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Defining LFIs and LFUs in extensions of infectious logics

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Abstract

The aim of this paper is to explore the peculiar case of *infectious logics*, a group of systems obtained generalizing the semantic behavior characteristic of the $\{\neg, \wedge, \vee\}$-fragment of the *logics of nonsense*, such as the ones due to Bochvar and Halldén, among others. Here, we extend these logics with classical negations, and we furthermore show that some of these extended systems can be properly regarded as Logics of Formal Inconsistency (LFIs) and Logics of Formal Undeterminedness (LFUs).

1. Introduction

From some time now, work in non-classical logics has brought to the light the possibility of having special operators –which can justifiably be called "recovery" operators– conceived as means to mark pieces of language which can indeed be used to infer classically.[1] In particular, da Costa's work with paraconsistent logics led to the possibility of having consistency operators, understood as means to mark pieces of language which can be used to infer consistently, and is therefore fundamental in this literature.

Later, the project of having an operator that helps recover classical reasoning was implemented beyond the realm of just paraconsistent logics. In (Marcos, 2005) the case of paracomplete logics equipped with a determinedness operator is considered. This task can be understood as being continued by (Martínez, 2007) and more explicitly by (Corbalán, 2012). Also extending this idea to paraconsistent and paracomplete logics, (da Costa & Loparic, 1984), (Omori & Waragai, 2011), (Carnielli & Rodrigues, 2015) and (Carnielli & Rodrigues, 2016) considered paraconsistent and paracomplete logics endowed with a classicality operator.

[1] As a referee points out this project is by no means a recent enterprise. On the contrary, both Bochvar and Halldén (on which more below) thought of themselves as enriching the classical connectives of the *Principia Mathematica* by adding means to mark pieces of language and identify them as well-behaved and non-problematic or non-pathological.

(Carnielli & Marcos, 2002) and (Carnielli, Coniglio, & Marcos, 2007) carried out this project with a brand new perspective. They produced a foundational study of paraconsistent logics that include a *consistency operator* –called hereafter Logics of Formal Inconsistency (LFIs, for short). Furthermore, they classified some paraconsistent logics between those where such an operator is *definable* and those where such an operator is not definable and has, therefore, to be introduced as a *primitive* connective. This division can be applied to logics having an undeterminedness operator –called hereafter Logics of Formal Undeterminedness (LFUs, for short)– and also to logics having both kind of operators –called Logics of Formal Inconsistency and Undeterminedness (LFIUs, for short) or Logics of Formal Classicality (LFCs, for short). In this paper, we will focus on the first two cases, i.e. on LFIs and LFUs.

The aim of the present paper is to *define* consistency and undeterminedness operators –and therefore to define LFIs and LFUs– in extensions of some peculiar non-classical logics, which will be called *infectious logics*.[2] These logics can be understood as a group of systems obtained by generalizing the semantic behavior present in the $\{\neg, \wedge, \vee\}$-fragment of *logics of nonsense*, such as Bochvar's logic from (Bochvar, 1938, 1981) and Halldén's logic from (Halldén, 1949). At the end of the present essay we will suggest that in this paper we built a huge collection of LFIs (and LFUs) that might be orthogonal to the hierarchy construed on top of the weakest LFI investigated so far, i.e. **mbC**.[3]

It is important to clarify which steps we will be following to define LFIs and LFUs in extensions of infectious logics. It is known that in many LFIs and LFUs the consistency or determinedness operators can be implemented to further define a *classical* negation. However, the reverse direction (that is, supplementing non-classical logics with a classical negation, and later defining consistency or undeterminedness operators with its help) is also worth investigating. It is this last path that concerns us here.

In detail, the route we will be following to fulfill the aim of this paper is as follows. In Section 2. we give a few technical preliminaries that will be useful for reading the paper. Then, in Section 3. we will give a precise definition of what infectious logics are, considering afterwards a large collection of infectious logics. Later, in Section 4. we will extend these infectious logics with a classical negation and we will moreover try to implement some definitions available in the literature[4] (or some subtle modifications of them) to define the consistency and undeterminedness operators. Finally, in Section 5. we will assess whether or not these infectious logics extended with classical negation can be genuinely regarded as LFIs and/or LFUs, providing some insight and examples for generating sound and complete proof-systems for them. We conclude in Section 6. mentioning some issues for further research.

2. Preliminaries

Throughout this paper we will handle logics mainly from a semantic point of view.[5] This implies that we will understand logical systems $\mathbf{L}$ as pairs $\langle \mathcal{L}, \vDash \rangle$ of an uninterpreted language $\mathcal{L}$ and a semantic consequence relation $\vDash$ induced, in turn, by a semantic structure $\mathcal{M}$, intended to interpret the language $\mathcal{L}$. As is common practice, we work with propositional languages $\mathcal{L}$ composed of a denumerable set of propositional variables Var and a set of connectives C, usually containing $\neg, \wedge, \vee$, where $\neg$ is a non-classical negation. We denote the set of well-formed formulae of the language $\mathcal{L}$

[2]It is important to remark that these logics have only non-classical connectives. They are, therefore, devoid of any definable or primitive consistency or undeterminedness operator.

[3]For more on the hierarchy of LFIs, see the recent monograph (Carnielli & Coniglio, 2016).

[4]In (da Costa & Arruda, 1976), (da Costa & Alves, 1977) and (da Costa & Loparic, 1984)

[5]Notation and terminology here follows closely the one employed in (Priest, 2008) and (Avron & Zamansky, 2011).

as $\mathsf{Form}_{\mathcal{L}}$, formulae of $\mathcal{L}$ with Greek letters $\alpha, \beta, \varphi, \psi$, etc., and set of formulae of the language with capital Greek letters Γ, Δ, Σ, etc.

We now consider a first approximation to giving a semantic interpretation to a propositional language. This approach will be generalized later, and it is this generalization that will provide us with the broader framework that we will be employing in this essay.

Definition 1. *A matrix for a language $\mathcal{L}$ is a structure $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where*

- *$\mathcal{V}$ is a non-empty set of truth values*
- *$\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$*
- *$\mathcal{O}$ is a set that contains for every n-ary connective $\diamond \in \mathcal{L}$, a n-ary truth-function $f^{\diamond} : \mathcal{V}^n \longrightarrow \mathcal{V}$*

Notice that $\langle \mathcal{V}, \mathcal{O} \rangle$ is an algebra of the same similarity type as $\mathcal{L}$, with universe $\mathcal{V}$ and a set of operations $\mathcal{O}$.

Definition 2 ((Humberstone, 2011))**.** *A matrix $\mathcal{M}$ is a submatrix of a matrix $\mathcal{M}'$ (notation $\mathcal{M} \sqsubseteq \mathcal{M}'$) if and only if $\langle \mathcal{V}, \mathcal{O} \rangle$ is a subalgebra of $\langle \mathcal{V}', \mathcal{O}' \rangle$ and $\mathcal{D} = \mathcal{D}' \cap \mathcal{V}$.*

Definition 3. *A valuation based on a matrix is a mapping $v : \mathsf{Form}_{\mathcal{L}} \longrightarrow \mathcal{V}$ such that for every n-ary connective $\diamond$ and every $\varphi_1, \ldots, \varphi_n \in \mathsf{Form}_{\mathcal{L}}$: $v(\diamond(\varphi_1, \ldots, \varphi_n)) = f^{\diamond}(v(\varphi_1), \ldots, v(\varphi_n))$*

In the context of a deterministic matrix, we say that the truth-value of a given formula has to be chosen deterministically out of a set of given options.[6]

It will be useful, however, to generalize the concept of a matrix to obtain the more general notion of a non-deterministic matrix or Nmatrix, taking the former (deterministic case) to be a limit case.[7]

Definition 4 ((Avron & Zamansky, 2011))**.** *A Nmatrix for a language $\mathcal{L}$ is a structure $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where*

- *$\mathcal{V}$ is a non-empty set of truth values*
- *$\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$*
- *$\mathcal{O}$ is a set that contains for every n-ary connective $\diamond \in \mathcal{L}$, a n-ary truth-function $f^{\diamond} : \mathcal{V}^n \longrightarrow 2^{\mathcal{V}} - \emptyset$*

Definition 5. *A valuation based on a Nmatrix is a mapping $v : \mathsf{Form}_{\mathcal{L}} \longrightarrow \mathcal{V}$ such that for every n-ary connective $\diamond$ and every $\varphi_1, \ldots, \varphi_n \in \mathsf{Form}_{\mathcal{L}}$: $v(\diamond(\varphi_1, \ldots, \varphi_n)) \in f^{\diamond}(v(\varphi_1), \ldots, v(\varphi_n))$*

In the context of a Nmatrix, we say that the truth-value of a given formula can be chosen non-deterministically out of a set of given options.

In what follows, we will talk of deterministic matrices and Nmatrices without any distinction, for in most cases the context will provide disambiguation. However, if needed, we will be explicit about the kind of structure at play. Notice also that sometimes we will need to add a subscript or

[6] It should be noticed that the set of truth-functions is closed under function composition. In reference texts such as (Humberstone, 2011), the truth-functions originally and explicitly included in the set are called "fundamental", whence the functions properly obtained by composition are called "derived" (see (Humberstone, 2011, 17)).

[7] The behavior of deterministic matrices can be mimicked in the context of Nmatrices by building, for each n-ary connective $\diamond$ that is interpreted in a deterministic matrix $\mathcal{M}$ by a function $f^{\diamond} : \mathcal{V}^n \longrightarrow \mathcal{V}$, a corresponding Nmatrix $\mathcal{M}'$ where that connective is interpreted by a function that has only singletons as its output values, i.e. $f'^{\diamond} : \mathcal{V}^n \longrightarrow \{\mathcal{A} \subseteq \mathcal{V} : |\mathcal{A}| = 1\}$. Analogously, valuations for Nmatrices can be seen as generalizations of valuations for deterministic matrices.

superscript to the functions $f^{\diamond}$, the (N)matrices $\mathcal{M}$ and their corresponding sets $\mathcal{V}, \mathcal{D}, \mathcal{O}$, and to the consequence relations $\vDash$, referring correspondingly to the system at play, in order to indicate that they are as in the intended semantics for the logic **L**.

With these details in mind, we define now the corresponding notion of semantic consequence.

Definition 6. *Let $\mathcal{M}$ be a matrix or Nmatrix. The single-conclusion consequence relation $\vDash$ induced by $\mathcal{M}$ is defined in the following way.*

A formula φ is a consequence of a set of formulae Γ (notation $\Gamma \vDash \varphi$) if for every valuation v, if $v(\gamma) \in \mathcal{D}$ for every $\gamma \in \Gamma$, then $v(\varphi) \in \mathcal{D}$.

It is useful, however, to consider the more general case where an argument can be regarded as having multiple conclusions, taking the single conclusion case to be a limit case of the latter.

Definition 7. *Let $\mathcal{M}$ a matrix or Nmatrix. The multiple-conclusion consequence relation $\vDash$ induced by $\mathcal{M}$ is defined in the following way.*

A set of formulae Δ are the multiple consequences of a set of formulae Γ (notation $\Gamma \vDash^{+} \Delta$) if for every valuation v, if $v(\gamma) \in \mathcal{D}$ for every $\gamma \in \Gamma$, then $v(\delta) \in \mathcal{D}$, for some $\delta \in \Delta$.

Finally, and since we are going to study potential LFIs and LFUs defined in extensions of infectious logics, we define LFIs and LFUs next.

Definition 8 ((Carnielli et al., 2007))**.** *A logic* **L** *is a Logic of Formal Inconsistency if and only if there is some possibly empty set of formulae $\circ(\alpha)$ depending on α such that the following conditions are met:*

There are some some Γ, α, β such that:

1. $\Gamma, \alpha, \neg\alpha \nvDash \beta$
2. $\Gamma, \circ(\alpha), \alpha \nvDash \beta$
3. $\Gamma, \circ(\alpha), \neg\alpha \nvDash \beta$

And for all Γ, α, β:

4. $\Gamma, \circ(\alpha), \alpha, \neg\alpha \vDash \beta$

Definition 9 ((Corbalán, 2012))**.** *A logic* **L** *is a Logic of Formal Undeterminedness if and only if there is a possibly empty set of formulae $\bigstar(\alpha)$ depending on α, such that the following conditions are met.*

There are some some Γ, α such that:

1. $\Gamma \nvDash \alpha, \neg\alpha$
2. $\Gamma \nvDash \bigstar(\alpha), \alpha$
3. $\Gamma \nvDash \bigstar(\alpha), \neg\alpha$

And for all Γ, α:

4. $\Gamma \vDash \bigstar(\alpha), \alpha, \neg\alpha$

Along these lines, when the sets $\circ(\alpha)$ and $\bigstar(\alpha)$ are singletons, they are usually called a *consistency* operator and an *undeterminedness* operator, respectively. The literature has it that consistency and inconsistency operators (respectively, determinedness and undeterminedness operators) can, or should, be obtained as some sort of negation of the other –where this negation is sometimes present in the object language, and sometimes is a negation of the meta-theory (see (Carnielli & Coniglio, 2016)).

3. Infectious Logics

Our aim in this paper is to determine if infectious logics extended with a classical negation render systems where consistency and/or undeterminedness operators are definable, such that the resulting systems can be properly regarded as a LFIs or LFUs.[8]

But, what are infectious logics? Intuitively, infectious logics can be understood as systems where there is a certain semantic value that *infects* every operation in which it takes part. Therefore, infectious values act according to the motto "one bad apple spoils the whole barrel".[9]

Yet, what might be a good reason for some semantic value to be transmitted or spread to the whole formula, once one of its parts has such semantic value? There have been, in fact, some cases in the literature where philosophers thought of conceptual motivations for some values to behave in this manner. We do not claim that this is a "desirable" feature of logical systems, but nevertheless we will review some of these motivations in Section 3..3 below. But first, it will be helpful to discuss the technical definition of an infectious logic.[10]

3..1 Technical definitions

Definition 10. *We say that a logic* $\mathbf{L} = \langle \mathcal{L}, \vDash \rangle$*, where* $\vDash$ *is induced by an Nmatrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is infectious if and only if there is a* $z \in \mathcal{V}$ *such that for every n-ary connective* $\diamond$ *with an associated truth-function* $f^{\diamond} \in \mathcal{O}$ *and for all* $v_1, \ldots, v_n \in \mathcal{V}$ *it holds that:*

$$\text{if } z \in \{v_1, \ldots, v_n\}, \text{ then } f^{\diamond}(v_1, \ldots, v_n) = \{z\}$$

When it is clear from the context, we will refer to the truth-values themselves as infectious. For example, in the context of the paraconsistent logic **PWK** below we will say that the truth-value $\mathfrak{b}'$ is infectious.

Remark 1. *Notice that the definition of an infectious logic, whose corresponding infectious value is z, requires the output of (the truth-function for) every n-ary connective to be a singleton, only when the infectious value z is included in the set of input values. In this sense, it might be said that the definition of infectious logic and infectious value, requires some sort of "deterministic" behavior of the underlying matrices.*

However, it should be noticed that when z is not included among the set of input values (the truth-function for) every n-ary connective is allowed to be as non-deterministic as one wants it to be. Therefore, it can be legitimately said that the definition of infectious logic and infectious value is compatible with non-deterministic semantics –and, hence does not depend on the semantics being deterministic.[11]

[8] To avoid confusion, let us state clearly that our aim is *not* to extend actual LFIs or LFUs with a classical negation. Instead, we want to extend logics that are neither LFIs nor LFUs with classical negation, so that consistency and/or undeterminedness operators become definable –having, as a result, proper LFIs and LFUs.

[9] Like many mottos, this motto might be somehow misleading and oversimplifying. For example, the use of the word "spoils" could lead the reader to think that infectious values have always a negative connotation, e.g. that they should always be taken to be *untrue*, which is not the case, as we shall see next.

[10] On a terminological side, it should be noted that the choice of the "infectious" denomination and the subsequent terminology is heavily inspired in the recent works (Ferguson, 2015b), (Ferguson, 2015a) where the term infectious and related uses are employed. Moreover, these definitions are also scrutinized under the magnifying glass of power algebras and power matrices in (Humberstone, 2014).

[11] Additionally, there might be other alternative senses in which a value z can be said to be infectious, in the context of non-deterministic semantics. For example, the case where every operation which has z as an input needs to have z among the possible outputs, but does not require z to be the *unique* output. Notwithstanding the interest of these generalizations we do not discuss them here for matters of space, hoping to explore them in future works. We would like to thank an anonymous referee for discussion with regard to this point.

Various remarkable infectious logics were presented in the history of non-classical logics. Perhaps the most famous are the $\{\neg, \wedge, \vee\}$-fragments (hereafter, the classical fragment(s)) of the paracomplete logic of nonsense (referred here as $\mathbf{B_3}$) due to (Bochvar, 1938, 1981) and the paraconsistent logic of nonsense (referred here as $\mathbf{H_3}$) due to (Halldén, 1949).[12] These infectious logics are intended to determine formally how logics *of* nonsense should work. Thus, in general, for both Bochvar and Halldén, if an input is meaningless there is no way, no operation where that input appears, where the output can be regarded as meaningful. Nonsense, in this kind of frameworks, infects everything it touches.[13]

It should be clear that, even if infectious logics are obtained as *formal generalizations* of the semantic behavior present in the classical fragment of logics of nonsense, working with the former does not make it mandatory to understand infectious values as nonsensical. Thus, we motivate the study of infectious logical formalisms as the study of systems that present a semantic behavior (though not a philosophical justification) reminiscent of the one that is characteristic of logics of nonsense. As we said before, for a philosophical discussion of what motivated scholars to embrace these formalisms, the reader is referred to read Section 3..3 and the references thereof.

Infectious logics form a vast collection that has ramifications in the realm of non-classical logics. As an example as to how wide this collection of distinct infectious logics can be, we consider here two basic infectious logics that will serve as the basis for generating more infectious systems –in fact, more than a dozen of them. These basic systems are the classical fragment of Bochvar's and Halldén's logics of nonsense. They constitute, respectively, different infectious logics –one of which is paracomplete, one of which is paraconsistent.

These fragments have been studied and presented in the literature independently, as Weak Kleene Logic $\mathbf{K_3^w}$ and Paraconsistent Weak Kleene Logic $\mathbf{PWK}$, respectively. We spell out their semantics below. It is easy to check that these logics are indeed infectious logics, according to Definition 10.

Definition 11. *Weak Kleene Logic* $\mathbf{K_3^w} = \langle \mathcal{L}, \vDash_{\mathbf{K_3^w}} \rangle$ *and Paraconsistent Weak Kleene Logic* $\mathbf{PWK} = \langle \mathcal{L}, \vDash_{\mathbf{PWK}} \rangle$ *are obtained by supplementing the same language with a different consequence relation, induced also by different logical matrices. The semantic structures* $\mathcal{M}_{\mathbf{K_3^w}} = \langle \mathcal{V}_{\mathbf{K_3^w}}, \mathcal{D}_{\mathbf{K_3^w}}, \mathcal{O}_{\mathbf{K_3^w}} \rangle$ *and* $\mathcal{M}_{\mathbf{PWK}} = \langle \mathcal{V}_{\mathbf{PWK}}, \mathcal{D}_{\mathbf{PWK}}, \mathcal{O}_{\mathbf{PWK}} \rangle$ *are obtained in the following way:*

- $\mathcal{V}_{\mathbf{K_3^w}} = \{\mathfrak{t}, \mathfrak{e}, \mathfrak{f}\}$ *and* $\mathcal{V}_{\mathbf{PWK}} = \{\mathfrak{t}, \mathfrak{b}', \mathfrak{f}\}$
- $\mathcal{D}_{\mathbf{K_3^w}} = \{\mathfrak{t}\}$ *and* $\mathcal{D}_{\mathbf{PWK}} = \{\mathfrak{t}, \mathfrak{b}'\}$
- $\mathcal{O}_{\mathbf{K_3^w}} = \{f^{\neg}_{\mathbf{K_3^w}}, f^{\wedge}_{\mathbf{K_3^w}}, f^{\vee}_{\mathbf{K_3^w}}\}$ *these functions being as detailed in (Table 1) when* $\mathfrak{u}$ *is interpreted as* $\mathfrak{e}$
- $\mathcal{O}_{\mathbf{PWK}} = \{f^{\neg}_{\mathbf{PWK}}, f^{\wedge}_{\mathbf{PWK}}, f^{\vee}_{\mathbf{PWK}}\}$ *these functions being as detailed in (Table 1) when* $\mathfrak{u}$ *is interpreted as* $\mathfrak{b}'$

It can be reasonably argued that these systems, and their subsystems therefore, constitute *logically weak* frameworks, given that some classically valid inferences that are usually taken to be very basic and non-problematic, fail in them. We portrait next the case of the failure of conjunction-elimination and disjunction-introduction.

[12] These fragments have been studied proof-theoretically by Marcelo Coniglio and Inés Corbalán in (Coniglio & Corbalán, 2012).

[13] We should clarify, however, that the systems $\mathbf{B_3}$ and $\mathbf{H_3}$ are not properly speaking infectious, since they are endowed with 'assignificativity' or 'nonsense' connectives, which take the nonsensical infectious values to classical values. However, as we remarked before, their classical fragments constitute infectious logics.

	$f^{\neg}_{\mathbf{K_3^w}}/\mathbf{PWK}$
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathbf{K_3^w}}/\mathbf{PWK}$	$\mathfrak{t}$	$\mathfrak{u}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{u}$	$\mathfrak{f}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{u}$	$\mathfrak{f}$

$f^{\vee}_{\mathbf{K_3^w}}/\mathbf{PWK}$	$\mathfrak{t}$	$\mathfrak{u}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{u}$	$\mathfrak{t}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{u}$	$\mathfrak{f}$

Table 1: Truth-tables for $\mathbf{K_3^w}$ and **PWK**

Remark 2. *Let* **L** *be a paraconsistent infectious logic, such that its infectious value z is designated. Therefore, $\alpha \wedge \beta \vDash \beta$ is not valid. For a counterexample, let $v(\alpha) = z$ and $v(\beta) \notin \mathcal{D}$. Thus, since z is infectious $v(\alpha \wedge \beta) = z \in \mathcal{D}$, which suffices to witness the failure.*[14]

Remark 3. *Let* **L** *be a paracomplete infectious logic, such that its infectious value z is undesignated. Therefore, $\beta \vDash \alpha \vee \beta$ is not valid. For a counterexample, let $v(\alpha) = z$ and $v(\beta) \in \mathcal{D}$. Thus, since z is infectious $v(\alpha \vee \beta) = z \notin \mathcal{D}$, which suffices to witness the failure.*

Thus, if LFIs and LFUs are to be built on top of these logics, it will be reasonable to expect that the resulting systems constitute logically *weak* LFIs and LFUs, in a sense that will be clarified by Propositions 17 and 18.

In what follows we will consider extending several infectious logics with a classical negation, in order to define consistency and determinedness operators with its aid, so that the resulting systems can be properly regarded as LFIs and LFUs defined in extensions of infectious logics. To do so, we will present a comprehensive collection of infectious logics that we will be working with. They will be, in fact, infectious subsystems of $\mathbf{K_3^w}$ and **PWK**.

3..2 A myriad of infectious logics

In order to generate a huge collection of infectious logics, we will start from a base of four logics, two of which are our paradigmatic infectious logics just mentioned and the other two are the widely known paracomplete Strong Kleene Logic $\mathbf{K_3}$ and the paraconsistent logic **LP**.[15] Let us recall the semantic presentation of the latter non-infectious systems.

Definition 12. *Strong-Kleene Logic* $\mathbf{K_3} = \langle \mathcal{L}, \vDash_{\mathbf{K_3}} \rangle$ *and Priest's Logic of Paradox* $\mathbf{LP} = \langle \mathcal{L}, \vDash_{\mathbf{LP}} \rangle$ *are obtained by supplementing the same language with a different consequence relation, induced also by different logical matrices. The corresponding semantic structures* $\mathcal{M}_{\mathbf{K_3}} = \langle \mathcal{V}_{\mathbf{K_3}}, \mathcal{D}_{\mathbf{K_3}}, \mathcal{O}_{\mathbf{K_3}} \rangle$ *and* $\mathcal{M}_{\mathbf{LP}} = \langle \mathcal{V}_{\mathbf{LP}}, \mathcal{D}_{\mathbf{LP}}, \mathcal{O}_{\mathbf{LP}} \rangle$ *are obtained in the following way:*

- $\mathcal{V}_{\mathbf{K_3}} = \{\mathfrak{t}, \mathfrak{n}, \mathfrak{f}\}$ *and* $\mathcal{V}_{\mathbf{LP}} = \{\mathfrak{t}, \mathfrak{b}, \mathfrak{f}\}$
- $\mathcal{D}_{\mathbf{K_3}} = \{\mathfrak{t}\}$ *and* $\mathcal{D}_{\mathbf{LP}} = \{\mathfrak{t}, \mathfrak{b}\}$
- $\mathcal{O}_{\mathbf{K_3}} = \{f^{\neg}_{\mathbf{K_3}}, f^{\wedge}_{\mathbf{K_3}}, f^{\vee}_{\mathbf{K_3}}\}$ *these functions being as detailed in (Table 2) when* $\mathfrak{i}$ *is interpreted as* $\mathfrak{n}$

[14] Notice that the validity of $\alpha, \beta \vDash \beta$ and the invalid character of $\alpha \wedge \beta \vDash \beta$ shows that these logics belong to the larger family of non-adjunctive systems, which comprises an entire collection of paraconsistent logics. This is interesting, for it shows that this non-adjunctive feature can be obtained as the result of the introduction of an infectious value. Moreover, some non-adjunctive logics (e.g. discussive logics) can be given some kind of frame semantics, whence it would be interesting to determine whether or not the paraconsistent infectious logics presented here can also enjoy semantics of this sort. Indeed, the work done in (Ciuni, 2015) can be understood as initiating this path. Discussing these matters is interesting, and we hope to embrace them in future research, but an exhaustive investigation will take us too far afield.

[15] Notice that what differentiates $\mathbf{K_3}$ and **LP** from some other well-known logics, more central to the literature about LFIs, such as **P1**, **LFI1** and **J3** is the fact that the former do not have means to define a proper classical negation, whereas the latter have many means to do so.

- $\mathcal{O}_{\mathbf{LP}} = \{f^{\neg}_{\mathbf{LP}}, f^{\wedge}_{\mathbf{LP}}, f^{\vee}_{\mathbf{LP}}\}$ *these functions being as detailed in (Table 2) when* $\mathfrak{i}$ *is interpreted as* $\mathfrak{b}$

	$f^{\neg}_{\mathbf{K_3/LP}}$
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{i}$	$\mathfrak{i}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathbf{K_3/LP}}$	$\mathfrak{t}$	$\mathfrak{i}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{i}$	$\mathfrak{f}$
$\mathfrak{i}$	$\mathfrak{i}$	$\mathfrak{i}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

$f^{\vee}_{\mathbf{K_3/LP}}$	$\mathfrak{t}$	$\mathfrak{i}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
$\mathfrak{i}$	$\mathfrak{t}$	$\mathfrak{i}$	$\mathfrak{i}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{i}$	$\mathfrak{f}$

Table 2: Truth-tables for $\mathbf{K_3}$ and $\mathbf{LP}$

Below, we present fifteen systems, fourteen of which are infectious, and spell out their semantics in the next paragraphs. We state, in the case of each logic *which* are the *non-classical* values characteristic of it.

For the sake of clarity, it should be pointed out that when some logic $\mathbf{L}$ is such that two allegedly infectious values belong to its set of truth values, for it to be an actual infectious logic, some of these values has to be infectious *properly speaking*, i.e. it has to be infectious over all the other values, and in particular over the other alleged infectious value.[16] This is the case, for example, of a logic intended to be a subsystem of e.g. $\mathbf{K_3^w}$ and $\mathbf{PWK}$. In this example, this implies that there are (at least) two infectious logics that have these values: one where $\mathfrak{b}'$ is infectious over $\mathfrak{e}$, and another where $\mathfrak{e}$ is infectious over $\mathfrak{b}'$. In fact, some systems appearing in the following list fall under these descriptions. When a logic has two values that are allegedly infectious, we will write as the last element the one that is infectious properly speaking, in order to highlight this fact.

We should also highlight that the style of the presentation here is inspired by the one given in (Priest, 2014), where many of these logics are generated and listed in a similar way. As suggested by an anonymous referee, these logics are endowed with names that point to the non-classical values included in them –something that might be more helpful for the reader than a mere arbitrary enumeration of these logics.

- $\emptyset$: $\mathbf{CL}$, Classical Logic.
- $\mathfrak{n}$: $\mathbf{K_3}$, Strong-Kleene Logic.
- $\mathfrak{b}$: $\mathbf{LP}$, Priest's Logic of Paradox.
- $\mathfrak{e}$: $\mathbf{K_3^w}$, Weak-Kleene Logic.
- $\mathfrak{b}'$: $\mathbf{PWK}$, Paraconsistent Weak-Kleene Logic.
- $\mathfrak{bn}$: $\mathbf{FDE}$, First-Degree Entailment Logic from Belnap and Dunn.
- $\mathfrak{b}'\mathfrak{e}$: $\mathbf{L}_{\mathfrak{b}'\mathfrak{e}}$, a sublogic of $\mathbf{K_3^w}$ and $\mathbf{PWK}$, where $\mathfrak{e}$ is infectious over $\mathfrak{b}'$.
- $\mathfrak{ne}$: $\mathbf{L}_{\mathfrak{ne}}$
- $\mathfrak{be}$: $\mathbf{S}_{\mathtt{fde}}$, a variant of $\mathbf{FDE}$, where $\mathfrak{n}$ is replaced by $\mathfrak{e}$. It was presented in (Deutsch, 1984), but has previously received alternative denominations in (Deutsch, 1977) and (Deutsch, 1981).[17]
- $\mathfrak{nb}'$: $\mathbf{L}_{\mathfrak{nb}'}$, a variant of $\mathbf{FDE}$, where $\mathfrak{b}$ is replaced by $\mathfrak{b}'$.[18]

[16](Humberstone, 2014) makes a parallel between this fact about infectious logics, and the uniqueness of zero elements in semigroups.

[17]It was later on discussed as the logic $\mathbf{AL}$ in (Oller, 1999).

[18]This system, despite its being dual to $\mathbf{S}_{\mathtt{fde}}$, did not appear previously in the literature, as far as we know.

- $\mathfrak{bb'}$: $\mathbf{L}_{\mathfrak{bb'}}$
- $\mathfrak{eb'}$: $\mathbf{L}_{\mathfrak{eb'}}$, a sublogic of $\mathbf{K_3^w}$ and **PWK**, where $\mathfrak{b'}$ is infectious over $\mathfrak{e}$.
- $\mathfrak{nb'e}$: $\mathbf{L}_{\mathfrak{nb'e}}$
- $\mathfrak{nb'e}$: $\mathbf{FDE}_{\varphi}$, a logic presented in (Priest, 2010).[19]
- $\mathfrak{bb'e}$: $\mathbf{L}_{\mathfrak{bb'e}}$
- $\mathfrak{neb'}$: $\mathbf{L}_{\mathfrak{neb'}}$
- $\mathfrak{nbb'}$: $\mathbf{L}_{\mathfrak{nbb'}}$
- $\mathfrak{beb'}$: $\mathbf{L}_{\mathfrak{beb'}}$
- $\mathfrak{nbb'e}$: $\mathbf{L}_{\mathfrak{nbb'e}}$
- $\mathfrak{nebb'}$: $\mathbf{L}_{\mathfrak{nebb'}}$

For the sake of exhaustivity, we give a detail of the characteristic matrices of these logics, in Definition 13. These can be read in a rather mechanical way and are spelled out here only to be rigorous. We also outline the relations between these systems in the Hasse diagram of Figure 1.

Definition 13. *We define all the characteristic matrices of the logics above as sub-matrices of at least one of the matrices* $\mathcal{M}_{\mathbf{L}_{\mathfrak{nbb'e}}} = \langle \mathcal{V}_{\mathbf{L}_{\mathfrak{nbb'e}}}, \mathcal{D}_{\mathbf{L}_{\mathfrak{nbb'e}}}, \mathcal{O}_{\mathbf{L}_{\mathfrak{nbb'e}}} \rangle$ *and* $\mathcal{M}_{\mathbf{L}_{\mathfrak{nebb'}}} = \langle \mathcal{V}_{\mathbf{L}_{\mathfrak{nebb'}}}, \mathcal{D}_{\mathbf{L}_{\mathfrak{nebb'}}}, \mathcal{O}_{\mathbf{L}_{\mathfrak{nebb'}}} \rangle$, *which are characteristic of the two incomparable systems* $\mathbf{L}_{\mathfrak{nbb'e}}$ *and* $\mathbf{L}_{\mathfrak{nebb'}}$, *respectively.*[20]

- $\mathcal{V}_{\mathbf{L}_{\mathfrak{nbb'e}}} = \mathcal{V}_{\mathbf{L}_{\mathfrak{nebb'}}} = \{\mathfrak{t}, \mathfrak{b}, \mathfrak{b'}, \mathfrak{n}, \mathfrak{e}, \mathfrak{f}\}$
- $\mathcal{D}_{\mathbf{L}_{\mathfrak{nbb'e}}} = \mathcal{D}_{\mathbf{L}_{\mathfrak{nebb'}}} = \{\mathfrak{t}, \mathfrak{b}, \mathfrak{b'}\}$
- $\mathcal{O}_{\mathbf{L}_{\mathfrak{nbb'e}}} = \{f^{\neg}_{\mathbf{L}_{\mathfrak{nbb'e}}}, f^{\wedge}_{\mathbf{L}_{\mathfrak{nbb'e}}}, f^{\vee}_{\mathbf{L}_{\mathfrak{nbb'e}}}\}$ *these functions being as detailed in (Table 3)*
- $\mathcal{O}_{\mathbf{L}_{\mathfrak{nebb'}}} = \{f^{\neg}_{\mathbf{L}_{\mathfrak{nebb'}}}, f^{\wedge}_{\mathbf{L}_{\mathfrak{nebb'}}}, f^{\vee}_{\mathbf{L}_{\mathfrak{nebb'}}}\}$ *these functions being as detailed in (Table 4)*

	$f^{\neg}_{\mathbf{L}_{\mathfrak{nbb'e}}}$
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{b'}$	$\mathfrak{b'}$
$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathbf{L}_{\mathfrak{nbb'e}}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b'}$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b'}$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b'}$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{e}$	$\mathfrak{b'}$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{f}$	$\mathfrak{b'}$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{b'}$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{f}$

$f^{\vee}_{\mathbf{L}_{\mathfrak{nbb'e}}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b'}$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b'}$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{t}$
$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b'}$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{b}$
$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{b'}$	$\mathfrak{e}$	$\mathfrak{b'}$
$\mathfrak{n}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b'}$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{n}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b'}$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$

Table 3: Truth-tables for the logic $\mathbf{L}_{\mathfrak{nbb'e}}$

We now define the rest of the matrices as submatrices of some or both of the incomparable matrices $\mathcal{M}_{\mathbf{L}_{\mathfrak{nbb'e}}}$ and $\mathcal{M}_{\mathbf{L}_{\mathfrak{nebb'}}}$.

Some of these submatrices include only one allegedly infectious value and are, thus, submatrices of *both* of the incomparable matrices $\mathcal{M}_{\mathbf{L}_{\mathfrak{nbb'e}}}$ and $\mathcal{M}_{\mathbf{L}_{\mathfrak{nebb'}}}$. In particular, $\mathcal{M}_{\mathbf{FDE}_{\varphi}}$

[19]This logic is referred as $\mathbf{FDE} \cap \mathbf{AI}_{\mathtt{fde}}$ in (Angell, 1977), and that it has also been discussed in (Daniels, 1990) with an homonym denomination to Deutsch's system of (Deutsch, 1984), whence e.g. in (Ferguson, 2014a) it is referred as $\mathbf{S}^{\star}_{\mathtt{fde}}$.

[20]Thanks to an anonymous referee for making the suggestion of presenting these systems in this more elegant way.

	$f^{\neg}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{b}'$	$\mathfrak{b}'$
$\mathfrak{n}$	$\mathfrak{n}$
$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{t}$

$f^{\wedge}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b}'$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b}'$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}'$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{f}$	$\mathfrak{b}'$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{b}'$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{b}'$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{f}$

$f^{\vee}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b}'$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}'$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{t}$
$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b}'$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{b}$
$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$	$\mathfrak{b}'$
$\mathfrak{n}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{b}'$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{n}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{b}'$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{b}$	$\mathfrak{b}'$	$\mathfrak{n}$	$\mathfrak{e}$	$\mathfrak{f}$

Table 4: Truth-tables for the logic $\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}$

is their $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{e}, \mathfrak{f}\}$-submatrix, $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'}}$ is their $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{b}', \mathfrak{n}, \mathfrak{f}\}$-submatrix, $\mathcal{M}_{\mathbf{S}_{\mathfrak{fde}}}$ is their $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{e}, \mathfrak{f}\}$-submatrix, $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}'}}$ is their $\{\mathfrak{t}, \mathfrak{b}', \mathfrak{n}, \mathfrak{f}\}$-submatrix, $\mathcal{M}_{\mathbf{L}_{\mathfrak{b}\mathfrak{b}'}}$ is their $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{b}', \mathfrak{f}\}$-submatrix and $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}}}$ is their $\{\mathfrak{t}, \mathfrak{n}, \mathfrak{e}, \mathfrak{f}\}$-submatrix.

Moreover, some of these submatrices include two *allegedly* infectious values, whence –as we said before– for it to induce an actual infectious logic, some of these values has to be infectious *properly speaking*, i.e. it has to be infectious over all the other values, and in particular over the other alleged infectious value. We, thus, have two family of matrices counting with the values $\mathfrak{e}$ and $\mathfrak{b}'$: on the one hand, those matrices where $\mathfrak{e}$ is infectious over $\mathfrak{b}'$, these will be submatrices of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'\mathfrak{e}}}$ but *not* of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$; on the other hand, those matrices where $\mathfrak{b}'$ is infectious over $\mathfrak{e}$, these will be submatrices of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$ but *not* of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'\mathfrak{e}}}$.

In particular, $\mathcal{M}_{\mathbf{L}_{\mathfrak{b}\mathfrak{b}'\mathfrak{e}}}$ is the $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{b}', \mathfrak{e}, \mathfrak{f}\}$-submatrix, $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}'\mathfrak{e}}}$ is the $\{\mathfrak{t}, \mathfrak{b}', \mathfrak{n}, \mathfrak{e}, \mathfrak{f}\}$-submatrix and $\mathcal{M}_{\mathbf{L}_{\mathfrak{b}'\mathfrak{e}}}$ is the $\{\mathfrak{t}, \mathfrak{b}', \mathfrak{e}, \mathfrak{f}\}$-submatrix of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'\mathfrak{e}}}$ but *not* of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$. Finally, $\mathcal{M}_{\mathbf{L}_{\mathfrak{b}\mathfrak{e}\mathfrak{b}'}}$ is the $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{b}', \mathfrak{e}, \mathfrak{f}\}$-submatrix, $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}'}}$ is the $\{\mathfrak{t}, \mathfrak{b}', \mathfrak{n}, \mathfrak{e}, \mathfrak{f}\}$-submatrix of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}}$ and $\mathcal{M}_{\mathbf{L}_{\mathfrak{e}\mathfrak{b}'}}$ is the $\{\mathfrak{t}, \mathfrak{b}', \mathfrak{e}, \mathfrak{f}\}$-submatrix but *not* of $\mathcal{M}_{\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'\mathfrak{e}}}$.

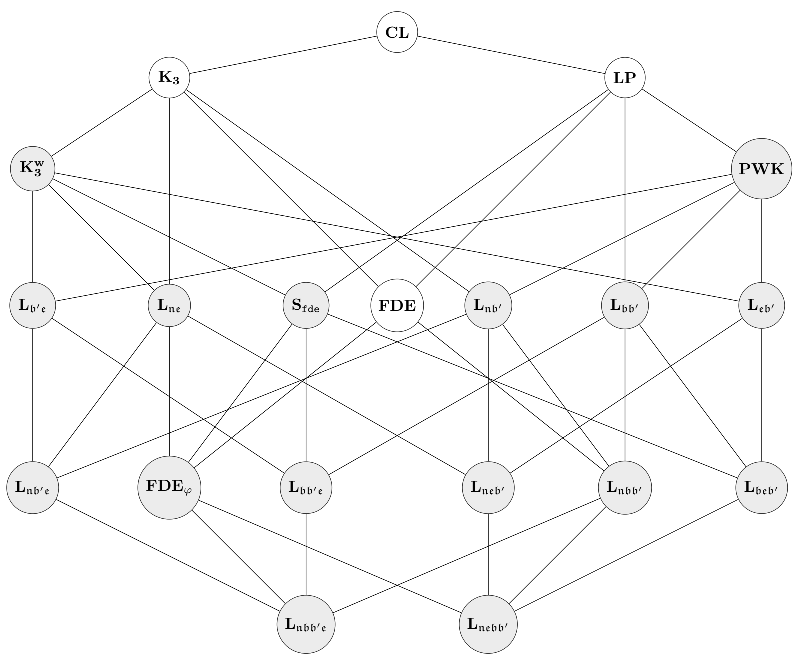

Figure 1: Hasse diagram of the inclusion ordering of the logics mentioned so far. Infectious logics are in grey.

However, infectious logics are not the most famous logical systems and thus, some readers might be interested in knowing whether or not it has been argued there are independent reasons for entertaining these systems. It should be nevertheless remembered that this essay is not intended to represent a defense of the philosophical justification for embracing infectious logics, but a technical exploration of the possibility of defining LFIs and LFUs in extensions of infectious logics. In any case, we hope to provide some insight on this conceptual matter in the following section.

3..3 Philosophical motivations for infectious logics found in the literature

In this Section we will briefly review two conceptual routes that led philosophers, logicians and mathematicians in the past to the study and application of infectious logics. With regard to both routes, paracomplete infectious logics have more or less monopolized the attention. However, we will point out, in each case, that paraconsistent infectious logics can be thought as enjoying dual properties.

The first route that led scholars in the past to focus their attention in infectious logics, concerns the study of the so-called '*cut-down*' operators and operations, analyzed by Fitting and Ferguson in (Fitting, 1994) and (Ferguson, 2015a), and of the dual case of what we call here '*track-down*' operators and operations. Fitting has proposed an epistemic interpretation of some subsystems of Weak Kleene logic $\mathbf{K_3^w}$, i.e. systems where the paracomplete value is *infectious*, in which groups of experts expressing their opinion on formulae serve as truth-values. In this vein, the truth-value of e.g. a conjunction $\varphi \wedge \psi$ is understood as a pair whose first element is the set of experts who assent to both φ and ψ, and the second element of the pair is the set of experts who oppose either φ or ψ.

In this reading, the indeterminate value should correspond to the case where experts lack any positive or negative opinion towards a certain formula. Thus, he goes on to say that some logics having infectious indeterminate values can be seen as systems adopting a 'cut-down' policy, i.e. as embracing the idea that with regard to a certain formulae e.g. $\varphi \wedge \psi$, we might want to 'cut down' by only considering people who have actually expressed an opinion (i.e. a positive *or* negative opinion) on both φ and ψ. This point of view can be rephrased by saying that adopting a cut-down policy amounts to the idea that *no positive or negative opinion can arise from a set that includes an indeterminate opinion.* It is shown in (Ferguson, 2015a) that embracing this approach induces the adoption of an infectious paracomplete logic that is equivalent to $\mathbf{S}_{\mathtt{fde}}$. Similar reasoning, with different truth-values at the starting point, might render cut-down interpretations for other paracomplete infectious logics that we presented here, but we do not discuss this issue in its full extent here, for it will take us too far afield.

Dualizing Fitting and Ferguson's proposal, an epistemic interpretation can be given for some subsystems of Paraconsistent Weak Kleene logic **PWK**, i.e. systems where the paraconsistent value is *infectious*. In Fitting's epistemic reading, the inconsistent value corresponds to the case where experts have both a positive and a negative opinion towards a certain formula. Thus, we think it is possible that some logics having infectious inconsistent values can be seen as systems adopting a 'track-down' policy, i.e. as embracing the idea that with regard to a certain formulae e.g. a conjunction $\varphi \wedge \psi$, we might want to 'track down' people who have actually expressed an inconsistent opinion (i.e a positive *and* a negative opinion) on either φ or ψ. This point of view can be rephrased by saying that adopting a track-down policy amounts to the idea that *no consistent opinion can arise from a set that includes an inconsistent opinion.* It is shown in (Szmuc, n.d.) that embracing this approach induces the adoption of an infectious paraconsistent logic that is equivalent to $\mathbf{L}_{\mathfrak{nb}'}$. Again, similar reasoning, with different truth-values at the starting point, might render track-down interpretations for other paraconsistent infectious logics that we presented here, but we do not discuss this issue in its full extent here, for it will take us too far afield.

To sum up, *if* someone actually agrees with the reasons provided before for adopting either a cut-down or a track-down policy, *then* these infectious logics might be useful to express their commitments.

The second route that led scholars in the past to focus their attention in infectious logics, concerns the study of *containment logics*, some of which were proposed in (Parry, 1933), analyzed in (Anderson & Belnap, 1975) and remarkably in (Ferguson, 2015b). Containment logics are logics where some inference holds only if certain containment (i.e. set-theoretic *inclusion*) principle between the set of propositional variables appearing in the premises Γ –call it $\mathsf{At}(\Gamma)$– and the set propositional variables appearing in the conclusion φ –call it $\mathsf{At}(\varphi)$– is respected.

On the one hand, the logics $\mathbf{S}_{\mathtt{fde}}$ and $\mathbf{FDE}_{\varphi}$ were discussed in (Ferguson, 2014a, 2015a) as logics with an analytic, conceptivist or proscriptivist consequence relation (see (Parry, 1933)), v.g. logics where all arguments are such that all the propositional variables appearing in the conclusion of an argument must have appeared in the premises. To this phenomenon we refer to as $\vDash$-*Proscriptive Containment*. We say, moreover, that a certain inference $\Gamma \vDash_{\mathbf{L}} \varphi$ carried out in the logic $\mathbf{L}$ enjoys the $\vDash$-Proscriptive Principle whenever

$$\Gamma \vDash_{\mathbf{L}} \varphi \text{ only if } \mathsf{At}(\varphi) \subseteq \mathsf{At}(\Gamma)$$

Concomitantly, we say that a logic $\mathbf{L}$ itself enjoys the $\vDash$-Proscriptive Principle only if every argument that is valid in it enjoys this principle. In light of Observation 1 appearing in (Ferguson, 2016), it could also be claimed that logics such as $\mathbf{L}_{\mathfrak{b}'\mathfrak{c}}$, $\mathbf{L}_{\mathfrak{n}\mathfrak{b}'\mathfrak{c}}$, $\mathbf{L}_{\mathfrak{b}\mathfrak{b}'\mathfrak{c}}$ and $\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'\mathfrak{c}}$ are systems of this kind.

On the other hand, containment systems where the $\vDash$-Proscriptive Principle is *reversed* can also be studied, in order to obtain a different containment principle, which we will call $\vDash$-Permissive Principle. That is to say, systems where variable inclusion between premises and conclusions is reversed, and what is required is not that every concept in the conclusion is mentioned in the premises, but that there is at least a nonempty subset of the premises such that every concept appearing in it appears in the conclusion.[21] The conclusion might feature other concepts, not present in any of the premises, though. To this phenomenon we refer to as $\vDash$-*Permissive Containment*.[22] We say, moreover, that a certain inference $\Gamma \vDash_{\mathbf{L}} \varphi$ carried out in the logic $\mathbf{L}$ enjoys the $\vDash$-Permissive Principle whenever

$$\Gamma \vDash_{\mathbf{L}} \varphi \text{ only if } \exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, \mathsf{At}(\Gamma') \subseteq \mathsf{At}(\varphi)$$

Concomitantly, we say that a logic $\mathbf{L}$ itself enjoys the $\vDash$-Permissive Principle only if every argument that is valid in it enjoys this principle. Generalizing the containment results obtained for **PWK** in (Ciuni & Carrara, 2016) and (Coniglio & Corbalán, 2012) among others (see e.g. Theorem 3.8 of (Ciuni & Carrara, 2016)), a dual of Observation 1 appearing in (Ferguson, 2016) can be obtained. Establishing this result will allow us to say that certain paraconsistent infectious logics –among them the logics $\mathbf{L}_{\mathfrak{c}\mathfrak{b}'}$, $\mathbf{L}_{\mathfrak{n}\mathfrak{b}'}$, $\mathbf{L}_{\mathfrak{n}\mathfrak{c}\mathfrak{b}'}$, $\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'}$, $\mathbf{L}_{\mathfrak{b}\mathfrak{c}\mathfrak{b}'}$ and $\mathbf{L}_{\mathfrak{n}\mathfrak{c}\mathfrak{b}\mathfrak{b}'}$– are systems with a reverse-analytic or permissivist consequence relation (as is proved in (Szmuc, n.d.)), v.g. there is some nonempty subset of the premises such that all of its propositional variables also appear in the conclusion[23]

Having reviewed this conceptual issues[24] we will next consider the prospect of extending infectious logics with appropriate connectives so that consistency and undeterminedness operators

[21] We *do not* intend to claim that this is the only way in which the $\vDash$-Proscriptive Principle can be reversed, but instead that this is an interesting way in which this can be done.

[22] As an anonymous referee points out, Richard Epstein has introduced a system called **DualD**, which is *dual* to Parry's logic **PAI** (see (Parry, 1933)) and which enjoys the Permissive Principle *with regard to an intensional implication connective.* For a presentation and discussion of Epstein's systems, see e.g. (Epstein, 1995).

[23] In particular, paraconsistent infectious logics that are also subsystems of paracomplete logics, enjoy this principle.

[24] There have been other discussions of infectious logics in the literature that we are unable to review here for matters of space. For example, Graham Priest's discussion of $\mathbf{FDE}_{\varphi}$ as the logic of the *catuskoti* and Thomas Ferguson's discussion of this logic as the logic of *faulty* Belnap computers in (Ferguson, 2016).

become definable. We will do this hoping to define proper LFIs and LFUs in extensions of infectious logics.

4. Defining consistency and undeterminedness operators in extensions of infectious logics

Let us no go back to our main aim: to define consistency and undeterminedness operators –and, thus, to define proper LFIs and LFUs– in extensions of infectious logics. In our discussion below we will be following closely the attempts made by da Costa and his collaborators to provide definitions for the consistency, inconsistency, determinedness and undeterminedness operators in (da Costa & Arruda, 1976), (da Costa & Alves, 1977) and (da Costa & Loparic, 1984).

We should emphasize that *we are not after new alternative definitions* of the consistency or undeterminedness operators. Our aim is rather different. We want to apply the previously given definitions or some suitable modifications thereof in a *new context* where they have not been discussed yet, i.e. in the context of *infectious* logics.

Were these attempts to succeed in these new contexts, this will support the claim that there is a huge collection of LFIs (and LFUs) built on top of infectious logics, that might be *orthogonal* to the hierarchy construed on top of the weakest LFI investigated so far –e.g. in (Carnielli & Coniglio, 2016)– namely, **mbC**.

4..1 Following da Costa and Loparic

In his works da Costa embraces the idea of *defining* a normality operator, which was later understood as a consistency operator and studied on its own by e.g. (Carnielli et al., 2007) and more recently in the comprehensive monograph (Carnielli & Coniglio, 2016). In this Section we will try to follow this definition both in letter and in spirit, when we try to extend infectious logics with a classical negation to define a consistency operator, in order to obtain new LFIs based on infectious logics.

The normality or consistency operator is defined by da Costa and his collaborators e.g. for the system C_1 as $\neg(\alpha \wedge \neg\alpha)$. A good question is whether or not this definition defines a consistency operator in the infectious logics described in Section 3..2. As is easy to check, when the symbol $\neg$ is taken to be the non-classical negation of the logics above, the formula $\neg(\alpha \wedge \neg\alpha)$ *does not define a consistency operator* in e.g. **PWK** and its subsystems presented above, or any infectious paraconsistent logic whatsoever.

Why does the formula $\neg(\alpha \wedge \neg\alpha)$ define a consistency operator in da Costa's case and not in our infectious case? To understand this, we must clarify how the negation $\neg$ being employed works in the case of da Costa, and how it works in our case. Thus, even though the paraconsistent negation $\neg$ of da Costa's systems has a non-classical behavior, it is one of a *non-deterministic* kind. Carnielli and Coniglio give us a more detailed explanation of this

> N. C. A. da Costa and E. H. Alves proposed [in (da Costa & Arruda, 1976) and (da Costa & Alves, 1977)] an original valuation semantics for C_1 over $\{\mathfrak{t}, \mathfrak{f}\}$(...) However, the paraconsistent negation $\neg$ has a non-deterministic behavior w.r.t. this semantics: in general, if one of such valuations assigns the value $\mathfrak{t}$ to a formula α, then the formula $\neg\alpha$ can receive either the value $\mathfrak{f}$ or the value $\mathfrak{t}$ (but not both) under the same valuation. That is: the truth value of α does not uniquely determine the truth-value of $\neg\alpha$. (Carnielli & Coniglio, 2016, p. 54, notation adjusted to fit ours)

Moreover, as remarked in e.g. (Carnielli & Coniglio, 2016, p. 147), the paraconsistent negation in da Costa's systems is not only non-deterministic, but in fact it is of a particularly strong non-deterministic kind. This is remarkably witnessed by the fact that the value of the formula $\neg(\alpha \wedge \neg\alpha)$ does not depend only on the value of the formula $\alpha \wedge \neg\alpha$, as it does in most logics. In da Costa's case, the value of the formula $\neg(\alpha \wedge \neg\alpha)$ is false if and only if the formulae α and $\neg\alpha$ have the same value, i.e. if they are in the non-classical case where they are both true.[25]

Thus, da Costa's systems allow the formula $\neg(\alpha \wedge \neg\alpha)$ to define a consistency operator because its semantics are indeed very strong algebraically speaking –when compared to finite matrix semantics. This has in fact been noticed not only by (Carnielli & Coniglio, 2016), but also previously by (Avron & Zamansky, 2011) among others.

On the other hand, the basic infectious logics we are dealing with are equipped with very simple finite matrix semantics, as discussed in Section 3..2. In the context of these simple matrices, the flexible behavior that the non-classical negation $\neg$ has in da Costa's systems cannot be mimicked –as proved in (Avron, 2007, Theorem 11). In fact, none of the infectious logics presented in Section 3..2 can define a consistency operator, neither with da Costa's definition, nor with any other definition. This is why, in order to define consistency and undeterminedness operators we need to *extend* these logics with suitable connectives –i.e. with a classical negation– in order to achieve this goal.

Nevertheless, even if we are not able to use the exact definition that da Costa's gave for the consistency operator, it certainly seems to us that his identification of non-contradiction with consistency constitutes the right approach to define this kind of operator. Thus, we shall look for a slight variation on this definition, that should honor its conceptual motivation and look as similar as possible to it.

To do this, our proposal below will be to use the formula $\sim(\alpha \wedge \neg\alpha)$ to define a consistency operator $\circ\alpha$, where $\sim$ represents a classical negation. It is no surprise that there is an obvious syntactical resemblance of this definition and that of da Costa's. Moreover, using a classical negation as the main connective of the *definiens* seems to be in phase with the spirit of da Costa's definition. For, as is remarked in (Omori & Sano, 2014), sometimes in the literature on LFIs when the formula $\neg(\alpha \wedge \neg\alpha)$ is given a special role to control the behavior of contradictions (just as the consistency of α), this formula can be proved to be equivalent to the formula $\sim(\alpha \wedge \neg\alpha)$, where $\sim$ is taken to be a *classical* negation.

Therefore, it seems fair to say that our definition of the consistency operator is *not* intended to be a new one, but instead is merely intended as a reformulation of da Costa's definition *in a new context*, i.e. the context of *an infectious logic*, extended with a *classical* negation.

Strictly analogous remarks apply to the case of the undeterminedness operator defined in (da Costa & Loparic, 1984) as $\neg(\alpha \vee \neg\alpha)$ and defined by us below as $\sim(\alpha \vee \neg\alpha)$. Thus, it also seems fair to say that our definition of the undeterminedness operator is not intended to be a new one, but instead is merely intended as a reformulation of da Costa's and Loparic's definition in a new context, i.e. the context of an infectious logic, extended with a classical negation.

Now, if we are about to extend infectious logics with a classical negation in order to define consistency and undeterminedness operators and if –moreover– classical negation is going to play an essential role in these definitions, it will be reasonable to spell out technically and conceptually what a classical negation is.

4..2 What is a classical negation?

To answer what a classical negation is, we draw inspiration from the recent discussion carried out in (De & Omori, 2015). There, the authors' first attempt is to consider a classical negation as being a unary operator $\sim$ that satisfies the next requirement:

[25] In (Carnielli & Coniglio, 2016) the semantic clause guaranteeing this is called (vCf).

Contra The sentence $\alpha \wedge \sim\alpha$ is always false *and not true*, and the sentence $\alpha \vee \sim\alpha$ is always true *and not false*

Nevertheless, this approach is quickly discarded, as it turns out to be unsuccessful. The reason is that it cannot be applied with full generality to all non-classical logics. Omori and De acknowledge that this criterion allows to uniquely determine an operation in the context of the well known four-valued logic **FDE** of Belnap and Dunn (see e.g. (Belnap, 1977)), but they also highlight the fact that it imposes an unsatisfiable requirement to any three-valued logic –be it paraconsistent or paracomplete. This is the case of the well-known systems **LP** and $\mathbf{K_3}$ (see (Asenjo & Tamburino, 1975), (Priest, 2008), (Kleene, 1952)) and of course of two of its subsystems, the (infectious) logics **PWK** and $\mathbf{K_3^w}$.[26]

Given this lack of generality of *Contra* as an adequate constraint for defining classical negations, a second attempt is made: embracing a more *liberal* approach. A classical negation is taken to be a unary operator $\sim$ that allows to form *classical* contradictions. This requires, of course, defining what classical contradictions are, which is done by means of the following definitions:

Classical contraries α and β are classical contraries if and only if: if α is true, then β is untrue; if β is true, then α is untrue.

Classical subcontraries α and β are classical subcontraries if and only if: if α is untrue, then β is true; if β is untrue, then α is true.

Classical contradictories α and β are classical contradictories if and only if they are classical contraries and classical subcontraries.

In a nutshell, this second attempt boils down to the idea that a classical negation is a unary operator $\sim$ that satisfies the next requirement

Liberal The sentence α is true if and only if the sentence $\sim\alpha$ is untrue; the sentence α is untrue if and only if the sentence $\sim\alpha$ is true.[27]

Since *Liberal* is a semantic (as opposed to a syntactic[28]) approach to what a classical negation is, what should be discussed is how should a semantic behavior of this sort of operation be formalized. In this section we explore how to give a general extensional semantics for it, by means of non-deterministic matrices.

Before moving further, let us comment, first, that we prefer to analyze such semantics from the point of view of non-deterministic frameworks, not because we think that there is some inherent philosophical understanding of classical negation that should only be represented by this formalism and not by its deterministic counterpart. However, as we clarified above, non-deterministic matrices generalize the case of deterministic matrices. We think that in logic, mathematics and philosophy there is a certain affinity for generality, and thus this approach honors that affinity.[29]

[26] To see why *Contra* cannot be applied to these three-valued logics:

> [S]uppose we have only three values, $\mathfrak{t}$, $\mathfrak{b}$, and $\mathfrak{f}$. (It will not matter whether $\mathfrak{b}$ is designated.) In order to meet *Contra*, a classical negation $\sim$ must take $\mathfrak{t}$ to $\mathfrak{f}$ and conversely. Now what to do with $\mathfrak{b}$? It can't go to $\mathfrak{f}$, lest $\alpha \vee \sim\alpha$ not always take $\mathfrak{t}$. And it can't go to $\mathfrak{b}$ or $\mathfrak{t}$ either, lest $\alpha \vee \sim\alpha$ not always take $\mathfrak{t}$ or $\alpha \wedge \sim\alpha$ not always take $\mathfrak{f}$. In other words, there is no operation satisfying *Contra* in a three-valued setting. (De & Omori, 2015, p. 5)

[27] Omori and De use 'not true' instead of the more succinct 'untrue'. For the sake of terminological unity with the rest of the paper, we prefer to employ the latter.

[28] For more on why a syntactic approach to the definition of classical negation should be discarded, see (De & Omori, 2015, Section 2)

[29] Additionally, this does not imply that we think there is no other reasonable semantics (e.g. possible-worlds semantics, etc.) for such a negation that can be explored, but instead that we prefer to focus on this simple treatment here.

Moreover, recall that since deterministic matrices are just particular cases of Nmatrices, when we give below a formal semantic constraint for an operation to be a classical negation, it would help to give it in a non-deterministic fashion, for the deterministic case can be seen as a limit case of the more general setting. For a full account of non-deterministic *classical* negation, its technical and philosophical motivations, see (Omori & Szmuc, n.d.).

Remark 4. *We will refer, regardless of $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ being a matrix or a Nmatrix, to the set $\mathcal{U} = \mathcal{V} - \mathcal{D}$ as the set of undesignated values. Additionally, for any $\mathcal{M}$ we will say that the set $\mathcal{D}$ is the set of truth-values representing the notion of 'truth' in $\mathcal{M}$, and the set $\mathcal{U}$ is the set of truth-values representing the notion of 'untruth' in $\mathcal{M}$.*

As a consequence of this remarks, we now give an account of *Liberal* in a non-deterministic fashion, formally defining next how a classical negation should work.

Definition 14. *Let a logic* **L** *be identified with the pair $\langle \mathcal{L}, \vDash \rangle$, where $\vDash$ is induced by a matrix (Nmatrix) $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that $|\mathcal{D}| = n$ and $|\mathcal{U}| = m$, for $n, m \in \mathbb{N}$. A (non-deterministic) classical negation for* **L** *is a unary operator $\sim$ with an associated truth-function $f^{\sim} : \mathcal{V} \longrightarrow 2^{\mathcal{V}} - \emptyset$, such that the following holds:*

$$f^{\sim}(x) \subseteq \mathcal{D} \textit{ if and only if } x \in \mathcal{U}$$

$$f^{\sim}(x) \subseteq \mathcal{U} \textit{ if and only if } x \in \mathcal{D}$$

These facts can be represented more graphically in the (non-deterministic) truth-table of (Table 5)

	$f^{\sim}$
a_1	$\mathcal{B}_1 \subseteq \mathcal{U}$
$\vdots$	$\vdots$
a_n	$\mathcal{B}_n \subseteq \mathcal{U}$
b_1	$\mathcal{A}_1 \subseteq \mathcal{D}$
$\vdots$	$\vdots$
b_m	$\mathcal{A}_m \subseteq \mathcal{D}$

Table 5: Truth-table for non-deterministic classical negation

Given this definition, an operation of this sort can be either fully (non-)deterministic or (non-)deterministic to some degree.[30] It would be interesting to generalize this remarks to infinitely-valued logics (e.g. fuzzy logics) but for the sake of brevity we do not comment on this case here.

Remark 5. *Let a logic* **L** *be as in Definition 14. Let the set* **CN** *of classical negations for $\mathcal{M}$ be the union of the sets of deterministic classical negations and non-deterministic classical negations for* **L***, denoted respectively by* **detCN** *and* **ndetCN***. Then, $|\mathbf{CN}| = (2^n - 1)^m (2^m - 1)^n$, $|\mathbf{detCN}| = m^n n^m$, $|\mathbf{ndetCN}| = [(2^n - 1)^m (2^m - 1)^n] - (m^n n^m)$.*

Before moving forward, let us consider some concerns that might be raised by the previous discussion of classical negation.

[30] In previous papers, like (De & Omori, 2015), the investigation was only focused on fully deterministic classical negations for many-valued logics, but left the other cases unexplored. This, in fact, shows that there is room to analyze the other cases, as is done in (Omori & Szmuc, n.d.), whose results are applied to this paper.

An anonymous referee suggests that that every many-valued logic is by principle not classical and hence, if a negation belongs to a many-valued logic it should not be called a classical negation. But, on the one hand, not all many-valued logics are non-classical. For, it is a well-known fact that any Boolean algebra characterizes classical logic, even those Boolean algebras that have more than two elements. Thus, a logic based on a matrix whose underlying algebra is a Boolean algebra of more than two elements would still characterize classical logic. Moreover, since this many-valued Boolean algebra has –by definition– a Boolean complement, this very Boolean complement can be legitimately thought as a classical negation, even if it appears in a many-valued setting.

Another take on the previous objection could be that it sounds odd to say that a logic that drops bivalence has a negation that deserves to be called classical. But, as Roman Suszko stated in (Suszko, 1977), every structural Tarskian many-valued propositional logic can be provided with a bivalent semantics. This is done, roughly, by mapping every truth-value x that belongs to the set of designated values –i.e. $x \in \mathcal{D}$– to the value "true", and every truth-value x that belongs to the set of undesignated values –i.e. $x \notin \mathcal{D}$– to the value "false". Thus, every structural Tarskian many-valued propositional logic enjoys this sort of *generalized bivalence*.

Furthermore, every many-valued propositional logic that is induced by a matrix (Nmatrix) is a structural Tarskian many-valued propositional logic (see e.g. (Avron & Zamansky, 2011)). Given all the logics discussed in this paper are many-valued propositional logics induced by some matrix (Nmatrix), then Suszko's reduction holds for all the logics discussed in this paper. As a consequence, all the logics presented in this paper enjoy this form of generalized bivalence.

To finally dissolve the aforementioned concern let us notice that if Suszko's reduction is applied to the logics in this paper, the semantic clauses for classical negation –as they are presented above in this Section– will look exactly like the semantic clauses for negation in two-valued (bivalent) classical logic.

Thirdly, it might be thought that although *Liberal* is an appealing characterization of a classical contradictory-forming operator, it does not seem to be compatible with a non-deterministic semantics.[31] To this we shall reply as follows. *Liberal* says that if α is true, then $\sim\alpha$ is untrue. But in non-classical settings there are many ways for a formula to be untrue. An untrue formula can be just false (and not true), it can be neither true nor false, etc. Thus, when we say that if α is true, then its classical negation $\sim\alpha$ is untrue, this does not entail that the formula in question must be untrue in this or that way (i.e. that it is just false, or that it is neither true nor false, etc.). As such, *Liberal* allows all these possible ways of being untrue and, therefore, is compatible with choosing the way in which a classical negation is untrue out of a certain number of possibilities.

An analogous argument applies for the case of untrue formulae α whose classical negation $\sim\alpha$ is, therefore, true. In non-classical settings there are many ways for a formula to be true. A true formula can be just true (and not false), it can be both true and false, etc. Thus, when we say that if α is untrue, then its classical negation $\sim\alpha$ is true, this does not entail that the formula in question must be true in this or that way (i.e. that it is just true, or that it is both true and false, etc.) As such, *Liberal* allows all these possible ways of being true and, therefore, is compatible with choosing the way in which a classical negation is true out of a certain number of possibilities. This establishes our conclusion, that *Liberal* is compatible with a non-deterministic semantics.

A fourth remark about classical negation is that, as showed in (De & Omori, 2015), there are in

[31] In which case, it could be argued that the following alternative criterion, compatible with non-deterministic semantics, should be adopted instead

Liberal* If the sentence α is true, then the sentence $\sim\alpha$ is deterministically untrue or non-deterministically untrue;
If the sentence α is untrue, then the sentence $\sim\alpha$ is deterministically true or non-deterministically untrue.

However, given the disjunction of being deterministically true (untrue) and being non-deterministically true (untrue) is just being true (untrue), this alternative and *Liberal* do indeed coincide. Thanks to an anonymous referee for discussion on this matter.

fact *many* operations satisfying *Liberal* for many-valued non-classical logics. This is not, however, a consequence of the introduction of non-deterministic semantics. As is noted in (De & Omori, 2015), *Liberal* does *not* uniquely determine a classical negation even in the case of non-classical three or four-valued logics such as $\mathbf{K_3}$ or **LP**. However, this is not taken as something that precludes the operations satisfying *Liberal* from being actual classical negations, but rather something that (philosophical and technical reasons for choosing one of them over the other aside) makes all of them equally considerable, in principle.

Finally, let us arrive at a more technical and controversial matter. It is usually considered that the standard way to express the "toggle" between truth and untruth, i.e. the mutually exclusive and jointly exhaustive character of truth and untruth with regard to classical negation, is via the logical principles of *Ex Contradictione Quodlibet* and the *Law of Excluded Middle*.

$$\text{(ECQ) } \alpha \wedge {\sim}\alpha \vDash \beta \qquad\qquad \text{(LEM) } \beta \vDash \alpha \vee {\sim}\alpha$$

Thus, in a non-classical logic extended with a classical negation, it should be required for the sentence $\alpha \wedge {\sim}\alpha$ to be always untrue and for the sentence $\alpha \vee {\sim}\alpha$ to be always true. We will show that the there are some difficulties with regard to this, in the context of extensions of infectious logics with classical negation.

Proposition 6. *If* $\mathbf{L} = \langle \mathcal{L}, \vDash \rangle$*, where* $\vDash$ *is induced by a matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is an infectious logic, such that* $z \in \mathcal{D}$ *is its infectious value, and* $\mathbf{L}$ *is extended with a classical negation* $\sim$*, then* ECQ *is invalid.*

Proof. Assume $z \in \mathcal{D}$, i.e. z is designated. Assume $v(\alpha) = z$ and $v(\beta) \notin \mathcal{D}$. Given z is infectious, if $v(\alpha) = z$ then $v(\alpha \wedge {\sim}\alpha) = z$, no matter what the undesignated value of $v({\sim}\alpha)$ is. Hence, $\alpha \wedge {\sim}\alpha \nvDash \beta$, ECQ is invalid. □

Proposition 7. *If* $\mathbf{L} = \langle \mathcal{L}, \vDash \rangle$*, where* $\vDash$ *is induced by a matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is an infectious logic, such that* $z \in \mathcal{U} = \mathcal{V} - \mathcal{D}$ *is its infectious value, and* $\mathbf{L}$ *is extended with a classical negation* $\sim$*, then* LEM *is invalid.*

Proof. Assume $z \notin \mathcal{D}$, i.e. z is undesignated. Assume $v(\alpha) = z$ and $v(\beta) \in \mathcal{D}$. Given z is infectious, if $v(\alpha) = z$ then $v(\alpha \vee {\sim}\alpha) = z$, no matter what the designated value of $v({\sim}\alpha)$ is. Hence, $\beta \nvDash \alpha \vee {\sim}\alpha$, LEM is invalid. □

Given the previous shortcomings of representing the exclusive and exhaustive character of classical negation at the object language, we will now propose a natural solution that solves the problem by expressing it in the meta-language.[32] To do so, we will need to adopt the multiple conclusion approach, outlined by Definition 7 of Section 2..

Secondly, we will propose to shift from expressing the exclusive and exhaustive character of truth and untruth with ECQ and LEM to expressing them with their meta-theoretical versions in a multiple conclusions setting. We will call these versions of these principles (ECQ^+) and (LEM^+).[33]

$$(\text{ECQ}^+) \ \alpha, {\sim}\alpha \vDash \beta \qquad\qquad (\text{LEM}^+) \ \beta \vDash \alpha, {\sim}\alpha$$

[32] An anonymous referee suggests considering Dummet's *Weak Law of Excluded Middle* $\beta \vDash {\sim}\alpha \vee {\sim}{\sim}\alpha$ and the *Weak Principle of Explosion* ${\sim}\alpha \wedge {\sim}{\sim}\alpha \vDash \beta$ from (Ferguson, 2014b). However, it is easy to see that given the possibly non-deterministic character of $\sim$, these are not guaranteed to hold. Let z be an infectious value (regardless if it is the case that $z \in \mathcal{D}$ or $z \notin \mathcal{D}$). In both cases, if $v(\alpha) = z$, this does not exclude the possibility of $v({\sim}{\sim}\alpha) = z$, whence the failure of these Weak versions of LEM and ECQ, in the respective cases.

[33] Notice that the former is canonically called the Principle of Explosion, while the latter does not enjoy a canonical denomination. The latter is sometimes called the Principle of Implosion in e.g. (Marcos, 2005) and (Corbalán, 2012), but this terminology might not be so felicitous, despite its duality with the case of Explosion, though. This is why we preferred to stick with the above symmetrical denominations (ECQ^+) and (LEM^+).

Remark 8. *If* $\mathbf{L} = \langle \mathcal{L}, \models^+ \rangle$*, where* $\models^+$ *is induced by a matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is an infectious logic, such that* $z \in \mathcal{D}$ *is its infectious value, and* $\mathbf{L}$ *is extended with a classical negation* $\sim$*, then* ECQ$^+$ *is valid.*

Proof. Follows in light of Definition 14. □

Remark 9. *If* $\mathbf{L} = \langle \mathcal{L}, \models^+ \rangle$*, where* $\models^+$ *is induced by a matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *is an infectious logic, such that* $z \in \mathcal{U} = \mathcal{V} - \mathcal{D}$ *is its infectious value, and* $\mathbf{L}$ *is extended with a classical negation* $\sim$*, then* LEM$^+$ *is valid.*

Proof. Follows in light of Definition 14. □

It could be argued that it seems strange to say that a negation for which e.g. ECQ is not valid is classical. To this we shall reply, on the one hand, that ECQ is not invalid because of the semantics of the negation, but because of the semantics of conjunction. As we showed above this feature can be explicitly highlighted by looking at the validity of the meta-theoretical version of ECQ, i.e. ECQ$^+$. As remarked by an anonymous referee, the assymetry between the invalid character of ECQ and the valid character of ECQ$^+$ lies in the non-adjunctive nature of paraconsistent infectious logics, not in the non-explosiveness of the extended systems. Similar remarks apply, *mutatis mutandis*, to the case of LEM and its meta-theoretical version LEM$^+$.

Having discussed these issues, we now proceed to consider the prospect of extending infectious logics with classical negation, in order to define consistency and undeterminedness operators, to finally obtain LFIs and LFUs.

4..3 Defining the operators

Proposition 10. *Let* $\mathbf{L}$ *be a paraconsistent logic that has some infectious value. Let* $\mathbf{L}$ *be extended with a classical negation* $\sim$.[34] *Then, in the context of the extended system, the formula* $\sim(\alpha \wedge \neg\alpha)$ *defines a consistency operator* $\circ\alpha$ *and the formula* $\sim\sim(\alpha \wedge \neg\alpha)$ *defines an inconsistency operator* $\bullet\alpha$*. As a consequence, such an extension of* $\mathbf{L}$ *is a Logic of Formal Inconsistency defined in an extension of an infectious logic.*

Proof. We have two cases: either (i) $v(\alpha) \in \mathcal{D}$ and $v(\neg\alpha) \in \mathcal{D}$, whence $v(\alpha \wedge \neg\alpha) \in \mathcal{D}$; or (ii) at least one of $v(\alpha) \notin \mathcal{D}$ or $v(\neg\alpha) \notin \mathcal{D}$, whence $v(\alpha \wedge \neg\alpha) \notin \mathcal{D}$.

For the case of the consistency operator the only important case is the first, where $v(\alpha)$ is such that $v(\alpha) \in \mathcal{D}$ and $v(\neg\alpha) \in \mathcal{D}$. Then, $v(\alpha \wedge \neg\alpha) \in \mathcal{D}$ and thus $v(\sim(\alpha \wedge \neg\alpha)) \notin \mathcal{D}$. If we set $\circ\alpha =_{def} \sim(\alpha \wedge \neg\alpha)$, then $\circ\alpha, \alpha, \neg\alpha \models \beta$. We leave the other case to the reader.

For the inconsistency operator, as above, it can be the case that (i) $v(\alpha \wedge \neg\alpha) \in \mathcal{D}$, or that (ii) $v(\alpha \wedge \neg\alpha) \notin \mathcal{D}$. In the former case, $v(\sim(\alpha \wedge \neg\alpha)) \notin \mathcal{D}$ and therefore $v(\sim(\sim(\alpha \wedge \neg\alpha)) \in \mathcal{D}$. In the latter case, $v(\sim(\alpha \wedge \neg\alpha)) \in \mathcal{D}$ and therefore $v(\sim(\sim(\alpha \wedge \neg\alpha)) \notin \mathcal{D}$, as expected. □

Remark 11. *Notice that a proper inconsistency operator* $\bullet\alpha$ *is not necessarily definable by putting any negation in front of* $\circ\alpha$*, in particular not by putting the usual non-classical negation* $\neg$*. Recall that* $\circ\alpha =_{def} \sim(\alpha \wedge \neg\alpha)$*, and let* $v(\alpha) \in \mathcal{D}$ *and* $v(\neg\alpha) \notin \mathcal{D}$*. Thus,* $v(\sim(\alpha \wedge \neg\alpha)) = v(\circ\alpha) \in \mathcal{D}$*. However, we do not know which designated value* $\circ\alpha$ *receives in this valuation, because* $\sim$ *can be non-deterministic. Hence, for all we know it could be the case that* $v(\circ\alpha) \in \mathcal{D}$ *and* $v(\neg \circ \alpha) \in \mathcal{D}$*, in which case* $\neg \circ \alpha$ *will not work as a proper inconsistency operator.*

[34]Notice that the extended system is not infectious anymore, particularly due to the involvement of classical negation that sends the infectious value to some *other* value.

Proposition 12. *Let* **L** *be a paracomplete logic that has some infectious value. Let* **L** *be extended with a classical negation* $\sim$*. Then, in the context of the extended system, the formula* $\alpha \vee \neg\alpha$ *defines a determinedness operator* $\star(\alpha)$ *and the formula* $\sim(\alpha \vee \neg\alpha)$ *defines an undeterminedness operator* $\bigstar$*. As a consequence, such an extension of* **L** *is a Logic of Formal Undeterminedness defined in an extension of an infectious.*

Proof. Dual to the case of Proposition 10. $\square$

Remark 13. *Notice that a proper undeterminedness operator* $\bigstar$ *is not necessarily definable by putting any negation in front of* $\star\alpha$*, in particular not by putting the usual non-classical negation* $\neg$*. Recall that* $\star\alpha =_{def} \alpha \vee \neg\alpha$*, and let* $v(\alpha) \notin \mathcal{D}$ *and* $v(\neg\alpha) \notin \mathcal{D}$*. Thus,* $v(\alpha \vee \neg\alpha) = v(\star\alpha) \notin \mathcal{D}$*. Therefore,* $v(\neg(\alpha \vee \neg\alpha)) \notin \mathcal{D}$*, in which case* $\neg\star\alpha$ *will not work as a proper undeterminedness operator.*

5. Defining LFIs and LFUs in extensions of infectious logics

We now state the next results, which follow straightforwardly from the previous propositions and definitions regarding infectious logics extended with any classical negation, be it deterministic or non-deterministic.

Corollary 14. *The systems*

- **PWK** *extended with any of its* 3 *classical negations can define a consistency operator* $\circ(\alpha)$ *and an inconsistency operator* $\bullet(\alpha)$ *with the help of the formulae of Proposition 10, and*
- $\mathbf{L}_{\mathfrak{bb}'}$ *extended with any of its* 7 *classical negations can define a consistency operator* $\circ(\alpha)$ *and an inconsistency operator* $\bullet(\alpha)$ *with the help of the formulae of Proposition 10,*

and are therefore LFIs defined in extensions of infectious logics.

The systems

- $\mathbf{K_3^w}$ *extended with any of its* 3 *classical negations can define a determinedness operator* $\star(\alpha)$ *and an undeterminedness operator* $\bigstar(\alpha)$*, with the help of the formulae of Proposition 12, and*
- $\mathbf{L}_{\mathfrak{nc}}$ *extended with any of its* 7 *classical negations can define a determinedness operator* $\star(\alpha)$ *and an undeterminedness operator* $\bigstar(\alpha)$*, with the help of the formulae of Proposition 12,*

and are therefore LFUs defined in extensions of infectious logics.

The systems

- $\mathbf{L}_{\mathfrak{b}'\mathfrak{c}}$ *extended with any of its* 81 *classical negations, and*
- $\mathbf{S}_{\mathtt{fde}}$ *extended with any of its* 81 *classical negations, and*
- $\mathbf{L}_{\mathfrak{nb}'}$ *extended with any of its* 81 *classical negations*
- $\mathbf{L}_{\mathfrak{cb}'}$ *extended with any of its* 81 *classical negations, and*
- $\mathbf{L}_{\mathfrak{nb}'\mathfrak{c}}$ *extended with any of its* 1323 *classical negations, and*
- $\mathbf{FDE}_{\varphi}$ *extended with any of its* 1323 *classical negations, and*

- $\mathbf{L}_{\mathfrak{b}\mathfrak{b}'\mathfrak{e}}$ *extended with any of its* 1323 *classical negations, and*
- $\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}'}$ *extended with any of its* 1323 *classical negations, and*
- $\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'}$ *extended with any of its* 1323 *classical negations, and*
- $\mathbf{L}_{\mathfrak{b}\mathfrak{e}\mathfrak{b}'}$ *extended with any of its* 1323 *classical negations, and*
- $\mathbf{L}_{\mathfrak{n}\mathfrak{b}\mathfrak{b}'\mathfrak{e}}$ *extended with any of its* 117649 *classical negations, and*
- $\mathbf{L}_{\mathfrak{n}\mathfrak{e}\mathfrak{b}\mathfrak{b}'}$ *extended with any of its* 117649 *classical negations*

can all define a consistency operator $\circ(\alpha)$*, an inconsistency operator* $\bullet(\alpha)$*, a determinedness operator* $\star(\alpha)$ *and an undeterminedness operator* $\bigstar(\alpha)$*, with the help of the formulae of Propositions 10 and 12.*

Therefore, they all constitute LFIs and LFUs defined in extensions of infectious logics.

These facts ultimately amount to the presentation of new LFIs and LFUs defined in extensions of infectious logics –indeed, of a new collection of LFIs and LFUs that might be orthogonal to the hierarchy built on top of **mbC** and similar systems by Carnielli, Coniglio, Marcos, and others.[35]

Earlier in this paper, we said that LFIs and LFUs built on top of infectious logics will probably constitute logically weak LFIs and LFUs. By that we meant that some inferences, such as disjunction-introduction and conjunction-elimination, which are taken to be very basic, fail in these systems. These can, nevertheless, be recovered with the help of the various operators that are definable with the aid of classical negation.

For the following results let us recall that we take **L** to be one of the logics above, identified with the pair $\langle \mathcal{L}, \vDash \rangle$ such that $\vDash$ is induced by the matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ and, furthermore, that $\mathcal{V} \subseteq \{\mathfrak{t}, \mathfrak{b}, \mathfrak{b}', \mathfrak{n}, \mathfrak{e}, \mathfrak{f}\}$ for every such set. Finally, when we say that $x, y \in \mathcal{V}$, we take it that these values interact between themselves and the classical truth values in the intended way; and that for the case of two allegedly infectious value, they interact as clarified in each of the following cases.

Proposition 15. *Let* **L** *one of the logics above that has a designated infectious value* z*. Let* **L** *be extended with a classical negation. By the above, the resulting system is a LFI which counts with a consistency operator* $\circ(\alpha)$ *definable as in Proposition 10. In such a system the following are the case:*

1. $\alpha \wedge \beta \nvDash \alpha$
2. $\circ\alpha, \alpha \wedge \beta \nvDash \alpha$
3. *However* $\circ\beta, \alpha \wedge \beta \vDash \alpha$

Proof. Let v be a valuation for **L** such that $v(\alpha) \notin \mathcal{D}$ and $v(\beta) = z$.

1. Then, given z is infectious, $v(\alpha \wedge \beta) = z$ and thus $\alpha \wedge \beta \nvDash \alpha$
2. Let v be as before. Then, by Proposition 10 $v(\circ\alpha) \in \mathcal{D}$ and thus $\circ\alpha, \alpha \wedge \beta \nvDash \alpha$
3. We assume that $\circ\beta$ and $\alpha \wedge \beta$ are true, and then show that α is true. Let, then, $v(\circ\beta) \in \mathcal{D}$ and $v(\alpha \wedge \beta) \in \mathcal{D}$. From the former we infer that either $v(\beta) \notin \mathcal{D}$ or $v(\neg\beta) \notin \mathcal{D}$. Thus, it follows that $v(\beta) \notin \{\mathfrak{b}, \mathfrak{b}'\}$. From this, together with the fact that $v(\alpha \wedge \beta) \in \mathcal{D}$, we can infer by looking at the truth-tables above and at Definition 13 that $v(\alpha) \in \mathcal{D}$.

[35]Moreover, the logic **FDE** extended with any of its classical negations can also define consistency, inconsistency, determinedness and undeterminedness operators. It is not an infectious logic, though.

□

Proposition 16. *Let* **L** *be one of the logics above that has an undesignated infectious value* z*. Let* **L** *be extended with a classical negation. By the above, the resulting system is a LFU which counts with a determinedness operator* ☆ *definable as in Proposition 12. In such a system the following are the case:*

1. $\alpha \nvDash \alpha \vee \beta$
2. $☆\alpha, \alpha \nvDash \alpha \vee \beta$
3. *However* $☆\beta, \alpha \vDash \alpha \vee \beta$

Proof. Let v be a valuation for **L** such that $v(\alpha) \in \mathcal{D}$ and $v(\beta) = z$.

1. Then, given z is infectious, $v(\alpha \vee \beta) = z$ and thus $\alpha \nvDash \alpha \vee \beta$
2. Let v be as before. Then, by Proposition 12 $v(☆\alpha) \in \mathcal{D}$ and thus $☆\alpha, \alpha \nvDash \alpha \vee \beta$
3. We assume that $☆\beta$ and α are true, and then show that $\alpha \vee \beta$ is true. Let, then, $v(☆\beta) \in \mathcal{D}$ and $v(\alpha) \in \mathcal{D}$. From the former we infer that $v(\beta \vee \neg\beta) \in \mathcal{D}$ and thus that either $v(\beta) \in \mathcal{D}$ or $v(\neg\beta) \in \mathcal{D}$. Thus, it follows that $v(\beta) \notin \{\mathfrak{n}, \mathfrak{e}\}$. From the latter, we infer that $v(\alpha) \in \{\mathfrak{t}, \mathfrak{b}, \mathfrak{b}'\}$, whence by looking at the truth-tables above and at Definition 13 this guarantees that $v(\alpha \vee \beta) \in \mathcal{D}$.

□

We show below how these logical shortcomings will be also reflected in the failure of some *propagation* and *retropropagation* properties (as called, for example, by (Corbalán, 2012) and (Carnielli & Coniglio, 2016)).

Proposition 17. *Let* **L** *one of the logics above that has a designated infectious value* $\mathfrak{b}'$*. Let* **L** *be supplemented with a classical negation. By the above, the resulting system is a LFI which counts with a consistency operator* $\circ$ *definable as in Proposition 10. In such a system the following are the case:*

1. $\circ\alpha \nvDash \circ(\alpha \wedge \beta)$
2. $\circ\alpha \nvDash \circ(\alpha \vee \beta)$
3. $\circ\alpha, \circ\beta \vDash \circ(\alpha \wedge \beta)$
4. $\circ\alpha, \circ\beta \vDash \circ(\alpha \vee \beta)$
5. *If* $\mathfrak{b} \in \mathcal{V}$*:* $\circ(\alpha \wedge \beta) \nvDash \circ\alpha \wedge \circ\beta$
6. *If* $\mathfrak{b} \in \mathcal{V}$*:* $\circ(\alpha \vee \beta) \nvDash \circ\alpha \wedge \circ\beta$
7. *If* $\mathfrak{b} \notin \mathcal{V}$*:* $\circ(\alpha \wedge \beta) \vDash \circ\alpha \wedge \circ\beta$
8. *If* $\mathfrak{b} \notin \mathcal{V}$*:* $\circ(\alpha \vee \beta) \vDash \circ\alpha \wedge \circ\beta$
9. $\circ(\alpha \wedge \beta) \vDash \circ\alpha, \circ\beta$
10. $\circ(\alpha \vee \beta) \vDash \circ\alpha, \circ\beta$

Proof. Straightforward from the above definitions. □

Proposition 18. *Let* **L** *one of the logics above that has an undesignated infectious value* $\mathfrak{e}$*. Let* **L** *be supplemented with a classical negation. By the above, the resulting system is a LFU which counts with an undeterminedness operator* $\bigstar(\alpha)$ *definable as in Proposition 12. In such a system the following are the case:*

1. *If* $\mathfrak{n} \in \mathcal{V}$*:* $\bigstar\alpha \nvDash \bigstar(\alpha \wedge \beta)$
2. *If* $\mathfrak{n} \in \mathcal{V}$*:* $\bigstar\alpha \nvDash \bigstar(\alpha \vee \beta)$
3. *If* $\mathfrak{n} \notin \mathcal{V}$*:* $\bigstar\alpha \vDash \bigstar(\alpha \wedge \beta)$
4. *If* $\mathfrak{n} \notin \mathcal{V}$*:* $\bigstar\alpha \vDash \bigstar(\alpha \vee \beta)$
5. $\bigstar\alpha, \bigstar\beta \vDash \bigstar(\alpha \wedge \beta)$
6. $\bigstar\alpha, \bigstar\beta \vDash \bigstar(\alpha \vee \beta)$
7. $\bigstar(\alpha \wedge \beta) \nvDash \bigstar\alpha \wedge \bigstar\beta$
8. $\bigstar(\alpha \vee \beta) \nvDash \bigstar\alpha \wedge \bigstar\beta$
9. $\bigstar(\alpha \wedge \beta) \vDash \bigstar\alpha, \bigstar\beta$
10. $\bigstar(\alpha \vee \beta) \vDash \bigstar\alpha, \bigstar\beta$

Proof. Straightforward from the above definitions. □

It might be the interest for the reader to have a proof system associated with the above infectious logics and, moreover, with their extensions with classical negations that constitute genuine LFIs and LFUs. Therefore, in what follows we will be giving details –based on techniques introduced by (Avron & Konikowska, 2005)– on how to provide calculi of signed formulae for all of the above systems.

5..1 Proof-theory

5..1.1 Preliminaries

Definition 15 ((Avron & Konikowska, 2005))**.** *Let* $\mathcal{V}$ *be a finite set of truth-values, let* $\mathcal{L}$ *be a propositional language whose set of well-formed formulae is* $\mathsf{Form}_{\mathcal{L}}$*, and let* $\mathcal{M}$ *be a Nmatrix for* $\mathcal{L}$ *with* $\mathcal{V}$ *as its set of truth values.*

- *A signed formula over* $\mathcal{L}$ *and* $\mathcal{V}$ *is an expression of the form* $a : \varphi$*, where* $a \in \mathcal{V}, \varphi \in \mathsf{Form}_{\mathcal{L}}$
- *A valuation* v *for* $\mathcal{M}$ *satisfies a signed formula* $a : \varphi$ *–in symbols* $v \vDash a : \varphi$*– if* $v(\varphi) = a$

Signed formulae will be denoted by $\varphi, \psi, \dots$ and sets of signed formulae by $\Omega, \Sigma, \dots$

Definition 16 ((Avron & Konikowska, 2005))**.** *In terms of satisfaction by a valuation, sets of signed formulae will be interpreted disjunctively.*

- *A valuation* v *for* $\mathcal{M}$ *satisfies a set of signed formulae* Ω *–in symbols* $v \vDash \Omega$*– if and only if it satisfies some signed formulae* $\varphi \in \Omega$
- *A set of signed formulae* Ω *is said to be valid in an Nmatrix* $\mathcal{M}$ *–in symbols* $\vDash_{\mathcal{M}} \Omega$*– if* $v \vDash \Omega$ *for every valuation* v *for* $\mathcal{M}$*.*

As before, with regard to a (N)matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we will refer to the set $\mathcal{U} = \mathcal{V} - \mathcal{D}$ as the set of undesignated values. Furthermore, for any set of truth-values $A \subseteq \mathcal{V}$, and any set of formulae $F \subseteq \mathsf{Form}_{\mathcal{L}}$, we denote $A : F = \{a : \psi \mid a \in A, \psi \in F\}$ (see (Avron & Konikowska, 2005, p. 6)).

Proposition 19 ((Avron & Konikowska, 2005))**.** *For any Nmatrix over $\mathcal{V}$ and any finite set of formulae Γ, Δ it happens that $\Gamma \vDash_{\mathcal{M}} \Delta$ if and only if the set of signed formulae $(\mathcal{U} : \Gamma) \cup (\mathcal{D} : \Delta)$ is valid in $\mathcal{M}$. In particular, a formula φ is valid in $\mathcal{M}$ if and only if the set $\mathcal{D} : \{\varphi\}$ is valid in $\mathcal{M}$.*

A deduction system $SF_{\mathcal{M}}$ based on finite sets of signed formulae for an n-valued Nmatrix $\mathcal{M}$ contains:

- **Axioms:** Each set of signed formulae containing $\{a : \varphi \mid a \in \mathcal{V}\}$, where φ is any formulae in $\mathsf{Form}_{\mathcal{L}}$
- **Rules:** For every $m-$ary connective $\diamond$ and any logical values $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_k \in \mathcal{V}$ such that $f^{\diamond}_{\mathcal{M}}(a_1, \ldots, a_m) = \{b_1, \ldots, b_k\}$, the rule:

$$\diamond\{b_1, \ldots, b_k\} \quad \frac{\Omega, a_1 : \varphi_1 \qquad \ldots \qquad \Omega, a_m : \varphi_m}{\Omega, b_1 : \diamond(\varphi_1, \ldots, \varphi_m), \ldots, b_k : \diamond(\varphi_1, \ldots, \varphi_m)}$$

Some "streamlining principles" for deleting, replacing and combining two rules with the same conclusion are presented in (Avron & Konikowska, 2005). These are applied to compress the calculus presented below, but we do not reproduce them here for matters of space. Although they incarnate natural and straightforward ideas, readers interested in a detailed account are referred to (Avron & Konikowska, 2005, p. 8).

- **Admissible Cuts:** The analogue of the well-known Cut-rule for ordinary (two-sided) sequents is the following generalized Cut-rule for sets of signed formulae –which, as proved in (Avron & Konikowska, 2005), are admissible in $SF_{\mathcal{M}}$:

$$\frac{\Omega \cup \{i : \varphi \mid i \in I\} \qquad \Omega \cup \{j : \varphi \mid j \in J\}}{\Omega} \text{ for } I, J \subseteq \mathcal{V}, I \cap J = \emptyset$$

Remark 20 ((Avron & Konikowska, 2005))**.** *The Weakening Rule is admissible in $SF_{\mathcal{M}}$.*

Definition 17 ((Avron & Konikowska, 2005))**.** *Let $\mathcal{M}$ be a Nmatrix, and let $SF_{\mathcal{M}}$ be a deduction system based on finite sets of signed formulae over the language $\mathcal{L}$ and the set of truth-values of $\mathcal{M}$.*

- *$SF_{\mathcal{M}}$ is complete for $\mathcal{M}$ if for all finite sets of formulae Γ, Δ it happens that $\Gamma \vDash_{\mathcal{M}} \Delta$ if and only if $\vdash_{SF_{\mathcal{M}}} (\mathcal{U} : \Gamma) \cup (\mathcal{D} : \Delta)$*
- *$SF_{\mathcal{M}}$ is weakly complete for $\mathcal{M}$ if for all formulae φ it happens that $\vDash_{\mathcal{M}} \varphi$ if and only if $\vdash_{SF_{\mathcal{M}}} (\mathcal{D} : \{\varphi\})$*
- *$SF_{\mathcal{M}}$ is fully complete for $\mathcal{M}$ if for all sets of formulae Ω it happens that $\vDash_{\mathcal{M}} \Omega$ if and only if $\vdash_{SF_{\mathcal{M}}} \Omega$*

Remark 21 ((Avron & Konikowska, 2005))**.** *Notice that by Proposition 19 full completeness implies completeness, which in turn implies weak completeness.*

Theorem 22 ((Avron & Konikowska, 2005))**.** *The calculus $SF_{\mathcal{M}}$ is sound and fully complete for $\mathcal{M}$*

Notice, of course, that the technique of (Avron & Konikowska, 2005) can be easily adapted to provide cut-free calculus of signed formulae for any of the infectious logics above, and for any of their extensions with classical negation.

In fact, we will be providing a concrete example of a calculus for the logic $\mathbf{L}^*_{\mathfrak{eb}'}$, i.e. the infectious logic $\mathbf{L}_{\mathfrak{eb}'}$ extended with a non-deterministic classical negation. For obvious reasons of space we cannot do the same for each of the logics described above.

5..1.2 An example

Let $\mathbf{L}^*_{\mathfrak{eb}'}$ be the extension of the logic $\mathbf{L}_{\mathfrak{eb}'}$ with a non-deterministic classical negation, i.e. the logic which is interprets the language $\mathcal{L}\cup\{\sim\}$ and is induced by the matrix $\mathcal{M}^*_{\mathbf{L}_{\mathfrak{eb}'}} = \langle \mathcal{V}_{\mathbf{L}_{\mathfrak{eb}'}}, \mathcal{D}_{\mathbf{L}_{\mathfrak{eb}'}}, \mathcal{O}^*_{\mathbf{L}_{\mathfrak{eb}'}}\rangle$, where $\mathcal{O}^*_{\mathbf{L}_{\mathfrak{eb}'}} = \mathcal{O}_{\mathbf{L}_{\mathfrak{eb}'}} \cup \{f^{\sim}_{\mathbf{L}^*_{\mathfrak{eb}'}}\}$. Since we defined the elements $\mathcal{V}_{\mathbf{L}_{\mathfrak{eb}'}}, \mathcal{D}_{\mathbf{L}_{\mathfrak{eb}'}}$ and $\mathcal{O}_{\mathbf{L}_{\mathfrak{eb}'}}$ in Definition 10, we just need to spell out the truth-function for the non-deterministic classical negation, i.e. $f^{\sim}_{\mathbf{L}^*_{\mathfrak{eb}'}}$. We do this in (Table 6) letting $\mathcal{U}_{\mathbf{L}_{\mathfrak{eb}'}} = \mathcal{V}_{\mathbf{L}_{\mathfrak{eb}'}} - \mathcal{D}_{\mathbf{L}_{\mathfrak{eb}'}}$.

	$f^{\sim}_{\mathbf{L}^*_{\mathfrak{eb}'}}$
$\mathfrak{t}$	$\mathcal{B}_1 \subseteq \mathcal{U}_{\mathbf{L}_{\mathfrak{eb}'}}$
$\mathfrak{b}'$	$\mathcal{B}_2 \subseteq \mathcal{U}_{\mathbf{L}_{\mathfrak{eb}'}}$
$\mathfrak{e}$	$\mathcal{A}_1 \subseteq \mathcal{D}_{\mathbf{L}_{\mathfrak{eb}'}}$
$\mathfrak{f}$	$\mathcal{A}_2 \subseteq \mathcal{D}_{\mathbf{L}_{\mathfrak{eb}'}}$

Table 6: Truth-table for non-deterministic classical negation for the logic $\mathbf{L}_{\mathfrak{eb}'}$

The deduction system $SF_{\mathbf{L}^*_{\mathfrak{eb}'}}$ for the logic $\mathbf{L}^*_{\mathfrak{eb}'}$, induced by the Nmatrix $\mathcal{M}^*_{\mathbf{L}_{\mathfrak{eb}'}}$ contains the following items:

- **Axioms:** Each set of signed formulae containing $\{a : \varphi \mid a \in \{\mathfrak{t}, \mathfrak{b}', \mathfrak{e}, \mathfrak{f}\}\}$, where φ is any formulae in $\mathsf{Form}_{\mathcal{L}\cup\{\sim\}}$
- **Rules:**

$$\neg\{\mathfrak{f}\}\ \frac{\Omega, \mathfrak{t} : \varphi}{\Omega, \mathfrak{f} : \neg\varphi} \qquad \neg\{\mathfrak{b}'\}\ \frac{\Omega, \mathfrak{b}' : \varphi}{\Omega, \mathfrak{b}' : \neg\varphi} \qquad \neg\{\mathfrak{e}\}\ \frac{\Omega, \mathfrak{e} : \varphi}{\Omega, \mathfrak{e} : \neg\varphi} \qquad \neg\{\mathfrak{t}\}\ \frac{\Omega, \mathfrak{f} : \varphi}{\Omega, \mathfrak{t} : \neg\varphi}$$

$$\wedge\{\mathfrak{t}\}\ \frac{\Omega, \mathfrak{t} : \varphi \qquad \Omega, \mathfrak{t} : \psi}{\Omega, \mathfrak{t} : \varphi\wedge\psi} \qquad \wedge\{\mathfrak{b}'\}\ \frac{\Omega, \mathfrak{b}' : \varphi, \mathfrak{b}' : \psi}{\Omega, \mathfrak{b}' : \varphi\wedge\psi}$$

$$\wedge\{\mathfrak{e}\}_1\ \frac{\Omega, \mathfrak{t} : \varphi \qquad \Omega, \mathfrak{e} : \psi}{\Omega, \mathfrak{e} : \varphi\wedge\psi} \qquad \wedge\{\mathfrak{e}\}_2\ \frac{\Omega, \mathfrak{e} : \varphi \qquad \Omega, \mathfrak{t} : \psi}{\Omega, \mathfrak{e} : \varphi\wedge\psi} \qquad \wedge\{\mathfrak{e}\}_3\ \frac{\Omega, \mathfrak{e} : \varphi \qquad \Omega, \mathfrak{e} : \psi}{\Omega, \mathfrak{e} : \varphi\wedge\psi}$$

$$\wedge\{\mathfrak{e}\}_4\ \frac{\Omega, \mathfrak{f} : \varphi \qquad \Omega, \mathfrak{e} : \psi}{\Omega, \mathfrak{e} : \varphi\wedge\psi} \qquad \wedge\{\mathfrak{e}\}_5\ \frac{\Omega, \mathfrak{e} : \varphi \qquad \Omega, \mathfrak{f} : \psi}{\Omega, \mathfrak{e} : \varphi\wedge\psi}$$

$$\wedge\{\mathfrak{f}\}_1\ \frac{\Omega, \mathfrak{t} : \varphi \qquad \Omega, \mathfrak{f} : \psi}{\Omega, \mathfrak{f} : \varphi\wedge\psi} \qquad \wedge\{\mathfrak{f}\}_2\ \frac{\Omega, \mathfrak{f} : \varphi \qquad \Omega, \mathfrak{t} : \psi}{\Omega, \mathfrak{f} : \varphi\wedge\psi} \qquad \wedge\{\mathfrak{f}\}_3\ \frac{\Omega, \mathfrak{f} : \varphi \qquad \Omega, \mathfrak{f} : \psi}{\Omega, \mathfrak{f} : \varphi\wedge\psi}$$

$$\vee\{\mathfrak{t}\}_1\ \frac{\Omega, \mathfrak{t} : \varphi \qquad \Omega, \mathfrak{f} : \psi}{\Omega, \mathfrak{t} : \varphi\vee\psi} \qquad \vee\{\mathfrak{t}\}_2\ \frac{\Omega, \mathfrak{f} : \varphi \qquad \Omega, \mathfrak{t} : \psi}{\Omega, \mathfrak{t} : \varphi\vee\psi} \qquad \vee\{\mathfrak{t}\}_3\ \frac{\Omega, \mathfrak{t} : \varphi \qquad \Omega, \mathfrak{t} : \psi}{\Omega, \mathfrak{t} : \varphi\vee\psi}$$

$$\vee\{\mathfrak{b}'\}\ \frac{\Omega, \mathfrak{b}' : \varphi, \mathfrak{b}' : \psi}{\Omega, \mathfrak{b}' : \varphi\vee\psi}$$

$$\vee\{\mathfrak{e}\}_1\ \frac{\Omega, \mathfrak{t} : \varphi \qquad \Omega, \mathfrak{e} : \psi}{\Omega, \mathfrak{e} : \varphi \vee \psi} \qquad \vee\{\mathfrak{e}\}_2\ \frac{\Omega, \mathfrak{e} : \varphi \qquad \Omega, \mathfrak{t} : \psi}{\Omega, \mathfrak{e} : \varphi \vee \psi} \qquad \vee\{\mathfrak{e}\}_3\ \frac{\Omega, \mathfrak{e} : \varphi \qquad \Omega, \mathfrak{e} : \psi}{\Omega, \mathfrak{e} : \varphi \vee \psi}$$

$$\vee\{\mathfrak{e}\}_4\ \frac{\Omega, \mathfrak{f} : \varphi \qquad \Omega, \mathfrak{e} : \psi}{\Omega, \mathfrak{e} : \varphi \vee \psi} \qquad \vee\{\mathfrak{e}\}_5\ \frac{\Omega, \mathfrak{e} : \varphi \qquad \Omega, \mathfrak{f} : \psi}{\Omega, \mathfrak{e} : \varphi \vee \psi}$$

$$\vee\{\mathfrak{f}\}\ \frac{\Omega, \mathfrak{f} : \varphi \qquad \Omega, \mathfrak{f} : \psi}{\Omega, \mathfrak{f} : \varphi \vee \psi}$$

$$\sim\{\mathfrak{e}, \mathfrak{f}\}\ \frac{\Omega, \mathfrak{t} : \varphi, \mathfrak{b}' : \varphi}{\Omega, \mathfrak{e} : \sim\varphi, \mathfrak{f} : \sim\varphi} \qquad \sim\{\mathfrak{t}, \mathfrak{b}'\}\ \frac{\Omega, \mathfrak{e} : \varphi, \mathfrak{f} : \varphi}{\Omega, \mathfrak{t} : \sim\varphi, \mathfrak{b}' : \sim\varphi}$$

- **Admissible Cuts:** The analogue of the well-known Cut-rule for ordinary (two-sided) sequents is the following generalized Cut-rule for sets of signed formulae –which as a corollary of (Avron & Konikowska, 2005, Theorem 4.6) is admissible in this calculus:

$$\frac{\Omega \cup \{i : \varphi \mid i \in I\} \qquad \Omega \cup \{j : \varphi \mid j \in J\}}{\Omega} \text{ for } I, J \subseteq \{\mathfrak{t}, \mathfrak{b}', \mathfrak{e}, \mathfrak{f}\}, I \cap J = \emptyset$$

Remark 23. *Being defined connectives in e.g.* $\mathbf{L}^*_{\mathfrak{e}\mathfrak{b}'}$ *the following rules for the consistency and the undeterminedness operators are derivable in the calculus* $SF_{\mathbf{L}^*_{\mathfrak{e}\mathfrak{b}'}}$ *–as is routine to check.*

$$\circ\{\mathfrak{t}, \mathfrak{b}'\}\ \frac{\Omega, \mathfrak{t} : \varphi, \mathfrak{e} : \varphi, \mathfrak{f} : \varphi}{\Omega, \mathfrak{t} : \circ\varphi, \mathfrak{b}' : \circ\varphi} \qquad \circ\{\mathfrak{e}, \mathfrak{f}\}\ \frac{\Omega, \mathfrak{b}' : \varphi}{\Omega, \mathfrak{e} : \circ\varphi, \mathfrak{f} : \circ\varphi}$$

$$\bigstar\{\mathfrak{e}, \mathfrak{f}\}\ \frac{\Omega, \mathfrak{t} : \varphi, \mathfrak{b}' : \varphi, \mathfrak{f} : \varphi}{\Omega, \mathfrak{e} : \bigstar\varphi, \mathfrak{f} : \bigstar\varphi} \qquad \bigstar\{\mathfrak{t}, \mathfrak{b}'\}\ \frac{\Omega, \mathfrak{e} : \varphi}{\Omega, \mathfrak{t} : \bigstar\varphi, \mathfrak{b}' : \bigstar\varphi}$$

Corollary 24. *The Soundness of the calculus is straightforward by looking at the Nmatrix in question. Moreover, as a Corollary of Theorem 22, we have that the calculus* $SF_{\mathbf{L}^*_{\mathfrak{e}\mathfrak{b}'}}$ *is fully complete for the logic* $\mathbf{L}^*_{\mathfrak{e}\mathfrak{b}'}$*, induced by the Nmatrix* $\mathcal{M}^*_{\mathbf{L}_{\mathfrak{e}\mathfrak{b}'}}$

6. Conclusions

In the previous sections we showed that there are logically or inferentially weak LFIs and LFUs that have not been discussed in the literature about these topics, until now. Moreover, we proved that many of these systems are built as extensions of subsystems of known infectious logics, that when extended with a classical negation (spelled out conceptually and formally in a precise way) can define proper consistency, inconsistency, determinedness and undeterminedness operators.

It remains to be analyzed how many of the resulting LFIs and LFUs are non-equivalent, and how many of them have interesting meta-theoretical properties, such as being Post-complete, maximal with regard to some systems (especially classical logic) or functionally complete. We leave these issues for further works, hoping to discuss them in the near future.

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