Ithaque : Revue de philosophie de l'Université de Montréal

Pour citer cet article : **Therrien, V. L. (2012) « Wittgenstein And Labyrinth Of 'Actual Infinity': The Critique Of Transfinite Set Theory », *Ithaque*, 10, p. 43-65.**

URL : http://www.revueithaque.org/fichiers/Ithaque10/Therrien.pdf

Wittgenstein And The Labyrinth Of 'Actual Infinity' : The Critique Of Transfinite Set Theory

Valérie Lynn Therrien*

Abstract

In order to explain Wittgenstein's account of the reality of completed infinity in mathematics, a brief overview of Cantor's initial injection of the idea into set-theory, its trajectory (including the Diagonal Argument, the Continuum Hypothesis and Cantor's Theorem) and the philosophic implications he attributed to it will be presented. Subsequently, we will first expound Wittgenstein's grammatical critique of the use of the term 'infinity' in common parlance and its conversion into a notion of an actually existing (completed) infinite 'set'. Secondly, we will delve into Wittgenstein's technical critique of the concept of 'denumerability' as it is presented in set theory as well as his philosophic refutation of Cantor's Diagonal Argument and the implications of such a refutation onto the problems of the Continuum Hypothesis and Cantor's Theorem. Throughout, the discussion will be placed within the historical and philosophical framework of the Grundlagenkrise der Mathematik *and Hilbert's problems.*

1. Introduction

The history of the concept of infinity is about as linear as the supposedly undenumerable 'line' of real numbers and Wittgenstein's critique of the idea of the infinite touches upon an 'infinity' of these 'moments'. In the history of the development of a scientifically definable idea of the infinite, only the contributions of Aristotle and Kant have been as important as those of the German mathematician

* L'auteure est étudiante au baccalauréat en philosophie (Université de Montréal).

Georg Cantor (1845-1918). In proving the uncountability of the continuum of real numbers, he effectively founded Set Theory (i.e. the mathematical study of infinity) – a theory which he had hoped would provide mathematics with its 'true Foundations'. Thus, in an effort to appreciate more clearly Wittgenstein's oft maligned critique of the actual mathematical infinite[1], we have chosen Wittgenstein's commentary on Cantor and his Theory of Transfinite Sets to be our Ariadne's thread ; in fact

> the problems posed by the mathematical theory of infinity founded by Cantor provide what might best be described as both the testing-ground and the touchstone of Wittgenstein's approach to the philosophy of mathematics. Herein lies the juncture between calculus and prose, and thus, the confrontation between mathematics and philosophy[2].

Furthermore, we shall also filter the discussion through two sieves : a) Cantor's notion of 'infinite sets' and b) Cantor's ideas on denumerability and cardinality. In each of these topics, Wittgenstein's contribution can be summed up as i- a grammatical clarification of mathematical prose ; ii- a conceptual elucidation of the relevant metamathematical underpinnings ; and iii- the dissolution of an outstanding mathematical problem through its passage into obsolescence. This will thus provide us with ample room to extrapolate Wittgenstein's stance on actual and possible infinity in mathematics from his writings as well as the received literature.

[1] Which in reality consists of a hodge-podge of jotted fragments mostly published posthumously. For brief speculations as to why Wittgenstein's was (perhaps wrongly) reviled by his philosophical peers and mathematical contemporaries, as well as why this customary position ought to be revisited in light of the modern constructivist position as well as advances in complexity theory and the theoretical computer sciences, see Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. vii-ix.

[2] Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 161.

2. Cantor And The Mathematics Of Infinity

The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds.
Cantor[3]

There is nothing quite like set theory to make even the most skeptical mathematician wax eloquent. In general, it is widely considered to be one of the greatest achievements in mathematics and almost all of modern mathematics can be formalized in terms of the set-theoretical approach. As such, it is often claimed that set theory constitutes the very foundations of mathematics[4]. Set theory is the mathematical study of the infinite. Cantor has rightly been called the founder of set theory when, in 1872, he unveiled his first axiomatic analysis of the topology of the transfinite set of real numbers ($\mathbb{R}$). Cantor was led to his transfinite set theory through his fundamental belief that infinity is *ipso facto* mathematically comprehensible. In fact, the finite and the transfinite were similarly structured and he saw the transfinite sets as 'reaching' towards an immanently Absolute infinity (God), which he denoted Ω[5]. In this

[3] Quoted in Rucker R. (1983) *Infinity and the Mind*, p. 46.
[4] Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 281.
[5] Dauben, J. W. (2005), "The Battle for Cantorian Set Theory", p. 230-231 ; Ferreirós, J. (2004), "The Motives behind Cantor's Set Theory – Physical, Biological, and Philosophical Questions", p. 62 and 68 ; Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", p. 6. As well, it is essential to understanding Cantor's thinking on the infinite to note that he perceived mathematics, metaphysics and theology to be inextricably intertwined : mathematics is the language of God's immanent reality. In this, he was philosophically heavily and self-avowedly indebted to Plato, Spinoza and Leibniz's idea of monads (Ferreirós, J. (2004), "The Motives behind Cantor's Set Theory – Physical, Biological, and Philosophical Questions", p. 59). It is of note that, for Cantor, each element in $\mathbb{R}$ corresponds to a real 'point' on the 'line' representing the number continuum and, furthermore, he conceived the infinity of the continuum to be 'given' through experience (Kanamori, A. (1996), "The

section we shall briefly review Cantor's major contributions to the idea of the actual mathematical infinite, with a special emphasis on those aspects that will later be taken up by Wittgenstein in his critique of the actual infinite in mathematics.

2.1. The Topology ℝ *And The Continuum Hypothesis*

A set is a Many that allows itself to be thought of as a One.
Cantor[6]

It was in 1874 that Cantor published his views on the existence of all of the real numbers[7] as well as his first uncountability proof by which he determined via *reductio ad absurdum* that if one assumed that ℝ is denumerable (like ℕ) and attempted to establish a bijective function[8] between ℝ and ℕ, then one is presented with a contradiction (given the Bolzano-Weierstrass theorem[9]). In other words, no matter how 'close' on the 'line' the elements in ℝ one assigns to those in ℕ, there are always more elements between these

Mathematical Development of Set Theory from Cantor to Cohen", p. 3). As a result, he is generally considered a Platonist – finite experience of the transfinite guides us towards God's immanently infinite reality, which we may then coherently reproduce through rigorous mathematical analysis.

[6] "Über Unendliche, Lineare Punktmannichfaltigkeiten" (1882). *Cf.* Cantor G. (1932), *Gesammelte Abhandlugen mathematischen und philosophischen Inhalts*, p. 204.

[7] In a paper obtusely christened "On a Property of the Totality of All Real Algebraic Numbers" – the property being the denumerability of the algebraic numbers – for his real purpose was to prove the *existence* and non-denumerable *nature* of the *transcendental numbers* (i.e., all of the real non-algebraic numbers) ! At the time, this was a very controversial position. For more information as to why Cantor may have purposefully downplayed the importance of his major discovery in favor of a more milquetoast presentation, *Cf.* Dauben, J. W. (2005), "The Battle for Cantorian Set Theory", p. 225-227.

[8] That is, to place the elements in ℝ in a one-to-one correspondence with those in ℕ.

[9] This theorem asserts the completeness of the ℝ continuum – in terms of sequences of nested closed intervals – that is, that there are no empty 'points' on the 'line' of real numbers.

two that one will have 'neglected' to assign. This was his revolutionary discovery of the non-denumerability of the continuum of real numbers. For Cantor, the non-denumerability of $\mathbb{R}$ implies necessarily that there are more elements in $\mathbb{R}$ than in $\mathbb{N}$ – that is, that $\mathbb{R}$ has a greater cardinality than $\mathbb{N}$[10]. But could he *prove* that the transfinite set $\mathbb{R}$ is really 'bigger' than the transfinite set $\mathbb{N}$?

Impelled by his meandering on the nature and dimension of space and infinity, Cantor established in 1878 that $\mathbb{R}''$ and $\mathbb{R}^n$ have the same cardinalities as $\mathbb{R}$[11]. These investigations subsequently led him to advance the Continuum Hypothesis : all transfinite sets are either countably infinite *or* have the power of the continuum. In the notation of Cantor's 1895-97 works on transfinite cardinals, the cardinality of $\mathbb{N}$ is denoted as $\aleph_0$ and the cardinality of $\mathbb{R}$ as $\aleph_1$, such that the Continuum Hypothesis is equivalent to stating that there exists no intermediate cardinality between $\aleph_0$ and $\aleph_1$[12]. All further developments in transfinite set theory can be seen as attempts to provide a definitive solution to the Continuum Problem – which to this day remains unresolved.

---

[10] Dauben, J. W. (2005), "The Battle for Cantorian Set Theory", p. 228 ; Ferreirós, J. (2004), "The Motives behind Cantor's Set Theory – Physical, Biological, and Philosophical Questions", p. 54 ; Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", p. 1 and p. 5. By contrast, the set of rational numbers ($\mathbb{Q}$) was proven by Cantor to be a countably transfinite set, that is, to be a set possessing the same cardinality as $\mathbb{N}$ – despite the fact that one intuitively feels that there are more elements in $\mathbb{Q}$ than in $\mathbb{N}$. In fact, this is not so, for $\mathbb{Q}$ (like other transfinite sets such as the set of 'all even numbers', the set of 'all the products of the number 3', the set of 'all the prime numbers', etc.) can be placed in a bijective function with $\mathbb{N}$.

[11] Ferreirós, J. (2004), "The Motives behind Cantor's Set Theory – Physical, Biological, and Philosophical Questions", p. 53 ; Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", *The Bulletin of Symbolic Logic*, vol. 2, n°1, p. 4.

[12] That is, there is no intermediate gradation in 'size' between those transfinite sets of cardinality ≡ $\mathbb{N}$ and those transfinite sets of cardinality ≡ $\mathbb{R}$.

2.2. The Transfinite Numbers, The Diagonal Argument And Cantor's Theorem

> *What I assert and believe to have demonstrated in this and earlier works is that following the finite there is a transfinite (which one could also call the supra-finite), that is an unbounded ascending ladder of definite modes, which by their nature are not finite but infinite, but which just like the finite can be determined by well-defined and distinguishable numbers.*
>
> Cantor[13]

In order to support his Continuum Hypothesis, Cantor was led to posit the existence of transfinite ordinal numbers to account for his essential notion of the 'well-ordering' of all transfinite sets[14]. In 1883[15], Cantor identified $\mathbb{N}$ (or $\aleph_0$) with the least denumerable infinite ordinal number ω and further posited the existence of uncountably many countably infinite ordinals to account for the infinity of ways in which a countably transfinite set may be considered well-ordered : ω, $\omega + 1$, $\omega + 2$, …, $\omega + \omega$ (= $\omega \cdot 2$), $\omega \cdot 2 + 1$, …, $\omega \cdot \omega$ (= ω^2), …, ω^3, …, ω^{ω}, …, $\omega^{\omega^{\omega}}$, …, ε_0, … *ad infinitum*[16]. And all of these transfinite

[13] Cantor, G. (1882), "Über Unendliche, Lineare Punktmannichfaltigkeiten". *Cf.* Cantor G. (1932), *Gesammelte Abhandlugen mathematischen und philosophischen Inhalts*, p. 174.

[14] Although he was only able to prove this of the countably transfinite sets. His postulate that uncountably transfinite sets (such as $\mathbb{R}$) were susceptible of being well-ordered was precisely one which proof eluded him, much to his consternation. As crucial as the notion of well-ordering is to Cantor's system, a discussion of this topic unfortunately lies outside the scope of this paper.

[15] Published under the title "Foundations of a General Theory of Manifolds" (although often simply referred to as the *Grundlagen*), this work also contains a large section on Cantor's philosophical and theological views on mathematics and the immanence of actual infinity. *Cf.* Cantor, G. (1883), *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.*

[16] The cardinality of $\mathbb{R}$ (or $\aleph_1$) he identified with the first uncountably infinite ordinal ω_1.

sets, he claimed, were of the same size, they all were ≡ ℕ[17]. But Cantor was not done with infinity. In particular, he wished to determine the existence, nature and size of the 'set of all sets' : his Absolute infinite (Ω) from which he felt the finite and the transfinite received their structure. For that, he would need to determine what – if anything – lay beyond the continuum.

Notably, in 1891, he published his second uncountability proof (the Diagonal Argument) which was the stone he used to prove the proverbial two birds : not only was ℝ non-denumerable through diagonalization[18] of the elements in ℕ, but also that for any set S, there exists a power-set[19] p(S) such that the latter has a greater cardinality than S. This was later to be known as Cantor's Theorem. Specifically, all power-sets of countably infinite sets are uncountably infinite (that is, if S ≡ $\aleph_0$ than p(S) ≡ $\aleph_1$). Applied to uncountably transfinite sets, this theorem affirms that there exists infinitely many cardinalities[20] (for if S ≡ $\aleph_1$, then p(S) ≡ $\aleph_2$, *ad infinitum*)[21]. Hence, if Cantor is right, then there exists an uncountable infinity of transfinite ordinal numbers (or, an uncountable infinity of countably and uncountably infinite sets) and a countable infinity of transfinite cardinal numbers (or, a countable infinity of infinitely big sets, each one infinitely bigger than the last) ! Already, one begins to get a sense

[17] Dauben, J. W. (2005), "The Battle for Cantorian Set Theory", p. 229 ; Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", p. 5 and p. 10.

[18] For an exposition of the method of diagonalization by which Cantor was able to ascertain that even a segment of the ℝ continuum contains more elements than in ℕ, see Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", p. 8.

[19] The power set of S being the set of 'all subsets of S', including the empty set and the set S itself.

[20] Or an infinity of sets each one 'bigger' than the last.

[21] In fact, between 1895 and 1897, Cantor was able to introduce an arithmetic of the transfinite cardinals that allowed him to establish that the continuum $\aleph_1$ (thus the size of ℝ, ℝ″ and $\mathbb{R}^n$) was equivalent to $2^{\aleph_0}$ (Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", p. 9-10).

for the "giddiness [that] attacks us when we think of certain theorems in set theory[22]".

3. Cantor's Metamathematics And The Crisis In The Foundation Of Mathematics

No one will drive us from the paradise that Cantor has created.
Hilbert[23]

However, Cantor's theorem also meant that there could be no actual/complete infinite set Ω[24]. Furthermore, from the well-ordering of a set, Cantor was not able to deduce the well-ordering of its power-set[25], which did not bode well for Cantor's fundamental belief that the continuum was well-ordered and therefore susceptible to infallible mathematical analysis. In 1900, Hilbert named Cantor's Continuum Problem as the most important modern mathematical problem, and the consistency of the idea of $\mathbb{R}$ as the second one. As well, in 1903 Russell published their respective opuses – which included their merciless visions of the logical paradoxes inherent to Cantor's 'paradise'[26].

What to make now of the transfinite sets ? Could they still be viewed as reaching towards Absolute infinity if one could not even coherently hypothesize its existence ? Had Cantor really rehabilitated actual mathematical infinity ? Or had he merely reified possible

[22] Wittgenstein, L. (1989), *Lectures on the Foundations of Mathematics*, p. 16.
[23] Hilbert D. (1925), "Über das Unendliche", p. 170.
[24] Or, as Cantor resigned himself to put it, Absolute infinity lay outside the scope of mathematics.
[25] Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", p. 9.
[26] Ferreirós, J. (2008), "The Crisis in the Foundations of Mathematics", p. 3-4. The most famous paradox being Russell's logical paradox viz. the 'set of all sets'. This paradox had already been discovered by Cantor in 1896 but had not yet been framed in terms of a purely logical paradox. The contradiction lies in the idea of constructing a largest ordinal number (Burali-Forti's paradox) or cardinal number (Cantor's paradox). Wittgenstein had much to say about these paradoxes as well. However, we do not need to investigate these in order to grasp the portent of his critique of actual infinity.

infinity ? Could the 'paradise' he created be saved ? Thus, Cantor's Theorem, as well as his general Theory of Transfinite Sets generated much controversy and many revisions. Platonists, formalists, logicists, intuitionists and constructivists presented warring philosophical views as to the true nature of mathematical 'reality', 'truth' and 'provability' – ultimately culminating in the great *Grundlagenkrise der Mathematik* (or 'foundational crisis in mathematics') of the 20th century. Ultimately, this debate was never resolved but rather petered out as interest in the sub-discipline waned in the wake of Gödel's incompleteness theorems of 1931[27].

4. Wittgenstein And The 'Labyrinth Of Infinity'

I would say, 'I wouldn't dream of trying to drive anyone from this paradise.' I would do something quite different : I would try to show you that it is not a paradise -- so that you'll leave of your own accord.

Wittgenstein[28]

There is no need to go on any further in an elucidation of the history of Transfinite Set Theory to grasp the scope of Wittgenstein's critique ; for, if Wittgenstein was right, then there shouldn't have been any further history of the actual infinite in mathematics, there was simply no *need* for this Foundational Crisis. We shall now examine Wittgenstein's contributions to this debate (as they specifically pertain to Cantor's), without seeking to canton him in any particular school of mathematical thought (this approach being faithful to his aforementioned avowedly idiosyncratic philosophical style and rejection of academic 'schools')[29]. Specifically, we shall limit

[27] Ferreirós, J. (2008), "The Crisis in the Foundations of Mathematics", p. 11-13.
[28] Wittgenstein, L. (1989), *Lectures on the Foundations of Mathematics*, p. 103.
[29] It is of note that any discussion of Wittgenstein's views on anything cannot bypass his quintessential approach to philosophy as technical

our purports to those concerning Wittgenstein's critique of the meaning of 'infinity' and 'denumerability', of Transfinite Sets, and of Cantor's Diagonal Argument.

4.1. The Infinity Of 'Infinite Sets'

> *It is senseless to speak of the* number of all objects.
>
> Wittgenstein[30]

To begin, Cantor examined the mathematical infinite through the postulate that the 'topology' of the infinite could be examined as it is given to human understanding : through sets[31]. Principally, Wittgenstein's qualms lay in i- Cantor's misappropriation of the colloquial use of the word 'infinity' to denote a determinate (or in principle determinable) quantity ; and ii- his dubious notion of 'infinity' as an entity that not only exists independently from the mathematician but that, furthermore, prepossesses quasi-physical attributes that render it susceptible to topologic analysis. These extrapolations will have obvious immediate ramifications on iii- the idea of the existence of $\mathbb{R}$ as an infinite totality.

---

pruning of philosophical 'problems' through grammatical and conceptual refinement. In fact, all of the critiques presented in this essay may be viewed as Wittgenstein's attempt to elucidate the metamathematical 'problems' involved in Cantor's Transfinite Set Theory through this method (Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 167, p. 176 and p. 198). After all, "[philosophy] is a battle against the bewitchment of our intelligence by means of language", *cf.* Wittgenstein, L. (1953), *Philosophical Investigations*, §109), and its "aim is : to teach you to pass from a piece of disguised nonsense to a piece of patent nonsense" (*cf. Ibid.*, §464).

[30] Wittgenstein, L. (2010), *Tractatus Logico-Philosophicus*, §4.1272.

[31] This is known as the 'set-theoretical approach'. It is of note that Cantor marshaled Transfinite Set Theory explicitly as a means for providing a mathematical foundation to his religious metaphysics of the Infinite (Ferreirós, J. (2004), "The Motives behind Cantor's Set Theory – Physical, Biological, and Philosophical Questions").

4.1.1. The Grammar Of 'Infinity'

If you can show that there are numbers bigger than the infinite, your head whirls. This might be the chief reason [set theory] was invented.
Wittgenstein[32]

Wittgenstein bemoans the conflagration, firstly, of the grammar of the 'finite' with the 'infinite' ; and, secondly, of the colloquially 'infinite' and the technically 'infinite'. Firstly, the logical syntax of 'infinity' is non-reducible to that of a 'numeral' : while the latter may serve as an adjective denoting a given quantity, the former may not[33]. "'Infinite class' and 'finite class' are different logical categories ; what can be significantly asserted of the one category cannot be significantly asserted of the other[34]." For example, Cantor's notation '$\omega + 2 > \omega + 1$' hoodwinks us into thinking that '>' here means the same thing as in '$n + 1 > n$' : to denote the relationship 'is of greater magnitude than'[35]. And, secondly, it makes no sense to speak of an 'infinite totality'[36] : it is a logical paradox – one borne out of the

[32] Wittgenstein, L. (1989), *Lectures on the Foundations of Mathematics*, p. 16.
[33] Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 183-184 ; Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 286 ; Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 165 ; Wittgenstein, L. (1975), *Philosophical Remarks*, §138 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, §38. This category mistake is the result of such vernacular misnomers that depict the 'infinite' as being merely a gigantic unit of length, such as stating that 'there exists an infinity of stars' or that 'the universe is infinitely big', the way one would state that 'Mary has 3 cats' or that 'this page is 1mm thick' (Wittgenstein, L. (1974), *Philosophical Grammar*, p. 463 ; Wittgenstein, L. (1975), *Philosophical Remarks*, §142 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, V, §19 and VII, §59).
[34] Wittgenstein, L. (1974), *Philosophical Grammar*, p. 464.
[35] Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 172-173.
[36] A term, when pondered, that proves to be an oxymoron – exactly the kind of muddled philosophical pseudo-problem ("academic philosophy in sheep's clothing : a pseudo-explanatory 'theory' having no redeeming applications to anything other than metaphysics", *cf.* Steiner, M. (2001), "Wittgenstein as his

previous grammatical muddle. To illustrate this inkling, Wittgenstein offers the following paradox : "Imagine a man whose life goes back for an infinite time and who says to us : 'I'm just writing down the last digit of π, and it's a 2'. [...] This seems utter nonsense, and a *reductio ad absurdum* of the concept of an infinite *totality*[37]."

The positive term 'finite' is inextricably linked to the concept of a 'totality' whilst the negative term 'infinite' is just as much to that of a 'boundless series'. In fact, it was one of Wittgenstein's chief intentions to show that 'infinity' could not be divorced from the idea of the *endless process*. It therefore makes no sense to speak of 'infinite numbers' or 'infinite sets'[38] ! Although, properly, it may be asserted only of *rule-governed series* that they may be either finite or infinite, nevertheless even then "[one] has [...] a concept of an infinite series but here that gives us at most a vague idea, a guiding light for the formation of a concept[39]", a mere sense that there is a process here that will not terminate. However, building on his logico-syntactical category mistake, Cantor derived from his idea mathematical operations that were logically impossible to compute, in an effort to

Own Worst Enemy : The Case of Gödel's Theorem", p. 270) that Wittgenstein abhors.

[37] Wittgenstein, L. (1975), *Philosophical Remarks*, §145. Along this vein, an unsubstantiated joke is often attributed to Wittgenstein : a man overhears "..., 9, 5, 1, 4, 1, 3. Done !" "What are you doing ?" "Oh, just reading all the digits of π backwards !".

[38] Lampert, T. (2008), "Wittgenstein on the Infinity of Primes", p. 69 ; Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 180-183 ; Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 175 and p. 197 ; Wittgenstein, L. (1975), *Philosophical Remarks*, §138-139 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §45. By that same token, it makes no sense to say that a set is finite, for if a thing is to be a set at all, it will by definition be completed, hence, finite. For this reason, Wittgenstein thought that "[in] a correct language there must not even be a temptation of raising the question whether a class is finite or infinite. [...] *'Infinite' is not a quantity*. The word 'infinite' has a different syntax from a number word." (Reproduced in : Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 164-165).

[39] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §16.

discern a concept that is necessarily unfathomable[40] ! This leads us right into the heart of Wittgenstein's logical dissection of Cantor's 'topology of infinite sets'. For this first category mistake is mirrored by a second one.

4.1.2. The Concept Of An 'Infinite Set'

> *I have always said you can't speak of* all *numbers, because there's no such thing as 'all numbers'.*
>
> Wittgenstein[41]

"The infinite number series is only the infinite possibility of finite series of numbers. It is senseless to speak of the *whole* infinite number series, as if it, too, were an extension[42]" : in fact, the main problem with Transfinite Set Theory is that, through its use of abstract symbolic notation, it conflagrates intensions with extensions – thus reifying mathematical objects that *simply do not exist.* A 'set' is nothing more than an abstract symbol for a list ('extension') generated by a rule ('intension'). In the case of a transfinite set, then, the 'infinite' intension is simply a recursive rule for calculating certain kinds of results – one that does not have an 'and then stop' at the end. However, while the rule may not have a proper end, the extension cannot be considered infinite simply because the extension is precisely only what we have written down on the list, what we have calculated ; the law yields *only* the endless process, not the endless extension[43].

---

[40] It is amenable to a proper understanding of Wittgenstein's thoughts on infinity to discern that his finitist assailment of infinity is *not* concerned with man's psychological or epistemological ability to 'seize' infinite totalities but, rather, with the logical syntax of the word 'infinity' as well as its mathematical applications. *cf.* Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 164, p. 180 and p. 197.

[41] Wittgenstein, L. (1975), *Philosophical Remarks,* §129.

[42] *Ibid.*, §144.

[43] Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 284 and p. 294 ; Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, 167 and 183 ; Wittgenstein, L. (1974) *Philosophical Grammar*, p. 457 and 461 ; Wittgenstein, L. (1975), *Philosophical Remarks*, §130, §142 and

For example, the 'set of all even numbers' is constructed by the recursive rule 'add 2' *ad nauseam*. Its proper extension is an enumeration : a list such as {2, 4, 6, 8, 10, ...}. This 'set of all even numbers' is not an actually infinite extension ; we could symbolize this set as '&' but this would still not also constitute an actual infinite set – although we might delude ourselves into thinking it was if it was presented to us as a true premise in an argument that otherwise works. Seen through this example, it is easy to see how describing 'types' of 'infinity' with abstract symbolic notation[44] deludes us into thinking that we have arrived at an actual infinite extension when in reality there are "only finite mathematical extensions[45]". The law *is* the potentially infinite series, but that doesn't mean that there is an actually infinite series ; nothing in the actual extension/list/enumeration has so far been 'revealed' to be actually infinite – all we have is a recursive intension/rule/law[46]. Of course

§157 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II §45.

[44] Such as Cantor's transfinite sets (e.g. $\mathbb{R}$), transfinite ordinals (e.g. ω) and transfinite cardinals (ℵ), which all purport to describe or denote various types of infinity. Of course, Cantor also initially believed in an all-encompassing 'complete' infinity : the Absolute, aka God (Ω). According to Wittgenstein, one ought to be wary of such abstract notations denoting various manifestations word-concept 'infinity', as they merely conceal its tenuous nature beneath a veneer of concretism.

[45] Wittgenstein, L. (1974), *Philosophical Grammar*, p. 469. The rest of the shroud of authenticity comes through the deployment of Transfinite Arithmetic, an example of which being the aforementioned delusion that '$\omega + 2 > \omega + 1$' ≡ '$n + 1 > n$'. Thus, this abstract notation does nothing but artificially strengthen this view that mathematics is discovering the properties of the actual mathematical infinity when, in reality, it only "builds on a fictitious symbolism, therefore on nonsense". Wittgenstein, L. (1975), *Philosophical Remarks,* §174.

[46] In fact, no mathematical proof could prove that there is an actual infinite series because "there isn't a dualism of the law and the infinite series obeying it" ! *Ibid.,* §180. Wittgenstein was led to his conclusions on the logical impossibility of infinite totalities and on the dichotomy between mathematical intensions and extensions partly through his many fascinatingly unorthodox musings on irrational numbers. Although a discussion of these lie outside the scope of this paper, it is certainly not at all

this has immediate ramifications on the idea of the 'set of all real numbers' ($\mathbb{R}$).

4.1.3. The Existence Of The Infinite Set $\mathbb{R}$ And Mathematical 'Reality'

> *[We] can't describe mathematics, we can only do it. (And that of itself abolishes every 'set theory').*
> Wittgenstein [47]

This is so because, for Wittgenstein, "mathematical truth is created, not discovered[48]." As such, there is no 'real' quasi-physical mathematical landscape that pre-exists Man, revealing itself through Set Theory. After all, mathematics is algorithmic, *not* descriptive metaphysics ; mathematics *is* the calculus[49]. For Wittgenstein, the main impetus behind the semantic and conceptual confusion at the heart of the metamathematical belief in the infinite set $\mathbb{R}$[50] is a metaphysical one : it produces a pleasant impression, a "giddiness [that] attacks us when we think of certain theorems in set theory[51]."

---

hors-propos to mention that Wittgenstein believed : a) that there were many different *kinds* of irrational numbers ; b) that not all of these kinds of irrationals could fly under the banner $\mathbb{R}$; and, most importantly for our purposes c) that there were not infinitely many irrational numbers in the specific sense that they formed a 'gapless' continuum. *Cf.* Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 195-199 ; Rodych, V. (1999), "Wittgenstein on Irrationals and Algorithmic Decidability" ; Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 183-192 ; Wittgenstein, L. (1974) *Philosophical Grammar*, p. 46, p. 471 and p. 473 ; Wittgenstein, L. (1975), *Philosophical Remarks*, §172, §§180-1, §183 and §186 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §29 and §33, and V, §34.

[47] Wittgenstein, L. (1975), *Philosophical Remarks*, §159.

[48] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, I, §168.

[49] Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 3-6 and p. 198 ; Wittgenstein, L. (1974), *Philosophical Grammar*, p. 368 and p. 468 ; Wittgenstein, L. (1975), *Philosophical Remarks*, §159.

[50] The all-encompassing mother-of-all infinite sets, all other infinite sets (such as $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{Z}$) being, at least theoretically, proper subsets of $\mathbb{R}$.

[51] Wittgenstein, L. (1989), *Lectures on the Foundations of Mathematics*, p. 16.

As such, he was convinced that if this aura could be dispelled and the Platonic pretenses shrugged, the inquiry into the 'Foundations' of mathematics would lose all its appeal[52]. It is thus patent that, for Wittgenstein, there could no more metaphysically than logically be an infinite set $\mathbb{R}$[53] – which solves the second of Hilbert's problems.

4.2. The Infinity Of The 'Infinite Cardinalities' Of 'Infinite Sets'

> *The riddle does not exist. If a question can be put at all, then it can also be answered.*
>
> Wittgenstein[54]

In Transfinite Set Theory, Cantor's Diagonal Argument is at the heart of his results about $\mathbb{R}$ and is generally understood as proving the existence of uncountably infinite sets (a necessary condition of the Continuum Hypothesis) and, as a corollary, the existence of an infinite hierarchical structure of the actual mathematical infinite (Cantor's Theorem). Here, Wittgenstein's qualms lie in i- Cantor's muddled idea of 'denumerability' and of what it means to be 'non-denumerable' ; and ii- whether his famous Diagonal Proof truly proves that some infinite sets are non-denumerable, or that some infinite sets have greater cardinalities than others. This will have some consequences on iii- the Continuum Hypothesis.

---

52 Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 17-18 ; Steiner, M. (2001), "Wittgenstein as his Own Worst Enemy : The Case of Gödel's Theorem", p. 269-271 ; Wittgenstein, L. (1989), *Lectures on the Foundations of Mathematics*, p. 16 Wittgenstein, L. (1974) *Philosophical Grammar*, p. 469-70. After all, set theory has no intra- or extra-mathematical applications. *Cf.* Steiner, M. (2001), "Wittgenstein as his Own Worst Enemy : The Case of Gödel's Theorem", p. 269. As such, "(i)t is simply not really necessary [...] to conjure up the picture of the infinite [of the enormously big] for how is this picture connected with the *calculus*." *Cf.* Wittgenstein, L. (1978) *Remarks on the Foundations of Mathematics*, II, §59.

53 Or $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{Z}$ for that matter.

54 Wittgenstein, L. (2010), *Tractatus Logico-Philosophicus*, §6.5.

4.2.1. The Grammar Of 'Denumerability'

The philosophy of mathematics consists in an exact scrutiny of mathematical proofs – not in surrounding mathematics with a vapor.
Wittgenstein[55]

According to Wittgenstein, "[it] should not have been called 'denumerable', but [...] 'numberable'. [...] For one cannot set out to enumerate the rational numbers, but one can perfectly well set out to assign numbers to them[56]." In other words, it makes no sense to say that a set like $\mathbb{N}$ is countably infinite, because it is not true that one can ever count them to completion[57]. However, we can set out to assign numbers to each element in the set, so long as we accept that it is a task that we cannot complete[58]. Hence, $\mathbb{N}$ is numberable – but it is not denumerable in Wittgenstein's sense[59].

As such, the converse term 'non-denumerable' has nothing to do with the concept of cardinality, or that of a 'greater infinity'[60]. In fact, on this view, the idea of construing infinite cardinal numbers to account for the sizes of denumerable and non-denumerable infinite sets is rendered utterly incongruous. It is precisely in this sense that

[55] Wittgenstein, L. (1974), *Philosophical Grammar*, p. 367.
[56] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, §15.
[57] The term 'denumerable' does indeed have the connotation that it is factually possible to enumerate the thing at hand, whereas the term 'numberable' merely states that it is factually possible to assign numbers to the thing at hand – which turns it into the *task* at hand.
[58] Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 294-295.
[59] Wittgenstein proposes an alternate definition : "I call number-concept X non-denumerable if it has been stipulated that, whatever numbers falling under this concept you arrange in a series, the diagonal number of this series is also to fall under that concept". (Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §10). What this would mean, is that denumerable and non-denumerable are horizontal categories (much like 'a priori' and 'a posteriori' are kinds of knowledge in the Kantian system), not hierarchical ones (much like 'animal' and 'mammal' do in the Aristotelian system).
[60] Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 196-197.

Wittgenstein presents Cantor's rapprochement of the transfinite sets with the transfinite cardinal numbers to be a particularly disingenuous brand of "hocus pocus[61]." We shall now summarily investigate Cantor's supposed proof of the non-denumerability of ℝ, assuming for argument's sake that Cantor's basic use of the terms 'infinite set' and 'non-denumerability' is cogent.

4.2.2. 'Benhind' The Diagonal Argument And 'Infinite Numbers'

The verbal expression casts only a dim general glow over the calculations but a calculation a brilliant light on the verbal expression.
Wittgenstein[62]

It is then Wittgenstein's categorical postulate that Cantor's Diagonal Argument *does not prove* either that some transfinite sets are 'bigger' than others, or Cantor's Theorem[63]. This is because being unable to place an 'infinite set' in one-to-one correspondence with its 'infinite subset' does not have the kind of implications on magnitude that Cantor concludes that it does : it neither confers 'non-equinumerosity' as it would for a 'finite set', nor non-denumerability (and it certainly doesn't establish the existence of transfinite cardinal numbers)[64] ! Rather, in establishing that the 'real numbers' cannot be placed in a series, the "proper purpose[65]" of the Diagonal Argument

[61] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §22.
[62] *Ibid.*, §7.
[63] It is in this sense that Wittgenstein claims that "[it] means nothing to say : "*Therefore* the X numbers are not denumerable". (Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §10) Where the non-denumerable is construed in the classical Cantorean sense of 'is greater in cardinality than the denumerable sets – because this claim simply does not follow from the argument.
[64] Lampert, T. (2008), "Wittgenstein on the Infinity of Primes", p. 69 ; Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 200-201.
[65] Steiner, M. (2001), "Wittgenstein as his Own Worst Enemy : The Case of Gödel's Theorem, p. 269 ; Wittgenstein, L. (1978) *Remarks on the Foundations of Mathematics*, II, §13 and §16.

ought to be that it proves that Cantor's 'real number series' is simply senseless.

Furthermore, the Diagonal Argument is nothing *more* than a constructive rule – that is, a rule for constructing certain kinds of numbers out of certain other kinds of numbers. It cannot generate an entirely different infinite *set* of numbers out of a first infinite set[66]. Thus, Cantor has only *actually* proved that he can construct a finite expansion through application of a rule-governed intension ; the constructive rule itself (even if infinitely applied) *will never yield* an infinite extension[67] ! The Diagonal Method can thus prove neither that $\mathbb{R}$ exists as a gapless continuum[68] (and, by extension, it cannot support the Continuum Hypothesis), nor that there is such a thing as an infinite cardinal (Cantor's Theorem)[69].

[66] Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics,* p. 178-181 and 195-196 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §3.

[67] Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 199 ; Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 295-296.

[68] For Wittgenstein, the concept of $\mathbb{R}$ as a continuous, gapless entity, which we mention as a corollary, is borne out by a semantic and conceptual confusion (*cf.* Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 209-210) : "[like] the enigma of Time for Augustine, the enigma of the continuum arises because language misleads us into applying to it a picture that doesn't fit". (Wittgenstein, L. (1974) *Philosophical Grammar*, p. 471) : "[a] misleading picture : 'The rational points lie close together on the number-line'" (*Ibid.*, p. 460) ; "[the] picture of the number line is an absolutely natural one up to a certain point ; that is to say so long as it is not used for a general theory of real numbers" (Wittgenstein, L. (1978) *Remarks on the Foundations of Mathematics,* V, §32), for "[the] straight line isn't composed of points" (Wittgenstein, L. (1975) *Philosophical Remarks*, §172), the "mathematical *rules* are the points" (Wittgenstein, L. (1974) *Philosophical Grammar,* p. 484) !

[69] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §33.

4.2.3. The Continuum Hypothesis, Cantor's Theorem And Mathematical Reality As Constructive Proof-Process

> [Only] in our verbal language [...] are there 'as yet unsolved problems' in mathematics.
>
> Wittgenstein[70]

This is because, for Wittgenstein, as mathematics is an algorithm, the only mathematical reality is in the constructive proof-process, *not the result*[71] – "[our] suspicion ought always to be aroused when a proof proves more than its means allow it" ; the Diagonal Argument is then a "puffed-up proof[72]" because, while it may be infinitely applied, the completed collection of all its results can never be called infinite in the sense that Cantor intended. Thus, both by highlighting the nature of the intension/extension double-helix and by questioning the putative link between the bijective function and equinumerosity at the heart of Cantor's set-theoretical approach[73], Wittgenstein's grammatical analysis has the effect of an axe dropping on the concretism of the Continuum Problem and Cantor's Theorem : Cantor's Theorem is logically false and the Continuum Hypothesis is not an 'unsolved problem' but, rather, merely a nonsensical pseudo-problem – which solves Hilbert's first problem[74].

---

[70] Wittgenstein, L. (1975), *Philosophical Remarks*, §159.

[71] Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 158-162 and p. 171-173 ; Wittgenstein, L. (1974), *Philosophical Grammar*, p. 369-370 and p. 374 ; Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §7.

[72] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §21.

[73] Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 181.

[74] Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 293. We leave it up to the reader to discern for himself whether Wittgenstein's logico-syntactical obliteration of Hilbert's first two major 'unsolved problems in mathematics' is convincing. It is of course of note that career mathematicians have historically almost exclusively either rejected or ignored Wittgenstein's approach. Of course, some authors have conjectured that this reception has mostly is due to base misapprehension of his basic intentions (Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics* ; Lampert, T. (2008), "Wittgenstein on the Infinity of Primes"). A special

5. Conclusion

It's almost unbelievable, the way in which a problem gets completely barricaded in by the misleading expressions which generation upon generation throw up for miles around it, so that it's become virtually impossible to get at it.
Wittgenstein[75]

To sum up, Wittgenstein's view of the algorithmic nature of mathematics is directly linked to his anti-Platonic and anti-Cantorean stance against the completed infinity[76]. As such, Wittgenstein's position on the status of infinity in mathematics closely related to Aristotle's ἄπειρον which holds that the infinite is intrinsically incomplete[77]. The logical syntax and conceptual analysis of 'infinity' therefore precludes any technical use other than as an adjective putatively describing a potentially infinite mathematical *process* – the possibility of constructing infinitely many series through the deployment of a recursive rule[78]. Of course, the term 'possible' is ambiguous : it purveys the connotation that what is 'possible' can become 'actual', given enough time[79] ; it is "one of the most deep rooted mistakes of philosophy to see possibility as a shadow of reality[80]." Thus, in order to shrug off any pretense of an actual

mention goes here to Daesuk Han for rigorously arguing that the 'extension' of the 'real numbers' is indeed a "homeless fiction" that has no application either inside the actual practice of the calculus, or outside (as a 'support' to real analysis). *Cf.* Han, D. (2010), "Wittgenstein and the Real Numbers".

75 Wittgenstein, L. (1974), *Philosophical Grammar*, p. 466.

76 Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 14 ; Steiner, M. (2001), "Wittgenstein as his Own Worst Enemy : The Case of Gödel's Theorem", p. 269.

77 Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, p. 181-183.

78 Lampert, T. (2008), "Wittgenstein on the Infinity of Primes", p. 69 ; Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", p. 290 ; Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, p. 197-198.

79 Wittgenstein, L. (1975), *Philosophical Remarks*, §141.

80 Wittgenstein, L. (1974), *Philosophical Grammar*, p. 283.

extensional doppelganger to each possible infinite rule : "[the] word infinite ought to be avoided in mathematics[81]."

Bibliography

Cantor, G. (1882), "Über Unendliche, Lineare Punktmannichfaltigkeiten", in Zermelo, E. (ed.), Cantor G. (1932), *Gesammelte Abhandlungen Mathematischen und Philosophischen Inhalts*, Berlin, Springer, p. 165-208. Reprint 1966, Hidesheim, Olms.

Cantor G. (1883), *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*. Leipzig, Teubner.

Dauben, J. W. (2005), "The Battle for Cantorian Set Theory", in Kinyon, Michael and van Brummelen, Glen (ed.), *Mathematics and the Historian's Craft : the Kenneth O. May Lectures*, New York, Springer, p. 221-241.

Ferreirós, J. (2004), "The Motives behind Cantor's Set Theory – Physical, Biological, and Philosophical Questions", *Science in Context*, vol 17, p. 49-83.

Ferreirós, J. (2008), "The Crisis in the Foundations of Mathematics", in Gowers, T (ed.), *Princeton Companion to Mathematics*, Princeton, Princeton University Press, p. 1-14.

Han, D. (2010), "Wittgenstein and the Real Numbers", *History and Philosophy of Logic*, vol. 31, August, p. 219-245.

Hilbert D. (1925), "Über das Unendliche Mathematische Annalen", vol. 96, p. 161-90.

Kanamori, A. (1996), "The Mathematical Development of Set Theory from Cantor to Cohen", *The Bulletin of Symbolic Logic*, vol. 2, n°1, p. 1-71.

Lampert, T. (2008), "Wittgenstein on the Infinity of Primes", *History and Philosophy of Logic*, vol. 29, February, p. 63-81.

Marion, M. (1998), *Wittgenstein, Finitism and the Foundations of Mathematics*, Oxford, Clarendon Press.

Rodych, V. (1999), "Wittgenstein on Irrationals and Algorithmic Decidability", *Synthese*, vol. 118, p. 279-304.

[81] Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, II, §58.

Rodych, V. (2000), "Wittgenstein's Critique of Set Theory", *The Southern Journal of Philosophy*, vol. 38, n°2, p. 281-319.

Rucker, R. (1983) *Infinity and the Mind*, New York, Bantham.

Shanker, S. G. (1987), *Wittgenstein and the Turning-Point in the Philosophy of Mathematics*, Albany, State University of New York Press.

Steiner, M. (2001) "Wittgenstein as his Own Worst Enemy : The Case of Gödel's Theorem", *Philosophia Mathematica* vol. 9, n°3, p. 257-279.

Wittgenstein, L. (1989), *Lectures on the Foundations of Mathematics*, C. Diamond (ed.), Chicago, Chicago University Press.

Wittgenstein, L. (1953), *Philosophical Investigations*, Oxford, Blackwell.

Wittgenstein, L. (1974), *Philosophical Grammar*, Oxford, Blackwell.

Wittgenstein, L. (1975), *Philosophical Remarks*, Oxford, Blackwell.

Wittgenstein, L. (1978), *Remarks on the Foundations of Mathematics*, Oxford, Blackwell.

Wittgenstein, L. and B. Russell (2010), *Tractatus Logico-Philosophicus*, transl. C. K. Ogden, New York, Cosimo Classics.