ON THE NOTION OF VALIDITY FOR THE BILATERAL CLASSICAL LOGIC

ANONYMOUS

ABSTRACT. This paper considers Rumfitt's *bilateral classical logic (BCL)*, which is proposed to counter Dummett's challenge to classical logic. First, agreeing with several authors, we argue that Rumfitt's notion of harmony, used to justify logical rules by a purely proof theoretical manner, is not sufficient to justify *coordination* rules in BCL purely proof-theoretically. For the central part of this paper, we propose a notion of proof-theoretical validity similar to Prawitz for BCL and proves that BCL is sound and complete respect to this notion of validity. The major difficulty in defining validity for BCL is that validity of positive $+A$ appears to depend on negative $-A$, and vice versa. Thus, the straightforward inductive definition does not work because of this circular dependance. However, Knaster-Tarski's fixed point theorem can resolve this circularity. Finally, we discuss the philosophical relevance of our work, in particular, the impact of the use of fixed point theorem and the issue of decidability.

1. INTRODUCTION

1.1. **Proof-theoretic semantics.** Proof-theoretic semantics [6, 15, 8, 13, 22] is a formal realization of inferential semantics [3, 5, 1, 2], using a natural-deduction proof system. In proof-theoretic semantics, the meaning of a sentence is determined by how the sentence is verified by formalized proofs.

However, the idea that all proofs equally contribute to the meaning of sentences has a serious difficulty, as made clear by the famous paradoxical operator: Prior's **tonk** [16]. In order to construct a proof-theoretic semantics, we should have some distinction between justifiable and unjustifiable proofs. Gentzen already sketches how we can modify the basic idea that proofs determine meaning, to avoid paradoxical operators.

> To every logical symbol &, $\vee$, $\forall$, $\exists$, $\to$, $\neg$, belongs precisely one inference figure which 'introduces' the symbol – as the terminal symbol of a formula – and which 'eliminates' it. The fact that the inference figures &-E and $\vee$-I each have two forms constitutes a trivial, purely external deviation and is of no interest. The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: in eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in a sense afforded it by the introduction of that symbol'.

In this citation, Gentzen associates the meaning of sentences with only the I-rules of the system and sees the E-rules as derivatives from them. E-rules are only justified if I-rules explain their use in proofs. This idea introduces the concept of justification of proofs and makes it possible to exclude connectives like **tonk**.

There are two approaches to the formalization of Gentzen's suggestion in the literature. Following Shroeder-Heister's terminology, we call the one local and the other global [21].

In the local approach, the primitive units of justification are rules. I-rules are self-justifying rules. E-rules are justified by showing that they are in a certain balance with I-rules. That balance is called harmony [6, pp.247-251]. Harmony is often defined by the possibility of leveling of local peaks, but many other criteria are proposed [21, 26].

In the global approach, the units of justification are arguments. Arguments are proof figures with arbitrary steps. Canonical proofs that end with meaning-conferring rules are prescribed as self-justifying. Non-canonical but valid proofs are defined as arguments which can be transformed to canonical proofs when supplied with valid proofs for their assumptions. The model of that transformation is a normalization procedure. The semantics is expressed in the form of the definition of valid arguments, which expresses proof-theoretical justifiability of proofs [6, pp.245-264][14].

A fixed *base system* defines the set of canonical proofs in the global approach. Base systems are proof systems where only atomic sentences can appear. They describe rules which are specific to each domain of discourse. A base system can be viewed as a proof-theoretic counterpart of valuation for atomic sentences. Logically valid arguments are defined as proofs valid under any base system.

1.2. **Classical proof-theoretic semantics.** Dummett and Prawitz, who started the study of proof-theoretic semantics, are constructivists. For this reason, many versions of proof-theoretic semantics for intuitionist logic have been given and studied. These semantics are sound for intuitionist logic but do not validate some classical laws. Dummett himself suggests that the proof-theoretic justifiability is an advantage of intuitionist logic to classical logic. Kürbis call the problem to give an adequate proof-theoretic semantics to classical logic *Dummett's challenge* (in the weaker form) [9]

There are many attempts to resolve Dummett's challenge. They can be distinguished into two groups: *atomistic inferentialists* and *nonatomistic inferentialists*. Atomistic inferentialists accept the atomistic conception of constants [11] that meaning of logical constants must be explained independently, without reference to other operators. Therefore, meaning-conferring rules for operators should not mention other constants. In Dummett's terminology, meaning-conferring rules should be pure. On the other hand, nonatomistic inferentialists allow interdependency of logical constants. They can use impure rules and syntactic definition of constants to explain the meaning of classical connectives.

Dummett argues that there seems to be no reduction procedure for justifying the classical natural deduction. Atomistic inferentialists agree with him and consider that it is difficult to give a proof-theoretic semantics for a typical formulation of classical logic. For this reason, some researchers introduce structural improvements of classical natural deduction to meet Dummett's challenge. With new proof structures and structural rules, new procedures for harmonizing classical operational rules or transforming classical non-canonical proofs into canonical proofs are introduced. For example, Read [17] introduces a multiple-conclusion system for classical logic. Rumfitt [18] introduces force operators for assertion and denial. Murzi [11] introduces higher-order assumptions.

Another approach allows interdependency of the meaning of logical constants. They can use impure rules and syntactic definition of constants to explain the meaning of classical connectives. Therefore, they often do not introduce new structures and explain the meaning of classical logic by the proof system commonly used. For example, Sandqvist [20] exploits the fact that the law of excluded middle can be justified proof-theoretically for atomic formulas if we assume that a base system only contains first-order rules, and uses negative translation to explain the meaning of logical constants. Milne [10] introduces impure introduction rules for $\neg$ and $\to$.

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B \vee D\end{array}}{(A \to B) \vee D} \to I \qquad\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ D\end{array}}{\neg A \vee D} \neg I$$

Milne reads those rules as defining the meaning of constants in disjunctive contexts. The meaning of $\vee$ is separately explained with the standard rules.

This paper concentrates on the atomistic inferentialist approach. In particular, we investigate Rumfitt's bilateral formulation of classical logic and gives his system a purely proof-theoretical justification.

1.2.1. *Rummfitt's bilateral classical logic.* Rumfitt proposes a *bilateral* formulation for classical logic *BCL* and argues that this formulation solves Dummett's challenge (at least he can be so interpreted) [18]. Bilateral formulations are formulations of logical systems in which we do not only assert or accept a proposition, but we also deny or reject it. In order to mark the force with which propositions are stated, propositions A can appear *positive* and *negative* contexts $+A$ or $-A$ respectively in proofs. We call $+$ and $-$ *force operators*. $\bot$ can also appear in a inferential step on par with assertion $+A$ or denial $-A$. $\bot$ is regarded as a punctuation mark in inference, not a propositional constant ($+\bot$ or $-A \to \bot$ is illegal). Like the empty succedent in sequents, it expresses a structure of inference, not in proposition.

BCL has operational rules and coordination rules. Operational rules are I-rules and E-rules for constants that appear in negative and positive contexts. For example, rules for $\vee$ is the following.

$$\dfrac{+A}{+A \vee B} + \vee I1 \qquad \dfrac{+B}{+A \vee B} + \vee I2 \qquad \dfrac{+A \vee B \quad \begin{array}{c}[+A]\\ \vdots\\ \alpha\end{array} \quad \begin{array}{c}[+B]\\ \vdots\\ \alpha\end{array}}{\alpha} + \vee E$$

$$\dfrac{-A \quad -B}{-A \vee B} - \vee I \qquad \dfrac{-A \vee B}{-A} - \vee E1 \qquad \dfrac{-A \vee B}{-B} - \vee E2$$

Coordination rules can be seen as structural rules for the newly introduced force operators. One of the coordination rules allows inference from $+A$ and $-A$ to $\bot$ (we call this rule the law of contradiction). The other allows to infer $+A$ when $\bot$ can be inferred from $-A$, and vice versa (we call this rule RAA). Rumfitt formalizes classical logic in this form and claims that inferences in BCL escape the problem that unilateral classical logic has from the viewpoint of the proof-theoretic semantics.

An important thing to note, but not explicitly pointed out by Rumfitt, is that not only operational rules but also coordination rules call for justification. Suppose we introduce a pair of rules governing the use of force operators.

$$\dfrac{\begin{array}{c}[+A]\\ \vdots\\ \bot\end{array}}{+A} + \qquad\qquad \dfrac{\begin{array}{c}[-A]\\ \vdots\\ \bot\end{array}}{-A} -$$

Those coordination rules differ from RAA just in the force of the conclusion. They, like operational rules for *tonk*, make a paradoxical inference possible together with the law of contradiction.

$$\dfrac{\dfrac{\dfrac{[+A]_1 \quad [-A]_0}{\bot}}{-A}\ 0 \qquad [+A]_1}{\dfrac{\bot}{+A}\ 1}$$

This inference deduces $+A$ from nothing, because all assumptions $[-A]_0$ and $[+A]_1$ are discharged. Therefore, there is an unacceptable coordination rule. Therefore, we need to explain why coordination rules in BCL are acceptable.

In justification of BCL, Rumfitt takes the local approach, but his argument is not satisfactory for proof-theoretic justification for coordination rules. For operational rules, he proves that I-rules and E-rules for logical constants are in harmony. For coordination rules, Rumfitt argues that the law of contradiction is justified from the theorem that the law of contradiction for complex propositions can be syntactically derived from atomic cases. However, for proof-theoretic validity of RAA, he seems to argue that proof-theoretic justifiability of operational rules is sufficient as a defense from Dummett's criticism of classical logic, and the positive argument for RAA rests on considerations on the concept of truth values [19], which is arguably not acceptable from the viewpoint of proof-theoretic semantics.

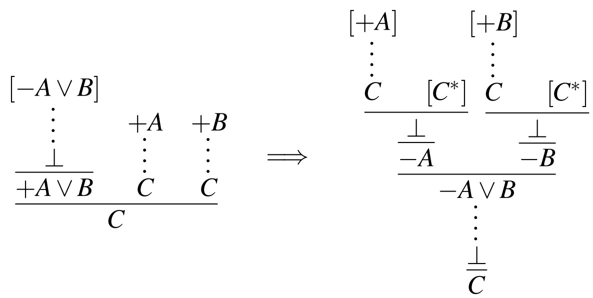
However, it is difficult to justify RAA in the same way as other rules proof-theoretically. Ferreira shows RAA for a complex proposition cannot be syntactically reduced to RAA for atomic propositions like the law of contradiction [7]. Generally, it is impossible to reduce the form of RAA with the disjunctive conclusion to the one with a simpler conclusion.

Example 1.

$$\dfrac{\begin{matrix}[-A\vee B]\\ \vdots \\ \bot\end{matrix}}{+A\vee B} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{[-A] \quad -B}{-A\vee B}\\ \vdots\\ \bot\end{matrix}}{+A}}{+A\vee B} \qquad \dfrac{\dfrac{\begin{matrix}\dfrac{-A \quad [-B]}{-A\vee B}\\ \vdots\\ \bot\end{matrix}}{+B}}{+A\vee B}}{\textit{how to discharge } -A \textit{ and } -B?}$$

In addition to that, Suzuki argues that it is impossible to justify RAA in the same way as operational rules [24]. If we are to take RAA as meaning-conferring I-rules and justify other rules with harmony, explanation of meaning runs into vicious circularity. It explains the proof of $+A$ by referring to the proofs of $-A$, and the proof of $-A$ by referring to the proofs of $+A$. The meaning of $+A$ and the meaning of $-A$ are interdependent. If we are to take E-rules as meaning-conferring and justify RAA, the problem is that showing harmony with RAA depends on the validity of RAA itself. For example, when leveling local peaks created by RAA and the disjunctive conclusion, the proof after reduction contains RAA for the formula C with unbounded complexity.

Example 2.



A possibility of leveling of local peaks with RAA rests on the validity of RAA for C. Therefore, the justification of RAA with harmony is circular. All the atomistic inferentialist we mentioned show operational rules are in harmony, but they do not show that newly introduced operators or rules that enable classical proofs are in harmony with other rules.

This paper gives a full justification for BCL. We take the global approach and define the set of valid arguments in BCL. Our semantics show all the proofs in BCL are valid and further. RAA, with other rules in BCL, is proof-theoretically justified. Furthermore, BCL is complete respect to our semantics. Our system also has the circular nature of RAA, because we see RAA as the meaning-conferring rule. In order to tackle the problem, we introduce the technique of fixed point construction.

1.3. **Our contribution and assumption.** In this paper, we follow the global approach, define a validity concept for a bilateral propositional-logic and show soundness and completeness of BCL and resolves the problem of Rumfitt's work. However, we make two assumptions. The first is that fixed-point construction is acceptable for the proof-theoretic semantics. Another is that the form of a base system is axiomatic. We discuss two assumptions after the proof of the main results (Sections 7.1 and 7.2).

Our definition of validity uses fixed-point construction, which is often used by studies of classical logic in computer science. This tool is necessary for overcoming the seemingly-circular nature of RAA. With this machinery, we can treat RAA as a kind of meaning-conferring rule, while inductively defining canonical proofs along logical complexity of sentences. The compositionality of semantics for logical connectives is an important property of a molecular semantics of language [4, 5].

2. Mathematical Preliminary

In this section, we introduce Knaster-Tarski Theorem [25].

Theorem 1. *Let $\mathscr{L}$ be a complete lattice, $f\colon \mathscr{L} \to \mathscr{L}$ be an increasing function and F be the set of all fixed points of f. Then, F forms a complete lattice. In particular, F is not empty.*

Proof. We only prove the existence of the greatest and least fixed points. Let

$$u := \bigvee \{x \in \mathscr{L} \mid x \leq f(x)\}. \tag{1}$$

Let $x \leq f(x)$. Then, $f(x) \leq f(u)$ by the definition of u and f being increasing. Because $x \leq f(x)$, $x \leq f(u)$. Thus, $u \leq f(u)$. Therefore, $f(u) \leq f(f(u))$. This means $f(u) \in \{x \in \mathscr{L} \mid x \leq f(x)\}$. We can conclude $f(u) \leq u$ and therefore, $f(u) = u$. Thus, u is the greatest fixed point.

Applying a similar construction to the dual lattice $\mathscr{L}^{\mathrm{op}}$, we obtain the least fixed point

$$l := \bigwedge\{x \in \mathscr{L} \mid f(x) \leq x\}. \tag{2}$$

□

This proof, although simple and straightforward, uses an impredicative definition, because we assume u is already contained in the set $\{x \in \mathscr{L} \mid x \leq f(x)\}$. We discuss the philosophical issues which arise by using an impredicative definition to define validity in Section 7.1.

We have induction principles on the least and greatest fixed points of f.

Lemma 1. *Let $P \subseteq \mathscr{L}$. Assume that $x \in P$ implies $f(x) \in P$. Further, if $(x_i)_{i\in I}$ are elements of P, $\bigwedge_{i\in I} x_i \in P$. Then, $u \in P$. Similarly, assume $\bigvee_{i\in I} x_i \in P$ if $(x_i)_{i\in I}$ are elements of P. Then, $l \in P$.*

Proof. Let $L := \{y \geq u \mid y \in P\}$. By assumption, $u_P := \bigwedge L \in P$. Because $f(u_P)$ satisfies P and $u \leq f(u_P)$, $f(u_P) \in L$. Thus $u_P \leq f(u_P)$. Therefore, $u_P \leq u$. By definition, $u \leq u_P$. Therefore, $u = u_P \in P$. The case for the least fixed point is proved similarly. □

3. Bilateral language and its formal systems

In this section, we introduce the *bilateral language* (abbreviated as BL) and the general notion of *formal systems* for BL.

Bilateral language is a language based on the idea that in classical logic, statements can have two linguistic forces, not only affirmation but also, denial. The system is most famously proposed by Rumfitt [18], but similar ideas appear in the other literature [12, 23]. In this paper, we call each representation of a linguistic act a *statement*, and its content a *proposition*. Formally, we define

Definition 1 (Proposition, Statement)**.** *Atomic propositions are denoted by symbols $p, q, p_1, \ldots$. Propositions $A, B, A_1, \ldots$ are defined by*

$$A := p \mid A \wedge A \mid A \vee A \mid \neg A \mid A \to A. \tag{3}$$

Statements $\alpha, \beta, \alpha_1, \ldots$ are defined by

$$\alpha := +A \mid -A. \tag{4}$$

In addition, a special symbol $\perp$ appears in derivations. $\perp$ should be understood as a punctuation symbol, not a statement.

Definition 2. *$+p$ and $-p$ for an atomic proposition a are called* atomic statements*. For a statement α, its* conjugate *α^* is defined as*

$$(+A)^* \equiv -A \qquad\qquad (-A)^* \equiv +A \tag{5}$$

Formal systems are arbitrary sets of *second-order* inference rules, that is, inferences which potentially discharge some assumptions.

Definition 3 ((Inference) rule)**.** *For a set of statements Γ and a statement α, $[\Gamma : \alpha]$ is called a* premise*. For a list of premises P and a statement α, $P \rhd \alpha$ is a* (second-order) rule*. We identify a rule $[\Gamma_0 : \alpha_0], \ldots, [\Gamma_n : \alpha_n] \rhd \alpha$ with the natural deduction rule below.*

$$\dfrac{\begin{matrix}[\Gamma_0] \\ \vdots \\ \alpha_0\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_n] \\ \vdots \\ \alpha_n\end{matrix}}{\alpha} \tag{6}$$

All statements in $\Gamma_0, \ldots, \Gamma_n$ are discharged. A rule composed only of atomic statements is called atomic. *A rule that does not discharge any assumption is called* first-order. *A rule without any premises is called* zeroth-order *or an* axiom.

For a set Γ of statements, we define the axiomatic system $\mathbf{Ax}(\Gamma)$ as $\{\rhd\alpha \mid \alpha \in \Gamma\}$. If a system consists of only atomic axioms, it is called an *atomic system*. In the following, we use $\mathbf{B}, \mathbf{B}'$ with indices as variables for atomic systems.

We consider the following set BCL of rules, which is divided into *logical rules* $\mathbf{L}$ and *coordination rules* $\mathbf{C}$, as a "natural" set of rules and show that all "valid" rules can be reduced to them in some sense. BCL is called *bilateral classical logic*.

Definition 4 (Logical rules)**.**

$$(7)\qquad \frac{+A \quad +B}{+A\wedge B}\ +\wedge I \qquad \frac{+A\wedge B}{+A}\ +\wedge E1 \qquad \frac{+A\wedge B}{+B}\ +\wedge E2$$

$$(8)\qquad \frac{-A}{-A\wedge B}\ -\wedge I1 \qquad \frac{-B}{-A\wedge B}\ -\wedge I2 \qquad \frac{-A\wedge B \quad \begin{array}{c}[-A]\\ \vdots \\ \alpha\end{array} \quad \begin{array}{c}[-B]\\ \vdots \\ \alpha\end{array}}{\alpha}\ -\wedge E$$

$$(9)\qquad \frac{+A}{+A\vee B}\ +\vee I1 \qquad \frac{+B}{+A\vee B}\ +\vee I2 \qquad \frac{+A\vee B \quad \begin{array}{c}[+A]\\ \vdots \\ \alpha\end{array} \quad \begin{array}{c}[+B]\\ \vdots \\ \alpha\end{array}}{\alpha}\ +\vee E$$

$$(10)\qquad \frac{-A \quad -B}{-A\vee B}\ -\vee I \qquad \frac{-A\vee B}{-A}\ -\vee E1 \qquad \frac{-A\vee B}{-B}\ -\vee E2$$

$$(11)\qquad \frac{-A}{+\neg A}\ +\neg I \qquad \frac{+\neg A}{-A}\ +\neg E$$

$$(12)\qquad \frac{+A}{-\neg A}\ -\neg I \qquad \frac{-\neg A}{+A}\ -\neg E$$

$$(13)\qquad \frac{\begin{array}{c}[+A]\\ \vdots \\ +B\end{array}}{+A\to B}\ +\to I \qquad \frac{+A\to B \quad +A}{+B}\ +\to E$$

$$(14)\qquad \frac{+A \quad -B}{-A\to B}\ -\to I \qquad \frac{-A\to B}{+A}\ -\to E1 \qquad \frac{-A\to B}{-B}\ -\to E1$$

Definition 5 (Coordination rules)**.**

$$(15)\qquad \frac{+A \quad -A}{\bot}\ \bot \qquad\qquad \frac{\begin{array}{c}[\alpha]\\ \vdots \\ \bot\end{array}}{\alpha^*}\ \mathrm{RAA}$$

In the rules $+\to I$ and RAA, the assumption α is discharged and no longer open.

Definition 6 (Argument skeleton)**.** *The set of* argument skeletons *of system* $\mathbf{S}$ *with* open assumptions Γ *and a conclusion* α *is defined inductively as follows.*

(1) *Each statement α is an argument skeleton with an open assumption α.*
(2) *Let s_i be argument skeletons with open assumptions Γ_i and a conclusion α_i for $i = 1, \ldots, n$. Let r be a rule $[\Delta_1 : \alpha_1], \ldots, [\Delta_1 : \alpha_n] \rhd \alpha$ of* $\mathbf{S}$*. If $\Gamma_i \backslash \Delta_i \subseteq \Sigma_i, i =$*

$1,\dots,n$, $r(s_1,\dots,s_n)$ *is an argument skeleton with open assumptions* $\Sigma_1,\dots,\Sigma_n$ *and a conclusion* α.

We identify an argument skeleton with a natural deduction style derivation. If s is an argument skeleton with open assumptions Γ *and a conclusion* α*, we write* $\Gamma \vdash_s \alpha$*. If s belongs to a system* $\mathbf{S}$, $\Gamma \vdash_{s,\mathbf{S}} \alpha$*. If there is an argument skeleton of a system* $\mathbf{S}$ *with open assumptions* Γ *and a conclusion* α*, we write* $\Gamma \vdash_{\mathbf{S}} \alpha$*. If an argument skeleton s has no open assumption, we call s closed. A* Subskeleton *of an argument skeleton s is an argument skeleton which is contained in s.*

Definition 7 (Supplementation)**.** *A supplementation function is a function that takes an arbitrary statement* α *and returns an argument skeleton with the conclusion* α*. For an supplementation function f and an argument skeleton s,* $s[f]$ *is an argument skeleton formed by replacing open assumptions* α *to an argument skeleton* $f(\alpha)$. $s[f]$ *is called an* supplementation *of s. A set* $\mathbf{D}$ *of argument-skeletons is supplementation-closed iff for all* $s \in \mathbf{D}$*, all the supplementations of s are in* $\mathbf{D}$. Composition $f * g$ *of supplementations f and g is defined as the map from statements* α *to* $g(\alpha)[f]$.

Let s_i *be an argument skeleton with a conclusion* α_i *for each* $i = 1,\dots,n$*. Let s be an argument skeleton with a conclusion* α *and open assumptions* $\alpha_1,\dots,\alpha_n$*. Then, the argument skeleton* s' *by substituting* $s_1,\dots,s_n$ *into* $\alpha_1,\dots,\alpha_n$ *in s is defined as* $s[f]$ *where f is the supplementation function which maps* α_i *to* s_i.

Lemma 2. $\Gamma \vdash_{\mathbf{S}} \alpha \iff \vdash_{\mathbf{S}\cup\mathbf{Ax}(\Gamma)} \alpha$

Proof. ($\Longrightarrow$) Let s be an argument skeleton from Γ to α of system $\mathbf{S}$. Clearly, s is also an argument skeleton of system $\mathbf{S}\cup\mathbf{Ax}(\Gamma)$. Let f be the supplementation function which maps each $\alpha_i \in \Gamma$ to the axiom which derives α_i. Then, $s[f]$ be the argument skeleton of the system $\mathbf{S}\cup\mathbf{Ax}(\Gamma)$ which derives α. ($\Longleftarrow$) Suppose that the right hand holds. Then, for some argument skeleton s of system $\mathbf{S}\cup\mathbf{Ax}(\Gamma)$, $\vdash_s \alpha$ holds. If we replace all the application in s of the rules in $\mathbf{Ax}(\Gamma)$ with open assumptions of the same statement, we get argument skeleton s' of the system $\mathbf{S}$ and $\Gamma \vdash_{s'} \alpha$. □

Lemma 3. $\Gamma \vdash_{\mathbf{S}} \alpha \iff$ *for any* $\mathbf{S}' \supseteq \mathbf{S}$ *s.t.* $\vdash_{\mathbf{S}'} \Gamma$, $\vdash_{\mathbf{S}'} \alpha$

Proof. ($\Longrightarrow$) Let s be an argument skeleton from Γ to α of system $\mathbf{S}$. Clearly, s is also an argument skeleton of system $\mathbf{S}'$. Because $\vdash_{\mathbf{S}'} \Gamma$, for each $\alpha_i \in \Gamma$, there is an argument skeleton s_i of $\mathbf{S}'$ which derives α_i. Let f be the supplementation function which maps α_i to s_i. Then, $s[f]$ is an argument skeleton of system $\mathbf{S}'$ which derives α. ($\Longleftarrow$) Let $S' := S \cup \mathbf{Ax}(\Gamma)$. Then, from the assumption $\vdash_{\mathbf{S}'} \alpha$ holds. By Lemma 2, $\Gamma \vdash_S \alpha$ holds. □

Definition 8 (Conversion, Justification, Argument)**.** *A* conversion *is defined as a function c from a supplementation-closed set* $\mathbf{D}$ *of argument-skeletons to argument skeletons such that c preserves conclusions, does not make any new open assumption and for all* $s \in \mathbf{D}$ *and all f,* $c(s)[f] = c(s[f])$*. A set of conversions j is called a* justification*. A pair* $a = \langle s, j\rangle$ *of an argument skeleton and a justification is called an* argument*. If s is closed, the argument is also called* closed.

Definition 9 (Substitution of arguments into open assumptions)**.** *Let* $a = \langle s_0, j_0\rangle$ *be an argument. Assume that s has open assumptions* $\alpha_1,\dots,\alpha_n$*. We also say that* $\alpha_1,\dots,\alpha_n$ *are open assumptions of a by abusing a notation. Let* $a_1 = \langle s_1, j_1\rangle,\dots,a_n = \langle s_n, j_n\rangle$ *be arguments of* $\alpha_1,\dots,\alpha_n$ *respectively. Then* substitution *of* $a_1,\dots,a_n$ *to* $\alpha_1,\dots,\alpha_n$ *in a is defined as an argument* $\langle s, j_0 \cup \dots \cup j_n\rangle$ *where s is an argument skeleton which obtained by substituting* $s_1,\dots,s_n$ *into* $\alpha_1,\dots,\alpha_n$ *in s.*

Definition 10 (Reduction)**.** *The set of conversion j induces a* reduction relation $\rightsquigarrow$.

(1) *s 1-step reduces to s' by c, denoted by $s \rightsquigarrow^1_c s'$, if s' results from s by transforming a subskeleton in s by c.*
(2) *s 1-step reduces to s' by j, denoted by $s \rightsquigarrow^1_j s'$, if for some $c \in j$ s 1-step reduces to s' by c.*
(3) *s reduces to s' by j, denoted by $s \rightsquigarrow_j s'$, if $s \rightsquigarrow^1_j s_0 \rightsquigarrow^1_j s_1 \cdots \rightsquigarrow^1_j s'$.*
(4) *For $a = \langle s, j\rangle, a' = \langle s', j'\rangle$, a reduces to a', denoted by $a \rightsquigarrow_j a'$, if $s \rightsquigarrow_j s'$ and $j' \supseteq j$*
(5) $\overline{\mathbf{A}} = \{\langle s, j\rangle \mid$ *for some a' in $\mathbf{A}$, $s \rightsquigarrow_j a'$*$\}$

If a no longer 1-step reduces, a is called a normal form*. If a can be reduced to a normal form, a is called* weakly normalizable*.*

We make argument skeletons of BCL arguments by the conversion defined below. The idea is to reduce every introduction-elimination pairs in the main branch of an inference, counting RAA as also an introduction rule and the law of contradiction as an elimination rule. Therefore, in addition to usual introduction/elimination pairs, RAA/contradiction pairs, RAA/$\wedge,\vee,\to,\neg$-elimination pairs and $\wedge,\vee,\to,\neg$-introduction/contradiction pairs are reduced.

(16) $$\frac{\dfrac{+A \quad +B}{+A\wedge B}}{+A} \Longrightarrow +A \qquad\qquad \frac{\dfrac{+A \quad +B}{+A\wedge B}}{+B} \Longrightarrow +B$$

(17) $$\frac{\dfrac{-A}{-A\wedge B} \quad \begin{matrix}[-A]\\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}[-B]\\ \vdots \\ \alpha\end{matrix}}{\alpha} \Longrightarrow \begin{matrix}-A\\ \vdots \\ \alpha\end{matrix} \qquad \frac{\dfrac{-B}{-A\wedge B} \quad \begin{matrix}[-A]\\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}[-B]\\ \vdots \\ \alpha\end{matrix}}{\alpha} \Longrightarrow \begin{matrix}-B\\ \vdots \\ \alpha\end{matrix}$$

(18) $$\frac{\dfrac{+A}{+A\vee B} \quad \begin{matrix}[+A]\\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}[+B]\\ \vdots \\ \alpha\end{matrix}}{\alpha} \Longrightarrow \begin{matrix}+A\\ \vdots \\ \alpha\end{matrix} \qquad \frac{\dfrac{+B}{+A\vee B} \quad \begin{matrix}[+A]\\ \vdots \\ \alpha\end{matrix} \quad \begin{matrix}[+B]\\ \vdots \\ \alpha\end{matrix}}{\alpha} \Longrightarrow \begin{matrix}+B\\ \vdots \\ \alpha\end{matrix}$$

(19) $$\frac{\dfrac{-A \quad -B}{-A\vee B}}{-A} \Longrightarrow -A \qquad\qquad \frac{\dfrac{-A \quad -B}{-A\vee B}}{-B} \Longrightarrow -B$$

(20) $$\frac{\dfrac{+A}{-\neg A}}{+A} \Longrightarrow +A$$

(21) $$\frac{\dfrac{-A}{+\neg A}}{-A} \Longrightarrow +A$$

(22) $$\frac{\dfrac{\begin{matrix}[+A]\\ \vdots \\ +B\end{matrix}}{+A\to B} \quad \begin{matrix}\vdots \\ +A\end{matrix}}{+B} \Longrightarrow \begin{matrix}\vdots \\ +A\\ \vdots \\ +B\end{matrix}$$

(23) $$\frac{\dfrac{+A \quad -B}{-A\to B}}{+A} \Longrightarrow +A \qquad\qquad \frac{\dfrac{+A \quad -B}{-A\to B}}{-B} \Longrightarrow -B$$

(24) $$\dfrac{\dfrac{+A \quad +B}{+A\wedge B} \qquad \dfrac{-A}{-A\wedge B}}{\bot} \quad\Longrightarrow\quad \dfrac{+A \quad -A}{\bot} \quad \dfrac{\dfrac{+A \quad +B}{+A\wedge B} \qquad \dfrac{-B}{-A\wedge B}}{\bot} \quad\Longrightarrow\quad \dfrac{+B \quad -B}{\bot}$$

(25) $$\dfrac{\dfrac{+A}{+A\vee B} \qquad \dfrac{-A \quad -B}{-A\vee B}}{\bot} \quad\Longrightarrow\quad \dfrac{+A \quad -A}{\bot} \quad \dfrac{\dfrac{+B}{+A\vee B} \qquad \dfrac{-A \quad -B}{-A\vee B}}{\bot} \quad\Longrightarrow\quad \dfrac{+B \quad -B}{\bot}$$

(26) $$\dfrac{\dfrac{\begin{array}{c}[+A]\\ \vdots\\ +B\end{array}}{+A\to B} \qquad \dfrac{\begin{array}{c}\vdots\\ +A\end{array} \quad -B}{-A\to B}}{\bot} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\vdots\\ +A\\ \vdots\\ +B\end{array} \quad -B}{\bot}$$

(27) $$\dfrac{\dfrac{\begin{array}{c}[\alpha]\\ \vdots\\ \bot\end{array}}{\alpha^*} \qquad \begin{array}{c}\vdots\\ \alpha\end{array}}{\bot} \quad\Longrightarrow\quad \begin{array}{c}\vdots\\ \alpha\\ \vdots\\ \bot\end{array}$$

(28) $$\dfrac{\dfrac{\begin{array}{c}[-A\wedge B]\\ \vdots\\ \bot\end{array}}{+A\wedge B}}{+A} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{[-A]}{-A\wedge B}\\ \vdots\\ \bot\end{array}}{+A} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}[-A\wedge B]\\ \vdots\\ \bot\end{array}}{+A\wedge B}}{+B} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{[-B]}{-A\wedge B}\\ \vdots\\ \bot\end{array}}{+B}$$

(29) $$\dfrac{\dfrac{\begin{array}{c}[+A\wedge B]\\ \vdots\\ \bot\end{array}}{-A\wedge B} \qquad \begin{array}{c}[-A]\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}[-B]\\ \vdots\\ \alpha\end{array}}{\alpha} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[-A]\\ \vdots\\ \alpha\end{array} \quad [\alpha^*]}{\bot}}{+A} \qquad \dfrac{\dfrac{\begin{array}{c}[-B]\\ \vdots\\ \alpha\end{array} \quad [\alpha^*]}{\bot}}{+B}}{+A\wedge B}\\ \vdots\\ \bot\end{array}}{\alpha}$$

(30) $$\dfrac{\dfrac{\begin{array}{c}[-A\vee B]\\ \vdots\\ \bot\end{array}}{+A\vee B} \qquad \begin{array}{c}[+A]\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}[+B]\\ \vdots\\ \alpha\end{array}}{\alpha} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[+A]\\ \vdots\\ \alpha\end{array} \quad [\alpha^*]}{\bot}}{-A} \qquad \dfrac{\dfrac{\begin{array}{c}[+B]\\ \vdots\\ \alpha\end{array} \quad [\alpha^*]}{\bot}}{-B}}{-A\vee B}\\ \vdots\\ \bot\end{array}}{\alpha}$$

(31) $$\dfrac{\dfrac{\begin{array}{c}[+A\vee B]\\ \vdots\\ \bot\end{array}}{-A\vee B}}{-A} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{[+A]}{+A\vee B}\\ \vdots\\ \bot\end{array}}{-A} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}[+A\vee B]\\ \vdots\\ \bot\end{array}}{-A\vee B}}{-B} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{[+B]}{+A\vee B}\\ \vdots\\ \bot\end{array}}{-B}$$

$$\cfrac{\cfrac{\begin{array}{c}[+\neg A]\\ \vdots\\ \bot\end{array}}{-\neg A}}{+A} \quad\Longrightarrow\quad \cfrac{\begin{array}{c}\cfrac{[-A]}{+\neg A}\\ \vdots\\ \bot\end{array}}{+A} \tag{32}$$

$$\cfrac{\cfrac{\begin{array}{c}[-\neg A]\\ \vdots\\ \bot\end{array}}{+\neg A}}{-A} \quad\Longrightarrow\quad \cfrac{\begin{array}{c}\cfrac{[+A]}{-\neg A}\\ \vdots\\ \bot\end{array}}{-A} \tag{33}$$

$$\cfrac{\cfrac{\begin{array}{c}[-A \to B]\\ \vdots\\ \bot\end{array}}{+A \to B} \qquad \begin{array}{c}\vdots\\ +A\end{array}}{+B} \quad\Longrightarrow\quad \cfrac{\begin{array}{c}\cfrac{\begin{array}{c}\vdots\\ +A\end{array} \quad [-B]}{-A \to B}\\ \vdots\\ \bot\end{array}}{+B} \tag{34}$$

$$\cfrac{\cfrac{\begin{array}{c}[+A \to B]\\ \vdots\\ \bot\end{array}}{-A \to B}}{+A} \quad\Longrightarrow\quad \cfrac{\begin{array}{c}\cfrac{\cfrac{\cfrac{[+A] \quad [-A]}{\bot}}{+B}}{+A \to B}\\ \vdots\\ \bot\end{array}}{+A} \tag{35}$$

$$\cfrac{\cfrac{\begin{array}{c}[+A \to B]\\ \vdots\\ \bot\end{array}}{-A \to B}}{-B} \quad\Longrightarrow\quad \cfrac{\begin{array}{c}\cfrac{[+B]}{+A \to B}\\ \vdots\\ \bot\end{array}}{-B} \tag{36}$$

We call the set of these reduction R.

4. VALIDITY

In this section, we define the notion of canonical validity and validity.

Definition 11 (Semantic Map)**.** *A semantic map $\mathscr{S}$ for α is a function sending an atomic system $\mathbf{B}$ to the set $\mathscr{S}(\mathbf{B})$ of closed weakly-normalizable arguments with the conclusion α. The semantic space $\mathscr{M}(\alpha)$ is the set of all the semantic maps for α. Because the sets of closed weakly-normalizable arguments with conclusion α forms a complete Boolean algebra, $\mathscr{M}(\alpha)$ forms a complete Boolean algebra by the ordering $\mathscr{S} \leq \mathscr{S}'$, where $\mathscr{S} \leq \mathscr{S}'$ holds iff for any atomic system $\mathbf{B}$, $\mathscr{S}(\mathbf{B}) \subseteq \mathscr{S}'(\mathbf{B})$.*

We restrict the semantic space to the set of weakly-normalizable arguments because we consider the argument which never reduces to a normal form as unfounded.

Definition 12 (Semantic Operators)**.** *We define* $\otimes, \iota(), \Rightarrow$ *as operators on semantic maps. Let* $\mathscr{S}_\alpha \in \mathscr{M}(\alpha)$ *and* $\mathscr{S}_\beta \in \mathscr{M}(\beta)$*.* $\langle s, j\rangle \in (\mathscr{S}_\alpha \otimes_\gamma \mathscr{S}_\beta)(\mathbf{B})$ *if* s *has the form*

$$\frac{\begin{array}{cc}\vdots\ \sigma_A & \vdots\ \sigma_B\\ \alpha & \beta\end{array}}{\gamma} \tag{37}$$

for $\langle \sigma_A, j\rangle \in \overline{\mathscr{S}_\alpha(\mathbf{B})}, \langle \sigma_B, j\rangle \in \overline{\mathscr{S}_\beta(\mathbf{B})}$*.* $\langle s, j\rangle \in (\iota_\gamma(\mathscr{S}_\alpha))(\mathbf{B})$ *if* s *has the form*

$$\frac{\begin{array}{c}\vdots\ \sigma_A\\ \alpha\end{array}}{\gamma} \tag{38}$$

for $\langle \sigma_A, j\rangle \in \overline{\mathscr{S}_\alpha(\mathbf{B})}$*.* $\langle s, j\rangle \in (\mathscr{S}_\alpha \Rightarrow_\gamma \mathscr{S}_\beta)(\mathbf{B})$ *if* s *has the form*

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\ \pi\\ \beta\end{array}}{\gamma} \tag{39}$$

and for any atomic extension $\mathbf{B}'$ *of* $\mathbf{B}$*, any extension* j' *of* j*, any* $\langle \sigma, j'\rangle \in \overline{\mathscr{S}_\alpha(\mathbf{B}')}$*, and any supplementation* f *s.t.* $f(\alpha) = \sigma$*,* $\langle s[f], j'\rangle \in \overline{\mathscr{S}_\beta(\mathbf{B}')}$*.*

Definition 13 (Canonical Validity)**.** *The canonical semantic map for* α*,* $[\![\alpha]\!]$*, is defined in the following way.* $[\![\bot]\!](\mathbf{B})$ *is defined as the following set.*

$$[\![\bot]\!](\mathbf{B}) := \{a : \textit{normal form} \mid a = \langle s, j\rangle, \vdash_{s,\mathbf{B}\cup\mathbf{AC}} \bot\} \tag{40}$$

We abbreviate $[\![\alpha]\!] \Rightarrow_{\alpha^*} [\![\bot]\!]$ *as* $[\![\alpha]\!]^*$*.*

Let the operator F_α *is defined as follows.*

$$G_\alpha(\mathscr{S}, f)(\mathbf{B}) = \mathscr{S}(\mathbf{B}) \cup f^*(\mathbf{B}) \tag{41}$$

$$F_\alpha(\mathscr{S}, \mathscr{R}, g) = G_\alpha(\mathscr{S}, G_{\alpha^*}(\mathscr{R}, g)) \tag{42}$$

for semantic maps $\mathscr{S}$ *for* α*,* $\mathscr{R}$ *for* α^**,* f *for* α^* *and* g *for* α*. Then,* F_α *is a monotone operator respect to* g*. Therefore, the operator* $F_\alpha(\mathscr{S}, \mathscr{R}, -)$ *has the least fixed point by Theorem 1. Let denote the least fixed point of* $F_\alpha(\mathscr{S}, \mathscr{R}, -)$ *by* $L_\alpha(\mathscr{S}, \mathscr{R})$*.*

For atomic propositions p*, let*

$$\mathscr{S}(\mathbf{B}) = \{a \mid \vdash_{a,\mathbf{B}} +p\} \tag{43}$$

$$\mathscr{R}(\mathbf{B}) = \{a \mid \vdash_{a,\mathbf{B}} -p\}. \tag{44}$$

$[\![+p]\!]$ *and* $[\![-p]\!]$ *are defined by*

$$[\![+p]\!] := L_{+p}(\mathscr{S}, \mathscr{R}) \tag{45}$$

$$[\![-p]\!] := G_{-p}(\mathscr{R}, [\![+p]\!]). \tag{46}$$

By this definition, $[\![+p]\!]$ *and* $[\![-p]\!]$ *satisfy the following equations.*

$$[\![+p]\!](\mathbf{B}) = \{a \mid \vdash_{a,\mathbf{B}} +p\} \cup [\![-p]\!]^*(\mathbf{B}) \tag{47}$$

$$[\![-p]\!](\mathbf{B}) = \{a \mid \vdash_{a,\mathbf{B}} -p\} \cup [\![+p]\!]^*(\mathbf{B}) \tag{48}$$

By similar constructions, we define $[\![+A \wedge B]\!]$*,* $[\![-A \wedge B]\!]$*,* $[\![+A \vee B]\!]$*,* $[\![-A \vee B]\!]$*,* $[\![+\neg A]\!]$*,* $[\![-\neg A]\!]$*,* $[\![+A \to B]\!]$ *and* $[\![-A \to B]\!]$ *which satisfy the following equations.*

$$[\![+A \wedge B]\!](\mathbf{B}) = ([\![+A]\!] \otimes_{+A\wedge B} [\![+B]\!])(\mathbf{B}) \cup [\![-A \wedge B]\!]^*(\mathbf{B}) \tag{49}$$

$$[\![-A \wedge B]\!](\mathbf{B}) = (\iota_{-A\wedge B}([\![-A]\!]))(\mathbf{B}) \cup (\iota_{-A\wedge B}([\![-B]\!]))(\mathbf{B}) \cup [\![+A \wedge B]\!]^*(\mathbf{B}) \tag{50}$$

$$[\![+A\vee B]\!](\mathbf{B}) = (\iota_{+A\vee B}([\![+A]\!]))(\mathbf{B}) \cup (\iota_{+A\vee B}([\![+B]\!]))(\mathbf{B}) \cup [\![-A\vee B]\!]^*(\mathbf{B}) \tag{51}$$

$$[\![-A\vee B]\!](\mathbf{B}) = ([\![-A]\!] \otimes_{-A\vee B} [\![-B]\!])(\mathbf{B}) \cup [\![+A\vee B]\!]^*(\mathbf{B}) \tag{52}$$

$$[\![+\neg A]\!](\mathbf{B}) = (\iota_{+\neg A}([\![-A]\!]))(\mathbf{B}) \cup [\![-\neg A]\!]^*(\mathbf{B}) \tag{53}$$

$$[\![-\neg A]\!](\mathbf{B}) = (\iota_{-\neg A}([\![+A]\!]))(\mathbf{B}) \cup [\![+\neg A]\!]^*(\mathbf{B}) \tag{54}$$

$$[\![+A\to B]\!](\mathbf{B}) = ([\![+A]\!] \Rightarrow_{+A\to B} [\![+B]\!])(\mathbf{B}) \cup [\![-A\to B]\!]^*(\mathbf{B}) \tag{55}$$

$$[\![-A\to B]\!](\mathbf{B}) = ([\![+A]\!] \otimes_{-A\to B} [\![-B]\!])(\mathbf{B}) \cup [\![+A\to B]\!]^*(\mathbf{B}) \tag{56}$$

We call arguments in $[\![\alpha]\!](\mathbf{B})$ canonically $\mathbf{B}$*-valid arguments of* α*.*

Definition 14 (Validity)**.** *Validity is defined by two steps: first, it is defined for closed arguments and then is extended to all arguments.*

(1) *A closed argument a with the conclusion α is* $\mathbf{B}$*-valid if $a \in \overline{[\![\alpha]\!](\mathbf{B})}$.*
(2) *An argument a with open assumptions $\alpha_1,\dots,\alpha_n$ is* $\mathbf{B}$*-valid if for any atomic extension $\mathbf{B}'$ of $\mathbf{B}$, any extension j' of j, and any valid closed arguments $a_1,\dots,a_n$ of $\alpha_1,\dots,\alpha_n$ respectively, substitution of $a_1,\dots,a_n$ to $\alpha_1,\dots,\alpha_n$ in a is $\mathbf{B}'$-valid.*

If an argument $\langle s, j\rangle$ is $\mathbf{B}$*-valid, we say that s is* $\mathbf{B}$*-valid with j. The argument s is logically valid with j if s is $\emptyset$-valid with j.*

Lemma 4. *If $\mathbf{B}' \supseteq \mathbf{B}$, $j' \supseteq j$, and $\langle s, j\rangle$ is a* $\mathbf{B}$*-valid or canonically* $\mathbf{B}$*-valid argument α, $\langle s, j'\rangle$ is $\mathbf{B}'$-valid or canonically $\mathbf{B}'$-valid argument of α respectively.*

Proof. $F_\alpha(\mathscr{S},\mathscr{R},g)$ is monotone respect to $\mathscr{S},\mathscr{R}$. Therefore, if $\mathscr{S} \subseteq \mathscr{S}'$ and $\mathscr{R} \subseteq \mathscr{R}'$, $F_\alpha(\mathscr{S},\mathscr{R},g) \le F_\alpha(\mathscr{S}',\mathscr{R}',g)$. Thus,

$$F_\alpha(\mathscr{S},\mathscr{R},L_\alpha(\mathscr{S}',\mathscr{R}')) \le F_\alpha(\mathscr{S}',\mathscr{R}',L_\alpha(\mathscr{S}',\mathscr{R}')) \tag{57}$$

$$\le L_\alpha(\mathscr{S}',\mathscr{R}'). \tag{58}$$

By construction (2) of $L_\alpha(\mathscr{S},\mathscr{R})$, $L_\alpha(\mathscr{S},\mathscr{R}) \le L_\alpha(\mathscr{S}',\mathscr{R}')$. Therefore, the monotonicity of canonic validity is proven by induction on α. Monotonicity of validity follows from the monotonicity of canonical validity. □

Lemma 5. *Let $\langle s, j\rangle$ be a* $\mathbf{B}$*-valid argument and f be a supplementation such that for any α in the set Γ of the open assumptions in s, there is a j_α and $\langle f(\alpha), j_\alpha\rangle$ is* $\mathbf{B}$*-valid. Then $\langle s[f], j \cup (\bigcup_{\alpha\in\Gamma} j_\alpha)\rangle$ is also* $\mathbf{B}$*-valid.*

Proof. We call a supplementation f $\mathbf{B}$-appropriate if $f(\alpha_i)$ is a closed $\mathbf{B}$-valid argument for each assumption α_i of domain of f.

Let $j_1 = j \cup (\bigcup_{\alpha\in\Gamma} j_\alpha)$. We need to prove that for any atomic extension $\mathbf{B}'$ of $\mathbf{B}$, any extension of j' of j_1 and any $\mathbf{B}'$-appropriate supplementation function g for $\langle s[f], j'\rangle$, $\langle s[f][g], j'\rangle$ is $\mathbf{B}'$-valid.

First, we prove that the composition $g * f$ is $\mathbf{B}'$-appropriate for $\langle s, j\rangle$. $\langle f(\alpha)[g], j'\rangle$ is a closed $\mathbf{B}'$-valid argument because $\langle f(\alpha), j_\alpha\rangle$ is $\mathbf{B}$-valid for each α and $\mathbf{B}' \supseteq \mathbf{B}$ and $j' \supseteq j_\alpha$. Therefore, $g * f$ is $\mathbf{B}'$-appropriate for $\langle s, j\rangle$.

By assumption, $\langle s[g * f], j'\rangle$ is $\mathbf{B}'$-valid. Therefore, $\langle s[f][g], j'\rangle$ is $\mathbf{B}'$-valid. □

5. Validity of BCL

Definition 15 (Rule Validity)**.** *A rule $r \equiv [\Gamma_1 : \alpha_1],\dots,[\Gamma_n : \alpha_n] \rhd \alpha$ is* $\mathbf{B}$*-valid with j if $\langle r(s_1,\dots,s_n), j'\rangle$ is $\mathbf{B}'$-valid for any $\mathbf{B}' \supseteq \mathbf{B}$, $j' \supseteq j$, and any $\mathbf{B}'$-valid arguments $\langle s_1, j'\rangle,\dots,\langle s_n, j'\rangle$. If a rule is* $\mathbf{B}$*-valid with justification j for any* $\mathbf{B}$ *or equivalently, $\emptyset$-valid with justification j, we call the rule* logically valid *with justification j.*

Theorem 2 (Validity of rules in BCL)**.** *All the rules in BCL are logically valid with the justification* **R**.

Proof. We can assume that each $\langle s_i, j' \rangle$ is closed except the assumptions in Γ_i because otherwise, we supplement the other assumptions by closed logically valid arguments.

We check each rule of BCL one by one. We start from coordination rules.

$$\dfrac{\begin{array}{c}\vdots\, s_1\\ \alpha\end{array} \quad \begin{array}{c}\vdots\, s_2\\ \alpha^*\end{array}}{\bot} \tag{59}$$

First we reduce s_1, s_2 to canonically valid arguments. It suffices to show that there is a reduction which leads to a logically valid argument of $\bot$ using **R**. We prove this by induction on α, using case analysis on the form of s_1 and s_2, ignoring symmetric cases. First, we consider the case in which the inference of s_1 is RAA.

$$\dfrac{\dfrac{\begin{array}{c}[\alpha^*]\\ \vdots\, s_1'\\ \bot\end{array}}{\alpha} \quad \begin{array}{c}\vdots\, s_2\\ \alpha^*\end{array}}{\bot} \tag{60}$$

This reduces

$$\begin{array}{c}\vdots\, s_2\\ \alpha^*\\ \vdots\, s_1'\\ \bot.\end{array} \tag{61}$$

Because s_1 and s_2 are **B**-valid, supplementing s_2 to α^* in s_1 yields a **B**-valid argument. Therefore the argument above is **B**-valid. Now, we assume that the last inferences of s_1 and s_2 are not RAA.

If α is atomic, by definition of canonical validity, s_1 and s_2 can be reduced to **B**-axioms. Therefore, (59) is also a $\mathbf{B} \cup \mathbf{AC}$-argument skeletons. Therefore, (59) is in $[\![\bot]\!](\mathbf{B})$. The proof of the base case of the induction is completed.

If $\alpha \equiv +A \wedge B$, because s_1 and s_2 are **B**-valid, we can transform the argument into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, t_1\\ +A\end{array} \quad \begin{array}{c}\vdots\, t_2\\ +B\end{array}}{+A \wedge B} \qquad \dfrac{\begin{array}{c}\vdots\, t_3\\ -A\end{array}}{-A \wedge B}}{\bot} \tag{62}$$

where t_1, t_2, t_3 are **B**-valid. Then, we reduce the argument into

$$\dfrac{\begin{array}{c}\vdots\, t_1\\ +A\end{array} \quad \begin{array}{c}\vdots\, t_2\\ -A\end{array}}{\bot} \quad . \tag{63}$$

By induction hypothesis, this argument is **B**-valid. The proof is similar for $\alpha \equiv +A \vee B, +A \to B, +\neg A$. For other cases, the proof is symmetric. Therefore, we prove the theorem for the case of the $\bot$ rule.

Next, we consider RAA. Let $\langle s, j\rangle, j \supseteq \mathbf{R}$ be a **B**-valid argument with an assumption α. We must show that

$$\frac{\begin{array}{c}[\alpha]\\ \vdots\; s\\ \bot\end{array}}{\alpha^*} \tag{64}$$

is **B**-valid. By definition, for any sentence α, $[\![\alpha]\!]^*(\mathbf{B}) \subseteq [\![\alpha^*]\!]$. Therefore, (64) is contained in $[\![\alpha^*]\!]$. The theorem is valid for the case of RAA.

Next, we consider all logical rules. The proof is immediate from the definition of canonical validity for the case of the introduction rules. Therefore, we concentrate on the elimination rules.

$$\frac{\begin{array}{c}\vdots\; s_1\\ +A\wedge B\end{array}}{+A} \tag{65}$$

Because s_1 is **B**-valid, it can be reduced to a canonically **B**-valid argument s_1'. Therefore, the whole argument can be reduced to either

$$\frac{\dfrac{\begin{array}{c}\vdots\; t_1\\ +A\end{array} \qquad \begin{array}{c}\vdots\; t_1\\ +B\end{array}}{+A\wedge B}}{+A} \tag{66}$$

which can be reduced to the **B**-valid argument t_1, or

$$\frac{\dfrac{\begin{array}{c}[-A\wedge B]\\ \vdots\; r_1\\ \bot\end{array}}{+A\wedge B}}{+A} \tag{67}$$

which can be reduced to

$$\frac{\begin{array}{c}\dfrac{[-A]}{-A\wedge B}\\ \vdots\; r_1\\ \bot\end{array}}{+A}\quad . \tag{68}$$

For any $\mathbf{B}' \supseteq \mathbf{B}$ and closed $\mathbf{B}'$-valid argument r_2,

$$\frac{\begin{array}{c}\vdots\; r_2\\ -A\end{array}}{-A\wedge B} \tag{69}$$

is (canonically) closed $\mathbf{B}'$-valid. Because r_1 is $\mathbf{B}'$-valid,

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\; r_2\\ -A\end{array}}{-A\wedge B}\\ \vdots\; r_1\\ \bot\end{array} \tag{70}$$

is $\mathbf{B}'$-valid. This implies

$$\dfrac{[-A]}{-A \wedge B} \atop {\vdots\, r_1 \atop \bot} \tag{71}$$

is $\mathbf{B}$-valid. By the definition of canonical validity,

$$\dfrac{\dfrac{[-A]}{-A \wedge B} \atop {\vdots\, r_1 \atop \bot}}{+A} \ . \tag{72}$$

is (canonically) $\mathbf{B}$-valid. The proof for the other elimination rules is similar.

□

Theorem 3 (Validity of BCL). *For any argument skeleton s in BCL and justification $\mathbf{R}$, $\langle s, R\rangle$ is logically-valid.*

Proof. By induction on s using Theorem 2. □

6. Completeness

This section proves the reverse of Theorem 2.

Theorem 4 (Rule Completeness (Prawitz's Conjecture)). *All rules logically valid with some j are derived from the rules of BCL.*

Let ϕ be a proposition. We say ϕ is $\mathbf{B}$-valid if there is a $\mathbf{B}$-valid argument of ϕ. Specifically, if $\mathbf{B} = \emptyset$, we say ϕ is logically valid. Assume the rule R has a form

$$\dfrac{\begin{matrix} [\Gamma_1] & & [\Gamma_m] \\ \vdots & & \vdots \\ \beta_1 & \cdots & \beta_m \end{matrix}}{\alpha} \tag{73}$$

where $\Gamma_i = \gamma_1, \dots, \gamma_{k_i}$. We modify R to the rule R' in which all α, $\beta_1, \dots, \beta_m$, $\Gamma_1, \dots, \Gamma_m$ have the positive sign by replacing statements δ with the negative sign by $+\neg\delta^*$. Negation rule derives R from R' and vice versa. Therefore, we assume that all statements which appear in the rule have a positive sign and identify the statements with its proposition. Let

$$\rho_{R'} \equiv \{(\gamma_1 \wedge \cdots \wedge \gamma_{k_1}) \to \beta_1 \wedge \cdots \wedge (\gamma_m \wedge \cdots \wedge \gamma_{k_m}) \to \beta_m\} \to \alpha. \tag{74}$$

It is easy to see that R' is derived using $\rho_{R'}$ by BCL and vice versa. Further, because BCL is classically complete, $\rho_{R'}$ can be a conjunction normal form. Let $\mathscr{V}(\rho_{R'})$ be the set of all assignments of classical truth values in all atomic propositions in $\rho_{R'}$. For each $v \in \mathscr{V}(\rho_{R'})$, $\mathbf{B}(v)$ is defined as

$$\mathbf{B}(v) := \{+p \mid v(p) = \text{true}\} \cup \{-p \mid v(p) = \text{false}\} \tag{75}$$

Lemma 6. *If $A \wedge B$ is $\mathbf{B}$-valid, A and B are $\mathbf{B}$-valid and vice versa.*

Proof. Using BCL, we can derive A and B from $A \wedge B$. Also, using BCL, we can derive $A \wedge B$ from A land B. □

Lemma 7. *Assume either* $[\![+A]\!](\mathbf{B})$ *or* $[\![-A]\!](\mathbf{B})$ *is non-empty. Similarly, Assume either* $[\![+B]\!](\mathbf{B})$ *or* $[\![-B]\!](\mathbf{B})$ *is non-empty. Then, either* $[\![+A\vee B]\!](\mathbf{B})$ *or* $[\![-A\vee B]\!](\mathbf{B})$ *is non-empty.*

Proof. If one of $[\![+A]\!](\mathbf{B}), [\![+B]\!](\mathbf{B})$ is non-empty, $[\![+A\vee B]\!](\mathbf{B})$ is non-empty. Otherwise, $[\![-A]\!](\mathbf{B})$ and $[\![-B]\!](\mathbf{B})$ are non-empty, therefore $[\![-A\vee B]\!](\mathbf{B})$. □

Lemma 8. *Assume either* $[\![+A]\!](\mathbf{B})$ *or* $[\![-A]\!](\mathbf{B})$ *is non-empty. Similarly, Assume either* $[\![+B]\!](\mathbf{B})$ *or* $[\![-B]\!](\mathbf{B})$ *is non-empty. If* $[\![+A\vee B]\!](\mathbf{B})$ *is non-empty, either* $[\![+A]\!](\mathbf{B})$ *or* $[\![+B]\!](\mathbf{B})$ *is non-empty.*

Proof. Assume that $[\![+A]\!](\mathbf{B})$ and $[\![+B]\!](\mathbf{B})$ are empty. Then, $[\![-A]\!](\mathbf{B})$ and $[\![-B]\!](\mathbf{B})$ are non-empty. Therefore, $[\![-A\vee B]\!](\mathbf{B})$ is non-empty. Therefore, $[\![\bot]\!](\mathbf{B})$ is non-empty. This contradicts the assumption that $[\![+A]\!](\mathbf{B})$ and $[\![+B]\!](\mathbf{B})$ are empty because if $\bot$ is canonically valid, any sentence is canonically valid. □

Corollary 1. *Let* $v \in \mathscr{V}(\rho_{R'})$ *and* $q_1,\dots,q_n$ *are either one of atomic sentences* $+p$ *in* $\rho_{R'}$ *or their negation* $+\neg p$*. If* $q_1 \vee \cdots \vee q_n$ *is* $\mathbf{B}(v)$*-valid, one* q_i *of* $q_1,\dots,q_n$ *is true by* v*.*

Proof. Repeatedly applying Lemmas 7 and 8, we can show that q_i, one of $q_1,\dots,q_n$ is $\mathbf{B}(v)$-valid. By definition, this means that q_i is true by v. □

Proof of Theorem 4. By Lemma 6 and Corollary 1, $\rho_{R'}$ is classically valid. Therefore, $\rho_{R'}$ can be derived by BCL. □

7. Discussion

7.1. **impredicativity.** Our semantics uses a fixed-point construction justified by an impredicative definition. An impredicative definition is often considered non-constructive. Therefore our semantics may not persuade the validity of classical logic if one takes a constructive point of view. However, the purpose of our semantics is to explain how classical logic functions, not to persuade someone who does not believe impredicative definitions. Although we appeal to an impredicative definition, we do not use the principle of excluded middle. Therefore our semantics is not trivial.

Dummett emphasizes that logic in a meta-theory can influence logic in the object language [6, pp.54-55]. Interestingly, our definition of validity and soundness proof of classical logic show that mathematics in a meta-theory also influences logic in the object-level.

7.2. **Axiomatic Base System.** Although Dummett's and Prawitz's work uses systems of first-order rules as base systems, we restrict base systems to be axiomatic. This restriction is mostly for technical simplification, but we claim that axiomatic base systems can capture important fragments of our inferential practice. For example, atomic statements of Peano arithmetic is decidable, verified by simple calculation. Therefore, the inferential practice of atomic arithmetic can be expressed by the set of verifiable atomic sentences and its decision procedure. Another possible example is the discourse about sense-data. It could be argued that sense-data sentences are verified just by sensation, not by inference. Axiomatic systems would express the proofs of sense-data atomic sentences. Our semantics justifies classical reasoning on these domains. Extension of our semantics to base systems with first-order or second-order rules is future work.

7.3. **Decidability.** Dummett motivates inferential semantics with the manifestation requirement. He argues that knowledge embodied by semantics should be publicly manifestable. His example of unmanifestable knowledge is understanding of bivalent truth-conditions for undecidable sentences. Dummett argues in favor of inferential semantics because, in inferential semantics, speakers can manifest their semantic understanding with discriminating valid proofs from invalid ones [5].

A possible problem in our approach is that, in our semantics, semantic knowledge may be beyond manifestation. In proof-theoretic semantics, the meaning of sentences is determined by their canonical proofs. It follows that semantic knowledge is knowledge of canonical proofs. However, there seems to be a tension between manifestability of knowledge of canonical proofs and our use of fixed point construction. Fixed point construction has infinite nature, and some people would think that makes our definition of canonical proofs unmanifestable.

However, we are not committed to the manifestation requirement because there are other motivations than the manifestation requirement for proof-theoretic semantics. Proof-theoretic semantics are use-theoretic, have light metaphysical commitment, and fit well with broader philosophical pictures like anti-representationalism or pragmatism[2, pp.1-44].

Even if we accept the manifestation requirement, and if we understand the manifestation requirement as the requirement to have an effective procedure to judge the speaker's semantic knowledge, all known global approach in proof-theoretic semantics does not yet provide such an algorithm. Our definition, like other definitions of validity in the literature, is along the style of Prawitz's and Dummett's. In Prawitz and Dummett's definition, canonical validity of an argument depends on the validity of the immediate sub-argument. Definition of validity is basically that $\mathbf{B}$-valid arguments have some reduction sequence that ends with a $\mathbf{B}'$-canonically valid argument, for any extended base system $\mathbf{B}'$ and any supplementation for hypotheses with $\mathbf{B}'$-valid arguments. That definition has quantifiers over infinite domains: reduction sequences, base systems, and supplementations. Therefore, it is not straightforward to give an effective way to judge whether a given argument is valid or not.

A version of Prawitz's approach does not have an effective decision procedure of validity. Consider a Turing machine T and its states $s_0, s_1, \ldots$. For a state of s, Rs means that the computation from s terminates. We consider the following system $\mathbf{S}(T)$. First, the base system $\mathbf{B}$ of $\mathbf{S}(T)$ consists of the single formula $R\omega$, where ω is the terminal state and a rule $Rs_1 \vdash Rs_2$ if s_1 is the next state of s_2. We use the usual introduction rules of $\to$. In addition to the usual elimination rule, we introduce the following"elimination" rule.

$$\frac{Rs' \to Rs}{Rs} \tag{76}$$

where s' is the next state of s. We introduce the reduction rule

$$\frac{\begin{matrix} Rs' \\ \vdots \\ Rs \end{matrix}}{\dfrac{Rs' \to Rs}{Rs}} \rightsquigarrow \begin{matrix} \dfrac{\dfrac{\dfrac{Rs''}{Rs'}}{Rs'' \to Rs'}}{Rs'} \\ \vdots \\ Rs \end{matrix} \tag{77}$$

where s'' is the next state of s'. Then, the validity of (76) is undecidable, because we do not know that the reduction sequence leads to $R\omega$.

The undecidability of (76) is nothing to do with classical logic because we only use intuitionistic axioms for implication. Therefore, most of Prawitz's style proof-theoretic semantics share undecidability of the validity.

7.4. **Motivation to use bilateral formulation.** Bilateral formulation of classical logic is required to reconcile the meaning conferring status of RAA and the inductive nature of our semantics. In the proof of soundness for classical logic, it is vital that RAA is a self-justifying and meaning-conferring rule in our semantics. In the unilateral formulation, RAA cannot take such a status. With only affirmation (+), RAA itself cannot even be formulated. The closest translation would be $[+\neg A : \bot] \rhd +A$ and $[+A : \bot] \rhd +\neg A$. The former rule, if seen as a meaning-conferring rule, stipulates that $[\![+A]\!](\mathbf{B})$ contains the elements of $([\![+\neg A]\!] \Rightarrow_{+A} [\![\bot]\!])(\mathbf{B})$. The set of canonical arguments for $+A$ is defined with dependence on the set of canonical arguments for $+\neg A$. Then the induction along logical complexity of formulas is impossible. Bilateralism makes it possible to formulate RAA for the conclusion A with just one sentence A ($\bot$ is a punctuation mark). Therefore, our classical validity can be defined by an inductive definition.

8. CONCLUSION AND FUTURE WORKS

This paper gives a solution to a problem in Rumfitt's work of the bilateral classical logic, which cannot justify all inferences. In particular, the validity of the rule RAA is left unexplained. To explain the validity of RAA, we introduce a fixed-point construction to the definition of the validity and argue that fixed-point construction is acceptable for proof-theoretic semantics. Then, we prove that the bilateral classical logic is sound and complete respect to our notion of validity. Finally, we discuss the philosophical significance of impredicativity, restriction to axiomatic base systems undecidability of our notion of validity.

Our future work includes extensions of our semantics to richer classical systems like classical predicate logic, classical modal logic, classical higher-order logic, and classical arithmetic. Besides, fixed-point constructions may be useful to define the validity of systems with inter-dependent constants.

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