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The *Cost of Consistency:* Information Economy in Paraconsistent Belief Revision

Rafael R. Testa

Abstract

By Belief Revision it is understood a system that logically explains the rational process of changing beliefs by taking into account a new piece of information. The most influential approach in this field of study, the AGM system, proposed by Alchourrón, Gärdenfors, and Makinson, postulates *rationality criteria* for different types of belief change. In this paper I shall assess the relationship between those criteria and argue for an opposition between the principles of *Information Economy* and *Consistency*. Furthermore, I shall argue that Paraconsistent Belief Revision manages to minimise this friction in the best possible way.

Keywords: Paraconsistent Belief Revision, formal consistency, non-contradiction, information economy, minimality.

1 Brief introduction of Belief Revision

This article is generally concerned with (i) the dynamic process of changing one's mind and (ii) the principles that describe such revisions. The core intuition I want to stress is that beliefs are not static. Rather, they evolve over time. This fact can come about in several scenarios. For instance, a reasoner may obtain previously unknown information by (a) either new observations or (b) experiments that reveal new facts about the world, or (c) by changes of one own interests. In each of these cases, either accepted beliefs have to be adapted according to the new information, or the latter has to be shaped according to the original set of information, or ignored and not be incorporated into it.

These situations arise in whichever way one deals with information, be it with beliefs, facts, rules, data, and so on. As long as information belong to certain domains of interest, Belief Revision is applicable to several fields of study, such as Artificial Intelligence (Nebel [28]), Software Engineering and Market Research (Williams [37]), Ontology and Web Semantic (Flouris [7]), Learning (Kelly [22]), Epistemology (Hendricks [20]), Philosophy of Science (Hansson [16]) and Rational Choice Theory (Arlo-Costa and Pedersen [3]), to name a few.

In this paper I shall focus primarily on the conceptual analysis of Belief Revision, making use of the technical aspects only if necessary. I shall analyse three core features of classical Belief Revision, specifically from the system AGM, and explore the limits of their mutual dependency. I claim that, if one strictly adheres to consistency, this mutual dependency can only be obtained under strict conditions. Otherwise, once these conditions undergo even minor changes, the dependency has to be reassessed significantly. In order to accommodate these results, I shall argue in favour of an underlying logic that provides a new perspective on the notion of consistency.

It is commonly held that AGM model of belief revision is an unusually simple and elegant system. Nevertheless, there still remain several problems in this field of research that are open to debate (Hansson [19]). I claim that most of these difficulties result from considering belief revision, AGM in particular, to be an adequate account of changes of mind. I follow Hansson [17] on the importance of advancing new definitions of the operation of revision. However, while Hansson, among other things, is more concerned with the notion of closure for revisions, I shall assume closure and argue against the notion of consistency. I view these results only to be an extension of the AGM model, rather than an attack on it.

In what follows, I shall start by providing a basic account of standard AGM-style models of belief revision. I will restrict my account according to my intends and purposes. Hence, a brief characterisation of criteria and operations will be introduced, which lay the foundations for the present paper.

1.1 The system AGM and its Criteria of Rationality

Consider the following situation. A reasoner is presented to a new piece of information that contradicts with her current set of beliefs. According to the AGM model, she has to retract some of the previously accepted beliefs, so that she can consistently accommodate the newly obtained information. Furthermore, the necessary changes within her set of beliefs are required to be as small as possible.

These two conditions set by the AGM model imply a theory based on notions of consistency and the economy of information. For instance, although the reasoner is required to give up a certain belief, she does not have to reject any other belief, whose sole justification is the rejected belief, as long as the resulting set of beliefs remains consistent. Furthermore, AGM-style belief revision assumes an idealized reasoner, referred to as *agent*. The agent is endowed with (i) a potentially infinite set of beliefs (ii) closed under logical consequence, which is called *epistemic state*. As long as the agent satisfies this condition, she can be either a human, a computer programme, or any system able to subscribe to beliefs and whose behavior can be expected to be *rational*. By rationality, the following criteria are to be considered (see [13]):

Consistency. Whenever it is possible, epistemic states should remain consistent;

Closure. Any proposition logically entailed by beliefs within an epistemic state belong to the epistemic state;

Information economy (minimality). Whenever changes occur in an epistemic state, loss of information should be kept at a minimum;

Entrenchment (preference relation). Beliefs held in higher regard should be retained at the expense of those held in lower regard.

These criteria settle how rational agents are expected to apply certain operations for the dynamics of their beliefs. It is important to remark that each of these criteria is, in fact, an intuitive notion. Accordingly, the formalization of these criteria, or better, the formal definitions that are introduced in order to capture these intuitions will bear significant consequences on how the system of belief revision may be interpreted.

The operations of a system of belief revision are supposed to formally represent what happens to a certain set of beliefs to which new information is introduced. The following are the most basic operations:

Contraction. Removal of a belief from the current epistemic state. Eventually, it is required to remove a certain belief, in order to ensure the success of the operation.

Expansion. Incorporation of a belief that is compatible with the current epistemic state. Hence, retraction of any existing belief is not required.

Revision. Incorporation of a belief that is incompatible with the current epistemic state. Therefore, removal of some existing beliefs are required in order to maintain consistency[1].

Each individual operation can be fully characterised by a set of formal postulates, grounded on the criteria stated above[2].

The system AGM, as sketched above, is successful in presenting a formal account of what may happen whenever new information is added to the original set of beliefs of an agent. Now, the success of its account depends on how the system formalizes the intuitive criteria stated above. Although it captures each of the criteria, its formalization sets up a picture that is rather artificial, due to their idealisation. Since the criteria are intuitive notions, it does not mean that AGM-style systems are the only possible interpretation. Rather, they allow different formalisations, according to the refinement of the intuitive notions[3]. So the obvious question is whether or not the same analysis of belief revision holds, once such a refinement is proposed.

[1]The name of the theory is due to the homonymous operation. It is often called in the texts of the area all operations in a broad sense.

[2]See David Makinson [24], for a more detailed account.

[3]An interesting discussion on that matter is made by Fermé [6].

2 On the Principles of Rationality

As mentioned above, the criteria that form the basis of the analysis of belief revision are intuitive and may be given a formal interpretation. Hence, the formal system of belief revision may depend on a semantic account of what each individual criterion means. Stated in this manner, many philosophical issues related to this theory may come to light. Thus, there are some logico-philosophical questions underlying the formal constructions of this system, such as the nature of rationality and how it may be adequately captured by formal accounts.

Due to the aims of the present paper, I shall consider the following similar question: What are the most basic logical and non-logical principles, assumed in rational belief revision? In order to treat this question in a adequate way, one has to consider several aspects. An interesting aspect, among these, is the role played by the notion of consistency.

In classical analyses of belief revision, epistemic states in which both a sentence and its negation are present are taken to be contradictory and called logically inconsistent. The very idea that inconsistency is an undesirable or even an impossible aspect is at the heart of classical analysis. Unsurprisingly, this intuition underlies the interpretation of rationality adopted by classical theories of belief revision.

In what follows, I shall address the question whether the criterion of consistency has to be considered the most basic of the principle in belief revision.

2.1 The Principle of Consistency

Many systems have focused on wiping out contradictions of belief sets. Others burden the system with notions such as temporality and alethic modalities, or any other mechanism by means of which they can deal with contradiction. These mechanisms allow these systems to either isolate or suppress a given contradiction. Nevertheless, all those systems seem to agree that an epistemic state, or, for that matter, a theory, in which a contradiction is present constitutes a puzzling situation that should be somehow resolved.

Interpretations like this are faced with an issue, however. In order to maintain consistency, such an interpretation looses its ability to account for the dynamic character of belief sets in a satisfactory way, as will be formally shown, in the next section. This is a puzzling result. If one wishes to preserve consistency, one has to restrict the system's capacity to account for the dynamical aspect of changes on the set of beliefs. If, on the other hand she wishes to provide a proper account of those dynamical aspects, the classical account of consistency cannot be maintained. This puzzle is particularly baffling, for it confronts two basic aspects which are supposed to lay down the foundations of belief revision.

The problem with the aforementioned puzzle is that one would prefer to account for both conflicting aspects. Would there be a way out? Previously, we have agreed that notions such as that of consistency are intuitive and may be subject to different approaches. Since the dynamics of changes in the sets of belief are a posit of the system, the only chance to solve this conundrum would be to provide a more fine grained analysis of the notion of consistency.

Furthermore, active avoidance of contradictory situations within epistemic states fails to see the advantages these situations actually provide. For instance, such situations increase the potential informational power of an epistemic state. In fact, they enable the acquisition of new information[4] as well as the increase of such information within the epistemic state. These are clear advantages to which scholars such as the dialogical logicians have shown to be sensitive. They have perceived that the presence of a contradiction can be understood as a relevant information to the system, for notwithstanding the interpretive preference one is forced to take some logical action.

Thus, even the classical minded scholar would have to admit that after applying the necessary changes for maintaining consistency she obtained a meta-logical information regarding that particular contradiction. In that sense, the lack of consistency is quite useful regardless of interpretive preferences. Hence, if a belief system can classify different aspects of consistency and propose a satisfactory treatment of it, without committing to the puzzle above, the advantages of such a system could be significant.

2.2 Principle of Information Economy

Another important principle related to these dynamics is the principle of*information economy*, also known as the *minimality* or *minimal change principle*. This principle asserts that for each possible outcome motivated by an operation on the epistemic states, the system should preferably adopt that outcome that minimizes the loss of information. That is to say, that the system is supposed to maximise the gains of each operation. In other words, the aim is to maximise the informational power of beliefs as well as the reliance on the resulting belief set which is akin, in a way, to preference relations or entrenchment.

It thus follows from this that, once retraction is considered, the resulting epistemic state should be the closest state possible to the original set of beliefs. As was mentioned above, this stricture seeks to maximise the outcome of the new epistemic state. According to it, information is much too valuable to allow unnecessary losses. After all, it comes at the expense of something else. So, if losses tend to be more significant, then incorporation should be avoided. Thus, this could be related to principles such as Ockham's razor.

[4]Kevin Kelly refers to this as the learning power, see [22].

The principle of minimality is taken to be valid. In fact, there is a general consensus about it. In spite of that, it is an altogether different problem to see how this consensus is reached. Many authors agree that this depends on the exact formulation of *minimal change*. In spite of the tension that comes through, I will not dwell on this matter. Rather, I wish to go on to explore the possibilities it provides, further on, in this paper.

There are several heuristics that measure the loss of information. These have been used in different ways[5]. Moreover, as was indicated above, different postulates do capture the intuition of information loss in different ways. That raises an obvious problem. If one wishes to suggest an account of this intuition and defend it as being adequate, the burden of proof falls onto this proposal. This, however, turns out to be rather challenging, for there are several such proposals present in the literature on that matter[6].

The main motivation for this debate is the fact that a strictly logical analysis is not enough to characterise such operations as, say, contraction. This is an indication of the rather limited expressive power the logical analysis has for grasping the intrinsic features of these operations. For instance, it has no means by which to decide upon what information needs to be abandoned during a change on the set of beliefs. According to Gärdenfors, the only viable solution would be to admit extra-logical information to the system (see [11]). The way in which these extra-logical informations are structured and used in the system determines the interpretation of the principle of minimal change. It, therfore, also determines a connection between belief revision and other areas such as counter-factual conditionals, defeasible inference and others (Makinson [26]).

In spite of the limitations that a strictly logical analysis of minimality is subjected to, its intuition still maintains its validity and may successfully be applied to the analysis. In fact, this principle is a posit of the systems and is fundamental to the analysis of the loss of information. Moreover, logical formalization may still be useful to some extend, at least for purposes of representing a loss of information. Thus, classical interpretations that actively try to avoid contradictions in their analysis are faced with the following conundrum. By not taking advantage of the informational power that contradictions hold and by demanding the retraction of some beliefs from the epistemic state, a strict adherence to consistency conflicts with the principle of information economy. This seems to be the case, at least at first glance. In this sense the consistency itself is an expensive principle.

[5]On this matter, there are several different proposals advanced in the literature. For some examples, see Alchourrón and Makinson [2], Fuhrmann [8], Gärdenfors and Makinson [12], Grove [14] and Makinson [24].

[6]For references on this debate, see Gärdenfors [11], Hansson [17], Makinson [25] and, more recently, Flouris [7] and Ribeiro [31] (concerning non- classical logics), among others.

2.3 Closure principle

In general, it is quite obvious that there are several issues regarding the underlying principles of rationality. The most striking feature so far is the difficulty that a strictly logical system has to properly assess these notions. As a matter of fact, it is rather clear that there is no single way in which a strictly logical interpretation can translate these notions. This does suggest that any such interpretation is highly dependent on one's personal intents and purposes.

This is in no way different with the next principle. Consider the following cases. It is possible to represent an epistemic state as either (i) a set of beliefs, closed under logical consequence (AGM), or as (ii) a finite subset of the language, which is not closed under logical consequence (i.e. a *base*, see [17]). According to case (i), any operation on an epistemic state needs to be a closed set.

When we make changes on a base, we have to temporarily ignore its logical consequences. This establishes a clear distinction between explicitly accepted beliefs and implicitly accepted beliefs, i.e. the logical consequence of the explicit beliefs that cannot be changed directly, but are nevertheless affected by a change in the base. On the other hand, in logically closed sets there is no such distinction.

Now, consider the following. Any model that assumes epistemic states to be closed under logical consequence is committed to an underlying logic. Consequently, it has to comply with all conventional features that are proper to a logic[7]. Notably, one consequence of AGM assumptions is the *explosion principle*, which states that there can be just one inconsistent epistemic state, i.e. the language itself.

The principle of closure entails that contradictory epistemic states are not informative and, thus, do incorporate unnecessary information. Alas, this goes against minimality. In order to maintain consistency, such situations had better be avoided. This is an indication, however, that consistency itself is fundamental to refrain from an unnecessary increase in information, due to classical explosion. So the problem seems to be within classical closure. Since closure is an important tool for incorporation of information[8], the issue cannot be with closure itself, but rather with its classicality, as just mentioned.

Although consistency, minimality, and closure are all important features of belief revision, they conflict with each other, as we have seen. In particular, it was stated that classical explosion lies at the heart of these issues. So could there be a way to avoid this? The answer may be given by paraconsistent logics. These logics do not aim to justify the existence of contradictions, but they enable us to maximise the system's

[7]In case of AGM, closure is obtained under all conventional logical connectives. The logic must be an expansion of Classical Propositional Logic, in which the connective of implication still satisfies the meta-theorem of deduction and its consequence relation Cn must be *standard*.

[8]The principle of closure is of philosophical significance, for it is sensitive to what is referred as a *doxastic commitment* in accepting beliefs' logical consequences (cf. Gärdenfors [9]).

informational power, while minimising superfluity, in my view.

3 On paraconsistency

Paraconsistent logics study contradictory yet non-trivial theories. Systems such as these focus on a distinction between asserting incompatible and opposing propositions, in an attempt to ensure the non-triviality of a theory. An interesting class of such systems is known as Logics of Formal Inconsistency (LFIs) (see [4]), that allows consistency to be introduced into the object language[9].

3.1 LFIs and Paraconsistent Belief Revision

Classically, contradictions within a body of knowledge, or theory, imply trivial results. Furthermore, if a theory is trivial, it contains contradictions. Thus, contradictoriness is held to be on a par with triviality. The obvious counterpart would be to correlate concepts of consistency and non-contradiction. However, paraconsistent logics challenge precisely this view. Moreover, once consistency is inserted into the language, the aforementioned correlation has to be further classified. It no longer can be the case that the presence of contradiction in a body of knowledge implies triviality. Rather, a theory becomes trivial only if, under the assumption of consistency, there are contradictions.

The novelty of LFIs is the introduction of a consistency operator $\circ$ such that $\circ\alpha$ denotes that α is consistent. Hence, for any LFI denoted by the consequence relation $\vdash$, the following applies:

(1) Explosion Principle $\alpha, \neg\alpha \vdash \beta$ does not hold in general. Instead,

(2) Gentle Explosion Principle $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ holds.

This is precisely the main distinction in AGM-like systems of Paraconsistent Belief Revision, see [34, 35]. Such systems assume as underlying logic the Logics of Formal Inconsistency[10].

3.2 Explosion principle and contradictory belief sets

We have already seen that classical explosion yields some difficulties. As a matter of fact, this classical principle inhibits interesting operations for AGM system of Belief Revision.

[9] In particular, it is possible to introduce consistency as a primitive notion. For further references, see [4]. For other interpretations, see [21, 29, 5].

[10] To avoid any misinterpretation, the terms consistency and inconsistency shall be used in the classical sense. To denote the interpretation of LFIs, I shall use specifically the concepts of formal consistency and formal inconsistency.

These operations can now be cashed out and represented within the Paraconsistent Belief Revision. I shall consider, for the present purposes, those that can be expressed within bases. To further elicit this, I shall consider the principle stated by Levi [23] that complex changes in beliefs can be reduced to simpler operations:

Principle of decomposition (Fuhrman [8]) Every revision operation is decomposable into a sequence of contractions and expansions.

This principle should not be understood as an iteration of contraction and expansion at a time. Instead, it is intended only to convey the idea that outcomes of complex operations may be equivalent to the results obtained by a succession of simple operations.

Classical AGM, however, is to some extend restricted by the rule of decomposition. As we have already noted, classical AGM models have to avoid the presence of contradictions within a given epistemic state, in order to satisfy consistency and closure. Now, consider a random new piece of information that presents itself to the system. If it is an instance of revision, it has informational content that would introduce a contradiction into the original state. So, since classical AGM rules out this possibility, it can only apply the principle of decomposition in one possible way, namely by first contracting an information and, afterwards, expanding the state by introducing a new one[11]. This is known as *internal revision.*

This is an obvious limitation of classical AGM Belief Revision, not only with regard to its capacity of analysis, but also with regard to its original purpose of providing an account of dynamic systems. Yet this is precisely the point. If classical AGM proposes to analyse dynamic systems, it cannot restrict the very feature it tries to understand. To some extend, this would be to admit a failure of its own proposal. Thus, if we wish to properly understand belief revision as the study of dynamic systems, we cannot adopt classical AGM as the preferred theory. The original system, however, admits the possibility of adopting different systems based on AGM[12], such that this is not a criticism of the system itself, but only of its rigid classical interpretation.

This difficulty of the rigid classical interpretation is a consequence of assuming the classical explosion principle. Were it not for this stricture, the order of the two sub-operations in case would not matter. So, since for dynamic systems the order of sub-operations cannot matter, classical explosion has to be denied. This would enable the system to consider first expansion and then contraction. This is known as *external revision*, see [18].

So far two different approaches were provided for revision. Initially, they are characterised only by their order of procedure, but they do differ also with regard both to

[11] Formally, revision, as intuitively stated, is represented by (i) contracting an information, which is equivalent to the negation of the new information that will be incorporated, and (ii) expand the resulting state by incorporating the new information.

[12] Cf. Gärdenfors [9]

the underlying intuitions and the logical properties, as pointed out by Hansson [19] for bases. This is necessary so, because while non-contradiction has to be adhered to in the case of internal revision, it is not an issue in the case of external revision. Internal revision has to admit, at least temporarily, a state of non-commitment to a certain situation, when neither the belief-representing sentence nor its negation are admitted by the system. On the other hand, the external revision operation, as was indicated above, does admit a contradiction within the epistemic state, when both a sentence and its negation are accepted temporarily[13].

In Hansson's analysis, the intuition for internal revision is more plausible, when there is a moment of hesitation in which neither the new belief nor its negation are accepted. As for the case of external revision, it is obvious that the new information should be accepted, though it is less obvious which prior belief should be abandoned. Hence, since new information may be incorporated satisfactorily to the epistemic state, the external revision operation becomes more plausible. This plausibility cannot be the case in classical AGM, as was indicated above.

The difference between internal and external revision, particularly considering the acceptance of contradictions in the latter, can be further characterised in Paraconsistent Belief Revision. In the following section, I shall give a survey of the formal construction of these operations within a paraconsistent setting, called AGM$\circ$ (cf. [35]). A reader more interested in the conceptual discussion may skip the following subsection without loss of content. She is just asked to bear in mind that paraconsistent belief revision allows meaningful reasoning with contradictory contents of belief.

3.3 Formal constructions of revisions

Let us assume a LFI **L**, such that **L** is an extension of mbC, mbC being the smallest LFI of the family presented in [4] that internalizes the consistency as described above. The deductively closed belief sets (or theories) of **L** are denoted by K, and are called epistemic states. $Cn(X)$ is the set of logical consequences of the belief set X. The language $\mathbb{L}$ of **L** is generated by the connectives $\wedge, \vee, \rightarrow, \neg, \circ$ and the constant f (*falsum*). The classical negation (strong negation) is defined, as usual, by $\sim \alpha =_{def} (\alpha \rightarrow f)$ and $\alpha \leftrightarrow \beta$ is an abbreviation of $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$.

In AGM$\circ$, the operation of contraction, $K - \alpha$, follows the one presented by [1], called *partial meet contraction*, which consists in:

1. Choose some maximal subsets of K that do not entail α.

[13] In a belief set closed under a LFI this temporary contradictory situation within an epistemic state is quite informative: it entails the formal non-consistency of the sentence α involved in the contradiction. In other words, the information that $\neg\circ\alpha$ as well as the relevant - non-trivial and not necessarily conflictive - logical consequences of both α and $\neg\alpha$.

2. Take the intersection of those sets.

Formally we have:

Definition 3.1 (remainder, [1]) *A set K' of beliefs is a maximal subset of K that doesn't entail α if and only if:*

(i) $K' \subseteq K$
(ii) $\alpha \notin K'$
(iii) *If* $K'' \subset K' \subseteq K$ *then* $\alpha \in Cn(K')$

The set of all belief sets that are maximal subsets of K that does not entails α is called remainder set, and is denoted by $K \bot \alpha$.

Definition 3.2 (selection function for AGM∘ contraction, [35]) *A selection function in K is a function γ such that for all α:*

1. $\emptyset \neq \gamma(K, \alpha) \subseteq K \bot \alpha$ *if* $\alpha \notin Cn(\emptyset)$ *and* $\circ\alpha \notin K$.
2. $\gamma(K, \alpha) = \{K\}$ *otherwise.*

Note that the AGM∘ system internalizes the notion of formal consistency during the contraction operation. Consistent beliefs are not liable to be retracted from the epistemic state. Alternatively, a yet different paraconsistent system, AGMp, is more general in its treatment of these issues, since it is not necessarily bounded to formal consistency. Hence, the mentioned operator is not taken into account in the case of contraction. Therefore, the clause $\circ\alpha \notin K$ is not assumed.

The partial meet contraction is the intersection of sets chosen by a selection function. Intuitively, the selection function chooses the beliefs held in higher regard by an agent, i.e. those more entrenched in the epistemic state. This is the extra-logical factor that is taken into account in a contraction.

Definition 3.3 (AGM∘ contraction)

$$K -_{\gamma} \alpha = \bigcap \gamma(K, \alpha)$$

The following postulates characterise precisely the formal construction for contraction and provide a more intuitive approach[14].

Definition 3.4 (postulates for AGM∘ contraction, [35]) *The contraction satisfies the following:*

[14]The result showing that a set of postulates fully characterises a construction is the representation theorem, a central result in AGM-like Belief Revision. For representation theorems of the AGM∘ system, see the references.

(closure) $K - \alpha = Cn(K - \alpha)$.
(success) *If* $\alpha \notin Cn(\emptyset)$ *and* $\circ\alpha \notin K$ *then* $\alpha \notin K - \alpha$.
(inclusion) $K - \alpha \subseteq K$.
(failure) *If* $\circ\alpha \in K$ *then* $K - \alpha = K$.
(relevance) *If* $\beta \in K \setminus K - \alpha$ *then exists* K' *such that* $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$ *e* $\alpha \in K' + \beta$

No notion of *failure* is assumed in the AGMp system, which presupposes a more general idea of paraconsistency.

By Levi identity, as in the classical model, revision $(K * \alpha)$ is to be understood as a construction with partial meet contraction, i.e. an internal partial meet revision.

Definition 3.5 (AGM∘ internal revision)

$$K *_\gamma \alpha = (K -_\gamma \neg\alpha) + \alpha = \{\bigcap \gamma(K, \neg\alpha)\} \cup \{\alpha\}$$

Definition 3.6 (postulates for AGM∘ internal revision, [35]) *An internal AGM∘ revision satisfies the following:*

(closure) $K * \alpha = Cn(K * \alpha)$.
(success) $\alpha \in K * \alpha$.
(inclusion) $K * \alpha \subseteq K + \alpha$.
(non-contradiction) *If* $\neg\alpha \notin Cn(\emptyset)$ *and* $\circ\neg\alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.
(failure) *If* $\circ\neg\alpha \in K$ *then* $K * \alpha = K + \alpha$
(relevance) *If* $\beta \in K \setminus K * \alpha$ *then exists* K' *such that* $K \cap K * \alpha \subseteq K' \subseteq K$ *and* $\neg\alpha \notin K'$, *but* $\neg\alpha \in K' + \beta$.

By reverse Levi identity, revision is to be understood as a construction with partial meet AGM∘ contraction.

Definition 3.7 (AGM∘ external revision)

$$K *_\gamma \alpha = (K + \alpha) -_\gamma \neg\alpha = \bigcap \gamma(\{K \cup \alpha\}, \neg\alpha)$$

Definition 3.8 (postulates for AGM∘ external revision, [35]) *An external revision satisfies the following:*

(closure) $K * \alpha = Cn(K * \alpha)$.
(success) $\alpha \in K * \alpha$.
(inclusion) $K * \alpha \subseteq K + \alpha$.
(non-contradiction) *If* $\neg\alpha \notin Cn(\emptyset)$ *and* $\sim \alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.
(failure) *If* $\sim\alpha \in K$ *then* $K * \alpha = \mathbb{L}$
(relevance) *If* $\beta \in K \setminus K * \alpha$ *then exists* K' *such that* $K * \alpha \subseteq K' \subseteq K + \alpha$ *and* $\neg\alpha \notin K'$, *but* $\neg\alpha \in K' + \beta$.
(pre-expansion) $(K + \alpha) * \alpha = K * \alpha$

3.4 Rationality criteria revisited

As in classical AGM system, the rationality criteria specify the postulates that revision operations should satisfy. To set the postulates of different operations, the AGM-like systems of Paraconsistent Belief Revision follow virtually the same criteria presented by Gärdenfors and Rott [13], with the due adaptations, as I have indicated above.

There is one issue that could be raised with respect to external revision: if consistency is assumed, then external revision cannot be carried out in a rational way. If consistency is assumed, then classical intuitions apply to revision. Thus, such a situation would clearly violate its own assumption. Hence, once consistency is adopted, the agent has to avoid contradictions through prior contraction, whenever this is possible. This follows from both the criterion, as well as from the logical strictures. It is now possible to return to the puzzle of consistency that we have discussed above.

This is where the economy of information becomes relevant. Should minimality have priority over consistency, then external revision is to be adopted. Given such a scenario, internal revision would imply too great a loss of information, since minimality does not hold contradictions to be harmful under such circumstances. In fact, in this particular case, if contraction were to be applied as the first operation of revision, this would indeed violate the assumed conditions.

From what can be gathered, there is some tension between the criteria of consistency and minimality. Different procedures have to be adopted once one decides upon the priority of either of these. Since no proper methods for this decision can be provided, these analyses of revision become strikingly similar to a game between these criteria. Putted on a scale of priority, one would outweigh the other. So the aim of the analysis is to maintain some form of equilibrium in accordance to economic rationality, which would not be possible within a rigid classical setting.

As with any form of game, there are gains and losses, according to the choice of criteria. Choosing consistency over minimality comes at the price of discarding information that could be relevant to the epistemic state and limiting the dynamics of belief changes. Inverting the priority, it would be reasonable to think that the maximal preservation of information and an exhaustive treatment of the dynamics of belief changes would come at the expense of consistency. But this does not have to be necessarily the case. One way to satisfy classical demands would be to interpret consistency as an output condition.

So far, I have focused on the difference between internal and external revision. Notwithstanding, the operation of external revision can be analysed within bases as a particular case of a generalised notion of this kind of operation. This generalised form of revision is called semi-revision (Hansson [15]). Thus, it should not come as a surprise to the reader that this is also applicable to Paraconsistent Belief Revision.

The operation of semi-revision distinguishes itself for temporarily accepting new information and, should there be a contradiction, the semi-revision can regain consistency in the resulting epistemic state by means of retracting one of the admitted beliefs. This means that this operation can retract the original or the newly added belief in order to respect the consistency principle. All in all, it is the removal of contradictions as an output condition. Consequently, this operation does not subscribe to the tacit principle of the *primacy of the new information*. Within the Systems of Paraconsistent Belief Revision that I advance, semi-revision becomes even more general. Instead of retracting either one or the other of two contradictory beliefs, it can isolate conflicting aspects of both and apply retraction only to these particular aspects. Thus, the priority (entrenchment) is established by the selection function, raising the economy of information to its uppermost limit[15].

The next subsection gives a survey on this operation.

3.5 Formal construction of semi-revision

Consider the formal language presented in the section 3.3. Consolidation is the operation that removes the contradictions from the epistemic state.

Definition 3.9 (postulates for AGM∘ consolidation, [35]) *A consolidation satisfies the following:*

(closure) $K! = Cn(K!)$.
(inclusion) $K! \subseteq K$.
(non-contradiction) *If* $K \neq \mathbb{L}$*, then* $K!$ *is not contradictory.*
(failure) *If* $K = \mathbb{L}$*, then* $K! = \mathbb{L}$.
(relevance) *If* $\beta \in K \setminus K!$ *then exists* K' *such that* $K! \subseteq K' \subseteq K$ *and* K' *is not contradictory, but* $K' + \beta$ *is contradictory.*

Just like contraction, in AGMp the *failure* is not valid.

A choice function over a remainder set is used. To do so, a generalization of remainder set is needed.

Definition 3.10 (generalization of remainder for sets, [35]) *Let* K *in* $\mathbf{L}$ *and* $A \subset \mathbb{L}$*. The set* $K \perp_P A \subseteq \wp(\mathbb{L})$ *is such that for all* $X \subseteq \mathbb{L}$*,* $X \in K \perp_P A$ *iff the following is the case:*

1. $X \subseteq K$
2. $A \cap Cn(K) = \emptyset$
3. *If* $X \subset X' \subseteq K$ *then* $A \cap Cn(X') \neq \emptyset$.

Consolidation considers a specific subset A, that is, the one that represents the totality of contradictory sentences in K, defined as follows:

[15] For a detailed account, see [34].

Definition 3.11 (contradictory set, [35]) *Let Ω_K be the set of contradictory sentences of K:*

$$\Omega_K = \{\alpha \in K : \text{ exists } \beta \in \mathbb{L} \text{ such that } \alpha = \beta \wedge \neg\beta\}$$

So the consolidation is defined as follows:

Definition 3.12 (AGM∘ consolidation, [35]) *A* consolidation function *over K is a function γ such that:*

1. *If $K \neq \mathbb{L}$ then $\emptyset \neq \gamma(K) \subseteq K \perp_P \Omega_K$*
2. *If $K = \mathbb{L}$ then $\gamma(K) = \{K\}$*

$$K!_\gamma = \bigcap \gamma(K)$$

The semi-revision can be defined as the generalization of external-revision, in which the choice for the removal is left to the selection function.

$$K?_\gamma\alpha = (K + \alpha)!_\gamma$$

The following postulates helps to better understand that operation.

Definition 3.13 (postulates for AGM∘ semi-revision, [35]) *The semi-revision satisfies the following:*

(closure) *$K?\alpha = Cn(K?\alpha)$.*
(inclusion) *$K?\alpha \subseteq K + \alpha$.*
(non-contradiction) *If $K \neq \mathbb{L}$ then $K?\alpha$ is not contradictory.*
(failure) *If $\sim\alpha \in K$ then $K?\alpha = \mathbb{L}$*
(relevance) *If $\beta \in K \setminus K?\alpha$ then exists K' such that $K?\alpha \subseteq K' \subseteq K + \alpha$ and K' is not contradictory but $K' + \beta$ is contradictory.*

3.6 Remarks

I have tried to show that AGM-style Paraconsistent Belief Revisions of the type outlined in this paper, can successfully avoid the major problems that a rigid classical interpretation of such systems has to face. By weakening the consistency conditions, it is now possible to fully understand dynamical systems and still adhere to all of the classical criteria that are required by AGM. Moreover, by prioritizing the classical criterion of minimality, it is not only possible to deal with contradictions, it is even a requirement. By accepting contradictions in a temporary state of reasoning, that is, the intermediate paraconsistent behaviour, and by restricting the requirement of consistence to the output of the analysis, the preservation of informative content is maximized, while the loss of information is kept at a minimum. Therefore, as I see it, the paraconsistent analysis of AGM-style Belief Revision is the one that sets out to preserve the classical criteria to the highest extend. Thus understood, Paraconsistent Belief Revision is as classical as dynamic systems allow. In other words, the classical minded reasoner has to be paraconsistent, if he wishes to subscribe to the classical criteria to the highest extend. Hence, the strict adherence to the strong classical consistency is too high a cost even to the classical reasoner.

4 Applicability of the results: Belief Revision as heuristics

The importance of the AGM system and, hence, its obvious choice as the theoretical background for the present paper is due to its simplicity and its expressive power. Furthermore, it was shown to be equivalent to a plethora of alternative proposals[16]. Many works in the literature use the AGM postulates to handle different logical concepts and intuitive notions of both formal and philosophical interest. For example, Gärdenfors [10] and Rott [32] dealt with non-monotonic logics with AGM. On a different matter, Witte [38] addressed the connection of AGM postulates with fuzzy sets, in general, Martin and Osherson [27] and several other authors relate the AGM concepts with Bayesian Epistemology, and Stalnaker [33] with Game Theory.

Due to its simplicity, Belief Revision can be seen as a heuristic tool that allows one to reach a better understanding of the concepts discussed in this article. It provides a big picture that explains the differences between the several forms of analyses.

5 Conclusions

Belief Revision does not, in fact, encompass all problems of rational reasoning. Nor does any other logical theory. It is quite obvious that the actual process of belief change is far more complex than any of the operations stated above. Instead, in order to obtain a more useful model, substantial simplifications impose themselves. This paper points out that classical Belief Revision imposes unnecessary strong criteria for revision. As I have shown, in accordance to an economic standard of rationality, paraconsistency models a more interesting system that is supposed to capture the classical criteria to their highest extend. If my proposal is correct, rigid classical analysis has to pay a high price in order to maintain its understanding of consistency. It is this interplay between

[16] Among such proposals, the following are especially interesting: for discussing partial meet selection function, see Alchourrón [1]; for epistemic entrenchment, Gärdenfors [12]; for safe contraction, Alchourrón and Makinson [2]; for systems of Grove's spheres, Adam Grove [14], to mention a few.

the different notions of consistency and their respective consequences within system of Belief Revision that I call the *cost of consistency*.

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Rafael R. Testa
Centre for Logic, Epistemology and the History of Science
State University of Campinas (UNICAMP)
R. Sérgio Buarque de Holanda 251, CEP 13083-859, Campinas, SP, Brazil
E-mail: rafaeltesta@gmail.com