# The Entropy Under Geometric Constraints; A Framework for Quantum Gravity?

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**Abstract**

A quantum theory of the Einstein field equation is derived by maximizing the entropy under the broadest geometric constraint. Quantum field theory and non-relativistic quantum mechanics are also derived under more specific geometric constraints. The origin of the Born rule is revealed. The wave-function collapse problem is dissolved. The key idea is to connect probability theory to geometry using the trace, then to use it to constrain the entropy. Specifically, the trace can be interpreted as the expectation value of the eigenvalues of the matrix times the dimension of the vector space; and the eigenvalues as the ratios of the distortion of the geometric transformation associated with the matrix. It then suffices to use the Lagrange multipliers method to maximize the entropy under the constraint of the trace. Instead of the typical Gibbs ensemble of statistical mechanics, we find as our main result a generalized Born rule applied to a wave-function and admitting the Einstein field equations as its equation of motion.

## 1 Introduction

First, the trace: we will use the trace to introduce a new form of constraint into statistical mechanics: the *geometric constraint*. The trace admits a probability interpretation[1] as the expectation value of the eigenvalues times the dimension of the vector space. It also connects to geometry as the eigenvalues are the ratio of the distortion of the geometric transformation associated with the matrix.

The constraint will be defined as follows:

$$\operatorname{tr} \overline{\mathbf{M}} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q) \tag{1}$$

where $\mathbf{M}$ is an arbitrary $n \times n$ matrix, and where $\mathbb{Q}$ is a sample space.

How can we use this constraint in statistical mechanics?

In statistical mechanics, the Gibbs measure is derived using the method of the Lagrange multipliers[2] by maximizing the entropy under constraints.

For instance, an energy constraint on the entropy:

$$\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q) \tag{2}$$

is associated to an energy-meter measuring the system and producing a series of energy measurement $E_1, E_2, \dots$ converging to an expectation value $\overline{E}$.

Another common constraint is that of the volume:

$$\overline{V} = \sum_{q\in\mathbb{Q}} \rho(q)V(q) \tag{3}$$

associated to a volume-meter also acting on the system by producing a sequence of measurements of the volume $V_1, V_2, \dots$ converging to an expectation value $\overline{V}$.

And of course the sum over the sample space must equal 1:

$$1 = \sum_{q\in\mathbb{Q}} \rho(q) \tag{4}$$

With two of these constraints, the typical system of statistical mechanics is obtained by maximizing the entropy using its corresponding Lagrange equation, and the method of the Lagrange multipliers:

$$\mathcal{L} = -k_B \sum_{q\in\mathbb{Q}} \rho(q)\ln\rho(q) + \lambda\left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \beta\left(\overline{E} - \sum_{q\in\mathbb{Q}} \rho(q)E(q)\right) \tag{5}$$

where $\lambda$ and $\beta$ are Lagrange multipliers.

Then solving $\frac{\partial\mathcal{L}}{\partial\rho} = 0$ for $\rho$, we get the Gibbs measure:

$$\rho(q,\beta) = \frac{1}{Z(\beta)}\exp(-\beta E(q)) \tag{6}$$

where

$$Z(\beta) = \sum_{q\in\mathbb{Q}} \exp(-\beta E(q)) \tag{7}$$

In the case of a geometric constraints on the entropy, we remove the constraint $\overline{E}$ and instead inject the constraint $\operatorname{tr}\overline{\mathbf{M}}$. We also use the Shannon entropy instead of the Boltzmann entropy (more on that choice in the discussion section). The corresponding Lagrange equation is:

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q)\ln\rho(q) + \lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\mathbf{M}(q)\right) \quad (8)$$

As we found, and as we will now investigate, solving $\frac{\partial\mathcal{L}}{\partial\rho} = 0$ for $\rho$ produces a quantum theory of gravity, in which the Einstein field equations are its equations of motions. Ordinary quantum field theory and non-relativistic quantum mechanics are also recovered as special cases.

# 2 Methods

## 2.1 Notation

- Typography: Sets, unless a prior convention assigns it another symbol, will be written using the blackboard bold typography (ex: $\mathbb{L}, \mathbb{W}, \mathbb{Q}$, etc.). Matrices will be in bold upper case (ex: $\mathbf{P}, \mathbf{M}$), whereas tuples, vectors and multi-vectors will be in bold lower case (ex: $\mathbf{u}, \mathbf{v}, \mathbf{g}$) and most other constructions (ex.: scalars, functions) will have plain typography (ex. $a, A$). The unit pseudo-scalar (of geometric algebra) will be $\mathbf{i}$. The imaginary number will be $i$. The identity matrix will be $\mathbf{I}$.

- Sets: The projection of a tuple $\mathbf{p}$ will be $\operatorname{proj}_i(\mathbf{p})$. As an example, let us denote the elements of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$ as $\mathbf{p} = (x, y)$. The projection operators are $\operatorname{proj}_1(\mathbf{p}) = x$ and $\operatorname{proj}_2(\mathbf{p}) = y$. If projected over a set, the results are $\operatorname{proj}_1(\mathbb{R}^2) = \mathbb{R}_1$ and $\operatorname{proj}_2(\mathbb{R}^2) = \mathbb{R}_2$. The size of a set $\mathbb{X}$ is $|\mathbb{X}|$.

  The symbol $\cong$ indicates a group isomorphism relation between two sets. The symbol $\simeq$ indicates equality if defined, or both undefined otherwise.

- Analysis: The asterisk $z^*$ denotes the complex conjugate of $z$.

- Matrix: The Dirac gamma matrices are $\gamma_0, \gamma_1, \gamma_2, \gamma_3$. The Pauli matrices are $\sigma_x, \sigma_y, \sigma_z$. The dagger $\mathbf{M}^\dagger$ denotes the conjugate transpose of $\mathbf{M}$. The commutator is defined as $[\mathbf{M}, \mathbf{P}] : \mathbf{MP} - \mathbf{PM}$ and the anti-commutator as $\{\mathbf{M}, \mathbf{P}\} : \mathbf{MP} + \mathbf{PM}$.

- Geometric Algebra: The basis elements of an arbitrary curvilinear geometric basis will be denoted $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ (such that $\mathbf{e}_\nu \cdot \mathbf{e}_\mu = g_{\mu\nu}$) and if they are orthonormal as $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n$ (such that $\hat{\mathbf{x}}_\mu \cdot \hat{\mathbf{x}}_\nu = \eta_{\mu\nu}$). A geometric algebra of $m$ dimensions over a field $\mathbb{F}$ is noted as $\mathbb{G}(m, \mathbb{F})$. The grades of a multi-vector will be denoted as $\langle\mathbf{v}\rangle_k$. Specifically, $\langle\mathbf{v}\rangle_0$ is a scalar, $\langle\mathbf{v}\rangle_1$ is a vector, $\langle\mathbf{v}\rangle_2$ is a bi-vector, $\langle\mathbf{v}\rangle_{n-1}$ is a pseudo-vector and $\langle\mathbf{v}\rangle_n$ is a pseudo-scalar. A scalar and a vector $\langle\mathbf{v}\rangle_0 + \langle\mathbf{v}\rangle_1$ is a para-vector, and a combination of even grades ($\langle\mathbf{v}\rangle_0 + \langle\mathbf{v}\rangle_2 + \langle\mathbf{v}\rangle_4 + \ldots$) or odd grades ($\langle\mathbf{v}\rangle_1 + \langle\mathbf{v}\rangle_3 + \ldots$) are even-multi-vectors or odd-multi-vectors, respectively.

Let $\mathbb{G}(2,\mathbb{R})$ be the two-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(2,\mathbb{R})$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, where $a$ is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ is a pseudo-scalar.

Let $\mathbb{G}(4,\mathbb{R})$ be the four-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(4,\mathbb{R})$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where $a$ is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

## 2.2 Geometric Constraints

**Definition 1** (Geometric Constraints)**.** *Let* $\mathbf{M}$ *be a* $n \times n$ *matrix and let* $\mathbb{Q}$ *be a statistical ensemble. Then, a geometric constraint is:*

$$\operatorname{tr} \overline{\mathbf{M}} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q) \tag{9}$$

## 2.3 Unitary Gauge (Recap)

Quantum electrodynamics is obtained by gauging the wave-function with $U(1)$. The $U(1)$ invariance results from the usage of the complex norm in ordinary quantum theory. A parametrization of $\psi$ over a differentiable manifold is required to support this derivation. Localizing the invariance group $\theta \to \theta(x)$ over said parametrization, yields the corresponding covariant derivative:

$$D_\mu = \partial_\mu + iqA_\mu(x) \tag{10}$$

where $A_\mu(x)$ is the gauge field.
If one then applies a gauge transformation to $\psi$ and $A_\mu$:

$$\psi \to e^{-iq\theta(x)}\psi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu \theta(x) \tag{11}$$

The covariant derivative is:

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi + iqA_\mu \psi && (12) \\ &\to \partial_\mu (e^{-iq\theta(x)}\psi) + iq(A_\mu + \partial_\mu \theta(x))(e^{-iq\theta(x)}\psi) && (13) \\ &= e^{-iq\theta(x)} D_\mu \psi && (14) \end{aligned}$$

Finally, the field is given as follows:

$$F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \tag{15}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to the potential one-form $A_\mu = A_\mu^\alpha T_\alpha$, and where $T_\alpha$ are the generators of the lie algebra of $U(1)$.

## 2.4 Geometric Representation of Matrices

The notation will significantly improved if we use a geometric representation of matrices, which we introduce now.

### 2.4.1 Geometric Representation of 2x2 real matrices

Let $\mathbb{G}(2, \mathbb{R})$ be the two-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(2, \mathbb{R})$ as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b} \tag{16}$$

where $a$ is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ is a pseudo-scalar.

Each multi-vector has a structure-preserving (addition/multiplication) matrix representation:

**Definition 2** (Geometric representation 2D)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{pmatrix} a+x & -b+y \\ b+y & a-x \end{pmatrix} \tag{17}$$

And the converse is also true; each $2 \times 2$ real matrix is represented as a multi-vector of $\mathbb{G}(2, \mathbb{R})$.

We can define the determinant solely using constructs of geometric algebra[3]. The determinant of $\mathbf{u}$ is:

**Definition 3** (Geometric Representation of the Determinant 2D)**.**

$$\begin{aligned} \det \quad : \quad \mathbb{G}(2, \mathbb{R}) &\longrightarrow \mathbb{R} \\ \mathbf{u} &\longmapsto \mathbf{u}^{\ddagger}\mathbf{u} \end{aligned} \tag{18}$$

where $\mathbf{u}^{\ddagger}$ is:

**Definition 4** (Clifford conjugate 2D)**.**

$$\mathbf{u}^{\ddagger} := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 \tag{19}$$

For example:

$$\det \mathbf{u} = (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) \tag{20}$$

$$= a^2 - x^2 - y^2 + b^2 \tag{21}$$

$$= \det \begin{pmatrix} a+x & -b+y \\ b+y & a-x \end{pmatrix} \tag{22}$$

Finally, we define the Clifford transpose:

**Definition 5** (Clifford transpose 2D)**.** *The Clifford transpose is the geometric analogue to the conjugate transpose. Like the conjugate transpose can be interpreted as a transpose followed by an element-by-element application of the complex conjugate, here the Clifford transpose is a transpose followed by an element-by-element application of the Clifford conjugate:*

$$\begin{pmatrix} \mathbf{u}_{00} & \dots & \mathbf{u}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{mn} \end{pmatrix}^{\ddagger} = \begin{pmatrix} \mathbf{u}_{00}^{\ddagger} & \dots & \mathbf{u}_{m0}^{\ddagger} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{nm}^{\ddagger} \end{pmatrix} \tag{23}$$

*If applied to a vector, then:*

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}^{\ddagger} = \begin{pmatrix} \mathbf{v}_1^{\ddagger} & \dots \mathbf{v}_m^{\ddagger} \end{pmatrix} \tag{24}$$

#### 2.4.2 Geometric Representation of 4x4 real matrices

Let $\mathbb{G}(4,\mathbb{R})$ be the two-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(4,\mathbb{R})$ as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{25}$$

where $a$ is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

Each multi-vector has a structure-preserving (addition/multiplication) matrix representation. Explicitly, the multi-vectors of $\mathbb{G}(4,\mathbb{R})$ are represented as follows:

**Definition 6** (Geometric representation 4D)**.**

$$\begin{aligned} & a + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\ & \quad + f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 + f_{13}\gamma_1 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 \\ & \quad + v_t\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + v_x\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v_y\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + v_z\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\ & \quad + b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \end{aligned}$$

$$\cong \begin{pmatrix} a + x_0 - if_{12} - iv_3 & f_{13} - if_{23} + v_2 - iv_1 & -ib + x_3 + f_{03} - iv_0 & x_1 - ix_2 + f_{01} - if_{02} \\ -f_{13} - if_{23} - v_2 - iv_1 & a + x_0 + if_{12} + iv_3 & x_1 + ix_2 + f_{01} + if_{02} & -ib - x_3 - f_{03} - iv_0 \\ -ib - x_3 + f_{03} + iv_0 & -x_1 + ix_2 + f_{01} - if_{02} & a - x_0 - if_{12} + iv_3 & f_{13} - if_{23} - v_2 + iv_1 \\ -x_1 - ix_2 + f_{01} + if_{02} & -ib + x_3 - f_{03} + iv_0 & -f_{13} - if_{23} + v_2 + iv_1 & a - x_0 + if_{12} - iv_3 \end{pmatrix} \tag{26}$$

And the converse is also true; each $4 \times 4$ real matrix is represented as a multi-vector of $\mathbb{G}(4,\mathbb{R})$.

In 4D as well we can define the determinant solely using constructs of geometric algebra[3]. The determinant of $\mathbf{u}$ is:

**Definition 7** (Geometric Representation of the Determinant 4D)**.**

$$\det \quad : \qquad \mathbb{G}(4,\mathbb{R}) \longrightarrow \mathbb{R} \tag{27}$$

$$\mathbf{u} \longmapsto \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} \tag{28}$$

where $\mathbf{u}^\ddagger$ is:

**Definition 8** (Clifford conjugate 4D)**.**

$$\mathbf{u}^\ddagger := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 + \langle \mathbf{u} \rangle_3 + \langle \mathbf{u} \rangle_4 \tag{29}$$

and where $\lfloor \mathbf{m} \rfloor_{\{3,4\}}$ is the blade-conjugate of degree 3 and 4 (flipping the plus sign to a minus sign for blade 3 and blade 4):

$$\lfloor \mathbf{u} \rfloor_{\{3,4\}} := \langle \mathbf{u} \rangle_0 + \langle \mathbf{u} \rangle_1 + \langle \mathbf{u} \rangle_2 - \langle \mathbf{u} \rangle_3 - \langle \mathbf{u} \rangle_4 \tag{30}$$

# 3 Result

## 3.1 Non-Relativistic Quantum Mechanics

We will now recover non-relativistic quantum mechanics also using the method of the Lagrange multipliers.

Instead of the Boltzmann entropy we will use the Shannon entropy:

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \tag{31}$$

What constraint will we use on this entropy?

In statistical mechanics we use "scalar" constraints on the entropy such as the energy-meter and the volume-meter. Such are sufficient to recover the Gibbs ensemble, but are insufficient to recover quantum mechanics. Let us introduce the "phase-invariant" constraint, which for a complex-phase, is defined as follows:

$$\operatorname{tr} \begin{bmatrix} 0 & -\overline{b} \\ \overline{b} & 0 \end{bmatrix} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{32}$$

where $\begin{bmatrix} a(q) & -b(q) \\ b(q) & a(q) \end{bmatrix} \cong a(q) + ib(q)$ is the matrix representation of the complex numbers. Like the energy-meter or the volume-meter, a phase-invariant instruments also produces a sequence of measurements converging to an expectation value, but such measurements have a phase-invariance. The trace here grants and enforces said phase-invariance.

The Lagrangian equation that maximizes the entropy subject to this constraint is:

$$
\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q)\ln(q) + \alpha\left(1 - \sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\operatorname{tr}\begin{bmatrix}0 & -\overline{b} \\ \overline{b} & 0\end{bmatrix} - \sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) \tag{33}
$$

Maximizing this equation for $\rho$ by posing $\frac{\partial\mathcal{L}}{\partial\rho(q)} = 0$, we obtain:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\rho(q)} &= -\ln\rho(q) - 1 - \alpha - \tau\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix} &(34)\\
0 &= \ln\rho(q) + 1 + \alpha + \tau\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix} &(35)\\
\implies \ln\rho(q) &= -1 - \alpha - \tau\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix} &(36)\\
\implies \rho(q) &= \exp(-1-\alpha)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) &(37)\\
&= \frac{1}{Z(\tau)}\det\exp\left(-\tau\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) &(38)
\end{aligned}
$$

where $Z(\tau)$ is obtained as follows:

$$
\begin{aligned}
1 &= \sum_{q\in\mathbb{Q}}\exp(-1-\alpha)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) &(39)\\
\implies (\exp(-1-\alpha))^{-1} &= \sum_{q\in\mathbb{Q}}\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) &(40)\\
Z(\tau) &\coloneqq \sum_{q\in\mathbb{Q}}\det\exp\left(-\tau\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) &(41)
\end{aligned}
$$

We note that the trace in the exponential drops down to a determinant, via the relation $\det\exp A \equiv \exp\operatorname{tr} A$.

Finally, we obtain:

$$
\begin{aligned}
\rho(\tau, q) &= \frac{1}{Z(\tau)}\det\exp\left(-\tau\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}\right) &(42)\\
&\cong |\exp -i\tau b(q)|^2 \qquad \text{Born rule} &(43)
\end{aligned}
$$

Renaming $\tau \to t/\hbar$ and $b(q) \to H(q)$ recovers the familiar form:

$$
\rho(q) = \frac{1}{Z}\left|\exp(-itH(q)/\hbar)\right|^2 \tag{44}
$$

or even more familiar:

$$\rho(q) = \frac{1}{Z} \left|\psi(q)\right|^2, \text{ where } \psi(q) = \exp(-itH(q)/\hbar) \tag{45}$$

This gives us a powerful method to recover quantum mechanics from first principle only by appealing to the instruments we have access to; in this case phase-invariant instruments. We will discuss an interpretation in the discussion.

### 3.2 Quantum Theory of Gravity

We will now investigate the most general geometric constraint:

$$\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \operatorname{tr}\mathbf{M}(q) \tag{46}$$

where $\mathbf{M}$ is an arbitrary $n \times n$ matrix.

The Lagrange equation used to maximize the entropy subject to this constraint is:

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q)\ln(q) + \alpha\left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau\left(\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}} \rho(q)\operatorname{tr}\mathbf{M}(q)\right) \tag{47}$$

where $\alpha$ and $\tau$ are the Lagrange multipliers.

Maximizing this equation for $\rho$ by posing $\frac{\partial\mathcal{L}}{\partial\rho(q)} = 0$, we obtain:

$$\frac{\partial\mathcal{L}}{\partial\rho(q)} = -\ln\rho(q) - 1 - \alpha - \tau\operatorname{tr}\mathbf{M}(q) \tag{48}$$

$$0 = \ln\rho(q) + 1 + \alpha + \tau\operatorname{tr}\mathbf{M}(q) \tag{49}$$

$$\implies \ln\rho(q) = -1 - \alpha - \tau\operatorname{tr}\mathbf{M}(q) \tag{50}$$

$$\implies \rho(q) = \exp(-1-\alpha)\exp(-\tau\operatorname{tr}\mathbf{M}(q)) \tag{51}$$

$$= \frac{1}{Z(\tau)} \det\exp(-\tau\mathbf{M}(q)) \tag{52}$$

where $Z(\tau)$ is obtained as follows:

$$1 = \sum_{q\in\mathbb{Q}} \exp(-1-\alpha)\exp(-\tau\operatorname{tr}\mathbf{M}(q)) \tag{53}$$

$$\implies (\exp(-1-\alpha))^{-1} = \sum_{q\in\mathbb{Q}} \exp(-\tau\operatorname{tr}\mathbf{M}(q)) \tag{54}$$

$$Z(\tau) \coloneqq \sum_{q\in\mathbb{Q}} \det\exp(-\tau\mathbf{M}(q)) \tag{55}$$

We note that the trace in the exponential drops down to a determinant, via the relation $\det \exp A \equiv \exp \operatorname{tr} A$.

The resulting probability measure is:

$$\rho(q,\tau) = \frac{1}{Z(\tau)} \det \exp(-\tau \mathbf{M}(q)) \tag{56}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} \det \exp(-\tau \mathbf{M}(q)) \tag{57}$$

Posing $\psi(q,\tau) = \exp(-\tau \mathbf{M}(q))$, we can write $\rho(q,\tau) = \det \psi(q,\tau)$, where the determinant acts as a "generalized Born rule", connecting in this case a general linear amplitude to a real number representing a probability.

As we will now see, the extra sophistication of the general linear amplitude along with the determinant as the "generalized Born rule" is sufficient to produce a quantum theory of gravity.

#### 3.2.1 General Linear Gauge

The fundamental invariance group of the general linear wave-function is the orientation-preserving general linear group $\mathrm{GL}^+(n,\mathbb{R})$. Like quantum electrodynamics (via the $U(1)$ gauge) is the archetypal example of QFT, here quantum gravity (via the $\mathrm{GL}^+(n,\mathbb{R})$ gauge) will be the archetypal example of our system.

Indeed, The exponential term $\exp(-\tau \mathbf{M}(\mathbf{p}))$ maps to a one-parameter subgroup of the orientation preserving general linear group:

$$\exp\colon \mathbf{M}_n(\mathbb{R}) \to \mathrm{GL}^+(n,\mathbb{R}) \tag{58}$$

and the Lagrange multiplier $\tau$ acquires the role of the evolution operator.

Gauging the $\mathrm{GL}(n,\mathbb{R})$ group is known to produce the Einstein field equations since the resulting $\mathrm{GL}(n,\mathbb{R})$-valued field can be viewed as the Christoffel symbols $\Gamma^\mu$, and the commutator of the covariant derivatives as the Riemann tensor. This is not a new result and dates back to 1956 by Utiyama[4], and to 1961 by Kibble[5].

The novelty here is that our wave-function is able to accommodate all transformations required by general relativity without violating probability conservation laws.

Due to our usage of the determinant, a general linear transformation:

$$\psi'(x) \to g\psi(x)g^{-1} \tag{59}$$

will leave the probability measure of the wave-function invariant, because

$$\det g\psi(x)g^{-1} = \det\psi(x) \tag{60}$$

The gauge-covariant derivative associated with this transformation is:

$$D_\mu\psi = \partial_\mu\psi - [iqA_\mu, \psi] \tag{61}$$

Finally, the field is given as follows:

$$R_{\mu\nu} = [D_\mu, D_\nu] \tag{62}$$

where $R_{\mu\nu}$ is the Riemann tensor.

The resulting Lagrangian is of course the Einstein-Hilbert action which, up to numerical constant, is:

$$S = \int \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d = \int \mathrm{d}^4\, x\sqrt{-g}R \tag{63}$$

Consequently, the equations of motion of our quantum field are the Einstein field equations.

#### 3.2.2 Dirac Spinors (A Special Case)

For this application, we will represent arbitrary $4 \times 4$ matrices with the general multi-vectors of $\mathbb{G}(4, \mathbb{R})$. The wave-function then is:

$$\psi = \exp(a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}) \tag{64}$$

where $a+\mathbf{x}+\mathbf{f}+\mathbf{v}+\mathbf{b}$ is the geometric algebra representation of an arbitrary $4 \times 4$ matrix.

We will now impose a group reduction from the general linear group to the spinor group. As such we pose $\mathbf{x} \to 0$ and $\mathbf{v} \to 0$.

The general linear wave-function reduces to:

$$\psi_{\mathbf{x}\to 0, \mathbf{v}\to 0} = \exp(a + \mathbf{f} + \mathbf{b}) \tag{65}$$

We recall that in 4D, the probability associated with our wave-function is given as follows:

$$\det\psi = \lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi = \exp 4a = \rho \tag{66}$$

but, since we eliminated $\mathbf{x} \to 0$ and $\mathbf{v} \to 0$, we can drop the blade inversion of degree 3, and the rule reduces to:

$$\det \psi = (\psi^{\ddagger})^{*}\psi^{*}\psi^{\ddagger}\psi = \exp 4a = \rho \tag{67}$$

Let us now recover the familiar Dirac theory.

First, we will expand the probability rule explicitly:

$$(\psi^{\ddagger})^{*}\psi^{*}\psi^{\ddagger}\psi = e^{a}e^{-\mathbf{b}}e^{-\mathbf{f}}e^{a}e^{-\mathbf{b}}e^{\mathbf{f}}e^{a}e^{\mathbf{b}}e^{-\mathbf{f}}e^{a}e^{\mathbf{b}}e^{\mathbf{f}} \tag{68}$$

Since the terms commute with each other, we can reorganize as follows:

$$= \underbrace{\left(e^{2a}e^{-2\mathbf{b}}e^{-2\mathbf{f}}\right)}_{\phi^{\ddagger *}}\underbrace{\left(e^{2a}e^{2\mathbf{b}}e^{2\mathbf{f}}\right)}_{\phi} \tag{69}$$

With the substitutions by $\phi^{\ddagger *}$ and $\phi$, we can then rewrite the probability density as $\phi^{\ddagger *}\phi = \rho$. Here, $\phi$ is the relativistic wave-function and $\rho$ is the Dirac current.

To see more clearly that this is indeed the case, we will adopt the geometric algebra notation of David Hestenes for the wave-function. The replacements are $e^{4a} := \rho, e^{4\mathbf{b}} := e^{ib}, e^{2\mathbf{f}} := R$, where $R$ is a rotor. We will also use $\widetilde{R}$ to designate the reverse of $R$, such that $\widetilde{R}R = I$. Thus:

$$\phi = e^{2a}e^{2\mathbf{b}}e^{2\mathbf{f}} = \rho^{1/2}e^{ib/2}R \tag{70}$$

$$:= (\rho e^{ib})^{\frac{1}{2}}R \tag{71}$$

$\phi$ is now identical to the David Hestenes' geometric algebra formulation of the relativistic wave-function[6].

We also define:

$$\bar{\phi} := (\rho e^{-ib})^{\frac{1}{2}}\widetilde{R}\gamma_0 \tag{72}$$

We can now obtain the full list of bilinear covariants:

Table 1: Bilinear covariants

| | Ours | Standard Form | Result |
|---|---|---|---|
| scalar | $\bar{\phi}\phi$ | $\langle\bar{\psi}\vert\,\vert\psi\rangle$ | $e_0\rho\cos b$ |
| vector | $\bar{\phi}\gamma_\mu\phi$ | $\langle\bar{\psi}\vert\,\gamma_\mu\vert\psi\rangle$ | $J_\mu$ |
| bivector | $\bar{\phi}I\gamma_\mu\gamma_\nu\phi$ | $\langle\bar{\psi}\vert\,i\gamma_\mu\gamma_\nu\vert\psi\rangle$ | $S$ |
| pseudo-vector | $\bar{\phi}\gamma_\mu I\phi$ | $\langle\bar{\psi}\vert\,\gamma_\mu\gamma_5\vert\psi\rangle$ | $s_\mu$ |
| pseudo-scalar | $\bar{\phi}I\phi$ | $\langle\bar{\psi}\vert\,i\gamma_5\vert\psi\rangle$ | $-e_0\rho\sin b$ |

Our results here are the same as those of David Hestenes'[6].

The wave-function can be parametrized over $\mathbb{R}^{3,1}$. Then it assigns an element of the spinor group to each event, admits the $U(1)$ local gauge symmetry, and thus constitutes the building block of a quantum field theory.

# 4 Foundation of Physics

We are now ready to begin investigating the main result as a general linear quantum theory, in full rigour. To this end, we will now introduce the *algebra of geometric observables* applicable to the general linear wave-function.

## 4.1 Axiomatic Definition of the Algebra, in 2D

Let $\mathbb{V}$ be an $m$-dimensional vector space over $\mathbb{G}(2,\mathbb{R})$. A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ iff the following holds:

1. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the bilinear map:

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \qquad \mathbb{V} \times \mathbb{V} &\longrightarrow \mathbb{G}(2,\mathbb{R}) \\ \langle \mathbf{u}, \mathbf{v} \rangle &\longmapsto \mathbf{u}^{\ddagger}\mathbf{v} \end{aligned} \tag{73}$$

   is positive-definite:

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \in \mathbb{R}_{>0} \tag{74}$$

2. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\psi}$, the function:

$$\rho(\psi(q), \boldsymbol{\psi}) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \psi(q)^{\ddagger}\psi(q) \tag{75}$$

   is positive-definite:

$$\rho(\psi(q), \boldsymbol{\psi}) \in \mathbb{R}_{>0} \tag{76}$$

We note the following comments and definitions:

- From (1) and (2) it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum to unity:

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{77}$$

- $\boldsymbol{\psi}$ is called a *natural* (or physical) state.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- $\rho(q, \boldsymbol{\psi})$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$, as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$, which leaves the sum of probabilities normalized (invariant):

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \mathbf{T}\boldsymbol{\psi}) = \sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{78}$$

  are the *natural* transformations of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{v} \in \mathcal{A}(\mathbb{V})$ :

$$\langle \mathbf{O}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{v} \rangle \tag{79}$$

  is called an observable.

- The expectation value of an observable $\mathbf{O}$ is:

$$\langle \mathbf{O} \rangle = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{80}$$

## 4.2 Observable in 2D — Self-Adjoint Operator

Let us now investigate the general case of an observable in 2D. A matrix $\mathbf{O}$ is an observable iff it is a self-adjoint operator; defined as:

$$\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle \tag{81}$$

$\forall \mathbf{u} \forall \mathbf{v} \in \mathbb{V}$.

**Setup:** Let $\mathbf{O} = \begin{pmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{pmatrix}$ be an observable. Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be 2 two-state vectors of multi-vectors $\boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{pmatrix}$ and $\boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{pmatrix}$. Here, the components $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, $\mathbf{o}_{11}$ are multi-vectors of $\mathbb{G}(2, \mathbb{R})$.

**Derivation:** 1. Let us now calculate $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle &= (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2) \\ &\quad + (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2) &(82) \\ &= \boldsymbol{\phi_1}^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\phi}_2 \\ &\quad + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\phi}_2 &(83) \end{aligned}$$

2. Now, $\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle &= \boldsymbol{\phi}_1^\ddagger (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2) + (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2) + (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 &(84) \\ &= \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\phi}_1 &(85) \end{aligned}$$

For $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$ to be realized, it follows that these relations must hold:

$$\mathbf{o}_{00}^{\ddagger} = \mathbf{o}_{00} \tag{86}$$

$$\mathbf{o}_{01}^{\ddagger} = \mathbf{o}_{10} \tag{87}$$

$$\mathbf{o}_{10}^{\ddagger} = \mathbf{o}_{01} \tag{88}$$

$$\mathbf{o}_{11}^{\ddagger} = \mathbf{o}_{11} \tag{89}$$

Therefore, it follows that it must be the case that $\mathbf{O}$ must be equal to its own Clifford transpose. Thus, $\mathbf{O}$ is an observable iff:

$$\mathbf{O}^{\ddagger} = \mathbf{O} \tag{90}$$

which is the equivalent of the self-adjoint operator $\mathbf{O}^{\dagger} = \mathbf{O}$ of complex Hilbert spaces.

## 4.3 Observable in 2D — Eigenvalues / Spectral Theorem

Let us show how the spectral theorem applies to $\mathbf{O}^{\ddagger} = \mathbf{O}$, such that its eigenvalues are real. Consider:

$$\mathbf{O} = \begin{pmatrix} a_{00} & a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12} \\ a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} & a_{11} \end{pmatrix} \tag{91}$$

It follows that $\mathbf{O}^{\ddagger} = \mathbf{O}$:

$$\mathbf{O}^{\ddagger} = \begin{pmatrix} a_{00} & a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12} \\ a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} & a_{11} \end{pmatrix} \tag{92}$$

This example is the most general $2 \times 2$ matrix $\mathbf{O}$ such that $\mathbf{O}^{\ddagger} = \mathbf{O}$.

The eigenvalues are obtained as follows:

$$0 = \det(\mathbf{O} - \lambda \mathbf{I}) = \det \begin{pmatrix} a_{00} - \lambda & a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12} \\ a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} & a_{11} - \lambda \end{pmatrix} \tag{93}$$

implies:

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12})(a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} + a_{11}) \tag{94}$$

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2) \tag{95}$$

finally:

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{96}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\} \tag{97}$$

We note that in the case where $a_{00} - a_{11} = 0$, the roots would be complex iff $a^2 - x^2 - y^2 + b^2 < 0$, but we already stated that the determinant of real matrices must be greater than zero because the exponential maps to the orientation-preserving general linear group— therefore it is the case that $a^2 - x^2 - y^2 + b^2 > 0$, as this expression is the determinant of the multi-vector. Consequently, $\mathbf{O}^\ddagger = \mathbf{O}$ — implies, for orientation-preserving transformations, that its roots are real-valued, and thus constitute a 'geometric' observable in the traditional sense of an observable whose eigenvalues are real-valued.

## 4.4 Left Action, in 2D

A left action on a wave-function : $\mathbf{T}\left|\psi\right\rangle$, connects to the bilinear form as follows: $\left\langle\psi\right| \mathbf{T}^\ddagger \mathbf{T} \left|\psi\right\rangle$. The invariance requirement on $\mathbf{T}$ is as follows:

$$\left\langle\psi\right| \mathbf{T}^\ddagger \mathbf{T} \left|\psi\right\rangle = \left\langle\psi|\psi\right\rangle \tag{98}$$

We are thus interested in the group of matrices such that:

$$\mathbf{T}^\ddagger \mathbf{T} = I \tag{99}$$

Let us consider a two-state system. A general transformation is:

$$\mathbf{T} = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \tag{100}$$

where $u, v, w, x$ are multi-vectors of 2 dimensions. The expression $\mathbf{G}^\ddagger \mathbf{G}$ is:

$$\mathbf{T}^\ddagger \mathbf{T} = \begin{pmatrix} v^\ddagger & u^\ddagger \\ w^\ddagger & x^\ddagger \end{pmatrix} \begin{pmatrix} v & w \\ u & x \end{pmatrix} = \begin{pmatrix} v^\ddagger v + u^\ddagger u & v^\ddagger w + u^\ddagger x \\ w^\ddagger v + x^\ddagger u & w^\ddagger w + x^\ddagger x \end{pmatrix} \tag{101}$$

For the results to be the identity, it must be the case that:

$$v^\ddagger v + u^\ddagger u = 1 \tag{102}$$

$$v^\ddagger w + u^\ddagger x = 0 \tag{103}$$

$$w^\ddagger v + x^\ddagger u = 0 \tag{104}$$

$$w^\ddagger w + x^\ddagger x = 1 \tag{105}$$

This is the case if

$$\mathbf{T} = \frac{1}{\sqrt{v^\ddagger v + u^\ddagger u}} \begin{pmatrix} v & u \\ -e^{\varphi} u^\ddagger & e^{\varphi} v^\ddagger \end{pmatrix} \tag{106}$$

where $u, v$ are multi-vectors of 2 dimensions, and where $e^{\varphi}$ is a unit multi-vector. Comparatively, the unitary case is obtained when the vector part of the multi-vector vanishes $\mathbf{x} \to 0$, and is:

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{pmatrix} a & b \\ -e^{i\theta} b^\dagger & e^{i\theta} a^\dagger \end{pmatrix} \tag{107}$$

We can show that $\mathbf{G}^\ddagger \mathbf{G} = I$ as follows:

$$\begin{aligned}
\implies \mathbf{T}^\ddagger \mathbf{T} &= \frac{1}{v^\ddagger v + u^\ddagger u} \begin{pmatrix} v^\ddagger & -e^{-\varphi} u \\ u^\ddagger & e^{-\varphi} v \end{pmatrix} \begin{pmatrix} v & u \\ -e^{\varphi} u^\ddagger & e^{\varphi} v^\ddagger \end{pmatrix} && (108) \\
&= \frac{1}{v^\ddagger v + u^\ddagger u} \begin{pmatrix} v^\ddagger v + u^\ddagger u & v^\ddagger u - v^\ddagger u \\ u^\ddagger v - u^\ddagger v & u^\ddagger u + v^\ddagger v \end{pmatrix} && (109) \\
&= I && (110)
\end{aligned}$$

In the case where $\mathbf{T}$ and $|\psi\rangle$ are $n$-dimensional, we can find an expression for it starting from a diagonal matrix:

$$\mathbf{D} = \begin{pmatrix} e^{x_1 \hat{\mathbf{x}} + y_1 \hat{\mathbf{y}} + i b_1} & 0 \\ 0 & e^{x_2 \hat{\mathbf{x}} + y_2 \hat{\mathbf{y}} + i b_2} \end{pmatrix} \tag{111}$$

where $\mathbf{T} = P \mathbf{D} P^{-1}$. It follows quite easily that $D^\ddagger D = I$, because each diagonal entry produces unity: $e^{-x_1 \hat{\mathbf{x}} - y_1 \hat{\mathbf{y}} - i b_1} e^{x_1 \hat{\mathbf{x}} + y_1 \hat{\mathbf{y}} + i b_1} = 1$.

## 4.5 Adjoint Action, in 2D

The left action case can recover at most the special linear group. For the general linear group itself, we require the adjoint action. Since the elements of $|\psi\rangle$ are matrices, in the general case, the transformation is given by adjoint action:

$$\mathbf{T} \, |\psi\rangle \, \mathbf{T}^{-1} \tag{112}$$

The bilinear form is:

$$(\mathbf{T} \, |\psi\rangle \, \mathbf{T}^{-1})^\ddagger (\mathbf{T} \, |\psi\rangle \, \mathbf{T}^{-1}) = (\mathbf{T}^{-1})^\ddagger \, \langle\psi| \, \mathbf{T}^\ddagger \mathbf{T} \, |\psi\rangle \, \mathbf{T}^{-1} \tag{113}$$

and the invariance requirement on $\mathbf{T}$ is as follows:

$$(\mathbf{T}^{-1})^{\ddagger} \langle\psi| \mathbf{T}^{\ddagger}\mathbf{T} |\psi\rangle \mathbf{T}^{-1} = \langle\psi|\psi\rangle \tag{114}$$

With a diagonal matrix, this occurs for general linear transformations:

$$\mathbf{D} = \begin{pmatrix} e^{a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1} & 0 & 0 \\ 0 & e^{a_2+x_2\hat{\mathbf{x}}+y_2\hat{\mathbf{y}}+ib_2} & 0 \\ 0 & 0 & \ddots \end{pmatrix} \tag{115}$$

where $\mathbf{T} = P\mathbf{D}P^{-1}$.

Taking a single diagonal entry as an example, the reduction is:

$$e^{-a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}\psi_1^{\ddagger}e^{a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1}e^{a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}\psi_1 e^{-a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1} \tag{116}$$

$$= e^{-a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}\psi_1^{\ddagger}e^{2a_1}\psi_1 e^{-a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1} \tag{117}$$

We note that $\psi^{\ddagger}\psi$ is a scalar, therefore

$$= \psi_1^{\ddagger}\psi_1 e^{2a_1}e^{-a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}e^{-a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1} \tag{118}$$

$$= \psi_1^{\ddagger}\psi_1 e^{2a_1}e^{-a_1}e^{-a_1} = \psi_1^{\ddagger}\psi_1 \tag{119}$$

## 4.6 Algebra of Geometric Observables, in 4D

We will now consider the general case for a vector space over $4 \times 4$ matrices.

Let $\mathbb{V}$ be a $m$-dimensional vector space over the $4\times 4$ real matrices. A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ iff the following holds:

1. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the quadri-linear form:

$$\begin{aligned} \langle \cdot, \cdot, \cdot, \cdot \rangle \quad : \qquad & \mathbb{V}\times\mathbb{V}\times\mathbb{V}\times\mathbb{V} \longrightarrow \mathbb{G}(4,\mathbb{R}) \\ & \langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \rangle \longmapsto \lfloor \mathbf{u}^{\ddagger}\mathbf{v} \rfloor_{3,4} \mathbf{w}^{\ddagger}\mathbf{x} \end{aligned} \tag{120}$$

   is positive-definite:

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \in \mathbb{R}_{>0} \tag{121}$$

2. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\psi}$, the function:

$$\rho(\psi(q), \boldsymbol{\psi}) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \lfloor \psi(q)^{\ddagger}\psi(q) \rfloor_{3,4} \psi(q)^{\ddagger}\psi(q) \tag{122}$$

   is positive-definite:

$$\rho(\psi(q), \boldsymbol{\psi}) \in \mathbb{R}_{>0} \tag{123}$$

We note the following properties, features and comments:

- $\boldsymbol{\psi}$ is called a *natural* (or physical) state.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- $\rho(\psi(q), \boldsymbol{\psi})$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ such as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$ which leaves the sum of probabilities normalized (invariant):

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \mathbf{T}\boldsymbol{\psi}) = \sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{124}$$

  are the *natural* transformations of $\boldsymbol{\psi}$.
- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{v} \forall \mathbf{w} \forall \mathbf{x} \in \mathbb{V}$ :

$$\langle \mathbf{Ou}, \mathbf{v}, \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{u}, \mathbf{Ov}, \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{u}, \mathbf{v}, \mathbf{Ow}, \mathbf{x} \rangle = \langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{Ox} \rangle \tag{125}$$

  is called an observable.
- The expectation value of an observable $\mathbf{O}$ is:

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \tag{126}$$

### 4.7 A Step Towards Falsifiable Predictions

Let us now list a number falsifiable predictions.

The main idea is that a general linear wave-function would allow a larger class of interference patterns than what is possible merely with complex interference. We note the work of B. I. Lev[7] treating the interference pattern associated with the geometric algebra formulation of the wave-function.

As a secondary idea, it is also plausible that an Aharonov–Bohm effect experiment on gravity[8] could detect a general linear phase.

An interference pattern follows from a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the application of the determinant:

$$\det(\mathbf{u} + \mathbf{v}) = \det \mathbf{u} + \det \mathbf{v} + \text{extra-terms} \tag{127}$$

The sum $\det \mathbf{u} + \det \mathbf{v}$ are a sum of probability and the extra terms represents the interference term.

We use the extra-terms to define a bilinear form using the dot product notation, as follows:

$$\cdot \quad : \qquad \mathbb{G}(2n,\mathbb{R}) \times \mathbb{G}(2n,\mathbb{R}) \longrightarrow \mathbb{R} \tag{128}$$

$$\mathbf{u}\cdot\mathbf{v} \longmapsto \frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) \tag{129}$$

For example in 2D, we have:

$$\mathbf{u} = a_1 + x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + b_1\mathbf{e}_{12} \tag{130}$$

$$\mathbf{v} = a_2 + x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + b_2\mathbf{e}_{12} \tag{131}$$

$$\implies \mathbf{u}\cdot\mathbf{v} = a_1a_2 + b_1b_2 - x_1x_2 - y_1y_2 \tag{132}$$

Iff $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$ then $\mathbf{u}\cdot\mathbf{v}$ is always positive, and therefore qualifies as a positive-definite inner product, but no greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$, whichever is greater; thus also satisfying the conditions of an interference term.

- In 2D the dot product is equivalent to this form:

$$\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) = \frac{1}{2}\left((\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v}\right) \tag{133}$$

$$= \mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v} - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v} \tag{134}$$

$$= \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} \tag{135}$$

- In 4D it is substantially more verbose:

$$\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) \tag{136}$$

$$= \frac{1}{2}\left(\lfloor(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})\rfloor_{3,4}(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} - \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\right) \tag{137}$$

$$= \frac{1}{2}\left(\lfloor\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}(\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}) - \dots\right) \tag{138}$$

$$\begin{aligned}
= {} & \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
& + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
& + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} \\
& + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} - \dots
\end{aligned} \tag{139}$$

$$
\begin{aligned}
&= \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} + \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{v} \\
&\quad + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} + \lfloor \mathbf{v}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{v}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} \qquad (140)
\end{aligned}
$$

Simpler version of this interference pattern are possible when the general linear group is reduced.

Complex interference:

For instance, a reduction to the circle group, likewise reduces the interference pattern to complex interference:

$$
|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2| \cos(\phi_1 - \phi_2) \qquad (141)
$$

Deep spinor interference:

A reduction to the spinor group, reduces the interference pattern to a "deep spinor rotation".

Consider a two-state wave-function (we note that $[\mathbf{f}, \mathbf{b}] = 0$):

$$
\psi = \psi_1 + \psi_2 = e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2} \qquad (142)
$$

The geometric interference pattern for a full general linear transformation in 4D is given by the product:

$$
\lfloor \psi^\ddagger \psi \rfloor_{3,4} \psi^\ddagger \psi \qquad (143)
$$

Let us start with the sub-product:

$$
\begin{aligned}
\psi^\ddagger \psi &= (e^{a_1} e^{-\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{-\mathbf{f}_2} e^{\mathbf{b}_2})(e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2}) \qquad (144) \\
&= e^{a_1} e^{-\mathbf{f}_1} e^{\mathbf{b}_1} e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_1} e^{-\mathbf{f}_1} e^{\mathbf{b}_1} e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2} \\
&\quad + e^{a_2} e^{-\mathbf{f}_2} e^{\mathbf{b}_2} e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{-\mathbf{f}_2} e^{\mathbf{b}_2} e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2} \qquad (145) \\
&= e^{2a_1} e^{2\mathbf{b}_1} + e^{2a_2} e^{2\mathbf{b}_2} + e^{a_1 + a_2} e^{\mathbf{b}_1 + \mathbf{b}_2} (e^{-\mathbf{f}_1} e^{\mathbf{f}_2} + e^{-\mathbf{f}_2} e^{\mathbf{f}_1}) \qquad (146)
\end{aligned}
$$

The full product is:

$$
\begin{aligned}
\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi &= \big(e^{2a_1}e^{-2\mathbf{b}_1}+e^{2a_2}e^{-2\mathbf{b}_2}+e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\big)\\
&\quad\times\big(e^{2a_1}e^{2\mathbf{b}_1}+e^{2a_2}e^{2\mathbf{b}_2}+e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1}\big) && (147)\\
&= e^{2a_1}e^{-2\mathbf{b}_1}e^{2a_1}e^{2\mathbf{b}_1}+e^{2a_1}e^{-2\mathbf{b}_1}e^{2a_2}e^{2\mathbf{b}_2}+e^{2a_1}e^{-2\mathbf{b}_1}e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\\
&\quad+e^{2a_2}e^{-2\mathbf{b}_2}e^{2a_1}e^{2\mathbf{b}_1}+e^{2a_2}e^{-2\mathbf{b}_2}e^{2a_2}e^{2\mathbf{b}_2}+e^{2a_2}e^{-2\mathbf{b}_2}e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\\
&\quad+e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{2a_1}e^{2\mathbf{b}_1}\\
&\qquad+e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{2a_2}e^{2\mathbf{b}_2}\\
&\qquad+e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1}) && (148)\\
&= e^{4a_1}+e^{4a_2}+2e^{2a_1+2a_2}\cos(2b_1-2b_2) && (149)\\
&\quad+e^{a_1+a_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})( && (150)\\
&\quad\ e^{2a_1}(e^{-\mathbf{b}_1+\mathbf{b}_2}+e^{\mathbf{b}_1-\mathbf{b}_2}) && (151)\\
&\qquad+e^{2a_2}(e^{\mathbf{b}_1-\mathbf{b}_2}+e^{-\mathbf{b}_1+\mathbf{b}_2})) && (152)\\
&\quad+e^{2a_1+2a_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})^2 && (153)\\
&= \underbrace{e^{4a_1}+e^{4a_2}}_{\text{sum}}+\underbrace{2e^{2a_1+2a_2}\cos(2b_1-2b_2)}_{\text{complex interference}}\\
&\quad+\underbrace{2e^{a_1+a_2}(e^{2a_1}+e^{2a_2})(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})(\cos(B_1-B_2))+e^{2A_1+2A_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2}+e^{-\mathbf{f}_2}e^{\mathbf{f}_1})^2}_{\text{deep spinor interference}} && (154)
\end{aligned}
$$

Finally, we stress that the general linear interference pattern occurs in context of quantum gravity, as ordinary quantum field theory reduces to typical complex interference.

# 5 Discussion

The complete correspondence between an ordinary system of statistical mechanics and our method is as follows:

Table 2: Correspondence

| Concept | Statistical Mechanics | Geometric Constraints (Our Method) |
|---|---|---|
| Entropy | Boltzmann | Shannon |
| Measure | Gibbs | Born rule on wave-function |
| Constraint | Energy meter | Phase-invariant instrument |
| Micro-state | Energy values | Possible measurements |
| Macro-state | Equation of state | Evolution of the wave-function |
| Experience | Ergodic | Message of measurements |

Let us discuss the correspondence.

In statistical mechanics, it is common to interpret constraints as instruments acting on the system. For instance, one can think of the constraint of an expected energy or volume value as an energy-meter or volume-meter producing a sequence of measurements converging towards said expected value.

In this work, we have introduced geometric constraints into statistical mechanics. Maximizing the entropy under geometric constraints induces various phase-invariances into the resulting probability measure whose complexity depends on the geometry. Specifically, the constraint $\operatorname{tr}\begin{bmatrix}0 & -\overline{b} \\ \overline{b} & 0\end{bmatrix} = \sum_{q\in\mathbb{Q}} \operatorname{tr}\rho(q)\begin{bmatrix}0 & -b(q) \\ b(q) & 0\end{bmatrix}$ induces a complex phase-invariance into the probability measure $\rho(q) = |\exp(-i\tau b(q))|^2$ giving rise to the Born rule and the wave-function, and the constraint $\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \operatorname{tr}\rho(q)\mathbf{M}(q)$ induces a general linear phase-invariance in the probability measure $\rho(q) = \det\exp(-\tau\mathbf{M}(q))$ giving rise to a quantum theory of gravity. In each cases, we can interpret the constraint as an instrument acting on the system. In the case of the complex phase we associate the constraint to a incidence counter measuring a particle or a photon, and in the case of the general linear phase, and specifically its group reduction to the Lorentz group, we associate the constraint and its phase-invariance to a interval measurement between two events.

The probabilistic interpretation of the wave-function along with the Born rule is entailed from its origins in statistical mechanics. The wave-function is also entailed, hence it is not taken as axiomatic. Rather, it is the registration of a measurement by an instrument along with the geometric constraints on the entropy that are the forefront. Since the wave-function is derived from the entropy of already registered measurements, it is never updated to a collapsed state; thus dissolving the collapse problem at the interpretational level.

The consequence is a minimal interpretation of quantum mechanics: *In nature, there exists instruments that record sequences of measurements on systems, those measurements are unique up to a phase, and the wave-function along with the Born rule are the entropy-maximizing measure constrained by those measurements.* This interpretation is minimal, completely factual and entirely free of all unfalsifiable redundancies: no need for many-worlds, no need to attribute an ontological existence to the wave-function, no need to appeal to a collapse upon measurements, etc. The pieces automatically fall into place and are all entailed by the method.

When the geometric constraint is arbitrary (any square matrix), the procedure yield a quantum theory of gravity, a wave-function of the general linear group and a Born rule extended to the determinant. The wave-function, if parametrized in $\mathbb{R}^{3,1}$, then represents an instruction, or superposition thereof, to transform the frame bundle at each event in space-time. Finally, gauging this group produces the Einstein field equations as the equations of motion of the quantum field. We also state that under a simple reduction from the general linear group to the spinor group (and remapping $\psi$ to $\phi$ in equation 69), the theory reduces to ordinary quantum field theory.

In the correspondence, the usage of the Shannon entropy instead of the Boltzmann entropy changes the experience from ergodic to a message (in the sense of the theory of communication of Claude Shannon[9]) of measurements. The receipt of such a message is interpreted as the registration of a 'click'[10] on a screen. We also note that the screen is an instrument that is geometrically extended, and the path of the particle or photon is also geometric. With this in mind, quantum physics (up to quantum gravity) can thus, within our method, be interpreted as the probability measure resulting from maximizing the entropy of a message of geometrically constrained measurements.

# References


[1] Makoto Yamashita (https://mathoverflow.net/users/9942/makoto yamashita). Geometric interpretation of trace. MathOverflow. URL:https://mathoverflow.net/q/46447 (version: 2016-05-17).

[2] Frederick Reif. *Fundamentals of statistical and thermal physics*. Waveland Press, 2009.

[3] Douglas Lundholm and Lars Svensson. Clifford algebra, geometric algebra, and applications. *arXiv preprint arXiv:0907.5356*, 2009.

[4] Ryoyu Utiyama. Invariant theoretical interpretation of interaction. *Physical Review*, 101(5):1597, 1956.

[5] Tom WB Kibble. Lorentz invariance and the gravitational field. *Journal of mathematical physics*, 2(2):212–221, 1961.

[6] David Hestenes. Spacetime physics with geometric algebra. *American Journal of Physics*, 71(7):691–714, 2003.

[7] I Bohdan et al. Wave function as geometric entity. *Journal of Modern Physics*, 2012, 2012.

[8] Chris Overstreet, Peter Asenbaum, Joseph Curti, Minjeong Kim, and Mark A Kasevich. Observation of a gravitational aharonov-bohm effect. *Science*, 375(6577):226–229, 2022.

[9] Claude Elwood Shannon. A mathematical theory of communication. *Bell system technical journal*, 27(3):379–423, 1948.

[10] John A Wheeler. Information, physics, quantum: The search for links. *Complexity, entropy, and the physics of information*, 8, 1990.