Liberating classical negation from falsity conditions

Damian Szmuc
Insituto de Investigaciones Filosóficas
CONICET-SADAF
Buenos Aires, Argentina
szmucdamian@conicet.gov.ar

Hitoshi Omori
Department of Philosophy I
Ruhr University Bochum
Bochum, Germany
hitoshi.omori@rub.de

***Abstract*—In one of their papers, Michael De and Hitoshi Omori observed that the notion of classical negation is not uniquely determined in the context of so-called Belnap-Dunn logic, and in fact there are 16 unary operations that qualify to be called classical negation. These varieties are due to different falsity conditions one may assume for classical negation. The aim of this paper is to observe that there is an interesting way to make sense of classical negation independent of falsity conditions. We discuss two equivalent semantics, and offer a Hilbert-style system that is sound and complete with respect to the semantics.**

***Index Terms*—Classical negation, Belnap-Dunn logic, falsity condition, non-deterministic semantics.**

I. Introduction

A. Background and aim

Many things are simple in classical propositional logic (**CL** hereafter). One such example is negation. However, the notion of classical negation becomes non-trivial if we deviate from **CL**. The prime example is the case of intuitionistic propositional logic. Indeed, it is not obvious at all how to expand the logic by a connective that can rightfully be regarded as a classical negation, without *collapsing* into **CL** (cf. [9]). The aim of this paper is to discuss the notion of classical negation building on the discussion carried out by Michael De and Hitoshi Omori (D&O hereafter) in [6], for the family of non-classical logics related to the four-valued logic of Nuel Belnap and Michael Dunn (**BD** hereafter). Our discussion of **BD** and its expansions with classical negation also follows the analysis by D&O which treats relational and many-valued semantics in tandem. The crucial difference from the previous discussion is that when dealing with relational semantics for **BD** and its expansions with classical negation, we *only* focus on the truth condition for negation, leaving the falsity condition untouched, taking the truth condition to be *sufficient* to represent the notion of untruth within the language. This strategy is inspired by Arnon Avron's work in [3], where a similar treatment is given to de Morgan negation in **BD**, focusing only on the truth condition for this negation, taking it to be sufficient to represent the notion of falsity within the language. As pointed out by Avron, his resulting relational semantics for de Morgan negation is equivalent to a four-valued semantics, albeit of a *non-deterministic* kind. Similarly, we will show that the relational semantics we propose for the expansion of **BD** with classical negation (where negation is only endowed with a truth but not a falsity condition) will have an equivalent representation in terms of non-deterministic four-valued semantics.

Before moving on, let us clarify that in what follows our languages $\mathcal{L}$ and $\mathcal{L}^+$ consist of sets $\{\sim, \wedge, \vee\}$ and $\{\neg, \sim, \wedge, \vee, \rightarrow\}$ of propositional connectives, respectively, and a denumerable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form and Form^+ the set of formulas defined as usual in $\mathcal{L}$ and $\mathcal{L}^+$ respectively. We denote a formula of $\mathcal{L}$ and $\mathcal{L}^+$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ and $\mathcal{L}^+$ by Γ, Δ, Σ, etc.

B. *Why* classical *negation?*

Let us consider the following two scenarios. First, an *inconsistent* scenario is a scenario when there is a sentence A such that A and $\sim A$ are both true and, thus, they are both suitable for being *asserted*. Second, an *incomplete* scenario is a scenario when there is a sentence A such that neither A nor $\sim A$ are true and, thus, they are both suitable for being *denied*. With the presence of these scenarios, scholars agree that if we are in an inconsistent scenario, the assertion of a negation like $\sim A$ does not express the denial of A and, analogously, if we are in an incomplete scenario, the denial of A is not expressed by the assertion of $\sim A$. Therefore, we need some device to help us model *denial* and this is the place in which classical negation, represented as $\neg A$, may play an important role.

Thus, a first reason for e.g. non-classical logicians to aim at extending their non-classical logics with a *classical negation* is its availability as a connective for representing *denial*. A second reason why non-classicist might be interested in classical notions can be extracted from D&O's discussion:

> We think it is even better to go one step further by having all classical notions expressible in the *object language itself*. What better way to preach to the gentiles in their own tongue? If a classical notion is coherently expressible in your language, why not help yourself to it? [6, p.826]

A third and more salient reason for non-classical logicians to be interested in classical negation is that—we will argue below—it can serve the role of representing the notion of *untruth* within the language, just like usually de Morgan negation represents the more familiar notion of *falsity* within the language. Attempts with this inspiration were already carried out in relation to intuitionistic and subintuitionistic logics in [7], [9], to relevant logic in [15], [16], and the previously mentioned approach of D&O to apply these ideas to non-classical many-valued logics such as **BD** in [6].

Finally, recall some of the motivations for the use of non-classical logics: database and information-based reasoning, digital circuits, programming languages, legal problems, linguistics and pragmatics models, and a long list of the most diverse phenomena. In such contexts, classical negation represents a useful notion too, as the following examples illustrate. For instance, regarding logic programming, a form of classical negation called *negation as failure* is associated with the derivation of the negation of p whenever p fails to be derived by the running algorithm (see e.g. [11]). Furthermore, concerning database reasoning, a similar form of classical negation called *default negation* is associated with the derivation of the negation of p whenever the information that p is not included in the database (see e.g. [2]). This latter form of classical negation can also be obtained in information-based structures associated with many-valued logics called bilattices (see e.g. [10]), where it is defined as the composition of the truth-inversion and the information-inversion operations on this structure (see e.g. [1]).

II. A natural generalization of CL: Belnap-Dunn Logic

As is well-known, **BD** is characterized in terms of the following truth-tables for negation, conjunction and disjunction, where $\mathbf{t}$ and $\mathbf{b}$ are the designated values of **BD**.

	$\sim$
t	**f**
b	**b**
n	**n**
f	**t**

$\wedge$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$\vee$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

This four-valued presentation might make less clear in what sense **BD** generalizes **CL**. Indeed, we only have two, not four, truth values in the intended semantics of **CL**. However, we can reformulate the semantics in terms of two-valued *relational* semantics, thanks to the discovery of Dunn (cf. [8]). More precisely, **BD** may be characterized as follows.

Definition 1: A **BD**-*interpretation* for $\mathcal{L}$ is a relation, r, between propositional variables and the values 1 and 0.[1] More precisely, $r \subseteq \mathsf{Prop} \times \{1, 0\}$. Given an interpretation, r, this is extended to a relation between all formulas and truth values by the following clauses:

- $\sim\!Ar1$ iff $Ar0$,
- $\sim\!Ar0$ iff $Ar1$,
- $A\wedge Br1$ iff $Ar1$ and $Br1$,
- $A \wedge Br0$ iff $Ar0$ or $Br0$,
- $A \vee Br1$ iff $Ar1$ or $Br1$,
- $A\vee Br0$ iff $Ar0$ and $Br0$.

Based on these, A is a **BD**-relational semantic consequence of Γ ($\Gamma \models^r_{\mathbf{BD}} A$) iff for every **BD**-interpretation r, if $Br1$ for all $B \in \Gamma$ then $Ar1$.

Remark 2: In view of the relational (or Dunn) semantics, there is a clear sense in which **BD** is a generalization of **CL**. Indeed, if we assume that the truth and falsity are both exclusive and exhaustive, then we obtain the semantics for **CL**. But, **BD** allows us to reason when the truth and falsity are not necessarily exclusive nor exhaustive.

[1]Note that existence and uniqueness restrictions are not placed over r.

Remark 3: Given a four-valued **BD** valuation v and a **BD**-relational interpretation r, the following correspondence

$v(A)$=$\mathbf{t}$ iff $Ar1$ and not$(Ar0)$
$v(A)$=$\mathbf{b}$ iff $Ar1$ and $Ar0$
$v(A)$=$\mathbf{n}$ iff not$(Ar1)$ and not$(Ar0)$
$v(A)$=$\mathbf{f}$ iff not$(Ar1)$ and $Ar0$

clarifies the relation between the four-valued semantics and the two-valued Dunn semantics. For a *mechanical* procedure describing how one can compute the truth and falsity conditions for connectives specified in terms of truth tables, see [17].

III. What is *classical* negation?

Now, as is well known, **BD** does *not* have the good old negation of **CL** which satisfies, among others, the *ex contradictione quodlibet* and the *tertium non datur*. But one might still be interested in having such an operation contrary to the motivation for relevant logics where **BD** was originally discussed. D&O, in [6], address this problem by discussing the following question in the context of **BD**: what is *classical* negation? Let us briefly review the arguments in [6].

The first attempt made in [6] to provide an answer to the above question suggests that a classical negation is a unary operator $\neg$ that satisfies the following requirement:

Contra

The sentence $A \wedge \neg A$ is always false *and never true*, and the sentence $A \vee \neg A$ is always true *and never false*.

This attempt, however, turns out to be unsuccessful, for it allows to uniquely determine an operation in the context of **BD**, but not in three-valued logics, such as Stephen Cole Kleene's three-valued logic $\mathbf{K_3}$ and Graham Priest's logic of paradox **LP**. In fact, it is remarked in [6, p. 830] that in three-valued logics it is *impossible* to define a unary operator respecting this condition.

The second attempt made in [6] is, therefore, a more *liberal* approach, where classical negation is taken to be a classical contradictory-forming unary operator $\neg$. This requires, of course, defining what a classical contradiction is. This is spelled out as follows.

- A and B are *classical contraries* iff: if A is true, then B is not true; if B is true, then A is not true.
- A and B are *classical subcontraries* iff: if A is not true, then B is true; if B is not true, then A is true.
- A and B are *classical contradictories* iff they are classical contraries and classical subcontraries.

This attempt boils down to a classical negation satisfying:

Liberal

A is true iff $\neg A$ is not true.

Thus it can be legitimately argued that classical negation *represents the idea of untruth within the language*.

Remark 4: This is not only the take that D&O in [6] have on what would amount to add a classical negation to a non-classical logic. For example, already in [15], [16], Robert Meyer and Richard Routley discussed the addition of classical negation to relevant logics, and in [9] Andreas Herzig and Luis Fariñas del Cerro discussed the addition of classical negation to intutionistic logic. More recently, D&O, in [7], discussed the addition of classical negation to subintuitionistic logic (i.e. subsystems of intuitionistic logic obtained by dropping various frame conditions). All of these attempts agree that classical negation is characterized by saying that $\neg A$ is true at x iff A is not true at x, where x is taken to be a situation, a point or a possible world, depending on each case. This confluence provides, for us, some evidence for the plausibility of *Liberal*.

In [6], D&O correctly note that *Liberal* provides only the *truth condition* of a classical negation without making any reference to a *falsity condition*. But, in the context of **BD** and related systems in which the relation between the truth and the falsity deviates from the classical one, *Liberal* is of little help in picking out a unique operation.

More precisely, given a many-valued non-classical logic, *Liberal* by itself does not necessarily secure a truth-function for classical negation. In many cases, further constraints are needed to pin down the different truth-functions available for such a connective. These constraints are the *falsity conditions*. For example, D&O observe that in the case of **BD**, there are 16 different falsity conditions available that, when conjoined with *Liberal*, render 16 different classical negations. Based on these considerations, D&O conclude in [6, p. 830] that:

> *Liberal* can therefore serve only as a *necessary* condition on classical negation. Indeed it can be seen as merely one component of a definition of classical negation that generalizes to a non-classical setting.

IV. Do we really need a falsity condition?

But is this really so? Is the *Liberal* condition only a necessary condition, not a sufficient one? As we previously observed, in the context of **CL**, where truth and falsity are mutually exclusive and jointly exhaustive, *Liberal* is enough to secure a unique operation. However, when we allow these conditions to be relaxed, as D&O point out, the *Liberal* condition does not pin down a *unique* truth-function. This can be, alternatively, seen as a situation where the lack of the falsity condition implies the impossibility to select a single operation satisfying the intended requirements.

This may seem to be somewhat puzzling especially for those who think that there is nothing more to classical negation than *Liberal*. Then, if so, how should we make sense of this? We believe that Avron's treatment of de Morgan negation deserves attention. In [3], Avron provided a quite distinctive analysis of *de Morgan negation*, exploring this notion in a systematic manner in the context of systems closely related to **BD**. Avron's main idea is that de Morgan negation is a logico-linguistic tool to represent "the idea of *falsehood* within the language", namely $\sim A$ is true iff A is false [3, p. 160].[2] Although Avron did not discuss his systems in *relational* terms, we spell out his idea saying that the distinctive feature of de Morgan negation is its *truth condition*: $\sim Ar1$ iff $Ar0$. Further implying that we do not need the falsity condition for $\sim$ to express "the idea of *falsehood* within the language".

Now, we can easily see that Avron's idea of focusing on the truth condition and dropping the falsity condition for a given operation, can be applied to the semantic analysis of other logical operations, and in particular of *classical negation*. More specifically, we can analogously say that classical negation is a tool to represent "the idea of *untruth* within the language", and even though *Liberal* amounts only to giving an analysis of the truth condition for classical negation, this will be necessary *and sufficient* to realize the idea.

In fact, in [6] the D&O confess that the issue of finding an appropriate falsity condition for this logical operation is treated in a tangential way, for they consider different alternatives that arise from orthogonal considerations such as symmetry between truth and falsity conditions. In other words, it seems that there is no falsity condition which can be said to be essential or necessary to characterize classical negation. Based on these, we now provide two ways to show *how* a classical negation equipped only with a truth condition (i.e. *Liberal*) can work perfectly well within a formal framework.

V. Liberal as *the* condition: How?

A. Relational semantics

We first introduce **BDLCN**, the expansion of **BD** with a liberal classical negation and the material conditional in terms of Dunn semantics.

Definition 5: A **BDLCN**-*interpretation* for $\mathcal{L}^+$ is a relation, r, between Form and $\{1, 0\}$ such that r satisfies the following clauses in addition to those for **BD**-interpretation:

- $\neg Ar1$ iff not($Ar1$)
- $A \to Br1$ iff not($Ar1$) or $Br1$

Based on these, A is a **BDLCN**-relational semantic consequence of Γ ($\Gamma \models_r A$) iff for every **BDLCN**-interpretation r, if $Br1$ for all $B \in \Gamma$ then $Ar1$.

Remark 6: Note that we added not only classical negation but also a conditional in introducing the system **BDLCN**. It is important that it is taken as a *primitive* connective, not *defined*. Indeed, if we take the conditional to be an abbreviation of $\neg A \vee B$, then we will have a different falsity condition, because of the falsity condition for disjunction.

B. Non-determinisitic semantics: a brief overview

Non-deterministic semantics is a natural generalization of the many-valued semantics.[3] We here review the central notions briefly.

Definition 7: A non-deterministic matrix (*Nmatrix*, for short) for a language $\mathcal{L}$ is a structure $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where

- $\mathcal{V}$ is a (non-empty) set of truth values;
- $\mathcal{D}$ is a (non-empty) proper subset of $\mathcal{V}$;
- $\mathcal{O}$ is a set that contains for every n-ary connective $\odot \in \mathcal{L}$, a n-ary truth-function $\widetilde{\odot} : \mathcal{V}^n \longrightarrow 2^{\mathcal{V}} \setminus \{\emptyset\}$.

[2] For a related consideration, see [18] in which systems introduced in [12], [13] are discussed from Avron's perspective.

[3] Non-deterministic semantics were systematically developed for the first time in [4]. For the details on non-deterministic semantics, see [5].

Remark 8: Note that many-valued semantics are just special cases of Nmatrices, where each n-ary connective $\odot$ is interpreted by a singleton-valued function $\widetilde{\odot}$. Moreover, in what follows when talking about the output of a non-determnistic truth-function, we will omit curly brackets to note e.g. $\{x, \ldots, y\}$ and we will instead write simply $x, \ldots, y$ hoping to make the truth-tables more readable.

When context allows for disambiguation, we will often conflate the connectives $\odot$ with their respective (non-deterministic) truth-functions $\widetilde{\odot}$. We now proceed to define the corresponding notions of valuation and semantic consequence.

Definition 9: A *valuation* for the logic in question is a mapping $v : \mathsf{Form}_{\mathcal{L}} \longrightarrow \mathcal{V}$ such that for every n-ary operator $\odot$ the following holds for every $A_1, \ldots, A_n \in \mathsf{Form}_{\mathcal{L}}$: $v(\odot(A_1, \ldots, A_n)) \in \widetilde{\odot}(v(A_1), \ldots, v(A_n))$

Definition 10: Given an Nmatrix $\mathcal{M}$, the semantic consequence relation induced by $\mathcal{M}$ is defined in the following way: A is the *semantic consequence* of a set of formulae Γ (notation $\Gamma \vDash_{\mathcal{M}} A$) if for every valuation v, if $v(B) \in \mathcal{D}$ for every $B \in \Gamma$, then $v(A) \in \mathcal{D}$.

C. *Another way to realize* Liberal

We now instantiate *Liberal* in terms of the non-deterministic semantics.[4]

Definition 11: A **BDLCN**-*valuation* is a function from Form^{+} to $\mathcal{V}$, induced by the following Nmatrix: $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$, $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$, and $\mathcal{O}$ includes the following truth functions in addition to those of **BD**:

A	$\neg A$
$\mathbf{t}$	$\mathbf{n}, \mathbf{f}$
$\mathbf{b}$	$\mathbf{n}, \mathbf{f}$
$\mathbf{n}$	$\mathbf{t}, \mathbf{b}$
$\mathbf{f}$	$\mathbf{t}, \mathbf{b}$

$A{\to}B$	$\mathbf{t}$	$\mathbf{b}$	$\mathbf{n}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{n}, \mathbf{f}$	$\mathbf{n}, \mathbf{f}$
$\mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{n}, \mathbf{f}$	$\mathbf{n}, \mathbf{f}$
$\mathbf{n}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$
$\mathbf{f}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$	$\mathbf{t}, \mathbf{b}$

Based on this, semantic consequence relation $\Gamma \models_4 A$ is defined as in Definition 10.

D. *A comparison between* BD+ *and* BDLCN

It is perhaps illuminating to closely observe the differences between **BDLCN** and **BD+** of [6]. These can be summarized as follows. First, while the former takes a non-deterministic stance towards formally representing classical negation in the context of **BD**, the latter take a deterministic stance towards this issue. Second, while the former can be taken as a means to summarize or conflate the many (indeed, sixteen) variants of classical negation complying with *Liberal* that can be considered in the context of **BD**, the latter only considers one of such alternatives, i.e. Boolean negation. Third, the former assumes no falsity condition whatsoever for classical negation, whereas the latter assumes a particular falsity condition for classical negation, i.e. that of Boolean negation (namely, that $\neg A$ is false iff A is not false).

Thus, in the context of **BDLCN** there are some inferences involving classical negation which are invalid, although they are valid in the context of the system **BD+**. One of such principles is the commutativity of classical and de Morgan, namely the equivalence $\neg{\sim}A \leftrightarrow {\sim}\neg A$.

[4]An early exploration of this technique can be found in [20].

The failure of such commutativity properties can be accounted for in several ways. First, it is important to notice that inferences of this sort are closely tied to the falsity condition attributed to classical negation. Thus, given no particular falsity condition is attached to classical negation in the context of **BDLCN** (as opposed to e.g. the case of **BD+**), it is expected that this and other inferences will fail to hold. Second, that as long as the non-deterministic classical negation in **BDLCN** validates *only* those inferences and principles valid in *all* of the deterministic classical negation studied by D&O, this is again expected to happen. For, as can be easily seen by observing e.g. the case of the exclusion negation (written as $\neg^{e}$ in [6, p.829]), such a deterministic classical negation fails to satisfy the commutativity laws. This is because some classical negations, such as the exclusion negation, do not allow to infer that $\neg A$ is false from the fact that ${\sim}A$ is untrue, precisely because A may be neither true nor false. Similarly, some classical negations do not allow to infer that ${\sim}A$ is untrue from the fact that $\neg A$ is false, precisely because A may be both true and false.

We do not believe that e.g. the failure of commutativity between classical and de Morgan negation and other properties gives a reason to prefer the non-determinstic reading over the deterministic reading *per se*. They can only provide reasons of this sorts, conditional on someone believing e.g. that commutativity between such negations or Contraposition as an inference rule are highly esteemed logical properties. But these issues are, in any case, orthogonal to the present discussion, which focuses on the particular question of whether or not classical negation can be made sense of (technically speaking) without appealing to any falsity conditions whatsoever.

VI. Proof system, soundness and completeness

We now introduce a Hilbert-style system and prove soundness and completeness results with respect to two semantics.

A. *Proof system*

Definition 12: The system $\mathcal{H}$**BDLCN** consists of the following axiom schemata and a rule of inference, where $A{\leftrightarrow}B$ abbreviates $(A{\to}B) \wedge (B{\to}A)$.

$A{\to}(B{\to}A)$ (A1)

$(A{\to}(B{\to}C)){\to}((A{\to}B){\to}(A{\to}C))$ (A2)

$((A{\to}B){\to}A){\to}A$ (A3)

$(A \wedge B){\to}A$ (A4)

$(A \wedge B){\to}B$ (A5)

$(C{\to}A){\to}((C{\to}B){\to}(C{\to}(A{\wedge}B)))$ (A6)

$A{\to}(A \vee B)$ (A7)

$B{\to}(A \vee B)$ (A8)

$(A{\to}C){\to}((B{\to}C){\to}((A{\vee}B){\to}C))$ (A9)

$A \vee \neg A$ (A10)

$(A \wedge \neg A){\to}B$ (A11)

${\sim}{\sim}A{\leftrightarrow}A$ (A12)

${\sim}(A{\wedge}B){\leftrightarrow}({\sim}A{\vee}{\sim}B)$ (A13)

${\sim}(A{\vee}B){\leftrightarrow}({\sim}A{\wedge}{\sim}B)$ (A14)

$$\frac{A \quad A{\to}B}{B} \text{ (MP)}$$

Finally, we write $\Gamma \vdash_{\mathbf{H}} A$ if there is a sequence of formulas $\langle B_1, \ldots, B_n, A\rangle$ ($n \geq 0$), called *a derivation*, such that every formula in the sequence either (i) belongs to Γ; (ii) is an axiom of $\mathcal{H}$**BDLCN**; (iii) is obtained by (MP) from formulas preceding it in the sequence. As usual, we write $\Gamma, A_1, \ldots, A_n \vdash_{\mathbf{H}} B$ for $\Gamma \cup \{A_1, \ldots, A_n\} \vdash_{\mathbf{H}} B$.

Remark 13: If we add the axioms ${\sim}\neg A{\leftrightarrow}\neg{\sim}A$ and ${\sim}(A{\to}B){\leftrightarrow}(\neg{\sim}A \wedge {\sim}B)$, then we obtain **BD+** introduced in [6]. Note also that our axiomatization is not independent. Indeed, (A3) is derivable in view of (A10), (A11) and others.

Proposition 14 (Deduction Theorem): For any $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}^{+}$, $\Gamma, A \vdash_{\mathbf{H}} B$ iff $\Gamma \vdash_{\mathbf{H}} A \rightarrow B$.

Proof: The left-to-right direction can be proved in the usual way, given axioms (A1) and (A2), and (MP) the sole rule of inference. For the other direction, we use (MP). □

We now turn to prove the soundness and completeness of $\mathcal{H}$**BDLCN** with respect to both semantics.

B. Soundness

Soundness results, for our cases, are straightforward.

Theorem 15 (Soundness): For any $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \vdash_{\mathbf{H}} A$ then $\Gamma \models_r A$.

Proof: By verifying that each instance of each axiom schema always relates to $\mathbf{t}$, and that (MP) preserves relation to $\mathbf{t}$. □

Theorem 16 (Soundness): For any $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \vdash_{\mathbf{H}} A$ then $\Gamma \models_4 A$.

Proof: By a straightforward verification that each instance of each axiom schema always takes a designated value, and that (MP) preserves designated values. Here we only spell out the details for the validity of (A10) and (A11).

Ad (A10) Assume $v(A)=\mathbf{t}$. Given the **BDLCN** truth-table for $\vee$, it is guaranteed that $v(A \vee \neg A) \in \mathcal{D}$. Similarly, if we assume $v(A)=\mathbf{b}$. Assume $v(A)=\mathbf{n}$. Given the **BDLCN** non-deterministic truth-tables for $\neg$, it is guaranteed that $v(\neg A) \in \mathcal{D}$. Whence, again, given the **BDLCN** truth-tables for $\vee$, it is guaranteed that $v(A \vee \neg A) \in \mathcal{D}$. Similarly, if we assume $v(A)=\mathbf{f}$. Thus, for all **BDLCN** valuations v, $v(A \vee \neg A) \in \mathcal{D}$.

Ad (A11) Suppose, for *reductio*, that there is a **BDLCN** valuation v such that $v((A \wedge \neg A) \rightarrow B) \notin \mathcal{D}$. Thus, either $v(A \wedge \neg A \rightarrow B)=\mathbf{n}$ or $v(A \wedge \neg A \rightarrow B)=\mathbf{f}$. By the **BDLCN** truth-table for $\rightarrow$, we obtain either $v(A \wedge \neg A)=\mathbf{t}$ or $v(A \wedge \neg A)=\mathbf{b}$.

If $v(A \wedge \neg A)=\mathbf{t}$, then by the **BDLCN** truth-table for $\wedge$, we obtain $v(A)=\mathbf{t}$ and $v(\neg A)=\mathbf{t}$. But, if $v(A)=\mathbf{t}$, then by the **BDLCN** truth-table for $\neg$, we have $v(\neg A) \in \{\mathbf{n}, \mathbf{f}\}$, which is absurd given our previous reasoning.

If $v(A \wedge \neg A)=\mathbf{b}$, then by the **BDLCN** truth-table for $\wedge$, we obtain either $v(A)=\mathbf{b}$ or $v(\neg A)=\mathbf{b}$. Now, if $v(A)=\mathbf{b}$, then by the **BDLCN** truth-table for $\neg$, we obtain $v(\neg A) \in \{\mathbf{n}, \mathbf{f}\}$. But, by the **BDLCN** truth-table for $\wedge$, we obtain $v(A \wedge \neg A)=\mathbf{f}$, which is absurd given our previous reasoning. Similarly, if $v(\neg A)=\mathbf{b}$, then by the **BDLCN** truth-table for $\neg$, we obtain $v(A) \in \{\mathbf{n}, \mathbf{f}\}$. But, by the **BDLCN** truth-table for $\wedge$, we have $v(A \wedge \neg A)=\mathbf{f}$, which is absurd given our previous reasoning. Thus, for all **BDLCN** valuations v, $v((A \wedge \neg A) \rightarrow B) \in \{\mathbf{t}, \mathbf{b}\}$.

We leave the further details to the reader. □

C. Completeness

We now turn to completeness. First, we introduce some standard terminologies.

Definition 17: A set Σ is a *theory* iff it is a deductively closed set of sentences, i.e. iff $\Sigma \vdash A$ implies $A \in \Sigma$. Σ is *prime* iff $A \vee B \in \Sigma$ implies $A \in \Sigma$ or $B \in \Sigma$. Σ is a *prime theory* if it is both. Finally, Σ is *non-trivial* if $A \notin \Sigma$ for some A.

Then the following lemma is well-known, and thus we will omit the details of the proof.

Lemma 18 (Lindenbaum): If $\Sigma \nvdash A$ then there is a prime theory, Δ, such that $\Sigma \subseteq \Delta$ and $\Delta \nvdash A$.

Moreover, the following lemma is useful and easy to prove.

Lemma 19: If Σ is a non-trivial prime theory, then (i) $\neg A \in \Sigma$ iff $A \notin \Sigma$, (ii) $A \rightarrow B \in \Sigma$ iff ($A \notin \Sigma$ or $B \in \Sigma$).

The following lemma is the key for the completeness with respect to the relational semantics.

Lemma 20: Let Σ be a non-trivial prime theory, and define a relation $r_0 \subseteq \mathsf{Form} \times \{1, 0\}$ as follows:

$$Ar_0 1 \text{ iff } A \in \Sigma \quad Ar_0 0 \text{ iff } {\sim} A \in \Sigma.$$

Then, r_0 is a **BDLCN**-interpretation.

Proof: The cases for $\sim, \wedge$ and $\vee$ are standard. For $\neg$ and $\rightarrow$, we make use of Lemma 19, as follows, respectively.

$\neg B r_0 1$ iff $\neg B \in \Sigma$ (by def. of r_0) iff $B \notin \Sigma$ (by (i) of Lemma 19) iff not ($B r_0 1$) (by def. of r_0).

$B \rightarrow C r_0 1$ iff $B \rightarrow C \in \Sigma$ (by def. of r_0) iff $B \notin \Sigma$ or $C \in \Sigma$ (by (ii) of Lemma 19) iff not ($B r_0 1$) or $C r_0 1$ (by def. of r_0).

This completes the proof. □

Theorem 21 (Completeness): For any $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_r A$ then $\Gamma \vdash_{\mathbf{H}} A$.

Proof: We prove the contrapositive. Suppose that $\Gamma \nvdash_{\mathbf{H}} A$. Then by Lemma 18, we have a non-trivial prime theory Σ_0 such that $\Gamma \subseteq \Sigma_0$ and $\Sigma_0 \nvdash A$. In view of Lemma 20, we can define a **BDLCN**-interpretation r_0. Since we have $C r_0 1$ for any $C \in \Gamma$ and not $A r_0 1$, we have $\Gamma \not\models_r A$. □

For the completeness with respect to the non-deterministic semantics, we make use of the following definition.

Definition 22: Let Σ be a non-trivial prime theory. Then, we define a function v_Σ from Form to $\mathcal{V}$ as follows:

$$v_\Sigma(B) := \begin{cases} \mathbf{t} & \text{if } \Sigma \vdash B \text{ and } \Sigma \nvdash {\sim} B, \\ \mathbf{b} & \text{if } \Sigma \vdash B \text{ and } \Sigma \vdash {\sim} B, \\ \mathbf{n} & \text{if } \Sigma \nvdash B \text{ and } \Sigma \nvdash {\sim} B, \\ \mathbf{f} & \text{if } \Sigma \nvdash B \text{ and } \Sigma \vdash {\sim} B. \end{cases}$$

The following lemma is the key for the completeness with respect to the non-deterministic semantics.

Lemma 23: If Σ is a non-trivial prime theory, then v_Σ is a well-defined **BDLCN**-valuation.

Proof: Note first that the well-definedness of v_Σ is obvious. Then the desired result is proved by induction on the number n of connectives.

(Base): for atomic formulas, it is obvious by the definition.

(Induction step): We split the cases based on the connectives. Here we only deal with $\neg$, $\sim$ and $\rightarrow$.

Case 1. If $B = \neg C$, then we have the following four cases.

Cases	$v_\Sigma(C)$	condition for C	$v_\Sigma(B)$	condition for B i.e. $\neg C$
(i)	$\mathbf{t}$	$\Sigma \vdash C$ and $\Sigma \nvdash {\sim} C$	$\mathbf{n}, \mathbf{f}$	$\Sigma \nvdash \neg C$
(ii)	$\mathbf{b}$	$\Sigma \vdash C$ and $\Sigma \vdash {\sim} C$	$\mathbf{n}, \mathbf{f}$	$\Sigma \nvdash \neg C$
(iii)	$\mathbf{n}$	$\Sigma \nvdash C$ and $\Sigma \nvdash {\sim} C$	$\mathbf{b}, \mathbf{t}$	$\Sigma \vdash \neg C$
(iv)	$\mathbf{f}$	$\Sigma \nvdash C$ and $\Sigma \vdash {\sim} C$	$\mathbf{b}, \mathbf{t}$	$\Sigma \vdash \neg C$

By induction hypothesis (IH), we have the conditions for C, and it is easy to see that the conditions for B i.e. $\neg C$ are provable in view of (i) of Lemma 19.

Case 2. If $B = {\sim} C$, then we have the following four cases.

Cases	$v_\Sigma(C)$	condition for C	$v_\Sigma(B)$	condition for B i.e. ${\sim} C$
(i)	$\mathbf{t}$	$\Sigma \vdash C$ and $\Sigma \nvdash {\sim} C$	$\mathbf{f}$	$\Sigma \nvdash {\sim} C$ and $\Sigma \vdash {\sim}{\sim} C$
(ii)	$\mathbf{b}$	$\Sigma \vdash C$ and $\Sigma \vdash {\sim} C$	$\mathbf{b}$	$\Sigma \vdash {\sim} C$ and $\Sigma \vdash {\sim}{\sim} C$
(iii)	$\mathbf{n}$	$\Sigma \nvdash C$ and $\Sigma \nvdash {\sim} C$	$\mathbf{n}$	$\Sigma \nvdash {\sim} C$ and $\Sigma \nvdash {\sim}{\sim} C$
(iv)	$\mathbf{f}$	$\Sigma \nvdash C$ and $\Sigma \vdash {\sim} C$	$\mathbf{t}$	$\Sigma \vdash {\sim} C$ and $\Sigma \nvdash {\sim}{\sim} C$

By IH, we have the conditions for C, and it is easy to see that the conditions for B i.e. $\sim C$ are provable by (A12).

Case 3. If $B{=}C{\to}D$, then we have the following three cases.

	$v_{\Sigma}(C)$	condition for C	$v_{\Sigma}(D)$	condition for D	$v_{\Sigma}(B)$	condition for B
(i)	any	—	$\mathbf{t}$ or $\mathbf{b}$	$\Sigma \vdash D$	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C{\to}D$
(ii)	$\mathbf{n}$ or $\mathbf{f}$	$\Sigma \nvdash C$	any	—	$\mathbf{t}, \mathbf{b}$	$\Sigma \vdash C{\to}D$
(iii)	$\mathbf{t}$ or $\mathbf{b}$	$\Sigma \vdash C$	$\mathbf{n}$ or $\mathbf{f}$	$\Sigma \nvdash D$	$\mathbf{n}, \mathbf{f}$	$\Sigma \nvdash C{\to}D$

By IH, we have the conditions for C and D, and we can see that the conditions for B i.e. $C{\to}D$ are provable by (ii) of Lemma 19. This completes the proof. □

Theorem 24 (Completeness): For any $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models_4 A$ then $\Gamma \vdash_{\mathbf{H}} A$.

Proof: We prove the contrapositive. Assume $\Gamma \nvdash_{\mathbf{H}} A$. Then, by Lemma 18, we have a non-trivial prime theory Π such that $\Gamma \subseteq \Pi$ and $A \notin \Pi$ and by Lemma 23, we can define a legal-**BDLCN**-valuation v_{Π} such that $v_{\Pi}(B) \in \mathcal{D}$ for every $B \in \Gamma$ and $v_{\Pi}(A) \notin \mathcal{D}$. Thus we have $\Gamma \not\models_4 A$. □

Remark 25: In view of the results established in this section, we obtain the equivalence of the two semantics, namely the non-deterministic semantics and the relational semantics. A more direct proof is also available through a general relation between Dunn semantics and a certain family of non-deterministic semantics carried out in [17]. Details will be kept for another occasion for the sake of brevity.

VII. Conclusion

In this paper, we studied classical negation by focusing on what we take to be its two core features. First, it is a logical tool intended to represent the idea of *untruth* within the language. Second, and because of that, it can be legitimately argued that $\neg A$ is true iff A is not true. This latter account, which is dubbed *Liberal* after the work of D&O in [6], entails that what is essential to characterize a classical negation is its truth condition, independent of any falsity condition. In the context of **CL**, where truth and falsity are mutually exclusive and jointly exhaustive, this allows to pin down a unique operation; but when we are in contexts where such constraints are relaxed, such as **BD**, *Liberal* does not single out a unique operation. For this reason, in [6] the authors concluded that *Liberal* can only be taken to be a necessary, but not a sufficient condition for a negation to be classical.

The main claim of the paper is that this is not the case. For the purpose of making the point clear, we provided clear formalizations of the expansion of **BD** with a classical negation, equipped with its truth condition. More specifically, we devised both a relational and a many-valued semantics. When taking into account of the many-valued perspective, we noticed that the lack of the falsity conditions required allowing the truth-function of classical negation to be non-deterministic and, thus, the corresponding semantics for the expansion of **BD** with a classical negation were given with the aid of non-deterministic semantics. Finally, we presented sound and complete Hilbert-style system for the expansions of **BD** with a classical negation.

There are a number of possible venues for further developments based on the present investigation. First, the first-order case deserves a proper exploration and a thorough examination of the differences that the non-deterministic case has with e.g. the work done in [19] and [14]. Second, our discussion focused only in the four-valued case, but many other non-classical logics remain to be explored, most saliently, infinitely-valued (e.g. fuzzy logics) and infectious logics (e.g. Weak Kleene logic, Bochvar's logic, Halldén's logic etc). We hope to investigate these matters in future work.

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