A Candidate for a Final Theory Is Identified as the Solution to an Optimization Problem on Entropy, Information and Measurements

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Abstract

In modern theoretical physics, the laws of physics are formulated as axioms (e.g., the Dirac–Von Neumann axioms, the Wightman axioms, and Newton's laws of motion). While axioms in modern logic hold true merely by definition, the laws of physics are entailed by measurements. This entailment reveals an opportunity to derive (rather than to posit) the laws of physics from first principle. It also reveals a redundancy in the foundations of physics. We propose an optimization problem relating to entropy, information and measurements as a mean to select the optimal predictive theory that respects this entailment. In consideration of the extreme generality of this optimization problem, we note that the solution we obtain is remarkable in its specificity and in its fitness for reality. Of primary interest, the solution reveals the general setting for physics to be the *general linear* Hilbert space, a generalization of the complex Hilbert space able to support arbitrary geometry. Below 4D, the solutions are vacuous. In 4D, the general linear Hilbert space naturally contains gravity for fermions and bosons from the quotient bundle FX/Spin^c(3,1), electromagnetism from the U(1)-bundle, the standard model from the gauge group SU(3)xSU(2)xU(1), and admits no freedom for anything else. Above 4D, the general linear Hilbert space fails to admit normalizable observables altogether, suggesting an intrinsic delimitation of the dimensionality of observable geometry and, by association, to that of spacetime.

1 Introduction

In logic, if an axiom is shown to be provable from the other axioms of the theory, it is considered redundant and is removed. As measurements are part of physics and the laws of physics are entailed by measurements, it follows that all axioms that pertain to the laws of physics (but obviously not those that pertain to measurements) are necessarily redundant. This argument holds irrespective of the perceived convenience and past successes of expressing the laws of physics as axioms.

Our goal is this work is to realize this newly found opportunity to *derive* from first principle (rather than to posit) the laws of physics.

As such, our axiomatic basis will relate solely to the measurements themselves. As for the laws of physics, they will be derived by solving an optimization problem.

To formulate the optimization problem in full rigour, a few concepts must be introduced.

First, let us consider the types of measurements that an observer can make in nature. For instance, the observer can use a meter to measure the distance between two points. The observer can also use a protractor to measure angles, or a clock to count the number of ticks between two events, etc. By listing the possible types, we are lead to identify a *general measurement pattern* representing the possible measurements of nature. In general, the pattern could include scalar measurements such as energy or volume measurements and geometric measurements such as those produced by protractors, and phase, boost, dilation, spin, and shear meters.

We will now produce the mathematically precise definition of this pattern, then we will discuss its intended usage and clarify its meaning. The construction of the general measurement pattern exploits the connection between geometry and probability via the trace. The trace of a matrix can be understood as the expected eigenvalue multiplied by the vector space dimension, and the eigenvalues as the ratios of the distortion of the linear transformation associated with the matrix[1].

Axiom 1 (The General Measurement Pattern of Nature)**.** *The general measurement pattern of nature is:*

$$\frac{1}{d}\operatorname{tr}\overline{\mathbf{u}} = \sum_{q\in\mathbb{Q}} \rho(q)\frac{1}{d}\operatorname{tr}\mathbf{u}(q) \tag{1}$$

where $\operatorname{tr}\mathbf{u}(q)$ *is an observable, where* $\operatorname{tr}\overline{\mathbf{u}}$ *is its average, and where* $\mathbf{u}$ *corresponds to a multivector of the geometric algebra* $\mathcal{G}(\mathbb{R}^{m,n})$ *such that* $d = m + n$*, where* ρ *is a probability measure, and where* $\mathbb{Q}$ *is a statistical ensemble.*

Since the multivectors of $\mathcal{G}(\mathbb{R}^2)$ and $\mathcal{G}(\mathbb{R}^{3,1})$ are group isomorphic to $\mathbb{M}(2,\mathbb{R})$ and $\mathbb{M}(4,\mathbb{R})$, respectively, we can understand the domain of the general measurement pattern to be that of general linear measurements. The use of multivectors instead of matrices merely singles out a preferred geometric representation of said general linear measurements.

Finally, we note that the trace of a multivector can be obtained by mapping the multivector to its matrix representation (Section 2), and taking its trace.

Now, we discuss its intended usage.

Measurement patterns are used as constraints in statistical mechanics to derive the Gibbs measure using Lagrange multipliers[2] by maximizing the entropy.

For instance, an energy constraint on the entropy is

$$\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q), \tag{2}$$

which is associated with an energy meter that measures the system's energy and produces a series of energy measurements $E_1, E_2, \ldots,$ convergent to an expectation value $\overline{E}$.

Another common constraint is related to the volume:

$$\overline{V} = \sum_{q \in \mathbb{Q}} \rho(q) V(q), \tag{3}$$

which is associated with a volume meter acting on a system and produces a sequence of measured volumes $V_1, V_2, \ldots,$ converging to an expectation value $\overline{V}$.

Moreover, the sum over the statistical ensemble must equal 1, as follows:

$$1 = \sum_{q \in \mathbb{Q}} \rho(q) \tag{4}$$

Using equations (2) and (4), a typical statistical mechanical system is obtained by maximizing the entropy using the corresponding Lagrange equation. The Lagrange multiplier method is expressed as:

$$\mathcal{L}(\rho, \lambda, \beta) = -k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right), \tag{5}$$

where λ and β are the Lagrange multipliers.

By solving $\frac{\partial \mathcal{L}(\rho, \lambda, \beta)}{\partial \rho} = 0$ for ρ, we obtain the Gibbs measure as:

$$\rho(q, \beta) = \frac{1}{Z(\beta)} \exp\bigl(-\beta E(q)\bigr), \tag{6}$$

where

$$Z(\beta) = \sum_{q \in \mathbb{Q}} \exp\bigl(-\beta E(q)\bigr). \tag{7}$$

In our optimization problem, Equation 2, a scalar measurement pattern, is replaced with Axiom 1, the general measurement pattern. In addition to energy or volume meters, we will have protractors, and phase, boost, dilation, spin, and shear meters.

In general, an optimization problem is the problem of finding the best solution from the solution space. Specifically, we wish to find the best predictive theory (best solution) out of all possible predictive theories (the solution space) constrained by the general measurement pattern (the constraint).

Before we continue further, let us discuss the familiar scientific procedure, so that we can understand precisely the problem that the general measurement pattern solves.

Why not simply use the scientific method; is it not the pinnacle of scientific optimization? For instance, could we not consider scientific fitness as a quality score, the space of all scientific theories as a solution space, and the scientific method as the means to establish fitness over the solution space? Alas, since scientific fitness is binary (fit/unfit), the scientific method is a decision problem, not an optimization problem. Furthermore, as pure mathematics is apriori unaware of the scientific fitness of any particular sets of axioms, it cannot assign them a scientific fitness score without feedback from nature. The problem can neither be solved for a maximum (not an optimization problem) nor be entirely defined mathematically (feedback from nature is required to establish scientific fitness).

Knowing that scientific fitness does not allow optimization (nor full mathematization), we may become concerned that any such attempt at optimization would necessarily fail and lose contact with reality. Afterall, what way could there possibly be of establishing the soundness of a scientific theory other than by testing its scientific fitness?

This is precisely the problem that the general measurement pattern addresses. Using it to constrain predictive theories guarantees contact with reality. The general measurement pattern is the strongest *reality constraint* that can be formulated. Comparatively, falsifiability allows, even encourages, predictive theories to overshoot the known domain of nature so as to make them potentially falsifiable, but the general measurement pattern neither allows overshooting nor undershooting — when used as a constraint, it demands that a predictive theory exactly fit to nature.

That is not to say that scientific fitness (and falsifiability) does not play a role in our proposal; it does, but this role is transposed away from the predictive theories (the solution space) to the pattern (the axiom). The general measurement pattern remains the subject of a scientific fitness test, because we must interact with nature to identify what this pattern is. If new unsupported types of measurements are ever found, the pattern must be altered to account for them. We only mean to say that once a pattern is identified, we can produce an optimization problem using a *mathematizable* quality score other than that of scientific fitness, because scientific fitness applies to the axiom and the solution space is exactly constrained to this axiom.

As hinted, for the quality score to contain a maximum, it cannot be binary. We must assign to each predictive theory a score in $[0, \infty[$.

As such, we will score each predictive theory by the quantity of information required to exactly specify a sequence of realized measurements (whose elements are selected from the ensemble of the theory). To help us understand why maximizing this information is the correct approach, let us now contrast two examples. Suppose Alice uses a predictive theory such that no additional information is required to specify the realized measurements. For this to be the case, her predictive theory must be a brute enumeration of those measurements. Such a theory has no explanatory power. Now, suppose with his theory, that Bob requires more information to specify the realized measurements, than he would had he used any other. This is the opposite of the first example; it *maximizes* explanatory power.

The above is one of a few ways to justify the approach. Another way to justify it is to understand that any information within the foundations of a physical theory that relates to realized measurements is axiomatically redundant. When we maximize this information, we identify the least axiomatically redundant predictive theory as the solution. It is the least redundant, because it requires the most amount of information (out of all feasible solutions) to specify those measurements.

We can now define the main theorem of our proposal.

Theorem 1 (The Fundamental Theorem of Physics)**.** *Physics is the solution to a maximization problem over all predictive theories on the quantity of information associated with the receipt of a message of realized measurements whose elements are randomly selected according to a probability measure constrained by the general measurement pattern of nature. The mathematical equation is:*

$$\underbrace{\mathcal{L}(\rho,\lambda,\boldsymbol{\tau})}_{\substack{\text{a}\\ \text{maximization}\\ \text{problem}}} = \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\text{over all predictive theories}} + \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{on the quantity of information}\\ \text{associated with the}\\ \text{construction of a message}}} \underbrace{+\boldsymbol{\tau}\left(\frac{1}{n}\operatorname{tr}\overline{\mathbf{u}}-\sum_{q\in\mathbb{Q}}\rho(q)\frac{1}{n}\operatorname{tr}\mathbf{u}(q)\right)}_{\substack{\text{whose elements are constrained by the}\\ \text{general measurement pattern of nature}}} \tag{8}$$

where λ and τ are Lagrange multipliers.

To summarize, we formulated a *scientific theory* of measurements, whose single axiom is the general measurement pattern of nature. This formulation can be used to define an optimization problem able to find the best predictive theory with conditions; to maintain contact with reality, the predictive theories are constrained by the general measurement pattern of nature. Finding the best solution amounts to finding the predictive theory with the highest explanatory power, itself quantifiable via the information required to exactly specify the realized measurements of nature from the theory.

To solve Theorem 1 for the explicit solution, the techniques of statistical mechanics will be used abundantly. It therefore aids the comprehension to identify and discuss the correspondence between ordinary statistical mechanics and our proposal.

Table 1: Correspondence

Constraint	Energy Constraint	General Measurement Constraint
Ontology	Ergodic system	Construction of a message
Entropy	Boltzmann	Shannon
Probability Measure	Gibbs	Born rule
Micro-state	Energy levels	Collapsed state
Lagrange multiplier	Temperature	Entropic flow of time

As we use the (relative) Shannon entropy instead of the Boltzmann entropy, the ontology has changed from ergodic systems to the receipt of a message (in the sense of the communication theory of Claude Shannon[3]) of realized measurements of nature. The receipt of such a message by an observer carries information; it is associated to the registration of a "click"[4] on a screen or an incidence counter. It corresponds to a measurement-event. The message of measurement represents the information missing from the observer's best predictive understanding of nature for the physical system to be canonically defined.

The manuscript is organized as follows:

The Methods section introduces tools using geometric algebra, based on the study by Lundholm et al. [5, 6]. Specifically, we use the notion of a determinant for multivectors and the Clifford conjugate for generalizing the complex conjugate. These tools enable the geometric expression of our results.

The Results section presents two solutions for the Lagrange equation. The first applies to an ensemble $\mathbb{Q}$ which is at most countably infinite, and the second applies to the continuum ($\sum \to \int$) where $\mathbb{Q}$ is uncountable.

In the Analysis section we inspect the solution. Of primary interest, the solution identifies a Hilbert space for the states of the system. In 0+1D, a complex Hilbert space is recovered, in which the solution is identical to non-relativistic quantum mechanics. To accommodate the states of all general linear measurements in 2D, the complex Hilbert space is not sufficient; instead a *general linear* Hilbert space is obtained, and in 3+1D a *double-copy* general linear Hilbert space is obtained. The 2D case contains gravity but is otherwise vacuous, whilst the 4D case contains gravity, electromagnetism and the standard model. Finally, we show that the general solution lacks normalizable observables beyond 4D, naturally limiting the dimensionality of spacetime.

2 Methods

2.1 Notation

- Typography:

Sets are written using the blackboard bold typography (e.g., $\mathbb{L}$, $\mathbb{W}$, and $\mathbb{Q}$) unless a prior convention assigns it another symbol.

Matrices are in bold uppercase (e.g., $\mathbf{P}$ and $\mathbf{M}$), tuples, vectors, and multivectors are in bold lowercase (e.g., $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{g}$), and most other constructions (e.g., scalars and functions) have plain typography (e.g., a, and A).

The unit pseudo-scalar (of geometric algebra), imaginary number, and identity matrix are $\mathbf{i}$, i, and $\mathbf{I}$, respectively.

- Sets:

  The projection of a tuple $\mathbf{p}$ is $\mathrm{proj}_i(\mathbf{p})$.

  As an example, the elements of $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$ are denoted as $\mathbf{p} = (x, y)$.

  The projection operators are $\mathrm{proj}_1(\mathbf{p}) = x$ and $\mathrm{proj}_2(\mathbf{p}) = y$;

  if projected over a set, the corresponding results are $\mathrm{proj}_1(\mathbb{R}^2) = \mathbb{R}_1$ and $\mathrm{proj}_2(\mathbb{R}^2) = \mathbb{R}_2$, respectively.

  The size of a set $\mathbb{X}$ is $|\mathbb{X}|$.

  The symbol $\cong$ indicates an isomorphism, and $\rightarrow$ denotes a homomorphism.

- Analysis:

  The asterisk $z^\dagger$ denotes the complex conjugate of z.

- Matrix:

  The Dirac gamma matrices are γ_0, γ_1, γ_2, and γ_3.

  The Pauli matrices are σ_x, σ_y, and σ_z.

  The dagger $\mathbf{M}^\dagger$ denotes the conjugate transpose of $\mathbf{M}$.

  The commutator is defined as $[\mathbf{M}, \mathbf{P}] : \mathbf{MP} - \mathbf{PM}$, and the anti-commutator is defined as $\{\mathbf{M}, \mathbf{P}\} : \mathbf{MP} + \mathbf{PM}$.

- Geometric algebra:

  The elements of an arbitrary curvilinear geometric basis are denoted as $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ (such that $\mathbf{e}_\nu \cdot \mathbf{e}_\mu = g_{\mu\nu}$), and $\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \ldots, \hat{\mathbf{x}}_n$ (such that $\hat{\mathbf{x}}_\mu \cdot \hat{\mathbf{x}}_\nu = \eta_{\mu\nu}$) if they are orthonormal.

  A geometric algebra of $m + n$D over field $\mathbb{F}$ is denoted as $\mathcal{G}(\mathbb{F}^{m,n})$.

  The grades of a multivector are denoted as $\langle \mathbf{v} \rangle_k$.

  Specifically, $\langle \mathbf{v} \rangle_0$ is a scalar, $\langle \mathbf{v} \rangle_1$ is a vector, $\langle \mathbf{v} \rangle_2$ is a bivector, $\langle \mathbf{v} \rangle_{n-1}$ is a pseudo-vector, and $\langle \mathbf{v} \rangle_n$ is a pseudo-scalar.

  A scalar and vector such as $\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_1$ form a para-vector; a combination of even grades $(\langle \mathbf{v} \rangle_0 + \langle \mathbf{v} \rangle_2 + \langle \mathbf{v} \rangle_4 + \ldots)$ or odd grades $(\langle \mathbf{v} \rangle_1 + \langle \mathbf{v} \rangle_3 + \ldots)$ form even or odd multivectors, respectively.

  Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

We can formulate a general multivector of $\mathcal{G}(\mathbb{R}^2)$ as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set.

Then, a general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be formulated as $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

The notation $\odot$ designates the Hadamard product, which is an entrywise product. For instance, consider the multivector $\mathbf{u} = a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}}\hat{\mathbf{y}}$ of $\mathcal{G}(\mathbb{R}^2)$, and consider $\boldsymbol{\tau} = \tau_a + \tau_x\hat{\mathbf{x}} + \tau_y\hat{\mathbf{y}} + \tau_b\hat{\mathbf{x}}\hat{\mathbf{y}}$, then $\boldsymbol{\tau} \odot \mathbf{u} = \tau_a a + \tau_x x\hat{\mathbf{x}} + \tau_y y\hat{\mathbf{y}} + \tau_b b\hat{\mathbf{x}}\hat{\mathbf{y}}$.

2.2 Geometric representation in 2D

Let $\mathcal{G}(\mathbb{R}^2)$ be the 2D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^2)$ is given as

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}, \tag{9}$$

where a is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.

Each multivector has a structure-preserving (addition/multiplication) matrix representation.

Definition 1 (2D geometric representation)**.**

$$a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong \begin{bmatrix} a+x & -b+y \\ b+y & a-x \end{bmatrix} \tag{10}$$

Thus, the trace of $\mathbf{u}$ is a.

The converse is also true: each 2×2 real matrix is represented as a multivector of $\mathcal{G}(\mathbb{R}^2)$.

In geometric algebra, the determinant[6] of a multivector $\mathbf{u}$ can be defined as:

Definition 2 (Geometric representation of the determinant 2D)**.**

$$\begin{aligned} \det \quad : \quad \mathcal{G}(\mathbb{R}^2) &\longrightarrow \mathbb{R} \\ \mathbf{u} &\longmapsto \mathbf{u}^\ddagger \mathbf{u}, \end{aligned} \tag{11}$$

where $\mathbf{u}^\ddagger$ is

Definition 3 (Clifford conjugate 2D)**.**

$$\mathbf{u}^\ddagger := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2. \tag{12}$$

For example,

$$\begin{aligned}
\det \mathbf{u} &= (a - \mathbf{x} - \mathbf{b})(a + \mathbf{x} + \mathbf{b}) && (13)\\
&= a^2 - x^2 - y^2 + b^2 && (14)\\
&= \det \begin{bmatrix} a + x & -b + y \\ b + y & a - x \end{bmatrix} && (15)
\end{aligned}$$

Finally, we define the Clifford transpose.

Definition 4 (2D Clifford transpose)**.** *The Clifford transpose is the geometric analog to the conjugate transpose, interpreted as a transpose followed by an element-by-element application of the complex conjugate. Likewise, the Clifford transpose is a transpose followed by an element-by-element application of the Clifford conjugate.*

$$\begin{bmatrix} \mathbf{u}_{00} & \dots & \mathbf{u}_{0n} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{mn} \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{u}_{00}^{\ddagger} & \dots & \mathbf{u}_{m0}^{\ddagger} \\ \vdots & \ddots & \vdots \\ \mathbf{u}_{m0} & \dots & \mathbf{u}_{nm}^{\ddagger} \end{bmatrix} \tag{16}$$

If applied to a vector, then

$$\begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}^{\ddagger} = \begin{bmatrix} \mathbf{v}_1^{\ddagger} & \dots \mathbf{v}_m^{\ddagger} \end{bmatrix} \tag{17}$$

2.3 Geometric representation in 3+1D

Let $\mathcal{G}(\mathbb{R}^{3,1})$ be the 3+1D geometric algebra over the real set.

A general multivector of $\mathcal{G}(\mathbb{R}^{3,1})$ can be written as:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}, \tag{18}$$

where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector, and $\mathbf{b}$ is a pseudo-scalar.

Similarly, each multivector has a structure-preserving (addition/multiplication) matrix representation.

The multivectors of $\mathcal{G}(\mathbb{R}^{3,1})$ are represented as follows:

Definition 5 (4D geometric representation)**.**

$$\begin{aligned}
&a + t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \\
&\quad + f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 + f_{13}\gamma_1 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 \\
&\quad + v_t\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + v_x\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v_y\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + v_z\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\
&\quad + b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3
\end{aligned}$$

$$\cong \begin{bmatrix} a+x_0-if_{12}-iv_3 & f_{13}-if_{23}+v_2-iv_1 & -ib+x_3+f_{03}-iv_0 & x_1-ix_2+f_{01}-if_{02} \\ -f_{13}-if_{23}-v_2-iv_1 & a+x_0+if_{12}+iv_3 & x_1+ix_2+f_{01}+if_{02} & -ib-x_3-f_{03}-iv_0 \\ -ib-x_3+f_{03}+iv_0 & -x_1+ix_2+f_{01}-if_{02} & a-x_0-if_{12}+iv_3 & f_{13}-if_{23}-v_2+iv_1 \\ -x_1-ix_2+f_{01}+if_{02} & -ib+x_3-f_{03}+iv_0 & -f_{13}-if_{23}+v_2+iv_1 & a-x_0+if_{12}-iv_3 \end{bmatrix} \tag{19}$$

Thus, the trace of $\mathbf{u}$ is a.

In 3+1D, we define the determinant solely using the constructs of geometric algebra[6].

The determinant of $\mathbf{u}$ is

Definition 6 (3+1D geometric representation of determinant)**.**

$$\det \quad : \quad \mathcal{G}(\mathbb{R}^{3,1}) \longrightarrow \mathbb{R} \tag{20}$$

$$\mathbf{u} \longmapsto \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}, \tag{21}$$

where $\mathbf{u}^\ddagger$ is

Definition 7 (3+1D Clifford conjugate)**.**

$$\mathbf{u}^\ddagger := \langle \mathbf{u} \rangle_0 - \langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 + \langle \mathbf{u} \rangle_3 + \langle \mathbf{u} \rangle_4, \tag{22}$$

and where $\lfloor \mathbf{u} \rfloor_{\{3,4\}}$ is the blade-conjugate of degrees three and four (the plus sign is reversed to a minus sign for blades 3 and 4)

$$\lfloor \mathbf{u} \rfloor_{\{3,4\}} := \langle \mathbf{u} \rangle_0 + \langle \mathbf{u} \rangle_1 + \langle \mathbf{u} \rangle_2 - \langle \mathbf{u} \rangle_3 - \langle \mathbf{u} \rangle_4. \tag{23}$$

3 Results

The Lagrange equation that defines our optimization problem is:

$$\mathcal{L}(\rho,\lambda,\tau) = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau \left(\frac{1}{d} \operatorname{tr} \overline{\mathbf{u}} - \sum_{q\in\mathbb{Q}} \rho(q) \frac{1}{d} \operatorname{tr} \mathbf{u}(q)\right), \tag{24}$$

where λ and τ are the Lagrange multipliers, and where $\mathbf{u}(q)$ is an arbitrary multivector of $d = m + n$ dimensions.

To maximize this equation for ρ, we use the criterion $\frac{\partial \mathcal{L}(\rho,\lambda,\tau)}{\partial \rho(q)} = 0$ as follows:

$$\frac{\partial \mathcal{L}(\rho,\lambda,\tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \frac{1}{d} \operatorname{tr} \mathbf{u}(q) \tag{25}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \frac{1}{d} \operatorname{tr} \mathbf{u}(q) \tag{26}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \frac{1}{d} \operatorname{tr} \mathbf{u}(q) \tag{27}$$

$$\implies \rho(q) = p(q) \exp(-1-\lambda) \exp\left(-\tau \frac{1}{d} \operatorname{tr} \mathbf{u}(q)\right) \tag{28}$$

$$= \frac{1}{Z(\boldsymbol{\tau})} p(q) \det \exp\left(-\tau \frac{1}{d} \mathbf{u}(q)\right) \tag{29}$$

where $Z(\tau)$ is obtained as:

$$1 = \sum_{q \in \mathbb{Q}} p(q) \exp(-1-\lambda) \exp\left(-\tau \frac{1}{d} \operatorname{tr} \mathbf{u}(q)\right) \tag{30}$$

$$\implies \big(\exp(-1-\lambda)\big)^{-1} = \sum_{q \in \mathbb{Q}} p(q) \exp\left(-\tau \frac{1}{d} \operatorname{tr} \mathbf{u}(q)\right) \tag{31}$$

$$Z(\boldsymbol{\tau}) \coloneqq \sum_{q \in \mathbb{Q}} p(q) \det \exp\left(-\tau \frac{1}{d} \mathbf{u}(q)\right) \tag{32}$$

The resulting probability measure is:

$$\rho(q,\tau) = \frac{1}{Z(\tau)} p(q) \det \exp\left(-\tau \frac{1}{d} \mathbf{u}(q)\right), \tag{33}$$

where

$$Z(\tau) = \sum_{q \in \mathbb{Q}} p(q) \det \exp\left(-\tau \frac{1}{d} \mathbf{u}(q)\right). \tag{34}$$

Finally, we can pose

$$\rho(q,\tau) = \frac{1}{Z(\tau)} \det \psi(q,\tau), \text{ where } \psi(q,\tau) = \exp\left(-\tau \frac{1}{d} \mathbf{u}(q)\right) \psi(q) \tag{35}$$

and where $p(q) = \det \psi(q)$.

Here, the determinant acts as a generalization of the Born rule, connecting a general linear amplitude to a real-valued probability.

3.1 Continuum case

In his original paper, Claude Shannon did not derive the differential entropy as a theorem: instead, he posited that the discrete entropy ought to be extended by replacing the sum with the integral:

$$-\sum_{q\in\mathbb{Q}} \rho(q) \ln \rho(q) \to -\int_{\mathbb{R}} \rho(x) \ln \rho(x)\, \mathrm{d}x \tag{36}$$

Unfortunately, it was later discovered that the differential entropy is not always positive, and neither is it invariant under a change of parameters. Specifically, it transforms as follows:

$$-\int_{\mathbb{R}} \rho(x) \ln \rho(x)\, \mathrm{d}x \to -\int_{\mathbb{R}} \tilde{\rho}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \ln \left(\tilde{\rho}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}x \tag{37}$$

$$= -\int_{\mathbb{R}} \tilde{\rho}(y) \ln \left(\tilde{\rho}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \right) \mathrm{d}y \tag{38}$$

Furthermore, due to an argument by Edwin Thompson Jaynes[7, 8], it is known not to be the correct limiting case of the Shannon entropy. Rather, the limiting case is relative entropy:

$$S = -\int_{\mathbb{R}} \rho(x) \ln \frac{\rho(x)}{p(x)}\, \mathrm{d}x \tag{39}$$

where $p(x)$ is the initial preparation.

The relative entropy, unlike the differential entropy, is invariant with respect to a change of parameter:

$$-\int_{\mathbb{R}} \rho(x) \ln \frac{\rho(x)}{p(x)}\, \mathrm{d}x \to -\int_{\mathbb{R}} \tilde{\rho}(y(x)) \frac{dy}{dx} \ln \frac{\tilde{\rho}(y(x)) \frac{dy}{dx}}{\tilde{p}(y(x)) \frac{dy}{dx}}\, \mathrm{d}x \tag{40}$$

$$= -\int_{\mathbb{R}} \tilde{\rho}(y) \ln \frac{\tilde{\rho}(y)}{\tilde{p}(y)}\, \mathrm{d}y \tag{41}$$

Let us also show that the normalization constraint is invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} \rho(x)\, \mathrm{d}x \to \int_{\mathbb{R}} \tilde{\rho}(y(x)) \frac{dy}{dx}\, \mathrm{d}x \tag{42}$$

$$= \int_{\mathbb{R}} \tilde{\rho}(y)\, \mathrm{d}y \tag{43}$$

Let us now investigate the differential observable. A differential observable is typically formulated as

$$\overline{O} = \int_{\mathbb{R}} O(x)\rho(x)\,\mathrm{d}x \tag{44}$$

But, this expression is not invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} O(x)\rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \tilde{O}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \tag{45}$$

$$= \int_{\mathbb{R}} \tilde{O}(y)\tilde{\rho}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}y \tag{46}$$

To correct this, we now introduce the relative (with respect to a reference) observable. For instance, if we stretch space by a factor of 2: $x \to 2x$, then the reference must also be stretched by the same amount for the observable to remain invariant. The consequence is that we observe a ratio:

$$\overline{M/R} = \int_{\mathbb{R}} \frac{M(x)}{R(x)}\rho(x)\,\mathrm{d}x \tag{47}$$

Where R is the reference and the ratio $\overline{O} = \overline{U/R}$ is observable.
We now show that it is invariant with respect to a change of parameter:

$$\int_{\mathbb{R}} \frac{M(x)}{R(x)}\rho(x)\,\mathrm{d}x \to \int_{\mathbb{R}} \frac{\tilde{M}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{R}(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}}\rho(y(x))\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x \tag{48}$$

$$= \int_{\mathbb{R}} \frac{\tilde{M}(y)}{\tilde{R}(y)}\rho(y)\,\mathrm{d}y \tag{49}$$

With these definitions, the Lagrange equation becomes:

$$\mathcal{L}(\rho,\lambda,\tau) = -\int_{\mathbb{R}} \rho(x)\ln\frac{\rho(x)}{p(x)}\,\mathrm{d}x + \lambda\left(1 - \int_{\mathbb{R}}\rho(x)\,\mathrm{d}x\right) + \tau\left(\frac{1}{d}\operatorname{tr}\frac{\overline{\mathbf{m}}}{\overline{\mathbf{r}}} - \int_{\mathbb{R}}\frac{1}{d}\operatorname{tr}\frac{\mathbf{m}(x)}{\mathbf{r}(x)}\rho(x)\,\mathrm{d}x\right) \tag{50}$$

Maximizing this equation with respect to ρ gives

$$\rho(x,\tau)\,|_a^b = \frac{1}{Z(\tau)}\int_a^b p(x)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(x)\right)\mathrm{d}x \tag{51}$$

where

$$Z(\tau) = \int_{\mathbb{R}} p(q)\det\exp\left(-\tau\frac{1}{d}\mathbf{u}(x)\right)\mathrm{d}x \tag{52}$$

where $\mathbf{u}(x) = \frac{\mathbf{m}(x)}{\mathbf{r}(x)}$.

The probability measure is now invariant with respect to a change of parameter:

$$\frac{\int_a^b p(x) \det \exp\left(-\tau \frac{1}{d} \frac{\mathbf{m}(x)}{\mathbf{r}(x)}\right) \mathrm{d}x}{\int_{\mathbb{R}} p(x) \det \exp\left(-\tau \frac{1}{d} \frac{\mathbf{m}(x)}{\mathbf{r}(x)}\right) \mathrm{d}x} \to \frac{\int_a^b \tilde{p}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \det \exp\left(-\tau \frac{1}{d} \frac{\tilde{\mathbf{m}}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{\mathbf{r}}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}}\right) \mathrm{d}x}{\int_{\mathbb{R}} \tilde{p}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x} \det \exp\left(-\tau \frac{1}{d} \frac{\tilde{\mathbf{m}}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}}{\tilde{\mathbf{r}}(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}}\right) \mathrm{d}x} \tag{53}$$

$$= \frac{\int_a^b \tilde{p}(y) \det \exp\left(-\tau \frac{1}{d} \frac{\tilde{\mathbf{m}}(y)}{\tilde{\mathbf{r}}(y)}\right) \mathrm{d}x}{\int_{\mathbb{R}} \tilde{p}(y) \det \exp\left(-\tau \frac{1}{d} \frac{\tilde{\mathbf{m}}(y)}{\tilde{\mathbf{r}}(y)}\right) \mathrm{d}y} \tag{54}$$

4 Analysis

We first demonstrate how a complex Hilbert space is produced in 0+1D, followed by a *general linear* Hilbert space in 2D, and then a *double-copy* general linear Hilbert space in 3+1D. We further show that the last two structures include gravity, while the last one additionally includes the standard model. As for the first case, it corresponds to non-relativistic quantum mechanics.

4.1 Phase-invariant measurements in 0+1D

In this subsection, which also serves as an introductory example, we recover non-relativistic quantum mechanics using the Lagrange multiplier method and a linear constraint on the relative Shannon entropy.

We recall that in statistical physics, the identification of β with the temperature involves the recovery of the equation of states, and the equality $\beta = 1/(k_B T)$. Similarly, here we will identify the role played by the Lagrange multiplier τ.

As previously mentioned, the relative Shannon entropy (in base e) is applied instead of the Boltzmann entropy to achieve the aforementioned goal.

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} \tag{55}$$

In statistical mechanics, we use scalar measurement constraints on the entropy, such as energy and volume meters, which are sufficient for recovering the Gibbs ensemble. However, applying such scalar measurement constraints is insufficient to recover quantum mechanics.

A *complex measurement constraint*, a subset of the linear measurement constraint invariant for a complex phase, is used to overcome this limitation. It is defined[1] as

$$\operatorname{tr}\begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0, \end{bmatrix} \tag{56}$$

We recall that $\begin{bmatrix} a(q) & -b(q) \\ b(q) & a(q) \end{bmatrix} \cong a(q) + ib(q)$ is the matrix representation of the complex numbers. In terms of multivectors, this constraint corresponds to the matrix representation of the pseudoscalar of $\mathcal{G}(\mathbb{R}^{0,1})$.

Similar to energy or volume meters, linear instruments produce a sequence of measurements that converge to an expected value but with phase invariance. In our framework, this phase invariance originates from the trace.

The Lagrangian equation that describes this optimization problem is:

$$\mathcal{L}(\rho,\lambda,\tau) = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau \left(\operatorname{tr}\begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{57}$$

This equation is maximized for ρ by imposing the condition $\frac{\partial \mathcal{L}(\rho,\lambda,\tau)}{\partial \rho(q)} = 0$. The following results are obtained:

$$\frac{\partial \mathcal{L}(\rho,\lambda,\tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{58}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{59}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{60}$$

$$\implies \rho(q) = p(q) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{61}$$

$$= \frac{1}{Z(\tau)} p(q) \det \exp\left(-\tau \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right), \tag{62}$$

where $Z(\tau)$ is obtained as:

[1]We may wonder why we take $n = 1$ (in Equation **??**) if the matrix is 2×2. Here, we only use the imaginary part of the complex numbers $a + ib \,|_{a\to 0} = ib$, making the constraint one-dimensional.

$$1 = \sum_{q\in\mathbb{Q}} p(q) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{63}$$

$$\implies \big(\exp(-1-\lambda)\big)^{-1} = \sum_{q\in\mathbb{Q}} p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{64}$$

$$Z(\tau) \coloneqq \sum_{q\in\mathbb{Q}} p(q) \det\exp\left(-\tau \begin{bmatrix} 0 & -E(q) \\ E(q) & 0. \end{bmatrix}\right) \tag{65}$$

The exponential of the trace is equal to the determinant of the exponential according to the relation $\det\exp\mathbf{A} \equiv \exp\operatorname{tr}\mathbf{A}$.

Finally, we obtain

$$\rho(q,\tau) = \frac{1}{Z(\tau)} p(q) \det\exp\left(-\tau \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{66}$$

$$\cong p(q)|\exp -i\tau E(q)|^2 \tag{67}$$

With the equality $\tau = t/\hbar$ (analogous to $\beta = 1/(k_B T)$) we recover the familiar form of

$$\rho(q,t) = \frac{1}{Z(t)} p(q) \Big|\exp\big(-itE(q)/\hbar\big)\Big|^2 . \tag{68}$$

or in general

$$\rho(q,t) = \frac{1}{Z} \big|\psi(q,t)\big|^2 , \text{ where } \psi(q,t) = \exp\big(-itE(q)/\hbar\big)\psi(q). \tag{69}$$

and where $|\psi(q)|^2 = p(q)$ is the initial preparation.

The time t here emerges as a Lagrange multiplier, which is the same manner in which T, the temperature, emerges in ordinary statistical mechanics. We may qualify t as a "thermal time" or as an "entropic flow".

We can show that the Dirac Von–Neumann axioms and the Born rule are satisfied.

To do so, we identify the wavefunction as a vector of a complex Hilbert space, and the partition function as its inner product, expressed as:

$$Z = \langle\psi|\psi\rangle . \tag{70}$$

As the solution is automatically normalized by the entropy-maximization procedure, the physical states are associated with the unit vectors, and the probability of any particular state is given by

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger\psi(q,t). \tag{71}$$

As the solution is invariant under unitary transformations, it can be transformed out of its eigenbasis, and the energy $E(q)$ is in general represented by a Hamiltonian operator as follows:

$$\big|\psi(t)\big\rangle = \exp\bigl(-it\mathbf{H}/\hbar\bigr)\,\big|\psi(0)\big\rangle \tag{72}$$

Any self-adjoint operator, defined as $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$, will correspond to a real-valued statistical mechanics observable if measured in its eigenbasis, thereby completing the equivalence.

The dynamics are governed by the Schrödinger equation, obtained by taking the derivative with respect to the Lagrange multiplier:

$$\frac{\partial}{\partial t}\big|\psi(t)\big\rangle = \frac{\partial}{\partial t}(\exp\bigl(-it\mathbf{H}/\hbar\bigr)\,\big|\psi(0)\big\rangle) \tag{73}$$

$$= -i\mathbf{H}/\hbar\exp\bigl(-it\mathbf{H}/\hbar\bigr)\,\big|\psi(0)\big\rangle \tag{74}$$

$$= -i\mathbf{H}/\hbar\,\big|\psi(t)\big\rangle \tag{75}$$

$$\implies \mathbf{H}\,\big|\psi(t)\big\rangle = i\hbar\frac{\partial}{\partial t}\,\big|\psi(t)\big\rangle \tag{76}$$

which is the Schrödinger equation.

Finally, the measurement postulate is imported as a direct consequence of $\rho(q,\tau)$ being a probability measure of statistical mechanics like any other; as it is parametrized over $\mathbb{Q}$, it describes the probability of finding the state at parametrization q upon measurement (in the continuum case, this is a Dirac delta).

Consequently, <u>all</u> axioms of non-relativistic quantum mechanics (including the Born rule and measurement postulate) have been reduced to a specific solution to our optimization problem which depends only on a single axiom regarding the measurement structure of nature. This demonstrates, at least in the case of non-relativistic quantum mechanics, that the axioms pertaining to the laws of physics (but not those relating to the measurement structure) are redundant.

4.2 Geometric Hilbert space in 2D

We now attack the 2D case.

We recall that the general solution to the optimization problem is:

$$\rho(q,\tau) = \frac{1}{Z(\tau)}\det\psi(q,\tau),\ \text{where}\ \psi(q,\tau) = \exp\left(-\tau\frac{1}{d}\mathbf{u}(q)\right)\psi(q) \tag{77}$$

and where $p(q) = \det \psi(q)$.

In 2D, the multivector $\mathbf{u}$ is in $\mathcal{G}(\mathbb{R}^2)$. It contains a scalar a, a vector $\mathbf{x}$ and a pseudoscalar $\mathbf{b}$, and can be written as $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$.

We also recall that the determinant in 2D can be expressed as $\det \mathbf{u} = \mathbf{u}^\ddagger \mathbf{u}$, where $\mathbf{u}^\ddagger$ is the Clifford conjugate of $\mathbf{u}$.

We note that in the following sections, we wish to investigate the geometric properties of the wavefunction, and ignoring all dynamical evolution. As such, we will normalize τ to 1. This does not affect the generality of our analysis, because what follows only depends on the impact of the multivector $\mathbf{u}$ and not τ. The dynamical evolution will be investigated in Section 4.21 where τ will be reintroduced.

Consequently, we can write the solution as:

$$\rho(q,\tau)\mid_{\tau\to 1} = \frac{1}{Z}\psi(q)^\ddagger\psi(q), \text{ where } \psi(q) = \exp\left(-\frac{1}{2}\mathbf{u}(q)\right)\psi_0(q) \tag{78}$$

and where $p_0(q) = \psi_0(q)^\ddagger \psi_0(q)$.

Rewriting the determinant as the 2D multivector norm allows us to use a notation similar to the bra-ket notation used in physics. It also allows us to represent an inner product over the general linear group analogously to how the complex norm is represented for complex Hilbert spaces.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^2)$.

A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

A) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the sesquilinear map

$$\begin{aligned} \langle \cdot,\cdot \rangle \quad : \qquad \mathbb{V}\times\mathbb{V} &\longrightarrow \mathcal{G}(\mathbb{R}^2) \\ \langle \mathbf{u},\mathbf{v}\rangle &\longmapsto \mathbf{u}^\ddagger \mathbf{v} \end{aligned} \tag{79}$$

is positive-definite such that for $\boldsymbol{\psi} \neq 0$, $\langle \boldsymbol{\psi},\boldsymbol{\psi}\rangle > 0$

B) $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$. Then, for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q)) = \frac{1}{\langle \boldsymbol{\psi},\boldsymbol{\psi}\rangle}\psi(q)^\ddagger\psi(q) \tag{80}$$

is either positive or equal to zero.

We note the following comments and definitions:

- From A) and B), it follows that $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the probabilities sum up to unity:

$$\sum_{\psi(q)\in\boldsymbol{\psi}} \rho(\psi(q)) = 1 \tag{81}$$

- $\boldsymbol{\psi}$ is called a *physical state*.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\psi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$, such that the sum of probabilities remains normalized.

  $$\langle \mathbf{T}\boldsymbol{\psi}, \mathbf{T}\boldsymbol{\psi} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{82}$$

  are the *physical transformations* of $\boldsymbol{\psi}$.
- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \in \mathbb{V}$ and $\forall \mathbf{v} \in \mathbb{V}$:

  $$\langle \mathbf{O}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{v} \rangle \tag{83}$$

  is called an observable.
- The expectation value of an observable $\mathbf{O}$ is

  $$\langle \mathbf{O} \rangle = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{84}$$

4.3 Geometric self-adjoint operator in 2D

The general case of an observable in 2D is shown in this section. A matrix $\mathbf{O}$ is observable if it is a self-adjoint operator defined as:

$$\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle \tag{85}$$

$\forall \boldsymbol{\phi} \in \mathbb{V}$ and $\forall \boldsymbol{\psi} \in \mathbb{V}$.

Setup: Let $\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix}$ be an observable.

Let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be two two-state multivectors $\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{bmatrix}$ and $\boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}$. Here, the components $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$, $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, $\mathbf{o}_{11}$ are multivectors of $\mathcal{G}(\mathbb{R}^2)$.

Derivation: 1. Calculate $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle$:

$$\begin{aligned}2\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle &= (\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2)^{\ddagger}\boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^{\ddagger}(\mathbf{o}_{00}\boldsymbol{\phi}_1 + \mathbf{o}_{01}\boldsymbol{\phi}_2) \\ &\quad + (\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2)^{\ddagger}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^{\ddagger}(\mathbf{o}_{10}\boldsymbol{\phi}_1 + \mathbf{o}_{11}\boldsymbol{\phi}_2) \qquad (86)\\ &= \boldsymbol{\phi_1}^{\ddagger}\mathbf{o}_{00}^{\ddagger}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{01}^{\ddagger}\boldsymbol{\psi}_1 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{00}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{01}\boldsymbol{\phi}_2 \\ &\quad + \boldsymbol{\phi}_1^{\ddagger}\mathbf{o}_{10}^{\ddagger}\boldsymbol{\psi}_2 + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{11}^{\ddagger}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{10}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{11}\boldsymbol{\phi}_2 \qquad (87)\end{aligned}$$

2. Next, calculate $\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$:

$$\begin{aligned}2\langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle &= \boldsymbol{\phi}_1^{\ddagger}(\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2) + (\mathbf{o}_{00}\boldsymbol{\psi}_1 + \mathbf{o}_{01}\boldsymbol{\psi}_2)^{\ddagger}\boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^{\ddagger}(\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2) + (\mathbf{o}_{10}\boldsymbol{\psi}_1 + \mathbf{o}_{11}\boldsymbol{\psi}_2)^{\ddagger}\boldsymbol{\phi}_1 \qquad (88)\\ &= \boldsymbol{\phi}_1^{\ddagger}\mathbf{o}_{00}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_1^{\ddagger}\mathbf{o}_{01}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{00}^{\ddagger}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{01}^{\ddagger}\boldsymbol{\phi}_1 \\ &\quad + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{10}\boldsymbol{\psi}_1 + \boldsymbol{\phi}_2^{\ddagger}\mathbf{o}_{11}\boldsymbol{\psi}_2 + \boldsymbol{\psi}_1^{\ddagger}\mathbf{o}_{10}^{\ddagger}\boldsymbol{\phi}_1 + \boldsymbol{\psi}_2^{\ddagger}\mathbf{o}_{11}^{\ddagger}\boldsymbol{\phi}_1 \qquad (89)\end{aligned}$$

To realize $\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}, \mathbf{O}\boldsymbol{\psi} \rangle$, the following relations must hold:

$$\mathbf{o}_{00}^{\ddagger} = \mathbf{o}_{00} \qquad (90)$$

$$\mathbf{o}_{01}^{\ddagger} = \mathbf{o}_{10} \qquad (91)$$

$$\mathbf{o}_{10}^{\ddagger} = \mathbf{o}_{01} \qquad (92)$$

$$\mathbf{o}_{11}^{\ddagger} = \mathbf{o}_{11}. \qquad (93)$$

Therefore, $\mathbf{O}$ must be equal to its own Clifford transpose, indicating that $\mathbf{O}$ is observable if

$$\mathbf{O}^{\ddagger} = \mathbf{O}, \qquad (94)$$

which is the geometric generalization of the self-adjoint operator $\mathbf{O}^{\dagger} = \mathbf{O}$ of complex Hilbert spaces.

4.4 Geometric spectral theorem in 2D

The application of the spectral theorem to $\mathbf{O}^{\ddagger} = \mathbf{O}$ such that its eigenvalues are real is shown below:

Consider

$$\mathbf{O} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}, \qquad (95)$$

Then $\mathbf{O}^{\ddagger}$ is

$$\mathbf{O}^{\ddagger} = \begin{bmatrix} a_{00} & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} \end{bmatrix}, \tag{96}$$

It follows that $\mathbf{O}^{\ddagger} = \mathbf{O}$
This example is the most general 2×2 matrix $\mathbf{O}$ such that $\mathbf{O}^{\ddagger} = \mathbf{O}$.
The eigenvalues are obtained as:

$$0 = \det(\mathbf{O} - \lambda \mathbf{I}) = \det \begin{bmatrix} a_{00} - \lambda & a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12} \\ a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} & a_{11} - \lambda \end{bmatrix}, \tag{97}$$

This implies that

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a - x\hat{\mathbf{x}}_1 - y\hat{\mathbf{x}}_2 - b\hat{\mathbf{x}}_{12})(a + x\hat{\mathbf{x}}_1 + y\hat{\mathbf{x}}_2 + b\hat{\mathbf{x}}_{12} + a_{11}) \tag{98}$$

$$0 = (a_{00} - \lambda)(a_{11} - \lambda) - (a^2 - x^2 - y^2 + b^2), \tag{99}$$

Finally,

$$\lambda = \{\frac{1}{2}\left(a_{00} + a_{11} - \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right), \tag{100}$$

$$\frac{1}{2}\left(a_{00} + a_{11} + \sqrt{(a_{00} - a_{11})^2 + 4(a^2 - x^2 - y^2 + b^2)}\right)\} \tag{101}$$

The roots would be complex if $a^2 - x^2 - y^2 + b^2 < 0$. Since $a^2 - x^2 - y^2 + b^2$ is the determinant of the multivector, the complex case is ruled out for orientation-preserving multivectors. Consequently, it follows that $\mathbf{O}^{\ddagger} = \mathbf{O}$ constitutes an observable with real-valued eigenvalues for orientation-preserving multivectors.

4.5 Invariant transformations in 2D

A left action on the wavefunction $\mathbf{T}|\psi\rangle$ connects to the bilinear form as $\langle\psi|\mathbf{T}^{\ddagger}\mathbf{T}|\psi\rangle$.
The invariance requirement on $\mathbf{T}$ is

$$\langle\psi|\mathbf{T}^{\ddagger}\mathbf{T}|\psi\rangle = \langle\psi|\psi\rangle. \tag{102}$$

Therefore, we are interested in the group of matrices that follow

$$\mathbf{T}^{\ddagger}\mathbf{T} = \mathbf{I}. \tag{103}$$

Let us consider a two-state system, with a general transformation represented by

$$\mathbf{T} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}, \tag{104}$$

where u, v, w, x are the 2D multivectors.

The expression $\mathbf{T}^{\ddagger}\mathbf{T}$ is

$$\mathbf{T}^{\ddagger}\mathbf{T} = \begin{bmatrix} v^{\ddagger} & u^{\ddagger} \\ w^{\ddagger} & x^{\ddagger} \end{bmatrix} \begin{bmatrix} v & w \\ u & x \end{bmatrix} = \begin{bmatrix} v^{\ddagger}v + u^{\ddagger}u & v^{\ddagger}w + u^{\ddagger}x \\ w^{\ddagger}v + x^{\ddagger}u & w^{\ddagger}w + x^{\ddagger}x \end{bmatrix} \tag{105}$$

For $\mathbf{T}^{\ddagger}\mathbf{T} = \mathbf{I}$, the following relations must hold:

$$v^{\ddagger}v + u^{\ddagger}u = 1 \tag{106}$$

$$v^{\ddagger}w + u^{\ddagger}x = 0 \tag{107}$$

$$w^{\ddagger}v + x^{\ddagger}u = 0 \tag{108}$$

$$w^{\ddagger}w + x^{\ddagger}x = 1 \tag{109}$$

This is the case if

$$\mathbf{T} = \frac{1}{\sqrt{v^{\ddagger}v + u^{\ddagger}u}} \begin{bmatrix} v & u \\ -e^{\varphi}u^{\ddagger} & e^{\varphi}v^{\ddagger} \end{bmatrix}, \tag{110}$$

where u, v are the 2D multivectors, and e^{φ} is a unit multivector.

Comparatively, the unitary case is obtained when the vector part of the multivector vanishes, i.e., $\mathbf{x} \to 0$, and we obtain

$$\mathbf{U} = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{bmatrix} a & b \\ -e^{i\theta}b^{\dagger} & e^{i\theta}a^{\dagger} \end{bmatrix}. \tag{111}$$

Here $\mathbf{T}$ is the geometric generalization (in 2D) of unitary transformations.

4.6 Gravity in FX/SO(2)

We will now investigate the quotient bundle associated with the structure reduction from $\mathrm{GL}^+(2, \mathbb{R})$ to $\mathrm{SO}(2)$.

Let X^2 be a smooth orientable real-valued manifold in 2D. We consider its tangent bundle TX and its associated frame bundle FX. Since X^2 is orientable, its structure group is $\mathrm{GL}^+(2, \mathbb{R})$. The action by our wavefunction, valued in $\exp \mathcal{G}(\mathbb{R}^2) \cong \exp \mathbb{M}(2, \mathbb{R})$ generates $\mathrm{GL}^+(2, \mathbb{R})$, and thus acts on FX. We now consider a reduction of the structure group of FX to $\mathrm{SO}(2)$.

Let us begin by investigating the cosets of $\mathrm{SO}(2)$ in $\mathrm{GL}^+(2, \mathbb{R})$. Let $g_1 \in \mathrm{GL}^+(2, \mathbb{R})$, $g_2 \in \mathrm{GL}^+(2, \mathbb{R})$ and $s \in \mathrm{SO}(2)$. We now identify the relation

$g_2 = g_1 s$. We also note $g_2^T = s^T g_1^T$. Finally, we note the product $g_2 g_2^T = g_1 s s^T g_1^T \implies g_2 g_2^T = g_1 g_1^T$. Since $g_1 g_1^T$ and $g_2 g_2^T$ are symmetric positive-definite 2×2 matrices, one verifies a diffeomorphism between $\mathrm{GL}^+(2,\mathbb{R})/\mathrm{SO}(2)$ and the inner products.

The global section of the quotient bundle FX/SO(2) is a tetrad field $h_\mu^a(x)$ and it associates to a Riemannian metric on X^2 via the identity $g_{\mu\nu} = h_\mu^a h_\nu^b \eta_{ab}$. The connection that preserves the structure SO(2) across the manifold are the metric connections[9], and with the additional requirement of no torsion, the connections reduce to the Levi-Civita connection. It has been shown recently[10] that the Goldstone fields associated with the quotient bundle have enough degrees of freedom to create a metric and a covariant derivative. Finally, the frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^2[11]. This is the geometric setting for gravity.

In this work, we have merely maximized the entropy of all possible geometric measurements, and we have arrived, without introducing any other assumptions, at a general linear quantum theory holding in the $\mathrm{GL}^+(2,\mathbb{R})$ group, whose symmetry breaks into a theory of gravity (FX/SO(2)) and into a quantum theory of the special orthogonal group (valued in SO(2)).

4.7 Wavefunction in SO(2)

With its structure reduced to SO(2), we thus arrived at a quantum theory of the special orthogonal group, where the wavefunction defines the action on a vector of the tangent space of the manifold as follows:

$$\psi(x,y)^\ddagger \hat{\mathbf{x}}_0 \psi(x,y) = \exp\left(\frac{1}{2}\mathbf{i}B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\mathbf{i}B(x,y)\right) \tag{112}$$

$$= \exp\left(\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right) \tag{113}$$

The expression $\exp\left(\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)$ maps $\hat{\mathbf{x}}_0$ to a curvilinear basis $\mathbf{e}_0$ via the application of the rotor and its reverse:

$$\exp\left(\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right)\hat{\mathbf{x}}_0 \exp\left(-\frac{1}{2}\hat{\mathbf{x}}_0\hat{\mathbf{x}}_1 B(x,y)\right) = \mathbf{e}_0 \tag{114}$$

Consequently, we have obtained a 2D relativistic wavefunction (with a Euclidean signature in this case). This is the 2D version of David Hestenes' geometric algebra formulation of the relativistic wavefunction. In the 3+1D case, we will see that the wavefunction has 6 generators for rotations and boosts and one generator for a complex phase.

4.8 Metric interference in 2D

We now consider a transformation $\mathbf{T}^{\ddagger}\mathbf{T} = \mathbf{I}$ and a wavefunction $|\psi\rangle = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$ such that a multivector $\mathbf{u}$ is mapped to a linear combination of two multivectors. Let us consider this transformation:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{u}-\mathbf{v} \end{bmatrix} \tag{115}$$

We can now investigate the probability:

$$\rho(\mathbf{u}+\mathbf{v}) = \frac{1}{Z} \det(\mathbf{u}+\mathbf{v}), \text{ where } Z = \det(\mathbf{u}+\mathbf{v}) + \det(\mathbf{u}-\mathbf{v}) \tag{116}$$

We proceed as follows:

$$\begin{aligned} \det(\mathbf{u}+\mathbf{v}) &= (\mathbf{u}+\mathbf{v})^{\ddagger}(\mathbf{u}+\mathbf{v}) && (117)\\ &= (\mathbf{u}^{\ddagger}+\mathbf{v}^{\ddagger})(\mathbf{u}+\mathbf{v}) && (118)\\ &= (\mathbf{u}^{\ddagger}\mathbf{u}+\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u}+\mathbf{v}^{\ddagger}\mathbf{v}) && (119)\\ &= \det\mathbf{u}+\det\mathbf{v}+\mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u} && (120)\\ &= \det\mathbf{u}+\det\mathbf{v}+\mathbf{u}\cdot\mathbf{v} && (121) \end{aligned}$$

where we have defined the dot product between multivectors as follows:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^{\ddagger}\mathbf{v}+\mathbf{v}^{\ddagger}\mathbf{u} \tag{122}$$

Since $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$, then $\mathbf{u}\cdot\mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$ (whichever is greater). Therefore, it also satisfies the conditions of an interference term capable of destructive and constructive interference.

In the case $\mathbf{x} \to 0$, the interference pattern reduces to a form identical to the unitary case:

$$\begin{aligned} \det\Big(\psi_1 e^{-\frac{1}{2}\mathbf{b}_1} + \psi_2 e^{-\frac{1}{2}\mathbf{b}_2}\Big) &= \det\psi_1 + \det\psi_2 + 2\psi_1\psi_2 e^{-\frac{1}{2}\mathbf{b}_1-\frac{1}{2}\mathbf{b}_2} && (123)\\ &= |\psi_1|^2 + |\psi_2|^2 + 2\psi_1\psi_2 e^{-\frac{1}{2}\mathbf{b}_1-\frac{1}{2}\mathbf{b}_2} && (124) \end{aligned}$$

whereas, in the general linear case, we would have

$$\begin{aligned} &\det\left(\psi_1 e^{-\frac{1}{2}(a_1+\mathbf{x}_1+\mathbf{b}_1)} + \psi_2 e^{-\frac{1}{2}(a_2+\mathbf{x}_2+\mathbf{b}_2)}\right) && (125)\\ &\quad= \det\psi_1 + \det\psi_2 + 2\psi_1\psi_2\left(e^{-\frac{1}{2}(a_1+\mathbf{x}_1+\mathbf{b}_1)} + e^{-\frac{1}{2}(a_2+\mathbf{x}_2+\mathbf{b}_2)}\right) && (126) \end{aligned}$$

which includes non-commutative effects in the interference pattern.

4.9 A *double-copy* geometric Hilbert space in 4D

In 2D, the determinant can be expressed using only the product $\psi^\ddagger\psi$, which can be interpreted as the inner product of two multivectors. This form allowed us to extend the complex Hilbert space to a *geometric* Hilbert space. We then found that the familiar properties of the complex Hilbert spaces were transferable to the geometric Hilbert space, eventually yielding a 2D gravitized quantum theory in the language of geometric algebra.

Although a similar correspondence exists in 4D, it is less recognizable because we need a *double-copy* inner product (i.e., $\rho = \lfloor \phi^\ddagger\phi \rfloor_{3,4}\phi^\ddagger\phi$) to produce a real-valued probability in 4D.

Thus, in 4D, we cannot produce an inner product as in the 2D case. The absence of a satisfactory inner product indicates no Hilbert space in the usual sense of a complete *inner product* vector space.

We aim to find a construction that supports the geometric wavefunction in 4D.

To build the right construction, a double-copy inner product of four terms is devised, superseding the inner product in the Hilbert space, mapping any four vectors to an element of $\mathcal{G}(\mathbb{R}^{3,1})$, and yielding a complete *double-copy* inner product vector space — or simply, a *double-copy* Hilbert space.

We note that the construction will be more familiar than it may first appear. Indeed, the familiar quantum mechanical features (linear transformations, unit vectors, and linear superposition in the probability measure, etc.) will be supported in the construction, and just as it did in 2D, it will also here break into a familiar inner-product Hilbert space whose Dirac current is invariant for SU(3)×SU(2)×U(1) and into a theory of gravity and of electromagnetism for charged fermions FX/Spin$^c(3,1)$.

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$.

A subset of vectors in $\mathbb{V}$ forms a double-copy algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, the double-copy inner product form

$$\begin{aligned} \langle \cdot,\cdot,\cdot,\cdot \rangle \quad : \qquad & \mathbb{V}\times\mathbb{V}\times\mathbb{V}\times\mathbb{V} \longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ & \langle \mathbf{u},\mathbf{w},\mathbf{y},\mathbf{z}\rangle \longmapsto \sum_{i=1}^{m} \lfloor u_i^\ddagger w_i \rfloor_{3,4} y_i^\ddagger z_i \end{aligned} \tag{127}$$

   is positive-definite when $\boldsymbol{\phi} \neq 0$; that is $\langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle > 0$

2. $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, then for each element $\phi(q) \in \boldsymbol{\phi}$, the function

$$\rho(\phi(q)) = \frac{1}{\langle \boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi},\boldsymbol{\phi}\rangle} \det \phi(q), \tag{128}$$

   is either positive or equal to zero.

We note the following properties, features, and comments:

- From A) and B), it follows that, $\forall \boldsymbol{\phi} \in \mathcal{A}(\mathbb{V})$, and the probabilities sum to unity.

$$\sum_{\phi(q)\in\boldsymbol{\phi}} \rho(\phi(q)) = 1 \tag{129}$$

- $\boldsymbol{\phi}$ is called a *physical state*.
- $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle$ is called the *partition function* of $\boldsymbol{\phi}$.
- If $\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle = 1$, then $\boldsymbol{\phi}$ is called a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\phi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\phi}$ such as $\mathbf{T}\boldsymbol{\phi} \to \boldsymbol{\phi}'$ makes the sum of probabilities normalized (invariant):

$$\langle \mathbf{T}\boldsymbol{\phi}, \mathbf{T}\boldsymbol{\phi}, \mathbf{T}\boldsymbol{\phi}, \mathbf{T}\boldsymbol{\phi} \rangle = \langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle \tag{130}$$

  are the *physical transformations* of $\boldsymbol{\phi}$.
- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{w} \forall \mathbf{y} \forall \mathbf{z} \in \mathbb{V}$:

$$\langle \mathbf{Ou}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{Ow}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w}, \mathbf{Oy}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{Oz} \rangle \tag{131}$$

  is called an observable.
- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle}{\langle \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi}, \boldsymbol{\phi} \rangle} \tag{132}$$

4.10 Wavefunction in 3+1D

In the David Hestenes' notation[12], the 3+1D wavefunction is expressed as:

$$\psi = \sqrt{\rho e^{-ib}} R, \tag{133}$$

where ρ represents a scalar probability density, e^{ib} is a complex phase, and R is a rotor.

Comparatively, our wavefunction in $\mathcal{G}(\mathbb{R}^{3,1})$ is:

$$\phi = e^{-\frac{1}{4}(a+\mathbf{x}+\mathbf{f}+\mathbf{v}+\mathbf{b})}\phi_0 \tag{134}$$

To recover David Hestenes' formulation of the wavefunction, it suffices to eliminate the terms $a \to 0$, $\mathbf{x} \to 0$ and $\mathbf{v} \to 0$, and to perform a substitution of the entries of the double-copy inner product (Equation 142), as follows:

$$\mathbf{w} \to \mathbf{u}^{\ddagger} \tag{135}$$

$$\mathbf{y} \to \mathbf{z}^{\ddagger} \tag{136}$$

As one of the copies is destroyed by the substitution, the double-copy inner product reduces to an inner product. Furthermore, with the elimination, the blade-3,4 conjugate is also reduced to the blade-4 conjugate, yielding

$$\langle \mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z} \rangle \to \langle \mathbf{u}, \mathbf{u}^{\ddagger}, \mathbf{z}^{\ddagger}, \mathbf{z} \rangle \cong \langle \mathbf{u}, \mathbf{z} \rangle = \sum_{i=1}^{m} \lfloor u_i^2 \rfloor_{2,4}(z_i^2) \tag{137}$$

Consequently, our wavefunction ϕ reduces to

$$\phi^2 = e^{-\frac{1}{2}(\mathbf{f}+\mathbf{b})}\phi_0^2 \tag{138}$$

This shows that the 3+1D wavefunction (comprising a rotor $R = e^{-\frac{1}{2}\mathbf{f}}$, a pseudo-scalar $e^{-\frac{1}{2}\mathbf{b}}$ and a prior probability $\phi_0^2 = \sqrt{\rho}$) is a sub-structure of the general $\mathcal{G}(\mathbb{R}^{3,1})$ wavefunction. The primary difference is that our formulation lives in a grade 2-4 geometric Hilbert space.

In this sub-structure, the observables are satisfied when

$$\lfloor \mathbf{O} \rfloor_{2,4} = \mathbf{O} \tag{139}$$

Let us now analyze the symmetry group of this wavefunction.

First, we note that the term $\mathbf{b}$ commutes with $\mathbf{f}$. They can be factored out as

$$e^{-\frac{1}{2}(\mathbf{f}+\mathbf{b})}\phi_0^2 = e^{-\frac{1}{2}\mathbf{b}}e^{-\frac{1}{2}\mathbf{f}}\phi_0^2 \tag{140}$$

Second, the term $\exp \mathbf{f}$ can be understood as the exponential map from the bivectors to the $\mathrm{Spin}_+(3,1)$ group and the term $\exp \mathbf{b}$ to $\mathrm{U}(1)$.

Finally, since $\mathrm{Spin}_+(3,1) \cap \exp \mathbf{b} = \{\pm 1\}$, it must be removed from the group product[13].

We conclude that the geometric components of the wavefunction correspond to the following group

$$\mathrm{U}(1) \times (\mathrm{Spin}_+(3,1)/\{\pm 1\}) \cong \mathrm{Spin}^c(3,1) \tag{141}$$

4.11 Geometric Hilbert space in 3+1D (broken symmetry)

The substitution given by Equation 137 yields the following algebra of geometric observables:

Let $\mathbb{V}$ be an m-dimensional vector space over $\mathcal{G}(\mathbb{R}^{3,1})$.

A subset of vectors in $\mathbb{V}$ forms an algebra of observables $\mathcal{A}(\mathbb{V})$ if the following holds:

1. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, the inner product form

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \qquad \mathbb{V} \times \mathbb{V} &\longrightarrow \mathcal{G}(\mathbb{R}^{3,1}) \\ \langle \mathbf{u}, \mathbf{w} \rangle &\longmapsto \sum_{i=1}^{m} \lfloor u_i^2 \rfloor_{2,4} w_i^2 \end{aligned} \tag{142}$$

   is positive-definite when $\boldsymbol{\psi} \neq 0$; that is $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle > 0$

2. $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, then for each element $\psi(q) \in \boldsymbol{\psi}$, the function

$$\rho(\psi(q)) = \frac{1}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \det \psi(q), \tag{143}$$

   is either positive or equal to zero.

We note the following properties, features, and comments:

- From A) and B), it follows that, $\forall \boldsymbol{\psi} \in \mathcal{A}(\mathbb{V})$, and the probabilities sum to unity.

$$\sum_{\psi(q) \in \boldsymbol{\psi}} \rho(\psi(q)) = 1 \tag{144}$$

- $\boldsymbol{\psi}$ is called a *physical state*.
- $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$ is called the *partition function* of $\boldsymbol{\phi}$.
- If $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = 1$, then $\boldsymbol{\psi}$ is called a unit vector.
- $\rho(q)$ is called the *probability measure* (or generalized Born rule) of $\psi(q)$.
- The set of all matrices $\mathbf{T}$ acting on $\boldsymbol{\psi}$ such as $\mathbf{T}\boldsymbol{\psi} \to \boldsymbol{\psi}'$ makes the sum of probabilities normalized (invariant):

$$\langle \mathbf{T}\boldsymbol{\psi}, \mathbf{T}\boldsymbol{\psi} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{145}$$

  are the *physical transformations* of $\boldsymbol{\psi}$.

- A matrix $\mathbf{O}$ such that $\forall \mathbf{u} \forall \mathbf{w} \in \mathbb{V}$:

$$\langle \mathbf{O}\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{O}\mathbf{w} \rangle \tag{146}$$

  is called an observable.

- The expectation value of an observable $\mathbf{O}$ is

$$\langle \mathbf{O} \rangle = \frac{\langle \mathbf{O}\boldsymbol{\psi}, \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle} \tag{147}$$

4.12 Gravity and electromagnetism in 3+1D

In 2D, we benefited from a coincidence of low dimensions, where the matrix representation of $\mathcal{G}(\mathbb{R}^2)$ was in $\mathbb{M}(2, \mathbb{R})$. As such, our wavefunction generated $\mathrm{GL}^+(2, \mathbb{R})$ which acted as the structure group of the frame bundle FX, and following a structure reduction from $\mathrm{GL}^+(2, \mathbb{R})$ to $\mathrm{SO}(2)$, a tetrad field was associated with the global section of the quotient bundle FX/SO(2) which led to a gravitized quantum theory.

In 4D, unlike in 2D where $\mathrm{SO}(2) = \mathrm{Spin}(2)$, the geometry of the wavefunction is not in SO but rather in Spin^c (since 4D also contains a pseudoscalar in addition to bivectors). And since Spin^c is not, in general, in GL^+, we cannot benefit from the same coincidences as in 2D.

Typically, to reach $\mathrm{Spin}(p, q)$ from the structure group $\mathrm{GL}(p+q)$, one would reduce $\mathrm{GL}(p+q)$ to $\mathrm{O}(p, q)$, then lift it to $\mathrm{Spin}(p, q)$. Here, however, we will use a different approach to get the spin connection.

Remarkably, 4D admits a coincidence that will allow us to embed the $\mathrm{Spin}^c(3, 1)$ group into the $\mathrm{GL}^+(4, \mathbb{R})$ group, then take its quotient $\mathrm{FX}/\mathrm{Spin}^c(3, 1)$ without having to lift to a larger geometric structure; our solution already contains what is necessary to take this quotient.

The coincidence comes from the standard classification of real Clifford algebra[14] and from the fact that $\exp(\mathbf{f} + \mathbf{b}) \cong \mathrm{Spin}^c(3, 1) \subset \exp \mathcal{G}(\mathbb{R}^{3,1})$. The diagram

$$\begin{array}{ccc} \mathcal{G}(\mathbb{R}^{3,1}) & \xrightarrow{f} & \mathbb{M}(4, \mathbb{R}) \\ \Big\downarrow{\scriptstyle exp} & & \Big\downarrow{\scriptstyle exp} \\ \exp \mathcal{G}(\mathbb{R}^{3,1}) & \xrightarrow{f} & \mathrm{GL}^+(4, \mathbb{R}) \end{array} \tag{148}$$

commutes by group homomorphisms. Since $\exp(\mathbf{f} + \mathbf{b}) \cong \mathrm{Spin}^c(3, 1) \subset \exp \mathcal{G}(\mathbb{R}^{3,1})$, the map f embeds $\mathrm{Spin}^c(3, 1)$ into $\mathrm{GL}^+(4, \mathbb{R})$. The inclusion of $\mathrm{Spin}^c(3, 1)$ in $\exp \mathcal{G}(\mathbb{R}^{3,1})$ is required to break the symmetry into exactly a theory of gravity and of electromagnetism for charged fermions and into a $\mathrm{Spin}^c(3, 1)$-valued quantum theory. We are now ready.

Let X^4 be a world manifold.

We first consider the tangent bundle TX along with its associated frame bundle FX. Our wavefunction acts on the frame bundle using the exponential map of multivectors $\exp \mathcal{G}(\mathbb{R}^{3,1}) \cong \exp \mathbb{M}(4,\mathbb{R})$ which generates $\mathrm{GL}^+(4,\mathbb{R})$.

The desired reduction is from $\exp \mathcal{G}(\mathbb{R}^{3,1})$ to the $\mathrm{Spin}^c(3,1)$ group. With its symmetry reduced, the wavefunction will assign an element of $\mathrm{Spin}^c(3,1)$ to each event $x \in X^4$. The connection that preserves the structure is a $\mathrm{Spin}^c(3,1)$ preserving connection. It relates to a theory of gravity and electromagnetism for charged fermions. We note that since $\mathrm{SO}(3,1) \times \mathrm{U}(1)$ is a quotient $\mathrm{Spin}^c(3,1)$, the cosets are further associable with the inner products. Thus, the global section of the quotient bundle $\mathrm{FX}/\mathrm{SO}(3,1)$ associates with a tetrad field that uniquely determines a pseudo-Riemannian metric. As for the U(1)-bundle, it is simply the geometric setting for electromagnetism. Finally, the frame bundle is a natural bundle that admits general covariant transformations, which are the symmetries of the gravitation theory on X^4[11]. This is the geometric setting for gravity and electromagnetism.

4.13 Dirac current

David Hestenes[12] defines the Dirac current in the language of geometric algebra as:

$$\mathbf{j} = \psi^\ddagger \gamma_0 \psi = \rho R^\ddagger \gamma_0 R = \rho e_0 = \rho v \tag{149}$$

where v is the proper velocity.

In our formulation, this relation also holds: the Dirac current represents the action of the wavefunction on the unit time-like vector in the tangent space on X^4. Specifically, the Dirac current is a statistically weighted Lorentz action on γ_0:

$$\begin{aligned}
\mathbf{j} &= \psi^\ddagger \gamma_0 \psi && (150)\\
&= e^{-\frac{1}{2}\mathbf{f}+\frac{1}{2}\mathbf{b}} \phi_0 \gamma_0 e^{\frac{1}{2}\mathbf{f}+\frac{1}{2}\mathbf{b}} \phi_0 && (151)\\
&= \phi_0^2 e^{-\frac{1}{2}\mathbf{f}} \gamma_0 e^{\frac{1}{2}\mathbf{f}} && (152)\\
&= \rho e_0 && (153)\\
&= \rho v && (154)
\end{aligned}$$

We now have all the tools required to construct particle physics by exhausting the remaining geometry of our solution.

4.14 SU(2) $\times$ U(1) group

Our wavefunction transforms as a group under multiplication. We now ask, what is the most general multivector $e^{\mathbf{u}}$ which leaves the Dirac current invariant?

$$\psi^\ddagger (e^{\mathbf{u}})^\ddagger \gamma_0 e^{\mathbf{u}} \psi = \psi^\ddagger \gamma_0 \psi \iff (e^{\mathbf{u}})^\ddagger \gamma_0 e^{\mathbf{u}} = \gamma_0 \tag{155}$$

When is this satisfied?

The bases of the bivector part $\mathbf{f}$ of $\mathbf{u}$ are $\gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3, \gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$. Among these, only $\gamma_1\gamma_2, \gamma_1\gamma_3$, and $\gamma_2\gamma_3$ commute with γ_0, and the rest anti-commute; therefore, the rest must be made equal to 0. Finally, the base $\gamma_0\gamma_1\gamma_2\gamma_3$ anti-commutes with γ_0 and cancels out.

Consequently, the most general exponential multivector of the form $e^{\mathbf{u}}$ where $\mathbf{u} = \mathbf{f} + \mathbf{b}$ which preserves the Dirac current is

$$e^{\mathbf{u}} = \exp\left(\frac{1}{2}F_{12}\gamma_1\gamma_2 + \frac{1}{2}F_{13}\gamma_1\gamma_3 + \frac{1}{2}F_{23}\gamma_2\gamma_3 + \frac{1}{2}\mathbf{b}\right) \tag{156}$$

We can rewrite the bivector basis with the Pauli matrices

$$\gamma_2\gamma_3 = \mathbf{i}\sigma_x \tag{157}$$

$$\gamma_1\gamma_3 = \mathbf{i}\sigma_y \tag{158}$$

$$\gamma_1\gamma_2 = \mathbf{i}\sigma_z \tag{159}$$

$$\mathbf{b} = \mathbf{i}b \tag{160}$$

After replacements, we obtain

$$e^{\mathbf{u}} = \exp\frac{1}{2}\mathbf{i}(F_{12}\sigma_z + F_{13}\sigma_y + F_{23}\sigma_x + b) \tag{161}$$

The terms $F_{23}\sigma_x + F_{13}\sigma_y + F_{12}\sigma_z$ and b are responsible for SU(2) and U(1) symmetries, respectively[15, 16].

4.15 SU(3) group

The invariance transformations identified by the 3+1D algebra of geometric observables (Equation 145) are $\mathbf{T}^\ddagger\mathbf{T} = \mathbf{I}$, $\mathbf{T}^\dagger\mathbf{T} = \mathbf{I}$ and $\lfloor\mathbf{T}\rfloor_{2,4}\mathbf{T} = \mathbf{I}$. In the first case, the identified evolution is bivectorial rather than unitary.

As we did for the SU(2) $\times$ U(1) case, we ask, in this case, what is the most general bivectorial evolution that leaves the Dirac current invariant?

$$\mathbf{f}^\ddagger \gamma_0 \mathbf{f} = \gamma_0 \tag{162}$$

where $\mathbf{f}$ is a bivector:

$$\mathbf{f} = F_{01}\gamma_0\gamma_1 + F_{02}\gamma_0\gamma_2 + F_{03}\gamma_0\gamma_3 + F_{23}\gamma_2\gamma_3 + F_{13}\gamma_1\gamma_3 + F_{12}\gamma_1\gamma_2 \tag{163}$$

Explicitly, the expression $\mathbf{f}^\ddagger\gamma_0\mathbf{f}$ is

$$\mathbf{f}^\ddagger\gamma_0\mathbf{f} = -\mathbf{f}\gamma_0\mathbf{f} = (F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2)\gamma_0 \tag{164}$$
$$+ (-2F_{02}F_{12} + 2F_{03}F_{13})\gamma_1 \tag{165}$$
$$+ (-2F_{01}F_{12} + 2F_{03}F_{23})\gamma_2 \tag{166}$$
$$+ (-2F_{01}F_{13} + 2F_{02}F_{23})\gamma_3 \tag{167}$$

For the Dirac current to remain invariant, the cross-product must vanish:

$$-2F_{02}F_{12} + 2F_{03}F_{13} = 0 \tag{168}$$
$$-2F_{01}F_{12} + 2F_{03}F_{23} = 0 \tag{169}$$
$$-2F_{01}F_{13} + 2F_{02}F_{23} = 0 \tag{170}$$

leaving only

$$\mathbf{f}^\ddagger\gamma_0\mathbf{f} = (F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2)\gamma_0. \tag{171}$$

Finally, $F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2$ must equal 1.

We note that we can re-write $\mathbf{f}$ as a 3-vector with complex components:

$$\mathbf{f} = (F_{01} + \mathbf{i}F_{23})\gamma_0\gamma_1 + (F_{02} + \mathbf{i}F_{13})\gamma_0\gamma_2 + (F_{03} + \mathbf{i}F_{12})\gamma_0\gamma_3 \tag{172}$$

Then, with the nullification of the cross-product and equating $F_{01}^2 + F_{02}^2 + F_{03}^2 + F_{13}^2 + F_{23}^2 + F_{12}^2$ to unity, we can understand the bivectorial evolution when constrained by the Dirac current to be a realization of the SU(3) group[16].

4.16 Satisfiability of geometric observables in 4D

In 4D, an observable must satisfy equation 131. Let us now verify that geometric observables are satisfiable in 4D. For simplicity, let us take m in equation 142 to be 1. Then,

$$\lfloor(\mathbf{O}u)^\ddagger w\rfloor_{3,4} y^\ddagger z = \lfloor u^\ddagger \mathbf{O} w\rfloor_{3,4} y^\ddagger z = \lfloor u^\ddagger w\rfloor_{3,4} (\mathbf{O}y)^\ddagger z = \lfloor u^\ddagger w\rfloor_{3,4} y^\ddagger \mathbf{O} z \tag{173}$$

where u_1, w_1, y_1 and z_1 are multivectors.

Let us investigate.

If $\mathbf{O}$ contained a vector, bivector, pseudo-vector, or pseudo-scalar, the equality would not be satisfied as these terms do not commune with the multivectors and cannot be factored out. The equality is satisfied if $\mathbf{O} \in \mathbb{R}$. Indeed, as a real value, $\mathbf{O}$ commutes with all multivectors and hence, can be factored out to satisfy the equality.

We thus find that observables are satisfied in the general 4D case. We also recall that in 3+1D, the observable reduces to $\lfloor\mathbf{O}\rfloor_{2,4} = \mathbf{O}$, which is also satisfiable.

4.17 Unsatisfiability of geometric observables in 6D and above

At dimensions of 6 or above, the corresponding observable relation cannot be satisfied. To see why, we look at the results[17] of Acus et al. regarding the 6D multivector norm. The authors performed an exhaustive computer-assisted search for the geometric algebra expression for the determinant in 6D; as conjectured, they found no norm defined via self-products. The norm is a linear combination of self-products.

The system of linear equations is too long to list in its entirety; the author gives this mockup:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \tag{174}$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \tag{175}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{176}$$

$$\langle 74 \text{ monomials} \rangle = 0 \tag{177}$$

The author then produces the special case of this norm that holds only for a 6D multivector comprising a scalar and a grade 4 element:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{178}$$

Even in this simplified special case, formulating a linear relationship for observables is doomed to fail. Indeed, the real portion of the observable cannot be extracted from the equation. We find that for any function f_i and g_i, the coefficients b_1 and b_2 will frustrate the equality:

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{179}$$

$$= b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \tag{180}$$

Equations 179 and 180 can only be equal if $b_1 = b_2$; however, the norm $s(B)$ requires both to be different. Consequently, the relation for observables in 6D is unsatisfiable even by real numbers.

Thus, in our solution, observables are satisfied in 6D.

Furthermore, since the norms involve more sophisticated systems of linear equations in higher dimensions, this result is likely to generalize to all dimensions above 6.

4.18 Defective probability measure in 3D and 5D

The 3D and 5D cases (and possibly all odd-dimensional cases of higher dimensions) contain a number of irregularities that make them defective for use in this framework. Let us investigate.

In $\mathcal{G}(\mathbb{R}^3)$, the matrix representation of a multivector is as follows:

$$\mathbf{u} = a + x\sigma_x + y\sigma_y + z\sigma_z + q\sigma_y\sigma_z + v\sigma_x\sigma_z + w\sigma_x\sigma_y + b\sigma_x\sigma_y\sigma_z \tag{181}$$

is

$$\mathbf{u} \cong \begin{bmatrix} a+ib+iw+z & iq-v+x-iy \\ iq+v+x+iy & a+ib-iw-z \end{bmatrix} \tag{182}$$

and the determinant is

$$\det \mathbf{u} = a^2 - b^2 + q^2 + v^2 + w^2 - x^2 - y^2 - z^2 + 2i(ab - qx + vy - wz) \tag{183}$$

The result is a complex-valued probability. Since a probability must be real-valued, the 3D case is defective in our solution and cannot be used. In theory, it can be fixed by defining a complex norm to apply to the determinant:

$$\langle \mathbf{u}, \mathbf{u} \rangle = (\det \mathbf{u})^\dagger \det \mathbf{u} \tag{184}$$

However, defining such a norm would entail a double-copy inner product of 4 multivectors, but the space is only 3D, not 4D (so why four?). It would also break the relationship between trace and probability that justified its usage in statistical mechanics.

Consequently, this case appears to us to be defective.

Perhaps, instead of $\mathcal{G}(\mathbb{R}^3)$ multivectors, we ought to use 3×3 matrices in 3D? Alas, 3×3 matrices do not admit a geometric algebra representation because they are not isomorphic with $\mathcal{G}(\mathbb{R}^3)$. Indeed, $\mathcal{G}(\mathbb{R}^3)$ has 8 parameters and 3×3 matrices have 9. 3×3 matrices are not representable geometrically in the same sense that 2×2 matrices are with $\mathcal{G}(\mathbb{R}^2)$.

In $\mathcal{G}(\mathbb{R}^{4,1})$, the algebra is isomorphic to complex 4×4 matrices. In this case, the determinant and probability would be complex-valued, making the case defective. Furthermore, 5×5 matrices have 25 parameters, but $\mathcal{G}(\mathbb{R}^{4,1})$ multivectors have 32 parameters.

4.19 The dimensions that admit observable geometry

Our solution is non-defective in the following dimensions:

- $\mathbb{R}$: This case corresponds to familiar statistical mechanics. The constraints are scalar $\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q)$, and the probability measure is the Gibbs measure $\rho(q) = \frac{1}{Z(\beta)} \exp\big(-\beta E(q)\big)$.
- $\mathbb{C} \cong \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$: This case corresponds to familiar non-relativistic quantum mechanics.

However, neither of these cases contain geometry. The only case that contains observable geometry are:

- $\mathcal{G}(\mathbb{R}^2)$: This case corresponds to the geometric quantum theory in 2D. Its $\mathrm{GL}^+(2)$ symmetry breaks into a theory of gravity FX/SO(2) and into a quantum theory valued in SO(2).
- $\mathcal{G}(\mathbb{R}^{3,1})$: This case is valid. Like the 2D case, it also corresponds to a geometric quantum theory. As such, its symmetry will break into a theory of gravity and a relativistic wavefunction. But unlike the 2D case, the wavefunction further admits an invariance with respect to the SU(2)×U(1) and SU(3) gauge groups.

In contrast, our solution is defective in the following dimensions:

- $\mathcal{G}(\mathbb{R}^3)$: In this case, the probability measure is complex-valued.
- $\mathcal{G}(\mathbb{R}^{4,1})$: In this case, the probability measure is complex-valued.
- 6D and above: For $\mathcal{G}(\mathbb{R}^n)$, where $n \geq 6$, no observables satisfy the corresponding observable equation, in general.

We may thus say that 3D and 5D fail to normalize, and 6D and above fail to satisfy observables. Consequently, in the general case of our solution, normalizable geometric observables cannot be satisfied beyond 4D. This suggests an intrinsic limit to the dimensionality of observable geometry and, by extension, to spacetime.

4.20 Metric interference in 3+1D

A geometric wavefunction would allow a larger class of interference patterns than complex interference. The geometric interference pattern includes the ways in which the geometry of a probability measure can interfere constructively or destructively and includes interference from rotations, phases, boosts, shears, spins, and dilations.

In the case of 4D *metric interference* (shown below), the interference pattern is associated with a superposition of elements of the group $\mathrm{Spin}^c(3,1)$, whose subgroup $\mathrm{SO}(3,1)$ associates with a superposition of inner products in the quotient.

It is possible that a *sensitive* Aharonov–Bohm effect experiment on gravity[18] could detect special cases of the geometric phase and interference patterns identified in this section.

An interference pattern follows from a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and the application of the determinant:

$$\det(\mathbf{u}+\mathbf{v}) = \det \mathbf{u} + \det \mathbf{v} + \mathbf{u} \cdot \mathbf{v} \tag{185}$$

The determinants $\det \mathbf{u}$ and $\det \mathbf{v}$ are a sum of probabilities, whereas the dot product term $\mathbf{u} \cdot \mathbf{v}$ represents the interference term.

Such can be obtained following a transformation of a wavefunction $|\psi\rangle = \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$ such that the multivectors are mapped to a linear combination of two multivectors:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{u}+\mathbf{v}\\ \mathbf{u}-\mathbf{v}\end{bmatrix} \tag{186}$$

The dot product defines a bilinear form.

$$\begin{aligned} \cdot \quad : \qquad \mathcal{G}(\mathbb{R}^{m,n}) \times \mathcal{G}(\mathbb{R}^{m,n}) &\longrightarrow \mathbb{R} && (187)\\ \mathbf{u}\cdot\mathbf{v} &\longmapsto \frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) && (188) \end{aligned}$$

If $\det\mathbf{u} > 0$ and $\det\mathbf{v} > 0$, then $\mathbf{u}\cdot\mathbf{v}$ is always positive, thereby qualifying as a positive-definite inner product, but not greater than either $\det\mathbf{u}$ or $\det\mathbf{v}$ (whichever is greater). Therefore, it also satisfies the conditions of an interference term.

In 2D, the dot product has this form:

$$\begin{aligned} &\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) && (189)\\ &= \frac{1}{2}\left((\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v}\right) && (190)\\ &= \mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v} - \mathbf{u}^\ddagger\mathbf{u} - \mathbf{v}^\ddagger\mathbf{v} && (191)\\ &= \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} && (192) \end{aligned}$$

In 3+1D, it has this form.

$$\begin{aligned} &\frac{1}{2}(\det(\mathbf{u}+\mathbf{v}) - \det\mathbf{u} - \det\mathbf{v}) && (193)\\ &= \frac{1}{2}\left(\lfloor(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v})\rfloor_{3,4}(\mathbf{u}+\mathbf{v})^\ddagger(\mathbf{u}+\mathbf{v}) - \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} - \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\right) && (194)\\ &= \frac{1}{2}\left(\lfloor\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}(\mathbf{u}^\ddagger\mathbf{u} + \mathbf{u}^\ddagger\mathbf{v} + \mathbf{v}^\ddagger\mathbf{u} + \mathbf{v}^\ddagger\mathbf{v}) - \dots\right) && (195) \end{aligned}$$

$$\begin{aligned} = {} &\lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\ &+ \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{u}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\ &+ \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v}\\ &+ \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{v} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{u} + \lfloor\mathbf{v}^\ddagger\mathbf{v}\rfloor_{3,4}\mathbf{v}^\ddagger\mathbf{v} - \dots \end{aligned} \tag{196}$$

$$
\begin{aligned}
&= \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} + \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{v} \\
&\quad + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} + \lfloor \mathbf{u}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} + \lfloor \mathbf{v}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{v} \\
&\quad + \lfloor \mathbf{v}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} + \lfloor \mathbf{v}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{v} + \lfloor \mathbf{v}^\ddagger \mathbf{v} \rfloor_{3,4} \mathbf{v}^\ddagger \mathbf{u} \qquad (197)
\end{aligned}
$$

We now consider simpler interference patterns.

Interference in 3+1D:

As seen previously, the substituted double-copy inner product reduces to an inner product (Equation 137). The interference pattern[19] is given as follows:

$$
\begin{aligned}
\det(\mathbf{u}+\mathbf{v}) &= \lfloor \mathbf{u}+\mathbf{v} \rfloor_{2,4} (\mathbf{u}+\mathbf{v}) && (198)\\
&= \lfloor \mathbf{u} \rfloor_{2,4} (\mathbf{u}+\mathbf{v}) + \lfloor \mathbf{v} \rfloor_{2,4} (\mathbf{u}+\mathbf{v}) && (199)\\
&= \lfloor \mathbf{u} \rfloor_{2,4} \mathbf{u} + \lfloor \mathbf{u} \rfloor_{2,4} \mathbf{v} + \lfloor \mathbf{v} \rfloor_{2,4} \mathbf{u} + \lfloor \mathbf{v} \rfloor_{2,4} \mathbf{v} && (200)\\
&= \det \mathbf{u} + \det \mathbf{v} + \lfloor \mathbf{u} \rfloor_{2,4} \mathbf{v} + \lfloor \mathbf{v} \rfloor_{2,4} \mathbf{u} && (201)
\end{aligned}
$$

Now replacing $\mathbf{u} = \rho_u e^{-\frac{1}{2}\mathbf{b}_u} e^{-\frac{1}{2}\mathbf{f}_u}$ and $\mathbf{v} = \rho_v e^{-\frac{1}{2}\mathbf{b}_v} e^{-\frac{1}{2}\mathbf{f}_v}$

$$
= |\rho_u|^2 + |\rho_v|^2 + \rho_u \rho_v \left(e^{\frac{1}{2}\mathbf{b}_u} e^{\frac{1}{2}\mathbf{f}_u} e^{-\frac{1}{2}\mathbf{b}_v} e^{-\frac{1}{2}\mathbf{f}_v} + e^{\frac{1}{2}\mathbf{b}_v} e^{\frac{1}{2}\mathbf{f}_v} e^{-\frac{1}{2}\mathbf{b}_u} e^{-\frac{1}{2}\mathbf{f}_u} \right) \qquad (202)
$$

Due to the presence of $\mathbf{f}$ and $\mathbf{b}$, the geometric richness of the interference pattern exceeds that of the 2D case. The term $\mathbf{f}$ associates with a non-commutative interference effect in the interference pattern, which distinguishes it from (the entirely commutative) complex interference and could presumably be identified experimentally in a properly constructed interference experiment.

4.21 The entropic flow of time and the problem of time

Finally, we elucidate the role of τ in the 2D and 4D cases.

We recall that in 0+1D, τ associated with the time t. We recall also that the Schrödinger equation was recovered by taking the derivative of the wavefunction with respect to t:

$$
i\hbar \frac{d}{dt} \left|\psi(t)\right\rangle = \mathbf{H} \left|\psi(t)\right\rangle \qquad (203)
$$

In both 2D and 4D, we can recover a Schrödinger-like equation also by deriving the wavefunction (with respect to τ).

First, let us do the 2D case.

A naive way to treat the dynamics would be to consider that τ constitutes a third dimension (2+1D). In this case, the 2D Schrödinger equation is

$$
\frac{\partial}{\partial \tau} \psi(x,y,\tau) = -\frac{1}{2} \mathbf{u}(x,y) \psi(x,y,\tau) \qquad (204)
$$

How are we to understand the dynamics?

Consider that in 0+1D, the non-relativistic Schrödinger equation generates in time rotations in the complex plane (i.e. $\exp it$ generates the U(1) group with $\exp E(q)$ as the magnitude) for the probability amplitude.

Likewise in 2D, τ generates a one-parameter group that causes the probability amplitude to cycle over the possible geometric configuration of the system in a manner that preserves the probabilities. The U(1) group is replaced with a one-parameter realization of the $\mathrm{GL}^+(2,\mathbb{R})$ group. As the quotient FX/SO(2) defines the tetrad field, this includes cycling over the possible metrics of the system, so long as they preserve the probabilities. This is completely analogous to how time cycles the probability amplitude over the U(1) group in non-relativistic quantum mechanics, except that the geometry the system cycles over is much richer.

The only problem with this naive story is that one has to introduce a third dimension to what should be 2D only. In 4D, we will not be able to add another time dimension to support τ, because spacetime is all there is to it.

Before we attack the 4D case, let us recall that the Hamiltonian in the non-relativistic Schrödinger (0+1D) equation can be made to depend on time. In this case, the equation is:

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \mathbf{H}(t)\left|\psi(t)\right\rangle \tag{205}$$

And its solution[20] is

$$\left|\psi(t)\right\rangle = \sum_n c_n \exp\left(-\frac{i}{\hbar}\int_0^t E_n(t')\,\mathrm{d}t'\right)|n,t\rangle \tag{206}$$

Let us now consider the 4D case. The corresponding Schrödinger equation would be

$$\frac{\partial}{\partial\tau}\psi(x,y,z,t,\tau) = -\frac{1}{4}\mathbf{u}(x,y,z,t)\psi(x,y,z,t,\tau) \tag{207}$$

But this would add an extra fifth dimension to spacetime, which is unwanted.

To resolve this, we begin by making a change of coordinate from $\psi(x,y,z,t)$ to $\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau)$, where τ is the proper time experienced by the observer. The equation becomes:

$$\frac{\partial}{\partial\tau}\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau) = -\frac{1}{4}\mathbf{u}(\tilde{x},\tilde{y},\tilde{z},\tau)\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau) \tag{208}$$

This form is very similar to the non-relativistic Schrödinger equation with a time-dependent Hamiltonian, as shown above. The solution will involve an integral over τ':

$$\tilde{\psi}(\tilde{x},\tilde{y},\tilde{z},\tau)=\sum_n c_n \exp\left(-\frac{1}{4}\int_0^\tau \mathbf{u}(\tilde{x},\tilde{y},\tilde{z},\tau')\,\mathrm{d}\tau'\right)\tilde{\psi}_n(\tilde{x},\tilde{y},\tilde{z},\tau) \tag{209}$$

A solution of this kind admits an arbitrary general linear geometry at every event of spacetime via $\mathbf{u}(\tilde{x},\tilde{y},\tilde{z},\tau)$. This symmetry, as shown before, can break into the FX/Spin^c(3,1) bundle, yielding a tetrad field. It can also support the Spin^c(3,1) geometry (and, in fact, anything else the general linear group supports). The evolution causes the probability amplitude of the wavefunction to cycle over the possible geometric configurations of the system as long as they preserve the probabilities.

This construction also allows for the definition of a time evolution, defined from the perspective of the observer and valid for both general relativity and quantum mechanics[21].

5 Conclusion

We have maximized the information associated with the receipt by the osberver of a message of measurement under the constraint of the general measurement pattern. The solution supports a geometry richer than what could previously be supported in either statistical physics or quantum mechanics alone. Accommodating all possible geometric measurements entails a geometric wavefunction, for which the Born rule is extended to the determinant. This substantially extends the opportunity to capture all fundamental physics within a single framework. The framework produces solutions for 2D and 4D in which general observables are normalizable. 4D stands out as the largest geometry that satisfies the conditions for having normalizable observables in the general case. A gravitized standard model results from the frame bundle FX of a world manifold, whose structure group is generated by $\exp\mathcal{G}(\mathbb{R}^{3,1})$ (which is group isomorphic to $\exp\mathbb{M}(4,\mathbb{R})$ and as such generates to $\mathrm{GL}^+(4,\mathbb{R})$ up to group isomorphism), undergoing symmetry breaking to $\mathrm{Spin}^c(3,1)$. The global sections of the quotient bundle $\mathrm{FX}/\mathrm{SO}(3,1)$ identify a pseudo-Riemannian metric. The connection is a Spin^c-preserving connection. The group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ is recovered in the broken symmetry and associates with the invariant transformations under the action of the wavefunction on a unit time-like vector of the tangent space, preserving the Dirac current. Finally, an interpretation of quantum mechanics, i.e., the optimization problem interpretation, is proposed; the structure of measurements acquires a foundational role, and the wavefunction is derived as a theorem. In this interpretation, it is considered that an observer receives or produces a message (theory of communication/Shannon entropy) of phase-invariant measurements, and the probability measure, maximizing the information of this message, is the geometric wavefunction accompanied by the geometric Born rule. The states of this wavefunction live in a geometric Hilbert space, which generalizes complex Hilbert space to arbitrary geometry. It is the only interpretation whose mathematical formulation is sufficiently precise to recover, by itself, the full machinery of quantum physics, proving interpretational completeness. Finally, as the solution to an optimization problem on information, we concluded that physics, distilled to its conceptually simplest expression, is the solution that provably makes realized measurements maximally informative to the observer. Equivalently, physics is the provable explanatory maximum for realized measurements.

6 Statements and Declarations

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