Non-transitive counterparts of every Tarskian logic

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Abstract

The aim of this article is to show that, just like in recent years Cobreros, Egré, Ripley and van Rooij provided a non-transitive counterpart of classical logic (meaning by this that all classically acceptable inferences are valid, but Cut and other metainferences are not) the same can be done for every Tarskian logic, with full generality. In order to establish this fact, we take a semantic approach, by showing that appropriate structures can be devised to characterize a non-transitive counterpart of every Tarskian logic, starting from the logical matrices that are usually taken to render them.

1. Background and aim

In recent years, scholars have extensively discussed a presentation of classical logic (CL, for short) exhibiting non-transitivity as its central trait. This presentation, due to Cobreros, Egré, Ripley, and van Rooij, was dubbed ST.[1] By this, it's meant that ST has the *same inferential validities* that the usual two-valued Boolean presentation of CL, despite *invalidating transi-*

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[1]For reference, the reader may consult some works of its original advocates in [4, 5, 20].

tivity in a specific way, to be discussed shortly. Our contribution in this article lies in showing that similar non-transitive counterpart systems can be introduced for any Tarskian logic whatsoever, presenting the same inferential validities but invalidating transitivity. For future reference, letting φ be a formula and Γ, Δ be sets of formulas, a logic L (or, more accurately, its consequence relation) $\vdash$ is *Tarskian* if and only if it satisfies Reflexivity ($\varphi \vdash \varphi$), Monotonicity (if $\Gamma \vdash \varphi$, then $\Gamma, \Delta \vdash \varphi$), and Transitivity—to be discussed shortly.

The manner in which ST violates the transitivity aspect of the otherwise general notion of logical consequence is understood (when speaking about the sole logical system and not the naive semantic theories that may be built on top of it) in two equivalent technical ways. Thus, inasmuch as it's presented semantically and proof-theoretically, via a sequent calculus, the metainference appearing below (acting as the surrogate for the rather abstract idea of transitivity for logical consequence), usually referred to as *Cut*, has to be invalid according to both of these standards.[2]

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}\ Cut$$

For past and future reference, just like an inference is an argument relating formulas, *a metainference is an argument relating inferences*. More formally, an inference $\Gamma \Rightarrow \Delta$ of a propositional language $\mathcal{L}$ is a pair $\langle \Gamma, \Delta \rangle$ where Γ and Δ are collections of formulas of $\mathcal{L}$, whereas a metainference is a pair $\langle S, s \rangle$ where the elements of S, and s itself, are inferences of $\mathcal{L}$.

In these regards, semantically speaking, in ST Cut is *locally invalid*, whereas proof-theoretically speaking it's *not a derivable rule*. On the one hand, local validity, taking into account a certain notion of logical conse-

[2]For more on the non-transitivity of the logic ST see, e.g., [1, 2, 6, 7].

quence X over a certain set of valuations V, stands for the preservation of satisfaction by a valuation v from the premises of the metainference in question, to the conclusion of the metainference in question—see [14].[3] In other words, if a valuation isn't a counterexample to any of the premises according to the definition X of logical validity, it must not be a counterexample to the conclusion. Derivability, on the other hand, stands for the provability in a calculus G of the conclusion of the metainference in question, whenever the premises of said metainference are supplementarily added as axioms to the calculus—see [7]. That is, whenever it's possible to prove the conclusion of the metainference in question in the system resulting from expanding G with the premises of the metainference.

Going back to our previously discussed non-transitive presentation of classical logic, both according to the semantics for ST and to its calculus, all the inferences valid in the two-valued Boolean presentation of CL are deemed as valid or provable, respectively. However, some metainferences—prime among them Cut—that are valid in the two-valued Boolean presentation of CL are not valid in ST. That is to say, not locally valid according to its semantics and not derivable according to its sequent calculus, on which more below.

How does ST achieve this? From the point of view of the proof theory, one we won't be delving too much into on this occasion, ST is presented via a Cut-free (notational variant) of the propositional fragment of Gentzen's sequent calculus for classical logic LK, introduced in [13]. More crucial to our goals is ST's semantic apparatus which consists of consider-

[3] A valuation is a function assigning each formula of the language a given truth-value from a carrier set. For example, strong Kleene valuations for the propositional language counting with $\neg$ (negation), $\wedge$ (conjunction) and $\vee$ (disjunction), to be discussed below are functions onto $\{1, .5, 0\}$ satisfying that $v(\neg\varphi) = 1 - v(\varphi)$, $v(\varphi \wedge \psi) = min(v(\varphi), v(\psi))$ and $v(\varphi \vee \psi) = max(v(\varphi), v(\psi))$.

ing the set of three-valued strong Kleene valuations from [15] and endowing it with the main ingredient, the so-called *strict-tolerant notion of logical consequence* discussed, e.g., in [4]. According to this, we may say that an inference $\Gamma \Rightarrow \Delta$ is strict-tolerant valid over the set of strong Kleene valuations if and only if there's no valuation v of this kind such that $v(\gamma) = 1$, for all $\gamma \in \Gamma$, and also $v(\delta) = 0$, for all $\delta \in \Delta$. More informally, the notion of validity characteristic of ST is just the absence of counterexamples where all the premises are true and all the conclusions are false—an approach dear to CL that here has different metainferential implications, because the valuations taken into account are of the three-valued strong Kleene kind instead of the two-valued Boolean ones.

Many interesting discussions have sprung from the literature revolving around ST and the present article intends to insert itself into that list. One of these debates, perhaps worth discussing, is Barrio, Pailos, and Szmuc's intention in [2] to *generalize the result incarnated in ST*—that it's possible to have a system with the same inferential but different metainferential validities **as** classical logic—by showing that it's possible to build *an infinite sequence of systems that are incrementally closer to classical logic*, in validating the same inferences, metainferences, metametainferences, and so on... although they are always different at some point of the so-called metainferential hierarchy of arguments.

A very intriguing philosophical takeaway that the authors extract from their results is that *logical systems may not be characterized by the set of inferences that are valid in them*. Briefly put, just like decades ago the characterization of logics with sets of theorems became outdated, as discussed in [9], we might now be in need of abandoning the inference-based approach in favor of taking also metainferences (and maybe metameta...inferences) into account. Be that as it may, one could be tempted to disregard these remarks given the fact that the necessity to come up with an identity criterion for logical systems that is sensible and fine-grained enough was elicited only by the presentation of a single case (that of classical logic) where we have a logical system and a non-transitive lookalike. It could be wise to stick to the old ways if this phenomenon isn't widespread and many, if not all of the other logical systems, are safe from exhibiting it too.

This is why it's important for us to mention another work whose objective is closer to ours, due to Melvin Fitting in [10]. In his work, the question is posed as to whether it's possible to provide non-transitive versions of other logical systems. This enigma is shown to have a definite answer, in giving semantics for non-transitive versions of the non-classical logics K3 of Kleene [15], LP of Priest [19], and E_{fde} of Belnap and Dunn [3, 8]. Our goal in this article is to show that the phenomenon characteristic of ST is indeed pervasive and that *every Tarskian logical system is bound to be affected* by it. For this purpose, we'll establish that every logical system L of a Tarskian kind has at least one non-transitive counterpart, understanding by this a logical system L^{nt} having the same valid inferences that L although different valid metainferences, with Cut being salient among them. To guarantee these results we'll be taking an exclusively semantic approach, leaving proof-theoretic approximations for another occasion.

2. From Tarskian logics to their counterparts

Although some level of technical machinery will be necessary to establish our results below, the underlying phenomenon is simple to grasp once the pieces are on the table. Thus, let us comprehend first what these pieces are and what are we supposed to do with them.

Now, if we are to meddle with Tarskian logics from a semantic point of view, we should be aware of how to obtain them semantically. In this respect, there's a key result for the semantical understanding of logical systems due to Wójcicki. It was proven by him in [22] that *every Tarskian logical system can be characterized by a collection of logical matrices*. Essentially, a logical matrix consists of a collection of truth values together with a set of truth functions for the connectives in the language (so, an algebra of the same similarity type as the language in question), together with a certain notion of logical consequence understood in terms of the preservation of certain designated truth values from premises to conclusion.

Interestingly enough, some logics may not need a multiplicity of logical matrices to arrive at its resident notion of logical consequence, but rather a single one featuring only finitely many truth values—see [17]. In some other cases, we may need a multiplicity, perhaps even an infinity, of logical matrices. What we are going to show is that given a certain Tarskian logic, characterized by a collection of logical matrices, it's possible to take that collection and extend each of its members in a certain way so as to keep the same inferences being considered valid according to the extended structure—despite the fact that the resulting logic will be non-transitive in the way mentioned above, thanks to working with p-consequence relations, to be detailed below. More importantly, in a certain way our manner of extending the target logical matrix is innocuous, for the newly added truth values will not be participating in the generation of any counterexamples that were not constructible with the pieces available in the original matrices. In doing this, we'll deal with infectious logics in connection with the ST phenomenon—see [21] and [18].

Definition 1. A *logical matrix* $\mathcal{M}$ for a language $\mathcal{L}$ is a pair $\langle \mathbf{A}, D\rangle$ where

$\mathbf{A}$ is an algebra of the same similarity-type that $\mathcal{L}$ with universe A, and $D \subseteq A$. Let $\mathbf{FOR}(\mathcal{L})$ be the algebra of formulas of $\mathcal{L}$, then a *$\mathcal{M}$-valuation* v is a homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$. Finally, a logical matrix $\mathcal{M}$ for $\mathcal{L}$ induces a *Tarskian consequence relation* $\vDash_{\mathcal{M}}$ as follows:

$$\Gamma \vDash_{\mathcal{M}} \Delta \text{ if and only if } \neg\exists v : v(\gamma) \in D \text{ for all } \gamma \in \Gamma \text{ and } v(\delta) \notin D \text{ for all } \delta \in \Delta$$

When we have a class $\mathbb{M}$ of logical matrices for a language $\mathcal{L}$, the Tarskian consequence relation $\vDash_{\mathbb{M}}$ is understood as $\bigcap\{\vDash_{\mathcal{M}} \mid \mathcal{M} \in \mathbb{M}\}$.

Theorem 2 (Wójcicki [22], Theorems 3.1.5, 3.1.6)**.** *For every Tarskian logic L formulated over the language $\mathcal{L}$, there's a (possibly infinite) class $\mathbb{M}$ of logical matrices for $\mathcal{L}$ such that $\vdash_L = \vDash_{\mathbb{M}}$.*

For our result, now, we need to introduce structures that relate to logics where transitivity may be locally invalid, just like logical matrices are related to Tarskian logics. This role is played by logical p-matrices, introduced by Frankowski in [11], that are actually a generalization of regular logical matrices—in a way that will be clear in the following definition. This kind of matrices induce p-consequence relations, as defined in [11], which are generalizations of Tarskian consequence relations where both Reflexivity and Monotonicity are valid, but Cut isn't necessarily so.

Definition 3. A *logical p-matrix* $\mathcal{M}$ for a language $\mathcal{L}$ is a structure $\langle \mathbf{A}, D_p, D_c \rangle$, where $\mathbf{A}$ is an algebra of the same similarity-type as $\mathcal{L}$ with universe A and $D_p, D_c \subseteq A$, where also $D_p \subseteq D_c$. When $D_p = D_c$, then $\mathcal{M}$ is a regular logical matrix. Let $\mathbf{FOR}(\mathcal{L})$ be the algebra of formulas of $\mathcal{L}$, then a *$\mathcal{M}$-valuation v* is a homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$. A p-matrix $\mathcal{M}$ induces a so-called p-consequence relation $\vDash_{\mathcal{M}}$ as follows:

$$\Gamma \vDash_{\mathcal{M}} \Delta \text{ if and only if } \neg\exists v : v(\gamma) \in D_p \text{ for all } \gamma \in \Gamma \text{ and } v(\delta) \notin D_c \text{ for all } \delta \in \Delta$$

When we have a class $\mathbb{M}$ of logical p-matrices for a language $\mathcal{L}$, the p-consequence relation $\vDash_{\mathbb{M}}$ is understood as $\bigcap\{\vDash_{\mathcal{M}} \mid \mathcal{M} \in \mathbb{M}\}$.

For a further piece of notation, consider a logical p-matrix $M = \langle \mathbf{A}, D_p, D_c \rangle$. We say that an M-valuation v *satisfies* the inference or sequent $\Gamma \vDash \Delta$ if it is not a counterexample to it, and we symbolize it as $v \vDash_M \Gamma \Rightarrow \Delta$.

Definition 4. Given an algebra $\mathbf{A}$, its *infectious extension* with an element $e \notin A$ is the algebra $\mathbf{A}[e]$, where for all n-ary operations ¶ of $\mathbf{A}[e]$ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{e\}$: $\P^{\mathbf{A}[e]}(a_1, \ldots, a_n) = e$ if $e \in \{a_1, \ldots, a_n\}$, and $\P^{\mathbf{A}[e]}(a_1, \ldots, a_n) = \P^{\mathbf{A}}(a_1, \ldots, a_n)$ otherwise.

The construction above was first introduced in [16]. Furthermore, a version of following result can also be found in [12].

Theorem 5. *For every consequence relation $\vDash_{\mathcal{M}}$ induced by a logical matrix $\mathcal{M} = \langle \mathbf{A}, D \rangle$ there's a corresponding logical p-matrix $\mathcal{M}[e] = \langle \mathbf{A}[e], D, D \cup \{e\} \rangle$ such that $\vDash_{\mathcal{M}[e]}$ has the same inferential validities **as** $\vDash_{\mathcal{M}}$.*

Proof. First, to prove that $\Gamma \vDash_{\mathcal{M}[e]} \Delta$ implies $\Gamma \vDash_{\mathcal{M}} \Delta$ we assume that $\Gamma \nvDash_{\mathcal{M}} \Delta$. Thus, there's an $\mathcal{M}$-valuation v witnessing that $v(\gamma) \in D$ for all $\gamma \in \Gamma$ and also $v(\delta) \notin D$ for all $\delta \in \Delta$. But, it's straightforward to notice, that every $\mathcal{M}$-valuation is also a $\mathcal{M}[e]$-valuation, and therefore v is also such that $v(\gamma) \in D$ for all $\gamma \in \Gamma$ and also $v(\delta) \notin D \cup \{e\}$ for all $\delta \in \Delta$. Whence, $\Gamma \nvDash_{\mathcal{M}[e]} \Delta$. Secondly, to prove that $\Gamma \vDash_{\mathcal{M}} \Delta$ implies $\Gamma \vDash_{\mathcal{M}[e]} \Delta$, assume that $\Gamma \nvDash_{\mathcal{M}[e]} \Delta$. This implies there's a $\mathcal{M}[e]$-valuation such that $v(\gamma) \in D$ for all $\gamma \in \Gamma$ and also $v(\delta) \notin D \cup \{e\}$ for all $\delta \in \Delta$. By the infectiousness of e, this further allows us to infer that for all propositional variables p appearing in Γ and Δ, $v(p) \neq e$. Thus, for all intents and purposes, restricted to Γ

and Δ, v is a $\mathcal{M}$-valuation witnessing that $v(\gamma) \in D$ for all $\gamma \in \Gamma$ and also $v(\delta) \notin D$ for all $\delta \in \Delta$. Whence, $\Gamma \nvDash_{\mathcal{M}} \Delta$. $\square$

Lemma 6. *For every consequence relation $\vDash_{\mathbb{M}}$ induced by a class $\mathbb{M}$ of regular logical matrices $\{\langle \mathbf{A}_1, D_1\rangle, \ldots, \langle \mathbf{A}_n, D_n\rangle, \ldots\}$ there's a corresponding class $\mathbb{M}[\vec{e}]$ of logical p-matrices $\{\langle \mathbf{A}_1[e_1], D_1, D_1 \cup \{e_1\}\rangle, \ldots, \langle \mathbf{A}_n[e_n], D_n, D_n \cup \{e_n\}\rangle, \ldots\}$ such that $\vDash_{\mathbb{M}[e]}$ has the same inferential validities **as** $\vDash_{\mathbb{M}}$.*

Proof. By Theorem 5, for every $\mathcal{M}_i \in \mathbb{M}$ and each corresponding $\mathcal{M}_i[e_i] \in \mathbb{M}[\vec{e}]$, we have that $\vDash_{\mathcal{M}_i} = \vDash_{\mathcal{M}_i[e_i]}$. From this, the result easily ensues. $\square$

Corollary 7. *Every Tarskian logic L has a non-transitive counterpart L^{nt}.*

Proof. Let L be a Tarskian logic. We know by Wójcicki's result (Theorem 2 above) there's a class of logical matrices $\mathbb{M}$ such that, without loss of generality, $\vDash_{\mathbb{M}} = \vdash_L$. By Theorem 5 and Lemma 6, we know there's a class of logical p-matrices $\mathbb{M}[\vec{e}]$ extending $\mathbb{M}$ appropriately with infectious elements such that $\vDash_{\mathbb{M}} = \vDash_{\mathbb{M}[\vec{e}]}$. Let L^{nt} be the p-logic such that $\vdash_{L^{nt}} = \vDash_{\mathbb{M}[\vec{e}]}$. Then, we have that $\vdash_L = \vdash_{L^{nt}}$. **Now, to witness the non-transitivity of L^{nt}, we appeal to the notation introduced after Definition 3. Consider a $\mathcal{M}_i[e_i]$-valuation v such that $v(p) \in D_i$, $v(q) \notin D_i \cup \{e_i\}$ and $v(r) = e_i$. There, we have that $v \vDash_{\mathcal{M}_i[e_i]} p \Rightarrow r$, $v \vDash_{\mathcal{M}_i[e_i]} r \Rightarrow q$, and yet $v \nvDash_{\mathcal{M}_i[e_i]} p \Rightarrow q$.** From this, it follows that the class $\mathbb{M}[\vec{e}]$ provides semantics for a non-transitive counterpart L^{nt} of the Tarskian logic L. $\square$

3. Final thoughts

Having proved that the ST phenomenon is in fact as widespread as it can be amongst logics, one may be tempted to wonder what conceptual conclusions are to be drawn from this.

For starters, we may say that if the existence of a non-transitive counterpart to CL pushed for a more fine-grained identity criterion for this logic, our results show that indeed this isn't only required because of what is affecting CL, but because of what is affecting every Tarskian logical system. However, some could say the complications elicited by the presence of non-transitive counterparts should be resolved by the identification of a logic with a set of valid inferences together with a set of valid metainferences. Nevertheless, the results above point to the fact that it's quite straightforward to mimic the argumentative move by Barrio, Pailos and Szmuc in [2]—where a sequence of systems that were incrementally closer to CL is presented, with the systems coinciding with it with respect to more and more metameta... inferences. For the purpose of this further generalization, certain non-reflexive systems associated to any Tarskian logic will need to be presented, in order to define such a hierarchy of logics that would be similar to the original logic in question. The details of the construction of such a hierarchy can be easily obtained by the construction in [2], adapting them to the present cases. Thus, this could all be used to support the idea that *the philosophical reflections on the need for refined identity criteria for logics are not only backed by what pertains to CL*, but rather to what affects any Tarskian logic, with full generality.

Finally, one could also ask for *a complete characterization of the non-transitive counterparts of any Tarskian logic*. Admittedly, the counterparts that we presented here were explicitly motivated by the addition of infectious values to the original structures rendering the logic in question. It may also be possible, though, to arrive at other non-transitive counterparts via some other considerations, be they semantic or proof-theoretic. An examination of these and other issues will have to wait for further research.[4]

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