathbb{N} -abstracts). Fortunately, though, I need place no weight on these divisive claims. Wholesale abstractionism faces two further objections that do not turn on intuitions such as (i1)–(i3).

2.5

The first objection is that the wholesale proposals lead to a version of Benacerraf’s multiple reduction problem. The original version concerns would-be set-theoretic reductions of arithmetic. Recall that, among other options, we may interpret arithmetic in set theory either by identifying natural numbers with z -ordinals (following Zermelo) or vN -ordinals (following von Neumann):

$$\begin{array}{ll} 0_z =_{\text{df}} \emptyset & 0_{\text{vN}} =_{\text{df}} \emptyset \\ (n+1)_z =_{\text{df}} \{n_z\} & (n+1)_{\text{vN}} =_{\text{df}} \{0_{\text{vN}}, 1_{\text{vN}}, \dots, n_{\text{vN}}\} \end{array}$$

The problem then comes out in a trilemma:

- ALL: each of the candidate interpretations is a faithful reduction, so that its identifications are genuine identities. When the candidate interpretations are Zermelo’s and von Neumann’s, this implies that,

⁴⁶This follows by compositionality again given that ‘ $0_{\mathbb{N}} = 0_{\mathbb{N}}$ ’ is true.

⁴⁷Fine himself is questioning (i2) here only if the items we ‘conceive of’ as set-like abstracts are indeed sets (and 0 is $0_{\mathbb{N}}$).

⁴⁸See also MacBride (2003, 129–30) and Shapiro (2006, 128–9).

for each natural number n , $n = n_z$ and $n = n_{vN}$. This horn leads to absurdity. A modicum of set theory demonstrates that $2_z \neq 2_{vN}$.

- **SOME**: some but not all of the candidate interpretations are faithful reductions. In this case, what reason have we to think $2_{\mathbb{N}} = 2_z$, as opposed to $2_{\mathbb{N}} = 2_{vN}$, or vice versa? After all, the two interpretations agree on the arithmetical properties of the natural numbers. In the absence of a ‘cogent reason’ to think that naturals are z -ordinals rather than vN -ordinals, or vice versa, Benacerraf rejects as ‘hardly tenable’ the position that the number–set identity facts are ‘unknowable truths’ (1965, 284).
- **NONE**: none of the candidate interpretations are faithful reductions. Having rejected **ALL** and **SOME**, Benacerraf concludes that natural numbers are neither z - nor vN -ordinals.

In its original form, the trilemma poses no immediate threat to abstractionism. Notwithstanding Frege’s later position, the core abstractionist view carries no commitment to reduce familiar mathematical objects to sets or extensions. But what about would-be reductions to *abstracts*? In the case of real analysis, for example, Shapiro’s sup-based interpretation has a natural dual:

CP* For any $X, Y \subseteq \mathbb{Q}$, TFAE:

$\inf X = \inf Y$; X and Y have the same rational lower bounds

Which, then, of the inf- and sup-reals provide the faithful reduction? The **ALL** option is unavailable to a proponent of **FG**, **CG**, or **MG**. According to these views—specifically **NCC**—there is no overlap between sup- and inf-reals (since they are associated with distinct equivalence classes). Since the two interpretations agree on questions posed in the language of real analysis, the **SOME** option is no more appealing than in Benacerraf’s original trilemma. This leaves the **NONE** option, which is to deny that real numbers are sup- or inf-abstracts.

2.6

The second objection is that the wholesale proposals unduly constrain mathematical freedom. Suppose a community seek to introduce two different notations for ordered pairs, $x < y$ and $x > y$, pronounced ‘ x -before- y ’ and ‘ x -after- y ’, and a projection predicate $\pi(z, i, p)$ —‘ z is the i th coordinate of p ’—subject to the following stipulations:⁴⁹

<-PAIR For any x, y, z , and w , TFAE: $x < y = z < w$; $x = z$ and $y = w$

>-PAIR For any x, y, z , and w , TFAE: $x > y = z > w$; $x = z$ and $y = w$

<-PROJ For any x, y, z , and any $i \in \{1, 2\}$, TFAE:

$\pi(z, i, x < y)$; $i = 1$ and $z = x$ OR $i = 2$ and $z = y$

>-PROJ For any x, y, z , and any $i \in \{1, 2\}$, TFAE:

⁴⁹The pronunciation guide for $<$, $>$, and π is intended to be suggestive but should not be taken to constrain the interpretation that abstraction may accord these expressions.

$$\pi(z, i, x > y); \quad i = 1 \text{ and } z = y \text{ OR } i = 2 \text{ and } z = x$$

By the lights of wholesale abstractionism, if all goes well, \langle -PAIR and \rangle -PAIR effectively function as unmixed postulates that introduce \langle -pairs and \rangle -pairs, and \langle -PROJ and \rangle -PROJ define the projection predicate π .

The trouble is that, according to FG, CG, and MG, at least one of these stipulations must fail. This time the objection relies only on the no-finer-than-FG assumption. Assuming NFF, $4 < 5$ and $4 > 5$ are identical (since they are both abstracted from the same specifications— x has value 4 and y value 5—according to the same unity relation). By \langle -PROJ and \rangle -PROJ, however, $4 < 5$ and $4 > 5$ are distinct (since they differ in their first coordinate).⁵⁰

Of course, once the community realize their mistake, it's open to them to sanitize their practice to bring it in line with NFF. But I'm reluctant to think that mathematics is thus metaphysically constrained. By mathematical standards, the community's stipulations—as they stand—provide a perfectly coherent set of axioms. Unless NFF is rejected, the abstractionist has no choice but to condemn the community's practice as flawed.⁵¹

To briefly take stock, a broad range of wholesale proposals are subject to three objections: FG, CG, and MG face the objection from intuition, from multiple reduction, and from mathematical freedom. Even if no importance is attached to the first objection, the other two provide plenty of motivation to consider alternatives.

3 A piecemeal solution

The basic idea behind the piecemeal response to the Caesar problem is straightforward: Caesar questions left unanswered by HP or other abstraction principles may be decided via further stipulations.

But how is this response to be developed? Linnebo (2018) is one of few abstractionists to take a piecemeal response seriously:

When our ancestors first confronted Caesar-style questions, they had a choice which way to go; and this choice played a role in shaping the concepts that they thereby forged. (160)

In his view, although speakers 'tend to operate' according to FG (168), 'exceptions are certainly possible and very likely even actual' (160). For example, Linnebo maintains that it's 'consistent' to identify #- and N-cardinals, in line with Cook and Ebert's intuition (166).

⁵⁰More fully, $\pi(4, 1, 4 < 5)$ holds and $\pi(4, 1, 4 > 5)$ fails because when x, y, z , and i respectively take the values 4, 5, 4, and 1, the first disjunct of \langle -PROJ holds and both disjuncts of \rangle -PROJ fail.

⁵¹Might a wholesale abstractionist reply by restricting NFF? Perhaps abstraction attempts based solely on unmixed postulates determine the content of mixed contexts, in accordance with NFF, but NFF may be overridden by *additional* stipulations. I find this view at least half-way congenial. This is because it goes a good way towards the piecemeal view I wish to defend. If the wholesale mechanism which ensures that ' $4 < 5 = 4 > 5$ ' expresses a truth can be bypassed in this way, then the content of mixed contexts may be settled by additional stipulations alongside unmixed postulates, just as the piecemeal view maintains. Indeed, what bar remains to overriding NFF by directly stipulating the content of mixed contexts? My main reservation is that this leaves the wholesale component of the view poorly motivated: if Caesar questions may be decided via stipulation, what need has the abstractionist for a category-based ontology, or similar?

Linnebo, however, does not present a general account of what can and cannot be successfully stipulated in a piecemeal fashion.⁵² This leaves some important questions unanswered. How *much* choice did our ancestors have? Could they also have chosen to identify #-cardinals with †-sets? Could they have chosen to identify numbers with Romans? What about MacBride’s worry about conflict with ‘antecedent facts’ or the Hale–Wright worry about incoherent stipulations? What is wanted, if the piecemeal response is to emerge as a serious candidate solution, is a systematic account of how, in the context of an abstractionist metasemantics, Caesar questions may be decided via stipulation. My aim in this section is to provide such an account. I argue that the piecemeal view that emerges has a robust response to the concerns from MacBride and Hale and Wright and that this view improves on its wholesale competitors when it comes to the objections from intuition, multiple reduction, and mathematical freedom.

To begin with, let’s continue to focus on Caesar questions concerning different kinds of abstract.⁵³ *Pace* the wholesale account, there is no meta-physical or metasemantic law that permits us to extract the content of mixed contexts from postulates governing unmixed ones. Instead, according to piecemeal abstractionism, a community is just as free to stipulate the content of mixed contexts as they are to stipulate the content of unmixed contexts. The three-step account of abstraction outlined in 1.1 is unchanged save that, in addition to unmixed postulates, an abstraction attempt may also be based on *mixed postulates* of the following form:

$\sigma:\rho$ For any $x \in \mathcal{D}_\sigma$ and $y \in \mathcal{D}_\rho$, TFAE: $\sigma x = \rho y$; $x \sim_{\sigma:\rho} y$

The postulate stipulates the content of a mixed context in terms of an antecedently expressible unity relation $\sim_{\sigma:\rho}$. As before, semantic values for the expanded lexicon are then selected subject to the constraint that they compositionally recover the sentential content stipulated by the attempt’s postulates.

3.1

Does the piecemeal view really improve on its wholesale competitors? Let me first return to the intuitions reported in Section 2. Assuming Caesar questions may be settled via stipulation, it’s straightforward to give answers in line with the intuitions (i1)–(i3) while maintaining the reduction theses (r1)–(r3). Suppose a community—Community 1—successfully lay down the following alongside the corresponding unmixed postulates:⁵⁴

$\mathfrak{n}:\#-1$ For any classes X and Y , TFAE:

$\#X = \mathfrak{n}Y$; X and Y are equinumerous and both finite

$\#:\dagger-1$ For any classes X and Y , TFAE: $\#X = \dagger Y$; \perp

⁵²One general constraint can be gleaned from Linnebo’s account (2018, 167): his discussion of the axiom he labels G makes clear that he endorses $\mathfrak{n}FF$ (the converse of G). This is a noteworthy difference from the piecemeal account I wish to recommend, which permits stipulations that run contrary to $\mathfrak{n}FF$ and $\mathfrak{n}CC$.

⁵³I return to Caesar himself in 3.3.

⁵⁴As usual, \perp stands for a trivial contradiction.

#:sup-1 For any class X and any $Y \subseteq \mathbb{Q}$, TFAE:

$\#X = \text{sup } Y$; there is a class $\{1_{\mathbb{Q}}, \dots, n_{\mathbb{Q}}\}$ of integral rationals that is equinumerous with X and has the same rational upper bounds as Y

According to $\mathfrak{N}:\#-1$, each finite $\#$ -cardinal $n_{\#}$ is identical to its \mathfrak{N} -counterpart $n_{\mathfrak{N}}$, as per (i1). The postulates $\#:\dagger-1$ and $\#:\text{sup}-1$ ensure that $\#$ -naturals are distinct from \dagger -sets but identical to the corresponding sup-reals. Given the desired identification with their marketplace mathematical counterparts, as per (r1)–(r3), it follows that $0_{\mathfrak{N}} \neq \emptyset$ and $0_{\mathfrak{N}} = 0_{\mathbb{R}}$, as per (i2) and (i3).

I concede (once again) that the intuitions here are open to question. But the key point is not that piecemeal abstractionism can accommodate *my intuitions* (or Cook and Ebert’s). Instead this approach is flexible enough to mirror the identity profile of natural and real numbers, whatever this may be. Suppose, for instance, that there is no overlap between natural and real numbers, contrary to (i3). Then a second community—Community 2—may seek to introduce disjoint $\#$ - and \mathfrak{N} -abstracts, by supplementing \mathfrak{N} and \mathfrak{C} with a different $\#:\text{sup}$ -postulate:

#:sup-2 For any class X and any $Y \subseteq \mathbb{Q}$, TFAE: $\#X = \text{sup } Y$; \perp

More generally, however naturalreal identity facts may pattern, it’s open to a piecemeal abstractionist to contend that the abstract realm is rich and varied enough to contain abstracts with the same identity profile.

3.2

All this free and easy stipulation brings me back to a MacBride-style objection:

Suppose that zero leads a double life. Suppose that in addition to leading its arithmetical existence $0_{\mathfrak{N}}$ is also the real number $0_{\mathbb{R}}$. In that case the stipulation that a sentence that says that $0_{\mathfrak{N}}$ is identical to $0_{\mathbb{R}}$ is false cannot succeed. For this sentence will be true and a true sentence cannot be stipulated to be false. Stipulation cannot suffice as a basis for determining that $0_{\mathfrak{N}}$ is distinct from $0_{\mathbb{R}}$.

Needless to say, stipulation cannot render identical things distinct. What is subject to stipulation is the meaning of expressions. It stretches the meaning of ‘stipulate’ to speak of a sentence’s *truth*-value being stipulated. However, MacBride’s locution is harmless provided it is understood in an explicitly extended sense: a sentence is *stipulated true* if, as a matter of stipulation, the sentence says that p and, as a matter of fact, it is the case that p ; similarly for ‘stipulated false’.⁵⁵

Would, then, zero leading a double life undermine Community 2’s ability to stipulate ‘ $0_{\#} = 0_{\text{sup}}$ ’ false, in this extended sense? The objection leaves the ambient interpretation unspecified. Clearly, if it’s to threaten Community 2’s stipulation, the occurrences of ‘false’ in the objection must

⁵⁵Analogous remarks apply to talk of ‘deciding’ Caesar questions via stipulation.

mean ‘false under i_2 ’ where i_2 is the interpretation resulting from Community 2’s abstraction attempt. What about ‘true’ and ‘says’? To avoid an obvious non-sequitur, their occurrences must be understood relative to the same interpretation—but is this i_2 or another interpretation? Let me consider the two options in turn.

Option 1: ‘True’ and ‘says’ mean truth-under- i_2 and says-under- i_2 . The key passage then reads as follows:

... the stipulation that a sentence that says under i_2 that $0_{\mathbb{N}}$ is identical to $0_{\mathbb{R}}$ is false under i_2 cannot succeed. For this sentence will be true under i_2 and a true-under- i_2 sentence cannot be stipulated to be false under i_2 ...

In response, a piecemeal abstractionist should simply deny that their abstraction attempt renders ‘ $0_{\#} = 0_{\text{sup}}$ ’ true under i_2 . After all, #:sup-2 stipulates that this sentence coincides in content with a trivial falsehood.

Option 2: ‘True’ and ‘says’ mean truth-under- j and says-under- j , for some interpretation j other than i_2 :

... the stipulation that a sentence that says under j that $0_{\mathbb{N}}$ is identical to $0_{\mathbb{R}}$ is false under i_2 cannot succeed. For this sentence will be true under j and a true-under- j sentence cannot be stipulated to be false under i_2 ...

In this case, the sentence in question—‘ $0_{\#} = 0_{\text{sup}}$ ’—may well be true under j . But there is no reason to think that a *sentence* that is true in one language cannot be stipulated false in another. Speakers of the two languages may respectively render the sentence true under j and false under i_2 by interpreting its abstract terms differently.

Either way, then, the identity of $0_{\mathbb{N}}$ and $0_{\mathbb{R}}$ is no obstacle to the success of Community 2’s abstraction attempt. Nonetheless, there is an important moral to be extracted from MacBride’s objection. Consider again Option 1: if $0_{\mathbb{N}}$ and $0_{\mathbb{R}}$ are identical, then to deny that ‘ $0_{\#} = 0_{\text{sup}}$ ’ is true under i_2 , the piecemeal abstractionist must also deny that this sentence *says* that $0_{\mathbb{N}} = 0_{\mathbb{R}}$ under i_2 . Indeed, on the piecemeal metasemantics, if Community 2’s abstraction attempt succeeds, semantic values are selected so that the compositionally determined content of ‘ $0_{\#} = 0_{\text{sup}}$ ’ is the content of a trivial falsehood (as per #:sup-2). Assuming that $0_{\mathbb{N}} = 0_{\mathbb{R}}$, this is achieved only if the relevant #- and sup-terms do *not* respectively refer to $0_{\mathbb{N}}$ and $0_{\mathbb{R}}$.⁵⁶ Consequently, on the proposed metasemantics, if zero leads a double life, Community 2’s stipulations ensure that they do not achieve a faithful reduction of both arithmetic and real analysis.⁵⁷

Similarly, the non-identity of $0_{\mathbb{N}}$ and $0_{\mathbb{R}}$ would be no bar to the success of Community 1’s stipulation. But in this case they cannot hope both to stipulate ‘ $0_{\#} = 0_{\text{sup}}$ ’ true and to hit upon the familiar naturals and reals as the referents of the corresponding #- and sup-terms. The moral here is that it is reduction rather than success that is hostage to ‘antecedent facts’. Assuming all goes well, both communities succeed in introducing natural-

⁵⁶This is because, assuming the standard semantics for identity, ‘ $0_{\#} = 0_{\text{sup}}$ ’ is true if its terms both refer to the same number. Similarly, ‘ $0_{\#} = 0_{\text{sup}}$ ’ is true on the supervaluationist semantics if ‘ $0_{\#}$ ’ and ‘ 0_{sup} ’ refer to $0_{\mathbb{N}}$ ($= 0_{\mathbb{R}}$) under each admissible interpretation.

⁵⁷Community 2, however, may yet go on to achieve *one* faithful reduction. See n. 61.

and *real-like* abstracts, which underwrite interpretations of arithmetic and real analysis. But it does not follow that they introduce the same abstracts. The different abstraction attempts accord different meanings to their abstract terms, so that at most one of the communities achieves a faithful reduction in both cases.

3.3

The discussion in the last section raises the spectre of multiple reductions once more. But before I come to that objection, it's helpful to return to Caesar himself. The piecemeal account is readily extended to contexts of the form ' $\sigma x = q$ ' and other atomic contexts that contain abstract terms: e.g., $R(\sigma x)$. In addition to the unmixed and mixed postulates introduced so far, a piecemeal abstraction attempt may also include mixed postulates of the following forms:

$$\begin{aligned} \sigma:q \quad & \text{For any } x \in \mathcal{D}_\sigma \text{ and } q \in \mathcal{D}_q, \text{ TFAE: } \sigma x = q; \quad x \sim_{\sigma:q} q \\ R \quad & \text{For any } x \in \mathcal{D}_\sigma, \text{ TFAE: } R(\sigma x); \quad \mathcal{I}_\sigma^R(x) \end{aligned}$$

In the first case, q is a first-order variable whose associated domain \mathcal{D}_q comprises some or all of the items quantified over pre-abstraction.⁵⁸ In the second case, $\mathcal{I}_\sigma^R(x)$ expresses what I will call an *instantiation relation*. The postulates stipulate the content of a mixed identity or atomic context in terms of an antecedently expressible unity or instantiation relation. Their relata are either specifications of the relevant abstracts, as usual, or—in the case of members of \mathcal{D}_q —the very items in the initial domain whose identity relations are in question. In the case of a polyadic atomic context, a polyadic postulate supplants the monadic one stated above.⁵⁹ As ever, semantic values are selected for the expanded lexicon that compositionally recover the stipulated sentential content.

The basic piecemeal response to the Caesar problem is that Caesar questions are answered via stipulations of this kind. Community 1 may ensure that their $\#$ -terms do not refer to Romans by supplementing their earlier stipulations with postulates such as the following:

$$\begin{aligned} \#:q-1 \quad & \text{For any class } X \text{ and Roman } q, \text{ TFAE: } \#X = q; \quad \perp \\ \text{Roman}\#-1 \quad & \text{For any class } X, \text{ TFAE: } \#X \text{ is Roman}; \quad \perp \end{aligned}$$

⁵⁸Officially, I assume that each first-order variable q is associated with a domain \mathcal{D}_q that is specified by a condition $\delta_q(x)$ and comprises members of the initial domain that satisfy this condition under the initial interpretation. For technical convenience, I assume that at least one \mathcal{D}_q is the whole initial domain.

⁵⁹The polyadic postulate takes the following form:

$$R \quad \text{For any } x_1 \in \mathcal{D}_{t_1}, \dots, x_n \in \mathcal{D}_{t_n}, \text{ TFAE: } R(t_1, \dots, t_n) \quad \mathcal{I}_{t_1, \dots, t_n}^R(x_1, \dots, x_n)$$

In the axiom, each term t_i takes one of two forms ($i = 1, \dots, n$): either (i) t_i is an abstract term $\sigma_i(x_i)$ and \mathcal{D}_{t_i} is the domain associated with σ_i ; or (ii) t_i and x_i are the same first-order variable and \mathcal{D}_{t_i} is the domain associated with this variable. Allowing for R to be the identity predicate, all the abstraction postulates considered in the text take this form (modulo infix notation).

The #-cardinals introduced by piecemeal abstraction may consequently have a rich nature extending beyond arithmetical properties. Community 1 ensure that the referents of their #-terms are distinct from Caesar and non-Roman. Further postulates may constrain them to be non-concrete, and so on.

Of course, to say a community *may* give stipulative answers to Caesar questions is not to say that they *must*. A piecemeal abstraction attempt, even if it answers a wide range of Caesar questions, may still say nothing about others. In these cases, it's reasonable to expect languages interpreted by piecemeal abstraction to manifest some indeterminacy. Different abstractionists may prefer different accounts of indeterminacy. But, to fix ideas, let me continue to operate with the supervaluationist approach outlined in 1.2. On this approach, following a successful piecemeal abstraction attempt, the admissible MT-interpretations are those that extend the initial one according to the attempt's postulates. In the limiting case, a community who introduce # with no postulates other than HP leave the reference of #-terms radically indeterminate, as before. In my view, however, if in fact many interesting Caesar questions are answered, it is because our use of number terms goes far beyond a commitment to HP . Community 1 take a step in this direction with postulates that render '#X = Caesar' false. Further postulates, providing an idealized reconstruction of more of our number-term practice, may further reduce indeterminacy, and decide further Caesar questions.

Of course, *our use* of number terms is just one way to go. On the piecemeal view I wish to recommend, there's nothing to stop Community 2 *identifying* some of their #-abstracts with Romans with a postulate such as the following:

#:q-2 For any class X and Roman q, TFAE:

#X = q; q is a dictator of the Roman republic such that the class of dictators succeeding q is equinumerous with X

It bears repeating, contrary to MacBride's objection, that the success of this stipulation is not hostage to whether the familiar numbers—the referents of *our* number terms—are Roman. But a different concern calls for brief comment. Assuming Community 2's attempt succeeds, can the *abstracts* introduced really include *flesh-and-blood* Romans?

Part of this concern is easily dealt with. The term 'abstract' is notoriously polysemous. But my present use is in no way opposed to concrete or spatiotemporal. I've been using '#-abstract' to apply to the putative referents of #-terms that result from a (salient) abstraction attempt. I take it that there is nothing to stop a linguistic community using #-terms, or any other syntactically singular terms, to refer to Romans. The only remaining question is whether this can be achieved by the top-down metasemantics posited by abstractionism. But why should it not? In fact, the mooted cases of abstract–Roman overlap are unusually favourable ones for abstraction to confer reference on #-terms. In these cases, there is no doubt that there is a unique dictator in the initial domain who stands in the postulate's unity relation to the class assigned to X. Consequently #:q-2, in effect, stipulates that the reference of #X for this assignment is to be selected so as to

render ‘ $\#X = q$ ’ true of the specified Roman. But this is little different to the clearly unproblematic stipulation that the referent of $\#X$ for this assignment is *to be* the specified Roman. Even if the ability of abstraction to secure reference is deemed questionable in other cases, the means by which $\#:q-2$ ensures that $\#$ -terms refer to Romans is akin to comparatively straightforward kinds of reference-fixing.

3.4

Turn now to the objection from multiple reductions. On the piecemeal view, it’s straightforward to identify sup- and inf-reals in the natural way. Community 1, for instance, may supplement cp and cp^* with the following:

sup:inf-1 For any $X, Y \subseteq \mathbb{Q}$, TFAE:

$$\sup X = \inf Y; \quad \{x - y : x \in X, y \in Y\} \text{ has least upper bound } 0_{\mathbb{Q}}$$

On its own, however, this observation is of limited help since another community is equally free to distinguish their analogues of these abstracts. Witness Community 2’s addition to cp and cp^* :

sup:inf-2 For any $X, Y \subseteq \mathbb{Q}$, TFAE: $\sup X = \inf Y; \quad \perp$

The multiplicity of would-be reductions leads back to Benacerraf’s trilemma. To avoid confusion, I will use subscripts to distinguish Community 1’s overlapping \sup_1 - and \inf_1 -reals and Community 2’s disjoint \sup_2 - and \inf_2 -reals. Which, then, of the \sup_1 -/ \inf_1 -reals, the \sup_2 -reals, and the \inf_2 -reals are the familiar reals? The ALL horn leads swiftly to absurdity. In the absence of a reason to privilege one of the interpretations, the SOME horn is as unpalatable as ever. This leaves the NONE horn, which amounts to denying that either community achieves a faithful reduction.

Without further addition, this is indeed the case. The postulates listed so far say nothing about real–sup–real identity. On the supervaluationist semantics, a successful abstraction attempt based on Community 1’s postulates accords no determinate truth-value to reduction theses such as ‘ $0_{\sup} = 0_{\mathbb{R}}$ ’.⁶⁰ This indeterminacy, however, may be relieved with further stipulations. Once Community 1 is equipped with quantifiers ranging over \mathbb{R} —the familiar reals—they may constrain the reference of their sup-terms so as to ensure they obtain a faithful reduction with a suitable $\sigma:q$ -postulate:

sup: γ -1 For any $X \subseteq \mathbb{Q}$ and $\gamma \in \mathbb{R}$, TFAE:

$$\sup X = \gamma; \quad X \text{ has the same rational upper bounds as } \{r \in \mathbb{Q} : r < \gamma\}$$

If successful, an attempt that includes this postulate, in addition to cp , cp^* and sup:inf-1, ensures that each sup- and inf-term refers to the corresponding real, so that, in particular, ‘ $0_{\sup} = 0_{\mathbb{R}}$ ’ comes out true. This gives the piecemeal abstractionist the means to blunt the SOME horn of

⁶⁰I assume here that ‘ $0_{\mathbb{R}}$ ’ is part of the lexicon prior to the cp -abstraction attempt.

the trilemma. Benacerraf’s sought-for ‘cogent reason’ to think that Community 1’s sup-terms refer to real numbers is that this follows from their stipulation on the proposed piecemeal metasemantics.⁶¹

It should not go unnoticed that a stipulation such as $\text{sup}:y-1$ is only available to a community equipped with quantification over the familiar reals. But I want to insist that this is unproblematic. If we come to quantify over real numbers by some means other than abstraction, $\text{sup}:y-1$ is analogous to $\#:q-2$. The latter postulate quantifies over Romans to settle that some $\#$ -terms refer to Romans, in a way that has already been argued to be unproblematic. The more interesting case arises if abstraction provides our most fundamental way to acquire quantification over real numbers. But, in this case, there’s nothing metaphysically special about the ‘familiar’ reals compared with the many other fields of real-like objects that may be introduced by abstraction. The familiar reals are simply the real-like abstracts that are *familiar to us*: the ones we happened to introduce and to associate with *our* terms, such as ‘real number’ and ‘ $0_{\mathbb{R}}$ ’. If these terms were ultimately interpreted via abstraction, then a faithful reduction requires only that sup-terms should pick out the right *abstracts* among those we’ve already introduced. And once we have come to quantify over these abstracts—whether they’re introduced via CP^* , D , or any other abstraction attempt—we can deploy a postulate such as $\text{sup}:y-1$ to constrain sup-terms to refer to the *very same abstracts*. The same goes, *mutatis mutandis*, for natural numbers, sets, and so on.

3.5

Two loose ends remain: the objection from mathematical freedom and the Hale-Wright worry about incoherent stipulations. On the wholesale view, the community who attempt to introduce $<$ -pairs and $>$ -pairs fail in some of their stipulations, which conflict with the no-finer-than-FG assumption, NFF . The basic piecemeal response to the objection from mathematical freedom is simply to reject NFF . On the piecemeal view, laying down notational variants of the same postulate to govern unmixed identity contexts does not prevent the community from coherently distinguishing pairs such as $4 < 5$ and $4 > 5$ via their stipulations governing other contexts.

This response, however, may provoke a Hale-Wright-type worry about incoherent stipulations. Even if the piecemeal account carries no commitment to NFF , why think it avoids other equally objectionable constraints on successful abstraction? After all, it’s not hard to multiply examples of incoherent abstraction attempts. Suppose, for instance, that a community supplement HP and FHP with the following postulate:

$\text{N}:\#-!$ For any X and Y , TFAE: $\text{N}X = \#Y$; X and Y are equinumerous

⁶¹ The situation is not so different for Community 2. But they face a choice. They may stipulate a verbatim copy of $\text{sup}:y-1$ to constrain their sup-terms to refer to the corresponding reals. Alternatively, they may lay down the dual of this postulate to ensure the same for their inf-terms. Given $\text{sup}:\text{inf}-2$, either stipulation constrains the dual terms to refer to real-like abstracts distinct from the familiar reals. Similarly, notwithstanding $\#:\text{sup}-2$, Community 2 may lay down postulates to obtain a faithful reduction of either arithmetic or real analysis even if $0_{\mathbb{N}} = 0_{\mathbb{R}}$.

This postulate conflicts with the transitivity of identity: $\mathfrak{N}(\mathbb{R}) = \mathfrak{N}(\mathbb{N})$ (by FHP) and $\mathfrak{N}(\mathbb{N}) = \#(\mathbb{N})$ (by $\mathfrak{N}:\#-$) but $\mathfrak{N}(\mathbb{R}) \neq \#(\mathbb{N})$ (by $\mathfrak{N}:\#-$).

The risk of incoherence is not a new problem for abstractionism, of course. In the case of an abstraction attempt based on a single unmixed postulate, there is a straightforward necessary condition for its success: its unity relation must be an equivalence relation. There's an analogous necessary condition for a piecemeal abstraction attempt to succeed in extending the initial interpretation according to its postulates.⁶² In this case, the attempt's postulates provide a patchwork of sort-specific unity relations. These in turn induce one or more *global unity relations*. A global unity relation \sim serves to identify and distinguish abstracts and individuals, of any sort, in accordance with the attempt's unity relations and the identity relations that hold between individuals in the initial domain. Analogously, for each predicate R , the patchwork of sort-specific instantiation relations induces one or more global instantiation relations \mathcal{I}^R .⁶³ A necessary condition for the attempt's success is then given by what I will call *the congruence condition*:⁶⁴ the attempt must induce a family of relations—a global unity relation \sim , together with a global instantiation relation \mathcal{I}^R for each predicate in the expanded lexicon—that is congruent, in the sense that \sim is an equivalence relation that is respected by each \mathcal{I}^R .⁶⁵ The attempt to lay down $\mathfrak{N}:\#-$ alongside HP and FHP fails because it violates this condition: any global unity relation it induces fails to be transitive.

Is there an equally straightforward sufficient condition for success? Neologicist abstractionists long ago gave up hope that anything like the congruence condition might give this success-condition. For this condition is met by attempts that include the postulate widely blamed for Russell's paradox:

BLV For any X and Y , TFAE: $\dagger X = \dagger Y$; X and Y are coextensive

⁶²The postulates take the form indicated in n. 59. Formally, an MT-interpretation \mathcal{J} may then be said to extend another \mathcal{I} according to some stipulated postulates if it meets the conditions stated in n. 23 (with the second suitably generalized): (i) \mathcal{J} is an extension of \mathcal{I} and (ii) for each stipulated postulate, when x_1, \dots, x_n are respectively assigned to members of $\mathcal{D}_{t_1}, \dots, \mathcal{D}_{t_n}$, the terms t_1, \dots, t_n are defined under \mathcal{J} , and the truth-value of the postulate's left-hand side under \mathcal{J} coincides with the truth-value of its right-hand side under \mathcal{I} .

⁶³Officially, given an initial MT-interpretation $\mathcal{I} = \langle M, I \rangle$, the relations \sim and \mathcal{I}^R are relations on the set—henceforth, M —comprising pairs of the form $\langle t, a \rangle$ where t is either a first-order variable q or an abstract term σx drawn from the expanded lexicon and a belongs to the corresponding domain, \mathcal{D}_σ or \mathcal{D}_q . Given an n -ary predicate R in the expanded lexicon, a relation \mathcal{I}^R is a *global instantiation relation induced by the attempt* if it meets two conditions: (i) for each postulate deployed in the attempt (of the form described in n. 59), \mathcal{I}^R holds of a sequence of pairs drawn from M , $\langle t_1, a_1 \rangle, \dots, \langle t_n, a_n \rangle$ iff the postulate's right-hand side holds of a_1, \dots, a_n under \mathcal{I} ; (ii) for terms t_1, \dots, t_n that are *already defined under \mathcal{I}* (when their free variables are respectively assigned to a_1, \dots, a_n), \mathcal{I}^R holds of a sequence of members of M , $\langle t_1, a_1 \rangle, \dots, \langle t_n, a_n \rangle$ iff $R(t_1, \dots, t_n)$ holds of a_1, \dots, a_n under \mathcal{I} . In the special case when R is the identity predicate, a relation \mathcal{I}^R meeting (i) and (ii) is a *global unity relation induced by the attempt*.

⁶⁴Proof sketch: Suppose, that there is an MT-interpretation \mathcal{J} that extends \mathcal{I} according to the attempt's postulates. Then the congruence condition is witnessed by a family of relations read off \mathcal{J} : for pairs in M (see n. 63), \mathcal{I}^R is defined to hold of $\langle t_1, a_1 \rangle, \dots, \langle t_n, a_n \rangle$ iff $R(t_1, \dots, t_n)$ holds of a_1, \dots, a_n under \mathcal{J} ; \sim is defined analogously.

⁶⁵Recall that \sim is respected by \mathcal{I}^R if whenever \mathcal{I}^R holds of a sequence a_1, \dots, a_n it also holds of any sequence b_1, \dots, b_n with $b_i \sim a_i$, for $i = 1, \dots, n$.

In response to what is known as the bad company problem posed by BLV and other problematic abstraction principles, abstractionists have typically sought to defend a more demanding condition for an abstraction attempt to succeed.⁶⁶

There's nothing in principle to stop a piecemeal abstractionist adopting the same strategy. But a danger of this approach, for either kind of abstractionist, is that an over-demanding success-condition may undermine other aspects of their programme. In the piecemeal case, my responses to the objections from intuition, multiple reduction, and mathematical freedom ultimately turn on the success of the communities' stipulations. They are free to introduce abstracts whose identity profiles accord with the reported intuitions, or which sustain the desired reductions, or which conform to the stipulated axioms, only if their attempts succeed.

The bad company problem, however, is another place where the abstractionist may do better to depart from neologicist orthodoxy. Even in the ur-bad case, there is no obstacle to extending an interpretation in accordance with BLV unless we assume—in line with the standard, *impredicative*, treatment of abstraction—that any abstracts introduced must fall within the domain of the pre-abstraction interpretation.⁶⁷ An alternative is to defend a *predicative* account of abstraction. On this view, the items introduced by abstraction are not assumed to belong to the initial domain, so that abstraction may (iterately) introduce 'new' items. This kind of *dynamic abstraction* rehabilitates abstraction based on BLV.⁶⁸

More generally, for any piecemeal abstraction attempt that meets the congruence condition, assuming that the pre-abstraction interpretation is encoded as an MT-interpretation, there is also an MT-interpretation that extends the initial one according to the attempt's postulates.⁶⁹ This opens the way for a piecemeal abstractionist to adopt the maximally liberal view, according to which the congruence condition is sufficient for an attempt to succeed.

⁶⁶See, for instance, Wright (1997) and Cook (2012).

⁶⁷Wright (1998) defends this kind of impredicativity by appealing to the (pre-abstraction) availability of quantification over absolutely everything (including every abstract).

⁶⁸Dynamic abstraction is defended by Studd (2016) and Linnebo (2018, ch. 3), who distinguish the domains associated with the new and old interpretations by working in a many-sorted or modal setting. The requirement imposed by Studd and Linnebo that the unity relation be 'stable' follows from the congruence condition.

⁶⁹Proof sketch: Suppose that the attempt is based on an initial MT interpretation, $\mathcal{I} = \langle M, I \rangle$, that induces a congruent family of unity/instantiation relations \sim / \mathcal{F}^R on the set M defined in n. 63. When a is a suitably typed entity or sequence of entities based on M , and t is either a first-order variable x or an abstract term σx , write $t^{\mathcal{I}}(a)$ for the reference of this term, if defined, under \mathcal{I} , when a is assigned as the value of x or x (i.e. $t^{\mathcal{I}}(a) = a$ or $I(\sigma)(a)$). It's then a routine exercise in model theory to verify that the following interpretation $\mathcal{J} = \langle N, J \rangle$ is a well-defined MT-interpretation that extends \mathcal{I} according to the attempt's postulates. The interpretation-function J is defined as follows for each constant c , operator σ , and predicate R in the expanded lexicon (picking $*$ outside the transitive closure of \mathcal{I}): (i) $J(c) = I(c)$; (ii) $(J(\sigma))(a)$ is either $(I(\sigma))(a)$ if this is defined under \mathcal{I} or $t^{\mathcal{I}}(b)$ if this is defined for $\langle t, b \rangle \in M$ with $\langle t, b \rangle \sim \langle \sigma x, a \rangle$; otherwise, if $\langle \sigma x, a \rangle \in M$, $(J(\sigma))(a)$ is $\langle *, \{ \langle t, b \rangle \in M : \langle t, b \rangle \sim \langle \sigma x, a \rangle \} \rangle$ and, in all other cases, $(J(\sigma))(a)$ is undefined; (iii) $J(R)$ is the following relation: $\{ \langle t_1^{\mathcal{J}}(a_1), \dots, t_n^{\mathcal{J}}(a_n) \rangle : \mathcal{F}^R(\langle t_1, a_1 \rangle, \dots, \langle t_n, a_n \rangle) \text{ for } \langle t_1, a_1 \rangle, \dots, \langle t_n, a_n \rangle \in M \}$. In the last clause, $t^{\mathcal{J}}(a)$ is defined as before, with $J(\sigma)$ replacing $I(\sigma)$. Finally, the domain of \mathcal{J} is defined as follows: $N =_{\text{df}} M \cup \{ t^{\mathcal{J}}(a) : \langle t, a \rangle \in M \}$.

If this is right, the orthodox neologicist version of abstractionism is doubly over-restrictive. An alternative version of abstractionism, dynamic and piecemeal, takes a more liberal view, both on the range of unity relations that may be successfully abstracted upon and on the variety of contexts whose content is open to direct stipulation. When it comes to the Caesar problem, I've argued that this provides a viable candidate solution, which has a robust response to the MacBride and Hale–Wright worries, and which improves on its wholesale rivals when it comes to the objections from intuition, multiple reduction, and mathematical freedom.

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