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BACKGROUND NOTIONS IN LATTICE THEORY AND GENERALIZED QUANTIFIERS*

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Many papers in this volume build on certain elementary notions of lattice theory and generalized quantifier theory; often, their empirical predictions derive directly from them. The goal of this chapter is to enable readers who have some background in formal semantics, but not in these particular areas, to appreciate the pertinent papers. But readers who are familiar with lattices and GQs may also find the discussion useful because, elementary as it is, it highlights certain aspects that other literature may not. On the other hand, precisely because this chapter is geared towards particular applications, it does not attempt to cover issues that a standard introduction would, when they do not seem directly relevant here.

The chapter consists of three parts. The first part familiarizes the reader with the relevant notions and their significance. The second is a set of problems. Some of them merely check the mastery of definitions, others touch on linguistic issues that are of theoretical relevance to the contents of this volume. The third part offers quite elaborate solutions. The gentle reader who is not in a problem solving mood is encouraged to read the problems and their solutions as if they were part of the main text.

*This chapter is based on my lecture notes for classes given at UCLA, the University of Budapest, and the 1993 LSA Linguistic Institute. I thank the participants of these courses for feedback. Work on the present version was partially supported by NSF grant #9222501.

A. Szabolcsi (ed.), Ways of Scope Taking, 1–27.

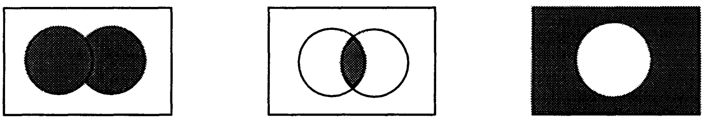
1 OPERATIONS IN PARTIALLY ORDERED SETS

1.1 Partially ordered sets: lattices, semi-lattices, Boolean algebras

Recall the basic set theoretical operations and their counterparts in the propositional calculus:

(1)

union: $A \cup B$	intersection: $A \cap B$	complement: $-A$
disjunction: $p \vee q$	conjunction: $p \wedge q$	negation: $\neg p$



What other operations are these related to? On what kind of entities can such operations be performed? What kind of structures do these entities form? These are the main questions we are going to ask.

The basic distinction to build on is between ordered and unordered sets. An unordered set is any set in the standard sense, e.g.,

(2) Unordered sets:

$$\begin{aligned} A &= \{\text{joe, ed, pat, sue}\} \\ B &= \{\emptyset, \{\text{joe}\}, \{\text{ed}\}, \{\text{pat}\}, \{\text{joe, ed}\}, \{\text{joe, pat}\}, \\ &\quad \{\text{ed, pat}\}, \{\text{joe, ed, pat}\}\} \\ C &= \{\text{joe, ed, pat, joe-and-ed, joe-and-pat, ed-and-pat,} \\ &\quad \text{joe-and-ed-and-pat}\} \end{aligned}$$

Sets become ordered if we explicitly assume some ordering relation on their members (whether or not there is a "natural ordering" that suggests itself anyway), e.g.,

(3) Ordered sets:[1]

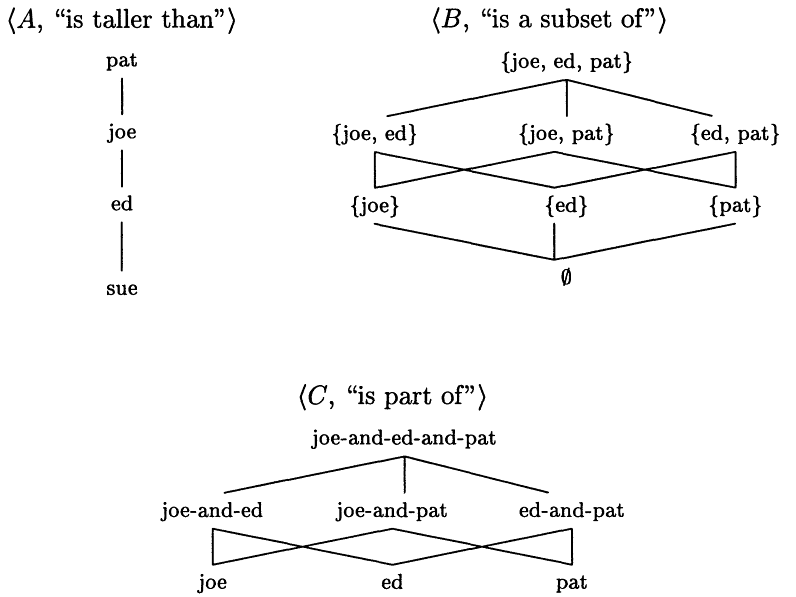
$\langle A$, "is taller than"$\rangle$	or	$\langle A$, "is likelier to cry than"$\rangle$
$\langle B$, "is a subset of"$\rangle$	or	$\langle B$, "has fewer elements than"$\rangle$
$\langle C$, "is part of"$\rangle$	or	$\langle C$, "is less happy than"$\rangle$

Clearly, different relations may order the same set differently. E.g., Joe may be taller than Ed (hence Joe $\geq_1$ Ed) but less likely to cry (Ed $\geq_2$ Joe). Or, {joe} is

[1] The non-atomic elements of C are called *collectives,* or *plural individuals,* or *sums.*

not a subset of {ed, pat} or vice versa (these two elements are not ordered with respect to each other by $\geq_3$) but has fewer elements ({ed, pat} $\geq_4$ {joe}). The two orderings may coincide, e.g., Joe is part of the collective Joe-and-Ed (Joe-and-Ed $\geq_5$ Joe) and may also be less happy on his own (Joe-and-Ed $\geq_6$ Joe). The ordering may be specified graphically, as in the Hasse-diagrams below. All lines can be read as upward arrows that point to the element ordered higher.

(4)



Two kinds of linguistic applications may be as follows. The elements of the set A are ordered with respect to an "extrinsic" property (in fact, these individuals cannot be ordered otherwise). Such an ordering may be invoked in the discussion of words like *even* (*Even Sue can reach this shelf* may be felicitous, because Sue herself is short relative to the others we are interested in). The elements of B and C can be ordered with respect to "intrinsic" properties such as "subset" and "part-of" as well as "extrinsic" ones. In this volume all linguistic applications happen to be of the "intrinsic" sort.

We now turn to more precise definitions. (Recall that R is reflexive iff $\forall x[Rxx]$, R is transitive iff $\forall xyz[(Rxy \;\&\; Ryz) \rightarrow Rxz]$, and R is anti-symmetrical iff $\forall xy[(Rxy \;\&\; Ryx) \rightarrow x = y]$.)

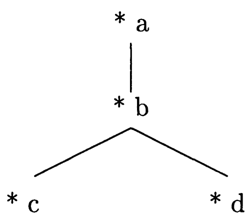
(5) A relation R is a *partial ordering* iff it is reflexive, transitive, and anti-symmetrical. A *partially ordered set* (partial order, or poset, for short) is any $\langle A, \leq \rangle$, where $\leq$ is a partial order.

The relations "larger than or equal to" and "subset of" are partial orderings. The relations "larger than" and "proper subset of" are strict orderings: they are not anti-symmetrical but asymmetrical.

How do we get to the desired operations from here, cf. (1)? They are definable in terms of partial ordering. The general lattice-theoretic names they come under are meet, join, and complement. Intersection is the realization of meet when applied to sets, and conjunction is meet when applied to propositions. Similarly, union is join for sets and disjunction is join for propositions; negation is complement for propositions.

(6) Let $\langle A, \leq \rangle$ be a poset. For any subset X of A,
a is a lower bound for X if for every element x of X, $a \leq x$.
The infimum of X, written $\bigwedge X$, is the *greatest lower bound* for X.
c is an upper bound for X if for every element x of X, $c \geq x$.
The supremum of X, written $\bigvee X$, is the *least upper bound* for X.

The lower bounds of the set X are elements of A (within X or outside X) which are smaller than or at best equal to all elements of X; the infimum is the greatest of these. Similarly for the least upper bound (supremum). E.g.,



The set of lower bounds for $\{a, b\}$ is $\{b, c, d\}$, of which b is the greatest.

(7) Let $a, b \in A$.

a. The *meet* of a and b, written $a \wedge b$, is the infimum of the 2-element set $\{a, b\}$.
Thus we have: $a \wedge b \leq a$ and $a \wedge b \leq b$.

b. The *join* of a and b, written $a \vee b$, is the supremum of the 2-element set $\{a, b\}$.
Thus we have: $a \vee b \geq a$ and $a \vee b \geq b$.

Meet is a special case of infimum: it is the infimum of some two-element set. Similarly for join and supremum.

Depending on what operations are available in a specific partially ordered set, we may have a Boolean algebra, a lattice, a meet or join semi-lattice, or none of these. "Available" means that the given poset is closed under that operation: whenever meet or join is applied to two elements of A, the result is also an element of A (the same for complement, which applies to one element). That is, these operations do not "lead out of" A.

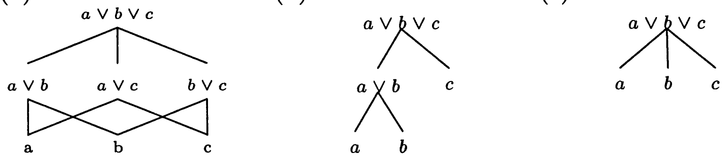
(8) A *lattice* defined in terms of partial ordering:
A lattice is a poset $\langle A, \leq \rangle$ which is closed under meet and join.
That is, for every $a, b \in A$, $\quad a \wedge b \in A \quad$ and $\quad a \vee b \in A$.
It follows that A is a lattice iff for any non-empty finite subset X of A,
$\bigwedge X \in A \quad$ and $\quad \bigvee X \in A$.

E.g., both $\langle A$, "taller than"$\rangle$ and $\langle B$, "subset of"$\rangle$ are lattices. $\langle C$, "part of"$\rangle$ is not: it does not have meet.

Lattices (as well as semi-lattices and Boolean algebras) can be equivalently defined in algebraic terms. E.g. a lattice is an algebra $\langle A, \wedge, \vee \rangle$, where $\wedge$ and $\vee$ are two-place operations satisfying idempotency, commutativity, associativity, and absorption. This otherwise important fact does not concern us, so it will not be dwelt on further.

(9) A *join semi-lattice* is the "upper half" of a lattice:
a poset $\langle A, \leq \rangle$ where for every $a, b \in A$, $\quad a \vee b \in A$.

(10) (a) (b) (c)



All three structures in (10) are join semilattices. (10a) is said to be "free," which means that whenever two distinct pairs of elements can possibly have distinct joins, they do have distinct joins. E.g., $\{a, b\}$ and $\{a, c\}$ have distinct joins; $\{a \vee b, a \vee c\}$ and $\{a \vee b, b \vee c\}$ do not have distinct joins, but they could not possibly have, either.

(11) A *meet semi-lattice* is the "lower half" of a lattice:
a poset $\langle A, \leq \rangle$ where for every $a, b \in A$, $\quad a \wedge b \in A$.

Mathematically, meet semilattices and join semilattices are the same thing, only the relation is inverted. Linguistically, it may be interesting to note that while there are many applications for join semilattices, I do not know of applications of meet semilattices. For instance, observe that (4c) is the same as (10a). The *and* that occurs in the definition of collectives is a join, not a meet.

(12) A lattice is *bounded* if it has a bottom element 0 and a top element 1.
For any a, $a \wedge 0 = 0$ and $a \wedge 1 = a$

For instance, the lattice in (4a) is bounded but the lattice of natural numbers is not, since it has no top (greatest) element.

(13) A *Boolean algebra* is a poset $\langle A, \leq \rangle$ which is closed under meet, join and (unique!) complement, where
$a \in A$ is a *complement* of $b \in A$ iff $a \wedge b = 0$ and $a \vee b = 1$.

For any set S, its powerset is the domain of a Boolean algebra. $\langle B$, "subset of"$\rangle$ is an example: B is the powerset of {joe, ed, pat} .

You may now want to check Problems (58) and (59).

What properties entail what others? Can a structure turn out to be closed under more operations than we stipulated? Yes! For many applications this does not matter: all we are interested in is that a certain operation is available. But if we claim that some linguistic phenomenon is explained by the fact that a certain operation is unavailable, matters like the following need to be paid close attention to.

(14) A lattice is *complete* iff for any (not just finite) subset X of A,
$\bigwedge X \in A$ and $\bigvee X \in A$.

Some facts: Every complete lattice is bounded (= has both 1 and 0). Every finite lattice is complete and bounded. Infinite lattices need not be complete or bounded.

(15) A join semi-lattice A is *complete* iff for any subset B, the supremum of B is in A.

(16) A join semi-lattice A is *complete#* iff for any non-empty subset B, the supremum of B is in A.

E.g., (10a) is complete#; if we add a bottom element, it becomes complete.

Some facts: Every complete join semi-lattice is a lattice; it is even a complete lattice, hence bounded. Not every complete# join semi-lattice is a lattice. Similarly, not every finite join semi-lattice is a lattice. See Problem (60).

1.2 Quantifiers and negation in Boolean terms

Finally, let us highlight the connection between the three Boolean operations and quantifiers. It is well-known that universal quantification reduces to conjunction, and existential quantification to disjunction over the elements of a finite universe. If the universe of discourse E is $\{a, b, c\}$, i.e. it contains Andy, Belinda, and Carl, then *Everyone walks* is the same as *Andy walks, Belinda walks, and Carl walks*; and *Someone walks* is the same as *Either Andy or Belinda or Carl walks.* That is,

(17) $\exists x[fx] = fa \vee fb \vee fc$

(18) $\forall x[fx] = fa \wedge fb \wedge fc$

Similarly for numerical quantifiers, negative quantifiers, and negation:

(19) $\exists_2 x[fx] = (fa \wedge fb) \vee (fa \wedge fc) \vee (fb \wedge fc)$

(20) $\neg\exists x[fx] = \neg(fa \vee fb \vee fc)$

(21) $\neg fa = a \in (E - \{x : fx\})$

Consider now the case when another quantifier is to take scope over the above, as in *Someone/everyone/no one read three books* on its object wide scope reading, for instance. An intermediate step is to define the property of being read by someone/everyone/no one. In present terms this can be spelled out as follows:

(22) $\{y : \exists x[r(x, y)]\} =$
$\{y : r(a, y) \vee r(b, y) \vee r(c, y)\} =$
$\{y : r(a, y)\} \cup \{y : r(b, y)\} \cup \{y : r(c, y)\}$

(23) $\{y : \forall x[r(x, y)]\} =$
$\{y : r(a, y) \wedge r(b, y) \wedge r(c, y)\} =$
$\{y : r(a, y)\} \cap \{y : r(b, y)\} \cap \{y : r(c, y)\}$

(24) $\{y : \neg\exists x[r(x, y)]\} =$
$E - \{y : r(a, y) \vee r(b, y) \vee r(c, y)\} =$
$E - \{y : r(a, y)\} \cup \{y : r(b, y)\} \cup \{y : r(c, y)\}$

That is, the narrow scope quantifier is cashed out in terms of the operations that define it.

2 GENERALIZED QUANTIFIERS

2.1 The elements of a GQ

Montague introduced generalized quantifiers into his grammar of English in order to be able to assign a uniform denotation to all noun phrases, whether they refer to single individuals or not. Going beyond this, GQ theory provides the tools for studying various semantic properties of quantifiers.

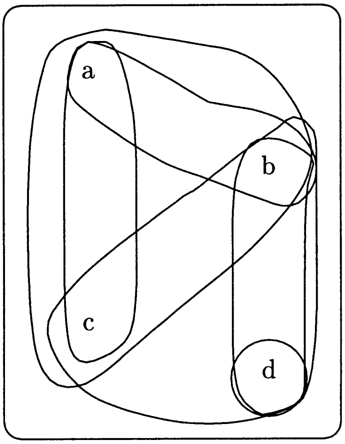
A generalized quantifier (henceforth, GQ) is not a syntactic object (an expression); it is a semantic object (something that expressions can denote). Specifically, a GQ is a set of properties, and noun phrases are claimed to denote such sets of properties. It is important to note that "property" is understood as nothing else than a set of individuals. E.g., if John, Bill, and Mary constitute the set of walkers, the property of walking is just {john, bill, mary}. In this sense, a GQ is a set of sets-of-individuals.

For instance, *every man* denotes the set of properties that every man has. The property of walking is in this set iff every man walks. Let us connect this to various terminologies and notations that are in use. In Montagovian terms, the denotation of *every man* is written as $\lambda P \forall x[\text{man}(x) \rightarrow P(x)]$. Here P is a variable of type $\langle e,t \rangle$: a variable over subsets of the universe. $\forall x[\ldots]$ is of type t. Hence the whole λ-expression is a function of type $\langle\langle e,t \rangle, t\rangle$. $\lambda P \forall x[\text{man}(x) \rightarrow P(x)]$ is the (characteristic function of the) set of properties every man has. Other ways of writing the same thing are $\lambda P[\text{MAN} \subseteq P]$ or $\{P : \text{MAN} \subseteq P\}$.

(At least) two men denotes the set of properties at least two men have, written as $\lambda P \exists x \exists y[x \neq y \;\&\; \text{man}(x) \;\&\; \text{man}(y) \;\&\; P(x) \;\&\; P(y)]$. Other ways of writing the same thing are: $\lambda P[|\text{MAN} \cap P| \geq 2]$ or $\{P : |\text{MAN} \cap P| \geq 2\}$.

Since GQs are sets (of sets of individuals), they have elements. For instance, *Every man walks* is true iff the set of walkers is an element of $[\![$*every man*$]\!]$, the GQ denoted by *every man.* When we are interested in (defining the conditions for) the truth of particular sentences, those sets that have "names" (that is, are denoted by the predicates in the sentence) are specifically interesting to us. However, when we are studying the GQs themselves, we are interested in all their elements and the structures they form. Hence no set is more interesting than the others. It is important to get into the habit of trading mnemonic names like *walk* for the corresponding sets and asking questions in the following form, "Is {john, bill, mary} an element of the quantifier denoted by *every man*?" (Yes, if the set of men is a subset of {john, bill, mary}.) For instance, consider a universe $E = \{a, b, c, d\}$ and some of its subsets (this example will be recycled in (41)):

(25) $\{a, b, c\} = \text{man}$ $\{d\} = \text{dog}$ $\{b, c, d\} = \text{jump}$
$\{a, b, c, d\} = \text{fat}$ $\{a, b\} = \text{run}$ $\{b, d\} = \text{laugh}$



On the other hand, the sets of all elements of a few quantifiers are as follows:

(26) $[\![\textit{at least two men}]\!] = \{P : |\text{man} \cap P| \geq 2\}$
$= \{\{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$

(27) $[\![\textit{every man}]\!] = \{P : \text{man} \subseteq P\} = \{\{a, b, c\}, \{a, b, c, d\}\}$

(28) $[\![\textit{no man}]\!] = \{P : \text{man} \cap P = \emptyset\} = \{\{d\}, \emptyset\}$

(29) $[\![\textit{andy and carl}]\!] = \{P : P(a) \ \& \ P(c)\} = \{P : \{a, c\} \subseteq P\}$
$= \{\{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$

In the previous section it was noted that quantifiers are reducible to Boolean operations. In GQ-theoretic terms (18) may be rephrased as follows. We have a universe with three humans, a, b, and c. $[\![\textit{Everyone}]\!]$, the set of properties everyone has, can be obtained by intersecting the sets of properties the individual humans have:

(30) $[\![\textit{everyone}]\!] = \{P : P(a)\} \cap \{P : P(b)\} \cap \{P : P(c)\}$

And similarly for the other quantifiers.

2.2 Determiners (DETS)

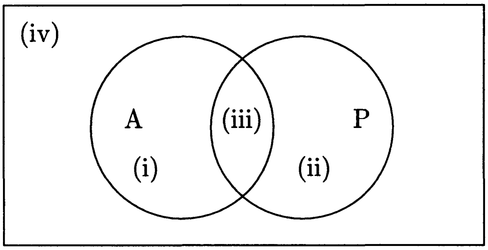
GQ theory does not concern itself only with GQs. It also deals with the denotations of determiners and with the denotations of noun phrases that are not exactly GQs (e.g. *himself*). Here we will not be directly concerned with determiners, but below is a small portion of necessary information.

In Montagovian terms, the denotation of *every* is written as $\lambda A \lambda P \forall x[A(x) \rightarrow P(x)]$. Here A is a variable of type $\langle e,t \rangle$, $\lambda P \forall x[\ldots]$ is of type $\langle\langle e,t\rangle,t\rangle$, hence the whole thing is of type $\langle\langle e,t\rangle,\langle\langle e,t\rangle,t\rangle\rangle$.

$\lambda A \lambda P \forall x[A(x) \rightarrow P(x)]$ is a function from properties to GQs or, equivalently, a relation between properties (A's and P's). Other ways of writing the same thing are, $\lambda A \lambda P[A \subseteq P]$ or $\{\langle A,P\rangle : A \subseteq P\}$.

Or, *two* denotes $\lambda A \lambda P \exists x \exists y[x \neq y \ \& \ A(x) \ \& \ A(y) \ \& \ P(x) \ \& \ P(y)]$. Other ways of writing the same thing: $\lambda A \lambda P[|A \cap P| \geq 2]$ or $\{\langle A,P\rangle : |A \cap P| \geq 2\}$.

Now consider the diagram below. It has four areas: (i) the individuals that have property A but not P, (ii) the individuals that have P but not A, (iii) the individuals that have both A and P, and (iv) the individuals that have neither A nor P.



Consider a sentence of the form DET$(A)(P)$. Do we need to check all four areas when we wish to determine whether it is true or false? It is an important empirical claim concerning natural language determiners (at least "simple" or "normal" ones) that they do not require the checking of all four areas. The following is a small but representative sample. The solidus in (c) indicates a fraction, and *n, m,* and *k* are natural numbers.

(31)	a.	At least two men walk.	$\lvert A \cap P \rvert \geq 2$
	b.	Every man walks.	$A \subseteq P$
	c.	Few men walk.	$\lvert A \cap P \rvert / \lvert A \rvert \leq n/m$ or $\lvert A \cap P \rvert \leq k$
	d.	No men walk.	$\lvert A \cap P \rvert = 0$

As the reader can easily check, none of these requires us to consider area (iv): their truth does not depend on non-walking non-men. We need not know anything beyond the properties explicitly mentioned: how big the surrounding universe is and what is going on in it are immaterial. A more surprising but equally intuitive fact is that none of these sentences requires us to check area (ii): their truth does not depend on walkers who are not men. On the other hand, (31b) and the first reading of (31c) require us to check (i): their truth is dependent on men who are not walkers. The irrelevance of areas (iv) and (ii) means that the two sets A and P do not play equal roles. The set A, the denotation of the noun that the determiner directly combines with, serves to restrict the universe to the largest part that can be possibly relevant: it serves as the determiner's restrictor. Natural language determiners are (overwhelmingly) restricted in this sense. Finally, observe that area (iii) is useful: (31a), (31d) and the second reading of (31c) require us to check only this.

Below are the definitions of the pertinent properties of determiners:

(32) DET has extension iff for any two universes E and E' where $A, B \subseteq E$ and $A, B \subseteq E'$, we have $D_E(A)(B) = D'_E(A)(B)$.

(33) DET is conservative iff $\text{DET}(A)(P) = \text{DET}(A)(A \cap P)$.

(34) DET is intersective iff $\text{DET}(A)(P) = \text{DET}(A \cap P)(P)$.

(35) DET is proportional iff $\text{DET}(A)(P)$ depends on $(A \cap P)/A$.

(36) DET is symmetrical iff $\text{DET}(A)(P) = \text{DET}(P)(A)$.

Two facts: A proportional DET cannot be symmetrical. If DET is conservative, symmetrical = intersective. See Problem (61).

Now back to GQs.

2.3 Live-on sets and witness sets

Conservativity is a property of determiners. Together with extension, it identifies DET's first argument as a restrictor set. A comparable notion for GQs is that of a live-on set.

(37) *Live-on*: A GQ lives on a set of individuals A if, for any set of individuals X,

$$X \in GQ \quad \text{iff} \quad (X \cap A) \in GQ.$$

(37) says that when a GQ lives on some set A, it makes no difference whether we check if a set X is an element of that GQ, or we check whether the intersection of X with A is an element of it; that is, we may safely restrict our attention to the smaller set $X \cap A$. What are a GQ's live-on sets? A linguistic way to check this is to instantiate the schema, as follows:

(38) More/fewer than two men run

$\leftrightarrow$ More/fewer than two men are *men* who run
$\leftrightarrow$ More/fewer than two men are *humans* who run
$\leftrightarrow$ More/fewer than two men are *existents* who run
but: $\not\leftrightarrow$ More/fewer than two men are *Frenchmen* who run

So ⟦*more than two men*⟧ and ⟦*fewer than two men*⟧ live on the set of men and its supersets. In general, the restrictor of the determiner is always a live-on set of the corresponding generalized quantifier. See Problem (62).

If we are interested in live-on sets as domains that we need not look beyond when checking the truth of a sentence, we do not need all of them: the smallest suffices and is thus the most efficient.[2]

In the above cases the restrictor set of DET is identical to the smallest set the GQ denoted by DET(A) lives on: A itself. So, do we need the notion of a smallest live-on set on top of the set with respect to which DET is conservative? The answer is Yes.

First, there are noun phrases that are not made up of a determiner and a noun, e.g., *John* and *John and Mary*. Here the question of what the determiner's restrictor is cannot arise. But the GQs that these noun phrases denote have run-of-the-mill smallest live-on sets: {john} and {john, mary}, respectively.

Second, the smallest live-on sets of some GQs are smaller than the restrictor sets of the corresponding determiners. Imagine a world in which the men are {john, bill, tim} and we are pointing at John and Bill:

(39) These two men run $\leftrightarrow$ These two men are *either John or Bill* and run

So, ⟦*these two men*⟧ lives on the set consisting of those two men who we are pointing at, which is smaller than the set of men. (This amounts to saying that *these two men* is interpreted as 'the two men I am pointing at'. Note though that while this interpretation is semantically justified, a syntactic analysis that mimics such a decomposition would not be.)

These discrepancies are understandable. Conservativity (with extension) may be regarded as a property of the syntax/semantics interface. It says that the syntactic unit that a determiner (or other two-place operator) directly combines with plays the semantic role of a restrictor, i.e. imposes a parallelism between syntax and interpretation. Live-on sets on the other hand are defined purely from the denotation of the noun phrase, without reference to its syntax and without requiring a direct syntactic correlate.

You may now want to tackle Problems (63)–(64).

With the notion of a smallest live-on set at hand, we may take another look at the elements of a GQ. What all the elements of a GQ are is characteristic of it; the individual elements themselves need not be. Take, for instance, the

[2] E. Keenan (p.c.) notes that the notion of a smallest live-on set is unproblematic as long as the universe is finite or at least our GQ does not crucially rely on infinity. But e.g. the intersection of the sets which ⟦*all but finitely many stars*⟧ lives on in an infinite universe is itself not a live-on set.

elements of ⟦*two men*⟧. They are those sets in the universe that contain two men—but note that they may as well contain tigers, stars, and forks.

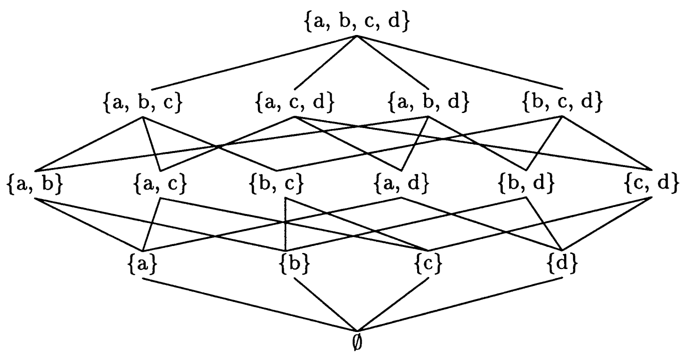
If we ask ourselves what sets the noun phrase "talks about", the elements of its GQ do not make a revealing choice. A natural alternative is to throw out the irrelevant beasts by restricting our attention to those elements that are also in the smallest live-on set:

(40) A set W is a *witness* of a GQ iff $W \in GQ$ and $W \subseteq SL(GQ)$, where $SL(GQ)$ is the smallest set the GQ lives on.

To compare elements and witnesses, we may consider a reincarnation of the four-element universe in (25). (41) is the Boolean algebra corresponding to its powerset. Its use is insightful, because it contains all subsets of the universe, not only those that have "mnemonic names"; and since it is partially ordered by the subset relation, it allows us to make inferences by simply going up or down in the diagram.

Recall that in our particular universe, a, b, c are men and d is a dog.

(41)



(42) a. $[\![\textit{more than one man}]\!] = \{\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},$
$\{a,c,d\},\{a,b,d\},\{b,c,d\},\{a,b,c,d\}\}$

b. the witnesses of $[\![\textit{more than one man}]\!] =$
$\{\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$

(43) a. $[\![\textit{fewer than two men}]\!] =$
$\{\{a\},\{b\},\{c\},\{d\},\{a,d\},\{b,d\},\{c,d\},\emptyset\}$

b. the witnesses of $[\![\textit{fewer than two men}]\!] = \{\{a\},\{b\},\{c\},\emptyset\}$

See Problems (65) through (67).

Some GQs have a unique witness. This may be empty: such is the case with ⟦*no man*⟧: its only witness is the empty set. Or, the unique witness may be non-empty; in this case it coincides with the smallest set that the GQ lives on. GQs with a non-empty unique witness are called principal filters, and their unique witness A the generator set:

(44) A GQ is a *principal filter* iff there is a set of individuals A such that A is not necessarily empty and for any set of individuals X,
$X \in GQ$ iff $A \subseteq X$.

Every man, these$_{\text{deictic}}$ *two men, Andy, Andy and Carl,* etc. all denote principal filters. These always "talk about" the same sets, their generators. In terms of (41):

(45) a. $[\![$*andy and carl*$]\!]$ is a principal filter generated by $\{a, c\}$:
$[\![$*andy and carl*$]\!] = \{P : \{a, c\} \subseteq P\}$

b. the smallest live-on set of $[\![$*andy and carl*$]\!]$ = unique witness of $[\![$*andy and carl*$]\!]$ = generator set of $[\![$*andy and carl*$]\!]$ = $\{a, c\}$

You may now want to think about Problems (68) through (72).

2.4 Monotonicity properties and witnesses

An important property of functions is what monotonicity type they belong to. Suppose the domain of a function f is a partially ordered set with, say, $a \geq b$. If f is upward monotonic, it preserves this ordering in its value: $f(a) \geq f(b)$. If f is downward monotonic, it reverses the ordering: $f(b) \geq f(a)$. If f is non-monotonic, it obliterates the ordering. Since GQs are functions (characteristic functions of sets of properties), their monotonicity can be examined.

(46) GQ is monotone increasing (= upward mon.):
$(A \in GQ \ \& \ A \subseteq B) \Rightarrow B \in GQ$.

(47) GQ is monotone decreasing (= downward mon.):
$(A \in GQ \ \& \ B \subseteq A) \Rightarrow B \in GQ$.

(48) GQ is non-monotone: neither increasing nor decreasing.

Some examples: *John, at least two men, every man* denote increasing GQs. *No men, fewer than six men* denote decreasing GQs. *John and nobody else* and *exactly two men* denote non-monotonic GQs. Here is a linguistic way to show these:

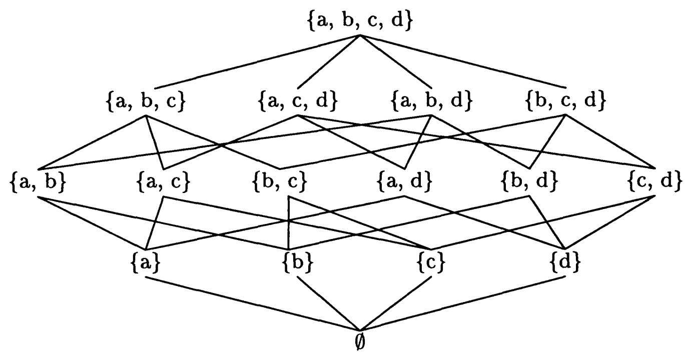
(49) (Every man runs & run ⊆ run or sit) ⇒ Every man runs or sits

(50) (Few men run or sit & run ⊆ run or sit) ⇒ Few men run

(51) (Exactly two men run & run $\subseteq$ run or sit) $\not\Rightarrow$
Exactly two men run or sit
(Exactly two men run or sit & run $\subseteq$ run or sit) $\not\Rightarrow$
Exactly two men run

A more general and also visualizable way to demonstrate monotonicity properties is to use a Boolean algebra as in (41), repeated here:

(41)



The algebra makes it easy to see, for instance, that if some set A is an element of $[\![$*at least two men*$]\!]$, i.e., has at least two men in it, then every set B that is larger than A (is above A in (41)) is also an element of this GQ; and conversely for, say, $[\![$*no man*$]\!]$:

(52) $[\![$*at least two men*$]\!]$ is monotone increasing:

for every A, B, ($A \in [\![$*at least two men*$]\!]$ & $B \supseteq A) \Rightarrow$
$B \in [\![$*at least two men*$]\!]$

(53) $[\![$*no man*$]\!]$ is monotone decreasing:

for every A, B, ($A \in [\![$*no man*$]\!]$ & $A \supseteq B) \Rightarrow B \in [\![$*no man*$]\!]$

See Problem (73).

The best known linguistic application of monotonicity properties has to do with the licensing of negative polarity items. We will be making crucial use of another type of consequence of monotonicity differences.

Let a noun phrase contain a determiner that provides information concerning cardinality, e.g., *two, at least two, more than two, at most two, less than two, exactly two.*

(54) If the GQ denoted by a cardinality-indicating noun phrase is monotone increasing, then

$$\text{DET}(A)(P) = \exists X[X \subseteq A \;\&\; |X| = \text{det-many} \;\&\; X \subseteq P]$$

That is, *At least two men walk* can be equivalently stated as, 'There is a set of individuals whose elements are all men, whose cardinality is at least two, and whose elements all walk.' Or, using witnesses, 'There is a witness of ⟦*at least two men*⟧ and its elements all walk.'

If, on the other hand, the GQ denoted by such a noun phrase is not increasing, then there is NO such equivalence. E.g., *Fewer than two men walk* is NOT equivalent to 'There is a set consisting of less than two men, and these men all walk'. Imagine the situation in which John walks, Bill walks, and Frank walks. The set {john} is surely one that has fewer than two elements, all of which are men and walk—but the existence of such a set does not make the sentence true here. The sentence does not allow us to ignore Bill and Fred, who also walk, but the proposed paraphrase allows us to ignore them. Or, *Exactly two men walk* is NOT equivalent to 'There is a set of individuals whose elements are all men, whose cardinality is exactly two, and whose elements all walk'. Imagine the same situation and pick the set to be {john, frank}, to see why not.

Note why this is so. The crucial property of upward monotonicity is that whatever is true in a small situation (say, one in which just two men walk) remains true when we embed that situation in a bigger one (in which three or more men walk). Neither downward monotonic nor non-monotonic quantifiers have this property, which means that to be safe, we must always look at the biggest possibly relevant situation.

The significance of these simple observations is that in the analysis of linguistic phenomena, one often wishes to associate existentially quantified sets with GQs. Great caution needs to be exercised in these cases. Either the phenomenon we are looking at is factually restricted to increasing GQs, or a maximality condition of some sort must be added to guarantee that no relevant individual gets ignored.

The relation between monotonicity and witnesses can be generally characterized as follows. Let W be a witness, and A the smallest live-on set, of GQ. Then,

(55) If GQ is monotone increasing, then for any X, $X \in GQ$ iff $\exists W[W \subseteq X]$.

E.g., *Two men run* is true iff there is a witness of ⟦*two men*⟧ whose members run.

(56) If GQ is monotone decreasing, then for any X, $X \in GQ$ iff $\exists W[(X \cap A) \subseteq W]$.

E.g., *Few men run* is true iff there is a witness of $[\![$*few men*$]\!]$ which contains all the men who run.

(57) If GQ is non-monotonic, then for any X, $X \in GQ$ iff $\exists W[(X \cap A) = W]$.

E.g., *Exactly two men run* is true iff there is a witness of $[\![$*exactly two men*$]\!]$ which equals all the men who run.

The observation in (54) is a special case of (55). On the other hand, the formulations in (56) and (57) ensure that we are looking at the maximal set: we are not "ignoring" anything. Just as (55) does not hold of decreasing GQs, (56) does not hold of increasing ones. MAN is a W, and the A, for $[\![$*at least two men*$]\!]$. Suppose that only one man runs. (RUN $\cap$ MAN) $\subseteq$ MAN does not entail that RUN $\in$ $[\![$*at least two men*$]\!]$.

Since W is a subset of smallest live-on set A anyway, in (55) we might have used $\exists W[W \subseteq (X \cap A)]$, to bear out the pattern common to the three cases: there exists a witness W that contains, is contained by, or equals $X \cap A$.

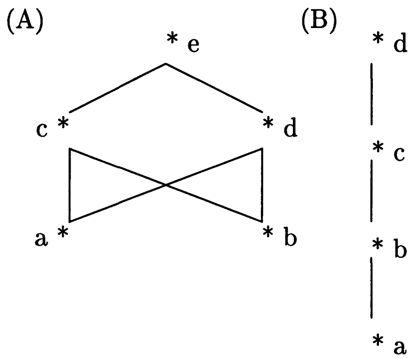
*

Finally, note that Section 1 of the next chapter (Beghelli et al. 1996) may be regarded as an extension of the present one: it is concerned with the use of witness sets in capturing some basic intuitions concerning scope.

3 PROBLEMS

Unmarked problems involve applying the definitions in the text. Those marked with an asterisk may require some creativity.

(58) Is (A) below a semi-lattice? Is (B) a Boolean algebra? Why?



(59) The structure in (10b) does not have elements labelled $b \vee c$ or $a \vee c$. How come it is still a join semi-lattice?

(60) Show that every *complete* semi-lattice A is a lattice. (Hints: Assume A is a meet semi-lattice. What is the $\bigwedge$ of the set of upper bounds of an arbitrary $X \subseteq A$? What is the $\bigvee$ of this set?)

(61) Show that for conservative DETs, symmetry = intersectivity.

(62) Show that $[\![$*more than one man*$]\!]$ and $[\![$*fewer than two men*$]\!]$ do not live on $\{a, b, d\}$.

(63) The textbook example of a "potential determiner" that is not conservative is *only.* Assume that *only men* is a noun phrase in *Only men run.* Demonstrate that *only* is not a conservative "determiner".

(64)* Formalize *Only men run* and *Only John and Bill run* using first order logic, and complete the following: If *only* was interpreted as ..., with restrictor ... and scope ..., it would turn out to be conservative. Are there linguistic arguments supporting this analysis?

(65) Is {John, Bill, Fido} (a) an element, (b) a witness of $[\![$*at least two men*$]\!]$?

(66) What are the witnesses of $[\![$*fewer than four men*$]\!]$ and of $[\![$*few men*$]\!]$ in the Boolean algebra (41)?

(67) The elements of a GQ are too big to be genuinely characteristic of it; the text suggests the use of witnesses. Couldn't we use minimal elements instead? The definition is this: X is a minimal element of GQ iff X is an element of GQ but ceases to be one if we take away even one individual from it. (Hint: What are the minimal elements of the GQs denoted by (i) *at least two men, more than one man, exactly two men,* (ii) *fewer than three men, at most one man, no man*? What are their witnesses?)

(68) (a) Is ⟦*no man*⟧ a principal filter? Why? (b) Is ⟦*every man*⟧ a principal filter? Why?

(69)* What set does *they* in (i) and (ii) refer to? Argue for your proposal with reference to whether they can be continued with *Perhaps there were others who did the same (i.e. both came in and were selling coke).* Formalize your proposal using notions introduced in 2.3.

(i) More than two people came in. They were selling coke.

(ii) At least two people came in. They were selling coke.

(70)* Is there a difference between the behavior of (i)–(ii) in (69) and that of (iii)? Sticking with the machinery of 2.3, come up with an interpretation for *two people* that makes the correct prediction without requiring a new rule, i.e. try to make (iii) a special case of (i)–(ii).

(iii) Two people came in. They were selling coke.

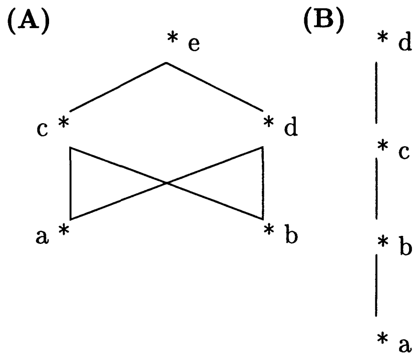
(71)* Compare the following sentences: (i) *A dog/every dog bit two women (you know, my neighbors)* and (ii) *A dog/every dog bit two or more/more than three women.* Do they have both a subject widest scope and an inverse, object wide scope reading? Set up a hypothesis that explains the data.

(72)* Examine now what readings *Every prof assigned more than two readings to three students* and *Every prof assigned three readings to more than two students* have. Do your findings make you change the hypothesis concerning inverse scope that was made in (71)?

(73) Is the smallest live-on set of (a) an increasing, (b) a decreasing, (c) a non-monotonic quantifier an element of that quantifier?

4 SOLUTIONS

(58) Is (A) below a semi-lattice? Is (B) a Boolean algebra? Why?



(a) No. The join is the least upper bound of a two-element set. $\{c, d, e\}$, the set of upper bounds of $\{a, b\}$, has no least element since c and d are equal. So there is no $a \vee b$ in A.

(b) No. B has no complements. Take b, for example. The complement of b ought to be another element of B (not a subset of B!). Now, $b \wedge a = a$ (and a is the bottom element 0) and $b \vee d = d$ (and d is the top element 1), but $a \neq d$. In fact, there is no element of B for which both equations would hold.

(59) The structure in (10b) does not have elements labelled $b \vee c$ or $a \vee c$. How come it is still a join semi-lattice?

Because $a \vee b \vee c$ is the least upper bound for $\{a, c\}$ and $\{b, c\}$: it is an upper bound, and there is no smaller upper bound in the structure. The fact that we could "imagine" a distinct $b \vee c$ does not matter: what matters is what elements the structure actually has.

(60) Show that every *complete* semi-lattice A is a lattice. (Hints: Assume A is a meet semi-lattice. What is the $\bigwedge$ of the set of upper bounds of an arbitrary $X \subseteq A$? What is the $\bigvee$ of this set?)

A being a complete meet semi-lattice means that not only every two-element subset, but any subset, of A has a greatest lower bound in A. What we need to show is that this guarantees that *every* subset also has a least upper bound. What is a least upper bound of $X \subseteq A$? It is the infimum of the set of upper bounds of X:

$$\bigvee X = \bigwedge(UB(X))$$

We can always define the set of upper bounds of X:

$$UB(X) = \{a \in A : \text{every } x \in X,\ a \geq x\}$$

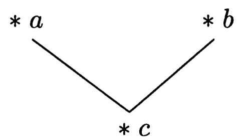
Since this is a subset of A, and A is complete, we know that its infimum is in A. So,

$$\bigwedge(UB(X)) \in A$$

Thus, if every subset has an infimum in A, then every subset has a supremum in A, too. The case of a join semi-lattice does not require new considerations: everything works the same, replacing $\bigwedge$ with $\bigvee$, upper by lower, etc.

This exercise highlights the fact that "greatest" (as in greatest lower bound) is defined using least upper bound, and "least" (as in least upper bound) is defined using greatest lower bound.

To give a concrete example of a meet semi-lattice that is not complete, consider



What subset lacks an infimum here? Well, the empty subset. Its lower bounds are a, b, and c since it is vacuously true of each of these that it is smaller than or equal to all the elements of the empty set. But this set of lower bounds $\{a, b, c\}$ has no greatest element, so the empty subset lacks an infimum. The missing top element would be the supremum of $\{a, b\}$, so its absence also prevents our structure from being a lattice, in accordance with the theorem just proved.

(61) Show that for conservative DETs, symmetry = intersectivity.

Symm: $D(A)(P) = D(P)(A)$
Int: $D(A)(P) = D(A \cap P)(P)$
Cons: $D(A)(P) = D(A)(A \cap P)$

Int ⇒ symm:	$D(A)(P) =$	by cons
	$D(A)(A \cap P) =$	by int
	$D(A \cap (A \cap P))(A \cap P) =$	by def. of $\cap$
	$D(A \cap P)(A \cap P) =$	by def. of $\cap$
	$D(P \cap (P \cap A))(P \cap A) =$	by int "reversed"
	$D(P)(P \cap A) =$	by cons
	$D(P)(A)$	Symm!

Symm $\Rightarrow$ int:

	$D(A)(P) =$	by symm
	$D(P)(A) =$	by cons
	$D(P)(P \cap A) =$	by symm
	$D(P \cap A)(P) =$	by def. of $\cap$
	$D(A \cap P)(P)$	Int!

(62) Show that $[\![$*more than one man*$]\!]$ **and** $[\![$*fewer than two men*$]\!]$ **do not live on** $\{a, b, d\}$**.**

$\{a, b, d\}$ has one of the men, c missing. Now, $\{a, c\}$ is an element of $[\![$*more than one man*$]\!]$, but $\{a, b, d\} \cap \{a, c\} = \{a\}$ is not. And conversely, $\{a, b, d\} \cap \{a, c\} = \{a\}$ is an element of $[\![$*fewer than two men*$]\!]$, although $\{a, c\}$ is not.

(63) The textbook example of a "potential determiner" that is not conservative is *only*. **Assume that** *only men* **is a noun phrase in** *Only men run.* **Demonstrate that** *only* **is not a conservative "determiner."**

Only men (if interpreted as a semantic constituent) is a GQ that does not live on the set of men at all, to wit:

Only men run $\not\Leftrightarrow$ Only men are *men* who run
Only men run $\Leftrightarrow$ Only men are *existents* who run

That *only* is not conservative is not very problematic: we can argue that it is simply not a determiner but a noun phrase modifier.

(64)* Formalize *Only men run* **and** *Only John and Bill run* **using first order logic, and complete the following: If** *only* **was interpreted as ... , with restrictor ... and scope ... , it would turn out to be conservative. Are there linguistic arguments supporting this analysis?**

$\forall x[\text{run}(x) \rightarrow \text{man}(x)]$
$\forall x[\text{run}(x) \rightarrow (x = \text{john} \vee x = \text{bill})]$

If *only* was interpreted as a universal quantifier, with the VP as its restrictor and the subject as its scope, it would turn out to be conservative:

Every runner is a man $\Leftrightarrow$ Every runner is a runner who is a man
Every runner is either John or Bill $\Leftrightarrow$ Every runner is a runner who is either John or Bill

(NB: We are not arguing that *only* is a determiner in syntax; we are arguing that semantically it is a conservative operator.)

This analysis is quite plausible. First, spelling out the contribution of *only* is obviously necessary anyway, and there is no reason why *only* should not be interpreted as 'all.' Second, *only* is a well-known focusing operator, that is, the phrase it combines with is the focus and the rest of the sentence is the focus frame. Since the focus–focus frame partition is assumed to be reflected in the syntax of Logical Form, and since many operators (e.g. adverbs of quantification) are assumed to have the focus frame as their restrictor, the same is natural in connection with *only.* So we are positing the following analogy:

Only MEN run	$\text{every}(_{\text{focus frame}}\{x : \text{run}x\})(_{\text{focus}}\{x : \text{man}x\})$
John always cites MEN	$\text{every}(_{\text{focus frame}}\{x : \text{John cites}x\})(_{\text{focus}}x : \text{man}x\})$

The only objection might be that *Only men run* requires the existence of runners, and the formula $\forall x[\text{run}(x) \rightarrow \text{man}(x)]$ does not. But this does not need to be specified in the meaning of *only*: it is generally assumed that sentences with focus presuppose that the property denoted by the focus frame is not empty. This analysis suggests that conservativity (or generally, domain restriction) may be far more pervasive than generally thought. It may be characteristic of all two-place operators, not only of determiners. This is natural if conservativity (domain restriction) indeed characterizes the syntax/semantics interface. This hypothesis suggests that the syntactic analyses of potential counterexamples should be checked and possibly recast.

(65) Is {John, Bill, Fido} (a) an element, (b) a witness of $[\![$*at least two men*$]\!]$**?**

(a) Yes, because the intersection of {John, Bill, Fido} with MAN has at least two members.

(b) No, because a witness of $[\![$*at least two men*$]\!]$ is an element of it that contains only men, and here we have a dog, too.

(66) What are the witnesses of $[\![$*fewer than four men*$]\!]$ **and of** $[\![$*few men*$]\!]$ **in the Boolean algebra (41)?**

(a) No element of this algebra has more than three men in it, so in this respect all qualify. But we need to discard those that contain d, the dog.

(b) *Few men* may mean either of two things. (i) 'fewer than a set number k'—if we set k as, say, 7, then the witnesses will be those sets that contain six men or less and no dog. This is independent of how many men we have in fact. (ii) 'few of the men'—we may stipulate that, say, 30% or less of the men counts as few of them; since we have 3 men, this will come down to 'at most one man'. So the witnesses are those sets that contain at most one man and no dog.

(67) The elements of a GQ are too big to be genuinely characteristic of it; the text suggests the use of witnesses. Couldn't we use minimal elements instead? The definition is this: X is a minimal element of GQ iff X is an element of GQ but ceases to be one if we take away even one individual from it. (Hint: What are the minimal elements of the GQs denoted by (i) *at least two men, more than one man, exactly two men,* (ii) *fewer than three men, at most one man, no man*? What are their witnesses?)

All the GQs in (i) have minimal elements consisting of exactly two men, and all the GQs in (ii) have the empty set as their unique minimal element. This indicates that the notion of a minimal element does not only eliminate irrelevant individuals but also eradicates "size" distinctions and therefore collapses noun phrases it should not. On the other hand, while witnesses eliminate individuals outside the restrictor, they retain "size" distinctions.

(68) (a) Is $[\![$*no man*$]\!]$ a principal filter? Why? (b) Is $[\![$*every man*$]\!]$ a principal filter? Why?

For GQ to be a principal filter, there must be a non-empty set A such that (i) if some X is an element of GQ, A is a subset of X, and (ii) if A is a subset of some X, X is an element of GQ.

(a) What sets come to mind in connection with $[\![$*no man*$]\!]$? Say, $\emptyset$ (its unique witness) and MAN (its smallest live-on set). $\emptyset$ satisfies (i), because it is a subset of any X, but not (ii), for the same reason (say, $\emptyset \subseteq$ WALK does not entail that no man walks). In addition, the generator set should be non-empty. So try MAN. MAN clearly does not satisfy (ii): MAN $\subseteq$ HUMAN does not entail that no man is a human. Indeed, the fact that the unique witness and the smallest live-on set differ already indicates that $[\![$*no man*$]\!]$ is not a principal filter.

(b) The smallest set $[\![$*every man*$]\!]$ lives on is MAN. The definition of *every* says that for any set X, $X \in [\![$*every man*$]\!]$ iff $X \in \{P : \text{MAN} \subseteq P\}$, and the latter is equivalent to MAN $\subseteq X$. MAN is also the GQ's unique witness. There may be models in which MAN is empty. The definition in the text is "modalized" in order to allow for this: it requires the set the GQ is preoccupied with to be not always empty. This practically allows us to ignore models without men. Alternatively, we might say that *every man* denotes a principal filter in those models where there are men. In any case, we take *every man* to be an uncontroversial principal filter.

(69)* What set does *they* in (i) and (ii) refer to? Argue for your proposal with reference to whether they can be continued with *Perhaps there were others who did the same (i.e. both came in and were*

selling coke). **Formalize your proposal using notions introduced in 2.3.**

(i) More than two people came in. They were selling coke.

(ii) At least two people came in. They were selling coke.

They refers back to all the people who came in. This is confirmed by the fact that the continuation "perhaps there were others ... " is no good: if all are referred to, there cannot be others. Noun phrases like *more than two people* are said to support only "maximal reference anaphora." A formalization may be: $SL(GQ) \cap VP$, where $SL(GQ)$ is the smallest set the subject of the first sentence lives on (here: MAN), and VP is the predicate of the first sentence (here: CAME IN).

(70)* Is there a difference between the behavior of (i)–(ii) in (69) and that of (iii)? Sticking with the machinery of 2.3, come up with an interpretation for ***two people*** **that makes the correct prediction without requiring a new rule, i.e. try to make (iii) a special case of (i)–(ii).**

(iii) Two people came in. They were selling coke.

Here the continuation "perhaps there were others ... " is good, so *they* cannot be referring to all the men who came in. It refers to just the two men the speaker was talking about in the first sentence. Noun phrases like *two people* are said to support "non-maximal reference anaphora." It might be argued that a specific (referential) interpretation of *two people* is what enables this reading; *more than/at least two people* does not seem to have a comparable interpretation.

We may formalize this referential interpretation by saying that *two men* (on this reading) denotes a principal filter; those two men who the speaker has in mind. The smallest live-on set of such a principal filter is smaller than the set of men: it contains just the two relevant individuals. The intersection of this set with VP is just the two-man set. So the formalization in (69) extends to this case and thus the example may speak in favor of a principal filter interpretation of *two men* (among other readings).

(71)* Compare the following sentences: (i) ***A dog/every dog bit two women (you know, my neighbors)*** **and (ii)** ***A dog/every dog bit two or more/more than three women.*** **Do they have both a subject wide scope and an inverse, object wide scope reading? Set up a hypothesis that explains the data.**

(i) can easily have both readings. (ii) does not easily have an object wide scope reading.

In (70) we have seen that specific indefinites may be regarded as denoting principal filters. In (i), the phrase *you know, my neighbors* suggests that we are dealing with a specific indefinite, too. The following descriptive hypothesis suggests itself. When the direct object receives a "specific" interpretation, it takes scope over the subject; when it does not, it cannot. *Two men* can be specific, because on one reading it denotes a principal filter; *two or more/more than three women* have no such readings.

(72)* Examine now what readings ***Every prof assigned more than two readings to three students*** **and** ***Every prof assigned three readings to more than two students*** **have. Do your findings make you change the hypothesis in (71)?**

The trick here is that we now have three quantifiers! The hypothesis in (71) can be checked by asking whether *three N* can take intermediate scope, that is, inverse scope inside the VP and still vary with the subject. Since the hierarchical order of VP-internal complements is a matter of debate, we check two sentences: in at least one of them *three N* must be taking inverse scope if it scopes highest inside the VP. So, are the following readings possible?

every prof > three students > more than two readings

and

every prof > three readings > more than two students

If yes, then on this construal *three students* and *three readings* do not denote principal filters. If they did, their witnesses could not vary with the individual professors. Also, if the given reading is available, then *three N* need not denote a principal filter in order to take inverse scope over the c-commanding *more than two N.*

The judgment seems to be that the critical reading is available. So the hypothesis in (71) is refuted. This example does not refute the assumption that *two men* can denote a principal filter. What it shows is that denoting a principal filter is not necessary for taking inverse scope.

This conclusion makes one want to go back and check if denoting a principal filter is strictly necessary for *two people* to support anaphora as in (iii) above. For instance, the following modified context is useful: *Every policeman reported that the following happened at 6 p.m. in the building he was watching. Two people entered. They were selling coke. Perhaps there were others who did the same ...* Indeed, it seems possible for the pairs to vary with the policemen and still support anaphora in the same way. This indicates that defining the antecedent as $SL(GQ) \cap VP$ is not a sufficiently general solution. (Indeed, a radically different treatment for the type of *two people* is proposed in Discourse Representation Theory.)

(73) Is the smallest live-on set of (a) an increasing, (b) a decreasing, (c) a non-monotonic quantifier an element of that quantifier?

The quantifiers $[\![$*at least one man*$]\!]$, $[\![$*fewer than two men*$]\!]$ and $[\![$*exactly one man*$]\!]$ have the same smallest live-on set: $[\![$*man*$]\!]$. *At least one man is a man* is true, *Fewer than two men are men* is false, and *Exactly one man is a man* is also false in the model (25), since we have three men who are all men. We see that its smallest live-on set may be too big to be an element of a decreasing or a non-monotonic quantifier. In the case of the increasing ones, we can be sure we cannot get into trouble: if, say, $\{a\} \in [\![$*at least one man*$]\!]$, then every superset of $\{a\}$ is.

REFERENCES

Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4(2):159–219.

Beghelli, Filippo, Dorit Ben-Shalom, and Anna Szabolcsi. 1996. Variation, distributivity, and the illusion of branching. In this volume.

Keenan, Edward, and Dag Westerståhl. 1994. Generalized quantifiers in linguistics and logic. In Johan van Benthem and Alice ter Meulen (Eds.), *Handbook of Logic and Linguistics*. Amsterdam: Elsevier. Forthcoming.

Landman, Fred. 1991. *Structures for Semantics*. Dordrecht: Kluwer.