

A Geometric Theory of Quantum Measurements Automatically Yielding the Standard Model and Gravity in 4D, with Disruptions Beyond

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Abstract

We present a geometric theory of quantum measurements, in which quantum probabilities and spacetime geometry intersect. To construct the theory, we utilize the 'Prescribed Measurement Problem,' an innovative algorithm that extends the entropy maximization problem – traditionally a concept in statistical physics – into the realms of quantum mechanics and geometry. This algorithm generates a unified probability measure that consistently assigns probabilities to both quantum and geometric measurements. Remarkably, this approach results in general relativity and the Standard Model emerging as intrinsic components of the theory. Intriguingly, the theory's consistency and coherence are strictly confined to four-dimensional spacetime. Efforts to extend it beyond these four dimensions encounter significant challenges and disruptions. This inherent limitation within the theory suggests a quantum-geometric barrier to the four-dimensionality of our universe, offering a unique perspective on the interplay between quantum mechanics and spacetime geometry.

1 Introduction

The reconciliation of quantum mechanics with general relativity remains a pivotal objective in the field of theoretical physics. In our previous work, we introduced the 'Prescribed Measurement Problem' (PMP) [1], a methodology anchored in the principles of statistical mechanics, focusing on entropy maximization constrained by observed measurement outcomes. Significantly, the PMP automatically reproduces the axioms of quantum mechanics as the optimal solution to an entropy maximization problem under a unitary-evolution energy constraint. In this paper, we present the natural generalization of the PMP, leading to a unified probability measure that accommodates all possible quantum and geometric measurements. Within this framework, general relativity and the Standard Model emerge not as superimposed axioms, but as inevitable consequences of the probability measure.

The Prescribed Measurement Problem (PMP) is directly inspired by the foundational principles of statistical mechanics, where theory is inherently constrained by a sequence of empirical measurements. Statistical mechanics exemplifies a natural PMP, where the aggregation of energy measurements informs the theoretical structure, leading to the derivation of the Gibbs measure.

Recapitulating this approach, statistical mechanics commences with an empirical sequence of energy measurements. These measurements, anticipated to converge to an average value \bar{E} , are utilized as defining constraints within the theoretical formulation:

$$0 = \bar{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \quad (1)$$

To derive a probability distribution, $\rho(q)$, that maximizes entropy while adhering to this constraint, the theory employs a Lagrange multiplier equation[2].

$$\mathcal{L}(\rho, \lambda, \beta) = \underbrace{-k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\bar{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right)}_{\text{Energy Measurement Constraint}} \quad (2)$$

Solving this yields the well-established Gibbs measure as the least biased probability measure for the constraint:

$$\rho(q) = \frac{\exp(-\beta E(q))}{\sum_{q \in \mathbb{Q}} \exp(-\beta E(q))} \quad (3)$$

Transitioning to quantum mechanics, the PMP framework demonstrates that quantum theory naturally arises from an entropy maximization problem, formulated with a sequence of measurement outcomes serving as constraints. This approach marks a significant departure from conventional interpretations that typically depend on postulating a wavefunction and other quantum mechanical constructs. Instead, the PMP proposes the use of measurement outcomes as the constraint, enabling the direct inference of both the initial state of the quantum system and its evolution over time.

However, quantum mechanics requires a more elaborate energy constraint than statistical mechanics. As such, the sequence of energy measurements is intrinsically related to the Hamiltonian, which presides over the system's unitary time evolution. This connection necessitates an adapted form of the energy constraint to encapsulate the proper attributes. Unlike the scalar energy constraints in statistical mechanics, quantum mechanics demands a matrix-based formulation to capture the phase information associated with quantum energy measurements, as mandated by the unitarily-evolving Hamiltonian.

To accommodate this requirement, we introduce a matrix-based unitary-evolution energy constraint, which is harmonious with unitary evolution and

respects the Born rule:

$$0 = \text{tr} \begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \text{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \quad (4)$$

This constraint is represented in matrix form and incorporates the system's energy characteristics and their phase without altering the probability measure. Special attention is given to the computation of the trace, which if prematurely simplified, would trivialize the constraint and, consequently, would eliminate the solution space of the constraint. Instead, this unitary-evolution energy constraint is integrated into the entropy maximization problem, commonly employed in statistical mechanics. The resulting solution not only aligns with the established quantum formalism but also simplifies it, rendering it as the most parsimonious formulations of quantum mechanics to date.

$$\mathcal{L} = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\text{Relative Shannon Entropy[3, 4]}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\tau \left(\text{tr} \begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \text{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right)}_{\text{Unitary-Evolution Energy Constraint}} \quad (5)$$

We solve for $\partial \mathcal{L}(\rho, \lambda, t)/\partial \rho = 0$ as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \quad (6)$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \quad (7)$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \quad (8)$$

$$\implies \rho(q) = p(q) \exp(-1 - \lambda) \exp \left(-\tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right) \quad (9)$$

$$= \frac{1}{Z(\tau)} p(q) \exp \left(-\tau \text{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right) \quad (10)$$

where $Z(\tau)$ is obtained as

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-1 - \lambda) \exp \left(-\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \quad (11)$$

$$\implies (\exp(-1 - \lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \quad (12)$$

$$Z(\tau) := \sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \text{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right) \quad (13)$$

The final result is:

$$\rho(q) = \frac{p(q) \exp \left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \right)}{\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix} \right)} \quad (14)$$

By utilizing fundamental equivalences and substituting $\tau = t/\hbar$ in a manner analogous to $\beta = 1/(k_B T)$, by noting that the trace drops down from the exponential into the determinant ($\exp \operatorname{tr} M = \det \exp M$), and that the determinant of such a matrix is equivalent to a complex norm, we can rearticulate this into its more commonly recognized form:

$$\rho(q) = \frac{p(q) \|\exp(-itE(q)/\hbar)\|}{\sum_{r \in \mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\|} \quad (15)$$

We can now reverse-engineer the uniquely determined solution space of this expression.

In our previous paper[1], we have shown that all 5 traditional axioms of quantum mechanics[5, 6] are provable from this solution:

- Axiom 1 **State Space:** Every physical system is associated with a complex Hilbert space, and the system's state is described by a unit vector (or ray) in that space.
- Axiom 2 **Observables:** Physical observables are represented by Hermitian operators acting on the Hilbert space.
- Axiom 3 **Dynamics:** The evolution of a quantum system over time is governed by the Schrödinger equation, with the Hamiltonian operator representing the total energy of the system.
- Axiom 4 **Measurement:** Upon measurement of an observable, the system collapses to one of the eigenstates of the corresponding operator, and the measured value is one of the eigenvalues.
- Axiom 5 **Probability Interpretation:** The probability of obtaining a particular measurement result is given by the squared magnitude of the projection of the state vector onto the corresponding eigenstate.

Let us see how each axiom is recovered.

The wavefunction is identified by "splitting the complex norm" into a complex number and its conjugate. It is envisioned as a vector in a complex Hilbert space, with the partition function acting as its inner product. Expressing the relation in those terms:

$$\sum_{q \in \mathbb{Q}} p(q) \|\exp(-itE(q)/\hbar)\| = Z = \langle \psi | \psi \rangle \quad (16)$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE(q_1)/\hbar) & & \\ & \ddots & \\ & & \exp(-itE(q_n)/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \quad (17)$$

and where $p(q)$ is the probability associated to the initial preparation of the wavefunction: $p(q_i) = \langle \psi_i(0) | \psi_i(0) \rangle$. The entropy-maximization procedure automatically normalizes the result which associates here to a unit vector (or more precisely, a ray). This demonstrates Axiom 1.

We now note that the energy constraint is unmodified by unitary transformations:

$$\langle \mathbf{E} \rangle = \langle \psi | \mathbf{H} | \psi \rangle = \langle \psi | U^\dagger \mathbf{H} U | \psi \rangle \quad (18)$$

Upon moving the solution out of its eigenbasis through unitary transformations, we find that energy, $E(q)$, generally transforms as an Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle \quad (19)$$

The dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t} |\psi(t)\rangle = \frac{\partial}{\partial t} (\exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle) \quad (20)$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle \quad (21)$$

$$= -i\mathbf{H}/\hbar |\psi(t)\rangle \quad (22)$$

$$\implies \mathbf{H} |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle \quad (23)$$

Which is the Schrödinger equation. This demonstrates Axiom 3.

The statistical ensemble \mathbb{Q} is defined such that the possible microstates $E(q)$ of the system corresponds to a specific eigenvalue of \mathbf{H} . An observation can thus be conceptualized as sampling from $\rho(q, t)$, with the post-collapse state being the occupied microstate q of \mathbb{Q} . Consequently, when an observation or measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of \mathbf{H} . Measured in the eigenbasis, the probability distribution is:

$$\rho(q, t) = \frac{1}{\langle \psi | \psi \rangle} (\psi(q, t))^\dagger \psi(q, t) \quad (24)$$

In scenarios where the probability measure $\rho(q, \tau)$ is described in a basis different from its eigenbasis due to a unitary transformation, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle \lambda_i | \psi \rangle|^2 \quad (25)$$

Here, $|\langle \lambda_i | \psi \rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. This demonstrates Axiom 4.

Any self-adjoint operator abides by the condition $\langle \mathbf{O}\psi | \phi \rangle = \langle \psi | \mathbf{O}\phi \rangle$. Measured in its eigenbasis, it aligns with a real-valued observable in statistical mechanics. This demonstrates Axiom 2.

Finally, we note that as the probability measure (Equation 15) reproduces the Born rule, Axiom 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. The constraint given by Equation 26 is now the formulation's sole axioms, and the five previous axioms are now its theorems.

2 Results

In this section, we present the natural generalization of the 'Prescribed Measurement Problem' (PMP), which culminates in the development of a geometric theory of quantum measurements. This advancement of the PMP framework effectively merges the principles of quantum mechanics with geometric considerations, providing a unique perspective on the convergence of these two crucial aspects of physics.

Our previous work on the PMP utilized the unitary-evolution energy constraint:

$$0 = \text{tr} \begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \text{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \quad (26)$$

which produced ordinary quantum mechanics.

We now extend this constraint to encompass the entire spectrum of geometric expressibility:

$$0 = \frac{1}{n} \text{tr} \overline{\mathbf{M}} - \sum_{q \in \mathbb{Q}} \rho(q) \frac{1}{n} \text{tr} \mathbf{M}(q) \quad (27)$$

Here, $\mathbf{M}(q)$ represents a $n \times n$ traceless matrix and where n corresponds to the dimension of the spacetime.

This generalized constraint is compatible with the Lagrange multiplier method of the PMP, reflecting the entropy maximization process seen in the introduction.

2.1 The Lagrange Multiplier Equation

The optimization of the probability measure is articulated through the Lagrange multiplier equation. This mathematical construct ensures that the derived probability measure maximizes the entropy while adhering to constraints imposed by the geometric measurement constraint.

The formal expression of the Lagrange multiplier equation is as follows:

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\tau \left(\frac{1}{n} \text{tr} \bar{\mathbf{M}} - \sum_{q \in \mathbb{Q}} \rho(q) \frac{1}{n} \text{tr} \mathbf{M}(q)\right)}_{\text{Geometric Measurement Constraint}} \quad (28)$$

This equation encapsulates the relative Shannon entropy, normalization constraint, and a geometric measurement constraint. The Shannon entropy quantifies the informational disparity between the probability distribution and measurement outcomes, the normalization constraint ensures the sum of probabilities equals unity, and the geometric measurement constraint ensures connection to empirical data of geometric measurements within a quantum system.

To identify the probability distribution that optimizes this Lagrange equation, the derivative with respect to ρ is calculated and set to zero:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \frac{1}{n} \text{tr} \mathbf{M}(q) \quad (29)$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \frac{1}{n} \text{tr} \mathbf{M}(q) \quad (30)$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \frac{1}{n} \text{tr} \mathbf{M}(q) \quad (31)$$

$$\implies \rho(q) = p(q) \exp(-1 - \lambda) \exp\left(-\tau \frac{1}{n} \text{tr} \mathbf{M}(q)\right) \quad (32)$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \frac{1}{n} \text{tr} \mathbf{M}(q)\right) \quad (33)$$

The partition function $Z(\tau)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-1 - \lambda) \exp\left(-\tau \frac{1}{n} \text{tr} \mathbf{M}(r)\right) \quad (34)$$

$$\implies (\exp(-1 - \lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \frac{1}{n} \text{tr} \mathbf{M}(r)\right) \quad (35)$$

$$Z(\tau) := \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\tau \frac{1}{n} \text{tr} \mathbf{M}(r)\right) \quad (36)$$

Consequently, the optimal probability distribution is given by:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \exp\left(-\frac{1}{n} \tau \text{tr} \mathbf{M}(r)\right)}}_{\text{Normalization Constant}} \underbrace{\exp\left(-\frac{1}{n} \tau \text{tr} \mathbf{M}(q)\right)}_{\text{Generalized Born Rule}} \underbrace{p(q)}_{\text{Prior}} \quad (37)$$

This formulation advances the Born rule to a generalized context, seamlessly integrating with the core principles of quantum mechanics and incorporating geometric measurements.

2.2 Linear Measurement Constraint in Two Dimensions

Our investigation into the linear measurement constraint commences in a two-dimensional (2D) framework, setting the stage for later extension into the more complex 3+1D spacetime.

In the 2D context, the measurement constraint is formulated as follows:

$$\frac{1}{2} \text{tr} \overline{\mathbf{M}} = \sum_{q \in \mathbb{Q}} \rho(q) \frac{1}{2} \text{tr} \mathbf{M}(q), \quad \text{where } \mathbf{M}(q) := \begin{bmatrix} x(q) & y(q) + b(q) \\ y(q) - b(q) & -x(q) \end{bmatrix} \quad (38)$$

In this expression, $b(q)$, $x(q)$, and $y(q)$ are scalar functions parameterized by q . These functions characterize the elements of a traceless matrix, representing 2D geometric measurements.

The probability distribution for this two-dimensional scenario, obtained by solving the Lagrange equation, is simplified to:

$$\rho(q) = \frac{1}{Z} \det \left(\exp \begin{bmatrix} x(q) & y(q) + b(q) \\ y(q) - b(q) & -x(q) \end{bmatrix} \right) p(q) \quad (39)$$

where

$$p(q) = \det \begin{bmatrix} a(q) + x(q) & y(q) + b(q) \\ y(q) - b(q) & a(q) - x(q) \end{bmatrix} = \det \varphi(q) \quad (40)$$

and where $\det \varphi(q) > 0$ is a precondition.

2.3 Inner Product

In constructing a Hilbert space for our probability measure, it is essential to express the determinant as an inner product. To do so, we begin by representing 2×2 real matrices as multivectors within the Clifford algebra framework:

$$\begin{bmatrix} a + x & y + b \\ y - b & a - x \end{bmatrix} \cong a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \quad (41)$$

The Clifford conjugate, denoted by † , is defined as:

$$(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})^\dagger = a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \quad (42)$$

Using this multivector representation, we establish that the determinant, when expressed through the geometric product, corresponds to an inner product of multivectors:

$$\mathbf{u}^\dagger \mathbf{u} = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) = a^2 - x^2 - y^2 + b^2 \quad (43)$$

We can verify that this inner product is equivalent to the determinant of the corresponding matrix:

$$\det \begin{bmatrix} a + x & y + b \\ y - b & a - x \end{bmatrix} = a^2 - x^2 - y^2 + b^2 \quad (44)$$

Furthermore, over the group of $GL^+(2, \mathbb{R})$ matrices, this inner product is positive definite (since the determinant is positive), and is equal to 0 only for the null multi-vector.

2.4 The General Linear Wavefunction Representation

The general linear wavefunction, denoted as φ , is expressible in a multivector column vector format, incorporating the algebraic elements characteristic of a 2×2 matrix. This allows for the representation of matrices within the geometric algebra framework, facilitating a seamless transition to a Hilbert space structure via the introduced inner product of multivectors. The wavefunction φ is represented as:

$$|\varphi\rangle\rangle = \frac{1}{\sqrt{Z}} \begin{bmatrix} a_1 + x_1 \hat{\mathbf{x}} + y_1 \hat{\mathbf{y}} + b_1 \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \\ \vdots \\ a_n + x_n \hat{\mathbf{x}} + y_n \hat{\mathbf{y}} + b_n \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \end{bmatrix} \quad (45)$$

The Clifford conjugate transpose (row vector representation) of φ is obtained by applying the Clifford conjugation to each multivector element:

$$\langle\langle\varphi| = \frac{1}{\sqrt{Z}} [a_1 - x_1 \hat{\mathbf{x}} - y_1 \hat{\mathbf{y}} - b_1 \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \quad \dots \quad a_n - x_n \hat{\mathbf{x}} - y_n \hat{\mathbf{y}} - b_n \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}] \quad (46)$$

This format enables the definition of an inner product within the Hilbert space as a sum of matrix determinants for each entry:

$$\langle\langle\varphi|\varphi\rangle\rangle = \frac{1}{Z} ((a_1^2 - x_1^2 - y_1^2 + b_1^2) + \dots + (a_n^2 - x_n^2 - y_n^2 + b_n^2)) \quad (47)$$

2.5 Self-Adjoint Operators in the General Linear Framework

Observables are characterized as self-adjoint operators. An observable, denoted as \mathbf{O} , must satisfy the condition:

$$\langle\langle\mathbf{O}\phi|\varphi\rangle\rangle = \langle\langle\phi|\mathbf{O}\varphi\rangle\rangle \quad (48)$$

For a two-state system, the observable \mathbf{O} is represented by a 2×2 matrix:

$$\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix} \quad (49)$$

where \mathbf{o}_{00} , \mathbf{o}_{01} , \mathbf{o}_{10} , and \mathbf{o}_{11} are multivectors, encapsulating the components of the observable.

In the Clifford algebra framework, the geometric product corresponds to matrix multiplication, resulting in:

$$\begin{aligned} \langle\langle \mathbf{O}\phi|\varphi\rangle\rangle &= (\mathbf{o}_{00}\phi_1 + \mathbf{o}_{01}\phi_2)^\dagger\varphi_1 + \varphi_1^\dagger(\mathbf{o}_{00}\phi_1 + \mathbf{o}_{01}\phi_2) \\ &\quad + (\mathbf{o}_{10}\phi_1 + \mathbf{o}_{11}\phi_2)^\dagger\varphi_2 + \varphi_2^\dagger(\mathbf{o}_{10}\phi_1 + \mathbf{o}_{11}\phi_2) \end{aligned} \quad (50)$$

$$\begin{aligned} &= \phi_1^\dagger\mathbf{o}_{00}^\dagger\varphi_1 + \phi_2^\dagger\mathbf{o}_{01}^\dagger\varphi_1 + \varphi_1^\dagger\mathbf{o}_{00}\phi_1 + \varphi_1^\dagger\mathbf{o}_{01}\phi_2 \\ &\quad + \phi_1^\dagger\mathbf{o}_{10}^\dagger\varphi_2 + \phi_2^\dagger\mathbf{o}_{11}^\dagger\varphi_2 + \varphi_2^\dagger\mathbf{o}_{10}\phi_1 + \varphi_2^\dagger\mathbf{o}_{11}\phi_2 \end{aligned} \quad (51)$$

$$\begin{aligned} \langle\langle \phi|\mathbf{O}\varphi\rangle\rangle &= \phi_1^\dagger(\mathbf{o}_{00}\varphi_1 + \mathbf{o}_{01}\varphi_2) + (\mathbf{o}_{00}\varphi_1 + \mathbf{o}_{01}\varphi_2)^\dagger\phi_1 \\ &\quad + \phi_2^\dagger(\mathbf{o}_{10}\varphi_1 + \mathbf{o}_{11}\varphi_2) + (\mathbf{o}_{10}\varphi_1 + \mathbf{o}_{11}\varphi_2)^\dagger\phi_2 \end{aligned} \quad (52)$$

$$\begin{aligned} &= \phi_1^\dagger\mathbf{o}_{00}\varphi_1 + \phi_1^\dagger\mathbf{o}_{01}\varphi_2 + \varphi_1^\dagger\mathbf{o}_{00}^\dagger\phi_1 + \varphi_2^\dagger\mathbf{o}_{01}^\dagger\phi_1 \\ &\quad + \phi_2^\dagger\mathbf{o}_{10}\varphi_1 + \phi_2^\dagger\mathbf{o}_{11}\varphi_2 + \varphi_1^\dagger\mathbf{o}_{10}^\dagger\phi_2 + \varphi_2^\dagger\mathbf{o}_{11}^\dagger\phi_2 \end{aligned} \quad (53)$$

For \mathbf{O} to be self-adjoint, the following conditions must hold:

$$\mathbf{o}_{00}^\dagger = \mathbf{o}_{00} \quad (54)$$

$$\mathbf{o}_{01}^\dagger = \mathbf{o}_{10} \quad (55)$$

$$\mathbf{o}_{10}^\dagger = \mathbf{o}_{01} \quad (56)$$

$$\mathbf{o}_{11}^\dagger = \mathbf{o}_{11} \quad (57)$$

This implies that \mathbf{O} is observable when $\mathbf{O}^\dagger = \mathbf{O}$, analogous to self-adjoint operators in complex Hilbert spaces where $\mathbf{O}^\dagger = \mathbf{O}$.

The most general form of an observable matrix \mathbf{O} in our framework is:

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \\ a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} & a_{11} \end{bmatrix} \quad (58)$$

2.6 Real Eigenvalues for Observables

In the context of geometric algebra, we explore the nature of eigenvalues associated with an observable matrix \mathbf{O} , emphasizing their real-valued characteristic. The eigenvalues are determined by solving the characteristic equation obtained from the matrix's determinant:

$$0 = \det(\mathbf{O} - \lambda\mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \\ a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} & a_{11} - \lambda \end{bmatrix}, \quad (59)$$

which expands to:

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2) \quad (60)$$

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2), \quad (61)$$

The resulting eigenvalues are:

$$\lambda = \left\{ \frac{1}{2} \left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)} \right), \right. \quad (62)$$

$$\left. \frac{1}{2} \left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)} \right) \right\} \quad (63)$$

It is important to recognize that if $a^2 - x^2 - y^2 + b^2 < 0$, the eigenvalues could potentially be complex. However, our focus is on multivectors and general linear matrices that preserve orientation, ensuring $a^2 - x^2 - y^2 + b^2 \geq 0$. Within these constraints, our observables invariably have real eigenvalues.

2.7 Probability-Preserving Transformations

In quantum mechanics, transformations that preserve the probability distribution of a system are fundamental. Such transformations are represented by operators \mathbf{T} which, when acting on a wavefunction $|\varphi\rangle$, fulfill the condition $\langle\langle\varphi|\mathbf{T}^\dagger\mathbf{T}|\varphi\rangle\rangle = 1$, indicating that $\mathbf{T}^\dagger\mathbf{T}$ functions as the identity operator I .

For a two-state quantum system undergoing a general transformation \mathbf{T} , represented in matrix form with 2D multivector components u, v, w , and x :

$$\mathbf{T} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}, \quad (64)$$

The Clifford conjugate product $\mathbf{T}^\dagger\mathbf{T}$ is given by:

$$\mathbf{T}^\dagger\mathbf{T} = \begin{bmatrix} v^\dagger & u^\dagger \\ w^\dagger & x^\dagger \end{bmatrix} \begin{bmatrix} v & w \\ u & x \end{bmatrix} = \begin{bmatrix} v^\dagger v + u^\dagger u & v^\dagger w + u^\dagger x \\ w^\dagger v + x^\dagger u & w^\dagger w + x^\dagger x \end{bmatrix} \quad (65)$$

For $\mathbf{T}^\dagger\mathbf{T}$ to satisfy the identity condition, the following must hold:

$$v^\dagger v + u^\dagger u = 1 \quad (66)$$

$$v^\dagger w + u^\dagger x = 0 \quad (67)$$

$$w^\dagger v + x^\dagger u = 0 \quad (68)$$

$$w^\dagger w + x^\dagger x = 1 \quad (69)$$

These conditions are met by:

$$\mathbf{T} = \frac{1}{\sqrt{v^\dagger v + u^\dagger u}} \begin{bmatrix} v & u \\ -e^\varphi u^\dagger & e^\varphi v^\dagger \end{bmatrix}, \quad (70)$$

where u and v are 2D multivectors, and e^φ is a unit multivector.

In the unitary case, where the vector part of the multivector diminishes ($x \rightarrow 0, y \rightarrow 0$), it simplifies to:

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta} b^\dagger & e^{i\theta} a^\dagger \end{bmatrix}. \quad (71)$$

Thus, \mathbf{T} represents a general linear extension of unitary transformations to two dimensions within the geometric algebra framework. This broadens the scope of unitary transformations, accommodating the intricate structure of multivectors.

2.8 General Linear Interference

Consider a two-state quantum system represented by the wavefunction $|\varphi\rangle$:

$$|\varphi\rangle = \frac{1}{\sqrt{Z}} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \quad (72)$$

where \mathbf{u} and \mathbf{v} are multivectors, and Z is a normalization constant.

We introduce an invariant transformation U satisfying $U^\dagger U = I$, such as the Hadamard transformation:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad (73)$$

Now applying U to $|\varphi\rangle$ yields a superposed state:

$$\frac{1}{\sqrt{Z}} \begin{bmatrix} \frac{1}{\sqrt{2}}\mathbf{u} + \frac{1}{\sqrt{2}}\mathbf{v} \\ \frac{1}{\sqrt{2}}\mathbf{u} - \frac{1}{\sqrt{2}}\mathbf{v} \end{bmatrix} \quad (74)$$

This superposition represents a combination of spacetime geometries that can exhibit interference effects upon measurement. The probability amplitude of the superposition is given by:

$$\frac{1}{2Z}(\mathbf{u}^\dagger + \mathbf{v}^\dagger)(\mathbf{u} + \mathbf{v}) = \frac{1}{2Z}(\mathbf{u}^\dagger\mathbf{u} + \mathbf{u}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{u} + \mathbf{v}^\dagger\mathbf{v}) \quad (75)$$

$$= \frac{1}{2Z}(\underbrace{\mathbf{u}^\dagger\mathbf{u} + \mathbf{v}^\dagger\mathbf{v}}_{\text{superposition}} + \underbrace{\mathbf{u}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{u}}_{\text{interference}}) \quad (76)$$

This decomposes into terms representing superposition ($\mathbf{u}^\dagger\mathbf{u} + \mathbf{v}^\dagger\mathbf{v}$) and interference ($\mathbf{u}^\dagger\mathbf{v} + \mathbf{v}^\dagger\mathbf{u}$). The interference complexity[7] exceeds that in systems described by complex wavefunctions, due to the multivector nature of \mathbf{u} and \mathbf{v} .

In cases where vector components are absent (φ with $\hat{\mathbf{x}}, \hat{\mathbf{y}} \rightarrow 0$), the interference reduces to the conventional complex interference pattern found in quantum mechanics.

2.9 Metric Operator

We introduce the metric operator \hat{g} , defined as:

$$\hat{g}\varphi = I\varphi I^{-1} \quad (77)$$

where $I = \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$ represents the pseudoscalar in two dimensions. This operator functions as a similarity transformation, altering the basis of the multivector φ .

The action of \hat{g} on a multivector inverses the sign of its vector part while preserving the scalar and bivector parts:

$$I(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})I^{-1} = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \quad (78)$$

We demonstrate \hat{g} as unitary and probability-preserving within the general linear group context:

$$(g\mathbf{u})^\dagger g\mathbf{u} = (I\mathbf{u}I^{-1})^\dagger I\mathbf{u}I^{-1} \quad (79)$$

$$= (I^{-1})^\dagger \mathbf{u}^\dagger I^\dagger I\mathbf{u}I^{-1} \quad (80)$$

$$= (-I^{-1})\mathbf{u}^\dagger (-I)I\mathbf{u}I^{-1}I^{-1}\mathbf{u}^\dagger I I\mathbf{u}I^{-1} \quad (81)$$

$$= e_2 e_1 \mathbf{u}^\dagger e_1 e_2 e_1 e_2 \mathbf{u} e_2 e_1 \quad (82)$$

$$= -e_2 e_1 \mathbf{u}^\dagger \mathbf{u} e_2 e_1 \quad (83)$$

$$= e_1 e_2 \underbrace{\mathbf{u}^\dagger \mathbf{u}}_{\text{scalar}} e_2 e_1 \quad (84)$$

$$= \mathbf{u}^\dagger \mathbf{u}. \quad (85)$$

This confirms the unitary nature of \hat{g} .

Furthermore, we show that I is also unitary:

$$(I\mathbf{u})^\dagger I\mathbf{u} = \mathbf{u}^\dagger (-I)I\mathbf{u} = \mathbf{u}^\dagger (-e_1 e_2 e_1 e_2)\mathbf{u} = \mathbf{u}^\dagger \mathbf{u} \quad (86)$$

We now apply \hat{g} to the inner product:

$$\mathbf{u}^\dagger \hat{g}\mathbf{u} = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \quad (87)$$

In matrix form, the multivectors and their transformations under \hat{g} are:

$$\mathbf{u}^\dagger = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a-x & -y-b \\ -y+b & a+x \end{bmatrix} \quad (88)$$

$$\hat{g}\mathbf{u} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a-x & -y+b \\ -y-b & a+x \end{bmatrix} \quad (89)$$

This representation shows that \mathbf{u}^\dagger is the transpose of $\hat{g}\mathbf{u}$, preserving the inner product within the $\text{SO}(2)$ group.

- For a general multivector, this inner product signifies the transition amplitude between \mathbf{u} and its transpose.
- For a vector $\mathbf{u} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$, it yields a scalar $\mathbf{u}^\dagger \hat{g}\mathbf{u} = x^2 + y^2$, representing the Euclidean norm.
- For a vector $\mathbf{u} = a + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$, the result is a scalar $\mathbf{u}^\dagger \hat{g}\mathbf{u} = a^2 + b^2$, analogous to the complex norm.

Consequently, this approach underscores how a common probabilistic structure can bridge the Euclidean norm, integral to geometry, with the complex norm, fundamental in quantum mechanics.

2.10 Transition to Gravity via the Metric Operator

Building upon the foundational role of the metric operator \hat{g} in our framework, we explore its implications in transitioning to gravity. Specifically, \hat{g} simplifies the inner product to a form exhibiting rotational invariance, as shown below:

$$(\hat{g}(L\varphi))^\dagger L\varphi = (L\varphi)^T L\varphi = \varphi^T L^T L\varphi = \varphi^T \varphi = g \implies L^T L = I, \quad (90)$$

where L is an $SO(2)$ transformation matrix. This equation illustrates that the wavefunction, if properly parametrized and when acted on by the metric operator, defines an $SO(2)$ -invariant inner product at each point on the manifold X^2 .

In examining the frame bundle FX of this manifold, we note that the action of the metric operator on the inner product reduces the structure group of FX from $GL^+(2, \mathbb{R})$ to $SO(2)$. This reduction induces a metric [8, 9, 10, 11, 12, 13, 14] on X^2 , as demonstrated when considering two vectors of the frame bundle $\mathbf{v} = x_v \hat{\mathbf{x}} + y_v \hat{\mathbf{y}}$ and $\mathbf{w} = x_w \hat{\mathbf{x}} + y_w \hat{\mathbf{y}}$. Applying the reduced inner product to those vector entails the following metric:

$$g = \begin{bmatrix} \langle \mathbf{v}, \mathbf{v} \rangle & \langle \mathbf{v}, \mathbf{w} \rangle \\ \langle \mathbf{w}, \mathbf{v} \rangle & \langle \mathbf{w}, \mathbf{w} \rangle \end{bmatrix} \quad (91)$$

The $SO(2)$ invariance guarantees the symmetry of this matrix, as $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

The resulting quotient, $FX/SO(2)$, comprises the global sections that constitute the Riemannian metrics on X^2 . This leads us to integrate these concepts with a gauge theory of gravity. To finalize the construction of the theory of gravity, we derive the connection by gauging the $SO(2)$ group as a promoted local symmetry, which is the invariant group of the inner product.

This approach enables us to express The Einstein-Hilbert Lagrangian using gauge-theoretical principles (for the connection) and by using the $FX/SO(2)$ bundle (for the metric):

$$\mathcal{L} = \sqrt{-g}R = \sqrt{-g}g^{\mu\nu} R_{\mu\nu}(\omega, e), \quad (92)$$

where $R_{\mu\nu}(\omega, e)$ is defined via the spin connection ω and tetrads e . This representation highlights the intrinsic link between the geometric properties of the metric operator and the foundational principles of gravity.

2.11 Linear Measurement Constraint in 3+1 Dimensions

In extending our analysis to a 3+1-dimensional spacetime framework, we adapt the measurement constraint to account for the complexities of this higher-dimensional space. The constraint for a 3+1D system is formalized as:

$$0 = \frac{1}{4} \text{tr } \overline{\mathbf{M}} - \sum_{q \in \mathbb{Q}} \rho(q) \frac{1}{4} \text{tr } \mathbf{M}(q), \quad (93)$$

where each $\mathbf{M}(q)$ is a traceless 4×4 matrix corresponding to state q .

In a 3+1-dimensional context, a 4×4 real matrix, \mathbf{M} , can be expressed using the real Majorana representation. Such a matrix has the general form:

$$\mathbf{M} = \begin{bmatrix} a + x - f_{02} + q & -z - f_{13} + w - b & f_{03} - f_{23} - p - v & t + y + f_{01} + f_{12} \\ -z - f_{13} + w + b & a - x - f_{02} - q & -t + y + f_{01} + f_{12} & f_{03} - f_{23} - p - v \\ f_{03} + f_{23} - p + v & t + y - f_{01} + f_{12} & a + x + f_{02} - q & -z - f_{13} - w + b \\ -t + y + f_{01} - f_{12} & -f_{03} - f_{23} - p + v & -z + f_{13} - w - b & a - x + f_{02} + q \end{bmatrix}, \quad (94)$$

In the case where tracelessness is required, it can be achieved by setting $a = 0$.

2.12 Introducing the "Double-Copy" Inner Product in 3+1 Dimensions

The matrix corresponding to a multivector in geometric algebra encompasses various grades, including scalar, vector, bivector, pseudo-vector, and pseudo-scalar components. It can be represented as follows:

$$\mathbf{M} \cong_a \quad (95)$$

$$+ t\hat{\mathbf{t}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \quad (96)$$

$$+ f_{01}\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}} \wedge \hat{\mathbf{z}} + f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \quad (97)$$

$$+ v\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + w\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + p\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + q\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \quad (98)$$

$$+ b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \quad (99)$$

In a 3+1-dimensional setting, a standard inner product is insufficient for defining a Hilbert space over the $\text{GL}^+(4, \mathbb{R})$ group. To address this, a more sophisticated "double-copy" inner product is proposed[15]:

$$\langle\langle \varphi | \varphi \rangle\rangle = [\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi. \quad (100)$$

The Clifford conjugate φ^\dagger alters the signs of the bivector and pseudo-vector components:

$$\varphi^\dagger = a + \mathbf{x} - \mathbf{f} - \mathbf{v} + \mathbf{b}. \quad (101)$$

Similarly, the blade 3-4 conjugate $[\psi]_{3,4}$ modifies the signs of the pseudo-vector and pseudo-scalar components:

$$[\varphi]_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b}. \quad (102)$$

When combined, these elements yield a scalar equivalent to the determinant of the associated 4×4 real matrix:

$$\det \varphi = [\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi. \quad (103)$$

This "double-copy" inner product methodology effectively captures the complex properties of higher-dimensional spacetime, pivotal for the coherent development of a geometric-quantum theory in 3+1D.

2.13 The Metric Operator \hat{g} in 3+1-Dimensions

Defined to operate on a multivector \mathbf{u} as $\hat{g}\mathbf{u} = I\mathbf{u}I^{-1}$, this operator, within a 3+1-dimensional context, transforms \mathbf{u} as follows:

$$\hat{g}\mathbf{u} = I(a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b})I^{-1} \quad (104)$$

$$= a + \mathbf{x} - \mathbf{f} - \mathbf{v} + \mathbf{b} \quad (105)$$

$$= \mathbf{u}^\dagger. \quad (106)$$

The operator effectively reverses the signs of the bivector and pseudo-vector components, yielding the Clifford conjugate of \mathbf{u} .

The influence of \hat{g} on the "double-copy" inner product is as follows:

$$[\varphi^\dagger \hat{g}\varphi]_{3,4}(\hat{g}\varphi)^\dagger\varphi = [(\varphi^\dagger)^2]_{3,4}\varphi^2 \quad (107)$$

$$= \tilde{\psi}^\dagger\psi \quad (108)$$

$$= g, \quad (109)$$

where ψ denotes φ^2 , $\tilde{\psi}$ represents the reverse of ψ , and ψ^\dagger is the blade-4 conjugate of ψ . Here, g represents the transition amplitude between $\tilde{\psi}^\dagger$ and ψ . This process reduces the inner product to a single-copy form, analogous to conventional quantum mechanical and geometrical inner products.

- The resultant inner product aligns with the probability density of the relativistic wavefunction in David Hestenes' geometric algebra formulation. In Hestenes' approach, the wavefunction is defined as $\psi = \sqrt{\rho}Re^{ib/2}$, where ρ is the probability density, R is a rotor, and $e^{ib/2}$ is a phase factor. In our formulation, the wavefunction is:

$$\psi = \exp\left(\frac{1}{2}(a + \mathbf{f} + \mathbf{b})\right), \quad (110)$$

and relates to David Hestenes' through $e^{a/2} = \sqrt{\rho}$, $e^{\mathbf{f}/2} = R$, and $\exp(\mathbf{b}/2) = e^{ib/2}$.

The interaction of the inner product in either representation can be expressed as:

$$\tilde{\psi}^\dagger\psi = \sqrt{\rho}\tilde{R}e^{-ib/2}\sqrt{\rho}Re^{ib/2} = \rho, \quad (111)$$

indicating that the reduced inner product fundamentally connects to the probability density in relativistic quantum theory.

- The inner product also produces the correct spacetime interval for vectors such as $\mathbf{v} = t\hat{\mathbf{t}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$:

$$\tilde{\mathbf{v}}^\dagger\mathbf{v} = (t\hat{\mathbf{t}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})(t\hat{\mathbf{t}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = t^2 - x^2 - y^2 - z^2 \quad (112)$$

Consequently, a singular inner product defines both the quantum theory and the geometry.

2.14 Theory of Gravity in 3+1D

Our 3+1D theory of gravity is embedded within the $\text{FX}/\text{Spin}^c(3,1)$ geometric structure. This approach diverges from the typical $\text{FX}/\text{SO}(3,1)$ framework found in ordinary gravity. Intriguingly, it is infeasible to construct a $\text{FX}/\text{SO}(3,1)$ or a $\text{FX}/\text{Spin}(3,1)$ theory of gravity in this context, as the metric operator leads to a reduction of the inner product not to the $\text{SO}(3,1)$ or $\text{Spin}(3,1)$ invariant groups, but rather to the $\text{Spin}^c(3,1)$ group. It is also worth noting that by a theorem of Hopf and Hirzebruch, closed orientable 4-manifolds always admit a spin^c structure.

The $\text{Spin}^c(3,1)$ gauge emerges as the invariance group essential for preserving the transition amplitude in relation to the metric operator. This is evidenced by:

$$\widetilde{(L\psi)}^\dagger L\psi = \widetilde{\psi}^\dagger \widetilde{L}^\dagger L\psi \implies \widetilde{L}^\dagger L = I, \quad (113)$$

where L represents a transformation matrix. The formulation $L = \exp(\mathbf{f} + \mathbf{b})$ aligns with $\text{Spin}^c(3,1)$ characteristics.

Acting on vectors $\mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{q}$ from the frames of the FX bundle, the reduced inner product establishes a metric:

$$g = \begin{bmatrix} \langle \mathbf{v}, \mathbf{v} \rangle & \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{w} \rangle & \langle \mathbf{v}, \mathbf{q} \rangle \\ \langle \mathbf{u}, \mathbf{v} \rangle & \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{w} \rangle & \langle \mathbf{u}, \mathbf{q} \rangle \\ \langle \mathbf{w}, \mathbf{v} \rangle & \langle \mathbf{w}, \mathbf{u} \rangle & \langle \mathbf{w}, \mathbf{w} \rangle & \langle \mathbf{w}, \mathbf{q} \rangle \\ \langle \mathbf{q}, \mathbf{v} \rangle & \langle \mathbf{q}, \mathbf{u} \rangle & \langle \mathbf{q}, \mathbf{w} \rangle & \langle \mathbf{q}, \mathbf{q} \rangle \end{bmatrix}. \quad (114)$$

The $\text{Spin}^c(3,1)$ invariance of the inner product ensures that this matrix is symmetric.

A $\text{Spin}^c(3,1)$ connection is characterized as:

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega_\mu^{ab} \Sigma_{ab} \psi + iq A_\mu \psi. \quad (115)$$

The gravitational theory requires calculating the curvature tensor:

$$R(u, v)\omega = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]})\omega, \quad (116)$$

where the connection is the aforementioned $\text{Spin}^c(3,1)$ connection.

The Lagrangian is then expressed as:

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\omega, e), \quad (117)$$

2.15 $\text{SU}(2) \times \text{U}(1)$

In exploring the particle physics implications of our framework, we focus on the conservation of the bilinear form $\bar{\psi} \gamma_0 \psi$ under Dirac dynamics. We introduce the

transformation $T = \exp(\mathbf{f} + \mathbf{b})$, following the work of Hestenes and Lasenby[16, 17]. Our goal is to investigate global gauges that preserve this invariance:

$$\tilde{\psi}\gamma_0\psi = \text{reverse}(\exp(\mathbf{f} + \mathbf{b})\psi)\gamma_0\exp(\mathbf{f} + \mathbf{b})\psi \quad (118)$$

$$= \tilde{\psi}\exp(-\mathbf{f} + \mathbf{b})\gamma_0\exp(\mathbf{f} + \mathbf{b})\psi. \quad (119)$$

Defining the following:

$$\mathbf{E} = f_{01}\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}} \wedge \hat{\mathbf{z}}, \quad (120)$$

$$\mathbf{B} = f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}}. \quad (121)$$

Exploring the exponential term further:

$$\exp(-\mathbf{E} - \mathbf{B} + b)\gamma_0\exp(\mathbf{E} + \mathbf{B} + b) = \gamma_0\exp(\mathbf{E} - \mathbf{B} - b)\exp(\mathbf{E} + \mathbf{B} + b) \quad (122)$$

$$= \gamma_0\exp(2\mathbf{E}). \quad (123)$$

The relation remains invariant when $\mathbf{E} = 0$. In this case, the exponential simplifies to:

$$\exp(f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}} + b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}}), \quad (124)$$

which can be reformulated as:

$$\exp(i(f_{12}\sigma_z + f_{13}\sigma_y + f_{23}\sigma_x + b)). \quad (125)$$

Consequently, the invariant transformation of the reduced inner product $(\tilde{\psi}^\dagger\psi)$, resonating with the $\text{SU}(2)\times\text{U}(1)$ gauge symmetry in Dirac dynamics.

2.16 SU(3)

After exploring the $\text{SU}(2)\times\text{U}(1)$ gauge, we turn our attention to the $\text{SU}(3)$ gauge. This gauge must be introduced as an evolution operator that leaves the Dirac dynamics invariant.

Consider transforming the wavefunction by a bivector, $\mathbf{f}\psi$. The invariance condition is given by:

$$(\widetilde{\mathbf{f}\psi})\gamma_0\mathbf{f}\psi = \tilde{\psi}\gamma_0\psi. \quad (126)$$

We need to solve the relation: $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$.

Let's define:

$$\mathbf{E} = f_{01}\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}} \wedge \hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}} \wedge \hat{\mathbf{z}}, \quad (127)$$

$$\mathbf{B} = f_{12}\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}} \wedge \hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}} \wedge \hat{\mathbf{z}}, \quad (128)$$

with $\mathbf{f} = \mathbf{E} + \mathbf{B}$. The relation then evolves as follows:

$$-(\mathbf{E} + \mathbf{B})\gamma_0(\mathbf{E} + \mathbf{B}) = \gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \quad (129)$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{B}^2) \quad (130)$$

$$= (f_{01}^2 + f_{02}^2 + f_{03}^2 + f_{12}^2 + f_{13}^2 + f_{23}^2)\gamma_0 \quad (131)$$

$$+ 2(-f_{02}f_{12} + f_{03}f_{13})\gamma_1 \quad (132)$$

$$+ 2(-f_{01}f_{12} + f_{03}f_{23})\gamma_2 \quad (133)$$

$$+ 2(-f_{01}f_{13} + f_{02}f_{23})\gamma_3 \quad (134)$$

The invariance is maintained if $f_{01}^2 + f_{02}^2 + f_{03}^2 + f_{12}^2 + f_{13}^2 + f_{23}^2 = 1$ and if the cross products vanish. The bilinear form's preservation under this transformation is the SU(3) invariance. Incorporating \mathbf{f} in the transformation, while not directly associated with probability preservation like the exponential term of the SU(2)xU(1), is essential to fully encapsulate the SU(3) symmetry characteristic of the Standard Model.

2.17 Standard Model Symmetries

The combination of our previous results leads to the formulation of the Standard Model's gauge group, SU(3)xSU(2)xU(1). This is achieved through the transformation $T = \mathbf{f}_1 \exp(\mathbf{f}_2 + \mathbf{b}_2)$, which maintains the invariance of Dirac dynamics as expressed by the relation $\widehat{T}\psi\gamma_0 T\psi = \tilde{\psi}\gamma_0\psi$. In this formulation, the exponential component is associated with the phase invariance of the probability measure, while the bivector part represents an invariant transformation. This unified approach encapsulates the essential symmetries of the Standard Model within a single transformation, aligning with the fundamental principles of gauge theory and quantum mechanics.

2.18 Dimensional Disruptions

Our methodology effectively supports quantum-geometric theory up to 4D spacetime. However, significant mathematical challenges arise when attempting to extend this model to higher dimensions. These challenges act as a natural mathematical barrier, indicating an inherent dimensional limit within quantum-geometry and intriguingly suggesting a rationale for why spacetime is 4-dimensional.

To illustrate, norms of multivector self-products have been established up to 5 dimensions[18]:

$$\text{CL}(1, 0) : \quad \varphi^\dagger \varphi \quad (135)$$

$$\text{CL}(2, 0) : \quad \varphi^\dagger \varphi \quad (136)$$

$$\text{CL}(3, 0) : \quad [\varphi^\dagger \varphi]_3 \varphi^\dagger \varphi \quad (137)$$

$$\text{CL}(3, 1) : \quad [\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi \quad (138)$$

$$\text{CL}(4, 1) : \quad ([\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi)^\dagger ([\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi) \quad (139)$$

Yet, in six dimensions and above, this pattern collapses. The research by Acus et al.[19] in 6D geometric algebra demonstrates that the determinant, usually defined through a norm via self-products, fails to extend into 6D. They could not define a norm in such terms. The blunt of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))). \quad (140)$$

This equation is not a self-product but a sum of two self-products, complicating the definition of self-adjoint operators.

Below 4D, self-adjointness is satisfiable through self-products, as evidenced below:

- In ordinary quantum mechanics, the relationship for self-adjoint operators is:

$$\langle \mathbf{O} \varphi | \psi \rangle = \langle \varphi | \mathbf{O} \psi \rangle \implies \mathbf{O}^\dagger = \mathbf{O}. \quad (141)$$

- In 2D geometric-quantum mechanics, it takes the form:

$$\langle \langle \mathbf{O} \varphi | \psi \rangle \rangle = \langle \langle \varphi | \mathbf{O} \psi \rangle \rangle \implies \mathbf{O}^\dagger = \mathbf{O}. \quad (142)$$

- In 4D, with a "double-copy" inner product, it evolves to:

$$\langle \langle \mathbf{O} \varphi | \psi | \phi | \varphi \rangle \rangle = \langle \langle \varphi | \mathbf{O} \psi | \phi | \varphi \rangle \rangle = \langle \langle \varphi | \psi | \mathbf{O} \phi | \varphi \rangle \rangle = \langle \langle \varphi | \psi | \phi | \mathbf{O} \varphi \rangle \rangle \implies \tilde{\mathbf{O}}^\dagger = \mathbf{O} \quad (143)$$

However, in 6D, establishing a self-adjoint relationship for observables is not feasible:

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (144)$$

$$\neq b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))). \quad (145)$$

In 6D, self-adjointness becomes untenable as the necessary conditions cannot be met using real numbers, except in the trivial case where $\mathbf{O} = 1$. This limitation possibly makes higher dimensions fundamentally unobservable in the quantum-geometric sense.

Additionally, odd-dimensional geometric algebra spaces introduce further complexities, as the norm shifts into the realm of complex numbers, diverging from the expected real-valued determinants of associated matrices: $\det \mathbf{u} \in \mathbb{C}$, thus impacting the real-valued nature expected of a probability measure.

These disruptions fundamentally limit the applicability of our model to 2D and the physically relevant 4D spacetime, suggesting a natural limitation on the dimensionality of spacetime within our quantum-geometric theory.

3 Conclusion

In conclusion, this paper advances the 'Prescribed Measurement Problem' (PMP) into a geometric theory of quantum measurements, seamlessly bridging the realms of quantum mechanics and spacetime geometry. Our findings reveal the PMP's exceptional ability to generate a general linear quantum theory that is mathematically valid and well-behaved, effectively generalizing quantum probabilities. This unified probability measure aligns coherently with both quantum and geometric measurements, leading to the natural emergence of general relativity and the Standard Model as intrinsic components of the theory. This research represents a significant step in reconciling quantum mechanics with general relativity, challenging and expanding conventional methodologies in theoretical physics, and potentially paving the way for groundbreaking insights and understandings in the field.

Statements and Declarations

Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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