A Gravitized Standard Model is Found as the Solution to the Problem of Maximizing the Entropy of All Linear Measurements

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Abstract

In modern theoretical physics, the laws of physics are directly represented with axioms (e.g., the Dirac–Von Neumann axioms, the Wightman axioms, and Newton's laws of motion). While the laws of physics are entailed by measurements, axioms (in modern logic) are not; rather, axioms hold true merely by definition. Motivated by this dissimilarity, we introduce a more suitable foundation than axioms to represent the laws of physics, then we make the case for its supremacy. In our model, the entailment is mirrored; measurements will be axiomatic, and the laws will be its theorems. Specifically, we introduce a maximization problem on the entropy of all linear measurements. Its sole solution is a gravitized quantum theory of the $\mathrm{GL}^+(n, \mathbb{R})$ symmetry. In 3+1D, a structure reduction breaks the model into both a gravitational theory and a quantum theory of the SU(2)xU(1) and SU(3) groups. Remarkably, the general solution fails to admit normalizable observables above 4 dimensions, suggesting a intrinsic limit to the dimensionality of observable spacetime.

1 Introduction

The physical laws in modern theoretical physics are expressed as axioms (e.g., the Dirac–Von Neumann axioms, the Wightman axioms, and Newton's laws of motion). The theorems provable by these axioms are the predictions of the theory. If laboratory measurements invalidate the predictions, the postulated laws are deemed falsified, and new laws are postulated.

In this scenario, it is the theorems (predictions) of the theory that are used (in concert with experiments) to invalidate its axioms (laws).

In logic, however, axioms define what is true in a theory. It follows obviously that its theorems cannot invalidate them.

Thus, there is a dissimilarity between using axioms in physics versus their use in logic.

Since the laws of physics require a more complex interplay between axioms, theorems, and their invalidations than the unidirectional entailment between axioms and theorems found in logic, the question of using axioms to express the laws of physics arises.

Motivated by this dissimilarity, we begun searching for a more appropriate logical formulation of the laws of physics, than as brute axioms. We intend to show that correcting the axiomatic entailment between the laws and measurements yields a significantly superior and optimized formulation of fundamental physics.

In our proposal, laboratory measurements entail the mathematical expression of those measurements, and it is this expression, not the laws of physics, that will constitute the axioms of our system. The laws of physics will be defined as the solution to a carefully crafted optimization problem on the entropy of all linear measurements.

The solution to this optimization problem is a novel and optimized formulation of fundamental physics. In 3+1D, it yields a gravitized quantum theory, whose symmetry breaks into a theory of gravitation part and into an SU(2)xU(1) and SU(3) quantum theory parts. Remarkably, the general solution cannot produce normalizable observables above 4D, suggesting an intrinsic limit to the dimensionality of observable spacetime. We interpret this tight configuration as suggestive of the power and efficiency of defining the laws of physics as the solution to a mathematical optimization problem, rather than as brute axioms.

In essence, from laboratory measurements, it is easier to "guess" the correct mathematical expression for all possible such measurements than to "guess" the right laws of physics. The distance one must travel in "guessing space" is much shorter for the former than the later, and this reduces the risk of running astray.

Our optimized formulation is unlikely to have been obtained by trial and error or traditional methods, making our optimization problem a key step in the derivation.

Corollaries that follow directly from our solution, such as the mathematical origin of the Born rule, the proof of the axioms of quantum physics, an identification of the correct interpretation of quantum mechanics, and a deprecation of the measurement/collapse problem, are also presented.

To define the problem rigorously, we first introduce the key structure that makes our approach possible: the *linear measurement constraint*. Next, we present its rationale.

The construction of the linear measurement constraint exploits the connection between geometry and probability via the trace. The trace of a matrix can be understood as the expected eigenvalue multiplied by the vector space dimension and the eigenvalues as the ratios of the distortion of the linear transformation associated with the matrix[1].

Let $\mathbb{Q}$ be a statistical ensemble, and let $\mathbf{M}$ be an arbitrary $n \times n$ matrix representing a linear transformation. The linear measurement constraint is defined as

$$\operatorname{tr} \overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q), \tag{1}$$

where $\operatorname{tr} \overline{\mathbf{M}}$ denotes the expected eigenvalue of the statistically weighted sum of the matrices $\mathbf{M}(q)$ parameterized over a statistical ensemble $\mathbb{Q}$.

Alternatively (and preferably), we may use geometric algebra to define the constraint as

$$\operatorname{tr} \overline{\mathbf{u}} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{u}(q), \tag{2}$$

where $\mathbf{u}$ is an arbitrary multivector of the real geometric algebra $\mathcal{G}(\mathbb{R}^n)$. The notation for this formulation is detailed in Section 2.

Although the constraint can be expressed by both approaches, using multivectors instead of matrices highlights the geometric characteristics of the method. In this case of geometric algebra, we will call the constraint *the geometric measurement constraint*. In Section 2, we will introduce isomorphisms between the linear measurement constraint and the geometric measurement constraint valid in 2D and 4D. As such we may use the terms interchangeably up to isomorphism and based on context.

Now, we discuss its rationale.

Constraints are used in statistical mechanics to derive the Gibbs measure using Lagrange multipliers[2] by maximizing the entropy.

For instance, an energy constraint on the entropy is

$$\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q) E(q), \tag{3}$$

which is associated with an energy meter that measures the system's energy and produces a series of energy measurements $E_1, E_2, \dots,$ converging to an expectation value $\overline{E}$.

Another common constraint is related to the volume:

$$\overline{V} = \sum_{q\in\mathbb{Q}} \rho(q) V(q), \tag{4}$$

which is associated with a volume meter acting on a system and produces a sequence of measured volumes $V_1, V_2, \dots,$ converging to an expectation value $\overline{V}$.

Moreover, the sum over the statistical ensemble must equal 1, as follows:

$$1 = \sum_{q\in\mathbb{Q}} \rho(q) \tag{5}$$

Using equations (3) and (5), a typical statistical mechanical system is obtained by maximizing the entropy using the corresponding Lagrange equation. The Lagrange multipliers method is expressed as

$$\mathcal{L} = -k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right), \quad (6)$$

where λ and β are the Lagrange multipliers.
Therefore, by solving $\frac{\partial \mathcal{L}}{\partial \rho} = 0$ for ρ, we obtain the Gibbs measure as

$$\rho(q, \beta) = \frac{1}{Z(\beta)} \exp\big(-\beta E(q)\big), \quad (7)$$

where

$$Z(\beta) = \sum_{q \in \mathbb{Q}} \exp\big(-\beta E(q)\big). \quad (8)$$

In our method, (3), a scalar measurement constraint, is replaced with $\operatorname{tr} \overline{\mathbf{M}}$, a linear measurement constraint. Instead of energy or volume meters, we have protractors, and boost, dilation, and shear meters.

As we found, the linear measurement constraint is compatible with the full machinery of statistical physics. The probability measure resulting from entropy maximization will preserve the expectation eigenvalue of these transformations up to a phase or symmetry group. For instance, based on our entropy maximization procedure, a statistical system measured exclusively using a protractor will carry a local rotation symmetry in the probability of the measured events.

By limiting the definition of constraints to scalar expressions, we believe that statistical physics failed to capture all measurements available in nature. Our linear measurement constraint redresses the situation and supports the totality of linear measurements that are possible in principle.

Finally, it is the Shannon entropy (in base e) that we maximize and not the Boltzmann entropy. The resulting probability measure quantifies the information associated with an observer's receipt of a message of measurements. The Shannon entropy does not change the mathematical equation for entropy (minus the Boltzmann constant); only the final interpretation is changed (further details on the interpretation of quantum mechanics resulting from this model are provided in section 6).

The corresponding Lagrange equation is

$$\mathcal{L} = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \tau \left(\operatorname{tr} \overline{\mathbf{u}} - \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{u}(q) \right), \quad (9)$$

and is sufficient to solve $\frac{\partial \mathcal{L}}{\partial \rho} = 0$ for ρ to obtain the solution, which is our main result.

The manuscript is organized as follows. The Methods section introduces tools using geometric algebra, based on the study by Lundholm et al. [3, 4]. Specifically, we use the notion of a determinant for multivectors and Clifford conjugate for generalizing the complex conjugate. These tools enable the geometric expression of the results.

The Results section presents two solutions for the Lagrange equation. The first is the recovery of standard non-relativistic quantum mechanics when reducing the matrix from an arbitrary matrix to a representation of the imaginary number. The second is the general case with an arbitrary matrix or multivector.

We then develop our initial results into a geometric foundation for physics in 2D and 3+1D, consistent with the general solution. We show in the general case that the model is a gravitized quantum theory whose principal symmetry is the $\mathrm{GL}^+(n, \mathbb{R})$ group. In 3+1D, the symmetry breaks into both gravity and into a quantum theory of the $\mathrm{SU}(2)\times\mathrm{U}(1)$ and $\mathrm{SU}(3)$ groups. Furthermore, we show how and why the general solution lacks normalizable observables beyond 4D.

Finally, the Discussion section provides an interpretation of quantum mechanics consistent with its newly revealed origin, namely the metrological interpretation. Central to this interpretation is the measure maximizing the Shannon entropy and constrained by linear measurements, which yields the wavefunction. This interpretation thus considers the information of measurements more fundamental than the now entirely derivable wavefunction. The end product is a theory that deprecates the measurement problem, supersedes it with a theory of instruments, and provides a plausible explanation for the origin of quantum mechanics in nature, connecting it entirely to entropy and linear mesurements.

2 Methods

2.1 Notation

- Typography:

  Sets are written using the blackboard bold typography (e.g., $\mathbb{L}$, $\mathbb{W}$, and $\mathbb{Q}$) unless a prior convention assigns it another symbol.

  Matrices are in bold uppercase (e.g., $\mathbf{P}$ and $\mathbf{M}$), tuples, vectors, and multivectors are in bold lowercase (e.g., $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{g}$), and most other constructions (e.g., scalars and functions) have plain typography (e.g., $a, and A$).

  The unit pseudo-scalar (of geometric algebra), imaginary number, and identity matrix are $\mathbf{i}$, i, and $\mathbf{I}$, respectively.

- Sets:

  The projection of a tuple $\mathbf{p}$ is $\mathrm{proj}_i(\mathbf{p})$.

As an example, the elements of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$ are denoted as $\mathbf{p} = (x, y)$.

The projection operators are $\text{proj}_1(\mathbf{p}) = x$ and $\text{proj}_2(\mathbf{p}) = y$;

if projected over a set, the corresponding results are $\text{proj}_1(\mathbb{R}^2) = \mathbb{R}_1$ and $\text{proj}_2(\mathbb{R}^2) = \mathbb{R}_2$, respectively.

The size of a set $\mathbb{X}$ is $|\mathbb{X}|$.

The symbol $\cong$ indicates a homomorphism.

- Analysis:

  The asterisk $z^\dagger$ denotes the complex conjugate of z.

- Matrix:

  The Dirac gamma matrices are γ_0, γ_1, γ_2, and γ_3.

  The Pauli matrices are σ_x, σ_y, and σ_z.

  The dagger $\mathbf{M}^\dagger$ denotes the conjugate transpose of $\mathbf{M}$.

  The commutator is defined as $[\mathbf{M}, \mathbf{P}] : \mathbf{MP} - \mathbf{PM}$, and the anti-commutator is defined as $\{\mathbf{M}, \mathbf{P}\} : \mathbf{MP} + \mathbf{PM}$.

- Geometric algebra:

  The elements of an arbitrary curvilinear geometric basis are denoted as $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ (such that $\mathbf{e}_\nu \cdot \mathbf{e}_\mu = g_{\mu\nu}$), and $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n$ (such that $\hat{\mathbf{x}}_\mu \cdot \hat{\mathbf{x}}_\nu = \eta_{\mu\nu}$) if they are orthonormal.

  A geometric algebra of $m + n$D over field $\mathbb{F}$ is denoted as $\mathcal{G}(\mathbb{F}^{m,n})$.

  The grades of a multivector are denoted as $\langle \mathbf{v} \rangle_k$.

  Specifically, $\langle \mathbf{v} \rangle_0$ is a scalar, $\langle \mathbf{v} \rangle_1$ is a vector, $\langle \mathbf{v} \rangle_2$ is a bivector, $\langle \mathbf{v} \rangle_{n-1}$ is a pseudo-vector, and $\langle \mathbf{v} \rangle_n$ is a pseudo-scalar.

  A scalar and vector such as $\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_1$ form a para-vector; a combination of even grades ($\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_2 + \langle \mathbf{v} \rangle_4 + \ldots$) or odd grades ($\langle \mathbf{v} \rangle_1 + \langle \mathbf{v} \rangle_3 + \ldots$) form even or odd multivectors, respectively.

  Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

  We can formulate a general multivector of $\mathcal{G}(\mathbb{R}^2)$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

  Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set.

  Then, a general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be formulated as $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

2.2 Geometric representation of matrices

2.2.1 Geometric representation in 2D

Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^2)$ is given as

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}, \tag{10}$$

where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

Each multivector has a structure-preserving (addition/multiplication) matrix representation.

Definition 1 (2D geometric representation)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{11}$$

The converse is also true: each 2×2 real matrix is represented as a multivector of $\mathcal{G}(\mathbb{R}^2)$.

In geometric algebra, the determinant[4] of a multivector $\mathbf{u}$ can be defined as

Definition 2 (Geometric representation of the determinant 2D)**.**

$$\begin{aligned} \det \quad : \quad & \mathcal{G}(\mathbb{R}^2) \longrightarrow \mathbb{R} \\ & \mathbf{u} \longmapsto \mathbf{u}^{\ddagger}\mathbf{u}, \end{aligned} \tag{12}$$

where $\mathbf{u}^{\ddagger}$ is

Definition 3 (Clifford conjugate 2D)**.**

$$\mathbf{u}^{\ddagger} := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2. \tag{13}$$

For example,

$$\det \mathbf{u} = (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) \tag{14}$$
$$= a^2 - x^2 - y^2 + b^2 \tag{15}$$
$$= \det \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{16}$$

Finally, we define the Clifford transpose.

Definition 4 (2D Clifford transpose)**.** *The Clifford transpose is the geometric analog to the conjugate transpose, interpreted as a transpose followed by an element-by-element application of the complex conjugate. Thus, the Clifford transpose is a transpose followed by an element-by-element application of the Clifford conjugate.*

$$\begin{bmatrix} \mathbf{u}_{00} & \cdots & \mathbf{u}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \cdots & \mathbf{u}_{mn} \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{u}_{00}^{\ddagger} & \cdots & \mathbf{u}_{m0}^{\ddagger} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \cdots & \mathbf{u}_{nm}^{\ddagger} \end{bmatrix} \tag{17}$$

If applied to a vector, then

$$\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{v}_1^{\ddagger} & \dots \mathbf{v}_m^{\ddagger} \end{bmatrix} \tag{18}$$

2.2.2 Geometric representation in 3+1D

Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be written as

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}, \tag{19}$$

where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

Similarly, each multivector has a structure-preserving (addition/multiplication) matrix representation.

The multivectors of $\mathcal{G}(\mathbb{R}^{3,1})$ are represented as follows:

Definition 5 (4D geometric representation)**.**

$$\begin{aligned} & a + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\ & \quad + f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 + f_{13}\gamma_1 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 \\ & \quad + v_t\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + v_x\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v_y\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + v_z\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\ & \quad + b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \end{aligned}$$

$$\cong \begin{bmatrix} a + x_0 - if_{12} - iv_3 & f_{13} - if_{23} + v_2 - iv_1 & -ib + x_3 + f_{03} - iv_0 & x_1 - ix_2 + f_{01} - if_{02} \\ -f_{13} - if_{23} - v_2 - iv_1 & a + x_0 + if_{12} + iv_3 & x_1 + ix_2 + f_{01} + if_{02} & -ib - x_3 - f_{03} - iv_0 \\ -ib - x_3 + f_{03} + iv_0 & -x_1 + ix_2 + f_{01} - if_{02} & a - x_0 - if_{12} + iv_3 & f_{13} - if_{23} - v_2 + iv_1 \\ -x_1 - ix_2 + f_{01} + if_{02} & -ib + x_3 - f_{03} + iv_0 & -f_{13} - if_{23} + v_2 + iv_1 & a - x_0 + if_{12} - iv_3 \end{bmatrix} \tag{20}$$

In this case, the converse is not true; that is, only a subset of a 4×4 complex matrices, namely, whose determinant is real-valued, can be represented as a multivector of $\mathcal{G}(\mathbb{R}^{3,1})$.

In 3+1D, we define the determinant solely using the constructs of geometric algebra[4].

The determinant of $\mathbf{u}$ is

Definition 6 (3+1D geometric representation of determinant)**.**

$$\det \quad : \quad \mathcal{G}(\mathbb{R}^{3,1}) \longrightarrow \mathbb{R} \tag{21}$$

$$\mathbf{u} \longmapsto \lfloor \mathbf{u}^{\ddagger}\mathbf{u} \rfloor_{3,4} \mathbf{u}^{\ddagger}\mathbf{u}, \tag{22}$$

where $\mathbf{u}^\ddagger$ is

Definition 7 (3+1D Clifford conjugate)**.**

$$\mathbf{u}^\ddagger \coloneqq \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 + \langle \mathbf{u} \rangle_3 + \langle \mathbf{u} \rangle_4, \tag{23}$$

and where $\lfloor \mathbf{u} \rfloor_{\{3,4\}}$ is the blade-conjugate of degrees three and four (the plus sign is reversed to a minus sign for blades 3 and 4)

$$\lfloor \mathbf{u} \rfloor_{\{3,4\}} \coloneqq \langle \mathbf{u} \rangle_0 + \langle \mathbf{u} \rangle_1 + \langle \mathbf{u} \rangle_2 - \langle \mathbf{u} \rangle_3 - \langle \mathbf{u} \rangle_4. \tag{24}$$

2.3 Measurement constraints

Definition 8 (The linear measurement constraint)**.** *Let $\mathbf{M}$ be an $n \times n$ matrix and $\mathbb{Q}$ be a statistical ensemble. The linear measurement constraint is*

$$\operatorname{tr} \overline{\mathbf{M}} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q), \tag{25}$$

$\operatorname{tr} \overline{\mathbf{M}}$ *denotes the expectation eigenvalue of the statistically weighted sum of matrices $\mathbf{M}(q)$, parameterized over ensemble $\mathbb{Q}$.*

Definition 9 (The geometric measurement constraint)**.** *Let $\mathbf{u}$ be a multivector of $\mathcal{G}(\mathbb{R}^{m,n})$ and $\mathbb{Q}$ be a statistical ensemble. The geometric mreasurement constraint is*

$$\operatorname{tr} \overline{\mathbf{u}} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{u}(q), \tag{26}$$

$\operatorname{tr} \overline{\mathbf{u}}$ *denotes the expectation eigenvalue of the statistically weighted sum of multivectors $\mathbf{u}(q)$, parameterized over ensemble $\mathbb{Q}$.*

3 Result

3.1 Non-relativistic quantum mechanics

In this subsection, which serves as an introductory example, we recover non-relativistic quantum mechanics using the Lagrange multiplier method and a linear constraint on the entropy.

As previously mentioned, the Shannon entropy (in base e) is applied instead of the Boltzmann entropy to achieve the aforementioned goal.

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \tag{27}$$

In statistical mechanics, we use scalar measurement constraints on the entropy, such as energy and volume meters, which are sufficient for recovering the Gibbs ensemble. However, applying such scalar measurement constraints is insufficient to recover quantum mechanics.

A *complex measurement constraint*, an invariant for a complex phase, is used to overcome this limitation. It is defined as

$$\operatorname{tr}\begin{bmatrix}0 & -\overline{b}\\ \overline{b} & 0\end{bmatrix} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0,\end{bmatrix} \tag{28}$$

where $\begin{bmatrix}a(q) & -b(q)\\ b(q) & a(q)\end{bmatrix} \cong a(q) + ib(q)$ is the matrix representation of the complex numbers.

Similar to energy or volume meters, linear instruments produce a sequence of measurements that converge to an expectation value but with a phase invariance. In our framework, this phase invariance originates from the trace.

The Lagrangian equation that maximizes the entropy subject to the complex measurement constraint is

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q)\ln(q) + \alpha\left(1 - \sum_{q\in\mathbb{Q}}\rho(q)\right) + \tau\left(\operatorname{tr}\begin{bmatrix}0 & -\overline{b}\\ \overline{b} & 0\end{bmatrix} - \sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right) \tag{29}$$

This equation is maximized for ρ by imposing the condition $\frac{\partial\mathcal{L}}{\partial\rho(q)} = 0$. The following results are obtained:

$$\frac{\partial\mathcal{L}}{\partial\rho(q)} = -\ln\rho(q) - 1 - \alpha - \tau\operatorname{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix} \tag{30}$$

$$0 = \ln\rho(q) + 1 + \alpha + \tau\operatorname{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix} \tag{31}$$

$$\implies \ln\rho(q) = -1 - \alpha - \tau\operatorname{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix} \tag{32}$$

$$\implies \rho(q) = \exp(-1-\alpha)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right) \tag{33}$$

$$= \frac{1}{Z(\tau)}\det\exp\left(-\tau\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right), \tag{34}$$

where $Z(\tau)$ is obtained as

$$1 = \sum_{q\in\mathbb{Q}} \exp(-1-\alpha)\exp\left(-\tau\,\mathrm{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right) \tag{35}$$

$$\implies \big(\exp(-1-\alpha)\big)^{-1} = \sum_{q\in\mathbb{Q}} \exp\left(-\tau\,\mathrm{tr}\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right) \tag{36}$$

$$Z(\tau) \coloneqq \sum_{q\in\mathbb{Q}} \det\exp\left(-\tau\begin{bmatrix}0 & -b(q)\\ b(q) & 0.\end{bmatrix}\right) \tag{37}$$

The exponential of the trace is equal to the determinant of the exponential according to the relation $\det\exp\mathbf{A} \equiv \exp\mathrm{tr}\,\mathbf{A}$.

Finally, we obtain

$$\rho(\tau,q) = \frac{1}{Z(\tau)}\det\exp\left(-\tau\begin{bmatrix}0 & -b(q)\\ b(q) & 0\end{bmatrix}\right) \tag{38}$$

$$\cong |\exp -i\tau b(q)|^2 \qquad \text{Born rule} \tag{39}$$

Renaming $\tau \to t/\hbar$ and $b(q) \to H(q)$ recovers the familiar form of

$$\rho(q) = \frac{1}{Z}\Big|\exp\big(-itH(q)/\hbar\big)\Big|^2. \tag{40}$$

or

$$\rho(q) = \frac{1}{Z}\big|\psi(q)\big|^2, \text{ where } \psi(q) = \exp\big(-itH(q)/\hbar\big). \tag{41}$$

Thus, we can show that all three Dirac Von–Neumann axioms and the Born rule are satisfied, revealing a possible origin of quantum mechanics as the solution to an optimization problem on entropy and linear measurements.

From (41), we can identify the wavefunction as a vector of some orthogonal space (here, a complex Hilbert space) and partition function as its inner product, expressed as

$$Z = \langle\psi|\psi\rangle\,. \tag{42}$$

As the result is automatically normalized by the entropy-maximization procedure, the physical states associates to the unit vectors, and the probability of any particular state is given by

$$\rho(q) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q))^\dagger\psi(q). \tag{43}$$

Finally, any self-adjoint matrix, defined as $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$, will correspond to a real-valued statistical mechanics observable, if measured in its eigenbasis, thereby completing the equivalence.

3.2 Probability measure of all linear measurements

Here, we use the linear measurement constraint in its full generality:

$$\operatorname{tr}\overline{\mathbf{M}} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\mathbf{M}(q), \tag{44}$$

where $\mathbf{M}(q)$ are arbitrary $n \times n$ matrices.

Notably, multivectors can be used instead of matrices. The derivation would remain the same.

The Lagrange equation used to maximize the entropy under this constraint is expressed as

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q)\ln(q) + \alpha\left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau\left(\operatorname{tr}\overline{\mathbf{M}} - \sum_{q\in\mathbb{Q}} \rho(q)\operatorname{tr}\mathbf{M}(q)\right), \tag{45}$$

where α and τ are the Lagrange multipliers.

Similarly, we maximize Equation (45) for ρ using the criterion $\frac{\partial\mathcal{L}}{\partial\rho(q)} = 0$ as follows:

$$\frac{\partial\mathcal{L}}{\partial\rho(q)} = -\ln\rho(q) - 1 - \alpha - \tau\operatorname{tr}\mathbf{M}(q) \tag{46}$$

$$0 = \ln\rho(q) + 1 + \alpha + \tau\operatorname{tr}\mathbf{M}(q) \tag{47}$$

$$\implies \ln\rho(q) = -1 - \alpha - \tau\operatorname{tr}\mathbf{M}(q) \tag{48}$$

$$\implies \rho(q) = \exp(-1-\alpha)\exp\bigl(-\tau\operatorname{tr}\mathbf{M}(q)\bigr) \tag{49}$$

$$= \frac{1}{Z(\tau)}\det\exp\bigl(-\tau\mathbf{M}(q)\bigr) \tag{50}$$

where $Z(\tau)$ is obtained as

$$1 = \sum_{q\in\mathbb{Q}} \exp(-1-\alpha)\exp\bigl(-\tau\operatorname{tr}\mathbf{M}(q)\bigr) \tag{51}$$

$$\implies \bigl(\exp(-1-\alpha)\bigr)^{-1} = \sum_{q\in\mathbb{Q}} \exp\bigl(-\tau\operatorname{tr}\mathbf{M}(q)\bigr) \tag{52}$$

$$Z(\tau) \coloneqq \sum_{q\in\mathbb{Q}} \det\exp\bigl(-\tau\mathbf{M}(q)\bigr) \tag{53}$$

The resulting probability measure is

$$\rho(q,\tau) = \frac{1}{Z(\tau)}\det\exp\bigl(-\tau\mathbf{M}(q)\bigr), \tag{54}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} \det \exp\big(-\tau \mathbf{M}(q)\big). \tag{55}$$

By defining $\psi(q,\tau) := \exp\big(-\tau \mathbf{M}(q)\big)$, we can express $\rho(q,\tau) = \det \psi(q,\tau)$, where the determinant acts as a "generalized Born rule," connecting, in this case, a general linear amplitude to a real-valued probability.

The sophistication of the general linear amplitude and determinant acting as a "generalized Born rule" will provide a platform to support fundamental physics.

Finally, let us remark the existence of a more general case where a Lagrange multiplier is assigned to each independent entry of matrix $\mathbf{M}(q)$. Then,

$$\rho(q,\tau) = \frac{1}{Z(\tau)} \det \exp\big(-\boldsymbol{\tau} \cdot \mathbf{M}(q)\big), \tag{56}$$

where $\boldsymbol{\tau}$ is an $n \times n$ matrix. The $\cdot$ operator assigns the first element of $\boldsymbol{\tau}$ to the first element of $\mathbf{M}(q)$, and so forth.

4 Analysis

This section analyses the main result as a general linear quantum theory. We introduce the *algebra of geometric observables* applicable to the general linear wavefunction.

An algebra of observables is introduced. The 2D definition of the algebra constitutes a special case reminiscent of the definitions of ordinary quantum mechanics yet includes gravity. The 3+1D case is significantly more sophisticated than the 2D case and is elucidated immediately after the 2D case analysis.

4.1 Axiomatic definition of the algebra in 2D

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^2)$.

A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

A) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the sesquilinear map

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \quad & \mathbb{V} \times \mathbb{V} \longrightarrow \mathcal{G}(\mathbb{R}^2) \\ & \langle \mathbf{u}, \mathbf{v} \rangle \longmapsto \mathbf{u}^\ddagger \mathbf{v} \end{aligned} \tag{57}$$

is positive-definite for $\boldsymbol{\psi}$, such that $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

B) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$. Then, for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q), \boldsymbol{\psi}) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \psi(q)^{\ddagger} \psi(q) \tag{58}$$

is positive-definite: $\rho(\psi(q), \boldsymbol{\psi}) > 0$

We note the following comments and definitions:

- From A) and B), it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum up to unity:

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{59}$$

- $\boldsymbol{\psi}$ is called a *natural* (or physical) state.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(q, \boldsymbol{\psi})$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$, such that the sum of probabilities remains normalized.

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \mathbf{T}\boldsymbol{\psi}) = \sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q), \boldsymbol{\psi}) = 1 \tag{60}$$

are the *natural* transformations of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \in \mathbb{V}$ and $\forall \mathbf{v} \in \mathbb{V}$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{v} \rangle \tag{61}$$

is called an observable.

- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{62}$$

4.2 Geometric self-adjoint operator in 2D

The general case of an observable in 2D is shown in this section. A matrix $\mathbf{O}$ is observable if it is a self-adjoint operator defined as

$$\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle \tag{63}$$

$\forall \boldsymbol{\phi} \in \mathbb{V}$ and $\forall \boldsymbol{\psi} \in \mathbb{V}$.

Setup: Let $\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix}$ be an observable.

Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be two two-state multivectors $\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix}$ and $\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}$. Here, the components $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, $\mathbf{o}_{11}$ are multivectors of $\mathcal{G}(\mathbb{R}^2)$.

Derivation: 1. Calculate $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle &= (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2)^{\ddagger}\boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^{\ddagger}(\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2) \\ &\quad + (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2)^{\ddagger}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^{\ddagger}(\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2) && (64) \\ &= \boldsymbol{\phi_1}^{\ddagger}\mathbf{o}_{00}^{\ddagger}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{01}^{\ddagger}\boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{00}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{01}\boldsymbol{\phi}_2 \\ &\quad + \boldsymbol{\phi}_1^{\ddagger}\mathbf{o}_{10}^{\ddagger}\boldsymbol{\psi}_2 + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{11}^{\ddagger}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{10}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{11}\boldsymbol{\phi}_2 && (65) \end{aligned}$$

2. Next, calculate $\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$:

$$\begin{aligned} 2\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle &= \boldsymbol{\phi}_1^{\ddagger}(\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2) + (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2)^{\ddagger}\boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^{\ddagger}(\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2) + (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2)^{\ddagger}\boldsymbol{\phi}_1 && (66) \\ &= \boldsymbol{\phi}_1^{\ddagger}\mathbf{o}_{00}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_1^{\ddagger}\mathbf{o}_{01}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{00}^{\ddagger}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{01}^{\ddagger}\boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{10}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{11}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{10}^{\ddagger}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{11}^{\ddagger}\boldsymbol{\phi}_1 && (67) \end{aligned}$$

To realize $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$, the following relations must hold:

$$\mathbf{o}_{00}^{\ddagger} = \mathbf{o}_{00} \tag{68}$$
$$\mathbf{o}_{01}^{\ddagger} = \mathbf{o}_{10} \tag{69}$$
$$\mathbf{o}_{10}^{\ddagger} = \mathbf{o}_{01} \tag{70}$$
$$\mathbf{o}_{11}^{\ddagger} = \mathbf{o}_{11}. \tag{71}$$

Therefore, $\mathbf{O}$ must be equal to its own Clifford transpose, indicating that $\mathbf{O}$ is an observable if

$$\mathbf{O}^{\ddagger} = \mathbf{O}, \tag{72}$$

which is the geometric generalization of the self-adjoint operator $\mathbf{O}^\dagger = \mathbf{O}$ of complex Hilbert spaces.

4.3 Geometric spectral theorem in 2D

The application of the spectral theorem to $\mathbf{O}^\ddagger = \mathbf{O}$ such that its eigenvalues are real is shown below:

Consider

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}, \tag{73}$$

Then $\mathbf{O}^\ddagger$ is

$$\mathbf{O}^\ddagger = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}, \tag{74}$$

It follows that $\mathbf{O}^\ddagger = \mathbf{O}$

This example is the most general 2×2 matrix $\mathbf{O}$ such that $\mathbf{O}^\ddagger = \mathbf{O}$.

The eigenvalues are obtained as

$$0 = \det(\mathbf{O} - \lambda \mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} - \lambda \end{bmatrix}, \tag{75}$$

This implies that

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12})(a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} + a_{11}) \tag{76}$$

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2), \tag{77}$$

Finally,

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{78}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\} \tag{79}$$

Notably, where $a_{00} - a_{11} = 0$, the roots would be complex if $a^2 - x^2 - y^2 + b^2 < 0$. Is this always the case? Note that the determinant of a 2D multivector must be greater than zero because of the exponential mapping to the orientation-preserving general linear group:

$$\exp \mathcal{G}(\mathbb{R}^{m,n}) \to \mathrm{GL}^+(n, \mathbb{R}) \tag{80}$$

Therefore, in this case, $a^2 - x^2 - y^2 + b^2 > 0$, which is the determinant of the multivector.

Consequently, under the orientation-preserving transformations, $\mathbf{O}^\ddagger = \mathbf{O}$ constitutes an observable with real-valued eigenvalues.

4.4 Left action in 2D

A left action on the wavefunction $\mathbf{T}\left|\psi\right\rangle$ connects to the bilinear form as $\left\langle\psi\right|\mathbf{T}^\ddagger\mathbf{T}\left|\psi\right\rangle$.

The invariance requirement on $\mathbf{T}$ is

$$\left\langle\psi\right|\mathbf{T}^\ddagger\mathbf{T}\left|\psi\right\rangle = \left\langle\psi|\psi\right\rangle . \tag{81}$$

Therefore, we are interested in the group of matrices that follow

$$\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}. \tag{82}$$

Let us consider a two-state system, with a general transformation represented by

$$\mathbf{T} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}, \tag{83}$$

where u, v, w, x are the 2D multivectors.

The expression $\mathbf{T}^\ddagger\mathbf{T}$ is

$$\mathbf{T}^\ddagger\mathbf{T} = \begin{bmatrix} v^\ddagger & u^\ddagger \\ w^\ddagger & x^\ddagger \end{bmatrix} \begin{bmatrix} v & w \\ u & x \end{bmatrix} = \begin{bmatrix} v^\ddagger v + u^\ddagger u & v^\ddagger w + u^\ddagger x \\ w^\ddagger v + x^\ddagger u & w^\ddagger w + x^\ddagger x \end{bmatrix} \tag{84}$$

For $\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}$, the following relations must hold:

$$v^\ddagger v + u^\ddagger u = 1 \tag{85}$$

$$v^\ddagger w + u^\ddagger x = 0 \tag{86}$$

$$w^\ddagger v + x^\ddagger u = 0 \tag{87}$$

$$w^\ddagger w + x^\ddagger x = 1 \tag{88}$$

This is the case if

$$\mathbf{T} = \frac{1}{\sqrt{v^\ddagger v + u^\ddagger u}} \begin{bmatrix} v & u \\ -e^\varphi u^\ddagger & e^\varphi v^\ddagger \end{bmatrix}, \tag{89}$$

where u, v are the 2D multivectors, and e^φ is a unit multivector.

Comparatively, the unitary case is obtained when the vector part of the multivector vanishes, i.e., $\mathbf{x} \to 0$, and we obtain

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta}b^\dagger & e^{i\theta}a^\dagger \end{bmatrix}. \tag{90}$$

Here $\mathbf{T}$ is the geometric generalization of unitary transformations.

4.5 Schrödinger equation in $\mathcal{G}(\mathbb{R}^2)$

First, the standard Schrödinger equation can be derived as follows.

In the bra-ket notation, we recall that a one-parameter group evolves as follows:

$$\exp(-it\mathbf{H}) \left|\psi(0)\right\rangle = \left|\psi(t)\right\rangle. \tag{91}$$

Thus, an infinitesimal displacement of t is obtained as follows:

$$\exp(-i\delta t\mathbf{H}) \left|\psi(\tau)\right\rangle = \left|\psi(\tau + \delta\tau)\right\rangle. \tag{92}$$

Now, we approximate the exponential into a power series as

$$\exp(-i\delta t\mathbf{H}) \left|\psi(\tau)\right\rangle \approx 1 - i\delta t\mathbf{H} \left|\psi(t)\right\rangle. \tag{93}$$

The process is continued as follows:

$$(1 - i\delta t\mathbf{H}) \left|\psi(t)\right\rangle = \left|\psi(t + \delta t)\right\rangle \tag{94}$$

$$\left|\psi(\tau)\right\rangle - i\delta t\mathbf{H} \left|\psi(t)\right\rangle = \left|\psi(t + \delta t)\right\rangle \tag{95}$$

$$-i\delta t\mathbf{H} \left|\psi(t)\right\rangle = \left|\psi(t + \delta t)\right\rangle - \left|\psi(t)\right\rangle \tag{96}$$

$$-i\mathbf{H} \left|\psi(t)\right\rangle = \frac{\left|\psi(t + \delta t)\right\rangle - \left|\psi(t)\right\rangle}{\delta t} \tag{97}$$

$$-i\mathbf{H} \left|\psi(t)\right\rangle = \frac{d\left|\psi(t)\right\rangle}{dt}. \tag{98}$$

which is the Schrödinger equation.

Returning to our result, we begin by eliminating the elements of $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ by posing $a \to 0, \mathbf{x} \to 0$:

$$\mathbf{u}\mid_{a\to 0,\mathbf{x}\to 0} = \mathbf{b} = \mathbf{i}b \tag{99}$$

This reduces $\mathrm{GL}^+(2, \mathbb{R})$ to $\mathrm{SO}(2)$.

Then, the left action matrix $\mathbf{T}$ becomes valued in $\langle\mathcal{G}(\mathbb{R}^2)\rangle_4$, and the Stone theorem on one-parameter groups applies. Consequently, we obtain

$$\mathbf{T}(\tau)\mid_{a\to 0,\mathbf{x}\to 0}= \exp(\mathbf{i}\tau\mathbf{O}) \tag{100}$$

where

$$(\mathbf{O}^\ddagger = \mathbf{O})\mid_{a\to 0,\mathbf{x}\to 0} \implies \mathbf{O}^\dagger = \mathbf{O} \tag{101}$$

The end result is mathematically similar to the Schrödinger equation (98):

$$-\mathbf{i}\mathbf{O}\left|\psi(\tau)\right\rangle = \frac{d\left|\psi(\tau)\right\rangle}{d\tau}, \tag{102}$$

and the wavefunction is $\psi(\tau) = \exp(-\tau\mathbf{i}\mathbf{O})$

Compared to the Schrödinger equation, here $\mathbf{i}$ is not an imaginary unit but a rotor in 2D. We recall that $\mathbf{i} = \hat{\mathbf{x}}_1\hat{\mathbf{x}}_2$ and that rotors $R = \exp\left(\frac{1}{2}\theta\mathbf{i}\right)$ are exponentials of bivectors.

We thus arrived at a quantum theory of geometry, visualized as follows:

$$\psi^\ddagger(\tau)\hat{\mathbf{x}}_0\psi(\tau) = \exp(\tau\mathbf{i}\mathbf{B})\hat{\mathbf{x}}_0\exp(-\tau\mathbf{i}\mathbf{B}) \tag{103}$$

$$= \exp(\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1\mathbf{B})\hat{\mathbf{x}}_0\exp(-\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1\mathbf{B}) \tag{104}$$

The expression $\exp(\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1\mathbf{B})\hat{\mathbf{x}}_0\exp(-\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1\mathbf{B})$ maps $\hat{\mathbf{x}}_0$ to a curvilinear basis $\mathbf{e}_0$ via the application of the rotor and its reverse:

$$\exp(\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1\mathbf{B})\hat{\mathbf{x}}_0\exp(-\tau\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1\mathbf{B}) = \mathbf{e}_0(\tau) \tag{105}$$

Here, we eliminated certain elements of $\mathrm{GL}^+(2,\mathbb{R})$ reducing it to $\mathrm{SO}(2)$, and found that the resulting geometry-valued Schrödinger equation is invariant in the $\mathrm{SO}(2)$ group.

4.6 Gravity in 2D (Sketch)

Roger Penrose argued "that the case for gravitizing quantum theory is at least as strong as that for quantizing gravity"[5].

We stress that a theory that would succeed at gravitizing the quantum does not need to quantize gravity (and vice-versa). Indeed, it seems reasonable to expect any consistent singular theory to be at most either, but not both.

Gravitizing the quantum (rather than quantizing gravity) is the approach of this study. Indeed, we made no changes to general relativity. Instead, our entropy maximization process produced a wavefunction valued in the orientation-preserving general linear group, whose geometric flexibility exceeds the familiar unitary wavefunction. It is within this extra flexibility that we will find gravity.

In the previous result, we bluntly eliminated elements $a \to 0$ and $\mathbf{x} \to 0$ of the group $\mathrm{GL}^+(2,\mathbb{R})$, reducing it to SO(2). How important are the eliminated terms? What if instead of eliminating them, we perform a structure reduction thus, recovering the SO(2) group as before, but also the space resulting from a quotient bundle?

Let us investigate.

Let X^2 be a smooth orientable real-valued manifold in 2D. We consider its tangent bundle TX and its associated frame bundle FX. Since X^2 is orientable, it possesses the structure group $\mathrm{GL}^+(2,\mathbb{R})$, which associates to a linear action by our wavefunction on the frame bundle of X^2. Our model assigns a quantum/statistical character to actions by said structure group on FX.

The structure group in $\mathrm{GL}^+(2,\mathbb{R})$ of FX can always be reduced to SO(2) yielding the geometric quantum theory of rotations identified in the previous section, but also yielding the global section of the quotient bundle FX/SO(2) which is a Riemmanian metric on X^2. The connection that preserves the structure SO(2) across the manifold is the Levi-Civita connection.

The frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^2[6].

We stress that the gravitized quantum theory holds before symmetry breaking (in the $\mathrm{GL}^+(2,\mathbb{R})$ group) and after symmetry breaking into theory of gravity part (General Relativity) and a theory of local quantum rotations (in SO(2)) part.

4.7 Gravity in 2D (another take)

David Hestenes [7] has formulated the wavefunction in the language of geometric algebra in 3+1D (we will introduce his notation in more detail in section 4.13).

The geometric formulation of the wavefunction in 2D, consistent with is formulation, is

$$\psi = \sqrt{\rho}\exp(\mathbf{i}b) \tag{106}$$

such that

$$\psi\psi^{\ddagger} = \sqrt{\rho}\exp(\mathbf{i}b)\sqrt{\rho}\exp(-\mathbf{i}b) = \rho \tag{107}$$

It is obtained from our formalism by eliminating $\mathbf{x}$ from $\mathbf{u}$ by posing $\mathbf{x} \to 0$. Thus, $\mathbf{u}\mid_{\mathbf{x}\to 0} = a + \mathbf{b}$.

The gravitational theory, in this case, would follow from this structure reduction $\mathrm{GL}^+(2,\mathbb{R})/(\mathbb{R}^+ \times \mathrm{SO}(2))$, yielding the Weyl connection to preserve this structure instead of the Levi-Civita connection.

Here, ρ can be seen as the prior (or initial) probability, and the Weyl connection preserves the weight of this prior (in addition to the rotation group) along the manifold.

4.8 Algebra of geometric observables in 3+1D

In this section, the general case in 3+1D is presented.

In 2D, the determinant can be expressed using only the product $\psi^\ddagger\psi$, which can be interpreted as the inner product of two vectors. This form allowed us to extend the complex Hilbert space to a *geometric Hilbert space*. We then found that the familiar properties of the complex Hilbert spaces were transferable to the geometric Hilbert space, eventually yielding a 2D gravitized quantum theory in the language of geometric algebra.

Although a similar correspondence exists in 3+1D, it is less recognizable because we need a *quartic-inner-product* (i.e., $\rho = \lfloor\phi^\ddagger\phi\rfloor_{3,4}\phi^\ddagger\phi$) to produce a real-valued probability in 3+1D.

Thus, in 3+1D, we cannot produce an inner product as in the 2D case. The absence of a satisfactory inner product indicates no Hilbert space in the usual sense of a complete *inner product* vector space.

We aim to find a construction that supports the general linear wavefunction in 3+1D.

To build the right construction, a quartic-inner-product of four terms is devised, replacing the inner product in the Hilbert space, mapping any four vectors to an element of $\mathcal{G}(\mathbb{R}^{3,1})$, and yielding a complete *quartic-inner-product* vector space.

We note that despite this modification, the familiar quantum mechanical features (linear transformations, unit vectors, observables as matrix or operators, and linear superposition in the probability measure, etc.) will be supported in the construction.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$.

A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall\boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, the quartic-inner-product form

$$\begin{aligned} \langle\cdot,\cdot,\cdot,\cdot\rangle \quad : \qquad \mathbb{V}\times\mathbb{V}\times\mathbb{V}\times\mathbb{V} &\longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ \langle\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}\rangle &\longmapsto \sum_{i=1}^{m}\lfloor u_i^\ddagger v_i\rfloor_{3,4} w_i^\ddagger z_i \end{aligned} \tag{108}$$

   is positive-definite when $\mathbf{u}=\mathbf{v}=\mathbf{w}=\mathbf{z}$; that is $\langle\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle > 0$

2. $\forall\boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\phi}$, the function

$$\rho(\psi(q),\boldsymbol{\phi}) = \frac{1}{\langle\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle}\det\phi(q), \tag{109}$$

   is positive-definite: $\rho(\phi(q),\boldsymbol{\phi}) > 0$

We note the following properties, features, and comments:

- From A) and B), it follows that, $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, and the probabilities sum to unity.

$$\sum_{\phi(q)\in\boldsymbol{\phi}} \rho(\phi(q), \boldsymbol{\phi}) = 1 \tag{110}$$

- $\boldsymbol{\phi}$ is called a *natural* (or physical) state.
- $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle$ is called the *partition function* of $\boldsymbol{\phi}$.
- If $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle = 1$, then $\boldsymbol{\phi}$ is called a unit vector.
- $\rho(\phi(q), \boldsymbol{\phi})$ is called the *probability measure* (or generalized Born rule) of $\phi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\phi}$ such as $\mathbf{T}\boldsymbol{\phi} \to \boldsymbol{\phi}'$ makes the sum of probabilities normalized (invariant):

$$\sum_{\phi(q)\in\boldsymbol{\phi}} \rho(\phi(q), \mathbf{T}\boldsymbol{\phi}) = \sum_{\phi(q)\in\boldsymbol{\phi}} \rho(\phi(q), \boldsymbol{\phi}) = 1 \tag{111}$$

  are the *natural* transformations of $\boldsymbol{\phi}$.
- A matrix $\mathbf{O}$ such that $\forall \mathbf{u}\forall \mathbf{v}\forall \mathbf{w}\forall \mathbf{z} \in \mathbb{V}$:

$$\langle \mathbf{Ou}, \mathbf{v}, \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{Ov}, \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{v}, \mathbf{Ow}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{Oz} \rangle \tag{112}$$

  is called an observable.
- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle}{\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle} \tag{113}$$

4.9 Geometric observables in 3+1D

In 4D, an observable must satisfy equation 112. For simplicity, let us take m in equation 108 to be 1. Then,

$$\lfloor (\mathbf{O}u)^{\ddagger} v \rfloor_{3,4} w^{\ddagger} z = \lfloor u^{\ddagger} \mathbf{O} v \rfloor_{3,4} w^{\ddagger} z = \lfloor u^{\ddagger} v \rfloor_{3,4} (\mathbf{O}w)^{\ddagger} z = \lfloor u^{\ddagger} v \rfloor_{3,4} w^{\ddagger} \mathbf{O} z \tag{114}$$

where u_1, v_1, w_1 and z_1 are multivectors.

Let us investigate.

If $\mathbf{O}$ contained a vector, bivector, pseudo-vector, or pseudo-scalar, the equality would not be satisfied as these terms do not commune with the equality multivectors and cannot be factored out. The equality is satisfied if $\mathbf{O} \in \mathbb{R}$. Indeed, as a real value, $\mathbf{O}$ commutes with all multivectors of equality and can be factored out to satisfy the equality.

We thus find that the observables are real-valued in the general 3+1D case.

At first, this may seem restrictive; comparatively, the observables in the 2D case were geometrically-valued $\mathbf{O}^\ddagger = \mathbf{O}$ and not merely real-valued. However, the geometric expressivity of the observables in 3+1D expands when reducing the structure (see Section 4.13).

We now identify the invariant transformations of probability measures:

$$\lfloor (\mathbf{T}u)^\ddagger \mathbf{T}v \rfloor_{3,4} (\mathbf{T}w)^\ddagger \mathbf{T}z = \lfloor u^\ddagger v \rfloor_{3,4} w^\ddagger z \tag{115}$$

$$\implies \lfloor u^\ddagger \mathbf{T}^\ddagger \mathbf{T} v \rfloor_{3,4} w^\ddagger \mathbf{T}^\ddagger \mathbf{T} z = \lfloor u^\ddagger v \rfloor_{3,4} w^\ddagger z \tag{116}$$

The measure is invariant when

1. $\mathbf{T}^\ddagger \mathbf{T} = \mathbf{I}$
2. $\mathbf{T}^\ddagger \mathbf{T} \in \mathbb{C}^{n\times n}$ and $(\mathbf{T}^\ddagger \mathbf{T})^\dagger \mathbf{T}^\ddagger \mathbf{T} = \mathbf{I}$ and $\mathbf{x} \to 0, \mathbf{v} \to 0$.
3. $\mathbf{T} \in \mathbb{C}^{n\times n}$ and $\mathbf{T}^\dagger \mathbf{T} = \mathbf{I}$ and $\mathbf{x} \to 0, \mathbf{v} \to 0$.

4.10 Geometric observables in 6D

Let us open a small parenthesis and investigates what happens in higher dimensions.

First, let us recap.

The observables in 3+1D must satisfy a more constraining equality relation than in 2D. This reduced the geometric expressivity that such observables could support. Specifically, in 2D the relation was satisfied for $\mathbf{O}^\ddagger = \mathbf{O}$ capturing the full general linear geometry, but was reduced to $\mathbf{O} \in \mathbb{R}$ in 3+1D, which is a tiny subset of the available geometry.

What happens if we increase the dimensions even further to 6 and above?

At dimensions of 6 or above, the corresponding observable relation cannot be satisfied. To see why, we look at the results[8] of Acus et al. regarding the 6-D multivector norm. They performed an exhaustive computer-assisted search for the geometric algebra expression for the determinant in 6D; as conjectured, they found no norm defined via self-products. However, the norm found was a linear combination of self-products.

The system of linear equations is too long to list in its entirety; the author gives this mockup:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{117}$$
$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{118}$$
$$\langle 74 \text{ monomials} \rangle = 0 \tag{119}$$
$$\langle 74 \text{ monomials} \rangle = 0 \tag{120}$$

The author then produces the special case of this norm that holds only for a 6D multivector comprising a scalar and grade 4 element:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{121}$$

Even in this simplified special case, formulating a linear relationship for observables is doomed to fail. Indeed, even the real portion of the observable cannot be extracted from the equation. We find that for any function f_i and g_i, the coefficient b_1 and b_2 will frustrate the equality:

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{122}$$
$$= b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{123}$$

Equations 122 and 123 can only be equal if $b_1 = b_2$; however, the norm $s(B)$ requires both to be different. Consequently, the relation for observables in 6D is unsatisfiable even by real numbers.

Thus, in our framework, the 6D geometry leads to the absence of observables. Since the norms involve more sophisticated systems of linear equations at higher dimensions, this result is likely to generalize to all dimensions above 6.

4.11 Defective probability measure in 3D and 5D

We can also rule out the 3D and 5D cases because the probability measure in these dimensions is not real but complex-valued, making them defective.

In $\mathcal{G}(\mathbb{R}^3)$, the matrix representation of a multivector

$$\mathbf{u} = a + x\sigma_x + y\sigma_y + z\sigma_z + q\sigma_y\sigma_z + v\sigma_x\sigma_z + w\sigma_x\sigma_y + b\sigma_x\sigma_y\sigma_z \tag{124}$$

is

$$\mathbf{u} \cong \begin{bmatrix} a + ib + iw + z & iq - v + x - iy \\ iq + v + x + iy & a + ib - iw - z \end{bmatrix} \tag{125}$$

and the determinant is

$$\det \mathbf{u} = a^2 - b^2 + q^2 + v^2 + w^2 - x^2 - y^2 - z^2 + 2i(ab - qx + vy - wz) \tag{126}$$

The result is a complex-valued probability. Since a probability must be real-valued, the 3D case is defective in our framework and cannot be used.

In $\mathcal{G}(\mathbb{R}^{4,1})$, the algebra is isomorphic to $\mathcal{G}(\mathbb{C}^{3,1})$ and to complex 4×4 matrices. In this case also the determinant and probability would be complex-valued, making the case defective.

4.12 Specialness of 4D

Our framework is non-defective only in the following dimensions:

- 0D: corresponds to the familiar (and classical) statistical mechanics. The constraints are scalar $\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q)$, and the probability measure is the Gibbs measure.
- 1D: the non-relativistic quantum mechanical case we recovered in the results section, using the matrix representation of the complex numbers.
- 2D: the geometric quantum theory discussed earlier.
- 4D: the case investigated in this subsection.

In contrast, our framework is defective in the following dimensions:

- 3D and 5D: the probability measure is complex-valued.
- 6D and above: no observables satisfy the corresponding observable equation, in general.

In our model, normalizable observables cease to exist beyond 4D. Consequently, 4D is simply the largest spacetime that captures all observable geometry.

4.13 Wavefunction

We now return to 4D.

In the David Hestenes' notation[7], the 3+1D wavefunction is expressed as

$$\psi = \sqrt{\rho e^{ib}} R, \tag{127}$$

where ρ represents a scalar probability density, e^{ib} is a complex phase, and R is a rotor expressed as the exponential of a bivector.

To recover David Hestenes' formulation of the wavefunction, it suffices to square our wavefunction and eliminate the terms $\mathbf{x} \to 0$ and $\mathbf{v} \to 0$:

$$\psi = \phi^2 \mid_{\mathbf{x} \to 0, \mathbf{v} \to 0} = e^{2a + 2\mathbf{f} + 2\mathbf{b}} = \sqrt{\rho e^{ib}} R \tag{128}$$

More rigorously, we can obtain this wavefunction via a reduction of the quartic-inner-product form (Equation 108). We perform the following substitutions:

$$\mathbf{v} \to \mathbf{u}^{\ddagger} \tag{129}$$

$$\mathbf{u} \to \mathbf{u} \tag{130}$$

$$\mathbf{z} \to \mathbf{w} \tag{131}$$

$$\mathbf{w} \to \mathbf{w}^{\ddagger} \tag{132}$$

Consequently, the quartic-inner-product form becomes a two-form (inner product). Since the multivectors are here reduced ($\mathbf{x} \to 0, \mathbf{v} \to 0$), the blade-3,4 conjugate is also reduced to the blade-4 conjugate.

$$\langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \rangle \to \langle \mathbf{u}, \mathbf{u}^{\ddagger}, \mathbf{w}^{\ddagger}, \mathbf{w} \rangle \cong \langle \mathbf{u}, \mathbf{w} \rangle = \sum_{i=1}^{m} ((u_i^2)^{\ddagger})^{\dagger} w_i^2 \tag{133}$$

This shows that the wavefunction is a statistical sub-ensemble of our general 3+1D ensemble. In this case the observables are satisfied when

$$\lfloor \mathbf{O} \rfloor_{2,4} = \mathbf{O} \tag{134}$$

Under the reduction given $\mathbf{x} \to 0, \mathbf{v} \to 0$, the observable relation captures the totality of the remaining geometry. Comparatively, in the quartic-inner-form case the observables satisfied Equation 114 only if real-valued. A structure reduction has increased the quantity of geometry that is observable. We thus arrive at the conclusion that the wavefunction is the largest statistical structure in 3+1D that is entirely observable geometrically.

We also notice that although the wavefunction lives in 4D it is a "2D-like statistical object"; that is, its statistical norm is given by an inner-product rather than a quartic-inner-form. We believe working in an inner-product framework, rather than a quartic-inner-form framework, is one of the key reason quantum theory resists gravitization (Section 4.14).

Let us analyze the symmetry group of this wavefunction.

First, we observed that the terms a and $\mathbf{b}$ commute with $\mathbf{f}$ and with each other. Thus, they can be factored out as

$$e^{2a+2\mathbf{f}+2\mathbf{b}} = e^{2a+2\mathbf{b}} e^{2\mathbf{f}} \tag{135}$$

Second, the term $\mathbf{f}$ can be understood as the exponential map from the $\mathfrak{so}(3,1)$ lie algebra to the $\mathrm{Spin}(3,1)$ group.

Consequently, the wavefunction represents the exponential map of the following lie algebra

$$\mathbb{R} \oplus \mathfrak{so}(3,1) \oplus \mathfrak{u}(1) \tag{136}$$

which associates to the following group

$$(\mathbb{R}^{+} \times \mathrm{Spin}(3,1) \times \mathrm{U}(1))/\mathbb{Z}_2 \cong \mathbb{R}^{+} \times \mathrm{Spin}^c(3,1) \tag{137}$$

as well as its double cover

$$\mathbb{R}^{+} \times \mathrm{Spin}(3,1) \times \mathrm{U}(1) \tag{138}$$

4.14 Gravity in 3+1D (Sketch)

The gravitational theory in 3+1D is defined in a manner to similar to the 2D csae.

Let X^4 be a world manifold. We consider the tangent bundle TX along with its associated frame bundle FX. The structure group of FX is $\mathrm{GL}^+(4,\mathbb{R})/\mathrm{SO}(3,1)$, which associates to a linear action by our wavefunction. The structure reduction $\mathrm{GL}^+(4,\mathbb{R})/\mathrm{SO}(3,1)$ entails a pseudo-Riemmanian metric in the global section of the quotient bundle. When followed by a structure lift to $\mathrm{Spin}^c(3,1)$ it is then entailed a U(1)-preserving spinlike connection.

If a prior is used, the structure is reduced to $\mathbb{R}^+ \times \mathrm{SO}(3,1)$ then lifted to $\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$. In this case, the associated spinlike connection preserves the $\mathbb{R}^+$ structure associated with the statistical prior, as well as the U(1) structure.

4.15 Dirac current

David Hestenes[7] defines the Dirac current in the language of geometric algebra as

$$\mathbf{j} = \psi^{\ddagger}\gamma_0\psi = \rho R^{\ddagger}\gamma_0 R = \rho e_0 = \rho v \tag{139}$$

where v is the proper velocity.

In our formulation, this relation also holds; the Dirac current represents the action of the wavefunction on the unit timelike vector, rotating it, and yielding a tangent space on X^4. Specifically, it is a statistically weighted Lorentz action on γ_0:

$$\begin{aligned}
\mathbf{j} &= \psi^{\ddagger}\gamma_0\psi && (140)\\
&= e^{2a-2\mathbf{f}+2\mathbf{b}}\gamma_0 e^{2a+2\mathbf{f}+2\mathbf{b}} && (141)\\
&= e^{4a}e^{-2\mathbf{f}}\gamma_0 e^{2\mathbf{f}} && (142)\\
&= \rho e_0 && (143)\\
&= \rho v && (144)
\end{aligned}$$

We now have all the tools required to construct particle physics by exhausting the remaining geometry of our model.

4.16 SU(2) × U(1) group

Our wavefunction $\psi = e^{2a+2\mathbf{f}+2\mathbf{b}}$ transforms as a group under multiplication. We now ask, what is the most general multivector $e^{\mathbf{u}}$ which leaves the Dirac current invariant?

$$\psi^{\ddagger}(e^{\mathbf{u}})^{\ddagger}\gamma_0 e^{\mathbf{u}}\psi = \psi^{\ddagger}\gamma_0\psi \iff (e^{\mathbf{u}})^{\ddagger}\gamma_0 e^{\mathbf{u}} = \gamma_0 \tag{145}$$

When is this satisfied?

The bases of the bivector part $\mathbf{f}$ of $\mathbf{u}$ are $\gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$. Among these, only $\gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$ commute with γ_0, and the rest anti-commute; therefore, the rest must be made equal to 0. Finally, the base $\gamma_0\gamma_1\gamma_2\gamma_3$ anti-commutes with γ_0 and cancels out.

Consequently, the most general exponential multivector of the form $e^{\mathbf{u}}$ where $\mathbf{u} = \mathbf{f} + \mathbf{b}$ which preserves the Dirac current is

$$e^{\mathbf{u}} = \exp(F_{12}\gamma_1\gamma_2 + F_{13}\gamma_1\gamma_3 + F_{23}\gamma_2\gamma_3 + \mathbf{b}) \tag{146}$$

We can rewrite the bivector basis with the Pauli matrices

$$\gamma_2\gamma_3 = \mathbf{i}\sigma_x \tag{147}$$
$$\gamma_1\gamma_3 = \mathbf{i}\sigma_y \tag{148}$$
$$\gamma_1\gamma_2 = \mathbf{i}\sigma_z \tag{149}$$
$$\mathbf{b} = \mathbf{i}b \tag{150}$$

After replacements, we obtain

$$e^{\mathbf{u}} = \exp \mathbf{i}(F_{12}\sigma_z + F_{13}\sigma_y + F_{23}\sigma_x + b) \tag{151}$$

The terms $F_{23}\sigma_x + F_{13}\sigma_y + F_{12}\sigma_z$ and b are responsible for $SU(2)$ and $U(1)$ symmetries, respectively. The details of this identification process are available in [9, 10].

4.17 SU(3) group

The invariance transformation identified by Equation 116 is $\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}$. The identified evolution was bivectorial rather than unitary.

As we did for the SU(2) × U(1) case, we ask, in this case, what is the most general bivectorial evolution which leaves the Dirac current invariant?

$$\mathbf{f}^\ddagger\gamma_0\mathbf{f} = \gamma_0 \tag{152}$$

where $\mathbf{f}$ is a bivector:

$$\mathbf{f} = F_{01}\gamma_0\gamma_1 + F_{02}\gamma_0\gamma_2 + F_{03}\gamma_0\gamma_3 + F_{23}\gamma_2\gamma_3 + F_{13}\gamma_1\gamma_3 + F_{12}\gamma_1\gamma_2 \tag{153}$$

Explicitly, the expression $\mathbf{f}^\ddagger\gamma_0\mathbf{f}$ is

$$\mathbf{f}^\ddagger\gamma_0\mathbf{f} = -\mathbf{f}\gamma_0\mathbf{f} = (F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2)\gamma_0 \tag{154}$$
$$+ (-2F_{02}F_{12} + 2F_{03}F_{13})\gamma_1 \tag{155}$$
$$+ (-2F_{01}F_{12} + 2F_{03}F_{23})\gamma_2 \tag{156}$$
$$+ (-2F_{01}F_{13} + 2F_{02}F_{23})\gamma_3 \tag{157}$$

For the Dirac current to remain invariant, the cross-product must vanish:

$$-2F_{02}F_{12} + 2F_{03}F_{13} = 0 \tag{158}$$
$$-2F_{01}F_{12} + 2F_{03}F_{23} = 0 \tag{159}$$
$$-2F_{01}F_{13} + 2F_{02}F_{23} = 0 \tag{160}$$

leaving only

$$\mathbf{f}^{\ddagger}\gamma_0\mathbf{f} = (F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2)\gamma_0. \tag{161}$$

Finally, $F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2$ must equal 1.

We note that we can re-write $\mathbf{f}$ as a 3-vector with complex components:

$$\mathbf{f} = (F_{01} + \mathbf{i}F_{23})\gamma_0\gamma_1 + (F_{02} + \mathbf{i}F_{13})\gamma_0\gamma_2 + (F_{03} + \mathbf{i}F_{12})\gamma_0\gamma_3 \tag{162}$$

Then, with the nullification of the cross-product, and equating $F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2$ to unity, we can understand the bivectorial evolution when constrained by the Dirac current to be a realization of the SU(3) group.

5 A Step toward falsifiable predictions

Several falsifiable predictions are listed below.

The main idea is that a general linear wavefunction would allow a larger class of interference patterns than complex interference. The general linear interference pattern includes ways in which the orientation-preserving general linear group can produce interfere under a linear transformation, including interference from from rotations, boosts, shear, and dilations.

It is plausible that an Aharonov–Bohm effect experiment on gravity[11] could detect up to the general linear phase and patterns identified in this section.

These patterns hold in the unbroken $\mathrm{GL}^+(4, \mathbb{R})$ and $\mathrm{GL}^+(2, \mathbb{R})$ symmetries.

An interference pattern follows from a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the application of the determinant:

$$\det(\mathbf{u} + \mathbf{v}) = \det \mathbf{u} + \det \mathbf{v} + \text{extra-terms} \tag{163}$$

The sum of the probability is $(\det \mathbf{u} + \det \mathbf{v})$. The "extra terms" represents the interference term.

We use the extra terms to define a bilinear form using the dot product notation.

$$\cdot \quad : \qquad \mathcal{G}(\mathbb{R}^{m,n}) \times \mathcal{G}(\mathbb{R}^{m,n}) \longrightarrow \mathbb{R} \tag{164}$$
$$\mathbf{u} \cdot \mathbf{v} \longmapsto \frac{1}{2}(\det(\mathbf{u} + \mathbf{v}) - \det \mathbf{u} - \det \mathbf{v}) \tag{165}$$

For example, in 2D, we have

$$\mathbf{u} = a_1 + x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + b_1\mathbf{e}_{12} \tag{166}$$
$$\mathbf{v} = a_2 + x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + b_2\mathbf{e}_{12} \tag{167}$$
$$\implies \mathbf{u}\cdot\mathbf{v} = a_1a_2 + b_1b_2 - x_1x_2 - y_1y_2. \tag{168}$$

If $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$, then $\mathbf{u}\cdot\mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$(whichever is greater). Therefore, it also satisfies the conditions of an interference term.

- In 2D, the dot product is equivalent to the form

$$\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) = \frac{1}{2}\Big((\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v}\Big) \tag{169}$$
$$= \mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v} - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v} \tag{170}$$
$$= \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} \tag{171}$$

- In 3+1D, it is substantially more complex:

$$\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) \tag{172}$$
$$= \frac{1}{2}\Big(\lfloor(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})\rfloor_{3,4}(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} - \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\Big) \tag{173}$$
$$= \frac{1}{2}\Big(\lfloor\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}(\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}) - \dots\Big) \tag{174}$$

$$\begin{aligned}
&= \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&\quad + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&\quad + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&\quad + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} - \dots
\end{aligned} \tag{175}$$

$$\begin{aligned}
&= \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&\quad + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&\quad + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\
&\quad + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u}
\end{aligned} \tag{176}$$

A simpler version of this interference pattern is possible when the general linear group is reduced.

Complex interference:

In 2D, reducing the general linear group to the circle group reduces the interference pattern to a complex interference.

$$|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2|\psi_1||\psi_2| \cos(\phi_1 - \phi_2) \tag{177}$$

Deep spinor interference:

Reducing to the spinor group reduces the interference pattern to a "deep spinor rotation."

Consider a two-state wavefunction (we note that $[\mathbf{f}, \mathbf{b}] = 0$)).

$$\psi = \psi_1 + \psi_2 = e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2} \tag{178}$$

The geometric interference pattern for a full general linear transformation in 4D is given by

$$\lfloor \psi^{\ddagger} \psi \rfloor_{3,4} \psi^{\ddagger} \psi. \tag{179}$$

Starting with the sub-product

$$\psi^{\ddagger}\psi = (e^{a_1} e^{-\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{-\mathbf{f}_2} e^{\mathbf{b}_2})(e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2}) \tag{180}$$

$$\begin{aligned} &= e^{a_1} e^{-\mathbf{f}_1} e^{\mathbf{b}_1} e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_1} e^{-\mathbf{f}_1} e^{\mathbf{b}_1} e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2} \\ &\quad + e^{a_2} e^{-\mathbf{f}_2} e^{\mathbf{b}_2} e^{a_1} e^{\mathbf{f}_1} e^{\mathbf{b}_1} + e^{a_2} e^{-\mathbf{f}_2} e^{\mathbf{b}_2} e^{a_2} e^{\mathbf{f}_2} e^{\mathbf{b}_2} \end{aligned} \tag{181}$$

$$= e^{2a_1} e^{2\mathbf{b}_1} + e^{2a_2} e^{2\mathbf{b}_2} + e^{a_1+a_2} e^{\mathbf{b}_1+\mathbf{b}_2} (e^{-\mathbf{f}_1} e^{\mathbf{f}_2} + e^{-\mathbf{f}_2} e^{\mathbf{f}_1}) \tag{182}$$

The full product is expressed as

$$\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi = \Big(e^{2a_1}e^{-2\mathbf{b}_1} + e^{2a_2}e^{-2\mathbf{b}_2} + e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})\Big)$$
$$\times\Big(e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_2}e^{2\mathbf{b}_2} + e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1}\Big) \tag{183}$$

$$= e^{2a_1}e^{-2\mathbf{b}_1}e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_1}e^{-2\mathbf{b}_1}e^{2a_2}e^{2\mathbf{b}_2} + e^{2a_1}e^{-2\mathbf{b}_1}e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})$$
$$+ e^{2a_2}e^{-2\mathbf{b}_2}e^{2a_1}e^{2\mathbf{b}_1} + e^{2a_2}e^{-2\mathbf{b}_2}e^{2a_2}e^{2\mathbf{b}_2} + e^{2a_2}e^{-2\mathbf{b}_2}e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})$$
$$+ e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{2a_1}e^{2\mathbf{b}_1}$$
$$+ e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{2a_2}e^{2\mathbf{b}_2}$$
$$+ e^{a_1+a_2}e^{-\mathbf{b}_1-\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})e^{a_1+a_2}e^{\mathbf{b}_1+\mathbf{b}_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1}) \tag{184}$$

$$= e^{4a_1} + e^{4a_2} + 2e^{2a_1+2a_2}\cos(2b_1 - 2b_2) \tag{185}$$
$$+ e^{a_1+a_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})(\tag{186}$$
$$e^{2a_1}(e^{-\mathbf{b}_1+\mathbf{b}_2} + e^{\mathbf{b}_1-\mathbf{b}_2}) \tag{187}$$
$$+ e^{2a_2}(e^{\mathbf{b}_1-\mathbf{b}_2} + e^{-\mathbf{b}_1+\mathbf{b}_2})) \tag{188}$$
$$+ e^{2a_1+2a_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})^2 \tag{189}$$

$$= \underbrace{e^{4a_1} + e^{4a_2}}_{\text{sum}} + \underbrace{2e^{2a_1+2a_2}\cos(2b_1 - 2b_2)}_{\text{complex interference}}$$
$$+ \underbrace{2e^{a_1+a_2}(e^{2a_1} + e^{2a_2})(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})(\cos(B_1 - B_2)) + e^{2A_1+2A_2}(e^{-\mathbf{f}_1}e^{\mathbf{f}_2} + e^{-\mathbf{f}_2}e^{\mathbf{f}_1})^2}_{\text{deep spinor interference}} \tag{190}$$

5.1 A model of $\mathrm{GL}^+(2,\mathbb{R})$ that lives in $\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$

Observing the interference patterns in the section above in 3+1D may pose a challenge, because they require the $\mathrm{GL}^+(4,\mathbb{R})$ symmetry to be unbroken.

An easier challenge, may be to realize an injection between $\mathrm{GL}^+(2,\mathbb{R})$ and $\mathbb{R}^+ \times \mathrm{Spin}^c(3,1)$, and then to witness the 2D version of the gravitized quantum theory.

Consider a wavefunction in 3+1D of this form

$$\psi = e^{A+F_{01}\gamma_0\gamma_1+F_{02}\gamma_0\gamma_2+F_{03}\gamma_0\gamma_3+F_{12}\gamma_1\gamma_2+F_{13}\gamma_1\gamma_3+F_{23}\gamma_2\gamma_3+B\gamma_0\gamma_1\gamma_2\gamma_3} \tag{191}$$

The following eliminations

$$F_{03} \to 0 \tag{192}$$
$$F_{12} \to 0 \tag{193}$$
$$F_{13} \to 0 \tag{194}$$
$$F_{23} \to 0 \tag{195}$$

along with the associations $\gamma_0\gamma_1 \to \sigma_x$, and $\gamma_0\gamma_2 \to \sigma_y$ causes ψ to be isomorphic to $\mathrm{GL}^+(2,\mathbb{R})$.

$$\psi = e^{A+F_{01}\sigma_x+F_{02}\sigma_y+B\gamma_0\gamma_1\gamma_2\gamma_3} \tag{196}$$

We recall that in 2D we were using the representation of the $\mathrm{GL}^+(2,\mathbb{R})$ group using the multivector $\mathbf{u} = \exp\big(a + x\sigma_x + y\sigma_y + b\sigma_x\sigma_y\big)$.

Using the reduced 3+1D norm (Equation 133), the observables of the injection satisfy $\lfloor \mathbf{O} \rfloor_{2,4} = \mathbf{O}$ which, in this case, is isomorphic to the same in 2D. The transformations are given as $\lfloor \mathbf{T} \rfloor_{2,4}\mathbf{T} = \mathbf{I}$ which, in this case, are also isomorphic to the same in 2D. Finally, the interference pattern is also isomorphic to the 2D case.

Consequently, it should be possible to construct a wavefunction with a prior ($\mathbb{R}^+$), with a pseudo-scalar ($\mathrm{U}(1)$), select a 2D structure (e.g. thin crystal, etc), and observe the unbroken $\mathrm{GL}^+(2,\mathbb{R})$ wavefunction behaviour.

6 Discussion

We recovered the foundations of quantum mechanics using the tools of statistical mechanics to maximize the entropy under the effect of a geometric measurement constraint. We also replaced the Boltzmann entropy with the Shannon entropy. We will now discuss the interpretation of our model in more details.

Contrary to multiple interpretations of quantum mechanics, the interpretation of statistical mechanics is singular, free of paradoxes, and without any measurement problem; automatically, this will carry over to our interpretation of quantum mechanics.

Definition 10 (Metrological interpretation)**.** *There exist instruments that record sequences of measurements on systems. These measurements are unique up to a geometric phase, and the Born rule (including its geometric generalization to the determinant) is the entropy-maximizing measure constrained by the expectation eigenvalue of these measurements.*

The Lagrange multiplier method, which maximizes the entropy subject to the geometric measurement constraint, is the mathematical backbone of this interpretation.

We now discuss the definition of the measuring apparatus entailed by this interpretation.

Integrating formally into physics the notion of an instrument or measuring apparatus has been a long-standing difficulty. One of the pitfalls is attributing too much "detailing" to this instrument (for instance, defining the instrument as a macroscopic system that amplifies quantum information), which increases the risk of capturing only a fraction of all possible instruments in nature. Fractional capture is to be avoided because the instruments are our only "eyes into nature"; consequently, the generality of their definition must be on a level similar to the laws of physics themselves, lest there would be no chance of deriving the laws of physics from measurements.

In statistical mechanics, instruments and their effects on systems are incorporated into the mathematical formalism. For instance, an energy or volume meter can produce a sequence of measurements whose average converges towards an expectation value, constituting a constraint on the entropy. However, the generalizability of this definition to all physical systems (including quantum and geometrical) was overlooked. This study capitalized on this definition and extended it appropriately.

The instrument is defined as follows:

Definition 11 (Instrument/Measuring Apparatus). *An instrument, or measuring apparatus, is a device that constrains the entropy of a message of measurements to an expectation eigenvalue (if the instrument is a scalar constraint, the expectation eigenvalue is the same as the expectation value).*

Nature allows geometrically richer measurements and instrumentations, which cannot be expressed with simple "scalar" or "phase-less" instruments. For instance, a protractor or boost meter also admit numerical measurements; however, they also contain geometric phase invariances, such as the rotational or Lorentz invariance, respectively. These invariances must be absorbed within the associated probability measure.

In the metrological interpretation, the existence of such instruments, not the wavefunction, is taken as axiomatic. The laws of physics are determined by the geometrical richness (invariance) of the instruments in nature.

This study interpreted the trace as the expectation eigenvalue of the eigenvalues of a matrix transformation multiplied by the dimension of the vector space. Maximizing the entropy under the constraint of this expectation eigenvalue introduces various phase invariances into the resulting probability measure, consistent with the available measuring apparatuses.

As we have seen, the constraint

$$\operatorname{tr} \begin{bmatrix} 0 & -\overline{b} \\ \overline{b} & 0 \end{bmatrix} = \sum_{q \in \mathbb{Q}} \operatorname{tr} \rho(q) \begin{bmatrix} 0 & -b(q) \\ b(q) & 0 \end{bmatrix} \tag{197}$$

induces a complex phase invariance into the probability measure $\rho(q) = \left|\exp\bigl(-i\tau b(q)\bigr)\right|^2$, which gives rise to the Born rule and wavefunction.

Moreover, the constraint

$$\operatorname{tr} \overline{\mathbf{M}} = \sum_{q \in \mathbb{Q}} \operatorname{tr} \rho(q) \mathbf{M}(q) \tag{198}$$

induces a general linear phase invariance in the probability measure $\rho(q) = \det \exp\bigl(-\tau \mathbf{M}(q)\bigr)$. The resulting probability measure supports a gravitized quantum theory.

In each case, we can interpret the constraint as an instrument acting on the system.

In the complex phase, we associate the constraint with an incidence counter measuring a particle or photon. Moreover, in the general linear case, we associate the constraint to a measure invariant with respect to natural transformations, such as measurements of the geometry of spacetime events.

The complete correspondence between an ordinary system of statistical mechanics and ours is as follows.

Table 1: Correspondence

Concept	Statistical Mechanics	Geometric Constraint (Ours)
Entropy	Boltzmann	Shannon
Measure	Gibbs	Born rule on wavefunction
Constraint	Energy meter	Phase-invariant instrument
Micro-state	Energy values	Possible measurements
Macro-state	Equation of state	Evolution of the wavefunction
Experience	Ergodic	Message of measurements

In the correspondence, using the Shannon entropy instead of the Boltzmann entropy changes the experience from ergodic to a message (in the sense of the communication theory of Claude Shannon[12]) of measurements.

The receipt of such a message by, say, an observer carries information; it is interpreted as registering a "click"[13] on a screen or other detecting instrument.

Using the Shannon entropy, quantum physics can be interpreted as the probability measure resulting from the entropy maximization of a message of geometrically invariant measurements an observer receives from measurement apparatuses.

The probabilistic interpretation of the wavefunction via the Born rule is inherited from statistical mechanics and results from maximizing the entropy under geometric constraints.

The wavefunction is also entailed, and consequently is not considered axiomatic. Instead, it is the receipt of a message of the measurements by an observer, along with the geometric constraints on the corresponding entropy, that is considered axiomatic.

The axioms of quantum mechanics are recoverable as theorems from the solution $\frac{\partial \mathcal{L}}{\partial \rho} = 0$ for ρ, where

$$\mathcal{L} = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \tau \left(\operatorname{tr} \overline{\mathbf{M}} - \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{M}(q) \right). \tag{199}$$

Now, let us discuss the wavefunction collapse problem:

Specifically, the mathematical foundation of quantum mechanics contains the following axiom: If the measurement of a quantity $\mathbf{O}$ on ψ gives the result o_n, then the state immediately after the measurement is given by the normalized projection of ψ onto the eigensubspace of o_n as

$$\psi \implies \frac{P_n \left|\psi\right\rangle}{\sqrt{\left\langle\psi\right| P_n \left|\psi\right\rangle}} \tag{200}$$

The difficulty of providing a mechanism to explain why this occurs is known as the wavefunction collapse problem.

The measurement-collapse problem is, in our framework, superseded as follows: Before deriving the wavefunction, measurements are assumed to have been registered by an instrument and associated with a geometric constraint, which is axiomatic. Registering new measurements, in this case, does not mean that a wavefunction has collapsed but implies that we need to adjust the constraints and derive a new wavefunction consistent with new measurements. Because the wavefunction is derived by maximizing the entropy constrained by the registered measurements, it never updates from an uncollapsed to a collapsed state. The collapse problem is a symptom of attributing an ontology to the wavefunction; however, the ontology belongs to the instruments and their measurements — not to the wavefunction.

Since our knowledge of nature comes from the available instruments, postulating these instruments (rather than the wavefunction) to be the axioms of physics makes the mathematics of physics entirely consistent with it being an empirical science. The foundations of physics have thus been cleaned of their Platonic defect, and their empirical purity has been restored.

The full correspondence is also consistent with the general intuition that *random information* must be axiomatic, as, by definition, it cannot be derived from any earlier principles. Ultimately, it is viable to consider *the message of random measurements*, rather than the wavefunction (a precise and deterministic mathematical equation), to be the axiomatic foundation of the theory. As shown, the latter can be derived from the former, but not vice versa, as suggested by the lack of a satisfactory mechanism for the wavefunction collapse in the usual interpretation. If a theory relies on random information, said theory ought always be repackaged such that this random information becomes its axiomatic foundation, lest we will go mad trying to interpret it.

6.1 Axioms of Physics

We propose that the laws of physics are ultimately entailed only and entirely by the following minimal axioms related to measurements.

Context 1 (Observability)**.** *Let q be the elements of a statistical ensemble $\mathbb{Q}$. Then $m : \mathbb{Q} \to \mathbb{R}$ is an observable of $\mathbb{Q}$. Then $\mathbb{Q}$ is observable.*

Context 2 (Comprehensibility)**.** *The experience of the observer in nature is defined as the receipt of a message* $\mathbf{m} \in (m(\mathbb{Q}))^n$ *of* n *measurements performed on* n *identical copies of* $\mathbb{Q}$*.*

Context 3 (Representativeness)**.** *Observations are* *representative* *of the limit: when* $|\mathbf{m}| \to \infty$*, then* $\overline{m} \in \mathbb{R}$ *(i.e., the average of these measurements converges towards a well-defined expectation value).*

Context 4 (Comprehensiveness)**.** *Observations are* *comprehensive* *in the limit: when* $|\mathbf{m}| \to \infty$*, then* $\mathbb{Q}$ *is well-defined (i.e., all the elements in* $\mathbb{Q}$ *are identified).*

Axiom 1 (Geometricity)**.** *A geometric measuring device constrains the entropy of a message of measurement as follows:*

$$\operatorname{tr} \overline{\mathbf{u}} = \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{u}(q) \tag{201}$$

where $m(q) := \operatorname{tr} \mathbf{u}(q)$ *is a possible measurement, and* $\mathbf{u}$ *corresponds to a multivector.*

Conjecture 1 (Geometric Closure)**.** *The geometric constraint is sufficiently restrictive to represent* *only* *the measurements that are possible in nature, yet sufficiently descriptive to represent* *all* *such measurements.*

Theorem 1 (Laws of Physics as a Theorem)**.** *Maximizing the entropy of a message of measurements constrained by a geometric measuring device yields a model of physics that maximizes the information acquired from said measurements:*

$$\mathcal{L} = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \tau \left(\operatorname{tr} \overline{\mathbf{u}} - \sum_{q \in \mathbb{Q}} \rho(q) \operatorname{tr} \mathbf{u}(q) \right). \tag{202}$$

Solving for $\partial \mathcal{L} / \partial \rho = 0$ *implies*

$$\rho(q, \tau) = \frac{1}{Z(\tau)} \det \exp\bigl(-\tau \mathbf{u}(q)\bigr), \tag{203}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} \det \exp\bigl(-\tau \mathbf{u}(q)\bigr). \tag{204}$$

where the Lagrange multiplier τ *represents the one-parameter group evolution of, in the general case, the orientation preserving general linear group* $\mathrm{GL}^+(n, \mathbb{R})$*.*

7 Conclusion

We proposed a geometric constraint to maximize the Shannon entropy. It is used to derive a probability measure that supports a geometry richer than what was previously used in statistical physics or quantum mechanics. This substantially extends the opportunity to capture all the modern physics phenomena within a single framework. The wavefunction of the general linear group is derived to accommodate all the possible linear measurements, and the Born rule is extended to the determinant. The framework produces a non-defective model for 0D, 1D, 2D, and 4D. 4D stands out as the largest space that includes all non-defective variations. A gravitational theory results from the $\mathrm{GL}^+(4,\mathbb{R})$ group undergoing symmetry breaking to the $\mathrm{Spin}^c(3,1)$ group. Furthermore, breaking the symmetry of the general linear wavefunction into the $\mathrm{Spin}^c(3,1)$ group reduces the quantum theory to the SU(2)× U(1) and SU(3) for its invariant transformations. Finally, an interpretation of quantum mechanics, i.e., the metrological interpretation, is proposed; the existence of instruments and the measurements they produce acquire the foundational role, and the wavefunction is derived as a theorem. In this interpretation, it is considered that an observer receives a message (theory of communication/Shannon entropy) of phase-invariant measurements, and the probability measure, maximizing the information of this message, is the (general linear) wavefunction accompanied by the (general linear) Born rule.

8 Statements and Declarations

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