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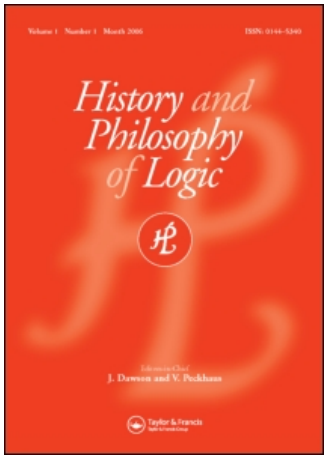

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What are logical notions?

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What are Logical Notions?

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In this manuscript, published here for the first time, Tarski explores the concept of logical notion. He draws on Klein's Erlanger Programm to locate the logical notions of ordinary geometry as those invariant under all transformations of space. Generalizing, he explicates the concept of logical notion of an arbitrary discipline.

1. Editor's introduction

In this article Tarski proposes an explication of the concept of logical notion. His earlier well-known explication of the concept of logical consequence presupposes the distinction between logical and extra-logical constants (which he regarded as problematic at the time). Thus, the article may be regarded as a continuation of previous work.

In Section 1 Tarski states the problem and indicates that his proposed explication shares features both with nominal (or normative) definitions and with real (or descriptive) definitions. Nevertheless, he emphasizes that his explication is not arbitrary and that it is not intended to 'catch the platonic idea'. In Section 2, in order to introduce the essential background ideas, Klein's Erlanger Programm for classifying geometrical notions is sketched using three basic examples: (1) the notions of metric geometry are those invariant under the similarity transformations; (2) the notions of descriptive geometry are those invariant under the affine transformations; and (3) the notions of topological geometry (topology) are those invariant under the continuous transformations. This illustrates the fact that as the family of transformations expands not only does the corresponding family of invariant notions contract but also, in a sense, the invariant notions become more 'general'. In Section 3 Tarski considers the limiting case of the notions invariant under all transformations of the space and he proposes that such notions be called 'logical'. Then, generalizing beyond geometry, a notion (individual, set, function, etc) based on a fundamental universe of discourse is said to be *logical* if and only if it is carried onto itself by each one-one function whose domain and range both coincide with the entire universe of discourse.

Tarski then proceeds to test his explication by deducing various historical, mathematical and philosophical consequences. All notions definable in *Principia mathematica* are logical in the above sense, as are the four basic relations introduced

by Peirce and Schröder in the logic of relations. No individual is logical: all numerical properties of classes are logical, etc. In Section 4 Tarski considers the philosophical question of whether all mathematical notions are logical. He considers two construals of mathematics—the type-theoretic construal due to Whitehead and Russell, and the set-theoretic construal due to Zermelo, von Neumann and others. His conclusion is that mathematical notions are all logical relative to the type-theoretic construal but not relative to the set-theoretic construal. Thus, no answer to the philosophical question of the reducibility of mathematical notions to logic is implied by his explication of the concept of logical notion.

2. Editorial treatment

The wording of this article reveals its origin as a lecture. On 16 May 1966 Tarski delivered a lecture of this title at Bedford College, University of London. A tape-recording was made and a typescript was developed by Tarski from a transcript of the tape-recording. On 20 April 1973 he delivered a lecture from the typescript as the keynote address to the Conference on the Nature of Logic sponsored by various units of the State University of New York at Buffalo. I made careful notes of this lecture and from them wrote an extended account which was published in the University newspaper (*The reporter*, 26 April 1973). Copies of the newspaper article were sent to Tarski and others. It was Tarski's intention to polish the typescript and to publish it as a companion piece to his 'Truth and proof' (*1969*). Over the next few years I had several opportunities to speak with Tarski and to reiterate my interest in having the lecture appear in print. In 1978 I began work on editing the second edition of Tarski's *Logic, semantics, metamathematics*, which finally appeared in December 1983 shortly after Tarski's death. During the course of my work with Tarski for that project, he said on several occasions that he wanted me to edit 'What are logical notions?', but it was not until 1982 that he gave me the typescript with the injunction that it needed polishing.

For the most part my editing consisted in the usual editorial activities of correcting punctuation, sentence structure and grammar. In some locations the typescript was evidently a transcript written by a non-logician. Occasionally there was a minor lapse (e.g. in uniformity of terminology). The bibliography and footnotes were added by me. The only explicit reference in the typescript is in Section 3 where the 1936 article by Lindenbaum and Tarski is mentioned. Of course, the greatest care was taken to guarantee that Tarski's ideas were fully preserved.

For further discussion and applications of the main idea of this paper see the book by Tarski and Steven Givant (*1987*), especially section 3.5 in chapter 3.

WHAT ARE LOGICAL NOTIONS?

Alfred Tarski

1. The title of my lecture is a question; a question of a type which is rather fashionable nowadays. There is another type of question you often hear: what is psychology, what is physics, what is history? Questions of this type are sometimes answered by specialists working in the given science, sometimes by philosophers of

science; the opinion of a logician is also asked from time to time as an alleged authority in such matters. Well, let me say that specialists working in a given science are usually the people least qualified to give a good definition of the science. It is a domain where you would normally expect an intelligent discussion from a philosopher of science. And a logician is certainly not an authority—he is not specially qualified to answer questions of this type. His role and influence are rather of a negative character—he offers criticism, he points out how vague a certain formulation is, how indefinite an account of a certain science is. In view of his negative approach to discussing definitions of other sciences, a logician must certainly be especially cautious when he discusses his own science and tries to say what logic is.

Answers to the question 'What is logic?' or 'What is such and such science?' may be of very different kinds. In some cases we may give an account of the prevailing usage of the name of the science. Thus in saying what is psychology, you may try to give an account of what most people who use this term normally mean by 'psychology'. In other cases we may be interested in the prevailing usage, not of all people who use a given term, but only of people who are qualified to use it—who are expert in the domain. Here we would be interested in what psychologists understand by the term 'psychology'. In still other cases our answer has a normative character: we make a suggestion that the term be used in a certain way, independent of the way in which it is actually used. Some further answers seem to aim at something very different, but it is very difficult for me to say what it is; people speak of catching the proper, true meaning of a notion, something independent of actual usage, and independent of any normative proposals, something like the platonic idea behind the notion. This last approach is so foreign and strange to me that I shall simply ignore it, for I cannot say anything intelligent on such matters.

Let me tell you in advance that in answering the question 'What are logical notions?' what I shall do is make a suggestion or proposal about a possible use of the term 'logical notion'. This suggestion seems to me to be in agreement, if not with all prevailing usage of the term 'logical notion', at least with one usage which actually is encountered in practice. I think the term is used in several different senses and that my suggestion gives an account of one of them.[1] Moreover, I shall not discuss the general question 'What is logic?' I take logic to be a science, a system of true sentences, and the sentences contain terms denoting certain notions, logical notions. I shall be concerned here with only one aspect of the problem, the problem of logical notions, but not for instance with the problem of logical truths.

2. The idea which will underlie my suggestion goes back to a famous German mathematician, Felix Klein. In the second half of the nineteenth century, Felix Klein did very serious work in the foundations of geometry which exerted a great influence on later investigations in this domain.[2] One problem which interested him was that of distinguishing the notions discussed in various systems of geometry, in various geometrical theories, e.g. ordinary Euclidean geometry, affine geometry, and topo-

1 It would be instructive to compare these remarks with those that Tarski makes in connection with his explications of truth in his *1935a* and of logical consequence in his *1936*, especially p. 420. See also Corcoran *1983*, especially pp. xx-xxii.

2 See, e.g., Klein *1872*.

logy. I shall try to extend his method beyond geometry and apply it also to logic. I am inclined to believe that the same idea could also be extended to other sciences. Nobody so far as I know has yet attempted to do it, but perhaps one can formulate using Klein's idea some reasonable suggestions to distinguish among biological, physical, and chemical notions.

Now let me try to explain to you very briefly Klein's idea. It is based upon a technical term 'transformation', which is a particular case of another term well known to everyone from high school mathematics—the term 'function'. A function or a functional relation is, as we all know, a binary relation r which has the property that whatever object x we consider there exists at most one object y to which x is in the relation r. Those x's for which such a y actually exists are called 'argument values'. The corresponding y's are called 'function values'. We also write $y=r(x)$; this is the normal function notation. The set of all argument values is called the 'domain of the function', the set of function values is called in *Principia Mathematica* the 'counter-domain', more often 'the range', of the function. So every function has its domain and its range. We often deal in mathematics with functions whose domain and range consist of numbers. However, there are also functions of other types. For instance we may consider functions whose domain and range consist of points. In particular, in geometry we deal with functions whose domain and range both coincide with the whole geometrical space. Such a function is referred to as a 'transformation' of the space onto itself. Moreover we often deal with functions which are one-one functions, with functions which have the property that to any two different argument values the corresponding function values are always different. We say that such a function establishes a one-one correspondence between its domain and its range. So a function whose domain and range both coincide with the whole space and which is one-one is called a one-one transformation of the space onto itself (more briefly, 'a transformation'). I shall now discuss transformations of ordinary geometrical space.

Now let us consider normal Euclidean geometry which again we all know from high school. This geometry was originally an empirical science—its purpose was to study the world around us. This world is populated with various physical objects, in particular with rigid bodies, and a characteristic property of rigid bodies is that they do not change shape when they move. Now every motion of such a rigid body corresponds to a certain transformation because a rigid body occupies one position when it starts moving and as a result of this motion occupies another position. Each point occupied by the rigid body at the beginning of the motion corresponds to a point occupied by the same body at the end of the motion. We have a functional relation. It is true that this is not a functional relation whose domain includes all points of the space, but it is known from geometry that it can always be extended to the whole space. Now what is characteristic about this transformation is that the distance between two points does not change. If x and y are at a certain distance and if $f(x)$ and $f(y)$ are the final points corresponding to x and y, then the distance between $f(x)$ and $f(y)$ is the same as that between x and y. We say that distance is *invariant* under this transformation. This is a characteristic property of motions of rigid bodies—if it did not hold, we would not call the body a rigid body.

As you see, we are naturally led in geometry to consider a special kind of transformation of this space, transformations which do not change the distance between points. Mathematicians have a bad habit of taking a term from other domains—from physics, from anthropology—and ascribing to it a related but different meaning. They have done this with the term 'motion'. They use the term 'motion' in a mathematical sense, in which it means simply a transformation in which distance does not change. So the motion of a particular physical object, a rigid body, results in a certain transformation; but to a mathematician motions are simply transformations which do not change distance. Such transformations are more properly called 'isometric transformations'.

Now, Klein points out that all the notions which we discuss in Euclidean geometry are invariant under all motions, that is, under all isometric transformations. Let me say again what we mean when we say that a notion is invariant under certain transformations. I use the term 'notion' in a rather loose and general sense, to mean, roughly speaking, objects of all possible types in some hierarchy of types like that in *Principia mathematica*. Thus notions include individuals (points in the present context), classes of individuals, relations of individuals, classes of classes of individuals, and so on. What does it mean, for instance, to say that a class of individuals is invariant under a transformation f? This means that x belongs to this class if and only if $f(x)$ also belongs to this class, in other words, that this class is carried onto itself by the transformation. What does it mean to say that a relation is invariant under a transformation f? This means that x and y stand in the relation if and only if $f(x)$ and $f(y)$ stand in the relation. We can easily extend the notion of invariance in a familiar way to classes of classes, relations between classes, and so on.

Now a close analysis of Euclidean geometry shows that all notions which we discuss there are invariant not only under motions, under isometric transformations, but under a wider class of transformations, namely under those transformations which geometers call 'similarity transformations'. These are transformations which do not all preserve distance, but which so to speak increase or decrease the size of a geometrical figure uniformly in all directions. More precisely, some similarity transformations do not preserve distance, but all preserve the ratio of two distances. If you have, for instance, three points, x, y, z, and if the distance from y to z is larger by 25% than the distance from x to y, then the result of a similarity transformation is again three points, $f(x)$, $f(y)$, $f(z)$, where the distance between $f(y)$ and $f(z)$ is 25% larger than the distance between $f(x)$ and $f(y)$. In other words, a triangle is transformed into a triangle which is similar to it, with the same angles and whose sides are proportionally larger or smaller. And it turns out that all properties which one discusses in Euclidean geometry are invariant under all possible similarity transformations. This means, incidentally, that we cannot discuss in Euclidean geometry the notion of a unit of measure. We should not ask such a geometer whether from the point of view of his discipline the metric system or a non-metric system is preferable. In Euclidean terms we cannot distinguish a metre from a yard; we cannot even distinguish a centimetre from a yard. Any two segments are "the same", since you can always transform them one into another by means of a similarity transformation.

Every Euclidean property that belongs to one segment belongs to every other segment as well.

Now Klein says that invariance under all similarity transformations is the characteristic property of the notions studied in metric geometry,[3] which is another term for ordinary Euclidean geometry. We can express this as a definition: a *metric* notion, or a notion of metric geometry, is simply a notion which is invariant under all possible similarity transformations. We could certainly imagine a discipline in which we would be interested in a narrower class of transformations, for instance only in isometric transformations, or only in transformations which preserve the distinction between being to the right and being to the left (a distinction which we are unable to make in our normal geometry), or between a motion which is clockwise from a motion which is counter-clockwise (again a distinction we cannot make in normal Euclidean geometry). But by narrowing down the class of permissible transformations we can make more distinctions, i.e. we widen the class of notions invariant under permissible transformations. The extreme case in this direction in geometry would be to single out four points, give them names, and to consider only those transformations which would leave these four points invariant. This would mean introducing a co-ordinate system, and we would be at a limit of the domain of geometry, i.e. at what is called analysis. Actually in this case there would be no permissible transformations except one "trivial" identity transformation.

On the other hand one can go in the opposite direction; instead of narrowing down the class of permissible transformations, and in this way widening the class of invariant notions, we can do the opposite, and widen the class of transformations. We can for instance include also transformations in which distance may change, but what is unchanged is mutual linear position of points. More precisely, if three points are on one line, then their images, after the transformation, are also on one line. If one point is between two other points, then its image is between the images of the two other points. One calls such transformations 'affine transformations'. Collinearity and betweenness are just two of the notions which are invariant under all transformations of this kind. The part of geometry where such notions are used is called affine geometry.[4] In this geometry we cannot distinguish, for example, one segment from another, indeed we cannot make any distinctions among triangles. Any two triangles are so to speak equal, that is, indistinguishable from the point of view of affine geometry. This means that we cannot point out any property in affine geometry which

3 Terminology in this field is not uniform, and Tarski's usage may not be familiar to some readers. The present terminology derives from Tarski *1935b*, where the term 'descriptive geometry' is used to indicate the part of ordinary Euclidean geometry based only on 'point' and 'between' (which Tarski refers to as 'the descriptive primitive'). The term 'metric geometry' is used to indicate all of ordinary Euclidean geometry (which, as Tarski notes, can be taken to be based only on 'point' and 'congruence'—a notion that Tarski calls 'the metric primitive'). In the same article Tarski indicates that descriptive geometry is a proper part of metric geometry in the sense that 'between' is definable from 'point' and 'congruence' while 'congruence' is not definable from 'point' and 'between'.

4 'Affine geometry' is in current use in exactly this sense. What Tarski calls 'affine geometry' here, he called 'descriptive geometry' in *1935b*. An affine transformation that is not a similarity can be obtained in plane geometry by a parallel projection of the plane onto a non-perpendicular, intersecting "copy" of itself. Concretely, the image of a suitably placed isosceles right triangle is scalene, but all images of triangles are triangles.

is possessed by one of the triangles but not by all others. In metric geometry we know many such properties, for example, the property of being equilateral, or of being right-angled. In affine geometry we cannot make any such distinctions. What we can distinguish is a triangle from a quadrangle, because no affine transformation could start with a triangle and lead to a quadrangle. So here we have an example of a wider class of transformations, and as a result of this, a narrower class of notions which are invariant under this wider class of transformations; the notions are fewer, and of a more "general" character.

We can go a step further. We can include, for instance, transformations in which even the betweenness relation is not preserved, and even transformations where points which lie on the same straight line are transformed into points lying on different lines. The characteristic thing which is preserved here is, roughly speaking, connectedness or closedness. A connected figure remains connected. A closed curve remains closed. Sometimes it is said, putting things "negatively" so to speak, that these transformations are those which do not "break up" or "tear apart". This is a very imprecise way of formulating it, but some of you probably have guessed what I have in mind; I have in mind the so-called continuous transformations, and the part of geometry, the geometrical discipline which deals with notions invariant under such transformations, is topology. In metric geometry we can distinguish one triangle from another; in affine geometry we cannot do so, but we can still distinguish between a triangle and, let us say, a quadrangle. But in topology we cannot distinguish between two polygons, or even between a polygon and a circle, because given a polygon, if we imagine it to be made of wire, we can always bend it in such a way as to obtain a circle, or any other polygon. Such a transformation will be continuous: we do not separate anything which was connected. What we can distinguish in topology is, for example, one triangle from two triangles. For a triangular wire can be bent into two triangles only if we break it into two parts and form a triangle from each part—and this would not be a continuous transformation.

3. Now suppose we continue this idea, and consider still wider classes of transformations. In the extreme case, we would consider the class of *all* one-one transformations of the space, or universe of discourse, or 'world', onto itself. What will be the science which deals with the notions invariant under this widest class of transformations? Here we will have very few notions, all of a very general character. I suggest that they are the logical notions, that we call a notion 'logical' if it is invariant under all possible one-one transformations of the world onto itself.[5] Such a suggestion perhaps sounds strange—the only way of seeing whether it is a reasonable suggestion is to

5 Apart from Mautner *1946*, which Tarski seems not to have known, this is, I believe, the first attempted application in English of Klein's Erlanger Programm to logic. However, in Silva *1945*, which is written in Italian, we find applications which anticipate essential elements of later model theory. Keyser (*1922*, 219) and Weyl (*1949*, 73) indicate in more or less vague terms the possibility of connections between logic and the Erlanger Programm. Tarski's papers from 1923 to 1938 (collected in Tarski *1983*) do not mention Felix Klein. The history of the influence of the Erlanger Programm on the development of logic remains to be written. Also needing investigation is the role of the Erlanger Programm in physics, especially relativity.

discuss some of its consequences, to see what it leads to, what we have to believe if we agree to use the term 'logical' in this sense.

A natural question is this: consider the notions which are denoted by terms which can be defined within any of the existing systems of logic, for instance *Principia mathematica*. Are the notions defined in *Principia mathematica* logical notions in the sense which I suggest? The answer is yes; this is a rather simple meta-logical result, formulated a long time ago (*1936*) in a short paper by Lindenbaum and myself. Though this result is simple, I think that it should be included in most logic textbooks, because it shows a characteristic property of what can be expressed by logical means. I am not going to formulate the result in a very exact way, but the essence of it is just what I have said. Every notion defined in *Principia mathematica*, and for that matter in any other familiar system of logic, is invariant under every one-one transformation of the 'world' or 'universe of discourse' onto itself.[6]

Next we look for examples of logical notions in a systematic way, starting with the simplest semantical categories[7] or types, and going on to more and more complicated ones. For instance, we can start with individuals, with objects of the lowest type, and ask: What are examples of logical notions among individuals? This means: What are examples of individuals which would be logical in the above sense? And the answer is simple: There are no such examples. There are no logical notions of this type, simply because we can always find a transformation of the world onto itself where one individual is transformed into a different individual. The simple fact that we can always define such a function means that on this level there are no logical notions.

If we proceed to the next level, to classes of individuals, we ask: What classes of individuals are logical in this sense? It turns out, again as a result of a simple argument, that there are exactly two classes of individuals which are logical, the universal class and the empty class. Only these two classes are invariant under every transformation of the universe onto itself.

If we go still further, and consider binary relations, a simple argument shows that there are only four binary relations which are logical in this sense: the universal relation which always holds between any two objects, the empty relation which never holds, the identity relation which holds only between "two" objects when they are identical, and its opposite, the diversity relation. So the universal relation, the empty relation, identity, and diversity—these are the only logical binary relations between individuals. This is interesting because just these four relations were introduced and discussed in the theory of relations by Peirce, Schröder, and other logicians of the nineteenth century. If you consider ternary relations, quaternary relations, and so on,

6 In his Buffalo lecture Tarski indicated that the present remarks apply to 'notions' taken in the narrow sense of sets, classes of sets, etc. but that the truth-functions, quantifiers, relation-operators, etc. of *Principia mathematica* can be construed as notions in the narrow sense and, so construed, the present remarks apply equally to them. For example, construing the truth-values T and F as the universe of discourse and the null set leads immediately to construing truth-functions as (higher-order) notions. Construals of this sort are familiar and natural to mathematicians, but they involve philosophical questions of the sort investigated by contemporary philosophers of logic.

7 In Tarski *1935a*, 'The Wahrheitsbegriff', there is an extended discussion of semantical categories (which properly include the 'types' treated by Whitehead and Russell). On p.215 Tarski attributes the concept of semantical categories to Husserl.

the situation is similar: for each of these you will have a small finite number of logical relations.

The situation becomes a little more interesting if you go to the next level, and consider classes of classes. Instead of saying 'classes of classes' we can say 'properties of classes', and ask: What are the properties of classes which are logical? The answer is again simple, even though it is quite difficult to formulate in a precise way. It turns out that the only properties of classes (of individuals) which are logical are properties concerning the number of elements in these classes. That a class consists of three elements, or four elements ... that it is finite, or infinite—these are logical notions, and are essentially the only logical notions on this level.

This result seems to me rather interesting because in the nineteenth century there were discussions about whether our logic is the logic of extensions or the logic of intensions. It was said many times, especially by mathematical logicians, that our logic is really a logic of extensions.[8] This means that two notions cannot be logically distinguished if they have the same extension, even if their intensions are different. As it is usually put, we cannot logically distinguish properties from classes. Now in the light of our suggestion it turns out that our logic is even less than a logic of extension, it is a logic of number, of numerical relations. We cannot logically distinguish two classes from each other if each of them has exactly two individuals, because if you have two classes, each of which consists of two individuals, you can always find a transformation of the universe under which one of these classes is transformed into the other. Every logical property which belongs to one class of two individuals belongs to every class containing exactly two individuals.

If you turn to more complicated notions, for instance to relations between classes, then the variety of logical notions increases. Here for the first time you come across many important and interesting logical relations, well known to those who have studied the elements of logic. I mean such things as inclusion between classes, disjointness of two classes, overlapping of two classes, and many others; all these are examples of logical relations in the normal sense, and they are also logical in the sense of my suggestion. This gives you some idea of what logical notions are. I have restricted myself to four of the simplest types, and discussed examples of logical notions only within these types. To conclude this discussion, I would like to turn to a question which has probably already occurred to some of you as you listened to my remarks.

4. The question is often asked whether mathematics is a part of logic. Here we are interested in only one aspect of this problem, whether mathematical notions are logical notions, and not, for example, in whether mathematical truths are logical truths, which is outside our domain of discussion. Since it is now well known that the whole of mathematics can be constructed within set theory,[9] or the theory of

8 See Whitehead and Russell *1910*, III (2).

9 Tarski is using the term 'set theory' here in a vague and general sense in which several distinct concrete theories may all qualify as set theory. In particular, the Whitehead–Russell theory of types and the (first-order) Zermelo–Fraenkel theory both qualify as set theory. It is to the point here to note that Tarski regarded the current variety of 'set theories' as only a small sample of what can usefully be developed in this field. In the Editor's introduction, 'set theory' is used in a narrower sense that contrasts with type theory.

classes, the problem reduces to the following one: Are set-theoretical notions logical notions or not? Again, since it is known that all usual set-theoretical notions can be defined in terms of one,[10] the notion of belonging, or the membership relation, the final form of our question is whether the membership relation is a logical one in the sense of my suggestion. The answer will seem disappointing. For we can develop set theory, the theory of the membership relation, in such a way that the answer to this question is affirmative, or we can proceed in such a way that the answer is negative.

So the answer is: 'As you wish'! You all know that as a result of the antinomies, basically Russell's Antimony, which appeared in set theory at the turn of the century, it was necessary to submit the foundations of set theory to a thorough investigation. One result of this investigation, which is by no means complete at this moment, is that two methods have been developed of constructing what can be saved from set theory after the crushing blow which it had suffered. One method is essentially the method of *Principia mathematica*, the method of Whitehead and Russell—the method of types. The second method is the method of people such as Zermelo, von Neumann, and Bernays—the first-order method. Now let us look to our question from the point of view of these two methods.[11]

Using the method of *Principia mathematica*, set theory is simply a part of logic. The method can be roughly described in the following way: we have a fundamental universe of discourse, the universe of individuals, and then we construct out of this universe of individuals certain notions, classes, relations, classes of classes, classes of relations, and so on. However, only the basic universe, the universe of individuals, is fundamental. A transformation is defined on the universe of individuals, and this transformation induces transformations on classes of individuals, relations between individuals, and so on. More precisely, we consider the universal class of the lowest type, and a transformation has this universal class as its domain and range. Then this transformation induces also a transformation whose domain and range is the universal class of the second type, the class of classes of individuals. When we speak of transformations of the 'world' onto itself we mean only transformations of the basic universe of discourse, of the universe of individuals (which we may interpret as the universe of physical objects, although there is nothing in *Principia mathematica* which compels us to accept such an interpretation). Using this method, it is clear that the membership relation is certainly a logical notion. It occurs in several types, for

10 This remark presupposes the convention that a given notion can be said to be definable in terms of one fixed notion if there is a definition (of the given notion) which uses no notions other than the following: (i) the one fixed notion, (ii) the universe of discourse, (iii) other notions already accepted as being logical. It is obvious, e.g., that there is no way to define the null set using the membership relation and absolutely nothing else. It should also be noted that Tarski says 'all *usual* set-theoretic notions' and not 'all set-theoretic relations'; there are uncountably many of the latter but only countably many definitions.

11 Tarski takes the first method to involve a higher-order underlying logic and the second method to involve a first-order underlying logic. It is possible of course to reconstrue type theory in a many-sorted first-order underlying logic, but this would be incompatible with the spirit and letter of this lecture. Likewise, it is possible to develop Zermelo's set theory in a higher-order logic. This too is incompatible with the spirit of this lecture—despite the historical fact that Zermelo may have done so himself. Incidentally, the historic papers establishing the two methods were published in the same year, 1908.

individuals are elements of classes of individuals, classes of individuals are elements of classes of classes of individuals, and so on. And by the very definition of an induced transformation it is invariant under every transformation of the world onto itself.

On the other hand, consider the second method of constructing set theory, where we have no hierarchy of types, but only one universe of discourse and the membership relation between its individuals is an undefined relation, a primitive notion. Now it is clear that this membership relation is not a logical notion, because as I mentioned before, there are only four logical relations between individuals, the universal relation, the empty relation, and the identity and diversity relations. The membership relation, if individuals and sets are considered as belonging to the same universe of discourse, is none of these relations; therefore, under this second conception, mathematical notions are not logical notions.

This conclusion is interesting, it seems to me, because the two possible answers correspond to two different types of mind. A monistic conception of logic, set theory, and mathematics, where the whole of mathematics would be a part of logic, appeals, I think, to a fundamental tendency of modern philosophers. Mathematicians, on the other hand, would be disappointed to hear that mathematics, which they consider the highest discipline in the world, is a part of something so trivial as logic; and they therefore prefer a development of set theory in which set-theoretical notions are not logical notions. The suggestion which I have made does not, by itself, imply any answer to the question of whether mathematical notions are logical.

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