An Extension of Heron's Formula to Tetrahedra, and the Projective Nature of Its Zeros

By TIMOTHY F. HAVEL

Retired Research Staff, Massachusetts Insitute of Technology

e-mail: tfhavel@gmail.com

(*Version dated 19 April 2022*)

Abstract

A natural extension of Heron's 2000 year old formula for the area of a triangle to the volume of a tetrahedron is presented. This extension gives the fourth power of the volume as a polynomial in six simple rational functions of the areas of its four faces and of its three medial parallelograms, which will be referred to herein as interior faces. Geometrically, these rational functions are the areas of the triangles into which the exterior faces are divided by the points at which the tetrahedron's in-sphere touches those faces. This leads to a conjecture as to how the formula likely extends to n-dimensional simplices for all $n > 3$. Remarkably, for $n = 3$ the zeros of the overall polynomial constitute a five-dimensional real semi-algebraic variety consisting almost entirely of collinear tetrahedra with vertices at infinite distances from one another. These unconventional Euclidean configurations can be identified with a quotient of the Klein quadric by an action of a group of reflections isomorphic to $\mathbb{Z}_2^4$, wherein four-point configurations in the finite affine plane constitute a distinguished three-dimensional subset. The paper closes by noting that the algebraic structure of the zeros in the finite affine plane naturally defines the associated 4-element, rank-3 chirotope, aka affine oriented matroid.

1. *Introduction*

Heron's formula for the squared area of a triangle is one of the oldest and most celebrated equations in classical Euclidean geometry [**23**, **39**]. It has been extended to n-dimensional simplices for all positive integers n via Cayley-Menger determinants, which similarly give the squared hyper-volumes as homogeneous polynomials in those n-simplices' edge lengths [**2**, **10**, **12**, **17**, **38**, **40**], but with one important difference: For $n = 2$ the 3-point determinant can be written as a product of four signed sums of the edge lengths, which is generally what is meant by "Heron's formula," whereas for $n > 2$ the Cayley-Menger determinants are polynomials in the *squared* edge lengths that do not factorize. As a result, the combinatorial geometry of their zeros is far less transparent than it is with Heron's formula, where one can see at a glance that there are exactly three ways in which a triangle can have an area of zero, depending on which one of its vertices lies on the edge spanned by the other two. Indeed three of the factors in Heron's formula are simply the deviations of the three triangle inequalities among the edge lengths from saturation, meaning from holding as equalities, while the fourth is a non-degeneracy condition that vanishes only if all three vertices coincide.

This paper presents a rather different, but geometrically natural, extension of Heron's formula to the tetrahedron. This extension gives the fourth power of the volume as a polynomial in six simple rational functions of seven areal magnitudes that are canonically associated with each and every tetrahedron. Four of these magnitudes are the areas of the usual four faces of the tetrahedron, while the remaining three are the areas of its *medial parallelograms*. Accordingly, the latter will be referred to herein as "interior faces." As will be shown in section 3 below, the exterior and interior areas together determine a non-degenerate tetrahedron up to isometry. The denominators of all six rational functions are just the exterior surface area of the tetrahedron, while each numerator factorizes into a product of two linear factors, one of which is a non-degeneracy condition and the other of which is the deviation from saturation of an areal generalization of the triangle inequality.

The significance of this extension lies not in providing yet-another means of calculating the volume of a tetrahedron *per se*, but in the rather surprising nature of the geometric insights it yields into all the ways in which a tetrahedron can become "flat." Almost all of the formula's zeros, in fact, correspond to collinear tetrahedra with vertices at infinite distances from one another, although these do not live in the projective completion of three-dimensional Euclidean space as it is usually conceived. The interpretation of these unconventional Euclidean configurations, and what they may have to tell us about the physical space in which we live, are questions of a kind generally seen as too obvious to even think about, and the main purpose of this paper is to challenge that assumption. Readers who doubt that such an exercise might be interesting are invited to consider the following innocent question:

How can the normal vectors of the usual four faces of a tetrahedron be coplanar but not collinear?

To make it subsequently clear that this extension is indeed geometrically natural, let us briefly revisit Heron's formula and its connexion to the in-circle of a triangle. Hence let a, b, c be the lengths of the edges of a triangle $\overline{\mathsf{ABC}}$ opposite to its vertices $\overline{\mathsf{A}}$, $\overline{\mathsf{B}}$, $\overline{\mathsf{C}}$, respectively, and let $s := \frac{1}{2}(a + b + c)$ be its semi-perimeter. Then the deviations of the three triangle inequalities from saturation are

$$u \ := \ \tfrac{1}{2}(-a + b + c) \ , \quad v \ := \ \tfrac{1}{2}(a - b + c) \ , \quad w \ := \ \tfrac{1}{2}(a + b - c) \ , \tag{1·1}$$

where the factor of $\frac{1}{2}$ was introduced so that $a = v+w$, $b = u+w$, $c = u+v$. These deviations have been called the Heron parameters of a triangle [**7**], and clearly determine it up to isometry. The Heron parameters are however not constrained by the triangle inequality, in that any $u, v, w \geq 0$ will yield distances that satisfy all three triangle inequalities among them. Together with $s = u + v + w$, they also enable the squared area of the triangle to be expressed simply as

$$\left|\overline{\mathsf{ABC}}\right|^2 \ = \ s\,uvw \ = \ \tfrac{1}{2}\,(u + v + w)\det\begin{bmatrix} 0 & u & v \\ u & 0 & w \\ v & w & 0 \end{bmatrix} \ . \tag{1·2}$$

Although this compact version of Heron's formula is well-known, the product uvw therein has not previously been viewed as a determinant. Nevertheless, an analogous 4×4 determinant will be found in its extension to tetrahedra.

As illustrated in Fig. 1, the Heron parameters are geometrically the distances from the vertices of the triangle to the *in-touch points* at which its in-circle "touches" its edges. They are also equal to the distances from the vertices to the *ex-touch points* at which the triangle's ex-circles touches its edges, as well as the lines through those edges.

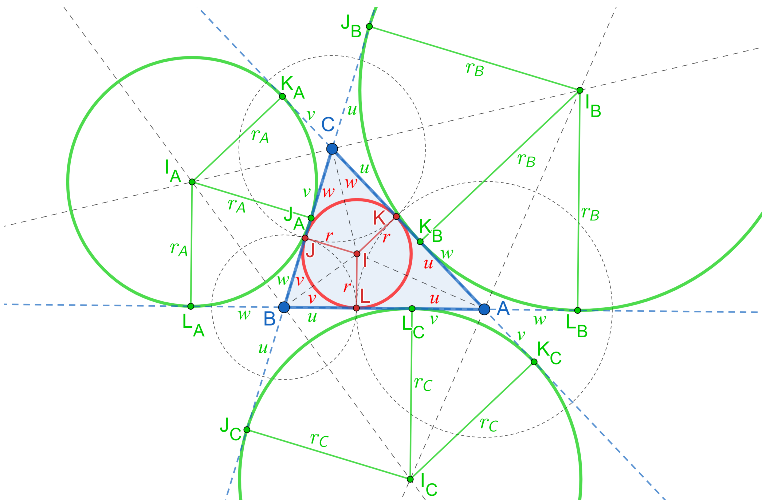

Fig. 1. The in-circle (red) and ex-circles (green) of a triangle (blue), along with the corresponding in-touch and ex-touch points at which they contact the lines through its edges. The distances from the vertices to the in-touch and ex-touch points are labelled by u, v, w in the corresponding colours. (NB: this and all the other figures in this paper were made using the `GeoGebra` dynamic geometry software [**32**].)

Analogously, the aforementioned rational functions in our extension are the areas of the three triangles into which each exterior face of a tetrahedron is divided by its in-touch point. There are twelve such areas but, just as occurs with the Heron parameters of a triangle, these *in-touch triangles* will be found to occur in congruent pairs, giving rise to only six independent areas. These will be referred to as the *natural parameters* of the tetrahedron. Additional parameters which depend on the natural parameters will be defined that are similarly related to the areas of the triangles into which the exterior faces are divided by their respective ex-touch points.

Like the tetrahedron itself, all these parameters are uniquely determined by the exterior and interior areas together. Expressing this geometric fact in algebraic terms will require us to take a bit of a detour through some very basic, though not very widely taught, vector geometry, to which we now turn.

2. *Areal relations from elementary vector algebra*

The nearly trivial relations among the inter-vertex vectors of a tetrahedron $\overline{\mathsf{ABCD}}$,

$$\overrightarrow{\mathsf{AB}} \;=\; \overrightarrow{\mathsf{AC}} + \overrightarrow{\mathsf{CB}} \;=\; \overrightarrow{\mathsf{AC}} - \overrightarrow{\mathsf{BC}} \;=\; \overrightarrow{\mathsf{CB}} - \overrightarrow{\mathsf{CA}} \;=\; \overrightarrow{\mathsf{DB}} - \overrightarrow{\mathsf{DA}}\,, \tag{2·1}$$

are the basis for much of what follows. An immediate consequence is that the cross product of the vectors between any two distinct pairs of vertices can be expanded as e.g.

$$\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}} \;=\; \overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}} - \overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}} \;=\; \overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{CD}} - \overrightarrow{\mathsf{BC}}\times\overrightarrow{\mathsf{CD}} \;=\; \overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}} - \overrightarrow{\mathsf{BC}}\times\overrightarrow{\mathsf{BD}}\,. \tag{2·2}$$

Up to sign, the cross products on the right are of course twice the areal vectors of the faces of the tetrahedron $\overline{\mathsf{ABCD}}$, which are just the outwards pointing normal vectors of those faces weighted by their areas, but what is the left-hand side? It can be viewed as four times the cross product of the vector between the midpoints of the edges $\overline{\mathsf{AC}}$ & $\overline{\mathsf{BC}}$

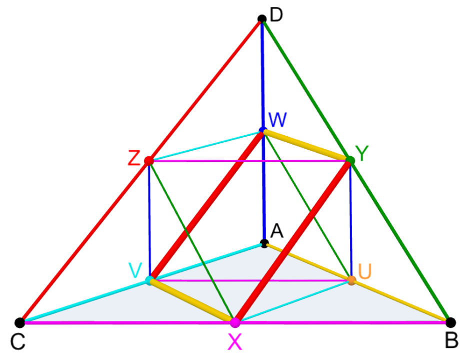

Fig. 2. The medial octahedron of the tetrahedron $\overline{\mathsf{ABCD}}$, with parallel line segments in space all having the same colour. Opposite pairs of edges of the octahedron $\overline{\mathsf{UVWXYZ}}$ have lengths equal to half that of the parallel edge of the tetrahedron, and its volume is half that of the tetrahedron itself. The medial parallelograms formed by pairs of parallel and congruent edges are $\overline{\mathsf{UVZY}}$, $\overline{\mathsf{UWZX}}$ & $\overline{\mathsf{VXYW}}$ (heavy lines); their diagonals $\overline{\mathsf{WX}}$, $\overline{\mathsf{VY}}$ & $\overline{\mathsf{UZ}}$, which correspond to the bimedians of the tetrahedron, were not drawn to reduce clutter.

and the vector between the midpoints of $\overline{\mathsf{AC}}$ & $\overline{\mathsf{AD}}$:

$$\Big(\tfrac{1}{2}\big(\overline{\mathsf{B}}+\overline{\mathsf{C}}\big)-\tfrac{1}{2}\big(\overline{\mathsf{A}}+\overline{\mathsf{C}}\big)\Big)\times\Big(\tfrac{1}{2}\big(\overline{\mathsf{A}}+\overline{\mathsf{D}}\big)-\tfrac{1}{2}\big(\overline{\mathsf{A}}+\overline{\mathsf{C}}\big)\Big) \;=\; \tfrac{1}{4}\,\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\;. \tag{2·3}$$

This is easily seen to be the same (up to sign) as the cross product of the vectors from the midpoint of any one of the edges $\overline{\mathsf{AC}}$, $\overline{\mathsf{AD}}$, $\overline{\mathsf{BC}}$, $\overline{\mathsf{BD}}$ to the midpoints of the other two of those edges sharing a vertex with the first. Thus $\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}$ is four times an areal vector of the medial parallelogram spanned by the midpoints of these four edges. Similar interpretations also hold for the cross products $\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{BD}}$ & $\overrightarrow{\mathsf{AD}}\times\overrightarrow{\mathsf{BC}}$. This is further clarified and expanded upon in Fig. 2.

In the following, the areas of the exterior faces will be denoted by

$$\big|\overline{\mathsf{ABC}}\big| \;=\; \tfrac{1}{2}\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| \;=\; \tfrac{1}{2}\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{BC}}\big\| \;=\; \tfrac{1}{2}\,\big\|\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{BC}}\big\| \tag{2·4}$$

etc., and the areas of the medial parallelograms (aka the interior faces) by

$$\begin{aligned}\big|\overline{\mathsf{AB|CD}}\big| \;=\; \tfrac{1}{4}\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\|\,, \quad \big|\overline{\mathsf{AC|BD}}\big| &\;=\; \tfrac{1}{4}\,\big\|\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{BD}}\big\| \\ \text{and}\quad \big|\overline{\mathsf{AD|BC}}\big| &\;=\; \tfrac{1}{4}\,\big\|\overrightarrow{\mathsf{AD}}\times\overrightarrow{\mathsf{BC}}\big\|\;.\end{aligned} \tag{2·5}$$

Then our first (new?) result is:

PROPOSITION 2·1. *The areas of the interior and exterior faces of a tetrahedron* $\overline{\mathsf{ABCD}}$ *satisfy a system of* 18 *linear inequalities, each of which involves one interior and two exterior faces. These may logically be grouped into six triples, with two triples for each interior face, a typical example of which is:*

$$4\,\big|\overline{\mathsf{AB|CD}}\big| \;=\; \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\| \;\le\; \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| + \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\| \;=\; 2\,\big|\overline{\mathsf{ABC}}\big| + 2\,\big|\overline{\mathsf{ABD}}\big| \tag{2·6 a}$$

$$2\,\big|\overline{\mathsf{ABC}}\big| \;=\; \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| \;\le\; \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\| + \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\| \;=\; 2\,\big|\overline{\mathsf{ABD}}\big| + 4\,\big|\overline{\mathsf{AB|CD}}\big| \tag{2·6 b}$$

$$2\,\big|\overline{\mathsf{ABD}}\big| \;=\; \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\| \;\le\; \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| + \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\| \;=\; 2\,\big|\overline{\mathsf{ABC}}\big| + 4\,\big|\overline{\mathsf{AB|CD}}\big| \tag{2·6 c}$$

Proof. Equation (2·6) follows immediately from the standard triangle inequality for vectors, $\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3 \implies \|\mathbf{v}_1\| \le \|\mathbf{v}_2\| + \|\mathbf{v}_3\|$, applied to the identity given by the first equality in Eq. (2·2) together with the other two identities obtained by swaping terms on the left- & right-hand sides. The remaining five triples of inequalities are obtained simply by permuting the labels $\mathsf{A}, \mathsf{B}, \mathsf{C}$ & D. $\square$

Note that these are inequalities amongst the areas of the parallelograms spanned by the inter-vertex vectors, *not* their lengths. For this reason, although they are technically "triangle inequalities," it seems more appropriate to call them *tetrahedron inequalities*. The deviations of these inequalities from saturation (times 2 or 4 to make the areas therein equal to the cross product norms) will henceforth be denoted by

$$\begin{aligned} \mathcal{T}_1[\mathsf{a},\mathsf{b}] &:= 2\,|\overline{\mathsf{abc}}| + 2\,|\overline{\mathsf{abd}}| - 4\,|\overline{\mathsf{ab|cd}}|\,, && (2{\cdot}7\,a) \\ \mathcal{T}_2[\mathsf{a},\mathsf{b}] &:= 4\,|\overline{\mathsf{ab|cd}}| + 2\,|\overline{\mathsf{abd}}| - 2\,|\overline{\mathsf{abc}}|\,, && (2{\cdot}7\,b) \\ \mathcal{T}_3[\mathsf{a},\mathsf{b}] &:= 4\,|\overline{\mathsf{ab|cd}}| + 2\,|\overline{\mathsf{abc}}| - 2\,|\overline{\mathsf{abd}}|\,, && (2{\cdot}7\,c) \end{aligned}$$

with corresponding non-degeneracy condition $\mathcal{T}_0[\mathsf{a},\mathsf{b}] := 2\,|\overline{\mathsf{abc}}| + 2\,|\overline{\mathsf{abd}}| + 4\,|\overline{\mathsf{ab|cd}}|$, where $\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\}$ with $|\{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\}| = 4$ and $\mathsf{c} < \mathsf{d}$ in alphabetic order.

Remark 2·1. Many additional, albeit weaker, linear inequalities among the seven facial areas can be derived by adding these deviations together, along with a great many more lower bounds on the areas having the form of inverse tetrahedron inequalities. An untypically well-known example is the upper bound on the area of any one exterior face given by the sum of the other three, e.g.

$$|\overline{\mathsf{ABC}}| \;\le\; |\overline{\mathsf{ABD}}| + |\overline{\mathsf{ACD}}| + |\overline{\mathsf{BCD}}|\,, \tag{2·8}$$

along with three others obtained by permuting the vertex labels [**33**, **47**]. These four inequalities are known to be necessary and sufficient for the existence of a tetrahedron exhibiting the given exterior areas [**44**].

The following identity is usually attributed to Hermann Minkowski [**43**].

LEMMA 2·2 (Minkowski's Identity). *The areal vectors of the exterior faces of a tetrahedron* $\overline{\mathsf{ABCD}}$ *(times* 2*) satisfy*

$$\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}} - \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}} - \overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}} \;=\; \mathbf{0}\,. \tag{2·9}$$

Proof. $\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}} = \big(\overrightarrow{\mathsf{BA}} + \overrightarrow{\mathsf{AC}}\big) \times \big(\overrightarrow{\mathsf{BA}} + \overrightarrow{\mathsf{AD}}\big) = \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}} - \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}$. $\square$

This extends to the seven faces together as follows.

PROPOSITION 2·3. *The areal vectors of the exterior faces (times* 4*) are equal to the following signed sums of the areal vectors of the interior faces (also times* 4*):*

$$\begin{aligned} \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}} + \overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}} &= 2\,\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}} && (2{\cdot}10\,a) \\ -\,\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}} + \overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}} &= 2\,\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}} && (2{\cdot}10\,b) \\ \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}} - \overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}} &= 2\,\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}} && (2{\cdot}10\,c) \\ -\,\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}} - \overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}} &= 2\,\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}} && (2{\cdot}10\,d) \end{aligned}$$

Proof. One can prove Eq. (2·10 a) simply as follows:

$$\begin{aligned} &\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}} + \overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}} \;= \\ &\qquad \overrightarrow{\mathsf{AB}} \times \big(\overrightarrow{\mathsf{AD}} - \overrightarrow{\mathsf{AC}}\big) + \overrightarrow{\mathsf{AC}} \times \big(\overrightarrow{\mathsf{AD}} - \overrightarrow{\mathsf{AB}}\big) + \overrightarrow{\mathsf{AD}} \times \big(\overrightarrow{\mathsf{AC}} - \overrightarrow{\mathsf{AB}}\big) \;=\; 2\,\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}} \end{aligned}$$

The proofs of the remaining identities are similar save for Eq. (2·10 *d*), where Minkowski's identity (2·9) is also needed. □

Remark 2·2. Applying the triangle inequality for vectors to these relations shows that the area of each exterior face is bounded from above by the sum of the interior areas. They also show that the areal vectors of the three interior faces (however oriented) determine those of the exterior faces which, as we shall see, suffices to determine the tetrahedron uniquely up to isometries. Finally, they show that the tetrahedron is equifacial if & only if the areal vectors of the interior faces are mutually orthogonal.

We now turn to the trigonometric relations among the areal vectors. While these formulae can only be ascribed to folklore, they are not given explicitly in otherwise comprehensive surveys of tetrahedral geometry from the early 20$^{\text{th}}$ century [**3**, **47**].

LEMMA 2·4 (The Areal Law of Cosines). *Given a tetrahedron* $\overline{\mathsf{ABCD}}$*, the areal vectors of its interior and exterior faces satisfy*

$$\begin{aligned}(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}})\cdot(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}) &= \tfrac{1}{2}\left(\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2\right)\\ &= \|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\cos(\varphi_{\mathsf{AB}})\ , \end{aligned}\tag{2·11}$$

where φ_{AB} *is the dihedral angle between* $\overline{\mathsf{ABC}}$ *&* $\overline{\mathsf{ABD}}$*, along with the five analogous relations obtained by permuting vertex labels.*

Proof. Dotting each side of the first equality in Eq. (2·2) with itself yields

$$\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2 = \|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2-2\,(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}})\cdot(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}})\ ,$$

from which the first line of Eq. (2·11) follows by rearrangement. The second line is just the geometric definition of the vector dot product in terms of the inter-vector angle, here the dihedral angle φ_{AB}. □

LEMMA 2·5 (The Areal Law of Sines). *Given a tetrahedron* $\overline{\mathsf{ABCD}}$*, its edge lengths, the areas of its exterior faces, and its volume satisfy*

$$\begin{aligned}\|\overrightarrow{\mathsf{AB}}\|\,\big|\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big| &= \|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\sin(\varphi_{\mathsf{AB}})\\ &= \tfrac{1}{2}\sqrt{\mathcal{T}_0[\mathsf{A},\mathsf{B}]\,\mathcal{T}_1[\mathsf{A},\mathsf{B}]\,\mathcal{T}_2[\mathsf{A},\mathsf{B}]\,\mathcal{T}_3[\mathsf{A},\mathsf{B}]}\ ,\end{aligned}\tag{2·12}$$

where φ_{AB} *is the dihedral angle as above, along with the five other relations obtained by permuting the vertex labels.*

Proof. The standard vector algebra identity $(\mathbf{p}\times\mathbf{q})\times(\mathbf{p}\times\mathbf{r}) = \big(\mathbf{p}\cdot(\mathbf{q}\times\mathbf{r})\big)\,\mathbf{p}$ implies

$$\big\|(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}})\times(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}})\big\|^2 = \|\overrightarrow{\mathsf{AB}}\|^2\,\big(\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big)^2\ ,$$

from which the first line of Eq. (2·12) is obtained via the usual formula for the norm of a cross product, i.e. $\|\mathbf{p}\times\mathbf{q}\| = \|\mathbf{p}\|\|\mathbf{q}\|\sin(\phi)$ where ϕ is the unsigned angle between $\mathbf{p}$ & $\mathbf{q}$. The second line of Eq. (2·12) then follows from the first line of Eq. (2·11) applied to the dot product in Lagrange's identity,

$$\big\|(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}})\times(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}})\big\|^2 = \|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2 - \Big((\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}})\cdot(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}})\Big)^2,$$

followed by factoring the resulting expression, taking its square-root, and applying the definitions in Eq. (2·7). □

Remark 2·3. It is also possible to derive a law of cosines for the dot product of the areal vectors of an interior and an exterior face, e.g.

$$\left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right) \cdot \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}}\right) \;=\; \frac{1}{2}\left(\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 - \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}}\right\|^2 - \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|^2\right), \tag{2·13}$$

and for the areal vectors of two interior faces, e.g. $\left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}}\right) \cdot \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}}\right) =$

$$\frac{1}{2}\left(\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2 - \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|^2 - \left\|\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right\|^2\right). \tag{2·14}$$

The well-known formula for the volume as 2/3 the area of an interior face $\left|\overline{\mathsf{AB}|\mathsf{CD}}\right|$ times the perpendicular distance between $\overline{\mathsf{AB}}$ & $\overline{\mathsf{CD}}$ (see e.g. Ex. 12 on pg. 91 of Ref. [**3**]) can also be viewed as a kind of areal law of sines, as can the rather lovely formula,

$$\left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}}\right) \cdot \left(\left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}}\right) \times \left(\overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}}\right)\right) \;=\; 2\left(\overrightarrow{\mathsf{AB}} \cdot \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right)\right)^2 . \tag{2·15}$$

Another formula which has also been called the law of cosines for a tetrahedron [**29**, **45**] (and hence our addition of the qualifier "areal" above) is:

LEMMA 2·6. *Given a tetrahedron* $\overline{\mathsf{ABCD}}$*, the areas of and dihedral angles between its exterior faces satisfy:*

$$\begin{aligned}\left\|\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right\|^2 \;=\;& \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|^2 + \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2 \\ & - 2\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|\cos(\varphi_{\mathsf{AB}}) - 2\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|\left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|\cos(\varphi_{\mathsf{AC}}) \\ & - 2\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|\left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|\cos(\varphi_{\mathsf{AD}})\end{aligned} \tag{2·16}$$

Proof. Simply solve Eq. (2·9) for $\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}$, then dot each side with itself and apply Lemma 2·4, taking into account that $\left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right) \cdot \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right) = -\left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AB}}\right) \cdot \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right)$. □

This leads to the algebraic identity that connects the interior and exterior areas.

PROPOSITION 2·7 (Yetter's Identity). *Given a tetrahedron* $\overline{\mathsf{ABCD}}$*, the areas of its interior and exterior faces satisfy*

$$\begin{aligned}&\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|^2 + \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right\|^2 \\ &\qquad = \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{CD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{BD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{BC}}\right\|^2 \end{aligned} \tag{2·17}$$

$$\iff \quad \Xi\left(2\left|\overline{\mathsf{ABC}}\right|, 2\left|\overline{\mathsf{ABD}}\right|, 2\left|\overline{\mathsf{ACD}}\right|, 2\left|\overline{\mathsf{BCD}}\right|, 4\left|\overline{\mathsf{AB}|\mathsf{CD}}\right|, 4\left|\overline{\mathsf{AC}|\mathsf{BD}}\right|, 4\left|\overline{\mathsf{AD}|\mathsf{BC}}\right|\right) = 0\,,$$

where Ξ *is the quadratic polynomial* $\Xi(a, b, c, d, e, f, g) := a^2 + b^2 + c^2 + d^2 - e^2 - f^2 - g^2$ *which, given without arguments, will always refer to the above polynomial in the twice the exterior and four times the interior facial areas.*

Proof. By adding and subtracting $\left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|^2 + \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2$ from the right-hand side of Eq. (2·16) and applying Lemma 2·4, it may be rewritten as

$$\begin{aligned}\left\|\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right\|^2 \;=\;& \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}} - \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}} + \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2 \\ & + \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}} - \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2 - \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right\|^2 - \left\|\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right\|^2 - \left\|\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right\|^2.\end{aligned}$$

But by Eq. (2·2) and its permutations, it is easily shown that the sum and differences of the cross products inside the norms in this equation are four times the areal vectors of the interior faces, whence Eq. (2·17) follows. □

This identity was given as Ex. 17 on pg. 294 of Altshiller-Court's 1935 text [**3**]. More recently, it has been extended by David N. Yetter to a family of identities connecting the "hyper-areas" of the faces & medial sections of simplices in all dimensions [**61**] (hence its attribution to him), although only the tetrahedral case will be needed in this paper.

3. *The areal Gram matrix*

The *areal Gram matrix* at any vertex of a tetrahedron $\overline{\mathsf{ABCD}}$, say $\overline{\mathsf{A}}$, plays a central role in what follows (extensions to n-dimensional spaces of constant curvature may be found in Ref. [**34**]). This is the 3×3 symmetric matrix $\mathbf{G}_{\mathsf{A}}$ consisting of the dot products of twice the outwards-pointing areal vectors of the three exterior faces meeting at the vertex $\overline{\mathsf{A}}$, and as such it is always positive semi-definite. Using the areal law of cosines (Lemma 2·4), this may be written as a matrix of polynomials in indeterminates representing the squared facial areas $F_{\mathsf{ABC}} \leftrightarrow 4\,|\overline{\mathsf{ABC}}|^2, \ldots, F_{\mathsf{AD|BC}} \leftrightarrow 16\,|\overline{\mathsf{AD|BC}}|^2$, namely:

$$\begin{bmatrix} F_{\mathsf{ABC}} & \frac{1}{2}\left(F_{\mathsf{AB|CD}} - F_{\mathsf{ABC}} - F_{\mathsf{ABD}}\right) & \frac{1}{2}\left(F_{\mathsf{AC|BD}} - F_{\mathsf{ABC}} - F_{\mathsf{ACD}}\right) \\ \frac{1}{2}\left(F_{\mathsf{AB|CD}} - F_{\mathsf{ABC}} - F_{\mathsf{ABD}}\right) & F_{\mathsf{ABD}} & \frac{1}{2}\left(F_{\mathsf{AD|BC}} - F_{\mathsf{ABD}} - F_{\mathsf{ACD}}\right) \\ \frac{1}{2}\left(F_{\mathsf{AC|BD}} - F_{\mathsf{ABC}} - F_{\mathsf{ACD}}\right) & \frac{1}{2}\left(F_{\mathsf{AD|BC}} - F_{\mathsf{ABD}} - F_{\mathsf{ACD}}\right) & F_{\mathsf{ACD}} \end{bmatrix} \longleftrightarrow$$

$$\mathbf{G}_{\mathsf{A}} \;:=\; \begin{bmatrix} \left\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\right\|^2 & -\left(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\right)\cdot\left(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\right) & -\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AB}}\right)\cdot\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right) \\ -\left(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\right)\cdot\left(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\right) & \left\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\right\|^2 & -\left(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\right)\cdot\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right) \\ -\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AB}}\right)\cdot\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right) & -\left(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\right)\cdot\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right) & \left\|\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right\|^2 \end{bmatrix} \tag{3·1}$$

Note that the negative signs before the dot products in the entries adjacent to the diagonal are due to the way these cross products are signed in Minkowski's identity (2·9), while the negative sign in the corner entries is due to the swap of the vectors in the cross product of the dot product's first factor that is needed to apply Eq. (2·11) directly.

The determinant of this matrix, henceforth the Grammian at $\overline{\mathsf{A}}$, will be denoted by $\Gamma_F[\mathsf{A}] \leftrightarrow \det(\mathbf{G}_{\mathsf{A}})$, and is a homogeneous cubic polynomial in the squared-area indeterminates F containing 30 terms. Analogous polynomials can be written down for the Grammians at the other three vertices $\Gamma_F[\mathsf{B}], \Gamma_F[\mathsf{C}]$ & $\Gamma_F[\mathsf{D}]$. They are related as follows.

LEMMA 3·1. *Given $F_{\mathsf{ABC}}, \ldots, F_{\mathsf{AD|BC}} \in \mathbb{R}$, the Grammians at the vertices satisfy*

$$\Gamma_F[\mathsf{A}] \;\equiv\; \Gamma_F[\mathsf{B}] \;\equiv\; \Gamma_F[\mathsf{C}] \;\equiv\; \Gamma_F[\mathsf{D}] \mod \hat{\Xi}_F\ , \tag{3·2}$$

where $\hat{\Xi}_F$ is the polynomial Ξ regarded as a linear form in the indeterminates F. Thus when the indeterminates satisfy Yetter's identity $\hat{\Xi}_F = 0$, the Grammians are all equal.

Proof. Using computer algebra, it is easily shown that

$$\begin{aligned}\Gamma_F[\mathsf{B}] \;-\; \Gamma_F[\mathsf{A}] \;=\; \frac{1}{4}\Big(&\left(F_{\mathsf{ACD}} - F_{\mathsf{BCD}}\right)\left(2\,F_{\mathsf{ABC}} + 2\,F_{\mathsf{ABD}} - F_{\mathsf{AB|CD}}\right) \\ &+\; \left(F_{\mathsf{ABC}} - F_{\mathsf{ABD}}\right)\left(F_{\mathsf{AC|BD}} - F_{\mathsf{AD|BC}}\right)\Big)\,\hat{\Xi}_F\ .\end{aligned}$$

Similar results are obtained for $\Gamma_F[\mathsf{C}] - \Gamma_F[\mathsf{A}]$ and $\Gamma_F[\mathsf{D}] - \Gamma_F[\mathsf{A}]$. □

The 2×2 principal minors of $\mathbf{G}_{\mathsf{A}}$ are also of interest, and will be denoted by

$$\Gamma_F[\mathsf{A};\mathsf{B}] \;:=\; \det\begin{bmatrix} F_{\mathsf{ABC}} & \frac{1}{2}\left(F_{\mathsf{AB|CD}} - F_{\mathsf{ABC}} - F_{\mathsf{ABD}}\right) \\ \frac{1}{2}\left(F_{\mathsf{AB|CD}} - F_{\mathsf{ABC}} - F_{\mathsf{ABD}}\right) & F_{\mathsf{ABD}} \end{bmatrix} \;=\; \Gamma_F[\mathsf{B};\mathsf{A}]\ , \tag{3·3}$$

with analogous definitions for $\Gamma_F[\mathsf{A};\mathsf{C}] = \Gamma_F[\mathsf{C};\mathsf{A}]$, $\Gamma_F[\mathsf{A};\mathsf{D}] = \Gamma_F[\mathsf{D};\mathsf{A}]$ and the other minors of the Gram matrices.

LEMMA 3·2. *Given any* $f_{\mathsf{ABC}}, \ldots, f_{\mathsf{AD|BC}} \in \mathbb{R}$, *and letting* $F_{\mathsf{abc}} := f_{\mathsf{abc}}^2$ *&* $F_{\mathsf{ab|cd}} := f_{\mathsf{ab|cd}}^2$ *for* $\mathsf{a}, \ldots, \mathsf{d} \in \{\mathsf{A}, \ldots, \mathsf{D}\}$ *with* $|\{\mathsf{a}, \ldots, \mathsf{d}\}| = 4$, *we have*

$$\Gamma_F[\mathsf{a};\mathsf{b}] \;=\; \tfrac{1}{4}\,\mathcal{T}_{0:f}[\mathsf{a},\mathsf{b}]\,\mathcal{T}_{1:f}[\mathsf{a},\mathsf{b}]\,\mathcal{T}_{2:f}[\mathsf{a},\mathsf{b}]\,\mathcal{T}_{3:f}[\mathsf{a},\mathsf{b}]\;, \tag{3·4}$$

where $\mathcal{T}_{1:f}[\mathsf{a},\mathsf{b}] := f_{\mathsf{abc}} + f_{\mathsf{abd}} - f_{\mathsf{ab|cd}}$, $\mathcal{T}_{2:f}[\mathsf{a},\mathsf{b}] := f_{\mathsf{ab|cd}} + f_{\mathsf{abd}} - f_{\mathsf{abc}}$, $\mathcal{T}_{3:f}[\mathsf{a},\mathsf{b}] := f_{\mathsf{ab|cd}} + f_{\mathsf{abc}} - f_{\mathsf{abd}}$ $(\mathsf{c} < \mathsf{d})$ *are the linear forms corresponding to the deviations of the tetrahedron inequalities from saturation as in Eq. (2·7), and* $\mathcal{T}_{0:f}[\mathsf{a},\mathsf{b}] := f_{\mathsf{abc}} + f_{\mathsf{abd}} + f_{\mathsf{ab|cd}}$ *are those of the associated nondegeneracy factors.*

Proof. Simply expand and compare the left and right-hand sides of Eq. (3·4). □

Note that the four factors in these formulae are analogous to those in Heron's formula, so these minors can likewise be written as 3-point Cayley-Menger determinants, albeit in one interior and two exterior areas rather than inter-vertex distances.

In order to see what these polynomials are geometrically, recall that the squared areas of the exterior faces of a tetrahedron $\overline{\mathsf{ABCD}}$ may be written in terms of its squared edge lengths $D_{\mathsf{AB}} \leftrightarrow |\overline{\mathsf{AB}}|^2$ etc. as 3-point Cayley-Menger determinants, e.g.

$$4\,|\overline{\mathsf{ABC}}|^2 \;\longleftrightarrow\; \Delta_D[\mathsf{A},\mathsf{B},\mathsf{C}] \;:=\; -\,\frac{1}{4}\det\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & D_{\mathsf{AB}} & D_{\mathsf{AC}} \\ 1 & D_{\mathsf{AB}} & 0 & D_{\mathsf{BC}} \\ 1 & D_{\mathsf{AC}} & D_{\mathsf{BC}} & 0 \end{bmatrix}\;. \tag{3·5}$$

The squared areas of the interior faces may also be written as polynomials in the squared edge lengths [**60**] via Cayley-Menger determinants, e.g.

$$16\,|\overline{\mathsf{AB|CD}}|^2 \;\longleftrightarrow\; \Delta_D[\mathsf{A},\mathsf{B}]\,\Delta_D[\mathsf{C},\mathsf{D}] \;-\; \Delta_D[\mathsf{A},\mathsf{B};\mathsf{C},\mathsf{D}]^2 \;=:\; \Delta_D[\mathsf{A},\mathsf{B}\,|\,\mathsf{C},\mathsf{D}]\;, \tag{3·6}$$

wherein the non-symmetric 2-point Cayley-Menger determinant is defined as

$$\Delta_D[\mathsf{A},\mathsf{B};\mathsf{C},\mathsf{D}] \;:=\; \frac{1}{2}\det\begin{bmatrix} 0 & 1 & 1 \\ 1 & D_{\mathsf{AC}} & D_{\mathsf{AD}} \\ 1 & D_{\mathsf{BC}} & D_{\mathsf{BD}} \end{bmatrix} \;\longleftrightarrow\; \overrightarrow{\mathsf{AB}} \cdot \overrightarrow{\mathsf{CD}}\;, \tag{3·7}$$

while $\Delta_D[\mathsf{A},\mathsf{B}] \;:=\; \Delta_D[\mathsf{A},\mathsf{B};\mathsf{A},\mathsf{B}] \;=\; D_{\mathsf{AB}}$ and similarly $\Delta_D[\mathsf{C},\mathsf{D}] \;=\; D_{\mathsf{CD}}$. Note that Eq. (3·6) can also be expressed via an elegant generalization of Cayley-Menger determinants to arbitrary medial sections of simplices developed by István Talata [**58**].

These relations allow us to convert polynomials in the squared facial areas into polynomials in the squared edge lengths by simple substitution.

PROPOSITION 3·3. *Given a Euclidean tetrahedron* $\overline{\mathsf{ABCD}}$, *the Grammians* $\Gamma_F[\mathsf{a}]$ *with* $\mathsf{a} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\}$, *when evaluated at* $F_{\mathsf{ABC}} = 4|\overline{\mathsf{ABC}}|^2, \ldots, F_{\mathsf{AD|BC}} = 16|\overline{\mathsf{AD|BC}}|^2$, *are all equal to* $t^4 := (6|\overline{\mathsf{ABCD}}|)^4$. *The* 2×2 *principal minors of the Gram matrices,* $\Gamma_F[\mathsf{a};\mathsf{b}]$ *with* $\mathsf{a},\mathsf{b} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\}$ *and* $\mathsf{a} \neq \mathsf{b}$, *likewise evaluated at these multiples of the squared areas in* $\overline{\mathsf{ABCD}}$, *are equal to* $|\overline{\mathsf{ab}}|^2\,t^2$.

Proof. On substituting for the six squared areas in $\Gamma_F[\mathsf{A}]$ using Eqs. (3·5) & (3·6), one obtains (preferrably with the aid of computer algebra) the square of the 4-point Cayley-Menger determinant,

$$\Gamma_F[\mathsf{A}]\,\big|_{F=\Delta_D} \;=\; \Delta_D[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]^2 \;\longleftrightarrow\; t^4\;, \tag{3·8}$$

and likewise for the Grammians at the other three vertices. Similarly, on substituting for the squared areas in the 2×2 minor $\Gamma_F[\mathsf{A};\mathsf{B}]$, one obtains

$$\Gamma_F[\mathsf{A};\mathsf{B}]\,\big|_{F=\Delta_D} \;=\; \Delta_D[\mathsf{A},\mathsf{B}]\,\Delta_D[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}] \;\longleftrightarrow\; |\overline{\mathsf{AB}}|^2\,t^2\;, \tag{3·9}$$

with analogous results for the other 2×2 minors of $\mathbf{G}_\mathsf{A}$ as well as those of the Gram matrices at the remaining three vertices. $\square$

Eq. (3·9) is of course just a restatement of the areal law of sines (2·12).

By first computing the volume t using Eq. (3·8) and then the edge lengths $|\overline{\mathsf{ab}}|$ from Eq. (3·9) and its analogues at the other edges, one obtains a simple proof that the areas of the seven faces of a non-degenerate Euclidean tetrahedron determine it up to isometry. This proof was first given, to this author's knowledge, in an unpublished paper posted on a remarkable online Blog by an amateur but dedicated geometer named Billy Don Sterling McConnell, apparently around 2012 (at the time of writing, this Blog was accessible at `http://daylateanddollarshort.com/bloog`). McConnell also noted that this calculation would succeed only if the Grammians were strictly positive and the 2×2 principal minors non-negative (and hence likewise strictly positive), i.e. the Gram matrices were all positive definite. Crane & Yetter subsequently derived the edge lengths from the areas using spherical trigonometry [**15**], but did not carefully identify the conditions the areas must satisfy in order for their calculation to succeed.

Once the edge lengths, however obtained, are available coordinates for the vertices can be computed by standard "multi-dimensional scaling" techniques based on the Gram matrices of the vectors along the edges at any vertex (see e.g. Ref. [**11**, **16**, **28**]). The proof given here instead computes vertex coordinates which reproduce the given areas directly from the areas themselves, without explicitly determining the edge lengths first.

THEOREM 3·4 (B. D. S. McConnell). *The seven real numbers* f_ABC, f_ABD, f_ACD, f_BCD, $f_\mathsf{AB|CD}$, $f_\mathsf{AC|BD}$, $f_\mathsf{AD|BC} \geq 0$ *are equal to the areas of the exterior (times* 2*) and interior (times* 4*) faces of a non-degenerate Euclidean tetrahedron* $\overline{\mathsf{ABCD}}$ *if & only if they satisfy Yetter's identity* $\Xi_f = 0$*, the* 18 *tetrahedron inequalities* $\mathcal{T}_f \geq 0$*, and yield a Grammian at* $\overline{\mathsf{A}}$ *(or any other vertex)* $\Gamma_{f^2}[\mathsf{A}] > 0$*. This tetrahedron is unique up to isometry.*

Proof. The necessity of the stated conditions were established above. To prove sufficiency, note these conditions together with Lemma 3·2 show that the Gram matrix $\mathbf{G}_\mathsf{A}$ computed from the areas via Eq. (3·1) with $F := f^2$ is positive definite by Sylvester's criterion. Hence coordinates for the cross-products it represents are obtained by diagonalizing it as $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, letting $\mathbf{V} := \mathbf{\Lambda}^{1/2}\,\mathbf{U}^\top$, and setting

$$\mathbf{p} := \mathbf{v}_1 \leftrightarrow \overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}} \;, \quad \mathbf{q} := \mathbf{v}_2 \leftrightarrow \overrightarrow{\mathsf{AD}} \times \overrightarrow{\mathsf{AB}} \;, \quad \mathbf{r} := \mathbf{v}_3 \leftrightarrow \overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}} \;, \tag{3·10}$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the columns of $\mathbf{V}$. The dot products among these coordinate vectors will then reproduce the matrix $\mathbf{G}_\mathsf{A}$ exactly. To convert the cross products' coordinates into those of their component vectors, observe first that the cross products of any three vectors $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ are the columns $\mathbf{r}, \mathbf{q}, \mathbf{p}$ of the adjugate matrix $\mathbf{Adj}[\mathbf{b}, \mathbf{c}, \mathbf{d}]$. Thus the well-known fact that the adjugate of the adjugate of a square matrix is the original matrix times its determinant, together with the fact that the determinant of the adjugate of a 3×3 matrix is the square of the determinant of the original matrix, establishes that the coordinates of the vertices of the tetrahedron $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are given by

$$\mathbf{a} \;=\; \mathbf{0}\,, \quad \mathbf{b} \;=\; \mathbf{p} \times \mathbf{q}\,/\,t\;, \quad \mathbf{c} \;=\; \mathbf{r} \times \mathbf{p}\,/\,t\;, \quad \mathbf{d} \;=\; \mathbf{q} \times \mathbf{r}\,/\,t\;, \tag{3·11}$$

where $t = \sqrt{|\det(\mathbf{V})|} = \sqrt[4]{\Gamma_F[\mathsf{A}]} > 0$. Uniqueness follows from the fact that the distances computed from these coordinates, when inserted into Eqs. (3·5), (3·6) and their permutations, reproduce the given squared areas, and the fact that the Jacobian of the mapping from squared distances to squared areas satisfies $\det(\mathbf{J}^\top\mathbf{J}) = 28\,\Delta_D[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]^4 > 0$. The details are omitted here for the sake of brevity (see Appendix D). $\square$

Note that $\Gamma_{f^2}[\mathsf{A}] = 0$ if any tetrahedron inequality saturates, so the given conditions imply $\mathcal{T}_f > 0$. The example $f_{\mathsf{ABC}} = 9$, $f_{\mathsf{ABD}} = 10$, $f_{\mathsf{ACD}} = 17$, $f_{\mathsf{BCD}} = 14$ & $f_{\mathsf{AB|CD}} = \sqrt{261}$, $f_{\mathsf{AC|BD}} = \sqrt{76}$, $f_{\mathsf{AD|BC}} = \sqrt{329}$ shows the Grammians can be negative even when Yetter's identity and all 18 tetrahedron inequalities are strictly satisfied.

Using Yetter's identity, B. D. S. McConnell has also over the span of better than three decades rewritten the polynomial $\Gamma_F[\mathsf{A}]$ in a variety of ingeneous ways to make it symmetric under vertex permutations and look in some sense more like Heron's formula, an exercise he refers to as "hedronometry." The analogies between his formulae and Heron's, however, are not very convincing, and they do not build upon the intimate connexion between Heron's formula and the in-circle of the triangle seen in Fig. 1. This in turn is the basis for the extension of Heron's formula that will now be derived.

4. *A natural extension of Heron's formula to tetrahedra*

The first step towards a formula for the volume of a tetrahedron that can justly be called a *natural* extension of Heron's formula is to find parameters which determine the tetrahedron in the same way that the Heron parameters were shown to determine a triangle in Section 1. Given that the Heron parameters are the lengths of the segments into which the sides of the triangle are divided by the in-touch points of its in-circle (Fig. 1), the obvious thing to do is to take these parameters to be the areas of the *in-touch triangles* into which the in-touch points $\overline{\mathsf{J}}$, $\overline{\mathsf{K}}$, $\overline{\mathsf{L}}$ & $\overline{\mathsf{N}}$ of the in-sphere of a tetrahedron $\overline{\mathsf{ABCD}}$ divide its exterior faces ("M" was reserved for the Monge point, although it plays no role here). This may be illustrated by a construction which parameterizes the set of all non-degenerate tetrahedra as follows:

(i) Choose a sphere of radius r, centered on e.g. the origin, as the in-sphere;

(ii) Choose four non-coplanar points $\overline{\mathsf{J}}$, $\overline{\mathsf{K}}$, $\overline{\mathsf{L}}$ & $\overline{\mathsf{N}}$ on this sphere such that the plane through any two of them and the center of the sphere separates the remaining two points; these four points will become the in-touch points of the tetrahedron (if eliminating rotational redundancy from the parameterization is of concern, this may be done by choosing the first point on say the z-axis, and the second in say the yz-plane);

(iii) Take the planes tangent to the sphere at these four points, and intersect them three-at-a-time to get the vertices $\overline{\mathsf{A}}$, $\overline{\mathsf{B}}$, $\overline{\mathsf{C}}$ & $\overline{\mathsf{D}}$ of the tetrahedron.

The results of this construction, carried out in the `GeoGebra` online dynamic geometry system [**32**], are shown in Fig. 3.

The first item of business is to establish the following:

LEMMA 4·1. *The twelve triangles into which the in-touch points divide the exterior faces of a tetrahedron occur in six congruent pairs, where each pair shares a common edge of the tetrahedron. Moreover, the line segment between each pair of in-touch points is perpendicular to the common edge of the two faces those in-touch points lie in.*

Proof. Using the point labels in Fig. 3, the first part of the lemma may be proven by noting that the vector from $\overline{\mathsf{A}}$ (say) to the in-center $\overline{\mathsf{I}}$ can be written in two ways, i.e.

$$\overrightarrow{\mathsf{AL}} + \overrightarrow{\mathsf{LI}} \;=\; \overrightarrow{\mathsf{AI}} \;=\; \overrightarrow{\mathsf{AN}} + \overrightarrow{\mathsf{NI}}\;, \tag{4·1}$$

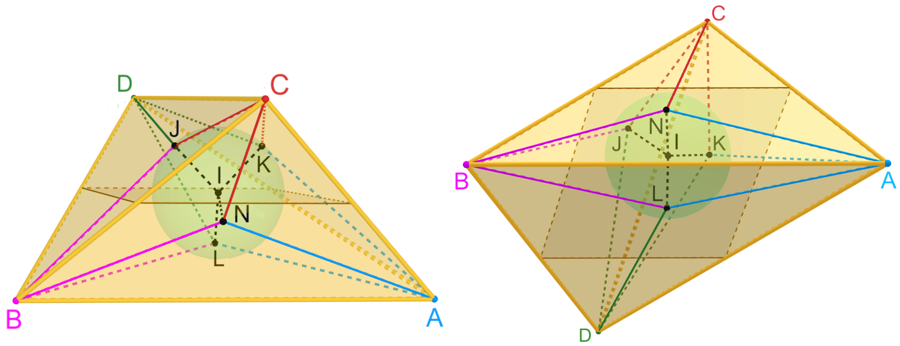

Fig. 3. Two perspectives on a generic tetrahedron $\overline{\mathsf{ABCD}}$, constructed as described in the main text, with its in-center $\overline{\mathsf{I}}$ and in-touch points $\overline{\mathsf{J}}$, $\overline{\mathsf{K}}$, $\overline{\mathsf{L}}$ & $\overline{\mathsf{N}}$ all labelled accordingly. The medial parallelogram separating $\overline{\mathsf{AB}}$ from $\overline{\mathsf{CD}}$ is drawn in light brown, and the line segments connecting each vertex to its three adjacent in-touch points are drawn using the same colour as the vertex. The pair of congruent triangles $\overline{\mathsf{ABN}}$ & $\overline{\mathsf{ABL}}$ can clearly be seen on the right, and it is evident that $\overline{\mathsf{AB}} \perp \overline{\mathsf{LN}}$.

from which it follows that

$$\left|\overline{\mathsf{AL}}\right|^2 + \left|\overline{\mathsf{LI}}\right|^2 + 2\,\overrightarrow{\mathsf{AL}} \cdot \overrightarrow{\mathsf{LI}} \;=\; \left|\overline{\mathsf{AN}}\right|^2 + \left|\overline{\mathsf{NI}}\right|^2 + 2\,\overrightarrow{\mathsf{AN}} \cdot \overrightarrow{\mathsf{NI}}\;. \tag{4·2}$$

But $\overrightarrow{\mathsf{AL}} \cdot \overrightarrow{\mathsf{LI}} = 0 = \overrightarrow{\mathsf{AN}} \cdot \overrightarrow{\mathsf{NI}}$ since $\overrightarrow{\mathsf{AL}}$ & $\overrightarrow{\mathsf{AN}}$ lie in the planes of the faces $\overline{\mathsf{ABD}}$ & $\overline{\mathsf{ABC}}$ resp., while $\overrightarrow{\mathsf{LI}}$ & $\overrightarrow{\mathsf{NI}}$ are perpendicular to those faces with a common length equal to the in-radius r by definition. This shows that $|\overline{\mathsf{AL}}| = |\overline{\mathsf{AN}}|$, and similarly $|\overline{\mathsf{BL}}| = |\overline{\mathsf{BN}}|$. Hence $\overline{\mathsf{ABL}}$ is congruent to $\overline{\mathsf{ABN}}$, as claimed. In an analogous fashion, one finds that all the distances from each vertex to the in-touch points on the three exterior faces incident to it are equal, i.e.

$$\begin{aligned} \left|\overline{\mathsf{AL}}\right| &= \left|\overline{\mathsf{AK}}\right| = \left|\overline{\mathsf{AN}}\right|\,, \quad & \left|\overline{\mathsf{BJ}}\right| &= \left|\overline{\mathsf{BL}}\right| = \left|\overline{\mathsf{BN}}\right|\,, \\ \left|\overline{\mathsf{CJ}}\right| &= \left|\overline{\mathsf{CK}}\right| = \left|\overline{\mathsf{CN}}\right|\,, \quad & \left|\overline{\mathsf{DJ}}\right| &= \left|\overline{\mathsf{DK}}\right| = \left|\overline{\mathsf{DL}}\right|\,, \end{aligned} \tag{4·3}$$

which implies the congruence of all the remaining pairs of triangles, where each pair contains an edge of the tetrahedron and the in-touch points of the two faces meeting in that edge.

The second part of the lemma is likewise easily proven using the orthogonality of the vectors $\overrightarrow{\mathsf{IL}}$ & $\overrightarrow{\mathsf{IN}}$ to the faces $\overline{\mathsf{ABD}}$ & $\overline{\mathsf{ABC}}$, resp., and hence to their common edge $\overline{\mathsf{AB}}$:

$$\overrightarrow{\mathsf{AB}} \cdot \overrightarrow{\mathsf{NL}} \;=\; \overrightarrow{\mathsf{AB}} \cdot \big(\overrightarrow{\mathsf{NI}} + \overrightarrow{\mathsf{IL}}\big) \;=\; \overrightarrow{\mathsf{AB}} \cdot \overrightarrow{\mathsf{NI}} + \overrightarrow{\mathsf{AB}} \cdot \overrightarrow{\mathsf{IL}} \;=\; 0 + 0 \;=\; 0\;. \tag{4·4}$$

The proofs for the pairs of in-touch triangles at the other five edges are similar. □

The areas of each of these 6 pairs of congruent triangles, times 2, will be taken as the tetrahedral analogues of the Heron parameters for a triangle, henceforth the *natural parameters* of the tetrahedron, specifically:

$$\begin{aligned} u &:= 2\left|\overline{\mathsf{ABL}}\right| = 2\left|\overline{\mathsf{ABN}}\right|\,, \quad & v &:= 2\left|\overline{\mathsf{ACN}}\right| = 2\left|\overline{\mathsf{ACK}}\right|\,, \\ w &:= 2\left|\overline{\mathsf{ADK}}\right| = 2\left|\overline{\mathsf{ADL}}\right|\,, \quad & x &:= 2\left|\overline{\mathsf{BCJ}}\right| = 2\left|\overline{\mathsf{BCN}}\right|\,, \\ y &:= 2\left|\overline{\mathsf{BDJ}}\right| = 2\left|\overline{\mathsf{BDL}}\right|\,, \quad & z &:= 2\left|\overline{\mathsf{CDJ}}\right| = 2\left|\overline{\mathsf{CDK}}\right|\,. \end{aligned} \tag{4·5}$$

Then because each exterior face of the tetrahedron is subdivided into three subfaces by the lines from the vertices of that face to its in-touch point, the natural parameters satisfy the following system of linear equations:

$$\begin{aligned} u+v+x &= 2\left|\overline{\mathsf{ABC}}\right| , & u+w+y &= 2\left|\overline{\mathsf{ABD}}\right| , \\ v+w+z &= 2\left|\overline{\mathsf{ACD}}\right| , & x+y+z &= 2\left|\overline{\mathsf{BCD}}\right| . \end{aligned} \tag{4·6}$$

The problem is that, unlike the triangle where the three Heron parameters are connected to the edge lengths by a non-singular system of three linear equations, here there are only four equations in the six unknowns u, v, w, x, y & z.

To obtain their values, let the areal vectors of the triangles $\overline{\mathsf{ABL}}$ etc. (times 2) be:

$$\begin{aligned} &\mathbf{u}_{\mathsf{AB;C}} := \overrightarrow{\mathsf{NB}}\times\overrightarrow{\mathsf{NA}} ,\ \ \mathbf{u}_{\mathsf{AB;D}} := \overrightarrow{\mathsf{LA}}\times\overrightarrow{\mathsf{LB}} ,\ \ \mathbf{v}_{\mathsf{AC;B}} := \overrightarrow{\mathsf{NA}}\times\overrightarrow{\mathsf{NC}} ,\ \ \mathbf{v}_{\mathsf{AC;D}} := \overrightarrow{\mathsf{KC}}\times\overrightarrow{\mathsf{KA}} , \\ &\mathbf{w}_{\mathsf{AD;C}} := \overrightarrow{\mathsf{KA}}\times\overrightarrow{\mathsf{KD}} ,\ \ \mathbf{w}_{\mathsf{AD;B}} := \overrightarrow{\mathsf{LD}}\times\overrightarrow{\mathsf{LA}} ,\ \ \mathbf{x}_{\mathsf{BC;A}} := \overrightarrow{\mathsf{NC}}\times\overrightarrow{\mathsf{NB}} ,\ \ \mathbf{x}_{\mathsf{BC;D}} := \overrightarrow{\mathsf{JB}}\times\overrightarrow{\mathsf{JC}} , \\ &\mathbf{y}_{\mathsf{BD;A}} := \overrightarrow{\mathsf{LB}}\times\overrightarrow{\mathsf{LD}} ,\ \ \mathbf{y}_{\mathsf{BD;C}} := \overrightarrow{\mathsf{JD}}\times\overrightarrow{\mathsf{JB}} ,\ \ \mathbf{z}_{\mathsf{CD;A}} := \overrightarrow{\mathsf{KD}}\times\overrightarrow{\mathsf{KC}} ,\ \ \mathbf{z}_{\mathsf{CD;B}} := \overrightarrow{\mathsf{JC}}\times\overrightarrow{\mathsf{JD}} . \end{aligned} \tag{4·7}$$

Note that the order of the factors in each cross-product has been chosen so as to ensure that these are all outwards-pointing vectors if the oriented volume of $\overline{\mathsf{ABCD}}$ is positive, or inwards-pointing if it is negative. Then the sum of the areal vectors of the two in-touch triangles sharing a common edge is e.g.

$$\begin{aligned} \mathbf{u}_{\mathsf{AB;D}} + \mathbf{u}_{\mathsf{AB;C}} &= \overrightarrow{\mathsf{LA}}\times\overrightarrow{\mathsf{LB}} - \overrightarrow{\mathsf{NA}}\times\overrightarrow{\mathsf{NB}} = \big(\overrightarrow{\mathsf{LN}}+\overrightarrow{\mathsf{NA}}\big)\times\big(\overrightarrow{\mathsf{LN}}+\overrightarrow{\mathsf{NB}}\big) - \overrightarrow{\mathsf{NA}}\times\overrightarrow{\mathsf{NB}} \\ &= \overrightarrow{\mathsf{NA}}\times\overrightarrow{\mathsf{LN}} + \overrightarrow{\mathsf{LN}}\times\overrightarrow{\mathsf{NB}} = \overrightarrow{\mathsf{LN}}\times\big(\overrightarrow{\mathsf{NB}} - \overrightarrow{\mathsf{NA}}\big) = \overrightarrow{\mathsf{LN}}\times\overrightarrow{\mathsf{AB}} . \end{aligned} \tag{4·8}$$

Since $\|\mathbf{u}_{\mathsf{AB;D}}\| = \|\mathbf{u}_{\mathsf{AB;C}}\| =: u$ and $\overrightarrow{\mathsf{LN}} \perp \overrightarrow{\mathsf{AB}}$ by Lemma 4·1, it follows that

$$\begin{aligned} &\big\|\overrightarrow{\mathsf{LN}}\times\overrightarrow{\mathsf{AB}}\big\|^2 = \|\overrightarrow{\mathsf{LN}}\|^2\,\|\overrightarrow{\mathsf{AB}}\|^2 = \|\mathbf{u}_{\mathsf{AB;D}}+\mathbf{u}_{\mathsf{AB;C}}\|^2 = \\ &\|\mathbf{u}_{\mathsf{AB;C}}\|^2 + \|\mathbf{u}_{\mathsf{AB;D}}\|^2 - 2\,\|\mathbf{u}_{\mathsf{AB;C}}\|\|\mathbf{u}_{\mathsf{AB;D}}\|\cos(\varphi_{\mathsf{AB}}) = 2\,u^2\big(1-\cos(\varphi_{\mathsf{AB}})\big) , \end{aligned} \tag{4·9}$$

where φ_{AB} is the dihedral angle between $\overline{\mathsf{ABC}}$ & $\overline{\mathsf{ABD}}$, and the "$-$" in front of the cosine is because φ_{AB} is the angle between $\mathbf{u}_{\mathsf{AB|C}}$ & $-\mathbf{u}_{\mathsf{AB|D}}$ (or vice versa). By the areal law of cosines (2·11), however, this cosine is also equal to

$$\cos(\varphi_{\mathsf{AB}}) = \frac{\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\|^2 + \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|^2 - \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\|^2}{2\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\|\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|} \tag{4·10 a}$$

$$\Longleftrightarrow\quad 1 - \cos(\varphi_{\mathsf{AB}}) = \frac{\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\|^2 - \big(\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| - \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|\big)^2}{2\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\|\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|} , \tag{4·10 b}$$

and plugging that into Eq. (4·9) gives

$$u^2 = \frac{\|\overrightarrow{\mathsf{LN}}\|^2\,\|\overrightarrow{\mathsf{AB}}\|^2\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\|\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|}{\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\|^2 - \big(\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| - \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|\big)^2} . \tag{4·11}$$

To finish the job, a formula for $\|\overrightarrow{\mathsf{LN}}\|^2$ is needed, and it is

$$\begin{aligned} \|\overrightarrow{\mathsf{LN}}\|^2 &= \|\overrightarrow{\mathsf{IN}} - \overrightarrow{\mathsf{IL}}\|^2 = \|\overrightarrow{\mathsf{IN}}\|^2 + \|\overrightarrow{\mathsf{IL}}\|^2 - 2\,\overrightarrow{\mathsf{IL}}\cdot\overrightarrow{\mathsf{IN}} = 2\,r^2\,\big(1+\cos(\varphi_{\mathsf{AB}})\big) \\ &= 2\,r^2\,\frac{\big(\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\| + \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|\big)^2 - \big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\big\|^2}{2\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\big\|\,\big\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\big\|} , \end{aligned} \tag{4·12}$$

where $r = \|\overrightarrow{\mathsf{IL}}\| = \|\overrightarrow{\mathsf{IN}}\|$ is the in-radius and the change in the sign of the cosine has the same explanation as above. This leads to the following relatively simple formula:

$$u \;=\; r\,\|\overrightarrow{\mathsf{AB}}\|\sqrt{\frac{1+\cos(\varphi_{\mathsf{AB}})}{1-\cos(\varphi_{\mathsf{AB}})}} \;=\; r\,\|\overrightarrow{\mathsf{AB}}\|\,\cot(\varphi_{\mathsf{AB}}/2) \;= \tag{4·13}$$

$$r\,\|\overrightarrow{\mathsf{AB}}\|\sqrt{\frac{\big(\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\big)^2-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2}{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2-\big(\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\big)^2}} \;=\; r\,\|\overrightarrow{\mathsf{AB}}\|\sqrt{\frac{\mathcal{T}_0[\mathsf{A},\mathsf{B}]\,\mathcal{T}_1[\mathsf{A},\mathsf{B}]}{\mathcal{T}_2[\mathsf{A},\mathsf{B}]\,\mathcal{T}_3[\mathsf{A},\mathsf{B}]}}$$

Here $\mathcal{T}_0[\mathsf{A},\mathsf{B}] \ge 0$ is the non-degeneracy factor and $\mathcal{T}_k[\mathsf{A},\mathsf{B}] \ge 0$ $(k = 1,2,3)$ are the deviations of the tetrahedron inequalities from saturation in Eq. (2·7), so the quantity in the square root is non-negative. Similar expressions can of course be derived for the other parameters v, w, x, y & z via the definitions given in Eq. (4·7).

This expression may be further simplified via the trigonometric identity $\cot(\vartheta/2) = \csc(\vartheta)+\cot(\vartheta)$, where the sine in $\csc(\varphi_{\mathsf{AB}})$ is obtained from the areal law of sines (2·12). It then follows from Eq. (4·13) that $u = \big(r\,\|\overrightarrow{\mathsf{AB}}\|\,/\sin(\varphi_{\mathsf{AB}})\big)\big(1+\cos(\varphi_{\mathsf{AB}})\big) =$

$$r\,\|\overrightarrow{\mathsf{AB}}\|\,\frac{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|}{\|\overrightarrow{\mathsf{AB}}\|\,\big|\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big|}\left(1+\frac{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2}{2\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|}\right)$$

$$= r\,\frac{2\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2}{2\,\big|\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big|} \tag{4·14}$$

$$= \frac{\big(\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\big)^2-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2}{2\,s} \;=\; \frac{\mathcal{T}_0[\mathsf{A},\mathsf{B}]\,\mathcal{T}_1[\mathsf{A},\mathsf{B}]}{2\,s}\,,$$

where the well-known relation $r = t/s := \big|\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big|/s$ was used on the last line with $s := 2\,|\overline{\mathsf{ABC}}|+2\,|\overline{\mathsf{ABD}}|+2\,|\overline{\mathsf{ACD}}|+2\,|\overline{\mathsf{BCD}}|$ equal to twice the exterior surface area.

This together with similar calculations for the remaining five parameters shows that the natural parameters can be expressed in terms of the seven areas as follows.

PROPOSITION 4·2. *The natural parameters of a tetrahedron* $\overline{\mathsf{ABCD}}$ *with* $s > 0$ *are:*

$$\begin{aligned} u &= \frac{\mathcal{T}_0[\mathsf{A},\mathsf{B}]\,\mathcal{T}_1[\mathsf{A},\mathsf{B}]}{2\,s}\,, \quad v = \frac{\mathcal{T}_0[\mathsf{A},\mathsf{C}]\,\mathcal{T}_1[\mathsf{A},\mathsf{C}]}{2\,s}\,, \quad w = \frac{\mathcal{T}_0[\mathsf{A},\mathsf{D}]\,\mathcal{T}_1[\mathsf{A},\mathsf{D}]}{2\,s}\,, \\ z &= \frac{\mathcal{T}_0[\mathsf{C},\mathsf{D}]\,\mathcal{T}_1[\mathsf{C},\mathsf{D}]}{2\,s}\,, \quad y = \frac{\mathcal{T}_0[\mathsf{B},\mathsf{D}]\,\mathcal{T}_1[\mathsf{B},\mathsf{D}]}{2\,s}\,, \quad x = \frac{\mathcal{T}_0[\mathsf{B},\mathsf{C}]\,\mathcal{T}_1[\mathsf{B},\mathsf{C}]}{2\,s}\,. \end{aligned} \tag{4·15}$$

Next, we will use $r\,s = t$ together with the alternative trigonometric identity $\tan(\vartheta/2) = \csc(\vartheta)-\cot(\vartheta) \Leftrightarrow \cot(\vartheta/2) = \sin(\vartheta)/\big(1-\cos(\vartheta/2)\big)$ to also compute u from Eq. (4·13) as $u =$

$$r\,\|\overrightarrow{\mathsf{AB}}\|\,\frac{\|\overrightarrow{\mathsf{AB}}\|\,\big|\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big|}{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|}\Bigg/\left(1-\frac{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2+\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2}{2\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|\,\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|}\right)$$

$$= r\,\frac{2\,\|\overrightarrow{\mathsf{AB}}\|^2\,\big|\overrightarrow{\mathsf{AB}}\cdot(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}})\big|\;/\;s}{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2-\big(\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\big)^2\;/\;s} \;=:\; r^2\,\frac{\|\overrightarrow{\mathsf{AB}}\|^2}{\tilde{u}}\,, \tag{4·16}$$

where $\tilde{u}$ is the natural parameter *inverse* to u, namely

$$\tilde{u} \;=\; \frac{\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2-\big(\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|-\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|\big)^2}{2\,s} \;=\; \frac{\mathcal{T}_2[\mathsf{A},\mathsf{B}]\,\mathcal{T}_3[\mathsf{A},\mathsf{B}]}{2\,s}\,. \tag{4·17}$$

Together with similar computations for the other five inverse parameters, this proves:

PROPOSITION 4·3. *If $s > 0$, the natural parameters inverse to u, v, w, x, y & z are:*

$$\begin{aligned} \tilde{u} &= \frac{\mathcal{T}_2[\mathsf{A},\mathsf{B}]\,\mathcal{T}_3[\mathsf{A},\mathsf{B}]}{2\,s}\,, \quad \tilde{v} = \frac{\mathcal{T}_2[\mathsf{A},\mathsf{C}]\,\mathcal{T}_3[\mathsf{A},\mathsf{C}]}{2\,s}\,, \quad \tilde{w} = \frac{\mathcal{T}_2[\mathsf{A},\mathsf{D}]\,\mathcal{T}_3[\mathsf{A},\mathsf{D}]}{2\,s}\,, \\ \tilde{z} &= \frac{\mathcal{T}_2[\mathsf{C},\mathsf{D}]\,\mathcal{T}_3[\mathsf{C},\mathsf{D}]}{2\,s}\,, \quad \tilde{y} = \frac{\mathcal{T}_2[\mathsf{B},\mathsf{D}]\,\mathcal{T}_3[\mathsf{B},\mathsf{D}]}{2\,s}\,, \quad \tilde{x} = \frac{\mathcal{T}_2[\mathsf{B},\mathsf{C}]\,\mathcal{T}_3[\mathsf{B},\mathsf{C}]}{2\,s}\,. \end{aligned} \tag{4·18}$$

Proposition 4·2 and the calculations leading up to Proposition 4·3 also establish:

COROLLARY 4·4. *The squared in-radius r^2 times the squared edge lengths are equal to the products of complementary pairs of natural & inverse natural parameters, specifically:*

$$\begin{aligned} r^2\left|\overline{\mathsf{AB}}\right|^2 &= u\tilde{u}\,, \qquad r^2\left|\overline{\mathsf{AC}}\right|^2 = v\tilde{v}\,, \qquad r^2\left|\overline{\mathsf{AD}}\right|^2 = w\tilde{w}\,, \\ r^2\left|\overline{\mathsf{CD}}\right|^2 &= z\tilde{z}\,, \qquad r^2\left|\overline{\mathsf{BD}}\right|^2 = y\tilde{y}\,, \qquad r^2\left|\overline{\mathsf{BC}}\right|^2 = x\tilde{x}\,. \end{aligned} \tag{4·19}$$

Note that $t^2\,|\overline{\mathsf{AB}}|^2 = s^2 u\,\tilde{u} = \frac{1}{4}\,\mathcal{T}_0[\mathsf{A},\mathsf{B}]\,\mathcal{T}_1[\mathsf{A},\mathsf{B}]\,\mathcal{T}_2[\mathsf{A},\mathsf{B}]\,\mathcal{T}_3[\mathsf{A},\mathsf{B}]$ etc. are the 2×2 minors of the Gram matrices, and that these are also equal to $\left(4\left|\overline{\mathsf{ABC}}\right|\left|\overline{\mathsf{ABD}}\right|\sin(\varphi_{\mathsf{AB}})\right)^2$ etc. by the areal law of sines (2·12).

The following further corollary summarizes some of the other algebraic identities which connect the natural and inverse natural parameters with the seven areas.

COROLLARY 4·5. *With everything defined as above, the following identities hold:*

$$2\,(u + v + w + x + y + z) \;=\; s + 2\,\Xi\,/\,s\,; \tag{4·20 a}$$

$$\begin{aligned} u - z &= \left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ABD}}\right| - \left|\overline{\mathsf{ACD}}\right| - \left|\overline{\mathsf{BCD}}\right|\,, \\ v - y &= \left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ACD}}\right| - \left|\overline{\mathsf{ABD}}\right| - \left|\overline{\mathsf{BCD}}\right|\,, \\ w - x &= \left|\overline{\mathsf{ABD}}\right| + \left|\overline{\mathsf{ACD}}\right| - \left|\overline{\mathsf{ABC}}\right| - \left|\overline{\mathsf{BCD}}\right|\,; \end{aligned} \tag{4·20 b}$$

$$\begin{aligned} u + v + x &= 2\left|\overline{\mathsf{ABC}}\right| + \Xi\,/\,(2s)\,, \quad u + w + y = 2\left|\overline{\mathsf{ABD}}\right| + \Xi\,/\,(2s)\,, \\ v + w + z &= 2\left|\overline{\mathsf{ACD}}\right| + \Xi\,/\,(2s)\,, \quad x + y + z = 2\left|\overline{\mathsf{BCD}}\right| + \Xi\,/\,(2s)\,; \end{aligned} \tag{4·20 c}$$

$$\begin{aligned} u + v + w &= \left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ABD}}\right| + \left|\overline{\mathsf{ACD}}\right| - \left|\overline{\mathsf{BCD}}\right| + \Xi\,/\,(2s)\,, \\ u + x + y &= \left|\overline{\mathsf{BCD}}\right| + \left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ABD}}\right| - \left|\overline{\mathsf{ACD}}\right| + \Xi\,/\,(2s)\,, \\ v + x + z &= \left|\overline{\mathsf{ACD}}\right| + \left|\overline{\mathsf{BCD}}\right| + \left|\overline{\mathsf{ABC}}\right| - \left|\overline{\mathsf{ABD}}\right| + \Xi\,/\,(2s)\,, \\ w + y + z &= \left|\overline{\mathsf{ABD}}\right| + \left|\overline{\mathsf{ACD}}\right| + \left|\overline{\mathsf{BCD}}\right| - \left|\overline{\mathsf{ABC}}\right| + \Xi\,/\,(2s)\,; \end{aligned} \tag{4·20 d}$$

$$\begin{aligned} \left(v + w + x + y - \Xi/s\right)^2 - 4\,uz &= 16\left|\overline{\mathsf{AB}|\mathsf{CD}}\right|^2\,, \\ \left(u + w + x + z - \Xi/s\right)^2 - 4\,vy &= 16\left|\overline{\mathsf{AC}|\mathsf{BD}}\right|^2\,, \\ \left(u + v + y + z - \Xi/s\right)^2 - 4\,wx &= 16\left|\overline{\mathsf{AD}|\mathsf{BC}}\right|^2\,; \end{aligned} \tag{4·20 e}$$

$$\begin{aligned} (u + \tilde{u})\,s &= 8\left|\overline{\mathsf{ABC}}\right|\left|\overline{\mathsf{ABD}}\right|, \quad (z + \tilde{z})\,s = 8\left|\overline{\mathsf{ACD}}\right|\left|\overline{\mathsf{BCD}}\right|, \\ (v + \tilde{v})\,s &= 8\left|\overline{\mathsf{ABC}}\right|\left|\overline{\mathsf{ACD}}\right|, \quad (y + \tilde{y})\,s = 8\left|\overline{\mathsf{ABD}}\right|\left|\overline{\mathsf{BCD}}\right|, \\ (w + \tilde{w})\,s &= 8\left|\overline{\mathsf{ABD}}\right|\left|\overline{\mathsf{ACD}}\right|, \quad (x + \tilde{x})\,s = 8\left|\overline{\mathsf{ABC}}\right|\left|\overline{\mathsf{BCD}}\right|; \end{aligned} \tag{4·20 f}$$

$$\begin{aligned} (u - \tilde{u})\,s/2 &= \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right) \cdot \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right), \quad (z - \tilde{z})\,s/2 = \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right) \cdot \left(\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right), \\ (\tilde{v} - v)\,s/2 &= \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right) \cdot \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right), \quad (\tilde{y} - y)\,s/2 = \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right) \cdot \left(\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right), \\ (w - \tilde{w})\,s/2 &= \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AD}}\right) \cdot \left(\overrightarrow{\mathsf{AC}} \times \overrightarrow{\mathsf{AD}}\right), \quad (x - \tilde{x})\,s/2 = \left(\overrightarrow{\mathsf{AB}} \times \overrightarrow{\mathsf{AC}}\right) \cdot \left(\overrightarrow{\mathsf{BC}} \times \overrightarrow{\mathsf{BD}}\right); \end{aligned} \tag{4·20 g}$$

$$2\,(\tilde{u} + \tilde{v} + \tilde{w} + \tilde{x} + \tilde{y} + \tilde{z})\,s \;=\; s^2 - 32\left(\left|\overline{\mathsf{AB}|\mathsf{CD}}\right|^2 + \left|\overline{\mathsf{AC}|\mathsf{BD}}\right|^2 + \left|\overline{\mathsf{AD}|\mathsf{BC}}\right|^2\right) - 4\,\Xi\,. \tag{4·20 h}$$

Proof. We will prove only Eq. (4·20 *a*), primarily to drive home the algebraic forms of the quantities involved, leaving the remaining formulae as a straightforward if tedious algebraic exercise. By Proposition 4·2, the definition of the $\mathcal{T}$'s in Eq. (2·7) and the definition of Ξ from Proposition 2·7, one obtains $2\,(u{+}v{+}w{+}x{+}y{+}z)\,s\;=$

$$\begin{aligned}
&4\left(\left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ABD}}\right|\right)^2 + 4\left(\left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ACD}}\right|\right)^2 + 4\left(\left|\overline{\mathsf{ABD}}\right| + \left|\overline{\mathsf{ACD}}\right|\right)^2 \\
&+ 4\left(\left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{BCD}}\right|\right)^2 + 4\left(\left|\overline{\mathsf{ABD}}\right| + \left|\overline{\mathsf{BCD}}\right|\right)^2 + 4\left(\left|\overline{\mathsf{ACD}}\right| + \left|\overline{\mathsf{BCD}}\right|\right)^2 \\
&- 32\left|\overline{\mathsf{AB|CD}}\right|^2 - 32\left|\overline{\mathsf{AC|BD}}\right|^2 - 32\left|\overline{\mathsf{AD|BC}}\right|^2 \\
&= 4\left(\left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ABD}}\right| + \left|\overline{\mathsf{ACD}}\right| + \left|\overline{\mathsf{BCD}}\right|\right)^2 + 2\,\Xi \;=:\; s^2 + 2\,\Xi\,.
\end{aligned}$$

The result now follows on dividing through by s. □

Remark 4·1. The analogues of the inverse natural parameters for a triangle are

$$\tilde u = \frac{(a-b+c)\,(a+b-c)}{4\,s}\,,\quad \tilde v = \frac{(-a+b+c)\,(a+b-c)}{4\,s}\,,\quad \text{and}\quad \tilde w = \frac{(-a+b+c)\,(a-b+c)}{4\,s}\,, \tag{4·21}$$

where $a, b, c \geq 0$ are its edge lengths and $s = (a+b+c)/2$ is its semi-perimeter. These may be shown to satisfy

$$\tilde u = vr/r_{\mathsf C} = wr/r_{\mathsf B}\,,\ \ \tilde v = ur/r_{\mathsf C} = wr/r_{\mathsf A}\,,\ \ \tilde w = ur/r_{\mathsf B} = vr/r_{\mathsf A}\,, \tag{4·22}$$

where r, $r_{\mathsf A} = rs/u$, $r_{\mathsf B} = rs/v$, $r_{\mathsf C} = rs/w$ are the radii of the in-circle and ex-circles tangent to the edge of length a, b, c respectively (cf. Fig. 1). For a tetrahedron, the relations between the in-radius and ex-radii given in e.g. Refs. [**47**, **59**], together with Eq. (4·20 *d*), show that these quantities satisfy

$$\begin{aligned}
r_{\mathsf A} &= \tfrac12\,rs/(u+v+w)\,, & r_{\mathsf B} &= \tfrac12\,rs/(u+x+y)\,, \\
r_{\mathsf C} &= \tfrac12\,rs/(v+x+z)\,, & r_{\mathsf D} &= \tfrac12\,rs/(w+y+z)\,.
\end{aligned} \tag{4·23}$$

where $r_{\mathsf A}$ is the radius of the ex-sphere tangent to the exterior face opposite $\overline{\mathsf A}$, etc. Furthermore, Corollary 4·5 can be used to show that

$$\begin{aligned}
(\tilde x{+}\tilde y{+}\tilde z)\,s &= 2\,(u{+}v{+}w)(x{+}y{+}z)\,, & (\tilde v{+}\tilde w{+}\tilde z)\,s &= 2\,(u{+}x{+}y)(v{+}w{+}z), \\
(\tilde u{+}\tilde w{+}\tilde y)\,s &= 2\,(v{+}x{+}z)(u{+}w{+}y)\,, & (\tilde u{+}\tilde v{+}\tilde x)\,s &= 2\,(w{+}y{+}z)(u{+}v{+}x),
\end{aligned} \tag{4·24}$$

and hence $2\left|\overline{\mathsf{ABC}}\right| = u+v+x = (\tilde u+\tilde v+\tilde x)\,r_{\mathsf D}/r$, etc. This suggests that the inverse natural parameters of a tetrahedron scaled by the ratios of the ex-radii to the in-radius may be interpreted as the areas of the triangles into which the ex-touch points $\overline{\mathsf J_{\mathsf A}}$, $\overline{\mathsf K_{\mathsf B}}$, $\overline{\mathsf L_{\mathsf C}}$ & $\overline{\mathsf N_{\mathsf D}}$ divide the tetrahedron's exterior faces, e.g.

$$\frac{r_{\mathsf D}}{r}\,\tilde u = 2\left|\overline{\mathsf{ABN_D}}\right|\,,\quad \frac{r_{\mathsf D}}{r}\,\tilde v = 2\left|\overline{\mathsf{ACN_D}}\right|\,,\quad \frac{r_{\mathsf D}}{r}\,\tilde x = 2\left|\overline{\mathsf{BCN_D}}\right|\,. \tag{4·25}$$

This hypothesis can easily be shown to hold numerically in randomly generated tetrahedra, thereby obtaining a "generic" proof of its correctness.

It is readily verified that the non-negativity of all the natural and inverse natural parameters is equivalent to all 18 tetrahedron inequalities holding, as long as the areas from which they were obtained are likewise non-negative. This is entirely analogous to the way in which the non-negativity of the Heron parameters of a triangle assure that the triangle inequalities are satisfied. In the case of the tetrahedron one also has Yetter's identity to deal with, but it turns out that this likewise occasions no difficulties.

Proposition 4·6. *The areas calculated from any values for the natural parameters via Corollary 4·5 always satisfy Yetter's identity.*

Proof. Simply use Eqs. (4·20 *c*) & (4·20 *e*) to substitute for the squared areas of the exterior & interior faces in the polynomial Ξ of Yetter's identity and simplify the result to get 0. □

Next, a similar process will be used to express the inverse natural parameters in terms of the natural parameters.

Lemma 4·7. *The inverse natural parameters of a tetrahedron* $\overline{\mathsf{ABCD}}$ *are given in terms of the natural parameters themselves by:*

$$\begin{aligned} \tilde{u} &= \frac{2\big((v+x)(w+y)-uz\big)}{s} & \qquad \tilde{z} &= \frac{2\big((v+w)(x+y)-uz\big)}{s} \\ \tilde{v} &= \frac{2\big((u+x)(w+z)-vy\big)}{s} & \qquad \tilde{y} &= \frac{2\big((u+w)(x+z)-vy\big)}{s} \\ \tilde{w} &= \frac{2\big((u+y)(v+z)-wx\big)}{s} & \qquad \tilde{x} &= \frac{2\big((u+v)(y+z)-wx\big)}{s} \end{aligned} \tag{4·26}$$

Proof. We will derive only the first of these formulae, since the others may be obtained in much the same fashion. From Eqs. (4·20 *a*), (4·20 *c*) & (4·20 *f*), we obtain

$$\tilde{u}\,s \;=\; 8\,\big|\overline{\mathsf{ABC}}\big|\big|\overline{\mathsf{ABD}}\big| - u\,s \;=\; 2\,(u{+}v{+}x)(u{+}w{+}y) \;-\; 2\,u\,(u{+}v{+}w{+}x{+}y{+}z)\ ,$$

which is readily verified to be s times the first formula in Eq. (4·26). □

Remark 4·2. Note that even when the natural parameters are all non-negative, these formulae can give negative values for one or more of the inverse natural parameters. Therefore, unlike the Heron parameters of a triangle, the natural parameters of a tetrahedron cannot be chosen arbitrarily subject to being merely non-negative.

The main result of this paper may now be stated as follows:

Theorem 4·8. *With everything defined as above, the volume* $\big|\overline{\mathsf{ABCD}}\big| = t/3!$ *of a tetrahedron* $\overline{\mathsf{ABCD}}$ *may be expressed in terms of its natural parameters* u, v, w, x, y, z *as*

$$\begin{aligned} t^4 \;&=\; s^2\left(2\,vwxy + 2\,uwxz + 2\,uvyz - u^2z^2 - v^2y^2 - w^2x^2\right) \qquad (4·27) \\ =:\; s^2\,\Omega(u,v,w,x,y,z) \;&=\; -4\,(u{+}v{+}w{+}x{+}y{+}z)^2\,\det\begin{bmatrix} 0 & u & v & w \\ u & 0 & x & y \\ v & x & 0 & z \\ w & y & z & 0 \end{bmatrix}, \end{aligned}$$

where $s = 2\,(u{+}v{+}w{+}x{+}y{+}z)$ *is twice the exterior surface area, and its in-radius is given by* $r^4 = t^4/s^4 = \Omega(u,v,w,x,y,z)/s^2$.

Proof. On dividing the formulae in Eq. (4·19) through by r^2 and substituting for the inverse natural parameters therein by the formulae from Eq. (4·26), one obtains

$$\begin{aligned}
|\overline{\mathsf{AB}}|^2 &= \frac{2\,u\big((v{+}x)(w{+}y)-uz\big)}{s\,r^2}\,, & |\overline{\mathsf{CD}}|^2 &= \frac{2\,z\big((v{+}w)(x{+}y)-uz\big)}{s\,r^2}\,,\\
|\overline{\mathsf{AC}}|^2 &= \frac{2\,v\big((u{+}x)(w{+}z)-vy\big)}{s\,r^2}\,, & |\overline{\mathsf{BD}}|^2 &= \frac{2\,y\big((u{+}w)(x{+}z)-vy\big)}{s\,r^2}\,,\\
|\overline{\mathsf{AD}}|^2 &= \frac{2\,w\big((u{+}y)(v{+}z)-wx\big)}{s\,r^2}\,, & |\overline{\mathsf{BC}}|^2 &= \frac{2\,x\big((u{+}v)(y{+}z)-wx\big)}{s\,r^2}\,.
\end{aligned} \tag{4·28}$$

Substituting these expressions for the squared distances into the usual 4-point Cayley-Menger determinant $\Delta_D[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}] \leftrightarrow t^2$ and factorizing the result then yields

$$t^2 \;=\; \frac{2\,(u{+}v{+}w{+}x{+}y{+}z)\,\Omega(u,v,w,x,y,z)^2}{\big(s\,r^2\big)^3}\;. \tag{4·29}$$

Multiplying this equation through by s^3r^6 and using the relation $r = t/s$ thus implies

$$t^8/s^3 \;=\; 2\,(u{+}v{+}w{+}x{+}y{+}z)\,\Omega(u,v,w,x,y,z)^2\;. \tag{4·30}$$

Multiplying through by s^3, using the relation $s = 2\,(u{+}v{+}w{+}x{+}y{+}z)$ from Eq. (4·20 a), and taking the square roots of both sides thus gives Eq. (4·27) as desired. □

Remark 4·3. Equation (4·27) can also be derived by using Eqs. (4·20 c) & (4·20 e) to substitute for the squared areas in the Grammian $\Gamma_F[\mathsf{A}]$, which is a little messier but has the advantage of also being valid in the degenerate case. On the other hand, if one converts $\Omega(u,v,w,x,y,z)$ into a rational function in the seven areas using Proposition 4·2, the numerator turns out to be a polynomial of total degree 8 in the areas containing 420 terms, which does not factorize. Since some of these terms contain odd powers of the exterior areas, those areas cannot be eliminated using Yetter's identity, but the interior areas occur in only even powers and hence can be. If for example one eliminates $|\overline{\mathsf{AD|BC}}|^2$, the resulting polynomial factorizes into the product of the square of the exterior surface area and a polynomial of total degree 6 in the remaining six areas containing mere 22 terms. (Curiously, this degree 6 polynomial is equal to 4 times a four-point Cayley-Menger determinant in the remaining six *areas*, wherein the exterior areas occupy the positions of the edge lengths in a quadrilateral and the two interior areas occupy the positions of its diagonals.) It turns out that this 22-term polynomial is the same as that which is obtained on eliminating $|\overline{\mathsf{AD|BC}}|^2$ from the Grammian of the areal vectors of the interior faces, namely

$$\det\begin{bmatrix}
\big\|\overrightarrow{\mathsf{AB}}{\times}\overrightarrow{\mathsf{CD}}\big\|^2 & \big(\overrightarrow{\mathsf{AB}}{\times}\overrightarrow{\mathsf{CD}}\big)\cdot\big(\overrightarrow{\mathsf{AC}}{\times}\overrightarrow{\mathsf{BD}}\big) & \big(\overrightarrow{\mathsf{AB}}{\times}\overrightarrow{\mathsf{CD}}\big)\cdot\big(\overrightarrow{\mathsf{AD}}{\times}\overrightarrow{\mathsf{BC}}\big)\\
\big(\overrightarrow{\mathsf{AB}}{\times}\overrightarrow{\mathsf{CD}}\big)\cdot\big(\overrightarrow{\mathsf{AC}}{\times}\overrightarrow{\mathsf{BD}}\big) & \big\|\overrightarrow{\mathsf{AC}}{\times}\overrightarrow{\mathsf{BD}}\big\|^2 & \big(\overrightarrow{\mathsf{AC}}{\times}\overrightarrow{\mathsf{BD}}\big)\cdot\big(\overrightarrow{\mathsf{AD}}{\times}\overrightarrow{\mathsf{BC}}\big)\\
\big(\overrightarrow{\mathsf{AB}}{\times}\overrightarrow{\mathsf{CD}}\big)\cdot\big(\overrightarrow{\mathsf{AD}}{\times}\overrightarrow{\mathsf{BC}}\big) & \big(\overrightarrow{\mathsf{AC}}{\times}\overrightarrow{\mathsf{BD}}\big)\cdot\big(\overrightarrow{\mathsf{AD}}{\times}\overrightarrow{\mathsf{BC}}\big) & \big\|\overrightarrow{\mathsf{AD}}{\times}\overrightarrow{\mathsf{BC}}\big\|^2
\end{bmatrix}, \tag{4·31}$$

which may be constructed from the areas using the areal law of cosines for the interior faces (2·14). This determinant in turn equals the sum of the exterior Grammians at the four vertices plus $\big(|\overline{\mathsf{ABC}}|^2 + |\overline{\mathsf{ABD}}|^2 + |\overline{\mathsf{ACD}}|^2 + |\overline{\mathsf{BCD}}|^2\big)\,\Xi^2$. In this way one can convert Eq. (4·27) into one that is equivalent modulo Ξ to the Grammians, but at the expense of losing the symmetry under vertex permutations or having to impose Yetter's identity as a constraint (or both).

One consequence of this Remark together with Proposition 4·6 and Theorem 3·4 is:

Corollary 4·9. *The natural parameters determine a non-degenerate Euclidean tetrahedron up to isometry when they are positive, the corresponding inverse natural parameters obtained via Lemma 4·7 are positive, and the fourth power of the volume as calculated from Eq. (4·27), or equivalently $\Omega(u, v, w, x, y, z)$, is likewise positive.*

Remark 4·4. Under the correspondence $u \leftrightarrow |\overline{\mathsf{AB}}|^2, \ldots, z \leftrightarrow |\overline{\mathsf{CD}}|^2$, the non-negativity of the negative determinant Ω in Eq. (4·27) is algebraically identical to Ptolemy's inequalities for the distances among four points in Euclidean space (Ref. [**10**], Ex. 2, pg. 80). As is well known, this inequality is saturated if & only if the four points in question lie on a circle in a plane or are collinear. This correspondence shows that the determinant factorizes into a product of factors each of which is linear in the products of the *square-roots* of "opposite" pairs of natural parameters, i.e. $\Omega(\breve{u}^2, \breve{v}^2, \breve{w}^2, \breve{x}^2, \breve{y}^2, \breve{z}^2) =$

$$\left(\breve{u}\breve{z} + \breve{v}\breve{y} + \breve{w}\breve{x}\right)\left(\breve{v}\breve{y} + \breve{w}\breve{x} - \breve{u}\breve{z}\right)\left(\breve{w}\breve{x} + \breve{u}\breve{z} - \breve{v}\breve{y}\right)\left(\breve{u}\breve{z} + \breve{v}\breve{y} - \breve{w}\breve{x}\right) , \tag{4·32}$$

where $\breve{u} := \sqrt{u}\,, \ldots, \breve{z} := \sqrt{z}$. Nevertheless, even when they determine a non-degenerate Euclidean tetrahedron, the natural parameters are *not* necessarily equal to the squared distances among four Euclidean points, because they can violate the triangle inequality or give a negative four-point Cayley-Menger determinant (as happens, for example, when $[u, v, w, x, y, z] = [2, 4, 1, 10, 5, 6]$). Therefore this algebraic correspondence does not extend to a geometrically meaningful relationship.

Remark 4·5. The expression of the quartic polynomial $\Omega(u, v, w, x, y, z)$ as a determinant nonetheless suggests that the formula does have another geometric interpretation. Specifically, it is well known that the matrix of squared distances among a set of points in Euclidean space can be interpreted as the Gram matrix of a set of vectors on the null cone of an indefinite space with signature $[-1, -1, \ldots, -1, +1]$, and normalized so that their inner product with a fixed null vector, which serves as the point-at-infinity of inversive geometry, is unity; this corresponds to the border of 1's in Cayley-Menger determinants [**22**, **52**, **53**]. Although this normalization is not applicable in the present situation, the rest of that geometric interpretation holds, in that the signature of the matrix in Eq. (4·27) is $[-1, -1, -1, +1]$. Because the interior of the null cone, projectively viewed, constitutes a model of hyperbolic space [**14**], it is likely that hyperbolic geometry will give deeper insights into the meaning of the formula (4·27).

Remark 4·6. Euler's theorem on homogeneous functions shows that $3\,\Delta_D[\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}] = \mathbf{d} \cdot \nabla_{\mathbf{d}} \Delta_D[\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}]$, where $\mathbf{d} := [D_{\mathsf{AB}}, \ldots, D_{\mathsf{CD}}]^\top$ is a vector of squared distances. The derivatives of the determinant in $\nabla_{\mathbf{d}} \Delta_D$, in turn, are the cofactors of the corresponding matrix, which Lemma 2·4 shows are the dot products of the areal vectors of the exterior faces. Thus it follows from Corollary 4·4, $r = t/s$ and Eq. (4·20 g) that

$$\begin{aligned} t^2 \;=\; \tfrac{1}{3}\,\mathbf{d} \cdot \nabla_{\mathbf{d}} \Delta_D[\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}] \;=\; \frac{s^3}{6\,t^2} \big(& u\tilde{u}\,(z - \tilde{z}) + v\tilde{v}\,(y - \tilde{y}) \\ &+ w\tilde{w}\,(x - \tilde{x}) + x\tilde{x}\,(w - \tilde{w}) + y\tilde{y}\,(v - \tilde{v}) + z\tilde{z}\,(u - \tilde{u})\big) , \end{aligned} \tag{4·33}$$

from which one obtains another formula for t^4 that is (outside of s) antisymmetric w.r.t. interchange of the natural and inverse natural parameters. On substituting for the inverse parameters using Lemma 4·7, one again arrives at Eq. (4·27).

Remark 4·7. A rather different formula which also relates the natural parameters to the volume of the tetrahedron is

$$s\,(u-z)\,(v-y)\,(w-x)\,(u{+}v{+}w)\,(u{+}x{+}y)\,(v{+}x{+}z)\,(w{+}y{+}z) \tag{4·34}$$

(cf. Eqs. (4·20 *a*), (4·20 *b*) & (4·20 *d*)). Unlike Eq. (4·27) this can be expanded into a polynomial in only the squared exterior areas, and hence may also be expressed as a polynomial in the squared distances. This later polynomial factorizes into a product of the four-point Cayley-Menger determinant and another factor, dubbed the "X-factor," of total degree 5 in the squared distances. By construction the X-factor vanishes whenever any one exterior area equals the sum of the other three, or the sum of any two exterior areas equals the sum of the other two (and hence, in particular, for equi-facial tetrahedra), but it is not necessarily non-negative even in the Euclidean case, and its full geometric interpretation remains an open problem.

The determinantal form of Eq. (4·27) immediately suggests a further extension to Euclidean spaces of dimension $n > 3$, as well. Clearly the analogues of the in-touch triangles of a tetrahedron for a Euclidean n-simplex are the "in-touch $(n-1)$-simplices" into which its facets are divided by their respective in-touch points. There are $n(n+1)$ of these, and it is reasonable to expect that they will also come in congruent pairs. Taking the "hyper-areas" of these pairs of $(n-1)$-simplices as the natural parameters of the n-simplex then leads to the following:

CONJECTURE 4·10. *The hyper-volume of an n-simplex $\overline{\mathsf{AB}\cdots\mathsf{Z}}$ is given in terms of its $(n+1)\,n/2$ natural parameters $u, v, \ldots, w, x, \ldots\, y, \ldots\, z, \ldots$ by*

$$\left(n!\left|\overline{\mathsf{AB}\cdots\mathsf{Z}}\right|\right)^{2(n-1)} = \\ (-1)^n\left(2\,(u{+}v{+}\cdots{+}z{+}\cdots)\right)^{n-1}\det\begin{bmatrix} 0 & u & v & \cdots & w \\ u & 0 & x & \cdots & y \\ v & x & 0 & \cdots & z \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w & y & z & \cdots & 0 \end{bmatrix}. \tag{4·35}$$

Note that the linear factor on the right contributes $(n-1)^2$ to the dimensionality, while the determinantal factor adds another $(n-1)(n+1)$, matching the total dimension of $2\,n\,(n-1)$ on the left.

Remark 4·8. By computing the relevant quantities numerically in multiple random 4-simplices, it has been numerically confirmed that the volumes of the 20 tetrahedra into which the in-touch points divide their facets are equal when they share a 2-face, and that putting these 10 numbers into Eq. (4·35) does indeed give the sixth power of 4! times their hyper-volumes.

5. *The Klein quadric and the combinatorial structure of the zeros*

It is geometrically clear that degenerate tetrahedra with areal Gram matrix at e.g. vertex $\overline{\mathsf{A}}$ of $\mathrm{rank}(\mathbf{G}_{\mathsf{A}}) = 1$ correspond to planar configurations, since the cross-products of the inter-vertex vectors in any such configuration are of course all collinear. The infinitely more common class of zeros for which $\mathrm{rank}(\mathbf{G}_{\mathsf{A}}) = 2$, however, do *not* correspond to planar tetrahedra, nor to any other configuration of points commonly considered in classical Euclidean geometry.

In order to gain some insight into what these are, we may express the squared distances in a three-point Cayley-Menger determinant in terms of the natural parameters just as was done with the four-point determinant in the proof of Theorem 4·8, obtaining e.g.

$$\begin{aligned}\Delta[\overline{\mathsf{A}},\overline{\mathsf{B}},\overline{\mathsf{C}}] \;&=\; -\frac{1}{4}\det\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & u\tilde{u}/r^2 & v\tilde{v}/r^2 \\ 1 & u\tilde{u}/r^2 & 0 & x\tilde{x}/r^2 \\ 1 & v\tilde{v}/r^2 & x\tilde{x}/r^2 & 0 \end{bmatrix} \;=\; \tfrac{1}{4}\,r^{-4}\,\Omega(u,v,x,\tilde{x},\tilde{v},\tilde{u}) \qquad (5{\cdot}1) \\ &=\; s^2\,\Omega(u,v,w,x,y,z)\,(u+v+x)^2\,/t^4 \;=\; (u+v+x)^2 \;=\; 4\left|\overline{\mathsf{ABC}}\right|^2 .\end{aligned}$$

On multiplying through by the fourth power r^4 of the in-radius we see that, in the limit of a degenerate tetrahedron for which $r^4 = \Omega/s^2 = 0$, all four of the three-point Cayley-Menger determinants in the complementary products vanish. Since the four-point Cayley-Menger determinant in the complementary products $r^6\Delta[\overline{\mathsf{A}},\overline{\mathsf{B}},\overline{\mathsf{C}},\overline{\mathsf{D}}] = r^6\,t^2$ also goes to zero, it follows that in any degenerate tetrahedron these complementary products are the squared distances among four points on a Euclidean line. In this same limit, however, the squared distances among the actual vertices of the tetrahedron diverge towards infinity as r^{-2} times the corresponding complementary product, providing that product is not zero. Thus these degenerate tetrahedra can be said to have collinear vertices separated by infinite distances but interior & exterior faces of finite, and generally non-zero, area.

These degenerate tetrahedra cannot be viewed simply as a quadruple of points on a line in the projective completion of Euclidean three-space, because it is possible for *all* the vertices in such configurations to be at infinite distances from each other whereas a line in that completion has only *one* point at infinity! Proposition 5·5 below will show, however, that these configurations determine a set of four vectors in $\mathbb{R}^1$ modulo inversion in the origin, while Remark 5·3 will describe how those vectors, together with the exterior surface area s, should uniquely determine the natural parameters and hence the configuration itself. This implies that, in this context, one should view the "plane" at infinity as a three-dimensional vector space instead of as a projective plane.

The proper interpretation of these unconventional Euclidean configurations within the framework of projective geometry, thereby reaffirming Arthur Cayley's claim that "projective geometry is all geometry" [**1**], will be left as a challenge to the experts in that field (Refs. [**5**, **6**] might be a good place to start). Instead, this section will seek to motivate the further study of such questions, by showing that the "generic" zeros of the formula (4·27) can be placed in a one-to-one correspondence with a certain quotient of the Klein quadric $\mathcal{K}$ by an action of a discrete group of reflections on the Plücker coordinates. It will further explore the combinatorial structure imposed on the set of zeros by all the possible coincidences among the aforementioned collinear vectors, and show (see also Appendix A) that when all four vectors are zero the three remaining degrees of freedom determine a four-point configuration in the special (scale dependent) affine plane.

We begin with a technical lemma which is needed in order to achieve these goals.

Lemma 5·1. *Given any $u,v,w,x,y,z \in \mathbb{R}$ with $s = 2\,(u+v+w+x+y+z) \neq 0$, let $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ be the inverse parameters obtained from Eq. (4·26). Then if any one of the products $u\tilde{u}$, $v\tilde{v}$, $w\tilde{w}$, $x\tilde{x}$, $y\tilde{y}$ or $z\tilde{z}$ vanishes, the polynomial Ω in Eq. (4·27) satisfies $\Omega(u,v,w,x,y,z) \le 0$. Conversely, given $u,v,w,x,y,z \ge 0$ with $\Omega(u,v,w,x,y,z) = 0$, the inverse natural parameters $\tilde{u},\tilde{v},\tilde{w},\tilde{x},\tilde{y},\tilde{z}$ from Eq. (4·26) are all non-negative, as are the complementary products $u\tilde{u}$, $v\tilde{v}$, $w\tilde{w}$, $x\tilde{x}$, $y\tilde{y}$ & $z\tilde{z}$.*

Proof. To prove the first statement, suppose for example $u\tilde{u} = 0$, so that either $u = 0$ or $\tilde{u} = 0$. In the former case we find that $\boldsymbol{\Omega}|_{u=0} = -(wx - vy)^2 \le 0$, whereas if $u \ne 0$ we may solve $\tilde{u} = 2\big((v+x)(w+y) - uz\big)/s = 0$ in Eq. (4·26) for $z^* = (v+x)(w+y)/u$, whence $\boldsymbol{\Omega}|_{z=z^*} = -(vw - xy)^2 \le 0$ as well. The proof if any of the other products vanish is of course similar.

To prove the second part, we solve $\boldsymbol{\Omega}(u,v,w,x,y,z) = 0$ for the product uz, obtaining

$$uz \;=\; wx + vy \,\pm\, 2\sqrt{vwxy} \;=\; \big(\check{w}\check{x} \,\pm\, \check{v}\check{y}\big)^2 \,. \tag{5·2}$$

If $s = 0$ the claim holds vacuously, and otherwise substituting this value of uz into Eq. (4·26) for $\tilde{u}$ yields

$$\tilde{u} \;=\; 2\big(vw + xy \,\pm\, 2\sqrt{vwxy}\big)/s \;=\; 2\big(\check{v}\check{w} \,\pm\, \check{x}\check{y}\big)^2/s \;\ge\; 0\,, \tag{5·3}$$

as desired. The proof for the remaining five products is again similar. □

Remark 5·1. Part two of Lemma 5·1 suggests that $u,v,w,x,y,z \ge 0$ & $\boldsymbol{\Omega}(u,v,w,x,y,z) \ge 0$ implies $\tilde{u},\tilde{v},\tilde{w},\tilde{x},\tilde{y},\tilde{z} \ge 0$, but that is presently a conjecture. A counterexample, if there is one, must have a Gram matrix $\mathbf{G}_{\mathsf{A}}$ with two negative eigenvalues.

The analysis of the "generic" zeros of $\boldsymbol{\Omega}$ that follows benefited greatly from an exposition of the relation between planar polygons and Grassmannians recently given by Cantarella *et al.* [**13**]. Specifically, when $\operatorname{rank}\big(\mathbf{G}_{\mathsf{A}}\big) \le 2$ the areal vectors of the exterior faces of a tetrahedron are coplanar, and when signed correctly sum to zero by Minkowski's identity (2·9) just like the edge vectors of a planar quadrilateral. (Thus it is possible to visualize these degenerate tetrahedra as planar quadrilaterals, but it should be noted that the vertices of such quadrilaterals do *not* correspond to the vertices of the tetrahedra themselves.) The analysis given in Cantarella *et al.* equates the components of four such vectors in $\mathbb{R}^2$ to the real & imaginary parts of the *squares* of four complex numbers $m_{\mathsf{A}}+in_{\mathsf{A}}, \ldots, m_{\mathsf{D}}+in_{\mathsf{D}}$, so that

$$\begin{aligned} \overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}} &= (m_{\mathsf{D}}^2 - n_{\mathsf{D}}^2)\,\mathbf{e}_1 + 2\,m_{\mathsf{D}}n_{\mathsf{D}}\,\mathbf{e}_2\,, & -\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}} &= (m_{\mathsf{C}}^2 - n_{\mathsf{C}}^2)\,\mathbf{e}_1 + 2\,m_{\mathsf{C}}n_{\mathsf{C}}\,\mathbf{e}_2\,, \\ \overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}} &= (m_{\mathsf{B}}^2 - n_{\mathsf{B}}^2)\,\mathbf{e}_1 + 2\,m_{\mathsf{B}}n_{\mathsf{B}}\,\mathbf{e}_2\,, & -\overrightarrow{\mathsf{BC}}\times\overrightarrow{\mathsf{BD}} &= (m_{\mathsf{A}}^2 - n_{\mathsf{A}}^2)\,\mathbf{e}_1 + 2\,m_{\mathsf{A}}n_{\mathsf{A}}\,\mathbf{e}_2\,, \end{aligned} \tag{5·4}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ is any orthonormal basis of their common plane (in practice, the basis obtained by diagonalizing $\mathbf{G}_{\mathsf{A}}$ is a convenient one to use). It follows that the norms of these cross products are given quite simply by

$$m_{\mathsf{A}}^2+n_{\mathsf{A}}^2 \;=\; 2\big|\overline{\mathsf{BCD}}\big|, \quad m_{\mathsf{B}}^2+n_{\mathsf{B}}^2 \;=\; 2\big|\overline{\mathsf{ACD}}\big|, \quad m_{\mathsf{C}}^2+n_{\mathsf{C}}^2 \;=\; 2\big|\overline{\mathsf{ABD}}\big|, \quad m_{\mathsf{D}}^2+n_{\mathsf{D}}^2 \;=\; 2\big|\overline{\mathsf{ABC}}\big|. \tag{5·5}$$

The fact that the vectors in Eq. (5·4) sum to $\mathbf{0}$ further implies that the vectors $\mathbf{m} := [m_{\mathsf{A}},\, m_{\mathsf{B}},\, m_{\mathsf{C}},\, m_{\mathsf{D}}]^\top$, $\mathbf{n} := [n_{\mathsf{A}},\, n_{\mathsf{B}},\, n_{\mathsf{C}},\, n_{\mathsf{D}}]^\top$ in $\mathbb{R}^4$ satisfy $\mathbf{m}\cdot\mathbf{n} = 0$ and $\|\mathbf{m}\|^2 = \|\mathbf{n}\|^2 = s/2$, where s is twice the exterior surface area as before. These vectors are unique up to improper rotations in the plane they span, and a rotation by an angle ϑ in their common plane corresponds to a rotation of the areal vectors by 2ϑ.

We will now derive two equivalent formulae for each of the three interior facial areas in terms of the components of these vectors.

LEMMA 5·2. *With everything defined as above:*

$$\begin{aligned} 16\,\big|\overline{\mathsf{AB|CD}}\big|^2 &= \big((m_{\mathsf{B}}+n_{\mathsf{A}})^2 + (m_{\mathsf{A}}-n_{\mathsf{B}})^2\big)\big((m_{\mathsf{B}}-n_{\mathsf{A}})^2 + (m_{\mathsf{A}}+n_{\mathsf{B}})^2\big) \\ &= \big((m_{\mathsf{D}}+n_{\mathsf{C}})^2 + (m_{\mathsf{C}}-n_{\mathsf{D}})^2\big)\big((m_{\mathsf{D}}-n_{\mathsf{C}})^2 + (m_{\mathsf{C}}+n_{\mathsf{D}})^2\big); \end{aligned} \tag{5·6 a}$$

$$\begin{aligned}16\left|\overline{\mathsf{AC|BD}}\right|^2 &= \left((m_\mathsf{D}+n_\mathsf{B})^2 + (m_\mathsf{B}-n_\mathsf{D})^2\right)\left((m_\mathsf{D}-n_\mathsf{B})^2 + (m_\mathsf{B}+n_\mathsf{D})^2\right) \\ &= \left((m_\mathsf{C}+n_\mathsf{A})^2 + (m_\mathsf{A}-n_\mathsf{C})^2\right)\left((m_\mathsf{C}-n_\mathsf{A})^2 + (m_\mathsf{A}+n_\mathsf{C})^2\right);\end{aligned} \tag{5·6 b}$$

$$\begin{aligned}16\left|\overline{\mathsf{AD|BC}}\right|^2 &= \left((m_\mathsf{D}+n_\mathsf{A})^2 + (m_\mathsf{A}-n_\mathsf{D})^2\right)\left((m_\mathsf{D}-n_\mathsf{A})^2 + (m_\mathsf{A}+n_\mathsf{D})^2\right) \\ &= \left((m_\mathsf{C}+n_\mathsf{B})^2 + (m_\mathsf{B}-n_\mathsf{C})^2\right)\left((m_\mathsf{C}-n_\mathsf{B})^2 + (m_\mathsf{B}+n_\mathsf{C})^2\right).\end{aligned} \tag{5·6 c}$$

Proof. To prove the first of the above formulae, we use Eq. (5·4) and the usual expression for the dot product of vectors in terms of their coordinates to obtain

$$-\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right)\cdot\left(\overrightarrow{\mathsf{BC}}\times\overrightarrow{\mathsf{BD}}\right) = (m_\mathsf{A}^2-n_\mathsf{A}^2)\,(m_\mathsf{B}^2-n_\mathsf{B}^2) + 4\,m_\mathsf{A}n_\mathsf{A}m_\mathsf{B}n_\mathsf{B}\ .$$

By the areal law of cosines (2·11) together with Eq. (5·5), however, $16\left|\overline{\mathsf{AB|CD}}\right|^2 =$

$$\begin{aligned}\left\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\right\|^2 &= \left\|\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right\|^2 + \left\|\overrightarrow{\mathsf{BC}}\times\overrightarrow{\mathsf{BD}}\right\|^2 - 2\left(\overrightarrow{\mathsf{AC}}\times\overrightarrow{\mathsf{AD}}\right)\cdot\left(\overrightarrow{\mathsf{BC}}\times\overrightarrow{\mathsf{BD}}\right) \\ &= \left(m_\mathsf{A}^2+n_\mathsf{A}^2\right)^2 + \left(m_\mathsf{B}^2+n_\mathsf{B}^2\right)^2 + 2\,(m_\mathsf{A}^2-n_\mathsf{A}^2)\,(m_\mathsf{B}^2-n_\mathsf{B}^2) + 8\,m_\mathsf{A}n_\mathsf{A}m_\mathsf{B}n_\mathsf{B} \\ &= \left((m_\mathsf{B}+n_\mathsf{A})^2 + (m_\mathsf{A}-n_\mathsf{B})^2\right)\left((m_\mathsf{B}-n_\mathsf{A})^2 + (m_\mathsf{A}+n_\mathsf{B})^2\right)\ .\end{aligned}$$

An analogous procedure, applied to the alternative expression $\|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{CD}}\|^2 = \|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}}\|^2 + \|\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}}\|^2 - 2\,(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AC}})\cdot(\overrightarrow{\mathsf{AB}}\times\overrightarrow{\mathsf{AD}})$, yields the second formula for $16\left|\overline{\mathsf{AB|CD}}\right|^2$. The remaining formulae can be established in an analogous fashion. □

These results allow us to express the natural and inverse natural parameters of a degenerate tetrahedron in terms of the m's & n's quite simply as follows.

Proposition 5·3. *Given a tetrahedron with volume* $\left|\overline{\mathsf{ABCD}}\right| = 0$ *and exterior surface area (times 2)* $s > 0$, *together with vectors* $\mathbf{m}, \mathbf{n} \in \mathbb{R}^4$ *as above, the natural parameters are given by*

$$\begin{aligned} u &= 2\,(m_\mathsf{C}n_\mathsf{D}-m_\mathsf{D}n_\mathsf{C})^2/s\ , & z &= 2\,(m_\mathsf{A}n_\mathsf{B}-m_\mathsf{B}n_\mathsf{A})^2/s\ , \\ v &= 2\,(m_\mathsf{B}n_\mathsf{D}-m_\mathsf{D}n_\mathsf{B})^2/s\ , & y &= 2\,(m_\mathsf{A}n_\mathsf{C}-m_\mathsf{C}n_\mathsf{A})^2/s\ , \\ w &= 2\,(m_\mathsf{B}n_\mathsf{C}-m_\mathsf{C}n_\mathsf{B})^2/s\ , & x &= 2\,(m_\mathsf{A}n_\mathsf{D}-m_\mathsf{D}n_\mathsf{A})^2/s\ ,\end{aligned} \tag{5·7}$$

while the inverse natural parameters are given by

$$\begin{aligned} \tilde u &= 2\,(m_\mathsf{C}m_\mathsf{D}+n_\mathsf{D}n_\mathsf{C})^2/s\ , & \tilde z &= 2\,(m_\mathsf{A}m_\mathsf{B}+n_\mathsf{B}n_\mathsf{A})^2/s\ , \\ \tilde v &= 2\,(m_\mathsf{B}m_\mathsf{D}+n_\mathsf{D}n_\mathsf{B})^2/s\ , & \tilde y &= 2\,(m_\mathsf{A}m_\mathsf{C}+n_\mathsf{C}n_\mathsf{A})^2/s\ , \\ \tilde w &= 2\,(m_\mathsf{B}m_\mathsf{C}+n_\mathsf{C}n_\mathsf{B})^2/s\ , & \tilde x &= 2\,(m_\mathsf{A}m_\mathsf{D}+n_\mathsf{D}n_\mathsf{A})^2/s\ .\end{aligned} \tag{5·8}$$

If $s = 0$, *of course, the natural and inverse natural parameters are all zero as well.*

Proof. Upon substituting for the exterior & interior areas in the expression for u from Proposition 4·2 and using Eqs. (5·5) & (5·6), we obtain

$$\begin{aligned} 2\,su &= \left(2\left|\overline{\mathsf{ABC}}\right| + 2\left|\overline{\mathsf{ABD}}\right|\right)^2 - 16\left|\overline{\mathsf{AB|CD}}\right|^2 = \\ &\left(m_\mathsf{D}^2+n_\mathsf{D}^2+m_\mathsf{C}^2+n_\mathsf{C}^2\right)^2 - \left((m_\mathsf{D}+n_\mathsf{C})^2+(m_\mathsf{C}-n_\mathsf{D})^2\right)\left((m_\mathsf{D}-n_\mathsf{C})^2+(m_\mathsf{C}+n_\mathsf{D})^2\right) \\ &= 4\,(m_\mathsf{C}n_\mathsf{D}-m_\mathsf{D}n_\mathsf{C})^2\ ,\end{aligned}$$

as desired. Similarly, upon substituting for the areas in the expression for $\tilde u$ from Proposition 4·3, we obtain

$$\begin{aligned} 2\,s\tilde u &= 16\left|\overline{\mathsf{AB|CD}}\right|^2 - \left(2\left|\overline{\mathsf{ABC}}\right| - 2\left|\overline{\mathsf{ABD}}\right|\right)^2 = \\ &\left((m_\mathsf{D}+n_\mathsf{C})^2+(m_\mathsf{C}-n_\mathsf{D})^2\right)\left((m_\mathsf{D}-n_\mathsf{C})^2+(m_\mathsf{C}+n_\mathsf{D})^2\right) - \left(m_\mathsf{D}^2+n_\mathsf{D}^2-m_\mathsf{C}^2-n_\mathsf{C}^2\right)^2 \\ &= 4\,(m_\mathsf{C}m_\mathsf{D}+n_\mathsf{D}n_\mathsf{C})^2\ ,\end{aligned}$$

The proofs of the expressions for the remaining natural and inverse natural parameters are analogous. □

The above expressions for the natural parameters involve the Plücker coordinates in the exterior product $\mathbf{m} \wedge \mathbf{n}$ of $\mathbf{m}, \mathbf{n} \in \mathbb{R}^4$; these will henceforth be denoted by

$$\begin{aligned} p_{\mathsf{AB}} &:= m_{\mathsf{A}} n_{\mathsf{B}} - m_{\mathsf{B}} n_{\mathsf{A}}\,, \quad & p_{\mathsf{AC}} &:= m_{\mathsf{A}} n_{\mathsf{C}} - m_{\mathsf{C}} n_{\mathsf{A}}\,, \quad & p_{\mathsf{AD}} &:= m_{\mathsf{A}} n_{\mathsf{D}} - m_{\mathsf{D}} n_{\mathsf{A}}\,, \\ p_{\mathsf{BC}} &:= m_{\mathsf{B}} n_{\mathsf{C}} - m_{\mathsf{C}} n_{\mathsf{B}}\,, \quad & p_{\mathsf{BD}} &:= m_{\mathsf{B}} n_{\mathsf{D}} - m_{\mathsf{D}} n_{\mathsf{B}}\,, \quad & p_{\mathsf{CD}} &:= m_{\mathsf{C}} n_{\mathsf{D}} - m_{\mathsf{D}} n_{\mathsf{C}}\,, \end{aligned} \tag{5·9}$$

so that $\mathbf{m} \wedge \mathbf{n} = [p_{\mathsf{AB}}, \ldots, p_{\mathsf{CD}}]^\top \in \mathbb{R}^6$. As is well known (see e.g. [**13**, **41**]), these satisfy the Plücker identity $p_{\mathsf{AB}} p_{\mathsf{CD}} - p_{\mathsf{AC}} p_{\mathsf{BD}} + p_{\mathsf{AD}} p_{\mathsf{BC}} = 0$, which in turn defines the Klein quadric $\mathcal{K} = \{\mathbf{p} \in \mathbb{R}^6 \mid p_1 p_6 - p_2 p_5 + p_3 p_4 = 0\}$. This notation, together with Proposition 5·3, allows us to express the products of opposite pairs of natural parameters as

$$u\,z \;=\; 4\,p_{\mathsf{AB}}^2\,p_{\mathsf{CD}}^2/s^2\;, \quad v\,y \;=\; 4\,p_{\mathsf{AC}}^2\,p_{\mathsf{BD}}^2/s^2\;, \quad w\,x \;=\; 4\,p_{\mathsf{AD}}^2\,p_{\mathsf{BC}}^2/s^2\;, \tag{5·10}$$

where the squared exterior surface area is given by $s^2/4 = \mathbf{m}^2\mathbf{n}^2 - (\mathbf{m} \cdot \mathbf{n})^2 = \|\mathbf{m} \wedge \mathbf{n}\|^2$.

The key to getting a correspondence between the natural parameters of degenerate tetrahedra and the Klein quadric, viewed non-projectively as a five-dimensional variety $\mathcal{K} \subset \mathbb{R}^6$, is to observe the similarity between the factors in Eq. (4·32), which factorizes the polynomial $\mathbf{\Omega}$ of Eq. (4·27) into four quadratic factors in the square-roots $\breve{u}, \ldots, \breve{z}$ of the natural parameters, and the Plücker identity. Which of these quadratic factors corresponds to the Plücker identity depends on the relative signs of the products of "opposite" pairs of Plücker coordinates computed from the natural parameters as above. Specifically, by Eq. (5·10) together with the Plücker identity itself,

$$\begin{aligned} &(s/2)\,\big(\mathrm{sign}(p_{\mathsf{AB}} p_{\mathsf{CD}})\,\breve{u}\breve{z} \;-\; \mathrm{sign}(p_{\mathsf{AC}} p_{\mathsf{BD}})\,\breve{v}\breve{y} \;+\; \mathrm{sign}(p_{\mathsf{AD}} p_{\mathsf{BC}})\,\breve{w}\breve{x}\big) \;= \\ &\qquad \mathrm{sign}(p_{\mathsf{AB}} p_{\mathsf{CD}})\,|p_{\mathsf{AB}} p_{\mathsf{CD}}| \;-\; \mathrm{sign}(p_{\mathsf{AC}} p_{\mathsf{BD}})\,|p_{\mathsf{AC}} p_{\mathsf{BD}}| \;+\; \mathrm{sign}(p_{\mathsf{AD}} p_{\mathsf{BC}})\,|p_{\mathsf{AD}} p_{\mathsf{BC}}| \\ &\qquad\qquad = \; p_{\mathsf{AB}}\,p_{\mathsf{CD}} \;-\; p_{\mathsf{AC}}\,p_{\mathsf{BD}} \;+\; p_{\mathsf{AD}}\,p_{\mathsf{BC}} \;=\; 0\;. \end{aligned} \tag{5·11}$$

Clearly there are eight possible combinations of signs for the three terms in this equation, each of which corresponds to one of the four factors in Eq. (4·32) or (equivalently) its negative vanishing. The two corresponding to the first factor in Eq. (4·32), namely those with $\mathrm{sign}(p_{\mathsf{AB}} p_{\mathsf{CD}}) = -\mathrm{sign}(p_{\mathsf{AC}} p_{\mathsf{BD}}) = \mathrm{sign}(p_{\mathsf{AD}} p_{\mathsf{BC}})$, only hold when all three products of opposite pairs of Plücker coordinates vanish simultaneously. In the following, these factors will be denoted by

$$\begin{aligned} \mathbf{\Omega}_0(\breve{u},\breve{v},\breve{w},\breve{x},\breve{y},\breve{z}) &:= \breve{u}\breve{z} + \breve{v}\breve{y} + \breve{w}\breve{x}\,, \quad & \mathbf{\Omega}_1(\breve{u},\breve{v},\breve{w},\breve{x},\breve{y},\breve{z}) &:= \breve{v}\breve{y} + \breve{w}\breve{x} - \breve{u}\breve{z}\,, \\ \mathbf{\Omega}_2(\breve{u},\breve{v},\breve{w},\breve{x},\breve{y},\breve{z}) &:= \breve{w}\breve{x} + \breve{u}\breve{z} - \breve{v}\breve{y}\,, \quad & \mathbf{\Omega}_3(\breve{u},\breve{v},\breve{w},\breve{x},\breve{y},\breve{z}) &:= \breve{u}\breve{z} + \breve{v}\breve{y} - \breve{w}\breve{x}\,. \end{aligned} \tag{5·12}$$

The main complication to be dealt with in fully defining the correspondence between the zeros of $\mathbf{\Omega}$ and $\mathcal{K}$ stems from the fact that the signs of the square-roots of the four complex numbers in Eq. 5·4 are arbitrary. To simplify the presentation we will restrict ourselves for now to zeros of $\mathbf{\Omega}$ where the natural parameters are all strictly positive and none of the Plücker coordinates vanish, so they correspond to points on the non-degenerate Klein quadratic $\mathcal{K}^* := \{\,\mathbf{p} \in \mathcal{K} \mid p_1, \ldots, p_6 \neq 0\,\}$. Then, starting from any given choice of signs for the square-roots of these four complex numbers, all the others are obtained by the reflecting the vectors $\mathbf{m}, \mathbf{n} \in \mathbb{R}^4$ in the subspaces orthogonal to the four coordinate axes. The group generated by these reflections is isomorphic to the direct product $\mathbb{Z}_2^4$ of four cyclic groups $\mathbb{Z}_2$ of order 2, and the action of this group on the Plücker coordinates $\mathbf{m} \wedge \mathbf{n}$ either maintains the signs of the products $p_{\mathsf{AB}} p_{\mathsf{CD}}$, $p_{\mathsf{AC}} p_{\mathsf{BD}}$, $p_{\mathsf{AD}} p_{\mathsf{BC}}$

or else changes all those signs identically. More specifically, starting with the Plücker coordinates $[p_{\mathsf{AB}}, p_{\mathsf{AC}}, p_{\mathsf{AD}}, p_{\mathsf{BC}}, p_{\mathsf{BD}}, p_{\mathsf{CD}}]$ obtained from any given choice of the square-roots' signs, it is easily verified that the 8 compositions of even numbers of these four reflections yield new coordinates of the form $[\epsilon_1 p_{\mathsf{AB}}, \epsilon_2 p_{\mathsf{AC}}, \epsilon_3 p_{\mathsf{AD}}, \epsilon_3 p_{\mathsf{BC}}, \epsilon_2 p_{\mathsf{BD}}, \epsilon_1 p_{\mathsf{CD}}]$ for $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, +1\}$, while the 8 produced by odd numbers of reflections are of the form $[\epsilon_1 p_{\mathsf{AB}}, \epsilon_2 p_{\mathsf{AC}}, \epsilon_3 p_{\mathsf{AD}}, -\epsilon_3 p_{\mathsf{BC}}, -\epsilon_2 p_{\mathsf{BD}}, -\epsilon_1 p_{\mathsf{CD}}]$. In all cases the *relative* signs of the products of opposite pairs are well defined, so the correspondence given by Eq. (5·11) between the signs of the Plücker coordinates and the factor of $\mathbf{\Omega}$ in Eq. (5·12) that vanishes is also well defined. In what follows, the 16-point sets generated by this action of $\mathbb{Z}_2^4$ on $\mathcal{K}^*$ will be denoted as $\mathbb{Z}_2^4 \circ \mathcal{K}^*$, and the corresponding quotient space by $\mathcal{K}^*/\mathbb{Z}_2^4$.

THEOREM 5·4. *The natural parameters $u, v, w, x, y, z > 0$ of a tetrahedron $\overline{\mathsf{ABCD}}$ with $\mathbf{\Omega}(u, v, w, x, y, z) = 0$ are in one-to-one correspondence with points in the quotient space $\mathcal{K}^*/\mathbb{Z}_2^4$ via Eq. (5·11), or equivalently, with 16-point subsets of $\mathcal{K}^*$ wherein these subsets are those generated by the group action $\mathbb{Z}_2^4 \circ \mathcal{K}^*$.*

Proof. First, note that the given conditions together with the second part of Lemma 5·1 imply that the inverse parameters $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}$ are non-negative, so the parameters u, v, w, x, y, z determine a proper but degenerate Euclidean tetrahedron. We may now define a corresponding set of Plücker coordinates as follows:

$$[p_{\mathsf{AB}}, p_{\mathsf{AC}}, p_{\mathsf{AD}}, p_{\mathsf{BC}}, p_{\mathsf{BD}}, p_{\mathsf{CD}}] \;:=\; \begin{cases} \sqrt{s/2}\,[\check{z},\ \check{y},\ \check{x},\ \ \check{w},\ -\check{v},\ -\check{u}] & \text{if } \mathbf{\Omega}_1 = 0; \\ \sqrt{s/2}\,[\check{z},\ \check{y},\ \check{x},\ \ \check{w},\ \ \check{v},\ \ \check{u}] & \text{if } \mathbf{\Omega}_2 = 0; \\ \sqrt{s/2}\,[\check{z},\ \check{y},\ \check{x},\ -\check{w},\ -\check{v},\ \ \check{u}] & \text{if } \mathbf{\Omega}_3 = 0; \end{cases}$$

(note that $\mathbf{\Omega}_0 > 0$ since $\check{u}, \ldots, \check{z}$ are strictly positive). These Plücker coordinates will satisfy the Plücker identity, and letting $\mathbb{Z}_2^4$ act on them as previously described will generate a set of 16 distinct points on $\mathcal{K}^*$ corresponding to the same $\mathbf{\Omega}_k = 0$ $(k = 1, 2, 3)$.

Conversely, given any set of 16 vectors of Plücker coordinates in $\mathbb{Z}_2^4 \circ \mathcal{K}^*$, the relative signs of the products of the three opposite pairs of coordinates will determine which of the factors $\mathbf{\Omega}_k$ $(k = 1, 2, 3)$ of $\mathbf{\Omega}$ vanishes, where the square-roots of the natural parameters in question are $\sqrt{2/s}$ times the coordinates' absolute values as in Proposition 5·3. It follows that these natural parameters satisfy $\mathbf{\Omega} = 0$ as desired; the non-negativity of the inverse natural parameters follows from the second part of Lemma 5·1 as before, so these natural parameters indeed correspond to a proper degenerate Euclidean tetrahedron. □

Remark 5·2. Note that these 16 point subsets of $\mathcal{K}^*$ will consist of 8 "antipodal" pairs related by an overall change of sign. In addition, given any point on the Klein quadric $\mathcal{K}$ for which one or more of the Plücker coordinates are zero, these sign changes will of course have no effect on those coordinates and hence some of the 16 points in $\mathbb{Z}_2^4 \circ \mathcal{K}$ will coincide. While exactly two non-opposite natural parameters cannot vanish unless $\mathbf{\Omega}(u, \ldots, z) < 0$, it is possible for exactly three non-opposite parameters to vanish, in which case all four factors in Eq. (5·12) will vanish and the signs of the three non-zero Plücker coordinates can be chosen arbitrarily, giving rise to only four antipodal pairs of points on $\mathcal{K}$. For example, if $[u, v, w, x, y, z] = [1, 1, 1, 0, 0, 0]$, then $p_{\mathsf{AB}} = p_{\mathsf{AC}} = p_{\mathsf{AD}} = 0$ while $p_{\mathsf{BC}}, p_{\mathsf{BD}}, p_{\mathsf{CD}} = \pm\sqrt{3}$ and $\operatorname{rank}(\mathbf{G}_{\mathsf{A}}) = 2$. The example $[u, v, w, x, y, z] = [0, 0, 0, 1, 1, 1]$, where $p_{\mathsf{AB}}, p_{\mathsf{AC}}, p_{\mathsf{AD}} = \pm\sqrt{3}$ while $p_{\mathsf{BC}} = p_{\mathsf{BD}} = p_{\mathsf{CD}} = 0$, shows that this can also happen when $\operatorname{rank}(\mathbf{G}_{\mathsf{A}}) = 1$.

Recall that when $\Omega = 0$ the products of complementary pairs of natural and inverse natural parameters equal the squared distances among four points on a line. Our next result shows that the positions of these points relative to the origin matters, so they are better seen as sets of four collinear vectors modulo inversion in the origin.

PROPOSITION 5·5. *Given a tetrahedron* $\overline{\mathsf{ABCD}}$ *for which* $\Omega(u, v, w, x, y, z) = 0$*, the complementary products* $u\tilde{u}$*, ...,* $z\tilde{z}$ *are the squared distances among at least one of the four quadruples of points given relative to the origin by the following four quadruples of vectors in* $\mathbb{R}^1$ *(wherein* $\check{s} := \sqrt{s/2}$*):*

$$\begin{aligned}
&\{\, \breve{u}\breve{v}\breve{w}/\check{s}, && \breve{u}\breve{x}\breve{y}/\check{s}, && \breve{v}\breve{x}\breve{z}/\check{s}, && \breve{w}\breve{y}\breve{z}/\check{s} \,\} && \text{if} \ \ \Omega_0(\breve{u}, \breve{v}, \breve{w}, \breve{x}, \breve{y}, \breve{z}) \ = \ 0\,; \\
&\{\, \breve{u}\breve{v}\breve{w}/\check{s}, && \breve{u}\breve{x}\breve{y}/\check{s}, && -\breve{v}\breve{x}\breve{z}/\check{s}, && -\breve{w}\breve{y}\breve{z}/\check{s} \,\} && \text{if} \ \ \Omega_1(\breve{u}, \breve{v}, \breve{w}, \breve{x}, \breve{y}, \breve{z}) \ = \ 0\,; \\
&\{\, \breve{u}\breve{v}\breve{w}/\check{s}, && -\breve{u}\breve{x}\breve{y}/\check{s}, && \breve{v}\breve{x}\breve{z}/\check{s}, && -\breve{w}\breve{y}\breve{z}/\check{s} \,\} && \text{if} \ \ \Omega_2(\breve{u}, \breve{v}, \breve{w}, \breve{x}, \breve{y}, \breve{z}) \ = \ 0\,; \\
&\{\, \breve{u}\breve{v}\breve{w}/\check{s}, && -\breve{u}\breve{x}\breve{y}/\check{s}, && -\breve{v}\breve{x}\breve{z}/\check{s}, && \breve{w}\breve{y}\breve{z}/\check{s} \,\} && \text{if} \ \ \Omega_3(\breve{u}, \breve{v}, \breve{w}, \breve{x}, \breve{y}, \breve{z}) \ = \ 0\,.
\end{aligned} \tag{5·13}$$

Proof. Let us define:

$$\begin{aligned}
\tilde{u}_\pm \ &:= \ \frac{2\,(\breve{v}\breve{w} \ \pm \ \breve{x}\breve{y})^2}{s}\,, \quad & \tilde{v}_\pm \ &:= \ \frac{2\,(\breve{u}\breve{w} \ \pm \ \breve{x}\breve{z})^2}{s}\,, \quad & \tilde{w}_\pm \ &:= \ \frac{2\,(\breve{u}\breve{v} \ \pm \ \breve{y}\breve{z})^2}{s}\,, \\
\tilde{x}_\pm \ &:= \ \frac{2\,(\breve{u}\breve{y} \ \pm \ \breve{v}\breve{z})^2}{s}\,, \quad & \tilde{y}_\pm \ &:= \ \frac{2\,(\breve{u}\breve{x} \ \pm \ \breve{w}\breve{z})^2}{s}\,, \quad & \tilde{z}_\pm \ &:= \ \frac{2\,(\breve{v}\breve{x} \ \pm \ \breve{w}\breve{y})^2}{s}\,.
\end{aligned} \tag{5·14}$$

Then if we regard $\tilde{u}, \ldots, \tilde{z}$ as rational functions of the natural parameters $u, \ldots, z$ as in Eq. (4·26), it is easily shown that when e.g. $\Omega_1(\breve{u}, \breve{v}, \breve{w}, \breve{x}, \breve{y}, \breve{z}) = 0$:

$$\begin{aligned}
\tilde{u}\,|_{uz=(\breve{v}\breve{y}+\breve{w}\breve{x})^2} \ &= \ \tilde{u}_- \,; \qquad & \tilde{z}\,|_{uz=(\breve{v}\breve{y}+\breve{w}\breve{x})^2} \ &= \ \tilde{z}_- \,; \\
\tilde{v}\,|_{vy=(\breve{u}\breve{z}-\breve{w}\breve{x})^2} \ &= \ \tilde{v}_+ \,; \qquad & \tilde{y}\,|_{vy=(\breve{u}\breve{z}-\breve{w}\breve{x})^2} \ &= \ \tilde{y}_+ \,; \\
\tilde{w}\,|_{wx=(\breve{u}\breve{z}-\breve{v}\breve{y})^2} \ &= \ \tilde{w}_+ \,; \qquad & \tilde{x}\,|_{wx=(\breve{u}\breve{z}-\breve{v}\breve{y})^2} \ &= \ \tilde{x}_+ \,.
\end{aligned}$$

It is now a matter of inspection to verify that, on multiplying these expressions for the inverse parameters by the corresponding natural parameters, the resulting complementary products are indeed the squared distances between the points specified on the second line of Eq. (5·13). The proofs for the remaining three cases are similar. □

Remark 5·3. It may be possible to derive a less-redundant, 2-to-1 parametrization of the set of generic degenerate tetrahedra by taking four distinct vectors $\alpha, \beta, \gamma, \delta \in \mathbb{R}^1$ such that $\alpha\beta\gamma\delta > 0$, together with the exterior surface area s, as the parameters, subject to s exceeding a certain lower bound determined by these four vectors. Their relative signs determine which factor Ω_k of Ω in Eq. (5·12) vanishes, which together with $uvw = \alpha^2$, $uxy = \beta^2$, $vxz = \gamma^2$, $wyz = \delta^2$ & $s = 2(u+v+w+x+y+z)$ should uniquely determine the natural parameters. The key is to note that these definitions imply

$$(\beta\gamma)/(\alpha\delta) \ = \ x/w\,, \qquad (\beta\delta)/(\alpha\gamma) \ = \ y/v\,, \qquad (\gamma\delta)/(\alpha\beta) \ = \ z/u\,, \tag{5·15}$$

which may be used to eliminate three of the natural parameters from $\Omega = 0$ and from the equation for s. This yields two linear equations in three unknowns, say u, v, w, which may be used to eliminate two of the three natural parameters from $uvw = \alpha^2$ resulting in a cubic equation for the remaining parameter. Numerical examples suggest that this cubic always has three real roots, and one of its non-negative roots should, when back-substituted into the linear equations, also yield non-negative values for the remaining natural parameters; moreover the same parameters should be obtained if all four vectors are negated. Further discussion of this approach may be found in Appendix C.

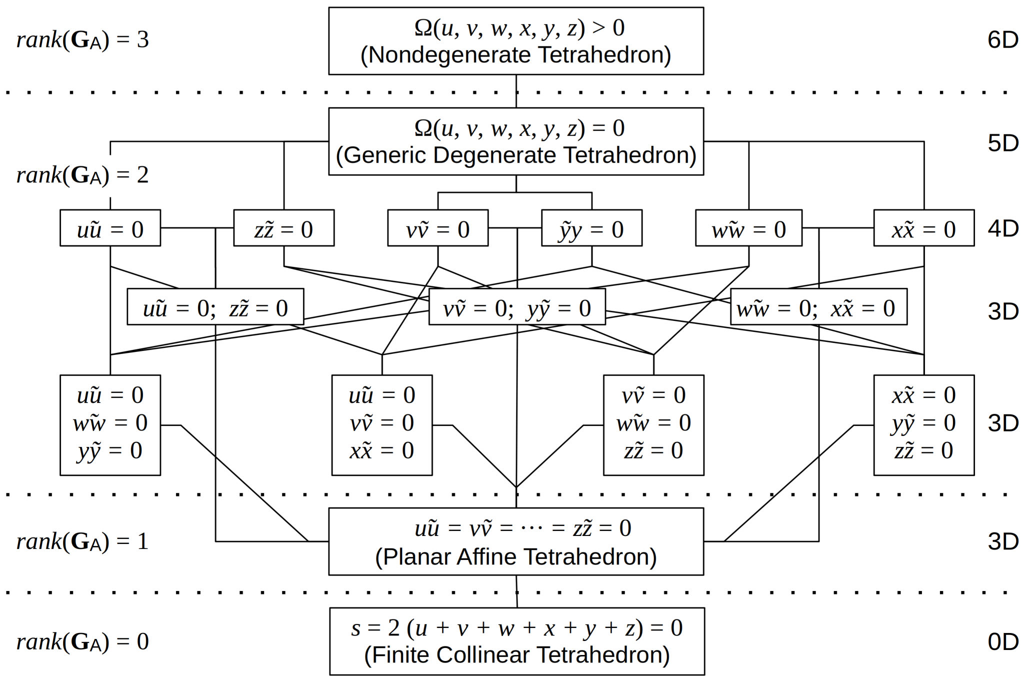

Fig. 4. The lattice of inclusions in $\mathbb{R}^6_{\geq 0}$ defined by $\boldsymbol{\Omega}(u, \ldots, z) = 0$ and all possible subsets of the equations $u\tilde{u} = 0, \ldots, z\tilde{z} = 0$ (see text). The ranks of the areal Gram matrix $\mathbf{G}_{\mathsf{A}}$ at the levels separated by dotted lines are shown on the left, while the generic dimensions of the sets defined by the equations at each level are shown on the right.

Note that if $\boldsymbol{\Omega}_0 = 0$ some triple of non-opposite parameters must vanish (e.g. $u, v, w = 0$; $u, v, x = 0$; etc.), in which case at most one of the vectors $\alpha, \beta, \gamma, \delta$ is non-zero, and that all these vectors are zero whenever any pair of opposite parameters vanishes (e.g. $u, z = 0$).

We now turn to the non-generic case in which one or more of the four collinear vectors coincide, or equivalently, some of the complementary products $u\tilde{u}$, $v\tilde{v}$ etc. vanish. The fact that "distance equals zero" is an equivalence relation implies that only certain combinations of the complementary products can vanish simultaneously, which in turn implies the lattice of inclusions depicted in Fig. 4 among the semi-algebraic sets defined by such subsets of the equations $u\tilde{u} = 0, \ldots, z\tilde{z} = 0$. Via Theorem 5·4, this combinatorial structure leads to an apparently novel stratification of the Grassmannian corresponding to $\mathcal{K}$, although that will not be further explored here.

While the semi-algebraic set in $\mathbb{R}^6_{\geq 0}$ defined by $\boldsymbol{\Omega} = 0$ & $\tilde{u}, \ldots \tilde{z} \geq 0$ is five-dimensional, requiring the product of any one complementary pair to vanish reduces the dimension to four. The four exterior areas can then be used as local coordinates on each of these six 4-dimensional sets since, as will now be shown, the three interior areas can be computed from them. Clearly one of the interior areas is given by the sum or a difference of two of the exterior areas, depending on which tetrahedron inequality saturates (i.e. $\mathcal{T}_k[\mathsf{a}, \mathsf{b}] = 0$) in order to make the corresponding complementary product vanish. To see how to obtain the other two, suppose for example that $u = 0 \;\Rightarrow\; 2\left|\overline{\mathsf{AB|CD}}\right| = \left|\overline{\mathsf{ABC}}\right| + \left|\overline{\mathsf{ABD}}\right|$. Upon using Yetter's identity $\Xi = 0$ to eliminate $\left|\overline{\mathsf{AD|BC}}\right|$ from the Grammian $\det(\mathbf{G}_{\mathsf{A}})$ and then eliminating $|\overline{\mathsf{AB|CD}}|$ from the result using that linear relation, we obtain

$$0 \;\le\; \det\!\big(\mathbf{G}_{\mathsf{A}}\big) \;=\; -16\Big(|\overline{\mathsf{ABC}}|\big(|\overline{\mathsf{ABD}}|^2-|\overline{\mathsf{BCD}}|^2\big)+|\overline{\mathsf{ABD}}|\big(|\overline{\mathsf{ABC}}|^2-|\overline{\mathsf{ACD}}|^2\big) \qquad (5{\cdot}16)$$
$$+\;4\,|\overline{\mathsf{AC|BD}}|^2\big(|\overline{\mathsf{ABC}}|+|\overline{\mathsf{ABD}}|\big)\Big)^2\,.$$

Thus the polynomial inside the main parentheses vanishes and can readily be solved to obtain $|\overline{\mathsf{AC|BD}}|^2$, whereupon $|\overline{\mathsf{AD|BC}}|^2$ may be obtained from Yetter's identity. In a similar fashion one can show that the exterior areas uniquely determine the interior whenever any single one of the tetrahedron inequalities saturates.

Requiring more than one complementary product to vanish immediately reduces the generic dimensionality of the set of degenerate tetrahedra consistent with that requirement to three. In contradistinction with intersection theory over algebraically closed fields, the dimension remains three even as additional complementary products are required to vanish; this, of course, is because these semi-algebraic sets intersect non-transversely in $\mathbb{R}^6$. In all of these cases Yetter's identity, a vanishing Grammian and the linear equations corresponding to the saturated tetrahedron inequalities can be solved to express the exterior areas in terms of the *interior*. Although a full proof would involve a fairly intricate case-by-case analysis, it appears that in all cases these expressions are simply signed sums of the three interior areas, i.e. linear combinations with coefficients equal to ± 1. These coefficients are determined by which combinations of the eighteen tetrahedron inequalities are required to saturate.

At the lowest level of the hierarchy of three-dimensional solutions all six complementary products vanish, meaning that $uvw = uxy = vxz = wyz$, and at least one $\mathcal{T}_k[\mathsf{a},\mathsf{b}] = 0$ in each of the six triples of tetrahedron inequalities. These six linear equations in the four unknown exterior areas generally do not have an exact solution, but when they do it is a signed sum of the interior areas as above which also satisfies Yetter's identity as well as $\mathbf{\Omega} = 0$. We close this section by proving, as indicated in Fig. 4, that this level of the lattice corresponds to those configurations for which $\operatorname{rank}\!\big(\mathbf{G}_{\mathsf{A}}\big) = 1$.

PROPOSITION 5·6. *Given a (not-necessarily-degenerate) Euclidean tetrahedron* $\overline{\mathsf{ABCD}}$ *with natural parameters* u, v, w, x, y, z *and inverse natural parameters* $\tilde u, \tilde v, \tilde w, \tilde x, \tilde y, \tilde z$*, the rank of the Gram matrix* $\mathbf{G}_{\mathsf{A}}$ *at vertex* $\overline{\mathsf{A}}$ *(or any other vertex) equals* 1 *if & only if*

$$u\tilde u \;=\; v\tilde v \;=\; w\tilde w \;=\; x\tilde x \;=\; y\tilde y \;=\; z\tilde z \;=\; 0\,. \qquad (5{\cdot}17)$$

Thus the planar situation is characterized by these complementarity relations between the natural and inverse natural parameters.

Proof. In proving this proposition, it is convenient to consider the full 4×4 Gram matrix of the outwards-pointing areal vectors of the exterior faces, namely $\mathbf{G}_{\mathsf{ABCD}} :=$

$$\begin{bmatrix} F_{\mathsf{ABC}} & \tfrac12\big(F_{\mathsf{AB|CD}}-F_{\mathsf{ABC}}-F_{\mathsf{ABD}}\big) & \tfrac12\big(F_{\mathsf{AC|BD}}-F_{\mathsf{ABC}}-F_{\mathsf{ACD}}\big) & \tfrac12\big(F_{\mathsf{AD|BC}}-F_{\mathsf{ABC}}-F_{\mathsf{BCD}}\big) \\ \tfrac12\big(F_{\mathsf{AB|CD}}-F_{\mathsf{ABC}}-F_{\mathsf{ABD}}\big) & F_{\mathsf{ABD}} & \tfrac12\big(F_{\mathsf{AD|BC}}-F_{\mathsf{ABD}}-F_{\mathsf{ACD}}\big) & \tfrac12\big(F_{\mathsf{AC|BD}}-F_{\mathsf{ABD}}-F_{\mathsf{BCD}}\big) \\ \tfrac12\big(F_{\mathsf{AC|BD}}-F_{\mathsf{ABC}}-F_{\mathsf{ACD}}\big) & \tfrac12\big(F_{\mathsf{AD|BC}}-F_{\mathsf{ABD}}-F_{\mathsf{ACD}}\big) & F_{\mathsf{ACD}} & \tfrac12\big(F_{\mathsf{AB|CD}}-F_{\mathsf{ACD}}-F_{\mathsf{BCD}}\big) \\ \tfrac12\big(F_{\mathsf{AD|BC}}-F_{\mathsf{ABC}}-F_{\mathsf{BCD}}\big) & \tfrac12\big(F_{\mathsf{AC|BD}}-F_{\mathsf{ABD}}-F_{\mathsf{BCD}}\big) & \tfrac12\big(F_{\mathsf{AB|CD}}-F_{\mathsf{ACD}}-F_{\mathsf{BCD}}\big) & F_{\mathsf{BCD}} \end{bmatrix}$$

where $F_{\mathsf{ABC}} = 4\,|\overline{\mathsf{ABC}}|^2, \ldots, F_{\mathsf{AD|BC}} = 16\,|\overline{\mathsf{AD|BC}}|^2$. As a Gram matrix amongst vectors in a Euclidean space this matrix is assured of being positive semi-definite, as are its 3×3 principal submatrices $\mathbf{G}_{\mathsf{A}}$, $\mathbf{G}_{\mathsf{B}}$, $\mathbf{G}_{\mathsf{C}}$ & $\mathbf{G}_{\mathsf{D}}$. It is also easily seen that $\mathbf{1} = [1,1,1,1]^\top$ is an eigenvector of $\mathbf{G}_{\mathsf{ABCD}}$ with eigenvalue $-\hat\Xi_F/2 = 0$ so that $\det\!\big(\mathbf{G}_{\mathsf{ABCD}}\big) = 0$. If $\operatorname{rank}\!\big(\mathbf{G}_{\mathsf{A}}\big) = 1$, then

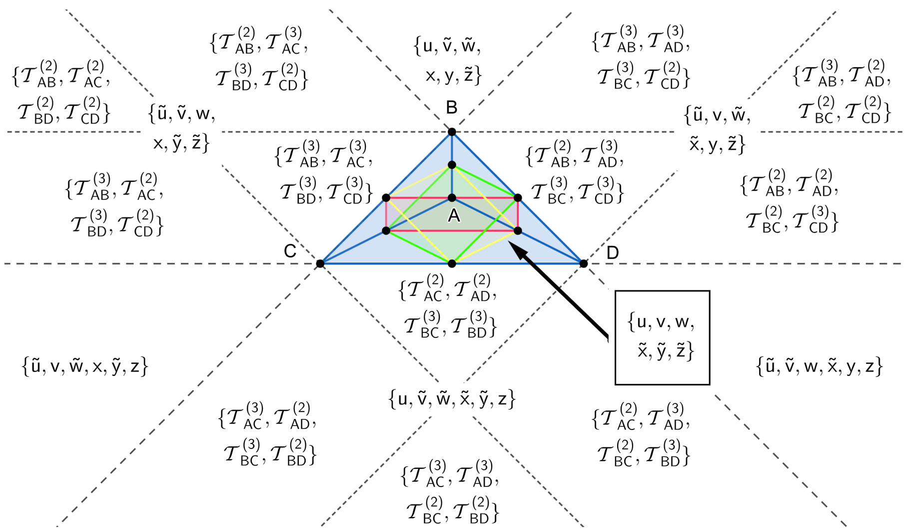

Fig. 5. Enumeration of the 16 classes of the zeros of $\mathbf{\Omega}(u, v, w, x, y, z)$ that jointly cover the set of four-point configurations in the affine plane. This enumeration was done by fixing the triangle $\overline{\mathsf{BCD}}$ and moving $\overline{\mathsf{A}}$ around while monitoring the deviations of all 18 tetrahedron inequalities from saturation, using the `GeoGebra` online dynamic geometry system [**32**]. (Note that, to make them fit, the deviations of the tetrahedron inequalities from saturation $\boldsymbol{\mathcal{T}}_k[\mathsf{a}, \mathsf{b}]$ are written here as $\boldsymbol{\mathcal{T}}^{(k)}_{\mathsf{ab}}$ for $\mathsf{a} \neq \mathsf{b} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ & $k = 0, 1, 2, 3$.) Placing vertex $\overline{\mathsf{A}}$ in each of the seven regions separated by the dashed lines through the edges of $\overline{\mathsf{BCD}}$ makes the sets of natural & inverse natural parameters indicated in the drawing vanish. The four regions wherein 3 non-opposite natural parameters vanish, or equivalently, 3 type (1) deviations $\boldsymbol{\mathcal{T}}^{(1)}_{\mathsf{ab}} = 0$, are those for which one vertex lies in the convex span of the other three (as seen in the drawing for $\overline{\mathsf{A}} \in \overline{\mathsf{BCD}} \Leftrightarrow u, v, w = 0$, along with the three medial, aka Varignon, parallelograms in red, green & yellow). The three regions wherein one opposite pair of natural parameters vanish when $\overline{\mathsf{A}}$ falls within them may be further divided into 4 subregions each, which are distinguished by which type (2) & (3) deviations vanish, namely $\boldsymbol{\mathcal{T}}^{(2)}_{\mathsf{ab}} = 0$ & $\boldsymbol{\mathcal{T}}^{(3)}_{\mathsf{ab}} = 0$. These subregions are separated by the dotted lines through each vertex of $\overline{\mathsf{BCD}}$ and parallel to its opposite edge.

the areal vectors of the three exterior faces meeting at $\overline{\mathsf{A}}$ are collinear, and Minkowski's identity (2·9) requires that the areal vector of the fourth exterior face $\overline{\mathsf{BCD}}$ also be collinear with those vectors. It follows that $\mathrm{rank}(\mathbf{G}_{\mathsf{ABCD}}) = 1 = \mathrm{rank}(\mathbf{G}_{\mathsf{B}}) = \mathrm{rank}(\mathbf{G}_{\mathsf{C}}) = \mathrm{rank}(\mathbf{G}_{\mathsf{D}})$, as well. Because the 2×2 minors of $\mathbf{G}_{\mathsf{ABCD}}$ are $\mathcal{T}_0[\mathsf{A}, \mathsf{B}]\mathcal{T}_1[\mathsf{A}, \mathsf{B}]\mathcal{T}_2[\mathsf{A}, \mathsf{B}]\mathcal{T}_3[\mathsf{A}, \mathsf{B}]/4 = s^2 u\tilde{u}$ etc., this establishes that Eq. (5·17) holds if $s > 0$. It also holds, of course, if $s = 2(u + \cdots + z) = 0$ and hence in general, as claimed.

Conversely, if Eq. (5·17) holds then all the 2×2 minors of $\mathbf{G}_{\mathsf{ABCD}}$ vanish, and since its determinant vanishes this matrix will have a rank of 1 if all its 3×3 minors vanish, i.e. $\det(\mathbf{G}_{\mathsf{A}}) = \det(\mathbf{G}_{\mathsf{B}}) = \det(\mathbf{G}_{\mathsf{C}}) = \det(\mathbf{G}_{\mathsf{D}}) = 0$. These determinants are equal modulo $\hat{\Xi}_F = 0$ by Lemma 3·1, whereas Eq. (3·1) shows that $\det(\mathbf{G}_{\mathsf{A}}) = t^4$, where $t^4 = s^2\,\mathbf{\Omega}(u, v, w, x, y, z)$ by Theorem 4·8. But Lemma 5·1 shows that if any one of the complementary products $u\tilde{u}, \ldots, z\tilde{z}$ vanish then $\mathbf{\Omega}(u, v, w, x, y, z) \leq 0$, while $\mathbf{\Omega}(u, v, w, x, y, z) \geq 0$ in the Euclidean case assumed here. It follows that both $\mathbf{\Omega}$ and all the 3×3 minors vanish as desired. $\square$

Remark 5·4. As illustrated in Fig. 5, only 7 of the $2^6 = 64$ possible combinations of natural & inverse natural parameters vanishing so as to satisfy Eq. (5·17) can actually occur in any (finite) planar configuration; these are exactly those for which the vectors $\alpha, \beta, \gamma, \delta \in \mathbb{R}^1$ in Remark 5·3 are all zero. Not surprisingly, they correspond to the seven uniform rank 3 *chirotopes* (aka affine oriented matroids) of four-point configurations in the affine plane [**9**]. This is entirely analogous to the way in which the $3!/2 = 3$ uniform rank 2 chirotopes (linear orders) for three points on a line correspond to the zeros of Heron's formula. The three combinations wherein every vertex is an extreme point of the convex hull may be further divided into four subclasses each, depending on exactly which combinations of tetrahedron inequalities saturate, and these in turn may be distinguished by their allowable sequences as defined by Goodman & Pollack [**26**, **27**]. In contrast to the situation for non-degenerate tetrahedra, the natural parameters or, equivalently, the seven areas, do not determine the configuration up to isometry when $\operatorname{rank}(\mathbf{G}_{\mathsf{A}}) = 1$, but only up to special (area preserving) affine transformations. Nevertheless, it is possible to define a canonical Euclidean representative which minimizes the radius of gyration about the centroid subject to reproducing the given areas.

The details of this analysis of the combinatorics of the rank 1 zeros, the existence of distances realizing the areas in the rank 1 case, the equations involved in the parametrization of the generic rank 2 zeros from Remark 5·3, and a full proof of Theorem 3·4 that characterizes the algebraic relations among the polynomials involved even if the indeterminates therein are not equal to Euclidean invariants in a non-degenerate Euclidean tetrahedron, may be found in the Appendices.

6. *Closing remarks*

This paper had two fairly big surprises in it. The first is that anything in it was new, especially considering the classical nature of its subject matter and the elementary techniques used to derive its results. This can be explained, at least in part, by the advent of computer algebra systems (such as the `SageMath` software package used for the calculations presented herein) which now enable an average student to accomplish many feats beyond even the grand masters of that distant era when low-dimensional Euclidean geometry was still at the cutting edge of mathematics research [**19**]. Vector algebra techniques, which the geometers of the late 19^{th} and early 20^{th} centuries seem to have largely relegated to the domain of physics, also proved enabling here. The most salient reason, however, is probably the fact that the Hamiltonians of classical physics depend on the distances between pairs of particles but not on the areas spanned by triples thereof (let alone the areas of medial parallelograms). As a result, many of the algebraic relations dealt with herein are not part of people's everyday experiences. Even if these relations have no physical relevance, however (?), they are mathematically quite straightforward, and with a little practice can even become intuitive.

The second, not entirely unrelated, surprise is of course the "projective" nature of almost all of the zeros of our extension of Heron's formula (4·27). Given that this extension has a pretty strong claim to being intrinsic to classical Euclidean geometry, it is difficult to argue that these collinear tetrahedra with vertices separated by infinite distances are not part of three-dimensional Euclidean space. Are they then also part of the Euclidean plane, and if not, where do they belong? (One might almost be tempted to ask "is Flatland a bigger place than anyone ever knew?" [**57**]!) As noted at the start of section

5, they cannot reasonably be embedded even in the projective completion of Euclidean space, and furthermore, having certain well-defined metrical properties, namely the areas of the seven faces, or equivalently, the six natural parameters, it seems unlikely they can be fully rationalized in purely projective geometric terms. Could a hyperbolic model of Euclidean space known as the "horosphere" be the right place to look for answers to such questions [**14**, **22**, **53**], analysed perhaps via the Cayley-Klein ansatz [**1**, **46**, **49**, **50**, **56**] (cf. Remark 4·5)?

Such interpretational issues aside, the results presented herein suggest a number of new lines of inquiry, the first of which is to work out a proof Conjecture 4·10 as to how the formula (4·27) extends to higher dimensions. Given its validity in 2 & 3 dimensions, and that numerical examples strongly suggest it also holds in 4, it would be surprising indeed if it failed in higher dimensions. The geometric relations it deals with, however, must have algebraic counterparts taking the form of rational functions in the "hyper-areas" of the n-simplices' facets and medial sections. Vector algebra only works in three dimensions, but higher dimensional generalizations are available, some of which actually predate it [**18**]. Today these are usually called Clifford algebras by mathematicians, although most of their users prefer the appellation *geometric algebra*, as did W. K. Clifford himself (even though these algebras are not covered in E. Artin's more recent book by that name [**4**]). They acquire particular power when applied to a vector space model of conformal geometry in which the Euclidean subgroup corresponds to the stabilizer of the point-at-infinity [**20**, **21**, **31**, **36**, **37**, **54**]. In that form, they should be ideally suited to the task of proving this conjecture.

Beyond that, it would be interesting to study the level sets defined by the equation $T = s^2\ \Omega(u,\ \ldots,\ z)$ for $T > 0$, and in particular if the distances therein can likewise be infinite (here the phenomena reviewed in Ref. [**24**] may be relevant). A purely algebraic challenge would be to invert the system of equations (4·26) so as to obtain the natural parameters as (roots of?) rational functions of the inverse parameters; the corresponding problem in the plane (cf. Remark 4·1) has the solution $u^2 = s\,\tilde{v}\tilde{w}/\tilde{u}$ etc., but it does not seem that s can be simply expressed in terms of $\tilde{u}$, $\tilde{v}$ & $\tilde{w}$ alone even in the plane. Some of the ideas herein might fruitfully be applied to the study of three-dimensional polyhedra based in their triangulations [**35**, **55**] (in this regard it is worth noting that a triangulation of any medial parallelogram of a tetrahedron is not generated by its standard barycentric subdivision). The connexions between the rank 1 zeros and the order-theoretic structure of planar 4-point configurations noted in Remark 5·4 (cf. Fig. 5) may also inspire further developments in discrete and combinatorial geometry [**48**, **53**, **62**]. Certainly, it will not be long before the computational commutative algebra community finds new directions in which to extend and generalize the elementary results presented herein [**41**, **42**].

Appendix A. The combinatorial structure of the rank 1 zeros

Proposition 5·6 above established that the $\mathrm{rank}(\mathbf{G}_{\mathsf{A}}) = 1$ situation is characterized by a complementarity relation between the natural and inverse natural parameters. Remark 5·4 and Fig. 5 then noted that only 7 of the $2^6 = 64$ combinations are can actually be realized in the Euclidean plane. This is because, a little more generally, only 16 of the $3^6 = 729$ combinations of $\mathcal{T}_k[\mathsf{a},\mathsf{b}] = 0$ ($k = 1, 2$ or 3 for all $\mathsf{a},\mathsf{b} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{C}\}$, $\mathsf{a} \neq \mathsf{b}$) constitute a consistent system of linear equations connecting the seven areas. An example

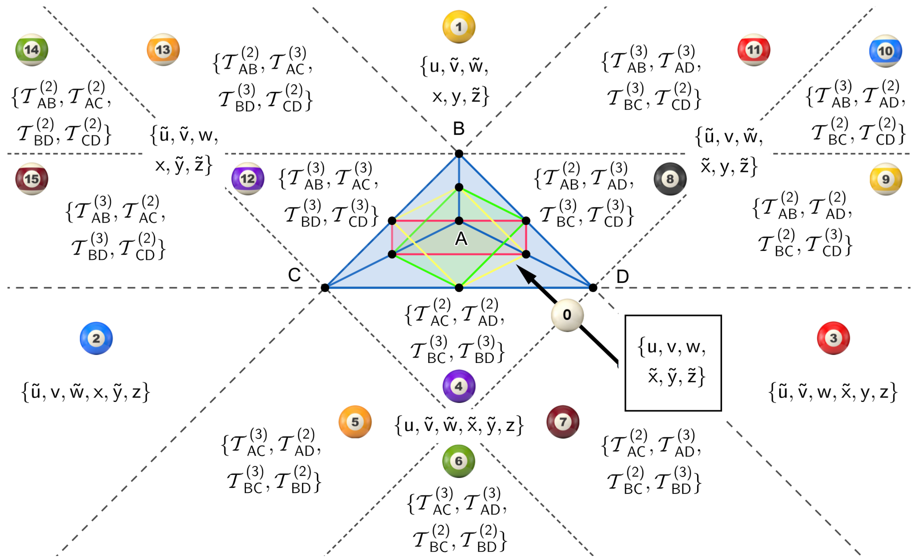

Fig. 6. The 16 classes of rank 1 zeros from Fig. 5, labelled by pool ball icons for easy reference.

of each one of these 16 combinations may be obtained by fixing the triangle $\overline{\mathsf{BCD}}$, and placing $\overline{\mathsf{A}}$ in one of the 16 regions into which the plane is divided by the lines through the triangle's edges together with the lines through each of its vertices and parallel to the opposite edge. To facilitate the ensuing discussion, Fig. 6 again shows these 16 regions, now labelled by the pool ball icons 0 through 15.

The 7 combinations of natural and inverse natural parameters vanishing so as to satisfy the complementarity relations will henceforth be referred to as "cases," to distinguish them from the full set of 16 combinations for which the equations $\mathcal{T}_k[\mathsf{a},\mathsf{b}] = 0$ are mutually consistent. In the 4 cases obtained by placing $\overline{\mathsf{A}}$ in the regions labelled by the pool balls 0 through 3, three non-opposite natural and three non-opposite inverse natural parameters vanish, while in the remaining 3 cases a pair of opposite natural parameters vanishes along with the four non-complementary inverse natural parameters. Geometrically, the first 4 cases are characterized by having one vertex in the convex span of the other three, while all four vertices are extreme points of their convex hull in the remaining 3 cases.

In the first 4 cases it is easily seen that the vanishing of each triple of non-opposite natural parameters ensures that $\Omega(u,v,w,x,y,z) = 0$, but in the latter 3 cases we have

$$\begin{aligned} u = z = 0 &\implies \Omega(u,v,w,x,y,z) = -(wx-vy)^2 \le 0\,, \\ v = y = 0 &\implies \Omega(u,v,w,x,y,z) = -(wx-uz)^2 \le 0\,, \\ w = x = 0 &\implies \Omega(u,v,w,x,y,z) = -(vy-uz)^2 \le 0\,. \end{aligned} \tag{A 1}$$

It follows that in the latter 3 cases there are likewise only three natural parameters that can be freely varied without making Ω negative. This shows that in the rank 1 situation the natural parameters do *not* determine a four-point configuration in the Euclidean plane up to isometry, but suggests they do so only up to area-preserving affine transformations. Since the special affine group is a 5-dimensional Lie group, the space of four-point configurations modulo its action has dimension $2 \cdot 4 - 5 = 3$ in accord with this expectation.

We will now show how each one of the 16 classes of four-point planar affine configurations illustrated in Fig. 6 can be parametrized by the areas of the interior faces (better known, in planar quadrilaterals, as Varignon parallelograms). We do this by expressing the area of each exterior face as a signed sum of the areas of the interior faces, where these linear relations hold only for the configuration class in question; it will also be shown that the seven areas together indeed determine the configuration up to special affine transformations. When $u = v = w = 0$, for example (specifically, the class with $\overline{\mathsf{A}} \in \overline{\mathsf{BCD}}$ as depicted in Fig. 6), the seven areas satisfy the system of six linear equations

$$\mathcal{T}_1[\mathsf{A},\mathsf{B}] \;=\; \mathcal{T}_1[\mathsf{A},\mathsf{C}] \;=\; \mathcal{T}_1[\mathsf{A},\mathsf{D}] \;=\; \mathcal{T}_3[\mathsf{C},\mathsf{D}] \;=\; \mathcal{T}_3[\mathsf{B},\mathsf{D}] \;=\; \mathcal{T}_3[\mathsf{B},\mathsf{C}] \;=\; 0\;. \tag{A 2}$$

Written out in terms of matrices, this system of equations is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} |\overline{\mathsf{ABC}}| \\ |\overline{\mathsf{ABD}}| \\ |\overline{\mathsf{ACD}}| \\ |\overline{\mathsf{BCD}}| \end{bmatrix} \;=\; 2 \begin{bmatrix} |\overline{\mathsf{AB|CD}}| \\ |\overline{\mathsf{AC|BD}}| \\ |\overline{\mathsf{AD|BC}}| \\ |\overline{\mathsf{AB|CD}}| \\ |\overline{\mathsf{AC|BD}}| \\ |\overline{\mathsf{AD|BC}}| \end{bmatrix}\;. \tag{A 3}$$

The left kernel of this matrix is spanned by the vectors $[1, 0, -1, -1, 0, 1]$ & $[0, 1, -1, 0, -1, 1]$, and since these are orthogonal to the right-hand side, these equations admit the exact solution:

$$\begin{bmatrix} |\overline{\mathsf{ABC}}| \\ |\overline{\mathsf{ABD}}| \\ |\overline{\mathsf{ACD}}| \\ |\overline{\mathsf{BCD}}| \end{bmatrix} \;=\; \begin{bmatrix} |\overline{\mathsf{AB|CD}}| + |\overline{\mathsf{AC|BD}}| - |\overline{\mathsf{AD|BC}}| \\ |\overline{\mathsf{AB|CD}}| - |\overline{\mathsf{AC|BD}}| + |\overline{\mathsf{AD|BC}}| \\ -\,|\overline{\mathsf{AB|CD}}| + |\overline{\mathsf{AC|BD}}| + |\overline{\mathsf{AD|BC}}| \\ |\overline{\mathsf{AB|CD}}| + |\overline{\mathsf{AC|BD}}| + |\overline{\mathsf{AD|BC}}| \end{bmatrix}\;. \tag{A 4}$$

It is easily shown that these four linear relations amongst the seven areas imply Yetter's identity Eq. (2·17). By performing similar analyses for each of the 16 classes illustrated in Fig. 6, one arrives at the 16 linear relations among the areas specified by the first 12 columns of Table 1.

The next three columns of Table 1 show that, even though the three signed sums of the interior areas given by

$$\Upsilon_1 \;:=\; \left|\overline{\mathsf{AC|BD}}\right| + \left|\overline{\mathsf{AD|BC}}\right| - \left|\overline{\mathsf{AB|CD}}\right|\,, \tag{A 5 a}$$

$$\Upsilon_2 \;:=\; \left|\overline{\mathsf{AD|BC}}\right| + \left|\overline{\mathsf{AB|CD}}\right| - \left|\overline{\mathsf{AC|BD}}\right|\,, \tag{A 5 b}$$

$$\Upsilon_3 \;:=\; \left|\overline{\mathsf{AB|CD}}\right| + \left|\overline{\mathsf{AC|BD}}\right| - \left|\overline{\mathsf{AD|BC}}\right|\,, \tag{A 5 c}$$

are not necessarily non-negative (and hence not the deviations from saturation of geometric inequalities), their signs constrain the possible values which the interior areas can assume on each quadruple of the 16 classes listed in Table 1. Subject to those constraints, the signs given in the first twelve columns of the table allow us to compute the exterior areas from the interior in each class. The last three columns in the table give the signs of the barycentric coordinates α_{B}, α_{C} & α_{D} of $\overline{\mathsf{A}}$ versus $\overline{\mathsf{BCD}}$, which are the same as the signs of Υ_1, Υ_2 & Υ_3 in classes 4 through 15. Because the absolute values of the barycentric coordinates of $\overline{\mathsf{A}}$ with respect to $\overline{\mathsf{BCD}}$ are the ratios of the external areas, i.e.

$$\alpha_{\mathsf{B}} \;:=\; \pm\,\frac{\left|\overline{\mathsf{ACD}}\right|}{\left|\overline{\mathsf{BCD}}\right|}\;,\quad \alpha_{\mathsf{C}} \;:=\; \pm\,\frac{\left|\overline{\mathsf{ABD}}\right|}{\left|\overline{\mathsf{BCD}}\right|}\;,\quad \alpha_{\mathsf{D}} \;:=\; \pm\,\frac{\left|\overline{\mathsf{ABC}}\right|}{\left|\overline{\mathsf{BCD}}\right|}\;, \tag{A 6}$$

Table 1. The first twelve columns list the signs of the coefficients ε_1, ε_2, $\varepsilon_3 = \pm 1$ in each of the signed sums of the interior facial areas $\varepsilon_1\,|\overline{\mathsf{AB}|\mathsf{CD}}| + \varepsilon_2\,|\overline{\mathsf{AC}|\mathsf{BD}}| + \varepsilon_3\,|\overline{\mathsf{AD}|\mathsf{BC}}|$ that equal one of the exterior facial areas $|\overline{\mathsf{ABC}}|$, $|\overline{\mathsf{ABD}}|$, $|\overline{\mathsf{ACD}}|$, $|\overline{\mathsf{BCD}}|$ in the 16 classes of planar tetrahedra illustrated in Fig. 6. The next three columns list the signs in the three signed sums of the interior facial areas Υ_1, Υ_2, Υ_3 from Eq. (A 5) for each class, while the last three columns give the signs of the barycentric coordinates α_{B}, α_{C}, α_{D} of $\overline{\mathsf{A}}$ vs. $\overline{\mathsf{BCD}}$. The horizontal lines of the table separate classes with differing chirotopes (see text).

Class	$\lvert\overline{\mathsf{ABC}}\rvert$			$\lvert\overline{\mathsf{ABD}}\rvert$			$\lvert\overline{\mathsf{ACD}}\rvert$			$\lvert\overline{\mathsf{BCD}}\rvert$			sign(Υ_k)			sign$(\alpha_?)$		
Identifier	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3	Υ_1	Υ_2	Υ_3	α_{B}	α_{C}	α_{D}
0	+	+	−	+	−	+	−	+	+	+	+	+	+	+	+	+	+	+
1	+	−	+	+	+	−	+	+	+	−	+	+	+	+	+	+	−	−
2	−	+	+	+	+	+	+	+	−	+	−	+	+	+	+	−	+	−
3	+	+	+	−	+	+	+	−	+	+	+	−	+	+	+	−	−	+
4	+	+	−	+	−	+	+	−	−	+	+	+	−	+	+	−	+	+
5	+	−	−	+	+	+	+	+	−	+	−	+	−	+	+	−	+	+
6	+	−	+	+	+	−	+	+	+	+	−	−	−	+	+	−	+	+
7	+	+	+	+	−	−	+	−	+	+	+	−	−	+	+	−	+	+
8	+	+	−	−	+	−	−	+	+	+	+	+	+	−	+	+	−	+
9	+	+	+	−	+	+	−	+	−	+	+	−	+	−	+	+	−	+
10	−	+	+	+	+	+	+	+	−	−	+	−	+	−	+	+	−	+
11	−	+	−	+	+	−	+	+	+	−	+	+	+	−	+	+	−	+
12	−	−	+	+	−	+	−	+	+	+	+	+	+	+	−	+	+	−
13	+	−	+	−	−	+	+	+	+	−	+	+	+	+	−	+	+	−
14	+	+	+	−	+	+	+	−	+	−	−	+	+	+	−	+	+	−
15	−	+	+	+	+	+	−	−	+	+	−	+	+	+	−	+	+	−

the interior areas together with those signs determine the configuration of the four points in the plane up to special affine transformations.

As is well known, the signs of the three barycentric coordinates determine which of the 7 realizable 4-point, rank 3 *chirotopes* (or affine oriented matroids) each class corresponds to [**9**, **48**]; these 7 super-classes are separated in Table 1 by horizontal lines. A glance at Fig. 6 shows that the 7 chirotope classes correspond exactly to which combinations of natural and inverse natural parameters vanish, as above. The full classification of four-point configurations into 16 classes thus constitutes a refinement of the chirotope classes, specifically a division of the 3 quadruples listed in the last 12 rows of Table 1 into 4 subclasses each. Goodman & Pollack [**26**, **27**] also defined a classification of affine point configurations which is finer than the chirotope one, based on the concept of "allowable sequences" of permutations of point labels. Each permutation is obtained by projecting the points onto an oriented line in the plane, while the sequence of permutations is obtained by rotating the line through 2π radians; such periodic sequences are uniquely defined by the configuration up to inversion of all the permutations therein and reversal of the overall sequence.

Based on an enumeration of the allowable sequences, again using the `GeoGebra` dynamic

Table 2. Table showing the periodic sequences of allowable permutations which are generated when $\overline{\mathsf{A}}$ is placed in the regions labeled by the pool balls 4, 5, 6 & 7 in Fig. 5, where each sequence starts from an orthogonal projection of the points into a horizontal line and continues up through the first inversion of the starting permutation (see text). Permutations which differ between the sequences are underlined for emphasis.

Class	Cyclic Sequence of Permutations up to First Inversion						
4	CBAD	BCAD	BCDA	BDCA	BDAC	DBAC	DABC
5	CBAD	BCAD	BCDA	BDCA	DBCA	DBAC	DABC
6	CBAD	CBDA	BCDA	BDCA	DBCA	DBAC	DABC
7	CBAD	CBDA	BCDA	BDCA	BDAC	DBAC	DABC

geometry system [**32**], it has been confirmed that the four subclasses into which each of the convex chirotope classes are divided (rows $4-15$ in Table 1) are distinguished by their allowable sequences. This is illustrated by the allowable sequences for classes $4-7$ which are shown in Table 2. It also appears that those allowable sequences are well defined for each class, in that moving the vertex $\overline{\mathsf{A}}$ around within each of the regions separated by dotted lines in Fig. 6 merely changes any given permutation to its predecessor or successor in the allowable sequence. This analysis suggests that more generally there may be a connection between the allowable sequences of affine configurations and the lines parallel to those spanned by pairs of points in the configuration, but through points other than the pair spanning that line. If so, this may serve to make the "combinatorial types" distinguished by allowable sequences up to relabeling more amenable to analysis than they currently seem to be, at least in comparison to the better known "order types" distinguished by chirotopes.

Appendix B. Canonical distances for rank 1 configurations

Given any values of the natural parameters for which the Gram matrix $\mathbf{G}_{\mathsf{A}}$ has rank 1, a specific *Euclidean* configuration will be determined generically by any two Euclidean but not affine invariants associated with the planar tetrahedron, for example any two of the distances between its vertices. It is also possible, however, to specify a *canonical* planar Euclidean tetrahedron which realizes the areas, which in turn are determined by the natural parameters via Corollary 4·5. This canonical planar tetrahedron is the one consistent with the given natural parameters and associated areas that minimizes the squared radius of gyration of the points about their centroid. By a well-known theorem of Lagrange [**25**], this in turn is equal to 1/16 times the sum of the six squared distances among the four vertices. It will now be shown how this problem can be solved by Lagrange's method of undetermined multipliers. As an interesting by-product, this shows that even though the squared areas cannot be realized by Cayley-Menger and Talata determinants in the general rank ≤ 2 case, they always can be in the rank 1 case.

To find such distances, consider the three-point instance of *Schönberg's quadratic form* [**10**, **17**, **51**]:

$$S_{\mathsf{BCD}}(\delta_{\mathsf{B}}, \delta_{\mathsf{C}}, \delta_{\mathsf{D}}) \;:=\; -\frac{1}{2}\begin{bmatrix} \delta_{\mathsf{B}} & \delta_{\mathsf{C}} & \delta_{\mathsf{D}} \end{bmatrix}\begin{bmatrix} 0 & D_{\mathsf{BC}} & D_{\mathsf{BD}} \\ D_{\mathsf{BC}} & 0 & D_{\mathsf{CD}} \\ D_{\mathsf{BD}} & D_{\mathsf{CD}} & 0 \end{bmatrix}\begin{bmatrix} \delta_{\mathsf{B}} \\ \delta_{\mathsf{C}} \\ \delta_{\mathsf{D}} \end{bmatrix}. \qquad \text{(B 1)}$$

It is well known [*op. cit.*] that $D_{\mathsf{BC}}, D_{\mathsf{BD}}, D_{\mathsf{CD}} \geq 0$ are the squared distances $|\overline{\mathsf{BC}}|^2$, $|\overline{\mathsf{BD}}|^2$, $|\overline{\mathsf{CD}}|^2$ among the vertices of a triangle $\overline{\mathsf{BCD}}$ in Euclidean space if & only if $S_{\mathsf{BCD}}(\delta_{\mathsf{B}}, \delta_{\mathsf{C}}, \delta_{\mathsf{D}}) \geq 0$ for all $\delta_{\mathsf{B}} + \delta_{\mathsf{C}} + \delta_{\mathsf{D}} = 0$. It is however seldom mentioned that, in this case, if $\overline{\mathsf{A}}$ & $\overline{\mathsf{A'}}$ are two points in the plane and $\alpha_{\mathsf{B}}, \alpha_{\mathsf{C}}, \alpha_{\mathsf{D}}$ & $\alpha'_{\mathsf{B}}, \alpha'_{\mathsf{C}}, \alpha'_{\mathsf{D}}$ are their barycentric coordinates w.r.t $\overline{\mathsf{BCD}}$ ($\alpha_{\mathsf{B}} + \alpha_{\mathsf{C}} + \alpha_{\mathsf{D}} = 1 = \alpha'_{\mathsf{B}} + \alpha'_{\mathsf{C}} + \alpha'_{\mathsf{D}}$), then

$$\delta_{\mathsf{B}} = \alpha_{\mathsf{B}} - \alpha'_{\mathsf{B}}, \;\; \delta_{\mathsf{C}} = \alpha_{\mathsf{C}} - \alpha'_{\mathsf{C}}, \;\; \delta_{\mathsf{D}} = \alpha_{\mathsf{D}} - \alpha'_{\mathsf{D}} \;\Longrightarrow\; |\overline{\mathsf{AA'}}|^2 = S_{\mathsf{BCD}}(\delta_{\mathsf{B}}, \delta_{\mathsf{C}}, \delta_{\mathsf{D}}) \;. \tag{B 2}$$

This implies, in particular, that the squared distances from the vertex $\overline{\mathsf{A}}$ to the vertices of $\overline{\mathsf{BCD}}$ itself are given by

$$\begin{aligned} |\overline{\mathsf{AB}}|^2 &= S_{\mathsf{BCD}}(\alpha_{\mathsf{B}} - 1, \alpha_{\mathsf{C}}, \alpha_{\mathsf{D}}) = S_{\mathsf{BCD}}(-\alpha_{\mathsf{C}} - \alpha_{\mathsf{D}}, \alpha_{\mathsf{C}}, \alpha_{\mathsf{D}}) \;, \\ |\overline{\mathsf{AC}}|^2 &= S_{\mathsf{BCD}}(\alpha_{\mathsf{B}}, \alpha_{\mathsf{C}} - 1, \alpha_{\mathsf{D}}) = S_{\mathsf{BCD}}(\alpha_{\mathsf{B}}, -\alpha_{\mathsf{B}} - \alpha_{\mathsf{D}}, \alpha_{\mathsf{D}}) \;, \\ |\overline{\mathsf{AD}}|^2 &= S_{\mathsf{BCD}}(\alpha_{\mathsf{B}}, \alpha_{\mathsf{C}}, \alpha_{\mathsf{D}} - 1) = S_{\mathsf{BCD}}(\alpha_{\mathsf{B}}, \alpha_{\mathsf{C}}, -\alpha_{\mathsf{B}} - \alpha_{\mathsf{C}}) \;, \end{aligned} \tag{B 3}$$

where the right-hand sides write the left in homogeneous form by substituting $1 = \alpha_{\mathsf{B}} + \alpha_{\mathsf{C}} + \alpha_{\mathsf{D}}$. This means that the sum of all six squared distances among the four vertices can be written as a function of the squared distances among the vertices of the triangle $\overline{\mathsf{BCD}}$ alone:

$$\begin{aligned} 16\, R_{\mathsf{G}} &= S_{\mathsf{BCD}}(-\alpha_{\mathsf{C}} - \alpha_{\mathsf{D}}, \alpha_{\mathsf{C}}, \alpha_{\mathsf{D}}) + S_{\mathsf{BCD}}(\alpha_{\mathsf{B}}, -\alpha_{\mathsf{B}} - \alpha_{\mathsf{D}}, \alpha_{\mathsf{D}}) \\ &\quad + S_{\mathsf{BCD}}(\alpha_{\mathsf{B}}, \alpha_{\mathsf{C}}, -\alpha_{\mathsf{B}} - \alpha_{\mathsf{C}}) + (\alpha_{\mathsf{B}} + \alpha_{\mathsf{C}} + \alpha_{\mathsf{D}})^2 \left(|\overline{\mathsf{BC}}|^2 + |\overline{\mathsf{BD}}|^2 + |\overline{\mathsf{CD}}|^2 \right) \\ &= 16\, \varrho_{\mathsf{BC}} |\overline{\mathsf{BC}}|^2 + 16\, \varrho_{\mathsf{BD}} |\overline{\mathsf{BD}}|^2 + 16\, \varrho_{\mathsf{CD}} |\overline{\mathsf{CD}}|^2 \;, \end{aligned} \tag{B 4}$$

where $R_{\mathsf{G}} = \left(|\overline{\mathsf{AB}}|^2 + \cdots + |\overline{\mathsf{CD}}|^2 \right)/16$ is the squared radius of gyration, and

$$\begin{aligned} \varrho_{\mathsf{BC}} &:= \tfrac{1}{16} \left(2\,\alpha_{\mathsf{B}}^2 + 2\,\alpha_{\mathsf{C}}^2 + \alpha_{\mathsf{D}}^2 + \alpha_{\mathsf{B}}\alpha_{\mathsf{C}} + 3\,\alpha_{\mathsf{B}}\alpha_{\mathsf{D}} + 3\,\alpha_{\mathsf{C}}\alpha_{\mathsf{D}} \right) , \\ \varrho_{\mathsf{BD}} &:= \tfrac{1}{16} \left(2\,\alpha_{\mathsf{B}}^2 + \alpha_{\mathsf{C}}^2 + 2\,\alpha_{\mathsf{D}}^2 + 3\,\alpha_{\mathsf{B}}\alpha_{\mathsf{C}} + \alpha_{\mathsf{B}}\alpha_{\mathsf{D}} + 3\,\alpha_{\mathsf{C}}\alpha_{\mathsf{D}} \right) , \\ \varrho_{\mathsf{CD}} &:= \tfrac{1}{16} \left(\alpha_{\mathsf{B}}^2 + 2\,\alpha_{\mathsf{C}}^2 + 2\,\alpha_{\mathsf{D}}^2 + 3\,\alpha_{\mathsf{B}}\alpha_{\mathsf{C}} + 3\,\alpha_{\mathsf{B}}\alpha_{\mathsf{D}} + \alpha_{\mathsf{C}}\alpha_{\mathsf{D}} \right) . \end{aligned} \tag{B 5}$$

Regarding R_{G} now as a function of D_{BC}, D_{BD} & D_{CD}, consider the Lagrangian for the minimization of $\frac{1}{2} R_{\mathsf{G}}$ subject to the constraint $\Delta_D[\mathsf{B}, \mathsf{C}, \mathsf{D}] = f^2_{\mathsf{BCD}}$, namely

$$L_\lambda(D_{\mathsf{BC}}, D_{\mathsf{BD}}, D_{\mathsf{CD}}) := \tfrac{1}{2} \left(\varrho_{\mathsf{BC}} D_{\mathsf{BC}} + \varrho_{\mathsf{BD}} D_{\mathsf{BD}} + \varrho_{\mathsf{CD}} D_{\mathsf{CD}} \right) - \lambda \left(\Delta_D[\mathsf{B}, \mathsf{C}, \mathsf{D}] - f^2_{\mathsf{BCD}} \right) \tag{B 6}$$

where λ is a Lagrange multiplier, f_{BCD} is the value of $|\overline{\mathsf{BCD}}|$ (times 2), and the barycentric coordinates α_{B}, α_{C}, α_{D} are obtained from the natural parameters via Corollary 4·5 together with the linear relations between the interior & exterior areas given in Table 1. Setting the gradient of L_λ w.r.t. D_{BC}, D_{BD}, D_{CD} to $\mathbf{0}$ gives

$$\mathbf{0} = \nabla L_\lambda = \frac{1}{2} \begin{bmatrix} \varrho_{\mathsf{BC}} \\ \varrho_{\mathsf{BD}} \\ \varrho_{\mathsf{CD}} \end{bmatrix} - \frac{\lambda}{2} \begin{bmatrix} D_{\mathsf{BD}} + D_{\mathsf{CD}} - D_{\mathsf{BC}} \\ D_{\mathsf{CD}} + D_{\mathsf{BC}} - D_{\mathsf{BD}} \\ D_{\mathsf{BC}} + D_{\mathsf{BD}} - D_{\mathsf{CD}} \end{bmatrix}, \tag{B 7}$$

and on adding these three equations together and solving for λ, one obtains

$$\lambda^* = \frac{\varrho_{\mathsf{BC}} + \varrho_{\mathsf{BD}} + \varrho_{\mathsf{CD}}}{D_{\mathsf{BC}} + D_{\mathsf{BD}} + D_{\mathsf{CD}}} \;. \tag{B 8}$$

Substituting this value of λ back into Eq. (B 7) and clearing denominators by multiplying through by $D_{\mathsf{BC}} + D_{\mathsf{BD}} + D_{\mathsf{CD}}$ then leads to the following linear system of equations:

$$\begin{bmatrix} 2\,\varrho_{\mathsf{BC}} + \varrho_{\mathsf{BD}} + \varrho_{\mathsf{CD}} & -\varrho_{\mathsf{BD}} - \varrho_{\mathsf{CD}} & -\varrho_{\mathsf{BD}} - \varrho_{\mathsf{CD}} \\ -\varrho_{\mathsf{BC}} - \varrho_{\mathsf{CD}} & \varrho_{\mathsf{BC}} + 2\,\varrho_{\mathsf{BD}} + \varrho_{\mathsf{CD}} & -\varrho_{\mathsf{BC}} - \varrho_{\mathsf{CD}} \\ -\varrho_{\mathsf{BC}} - \varrho_{\mathsf{BD}} & -\varrho_{\mathsf{BC}} - \varrho_{\mathsf{BD}} & \varrho_{\mathsf{BC}} + \varrho_{\mathsf{BD}} + 2\,\varrho_{\mathsf{CD}} \end{bmatrix} \begin{bmatrix} D_{\mathsf{BC}} \\ D_{\mathsf{BD}} \\ D_{\mathsf{CD}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{B 9}$$

The right-kernel of this matrix is spanned by $[\varrho_{\mathsf{BD}} + \varrho_{\mathsf{CD}}\,,\ \varrho_{\mathsf{BC}} + \varrho_{\mathsf{CD}}\,,\ \varrho_{\mathsf{BC}} + \varrho_{\mathsf{BD}}]^\top$, so the optimum squared distances are

$$D^*_{\mathsf{BC}} \;=\; \zeta\,(\varrho_{\mathsf{BD}} + \varrho_{\mathsf{CD}})\,, \quad D^*_{\mathsf{BD}} \;=\; \zeta\,(\varrho_{\mathsf{BC}} + \varrho_{\mathsf{CD}})\,, \quad D^*_{\mathsf{CD}} \;=\; \zeta\,(\varrho_{\mathsf{BC}} + \varrho_{\mathsf{BD}}) \tag{B 10}$$

for some $\zeta > 0$. Its value may be found by substituting these values of the squared distances into the constraint, which gives

$$\Delta^*_D[\mathsf{B},\mathsf{C},\mathsf{D}] \;=\; \zeta^2\,(\varrho_{\mathsf{BC}}\varrho_{\mathsf{BD}} + \varrho_{\mathsf{BC}}\varrho_{\mathsf{CD}} + \varrho_{\mathsf{BD}}\varrho_{\mathsf{CD}}) \;=\; f^2_{\mathsf{BCD}}\,, \tag{B 11}$$

so that $\zeta = f_{\mathsf{BCD}}\,/\sqrt{\varrho_{\mathsf{BC}}\varrho_{\mathsf{BD}} + \varrho_{\mathsf{BC}}\varrho_{\mathsf{CD}} + \varrho_{\mathsf{BD}}\varrho_{\mathsf{CD}}}$. Once the distances within $\overline{\mathsf{BCD}}$ are known, the remaining optimum distances can be computed from Eq. (B 3).

Note that even though the process by which this solution was obtained singled out $\overline{\mathsf{BCD}}$ as a barycentric basis, it remains canonical in that the same squared distances would've been found for any other choice of basis. Just as importantly, the 3-point Cayley-Menger and $(2,2)$-Talata determinants computed from those squared distances will all match the squares of the areas obtained directly from the natural parameters as above. Finally, if so desired Cartesian coordinates which realize these squared distances can be computed via standard eigenvalue methods [**16**, **17**], and the areas computed from these coordinates will also match those obtained directly from the natural parameters. It follows that in the $\mathrm{rank}(\mathbf{G}_{\mathsf{A}}) = 1$ case the natural parameters not only determine a four-point planar configuration up to special affine transformation, but also determine a unique Euclidean configuration which minimizes the squared radius of gyration R_{G} subject to reproducing the areas as calculated from those natural parameters.

Because the set of all sets of four points in the Euclidean plane modulo isometries is five-dimensional just like the set of all degenerate tetrahedra (cf. Fig. 4), it is tempting to speculate that the above canonical map can be extended to a canonical map between these two five-dimensional sets. This would be much more in accord with the intuition, which has been taken for granted by mathematicians throughout history, that the boundary of the six-dimensional set of non-degenerate tetrahedra is the set of all quadruples of points in the Euclidean plane. This possibility seems interesting enough to be stated formally as a conjecture:

Conjecture B1. *The projection of the set of quadruples of points in the Euclidean plane onto those for which the radius of gyration attains its unique minimum, subject to maintaining the areas of the triangles and Varignon parallelograms of the quadruple, can be extended to a one-to-one mapping of that set onto the set of all degenerate tetrahedra, as defined by the zeros of* $\mathbf{\Omega}$*; moreover this mapping is or can be chosen to be canonical, so these two sets can be identified.*

Appendix C. Towards a 2-to-1, near-global parametrization of the zeros

This appendix presents the details of the 2-to-1 parametrization of the "generic" zeros of $\mathbf{\Omega}$ that was briefly described in Remark 5·3, along with some alternatives thereto. It is based upon the values of the vectors α, β, γ & $\delta \in \mathbb{R}^1$ introduced in that remark, together with the exterior surface area s which, when treated in this way as a parameter, will be denoted by ς. Here, "generic" means that $\alpha^2, \beta^2, \gamma^2, \delta^2$ are all distinct and non-zero, subject to the constraint $\alpha\beta\gamma\delta > 0$ (see Proposition 5·5). Then upon using Eq. (5·15) to eliminate x, y, z from $\mathbf{\Omega}$, we get

$$\begin{aligned}\Omega(u,v,w) \;&:=\; \alpha^2\beta^2\gamma^2\delta^2\,\Omega(u,v,w,w\,\beta\gamma/\alpha\delta,v\,\beta\delta/\alpha\gamma,u\,\gamma\delta/\alpha\beta) \;= \\ &(\gamma\delta\,u+\beta\delta v+\beta\gamma w)(-\gamma\delta\,u+\beta\delta v+\beta\gamma w)(\gamma\delta\,u-\beta\delta v+\beta\gamma w)(\gamma\delta\,u+\beta\delta v-\beta\gamma w) \\ &\qquad =:\; \Omega_0(u,v,w)\,\Omega_1(u,v,w)\,\Omega_2(u,v,w)\,\Omega_3(u,v,w)\;.\end{aligned} \tag{C 1}$$

Similarly, on using Eq. (5·15) to eliminate x, y, z from the expression $2(u+\cdots+z)$ for s, we get

$$\begin{aligned}\varsigma \;=\; \Sigma(u,v,w)/(\alpha\beta\gamma\delta) \;&:= \\ &\big(2\,(\alpha\beta+\gamma\delta)\,\gamma\delta\,u+2\,(\alpha\gamma+\beta\delta)\,\beta\delta\,v+2\,(\alpha\delta+\beta\gamma)\,\beta\gamma\,w\big)/(\alpha\beta\gamma\delta)\;.\end{aligned} \tag{C 2}$$

Via Proposition 5·5, the relative signs of the vectors α,β,γ & δ tells us which one of the factors $\Omega_k(\check u,\check v,\check w,\check x,\check y,\check z)$ $(k=1,2,3)$ defined in Eq. (5·12) will vanish. It turns out that this factor always corresponds to what is obtained on substituting for these vectors in $\Omega_0(u,v,w)$ above by the signed products in Eq. (5·13). Thus the equation $\Omega_0(u,v,w)=0$ may be used to eliminate w from the equation $\Sigma(u,v,w)-\alpha\beta\gamma\delta\,\varsigma=0$, obtaining

$$\begin{aligned}0 \;&=\; \mathrm{Res}(\Omega_0(u,v,w),\,\Sigma(u,v,w)-\alpha\beta\gamma\delta\,\varsigma;\;w) \\ &=\; \beta\gamma\delta\,\Big(2\,\big(\gamma\,(\alpha-\gamma)(\beta-\delta)\,u\;+\;\beta\,(\alpha-\beta)(\gamma-\delta)\,v\;-\;\alpha\beta\gamma\,\varsigma\big)\Big),\end{aligned} \tag{C 3}$$

where "Res" is the resultant of its first two arguments with respect to the third. The overall factor of $\beta\gamma\delta$ is non-zero by our generic assumption and may be dropped. On also eliminating w from the equations $\Omega_0(u,v,w)=0$ and $\alpha^2-uvw=0$, one obtains

$$0 \;=\; \mathrm{Res}(\alpha^2-uvw,\,\Omega_0(u,v,w);\;w) \;=\; \delta\,(\gamma\,u+\beta\,v)\,uv\;+\;\alpha^2\beta\gamma\;. \tag{C 4}$$

Finally, upon taking the resultant of the right-hand sides of Eqs. (C 3) & (C 4) w.r.t. v and dropping the overall non-zero factors, we obtain a cubic equation for u:

$$\begin{aligned}0 \;=\; \Psi(u) \;=\; a_\Psi u^3+b_\Psi u^2+c_\Psi u+d_\Psi \;&:=\; 4\,\gamma\delta\,(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)\,u^3 \\ &-\,2\,\alpha\beta\gamma\delta\,\varsigma\,\big((\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)\big)u^2 \\ &+\,\alpha^2\beta^2\gamma\delta\,\varsigma^2\,u\;+\;4\,\alpha^2\beta^2(\alpha-\beta)^2(\gamma-\delta)^2 \\ =\; \big(2\,(\alpha-\delta)(\beta-\gamma)\,u-\alpha\beta\,\varsigma\big)&\big(2\,(\alpha-\gamma)(\beta-\delta)\,u-\alpha\beta\,\varsigma\big)+d_\Psi\end{aligned} \tag{C 5}$$

Clearly the constant term is positive, but the signs of the remaining coefficients depends on the values of $\alpha,\beta,\gamma,\delta$, although they are unchanged on negating all of them.

The discriminant of this cubic is a quadratic polynomial in ς^3, i.e.

$$\begin{aligned}&18\,a_\Psi b_\Psi c_\Psi d_\Psi-4\,(b_\Psi)^3d_\Psi+(b_\Psi c_\Psi)^2-4\,a_\Psi(c_\Psi)^3-27\,(a_\Psi d_\Psi)^2 \;= \\ &4\,\alpha^6\beta^6\gamma^4\delta^4(\alpha-\beta)^2(\gamma-\delta)^2\,\varsigma^6+64\,\alpha^5\beta^5\gamma^3\delta^3\big((\alpha-\gamma)(\beta-\delta)+(\alpha-\delta)(\beta-\gamma)\big) \\ &\cdots\big((\alpha-\beta)(\gamma-\delta)+(\alpha-\gamma)(\beta-\delta)\big)\big((\alpha-\beta)(\gamma-\delta)-(\alpha-\delta)(\beta-\gamma)\big)\varsigma^3 \\ &-\,6912\,\alpha^4\beta^4\gamma^2\delta^2(\alpha-\beta)^4(\alpha-\gamma)^2((\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^4\end{aligned} \tag{C 6}$$

Since the constant term is negative while the leading coefficient is positive, this polynomial has a single positive root, and for all ς^3 exceeding this root of the cubic in Eq. (C 5) will have three real roots. This is what has consistently been observed in numerical examples, which in turn suggests that the parametrization is only valid when ς observes this lower bound. Unfortunately, the single real root that exists when ς is below this bound can be positive, so it is difficult to eliminate this possibility *a priori*. Even when the discriminant is positive, however, the cubic can have more than one positive root, although presumably only one of them will yield non-negative natural parameters upon back-substitution.

One alternative to using the signed values of $\alpha, \beta, \gamma, \delta$ as above is to use their absolute values, and assume that one of the factors $\mathbf{\Omega}_1, \mathbf{\Omega}_2$ or $\mathbf{\Omega}_3$ in Eq. (C 1) vanishes. A similar process of elimination then leads to a cubic $\mathbf{\Psi}$ in one of the natural parameters for which $a_\Psi, c_\Psi, d_\Psi > 0$ but $b_\Psi < 0$. The discriminant of this cubic is again a quadratic polynomial in ς^3, and numerical examples again suggest that this discriminant is always positive. In this case, however, the numerical examples also suggest that the largest root is the correct one, as well as the one returned by Cardano's formula. This should facilitate further analysis, but would require that the analysis be performed separately for $\mathbf{\Omega}_1 = 0$, $\mathbf{\Omega}_2 = 0$ & $\mathbf{\Omega}_3 = 0$.

Yet another approach is to use Eq. (5·15) to eliminate one of each of the three pairs of opposite natural parameters from the equations $\mathbf{\Omega} = 0$ & $\varsigma - 2\,(u{+}v{+}w{+}x{+}y{+}z) = 0$, keeping only the $\mathbf{\Omega}_0$ factor in the former case, and obtaining two sets of eight linear equations each in just three non-opposite natural parameters. The matrix of this system of 16 linear equations in the six natural parameters can be shown to have rank 5, and hence by adding a multiple ρ of the right-kernel (as a vector of polynomials in $\alpha, \beta, \gamma, \delta$) to a particular solution (depending on ς) one obtains a one-parameter set of candidate solutions to e.g. $\alpha^2 - uvw = 0$. The resulting cubic in the dimensionless variable ρ again appears to always have a positive discriminant, providing ς is large enough. While this approach is somewhat more complicated, it has the advantage of yielding values for all the natural parameters directly, without back-substitution, which should facilitate selection of the correct root of the cubic in each case.

Whether any of these analyses can provide deeper insight into the structure of the zeros of $\mathbf{\Omega}$ remains an open question.

Appendix D. The polynomial map from squared distances to squared areas

The assertion made at the end of the proof of Theorem 3·4, namely that the squared areas F of a non-degenerate tetrahedron $\overline{\mathsf{ABCD}}$ determine it uniquely up to isometry, rests on the fact that the polynomial map from the squared distances $\boldsymbol{D}$ to those squared areas is non-singular precisely when the four-point Cayley-Menger determinant $\Delta_{\boldsymbol{D}}[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}] \neq 0$. This appendix will prove a stronger result, which is that this characteristic of the polynomial map holds for arbitrary values of the indeterminates $\boldsymbol{D}$, and not just for Euclidean squared distances. Moreover, it will show that any vector of "squared areas" $\mathbf{f} \in \mathbb{R}^7$ is either in the range of this polynomial map, or its negative, precisely when Yetter's identity $\hat{\Xi}_{\mathbf{f}} = 0$ is satisfied. Finally, it shows that it is in the range of the polynomial map whenever the Grammian $\Gamma_{\mathbf{f}}[\mathsf{A}]$ at A (or any other vertex) is positive, and in the range of its negative when $\Gamma_{\mathbf{f}}[\mathsf{A}]$ is negative.

Theorem D1. *Let $\mathbf{p}_\pm$ be the two quadratic polynomial mappings from the semi-algebraic set $\mathcal{D} := \{\, \mathbf{d} \in \mathbb{R}^6 \mid \Delta_{\mathbf{d}}[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}] \neq 0 \,\}$ into $\mathbb{R}^7$ that are given by*

$$\mathbf{p}_\pm(\mathbf{d}) \;:=\; \pm\big[\Delta_{\mathbf{d}}[\mathsf{A},\mathsf{B},\mathsf{C}]\,,\, \Delta_{\mathbf{d}}[\mathsf{A},\mathsf{B},\mathsf{D}]\,,\, \Delta_{\mathbf{d}}[\mathsf{A},\mathsf{C},\mathsf{D}]\,,\, \Delta_{\mathbf{d}}[\mathsf{B},\mathsf{C},\mathsf{D}]\,,\; \Delta_{\mathbf{d}}[\mathsf{A},\mathsf{B}\,|\,\mathsf{C},\mathsf{D}]\,,\, \Delta_{\mathbf{d}}[\mathsf{A},\mathsf{C}\,|\,\mathsf{B},\mathsf{D}]\,,\, \Delta_{\mathbf{d}}[\mathsf{A},\mathsf{D}\,|\,\mathsf{B},\mathsf{C}]\,\big]^\top. \qquad \text{(D 1)}$$

Then if Yetter's identity $\hat{\Xi}_{\mathbf{f}} = 0$ is satisfied by the squared areas $\mathbf{f} = [F_{\mathsf{ABC}}, \ldots, F_{\mathsf{AD|BC}}]^\top$, *there exist squared distances* $\mathbf{d} = [D_{\mathsf{AB}}, \ldots, D_{\mathsf{CD}}]^\top \in \mathcal{D}$ *such that*

$$\mathbf{f} \;=\; \mathbf{p}_+(\mathbf{d}) \quad \textit{if} \quad \Gamma_{\mathbf{f}}[\mathsf{A}] \;>\; 0\,, \qquad \textit{or} \quad \mathbf{f} \;=\; \mathbf{p}_-(\mathbf{d}) \quad \textit{if} \quad \Gamma_{\mathbf{f}}[\mathsf{A}] \;<\; 0\,, \tag{D 2}$$

where $\Gamma_{\mathbf{f}}[\mathsf{A}]$ *is the Grammian at* A. *In other words, Yetter's identity implicitly defines a linear subspace in* $\{\, \mathbf{f} \in \mathbb{R}^7 \mid \Gamma_{\mathbf{f}}[\mathsf{A}] \neq 0 \,\}$ *that is explicitly parametrized by the disjoint union of the images* $\mathbf{p}_\pm(\mathcal{D})$ *of the mappings* $\mathbf{p}_\pm$.

Proof. Let $\mathbf{Q}_{\mathsf{abc}}$ & $\mathbf{Q}_{\mathsf{ab|cd}}$ $\big(\mathsf{a,b,c,d} \in \{\mathsf{A,B,C,D}\} \mid |\{\mathsf{a,b,c,d}\}| = 4\big)$ be the matrices of the quadratic forms defined by the Cayley-Menger & Talata determinants relative to the order of the squared distances in $\mathbf{d}$, e.g.

$$\mathbf{Q}_{\mathsf{BCD}} \;:=\; \frac{1}{4}\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-1&1&1\\ 0&0&0&1&-1&1\\ 0&0&0&1&1&-1 \end{bmatrix}, \qquad \mathbf{Q}_{\mathsf{AB|CD}} \;:=\; \frac{1}{4}\begin{bmatrix} 0&0&0&0&0&2\\ 0&-1&1&1&-1&0\\ 0&1&-1&-1&1&0\\ 0&1&-1&-1&1&0\\ 0&-1&1&1&-1&0\\ 2&0&0&0&0&0 \end{bmatrix}. \tag{D 3}$$

Then the vector-valued functions $\mathbf{p}_\pm$ can be written as

$$\begin{aligned} \mathbf{p}_\pm(\mathbf{d}) \;=\; \pm\Big[\mathbf{d}^\top\mathbf{Q}_{\mathsf{ABC}}\,\mathbf{d},\; \mathbf{d}^\top\mathbf{Q}_{\mathsf{ABD}}\,\mathbf{d},\; \mathbf{d}^\top\mathbf{Q}_{\mathsf{ACD}}\,\mathbf{d},\; \mathbf{d}^\top\mathbf{Q}_{\mathsf{BCD}}\,\mathbf{d},& \\ \mathbf{d}^\top\mathbf{Q}_{\mathsf{AB|CD}}\,\mathbf{d},\; \mathbf{d}^\top\mathbf{Q}_{\mathsf{AC|BD}}\,\mathbf{d},\; \mathbf{d}^\top\mathbf{Q}_{\mathsf{AD|BC}}\,\mathbf{d}\Big]^\top&\,. \end{aligned} \tag{D 4}$$

Now consider the pair of least-squares problems:

$$\min_{\mathbf{d}\in\mathcal{D}}\left(\tfrac{1}{2}\,\Sigma_\pm^2(\mathbf{d})\right), \quad \text{where} \quad \Sigma_\pm^2(\mathbf{d}) \;:=\; \|\mathbf{p}_\pm(\mathbf{d}) - \mathbf{f}\|^2\,. \tag{D 5}$$

The Jacobians $\mathbf{J}_\pm = \mathbf{J}_\pm(\mathbf{d})$ of $\frac{1}{2}\,\Sigma_\pm^2$ will have the vectors $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{ABC}}$, $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{ABD}}$, $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{ACD}}$ & $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{BCD}}$ as their first four rows, and $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{AB|CD}}$, $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{AC|BD}}$ & $\pm\mathbf{d}^\top\mathbf{Q}_{\mathsf{AD|BC}}$ as their last three rows. As is well-known [**8**], the change in squared distances $\boldsymbol{\delta}\mathbf{d}$ obtained by applying the Gauss-Newton method to these least-squares problems satisfies the normal equations

$$\mathbf{J}_\pm^\top\mathbf{J}_\pm\,\boldsymbol{\delta}\mathbf{d} \;=\; -\,\mathbf{J}_\pm^\top\,\boldsymbol{\delta\Sigma}_\pm\,, \tag{D 6}$$

where $\boldsymbol{\delta\Sigma}_\pm = \boldsymbol{\delta\Sigma}_\pm(\mathbf{d}) := \mathbf{p}_\pm(\mathbf{d}) - \mathbf{f}$ are the vectors of residuals. The right-hand side is the negative gradient of each sum-of-squares $\frac{1}{2}\nabla\Sigma_\pm^2$, and it will vanish if & only if $\Sigma_\pm(\mathbf{d})^2 = 0$ or $\boldsymbol{\delta\Sigma}_\pm(\mathbf{d})$ lies in the generically one-dimensional right null space of $\mathbf{J}_\pm^\top$. Written out in full, these matrices are given by:

$$\pm\,2\,\mathbf{J}_\pm^\top(\mathbf{d}) \;=\; \left[\begin{matrix} -D_{\mathsf{AB}}+D_{\mathsf{AC}}+D_{\mathsf{BC}} & -D_{\mathsf{AB}}+D_{\mathsf{AD}}+D_{\mathsf{BD}} & 0 & 0 & \cdots \\ D_{\mathsf{AB}}-D_{\mathsf{AC}}+D_{\mathsf{BC}} & 0 & -D_{\mathsf{AC}}+D_{\mathsf{AD}}+D_{\mathsf{CD}} & 0 & \cdots \\ 0 & D_{\mathsf{AB}}-D_{\mathsf{AD}}+D_{\mathsf{BD}} & D_{\mathsf{AC}}-D_{\mathsf{AD}}+D_{\mathsf{CD}} & 0 & \cdots \\ D_{\mathsf{AB}}+D_{\mathsf{AC}}-D_{\mathsf{BC}} & 0 & 0 & -D_{\mathsf{BC}}+D_{\mathsf{BD}}+D_{\mathsf{CD}} & \cdots \\ 0 & D_{\mathsf{AB}}+D_{\mathsf{AD}}-D_{\mathsf{BD}} & 0 & D_{\mathsf{BC}}-D_{\mathsf{BD}}+D_{\mathsf{CD}} & \cdots \\ 0 & 0 & D_{\mathsf{AC}}+D_{\mathsf{AD}}-D_{\mathsf{CD}} & D_{\mathsf{BC}}+D_{\mathsf{BD}}-D_{\mathsf{CD}} & \cdots \end{matrix}\right.$$

$$\left.\begin{matrix} \cdots & 2\,D_{\mathsf{CD}} & -D_{\mathsf{AB}}+D_{\mathsf{AD}}+D_{\mathsf{BC}}-D_{\mathsf{CD}} & -D_{\mathsf{AB}}+D_{\mathsf{AC}}+D_{\mathsf{BD}}-D_{\mathsf{CD}} \\ \cdots & -D_{\mathsf{AC}}+D_{\mathsf{AD}}+D_{\mathsf{BC}}-D_{\mathsf{BD}} & 2\,D_{\mathsf{BD}} & D_{\mathsf{AB}}-D_{\mathsf{AC}}-D_{\mathsf{BD}}+D_{\mathsf{CD}} \\ \cdots & D_{\mathsf{AC}}-D_{\mathsf{AD}}-D_{\mathsf{BC}}+D_{\mathsf{BD}} & D_{\mathsf{AB}}-D_{\mathsf{AD}}-D_{\mathsf{BC}}+D_{\mathsf{CD}} & 2\,D_{\mathsf{BC}} \\ \cdots & D_{\mathsf{AC}}-D_{\mathsf{AD}}-D_{\mathsf{BC}}+D_{\mathsf{BD}} & D_{\mathsf{AB}}-D_{\mathsf{AD}}-D_{\mathsf{BC}}+D_{\mathsf{CD}} & 2\,D_{\mathsf{AD}} \\ \cdots & -D_{\mathsf{AC}}+D_{\mathsf{AD}}+D_{\mathsf{BC}}-D_{\mathsf{BD}} & 2\,D_{\mathsf{AC}} & D_{\mathsf{AB}}-D_{\mathsf{AC}}-D_{\mathsf{BD}}+D_{\mathsf{CD}} \\ \cdots & 2\,D_{\mathsf{AB}} & -D_{\mathsf{AB}}+D_{\mathsf{AD}}+D_{\mathsf{BC}}-D_{\mathsf{CD}} & -D_{\mathsf{AB}}+D_{\mathsf{AC}}+D_{\mathsf{BD}}-D_{\mathsf{CD}} \end{matrix}\right]. \tag{D 7}$$

Using computer algebra it is readily shown that $\det\big(\mathbf{J}_\pm^\top(\mathbf{d})\,\mathbf{J}_\pm(\mathbf{d})\big) = 28\,\Delta_{\mathbf{d}}\,[\mathsf{A,B,C,D}]^4 \neq 0$, so the Jacobians $\mathbf{J}_\pm$ are of full rank $= 6$ and the mappings $\mathbf{p}_\pm : \mathcal{D} \to \mathbb{R}^7$ are injective. It is also easily seen that the vector $\mathbf{n} := [1,\ 1,\ 1,\ 1, -1, -1, -1]^\top$ always lies in these Jacobians' left null space, meaning that $\mathbf{J}_\pm^\top(\mathbf{d})\,\mathbf{n} = \mathbf{0}$ for all $\mathbf{d} \in \mathcal{D}$. Thus the residual vector $\boldsymbol{\delta\Sigma}_\pm$ at any critical point $\mathbf{d}_\pm^*$ of $\Sigma_\pm^2$ can be written as $\sigma_\pm\,\mathbf{n}$ for some $\sigma_\pm \in \mathbb{R}$, but it can also be written as $\boldsymbol{\delta\Sigma}_\pm(\mathbf{d}_\pm^*) = \mathbf{p}_\pm(\mathbf{d}_\pm^*) - \mathbf{f}$. By expanding the corresponding Cayley-Menger & Talata determinants, it is easily shown that $\mathbf{p}_\pm(\mathbf{d})$ always satisfies Yetter's identity, i.e. $\mathbf{n}^\top\mathbf{p}_\pm = 0$, and hence

$$\Sigma_\pm^2 \;=\; \left\|\boldsymbol{\delta\Sigma}_\pm\right\|^2 \;=\; \mathbf{n}^\top\mathbf{n}\,\sigma_\pm^2 \;=\; 7\,\sigma_\pm^2 \;=\; \sigma_\pm\,\mathbf{n}^\top\left(\mathbf{p}_\pm(\mathbf{d}_\pm^*) - \mathbf{f}\right) \;=\; -\,\sigma_\pm\,\mathbf{n}^\top\,\mathbf{f}\;. \tag{D 8}$$

It follows that the residual vanishes at every critical point of $\Sigma_\pm^2$ if & only if $\mathbf{n}^\top\,\mathbf{f} = 0$, i.e. the given squared areas $\mathbf{f}$ satisfy Yetter's identity $\hat{\Xi}_{\mathbf{f}} = 0$.

This shows that when $\hat{\Xi}_{\mathbf{f}} = 0$ every critical point $\mathbf{d}_\pm^*$ of $\Sigma_\pm^2$ is a global minimum with value $\Sigma_\pm^2(\mathbf{d}_\pm^*) = 0$, but it does not establish the existence of a critical point, and indeed it is easy to find examples of $\mathbf{f}$'s for which numerical evidence indicates that either Σ_+ or Σ_- has no critical points, in that minimization of Σ_+^2 or Σ_-^2 the drives entries in $\mathbf{d}$ towards $\pm\infty$. It will now be shown, however, that for every $\mathbf{f} \in \mathbb{R}^7$ with $\mathbf{n}^\top\mathbf{f} = 0$ and $\Gamma_{\mathbf{f}}\,[\mathsf{A}] \neq 0$ exactly one of the functions Σ_+^2 or Σ_-^2 defined by $\mathbf{f}$ has a critical point. Towards that end, first note that by Euler's theorem on homogeneous functions,

$$\mathbf{p}_\pm(\mathbf{d}) \;=\; \tfrac{1}{2}\,\mathbf{J}_\pm(\mathbf{d})\,\mathbf{d} \;=\; \pm\,\tfrac{1}{2}\,\mathbf{J}_+(\mathbf{d})\,\mathbf{d} \tag{D 9}$$

for all $\mathbf{d} \in \mathcal{D}$, and since every critical point $\mathbf{d}_\pm^*$ of $\Sigma_\pm^2$ achieves a residual $\delta\Sigma_\pm(\mathbf{d}_\pm^*) = \mathbf{0}$,

$$\mathbf{f} \;=\; \mathbf{p}_\pm^* \;:=\; \mathbf{p}_\pm(\mathbf{d}_\pm^*) \;=\; \tfrac{1}{2}\,\mathbf{J}_\pm(\mathbf{d}_\pm^*)\,\mathbf{d}_\pm^* \;=:\; \tfrac{1}{2}\,\mathbf{J}_\pm^*\,\mathbf{d}_\pm^*\;. \tag{D 10}$$

Now consider the Cayley-Menger determinants in the squared *areas*, which are given by

$$\Delta_{\mathbf{f}}\,[\mathsf{a},\mathsf{b}] \;:=\; \tfrac{1}{2}\left(F_{\mathsf{abc}}F_{\mathsf{abd}} + F_{\mathsf{abc}}F_{\mathsf{ab|cd}} + F_{\mathsf{abd}}F_{\mathsf{ab|cd}}\right) - \tfrac{1}{4}\left(F_{\mathsf{abc}}^2 + F_{\mathsf{abd}}^2 + F_{\mathsf{ab|cd}}^2\right) \tag{D 11}$$

for $\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\}$ with $|\{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\}| = 4$. Let $\mathbf{Q}_{\mathsf{ab}}$ be the 7×7 matrices of these six quadratic forms versus the order of the squared areas in $\mathbf{f}$, e.g.

$$\mathbf{Q}_{\mathsf{AB}} \;:=\; \frac{1}{4}\begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 & 0\\ 1 & -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\;\ldots,\;\mathbf{Q}_{\mathsf{CD}} \;:=\; \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{D 12}$$

At a critical point $\mathbf{d}_\pm^*$ these quadratic forms are equal to

$$\mathbf{f}^\top\mathbf{Q}_{\mathsf{ab}}\,\mathbf{f} \;=\; \mathbf{p}_\pm^{*\top}\mathbf{Q}_{\mathsf{ab}}\,\mathbf{p}_\pm^* \;=\; \tfrac{1}{4}\,\mathbf{d}_\pm^{*\top}\mathbf{J}_\pm^{*\top}\mathbf{Q}_{\mathsf{ab}}\,\mathbf{J}_\pm^*\,\mathbf{d}_\pm^* \;=\; D_{\mathsf{ab}}^*\,\Delta_{\mathbf{d}_\pm^*}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]\;, \tag{D 13}$$

where D_{ab}^* is the associated entry of $\mathbf{d}_\pm^*$ and the last equality may readily be proven by applying any computer algebra program to the definitions (cf. Eq. (2·12)). Given any $\delta \neq 0$, we now set $D_{\mathsf{ab}} := \delta\,\mathbf{f}^\top\mathbf{Q}_{\mathsf{ab}}\,\mathbf{f}$ for $\mathsf{a},\mathsf{b} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}\}$ $(\mathsf{a} \neq \mathsf{b})$, and denote the result of substituting these values of D_{ab} in $\Delta_{\mathbf{d}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]$ by $\Delta_{\delta;\mathbf{f}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]$. Then it is easily seen that equations (D 13) will be satisfied when

$$\delta \;=\; \delta^* \;:=\; \frac{1}{\sqrt[4]{\Delta_{1;\mathbf{f}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]}}\;. \tag{D 14}$$

which of course requires that $\Delta_{1;\mathbf{f}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}] > 0$. It turns out, however, that if one eliminates F_{BCD} from $\Delta_{1;\mathbf{f}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]$ using Yetter's identity $F_{\mathsf{BCD}} = F_{\mathsf{AB|CD}} + F_{\mathsf{AC|BD}} + F_{\mathsf{AD|BC}} - F_{\mathsf{ABC}} - F_{\mathsf{ABD}} - F_{\mathsf{ACD}}$, it factorizes to $\Gamma_{\mathbf{f}}\,[\mathsf{A}]^2$. Therefore $\Delta_{1;\mathbf{f}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]$ is indeed strictly positive by hypothesis, and we have satisfied the necessary conditions for a critical point in Eq. (D 13).

Finally, upon setting the entries of $\mathbf{d}^*$ to $D^*_{\mathsf{ab}} = \delta^* \mathbf{f}^\top \mathbf{Q}_{\mathsf{ab}} \mathbf{f}$ in the given order, and again eliminating F_{BCD} using Yetter's identity as above, one finds that the Cayley-Menger & Talata determinants factorize as

$$\pm\, \mathbf{p}_\pm(\mathbf{d}^*) \;=\; \mathbf{p}_+(\mathbf{d}^*) \;=\; (\delta^*)^2\, \Gamma_{\mathbf{f}}\,[\mathsf{A}]\, \mathbf{f}\ . \tag{D 15}$$

Since $\left|\Gamma_{\mathbf{f}}\,[\mathsf{A}]\right| = \Delta_{1;\mathsf{f}}\,[\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D}]^{1/2} = (\delta^*)^{-2}$, it follows that the defining equation for a critical point $\mathbf{p}_\pm(\mathbf{d}^*) = \mathbf{f}$ is satisfied by $\mathbf{p}_+$ when $\Gamma_{\mathsf{f}}\,[\mathsf{A}] > 0$, or by $\mathbf{p}_-$ when $\Gamma_{\mathsf{f}}\,[\mathsf{A}] < 0$, as desired. $\square$

Because $\Sigma^2_\pm(\mathbf{d}) = \Sigma^2_\pm(-\mathbf{d})$ for all $\mathbf{d} \in \mathcal{D}$, the functions $\Sigma^2_\pm$ would have $\mathbf{0}$ as a saddle point but for $\mathbf{0} \notin \mathcal{D}$. Other critical points not in $\mathcal{D}$ are easy to construct simply by setting the entries of $\mathbf{d}$ to the squared distances in a random planar tetrahedron, finding a left null vector $\mathbf{n}_\perp$ of $\mathbf{J}_\pm(\mathbf{d})$ orthogonal to $\mathbf{n}$, and setting $\mathbf{f}_\pm = \mathbf{p}_\pm(\mathbf{d}) + \nu\, \mathbf{n}_\perp$ for any $\nu \in \mathbb{R}$. Such vectors $\mathbf{f}_\pm$ will satisfy $\mathbf{n}^\top \mathbf{f}_\pm = 0$ & $\Gamma_{\mathbf{f}_\pm}\,[\mathsf{A}] \neq 0$ unless $\nu = 0$, and $\mathbf{d}$ will generally also be a saddle point of $\Sigma^2_\pm$ extended to $\mathbb{R}^6$. Although most vectors $\mathbf{f} \in \mathbb{R}^7$ with $\mathbf{n}^\top \mathbf{f} = 0$ & $\Gamma_{\mathbf{f}}\,[\mathsf{A}] = 0$ will not be equal to the Cayley-Menger & Talata determinants for any $\mathbf{d} \in \mathbb{R}^6$, it was shown in Appendix B that they are when the entries of $\mathbf{f}$ are the squared areas in a planar tetrahedron.

REFERENCES

[**1**] NOBERT A'CAMPO and ATHANASE PAPADOPOULOS, *On Klein's So-called Non-Euclidean Geometry*, Sophus Lie and Felix Kelin: The Erlangen Program and its Impact in Mathematics and Physics, Europ. Math. Soc., 2015, pp. 91–136.

[**2**] ABDO Y. ALFAKIH, *Euclidean distance matrices and their applications in rigidity theory*, Springer-Verlag, 2018.

[**3**] NATHAN ALTSHILLER-COURT, *Modern Pure Solid Geometry*, The MacMillan Co., NY, 1935, Reprinted by Chelsea Publ. Co., Bronx, NY. 1964.

[**4**] EMIL ARTIN, *Geometric Algebra*, Wiley Intersci. Publ., 1988.

[**5**] ERIC BABSON, PAUL E. GUNNELLS, and RICHARD SCOTT, *A smooth space of tetrahedra*, Adv. Math. **165** (2002), 285–312.

[**6**] ———, *Geometry of the tetrahedron space*, Adv. Math. **204** (2006), 176–203.

[**7**] ALAN F. BEARDON and PAUL STEPHENSON, *The Heron parameters of a triangle*, Math. Gazette **99** (2015), no. 545, 205–212.

[**8**] AKE BJORCK, *Nonlinear Least Squares Problems*, ch. 9 in *Numerical Methods for Least Squares Problems*, SIAM, Philadelphia, PA, 1996.

[**9**] A. BJÖRNER, M. LAS VERGNAS, B. STURMFELS, N. WHITE, and G. ZIEGLER, *Oriented Matroids (2nd ed.)*, Encyclopedia of Mathematics and Its Applications, vol. 46, Cambridge Univ. Press, UK, 1999.

[**10**] LEONARD M. BLUMENTHAL, *Theory and Applications of Distance Geometry*, Oxford Univ. Press, 1953, Reprinted by Chelsea Publ. Co., Bronx, NY, 1970.

[**11**] I. BORG and P. J. F. GROENEN, *Modern Multidimensional Scaling: Theory and Applications (2nd ed.)*, Springer-Verlag, 2005.

[**12**] JOHN C. BOWERS and PHILIP L. BOWERS, *A Menger redux: Embedding metric spaces isometrically in Euclidean space*, Am. Math. Monthly **124** (2017), no. 7, 621–636.

[**13**] Jason Cantarella, Tom Needham, Clayton Shonkwiler, and Gavin Stewart, *Random triangles and polygons in the plane*, Amer. Math. Monthly **126** (2019), no. 2, 113–134.

[**14**] H. S. M. COXETER, *Non-Euclidean Geometry*, Math. Assoc. Am., 1998.

[**15**] LOUIS CRANE and DAVID N. YETTER, *Tetrahedral geometry from areas*, Am. Math. Monthly **127** (2020), no. 8.

[**16**] GORDON M. CRIPPEN and TIMOTHY F. HAVEL, *Stable calculation of coordinates from distance information*, Acta Cryst. **A34** (1978), 282–284.

[**17**] ———, *Distance Geometry and Molecular Conformation*, John Wiley & Sons, 1988.

[**18**] MICHAEL J. CROWE, *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, Notre Dame Press, 1967, Reprinted by Dover Publ., 1985.

[**19**] PHILIP J. DAVIS, *The rise, fall, and possible transfiguration of triangle geometry: A mini-history*, Amer. Math. Monthly **102** (1995), no. 3, 204–214.

[**20**] CHRIS DORAN and ANTHONEY LASENBY, *Geometric Algebra for Physicists*, Cambridge Univ. Press, (2003).

[**21**] LEO DORST, DANIEL FONTIJNE, and STEPHEN MANN, *The conformal model: Operational Euclidean geometry*, ch. 13 in "Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry," Elsevier, Amsterdam, NL, 2009.

[**22**] ANDREAS W. M. DRESS AND TIMOTHY F. HAVEL, *Distance geometry and geometric algebra*, Found. Phys. **23** (1993), 1357–1374.

[**23**] WILLIAM DUNHAM, *Journey through Genius: The Great Theorems of Mathematics*, John Wiley & Sons, 1990.

[**24**] ALAN H. DURFEE, *Five definitions of critical point at infinity*, Prog. Math. **162** (1998), 345–360.

[**25**] M. GIDEA and C. NICULESCU, *A brief account on Lagrange's algebraic identity*, Math. Intell. **34:3** (2012), 55–61.

[**26**] ———, *Multidimensional sorting*, SIAM J. Comput. **12** (1983), no. 3, 484–507.

[**27**] ———, *The complexity of point configurations*, Discrete Appl. Math. **31** (1990), 167–180.

[**28**] J. C. GOWER and G. B. DIJKSTERHUIS, *Procrustes Problems*, Oxford Univ. Press., 2004.

[**29**] MARSHALL HAMPTON, *Cosines and Cayley, triangles and tetrahedra*, Amer. Math. Monthly **121** (2014), no. 10, 937–941.

[**30**] THOMAS HAWKINS, *Emergence of the Theory of Lie Groups*, Springer-Verlag, 2000.

[**31**] DAVID HESTENES and GARRET SOBCZYK, *Clifford Algebra to Geometric Calculus*, D. Reidel Publ. Co., 1984.

[**32**] INTERNATIONAL GEOGEBRA INST., *GeoGebra online dynamic geometry system (version 5)*, 2021.

[**33**] M. S. KLAMKIN, *Vector proofs in solid geometry*, Am. Math. Monthly **77** (1970), no. 10, 1051–1065.

[**34**] SIMON L. KOKKENDORFF, *Polar duality and the generalized law of sines*, J. Geom. **86** (2006), 140–149.

[**35**] CARL W. LEE and FRACISCO SANTOS, *Subdivisions and triangulations of polytopes*, Handbook of Discrete and Computational Geometry (3rd ed.) (J. E. Goodman, J. O'Rourke, and C. D. Tóth, eds.), CRC Press, Boca Raton, FL, 2017, pp. 415–447.

[**36**] HONGBO LI, *Invariant Algebras and Geometric Reasoning*, World Scientific Publishing Co., 2008.

[**37**] HONGBO LI, DAVID HESTENES, and ALYN ROCKWOOD, *Generalized homogeneous coordinates for computational geometry*, in "Geometric Computing with Clifford Algebras" (G. Sommer, ed.), Springer-Verlag, Berlin & Heidelberg, 2001, pp. 27–59.

[**38**] LEO LIBERTI and CARLIE LAVOR, *Euclidean Distance Geometry: An Introduction*, Springer-Verlag, 2017.

[**39**] LEO LIBERTI and CARLILE LAVOR, *Six mathematical gems from the history of distance geometry*, Intl. Trans. in Op. Res. **23** (2016), 897–920.

[**40**] KARL MENGER, *New foundation of Euclidean geometry*, Am. J. Math. **53** (1931), 721–745.

[**41**] MATEUSZ MICHAŁEK and BERND STURMFELS, *Invitation to Non-linear Algebra*, Am. Math. Soc., 2021.

[**42**] EZRA MILLER and BERND STURMFELS, *Combinatorial Commutative Algebra*, Springer-Verlag, 2005.

[**43**] H. MINKOWSKI, *Volumen und Oberfläche*, Math. Ann. **57** (1903), no. 4, 447–495.

[**44**] D. S. MITRINOVIC, J. E. PECARIĆ, and V. VOLENEC, *Recent Advances in Geometric Inequalities*, ch. XIX: Inequalities for Tetrahedra, pp. 545–575, Kluwer Acad. Publ. (now Springer-Verlag), 1989.

[**45**] M. A. MURRAY-LASSO, *On the generalization of the law of cosines of triangles to three and more dimensions*, Ingeniería Investigación y Tecnología **3** (2002), no. 4, 169–183.

[**46**] ARKADY L. ONISHCHIK and ROLF SULANKE, *Projective and Cayley-Klein Geometries*, Springer-Verlag, 2006.

[**47**] G. RICHARDSON, *The trigonometry of the tetrahedron*, Math. Gazette (1902), 149–158.

[**48**] J. RICHTER-GEBERT and G. M. ZIEGLER, *Oriented matroids*, in "Handbook of Discrete and Computational Geometry" (3rd ed.) (J. E. Goodman, J. O'Rourke, and C. D. Tóth, eds.), CRC Press, Boca Raton, FL, 2017, pp. 159–184.

[**49**] JÜRGEN RICHTER-GEBERT, *Perspectives on Projective Geometry*, Springer-Verag, 2011.

[**50**] TONY ROBBIN, *Shadows of Reality: The Fourth Dimension in Relativity, Cubism, and Modern Thought*, Yale Univ. Press, New Haven and London, 2006.

[**51**] I. J. SCHOENBERG, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. Math. **38** (1937), 787–793.

[**52**] JOHAN J. SEIDEL, *Distance-geometric development of two-dimensional Euclidean, hyperbolical and spherical geometry*, Simon Stevin: Wis-en Natuurkundig Tijdschrift, **29** (1952), 32–50, 65–76.

[**53**] ———, *Discrete non-Euclidean geometry*, in "Handbook of Incidence Geometry: Buildings and Foundations" (F. Buekenhout, ed.), Elsevier B.V., 1995, pp. 843–920.

[**54**] GARRET SOBCZYK, *New Foundations for Mathematics: The Geometric Concept of Number*, Birkauser (now Springer-Verlag), 2013.

[**55**] I. KH. SABITOV, *A generalized Heron–Tartaglia formula and some of its consequences*, Sb. Math. **189** (1998), 1533–1561.

[**56**] D. M. Y. SOMMERVILLE, *An Introduction to the Geometry of N Dimensions*, Methuen & Co. Ltd., London, UK, 1929.

[**57**] IAN STEWART, *The Annotated Flatland: A Romance of many Dimensions, by Edwin A. Abbott*, Perseus Book Group, Philadelphia, PA, 2002, Paperback version published 2008.

[**58**] ISTVÁN TALATA, *A volume formula for medial sections of simplices*, Discrete Comput. Geom. **30** (2003), 343–353.

[**59**] A. A. TODA, *Radii of the inscribed and escribed spheres of a simplex*, Intl. J. Geom. **3** (2014), no. 2, 5–13.

[**60**] DAVID N. YETTER, *The area of the medial parallelogram of a tetrahedron*, Am. Math. Monthly **106** (1998), no. 10, 956–958.

[**61**] ———, *On a formula relating volumes of medial sections of simplices*, Discrete Comput. Geom. **43** (2010), 339–345.

[**62**] GÜNTER M. ZIEGLER, *Lectures on Polytopes*, Springer-Verlag (7th printing), 2000.

If this [the Mysterium cosmographicum] is published, others will perhaps make discoveries I might have reserved for myself. But we are all ephemeral creatures (and none more so than I). I have, therefore, for the Glory of God, who wants to be recognized from the book of Nature, that these things may be published as quickly as possible. The more others build on my work the happier I shall be.

Johannes Kepler, 1595

Every formula which expresses a law of nature is a hymn of praise to God.

Maria Mitchell, date unknown

As to the need of improvement there can be no question whilst the reign of Euclid continues. My own idea of a useful course is to begin with arithmetic, and then not Euclid but algebra Next, not Euclid, but practical geometry, solid as well as plane; not demonstration, but to make acquaintance. Then not Euclid, but elementary vectors, conjoined with algebra, and applied to geometry. Addition first; then the scalar product. Elementary calculus should go on simultaneously, and come into the vector algebraic geometry after a bit. Euclid might be an extra course for learned men, like Homer. But Euclid for children is barbarous.

Oliver Heaviside, 1893

Wir mussen wissen. Wir werden wissen.

from the Tomb of David Hilbert