God and the Numbers

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Abstract

According to Augustine, abstract objects are ideas in the mind of God. Because numbers are a type of abstract object, it would follow that numbers are ideas in the mind of God. Call such a view the *Augustinian View of Numbers* (AVN). In this paper, I present a formal theory for AVN. The theory stems from the symmetry conception of God as it appears in Studtmann (2021). I show that the theory in Studtmann's paper can interpret the axioms of Peano Arithmetic minus the induction schema. This fact allows for the development of arithmetic in a natural way. The development eventuates in a theory that can interpret Peano arithmetic. The conception of God that emerges by the end of the discussion is a conception of an infinite, ineffable, self-cause that contains objects that not only serve as numbers but also encode information about each other.

1 Introduction

According to Augustine, abstract objects are ideas in the mind of God. Because numbers are a type of abstract object, it would follow that numbers are ideas in the mind of God. Let us call such a view the *Augustinian View of Numbers* (AVN). Although it has proven attractive to many, the Augustinian doctrine has remained suggestive but critically imprecise. This paper offers a first step toward remedying this shortcoming by presenting a formal theory of AVN.

The formal theory presupposes the symmetry conception of God.[1] According to the symmetry conception, God is universally symmetrical with respect to set membership. One important provable consequence of God's being universally symmetrical with respect to set membership is that God is identical to her essence. In fact, it is not hard to prove that being universally symmetrical with respect to set membership is equivalent to being identical to one's essence. (A proof appears in the appendix.) Such an equivalence makes the theory in this paper particularly relevant to those theistic traditions, for instance, the Thomistic tradition, according to which God is identical to her essence. As I show, the symmetry conception, and hence identity to essence, entails a formal representation of AVN, a view which Aquinas correctly argues conflicts with divine simplicity. A simple mind can, according to Aquinas, comprehend a multiplicity of ideas, but it cannot contain them.[2] Although there are several informal arguments against divine simplicity,[3] the theory in this paper provides a formal argument that anyone committed to the identity of God and her essence should reject divine simplicity and adopt AVN instead.

I develop the formal theory of AVN in two steps. I begin by restating the five axioms that form the basis of the symmetry conception of God. I call the theory that is the deductive closure of the set containing those axioms G. As shall become apparent, the axioms in G contain the concepts of God, Essence, Being, and non-Being. Moreover, they entail that God is identical to her essence. I first show that G can interpret the first-order axioms of Peano arithmetic. That interpretation proceeds by way of three definitions. First, zero is defined as Being. Second, the successor function is defined as the essence function. And finally, numbers are given a Fregean definition. According to Frege, a number is an object that instantiates all properties that zero instantiates and that are hereditary with respect to the successor function. The definition I propose identifies a number with all the sets that

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[1] Paul Studtmann, "The Divine Fractal: 1st Order Extensional Theology," *Philosophia*, (July 2021): 285–305.

[2] Thomas Aquinas, *Summa Theologiae* 1 (Q7, A3; Q15, A2).

[3] William F. Vallicella, "Divine Simplicity," *The Stanford Encyclopedia of Philosophy* (Winter 2023 Edition), Edward N. Zalta, ed., https://plato.stanford.edu/archives/win2023/ entries/divine-simplicity/.

are contained by any set that contains Being and is hereditary with respect to the essence function. As I show, such definitions along with the five axioms in G entail the first-order Peano axioms along with a first-order set-theoretic version of induction. The definition and axioms also allow one to prove that all the numbers are in God. I then extend the theory by introducing a comprehension schema. As shall become apparent, the comprehension schema results from restricting the Axiom Schema of Separation from ZF to God. Within the resulting theory, it is possible to separate out any definable subset from God. It is straightforward, then, to show that the resulting theory interprets Peano arithmetic.

Unlike standard set-theoretic representations of the naturals such as the Von Neumann finite ordinals, the sets in God contain an infinite number of sets. Because each set in God, except for Being, is an essence, each one contains all the sets that contain the set of which it is an essence. Hence, because each set in God in addition to being an essence functions as a number, each number contains all the sets that contain its predecessor. So, for instance, because the set of odd numbers contains 1, 2 contains the set of odds as does every even number. If one uses the language of encoding and properties, one could say that the number 2 encodes all the properties of the number 1. Hence, each set in God plays a dual role. As a number, each is contained by an infinite number of sets. And as an essence, each contains all and only those sets that contain its predecessor. As a result of this dual role, the sets in God's mind can be seen as both numbers and as ideas. In their relations to each other, they have all the properties that numbers have; in their relation to their predecessors, they, like ideas, encode the properties of their predecessors.

The rest of the paper is structured as follows. I begin in section 2 by discussing some methodological issues that this introduction already raises. The fact that I am proposing a non-well-founded extensional theory as a metaphysical theory runs against very well-motivated orthodoxies among analytic metaphysicians according to which (i) an adequate theory of properties must be intensional and (ii) sets are well-founded. The discussion in the first section explains how the theory in this paper should be understood given the motivations for those two orthodoxies. Discussing these methodological issues leads naturally to a brief discussion of the substantive metaphysical theses in this paper. In section 3, I state the five axioms that form the basis of the symmetry conception of God and then provide an informal characterization of the structure that they entail. In section 4, I prove that with explicit definitions of zero, successor, and number the axioms entail the first-order Peano axioms and a first-order set-theoretic version of induction. I then add a comprehension schema.

2 Some Preliminaries[4]

There are two features of the theory in this paper that set it apart methodologically from much of current analytic metaphysics: it is extensional, and it is non-well-founded. Discussing the reasons for these bits of heterodoxy should help situate the view relative to other main positions within contemporary metaphysics and should help to rebut any initial objections to the theory based on these methodological idiosyncrasies. Moreover, clearing up the methodological issues will naturally lead to a brief informal discussion of the substantive metaphysical theses in this paper.

Let us begin with extensionality. There are well-known and compelling arguments for intensionalism in a theory of properties.[5] The arguments ultimately rely on the possibility that two properties that are in fact co-extensive could have different extensions. Although having a heart and having a kidney are in fact co-extensive, they could have different extensions and so are not the same property. One might think therefore that the theory in this paper is suspect from the start, since it includes an axiom of extensionality. At least from the viewpoint of a philosopher trying to provide a metaphysics of properties, one that must respect robust intuitions about property identity, an axiom of extensionality is problematic.

In response to this objection, one can restrict the scope of the properties that the theory is about to those that have their extensions necessarily. Given the nature of the intensionalist arguments, such properties are immune from the intensionalist critique. In addition to avoiding the intensionalist critique, such a restriction serves to isolate a very important set of properties, especially from the viewpoint of metaphysics. For, it is plausible that the properties that serve to structure modal space

[4]I would like to thank Graham Oddie for very insightful suggestions about these methodological issues.

[5]Richard Montague, Formal Philosophy: Selected Papers of Richard Montague (New Haven, CT: Yale University Press, 1974). George Bealer, Quality and Concept (Oxford: Clarendon Press, 1982). David Lewis, On the Plurality of Worlds (Oxford: Blackwell, 1986).

are those that have their extensions necessarily. By framing the view as a view about properties that have their extensions necessarily, it is possible to justify the non-well-founded aspect of the theory. Just about all philosophers and mathematicians, for good reason, accept well-founded set theory.[6] So, let us for the sake of argument accept that sets are well-founded. Well-foundedness nonetheless seems an unnatural restriction on properties. For, there are proper- ties that intuitively instantiate themselves. Some such properties, for instance, the property of being a property that has its extension necessarily, have a non-trivial relationship to foundational metaphysical issues.

It is important to note that the non-well-foundedness in this paper consists in a removal of the Axiom of Foundation and not, as in Aczel's anti-foundational set theory, the replacement of the Axiom of Foundation with an assertion that every accessible directed pointed graph corresponds to a set.[7] The non-well-foundedness of the theory represents an increase in generality. Non-well-founded set theory is thereby an ideal candidate for a theory of properties that are at the foundation of a metaphysical enterprise. Its non-well-foundedness allows for a more general theory of properties than is possible within well-founded set theory. And its extensionality serves to restrict the properties to those of particular interest to metaphysicians. Of course, self-predication of the sort that non-well-foundedness allows raises the specter of paradox. Some consistent way of comprehending non- well-founded sets must be specified. Holmes discusses at length two mathematically serviceable ways of introducing such comprehension schemata: Quine's NF and Positive Set Theory.[8] For the purposes of the theory in this paper, I appeal to a much weaker comprehension schema than either of those in NF or Positive Set Theory, one that is motivated by the fact that the theory is a theory of God, not a general theory of sets.

With these methodological preliminaries in place, it is possible to present informally the main metaphysical hypotheses that I will formalize in the next section. The first metaphysical hypothesis is that the property of being a property that has its extension necessarily exists and has its extension necessarily. Because the theory in this paper is restricted to properties that have their extensions necessarily, the first hypothesis can be expressed by an axiom asserting the existence of a universal set, or what I call *Being*. The second metaphysical hypothesis is that the empty property, what I call *non-Being* exists and has its extension necessarily. Once again, because the theory is restricted to properties that have their extensions necessarily, such a hypothesis can be expressed by an axiom asserting the existence of the empty set. The third metaphysical hypothesis is that every property that has its extension necessarily has an essence that is a property that has its extension necessarily. For the purposes of the theory, an essence of a set F is the set that contains all the sets that contain F. In property-theoretic language, the essence of F is the property that is instantiated by all and only those properties that F instantiates. Such a hypothesis is a property-theoretic analog of the Leibnizian claim that every individual has a complete individual concept, except unlike the Leibnizian claim it is iterative. Because a property essence is a property, a property's essence has an essence, and so on. As with Leibnizian essences, it is natural to think that the essences in the theory have explanatory power. Because an essence of F contains all of F's properties, it can serve as the explanation of F's having the properties it does.

Within such a framework, one can naturally ask whether there is some property that is identical to its own essence. If there is, it would explain why it has the properties it does. If one of those properties is Being, which for the purpose of the theory is the property of having one's extension necessarily, and if any property that has its extension necessarily exists necessarily, then such a property would explain its own necessary existence. The property in question would in other words be a self-cause, which is one of the characteristics of God as conceived in several theological traditions. Moreover, because essences uniquely characterize the object of which they are an essence, it is natural to think of them as representational: an essence of a property F represents F. A property that is its own essence therefore represents itself. Hence, it conforms to the Aristotelian conception of God, namely that God is thought thinking itself.[9] Because the object defined in this paper has characteristics that have traditionally been associated with God, calling such an object 'God' seems warranted. If a philosopher who has a bias toward particularity in the divine refuses to acknowledge that the property discussed in this

[6]However, see Randall Holmes, "Alternative Axiomatic Set Theories," *The Stanford Encyclopedia of Philosophy* (Winter 2021 Edition), Edward N. Zalta, ed., https://plato. stanford.edu/archives/win2021/entries/settheory-alternative/, for arguments in favor of non-well-founded sets.

[7]Peter Aczel, *Non-well-founded Sets* (Stanford, CA: CSLI Publications, 1988).

[8]Holmes, "Alternative Axiomatic Set Theories," op. cit.

[9]Aristotle, *Metaphysics* (1074b 32–34).

paper could be God simply because that would make God a property, so be it. Call her 'the Beautiful', with the understanding that the Beautiful, in addition to being an aesthetic and moral principle, is the ultimate metaphysical principle, one that has the structure of thought thinking itself.

The final metaphysical hypothesis in this paper is that a property that is identical to its essence exists. Although it is possible to frame an axiom that asserts such a claim directly, I show in the appendix that the existence of an object that is identical to its essence is equivalent to a simpler assertion, one that makes explicit a connection between the property defined in this paper and value, namely that the property in question is universally symmetrical with respect to the instantiation relation. Such an equivalence allows one to define God as that unique Being that is universally symmetrical with respect to set membership.[10] Although such a definition may not be initially intuitive, the equivalence between universal symmetry and identity to essence entails that universal symmetry characterizes God as she has been understood by several prominent theologians in both the Islamic and Christian traditions.

3 The Symmetry Conception of God

The symmetry conception of God depends on the following five axioms, which I call respectively *Extensionality*, *the Being Axiom*, *the Non-Being Axiom*, *Essence*, and *the God Axiom*.

$$(\forall x)(\forall y)(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y) \tag{1}$$

$$(\forall x)(x \in Being) \tag{2}$$

$$(\forall x)(x \notin Non - Being) \tag{3}$$

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow y \in x) \tag{4}$$

$$(\forall x)(x \in God \leftrightarrow God \in x) \tag{5}$$

Extensionality is part of any set theory. Each of the other four axioms corresponds to an important metaphysical concept: Being, Non-being, Essence, and God. To see the structure that is entailed it is helpful to begin with the first four of the above axioms.[11] The first thing to note is that the Essence Axiom and the Non-being Axiom jointly entail an infinite progression of sets. Let '$E(x)$' denote the essence of x and '$\emptyset$' denote Non-being. Then, the two axioms entail the existence of Non-being, $\emptyset$, the existence of the essence of Non-being, $E(\emptyset)$, the existence of the essence of the essence of Non-being, $E(E(\emptyset))$, and so on. Likewise, the Essence Axiom and the Being Axiom entail an infinite progression of sets: *Being*, $E(Being)$, $E(E(Being)$, and so on. For the ease of expression, I will call any set that is part of the progression of essences stemming from Non-being a 'Non-being essence' and any set that is part of the progression of essences stemming from Being a 'Being essence'. I will also employ the following notation—$E^n(x)$—to stand for the essence function applied n times repeatedly beginning with x. So, for instance, $E^3(\emptyset) = E(E(E(\emptyset)))$. In the limit when $n = 0$, $E^n(x) = x$.

Because the theory is first order, there are uncountable models of the axioms. But in the minimal countable model, which corresponds to the provable instances of the axioms, the Non-being essences are all finite sets whose members are Being essences. Each Non-being essence, $E^n(\emptyset)$, contains all the Being essences $E^m(Being)$ such that $m < n$. So, for instance, $E^1(\emptyset)$ contains $E^0(Being)$, $E^2(\emptyset)$ contains $E^0(Being)$ and $E^1(Being)$, and so on. The Being essences are all infinite sets. Each Being essence contains every Being essence. In addition, each Being essence, $E^m(Being)$, contains every Non-being essence, $E^n(\emptyset)$ such that $n < m$. The following is a visual representation of the first several Being and Non-being essences.

$$\begin{aligned}
E^0(Being) &= \{E^0(Being), E^1(Being), E^2(Being)... \quad E^0(\emptyset), E^1(\emptyset), E^2(\emptyset), E^3(\emptyset)...\} \\
E^1(Being) &= \{E^0(Being), E^1(Being), E^2(Being)... \quad E^1(\emptyset), E^2(\emptyset), E^3(\emptyset)...\} \\
E^2(Being) &= \{E^0(Being), E^1(Being), E^2(Being)... \quad E^2(\emptyset), E^3(\emptyset)...\} \\
E^3(Being) &= \{E^0(Being), E^1(Being), E^2(Being)... \quad E^3(\emptyset)...\}
\end{aligned}$$

[10] Studtmann, "The Divine Fractal," op. cit., discusses the need for an assertion of uniqueness in the definition and suggests a definition that does not require it.

[11] Proofs of theorems that characterize the structure occur in Studtmann, "The Divine Fractal," op. cit.

$$
\begin{aligned}
&E^0(\emptyset) = \{\} \\
&E^1(\emptyset) = \{E^0(Being)\} \\
&E^2(\emptyset) = \{E^0(Being), E^1(Being)\} \\
&E^3(\emptyset) = \{E^0(Being), E^1(Being), E^2(Being)\} \\
&E^4(\emptyset) = \{E^0(Being), E^1(Being), E^2(Being), E^3(Being)\}
\end{aligned}
$$

In this structure, the Being essences progressively lose more and more of the Non-being essences. So, for instance, $E^0(Being)$ contains everything, both all the Being essences and all the Non-being essences. $E^1(Being)$ contains all but one thing: It contains all the Being essences as well as all the Non-being essences except $E^0(\emptyset)$. $E^2(Being)$ contains everything but two things. And so on. It is as if the progression of Being essences is progressively drained of the Non-being essences. Were one to take such a progression out to infinity, one would reach a set that contains all the Being essences and none of the Non-being essences. In other words, $E^\omega(Being) = \{E^m(Being) : m\, is\, a\, natural\, number\}$. The progression of Non-being essences, on the other hand, does not consist in a successive loss of sets but rather a successive gaining of sets. $E^0(\emptyset)$, that is, Non-being, contains nothing, $E^1(\emptyset)$ contains one set, namely $E^0(Being)$. $E^2(\emptyset)$ contains two sets, namely $E^0(Being)$ and $E^1(Being)$. And so on. Were one to take such a progression out to infinity, one would again reach the set that contains all of the Being essences: $E^\omega(\emptyset) = \{E^m(Being) : m\, is\, a\, natural\, number\}$. Hence, $E^\omega(Being) = E^\omega(\emptyset)$. What can be called 'the point at infinity' for both the Being and Non-being essences is the set that contains all the Being essences.

When one considers the God Axiom in addition to the other four axioms, the structure includes a set, God, that is identical to its essence and contains all and only the Being essences, all of which contain it. The following is a visual representation of God and the first four Being essences that she contains.

$$
\begin{aligned}
God = \{&E^0(Being) = \{God, E^0(Being), E^1(Being), E^2(Being)... \quad E^0(\emptyset), E^1(\emptyset), E^2(\emptyset), E^3(\emptyset)...\}, \\
&E^1(Being) = \{God, E^0(Being), E^1(Being), E^2(Being)... \quad E^1(\emptyset), E^2(\emptyset), E^3(\emptyset)...\}, \\
&E^2(Being) = \{God, E^0(Being), E^1(Being), E^2(Being)... \quad E^2(\emptyset), E^3(\emptyset)...\}, \\
&E^3(Being) = \{God, E^0(Being), E^1(Being), E^2(Being)... \quad E^3(\emptyset)...\}, \\
&...\}
\end{aligned}
$$

This informal presentation should make plausible the thought that the Being essences form a series in God that is isomorphic to the set of natural numbers structured by the successor function. The series begins with Being and then progresses by repeated applications of the essence function. This suggests that one can identify the successor function with the essence function, and zero with Being. In the next section, I prove that with a set-theoretic version of Frege's definition of number, such identifications allow one to deduce the first-order Peano axioms and a first-order set-theoretic version of the induction schema.

Before proceeding to the extension of the theory, it is worth pointing out an interesting conception of the counting process that emerges from the above set. The process begins with Being. It may seem odd to think of a counting process as beginning with Being. For, it is very intuitive to think of a counting process as beginning with nothing. Hence, it is intuitive and commonplace to identify zero with the empty set. But Being and the empty set are conceptually related. Whereas the empty set does not contain anything and lacks everything, Being does not lack anything and contains everything. Moreover, Being, as so far characterized, is almost universally symmetrical. It is contained by every set that contains it with one exception—the empty set. The failure of Being to be universally symmetrical and hence identical to its essence, therefore, results from its containing the set that is typically identified as zero.

Suppose then that a counting process begins with Being and at each step of the process the one asymmetrical set is removed. One can think of the process as removing the one imperfection of the set. So, the first step in the process involves removing the empty set from Being. The resulting set is

$E^1(Being)$, which has only a single instance of asymmetry, namely $E^1(\emptyset)$, which contains $E^0(Being)$ but not $E^1(Being)$. So, $E^1(\emptyset)$ would have to be removed from $E^1(Being)$. The set that results is $E^2(Being)$. And so on. Moreover, the sets that are removed from the Being essences are the essences of Non-being, which increase in size, at each step gaining an additional Being essence, as if the Being essences are externalized into an ever-increasing finite realm. And like the Being essences, the Non-being essences each exhibit a single asymmetry. $E^1(\emptyset)$, for instance, contains $E^0(Being)$, which contains it, but does not contain $E^1(Being)$, which also contains it. The attempt to remove the asymmetry from the Non-being essence involves adding a Being essence. For instance, to remove the asymmetry from $E^1(\emptyset)$ would require adding $E^1(Being)$. Such an addition results in $E^2(\emptyset)$, which also exhibits a single asymmetry. At each step in the process, eliminating the only asymmetry in both the Being and Non-being essences leads inexorably, in Sisyphean fashion, from one essence to the next, a striving to eliminate an imperfection that always yields yet another imperfection. And though an asymmetry is always present in both the Being essences and Non-being essences, both the Being and Non-being essences become more godlike, the former containing more and more infinite sets and the latter containing fewer and fewer finite sets as the process continues. God, then, stands as a point at infinity toward which the process aims.

4 Extending the Symmetry Conception of God

Demonstrating a connection between the five axioms so far discussed and Peano arithmetic proceeds by way of definitions of zero, successor, and number. The definitions for zero and successor have been foreshadowed by the discussion of the counting process described above. Zero is defined as Being, and the successor function is defined as the essence function.

$$0 = Being \tag{6}$$

$$(\forall x)S(x) = E(x) \tag{7}$$

With these definitions of zero and successor, it is possible to propose a Fregean definition of number. According to Frege, a number is an object that possesses all the properties that zero possesses and that are hereditary with respect to the successor function. I propose a similar definition with one crucial difference. Instead of a second-order definition, I propose a first-order set-theoretic definition. The resulting definition is a first-order version of Frege's definition of number with Being taking the place of zero and the essence function taking the place of the successor function.

$$(\forall k)(k\,is\,a\,number \leftrightarrow (\forall x)[(Being \in x\ \ \&\ \ (\forall w)(\forall y)((w \in x\ \ \&\ \ y = E(w)) \to y \in x)) \to k \in x)] \tag{8}$$

As I now show, it is easy to prove from (1)-(8) the first-order Peano axioms as well as a first-order set-theoretic version of induction. It is also easy to prove an additional theorem which shows that the theory in this paper avoids the Julius Caesar problem. It is worth noting that the following Theorems 1 and 5 follow directly from the definition of number in (8). This is directly analogous to the original derivations in Frege's system. It is also worth noting that the following Theorem 4 follows from (8) and the *Essence Axiom*. Frege, by contrast, introduced his axiom schema to prove the analogous theorem in his system.

Theorem 1. *Being is a number.*

Proof. This is immediate from (8). □

Theorem 2. $(\forall x)(Being \neq E(x))$

Proof. Suppose for reductio that there is an x such that $E(x) = Being$. Then, by the *Essence Axiom*, every set in *Being* contains x. *Being* contains the empty set. The empty set does not contain y for any y. Hence, it is not the case that every set in *Being* contains x. Hence, there is no x such that $E(x) = Being$. □

Theorem 3. $(\forall x)(\forall y)(E(x) = E(y) \to x = y)$

Proof. Suppose $x \neq y$. Then there is a set, w, such that either w is a member of x and is not a member of y, or w is a member of y and is not a member of x. Suppose w is a member of x and not of y. Then, by the *Essence Axiom*, $E(w)$ contains x and not y. Hence, by the *Essence Axiom*, $E(x)$ contains $E(w)$, and $E(y)$ does not. Hence, by *Extensionality*, $E(x) = E(y)$. Suppose w is a member of y and not of x. Then, by the *Essence Axiom*, $E(w)$ contains y and not x. Hence, by the *Essence Axiom*, $E(y)$ contains $E(w)$, and $E(x)$ does not. Hence, by *Extensionality*, $E(x) \neq E(y)$. □

Theorem 4. $(\forall x)(x\, is\, a\, number \rightarrow (\exists y)(y = E(x) \;\;\&\;\; y\, is\, a\, number))$

Proof. Suppose that $x\, is\, a\, number$. By the *Essence Axiom*, there is a y such that y is the essence of x. By (8), x is contained by any set that contains *Being* and is hereditary with respect to the essence function. Any set that contains x and is hereditary with respect to the essence function contains the essence of x. Hence, y is contained by any set that contains *Being* and is hereditary with respect to the essence function. Hence, $y\, is\, a\, number$. □

Theorem 5. $(\forall x)((Being \in x \;\;\&\;\; (\forall w)(\forall y)((w \in x \;\;\&\;\; y = E(w)) \rightarrow y \in x)) \rightarrow (\forall y)(y\, is\, a\, number \rightarrow y \in x))$

Proof. This is immediate from (3). □

In addition to the above theorems, the following theorem can also be proven.

Theorem 6. $(\forall k)(k\, is\, a\, number \rightarrow k \in God)$

Proof. Suppose that $k\, is\, a\, number$. Hence, k is in every set that contains *Being* and is hereditary with respect to the essence function. *God* contains *Being* and is hereditary with respect to the essence function. That *God* contains *Being* follows from the fact that *Being* contains God and the symmetry of *God*. That *God* is hereditary with respect to the essence function can be proven as follows. Suppose that x is in *God* and that y is the essence of x. By *Essence*, y contains *God*. By the symmetry of *God*, *God* contains y. Hence, k is in *God*. □

Theorem 6 shows that the theory in this paper avoids the Julius Caesar problem. Because Julius Caesar is not in God, it follows from Theorem 6 that Julius Caesar is not a number.

Let G^+ be the deductive closure of $(1)-(8)$. The above theorems of G^+ show that the sets in God, that is, the Being essences, are ordered like the natural numbers. Hence, numbers are objects in the mind of God. As already noted, however, any Being essence has an internal structure that depends on the properties of its predecessor. Because, for instance, 2 is the essence of 1, 2 contains all and only those sets that contain 1. In this way, the Being essences also function like ideas in the mind of God—each one encodes all the properties of its predecessor. But we have not yet introduced into the formal theory properties for the Being essences to encode. As a result, G^+ does not contain Peano arithmetic. Although it contains a first-order set-theoretic version of induction, to incorporate Peano arithmetic it must be extended to include the existence of the various properties of the numbers. And this requires introducing a comprehension schema.

There are two well-studied non-well-founded set theories with different comprehension schemata: Quine's NF and Positive Set Theory.[12] Each of these theories is a general theory of sets with its own underlying motivations. The comprehension schemata are meant both to reflect the underlying motivation and to avoid the set-theoretic paradoxes. Unlike Quine's NF or Positive Set Theory, the theory in this paper is not meant to be a general theory of sets but rather a theory of God. A comprehension schema for the theory should of course avoid the paradoxes. But it should also reflect the underlying motivation for the theory. The following comprehension schema, where Φ is any formula (in a language with logical symbols and the set-membership sign) that has one free occurrence of y and that does not contain a free occurrence of x, does both.

$$(\exists x)(\forall y)(y \in x \leftrightarrow (y \in God \;\;\&\;\; \Phi(y))) \tag{9}$$

This schema is the axiom schema of separation from *ZFC* restricted to God. The restriction to God is motivated by the fact that the theory is a theory about God. The appeal to the form of the schema is motivated in the first instance by the idea that all the subsets of God exist, which can be motivated by

[12] Holmes, "Alternative Axiomatic Set Theories," op. cit.

appeal to the plenitude of God, and in the second instance by the fact that it avoids the paradoxes. As is known, the full *Axiom Schema of Separation* is inconsistent with the existence of a universal set—one need only separate out the Russell set from the universal set to derive a contradiction. Because CS is restricted to the sets in God, it is consistent with the existence of a universal set.[13]

Let G^{++} be G^{+} enhanced with every instance of the comprehension schema in (9). G^{++} interprets Peano arithmetic because any instance of the first-order induction schema corresponds to a conjunction of the first-order set-theoretic version of induction and the appropriately chosen instance of the comprehension schema. It is important to note that G^{++} does not yet interpret what is typically called *Full Peano Arithmetic*, which usually includes recursive axioms for addition and multiplication, as well as instances of the comprehension schema that contain symbols for them. Therefore, one can extend the theory further by including these as axioms. If the goal of this paper were to reduce arithmetic to G^{++}, essentially reducing arithmetic to theology, then adding such axioms would be unacceptable. However, since the aim has been to articulate a metaphysical/theological theory within mathematics, the addition of these mathematical axioms is beneficial. This allows for a traditional understanding of metaphysics according to which it provides the foundation for mathematical truths. Nevertheless, this foundation must be supplemented with explicitly mathematical axioms to form a functional mathematical theory. G^{++} offers such a foundation. It describes an entity with the structure of the natural numbers and is substantial enough to interpret a significant sub-theory of arithmetic, but it must be augmented with explicit mathematical definitions to interpret a widely used arithmetical theory. In essence, the relationship between theology and arithmetic can now be encapsulated in a slogan: arithmetic is God plus recursion.

It should be noted that G^{++} has one idiosyncratic feature: it contains the proposition that the successor function has a fixed point, namely God. Because no number is the fixed point of the successor function, it follows that God is not a number. It of course seems strange from an arithmetical point of view that the successor function has a fixed point that is not a number. But such a fact is of metaphysical importance. God's being a fixed point of the successor function is equivalent to her being identical to her essence. As noted above, because God is identical to her essence, she explains her own necessary existence. Because identity to essence is equivalent to universal symmetry with respect to the membership relation, and because well-founded set theories cannot admit symmetrical membership relations, G^{++} has a feature that well-founded theories not only lack but cannot contain, on pain of contradiction: It explains the necessary existence of the object that the theory is about.

5 Conclusion

In this paper, I have developed a formal theory that represents the Augustinian View of Numbers (AVN). The theory begins with five axioms within non-well-founded set theory that form the basis of the symmetry conception of God. I showed that it is possible to extend the theory by adding a first-order version of Frege's definition of number. The resulting theory contains Peano arithmetic and a first-order set-theoretic version of induction. The theory can then be extended by adding a comprehension schema that has the form of the Axiom Schema of Separation restricted to God. Such an extension serves to characterize the properties in God's mind and allows the resulting theory to interpret Peano arithmetic.

The formal theory provides a formal representation of AVN. The sets that God contains can be thought of as both numbers and ideas. Their status as numbers results from the fact that they form a progression that is isomorphic to the natural numbers. Their status as ideas results from the fact that each is the essence of, and so uniquely represents, its predecessor. Adding a comprehension schema that guarantees the existence of the arithmetically definable sets fills up, so to speak, the ideas in God's mind with the properties that their predecessors have. For instance, 3, that is, $E^3(Being)$, contains the set of primes, since 2 is prime. 5, on the other hand, does not.

The radical difference between this conception of the numbers and the well-founded conception should be evident. One notable difference between the two stems from the location of the properties of the numbers. In the well-founded conception of sets, the properties of the numbers are contained by the power set of ω. According to the theory in this paper, the properties of the numbers are contained

[13] To see this, suppose one forms the instance of *CS* that corresponds to the Russell set: $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in God \ \& \ y \notin y))$. One can then derive: $A \in A \leftrightarrow (A \in God \ \& \ A \notin A)$. It follows from this that $A \notin God$

by other numbers. This difference between the two conceptions has a consequence for the doctrine of the ineffability of the divine.

It has long been known that an infinite set can be comprehended by way of a rule that tells one how to proceed constructing it at each point. This is how one can comprehend, for instance, the set of finite Von Neumann ordinals. An understanding of that set is contained in the construction rule $x \cup \{x\}$. Though infinite, the set of all and only finite Von Neumann ordinals is most certainly effable. The Being essences, on the other hand, are not likewise effable. Each Being essence contains sets that occur at every level of the arithmetical hierarchy depending on whether its predecessor is contained in that set. But there is no rule that can determine that membership relation. To the extent that an infinite set that is not effectively enumerable is ineffable, in the mind of God one meets ineffability at the number 1. The view of God that is contained in G^{++} is thus a view of an infinite, ineffable, self-representational, necessarily existent, universally symmetrical self-cause that contains objects that not only serve as numbers but also encode information about each other. Not only can such an entity be defined within first-order extensional set theory, but its general features are provable by way of extensional inferences. It is philosophically significant that a robust conception of God that conforms to the Augustinian tradition can be articulated within a first-order axiomatized extensional framework and that the theory that describes such a God interprets Peano arithmetic. Some conceptions of God may exist in a magisterium that does not overlap with math and science. The conception of God in this paper is not one of them.

Appendix

Theorem 7. $(\forall y)(\forall x)((x \in y \leftrightarrow y \in x) \leftrightarrow y = E(y))$

Proof. First, assume the symmetry of y: $(\forall x)(x \in y \leftrightarrow y \in x)$. Suppose $w \in y$. By the symmetry of y, $y \in w$. By the *Essence Axiom*, $w \in E(y)$. Suppose $w \in E(y)$. By the *Essence Axiom*, $y \in w$. By the symmetry of y, $w \in y$. Hence, $w \in y \leftrightarrow w \in E(y)$. Hence,$(\forall x)(x \in y \leftrightarrow x \in E(y))$. By *Extensionality*, $y = E(y)$ Next, assume $y = E(y)$. Suppose $w \in y$. By identity, $w \in E(y)$. By the *Essence Axiom* $y \in w$. Suppose $y \in w$. By the *Essence Axiom*, $w \in E(y)$. By identity, $w \in y$. Hence, $w \in y \leftrightarrow y \in w$. Hence, $(\forall x)(x \in y \leftrightarrow y \in x)$. Hence, $(\forall y)(\forall x)((x \in y \leftrightarrow y \in x) \leftrightarrow y = E(y))$. □