Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.12785/jsapl/020201

A Merton Model of Credit Risk with Jumps

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Received: 13 Oct. 2014, Revised: 25 Nov. 2014, Accepted: 29 Nov. 2014
Published online: 1 May 2015

Abstract: In this note we consider a Merton model for default risk, where the firm's value is driven by a Brownian motion and a compound Poisson process.

Keywords: Merton model, default risk, default probability, processes with jumps

1 Introduction

Various models of Merton's type for credit risk have been studied so far (refer [1] to [8]). This paper aims to our recent results, where a model driven by a jump process is studied in [9] and another model governed by a jumps-diffusion is investigated in [10]. Suppose that the asset value V_t of a company, under a risk neutral measure, is given by the following differential equation

$$dV_t = (r - \beta\lambda)V_t dt + \sigma V_t dW_t + V_{t-} dQ_t, \tag{1.1}$$

where W_t is a standard Brownian motion, $Q(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process, $N(t)$ is a Poisson process with intensity $\lambda > 0$, Y_i's are independent and identically distributed random variables with $E(Y_i) = \beta$. All of these processes are supposed to be considered under the risk neutral measure. In (1.1), r is the interest rate, $\sigma > 0$ is a constant and N_t expresses the number of jumps of Q_t while Y_i is the i-th jump size of $Q(t)$.
The model (1.1) reflects a fact that, the firm's value can change randomly not only in a continuous way but also in a cumulatively discrete fashion.
We will study on the probability of default of the company when its value V_t is less than some debts.

2 Case of one debt L

A bankruptcy situation will occur at some time t when the company asset value is less than a debt L. And the problem is how to calculate the default probability $P(V_t < L)$.
It is known that the solution of (1.1) is given by (see [7])

$$V_t = V_0 \exp[\sigma W_t + (r - \beta\lambda - \frac{\sigma^2}{2})t] \prod_{i=1}^{N_t} (Y_i + 1). \tag{2.1}$$

We see that

$$\ln V_t = \ln V_0 + \sigma W_t + (r - \beta\lambda - \frac{\sigma^2}{2})t + \sum_{i=1}^{N_t} \ln(Y_i + 1).$$

And the event $\{V_t < L\}$ or $\{\ln V_t < \ln L\}$ means that

$$\sigma W_t + Z_t < x_t, \tag{2.2}$$

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where

$$x_t = \ln L - (r - \beta\lambda - \frac{\sigma^2}{2})t - \ln V_0, \tag{2.3}$$

$$Z_t = \sum_{i=1}^{N_t} U_i, \quad \text{with } U_i = \ln(1+Y_i). \tag{2.4}$$

Z_t is also a compound Poisson process where U_i's are i.i.d. random variables.
We calculate first the characteristic function $\Psi_{Z_t}(s)$ of Z_t. Z_t is also a compound Poisson process where U_i are i.i.d. random variables.
We recall first the characteristic function $\Psi_{Z_t}(s)$ of Z_t:

$$\begin{aligned}
\Psi_{Z_t}(s) &= E(e^{isZ_t}) \\
&= \sum_{j=0}^{\infty} E\left(e^{isZ_t}|N_t = j\right)P(N_t = j) \\
&= \sum_{j=0}^{\infty} E\left(e^{is(U_1+\ldots+U_j)}\right)P(N_t = j) \\
&= \sum_{j=0}^{\infty} \left(Ee^{isU_1}\ldots Ee^{isU_j}\right)P(N_t = j) \\
&= \sum_{j=0}^{\infty} (\psi_U(s))^j \frac{(\lambda t)^j}{j!} e^{-\lambda t} = \exp[\lambda t(\psi_U(s) - 1)]
\end{aligned} \tag{2.5}$$

where $\psi_U(s)$ is the common characteristic function of U_i's.
It is known also that, for a compound Poisson process as Z_t we have $\mu(t) = EZ_t = \lambda tE(U_i) = \lambda tE\ln(1+Y_i) = \lambda tm$; $\sigma^2(t) = VarZ_t = \lambda tE(U_i^2) = \lambda tE[\ln(1+Y_i)]^2 = \lambda t\gamma^2$, where $E\ln(1+Y_i) = m$ and $E[(\ln(1+Y_i))^2] = \gamma^2$.

Denote by $\overline{Z}_t$ the normalization of Z_t

$$\overline{Z}_t = \frac{Z_t - \mu(t)}{\sigma(t)}.$$

And we will show that Z_t has an approximately normal distribution.
Indeed, according to the Taylor expansion for characteristic function

$$\psi_U(s) = \sum_{k=0}^{\infty} \frac{(is)^k}{k!} E|U|^k,$$

we can write

$$\psi_U(s) = 1 + ism - \frac{\gamma^2}{2}s^2 + o(s^2). \tag{2.6}$$

Now we compute the characteristic function of $\overline{Z}_t = \frac{1}{\sigma(t)}Z_t - \frac{\mu(t)}{\sigma(t)}$,

$$\Psi_{\overline{Z}_t}(s) = e^{-is\frac{\mu(t)}{\sigma(t)}}\Psi_{Z_t}(s/\sigma(t)).$$

Taking account of (2.5) and (2.6) we have

$$\begin{aligned}
\Psi_{\overline{Z}_t}(s) &= e^{-is\frac{\mu(t)}{\sigma(t)}}\exp[\lambda t(\psi_U(s/\sigma(t)) - 1)] \\
&= e^{-is\frac{\mu(t)}{\sigma(t)}}\exp[i\lambda tm\frac{s}{\sigma(t)} - \frac{\lambda t\gamma^2}{\sigma(t)^2}\frac{s^2}{2} + o(\frac{s^2}{t})] \\
&= e^{-is\frac{\mu(t)}{\sigma(t)}}\exp[is\mu(t)/\sigma(t) - \frac{\sigma^2(t)}{\sigma^2(t)}\frac{s^2}{2} + o(\frac{s^2}{t})] \\
&= \exp(\frac{-s^2}{2} + o(\frac{s^2}{t})), \quad \text{as } t \to \infty.
\end{aligned}$$

Then $\overline{Z}_t \simeq \mathcal{N}(0,1)$ or $Z_t \simeq \mathcal{N}(\mu(t),\sigma(t)^2)$, where $\mu(t) = \lambda tE\ln(1+Y_i)$, $\sigma(t) = \sqrt{\lambda tE[\ln(1+Y_i)]^2} = \sqrt{\lambda t}\gamma$.
Now we can consider $\sigma W_t + Z_t$ as a sum of two independent normal random variables for each t large enough, so it has also a normal distribution with mean

$$\mu^*(t) = \mu(t) = \lambda tE\ln(1+Y_i)$$

and variance

$$\sigma^*(t) = \sigma^2 + \sigma^2(t) = \sigma^2 + \lambda tE[\ln(1+Y_i)]^2,$$

where $\sigma > 0$ is a known constant as in (1.1).
And

$$P(\sigma W_t + Z_t < x_t) \approx \Phi(\frac{x_t - \mu^*(t)}{\sigma^*(t)}), \tag{2.7}$$

where $\Phi(x)$ is the standard normal distribution function.
We are now in the position to state the follow theorem.
Theorem 2.1 *The default probability can be approximated by*

$$P_{default} \approx \frac{1}{\sigma^*(t)\sqrt{2\pi}}\int_{-\infty}^{x_t} e^{-(u-\mu^*(t))^2/2\sigma^{*2}(t)}du, \tag{2.8}$$

where

$$x_t = \ln L - (r - \beta\lambda - \sigma^2/2)t - \ln V_0$$
$$\mu^*(t) = \lambda tE\ln(1+Y_i), \quad \sigma^*(t) = \lambda tE[\ln(1+Y_i)^2].$$

3 Case of many liabilities $L_1, L_2, \ldots, L_m$

Now we consider the case where the company faces up numerous debts $L_1, L_2, \ldots, L_m$ that should be paid at times t_1, t_2,...,t_m respectively, with $t_1 < t_2 < \ldots < t_m = T$.
The company will jump into default position before the time T if and only if at one of time t_i $(i = 1,2,\ldots,m)$, it happens that

$$V_{t_i} < L_i.$$

So the probability of default before T is

$$P_{default}(0,T) = 1 - P(V_{t_i} > L_i, \forall t_i).$$

Denote $L = \max\{L_1,\ldots,L_m\}$ It is easy to see that for all $t_i (i = 1,\ldots,m)$ we have

$$(V_{t_i} > L_i) \supset (V_{t_i} > L).$$

Then

$$P_{default}(0,T) \leq 1 - P(V_{t_i} > L, \forall t_i). \tag{3.1}$$

Put $X_t = \sigma W_t + Z_t$, where, as before $Z_t = \sum_{i=1}^{N_t} U_i$, $U_i = \ln(1+Y_i)$. The inequality $V_{t_i} > L$ is equivalent to

$$X_{t_i} = \sigma W_{t_i} + Z_{t_i} > \ln L - \ln V_0 - (r - \beta\lambda - \frac{\sigma^2}{2})t_i := x_{t_i}.$$

Consider the event

$$A = \{V_{t_i} > L, \forall t_i\} = \bigcap_{i=1}^{m}\{X_{t_i} > x_{t_i}\}. \tag{3.2}$$

Then

$$P_{default}(0,T) \leq 1 - P(A).$$

It is known that a compound Poisson process is a process of independent increments. The processes (W_t) and (Z_t) are independent and both are of independent increments, so is the process $X_t = \sigma W_t + Z_t$.
Denoting by A_i the event $\{X_{t_i} > x_i\}, i = 1,2,\ldots,m$ we can see that

$$A_1 = \{X_{t_1} > x_{t_1}\} = \{X_{t_1} - X_0 > x_{t_1}\},$$

$$A_2=\{X_{t_2}>x_{t_2}\}=\{X_{t_2}-X_{t_1}>x_{t_2}-X_{t_1}\}\supset\{X_{t_2}-X_{t_1}>x_{t_2}-x_{t_1}\},$$

if A_1 occurs.

$$\ldots$$

$$A_m=\{X_{t_m}>x_{t_m}\}=\{X_{t_m}-X_{t_{m-1}}>x_{t_m}-X_{t_{m-1}}\}\supset\{X_{t_m}-X_{t_{m-1}}>x_{t_m}-x_{m-1}\},$$

if $A_1,\ldots A_{m-1}$ occur.

Put $B_i=\{X_{t_i}-X_{t_{i-1}}>x_{t_i}-x_{t_{i-1}}\}$ for $i=1,2,\ldots,m$ and $x_0=0$ by convention. It follows that

$$\bigcap_{i=1}^{m}B_i\subset\bigcap_{i=1}^{m}A_i=A.$$

Because of the independence of increments we have

$$P(A)\geq P(\bigcap_{i=1}^{m}B_i)=\prod_{i=1}^{m}P(B_i), \tag{3.3}$$

And by definition of B_i,

$$\begin{aligned}P(B_i)&=P(X_{t_i}-X_{t_{i-1}}>x_{t_i}-x_{t_{i-1}})\\&=P(\sigma(W_{t_i}-W_{t_{i-1}})+(Z_{t_i}-Z_{t_{i-1}})>x_{t_i}-x_{t_{i-1}}).\end{aligned} \tag{3.4}$$

Put $\overline{X}_i=X_{t_i}-X_{t_{i-1}}$, $\overline{W}_i=\sigma(W_{t_i}-W_{t_{i-1}})$ and $\overline{Z}_i=Z_{t_i}-Z_{t_{i-1}}$, where Z_t is defined as in (2.4). The random variable $\overline{W}_i$ has normal distribution $\mathcal{N}(0,\sigma^2(t_i-t_{i-1}))$. The random variable $\overline{Z}_i=\sum_{k=N_{t_{i-1}}+1}^{N_{t_i}}U_k$ has the same distribution with that of $\sum_{k=1}^{N_{t_i-t_{i-1}}}U_k$ since U_i's are i.i.d and N_t is a process of stationary and independent increments.
We can see that the distribution of $\overline{Z}_i$ is given by

$$\begin{aligned}F_{\overline{Z}_i}(z)=P(\overline{Z}_i\leq z)&=\sum_{n=0}^{\infty}P(N_{t_i-t_{i-1}}=n)P(\overline{Z}_i\leq z/N_{t_i-t_{i-1}}=n)\\&=\sum_{n=0}^{\infty}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}P(\overline{Z}_i\leq z/N_{t_i-t_{i-1}}=n)\\&=\sum_{n=1}^{\infty}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}P(\sum_{k=1}^{n}U_i\leq z)\\&=\sum_{n=1}^{\infty}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}F_U^{*n}(z),\end{aligned} \tag{3.5}$$

where F_U^{*n} is the n fold convolution of common distribution of U_k's.
Suppose now that U_i's are continuous random variables, so are Z_i's and $\overline{Z}_i$'s. Then the density function of $\overline{X}_i=\overline{W}_i+\overline{Z}_i$ is

$$\begin{aligned}f_{\overline{X}_i}(x)=f_{\overline{W}_i}\star f_{\overline{Z}_i}(x)&=\int_{-\infty}^{\infty}f_{\overline{W}_i}(x-z)f_{\overline{Z}_i}(z)dz\\&=\frac{1}{\sigma\sqrt{2\pi(t_i-t_{i-1})}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})}\right]f_{\overline{Z}_i}(z)dz,\end{aligned} \tag{3.6}$$

where $f_{\overline{Z}_i}(z)=\frac{d}{dz}F_{\overline{Z}_i}(z)$ is the density function of $\overline{Z}_i$.
Now we have

$$P(B_i)=1-\int_{-\infty}^{x_{t_i}-x_{t_{i-1}}}f_{\overline{X}_i}(x)dx,$$

where $f_{\overline{X}_i}(x)$ is defined by (3.6).
And so, the following assertion is ready to be stated:

Theorem 3.1 *If U_i's are continuous random variables then the probability of default before T is estimated by*

$$P_{default}(0,T) \leq 1-\prod_{i=1}^{m}\left(1-\int_{-\infty}^{x_{t_i}-x_{t_{i-1}}}\left[\frac{1}{\sigma\sqrt{2\pi(t_i-t_{i-1})}}\times \times\int_{-\infty}^{\infty}\exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})}\right]f_{\overline{Z}_i}(z)dz\right]dx\right), \quad (3.8)$$

where

$$x_{t_i} = \ln L - \ln V_0 - (r-\beta\lambda-\frac{\sigma^2}{2})t_i \quad (3.9)$$

and

$$f_{\overline{Z}_i}(z) = \sum_{n=0}^{\infty}\frac{d}{dz}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}P(\overline{Z}_i \leq z/N_{t_i-t_{i-1}}=n). \quad (3.10)$$

4 Particular cases of Theorem 3.1

We consider some particular cases for distribution of U_k's.

4.1. Case of normal random variables

Suppose that $U=U_k \sim \mathcal{N}(0,1)$ then we have $\sum_{k=1}^{n}U_k \sim \mathcal{N}(0,n)$ with density function $\frac{1}{\sqrt{2\pi n}}e^{-z^2/2n}$ and the density of $\overline{Z}_i$ is

$$f_{\overline{Z}_i}(z) = \frac{1}{\sqrt{2\pi n}}\sum_{n=1}^{\infty}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}e^{-z^2/2n} \quad (4.1)$$

From (3.8) and (4.1) we have

$$P_{default}(0,T) \leq 1-\prod_{i=1}^{m}\left(1-\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}(t_i-t_{i-1})^n e^{-\lambda(t_i-t_{i-1})}\frac{1}{2\pi\sigma\sqrt{n(t_i-t_{i-1})}}\times \times\int_{-\infty}^{x_{t_i}-x_{t_{i-1}}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})}-\frac{z^2}{2n}\right]dzdx\right). \quad (4.2)$$

4.2. Case of exponential random variable U_k with parameter $\nu > 0$

We know that if $U_k \sim \exp(\nu)$ then $\sum_{k=1}^{n}U_k \sim Gamma(n,\nu)$ with the density function

$$\frac{z^{n-1}e^{-z/\nu}}{\nu^n\Gamma(n)},$$

where Γ is Gamma function. Then

$$f_{\overline{Z}_i}(z) = \sum_{n=1}^{\infty}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}\frac{z^{n-1}e^{-z/\nu}}{\nu^n\Gamma(n)}.$$

We can see the estimation in (3.8):

$$\begin{aligned}P_{default}(0,T) &\leq 1-\prod_{i=1}^{m}\left(1-\int_{-\infty}^{x_{t_i}-x_{t_{i-1}}}\frac{1}{\sigma\sqrt{2\pi(t_i-t_{i-1})}}\int_{0}^{\infty}\exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})}\right]\times \times\sum_{n=1}^{\infty}\frac{\lambda^n(t_i-t_{i-1})^n}{n!}e^{-\lambda(t_i-t_{i-1})}\frac{z^{n-1}e^{-z/\nu}}{\nu^n\Gamma(n)}\right]dzdx\right)\\ &= 1-\prod_{i=1}^{m}\left(1-\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}(t_i-t_{i-1})^n e^{-\lambda(t_i-t_{i-1})}\frac{1}{\sigma\sqrt{2\pi(t_i-t_{i-1})}}\times \times\int_{-\infty}^{x_{t_i}-x_{t_{i-1}}}\int_{0}^{\infty}\exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})}-\frac{z}{\nu}\right]\frac{z^{n-1}}{\nu^n\Gamma(n)}dzdx\right). \quad (4.3)\end{aligned}$$

5 When $U = U_k$'s are general discrete random variables

In this case we have

$$\begin{aligned} P(\overline{Z}_i = z) &= P(\sum_{k=1}^{N_{t_i-t_{i-1}}} U_k = z) = \sum_{n=1}^{\infty} P(N_{t_i-t_{i-1}} = n) P(\sum_{k=1}^{N_{t_i-t_{i-1}}} U_k = z / N_{t_i-t_{i-1}} = n) \\ &= \sum_{n=1}^{\infty} P(N_{t_i-t_{i-1}} = n) P(\sum_{k=1}^{n} U_k = z) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(\sum_{k=1}^{n} U_k = z). \end{aligned} \tag{5.1}$$

Denote by $\mathscr{L}$ the set of all possible values of $\overline{Z}_i \equiv^d \sum_{k=1}^{N_{t_i-t_{i-1}}} U_k$. So that

$$\begin{aligned} P(\overline{X}_i < x) &= P(\sigma \overline{W}_i + \overline{Z}_i < x) = \sum_{z \in \mathscr{L}} P(\sigma \overline{W}_i < x - z) P(\overline{Z}_i = z) \\ &= \sum_{z \in \mathscr{L}} \sum_{n=1}^{\infty} \int_{-\infty}^{x-z} \frac{1}{\sigma \sqrt{2\pi (t_i - t_{i-1})}} \exp[-\frac{u^2}{2\sigma^2 (t_i - t_{i-1})}] \times \\ &\times \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(\sum_{k=1}^{n} U_k = z) du. \end{aligned} \tag{5.2}$$

The default probability in this case is estimated by

$$\begin{aligned} P_{default}(0,T) \le 1 - \prod_{i=1}^{m} \Bigg(1 - \sum_{z \in \mathscr{L}} \sum_{n=1}^{\infty} \frac{1}{\sigma \sqrt{2\pi (t_i - t_{i-1})}} \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \exp[-\frac{(x-z)^2}{2\sigma^2 (t_i - t_{i-1})}] dx \times \\ \times \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(\sum_{k=1}^{n} U_k = z) \Bigg). \end{aligned} \tag{5.3}$$

6 U is Poisson random variable with parameter $\beta > 0$

If $U = U_k \sim Poisson(\beta)$ then

$$\sum_{k=1}^{n} U_k \sim Poisson(n\beta)$$

with mass probability

$$p_z = P(\sum_{k=1}^{n} U_k = z) = e^{-n\beta} \frac{(n\beta)^z}{z!}, \quad z = 0, 1, 2, \ldots$$

Then

$$\begin{aligned} P_{default}(0,T) &\le 1 - \prod_{i=1}^{m} \Bigg(1 - \sum_{z=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\sigma \sqrt{2\pi (t_i - t_{i-1})}} \times \\ &\times \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \exp[-\frac{x^2}{2\sigma^2 (t_i - t_{i-1})} - n\beta] dx \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} \frac{(n\beta)^z}{z!} \Bigg) \\ &= 1 - \prod_{i=1}^{m} \Bigg(1 - \sum_{z=0}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} \frac{(n\beta)^z}{z!} \frac{1}{\sigma \sqrt{2\pi (t_i - t_{i-1})}} \times \\ &\times \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \exp[-\frac{x^2}{2\sigma^2 (t_i - t_{i-1})} - n\beta] dx \Bigg). \end{aligned} \tag{6.1}$$

Acknowledgement

We would like to thank referees for their valuable comments from that some corrections have been made.

References

[1] T. Berg, From actual to risk-neutral default probabilities: Merton and Beyond, The Journal of Credit Risk, **6**(2010), no. 1, 55-86.

[2] T. Bielecki, M. Jeanblanc and M. Rutkowski, Credit Risk Modeling, Center for Study of Insurance and Finance, Osaka University, 2009.

[3] H. Bystrom, Merton for Dummies: A Flexible Way of Modelling Default Risk, Research Paper Series **112** (2007), Quantitative Finance Research Centre, University of Technology, Sydney.

[4] R. Cont and P. Tankov, Financial Modelling With Jump Processes, Chapman and Hall, CRC Press, 2003.

[5] K. Giesecke and R. G. Lisa, Forecasting Default in Face of Uncertainty, The Journal of Derivatives, Fall (2004), 11-25.

[6] M. Roger, Merton Robert C. on putting theory into practice, CFA Magazine, July-August (2004), 34-37.

[7] S. R. Shreve , Stochastic Calculus for Finance II, Springer, 2003.

[8] Y. Wang, Structural Credit Risk Modeling: Merton and Beond, Risk Managements, June (2009), 30-33.

[9] H. T. Phuong Thao and T. H. Thao, A Note on A Model of Merton Type for Valuing Default Risk, Applied Mathematical Sciences, **6**(2012), no. 89-92, 4457-4461.

[10] H. T. Phuong Thao, Valuing Default Risk for Assets Value Jumps Processes, East-West J. of Mathematics, **15**(2013), no. 2, 101-106.