The Entropy Under Geometric Constraint; A Plausible Framework for Quantum Gravity and Particle Physics?

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Abstract

A quantum theory of the Einstein field equation is derived by maximizing the entropy under geometric constraints. Then, it is shown that ordinary quantum field theory up to the electroweak force naturally emerges from it. The origin of the Born rule and of the wave-function are revealed. Finally, the wave-function collapse problem is dissolved and a paradox-free interpretation of quantum mechanics is provided. The key idea is to understand the trace as connecting geometry to probability theory. Indeed, the trace can be seen as the expectation value of the eigenvalues of the matrix times the dimension of the vector space; and the eigenvalues as the ratios of the distortion of the geometric transformation associated with the matrix. Usage of the trace as a geometric constraint within statistical mechanics extends its scope to include all geometric and all quantum systems. Instead of the Gibbs measure, statistical mechanics produces as the measure a generalized Born rule and wave-function admitting the Einstein field equations as its equation of motion from which elements of particle physics also follow.

1 Introduction

We introduce a new form of constraint into statistical mechanics which we call the *geometric constraint*. This constraint extends the scope of statistical mechanics to all geometric systems and all quantum systems.

Before we start, let us note that a link between entropy and gravity has been investigated by others. For instance, Jacobson[1] describes a relation between the Einstein field equations and an area law for entropy. We also note the results of Bekenstein[2] where the laws of black hole thermodynamics are introduced. Finally, we note the work of Erik Verlinde suggesting that gravity is an entropic force[3].

Our method is different in that it is also able to recover quantum systems in addition to geometric systems. The key is to use an interpretation of the trace which connects geometry and probability theory. What is this interpretation?

The trace admits a probability interpretation[4] as the expectation value of the eigenvalues times the dimension of the vector space. It also connects to geometry as the eigenvalues are the ratio of the distortion of the geometric transformation associated with the matrix. As such, it is able to connect geometry to probability theory.

The constraint will be defined as follows:

$$\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\mathbf{M}(q) \tag{1}$$

where $\mathbf{M}$ is an arbitrary $n \times n$ matrix, and where $\mathbb{Q}$ is a statistical ensemble. Here, $\operatorname{tr}\overline{\mathbf{M}}$ denotes the expectation value of the statistically weighted sum of matrices $\mathbf{M}(q)$ parametrized over the ensemble $\mathbb{Q}$.

Using this equation as a constraint in statistical mechanics is a claim that we can observe the distortions produced by any geometric transformations in nature, and that the permissible statistics preserve the expectation value of these distortions.

Generally, how are constraints used in statistical mechanics?

Let us do a quick recap.

In statistical mechanics, the Gibbs measure is derived using the method of the Lagrange multipliers[5] by maximizing the entropy under constraints.

For instance, an energy constraint on the entropy:

$$\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q) E(q) \tag{2}$$

is associated to an energy-meter measuring the system and producing a series of energy measurement $E_1, E_2, \ldots$ converging to an expectation value $\overline{E}$.

Another common constraint is that of the volume:

$$\overline{V} = \sum_{q\in\mathbb{Q}} \rho(q) V(q) \tag{3}$$

associated to a volume-meter also acting on the system by producing a sequence of measurements of the volume $V_1, V_2, \ldots$ converging to an expectation value $\overline{V}$.

And of course the sum over the statistical ensemble must equal 1:

$$1 = \sum_{q\in\mathbb{Q}} \rho(q) \tag{4}$$

With equations (2) and (4), the typical system of statistical mechanics is obtained by maximizing the entropy using its corresponding Lagrange equation, and the method of the Lagrange multipliers:

$$\mathcal{L} = -k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right) \tag{5}$$

where λ and β are Lagrange multipliers.
Then solving $\frac{\partial \mathcal{L}}{\partial \rho} = 0$ for ρ, we get the Gibbs measure:

$$\rho(q, \beta) = \frac{1}{Z(\beta)} \exp(-\beta E(q)) \tag{6}$$

where

$$Z(\beta) = \sum_{q \in \mathbb{Q}} \exp(-\beta E(q)) \tag{7}$$

In the case of a geometric constraints on the entropy, we use the constraint $\operatorname{tr} \overline{\mathbf{M}}$ instead of (2). Following this replacement, the interpretation of the measure will be slightly altered; the resulting probability measure will be interpreted as quantifying the information associated with the receipt of a message of measurements. As such, we replace the Boltzmann entropy with the Shannon entropy. This replacement does not change the form of the mathematical equations, only the final interpretation (discussion, section 5). The corresponding Lagrange equation is:

$$\mathcal{L} = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \tau \left(\operatorname{tr} \overline{\mathbf{M}} - \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q) \right) \tag{8}$$

It now suffices to solve $\frac{\partial \mathcal{L}}{\partial \rho} = 0$ for ρ to obtain the solution.
Let us now introduce our methods.

2 Methods

2.1 Notation

- Typography: Sets, unless a prior convention assigns it another symbol, will be written using the blackboard bold typography (ex: $\mathbb{L}, \mathbb{W}, \mathbb{Q}$, etc.). Matrices will be in bold upper case (ex: $\mathbf{P}, \mathbf{M}$), whereas tuples, vectors and multi-vectors will be in bold lower case (ex: $\mathbf{u}, \mathbf{v}, \mathbf{g}$) and most other constructions (ex.: scalars, functions) will have plain typography (ex. a, A). The unit pseudo-scalar (of geometric algebra) will be $\mathbf{i}$. The imaginary number will be i. The identity matrix will be $\mathbf{I}$.

- Sets: The projection of a tuple $\mathbf{p}$ will be $\mathrm{proj}_i(\mathbf{p})$. As an example, let us denote the elements of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$ as $\mathbf{p} = (x, y)$. The projection operators are $\mathrm{proj}_1(\mathbf{p}) = x$ and $\mathrm{proj}_2(\mathbf{p}) = y$. If projected over a set, the results are $\mathrm{proj}_1(\mathbb{R}^2) = \mathbb{R}_1$ and $\mathrm{proj}_2(\mathbb{R}^2) = \mathbb{R}_2$. The size of a set $\mathbb{X}$ is $|\mathbb{X}|$.

  The symbol $\cong$ indicates a group isomorphism relation between two sets. The symbol $\simeq$ indicates equality if defined, or both undefined otherwise.

- Analysis: The asterisk z^* denotes the complex conjugate of z.

- Matrix: The Dirac gamma matrices are $\gamma_0, \gamma_1, \gamma_2, \gamma_3$. The Pauli matrices are $\sigma_x, \sigma_y, \sigma_z$. The dagger $\mathbf{M}^\dagger$ denotes the conjugate transpose of $\mathbf{M}$. The commutator is defined as $[\mathbf{M}, \mathbf{P}] : \mathbf{MP} - \mathbf{PM}$ and the anti-commutator as $\{\mathbf{M}, \mathbf{P}\} : \mathbf{MP} + \mathbf{PM}$.

- Geometric Algebra: The basis elements of an arbitrary curvilinear geometric basis will be denoted $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ (such that $\mathbf{e}_\nu \cdot \mathbf{e}_\mu = g_{\mu\nu}$) and if they are orthonormal as $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \dots, \hat{\mathbf{x}}_n$ (such that $\hat{\mathbf{x}}_\mu \cdot \hat{\mathbf{x}}_\nu = \eta_{\mu\nu}$). A geometric algebra of m dimensions over a field $\mathbb{F}$ is noted as $\mathbb{G}(m, \mathbb{F})$. The grades of a multi-vector will be denoted as $\langle \mathbf{v} \rangle_k$. Specifically, $\langle \mathbf{v} \rangle_0$ is a scalar, $\langle \mathbf{v} \rangle_1$ is a vector, $\langle \mathbf{v} \rangle_2$ is a bi-vector, $\langle \mathbf{v} \rangle_{n-1}$ is a pseudo-vector and $\langle \mathbf{v} \rangle_n$ is a pseudo-scalar. A scalar and a vector $\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_1$ is a para-vector, and a combination of even grades ($\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_2 + \langle \mathbf{v} \rangle_4 + \dots$) or odd grades ($\langle \mathbf{v} \rangle_1 + \langle \mathbf{v} \rangle_3 + \dots$) are even-multi-vectors or odd-multi-vectors, respectively.

  Let $\mathbb{G}(2, \mathbb{R})$ be the two-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(2, \mathbb{R})$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ is a pseudo-scalar.

  Let $\mathbb{G}(4, \mathbb{R})$ be the four-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(4, \mathbb{R})$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

2.2 Geometric Constraints

Definition 1 (Geometric Constraints)**.** *Let $\mathbf{M}$ be a $n \times n$ matrix and let $\mathbb{Q}$ be a statistical ensemble. Then, this equality constraint:*

$$\operatorname{tr} \overline{\mathbf{M}} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q) \tag{9}$$

is called a geometric constraint.

2.3 Geometric Representation of Matrices

The notation will significantly improved if we use a geometric representation of matrices, which we introduce now.

2.3.1 Geometric Representation of 2x2 real matrices

Let $\mathbb{G}(2,\mathbb{R})$ be the two-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(2,\mathbb{R})$ as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b} \tag{10}$$

where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ is a pseudo-scalar.

Each multi-vector has a structure-preserving (addition/multiplication) matrix representation:

Definition 2 (Geometric representation 2D)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{11}$$

And the converse is also true; each 2×2 real matrix is represented as a multi-vector of $\mathbb{G}(2,\mathbb{R})$.

We can define the determinant solely using constructs of geometric algebra[6]. The determinant of $\mathbf{u}$ is:

Definition 3 (Geometric Representation of the Determinant 2D)**.**

$$\begin{aligned} \det \quad : \quad & \mathbb{G}(2,\mathbb{R}) \longrightarrow \mathbb{R} \\ & \mathbf{u} \longmapsto \mathbf{u}^\ddagger \mathbf{u} \end{aligned} \tag{12}$$

where $\mathbf{u}^\ddagger$ is:

Definition 4 (Clifford conjugate 2D)**.**

$$\mathbf{u}^\ddagger \coloneqq \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 \tag{13}$$

For example:

$$\det \mathbf{u} = (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) \tag{14}$$

$$= a^2 - x^2 - y^2 + b^2 \tag{15}$$

$$= \det \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{16}$$

Finally, we define the Clifford transpose:

Definition 5 (Clifford transpose 2D)**.** *The Clifford transpose is the geometric analogue to the conjugate transpose. Like the conjugate transpose can be interpreted as a transpose followed by an element-by-element application of the* *complex conjugate, here the Clifford transpose is a transpose followed by an element-by-element application of the Clifford conjugate:*

$$\begin{bmatrix} \mathbf{u}_{00} & \dots & \mathbf{u}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{mn} \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{u}_{00}^{\ddagger} & \dots & \mathbf{u}_{m0}^{\ddagger} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{nm}^{\ddagger} \end{bmatrix} \tag{17}$$

If applied to a vector, then:

$$\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{v}_1^{\ddagger} & \dots \mathbf{v}_m^{\ddagger} \end{bmatrix} \tag{18}$$

2.3.2 Geometric Representation of 4x4 real matrices

Let $\mathbb{G}(4,\mathbb{R})$ be the two-dimensional geometric algebra over the reals. We can write a general multi-vector of $\mathbb{G}(4,\mathbb{R})$ as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{19}$$

where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

Each multi-vector has a structure-preserving (addition/multiplication) matrix representation. Explicitly, the multi-vectors of $\mathbb{G}(4,\mathbb{R})$ are represented as follows:

Definition 6 (Geometric representation 4D)**.**

$$\begin{aligned} & a + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\ & \quad + f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 + f_{13}\gamma_1 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 \\ & \quad + v_t\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + v_x\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v_y\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + v_z\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\ & \quad + b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \end{aligned}$$

$$\cong \begin{bmatrix} a + x_0 - if_{12} - iv_3 & f_{13} - if_{23} + v_2 - iv_1 & -ib + x_3 + f_{03} - iv_0 & x_1 - ix_2 + f_{01} - if_{02} \\ -f_{13} - if_{23} - v_2 - iv_1 & a + x_0 + if_{12} + iv_3 & x_1 + ix_2 + f_{01} + if_{02} & -ib - x_3 - f_{03} - iv_0 \\ -ib - x_3 + f_{03} + iv_0 & -x_1 + ix_2 + f_{01} - if_{02} & a - x_0 - if_{12} + iv_3 & f_{13} - if_{23} - v_2 + iv_1 \\ -x_1 - ix_2 + f_{01} + if_{02} & -ib + x_3 - f_{03} + iv_0 & -f_{13} - if_{23} + v_2 + iv_1 & a - x_0 + if_{12} - iv_3 \end{bmatrix} \tag{20}$$

And the converse is also true; each 4×4 real matrix is represented as a multi-vector of $\mathbb{G}(4,\mathbb{R})$.

In 4D as well we can define the determinant solely using constructs of geometric algebra[6]. The determinant of $\mathbf{u}$ is:

Definition 7 (Geometric Representation of the Determinant 4D)**.**

$$\det \quad : \qquad \mathbb{G}(4,\mathbb{R}) \longrightarrow \mathbb{R} \tag{21}$$

$$\mathbf{u} \longmapsto \lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u} \tag{22}$$

where $\mathbf{u}^{\ddagger}$ is:

Definition 8 (Clifford conjugate 4D)**.**

$$\mathbf{u}^{\ddagger} \coloneqq \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 + \langle \mathbf{u} \rangle_3 + \langle \mathbf{u} \rangle_4 \tag{23}$$

and where $\lfloor \mathbf{m} \rfloor_{\{3,4\}}$ is the blade-conjugate of degree 3 and 4 (flipping the plus sign to a minus sign for blade 3 and blade 4):

$$\lfloor \mathbf{u} \rfloor_{\{3,4\}} \coloneqq \langle \mathbf{u} \rangle_0 + \langle \mathbf{u} \rangle_1 + \langle \mathbf{u} \rangle_2 - \langle \mathbf{u} \rangle_3 - \langle \mathbf{u} \rangle_4 \tag{24}$$

2.4 Unitary Gauge (Recap)

Quantum electrodynamics is obtained by gauging the wave-function with $U(1)$. The $U(1)$ invariance results from the usage of the complex norm in ordinary quantum theory. A parametrization of ψ over a differentiable manifold is required to support this derivation. Localizing the invariance group $\theta \to \theta(x)$ over said parametrization, yields the corresponding covariant derivative:

$$D_\mu = \partial_\mu + iqA_\mu(x) \tag{25}$$

where $A_\mu(x)$ is the gauge field.

If one then applies a gauge transformation to ψ and A_μ:

$$\psi \to e^{-iq\theta(x)}\psi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu\theta(x) \tag{26}$$

The covariant derivative is:

$$D_\mu\psi = \partial_\mu\psi + iqA_\mu\psi \tag{27}$$

$$\to \partial_\mu(e^{-iq\theta(x)}\psi) + iq(A_\mu + \partial_\mu\theta(x))(e^{-iq\theta(x)}\psi) \tag{28}$$

$$= e^{-iq\theta(x)}D_\mu\psi \tag{29}$$

Finally, the field is given as follows:

$$F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \tag{30}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to the potential one-form $A_\mu = A_\mu^\alpha T_\alpha$, and where T_α are the generators of the lie algebra of $U(1)$.

3 Result

3.1 Non-Relativistic Quantum Mechanics

We will now recover non-relativistic quantum mechanics by using the method of the Lagrange multipliers and a geometric constraint.

Instead of the Boltzmann entropy we will use the Shannon entropy:

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \tag{31}$$

What constraint will we use on this entropy?

In statistical mechanics we use "scalar" constraints on the entropy such as the energy-meter and the volume-meter. Such are sufficient to recover the Gibbs ensemble, but are insufficient to recover quantum mechanics. Let us introduce the "phase-invariant" constraint, which for a complex-phase, is defined as follows:

$$\operatorname{tr} \begin{bmatrix} 0 & -\overline{b} \\ \overline{b} & 0 \end{bmatrix} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{32}$$

where $\begin{bmatrix} a(q) & -b(q) \\ b(q) & a(q) \end{bmatrix} \cong a(q) + ib(q)$ is the matrix representation of the complex numbers. Like the energy-meter or the volume-meter, a phase-invariant instruments also produces a sequence of measurements converging to an expectation value, but such measurements have a phase-invariance. The trace here grants and enforces said phase-invariance.

The Lagrangian equation that maximizes the entropy subject to this constraint is:

$$\mathcal{L} = -\sum_{q \in \mathbb{Q}} \rho(q) \ln(q) + \alpha \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \tau \left(\operatorname{tr} \begin{bmatrix} 0 & -\overline{b} \\ \overline{b} & 0 \end{bmatrix} - \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \right) \tag{33}$$

Maximizing this equation for ρ by posing $\frac{\partial \mathcal{L}}{\partial \rho(q)} = 0$, we obtain:

$$\frac{\partial \mathcal{L}}{\partial \rho(q)} = -\ln \rho(q) - 1 - \alpha - \tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{34}$$

$$0 = \ln \rho(q) + 1 + \alpha + \tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{35}$$

$$\implies \ln \rho(q) = -1 - \alpha - \tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{36}$$

$$\implies \rho(q) = \exp(-1-\alpha) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{37}$$

$$= \frac{1}{Z(\tau)} \det \exp\left(-\tau \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{38}$$

where $Z(\tau)$ is obtained as follows:

$$1 = \sum_{q \in \mathbb{Q}} \exp(-1-\alpha) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{39}$$

$$\implies (\exp(-1-\alpha))^{-1} = \sum_{q \in \mathbb{Q}} \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{40}$$

$$Z(\tau) := \sum_{q \in \mathbb{Q}} \det \exp\left(-\tau \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{41}$$

We note that the trace in the exponential drops down to a determinant, via the relation $\det \exp A \equiv \exp \operatorname{tr} A$.

Finally, we obtain:

$$\rho(\tau, q) = \frac{1}{Z(\tau)} \det \exp\left(-\tau \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix}\right) \tag{42}$$

$$\cong |\exp -i\tau b(q)|^2 \qquad \text{Born rule} \tag{43}$$

Renaming $\tau \to t/\hbar$ and $b(q) \to H(q)$ recovers the familiar form:

$$\rho(q) = \frac{1}{Z} \left|\exp(-itH(q)/\hbar)\right|^2 \tag{44}$$

or in even more familiar form:

$$\rho(q) = \frac{1}{Z} \left|\psi(q)\right|^2, \text{ where } \psi(q) = \exp(-itH(q)/\hbar) \tag{45}$$

We can show from this that all three Dirac Von-Neumann axioms are satisfied, along with the Born rule; thus providing an origin story for quantum mechanics linked to entropy and geometry.

Indeed, from (45) we can identify the wave-function as the vector of some orthogonal space and the partition function as its inner product:

$$Z = \langle\psi|\psi\rangle \tag{46}$$

Normalized, the physical states are its unit vectors. The probability of any particular state is:

$$\rho(q) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q))^\dagger\psi(q) \tag{47}$$

Finally, any self-adjoint matrix, defined as $\langle\mathbf{O}\psi|\psi\rangle = \langle\psi|\mathbf{O}\psi\rangle$ will correspond to a real-valued statistical mechanics observable iff measured in its eigenbasis.

The equivalence is complete.

3.2 Quantum Theory of Gravity

We will now investigate the arbitrary geometric constraint:

$$\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\mathbf{M}(q) \tag{48}$$

where $\mathbf{M}$ is an arbitrary $n\times n$ matrix.

The Lagrange equation used to maximize the entropy subject to this constraint is:

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}}\rho(q)\ln(q) + \alpha\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\mathbf{M}(q)\right) \tag{49}$$

where α and τ are the Lagrange multipliers.

Maximizing this equation for ρ by posing $\frac{\partial\mathcal{L}}{\partial\rho(q)} = 0$, we obtain:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\rho(q)} &= -\ln\rho(q) - 1 - \alpha - \tau\operatorname{tr}\mathbf{M}(q) && (50)\\
0 &= \ln\rho(q) + 1 + \alpha + \tau\operatorname{tr}\mathbf{M}(q) && (51)\\
\implies \ln\rho(q) &= -1 - \alpha - \tau\operatorname{tr}\mathbf{M}(q) && (52)\\
\implies \rho(q) &= \exp(-1-\alpha)\exp(-\tau\operatorname{tr}\mathbf{M}(q)) && (53)\\
&= \frac{1}{Z(\tau)}\det\exp(-\tau\mathbf{M}(q)) && (54)
\end{aligned}$$

where $Z(\tau)$ is obtained as follows:

$$1 = \sum_{q \in \mathbb{Q}} \exp(-1-\alpha) \exp(-\tau \operatorname{tr} \mathbf{M}(q)) \tag{55}$$

$$\implies (\exp(-1-\alpha))^{-1} = \sum_{q \in \mathbb{Q}} \exp(-\tau \operatorname{tr} \mathbf{M}(q)) \tag{56}$$

$$Z(\tau) \coloneqq \sum_{q \in \mathbb{Q}} \det \exp(-\tau \mathbf{M}(q)) \tag{57}$$

We note that the trace in the exponential drops down to a determinant, via the relation $\det \exp A \equiv \exp \operatorname{tr} A$.

The resulting probability measure is:

$$\rho(q, \tau) = \frac{1}{Z(\tau)} \det \exp(-\tau \mathbf{M}(q)) \tag{58}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} \det \exp(-\tau \mathbf{M}(q)) \tag{59}$$

Posing $\psi(q, \tau) = \exp(-\tau \mathbf{M}(q))$, we can write $\rho(q, \tau) = \det \psi(q, \tau)$, where the determinant acts as a "generalized Born rule", connecting in this case a general linear amplitude to a real number representing a probability.

It is the sophistication afforded by the general linear amplitude along with the determinant as the "generalized Born rule" that allows for this solution to be a quantum theory of gravity.

Let us derive this theory as a general linear gauge theory.

3.2.1 General Linear Gauge

The fundamental invariance group of the general linear wave-function is the orientation-preserving general linear group $\mathrm{GL}^+(n, \mathbb{R})$. Like quantum electrodynamics (via the $U(1)$ gauge) is the archetypal example of QFT, here quantum gravity (via the $\mathrm{GL}^+(n, \mathbb{R})$ gauge) will be the archetypal example of our system.

The exponential term $\exp(-\tau \mathbf{M}(q))$ maps to a one-parameter subgroup of the orientation preserving general linear group:

$$\exp \colon \mathbf{M}_n(\mathbb{R}) \to \mathrm{GL}^+(n, \mathbb{R}) \tag{60}$$

Gauging the $\mathrm{GL}(n, \mathbb{R})$ group is known to produce the Einstein field equations since the resulting $\mathrm{GL}(n, \mathbb{R})$-valued field can be viewed as the Christoffel symbols Γ^μ, and the commutator of the covariant derivatives as the Riemann tensor. This is not a new result and dates back to 1956 by Utiyama[7], and to 1961 by Kibble[8].

The novelty here is that our wave-function is able to accommodate all transformations required to realize general relativity without violating probability conservation laws.

Due to our usage of the determinant, a general linear transformation:

$$\psi'(x) \to g\psi(x)g^{-1} \tag{61}$$

will leave the probability measure of the wave-function invariant, because

$$\det\bigl(g\psi(x)g^{-1}\bigr) = \det\psi(x) \tag{62}$$

The gauge-covariant derivative associated with this transformation is:

$$D_\mu\psi = \partial_\mu\psi - [iqA_\mu, \psi] \tag{63}$$

Finally, the field is given as follows:

$$R_{\mu\nu} = [D_\mu, D_\nu] \tag{64}$$

where $R_{\mu\nu}$ is the Riemann tensor.

The resulting Lagrangian is of course the Einstein-Hilbert action which, up to numerical constant, is:

$$S = \int \epsilon_{abcd}R^{ab} \wedge e^c \wedge e^d = \int \mathrm{d}^4\, x\sqrt{-g}R \tag{65}$$

Consequently, the equations of motion of our quantum field are the Einstein field equations.

4 Foundation of Physics

We are now ready to begin investigating the main result as a general linear quantum theory, in full rigour. To this end, we will now introduce the *algebra of geometric observables* applicable to the general linear wave-function.

The 2D case constitutes a special case whose tools and concepts maps directly with those of ordinary quantum mechanics. The 4D case is significantly less intuitive, but nonetheless will also be investigated.

4.1 Axiomatic Definition of the Algebra, in 2D

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathbb{G}(2, \mathbb{R})$. A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ iff the following holds:

A) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the sesquilinear map:

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \qquad \mathbb{V} \times \mathbb{V} &\longrightarrow \mathbb{G}(2, \mathbb{R}) \\ \langle \mathbf{u}, \mathbf{v} \rangle &\longmapsto \mathbf{u}^{\ddagger} \mathbf{v} \end{aligned} \tag{66}$$

is positive-definite when $\mathbf{u} = \mathbf{v}$; that is $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

B) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\psi}$, the function:

$$\rho(\psi(q), \boldsymbol{\psi}) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \psi(q)^{\ddagger} \psi(q) \tag{67}$$

is positive-definite: $\rho(\psi(q), \boldsymbol{\psi}) > 0$

We note the following comments and definitions:

- From A) and B) it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum to unity:

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{68}$$

- $\boldsymbol{\psi}$ is called a *natural* (or physical) state.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(q, \boldsymbol{\psi})$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$, as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$, which leaves the sum of probabilities normalized (invariant):

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \mathbf{T}\boldsymbol{\psi}) = \sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{69}$$

are the *natural* transformations of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{v} \in \mathcal{A}(\mathbb{V})$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{v} \rangle \tag{70}$$

is called an observable.

- The expectation value of an observable $\mathbf{O}$ is:

$$\langle \mathbf{O} \rangle = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{71}$$

4.2 Observable in 2D — Self-Adjoint Operator

Let us now investigate the general case of an observable in 2D. A matrix $\mathbf{O}$ is an observable iff it is a self-adjoint operator; defined as:

$$\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi}\rangle \tag{72}$$

$\forall \mathbf{u} \forall \mathbf{v} \in \mathbb{V}$.

Setup: Let $\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix}$ be an observable. Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be 2 two-state vectors of multi-vectors $\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix}$ and $\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}$. Here, the components $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, $\mathbf{o}_{11}$ are multi-vectors of $\mathbb{G}(2, \mathbb{R})$.

Derivation: 1. Let us now calculate $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle$:

$$\begin{aligned} 2\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle &= (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger(\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2) \\ &\quad + (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2)^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger(\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2) & (73) \\ &= \boldsymbol{\phi}_\mathbf{1}{}^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\phi}_2 \\ &\quad + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\phi}_2 & (74) \end{aligned}$$

2. Now, $\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi}\rangle$:

$$\begin{aligned} 2\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi}\rangle &= \boldsymbol{\phi}_1^\ddagger(\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2) + (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger(\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2) + (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2)^\ddagger \boldsymbol{\phi}_1 & (75) \\ &= \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{00}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_1^\ddagger \mathbf{o}_{01}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{00}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{01}^\ddagger \boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{10}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^\ddagger \mathbf{o}_{11}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^\ddagger \mathbf{o}_{10}^\ddagger \boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^\ddagger \mathbf{o}_{11}^\ddagger \boldsymbol{\phi}_1 & (76) \end{aligned}$$

For $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi}\rangle$ to be realized, it follows that these relations must hold:

$$\mathbf{o}_{00}^\ddagger = \mathbf{o}_{00} \tag{77}$$
$$\mathbf{o}_{01}^\ddagger = \mathbf{o}_{10} \tag{78}$$
$$\mathbf{o}_{10}^\ddagger = \mathbf{o}_{01} \tag{79}$$
$$\mathbf{o}_{11}^\ddagger = \mathbf{o}_{11} \tag{80}$$

Therefore, it follows that it must be the case that $\mathbf{O}$ must be equal to its own Clifford transpose. Thus, $\mathbf{O}$ is an observable iff:

$$\mathbf{O}^\ddagger = \mathbf{O} \tag{81}$$

which is the equivalent of the self-adjoint operator $\mathbf{O}^\dagger = \mathbf{O}$ of complex Hilbert spaces.

The extra geometric sophistication of this geometric observable allows the probability measure to retain invariance over a larger class of geometric transformations than what is possible merely with unitary transformation. These transformations are sufficiently flexible to support gravity while retaining valid observable statistics.

4.3 Observable in 2D — Eigenvalues / Spectral Theorem

Let us show how the spectral theorem applies to $\mathbf{O}^\ddagger = \mathbf{O}$, such that its eigenvalues are real. Consider:

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12} \\ a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} & a_{11} \end{bmatrix} \tag{82}$$

It follows that $\mathbf{O}^\ddagger = \mathbf{O}$:

$$\mathbf{O}^\ddagger = \begin{bmatrix} a_{00} & a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12} \\ a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} & a_{11} \end{bmatrix} \tag{83}$$

This example is the most general 2×2 matrix $\mathbf{O}$ such that $\mathbf{O}^\ddagger = \mathbf{O}$.

The eigenvalues are obtained as follows:

$$0 = \det(\mathbf{O} - \lambda\mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12} \\ a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} & a_{11} - \lambda \end{bmatrix} \tag{84}$$

implies:

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\mathbf{e}_1 - y\mathbf{e}_2 - b\mathbf{e}_{12})(a + x\mathbf{e}_1 + y\mathbf{e}_2 + b\mathbf{e}_{12} + a_{11}) \tag{85}$$

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2) \tag{86}$$

finally:

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{87}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\} \tag{88}$$

We note that in the case where $a_{00} - a_{11} = 0$, the roots would be complex iff $a^2 - x^2 - y^2 + b^2 < 0$, but we already stated that the determinant of real matrices must be greater than zero because the exponential maps to the orientation-preserving general linear group— therefore it is the case that $a^2-x^2-y^2+b^2>0$, as this expression is the determinant of the multi-vector. Consequently, $\mathbf{O}^\ddagger=\mathbf{O}$ — implies, for orientation-preserving transformations, that its roots are real-valued, and thus constitute a 'geometric' observable in the traditional sense of an observable whose eigenvalues are real-valued.

4.4 Left Action, in 2D

A left action on a wave-function : $\mathbf{T}\left|\psi\right\rangle$, connects to the bilinear form as follows: $\left\langle\psi\right|\mathbf{T}^\ddagger\mathbf{T}\left|\psi\right\rangle$. The invariance requirement on $\mathbf{T}$ is as follows:

$$\left\langle\psi\right|\mathbf{T}^\ddagger\mathbf{T}\left|\psi\right\rangle=\left\langle\psi|\psi\right\rangle \tag{89}$$

We are thus interested in the group of matrices such that:

$$\mathbf{T}^\ddagger\mathbf{T}=I \tag{90}$$

Let us consider a two-state system. A general transformation is:

$$\mathbf{T}=\begin{bmatrix}u & v\\ w & x\end{bmatrix} \tag{91}$$

where u,v,w,x are multi-vectors of 2 dimensions. The expression $\mathbf{G}^\ddagger\mathbf{G}$ is:

$$\mathbf{T}^\ddagger\mathbf{T}=\begin{bmatrix}v^\ddagger & u^\ddagger\\ w^\ddagger & x^\ddagger\end{bmatrix}\begin{bmatrix}v & w\\ u & x\end{bmatrix}=\begin{bmatrix}v^\ddagger v+u^\ddagger u & v^\ddagger w+u^\ddagger x\\ w^\ddagger v+x^\ddagger u & w^\ddagger w+x^\ddagger x\end{bmatrix} \tag{92}$$

For the results to be the identity, it must be the case that:

$$v^\ddagger v+u^\ddagger u=1 \tag{93}$$

$$v^\ddagger w+u^\ddagger x=0 \tag{94}$$

$$w^\ddagger v+x^\ddagger u=0 \tag{95}$$

$$w^\ddagger w+x^\ddagger x=1 \tag{96}$$

This is the case if

$$\mathbf{T}=\frac{1}{\sqrt{v^\ddagger v+u^\ddagger u}}\begin{bmatrix}v & u\\ -e^\varphi u^\ddagger & e^\varphi v^\ddagger\end{bmatrix} \tag{97}$$

where u,v are multi-vectors of 2 dimensions, and where e^φ is a unit multi-vector. Comparatively, the unitary case is obtained when the vector part of the multi-vector vanishes $\mathbf{x}\to 0$, and is:

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta} b^\dagger & e^{i\theta} a^\dagger \end{bmatrix} \tag{98}$$

We can show that $\mathbf{G}^\ddagger \mathbf{G} = I$ as follows:

$$\implies \mathbf{T}^\ddagger \mathbf{T} = \frac{1}{v^\ddagger v + u^\ddagger u} \begin{bmatrix} v^\ddagger & -e^{-\varphi} u \\ u^\ddagger & e^{-\varphi} v \end{bmatrix} \begin{bmatrix} v & u \\ -e^{\varphi} u^\ddagger & e^{\varphi} v^\ddagger \end{bmatrix} \tag{99}$$

$$= \frac{1}{v^\ddagger v + u^\ddagger u} \begin{bmatrix} v^\ddagger v + u^\ddagger u & v^\ddagger u - v^\ddagger u \\ u^\ddagger v - u^\ddagger v & u^\ddagger u + v^\ddagger v \end{bmatrix} \tag{100}$$

$$= I \tag{101}$$

In the case where $\mathbf{T}$ and $|\psi\rangle$ are n-dimensional, we can find an expression for it starting from a diagonal matrix:

$$\mathbf{D} = \begin{bmatrix} e^{x_1 \hat{\mathbf{x}} + y_1 \hat{\mathbf{y}} + i b_1} & 0 \\ 0 & e^{x_2 \hat{\mathbf{x}} + y_2 \hat{\mathbf{y}} + i b_2} \end{bmatrix} \tag{102}$$

where $\mathbf{T} = P\mathbf{D}P^{-1}$. It follows quite easily that $D^\ddagger D = I$, because each diagonal entry produces unity: $e^{-x_1 \hat{\mathbf{x}} - y_1 \hat{\mathbf{y}} - i b_1} e^{x_1 \hat{\mathbf{x}} + y_1 \hat{\mathbf{y}} + i b_1} = 1$.

4.5 Adjoint Action, in 2D

The left action case can recover at most the special linear group. For the general linear group itself, we require the adjoint action. Since the elements of $|\psi\rangle$ are matrices, in the general case, the transformation is given by adjoint action:

$$\mathbf{T} |\psi\rangle \mathbf{T}^{-1} \tag{103}$$

The bilinear form is:

$$(\mathbf{T} |\psi\rangle \mathbf{T}^{-1})^\ddagger (\mathbf{T} |\psi\rangle \mathbf{T}^{-1}) = (\mathbf{T}^{-1})^\ddagger \langle\psi| \mathbf{T}^\ddagger \mathbf{T} |\psi\rangle \mathbf{T}^{-1} \tag{104}$$

and the invariance requirement on $\mathbf{T}$ is as follows:

$$(\mathbf{T}^{-1})^\ddagger \langle\psi| \mathbf{T}^\ddagger \mathbf{T} |\psi\rangle \mathbf{T}^{-1} = \langle\psi|\psi\rangle \tag{105}$$

With a diagonal matrix, this occurs for general linear transformations:

$$\mathbf{D} = \begin{bmatrix} e^{a_1 + x_1 \hat{\mathbf{x}} + y_1 \hat{\mathbf{y}} + i b_1} & 0 & 0 \\ 0 & e^{a_2 + x_2 \hat{\mathbf{x}} + y_2 \hat{\mathbf{y}} + i b_2} & 0 \\ 0 & 0 & \ddots \end{bmatrix} \tag{106}$$

where $\mathbf{T} = P\mathbf{D}P^{-1}$.

Taking a single diagonal entry as an example, the reduction is:

$$e^{-a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}\psi_1^{\ddagger}e^{a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1}e^{a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}\psi_1 e^{-a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1} \tag{107}$$

$$= e^{-a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}\psi_1^{\ddagger}e^{2a_1}\psi_1 e^{-a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1} \tag{108}$$

We note that $\psi^{\ddagger}\psi$ is a scalar, therefore

$$= \psi_1^{\ddagger}\psi_1 e^{2a_1}e^{-a_1+x_1\hat{\mathbf{x}}+y_1\hat{\mathbf{y}}+ib_1}e^{-a_1-x_1\hat{\mathbf{x}}-y_1\hat{\mathbf{y}}-ib_1} \tag{109}$$

$$= \psi_1^{\ddagger}\psi_1 e^{2a_1}e^{-a_1}e^{-a_1} = \psi_1^{\ddagger}\psi_1 \tag{110}$$

4.6 Algebra of Geometric Observables, in 4D

We will now consider the general case for a vector space over 4×4 matrices.

In 2D we were able to extend the complex Hilbert space to a "geometric Hilbert space" and we found that the familiar properties of complex Hilbert spaces were translatable to the geometry of the general linear group; essentially it amounted to changing a few symbols and tweaking a few definitions.

In 4D we will not have this benefit.

The main roadblock is that unlike the 2D case whose determinant is given by $\psi^{\ddagger}\psi$, and thus can be interpreted as an inner product of two vectors, in 4D we need four multiplicand $\lfloor\psi^{\ddagger}\psi\rfloor_{3,4}\psi^{\ddagger}\psi$. As such, we are unable to produce a sesquilinear form of the inner product as we did for the 2D case. Since there is no satisfactory inner product, therefore there is no Hilbert space in the usual sense of a complete inner product space.

Nonetheless, the "features" of quantum mechanics (wave-function measurements, linear transformations, observables as matrix or operators, interference patterns in the probability measure, etc) remain present in the 4D case.

So if not a complete inner product space, what space supports the general linear wave-function in 4D?

We can create a "tensor extension" to the Hilbert space. In this case the role of the inner product is adopted by a rank 4 tensor linking four vectors to an element of $\mathbb{G}(4,\mathbb{R})$. In this environment the typical concepts of quantum mechanics have equivalences, and the sophistication of rank 4 tensor Hilbert space allows the wave-function to accommodate all transformation required by general relativity while retaining consistent probabilities for observables.

Let $\mathbb{V}$ be a m-dimensional vector space over the 4×4 real matrices. A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ iff the following holds:

1. $\forall\boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the quadri-sesquilinear form:

$$\begin{aligned}\langle\cdot,\cdot,\cdot,\cdot\rangle \quad : \quad & \mathbb{V}\times\mathbb{V}\times\mathbb{V}\times\mathbb{V} \longrightarrow \mathbb{G}(4,\mathbb{R}) \\ & \langle\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{x}\rangle \longmapsto \sum_{i=1}^{m}\lfloor u_i^{\ddagger}v_i\rfloor_{3,4}w_i^{\ddagger}x_i\end{aligned} \tag{111}$$

is positive-definite when $\mathbf{u} = \mathbf{v} = \mathbf{w} = \mathbf{x}$; that is $\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

2. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\psi}$, the function:

$$\rho(\psi(q), \boldsymbol{\psi}) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \det \psi(q) \tag{112}$$

is positive-definite: $\rho(\psi(q), \boldsymbol{\psi}) > 0$

We note the following properties, features and comments:

- From A) and B) it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum to unity:

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{113}$$

- $\boldsymbol{\psi}$ is called a *natural* (or physical) state.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(\psi(q), \boldsymbol{\psi})$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ such as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$ which leaves the sum of probabilities normalized (invariant):

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \mathbf{T}\boldsymbol{\psi}) = \sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{114}$$

are the *natural* transformations of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{v} \forall \mathbf{w} \forall \mathbf{x} \in \mathbb{V}$:

$$\langle \mathbf{Ou}, \mathbf{v}, \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{u}, \mathbf{Ov}, \mathbf{w}, \mathbf{x} \rangle = \langle \mathbf{u}, \mathbf{v}, \mathbf{Ow}, \mathbf{x} \rangle = \langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{Ox} \rangle \tag{115}$$

is called an observable.

- The expectation value of an observable $\mathbf{O}$ is:

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \tag{116}$$

Let us now recover quantum field theory and particles physics.

4.6.1 The SU(2)xU(1) Sector

Let $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$ be an arbitrary multi-vector of $\mathbb{G}(4, \mathbb{R})$, let $\mathbf{M}$ be its matrix representation, and let ψ be the general linear wave-function.

We now introduce the following definitions:

$$\phi(q) := \psi(q)^2 \tag{117}$$

$$\tilde{\phi}(q) := (\psi(q)^2)^{\ddagger} \tag{118}$$

In the David Hestenes' notation[9], the quantities ϕ and $\tilde{\phi}$ are intended to represent the wave-function and its reverse. However, here ϕ is allowed to be the exponential of an arbitrary multi-vector $\mathbf{u}$; unless restricted, the Dirac current will not be invariant with respect to the set of all possible transformations $\phi \mapsto e^{\mathbf{u}}\phi$, and the definition of the Dirac current $\tilde{\phi}\gamma_0\phi$ will not be realized.

We now wish to restrict the set of multi-vectors $e^{\mathbf{u}}$ to those which realizes the David Hestenes' definition of the current and remains invariant upon a transformation.

Let us investigate.

We note that $\mathbf{x}$ and $\mathbf{v}$ anti-commute with γ_0, and therefore must be equal to 0 as they would otherwise not cancel out. We also note that the bi-vectors of $\mathbf{u}$ have basis $\gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3$; of those, only $\gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3$ commute with γ_0; the rest must therefore be equal to 0. Finally, the pseudo-scalar anti-commutes with γ_0 but this is fine as it must cancel in the Dirac current. The most general multi-vector which realizes the definition is:

$$\mathbf{u} \to a + F_{12}\gamma_1\gamma_2 + F_{13}\gamma_1\gamma_3 + F_{23}\gamma_2\gamma_3 + b\gamma_0\gamma_1\gamma_2\gamma_3 \tag{119}$$

We can better see its physical significance by noting that $\gamma_1\gamma_2 = I\sigma_3$, $\gamma_1\gamma_3 = I\sigma_2$ and $\gamma_2\gamma_3 = I\sigma_1$. The resulting multi-vector is unitary and is equal to:

$$U = e^{\mathbf{u}} = e^{\frac{1}{2}I(F_{23}\sigma_1 + F_{13}\sigma_2 + F_{12}\sigma_3 + b)} \tag{120}$$

The terms $F_{23}\sigma_1 + F_{13}\sigma_2 + F_{12}\sigma_3$ are responsible for a $SU(2)$ symmetry and b for a $U(1)$ symmetry; and this is simply the electroweak sector[10, 11].

We can now interpret the electroweak sector to be the product of quantum gravity "casted" into the sub-algebra $\tilde{\phi}$ and ϕ; the "casting" reduces the set of all multi-vector transformations $\phi = e^{\mathbf{u}}$ otherwise permissible in quantum gravity to only those which leaves the Dirac current $\tilde{\phi}\gamma_0\phi$ invariant. The resulting multi-vectors form the $SU(2) \times U(1)$ group.

4.7 A Step Towards Falsifiable Predictions

Let us now list a number falsifiable predictions.

The main idea is that a general linear wave-function would allow a larger class of interference patterns than what is possible merely with complex interference. We note the work of B. I. Lev[12] treating the interference pattern associated with the geometric algebra formulation of the wave-function.

As a secondary idea, it is also plausible that an Aharonov–Bohm effect experiment on gravity[13] could detect a general linear phase.

An interference pattern follows from a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the application of the determinant:

$$\det(\mathbf{u}+\mathbf{v}) = \det\mathbf{u} + \det\mathbf{v} + \text{extra-terms} \tag{121}$$

The sum $\det\mathbf{u}+\det\mathbf{v}$ are a sum of probability and the extra terms represents the interference term.

We use the extra-terms to define a bilinear form using the dot product notation, as follows:

$$\begin{aligned} \cdot \quad : \qquad \mathbb{G}(2n,\mathbb{R}) \times \mathbb{G}(2n,\mathbb{R}) &\longrightarrow \mathbb{R} && (122)\\ \mathbf{u}\cdot\mathbf{v} &\longmapsto \frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) && (123) \end{aligned}$$

For example in 2D, we have:

$$\begin{aligned} \mathbf{u} &= a_1 + x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + b_1\mathbf{e}_{12} && (124)\\ \mathbf{v} &= a_2 + x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + b_2\mathbf{e}_{12} && (125)\\ \implies \mathbf{u}\cdot\mathbf{v} &= a_1a_2 + b_1b_2 - x_1x_2 - y_1y_2 && (126) \end{aligned}$$

Iff $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$ then $\mathbf{u}\cdot\mathbf{v}$ is always positive, and therefore qualifies as a positive-definite inner product, but no greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$, whichever is greater; thus also satisfying the conditions of an interference term.

- In 2D the dot product is equivalent to this form:

$$\begin{aligned} \frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) &= \frac{1}{2}\left((\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v}\right) && (127)\\ &= \mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v} - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v} && (128)\\ &= \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} && (129) \end{aligned}$$

- In 4D it is substantially more verbose:

$$\frac{1}{2}(\det(\mathbf{u}+\mathbf{v})-\det\mathbf{u}-\det\mathbf{v}) \tag{130}$$

$$=\frac{1}{2}\left(\lfloor(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})\rfloor_{3,4}(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})-\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}-\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\right) \tag{131}$$

$$=\frac{1}{2}\left(\lfloor\mathbf{u}^\ddagger\mathbf{u}+\mathbf{u}^\ddagger\mathbf{v}+\mathbf{v}^\ddagger\mathbf{u}+\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}(\mathbf{u}^\ddagger\mathbf{u}+\mathbf{u}^\ddagger\mathbf{v}+\mathbf{v}^\ddagger\mathbf{u}+\mathbf{v}^\ddagger\mathbf{v})-\dots\right) \tag{132}$$

$$\begin{aligned}=&\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\&+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\&+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\&+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}-\dots\end{aligned} \tag{133}$$

$$\begin{aligned}=&\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\&+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\&+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\&+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v}+\lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}\end{aligned} \tag{134}$$

Simpler version of this interference pattern are possible when the general linear group is reduced.

<u>Complex interference</u>:

For instance, a reduction to the circle group, likewise reduces the interference pattern to complex interference:

$$|\psi_1+\psi_2|^2=|\psi_1|^2+|\psi_2|^2+2|\psi_1||\psi_2|\cos(\phi_1-\phi_2) \tag{135}$$

<u>Deep spinor interference</u>:

A reduction to the spinor group, reduces the interference pattern to a "deep spinor rotation".

Consider a two-state wave-function (we note that $[\mathbf{f},\mathbf{b}]=0$):

$$\psi=\psi_1+\psi_2=e^{a_1}e^{\mathbf{f}_1}e^{\mathbf{b}_1}+e^{a_2}e^{\mathbf{f}_2}e^{\mathbf{b}_2} \tag{136}$$

The geometric interference pattern for a full general linear transformation in 4D is given by the product:

$$\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi \tag{137}$$

Let us start with the sub-product:

$$\psi^\ddagger\psi = (e^{a_1}e^{-\mathbf{f}_1}e^{\mathbf{b}_1} + e^{a_2}e^{-\mathbf{f}_2}e^{\mathbf{b}_2})(e^{a_1}e^{\mathbf{f}_1}e^{\mathbf{b}_1} + e^{a_2}e^{\mathbf{f}_2}e^{\mathbf{b}_2}) \tag{138}$$

$$= e^{a_1}e^{-\mathbf{f}_1}e^{\mathbf{b}_1}e^{a_1}e^{\mathbf{f}_1}e^{\mathbf{b}_1} + e^{a_1}e^{-\mathbf{f}_1}e^{\mathbf{b}_1}e^{a_2}e^{\mathbf{f}_2}e^{\mathbf{b}_2} + e^{a_2}e^{-\mathbf{f}_2}e^{\mathbf{b}_2}e^{a_1}e^{\mathbf{f}_1}e^{\mathbf{b}_1} + e^{a_2}e^{-\mathbf{f}_2}e^{\mathbf{b}_2}e^{a_2}e^{\mathbf{f}_2}e^{\mathbf{b}_2} \tag{139}$$

$$= e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_2}e^{2\mathbf{b}_2} + e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1}) \tag{140}$$

The full product is:

$$\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi = \left(e^{2a_1}e^{-2\mathbf{b}_1} + e^{2a_2}e^{-2\mathbf{b}_2} + e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\right) \times \left(e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_2}e^{2\mathbf{b}_2} + e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1}\right) \tag{141}$$

$$\begin{aligned}
&= e^{2a_1}e^{-2\mathbf{b}_1}e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_1}e^{-2\mathbf{b}_1}e^{2a_2}e^{2\mathbf{b}_2} + e^{2a_1}e^{-2\mathbf{b}_1}e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\\
&\quad + e^{2a_2}e^{-2\mathbf{b}_2}e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_2}e^{-2\mathbf{b}_2}e^{2a_2}e^{2\mathbf{b}_2} + e^{2a_2}e^{-2\mathbf{b}_2}e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\\
&\quad + e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{2a_1}e^{2\mathbf{b}_1}\\
&\qquad + e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{2a_2}e^{2\mathbf{b}_2}\\
&\qquad + e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})
\end{aligned} \tag{142}$$

$$= e^{4a_1} + e^{4a_2} + 2e^{2a_1+2a_2}\cos(2b_1 - 2b_2) \tag{143}$$

$$+ e^{a_1+a_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})(\tag{144}$$

$$e^{2a_1}(e^{-\mathbf{b}_1+\mathbf{b}_2} + e^{\mathbf{b}_1-\mathbf{b}_2}) \tag{145}$$

$$+ e^{2a_2}(e^{\mathbf{b}_1-\mathbf{b}_2} + e^{-\mathbf{b}_1+\mathbf{b}_2})) \tag{146}$$

$$+ e^{2a_1+2a_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})^2 \tag{147}$$

$$\begin{aligned}
&= \underbrace{e^{4a_1} + e^{4a_2}}_{\text{sum}} + \underbrace{2e^{2a_1+2a_2}\cos(2b_1 - 2b_2)}_{\text{complex interference}}\\
&\quad + \underbrace{2e^{a_1+a_2}(e^{2a_1} + e^{2a_2})(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})(\cos(B_1 - B_2)) + e^{2A_1+2A_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})^2}_{\text{deep spinor interference}}
\end{aligned} \tag{148}$$

Finally, we stress that the general linear interference pattern occurs in context of quantum gravity, as ordinary quantum field theory reduces to typical complex interference.

5 Discussion

In contrast to the multiple interpretations quantum mechanics, the interpretation of statistical mechanics is free of paradoxes and is singular. Here, we have recovered the foundations of quantum mechanics as a theorem of statistical mechanics; its interpretation will now be inherited from statistical mechanics.

The complete correspondence between an ordinary system of statistical mechanics and our method is as follows:

Table 1: Correspondence

Concept	Statistical Mechanics	Geometric Constraints (Our Method)
Entropy	Boltzmann	Shannon
Measure	Gibbs	Born rule on wave-function
Constraint	Energy meter	Phase-invariant instrument
Micro-state	Energy values	Possible measurements
Macro-state	Equation of state	Evolution of the wave-function
Experience	Ergodic	Message of measurements

Let us discuss the correspondence.

In statistical mechanics, one first assumes that an instrument measures a system. The action of this instrument is then interpreted as a constraint on the entropy. For instance, one can think of the constraint of an expected energy or volume value as an energy-meter or volume-meter producing a sequence of measurements whose average converges towards an expectation value.

In this work, we have introduced geometric constraints into statistical mechanics. Maximizing the entropy under geometric constraints induces various phase-invariances into the resulting probability measure whose sophistication depends on the geometry. Specifically, the constraint $\operatorname{tr}\begin{bmatrix}0 & -\overline{b}\\ \overline{b} & 0\end{bmatrix} = \sum_{q\in\mathbb{Q}} \operatorname{tr}\rho(q)\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}$ induces a complex phase-invariance into the probability measure $\rho(q) = |\exp(-i\tau b(q))|^2$ giving rise to the Born rule and the wave-function, and the constraint $\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \operatorname{tr}\rho(q)\mathbf{M}(q)$ induces a general linear phase-invariance in the probability measure $\rho(q) = \det\exp\left(-\tau\mathbf{M}(q)\right)$ giving rise to a quantum theory of gravity. In each cases, we can interpret the constraint as an instrument acting on the system. In the case of the complex phase we associate the constraint to a incidence counter measuring a particle or a photon, and in the case of the general linear phase we associate the constraint to a measure that is invariant with respect to all changes of coordinates, and specifically its group reduction to the Lorentz group, we associate the constraint and its phase-invariance to a measurement of a geometric transformation that is Lorentz invariant.

The probabilistic interpretation of the wave-function along with the Born rule is inherited from its origins in statistical mechanics. The wave-function is also entailed, hence it is not taken as axiomatic. Rather, it is the registration of a measurement by an instrument along with the geometric constraints on the entropy that are axiomatic. Since the wave-function is derived from the entropy of already registered measurements, it is never updated to a collapsed state; thus dissolving the collapse problem at the interpretational level. The collapse problem is a symptom of attributing an ontology to the wave-function; but the ontology belongs to the instruments, and the wave-function is a measure derived consistently with the measurements that have been made. It does not entail future measurements, only it is consistent with previous measurements.

The consequence is a minimal and paradox-free interpretation of quantum mechanics: *In nature, there exists instruments that record sequences of measurements on systems, those measurements are unique up to a phase, and the wave-function along with the Born rule are the entropy-maximizing measure constrained by those measurements.* This interpretation is minimal, completely factual and entirely free of all unfalsifiable redundancies: no need for many-worlds, no need to attribute an ontological existence to the wave-function, no need to appeal to a collapse upon measurements, etc.

Let us know discuss the extension to the general linear amplitude.

When the geometric constraint is arbitrary (any square matrix), the procedure yield a quantum theory of gravity, a wave-function of the general linear group and a Born rule extended to the determinant. The wave-function, if then parametrized in $\mathbb{R}^{3,1}$, represents an instruction, or superposition thereof, to transform the frame bundle at each event in space-time. Finally, gauging this group produces the Einstein field equations as the equations of motion of the quantum field. We also state that casting the general linear wave-function into the definition of the Dirac current, reduces the theory to a quantum field theory of the $SU(2) \times U(1)$ group, thus recovering a subset of particle physics.

In the correspondence, the usage of the Shannon entropy instead of the Boltzmann entropy changes the experience from ergodic to a message (in the sense of the theory of communication of Claude Shannon[14]) of measurements. The receipt of such a message is interpreted as the registration of a 'click'[15] on a screen or other detecting instrument. We also note that the screen is an instrument that is geometrically extended, and the path of the particle or photon is also geometric. With this in mind, quantum physics (up to quantum gravity) acquires a conceptually very simple expression; it can be interpreted as the probability measure resulting from maximizing the entropy of a message of geometrically constrained measurements.

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