TOPOLOGY AND LEIBNIZIAN

PRINCIPLES OF THE IDENTITY OF INDISCERNIBLES

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1. Introduction. Leibniz's Principle of the Identity of Indiscernibles (hereafter called PII) is usually formulated as follows: If, for every property *F*, object *x* has *F* if and only if object *y* has *F*, then *x* is identical to *y*. In symbolic notation:

(1.1) Definition. (Leibniz's Principle of the Identity of Indiscernibiles).

(PII) $\forall x,y\ \forall F\ (Fx \leftrightarrow Fy) \rightarrow x = y.$

(Forrest (2010)).♦

The logically equivalent transposition of (PII) is sometimes called the Principle of Dissimilarity of the Diverse (PDD), namely: If *x* and *y* are distinct objects, then there is at least one property F that *x* has and *y* does not, or that y has and x does not have. In symbolic notation:

(1.2) Definition. (The Principle of the Dissimilarity of the Diverse).

(PDD) $\forall x, y\ (x \neq y \rightarrow \exists F\ ((Fx \text{ and not } Fy) \text{ or } (Fy \text{ and not } Fx)))$

(Forrest (2010)). ♦

In this paper we consider PII and PDD as different versions of the same principle. Indeed, PDD is often better suited for the purposes of this paper than PII. Nevertheless, in order to keep matters as simple as possible we will refer to it also as PII.[1]

[1] Leibniz's original formulation of the principle in *Monadology* runs as follows: „There are never in nature two beings which are perfectly alike and in which it would not be possible to find a difference that is internal or founded upon an intrinsic denomination." (Leibniz 1714, section 9).

The status of PII is is controversial, to put it mildly. According to some, it is false, to others, it is trivially true. Still others claim that there are various versions of PII, some of them (trivially) true, others (trivially) false. These diverging convictions indicate that not all people understand the same as “Leibniz’s principle of the identity of indiscernibles”. This is indeed the case. For instance, if one accepts quantification over all properties whatsoever, then no Leibniz principle is needed at all. As Whitehead and Russell put it:

> It should be observed that by „indiscernibles“ [Leibniz] cannot have meant two objects which agree as to *all* their properties, for one of the properties of x is to be identical with x, and therefore this property would necessarily belong to y if x and y agreed in *all* their properties. Some limitation of the common properties necessary to make things indiscernible is therefore implied by the necessity of an axiom. ... [W]e may suppose the common properties required for indiscernibility to be limited to predicates. ... PM (Introduction, p. 57):

More explicitely, in *Principia Mathematica* Whitehead and Russell restricted the domain over which the quantifier in (PII) runs to predicates, excluding more general predicate functions. Also in *Introduction to Mathematical Philosophy* Russell pointed out that a predicate function such as „being identical with an object x“ should not be admitted, since otherwise Leibniz’s principle becomes trivially true. Quite generally, philosophers dealing with Leibniz’s principle are confronted with what may be called the „limitation problem“, i.e., the problem of giving criteria how to restrict the domain of properties to be taken into account for Leibniz’s principle.

Russell and Whitehead’s symbolic approach copes with the “limitation problem” by restricting the admissible properties in PII to those properties that corresponded to symbols of a certain level.

The problem is, of course, what is to be understood by „intrinsic denomination“. In this paper we interpret „intrinsic denomination“ as „monadic property“. This is sometimes called the „strong“ version of the principle. For other interpretations of Leibniz’s principle that take into account not only 1-place predicates, but also n-ary relations, see Saunders (2003), Caulton and Butterfield (2012).

In this paper properties will not be represented by symbols of some kind, our approach is a topological approach. According to it, admissible properties for Leibniz's principle are those that can be represented as certain well-formed subsets of geometrically or, more generally, topologically structured conceptual spaces. The topological account uses topological structures for determining the domain of properties over which the quantifier of Leibniz's principle is to run.

Although there is virtual unanimity among philosophers that for a non-trivial PII the domain of properties over which the universal quantifier runs has to be restricted in some way or other, it is far from clear, how this should be done. Thus it is not very surprising that quite a few versions of PII are discussed in the literature - each characterized by its own specific class of admissible properties. For instance, in Rodriguez-Pereyra (2006), there are distinguished not less than four different principles (see below). He argues that on pain of trivialisation identity must be excluded from PII. More precisely Rodriguez-Pereyra proposes to distinguish four different versions of PII:

(1.3) Different Versions of Leibniz's Principle PII (Rodriguez-Pereyra 2006).

PII1: No two (different) things share all their intrinsic properties.

PII2: No two (different) things share all their pure properties.

PII2.5: No two (different) things share all their non-trivialising properties.

PII3: No two (different) things share all their properties. ♦

The concepts of "intrinsic", "pure", and "non-trivialising properties" are defined in such way that the following chain of implications holds:

(1.4) $\quad$ PII1 $\Rightarrow$ PII2 $\Rightarrow$ PII2.5 $\Rightarrow$ PII3

The version PII3, without any restriction on the admissibility of properties, is considered as trivial. This sounds plausible, since PII3 allows to quantify over the properties of identity. According to Rodriguez-Pereyra, philosophers have typically claimed that PII1, PII2, and PII3 are the only versions of PII. Against this standard wisdom, he claims that there is still another (non-trivial) version of PII, namely, PII2.5. In the following we show that PII2.5 is to be considered as a red herring. There are many different versions of PII defined by different topological structures.

Natural, or more generally, relevant, properties, which should carve nature at its joins, as is often said, have to be sparse (cf, Lewis (1986, Chapter 1.5, pp. 66 - 67). The properties defined by a "good" topology turn out to be sparse in that sense. More precisely we show that topologically defined properties are sparse and stable properties (cf Duhem (1906), p. 143).[2]

According to Russell, PII is a metaphysical principle that may well be false, depending on the nature of the world, to which it refers:

> There might quite well, as a matter of abstract logical possibility, be two things which had exactly the same predicates, in the narrow sense in which we have been using the word "predicate." ... In the actual world there seems no way of doubting its empirical truth as regards particulars, owing to spatio-temporal differentiation: no two particulars have exactly the same spatial and temporal relations to all other particulars. But this is, as it were, an accident, a fact about the world in which we happen to find ourselves. Russell (1919, 193)

Russell's sense for "abstract logical possibilities" is confirmed by contemporary quantum theory. According to French and others PII turns out to be wrong for the quantum domain (cf. French and Rickles (2009), Caulton and Butterfied (2012)).

Moreover, there may be different versions of the PII, since one may argue for different sets of admissible properties over which the quantifiers of Leibniz's principle are assumed to run. These conceptual possibilities do not exclude, of course, that there are

[2] The exclusion of the so-called properties of identity is not unanimously accepted: for instance, Evans in his argument against vague existence relies on this kind of properties to refute vague existence. He relied on the property "vaguely identical".

philosophers, who reject any restriction of the domain of properties over which the quantifier PII might range. Williamson (2014) argues that any good version of Leibniz's principle should quantify over all properties, "however unnatural" they are (ibid., 226). One may deny that such a "strong logic" provides a "good version" of Leibniz's principle, rather, one may consider such a strong version as a trivial and uninteresting one (cf. Gonzalez-Pereyra 2006).

The question is, what does it mean to be an "interesting" version of PII? A vague answer is that an interesting version must not be too strong (and therefore trivially true) nor too weak (and therefore false). This answer is certainly not fully satisfying. In this paper we argue that topology may serve as a framework for formulating a wealth of possible restrictions for "reasonable" classes of admissible properties that lead to a variety of Leibniz's principles. PII becomes a hypothesis that may be wrong in one possible world and true in another.[3]

The outline of this paper is as follows: To set the stage, in the next (preparatory) section we define what property systems are to be. For every property system one may set up its own specific version of PII. In the section 3 we recall the rudiments of set-theoretical topology. Particularly, we deal with some of the better known separation axioms (cf. Steen and Seebach (1978)). These axioms describe the various possibilities of how different points (objects) may differ topologically. As we will show these different possibilities directly translate into different ways of how objects may differ in the properties they instantiate.

[3] Gärdenfors's geometrical characterization of admissible properties in terms of the geometrical concept of convexity may be considered as a very special case of the topological account (cf. Gärdenfors (2014)). Indeed, the notions of topology and convexity are closely related. For an authoritative detailed presentation of the relations between the mathematical theories of topological and convex structures, see (van de Vel (1993)). For the moment, we need not go into this in more detail, see section 5.

The very key observation of this paper is that PII is nothing but the topological separation axiom T_0 in disguise.[4] More precisely, PII is just T_0 for appropriately defined topological properties. This fact suggests the following train of thought: The axiom T_0 is just the simplest member of a large (and still growing) family of separation axioms $\{T_i: 0 \leq i\}$, it therefore seems plausible to assume that at least some of the various other separation axioms T_i $(i > 0)$ may give rise to various novel Leibnizian principles that deserve to be studied in metaphysics. The main aim of the central section 3 of this paper is to show that this is indeed the case.

In section 4 we point out that topologically defined properties can be interpreted as stable properties that are tolerant with respect to sufficiently small alterations of the objects that instantiate them. Already Duhem and Poincaré argued that stable properties are an epistemologically particularly important class of properties: Stability requires that, if an object x has a property a, then an object y very similar to x should also have the property a. Otherwise it cannot reliably be determined whether x i instantiates a or not. This is indeed the case for topologically defined properties: if an object x has property a ($x \in a$) there is always an open neighborhood U(x) of x such that $y \in U(x) \subseteq a$, i.e., the property a is stable with respect to sufficiently small variations of x.[5]

Another nice feature of topologically defined properties is that for "good" topologies identity properties such as "the property of being identical with the object x" for $x \in X$ fail to qualify as topologically acceptable "stable" properties.

In section 5 we point out that topological structures should not be considered as having a monopoly on defining "good" or admissible properties. There may well be other structures that serve this purpose. A case in question are convex structures that have been used by Gärdenfors and others to characterize "good" or "natural" properties as

[4] Or, more precisely, its logically equivalent transposition PDD.

[5] For some further steps toward a topological epistemology, see Schulte and Juhl (1996).

represented by convex sets of conceptual spaces (cf. Gärdenfors (2014)). Indeed, Gärdenfors's conceptual spaces approach implicitly relies on elementary topological considerations. It is expedient to make the relations between topological and convex structures more explicit, or so we will argue in this section. In section 6 we conclude with some general remarks on the prospects of "topological" epistemology and metaphysics.

2. Property Systems. Properties are essential ingredients of Leibniz's principle of identity. Hence it is expedient first of all to clarify the concept of "system of properties" to be used in this paper.

Properties are gregorious creatures, they do no come in isolation but as systems of concepts. A system of properties is assumed not to be just a set of properties, but rather a set of properties *cum* entailment relation $\leq$. If a and b are properties, then $a \leq b$ obtains if having the property a entails having the property b, or, equivalently, if an object x instantiates the property a then x also instantiates the property b.

If D is a system of properties for which an entailment relation $\leq$ is defined, then $(D, \leq)$ should have at least the structure of a partial order, i.e., $\leq$ should be reflexive, transitive, and anti-symmetric. But this does not suffice. Property systems should have at least the structure of distributive lattices $(D, \leq)$, the bottom element 0 of which is to be interpreted as the "impossible property" that no possible object instantiates, and the top element 1 as the property that every possible object instantiates, the partial order relation $\leq$ is to be interpreted as entailment, i.e., $a \leq b$ holds iff an object g has the property a it also has the property b.

Property systems should support the logical operations of conjunction and disjunction of properties. That is to say, if a and b are properties, there should be properties $(a \wedge b)$ and $(a \vee b)$ that satisfy the familiar features adscribed to the operators $\wedge$ (AND) of conjunction and $\vee$ (OR) of adjunction in standard logic, in particular, the distributive laws should hold for them:

<u>(2.1) Definition</u>. Let $(D, \leq)$ be a lattice. D is a distributive lattice if and only if for a, b, c $\in$ D the following equations hold:

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.♦

<u>(2.2) Definition</u>. Let $(D, \leq)$ be a distributive lattice of properties. A subset $P \subseteq D$ is a (proper) filter iff $0 \notin P$ and it satisfies the following two conditions:

(1) P is an upper set, i.e., if $a \in P$ and $a \leq b$ then $b \in P$.

(2) If a, $b \in P$, then $a \wedge b \in P$.

(3) A filter P is a prime filter iff $a \vee b \in P$ entails that either $a \in P$ or $b \in P$. The set of filters of D is denoted by Filt(D), and the set of prime filter by PrimeFilt(D).

(4) A filter is P called a maximal filter iff there is no (proper) filter Q strictly larger than P. ♦

Now assume that g be an object and $P(g) \subseteq D$ a set of properties attributed to g. It is natural to ask which requirements P(g) should satisfy in order to be a plausible candidate for the set of properties instantiated by an object g? The following requirements for P(g) seem plausible:

<u>(2.3) Definition</u>. Let $(D, \leq)$ be a distributive property lattice and G be a set of objects. A property distribution (G, f, D) is a function $G \xrightarrow{f} \mathrm{Filt}(D)$ that associates with $g \in G$ a prime filter $f(g) \subseteq D$. The set f(g) is to be interpreted as the set of properties attributed to g by the distribution f.♦

This definition is motivated by the following considerations:

(1) Assume $a \in D$ and $a \leq b$ and $a \in f(g)$. Then $a \leq b$ means that any object that instantiates a also instantiates b. Hence $b \in F(g)$. In other words, if $f(g)$ is to be considered as the set of properties instantiated by an object g it should be an upper set.

(2) If g instantiates the properties a and b it should instantiate the conjunctive property $a \wedge b$. Hence, if $f(g) \subseteq D$ is to be the set of properties of an object g it has to be a filter, i.e., $a, b \in f(g)$ entails $a \wedge b \in f(g)$.

(3) Finally, in order to be a plausible candidate of a property set of an object g, has to be a prime filter of D, assume that g has the property $a \vee b$. Then, according to our familiar (classical) understanding of the adjunction "v" an object g that has the property $(a \vee b)$ should have either the property a or the property b.♦

After these preparations we can formulate a Leibnizian principle of identity of indiscernibles as follows:

<u>(2.4) Definition</u>. Let (G, f, D) be a property system in the sense of (2.3). A Leibnizian principle of indistinguishability of indiscernibles PII holds for (G, f, D) iff the following equivalence holds:

(PII) $(g)(g')((a)(a \in f(g) \Leftrightarrow a \in f(g')) \Rightarrow (g = g'))$, $(g, g' \in G, a \in H)$

Informally stated, PII holds iff, when two objects g and g' instantiate the same properties of G via the property distribution f, then they are equal, and vice versa.♦

For the purposes of this paper it is more expedient to deal with the logically equivalent transposition PDD of PII:

(PDD) $(g)(g')(g \neq g' \Rightarrow \exists a(a \in f(g) \ \& \ a \notin f(g') \text{ OR } a \in f(g') \ \& \ a \notin f(g))$

Informally expressed: If the objects g and g' are not equal, then there is a property a such that either g has a and g' does not have a, or g' has a and g does not have it. In the rest of this paper (PII) and (PDD) are identified.

Now we are going to show that topology yields a wealth of property distributions that may or may not satisfy Leibniz's PII. Moreover, taking into account the specific structural features of the topology considered, one can define various Leibnizian principles of the identity of indiscernibles. It will turn out that the familiar principle PII is just the simplest and weakest member of a large family of principles of the identity of indiscernibles.

3. Property Distributions defined by Topology. In this section we recall the rudiments of topology that are necessary for a novel topological interpretation of Leibniz's principle PII. Our aim is to show that topology gives rise to a variety of property systems (in the sense of section 2) for which a variety of non-trivial Leibniz principles of can be defined. Depending on the specific structures of these topologies the corresponding principles PII turn out to be valid or not. Since topology, in contrast to elementary logic and set theory, not yet belongs to the philosopher's standard tool kit, let us start right-on with some basic definitions:

(3.1) Definition. Let X be a set with power set by PX. A topological space is a relational structure (X, OX) with $OX \subseteq PX$ satisfying the following requirements:

(1) $\emptyset, X \in OX$.

(2) Finite intersections and arbitrary unions of elements of OX are elements of OX.♦

If X has more than one element many different topological structures OX exist on X. In particular, there are two extreme topological structures (X, O_0X) and (X, O_1X) defined by

$$O_0X := \{\emptyset, X\} \qquad \text{and} \qquad O_1X := PX.$$

With respect to set-theoretical inclusion $\subseteq$ all topological structures (X, OX) on X lie between these two topologies:

$$O_0X \subseteq OX \subseteq O_1X.$$

Now we are going to show that topological spaces (X, OX) define in a natural way property systems in the sense of (2.3). We only need one further definition:

<u>(3.2) Definition</u>. Let (X, OX) be a topological space, $x \in X$. The neighborhood system N(x) of x is defined by $N(x) := \{a; x \in a \text{ and } a \in OX\}$.♦

<u>(3.3) Proposition</u>. Let (X, OX) be topological space. For every $x \in X$ denote by N(x) the set of open subsets of X that contain x. The the map $X \xrightarrow{f} Filt(O(X))$ defined by $f(x) := N(x)$ is a property distribution for the elements $x \in X$, i.e., N(x) is a prime filter.

<u>Proof</u>: First we have to show that N(x) is a filter for all $x \in X$ on OX. Clearly N(x) is an upper set, i.e., if $x \in a \in N(x)$ and $a \subseteq b \in OX$, then $x \in b$. Hence $b \in N(x)$. If $x \in a$ and $x \in b$ then $x \in a \cap b$, therefore $a \cap b \in N(x)$, i.e., N(x) is a filter. To show that N(x) is a prime filter assume that $a \vee b \in N(x)$. This means that $x \in a \vee b = a \cup b$. Hence $x \in a \cup b$, and therefore $x \in a$ or $x \in b$. Hence $a \in N(x)$ or $b \in N(x)$. Thus N(x) is a prime filter.♦

One may ask whether the N(x) ar not only prime filters, but even maximal filters. Elementary examples, already available for Euclidean spaces (E, OE) show that in general the neighborhood filters N(x) are not maximal filters, i.e., the N(x) are not complete property distributions for the objects x. That is to say it may happen that two different objects x and y can be distinguished by their property distributions N(x) and N(y), respectively, but nevertheless, there may be properties v and w of x and y, not belonging to N(x) and N(y), respectively. On the other hand, it is well known that for distributive lattices maximal filters are prime (cf. Johnstone 1982, (Chapter I, 2.4, Corollary, p.14))

If the property system (D, $\leq$) is not a Boolean system, but, as is generally the case for topological systems, only Heyting, then there may be an object g and a property a such that g neither instantiates a nor the Heyting complement a* of a, i.e., a, a* $\notin$ f(g). If this happens, g may be considered as a borderline case of a as is suggested by the special case when H is the algebra OX of open subsets of a topological space (X, OX). In this case one has a partition of the set X by X = a $\cup$ a* $\cup$ bd(a) with x $\in$ bd(a). By definition of bd(a) every open neighborhood U(x) of x has a non-empty intersection with a. Hence in every neighborhood of x there is a y $\in$ U(x) $\cap$ a. Therefore x may be considered as a borderline case of a.

<u>(3.4) Definition</u>. Let (X, OX) be a topological space with the topological property system (X, n, Filt(OX)), defined by f(x) := N(x), N(x) the neighborhood system of x. Then (X, n, Filt(OX)) satisfies PII iff

(PII) $(x)(x')(x \neq x' \Rightarrow \exists a(a \in N(x) \ \& \ a \notin N(x') \text{ OR } a \in N(x') \ \& \ a \notin N(x)).$ ♦

It is natural to ask for which topologies OX the canonical property distribution (X, n, Filt(OX)) satisfies (PII).[6]

Let us consider first the trivial topologies O_0X and O_1X. It is easily checked that PII is false for OX = O_0X, and that PII holds for OX = O_1X. This is not very exciting. The question of whether PII holds or does not hold becomes more interesting, when we consider some non-trivial topology OX strictly between O_0X and O_1X. As can be easily checked by examples of finite topological spaces (X, OX) indeed some of the inter–

[6] In order to avoid trivialities, in the following it is assumed throughout that the cardinality of X is strictly larger than 1.

mediate topological structures strictly between O_0X and O_1X do satisfy PII, and others do not. Before this question is treated in full generality, we may note that the real line **R** endowed with the standard Euclidean topology O**R**, does satisfy PII:

<u>(3.5) Example</u>. The standard Euclidean topology O**R** of the real line **R** is generated by open intervals (a, b) := {x; a < x < b}. Two distinct points a and b can be separated by appropriate small open intervals U(a) and U(b) that are disjoint to each other. Hence the property distribution (**R**, n, Filt(O**R**)) satisfies PII.♦

Asking whether a topological space (X, OX) satisfies the topological Leibniz principle (3.4) may appear, at first view, to be a somewhat contrived hybrid of a philosophical and a topological question. Actually, it is not. Rather, this question has attracted conside–rable interest among topologists since the beginnings of topology more than one hundred years ago (cf. Steen and Seebach (1978), or any textbook on topology). More precisely, we will show in a moment that the validity of PII in the sense of (3.4) is equivalent to the simplest topological separation axiom T_0:

<u>(3.5) Definition</u>. Let (X, OX) be a topological space, $x, y \in X$, and $a \in OX$. The topological space (X, OX) is a T_0-space iff there exists an open set $a \in OX$ such that either $x \in a$ and $y \notin a$, or $x \notin a$ and $y \in a$.♦

The axiom T_0 is often considerated as the minimal requirement that "good" topological spaces have to satisfy. That is to say, a topological space (X, OX) should at least satisfy T_0 if it is to be considered as a honest topological space. To obtain more specific results, however, one has to assume the validity of certain axioms T_i ($i > 0$) stronger than T_0 (Steen and Seebach Jr. (1978)).

The following proposition opens the gate for constructing a close connection between topological separation axioms and Leibnizian principles of the identity of indiscernibles:

<u>(3.6) Proposition</u>. Let (X, OX) be a topological space and (X, n, Filt(OX)) the corresp–onding property distribution. (X, OX) satisfies the separation axiom T_0 if and only if the topological property distribution (X, n, Filt(OX)) satisfies Leibniz's law PII.

<u>Proof</u>. The proof is just a matter of checking the pertinent definitions. Let (X, OX) be a T_0 –space, and x, y $\in$ X two distinct points. By definition either there is an open set a $\in$ OX such that x $\in$ a and y $\notin$ a, or there is an open set b $\in$ OX such that x $\notin$ b and y $\in$ b. This is to say that either there is a property a in n(x) that is not in n(y), or there is a property b in n(y) that is not in n(x). This is equivalent with the assertion that there is a property that distinguises x from y, see (1.2). In other words, PII is valid. On the other hand, from the assumption that PII is valid it immediately follows that (X, OX) satisfies T_0.♦

Proposition (3.6) offers a novel perspective on Leibnizian principles of identity: The equivalence of PII and T_0, and the fact that T_0 is just the first member of a large (and still growing) family of separation axioms T_i, suggests that not only T_0 but other members T_i of the family of separation axioms as well can be interpreted in terms of (generalized) Leibnizian principles of identity of indiscernibles. This is indeed the case as will shown in some detail in the following. We treat in more detail only the separation axioms T_1 and T_2. The axiom T_0 is sometimes called the Kolmogoroff axiom, T_1 the Frechet axiom, and T_2 is usually called the Hausdorff axiom.

<u>(3.7) Definition</u>. Let (X, OX) be a topological space, and x, y $\in$ X, a $\in$ OX. The topological space (X, OX) is a T_1-space iff there exist open sets a, b $\in$ OX such that x $\in$ a, and y $\in$ b, such that x $\notin$ b and y $\notin$ a. ♦

(3.7)* Definition. Let (G, f, D) be a property distribution. Then (G, f, D) satisfies the Leibnizian principle ($PIIT_1$) iff for all distinct objects g and g' there is either a property a $\in$ f(g) that is not contained in f(g') or there is a property b $\in$ f(g') that is not in f(g). ♦

(3.7)** Proposition. A topological space (X, OX) is a T_1-space iff the topological property distribution (X, n, Filt(OX)) satisfies ($PIIT_1$). ♦

(3.8) Definition. Let (X, OX) be a topological space, x, y $\in$ X, and a $\in$ OX. The topological space (X, OX) is a T_2-space iff there are disjoint open sets a, b $\in$ OX such that x $\in$ a and y $\in$ b. ♦

(3.8)* Definition. Let (G, f, D) be a property distribution. Then (G, f, D) satisfies the Leibnizian principle ($PIIT_2$) iff for all distinct objects g and g' there are properties a and b, respectively such that g has property a and all elements that that have property a lack property b, and g' has property b and all elements that have b lack property a. ♦

(3.8)** Proposition. A topological space (X, OX) is a T_2-space. iff the topological property distribution (X, n, Filt(OX)) satisfies ($PIIT_2$). ♦

The principles ($PIIT_0$), ($PIIT_1$) and ($PIIT_2$) are not equivalent. This is seen by constructing appropriate topological spaces (X, OX) that satisfy only one of these axioms T_0, T_1, and T_2 , respectively. Thereby one can show that T_0, T_1, and T_2 are non-equivalent, more precisely, the following strict chain of implications holds:

$$T_2 \Rightarrow T_1 \Rightarrow T_0$$

(see Steen and Seebach Jr. (1978) or any textbook of topology). This entails that the corresponding chain of identity principles ($PIIT_i$) holds:

$$PIIT_2 \Rightarrow PIIT_1 \Rightarrow PIIT_0 = PII$$

To be sure, not only T_1 and T_2 may be used to generate novel Leibnizian identity principles but any other of the many separation axioms T_i may be used for this purpose. Separation axiom T_i determine the behaviour of topological spaces to a large extent. It seems plausible to conjecture that the corresponding principles of identity ($PIIT_i$) determine the metaphysical features of the worlds for which they hold. Recently, in particular "weak separation axioms" between T_o and T_1 have been studied extensively (cf. REF, REF). To indicate their logical position they have labeled by appropriate rational numbers such $T_{1/4}$, $T_{1/3}$ $T_{1/2}$, etc. so that a strict implicative chain of separation axioms

(3.9) $$T_0 \Leftarrow T_{1/4} \Leftarrow T_{1/3} \Leftarrow T_{1/2} \Leftarrow T_1 \Leftarrow T_2 \Leftarrow T_{2,5} \ldots$$

exists to which a strict implicative chain of identity principles of indiscernibles

(3.9)* $$PII = PIIT_0 \Leftarrow PIIT_{1/4} \Leftarrow PIIT_{1/3} \Leftarrow PIIT_{1/2} \Leftarrow PIIT_1 \Leftarrow PIIT_2 \Leftarrow PIIT_{2,5}$$

corresponds. It would be an interesting philosophical question which identity principle under which metaphysical assumptions is a plausible principle and which is not. For reasons of space this topic cannot be treated in this paper. Rather, we are content to state that the several principles are indeed non-equivalent and therefore offer a realm of possibilities.

<u>4. Stable Properties</u>. From an empirical point of view, a meaningful property must be stable under slight changes of the circumstances under which it may obtain, otherwise it cannot be reliably measured and makes no empirical sense. After all, it is well-known that exactly the same experiment can never be repeated due to the approximative character of all our experimental settings and all our measurements. More generally, only properties that do not completely depend on the specifics of a singular situation can be

intersubjectively valid and epistemologically meaningful. Conceiving properties as represented by the open set of an appropriate topology takes into account this fact quite naturally: If an object x has the open property a, i.e., $x \in a$, then all objects y that are sufficiently similar to x, i.e., that are located in a sufficiently small neighborhood U(x) of x, also instantiate the property a since $U(x) \subseteq a$. The problem of stability is not an artefact of the elusive concept of topology, it may be formulated mathematically rigorous by the following well-known example:

> The mathematician may ask himself the following question: The positions and velocities of the bodies forming the solar system being what they are today, will they all continue indefinitely to turn around the sun? Will it not, on the contrary, probably come about that one of these bodies will finally escape from the swarm of its companions and get lost in the immensity of space? This question constitutes the problem of the stability of the solar system .... (Duhem 1991 (1906), p. 142)

The stability of concepts is generally relevant for (scientific) knowledge, it concerns the relevance and usefulness of mathematical (and more generally of exact) arguments, as was already emphasized by Duhem:

> Indeed, a mathematical deduction is of no use to the physicist so long as it is limited to asserting that a given rigorously true proposition has for its consequence the rigorous accuracy of some such other proposition. To be useful to the physicist, it must still be proved that the second proposition remains approximately exact when the first is only approximately true. (Duhem (1991, p. 143)

Discussing a fascinating example of „an example of mathematical deduction that can never be utilized“ (ibid., p. 138), due to Hadamard, Duhem points out that the exact deductions of ordinary mathematics are of limited use in empirical science. To cope with the inevitable vagueness of our experiments and measurements we have to rely on what he calls „mathematics of approximation“, and admonishes his readers:

> But let us not be deceived about it; this „mathematics of approximation“ is not a simpler and cruder form of mathematics. On the contrary, it is a more thorough and more refined form of mathematics ... . (Duhem 1991, p. 143)

Duhem is certainly correct about the difficulty of the “mathematics of approximation”. The introduction of topological structures can only be a first step. But at least open properties are tolerant with respect to small changes. They satisfy a kind of the tolerance principle (or continuity) in the sense that if an object x instantiates a then there is a small neighborhood U(x) of x such that all objects contained in U(x) also instantiate a. In other words, a topologically defined property a is tolerant with respect to sufficiently small changes. This tolerance or stability is built into the concept of open sets whereby topology is a particularly apt device for modeling epistemically relevant (and irrelevant) properties. For instance, the nowhere dense properties can be characterized as spurious properties since they are represented by nowhere dense sets.[7] Approximativity and vagueness-tolerance are important for scientific knowledge and ordinary cognition as well. The recognition of this fact amounts to saying farewell to a venerable philosophical dream, as Gärdenfors recently put it:

> Philosophers since Leibniz have dreamed of constructing a precise language where all vagueness is eliminated, where „every misunderstanding should be nothing more than a miscalculation“ and where it would suffice for scien–tists „to take their pencils in their hands, to sit down to their slates, and to say to each other ... let us calculate.“ Vagueness is, however, not a bug but a design feature of language. ... There are good reasons ... why language contains vague terms (Gärdenfors 2014, 44/55)

In the next section we will deal with Gärdenfors’s account in some more detail. In the rest of this section we deal with some aspects of the topologically defined stability of concepts in general.

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[7] A subset of a topological space (X, OX) is nowhere dense iff intcl(A) = Ø, see Schulte and Juhl (1996).

First, topological stability as defined above enjoys some nice "logical" features due to the fact that the set OX of open sets of a topological structure (X, OX) is a (complete) Heyting algebra: if a and b are stable properties then their conjunction $a \wedge b$ and their adjunction $a \vee b$ are stable properties. This renders the realm of stable properties a logically or algebraically very well behaved domain.

Another philosophically relevant virtue of topologically defined properties is that identity properties are naturally excluded from the domain of admissible properties. Or, more precisely, identity properties only occur for very rare topological structures. Here, it should be noted, "rare" is defined in purely topological terms. This avoids the undesired feature that the principle of the identity or indiscernibles PII becomes trivially true. A compelling topological argument against the admissibility of identity properties goes as follows. Consider an identity property such as „being identical with the object x". By definition there is only one object that has the property of being identical with x, namely, x itself. In other words, if this property were accepted, the singleton {x} – as a haecceitist property of the object x - would have to be open with respect to the topological structure OX.[8] Given a topological space (X, OX), a point $x \in X$ with $\{x\} \in OX$ is called an „isolated point". Topologically spoken, isolated points are singularities, well-behaved „natural" topologies don't possess them. This is suggested by the fact that the familiar topological structures of Euclidean spaces (E, OE) and their direct derivatives such as topological manifolds lack isolated points, and, on the other hand, that for all X the trivial topological space (X, PX) entirely consists of isolated points.[9]

In sum, if we want to exclude universes for which identity properties are admissible topology yields a compelling argument: a universe that allows for identity properties

[8] For detailed discussion of various versions of haeceitism, see Lewis (1986) and Caulton and Butterfield (2012)).

[9] This is, of course, not to say that topological spaces without isolated points are always well-behaved in the light of our topological intuitions.

necessarity has a topology that for good topological reasons can be characterized as unnatural.

Another argument in favor of the thesis that topology may be considered as an organon for singling out "good" properties we show that topologically stable properties can be naturally interpreted in Lewis's account of "world properties".

According to Lewis, a property of an object (for instance, "being a talking donkey") is defined as a function mapping a world w to the set {x; x has property d and x is an object of w}, which succinctly may denoted by $W \xrightarrow{d} \{x;\ x \text{ has property } d \text{ at } w\}$. d(w) := {x; x is an object existing in w that has property d} (Lewis 1986, p. 53). If d is the property of being a talking donkey, then the function d, which assigns to each world w the set d(p) of talking donkeys that exist at that world. A world w is a talking-donkey-world iff $d(w) \neq \emptyset$. An object x that exists at w is a talking donkey iff $x \in d(w.)$ In sum, the world property d is identified with a certain subset of the Cartesian product W x $\{d(w);\ d(w) \subseteq \{x;\ x \text{ exists at } w\}$. Then Leibniz's principle for properties of worlds says that two worlds x and y are identical if and only if they have the same world-properties. For a world w Leibniz's PII says that two things x and y existing at that world are identical iff for all properties $W \xrightarrow{p} p(w)$ one has $x \in p(w) \Leftrightarrow y \in p(w)$.

Now assume that W is endowed with some topological structure OW, such that (W, OW) is a topological space. Then we define:

<u>(4.1) Definition</u>. Let d be a property defined as a function $W \xrightarrow{d} \{x;\ x \text{ has property } d \text{ at } w\}$. Then d is stable at $w \in W$ iff w has an open neighborhood U(w) such that $\forall w' \in U(w)\ (d(w) \neq \emptyset \Rightarrow d(w') \neq \emptyset))$. Informally, the property d is stable at w iff for all sufficiently similar worlds w' the extension of d at w' is empty iff the extension of d at w is empty.♦

Applied to the property of being a talking donkey this means that if w is a talking donkey world iff then every world w' that is sufficiently similar to w is also a talking-donkey word. That is, if there is a talking donkey at w there is a talking donkey at all worlds w' sufficiently similar to w.

5. Beyond Topology. Topology is not the only device to define well-behaved property systems that give rise to non-trivial topological identity principles PIIT. fThere may be other "spatial" structures that may be used for this purpose. Indeed, another, meanwhile well establishd approach to deal with the problem of characterizing "good properties" (and, more generally "good (relational) concepts") geometrically is the approach of *Conceptual Spaces* originated by Gärdenfors (cf. Gärdenfors (2014). According to Gärdenfors, a "good" or "natural property" is to be represented by a convex region of a conceptual space. For the following it is sufficient to consider some elementary examples. Let us assume that the conceptual space C is a Euclidean space (or some derivative of it, e.g., a locally Euclidean manifold such as the well-known Color spindle, (cf. Gärdenfors (2014, p.23), (Gärdenforst (1990)). Then, following Gärdenfors, convex regions as representatives of natural properties in conceptual spaces may be informally defined as follows:

(5.1) Definition (Gärdenfors (2014, 29). A region R is convex means that for any two points x and y in R, all points between x and y are also in R. A natural property is represented by convex region of a conceptual space D.♦

Then his intuitive motivation for the convexity requirement is the following:

> [I]f some objects located at x and y in relation to some domain are both examples of a property, then any object that is located between x and y with respect to the

same domain will also be an example of the property. Although not all domains in a conceptual space may have a metric, I assume that the notion of betweenness is defined for all domains. This will make it possible to apply the thesis about properties generally. (Gärdenfors 2014, 29)

From this elucidation it transpires that the fundamental notion of Gärdenfors's conceptual spaces approach is the concept of "betweenness" that is taken as primitive. Or better, it is implicitly assumed that "betweenness" in general behaves more or less in the same way as it does in the special case of Euclidean space E and its vectorspace structure, where betweenness is defined via the concept of line segment: Given two different points a, b $\in$ E the line interval between a and b is defined as the set of points

(5.2) $$I(a, b) := \{x;\ x = \lambda a + (1-\lambda)b;\ 0 \leq \lambda \leq 1\}.$$ (vector space interval)

Actually, a reasonable notion of betweenness can be defined without reference to any vectorspace structure. For instance, given a metrical space (X, d), one may define the geodesic interval $I_{gd}(a, b)$ of elements between a and b as

(5.3) $$I_{gd}(a, b) := \{x;\ d(a, x) + d(x, b) = d(a, b)\}$$ (geodesic interval)

Geodesic intervals $I_{gd}(a, b)$ heavily depend on the specfics of the metric d. Even topologically equivalent metrics d and d' may yield quite different intervals $I_{gd}(a, b)$ and $I_{gd'}(a, b)$. This is shown by the Euclidean metric d and the Manhattan (or "taxi-driver") metric d' of the Euclidean plane E that are topologically equivalent: As is easily seen the interval $I_{gd'}(a, b)$ of the metric d' is given by the rectangle with vertices a and b and sides parallel to the coordinate axes x and y; in case a and b coincide in one coordinate $I_{gd'}(a, b)$ and $I_{gd}(a, b)$ coincide.

Indeed, as is shown in the general theory of interval structures (see van de Vel 1993 (Chapter I, in particular see §4 and §7), intervals and betweenness may be defined quite independently from metrical structures. For the following, in particular to elucidate the close relation between topological and convex structures, it is expedient to recall the explicit general definition of betweenness:

<u>(5.3) Definition (van de Vel 1993, 4.1, p.71).</u> Let X be a set and X x X —I—>PX be a function with the following properties:

(1) Extensive Law: a, b ∈ X ⇒ a, b ∈ I(a, b).

(2) Symmetry Law: I(a, b) = I(b, a).

Then I is called an interval operator on X and I(a, b) the interval (or line segment) between a and b. If x ≠ a, b and x ∈ I(a, b), then x is said to be (strictly) between a and b. The resulting pair (X, I) is called an interval space. ♦

Evidently, a line segment of a vectorspace V defined by (5.2) defines an interval structure on V. As is shown in the pertinent literature, there is a profusion of interval spaces (X, I) that do not arise from vectorspace structures via (5.2). Up to now, the conceptual space approach has not been very explicit about its formal foundations. Rather, it is content with a rather vague concept of betweenness that is based on some intuitive ideas strongly influenced by the special example of Euclidean intervals. This is not the place to go into the details of the theory of interval spaces, we are content to point out that interval spaces give rise to convex structures in a natural way, whereby a close relation between topological structures and convex structures comes to the fore:

<u>(5.4) Definition</u>. Let (X, I) be an interval space, A ∈ PX. The operator PX——cv——>PX is defined by cv(A) := {z; ∃x, ∃y ∈ A and z ∈ I(x, y)}.

(1) The set A is called convex iff for all a, b ∈ A the interval I(a, b) ⊆ A.

(2) The convex hull cv(A) is defined as cv(A) := {x; ∃a, b ∈ A and x ∈ I(a, b)}.

(3) The set A is called convex iff A = cv(A).

(4) For a finite set F = {p_1, ..., p_n} the convex hull cv(F) is called an n-polytope.♦

One easily proves that the operator cv is a closure operator in the sense of the following definition (cf. (van de Vel 1993, 3):

(5.5) Definition. An operator PX——c——>PX is a closure operator on X iff it satisfies the following requirements:

(Monotone Law) $A \subseteq B$ implies $c(A) \subseteq c(B)$.
(Extensive Law) $A \subseteq c(A)$.
(Idempotent Law) $c(c(A)) = c(A)$.
(Normalization Law) $c(\emptyset) = \emptyset$.♦

The operator cv defined in (5.4) is even a convex closure operator satisfying the further condition

(5.6) Definition (van de Vel 1993, p.3). Let (X, I be an interval space with closure operator cv as defined in (5.4). The operator cv is a convex closure operator iff it satisfies the following condition: If $K \subseteq PX$ is a totally ordered set of convex subsets of X then the union $\cup K$ of K is also a convex set (as defined in (5.4)(3)). A set X endowed with a convex operator cv is called a convex structure (X, cv).♦

Now we are ready to render explicit the close relation between convex structures and topological structures. We only need Kuratowski's well-known result that a topological structure on a set X can be defined not only by the lattice OX of open sets but equivalently also by a topological (or "Kuratowski") closure operator. In more detail this is spelt out as follows: Let (X, OX) be a topological space, and denote the set of set-theoretical complements of elements of OX by CX. Then the topological (Kuratowski) closure operator PX——ct——>PX is defined as $ct(A) := \cap\{B;\ A \subseteq B,\ B \in CX\}$, $A \in PX$. The operator ct is a closure operator in the sense of (5.5). Moreover, it satisfies the following further condition:

(5.7) Definition. An operator PX——ct——>PX is a topological closure operator iff it is a closure operator that satisfies for all A, B ∈ PX the further condition:

$$ct(A \cup B) = ct(A) \cup ct(B).\blacklozenge$$

From (5.6) and (5.7) it transpires that topological and convex structures are conceptually rather similar, both are special cases of closure structures. If both structures are defined on the same underlying set X, it is natural to ask in what sense They may even show up simultaneously on the same underlying set X. An elementary example is Euclidean space E and its canonical topological and convex structures. In is interesting to ask how the topological and the convex structure of E should be related to each other in order to be compatible:

(5.8) Definition. A topological structure (X, OX) and a convex structure (X, cv) on the same set X are compatible iff all polytopes cv(F) (see (5.4.)(4)) of the convex operator cv are closed with respect to the topological closure operator ct defined by OX.♦

As is easily checked the canonical topological and convex structures of Euclidean spaces E are compatible to each other in the sense of (5.8). Browsing through the existing literature of conceptual spaces it transpires that Euclidean space and the Euclidean interval structure is considered by many authors as a generic example (cf. Gärdenfors (2014)).[10]

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[10] More precisely, an essential ingredient of Gärdenfors's account are the so called Voronoi diagrams for conceptual spaces. They are constructed from the interval structures of the conceptual spaces and are to represent systems of "good" concepts. For special interval structures, e.g., for those defined by a Euclidean metric, the cells of the Voronoi diagrams are indeed convex regions. According to Gärdenfors this is crucial for the learnability of concepts since learnability is closely related to the fact that they can be represented by convex regions. Convexity of Voronoi cells may be considered as problematic insofar as for other (topologically equivalent) metrics such as the Manhattan metric, the cells of the corresponding Voronoi diagrams may not be convex anymore.

It seems plausible that good conceptual spaces should be endowed with appropriate convex-topological structure to zeroing-in the class of natural properties. As many examples of conceptual spaces show this is indeed the case.

6. Concluding Remarks. For a set X with more than one point topological structures (X, OX) abound. In order to tap the sources of topology it does not suffice to consider just any topological structure. Particularly, the trivial topologies (X, O_0X) and (O_1X) are quite uninteresting. The point is to find illuminating topologies for one's specific purposes. This is usually not an easy task.

Moreover, often topological structures are not the only candidates for zeroing-in "good" or "natural" properties. This is shown by the conceptual space approach. Conceiving topological structures and convex structures both as closure structures may help draw attention to the fact that they may be combined to obtain stronger results than can be obtained by using only one of them in an isolation. This does not only hold for the task of finding interesting versions of Leibniz's principle of the identity of indiscernibles but also for but also for the more general project of singling out "good" or "natural" properties for a variety of conceptual spaces.

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