Fitting Feelings and Elegant Proofs: on the psychology of aesthetic evaluation in mathematics

Abstract

This paper explores the role of aesthetic judgements in mathematics by focussing on the relationship between the epistemic and aesthetic criteria employed in such judgements, and on the nature of the psychological experiences underpinning them. I claim that aesthetic judgements in mathematics are plausibly understood as expressions of what I will call 'aesthetic-epistemic feelings' that serve a genuine cognitive and epistemic function. I will then propose a naturalistic account of these feelings in terms of sub-personal processes of representing and assessing the relation between cognitive processes and certain properties of the stimuli at which they are directed.

Introduction

Although the use of aesthetic judgements in the practice of mathematics is commonplace and has generally been taken at face value by mathematicians and philosophers alike, there are various obstacles to interpreting such claims literally, as genuinely *aesthetic*. Moreover, there are some relatively strong reasons for holding that the claims at issue are really 'masked' epistemic rather than aesthetic assessments.[1] This particular debate perhaps deserves further reflection, yet there is some risk of it descending into an irresolvable – perhaps even purely semantic – dispute concerning the demarcation of aesthetic *vis-a-vis* epistemic domains of experience and discourse. Accordingly, this paper will touch on it only insofar as it is relevant to exploring what I take to be an ultimately more fruitful pursuit, that of using mathematics to understand some important connections and interactions between the aesthetic and epistemic.[2] To this end, my primary aim in this paper is to examine the nature of the psychological experiences that lead mathematicians to make *prima facie* aesthetic assessments of proofs and theorems.[3] This will lead to some surprising and significant consequences for how we should envisage the relation between aesthetic and epistemic criteria in mathematical reasoning, and the role of aesthetic experience in cognition more generally.

Ultimately, I will aim to justify the role of *bona fide* aesthetic judgements in mathematics by showing how they are expressions of what I will call 'aesthetic-epistemic feelings' that serve a genuine cognitive and epistemic function. I will then propose a naturalistic account of these aesthetic-epistemic feelings. Specifically, I will suggest that they are all determinate variations of what I will call the 'feeling of fittingness', which is a determinable experiential manifestation of certain sub-personal processes of representing and assessing the relation between certain cognitive processes and certain properties of the stimuli at which they are directed. This explanation will, moreover, help to show why, in principle, anything can be aesthetically appreciated, and how we can justify the role of aesthetic experiences, judgements, and criteria in mathematics.

1. Aesthetic and Epistemic Criteria

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[1] See especially Harré (1958); Kivy (1991); McAllister (1996); Todd (2008); Aberdein & Inglis (2014) for discussion. It should be noted that although these discussions focus primarily on aesthetic judgements in science, relevantly similar considerations apply to mathematics.

[2] I will refer throughout to both aesthetic (or epistemic) judgements and experiences. I shall be assuming that judgements are, in part, the expressions of such experiences and so, to save repetition, I shall sometimes use 'judgement' and 'experience' interchangeably where the context should make clear that I'm referring to both.

[3] I will concentrate only on these two mathematical 'objects' though I hope that my conclusions will have broader implications for the practice of mathematics generally.

Even if we do not wish to get unduly bogged down in sceptical concerns about the genuinely aesthetic nature of the relevant claims in mathematics, in order to understand their prominent place in mathematical practice we do need to ponder how it is in fact possible for aesthetic judgements – *qua* aesthetic – to play any significant role in the assessment of things such as proofs and theorems, *qua* proofs and theorems.[4]

Naturally, the question of how to demarcate the domain of the aesthetic – to give, for example, the necessary and sufficient conditions for an experience or judgement or property to count as aesthetic – has received as much philosophical attention as it has proved relatively intractable. So it is useful to begin as uncontroversially as possible and state that whatever aesthetic experience is, *prima facie* it is not the same thing – it is not co-extensive with – moral pleasure, mere intellectual satisfaction, comic pleasure, the pleasure of financial (or other) personal gain, or perhaps (and more controversially) even mere sensory, hedonic pleasure, like having a nice warm bath or drinking a fine wine. It has been intimately, if not exclusively, tied to the sensory-intellectual pleasures paradigmatically (by which I mean indisputably) provided by the arts, but also to our appreciative experiences of nature.

The various intuitions at play here have lead philosophers to claim, for example, that aesthetic experience is a *sui generis* state, and that aesthetic value is the value that an object or experience has intrinsically; or in a famous if not entirely pellucid phrase, *for its own sake alone*. Aesthetic value is essentially non-instrumental.

These ideas are all more or less captured in the concept of disinterestedness, which has been a persistent one in philosophical aesthetics ever since Kant, and has acquired two main aspects that make philosophers immediately suspicious of the aesthetic claims of mathematicians. The first is the idea that aesthetic judgements are unconcerned with truth and utilitarian ends, they must not be based on cognitive interest in attaining knowledge about something nor with assessing an object in relation to some purpose which it serves. These thoughts give rise to the second aspect, encapsulated in the idea that aesthetic appreciation concerns taking disinterested pleasure in some object for its own sake, without regard to ulterior purposes or the satisfaction of further, extraneous desires, such as cognitive understanding or sensual gratification. In this way, the aesthetic and cognitive, logico-epistemic evaluations of a theorem or proof are completely independent of one another. Their aesthetic value bears no relation to their empirical/epistemic worth or utility. (Stolnitz, (1961); McAllister (1991), (1996); Todd (2008))

Despite some inevitable murkiness and the many disputes surrounding them, such notions are thought by many philosophers to play some important role in characterizing what makes an experience an aesthetic one. [5]

There is no space here to provide a comprehensive list of the aesthetic claims that prominent mathematicians have made or that are regularly in evidence in mathematical practice, and in any case these can be readily found in a number of sources.[6] It is useful, however, to divide the aesthetic claims of mathematicians into three rough categories.

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[4] The sceptical line of argument about to be given is taken directly from Todd (2008)

[5] It is naturally impossible to give an authoritative or comprehensive list of such discussions, but see for example: Kant (1790/1987); Sibley (195; 1965); Budd (1995); Zangwill (2001).

[6] For ready examples see: Hardy (1940); Engler (1990); Kivy (1991); McAllister (2005); Todd (2008); Aberdein & Inglis (2014).

First, *appreciative claims* constitute the mainstay of the kinds of aesthetic judgements we are concerned with, claims such as: 'The theorem is beautiful (ugly)'; 'the proof is elegant (inelegant/clumsy')'. Second are what I will call *criterial claims*, and they come in two types. On the one hand we find comparative claims involving an appeal to aesthetic criteria as grounds for deciding which theorem or proof is, in some yet to be determined sense, better or to be preferred.[7] For example: 'the proof is better *because* it is elegant'; 'the theorem should be preferred *because* it is beautiful'. On the other hand, there are claims about the criteria constituting and/or justifying the aesthetic appraisal: 'the proof is elegant *in virtue of* its simplicity, precision, concision; 'the theorem is beautiful *in virtue of* its unexpected generality, depth, symmetry.' Finally, we find *discovery claims* that play a role in applied mathematics and physical theories, such as: 'the beauty of the equations is a sure sign the theory is true'. (Dirac)

Most relevant for our purposes in exploring the relations between aesthetic and epistemic judgements are criterial claims; that is, claims in which an aesthetic property is attributed to some mathematical entity on the basis of some other property or properties. Providing a comprehensive list of such claims is, unsurprisingly, a difficult task[8], but it suffices for the moment to note a handful of features that seem to be regularly appealed to, amongst which we find the following:

- symmetry
- simplicity
- economy
- insightfulness
- significance
- breadth
- depth
- seriousness
- rigour
- efficiency

The first thing to note is that at least some, and perhaps many, of the criteria appealed to in mathematics to explain and/or justify aesthetic assessments of the objects of mathematics appear to be used in similar fashion in 'paradigm' aesthetic contexts, in particular the appreciation of artworks. So, for example, it is not unusual to be told that a work of art can be beautiful (at least partly) in virtue of the symmetry of its forms, and its insightfulness or depth; or it can perhaps be elegant (at least partly) in virtue of its simplicity and rigour. The second thing to note is that we are now entering some difficult terrain, for leaving aside indisputably aesthetic adjectives such as 'beautiful' or 'elegant', the criterial vocabulary used to explain and justify aesthetic judgements can be interpreted either as itself aesthetic or non-aesthetic, and as evaluative or non-evaluative. This poses a problem for interpreting the purported aesthetic criteria involved in mathematics.

For example, describing some object as simple, or symmetrical – say the pencil I have in front of me – might just be to state an evident fact about it. Arguably, however, mere descriptive judgements need not be accompanied by any affective experience of the kind

[7] It is important to note that judging a proof as better is not necessarily the same thing as preferring it. For example, a proof might be preferred in virtue of being simpler to comprehend, and yet a more complex version might be better insofar as it, for example, sheds light on other mathematical areas or problems.
[8] Aberdein & Inglis (2014) provide the most comprehensive treatment of which I'm aware.

that might naturally lead one to call that experience generally valuable (or evaluative), or more specifically, aesthetic. In this light, it can appear that the *way in which* the criteria cited above are appealed to in order to ground aesthetic judgements in mathematics – or to put it another way, the function of these properties in the context of mathematical evaluation – can begin to seem significantly different to the work they do in purportedly paradigmatic aesthetic contexts. To put it simply, they, and hence the 'aesthetic' judgements they supposedly ground, might seem to be functioning epistemically rather than aesthetically.[9]

This is because the relation of the supposedly aesthetic judgements to their criteria that occur in mathematics seems to contrast markedly with the corresponding relation in paradigmatic aesthetic contexts. For instance, there don't seem to be examples of the criteria offered above being used to justify negative aesthetic judgements, as is possible in the case of art. Could a clumsy proof or ugly theorem be so *in virtue of* its symmetry or simplicity? Second, in contrast to the case of art, there appear to be relatively few disagreements over whether a proof is elegant or not, or whether a particular theorem is beautiful; although there are clear cases of disagreements over comparative aesthetic judgements; for example, whether $E = mc^2$ is more beautiful than $a^2 + b^2 = c^2$.[10] Third, aesthetic terms used in mathematics appear to be regularly applied in virtue of a relatively limited range of properties, and there is, perhaps relatedly, a correspondingly limited *pure* aesthetic vocabulary in play.

Proofs and theorems are regularly described as elegant (e.g. Cantor's Diagnoal Proof) or clumsy (e.g. computer generated proofs such as that of the Four Colour Theorem), and ugly or beautiful, but can they also, for instance, be cute, kitsch, or graceful – assuming these latter are in some way purely aesthetic?[11] Proofs can, after all, be funny, cool, ingenious, or ridiculous, as the following example (possibly!) shows:

1) At any party, there are at least two people with the same number of friends there.

Proof (cool? Ingenious?): picture everyone at the party wearing a t-shirt displaying how many of their friends are present. The number of possible t-shirts is one less than the number of attendees, so there must be a double somewhere.[12]

How, then, are we to explain these differences? The sceptic has a ready explanation. We begin by noting that the criteria cited in support of aesthetic judgements in mathematics tend to be equally criteria of what I will call 'epistemic' adequacy.[13] The key contrast here is with the purportedly primary aesthetic and non-epistemic goal of art, as well as with the apparently entirely non-epistemic aesthetic appreciation of nature.

The sceptical conclusion to be drawn from these observations is that what appear to be aesthetic claims may often be, if perhaps not always are, really masked 'epistemic'

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[9] For a much fuller treatment of this sceptical line see Todd (2008). See also Rota (1997). For an account of aesthetic criteria that seems incompatible with the role they are held to pay in mathematics see Sibley (1959; 1965).

[10] See McAllister 1996 for discussion.

[11] Cf. Aberdein & Inglis, who suggest the use of a much greater range of criteria than I imply here. I will turn to this below.

[12] Assuming that everyone at the party has at least one friend at the party. For examples of ridiculous proofs see G.Linerholm (1971). https://www.quora.com/What-are-some-of-the-most-ridiculous-proofs-in-mathematics

[13] For discussion see: Osborne (1984); Engler (1990); Penrose (1974); Gowers (2002); Hardy (1940)

functional ones. The aesthetic terms at issue – 'beautiful' and 'elegant' and so on – are being used in a 'quasi', or metaphorical way. More strongly, it might even be claimed that the apparently aesthetic terms and concepts being deployed by mathematicians could in principle be wholly reduced to the non-aesthetic, logico-epistemic truth-functional terms or properties that serve as their criteria of application. In other words, when mathematicians say that a proof is elegant, they are really referring to its qualities of, say, simplicity, precision, efficiency, and so on. The supposed aesthetic judgements appealed to in maths can thus be taken to refer to epistemic values in mathematics – for example, 'ease of understanding', 'comprehensibility', 'enlightening' and so on – which naturally explains the many proposed explanations of mathematical beauty and elegance by mathematicians themselves in just these terms. (e.g. Hardy (1940); Cf. Rota (1997))

The 'aesthetic' judgements in maths are thus too closely correlated with its primary epistemic function to warrant being called unequivocally aesthetic, especially in light of the apparently purely descriptive properties appealed to as warrants for them. We are therefore to some extent justified in scepticism about the purported aesthetic judgements, and the experiences that ground them, in mathematics.[14]

This is not to deny that mathematicians have genuine affective experiences that they characterise as aesthetic when motivated to call proofs and theorems beautiful. And it is not even to deny that simplicity, symmetry, order, precision, illumination, depth etc. can be objects of genuine aesthetical appreciation. But it is to insist that it is more natural in the context of mathematics to interpret the relevant affective experiences as involving, say, intellectual satisfaction in solving problems or gaining understanding. When a mathematician exclaims that the theorem is 'beautiful', she may be doing no more than expressing pleasure in it being a mathematically 'good' theorem, where this can be taken as meaning epistemically useful, significant, surprising, enlightening, unifying; and where she appeals to non-aesthetic epistemic properties to support her judgement. It's not an aesthetic experience, but an intellectual one.

Take, for example, this claim about the equation: 1=.9999999...

"I love how simple it is — everyone understands what it says — yet how provocative it is. Many people don't believe it could be true. It's also beautifully balanced. The left side represents the beginning of mathematics; the right side represents the mysteries of infinity." (Steven Strogatz http://www.livescience.com/26680-greatest-mathematical-equations.html).

Nonetheless, it remains doubtful that these points are enough to entail that the 'aesthetic' claims made in mathematics are not genuine. Indeed, we might wonder why the reasons adduced above to support the sceptical view of aesthetic judgements in mathematics could not instead be used to conclude that the initial intuitions, and the requirements that Sibley, Kant and others have placed on aesthetic judgement, are mistaken. Moreover, some of these reasons seem to be hostage to the empirical evidence concerning the criteria mathematicians do in fact appeal to in justifying or explaining their aesthetic judgements. Some of this evidence seems to show both that the range of criteria I have discussed is much less limited than I've suggested, and more tellingly, that the relationship between aesthetic judgements and their criteria in mathematics is not nearly as straightforward.

[14] As noted earlier, see Todd (2008) for a fuller discussion of this sceptical line of attack.

Inglis and Aberdein (2015), for example, have recently provided a comprehensive study of this issue and directly addressed the sceptical line of argument I have been rehearsing. They investigate the deployment of 80 different adjectives used in the context of assessing proofs and concluded, amongst other things, that there are no significant correlations between judgements of beauty and judgements of simplicity.[15]
We might also turn to empirical evidence, in the form of neuroscientific research, to answer the question of how we decide when a response is aesthetic and when it isn't. Semir Zeki et al. (2014) have directly tackled mathematical experiences of beauty, focussing on a list of mathematical formulae. Zeki et al found some agreement about the aesthetic judgements of certain formulae. They found, for example, that the formulas most consistently rated as beautiful were both before and during the fMRI scans were Euler's identity $(1 + e^{i\pi} = 0)$ and that the one most consistently rated as ugly was Ramanujan's infinite series for $1/\pi$.

The experimenters were focussing on the hypothesis that such experiences correlate with activity in the same part of the 'emotional brain' – specifically A1 of the Orbito Medial Frontal Cortex (oMFC) – as experiences of beauty towards other paradigmatic aesthetic stimuli, such as visual artworks and music.[16] The results in favour of a positive conclusion are, however, tempered by the observation that this area is also active in a variety of conditions, including experiences relating to pleasure, reward and hedonic states. As they put it:

"This is perhaps not surprising, because the three merge into one another, without clear boundaries between them; neurologically, activity in mOFC correlates with all three experiences thus reflecting, perhaps, the difficulty of separating these experiences subjectively. The imperfect distinction between the three is also reflected in the positive post-scan answers given by the mathematical subjects to the question whether they experienced pleasure, satisfaction or happiness when viewing equations that they had rated as beautiful. Whether one can ever experience beauty without at the same time experiencing a sense of pleasure and/or reward is doubtful." (p.9)

Now this finding might explain why there is an overlap – between positive aesthetic experiences, hedonic pleasure, intellectual pleasure – in subjective reports about the nature of subjects' responses to stimuli that may give them different types of positive experiences. Unfortunately, however, it thereby fails to show definitively that the experiences being studied are really experiences of beauty; that is, given the concerns so far raised, the study itself simply begs the question as to the nature of the experiences being studied. One lesson to be drawn here is that, without an independent theory about how we demarcate the conceptual boundaries regarding different types of experiences, simply looking at the brain won't decide the issue for us. Nonetheless, Zeki's study is not without interest or promise, for it may be that it provides at least some defeasible reason

[15] I have, however, two concerns about this research. First, the 80 adjectives were chosen by the researchers and given to the mathematicians to apply to proofs, leaving it open whether in actual practice mathematicians do indeed invoke such a range. Second, it's not clear that their research does much to counter the idea that the purported aesthetic judgements are really epistemic in nature. Indeed, they themselves hint as much, while also noting that there do appear to be "adjectives which reside at the conjunction of these two dimensions" – e.g. proofs which are 'enlightening', 'fruitful', and 'insightful'. (p.16) I will return to this issue below.

[16] They note that: "Past brain imaging studies exploring the neurobiology of beauty have shown that the experience of *visual* (Kawabata and Zeki, 2004), *musical* (Blood et al., 1999; Ishizu and Zeki, 2011), and *moral* (Tsukiura and Cabeza, 2011) beauty all correlate with activity in a specific part of the emotional brain, field A1 of the medial orbito- frontal cortex (mOFC), which probably includes segments of Brodmann Areas (BA) 10, 12 and 32 (see Ishizu and Zeki, 2011 for a review)." (p.8)

for thinking that perhaps (some?) aesthetic experiences can indeed overlap in important ways with (some?) intellectual, cognitive, or epistemic experiences.

This idea is what I will spend the rest of the paper exploring. It is worth first recalling, however, that the traditional idea of the autonomous nature of aesthetic experience, of aesthetic interest as being an interest in an object for its own sake alone, stems largely from Kant's lengthy and complex work on aesthetic judgement, *The Critique of Judgement*. In this work, Kant was explicitly averse to the idea that mathematics could be the object of genuine experiences of beauty, which for him essentially depended on feeling, rather than cognition. Nonetheless, as Christian Wenzel (2001) has shown, in earlier writings Kant allowed that certain epistemic properties involved in understanding, such as symmetry and harmony, could be aesthetically appreciated and hence render their objects beautiful. As such, Kant at one time held a position that appears to be similar to the one I defend in this paper. This was because, Wenzel argues, he had yet to make a clear separation between subjective and objective purposiveness (roughly speaking, between aesthetic and epistemic value):

"mathematical figures are not beautiful, but demonstrations in geometry can be beautiful due to their shortness, their completeness, their natural light, and their suitability for an easier understanding. It is the pleasure we take in the ease of proofs which makes them appear beautiful to us" (Quoted in Wenzel 2001: 426)

In a recent paper, Angela Breitenbach (2013) argues that Kant's own account of aesthetic judgement can accommodate mathematical beauty, specifically by showing how it involves a non-conceptual response felt in light of our own creative activities involved in the process of mathematical reasoning. Beauty, on this view, is not a feature attributable to mathematical properties, but is experienced in the process of demonstrating mathematical theorems through a creative act of the imagination. Specifically, it involves the awareness of a fit between our intellectual capacities, and the awareness of this fit or harmony is experienced as a feeling of pleasure. As such, she argues, the feeling of beauty is elicited by the process of mathematical demonstration rather than by the finished product; namely, the proof.[17] Recently, Alix Cohen (forthcoming) has shown that we should think of beauty as connected to cognition in a number of important and meaningful ways. In particular, she argues that far from portraying cognition as opposed to beauty, not only can cognition be beautiful for Kant, its beauty is epistemically valuable. Thus the aesthetic dimension of cognition has a rightful place in Kant's epistemic framework.

Without entering into Kantian exegesis, I would like to take up the idea that aesthetic values ccould serve an epistemic function by connecting it to some recent psychological and philosophical literature on meta-cognition. But first it is crucial to note two challenges arising from this picture that the kind of account of mathematical beauty I will offer must also eventually tackle.

First, the intentional object of the experience on this Kantian account seems to be one's own cognitive state, not the object about which we make the actual judgements. But that seems wrong. After all, I don't find my own reasoning processes beautiful. I find the mathematics beautiful. Second, why think this feeling is a genuine aesthetic feeling, rather than, say, a feeling of mere intellectual satisfaction? Kant's answer is stipulative, given his

[17] For another, compelling attempt to show that Kant's philosophy has the resources to deal with cases of beautiful mathematics in spite of the fact that he himself may not have seen it, see Wenzel (2013).

theory about the nature of aesthetic judgement, but without buying into Kant's own rather idiosyncratic account of aesthetic experience the question remains to be answered.

2. Epistemic Feelings

A number of psychologists have recently become interested in a range of phenomena that have come to be known as epistemic feelings. That is, there are certain quite common affective conscious states, felt to be positively or negatively valenced, that arise in what can be broadly referred to as epistemic contexts. Drawing on a long list compiled by Arango-Munoz and Michaelian (2014), some of the feelings relevant to our present topic would include, for example:

- **The feeling of knowing:** Koriat (2000)= the 'feeling of knowing' is in some way a measure of the accessibility of the knowledge one has, correct or incorrect.
- **The feeling of understanding**: A feeling of intellectual satisfaction that motivates the endorsement of an explanation, a sense that we have achieved an understanding of a phenomenon that was not clearly understood before (Gopnik, 1998, 2000; Trout, 2002, 2007). This is sometimes called the "ah ha" feeling (Mangan, 2001) or the eureka feeling.
- **The feeling of rightness (contrasted with the feeling or error)**: The subjective experience that the execution of a mental action (e.g., reasoning or decision making) was successful (Mangan, 1993, 2001; Thompson, 2009; Thompson et al. 2011).
- **The feeling of certainty (contrasted with the feeling of uncertainty):** The subjective awareness of the precision of a mental representation or piece of information (Smith 2009; Bach and Dolan 2012; and see Dokic, this issue).

The primary focus of the psychological literature has centred on the nature and function of such feelings, and whether they are in fact heuristically valuable in assessing the information they seem to concern. Although there is naturally much debate concerning both issues, there seems to be some consensus that epistemic feelings are the valenced experienced manifestation of some kind of quick and dirty, low-level, sub-personal mechanism that monitors the performance of different cognitive processes. There is evidence for neural correlates of epistemic feelings, and there is evidence that some epistemic feelings do play a justificatory role in, for example, accurately predicting future cognitive performance, and in acting as a stimulus to judgement.[18] In other words, my feeling of uncertainty or of knowing will incline me to (and arguably give me reason to) judge that I really am uncertain or that I do know.

Before looking further at the details, we can note that a few philosophers have also taken feelings seriously as playing an important role in cognition. For example, in discussing the so-called Frame Problem for AI, Ronald deSousa (2008) argues that feelings and emotions serve to stop the potentially endless regress of deliberation by revealing patterns of saliency among objects of attention and strategies. And Christopher Hookway (2008) has argued that affective states in general embody evaluations that appeal to standards and principles of reasoning (for example, concerning inductive reasoning) that we may not be able to articulate. He claims that even if one can offer no reasons to justify one's acceptance of some inference, they may possess an *immediate phenomenology* of being compelling, as having epistemic salience, which is an affective state

[18] See Arango-Munoz & Michaelian (2014) and Arango-Munoz (2014) for discussion.

rather than a cognitive one. (Cf. Elgin 2002; Thagard 2008).[19]
There are three main questions that we need to address: (1) What are the intentional objects of epistemic feelings? (2) What is their role in mathematical reasoning? (3) What is their connection to aesthetic experience?

Let's begin with the well-studied feeling of knowing.[20] This is an experience we have whenever we are asked to recall a piece of information that seems to us to indicate that we possess the information without us (as yet) being able to recall the information. The classic example is in certain game shows where the participants are required to press a buzzer before they even have time to recall the relevant information. Psychological studies of memory retrieval suggest that subjects can accurately determine whether they are going to be able to recall the information by means of the FOK before they even try to recall it.[21]

As noted above, psychologists have proposed that the origin of epistemic feelings is low-level metacognition, which involves a sub-personal mechanism that monitors the performance of different cognitive processes. There is debate about how exactly to understand the nature of such a mechanism, but one plausible way is to think of it as a heuristic device that evaluates mental activity by reference to external conditions and/or properties of the relevant cognitive processing – such as fluency or familiarity – as well as by reference to some salient concepts and theories, as the conceptual nature of some E-feelings suggests.[22]

What, then, are such feeling actually about? What is their intentional content? Are they about our sub-personal process or about the objects of those processes, or about the relation between the two? Determining the content of such feelings is not straightforward.[23] This is partly because they seem to have multiple intentional objects, and partly because some cases of epistemic feeling seem to arise where the available grounds or evidence for the relevant judgement are missing or not evident, whereas others do not. Indeed this distinction is not often clearly marked in the psychological discussions.

For example, a mathematician may feel confident or not about the steps of a proof, they may feel that there is an error somewhere, or they may feel certain that it is correct. In such cases, the subject is conscious about the result and feels that it is right or wrong, but may not be not conscious about why he or she feels that it is right or wrong. The reasons are opaque. The object of the feeling in such cases seems to be the whole mental process of solving the mathematical problem *and* the result. (See Arango-Munoz 2015) On the other hand, one might have the feeling of understanding in response to a difficult proof in virtue of the fact that all of the steps and their relations are transparent.

Arguably, it seems that epistemic feeling have a dual intentional content; on the one hand, they are in some way about our own cognitive states– such as our capacity to know

[19] These lines of thought are compatible with recent philosophical accounts of the emotions as states or processes whose function is largely to disclose certain evaluative states of affairs, and which is accomplished primarily via their experiential phenomenology. See Deonna and Teroni (2012) for discussion.
[20] The *locus classicus* is Koriat (2000).
[21] See Reder (1987), (1988), (1996).
[22] See Arango-Munoz (2014)
[23] There is some debate in philosophy about whether feelings are the kinds of states that can possess representational content, conceptual or non-conceptual, but I will leave this issue aside.

or remember something – and on the other hand they are about the actual specific piece of information or object concerned. And yet, in resulting from the monitoring of sub-personal dispositions, epistemic feelings are apparently not themselves sensitive to the content of the processes they monitor but only to how well or poorly these processes are performing or will perform, given certain properties of the stimulus and its relation to us; properties such as fluency and familiarity, as we'll see below.

So, I suggest for the moment that the best way to think of epistemic feelings' intentional content is as follows: they are the result of a mechanism directed at the content and functioning of our own cognitive processes, and are themselves felt reactions to the manner – well or badly, efficiently or inefficiently – in which it is proceeding *as well as to* the object of the processing. Indeed, it is hard to understand how one could monitor the functioning of a process without also attending to the object in virtue of which this process is functioning well or badly. The affective, valenced component of epistemic feelings also seems to play an important role here in evaluating our mental processes, as doing well or badly in relation to the task at issue.[24]

Phenomenologically, however, I contend that the main focus of our epistemic feelings will often be simply the relevant object or task at which our cognitive processes are directed, which explains why we attribute aesthetic features to it, rather than to our own mental states. This will be because, after all, the object or task is what grabs most or all of our attention. For, on the one hand, it takes some cognitive effort to attend to our own mental processes, while on the other hand, such attention will normally be seen as irrelevant to the task in hand, for example doing a proof. I will have more to say about the details below, but if this is right then we can avoid the problem identified in our discussion of Kant's theory above, namely the idea we don't find our own cognitive processes beautiful. Even if they are ultimately what *causes* the feeling of beauty they are not normally its object.

Turning to our next question, what is the role of epistemic feelings in mathematics? The feelings of error and certainty, of confidence and understanding all clearly play a central role in mathematical reasoning, both in doing and assessing proofs (assuming these activities can be differentiated)[25] and in grasping theorems.[26] Consider, for example, cases or mathematical fallacies committed in proofs that may lead to certainty about the conclusion, yet where the feeling of certainty is, in such cases, misleading in some way (and sometimes even in cases where the right result is reached). A well known example of this, albeit one that presupposes a relatively limited amount of mathematical expertise would be the proof that 1=2:

$$a\,b = a^2 \qquad (1)$$

[24] Cf Dokic (2012); Proust (2009a). I thus disagree with Arango-Munoz's (2014) claim that epistemic feelings have mental dispositions as their content.

[25] Arguably, assessing proofs may require, in some way, doing the proof for oneself.

[26] See Arango-Munoz (2015); Reber (2008); Zeki (2014). This would also be especially unsurprising if certain positions about mathematics were true: for example, fallibilism (Lakatos 1976), humanism (Hersh 1997) and social constructivism (Ernest 1998). One could perhaps aim to give a general taxonomy of the various epistemic feelings and their different roles in the context of mathematical practice. It might, for example, be the case that the activity of doing and assessing proofs is more likely to elicit feelings of error or certainty than of understanding, whereas the state of grasping a theorem – perhaps when combined with a comprehension of its proof(s) – is an obvious candidate for the feeling of understanding. I do not want to play this game here, however, and in any case I suspect that the range of phenomena and phenomenology will be far more complex than this simple taxonomy suggests.

$$a\,b - b^2 = a^2 - b^2 \quad (2)$$
$$b\,(a-b) = (a+b)\,(a-b) \quad (3)$$
$$b = a + b \quad (4)$$
$$b = 2\,b \quad (5)$$
$$1 = 2. \quad (6)$$

Or consider the anomalous cancellation fallacy:

$$\frac{16}{64} = \frac{1\not{6}}{\not{6}4} = \frac{1}{4}.$$

Nonetheless, one recent study has shown that certain gut feelings, such as the feeling of error, can be reliable indicators of the accuracy of one's own mental performances in mathematical reasoning tasks. Arango-Munoz et al. (2015) showed, for example, that the feeling of error was strongly correlated with arithmetic errors in a Number Bisection Task, where the instructions and time restriction imposed ensured that the answers to the questions were the first that quickly and intuitively came to participants' awareness. This suggests, in the words of the authors, that "this type of feeling-based metacognition provided participants with accurate assessments of their ongoing cognitive processes without the necessity of effortful and analytical thinking". (31)

One of the crucial factors that appears to have some influence on the elicitation of feelings of error or confidence is the fluency of the task, where fluency is, roughly, the experienced ease with which mental content is processed. (R. Reber, Schwarz, & Winkielman, 2004). As Reber et al. (2008) note, many studies have demonstrated that stimuli processed with greater ease elicit more positive affect (R. Reber, Winkielman, & Schwarz, 1998; Whittlesea, 1993; Winkielman, Halberstadt, Fazendeiro, & Catty, 2006), and that fluency plays a role in the rejection of mathematical theories that are difficult to understand (McColm, 2007). Processing fluency increases either through former exposure, such as with stimulus repetition (e.g., Jacoby & Dallas, 1981) and associative learning (e.g., Posner & Keele, 1968; A. S. Reber, 1967), which renders stimuli familiar, or through stimulus features, such as simplicity (Garner, 1974) and symmetry (Palmer, 1991; Royer, 1981), that facilitate perceptual processing.

Reber et al. (2008) set out to study the role of fluency in mathematical reasoning and its relationship to certain properties of stimuli – such as simplicity or symmetry – that have also been implicated in experiences of beauty. So, the aim of this study was precisely on our topic, the purported role of beauty in mathematical reasoning. They examined the use of symmetry as a cue for correctness in an arithmetic verification task by manipulating the symmetry of sets of dot pattern addition equations. They found that speeded decisions about the correctness of these equations led to higher endorsements for both correct and incorrect equations when the addend and sum dot patterns were symmetrical. That is, people without enough time to analyse the problem instead use heuristic cues – the presence of symmetry – in their assessment of the correctness of a proposed solution. Therefore, they argue, this effect does not seem to be due to the fact that symmetry facilitates calculation or estimation.

These are intriguing findings, but we should be cautious, for given the sceptical concerns outlined at the start of this paper, we need further reason for thinking that the fluency-relevant properties used in this and other studies are functioning aesthetically here. This

thus brings us to our third question – how the epistemic feelings we've been examining are related to aesthetic experience – which is the topic of the next section.

3. Aesthetic Feelings

In an earlier study Reber et al. (2004) also identified processing fluency at work in purported judgements of beauty, thus promising to connect judgements of truth and judgements of beauty together via the medium of fluency and via epistemic feelings. They proposed that beauty, and so judgements of beauty, might be grounded in the processing experiences of the perceiver, which are in part a function of stimulus properties of the objects to which beauty is attributed. This idea aligns nicely with the complex, multi-dimensional intentionality of epistemic feelings I alluded to earlier and can, I will argue, make sense of the role of aesthetic judgements in mathematics, and indeed in cognition more generally.

Specifically, Reber et al. appealed to previous empirical experiments as well as their own studies to argue that "aesthetic pleasure is a function of the perceiver's processing dynamics: The more fluently perceivers can process an object, the more positive their aesthetic response." (p. 364) They examined a number of variables taken to influence aesthetic judgments, such as figural goodness, figure-ground contrast and clarity, stimulus repetition and familiarity, symmetry, and prototypicality, and tried to trace their effects to changes in processing fluency. They also looked at the role of other variables, like visual or semantic priming, that appear to influence judgments of aesthetic pleasure. The range of studies cited and performed is too large to discuss in any length here but, in their own words, the main findings were the following:

> "First, objects differ in the fluency with which they can be processed. Features that facilitate fluent processing include all the core features identified in objectivist theories of beauty, like goodness of form, symmetry, and figure- ground contrast, as well as variables that have not received attention in traditional theories of aesthetic pleasure, like perceptual and conceptual priming procedures. Second, processing fluency is itself hedonically marked and high fluency is subjectively experienced as positive, as indicated by psychophysiological findings. Third, the affective response elicited by processing fluency feeds into judgments of aesthetic appreciation, unless the informational value of the experience is called into question. Finally, the impact of fluency is moderated by expectations and attribution." (377)

Some features of their study and related studies are important to note. First, the stimuli for many of the experiments – with the exception of priming experiments – are often simple visual figures, rather than sophisticated visual artworks, and accordingly preference is given to visual stimuli. Second, the experiments depend upon the self-report of subjects concerning their affective responses, including judgements of pleasure and preference, but rarely of beauty, as we'll note below. Third, the authors claim that although aesthetic preferences *always* depend on fluency, experiences of fluency can be influenced by both biological equipment and socialization. This is important in explaining the differences between, for example, novice and expert preferences and a number of other variables resulting from the experiments. This finding fits too with other research in showing that epistemic-aesthetic feelings seem to be cognitively penetrable: that is, they can be influenced by top-down processing involving certain background knowledge, assumptions and expectations. For example, complexity may sometimes be preferred to simplicity by experts because it facilitates access to the meaning of the stimulus. "That is, a decrease in perceptual fluency due to complexity may be outweighed by an increase in conceptual fluency due to meaningfulness." (373)

They conclude: "Beauty is in the processing experiences of the beholder, but these

processing experiences are themselves, in part, a function of objective stimulus properties and the history of the perceiver's encounters with the stimulus. Hence, beauty appears to be "in the interaction" between the stimulus and the beholder's cognitive and affective processes." (378)

Again, we must ask: even if the experiences and attendant judgements of pleasure, liking and preference are, as they argue, based on feelings of fluency, are these experiences and judgements aesthetic? The researchers themselves note that the participants in the experiments are rarely asked to judge beauty as such, and the hypothesis is that judgements of preference, liking and beauty are closely related.

Although this might seem to mirror the same sceptical problems we have already encountered, and the same criticism directed at Zeki's study referred to earlier, I think we are on much safer ground here in including these judgements of very basic preferences and likings within the domain of aesthetic interest. What else could explain the pleasurable experience in simple visual figures, symmetrical arrangements of shapes, simple melodies and other objects? There does not seem to be a competing pleasure – intellectual or moral – that might be doing the work here. Indeed, it is arguable that the best way to empirically examine judgements of beauty and aesthetic experiences generally is not to focuss on the sophisticated judgements of experts deploying high-level knowledge, understanding, and evaluative culture-laden and theory-laden principles in relation to complex and culturally meaningful artworks. For this is just to invite confusions and conflations about the very nature of the responses involved, as it is here that we are most likely to encounter many competing interests, values and experiences.

Crucially, all of the relevant aesthetic judgements studied by these researchers seem to depend upon, and arise out of feelings of fluency, which is the experienced ease of processing of some stimulus and its properties. Recall from the last section that fluency too can be seen as underpinning the kinds of epistemic feelings and judgements operative in mathematical reasoning and other cognitive domains. Reber at al. (2008) thus give the following options concerning the possible relations between beauty, fluency, and truth:

"Theoretically, there are at least three alternatives: First, beauty— which is correlated with processing fluency—may be used as a cue for truth. This comes close to what mathematicians and scientists such as Chandrasekhar (1987) and Hadamard (1954) claimed. Second, processing fluency may influence perceived beauty, which in turn may be used for judging truth... Third, in accordance with the processing fluency view advocated by R. Reber et al. (2004), processing fluency may influence both perceptions of beauty and judgments of truth, two activities that are correlated because of a common underlying mechanism. Beauty in this case would not be involved causally in assessing truth." (1177)

It seems to me, however, that the first two options are more or less equivalent insofar as occurrent experiences of beauty are interpreted by subjects as being relevant to the epistemic claims of the relevant domain being judged. The third view may offer a plausible account of the underlying mechanism, but it's not clear why beauty – or more accurately, the experience of beauty – could not then be causally involved in assessing truth.

In any case, even if we have identified fluency as at least one central element constituting aesthetic and epistemic feelings, why think that the feelings operative in mathematics are aesthetic rather than epistemic? The sceptic might, indeed, remain unmoved. But I would contend that, if the implications of the above psychological studies are right, then

drawing any kind of a priori conceptual boundary between epistemic and aesthetic experience in mathematics looks increasingly strained in light of the empirical evidence. What we appear to be confronted with across domains are experiences of the *same kind* – let's say for the moment experiences of fluent processing, perhaps amongst other things – that may be thought of as epistemic rather than aesthetic, or vice versa, depending on the particular kinds of objects and cognitive tasks taking place in response to them.[27] We can motivate this idea further by pointing out some characteristics of epistemic feelings that look very much like aesthetic characteristics.

First, epistemic feelings, like aesthetic experiences, are valenced. They are experienced as positive or negative. This, plausibly, is a result of them being manifestations of some sort of meta-cognitive monitoring of how well or badly, efficiently or inefficiently, our cognitive process are engaging with an object or task. Second, epistemic feelings, like aesthetic experiences, can be and perhaps often are quick and dirty responses that are opaque, at least initially, to the reasons that ground them. Third, let's not forget, in the context of mathematical practice these epistemic feelings are often characterised as aesthetic. The nature of the experiences is, to the subjects undergoing them, experienced as being of the same (rough and ready) kind as those present in paradigmatic aesthetic contexts.[28] Finally, as we have seen, many of the criteria and stimulus properties occur as the objects and causes of both aesthetic and epistemic experiences – symmetry, simplicity, fluency, order, clarity and so on.

So, I contend, we can think of (some) epistemic feelings as possessing aesthetic attributes, or of (some) aesthetic feelings as exhibiting epistemic attributes, or of a range of feelings that are jointly aesthetic-epistemic in nature. Let's look at the various options here in more detail, but I will suggest that, at least from an explanatory point of view there is little to choose between them and they may in fact be perfectly compatible.

Although much emphasis has been placed on the feeling of fluency in the preceding discussion, and this can explain an impressive number of different epistemic-aesthetic feelings in various domains, including mathematics, I don't think it's sufficient to account, even in a broad way, for all such aesthetic-epistemic experiences. In part this is because, despite our discussion of the intentional objects of epistemic feelings earlier, I think it fails to give due weight to certain aesthetically valuable experiences that are not obviously best characterised in terms of the fluency with which we process them. The two notions I want to concentrate on here are those of 'understanding' and 'fit'.

The Aha! moment that encompasses a surprising discovery, or simply reflects the fact that some impressive intellectual achievement or insight has been accomplished need not, it seems, be the result of a feeling of fluency, although no doubt some feelings of understanding resemble a feeling of fluency in the sense that we have now succeeded in processing whatever it was we were supposed to process. The pleasure of understanding in mathematics and in other aesthetic contexts seems often to have something to do with the notions of harmony or fit rather than fluency *per se*. And here we find, as many have pointed out, come striking continuities between art (especially music) and mathematics,

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[27] Of course, there is now an issue here about whether interpreting experiences changes their nature so that they could not, in this case, continue to be of the same kind, and I will address this below.

[28] For example, in response to Ramanujan's work on identities, Watson said that it gave him "a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of Cappella Medicee and see before me the austere beauty of "Day", "Night", "Evening" and "Dawn" which Michelangelo has set over the tombs of Giuliano d' Medici and Lorenzo d' Medicii". (Quoted in Chandrasekhar 1987: 61).

in terms of the appreciation of patterns, connections, symmetries and harmonies common to each. As Peter Kosso (2002) notes, for instance, in the context of science: 'the hallmarks of scientific understanding are similar to an aesthetic feature associated with literature, music, and the visual arts. It is the feature described as coherence, harmony, and *inevitability of fit.* Aesthetics thus plays an epistemic role in science as an indication of understanding.'(39)[29]

This notion of fit, as Wittgenstein long ago remarked, seems central to many aesthetic experiences, although it is very difficult to articulate precisely. Nonetheless, I think that there is a distinctive type of feeling that accompanies the 'perception' of fittingness and it is very closely related to, or perhaps a variant of the feeling of understanding.

Timothy Gowers (2002) captures something like this idea in his discussion of one particular proof of the impossibility of tiling an eight-by-eight grid – with opposing corners removed – evenly with domino-shaped tiles. Taking a chessboard, Gowers notes that the two opposite corner squares will have the same colour (e.g. black), which once removed will leave the chessboard with 32 white squares and 30 black ones. Each domino covers exactly one square of each colour. So, once you have put down 30 dominoes, you will be left with two white squares that you won't be able to cover. As Gowers remarks of this particular proof, it captures something of what mathematicians mean by describing proofs as 'elegant', 'beautiful', or 'witty' and resembles in certain respects the appreciation of music:

"[it] illustrates very well how a proof can offer more than just a guarantee that a statement is true...[It] depends on a single idea, which, though unexpected, seems very natural as soon as one has understood it...Mathematical proofs can provide a similar pleasure [to music] with sudden revelations, unexpected yet natural ideas, and intriguing hints that there is more to be discovered." (51-2) (Cf. Glynn 2010: 3-4)

Here, I contend, we have a genuine hybrid epistemic-aesthetic feeling, and one that cannot be reduced to feelings of fluency. Of course, in the naturalistic spirit at which this paper has been aiming, this feeling requires empirical investigation, and it does not currently figure on the lists of epistemic feelings that psychologists investigate. Such an investigation should, I think, be initially guided by the following strong hypothesis:

HYPOTHESIS: All epistemic feelings and aesthetic experiences are determinate variations of the 'feeling of fittingness', which is a determinable experiential manifestation of certain sub-personal processes of representing and assessing the relation between certain cognitive processes and certain properties of the stimuli at which they are directed. When this relation exhibits, among other thing: a) coherence b) breadth of scope, b) simplicity, and c) fluency we have what I will call 'cognitive consonance'. It is this consonance that is manifested as a feeling of fittingness, and this feeling is jointly aesthetic-epistemic in nature.

Note that the claim is a strong one because it encompasses *all* epistemic and aesthetic feelings. It does not say, however, that all aesthetic feelings *are* epistemic feelings, or vice versa. Recall the point made earlier that experiences of the *same kind* may be thought of and interpreted as epistemic rather than aesthetic, or vice versa, depending on the particular kinds of objects and cognitive tasks taking place in response to them. So, for example, a mathematician experiencing the feeling of understanding might think of and report this naturally as an aesthetic experience, but when pressed by an annoying

[29] Cf. Root-Bernstein (2002); Lipton (2004): ch. 9; Arnheim (1996).

philosopher might characterise it instead as an epistemic experience. And then, after a period of further reflection, might call it instead an aesthetic-epistemic experience.

In general, and put very roughly, I'm inclined to think that if the object of the experience is categorized under a broadly intellectual/cognitive domain then the resulting feeling of fittingness will take on an epistemic phenomenal tinge and be interpreted as an epistemic feeling. If the domain is seen as aesthetic, however, then the way in which the experience is characterised by the subject will change accordingly. Mathematics is interesting precisely because it seems to turn this relationship on its head, which is of course why we have the problem of interpreting and justifying the relevant 'aesthetic' judgements in the first place.

This might seem to complicate the issue because interpreting experiences can arguably change their nature, by changing how they feel. But, in fact, I do not think this really threatens the claim, because we are still concerned with a determinable feeling that is fundamentally epistemic-aesthetic in nature. For that reason, I think it makes no difference, in explanatory terms, for any determinate of this determinable experience whether we think of it as an epistemic feeling that possesses aesthetic attributes, or as an aesthetic feeling that exhibits some epistemic attributes.

From the point of view of phenomenology, however, things look a little different, and it's here that we can explain the relationship between determinable and determinates. Roughly, how the feeling of fittingness will be manifested in experience to become a determinate experience will depend on a large number of variables, including but not limited to, the particular properties of the object, how the object is perceived, conceptualised and categorised, and the particular psychological make-up of the subject. In other words, whether an experience is felt to be one of elegance or gracefulness or clumsiness, to be aesthetic or intellectual in nature, to be pleasurable or not, will depend on all those complex factors that go into any conceptual analysis of the nature of aesthetic judgement, vocabulary, and experience.

Thankfully it is beyond the scope of the present paper to offer any further thoughts on this, other than to note that: i) it may turn out, for example, that 'elegance' is generally applied to 'objects' exhibiting properties like simplicity and economy, gracefulness to objects that exhibit certain kinds of movement, beauty to almost anything that arouses a positive aesthetic experience; ii) the difficult in analysing aesthetic terms like this just is reflection of the difficulty in articulating, in aesthetic experience generally, what properties are responsible for one's reactions.

Naturally, these are highly speculative remarks, but I think they do offer a promising starting point for further empirical investigation of aesthetic-epistemic experiences, and they do, I hope, give a plausible non-sceptical account of the role of aesthetic judgements in mathematics, and one that possesses both some aesthetic as well as epistemic virtues.[30]

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