God and the Numbers

Abstract:

According to Augustine, abstract objects are ideas in the Mind of God. Because numbers are a type of abstract object, it would follow that numbers are ideas in the Mind of God. Let us call such a view the *Augustinian View of Numbers* (AVN). In this paper, I present a formal theory for AVN. The theory stems from the symmetry conception of God as it appears in Studtmann (2021). I show that Robinson’s Arithmetic is a conservative extension of the axioms in Studtmann’s paper. The extension is made possible by identifying the set of natural numbers with God, 0 with Being, and the successor function with the essence function. The resulting theory can then be augmented to include Peano Arithmetic by adding a set-theoretic version of induction and a comprehension schema restricted to arithmetically definable properties. In addition to these formal matters, the paper provides a characterization of the mind of God. According to the characterization, the Being essences that constitute God’s mind act as both numbers and representations – each has all the properties of some number and encodes all the properties of that number’s predecessor. The conception of God that emerges by the end of the discussion is a conception of an infinite, ineffable, axiologically and metaphysically ultimate entity that contains objects that not only serve as numbers but also encode information about each other.

“The ideas are certain original and principle forms of things, i.e., reasons, fixed and unchangeable, which are not themselves formed and, being thus eternal and existing always in the same state, are contained in the Divine Intelligence.” (Augustine, *De Diversis Quaestionibus LXXXIII)*

According to Augustine, abstract objects are ideas in the Mind of God. Because numbers are a type of abstract object, it would follow that numbers are ideas in the Mind of God. Let us call such a view the *Augustinian View of Numbers* (AVN). Although it has proven attractive to many, the Augustinian doctrine that abstract ideas are contained in the mind of God has remained suggestive but critically imprecise. This paper offers a first step towards remedying this shortcoming by presenting a formal theory of AVN.

The formal theory presented presupposes a *symmetry conception of God.* (Studtmann 2021) Briefly, according to this conception, God is universally symmetrical with respect to set-membership. One important provable consequence of God’s being universally symmetrical with respect to set membership is that God is identical with God’s essence. In fact, it is not hard to prove that being universally symmetrical with respect to set membership is *equivalent* to being identical to one’s essence. (A proof appears in the appendix.) Such an equivalence makes the theory in this paper particularly relevant to those theistic traditions, for instance the Thomistic tradition, according to which God is identical to her essence. As I shall show, the symmetry conception, and hence identity to essence, entails a formal representation of AVN, a view which Aquinas correctly argues conflicts with divine simplicity. A simple mind can, according to Aquinas, comprehend a multiplicity of ideas, but it cannot *contain* them (*ST* 1, Q7, A3, *ST* I Q15, A2). It would seem, then, that anyone committed to the identity of God and her essence should abandon divine simplicity and adopt AVN instead.[[1]](#footnote-1)

Before beginning, two methodological matters should be addressed. First, the mathematical theory in this paper belongs to non-well-founded set theory. It is thus a departure from the standard iterative conception of a set. There are of course important philosophical issues concerning the virtues of well-founded versus non-well-founded set theories. It is not the purpose of this paper to enter those debates explicitly. Rather, the theory in this paper should be seen as an analysis within non-well-founded set theory of certain metaphysical concepts rather than consisting in any argument about the superiority of non-well-founded set theory to its alternatives. Still, to the extent that a theory’s ability to define fundamental metaphysical concepts such as God counts in its favor, the theory in this paper can be considered an indirect argument for non-well-founded set theory. In addition, I do not take a stand as to a general non-well-founded set-theoretic comprehension schema. As will become apparent, I introduce a comprehension schema but restrict it to arithmetical properties. Hence, it is weaker than either of the two comprehension schemata for the two mathematically serviceable non-well founded set theories, namely Positive Set Theory and Quine’s NF (Holmes 2017). Although the theory in this paper is weaker than either positive set theory or Quine’s NF, the non-well-founded character of the theory is crucial. Both God and those sets that God contains contain sets that contain them. Hence, the membership relation violates the axiom of Foundation.

Second, one may object that whatever mathematical interest the theory in this paper may have, the theory does not have any theological interest because it entails that God is a set. Because God is not and cannot be a set, the objection continues, the theory is false and so of no genuine theological interest. A similar sort of objection has been raised to certain theories of properties (Bealer 1980). According to the objection, a property cannot be a function from possible worlds to sets, since that would entail that when I savor the taste of pineapple, I am savoring a function. But certainly, one might object, I cannot savor a function. There are at least two plausible responses to this sort of objection (Oddie 2001). First, one might insist that the identification of properties with functions from possible worlds to sets is a proper reduction and hence that, appearances notwithstanding, one does savor a function when savoring the taste of pineapple. Although such a response is certainly possible with respect to the understanding of God in this paper – I could claim that God is a set – there is a second, less extreme, response to the objection that does not require holding that everything is a set. Just as one might in the case of properties argue that the appeal to functions from intension to extensions is a way of understanding the structure of properties – whatever properties end up being – so one might use the theory in this paper to illuminate the logical structures inherent in the concept of identity to essence. For instance, the theory in this paper shows that if God is identical to her essence, then, whatever ontological category she occupies, if any, she has an internal structure that includes an infinite number of infinite objects.

I develop the formal theory of AVN in two steps. I begin by restating the five axioms that form the basis of the Symmetry Conception of God. I call the set containing those axioms . As shall become apparent, the axioms in G contain the concepts of God, Essence, and Being. Moreover, as demonstrated in Studtmann (2021), they entail that God is identical to her essence.

The first step involves showing that G can be conservatively extended to include the axioms of Robinson’s Arithmetic (Q). That extension involves identifying the three mathematical concepts in the Peano Axioms – Number, Successor, and Zero – with the metaphysical concepts of God, Essence, and Being as they appear in G. With such identifications, it is easy to show that the resulting theory contains the axioms of Peano Arithmetic minus the Induction Schema. One can thus conservatively add recursive definitions of multiplication and addition as well as an explicit definition of less-than to ; and the resulting theory, , contains the axioms of Robinson’s Arithmetic (Q).

The second step in the formal development is to extend to a theory,, that includes all Peano Arithmetic by adding a first-order set-theoretic version of induction as well as a comprehension schema that guarantees the existence of all arithmetically definable sets.

These formal mathematical developments, interesting as they may be in their own right, ultimately serve a larger purpose, which is to argue that the symmetry conception of God entails a theory that characterizes AVN. As shall become apparent, the conception of God that contains is rather remarkable. Indeed, two mathematical facts about God show that the symmetry conception of God characterizes an entity that arguably satisfies Schellenberg’s criteria for metaphysical and axiological ultimacy (Schellenberg 2005, 2007, 2009). This metaphysical ultimacy stems from the fact that universal symmetry is equivalent to identity to essence. Axiological ultimacy, according to Schellenebrg, requires an entity to exhibit some value in an unsurpassable way. According to the symmetry conception of God, God is *universally* symmetrical with respect to the most fundamental mathematical relation, namely the set-membership relation. I won’t pursue to what extent and in what circumstances symmetry is indeed a value, though I will mention that there is empirical evidence that symmetry is deeply connected to both moral and aesthetic judgements (Jacobsen, et. al., 2005). To be universally symmetrical is to be as symmetrical as possible. To the extent that symmetry is a value, therefore, universal symmetry entails having a value in an unsurpassable way. The mathematical equivalence of universal symmetry and identity to essence is a way of demonstrating the equivalence of metaphysical and axiological ultimacy.

Although any entity that satisfies the symmetry conception of God is both metaphysically and axiologically ultimate, it is not, as some theologians have held, simple. Rather, it contains an infinite number of infinite sets. The sets can be thought of as generated by repeated applications of the essence of function beginning with Being. In other words, using ‘E(x)’ for the essence of x, God contains the series of sets: Being, E(Being), E(E(Being)), and so on. Such a series is isomorphic to the Natural Numbers structured by the successor function, which makes the extension of G to Peano Arithemtic very natural. Unlike the Von Neumann finite ordinals, however, the sets in in God contain an infinite number of sets.

Because each set in God, except for Being, is an essence, each one contains all the sets that contain the set of which it is an essence. Hence, because each set in God in addition to being an essence functions as a number, each number contains all the sets that contain its predecessor. So, for instance, because the set of odd numbers contains 1, 2 contains the set of odds. Indeed, every even number contains the set of odd numbers and vice versa. If one uses the language of encoding and properties, one could say that the number 2 encodes all the properties of the number 1. Hence, each set in God plays a dual role. As a number, each is contained by an infinite number of sets. And as an essence, each contains all and only those sets that contain its predecessor. As a result of both this dual role of the sets in God’s mind and the identity of God to her essence, the sets in God’s mind can be seen as both numbers and as ideas. In their relations to each other, they have all the properties that numbers have; in their relation to their predecessors, they, like ideas, encode the properties of their predecessors; and because they are contained in a set that is identical to its essence and universally symmetrical, they are part of the internal structure of a metaphysically and axiologically ultimate object.

The rest of the paper is structured as follows. In section I, I state the five axioms that form the basis of the symmetry conception of God followed by an informal characterization of the structure that they entail. In section II, I move to the first stage of the formal development. I do so by proving four theorems that together with explicit definitions of successor, number and zero entail Peano Arithmetic minus the Induction Schema. From there it is straightforward to add recursive definitions of multiplication and addition as well as a definition of the less-than relation. Finally, I present the second step in the formal development of the theory by adding a first-order induction axiom and a comprehension schema.

*Section I*

*The Symmetry Conception of God*

The symmetry conception of God depends on the following five axioms within non-well-founded set theory.

1. Extensionality:
2. Being:
3. Non-Being:
4. Essence:
5. God:

The axiom of Extensionality is part of any set theory. Each of the other four axioms corresponds to an important metaphysical concept: Being, non-Being, Essence, and God. To see the structure that is entailed it is helpful to begin with the first four axioms.[[2]](#footnote-2) The first thing to note is that the Essence Axiom and the non-Being Axiom jointly entail an infinite progression of sets. Let ‘E(x)’ denote the essence of x and ‘∅’ denote non-Being. Then, the two axioms entail the existence of non-Being, ∅, the existence of the essence of non-Being, E(∅), the existence of the essence of the essence of non-Being, E(E(∅)), and so on. Likewise, the Essence axiom and the Being axiom entail an infinite progression of sets: Being, E(Being), E(E(Being), and so on. For the ease of expression, I will call any set that is part of the progression of essences stemming from non-Being a ‘non-Being essence’ and any set that is part of the progression of essences stemming from Being a ‘Being essence’. I will also employ the following notation – En(x) – to stand for the essence function applied n times repeatedly beginning with x. So, for instance, E3(∅) = E(E(E(∅))). In the limit when n=0, En(x)=x.

In the intended structure the non-Being essences are all finite sets whose members are Being essences. Each non-Being essence, En(∅) contains all the Being essences Em(Being) such that . So, for instance, E1(∅) contains E0(Being), E2(∅) contains E0(Being) and E1(Being), and so on. The Being essences are all infinite sets. Each Being essence contains every Being essence. In addition, each Being essence Em(Being) contains every non-Being essence En(∅) such that . The following is a visual representation of the first several Being and non-Being essences.

E0(Being) = {E0(Being), E1(Being), E2(Being)… E0(∅), E1(∅), E2(∅), E3(∅)…}

E1(Being) = {E0(Being), E1(Being), E2(Being)… E1(∅), E2(∅), E3(∅)…}

E2(Being) = {E0(Being), E1(Being), E2(Being)… E2(∅), E3(∅)…}

E3(Being) = {E0(Being), E1(Being), E2(Being)… E3(∅)…}

E0(∅) = {}

E1(∅) = {E0(Being)}

E2(∅) = {E0(Being), E1(Being)}

E3(∅) = {E0(Being), E1(Being), E2(Being)}

E4(∅) = {E0(Being), E1(Being), E2(Being), E3(Being)}

In this structure, the Being Essences progressively lose more and more of the non-Being essences. So, for instance, E0(Being) contains everything, both all the Being essences and all the non-Being essences. E1(Being) contains all but one thing: It contains all the Being essences as well as all the non-Being essences except E0(∅). E2(Being) contains everything but two things. And so on. It is as if the progression of Being essences is progressively drained of the non-Being essences. Were one to take such a progression out to infinity, one would reach a set that contains all the Being essences and none of the non-Being essences. In other words, Eω(Being)= {Em(Being) | m is a non-negative integer}.The progression of non-Being essences, on the other hand, does not consist in a successive loss of sets but rather a successive gaining of sets. E0(∅), i.e., non-Being, contains nothing, E1(∅) contains one set, namely E0(Being). E2(∅) contains two sets, namely E0(Being) and E1(Being). And so on. Were one to take such a progression out to infinity, one would again reach the set that contains all of the Being essences: Eω(∅)= {Em(Being) | m is a positive integer}. Hence, Eω(Being) = Eω(∅). What can be called ‘the point at infinity’ for both the Being and non-Being essences is the set that contains all the Being essences.

When one considers the God axiom in addition to the other four axioms, the structure includes a set, *God*, that is identical to its essence and contains all and only the Being Essences all of which contain it. The following is a visual representation of God and the first four Being essences that she contains.

God = {

E0(Being) = {God, E0(Being), E1(Being), E2(Being)… E0(∅), E1(∅), E2(∅), E3(∅)…}

E1(Being) = {God, E0(Being), E1(Being), E2(Being)… E1(∅), E2(∅), E3(∅)…}

E2(Being) = {God, E0(Being), E1(Being), E2(Being)… E2(∅), E3(∅)…}

E3(Being) = {God, E0(Being), E1(Being), E2(Being)… E3(∅)…}

…}

Even this informal presentation should make plausible the thought that the Being essences form a series in God that is isomorphic to the set of natural numbers structured by the successor function. The series begins with Being and then progresses by repeated applications of the essence function. This suggests that one can identify the set of natural numbers with God, the successor function with the essence function, and the natural numbers with the Being essences. In the next section, I prove that such identifications allow one to deduce the axioms of Peano Arithmetic minus the induction schema.

Before proceeding to the extension of the theory, it is worth pointing out an interesting conception of the counting process that emerges from the above set. The process can already be seen from the above depiction of God. Consider first Being. Being is almost universally symmetrical. It is contained by every set that contains it with one exception – the empty set. Suppose then that one begins with Being and at each step of the process removes the one asymmetrical set. One can think of the process as removing the one imperfection of the set. So, the first step in the process would involve removing the empty set from Being. The resulting set is E1(Being), which has only a single instance of asymmetry, namely E1(∅), which contains E0(Being) but not E1(Being). So, that set would have to be removed from E1(Being). And the set that results is E2(Being). And so on. At each step, eliminating the only asymmetry in the set leads inexorably from one Being essence to the next. And though an asymmetry is always present at each moment of the process, the Being essences do become more godlike, containing fewer and fewer finite sets as the process continues. God, then, stands as a point at infinity toward which the process aims, though interestingly enough she is also contained in each of the sets that are part of the process.

*Section II*

*Extending the Symmetry Conception to Include Peano Arithmetic*

Demonstrating a connection between the five axioms so far discussed and Peano Arithmetic proceeds by way of four theorems.

Theorem 1:

Proof: By the Being axiom and the God axiom, Being exists, God exists, and . Hence, by the symmetry of God, .

Theorem 2:

Proof: Suppose for reductio that there is an x such that E(x) = Being. Then by the definition of an essence every set in Being contains x. Being contains the empty set. The empty set does not contain y for any y. Hence, it is not the case that every set in Being contains x. Hence, there is no x such that

Theorem 3:

Proof: Suppose that . Then, by the Essence Axiom, there exists a y such that . By the essence axiom, y contains all and only those sets, z, that contain x. God contains x. Hence, y contains God. By the symmetry of God, God contains y.

Theorem 4:

Proof: Suppose x y. Then, there is a set, w, such that either w is a member of x and is not a member of y, or w is a member of y and is not a member of x. Suppose w is a member of x and not of y. Then, by the Essence Axiom, E(w) contains x and not y. Hence, by the Essence Axiom, E(x) contains E(w) and E(y) does not. Hence, by Extensionality, E(x ). Suppose w is a member of y and not of x. Then, by the Essence Axiom, E(w) contains y and not x. Hence, by the Essence Axiom, E(y) contains E(w) and E(x) does not. Hence, by Extensionality, .

We can now add to G the following explicit definitions of Number, Successor, and 0.

(I)

(II)

(III) .

It is easy to see that I-III and the above four theorems entail the following four Peano Axioms.

With the four axioms characterizing the successor function in hand, it is possible to extend the theory conservatively by adding the following recursive definitions of addition and multiplication as well as the standard explicit definition of less-than.

Let be the deductive closure of the axioms in G as well as (I)-(VIII). contains Q. Because the additions to G are all conservative, Q is contained in a conservative extension of G. It should be clear that G is stronger than Q from the fact that in G it is provable that the successor function has a fixed point. That of course seems strange from an arithmetical point of view. But once one sees that such a number is God, one can see exactly why God transcends all finite numbers and any counting process. Moreover, once one sees that it is the Being essences that God transcends, one can also see the wisdom in Plato’s pronouncement that the Good transcends Being in both dignity and power.

These theorems establish that the Being essences are ordered like the naturals by the essence function. The Being essences can thus be considered numbers in the Mind of God. As already noted, however, any Being essence has an internal structure that depends on the properties of its predecessor. Because, for instance, 2 is the essence of 1, 2 contains all and only those sets that contain 1. In this way, the Being essences also function like ideas in the mind of God – each one encodes all the properties of its predecessor. In order to derive the existence of properties for the numbers to encode, however, we must augment in two ways.

First, we add a first order set theoretic version of induction :

Second, we add all instances of the following comprehension schema ():

, with an arithmetical formula with one free variable, y.

Call the resulting theory . interprets PA. In addition to containing PA, also characterizes a being that is identical to its essence. As should by now be clear, such a being contains an infinite number of infinite sets. In their relations to each other, the sets are structured like the natural numbers and so have (at least) all the first order arithmetically definable properties of the natural numbers. (One might of course strengthen the comprehension schema to include other properties.) In addition, each infinite set in the mind of God contains all the arithmetically definable properties of its unique predecessor.

Conclusion

In this paper, I have developed a formal theory that represents an Augustinian View of the Numbers (AVN). The theory begins with five axioms within non-well-founded set theory that form the basis of the symmetry conception of God. I showed that it is possible to conservatively extend the theory by adding explicit definitions of number, successor, and zero in terms of God, essence, and Being. The resulting theory contains Peano Arithmetic minus the Induction Schema. Such a fact allows one to conservatively extend the theory by adding recursive definitions of multiplication and addition as well as an explicit definition of the less-than relation. The resulting theory contains Robinson’s Arithmetic (Q). The theory can then be further non-conservatively extended to include all Peano Arithmetic (PA) by adding a first-order set-theoretic formulation of induction along with a comprehension schema that guarantees the existence of all and only those sets within the first-order arithmetical hierarchy.

In addition to containing PA, also provides a formal representation of AVN. The sets that God contains can be thought of as both numbers and ideas. Their status as numbers results from the fact that they form a progression that is isomorphic to the Naturals. Their potential to be ideas results from the fact that each is the essence of its predecessor. When one adds both induction and a comprehension schema that guarantees the existence of the arithmetically definable sets, the ideas in God’s mind are filled up, so to speak, with the arithmetical properties that their predecessors have. For instance, 3, i.e. E(E(E(0))), contains the set of primes, since 2 is prime. 5, on the other hand, does not. Of course, being prime is a decidable property. Not all properties are. Because the comprehension schema extends to all arithmetically definable sets, the Being essences contain not just the decidable properties but properties at every level of arithmetical complexity.

The radical difference between this conception of the numbers and the well-founded conception should be evident. One notable difference between the two stems from the location, so to speak, of the properties of the numbers. In the well-founded conception of sets, the properties of the numbers are contained by the power set of . According to the theory in this paper, the properties of the numbers are contained by other numbers that are contained by God. There is in addition a difference between the two conceptions that has a consequence for the doctrine of the ineffability of the divine essence.

It has long been known that an infinite set can be comprehended by way of a rule that tells one how to proceed constructing it at each point. This is how one can comprehend for instance the set of finite Von Neumann ordinals. An understanding of that set is contained in the construction rule x U {x}. Though infinite, the set of all and only finite Von Neumann ordinals is most certainly effable. The Being essences, on the other hand, are not likewise effable. Each Being essence contains sets that occur at every level of the arithmetical hierarchy depending on whether its predecessor is contained in that set. But there is of course no rule that can determine that membership relation. To the extent that an infinite set that is not effectively enumerable is ineffable, in the mind of God, one meets ineffability at the number 1.

The view of God that is contained in is thus a view of an infinite, ineffable, axiologically and metaphysically ultimate entity that contains objects that not only serve as numbers but also encode information about each other. Such an entity can not only be defined within first-order extensional set theory, but its general features are provable by way of extensional inferences from axioms that also entail PA. With that said, it may be that the theory does not go far enough. It may be that God has properties, for instance omnipotence and personhood, that are not provable from the symmetry conception alone. And it may be that understanding God’s relation to those properties requires intensional (or some other) concepts. But it is surely significant that a robust conception of God that conforms to the Augustinian tradition can be articulated within a 1st-order axiomatized extensional framework and that the theory within which such a God is defined can be conservatively extended to a theory of arithmetic, Q, that is presupposed by any mathematically sophisticated science. Some conceptions of God may exist in a magisterium that does not overlap with math and science. The theory in this paper is not one of them.

*Appendix*

Theorem: Being universally symmetrical with respect to set membership is equivalent to being identical to one’s essence.

First, assume God is universally symmetrical with respect to set membership. Suppose x ∈ God. Then, by the symmetry of God, God ∈ x. Therefore, by the definition of an essence, x ∈ E(God). Suppose x ∈ E(God). Then, by the definition of an essence, God ∈ x. Therefore, by the symmetry of God, x ∈ God. Therefore, for any set, x, x ∈ God ≡ x ∈ E(God). Therefore, by extensionality, God = E(God).

Second, assume that God is identical to her essence, i.e. that God = E(God). Suppose that x ∈ God. Then, by the identity of God to her essence, x ∈ E(God). Therefore, by the definition of an essence, God ∈ x. Suppose that God ∈ x. Then by the definition of an essence, x ∈ E(God). Therefore, by the identity of God and E(God), x ∈ God. Therefore, for any set, x, x ∈ God ≡ God ∈ x.

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1. There are of course other reasons for rejecting divine simplicity. For a good overview, *cf.* <https://plato.stanford.edu/entries/divine-simplicity/> [↑](#footnote-ref-1)
2. Proofs of theorems that characterize the structure occur in Studtmann’s original paper. [↑](#footnote-ref-2)